The Facets of Geometry
Algebraic, Differential, Complex Analytic & Categorical

(Under heavy construction!!)

October 19, 2023

I follow here the footing of thy feet,
That with thy meaning, so I may the rather meet.

Todo list

Contents

Part I

The Algebraic Viewpoint

Chapter 1

Foundational Algebraic Geometry

Contents

1.1   A guiding example

Let X be a compact Hausdorff topological space. In this section we would like to portray the main point of scheme theory in the case of space X, that is, one can study the geometry over "base" space completely by studying the algebra of ring of suitable functions over it. In particular, we would like to establish the following result.

Proposition 1.1.0.1. *Let X be a compact Hausdorff topological space. Denote R to be the ring of continuous real-valued functions on X under pointwise addition and multiplication and denote $\mathrm{mSpec}\,(R)$ to be the set of maximal ideals of R. Then,*

1. We have a set bijection:

$$\mathrm{mSpec}\,(R) \cong X.$$

2. We have that $\mathrm{mSpec}\,(R)$ and X are isomorphic as topological spaces:

$$\mathrm{mSpec}\,(R) \cong X$$

where $\mathrm{mSpec}\,(R)$ is given its Zariski topology.

Proof. 1. Let $x \in X$ be an arbitrary point. Denote $\mathfrak{m}_x := \{f \in R \mid f(x) = 0\}$ to be the vanishing ideal of point x. This ideal is maximal because the quotient $R/\mathfrak{m}_x \cong \mathbb{R}$ via the map $f + \mathfrak{m}_x \mapsto f(x)$. Indeed, it is a valid ring homomorphism and is surjective by virtue of the continuous map constant at a point in $\mathbb{R}$. Moreover, if $f(x) = g(x)$ for $f, g \in R$, then $f - g \in \mathfrak{m}_x$ and hence $f + \mathfrak{m}_x = g + \mathfrak{m}_x$, so it is injective as well. Now consider the function:

$$\varphi : X \to \mathrm{mSpec}\,(R)$$
$$x \mapsto \mathfrak{m}_x.$$

We claim that φ is bijective. To see injectivity, suppose $\mathfrak{m}_x = \mathfrak{m}_y$ for $x, y \in X$. Then, we have that $R/\mathfrak{m}_x = R/\mathfrak{m}_y \cong \mathbb{R}$. This tells us that for each $f \in R$, $f(x) = f(y) \in \mathbb{R}$. Now assume that $x \neq y$. Since X is T_1, therefore $\{x\}, \{y\}$ are two disjoint closed subspaces of X. Then, by **Urysohn's lemma** (we have that X is compact Hausdorff), we get that there exists a continuous $\mathbb{R}$-valued function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(y) = 1$, a contradiction. Hence $x = y$.

Next, we wish to establish the surjectivity of φ. For that, take any maximal ideal $\mathfrak{m} \in \mathrm{mSpec}\,(R)$. We wish to show that there exists $x \in X$ such that $\mathfrak{m} = \mathfrak{m}_x$. Assume to the contrary that for all $x \in X$, there exists $f_x \in \mathfrak{m}$ such that $f_x \notin \mathfrak{m}_x$. Indeed getting a contradiction here will do the job by maximality of $\mathfrak{m}_x$. Now, let $U_x \subseteq X$ be the open set such that $f_x(y) \neq 0$ for each $y \in U_x$. We thus have an open cover $\{U_x\}$ of X and by compactness we get $U_{x_1}, \ldots, U_{x_n}$ covers X. Now, since X is compact Hausdorff, therefore there exists a **partition of unity** subordinate to $\{U_{x_i}\}$. That is, we have $g_i : X \to \mathbb{R}$ where $\mathrm{Supp}\,(g_i) \subseteq U_{x_i}$ such that $1 = g_i f_{x_i} + \cdots + g_n f_{x_n}$. Since $f_{x_i} \in \mathfrak{m}$ for each i, therefore $1 = g_i f_{x_i} + \cdots + g_n f_{x_n} \in \mathfrak{m}$, hence $\mathfrak{m} = R$, a contradiction.

2. _____ □

> Complete the pr
> of Theorem 1.1.
> Chapter 1.

Remark 1.1.0.2. An important corollary of the above result is that we can actually distinguish between the points of X by looking at maximal ideals of R; for $x, y \in X$, $x \neq y$ if and only if $\mathfrak{m}_x \neq \mathfrak{m}_y$. This is interesting because a fundamental goal of algebraic geometry is to study geometric properties of varieties over an algebraically closed field k and dominant maps between them. A fundamental equivalence tells that this is equivalent to studying the ring of regular functions over such a variety. Moreover, this ring recovers the important topology on the variety (there can be atleast two topologies on the variety if we are in, say $\mathbb{C}$). Hence one motivation to undergo this switch of viewpoint, where we try to do everything algebraically is that 1) we can completely recover the points of the variety and the relevant topology on it and that 2) we have a broad generalization of algebro-geometric techniques and constructions to an arbitrary commutative and unital ring R.

Caution 1.1.0.3. While in the sequel we will encounter spaces which are compact, it would rarely be the case that the spaces will be Hausdorff. However, if one notices the way Hausdorff property is used in the above result, then one can see that if we somehow makes sure that the space X constructed out of a ring R is such that every point of X can be "distinguished" by functions on X in R, then you don't need Hausdorff property. This is precisely what will happen.

1.2   Affine schemes and basic properties

Let us first swiftly give an account of basic global constructions in scheme theory. The foundational philosophy of scheme theory is to handle a space completely by the ring of globally defined *nice* functions on it. This is taken to an unprecedented extreme by the definition of an affine scheme, which tells us that one can even do geometry on the base space by the knowledge of globally defined functions on the base space alone; you can indeed *reconstruct* the base space! So, we begin with a general ring R and construct a topological space $\mathrm{Spec}\,(R)$. The way we will define its points is by thinking of each point of this base space $\mathrm{Spec}\,(R)$ as that subset of R, each of whose function becomes zero at a common point. One then sees that these are exactly the prime ideals of R. Hence, the base space $\mathrm{Spec}\,(R)$ is:

$$\mathrm{Spec}\,(R) := \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal of } R\}.$$

Next thing we wish to do is to actually get a *space* structure on this constructed base space, that is, a topology on $\mathrm{Spec}\,(R)$. This is, again, given with the help of the ring R. In particular, we give a topology on $\mathrm{Spec}\,(R)$ where every closed set is given by the zero locus of collections of functions $S \subseteq R$, that is, $V(S) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid \mathfrak{p} \supseteq S\} = \{x \in \mathrm{Spec}\,(R) \mid f(x) = 0 \; \forall f \in S\}$ where the last equation tells one how to think about the definition of $V(S)$. This is known as Zariski topology on $\mathrm{Spec}\,(R)$ and is defined by the following:

$$A \subseteq \mathrm{Spec}\,(R) \text{ is closed } \iff A = V(S) \text{ for some } S \subseteq R.$$

After defining the topology on $\mathrm{Spec}\,(R)$, one is interested in interested in understanding the set of all *germs* of functions at a point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. What are germs of functions at a point? Well, heuristically, they are all possible ways a function can *look* different at the given point. So for this, we have to atleast gather all those functions in R which takes different values at point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. Clearly this is given by the quotient domain $R/\mathfrak{p}$. Now from this, we construct the *residue field* of $\mathrm{Spec}\,(R)$ at point $\mathfrak{p}$, denoted $\kappa(\mathfrak{p}) := (R/\mathfrak{p})_{\langle 0 \rangle}$, that is, the fraction field of domain $R/\mathfrak{p}$. What does this $\kappa(\mathfrak{p})$ denotes geometrically? Well, it denotes the field of all different values a function can take at point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. Now, if that is the case, then one sees that if one takes any function $f \in R$, then "evaluating" f at $\mathfrak{p}$ should yield a point $f(\mathfrak{p})$ in $\kappa(\mathfrak{p})$. Indeed, we have the natural quotient maps:

$$R \to R/\mathfrak{p} \to \kappa(\mathfrak{p}).$$

So one should see

$\kappa(\mathfrak{p})$ *as the field of possible values that a function* $f \in R$ *can take at point* $\mathfrak{p}$.

However, we have not yet made the set of germs at a point $\mathfrak{p}$. The relation between two functions of having equal germs on R at a point $\mathfrak{p}$ is given by the heuristic that $f, g \in R$ should become equal in some open neighborhood around $\mathfrak{p}$. Since we have a topology on $\mathrm{Spec}\,(R)$, so one can actually do this formally. One will then see this that the set of all germs at point $\mathfrak{p}$ are actually all rational functions of R definable at $\mathfrak{p}$, that is, heuristically, f/g with $g(\mathfrak{p}) \neq 0$ for $f, g \in R$. This in our language turns out to be all the symbols of the form f/g with $g \notin \mathfrak{p}$. This is exactly the local ring $R_\mathfrak{p}$, the localization of the ring R (seen as ring of functions over $\mathrm{Spec}\,(R)$) at the point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. So

$$germs\ of\ functions\ of\ R\ at\ \mathfrak{p}\ is\ R_\mathfrak{p}.$$

We will expand more on this when we will talk about the structure sheaf of $\mathrm{Spec}\,(R)$.

Let us now see a basic but important dictionary between the topology of space $\mathrm{Spec}\,(R)$ and the algebra of ideals of R:

Lemma 1.2.0.1. *Let R be a ring. We then have the following:*
1. *If $\mathfrak{a}$, $\mathfrak{b}$ are two ideals of R, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*
2. *If $\{\mathfrak{a}_n\}$ is a collection of ideals of R, then $V\left(\sum_n \mathfrak{a}_n\right) = \bigcap_n V(\mathfrak{a}_n)$.*
3. *If $\mathfrak{a}, \mathfrak{b}$ are two ideals of R, then $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{a} \supseteq \sqrt{b}$.*

Proof. 1. First, let us see that $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Take any $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. Suppose $p \notin V(\mathfrak{a})$ and $\mathfrak{p} \notin V(\mathfrak{b})$. Then there exists $f \in \mathfrak{a}$, $g \in \mathfrak{b}$ such that $fg \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Thus, $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$, a contradiction in both cases. Second, it is easy to see that $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ as if either $p \supseteq \mathfrak{a}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, then since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}$, therefore $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$.
2. Let $\mathfrak{p} \supseteq \sum_n \mathfrak{a}_n$. Since ideals are abelian groups so the sum contains each $\mathfrak{a}_n$, hence $\mathfrak{p} \supseteq \mathfrak{a}_n$ for each n, and so $\mathfrak{p} \in \bigcap_n V(\mathfrak{a}_n)$. Conversely, if $\mathfrak{p} \supseteq \mathfrak{a}_n$ for each n, then $\mathfrak{p} = \sum_n \mathfrak{p} \supseteq \sum_n \mathfrak{a}_n$.
3. (L $\implies$ R) Since each prime ideal containing $\mathfrak{a}$ also contains $\mathfrak{b}$, therefore the intersection of all prime ideals containing $\mathfrak{a}$ will contain the intersection of all prime ideals containing $\mathfrak{b}$. (R $\implies$ L) Take any prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$. Since $\sqrt{a} \supseteq \sqrt{b}$, therefore $\mathfrak{p} \supseteq \mathfrak{b}$. $\qquad \square$

1.2.1   Topological properties of $\mathrm{Spec}\,(R)$

Let us begin by an algebraic characterization of irreducible closed subspaces of $\mathrm{Spec}\,(R)$.

Lemma 1.2.1.1. *Let R be a ring and $X \hookrightarrow \mathrm{Spec}\,(R)$ be a closed subspace. Then the following are equivalent:*
1. *X is irreducible.*
2. *There is a unique point $\mathfrak{p} \in \mathrm{Spec}\,(R)$ such that $X = V(\mathfrak{p})$.*
One calls the point $\mathfrak{p}$ the generic point of the irreducible closed subspace X[1].

Proof. (1. $\Rightarrow$ 2.) Since X is closed therefore $X = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R. If we assume that $X \neq V(\mathfrak{p})$ for each prime $\mathfrak{p} \subseteq R$, then this holds true for points $\mathfrak{p} \in X$ as well. Hence take $\mathfrak{p} \in X$ and consider the proper closed subset $V(\mathfrak{p}) \subsetneq X$. Let $\mathfrak{q} \notin V(\mathfrak{p})$. Then, $V(\mathfrak{q}) \subsetneq X$ as well. Hence we get that $V(\mathfrak{p}) \cup V(\mathfrak{q}) = V(\mathfrak{a})$, which stands in contradiction to the fact that X is irreducible. Hence there exists a prime $\mathfrak{p} \in \mathrm{Spec}\,(R)$ such that $X = V(\mathfrak{p})$.

---

[1]Such spaces where every irreducible closed set has a unique generic point are called sober spaces.

Uniqueness is quite clear.

(2. $\Rightarrow$ 1.) Suppose $Y = V(\mathfrak{a})$ and $Z = V(\mathfrak{b})$ are two closed subspaces of $X = V(\mathfrak{p})$ such that $X = Y \cup Z = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$ (Lemma 1.2.0.1). Assume that Y, Z are proper inside X. Then, there are two points $\mathfrak{q}_1 \in Y \setminus Z$ and $\mathfrak{q}_2 \in Z \setminus Y$. Algebraically, this is equivalent to saying that $\mathfrak{q}_1 \supseteq \mathfrak{a}$, $\mathfrak{q}_1 \not\supseteq \mathfrak{b}$ and $\mathfrak{q}_2 \supseteq \mathfrak{b}$, $\mathfrak{q}_2 \not\supseteq \mathfrak{a}$. It follows that $\mathfrak{q}_1 \cap \mathfrak{q}_2$ is also a prime ideal which contains $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. Since $X = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{p}) \ni \mathfrak{p}$, hence it follows that $\mathfrak{q}_1 \cap \mathfrak{q}_2 \supseteq \mathfrak{p}$ as it already contains $\mathfrak{a}\mathfrak{b}$. Thus $\mathfrak{q}_1 \cap \mathfrak{q}_2 \in V(\mathfrak{a}) \cap V(\mathfrak{b}) \subseteq V(\mathfrak{p})$. Since $V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b})$, hence it follows that $\mathfrak{q}_1 \cap \mathfrak{q}_2 \supseteq \mathfrak{a}, \mathfrak{b}$, which implies in particular that $\mathfrak{q}_1 \supseteq \mathfrak{a}, \mathfrak{b}$, a contradiction. $\qquad\square$

Remark 1.2.1.2. The main idea of the above proof has been to first translate the topological condition to algebraic, and then using the critical observation that the closed subspace $V(\mathfrak{p})$ contains point $\mathfrak{p}$ itself.

A simple corollary of above gives all closed points of an affine scheme.

Lemma 1.2.1.3. *Let R be a ring. Then*

$$\{\text{Closed points of } \operatorname{Spec}(R)\} \cong \{\text{Maximal ideals of } R.\}$$

Proof. Follows immediately from Lemma 1.2.1.1. $\qquad\square$

Let us next observe a simple but important observation about topology of $\operatorname{Spec}(R)$.

Lemma 1.2.1.4. *Let R be a ring. For $f \in R$, define $\operatorname{Spec}(R)_f := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \notin \mathfrak{p}\}$. Then,*

1. *$\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)$ is an open set and such open sets form a basis of the Zariski topology on $\operatorname{Spec}(R)$.*
2. *$\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)_g$ if and only if $f \in \sqrt{Rg}$.*

Proof. 1. Clearly $\operatorname{Spec}(R)_f = \operatorname{Spec}(R) \setminus V(f)$ where we know that $V(f) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \in \mathfrak{p}\}$. Hence X_f is open. It is also clear that if $U \subseteq \operatorname{Spec}(R)$ is open, then $\operatorname{Spec}(R) \setminus U = V(\mathfrak{a})$ is closed and hence $U = \bigcup_{f \in \mathfrak{a}} \operatorname{Spec}(R)_f$. Further, $\operatorname{Spec}(R) = \operatorname{Spec}(R)_1$ and $\emptyset = \operatorname{Spec}(R)_0$.

2. This follows from the following equivalences. Let $\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)_g$, then we get the following (we implicitly use Hilbert Nullstellensatz)

$$\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)_g \iff f(\mathfrak{p}) \neq 0 \implies g(\mathfrak{p}) \neq 0 \iff g(\mathfrak{p}) = 0 \implies f(\mathfrak{p}) = 0 \iff V(g) \subseteq V(f)$$
$$\iff \sqrt{Rg} \supseteq \sqrt{Rf} \supseteq Rf \iff f \in \sqrt{Rg}.$$

This completes the proof. $\qquad\square$

Next we observe the equivalent formulation of partitions of unity in the context of algebra.

