An Exhaustive (hopefully) list of Probability Distributions

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1 Basic Definitions

This section provides the basic definitions of various statistical quantities which a particular probability distribution can have.

1. **Probability Density/Mass Function** : A PDF(PMF, in case of disrete random variables) describes the distribution of the likelihood of the outcome of a Random Variable over it's sample space or support.

2. **Cumulative Distribution Function** : A CDF describes the probability of the outcome of a Random Variable falling less than a particular value.

   Consider A Continuous Random variable X, let it's PDF be denoted by $P(.)$, then it's CDF can be written as:

   $$F_X(x) = P(X \leq x)$$

   Some properties of CDF are:

   (a) $F_X(x)$ is non-decreasing and hence only has jump discontinuities.

   (b) $\lim_{x\to\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$

   (c) CDF is right continuous, which means that:

   $$\lim_{h\to 0} F_X(x+h) = F_X(x) \; \forall x \in \mathbb{R}$$

   (d) However, CDF is not left continous, but is related to left ϵ neighborhood by the following:

   $$\lim_{h\to 0} F_X(x-h) = F_X(x) - P(X = x) \; \forall x \in \mathbb{R}$$

3. **Expected Value** : This is the probability weighted average of a given random variable X. In physical sense, it represents the mean of a large number of independent results of the X.

   Mathematically, consider a Random Variable X whose support is the set S_X, then, two cases arise:

   (a) If X is Discrete, then:

   $$E\left[g(X)\right] = \sum_{x \in S_X} g(x) f_X(x)$$

provided,

$$\sum_{x \in S_X} g(x) f_X(x) < \infty$$

where $f_X(x)$ is the PMF of DRV X.

(b) If X is Continuous, then:

$$E\left[g(X)\right] = \int_{x \in S_X} g(x) f_X(x) dx$$

provided,

$$\int_{x \in S_X} g(x) f_X(x) dx < \infty$$

2 Probability Distributions

2.1 The Uniform Distribution

The distribution describes an experiment where there is an arbitrary outcome that lies between certain bounds a and b.

1. Notation : $U(a, b)$ or $Unif(a, b)$
2. Parameters : $-\infty < a < b < \infty$
3. Support : $x \in [a, b]$
4. Probability Distribution Function $f_X(.)$:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

5. Cumulative Distribution Function $F_X(.)$:

   For CDF, one needs to calculate cumulative probability coming up to this point, as shown below:-

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)dx$$
$$F_X(x) = \int_a^b \frac{1}{b-a}dx$$
$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x}{b-a}, & x \in [a,b] \\ 1, & x > b \end{cases}$$

6. Expected Value (μ): $\mathbb{E}_X(x) = \mu$ is the probability weighted average of all possible values of Random Variable X. It is calculated as shown below:-

$$\begin{aligned} \mathbb{E}_X(x) &= \int_{-\infty}^{\infty} x f_X(x)dx \\ &= \int_a^b x\frac{1}{b-a}dx \\ &= \frac{1}{b-a}\int_a^b xdx \\ &= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b \\ &= \frac{b^2-a^2}{2(b-a)} \\ &= \frac{a+b}{2} \end{aligned}$$

7. Variance (σ): This is just the Expected Value of $(X - \mathbb{E}(X))^2$.

$$\begin{aligned}
\sigma^2 &= \mathbb{E}\left((X - \mathbb{E}(X))^2\right) \\
&= \mathbb{E}\left(X^2 + \mathbb{E}(X)^2 - 2X\mathbb{E}(X)\right) \\
&= \mathbb{E}(X^2) + \mathbb{E}(\mathbb{E}(X)^2) - \mathbb{E}(2X\mathbb{E}(X)) \\
&= \frac{b^3 - a^3}{3(b-a)} + \frac{(a+b)^2}{4} - 2\frac{a+b}{2}\left(\frac{a+b}{2}\right) \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

8. Moment Generating Function ($M_X(t)$): This is the expected value of e^{tX}

$$\begin{aligned}
MGF(X) = \mathbb{E}\left(e^{tX}\right) &= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\
&= \frac{1}{b-a} \int_a^b e^{tx} dx \\
&= \frac{1}{b-a} \left[\frac{e^{tx}}{t}\right]_a^b \\
&= \frac{e^{tb} - e^{ta}}{t(b-a)}
\end{aligned}$$