Lemma 1.2.1.5. *Let R be a ring. Then,*

1. If $U \hookrightarrow \operatorname{Spec}(R)$ is any open set given by $U = \bigcup_{f \in S} \operatorname{Spec}(R)_f$ for some subset $S \subseteq R$, then

$$\operatorname{Spec}(R) \setminus U = V\left(\sum_{f \in S} Rf\right).$$

2. $\operatorname{Spec}(R) = \bigcup_{f \in S} \operatorname{Spec}(R)_f$ for some $S \subseteq R$ if and only if the ideal of R generated by S is the whole of R.

Proof. 1. Let $U \hookrightarrow \operatorname{Spec}(R)$ be an open set. Then, $\mathfrak{p} \in \operatorname{Spec}(R) \setminus U \iff \mathfrak{p} \notin U \iff \forall f \in S, \ \mathfrak{p} \notin \operatorname{Spec}(R)_f \iff \forall f \in S, \ f \in \mathfrak{p} \iff \mathfrak{p} \supseteq S \iff \mathfrak{p} \in V(S)$.
2. Follows from 1. $\qquad \square$

We next have an interesting observation that $\operatorname{Spec}(R)$ are always quasicompact[2].

Lemma 1.2.1.6. *Let R be a ring. Then $\operatorname{Spec}(R)$ is quasicompact.*

Proof. Take any arbitrary basic open cover $\bigcup_{f \in S} \operatorname{Spec}(R)_f$ for some $S \subseteq R$. Then by Lemma 1.2.1.5, 2, we get that $\sum_{f \in S} Rf \ni 1$ and hence there are $f_1, \ldots, f_n \in S$ such that $g_1 f_1 + \ldots g_n f_n = 1$ for some $g_i \in R$. Hence $\operatorname{Spec}(R) \setminus \bigcup_{i=1}^{n} = V(f_1, \ldots, f_n) = V(R) = \emptyset$. $\quad \square$

Next, we see the topological effects on space $\operatorname{Spec}(R)$ of Noetherian hypothesis on ring R. In particular, we see that the space $\operatorname{Spec}(R)$ itself becomes *noetherian topological space*, that is, it's closed sets satisfies descending chain condition.

Lemma 1.2.1.7. *Let R be a ring. If R is noetherian, then $\operatorname{Spec}(R)$ is noetherian.*

Proof. Use $V(-)$ and $I(-)$, where $I(Y) = \{f \in R \mid f \in \mathfrak{p} \forall \mathfrak{p} \in Y\}$. Rest is trivial. $\qquad \square$

We next discuss few things about the irreducible subsets of a closed set of $\operatorname{Spec}(R)$. Let $F \hookrightarrow \operatorname{Spec}(R)$ be a closed subset. Then we can contemplate irreducible subsets of F. Clearly, each irreducible subset has to be in a maximal irreducible subset, which are called *irreducible components* of $\operatorname{Spec}(R)$. We have few basic observations about irreducible components

Lemma 1.2.1.8. *Let R be a ring and F be a closed subset of $\operatorname{Spec}(R)$. Then,*
1. *Each irreducible component of F is closed.*
2. *There are only finitely many irreducible components of F.*
3.

$$\{\textit{Irreducible components of } \operatorname{Spec}(R)\} \cong \{\textit{Closed sets } V(\mathfrak{p}), \ \mathfrak{p} \textit{ is minimal prime}\}.$$

Proof. Statements 1. follows from Lemma 1.2.1.1. Statement 2. follows from Lemma 1.2.1.6. For 3. we see that if there exists a prime $\mathfrak{p}' \subsetneq \mathfrak{p}$, then $V(\mathfrak{p}') \supsetneq V(\mathfrak{p})$. But by Lemma 1.2.1.1, $V(\mathfrak{p})$ is no longer an irreducible component, a contradiction. $\qquad \square$

---

[2] it is customary in algebraic geometry to call the topological compactness as quasi-compactness; compactness in algebraic geometry historically means Hausdorff *and* topological compactness.

Note that we are already in a position to prove some algebraic statements using topological arguments, as the following lemma shows.

Lemma 1.2.1.9. *Let A be a ring and let $a_1, \ldots, a_n \in A$ generate the unit ideal in A. Then for all $m > 0$, the collection $a_1^m, \ldots, a_n^m \in A$ also generates the unit ideal in A.*

Proof. From Lemma 1.2.1.6, 2, it follows that $\{D(a_i)\}_{i=1,\ldots,n}$ covers $\operatorname{Spec}(A)$. Since for any $a \in A$, the basic open $D(a) \subseteq \operatorname{Spec}(A)$ is equal to $D(a^m)$ as a prime $\mathfrak{p}$ doesn't contain a if and only if it doesn't contain any of its power. Consequently, we get that $\{D(a_i^m)\}_{i=1,\ldots,n}$ also forms a basic open cover of $\operatorname{Spec}(A)$. An application of Lemma 1.2.1.6, 2 again proves the result. $\square$

1.2.2   The structure sheaf $\mathcal{O}_{\operatorname{Spec}(R)}$

The next important thing we want to consider on $\operatorname{Spec}(R)$ is a sheaf of suitable nice functions over it. This sheaf will be of utmost importance as it will not be treated as an additional structure, but will be an integral part (in-fact, the most important part) of the definition of an affine scheme.

The question now is, *what are* nice functions over $\operatorname{Spec}(R)$ whose sheaf we should take. We turn to classical algebraic varieties for that (one may skip the following if he/she find himself/herself to be brave enough to face the abstraction of the structure sheaf). See Section 1.5 for more details.

Example 1.2.2.1. (*Structure sheaf of an algebraic variety*) Let k be an algebraically closed field. An important aspect of varieties is their morphism. We will display this only in the affine case. Let X, Y be two affine varieties. To define a morphism between X and Y, we would first need to understand the notion of *regular functions* over any variety X. A function $\varphi : X \to k$ is said to be regular if it is locally rational. That is, for each $p \in X$, there exists an open set $U \ni p$ of X and there exists two polynomials $f, g \in k[x_1, \ldots, x_n]$ such that $g(q) \neq 0 \forall q \in U$ and $\varphi|_U = f/g$. It then follows that a regular function is continuous when X and k are equipped with its Zariski topology (Lemma 3.1, [**??**] [Hartshorne]). We now define morphism of affine varieties.

A function $\varphi : X \to Y$ is said to be a morphism of varieties if
1. $\varphi : X \to Y$ is continuous,
2. for each open set $V \subseteq Y$ and a regular map $f : V \to k$, the map $f \circ \varphi$ as below

is also a regular map.

Hence the main part of the data of a variety is the locally defined regular maps. This is what we will take as our motivation in defining the structure sheaf over $\operatorname{Spec}(R)$, as this example tells us to take care of these local functions to the base field. A question that may

arise from this discussion is how are we going to define a regular map from an open set $U \hookrightarrow \mathrm{Spec}\,(R)$ when we don't even have a field. The answer is, as we discussed previously, to work with residue field at a point instead.

We now start to define the structure sheaf of $\mathrm{Spec}\,(R)$. First, let us give the following lemma, which reduces the burden of construction only to basis elements of $\mathrm{Spec}\,(R)$.

Lemma 1.2.2.2. *Let X be a topological space and $\mathcal{B}$ be a basis. Let F be an assignment over sets of $\mathcal{B}$ which satisfies sheaf conditions for it. Then, F extends to a sheaf $\mathcal{F}$ over X.*

Proof. The main observation here is that we can find the stalk of $\mathcal{F}$ at each point x just by the knowledge of F, because of the basis $\mathcal{B}$. Take any point $x \in X$. We see that we can get the stalk $\mathcal{F}_x$ as follows:

$$\mathcal{F}_x := \varinjlim_{x \in B \in \mathcal{B}} F(B).$$

Once we have the stalks, we can define the sections of $\mathcal{F}$ quite easily as follows. Let $U \subseteq X$ be an open set. Then $\mathcal{F}(U)$ is defined to be the subset of $\prod_{x \in U} \mathcal{F}_x$ of those elements (s_x) where there exists a basic open cover $\{B_i\}$ of U and there exists elements $s_i \in F(B_i)$ such that $s_x = (s_i)_x$ for each $x \in B_i$. One can check that this satisfies the conditions of a sheaf. $\square$

Construction 1.2.2.3. (*The $\mathcal{O}_{\mathrm{Spec}(R)}$*) Let R be a ring. By virtue of Lemma 1.2.2.2, we will define $\mathcal{O}_{\mathrm{Spec}(R)}$ only on basic open sets of the form $\mathrm{Spec}\,(R)_f$. Let $X := \mathrm{Spec}\,(R)$. Motivated by Example 1.2.2.1, take a basic open set $X_f \hookrightarrow X$ for some $f \in R$ and then we wish to consider *rational functions* over X_f. This means those functions of the form g/h for $g, h \in R$ such that $h(\mathfrak{p}) \neq 0 \forall \mathfrak{p} \in X_f$. This is equivalent to demanding that $h \notin \mathfrak{p} \forall \mathfrak{p} \in X_f$, that is, $X_h \supseteq X_f$. This is again equivalent to stating that $f \in \sqrt{Rh}$ by Lemma 1.2.1.4, 2. Hence $f^n = ah$ for some $n \in \mathbb{N}$ and $a \in R$. Thus, we see that the notion of rational functions over X_f is equivalent to all functions of the form g/f^n where $g \in R$ and $n \in \mathbb{N}$. Commutative algebra has an apt name for this, that is, the localization of R at f denoted by $R_f := \{a/f^n \mid a \in R, \ n \in \mathbb{N}\}$ which is again a ring by natural operation on fractions (see Special Topics, **??**). Thus, we should define the sections over X_f as:

$$\mathcal{O}_X(X_f) := R_f.$$

We would not verify the sheaf axioms here as it is a tedious but straightforward calculation. The sheaf $\mathcal{O}_X$ thus formed is called the structure sheaf on the space X. One should think of the sheaf $\mathcal{O}_X$ as natural as the ring R itself. In particular we will see in the next section that it indeed is the case.

Next, we would like to see the stalks of this sheaf $\mathcal{O}_X$. To understand this, we would have to understand the maps on sections induced by $X_f \hookrightarrow X_g$. As we saw earlier, this is equivalent to stating that $f^n = ag$ for some $n \in \mathbb{N}$ and $a \in R$. Hence, the induced map on sections are the restriction maps of the sheaf and is given by

$$\rho_{X_g, X_f} : R_g = \mathcal{O}_X(X_g) \longrightarrow \mathcal{O}_X(X_f) = R_f$$
$$b/g^m \longmapsto ba^m/a^m g^m = ba^m/f^{nm}.$$

We are now ready to calculate the stalk. Take any point $x \in X$. The stalk becomes:

$$
\begin{aligned}
\mathcal{O}_{X,x} &:= \varinjlim_{x \in X_f} \mathcal{O}_X(X_f) \\
&= \varinjlim_{x \in X_f} R_f \\
&= \varinjlim_{f \notin x} R_f \\
&= R_x
\end{aligned}
$$

where the last equality follows from a small colimit calculation (which should really be thought of as a definition). Hence $\mathcal{O}_X$ is a sheaf whose stalks are local rings. So we have a complete description of the sheaf $\mathcal{O}_X$ when $X = \mathrm{Spec}\,(R)$.

We finally define an affine scheme.

Definition 1.2.2.4. (**Affine scheme**) Let R be a ring. Then the pair $(\mathrm{Spec}\,(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is called an affine scheme.

Remark 1.2.2.5. (*Evaluation of functions*) Let $(\mathrm{Spec}\,(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. As noted earlier, we now see how all rational functions over $\mathrm{Spec}\,(R)$ are exactly the elements of R. In particular, since $\Gamma(\mathrm{Spec}\,(R), \mathcal{O}_{\mathrm{Spec}(R)}) = R_1 = R$. Hence if we interpret $\mathcal{O}_{\mathrm{Spec}(R)}$ as the sheaf of regular maps over $\mathrm{Spec}\,(R)$, then R itself appears as the globally defined regular maps.

Now take global map $f \in R$ and any point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. We can "evaluate" f at $\mathfrak{p}$ via the following composite (note that $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \cong (R_{\mathfrak{p}})_0$, the last one is the fraction field of $R_{\mathfrak{p}}$ obtained by localizing at 0 ideal):

$$
\Gamma(\mathrm{Spec}\,(R), \mathcal{O}_{\mathrm{Spec}(R)}) \longrightarrow \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}} \longrightarrow \kappa(\mathfrak{p})
$$

where the first map on the left is the inclusion into the direct limit and the map on right is the natural quotient map. Algebraically, we have the following maps

$$
R \longrightarrow R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}
$$

given by

$$
f \longmapsto \frac{f}{1} \longmapsto \frac{f}{1} + \mathfrak{p}R_{\mathfrak{p}},
$$

where $f/1 + \mathfrak{p}R_{\mathfrak{p}}$ denotes the class of all those functions in the stalk $\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}} = R_{\mathfrak{p}}$ which takes same value at $\mathfrak{p}$ as f does.

For completeness' sake, we give a description of the section of the sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ on any open set $U \subseteq \mathrm{Spec}\,(R)$.

Lemma 1.2.2.6. *Let R be a ring and $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ the associated affine scheme. Let $U \subseteq \operatorname{Spec}(R) =: X$ be an open set. Then,*

$$\mathcal{O}_X(U) = \left\{ (s_{\mathfrak{p}}) \in \prod_{\mathfrak{p} \in U} R_{\mathfrak{p}} \mid \forall \mathfrak{p} \in U, \exists \text{ basic open } X_g \ni \mathfrak{p} \ \& \ f/g^n \in R_g \ \text{s.t.} \ s_{\mathfrak{q}} = f/g^n \forall \mathfrak{q} \in X_g \right\}.$$

More concretely, we have

$$\mathcal{O}_X(U) = \left\{ s: U \to \coprod_{\mathfrak{p} \in U} R_{\mathfrak{p}} \mid \forall \mathfrak{p} \in U, \ s(\mathfrak{p}) \in R_{\mathfrak{p}} \ \& \ \exists \text{ open } \mathfrak{p} \in V \subseteq U \ \& \ f, g \in R \ \text{s.t.} \ \forall \mathfrak{q} \in V, \ g \notin \mathfrak{q} \ \& \ s(\mathfrak{q}) = f/g \right\}.$$

Proof. Follows from Lemma 1.2.2.2 and Construction 1.2.2.3. □

Ring morphisms and $\operatorname{Spec}(-)$

We now discuss some properties of ring morphisms and the associated map of affine schemes.

Lemma 1.2.2.7. [3] *Let A be a ring and $f \in A$. Then, $D(f) \subseteq \operatorname{Spec}(A)$ is empty if and only if f is nilpotent.*

Proof. Both sides follow immediately from the Lemma 18.1.2.9. □

We further obtain the following two results which corresponds to what happens on the level of sheaves.

Proposition 1.2.2.8. [4] *Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ be two affine schemes and $\varphi : A \to B$ be a morphism of rings.*
   1. *The ring map $\varphi : A \to B$ is injective if and only if the corresponding map of schemes $f : Y \to X$ yields injective map of structure sheaves, that is, $f^{\flat} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective.*
   2. *If $\varphi : A \to B$ is injective, then $f : Y \to X$ is dominant[5].*
   3. *The ring map $\varphi : A \to B$ is surjective if and only if the corresponding map of schemes $f : Y \to X$ is a closed immersion.*

Proof. 1. (L $\Rightarrow$ R) It suffices to show that $f^{\flat}$ is an injective map over basic opens of X. Pick any $g \in A$ and consider the basic open $D(g) \subseteq X$. We wish to show that the map

$$f^{\flat}_{D(g)} : \mathcal{O}_X(D(g)) \longrightarrow \mathcal{O}_Y(f^{-1}(D(g)))$$

is an injective homomorphism. Indeed, we first observe that $\mathcal{O}_X(D(g)) \cong A_g$ and $f^{-1}(D(g)) = D(\varphi(g))$, so that $\mathcal{O}_Y(D(\varphi(g))) \cong B_{\varphi(g)}$. It follows that the map $f^{\flat}_{D(g)} : A_g \to B_{\varphi(g)}$ is the localization map

$$\varphi_g : A_g \longrightarrow B_{\varphi(g)}$$
$$\frac{a}{g^n} \longmapsto \frac{\varphi(a)}{\varphi(g)^n}.$$

---

[3]Exercise II.2.18, a of Hartshorne.
[4]Exercise II.2.18 b,c,d of Hartshorne.
[5]that is, f has dense image.

We wish to show that the above map is injective. If $\varphi(a)/\varphi(g)^n = 0$, then for some $k \in \mathbb{N}$ we have $\varphi(g)^k \varphi(a) = 0$. It follows by injectivity of φ that $g^k a = 0$ in A. Consequently, we can write

$$\frac{a}{g^n} = \frac{ag^k}{g^{n+k}} = 0.$$

(R $\Rightarrow$ L) As a sheaf map is injective if and only if the kernel sheaf is zero (Theorem 22.2.0.7), where the latter is equivalent to the fact that every map on sections is injective. Consequently, over X, we get

$$f_X^\flat : \Gamma(\mathcal{O}_X, X) \longrightarrow \Gamma(\mathcal{O}_Y, Y)$$

Since $\Gamma(\mathcal{O}_X, X) \cong A$ and $\Gamma(\mathcal{O}_Y, Y) \cong B$, and the map $f_X^\flat : A \to B$ is just φ itself, therefore we are done.

2. We wish to show that for any basic non-empty open $D(g) \subseteq X$ for $g \in A$, the intersection $D(g) \cap f(Y)$ is non-empty. We have the following equalities:

$$\begin{aligned}
D(g) \cap f(Y) &= \{\mathfrak{p} \in X \mid \mathfrak{p} \in f(Y) \ \& \ g \notin \mathfrak{p}\} \\
&= \{\varphi^{-1}(\mathfrak{q}) \in X \mid \mathfrak{q} \in Y, \ g \notin \varphi^{-1}(\mathfrak{q})\} \\
&= \{\varphi^{-1}(\mathfrak{q}) \in X \mid \mathfrak{q} \in Y, \ \varphi(g) \notin \mathfrak{q}\} \\
&= f(D(\varphi(g))).
\end{aligned}$$

Conequently, $D(g) \cap f(Y)$ is non-empty if and only if $D(\varphi(g))$ is non-empty, which in turn implies by Lemma 1.2.2.7 that $D(g) \cap f(Y)$ is non-empty if and only if $\varphi(g)$ is not nilpotent. As g is not nilpotent because $D(g)$ is not empty, therefore $\varphi(g)$ is not nilpotent as φ is injective.

3. (L $\Rightarrow$ R) Let $\varphi : A \to B$ be surjective and $I \leq A$ be the kernel. We wish to show that $f : Y \to X$ is a closed immersion. For that, we first need to show that f is a topological closed immersion, that is its image is closed and is homeomorphic to it. We claim that $f(Y) = V(I) \subseteq X$. Indeed, for any $\varphi^{-1}(\mathfrak{q}) \in f(Y)$, we have that $I \subseteq \varphi^{-1}(\mathfrak{q})$. Thus, $f(Y) \subseteq V(I)$. Conversely, for any $\mathfrak{p} \in V(I)$, as φ is surjective and $\mathfrak{p}$ contains I, therefore $\varphi(\mathfrak{p}) \in Y$ is a prime ideal such that $\varphi^{-1}(\varphi(\mathfrak{p})) = \mathfrak{p}$, so that $\mathfrak{q} = \varphi(\mathfrak{p}) \in Y$ is such that $f(\varphi(\mathfrak{p})) = \mathfrak{p}$, hence $\mathfrak{p} \in f(Y)$.

Next, we wish to show that f is homeomorphic to its image. It suffices to show that $f : Y \to f(Y)$ is a closed mapping. But this is immediate by the fact that a surjective map $\varphi : A \to B$ with kernel I induces an order preserving isomorphism of ideals of A containing I and ideals of B by mapping ideals of B to those of A containing I via φ^{-1}. Alternatively, one can see that $A/I \cong B$ and $\mathrm{Spec}\,(A/I) \cong V(I) = f(Y)$, therefore application of $\mathrm{Spec}\,(-)$ functor would do the job.

Next, we wish to show that $f^\flat : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective. We can check this on a basis of X. Let $D(g) \subseteq X$ for some $g \in A$. Indeed, for $t \in (f_* \mathcal{O}_Y)(D(g)) = \mathcal{O}_Y(D(\varphi(g)) \cong B_{\varphi(g)}$, we wish to find an open covering of $D(g)$ say U_i and $s_i \in \mathcal{O}_X(U_i)$ such that $f_{U_i}^\flat(s_i) = t|_{U_i}$ for each i. Indeed, the open set $D(g)$ as its own covering will suffice here as $\mathcal{O}_X(D(g)) \cong A_g$

and the map $f^\flat_{D(g)} = \varphi_g : A_g \to B_{\varphi(g)}$. As φ is surjective, therefore for $t = b/\varphi(g)^n \in B_{\varphi(g)}$, we obtain $a \in A$ such that $\varphi(a) = b$ and thus a/g^n is mapped by φ_g to $b/\varphi(g)^n$, as required.

$(\text{R} \Rightarrow \text{L})$ Let $f : Y \to X$ be a closed immersion. We wish to show that $\varphi : A \to B$ is surjective. Pick $b \in B$. We wish to show that there exists $a \in A$ such that $\varphi(a) = b$. As the sheaf map $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, therefore there exists a basic open covering (which will be finite by quasi-compactness of affine schemes, Lemma 1.2.1.6) namely $\{D(a_i)\}_{i=1,\dots,n}$ of X together with sections $s_i \in \mathcal{O}_X(D(a_i))$ such that $f^\flat_{D(a_i)}(s_i) \in \mathcal{O}_Y(f^{-1}(D(a_i)))$ is the restriction of $b \in \Gamma(\mathcal{O}_Y, Y)$ to $D(\varphi(a_i))$, namely $\rho_{X,D(\varphi(a_i))}(b)$. As we have $\mathcal{O}_X(D(a_i)) \cong A_{a_i}$, $\mathcal{O}_Y(f^{-1}(D(a_i))) = \mathcal{O}_Y(D(\varphi(a_i))) \cong B_{\varphi(a_i)}$ and that the restriction $\rho_{Y,D(\varphi(a_i))} : \Gamma(\mathcal{O}_X, X) \to \mathcal{O}_X(D(a_i))$ is just the natural localization map $A \to A_{a_i}$, therefore we may identify $s_i = \frac{c_i}{a_i^{k_i}} \in A_{a_i}$ and $\rho_{X,D(a_i)}(b) = \frac{b}{1} \in B_{\varphi(a_i)}$. Consequently, we have for each $i = 1, \dots, n$ the following equation in $B_{\varphi(a_i)}$

$$\frac{b}{1} = \frac{\varphi(c_i)}{\varphi(a_i)^{k_i}}.$$

It follows that we obtain an equation of the form

$$\varphi(a_i^{m_i})b = \varphi(c_i a^{l_i})$$

for some $m_i, l_i \geq 0$. Taking $M = \max_i m_i$, we obtain

$$\varphi(a_i^m)b = \varphi(d_i) \tag{$*$}$$

for some $d_i \in A$.

, 2, the collection $\{a_i\}_{i=1,\dots,n}$ generates the unit ideal in A. By Lemma 1.2.1.9, it follows that the collection $\{a_i^m\}_{i=1,\dots,n}$ also generates the unit ideal in A. Consequently, we have $r_1 a_1^m + \cdots + r_n a_n^m = 1$ for some $r_i \in A$. Using this in $(*)$, we yield

$$b = \varphi\left(\sum_{i=1}^n r_i d_i\right),$$

as required[6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

1.2.3 $\mathcal{O}_{\text{Spec}(R)}$-modules

As we pointed out in Construction 1.2.2.3, the structure sheaf $\mathcal{O}_{\text{Spec}(R)}$ should really be thought of as natural as the ring R itself. This way of thought will be justified in this section, where we will see that, just like we can understand a ring by understanding the category of R-modules, we can understand the structure sheaf $\mathcal{O}_{\text{Spec}(R)}$ by understanding the category of soon to be constructed $\mathcal{O}_{\text{Spec}(R)}$-modules.

Let R be a ring and M be an R-module. Just like we underwent a "geometrification" to go from ring R (algebra) to the locally ringed space $\text{Spec}(R)$ (geometry), we will also "geometrify" the notion of an R-module. This will yield us a sheaf $\widetilde{M}$ over $\text{Spec}(R)$.

---

[6]Note that in the whole proof, we didn't even required the fact that $f : Y \to X$ is also a topological closed immersion!

Definition 1.2.3.1. $(\widetilde{M})$ Let R be a ring and M be an R-module. The following presheaf on $X := \operatorname{Spec}(R)$ generated by the following definition on basic opens

$$X_f \longmapsto \widetilde{M}(X_f) := M_f = M \otimes_R R_f$$

and restrictions given by

$$(X_f \hookrightarrow X_g) \longmapsto M \otimes_R R_g \overset{\operatorname{id} \otimes \rho_{X_g, X_f}}{\to} M \otimes_R R_f$$

defines a unique sheaf on $\operatorname{Spec}(R)$ corresponding to R-module M denoted $\widetilde{M}$.

The above construction gives the sheaf $\widetilde{M}$ over R a structure of an $\mathcal{O}_{\operatorname{Spec}(R)}$-module, that is, a sheaf $\mathcal{F}$ of abelian groups where for each open $U \subseteq \operatorname{Spec}(R)$ the group $\mathcal{F}(U)$ is a $\mathcal{O}_{\operatorname{Spec}(R)}(U)$-module. Since $\widetilde{M}(X_f) = M \otimes_R R_f$ is an $\mathcal{O}_X(X_f) = R_f$-module, therefore $\widetilde{M}$ are basic examples of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules.

A map $\eta : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules is just a sheaf morphism where for each inclusion $U \hookrightarrow V$ of $\operatorname{Spec}(R)$, we get that the following commutes

$$
\begin{array}{ccc}
\mathcal{F}(V) & \overset{\eta_V}{\longrightarrow} & \mathcal{G}(V) \\
\downarrow & & \downarrow \\
\mathcal{F}(U) & \underset{\eta_U}{\longrightarrow} & \mathcal{G}(U)
\end{array}
$$

where the top horizontal map is a $\mathcal{O}_{\operatorname{Spec}(R)}(V)$-module homomorphism, bottom horizontal is a $\mathcal{O}_{\operatorname{Spec}(R)}(U)$-module homomorphism and the verticals are the restriction map of sheaves $\mathcal{F}$ and $\mathcal{G}$, which are also module homomorphisms w.r.t. $\mathcal{O}_{\operatorname{Spec}(R)}(V) \to \mathcal{O}_{\operatorname{Spec}(R)}(U)$. The latter has the following meaning. If M is an R-module and N is an S-module, then a map $\phi : M \to N$ is a module homomorphism w.r.t $f : R \to S$ if $\phi(r \cdot m) = f(r) \cdot \phi(m)$.

We thus get a functor

$$\widetilde{-} : \mathbf{Mod}(R) \longrightarrow \mathbf{Mod}(\mathcal{O}_{\operatorname{Spec}(R)})$$
$$M \longmapsto \widetilde{M}$$
$$f : M \to N \longmapsto \tilde{f} : \widetilde{M} \to \widetilde{N}$$

where $\tilde{f}_{X_f} : M_f \to N_f$ is given by localization. We may denote $\widetilde{\mathbf{Mod}}(\mathcal{O}_{\operatorname{Spec}(R)}) \hookrightarrow \mathbf{Mod}(\mathcal{O}_{\operatorname{Spec}(R)})$ to be the full subcategory of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules of the form $\widetilde{M}$.

An explicit form of the sheaf $\widetilde{M}$ can be obtained by expanding the definition of the sheaf we obtain from it's definition on the basis.

Lemma 1.2.3.2. *Let M be an R-module and consider the associated $\mathcal{O}_{\operatorname{Spec}(R)}$-module $\widetilde{M}$. For any open $U \subseteq \operatorname{Spec}(R)$, we have*

$$\widetilde{M}(U) \cong \left\{ s : U \to \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}} \mid \forall \mathfrak{p} \in U,\ s(\mathfrak{p}) \in M_{\mathfrak{p}}\ \&\ \exists\ open\ \mathfrak{p} \in V \subseteq U\ \&\ \exists m \in M, f \in R\ \ s.t.\ \ \forall \mathfrak{q} \in V,\ f \notin \mathfrak{q}\ \&\ s(\mathfrak{q}) = m/f \right\}$$

Proof. Follows from Remark 22.1.0.4. □

We now collect properties of $\widetilde{M}$ below.

Proposition 1.2.3.3. *Let R be a ring and M, N, M_i be R-modules for $i \in I$,*

1. *$\left(\widetilde{M}\right)_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$,*
2. *$\widetilde{M}(\mathrm{Spec}\,(R)_f) \cong M_f$ for all $f \in R$,*
3. *$\Gamma(\widetilde{M}, \mathrm{Spec}\,(R)) \cong M$.*

Proof. Statement 1 follows from the alternate definition given in Lemma 1.2.3.2. Indeed one considers the function

$$\varphi : \left(\widetilde{M}\right)_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}}$$
$$(U, s)_{\mathfrak{p}} \longmapsto s(\mathfrak{p}).$$

One immediately sees this is R-linear. Injectivity and surjectivity is then also trivially checked by the above cited lemma.

Statements 3 follows from statement 2 by setting $f = 1$ and statement is just the Definition 1.2.3.1. □

We can also understand how $\mathcal{O}_{\mathrm{Spec}(R)}$-modules behave under morphism of affine schemes (see direct and inverse image of modules at Section 6.5)

Lemma 1.2.3.4. [7] *Let $f : \mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ be a morphism of affine schemes associated to map $\varphi : R \to S$ of rings. Then,*

1. *if N is an S-module, then $f_*\widetilde{N} \cong {}_R\widetilde{N}$ where ${}_RN$ is the R-module obtained by restriction of scalars by φ,*
2. *if M is an R-module, then $f^*\widetilde{M} \cong \widetilde{(S \otimes_R M)}$ where $S \otimes_R M$ is the S-module obtained by extension of scalars by φ.*

Proof. The proof is routine with main observation being the facts that for $g \in R$, we have $({}_RN)_g \cong N_{\varphi(g)}$ and for $\mathfrak{q} \in \mathrm{Spec}\,(S)$, we get the natural isomorphism $(f^*\widetilde{M})_{\mathfrak{q}} \cong \widetilde{(S \otimes_R M)}_{\mathfrak{q}}$. □

Theorem 1.2.3.5. *Let R be a ring. There is an equivalence of categories between those of R-modules and $\mathcal{O}_{\mathrm{Spec}(R)}$-modules of the form $\widetilde{M}$:*

$$\mathbf{Mod}(R) \underset{\Gamma(X,-)}{\overset{\widetilde{(-)}}{\underset{\equiv}{\rightleftarrows}}} \widetilde{\mathbf{Mod}}(\mathcal{O}_{\mathrm{Spec}(R)})$$

which moreover satisfies the following properties

1. *$\widetilde{(-)}$ is an exact functor; if $0 \to M' \to M \to M'' \to 0$ is exact, then $0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$ is exact,*
2. *$\widetilde{(-)}$ preserves tensor product; $\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$,*

---

[7]We will call it the *globalized* extension and restriction of scalars.

3. $\widetilde{(-)}$ *preserves coproducts;* $\widetilde{\bigoplus_{i \in I} M_i} \cong \bigoplus_{i \in I} \widetilde{M_i}.$

Proof. Let $X = \mathrm{Spec}\,(R)$. Consider the following map

$$\mathrm{Hom}_R\,(M, N) \to \mathrm{Hom}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$$
$$f : M \to N \mapsto \widetilde{f} : \widetilde{M} \to \widetilde{N}$$
$$\eta_X : M \to N \leftarrow\!\shortmid\ \eta : \widetilde{M} \to \widetilde{N}$$

Now, beginning from η, we may show that $(\widetilde{\eta_X})_{X_g} = \eta_{X_g}$ for some basic open $X_g \hookrightarrow X$. The result follows from the fact that $\eta : \widetilde{M} \to \widetilde{N}$ is completely characterized by the map on global sections $\eta_X : M \to N$ from the following square

$$
\begin{array}{ccc}
M_g & \xrightarrow{\ \eta_{X_g}\ } & N_g \\
\uparrow & & \uparrow \\
M & \xrightarrow[\ \eta_X\ ]{} & N
\end{array}
$$

where the verticals are restrictions morphisms w.r.t $R \to R_g$ and the top horizontal is R_g-module homomorphism and bottom is R-module homomorphism.

For statement 1, by Theorem 22.2.0.8, the question is local in nature. We deduce the result then from Lemma 18.1.2.2.

For statement 2, we proceed as follows. To define an isomorphism

$$\varphi : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_R N}$$

we need only define a map from the presheaf F given by $U \mapsto \widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U)$ to $\widetilde{M \otimes_R N}$ such that on basic open sets, we have an isomorphism. Indeed, let $D(f) \subseteq \mathrm{Spec}\,(R)$ be an open set for some $f \in R$. We define

$$\varphi_U : M_f \otimes_{R_f} N_f \overset{\cong}{\to} (M \otimes_R N)_f$$

as the obvious natural isomorphism. One checks that this does define φ to be a sheaf map.

For statement 3, as $\widetilde{(-)}$ is a left adjoint, therefore it preserves all colimits. $\qquad\square$

Remark 1.2.3.6. We will later see that on affine schemes $\mathrm{Spec}\,(R)$, the category $\widetilde{\mathbf{Mod}}(\mathcal{O}_{\mathrm{Spec}(R)})$ is precisely the category of quasicoherent $\mathcal{O}_{\mathrm{Spec}(R)}$-modules, which is a class of modules of utmost importance in algebraic geometry.

1.3   Schemes and basic properties

We can now define scheme to be a locally ringed space (see Foundational Geometry, 6) with an affine open covering.

Definition 1.3.0.1. (Schemes) A locally ringed space $(X, \mathcal{O}_X)$ is a scheme if there exists an open affine cover $\{(\mathrm{Spec}\,(R_i), \mathcal{O}_{\mathrm{Spec}(R_i)})\}$ of $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X|\mathrm{Spec}(R_i)} \cong \mathcal{O}_{\mathrm{Spec}(R_i)}$.

As we go along in understanding schemes, it will be more and more apparent the need of sheaf language to talk about the "generalized functions" over the scheme X. Indeed, there is a fine interrelationship between the *space structure* of the scheme $(X, \mathcal{O}_X)$ (that is, the topological space X) and the *function structure* on the scheme (that is, the sheaf of functions $\mathcal{O}_X$). A big part of learning scheme theory is to understand and use this relationship between them.

We will now bring some global topological properties of schemes which reflect their affine origins. An analogue of Lemma 1.2.1.1 holds in the general case of schemes.

Lemma 1.3.0.2. [8] *Let X be a scheme. The following are equivalent.*

1. *$S \subseteq X$ is a closed irreducible subset.*
2. *There exists a point $x \in S$ such that $\overline{\{x\}} = S$.*

Proof. (1. $\Rightarrow$ 2.) Let U be an affine open in X intersecting S. Then $U \cap S$ is an open subset of S. As open subsets of irreducibles are dense, therefore $U \cap S$ is dense in S. Consequently, it suffices to show that there exists a point $x \in U \cap S$ such that $\overline{\{x\}} = U \cap S$. As open subsets of irreducibles are irreducible, therefore $U \cap S$ is irreducible. Replacing X by U, we may assume X is affine. The result then follows by Lemma 1.2.1.1.

(2. $\Rightarrow$ 1.) Since $x \in U$ for some open affine $U \subset X$, thus, $x \in U \cap S$. Since $U \cap S \subseteq U$ and U is open, therefore closure of $\{x\}$ in U is same as closure of $\{x\}$ in X. Now, $\overline{\{x\}} = S$ but $\overline{\{x\}} \subseteq U$. It thus follows that $S \subseteq U$ and hence S is in an open affine. The result follows by Lemma 1.2.1.1. $\qquad\square$

Every open subspace of a scheme is a scheme.

Lemma 1.3.0.3. *Let X be a scheme and $U \subseteq X$ be an open subspace. Then $(U, \mathcal{O}_{X|U})$ is a scheme.*

Proof. Since for an affine scheme $\mathrm{Spec}\,(R)$, the basic open $\mathrm{Spec}\,(R)_f \cong \mathrm{Spec}\,(R_f)$ for $f \in R$, therefore for an open subspace $U \subseteq X$ and an affine open cover $\{U_i\}$ of X, $U_i \cap U$ is open in U_i and thus covered by affines of the form $\mathrm{Spec}\,(R_f)$. $\qquad\square$

Write **Sch** to be the category of schemes and **Sch**$/S$ to be the category of schemes over S. Morphisms of schemes is merely the same concept as that of morphism of locally ringed spaces (see Foundational Geometry, Chapter 6).

Definition 1.3.0.4. (**Map of schemes**) Let X and Y be two schemes. A map of underlying locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a map of schemes. In a more expanded form, $f : X \to Y$ is a continuous map and $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a map of sheaves such that the induced map (see Topics in Sheaf Theory, Chapter 22) on stalks for each $x \in X$

$$f_x^\sharp : \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X, x}$$

is a map of local rings, i.e., $(f_x^\sharp)^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,f(x)}$.

---

[8]Exercise II.2.9 of Hartshorne.

An important theorem in global study of schemes is a complete characterization of schemes over $\mathrm{Spec}\,(R)$, which is of-course of paramount importance.

Theorem 1.3.0.5. *Let X be a scheme and R be a ring. Then, there's a natural bijection*

$$\mathrm{Hom}_{\mathbf{Sch}}\,(X, \mathrm{Spec}\,(R)) \cong \mathrm{Hom}_{\mathbf{Ring}}\,(R, \Gamma(X, \mathcal{O}_X)).$$

In other words, we have the following adjunction[9]

$$\mathbf{Sch} \; \overset{\Gamma(-)}{\underset{\mathrm{Spec}(-)}{\xrightleftharpoons{\quad\perp\quad}}} \; \mathbf{Ring}^{\mathrm{op}} \; .$$

Proof. The proof will be played out in two steps. In the first one we will show the candidates for the unit and counit of this adjunction. In the second play we will show that they indeed satisfy the required triangle identities.

<center>**Act 1** : *The units and counits.*</center>

Let us first define the simpler one of them, the counit. For any $R \in \mathbf{Ring}$, we define a natural transformation $\epsilon : \mathrm{id}_{\mathbf{Ring}} \to \Gamma \circ \mathrm{Spec}\,()$ given by (note how we adjusted for the contravariant nature of $\mathrm{Spec}\,(-)$ and $\Gamma(-)$)

$$\epsilon_R : R \longrightarrow \Gamma(\mathrm{Spec}\,(R)) \cong R$$
$$f \longmapsto f.$$

Thus, $\epsilon_R = \mathrm{id}_R$. Hence, $\epsilon = \mathrm{id}_{\mathbf{Ring}^{\mathrm{op}}}$.

Next, we define the more intricate part, which is the unit. Take any scheme $X \in \mathbf{Sch}$. We define $\eta : \mathrm{id}_{\mathbf{Sch}} \to \mathrm{Spec}\,(\Gamma)$ on X by

$$\eta_X : X \longrightarrow \mathrm{Spec}\,(\Gamma(X))$$
$$x \longmapsto \mathfrak{p} = \eta_X(x) := \{f \in \Gamma(X) \mid f_x \in \mathfrak{m}_x\}.$$

Moreover, the map on structure sheaves is given by

$$(\eta_X)^\flat : \mathcal{O}_{\mathrm{Spec}(\Gamma(X))} \longrightarrow (\eta_X)_* \mathcal{O}_X$$

where as the map on global sections we keep it id and on a basic open $\mathrm{Spec}\,((\Gamma(X))_f)$ this is defined on sections by

$$(\eta_X)^\flat_{\mathrm{Spec}\left((\Gamma(X))_f\right)} : \Gamma(X)_f \cong \mathcal{O}_{\mathrm{Spec}(\Gamma(X))}((\mathrm{Spec}\,(\Gamma(X)))_f) \longrightarrow \mathcal{O}_X(\eta_X^{-1}((\mathrm{Spec}\,((\Gamma(X))))_f))$$

by the unique map that is obtained in the following diagram

<center>
$\Gamma(X)_f \dashrightarrow \mathcal{O}_X(\eta_X^{-1}((\mathrm{Spec}\,(\Gamma(X)))_f))$

$\uparrow \qquad \nearrow_{\rho}$

$\Gamma(X)$
</center>
,

where, indeed, $f \in \Gamma(X)$ is mapped to to an unit element in $\mathcal{O}_X(\eta_X^{-1}((\mathrm{Spec}\,(\Gamma(X)))_f$ because of the following simple lemma:

---

[9]This is also sometimes called the *algebra-geometry duality* or the *fundamental duality of algebraic geometry*.

(*) For a locally ringed space $(X, \mathcal{O}_X)$ and an open subspace $U \subseteq X$, $f \in \mathcal{O}_X(U)$ is a unit if and only if $f_x \notin \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ for all $x \in U$.

This construction has the following properties and we give the main idea which drives each one of them.

1. $\eta_X(x)$ *is a prime ideal of* $\Gamma(X)$: This follows from $\mathfrak{m}_x$ being a maximal (hence prime) ideal of $\mathcal{O}_{X,x}$.

2. η_X *is continuous* : Working with basis and reducing to assumption that $X = \mathrm{Spec}\,(S)$ is affine, we reduce to showing that $\{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid f_\mathfrak{p} \notin \mathfrak{m}_\mathfrak{p}\}$ is open, which is true as it is equal to $(\mathrm{Spec}\,(S))_f$.

3. $\eta : \mathrm{id}_{\mathbf{Sch}} \to \mathrm{Spec}\,() \circ \Gamma$ *is a natural transformation* : We wish to show that commutativity of the natural square. For a map of schemes $f : X \to Y$, this reduces to showing that

$$\forall x \in X, \ \eta_Y(f(x)) = (f_Y^\flat)^{-1}(\eta_X(x)).$$

This further follows from the observation that for $g \in \Gamma(Y)$, $f_Y^\flat(g) \in \mathfrak{m}_x \iff f_x(g_{f(x)}) \in \mathfrak{m}_x$ and the latter is clearly true by the definition of maps of locally ringed spaces, where $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is the map on stalks.

Hence, we have obtained a map of schemes $(\eta_X, \eta_X^\flat) : X \to \mathrm{Spec}\,(\Gamma(X))$. This is our candidate for the unit of the adjunction.

Act 2 : η *and* ϵ *satisfies the triangle identities.*

It follows that we wish to show that the following two diagrams commute:

$$
\begin{array}{ccc}
\Gamma(X) \xleftarrow{\Gamma\eta_X} \Gamma(\mathrm{Spec}\,(\Gamma(X))) \cong \Gamma(X) & \qquad & \mathrm{Spec}\,(R) \xrightarrow{\eta_{\mathrm{Spec}(R)}} \mathrm{Spec}\,(\Gamma(\mathrm{Spec}\,(R))) \cong \mathrm{Spec}\,(R) \\
\mathrm{id}_{\Gamma(X)} \searrow \quad \uparrow \epsilon_{\Gamma(X)} & & \mathrm{id}_{\mathrm{Spec}(R)} \searrow \quad \downarrow \mathrm{Spec}(\epsilon_R) \\
\Gamma(X) & & \mathrm{Spec}\,(R)
\end{array}
$$

<center>in Ring   in Sch</center>

This follows from a simple unraveling of the maps involved in the diagram as defined in Act 1.

$\square$

Corollary 1.3.0.6. *The above adjunction restricts to the following equivalence of categories:*

$$\mathbf{AfSch} \underset{\mathrm{Spec}(-)}{\overset{\Gamma(-)}{\underset{\simeq}{\rightleftarrows}}} \mathbf{Ring}^{\mathrm{op}} .$$

$\square$

Corollary 1.3.0.7. *Let* X *be a scheme over* $\mathrm{Spec}\,(R)$ *for a ring* R. *Then, for any open affine* $\mathrm{Spec}\,(S) \subseteq X$, S *is an* R-*algebra. Consequently, all stalks* $\mathcal{O}_{X,p}$ *are* R-*algebras.* $\square$

1.3.1   Basic properties

We can now observe some more basic properties. First, the behaviour of maps with respect to schemes over k and residue fields.

Lemma 1.3.1.1. *Let $f : X \to Y$ be a map of schemes over k. If $p \in X$ is such that $\kappa(p) = k$, then $\kappa(f(p)) = k$.*

Proof. On the stalks, we get the map

$$
\begin{array}{ccc}
\mathcal{O}_{Y,f(p)} & \xrightarrow{\ f_p^\sharp\ } & \mathcal{O}_{X,p} \\
 & k &
\end{array}
\qquad .
$$

As $f_p^\sharp$ is a local k-algebra homomorphism, therefore by quotienting with the respective ideals, we obtain

$$
\begin{array}{ccc}
\kappa(f(p)) & \longrightarrow & k \\
 & k & \text{id}
\end{array}
\qquad .
$$

The result then follows. $\qquad\square$

Local rings at non-closed points

Let X be an arbitrary scheme and $p \in X$ be a non-closed point. One can show that the local ring $\mathcal{O}_{X,p}$ is obtained by localizing local rings at closed points. Indeed, we have the following simple observation in this direction.

Lemma 1.3.1.2. *Let X be a scheme and $p \in X$ be a non-closed point. Then, $\mathcal{O}_{X,p}$ is isomorphic to localization of a local ring $\mathcal{O}_{X,x}$ at a prime ideal, where $x \in X$ is a closed point.*

Proof. Let $p \in X$ be a non-closed point and $U = \mathrm{Spec}\,(A)$ be an open affine containing p. Consequently, p corresponds to a prime ideal $\mathfrak{p} \lneq A$ which is not maximal. Let $\mathfrak{m} \lneq A$ be a maximal ideal containing $\mathfrak{p}$ and let $m \in U$ be the corresponding closed point in X. As $\mathcal{O}_{X,p} \cong A_\mathfrak{p}$ and $\mathcal{O}_{X,m} \cong A_\mathfrak{m}$, and since $(A_\mathfrak{m})_{\mathfrak{p}_\mathfrak{m}} \cong A_\mathfrak{p}$, therefore we have that $\mathcal{O}_{X,p}$ is obtained by localizing $\mathcal{O}_{X,m}$ at a prime ideal, as required. $\qquad\square$

Using ideas similar to above, we can also prove the following simple result.

Lemma 1.3.1.3. *Let X be an integral scheme and $\eta \in X$ be a non-closed point. Then the fraction field of $\mathcal{O}_{X,\eta} \cong K(X)$ where $K(X)$ is the function field of X.* $\qquad\square$

Non-vanishing locus of a global section

We next see that how a global section of a scheme defines an open set which is the set of those points where that element, when treated as a function, is non-zero. One then finds what the ring of functions over this open set looks like. First, for any scheme X and any $f \in \Gamma(\mathcal{O}_X, X)$, define the *non-vanishing locus* of f by

$$X_f := \{x \in X \mid f \notin \mathfrak{m}_{X,x}\}.$$

We first have the following simple result about non-vanishing locus.

Lemma 1.3.1.4. *Let* $f : X \to \operatorname{Spec}(B)$ *be a scheme over a ring* B *and let* $g \in B$. *Let* $\varphi : B \to \Gamma(\mathcal{O}_X, X)$ *be the map induced on the global sections. Then,*

$$f^{-1}(D(g)) = X_{\varphi(g)}.$$

Proof. Observe that $x \in X_{\varphi(g)}$ if and only if $\varphi(g)_x \notin \mathfrak{m}_{X,x}$. As we have the following commutative square

$$\begin{array}{ccc} B & \xrightarrow{\ \varphi\ } & \Gamma(\mathcal{O}_X, X) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\operatorname{Spec}(B),f(x)} & \xrightarrow[f_x^\sharp]{} & \mathcal{O}_{X,x} \end{array}$$

where vertical arrows are image into the stalk, therefore we deduce that $\varphi(g)_x \notin \mathfrak{m}_{X,x}$ if and only if $f_x^\sharp(g_x) \notin \mathfrak{m}_{X,x}$. As $f_x^\sharp$ is a local ring homomorphism, therefore $f_x^\sharp(g_x) \notin \mathfrak{m}_{X,x}$ if and only if $g_x \notin \mathfrak{m}_{\operatorname{Spec}(B),f(x)} = f(x)B_{f(x)}$. As $B \to \mathcal{O}_{\operatorname{Spec}(B),f(x)}$ is just localization map $B \to B_{f(x)}$, therefore $g_x \notin f(x)B_{f(x)}$ if and only if $g \notin f(x)$, that is $f(x) \in D(g)$. This completes the proof. $\qquad\square$

Proposition 1.3.1.5. [10] *Let* X *be a scheme and* $f \in \Gamma(\mathcal{O}_X, X)$.
1. *Let* $U = \operatorname{Spec}(A)$ *be an affine open subset of* X *and denote* $\bar{f} = \rho_{X,U}(f)$. *Then,* $U \cap X_f = D(\bar{f})$. *Consequently,* $X_f \subseteq X$ *is open.*
2. *Let* X *be quasicompact and* $a \in \Gamma(\mathcal{O}_X, X)$ *such that* $\rho_{X,X_f}(a) = 0$. *Then,* $f^n a = 0$ *in* $\Gamma(\mathcal{O}_X, X)$ *for some* $n > 0$.
3. *Let* X *admit an affine open cover* U_i *such that* $U_i \cap U_j$ *is quasicompact. If* $b \in \mathcal{O}_X(X_f)$, *then there exists* $a \in \Gamma(\mathcal{O}_X, X)$ *and* $n > 0$ *such that* $f^n b = \rho_{X,X_f}(a)$ *in* $\mathcal{O}_X(X_f)$.
4. *There is an isomorphism of rings* $\Gamma(\mathcal{O}_{X_f}, X_f) \cong (\Gamma(\mathcal{O}_X, X))_f$.

Proof. 1. We wish to show that $\{x \in U \mid \bar{f}_x \notin \mathfrak{m}_{X,x}\} = \{x \in U \mid \bar{f} \notin x\}$, where $x \in U$ in latter is treated as a prime ideal of A. The side $" \subseteq "$ folloes from the fact that for $x \in U$, we have $\mathcal{O}_{X,x} \cong A_x$, $\mathfrak{m}_{X,x} \cong xA_x$ and the fact that the map into stalks $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is given by the canonical map $A \to A_x$, $a \mapsto a/1$. One further would need the commutativity of the following diagram:

$$\begin{array}{ccc} & & \mathcal{O}_{X,x} \\ & \nearrow & \uparrow \\ \mathcal{O}_X(U) & \longleftarrow & \Gamma(\mathcal{O}_X, X) \end{array} \ .$$

---

[10] Exercise II.2.16 of Hartshorne.

The side " $\supseteq$ " also follows from the commutativity of the above triangle together with the canonical isomorphisms of the local ring and its maximal ideal.

2. □

Locality of isomorphism on target

We now show a rather simple result on locality of isomorphism on target, but it is quite useful in scenarios where one understands the map well on individual opens of target but not on the global level.

Proposition 1.3.1.6. *Let $f : X \to Y$ be a map of schemes and $Y = \bigcup_{i \in I} U_i$ be an open cover of Y such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is an isomorphism. Then, f is an isomorphism.*

Proof. □

Criterion for affineness

We now show a useful criterion for a scheme to be affine. This also portrays the power of previous result on locality of isomorphism.

Proposition 1.3.1.7. *Let X be a scheme and denote $A = \Gamma(\mathcal{O}_X, X)$. Then the following are equivalent:*
1. *X is affine,*
2. *there exists $f_1, \ldots, f_r \in A$ such that X_{f_i} are open affine subsets of X and $\langle f_1, \ldots, f_r \rangle = A$.*

Proof. □

1.4   First notions on schemes

Having defined schemes, our next goal is to bring to light some of the obvious definitions that one can make on them. In some sense, having made the general definition of schemes, we are now trying to go back to try and find where does varieties lie in this big world of **Sch**. Indeed, we will see that the definitions introduced in the following few sections are bringing us ever closer to define varieties as certain type of schemes, which will thus enable us to bring to light the most important geometric notions on varieties.

Write the notion
and examples, Chapter 1.

1.4.1   Noetherian schemes

Definition 1.4.1.1. (**Noetherian schemes**) A scheme X is called *locally noetherian* if there exists an affine open cover $X = \cup_{i \in I} U_i$ where each $U_i = \operatorname{Spec}(A_i)$ where A_i is a noetherian ring. If moreover, X is quasicompact, then X is called *noetherian*.

Remark 1.4.1.2. Since $X = \operatorname{Spec}(A)$ is already quasi-compact (Lemma 1.2.1.6), therefore for affine schemes X, the notion of locally noetherian and noetherian are equal.

The only immediately important result about such schemes that one needs is that an affine scheme is noetherian if and only if the obvious thing happens.

Lemma 1.4.1.3. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then, the following are equivalent:*
1. *X is a noetherian scheme.*
2. *A is a noetherian ring.*

Proof. (2. $\Rightarrow$ 1.) This follows from Remark 1.4.1.2 and the fact that localization of noetherian rings are noetherian.
(1. $\Rightarrow$ 2.) Let X be noetherian. Then there is an affine open cover of X by spectra of noetherian rings. Pick any ideal $I \leq A$. We shall show it is finitely generated. There is a finite cover $\{\operatorname{Spec}(A_{f_i})\}_{i=1}^n$ of $\operatorname{Spec}(A)$ where A_{f_i} are noetherian and $f_i \in A$. Hence we have that the ideal IA_{f_i} of A_{f_i} is finitely generated for all $i = 1, \ldots, n$. By Lemma 1.2.1.5, 2, we see that $f_1, \ldots, f_n$ generate the whole ring A. The result then follows by Lemma 18.1.2.10. $\square$

Example 1.4.1.4. By the Lemma 1.4.1.3, we observe that any of the variety over a field is a noetherian scheme (technically, we are identifying the affine variety with its associated scheme, see Section **??**, Schemes associated to varieties). So any of your favorite variety

$$\operatorname{Spec}\left(\frac{k[x, y, z]}{x^2 + y^2 - z^3 - 1}\right), k \text{ is algebraically closed}$$

gives a (*is* a) noetherian scheme.

1.4.2   Reduced, integral schemes and function field

The following are the definitions required, which are clearly geometric in nature.

Definition 1.4.2.1. (**Reduced and integral schemes**) A scheme X is said to be *reduced* if local rings $\mathcal{O}_{X,x}$ for all $x \in X$ is a reduced ring; have no nilpotents. A scheme X is said to be *integral* if it is reduced and irreducible as a topological space.

The one basic result that must be seen about these two types of schemes is that they are characterized by algebraic properties of local sections. Thus being reduced or integral, while defined geometrically, is concretely controlled by the algebraic properties of the structure sheaf.

Lemma 1.4.2.2. *Let X be a scheme. Then,*

  1. X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for each open set $U \subseteq X$ [11].

  2. X is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for each open set $U \subseteq X$.

Proof. 1. (L $\Rightarrow$ R) Suppose for some open $U \subseteq X$ there exists a section $f \in \mathcal{O}_X(U)$ which is nilpotent. Using the homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ given by $s \mapsto s_x$, we see that $f_x \in \mathcal{O}_{X,x}$ is a nilpotent element.

(R $\Rightarrow$ L) Suppose X is not reduced. Hence for some germ $f_x \in \mathcal{O}_{X,x}$ at some point $x \in X$ is a nilpotent where $f \in \mathcal{O}_X(U)$ for some open $x \in U \subseteq X$. Since $f_x^n = 0$ for some $n \in \mathbb{N}$, we get that $f^n = 0$ for some open $W \subseteq U$. Thus $\rho_{U,W}(f) \in \mathcal{O}_X(W)$ is a nilpotent element[12].

2. (L $\Rightarrow$ R) Pick any open $U \subseteq X$. We wish to show that $\mathcal{O}_X(U)$ is an integral domain. In other words, we wish to show the proposition for the open subscheme $(U, \mathcal{O}_{X|U})$. Replacing X by U, we reduce to showing $\mathcal{O}_X(X)$ is an integral domain. So let $f, g \in \mathcal{O}_X(X)$ be such that $fg = 0$. We wish to show that either $f = 0$ or $g = 0$. Suppose neither f nor g is 0 but $fg = 0$. It follows from Lemma 1.2.0.1, 1, that $V(f)$ and $V(g)$ covers X and hence by irreducibility of X, either $V(f) = 0$ or $V(g) = 0$, that is, $f = 0$ or $g = 0$.

(R $\Rightarrow$ L) We first need to show that X is reduced. Indeed, by 1. it follows immediately as integral domains are reduced. We then wish to show that X is irreducible. Indeed, if there are two open subsets of X say $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \emptyset$, then we claim that $\mathcal{O}_X(U_1 \cup U_2) \cong \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$. Since both $\mathcal{O}_X(U_1), \mathcal{O}_X(U_2)$ have 0 and 1, thus $\mathcal{O}_X(U_1 \cup U_2)$ will have a zero-divisor, a contradiction. Indeed, consider the following homomorphism, denoting $U := U_1 \cup U_2$

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$$
$$s \longmapsto (\rho_{U,U_1}(s), \rho_{U,U_2}(s)).$$

This is injective by locality axiom and surjective by gluing axiom of sheaves. $\qquad \square$

Corollary 1.4.2.3. *Let X be a scheme. If X is integral, then all local rings $\mathcal{O}_{X,x}$ are integral domains.*

Proof. Use Lemma 1.4.2.2, 2 together with the fact that localization of integral domains is an integral domain. $\qquad \square$

Corollary 1.4.2.4. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then X is integral if and only if A is an integral domain.*

---

[11]Exercise II.2.3.a of Hartshorne.

[12]This is a very inefficient way of using the equality on stalks. Indeed, two germs are equal if and only if the representatives are equal on some common shrinking of their domains. This is how usually people work with stalks without being overly full of symbols.

Proof. Use Lemma 1.4.2.2, 2 on global sections together to get one side. For the "only if" side, stalks are reduced as they are integral (localizations of A) and X is irreducible as for any $V(\mathfrak{a}) \cup V(\mathfrak{b}) = X$, we have $V(\mathfrak{ab}) = X$ and thus $\mathfrak{ab} \subseteq \mathfrak{n}$ where $\mathfrak{n}$ is the intersection of all prime ideals, the nilradical (Lemma 18.1.2.9). Since A is integral, therefore $\mathfrak{o}$ is prime as well and hence $\mathfrak{n} = 0$, making $\mathfrak{ab} = 0$. Since A is integral, hence $\mathfrak{a} = \mathfrak{o}$ or $\mathfrak{b} = \mathfrak{o}$. $\qquad \square$

Remark 1.4.2.5. (*Function field of an integral scheme*) Let X be an integral scheme. Since X is irreducible as a topological space, therefore there is a generic point η in X, i.e. a point whose closure is the whole of X (Lemma 1.3.0.2). Now let $\mathrm{Spec}\,(A) \subseteq X$ be an affine open such that $\eta \in \mathrm{Spec}\,(A)$. Thus, η is a generic point of $\mathrm{Spec}\,(A)$ as well. Hence η corresponds to the zero ideal of A, which is indeed an integral domain from Lemma 1.4.2.2, 2. Since $\mathcal{O}_{X,\eta} \cong \mathcal{O}_{\mathrm{Spec}(A),\eta} = A_{\mathfrak{o}}$, therefore $\mathcal{O}_{X,\eta}$ is a field, called the *function field of the integral scheme* X and is in particular given by field of fractions of any domain A such that open $\mathrm{Spec}\,(A)$ contains η. We denote the function field of X as $K(X)$[13].

Using the fact that the generic point of an integral scheme X will be in every non-empty open set, we can make some fascinating observations about the function field $K(X)$, which thus justifies its name.

Lemma 1.4.2.6. *Let X be an integral scheme with function field $K(X)$. Then for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is contained in $K(X)$.*

Proof. Let $x \in X$, $\eta \in X$ be the generic point and $U = \mathrm{Spec}\,(A)$ be an open affine in X. By Lemma 1.4.2.2, 2, A is a domain. Clearly, $\eta \in U$ and it corresponds to the zero ideal $\mathfrak{o} \lneq A$. Further we have $\mathcal{O}_{X,x} \cong A_{\mathfrak{p}}$, $\mathfrak{p} \in U$ is equal to the point $x \in U$. By definition $K(X) = A_{\mathfrak{o}}$. The result follows by observing that $A_{\mathfrak{p}} \subseteq A_{\mathfrak{o}}$. $\qquad \square$

Example 1.4.2.7. ($\mathrm{Spec}\,(\mathbb{Z})$) Since $\mathbb{Z}$ is an integral domain, therefore by Corollary 1.4.2.4, $X = \mathrm{Spec}\,(\mathbb{Z})$ is an integral scheme. Clearly, X as a topological space consists of all prime numbers and a generic point given by the zero ideal $\mathfrak{o}$. Further, the topology is thus given by cofinite topology. At the level of stalks, we have that for a prime $\mathfrak{p} \in X$, $\mathcal{O}_{X,\mathfrak{p}} \cong \mathbb{Z}_{\mathfrak{p}}$ and we can describe $\mathbb{Z}_{\mathfrak{p}}$ as all those rationals whose denominator is not a multiple of prime p where $\mathfrak{p} = \langle p \rangle$ as $\mathbb{Z}$ is a PID (it's ED). Clearly, localizing X at the generic point $\mathfrak{o}$ would yield $\mathcal{O}_{X,\mathfrak{o}} \cong \mathbb{Q}$. More fascinatingly, for a prime $\mathfrak{p} = \langle p \rangle$ in X, the residue field at point $\mathfrak{p}$ is $\kappa(\mathfrak{p}) = \mathbb{Z}_{\mathfrak{p}}/\mathfrak{p}\mathbb{Z}_{\mathfrak{p}} \cong \mathbb{F}_p$, the finite field with p elements!

Now for any affine scheme $\mathrm{Spec}\,(A)$, consider a map $f : X \to \mathrm{Spec}\,(\mathbb{Z})$. By the fact that $\mathbb{Z}$ is initial in category of rings, therefore $\mathrm{Spec}\,(\mathbb{Z})$ is terminal in the category of affine schemes (Corollary 1.3.0.6). Since any scheme is locally affine, it further follows that $\mathrm{Spec}\,(\mathbb{Z})$ is terminal in the category of schemes.

We now introduce a concept which will be used while discussing divisors.

Definition 1.4.2.8. (**Center of a valuation**) Let X be an integral scheme with function field K and $v : K \to G$ be a valuation over K with valuation ring $R \subset K$. A center of v is defined to be a point $x \in X$ such that R dominates $\mathcal{O}_{X,x}$ in K (see Definition 18.9.0.5).

---

[13]Exercise II.3.6 of Hartshorne.

1.4.3   (Locally) finite type schemes over k

This section is the beginning of a theme which we would like to understand intimately, schemes over a field. This is because most of the schemes we will encounter in nature will be varieties whose coordinate rings would be algebras over a field. Here we first understand in scheme language the first thing about coordinate rings of varieties over k, the fact that they are finitely generated as an k-algebra. Indeed, this is what we seek from the following definition.

Definition 1.4.3.1. (Finite and locally finite type schemes over a field) Let k be a field and let $X \to \operatorname{Spec}(k)$ be a scheme over k. Then X is said to be locally finite type if there exists an affine open covering $\{\operatorname{Spec}(A_i)\}_{i \in I}$ of X such that each A_i is a finitely generated k-algebra. Moreover, X is said to be finite type if X is locally finite type and quasi-compact.

Example 1.4.3.2. Our ɦyperboloid of one sheet (introduced in Example 1.5.1.3) has the following coordinate ring:

$$\frac{k[x, y, z]}{I(V(p))}$$

where $p(x, y, z) = x^2 + y^2 - z^2 - 1$, where we have chosen $a = b = c = 1$ for simplicity. Let $\mathfrak{h} := I(V(p))$. Clearly $\operatorname{Spec}(k[x, y, z]/\mathfrak{h})$ is a finite type k-scheme.

Great thing about the above definition is that it really doesn't depend on the affine open cover that is chosen.

Lemma 1.4.3.3. *Let k be a field and X be a k-scheme. Then the following are equivalent.*
   1. X is of locally finite type over k.
   2. For all open affine $U \hookrightarrow X$, the ring $\mathcal{O}_X(U)$ is finitely generated k-algebra.

Proof. (2. $\Rightarrow$ 1.) Immediate.
(1. $\Rightarrow$ 2.) We shall use Lemma 18.1.2.11 for this.   □

1.4.4   Subschemes and immersions

These notions are important in what is to come next.

Definition 1.4.4.1. (Open subscheme) Let X be a scheme. An open set $U \subseteq X$ has a canonical scheme structure, given by $(U, \mathcal{O}_{X|U})$. We call $(U, \mathcal{O}_{X|U})$ an open subscheme of X.

Indeed, locally U will look affine via the open affine cover of X. We can relativize this notion to define open immersions.

Definition 1.4.4.2. (Open immersion) A map $f : X \to Y$ of schemes is said to be an open immersion if $f : X \to f(X)$ is a homeomorphism, $f(X) \subseteq Y$ is open and $f^\flat_{|f(X)} : \mathcal{O}_{Y|f(X)} \to (f_* \mathcal{O}_X)_{|f(X)}$ is an isomorphism.

We observe that for any point in an intersection of open subschemes is contained in some special open subscheme. This is a very important result as this will be used as a technical tool to allow passage from one open affine with certain properties to another open affine, all the time while handling only basic open sets.

Lemma 1.4.4.3. *Let* $U = \operatorname{Spec}(A), V = \operatorname{Spec}(B) \hookrightarrow X$ *be two affine open subsets. For each* $x \in U \cap V$, *there exists an affine open subset* $x \in W \hookrightarrow U \cap V$ *such that* $W = \operatorname{Spec}(A_f)$ *and* $W = \operatorname{Spec}(B_g)$ *for some* $f \in A$ *and* $g \in B$.

Proof. By replacing B by B_g for some $g \in B$, we may assume that $x \in V \subseteq U$. Consequently, let $f \in A$ be such that $D_U(f) \subseteq V$ and contains x, where $D_U(f) = \{\mathfrak{p} \in U \mid f \notin \mathfrak{p}\}$. We thus have $x \in D_U(f) \subseteq V \subseteq U$. Consider the restriction $h = \rho_{U,V}(f) \in \mathcal{O}_X(V) = B$. We claim that $D_V(h) = D_U(f)$. Denote $\varphi : A \to B$ obtained by $V \subseteq U$. We then have that $\rho_{U,V} = \varphi$ and $h = \varphi(f)$. Thus $\mathfrak{q} \in D_V(h) \iff h \notin \mathfrak{q} \iff \varphi(f) \notin \mathfrak{q} \iff f \notin \varphi^{-1}(\mathfrak{q})$. As for each $\mathfrak{p} \in D_U(f)$ is $\varphi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in V$, therefore we are done. $\square$

Closed subschemes are defined in not that obvious way in which we have defined open subschemes, but at any rate, they are natural. Let X be a scheme. Suppose a closed subset $C \hookrightarrow X$ intersects some collection of affine opens $\{\operatorname{Spec}(A_i)\}$ and moreover it happens that $C \cap \operatorname{Spec}(A_i) = C \cap \operatorname{Spec}(A_j)$ for $i \neq j$[14]. Now clearly $C \cap \operatorname{Spec}(A_i) = \operatorname{Spec}(A_i/\mathfrak{a}_i)$ and $C \cap \operatorname{Spec}(A_j) = \operatorname{Spec}(A_j/\mathfrak{a}_j)$ for some ideals $\mathfrak{a}_i \subseteq A_i$ and $\mathfrak{a}_j \subseteq A_j$. Hence, we get two different structure sheaves $\mathcal{O}_{\operatorname{Spec}(A_i/\mathfrak{a}_i)}$ and $\mathcal{O}_{\operatorname{Spec}(A_j/\mathfrak{a}_j)}$ on on open subset of C. Thus we have to systematically track such identifications in order to define a unique scheme structure on the closed set C. Indeed, we take the help of the rich amount of constructions that we can make on the category of sheaves over a space (for more information, see Section 6.5).

We first define closed immersions.

Definition 1.4.4.4. (Closed immersions) A map $f : X \to Y$ of schemes is a closed immersion if $f : X \to f(X)$ is a homeomorphism, $f(X) \subseteq Y$ is closed and $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a surjective map.

Remark 1.4.4.5. Let $f : X \to Y$ be a closed immersion, so that $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. This is equivalent to saying that for each point $x \in X$, the map on stalks (see Theorem 22.2.0.6 and Lemma 22.4.0.5)

$$f^\flat_{f(x)} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is surjective. Observe that the above map is NOT the usual map on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Further observe that since $f^\flat$ is surjective, therefore we have an ideal (see Section 6.5, Global algebra for more details) $\mathcal{I} = \operatorname{Ker}\left(f^\flat\right) \leq \mathcal{O}_Y$. We will later see that a closed subscheme is completely determined by this ideal sheaf and in-fact these ideal sheaves gives us a family of good examples of what will later be called quasicoherent modules over a scheme.

A closed subscheme is then defined to be an isomorphism class of closed immersions.

---

[14]This may not always seem to be the case for an arbitrary closed set C, but we will later show that it is.

Definition 1.4.4.6. (**Closed subscheme & ideal sheaf**) Let Y be a scheme. A closed subscheme of Y is an isomorphism class of closed immersions over Y. That is, a closed subscheme is the class $[f : X \to Y]$ of closed immersions where two closed immersions $f : X \to Y$ and $f' : X' \to Y$ are identified if there is an isomorphism $X \overset{\cong}{\to} X'$ such that the following commutes

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;\cong\;\;} & X' \\
& f \searrow \quad \swarrow f' & \\
& Y &
\end{array}
\quad .
$$

For a closed subscheme $f : X \to Y$, we define kernel of $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ to be the ideal sheaf corresponding to the closed subscheme f.

Remark 1.4.4.7. Note that this definition is not "unnatural" as every closed immersion $f : X \to Y$ defines a closed set $f(X) \subseteq Y$ and a scheme structure over it. We then just define a closed subscheme to be the data of this closed set together with its scheme structure that is given by f. Clearly to make such a definition via immersions, we would need to identify those immersions which give same scheme structure on $f(X) \subseteq Y$.

We first understand closed subscheme structures in affine schemes.

Lemma 1.4.4.8. *Let $X = \mathrm{Spec}\,(R)$ be an affine scheme. Then every ideal $\mathfrak{a} \leq R$ defines a closed subscheme of X.*

Proof. Consider the closed set $Y = V(\mathfrak{a}) \subseteq X$. We endow Y with a scheme structure given by the isomorphism $Y \cong \mathrm{Spec}\,(R/\mathfrak{a})$. Now the inclusion map $i : (Y, \mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a})}) \to X$ is clearly a topological closed immersion. Further, $i^\flat : \mathcal{O}_{\mathrm{Spec}(R)} \to i_*\mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a})}$ is given on stalks (see Lemma 22.4.0.5) at point $x \in Y$ as $\mathcal{O}_{\mathrm{Spec}(R),x} \to \mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a}),x}$ which is just $R_x \to (R/\mathfrak{a})_x$ which is surjective. Thus, $\mathfrak{a}$ defines a closed subscheme structure on Y. $\square$

It is important to note that any other ideal $\mathfrak{b} \leq R$ such that $V(\mathfrak{a}) = V(\mathfrak{b})$ will define a possibly different closed subscheme structure on the underlying topological space. This is another example of the phenomenon that algebra has much more finer control over the geometric situation at hand. For example, for $X = \mathrm{Spec}\,(k[x])$, we have $\mathfrak{a}_n = \langle x^n \rangle$ and note that $V(\mathfrak{a}_n) = \{\langle x \rangle\} \subseteq X$. But each ideal $\mathfrak{a}_n$ defines a new closed subscheme structure on the same point $\langle x \rangle \in X$.

Closed subschemes and ideal sheaves

We now study closed subschemes of arbitrary schemes. To read the following results, see Section 1.9 on quasicoherent modules.

Proposition 1.4.4.9. *Let X be a scheme.*
  1. If $\mathcal{I} \leq \mathcal{O}_X$ is the ideal sheaf of a closed subscheme $Y \hookrightarrow X$, then $\mathcal{I}$ is a quasicoherent $\mathcal{O}_X$-module. If further X is Noetherian, then $\mathcal{I}$ is coherent.
  2. If $\mathcal{I} \leq \mathcal{O}_X$ is an ideal of $\mathcal{O}_X$ such that it is quasicoherent, then $\mathcal{I}$ determines a unique closed subscheme $Y \hookrightarrow X$ where Y is given by $\mathrm{Supp}\,(\mathcal{O}_X/\mathcal{I})$.

3. Consequently, we have a correspondence

$$\{\textit{All quasicoherent ideal sheaves } \mathcal{I} \leq \mathcal{O}_X \textit{ upto isomorphism}\} \cong \{\textit{All closed subschemes } Y \hookrightarrow X\}.$$

Proof. 1. This follows from the following facts; closed subschemes are quasicompact separated maps, that direct image of quasicoherent is quasicoherent for such maps and that kernels of maps of quasicoherent modules is quasicoherent. The second statement follows from reducing to affine and using the fact that we know all quasicoherent modules over affine.

2. Pick an ideal sheaf $\mathcal{I} \leq \mathcal{O}_X$ which is quasicoherent and let $Y = \mathrm{Supp}\,(\mathcal{O}_X/\mathcal{I}) := \{x \in X \mid \mathcal{O}_{X,x}/\mathcal{I}_x \neq 0\}$. Then consider $i : (Y, \mathcal{O}_X/\mathcal{I}) \hookrightarrow (X, \mathcal{O}_X)$. It is straightforward to see that the kernel of $i^\flat$ is exactly $\mathcal{I}$. We wish to show that this is a topological closed immersion and that the map $i^\flat$ is surjective. Clearly i is homeomorphic to its image, thus we need only show that its image is a closed set. This is a local property, so let $X = \mathrm{Spec}\,(R)$, so that $\mathcal{I} = \widetilde{\mathfrak{a}}$ for an ideal $\mathfrak{a} \leq R$. Now $Y = \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid (R/\mathfrak{a})_\mathfrak{p} \neq 0\} = \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\} = V(\mathfrak{a})$. Thus i is a topological closed immersion. Now the surjectivity of the map $i^\flat : \mathcal{O}_X \to i_*\mathcal{O}_X/\mathcal{I}$ follows from going to stalks via Lemma 22.4.0.5. The uniqueness of $(Y, \mathcal{O}_X/\mathcal{I})$ w.r.t. $\mathcal{I}$ is clear. $\qquad\square$

Note that the main use of quasicoherence of $\mathcal{I}$ in statement 2 was to make sure that the support of $\mathcal{O}_X/\mathcal{I}$ is indeed closed. We have a straightforward, but important corollary.

Corollary 1.4.4.10. *Let $X = \mathrm{Spec}\,(A)$ be an affine scheme. We have the following bijection*

$$\{\textit{Closed subschemes } Y \hookrightarrow X\} \mathrel{\substack{\xrightarrow{\mathcal{I} \mapsto \Gamma(\mathcal{I}, X)} \\ \cong \\ \xleftarrow{\;(\mathrm{Spec}(A/\mathfrak{a}), \widetilde{A/\mathfrak{a}}) \hookleftarrow \mathfrak{a}\;}}} \{\textit{Ideals } \mathfrak{a} \leq A\}/\cong\ .$$

Note that $\widetilde{A/\mathfrak{a}} \cong \mathcal{O}_{\mathrm{Spec}(A/\mathfrak{a})}$.

Proof. Follows immediately from Proposition 1.4.4.9 and Corollary 1.9.2.5. $\qquad\square$

1.5   Varieties

Most examples of schemes that we will encounter in the wild are quasi-projective/affine varieties. Therefore, we first cover them in a semi-classical setting not involving schemes. We will then show how to interpret them as finite type separated integral schemes over the base field. This will enable us to use the machinery we will be developing for schemes in the study of varieties. Indeed, by the end of this section, we will comfortably replace the definition of a variety to mean a separated, integral finite type scheme over an algebraically closed field.

1.5.1   Varieties over an algebraically closed field-I

We define varieties as zero sets of certain polynomials over an algebraically closed field k. We assume that the reader is aware of the Zariski topology that is present over $\mathbb{A}_k^n$. Let us first give the classical version of affine varieties.

Definition 1.5.1.1. (**Affine algebraic variety**) Let k be an algebraically closed field and let $\mathbb{A}_k^n$ be the affine n-space. An affine algebraic variety is an irreducible closed subset of $\mathbb{A}_k^n$.

We recall that the Hilbert Nullstellensatz further tells us that for any ideal $\mathfrak{a} \leq k[x_1, \ldots, x_n]$, the zero set of the ideal $Z(\mathfrak{a}) \subseteq \mathbb{A}_k^n$ is such that the ideal it generates is equal to the radical of the ideal, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Let $A \subseteq \mathbb{A}_k^n$ be an affine algebraic set. Then, the *affine coordinate ring* of A is defined to be the following finitely generated k-algebra

$$k[A] := \frac{k[x_1, \ldots, x_n]}{I(A)}$$

where $I(A) \leq k[x_1, \ldots, x_n]$ is the ideal generated by A. An important simple lemma to keep in mind for future is the following.

Lemma 1.5.1.2. *Let k be an algebraically closed field. Then B is a finitely generated k-algebra without nilpotent elements if and only if B is an affine coordinate ring of an algebraic set.*

Proof. One side is trivial and the other uses Nullstellensatz. □

Example 1.5.1.3. (*Hyperboloid of one sheet*) A recurring example that we choose to study in this notebook, amongst the others, is the hyperboloid of one sheet. This is given by the following equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

In the affine space over $\mathbb{R}$, $\mathbb{A}_{\mathbb{R}}^3$, we can draw it as shown in Figure 1.1.

We may simply call it a hyperboloid. This hyperboloid determines an affine variety given
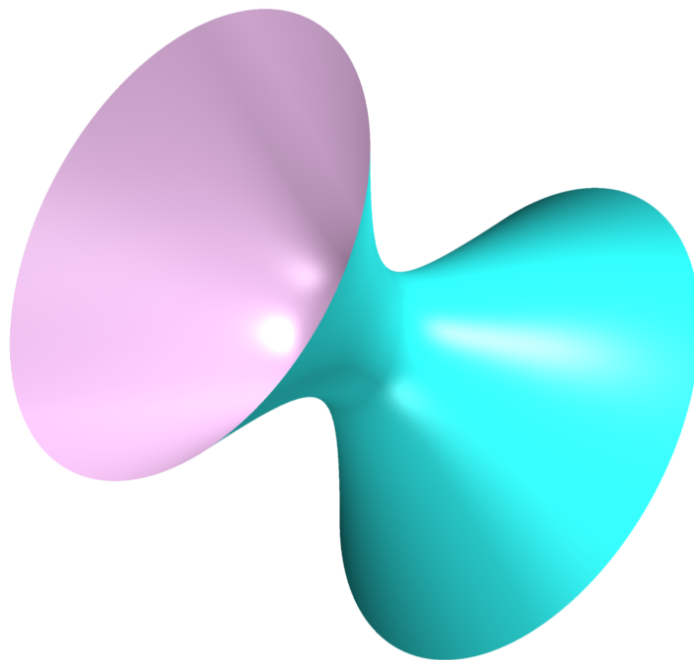
Figure 1.1: A hyperboloid of one sheet as a subvariety of $\mathbb{A}^3_{\mathbb{R}}$. The parameters are $a = 1.05, b = 1.05, c = 1$.

by the zero set of the polynomial

$$p(x, y, z) = x^2/a^2 + y^2/b^2 - z^2/c^2 - 1 \in k[x, y, z]$$

for any field k. Let $X = V(p) \subseteq \mathbb{A}^3_k$. The coordinate ring is given by

$$k[X] = \frac{k[x, y, z]}{I(V(p))}.$$

As we shall see, we will associate to the above variety $(X, \mathcal{O}_X)$ a scheme by considering the spectrum of the coordinate ring, $\mathrm{Spec}\,(k[X])$.

We will understand this fantastic example in much more detail as we develop more tools to handle it.

We now define projective varieties. Consider an algebraically closed field. Then the *projective n-space* is defined to be the quotient $\mathbb{P}^n_k := \mathbb{A}^{n+1}_k / \sim$ where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ if and only if there exists $\lambda \in k^\times$ such that $a_i = \lambda b_i$ for all $i = 0, 1 \ldots, n$. A point of $\mathbb{P}^n_k$ is denoted by $[a_0 : \cdots : a_n]$ and this presentation of the point is called the homogeneous coordinates of the point. Assuming that the reader is aware about graded rings and the natural grading of $k[x_0, \ldots, x_n]$, we observe that we can talk about the *zeroes of a homogeneous polynomial* $p(X) \in k[x_0, ; x_n]$ as follows:

$$Z(p) := \{P \in \mathbb{P}^n_k \mid p(P) = 0\}.$$

Indeed, one observes that a homogeneous polynomial is zero at a point $P \in P_k^n$ in a manner which is independent of the choice of representation of P in terms of the homogeneous coordinates of P. With this in our hand, we further define the zero set of a homogeneous ideal $\mathfrak{a} \leq k[x_0, \ldots, x_n]$ as

$$Z(\mathfrak{a}) := \{P \in \mathbb{P}_k^n \mid f(P) = 0 \forall f \in T_\mathfrak{a}\}$$

where $T_\mathfrak{a}$ is the set of all homogeneous elements of $\mathfrak{a}$. Remember that an ideal in a graded ring is homogeneous if and only if it is generated by the set of all of its homogeneous elements.

Lemma 1.5.1.4. *Let k be a field. Then*
   1. *For any two homogeneous ideals $\mathfrak{a}, \mathfrak{b} \leq k[x_0, \ldots, x_n]$, we have $Z(\mathfrak{ab}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$.*
   2. *For any family of homogeneous ideals $\{\mathfrak{a_i}\}\{i \in I\}$, we have $\cap_{i \in I} Z(\mathfrak{a_i}) = Z(\sum_{i \in I} \mathfrak{a_i})$.*

Proof. Straightforward unravelling of definitions. $\square$

Therefore we obtain a topology on $\mathbb{P}_k^n$ where a set $Y \subseteq \mathbb{P}_k^n$ is closed if and only if $Y = Z(\mathfrak{a_i})$ for a homogeneous ideal $\mathfrak{a_i}$ of $k[x_0, \ldots, x_n]$. This is called the Zariski topology of $\mathbb{P}_k^n$.

Definition 1.5.1.5. (Projective algebraic variety) Let k be an algebraically closed field. An irreducible algebraic set of $\mathbb{P}_k^n$ is said to be a projective algebraic variety in $\mathbb{P}_k^n$.

Let $V \subseteq \mathbb{P}_k^n$ be a projective algebraic variety. Then the *ideal generated by V* in $k[x_0, \ldots, x_n]$ is $I(V)$ which is the ideal generated by the following set of homogeneous polynomials: $\{f \in k[x_0, \ldots, x_n] \mid f$ is homogeneous and $f(P) = 0\}$.

For a projective algebraic set $Y \subseteq \mathbb{P}_k^n$, we define its *homogeneous coordinate ring* to be the following k-algebra

$$k[Y] := \frac{k[x_0, \ldots, x_n]}{I(Y)}$$

where $I(Y)$ is the homogeneous ideal of Y.

We now show that how the projective space $\mathbb{P}_k^n$ is covered by $n+1$ copies of affine space $\mathbb{A}_k^n$. Before that we discuss few maps which allows us to treat affine case projectively.

Homogenization and dehomogenization

One way to move back and from affine to projective setting is to use to fundamental functions between $k[y_1, \ldots, y_i, \ldots, y_n]$ and $k[x_0, \ldots, x_n]_h$.

Definition 1.5.1.6. ((De)homogenization) Let k be an algebraically closed field and let $A := k[y_1, \ldots, y_n]$ and $B := k[x_0, \ldots, x_n]_h$, the set of all homogeneous polynomials in

$k[x_0, \ldots, x_n]$. Consider the following two functions

$$d_i : B \longrightarrow A$$
$$f(x_0, \ldots, x_n) \longmapsto f(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$
$$h_i : A \longrightarrow B$$
$$g(y_1, \ldots, y_n) \longmapsto x_i^e g\left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

where e is the degree of g and $i = 0, \ldots, n$. The map h_i is called the ith-homogenization map and d_i is called the ith-dehomogenization map.

Using this, we can establish the result in question.

Proposition 1.5.1.7. *Let k be an algebraically closed field and consider the projective n-space over k, $\mathbb{P}_k^n$. Then, there exists $n + 1$ open subspaces say $U_i \subseteq \mathbb{P}_k^n$, such that $\mathbb{P}_k^n = \bigcup_{i=0}^n U_i$ and for each i, U_i is homeomorphic to $\mathbb{A}_k^n$.*

Proof. Consider the $n + 1$ open subspaces of $\mathbb{P}_k^n$ as follows:

$$U_i := \mathbb{P}_k^n \setminus H_i$$

where $H_i = Z(\langle x_i \rangle)$ is the algebraic set obtained by all those points whose i^{th} homogeneous coordinate is zero. Now consider the map

$$\varphi_i : U_i \longrightarrow \mathbb{A}_k^n$$
$$[a_0 : \cdots : a_n] \longmapsto \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}\right).$$

One can check that this pulls closed sets to closed sets by using the ith-homogenization map. Conversely, one can define the map

$$\theta_i : \mathbb{A}_k^n \longrightarrow U_i$$
$$(a_1, \ldots, a_n) \longmapsto (a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$$

and this can again be checked to be continuous by an application of ith dehomogenization map. $\qquad\square$

Corollary 1.5.1.8. *Let k be an algebraically closed field and $Y \subseteq \mathbb{P}_k^n$ be a projective algebraic variety. Then, in the notation of Proposition 1.5.1.7, for each $i = 0, \ldots, n$, $Y \cap U_i$ is an affine algebraic variety.*

Proof. This follows from the observation that $Y \cap U_i$ is a closed set of $U_i \cong \mathbb{A}_k^n$. The irreducibility follows from the fact that open subsets of irreducible spaces are irreducible. $\quad\square$

Properties of algebraic sets in $\mathbb{P}_k^n$

We now present some basic properties of algebraic sets in $\mathbb{P}_k^n$.

Lemma 1.5.1.9. *(Homogeneous Nullstellensatz) Let k be an algebraically closed field and let $\mathfrak{a} \le k[x_0, \dots, x_n]$ be a homogeneous ideal. Then,*

$$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

Proof. Denote by $V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$ to be the vanishing set of $\mathfrak{a}$ in the affine $n+1$-space. This is called the *affine cone* of the ideal $\mathfrak{a}$ in $\mathbb{A}^{n+1}$. Then observe that $I(Z(\mathfrak{a})) \hookrightarrow I(V(\mathfrak{a}))$. By affine Nullstellensatz, it follows that $I(Z(\mathfrak{a})) \subseteq \sqrt{a}$. The converse is straightforward.   □

Some consequences of the homogeneous nullstellensatz yields us the familiar results as in the affine case.

Lemma 1.5.1.10. *Let k be an algebraically closed field and consider the projective n-space $\mathbb{P}_k^n$. Then,*

  1. There is a bijection

$$\{\textit{All algebraic sets } Y \subseteq \mathbb{P}_k^n\} \quad \overset{I}{\underset{Z}{\rightleftarrows}} \quad \{\textit{All homogeneous radical ideals of } k[x_0, \dots, x_n]\} \ .$$

  2. An algebraic set $Y \subseteq \mathbb{P}_k^n$ is irreducible if and only if $I(Y)$ is a prime ideal in $k[x_0, \dots, x_n]$.
  3. $\mathbb{P}_k^n$ is a projective algebraic variety.

Remark 1.5.1.11. A corollary of the above lemma is that one can look at projective algebraic varieties in $\mathbb{P}_k^n$ akin to homogeneous prime ideals in $k[x_0, \dots, x_n]$, thus telling us another hint at how the idea of schemes might have looked back in the days.

Proof of Lemma 1.5.1.10. 1. This is a direct consequence of homogeneous nullstellensatz (Lemma 1.5.1.9) and the fact that $Z(I(Y)) = \bar{Y}$ for any $Y \subseteq \mathbb{P}_k^n$.

2. (L $\Rightarrow$ R) Suppose $Y = Z(\mathfrak{a})$ is irreducible and $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ is not prime. Then there exists $f, g \in\notin \mathfrak{a}$ such that $fg \in \sqrt{\mathfrak{a}}$. Consider the ideals $\mathfrak{b} := \langle f, \sqrt{\mathfrak{a}} \rangle$ and $\mathfrak{c} := \langle g, \sqrt{\mathfrak{a}} \rangle$. We then observe that $Z(\mathfrak{b}), Z(\mathfrak{c}) \subseteq Z(\mathfrak{a})$ and $Z(\mathfrak{b}) \cup Z(\mathfrak{c}) = Z(\mathfrak{b}\mathfrak{c}) = Z(\sqrt{\mathfrak{a}}) = Z(\mathfrak{a})$, where we have used Lemma 1.5.1.9 in the second last equation and the fact that $fg \in \sqrt{a}$ in third last. This yields a contradiction to the irreducibility of Y.
(R $\Rightarrow$ L) Suppose $I(Y)$ is prime but Y is not irreducible. Consequently, there are proper closed sets $Y_1, Y_2 \subseteq Y$ such that $Y_1 \cup Y_2 = Y$. Further, we obtain that $I(Y_i) \ge I(Y)$ for each $i = 1, 2$. It then follows that there exists $f_i \in I(Y_i) \setminus I(Y)$ such that $f_i \notin I(Y_j)$, $j \ne i$. Consequently, we have $f_1 f_2 \in k[x_0, \dots, x_n]$ such that $f_1 f_2(P) = f_1(P) f_2(P) = 0$ for all $P \in Y$, as $Y = Y_1 \cup Y_2$. We thus have a contradiction to primality of $I(Y)$.

3. Since $I(\mathbb{P}_k^n) = I(Z(\mathfrak{o})) = \sqrt{\mathfrak{o}} = \mathfrak{o}$, then by 2., $\mathbb{P}_k^n$ is irreducible. Note we have used the fact that $k[x_0, \dots, x_n]$ is an integral domain.   □

One of the reasons that one might be interested in projective varieties is that they "compactify" the question at hand, that is, there are no "missing points" in the ambient space. We will see more into this when we will see projective morphisms and invertible modules, but for now, it is good to keep in mind that reframing your question in the projective spaces/varieties may give you more handle (and of-course, machines) to solve the question at hand. In the same vein, we now see that every affine variety can be embedded compactly into a projective space, and this embedding is called the projective closure of the affine variety.

Definition 1.5.1.12. (**Projective closure of affine varieties**) Let k be an algebraically closed field and consider an affine variety $X \subseteq \mathbb{A}_k^n$. For any $i = 0, \ldots, n$, consider the homeomorphism

$$\theta_i : \mathbb{A}_k^n \longmapsto U_i$$
$$(a_1, \ldots, a_n) \longmapsto [1 : a_1 : \cdots : a_n],$$

as we considered in Proposition 1.5.1.7. Then, the ith-projective closure of X into $\mathbb{P}_k^n$ is given by the closure $\overline{\theta_i(Y)} \subseteq \mathbb{P}_k^n$ as a subspace in $\mathbb{P}_k^n$. We will usually say the 0th projective closure of X to be simply the *projective closure of X*.

Consider an affine variety $X \subseteq \mathbb{A}_k^n$ and consider $\overline{X} \subseteq \mathbb{P}_k^n$ to be the projective closure of X. Let $I(X) \leq k[y_1, \ldots, y_n]$ be the affine ideal of X and let $I(\overline{X}) \leq k[x_0, \ldots, x_n]$ be the homogeneous ideal of projective closure. A natural question is that how the homogeneous ideal $I(\overline{X})$ is connected to the affine ideal $I(X)$. The following proposition answers that.

Proposition 1.5.1.13. *Let k be an algebraically closed field and $X \subseteq \mathbb{A}_k^n$ be an affine variety. Let $I(X) \leq k[y_1, \ldots, y_n]$ be the affine ideal of X and let $I(\overline{X}) \leq k[x_0, \ldots, x_n]$ be the homogeneous ideal of projective closure. Then,*

$$I(\overline{X}) = \langle h_0(I(X)) \rangle$$

where $h_0 : k[y_1, \ldots, y_n] \rightarrow k[x_0, \ldots, x_n]$ is the 0th homogenization function (Definition 1.5.1.6).

Proof. Since $X \subseteq \mathbb{A}_k^n$ is irreducible and closure of irreducible is irreducible, therefore $\overline{X} \subseteq \mathbb{P}_k^n$ is irreducible. It would thus suffice to show that

$$\overline{X} = Z(h_0(I(X))).$$

Indeed, this would imply that $h_0(I(X))$ is a homogeneous prime ideal by Lemma 1.5.1.10, 1, thus applying $I(-)$ would yield the result. We therefore show the above equality. Consider any closed set $Y \supseteq X$ in $\mathbb{P}_k^n$. We then wish to show that $Y \supseteq Z(h_0(I(X)))$. Since $Y \subseteq \mathbb{P}_k^n$ is closed, therefore $Y = Z(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a}$ in $k[x_0, \ldots, x_n]$. It would thus suffice to show that

$$\mathfrak{a} \hookrightarrow h_0(I(X)).$$

It would further suffice to show the above inclusion only for homogeneous elements, as $\mathfrak{a}$ is generated by homogeneous elements. Consequently, pick any homogeneous polynomial $f \in \mathfrak{a}$. We can write

$$f(x_0, \ldots, x_n) = x_0^e g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

for some $g \in k[y_1, \ldots, y_n]$ and $e = \deg f$. In other words, $f = h_0(g)$. Now since $Y \supseteq X$, therefore $f(P) = 0 \forall P \in X \subseteq \mathbb{P}_k^n$, that is, if $P = [1 : a_1, \ldots, a_n] \in X$, then $f(1, a_1, \ldots, a_n) = 0$ and thus $g(a_1, \ldots, a_n) = 0$. Hence $g \in I(X) \leq k[y_1, \ldots, y_n]$. Thus $f = h_0(g)$ where $g \in I(X)$, that is, $f \in h_0(I(X))$, as required. $\square$

Dimension, hypersurfaces and complete intersections

Let us first understand how the notion of dimension plays out with the Krull dimension of homogeneous coordinate ring of a projective variety.

Proposition 1.5.1.14. *Let k be an algebraically closed field and $X \subseteq \mathbb{P}_k^n$ be a projective k-variety. Then,*
   1. $\dim k[X] = \dim X + 1$,
   2. $\dim X = \dim U_i \cap X$ where $U_i \subseteq X$ is an affine open subset as in Proposition 1.5.1.7, for all $i = 0, \ldots, n$.

Proof. We will prove the two statements together. The main technique here is, as usual, to reduce the computations to one of the affine patches. Let $U_i \subseteq \mathbb{P}_k^n$ be the hyperplane where $x_i \neq 0$. We know that U_is covers $\mathbb{P}_k^n$ and each U_i is isomorphic to $\mathbb{A}_k^n$. Denote $X_i = U_i \cap X$ so that X_i is an open subvariety of X. Further denote $k[X]^h$ to be the homogeneous coordinate ring of X and $k[X_i]^a$ the affine coordinate ring of X_i. Note that $k[X_i]^a = k[x_0, \ldots, \hat{x}_i, \ldots, x_n]/d_i I(X)$ where d_i is the ith dehomogenisation map. We would now like to note two things to move forward:
   1. $\dim X = \dim X_j$ for some $j = 0, \ldots, n$,
   2. $k[X]_{x_i}^h \cong k[X_i]^a[x_i, 1/x_i]$[15].
The first statement is immediate from the fact that $\dim Y = \sup_i \dim U_i$ for any space Y with U_i an open covering. The second statement is the heart of the proof. Indeed, consider the map $k[X]_{x_i}^h \to k[X_i]^a[x_i, 1/x_i]$ which takes an element f/x_i^n and treats it as a polynomial in $x_i, 1/x_i$ with coefficients in $k[X_i]^a$. One immediately checks all the necessary conditions to ensure that this is an isomorphism.

Observe that if K/k is algebraic, then $K(x)/k(x)$ is algebraic. It follows that $\operatorname{trdeg} k[X_i]^a[x_i, 1/x_i] = 1 + \operatorname{trdeg} k[X_i]^a$. We now complete the proof. We may assume $\dim X = \dim X_0$. Consequently, via Proposition 1.5.3.10, 6 and Theorem 18.7.2.1, we obtain the following equalities:

$$\dim k[X]^h = \operatorname{trdeg} k[X]^h = \operatorname{trdeg} k[X]_{x_0}^h = \operatorname{trdeg} k[X_0]^a[x_0, 1/x_0] = 1 + \operatorname{trdeg} k[X_0]^a$$
$$= 1 + \dim k[X_0]^a = 1 + \dim X_0 = 1 + \dim X.$$

---

[15]This statement can be seen as a generalization of Lemma 1.5.3.11.

The statement 2. follows from the following equalities:

$$\dim X_i = \dim k[X_i]^a = \operatorname{trdeg} k[X_i]^a_{x_i} = \operatorname{trdeg} k[X_i]^a[x_i, 1/x_i] - 1 = \operatorname{trdeg} k[X]^h_{x_0} - 1$$
$$= \operatorname{trdeg} k[X]^h - 1 = \dim X + 1 - 1 = \dim X.$$

$\square$

We would now like to establish the following result, which will later motivate the definition of Weil divisors and of complete intersections.

Lemma 1.5.1.15. *Let k be an algebraically closed field and $X \subseteq \mathbb{P}^n_k$ be a projective k-variety. Then, the following are equivalent*
1. *$\dim X = n - 1$.*
2. *The homogeneous ideal $I(X) \leq k[x_1, \ldots, x_n]$ is generated by a single irreducible homogeneous polynomial.*

Proof. (1. $\Rightarrow$ 2.) By Proposition 1.5.1.14, 1, we have $\dim k[X] = n$, where $k[X] = k[x_0, \ldots, x_n]/I(X)$. By Theorem 18.7.2.2, we have $\operatorname{ht} I(X) = 1$. Since any height 1 prime ideal of a UFD is principal, therefore $I(X)$ is principal. Since $I(X)$ is homogeneous, therefore the statement 2. follows.
(2. $\Rightarrow$ 1.) By Proposition 1.5.1.14, 2 and Theorem 18.7.2.2, we have

$$\dim X = \dim X_0 = \dim k[X_0]^a = n - \operatorname{ht} d_0(I(X)).$$

We need only show that $\operatorname{ht} d_0(I(X)) = 1$. Since $I(X) = \langle p(x_0, \ldots, x_n) \rangle$, therefore $d_o(I(X)) = \langle p(1, x_1, \ldots, x_n) \rangle$. Since $k[x_0, \ldots, x_n]$ is a UFD and an easy observation about UFDs yields that height 1 prime ideals are exactly principal prime ideals, therefore the result follows. $\square$

Cones

d-uple embedding

Veronese surface

Segre embedding

To complete co[n]
uple, Veronese,
and quadrics, C[
ter 1.

1.5.2 Morphism of varieties

We have defined affine and projective varieties so far. One would often, however, would like to know whether a subset of $\mathbb{A}^n$ or $\mathbb{P}^n$ is an open subspace of some affine or projective variety. Due to to this need, we define the following.

Definition 1.5.2.1. (Quasi-affine/projective variety) A subset X of $\mathbb{A}^n$ or $\mathbb{P}^n$ is said to be quasi-affine or quasi-projective if X is an open subset of an affine or projective variety, respectively.

Let X be a quasi-affine or projective variety. From our knowledge of geometry, we know that in a real C^α-manifold M, the right type of functions are those which are defined on open subsets of M as C^α-maps to $\mathbb{R}$, where the latter is treated as a C^α-manifold. Consequently, we are interested in the same type of maps to the affine line $\mathbb{A}^1_k$.

Definition 1.5.2.2. (**Regular maps**) This notion is defined differently for quasi-affine and quasi-projective varieties.

1. Let X be a quasi-affine variety. A function

$$\varphi : X \to \mathbb{A}^1_k$$

is said to be a *regular function* if for all $P \in X$, there exists an open subset $U \subseteq X$ such that $\varphi|_U = g/h$ where $g, h \in k[x_1, \ldots, x_n]$ and $h(P) \neq 0 \, \forall P \in U$.

2. Let X be a quasi-projective variety. A function

$$\varphi : X \to \mathbb{A}^1_k$$

is said to be a *regular function* if for all $P \in X$, there exists an open subset $U \subseteq X$ such that $\varphi|_U = g/h$ where $g, h \in k[x_0, \ldots, x_n]$ are homogeneous polynomials of same degree and $h(P) \neq 0 \, \forall P \in U$. Note that this defines a valid function to the affine line.

Indeed, regular maps are continuous.

Lemma 1.5.2.3. *Let X be a quasi-affine or quasi-projective variety and $\varphi : X \to \mathbb{A}^1_k$ be a regular function. Then φ is continuous.*

Proof. The Zariski topology on $\mathbb{A}^1_k$ is the cofinite topology, hence any closed set in $\mathbb{A}^1_k$ is a finite union of points of k. It thus suffices to show that for any $a \in k$, $Y := \varphi^{-1}(a) \subseteq X$ is closed. Since checking a set is closed is local in X, that is, $Y \subseteq X$ is closed if and only if there exists an open covering of X, say $\{U_\alpha\}$ such that $U_\alpha \cap Y$ is closed in U_α. We may thus replace X by an open subset of X where φ is represented as g/h for $g, h \in k[y_1, \ldots, y_n]$ (in $k[x_0, \ldots, x_n]$, homogeneous and of same degree in the projective case). Consequently, $\varphi^{-1}(a) \subseteq X$ is given by $\{P \in X \mid (g - ah)(P) = 0\}$ which in other words is $Z(g - ah)$ ($g - ah$ is homogeneous in the projective case). Thus $\varphi^{-1}(a)$ is closed, as required. $\square$

A simple corollary of above is the first striking result one learns in complex analysis for holomorphic maps (see Proposition 12.2.3.10).

Lemma 1.5.2.4. *(Identity principle) Let $\varphi, \xi : X \to \mathbb{A}^1_k$ be two regular maps over a quasi-affine or quasi-projective variety X. Then, $\varphi = \xi$ if and only if there exists an open set $U \subseteq X$ such that $\varphi = \xi$ over U.*

Proof. L $\Rightarrow$ R is easy. For R $\Rightarrow$ L, observe that for $\phi := \varphi - \xi$ is continuous by Lemma 1.5.2.3. Further, the set $\phi^{-1}(0) \subseteq X$ is closed and contains U. Since $\phi^{-1}(0) \supseteq U$ and U is an open set of an irreducible space, therefore U is dense in X. Consequently, $\phi^{-1}(0)$ is a closed and dense in X, hence is equal to X. $\square$

We now define varieties in general.

Definition 1.5.2.5. (**Varieties**) Let k be an algebraically closed field. A variety over k is defined to be a quasi-affine or a quasi-projective variety in $\mathbb{A}^n_k$ or $\mathbb{P}^n_k$, respectively.

The notion of morphism of varieties is then given by functions which pulls regular functions back by pre-composition.

Definition 1.5.2.6. (**Map of varieties**) Let k be an algebraically closed field and let X, Y be two varieties over k. A map of varieties is a continuous function $f : X \to Y$ such that for any open set $V \subseteq Y$ and any regular function $\varphi : Y \to \mathbb{A}^1_k$, the function

$$\varphi \circ f : \varphi^{-1}(V) \to \mathbb{A}^1_k$$

is a regular function on the open set $\varphi^{-1}(V)$ of X. We may also call a map of varieties a *morphism of varieties.*

We therefore obtain the category of varieties over k, whose objects are varieties over k and arrows are maps of varieties. We will denote this category by

$$\mathbf{Var}_k.$$

Just like in topological spaces, it is not true in general that a bijective continuous map is a homeomorphism, similarly it is not true in general that a bijective map of varieties is an isomorphism of varieties, as the following example shows.

Example 1.5.2.7. Consider the affine line $\mathbb{A}^1_k$ and consider the affine variety $X := Z(y^2 - x^3) \subseteq \mathbb{A}^2_k$. The function

$$f : \mathbb{A}^1_k \longrightarrow X$$
$$t \longmapsto (t^2, t^3)$$

is a map of varieties as for any open set $U \subseteq X$ and regular map $\varphi : X \to \mathbb{A}^1_k$, the composite $\varphi \circ f : \varphi^{-1}(U) \to \mathbb{A}^1_k$ is given by $t \mapsto \varphi(t^2, t^3)$ and then the regularity of this composite can be seen to be a result of regularity of φ. Further note that f induces an inverse continuous function

$$f^{-1} : X \longrightarrow \mathbb{A}^1_k$$
$$(a, b) \longmapsto ba^{-1}.$$

Thus, $\mathbb{A}^1_k$ and X are homeomorphic as topological spaces. However, as varieties, they can not be isomorphic. Indeed, we shall soon see that coordinate rings are invariant of affine varieties and in our case $\mathbb{A}^1_k$ has $k[x]$ as its coordinate ring whereas X has $k[x, y]/\langle y^2 - x^3 \rangle$ as its coordinate ring. These are not isomorphic as one is PID and the other is not.

We now construct some more algebraic gadgets on top of varieties and will prove how they will turn out to be invariants of the varieties under question. We have already seen one, the coordinate ring. We will now see the construction of others and we shall do it in a manner so that it is amenable to generalization to schemes, as is studied elsewhere in this chapter.

1.5.3 Varieties as locally ringed spaces

See Chapter 6, Foundational Geometry, for background on locally ringed spaces and basic global algebra. In this section, we would like to interpret varieties as locally ringed spaces, so that we can understand later that how a variety can be interpreted as a scheme. Clearly, for a variety X, we already have an underlying topological space X itself. To give X the structure of a locally ringed space, we need to consider a sheaf over X. We shall use regular functions over open sets of X for that.

Definition 1.5.3.1. (Structure sheaf of a variety) Let k be an algebraically closed field and X be a variety over k. For each open set $U \subseteq X$, consider the following set

$$\mathcal{O}_X(U) := \{f : U \to \mathbb{A}^1_k \mid f \text{ is regular}\}.$$

Further, for open $V \subseteq U$ in X, consider the function

$$\rho_{U,V} : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$$
$$f \longmapsto f|_V .$$

This defines a sheaf of sets, as the following lemma shows.

Lemma 1.5.3.2. *The assignment $\mathcal{O}_X$ on open sets of a k-variety X as defined in Definition 1.5.3.1 defines a sheaf of sets over X.*

Proof. The locality axiom is straightforward as $\mathcal{O}_X(U)$ is a collection of functions, which thus can be checked locally for equality. It thus suffices to show that $\mathcal{O}_X$ satisfies the gluing axiom. Pick any open set U, an open covering $\{U_i\}_{i \in I}$ of U and a matching family $f_i \in \mathcal{O}_X(U_i)$ for each $i \in I$, that is $\rho_{U_i, U_i \cap U_j}(f_i) = \rho_{U_j, U_i \cap U_j}(f_j)$ for each $i, j \in I$. Consequently, we define $f : U \to \mathbb{A}^1_k$ given by $x \mapsto f_i(x)$ if $x \in U_i$. This is a well-defined function by the matching condition and further f is a regular function as for each point $x \in U$, f can be written as a rational function in some open neighborhood around x (essentially by regularity of f_is). Consequently, $\mathcal{O}_X$ is a sheaf. $\qquad\square$

Further, $\mathcal{O}_X$ is a sheaf of k-algebras if X is a k-variety.

Lemma 1.5.3.3. *Let k be an algebraically closed field and consider a k-variety X. The structure sheaf $\mathcal{O}_X$ of X is a sheaf of k-algebras.*

Proof. Indeed, $\mathcal{O}_X$ is a ring by point-wise addition and multiplication. Further, its a k-algebra via the injective ring homomorphism

$$k \hookrightarrow \mathcal{O}_X(U)$$
$$c \mapsto c : U \to \mathbb{A}^1_k$$

where c is treated as the constant rational map. $\qquad\square$

Hence, $(X, \mathcal{O}_X)$ is a k-ringed space. We now show that it is locally k-ringed.

Lemma 1.5.3.4. *Let k be an algebraically closed field and let X be a k-variety. Then, for all points $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.*

Proof. We wish to show that $\mathcal{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x \leq \mathcal{O}_{X,x}$. Consider the set

$$\mathfrak{m}_x := \{(U, f) \in \mathcal{O}_{X,x} \mid f(x) = 0\}.$$

It then easily follows that $\mathfrak{m}_x$ an ideal and consequently is a maximal ideal because $\mathcal{O}_{X,x} \setminus \mathfrak{m}_x$ is jut the set of all units of $\mathcal{O}_{X,x}$. $\qquad\square$

Remark 1.5.3.5. We have thus established that for any k-variety X we obtain a locally k-ringed space $(X, \mathcal{O}_X)$. We now observe how the data of a morphism of varieties can be represented as data of a morphism of underlying locally ringed spaces.

The notion of morphism of locally ringed spaces is elucidated in Definition 6.1.0.2.

Lemma 1.5.3.6. *Let k be an algebraically closed field and X, Y be two k-varieties. Then, there is an injective inclusion*

$$\operatorname{Hom}_{\mathbf{Var}_k}(X, Y) \hookrightarrow \operatorname{Hom}_{\mathbf{LRSpace}}(X, Y).$$

Proof. Indeed, consider the map

$$\theta : \operatorname{Hom}_{\mathbf{Var}_k}(X, Y) \hookrightarrow \operatorname{Hom}_{\mathbf{LRSpace}}(X, Y)$$
$$f : X \to Y \longmapsto (f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

where $\theta(f)$ has the underlying continuous map same as f but the map on sheaves, $f^\flat : \mathcal{O}_Y \to f_* \mathcal{O}_X$, is given on sections as follows: let $V \subseteq Y$ be an open set, then the map on sections over V is

$$f_V^\flat : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V))$$
$$(V, \varphi) \longmapsto (f^{-1}(V), \varphi \circ f).$$

The fact that $f^\flat$ as defined above is indeed a sheaf morphism is straightforward. We thus need only show that the adjoint map $f^\sharp$ of the above defines a map on stalks which is local. For this, we need only observe how the comorphism, $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$, as defined in Definition 6.1.0.2, in this case turns out to be the following mapping

$$(V, \varphi)_x \longmapsto (f^{-1}(V), \varphi \circ f)_x.$$

Now if $(V, \varphi)_x \in \mathfrak{m}_{Y,f(x)}$, then $\varphi(f(x)) = 0$ by definition. Thus $(f^{-1}(V), \varphi \circ f) \in \mathfrak{m}_{X,x}$. With this, the fact that θ is injective is straightforward. $\square$

Remark 1.5.3.7. We therefore have an inclusion

$$\mathbf{Var}_k \hookrightarrow \mathbf{LRSpace}.$$

Indeed, we now show that the notion of isomorphisms coincide here.

We will now define various algebraic gadgets out of the structure sheaf $\mathcal{O}_X$ of a variety X. Indeed, to some extent, that's the goal of algebraic geometry in general.

We now define an important field corresponding to each variety X, called its function field.

Definition 1.5.3.8. (**Function field of a variety**) Let k be an algebraically closed field and X be a k-variety. The function field of X, denoted $K(X)$, is obtained as the quotient of the set $\cup_{U \supseteq X, \text{ open}} \cup_{(U,\varphi) \in \mathcal{O}_X(U)} (U, f)$ by the following relation

$$(U, \varphi) \sim (V, \phi) \iff \exists \text{ open } W \subseteq U \cap V \text{ s.t. } \rho_{U,W}(\varphi) = \rho_{V,W}(\phi).$$

Indeed, this has an addition and a multiplication given by restriction to the open sets where they agree. This is further a field as any non-zero element $[(U, f)]$ can be inverted in a small enough open set $W \subseteq U$ (which will be non-empty as X is irreducible) where f is non-zero (otherwise the class $[(U, f)]$ is identically zero).

Remark 1.5.3.9. Note that we have the following ring homomorphisms for any k-variety X and $x \in X$

$$\Gamma(\mathcal{O}_X, X) \longrightarrow \mathcal{O}_{X,x} \longrightarrow K(X)$$
$$(X, \varphi) \longmapsto (X, \varphi)_x \longmapsto [(X, \varphi)].$$

In-fact, both these are injective by a simple use of the identity principle (Lemma 1.5.2.4). In this way, algebraic gadgets start taking a hold onto the geometry of varieties, which we will see further in this chapter.

We now give two results; one for affine and one for projective; which shows how the three algebraic gadgets introduced in Remark 1.5.3.9 can be realized more algebraically.

Proposition 1.5.3.10. *Let k be an algebraically closed field and let X be an affine k-variety. Let $\mathfrak{m}_p = \{f \in k[X] \mid f(p) = 0 \text{ as a regular function}\}$. Then,*
1. *$\mathfrak{m}_p$ is a maximal ideal of $k[X]$ for every point $p \in X$,*
2. *$\mathrm{mSpec}\,(k[X]) \cong X$ as sets,*
3. *$k[X]_{\mathfrak{m}_p} \cong \mathcal{O}_{X,p}$,*
4. *$k[X]_{\langle 0 \rangle} \cong K(X)$,*
5. *$\Gamma(\mathcal{O}_X, X) \cong k[X]$,*
6. *$\dim X = \mathrm{trdeg}\, K(X)/k$[16],*
7. *$\dim X = \dim \mathcal{O}_{X,p}$ for all $p \in X$[17].*

Proof. We give the main ideas of each. The main idea in the latter parts is to embed all the relevant rings inside the function field and do the relevant algebra there.
1. Since there is a correspondence between radical ideals of $k[X]$ and algebraic sets of X and since the correspondence is antitone, therefore minimal algebraic sets (point $p \in X$) of X correspond to maximal ideals of $k[X]$ vanishing at p. The result then follows.
2. This follows from 1. Explicitly, one considers the mapping $p \in X \mapsto \mathfrak{m}_p$.
3. Consider the canonical mapping

$$k[X]_{\mathfrak{m}_p} \longrightarrow \mathcal{O}_{X,p}$$
$$\frac{f}{g} \longmapsto (X \setminus Z(g), f/g)_p$$

    where $g(p) \neq 0$ (so $g \notin \mathfrak{m}_p$). This is a homomorphism by the fact that $Z(f) \cup Z(g) = Z(fg)$. This is injective because if $f/g = 0$, then $f = 0$ on some open subset

---

[16]Thus the function field $K(X)/k$ holds important global information about the algebra and geometry of X.

[17]Thus the notion of dimension of varieties is detectable at the level of stalks. This is because, as the proof and the statement 3 shows, the local ring $\mathcal{O}_{X,p}$ holds almost all relevant information about the coordinate ring.

$W \subseteq X \setminus Z(g)$. By an application of identity principle (Lemma 1.5.2.4), the injectivity follows. For surjectivity, observe that for any $(U, f)_p \in \mathcal{O}_{X,p}$, we can represent it by the rational function that f looks like around p, so $(U, f)_p = (W, g/h)_p$ where g/h is a rational function. Consequently, $g/h \mapsto (X \setminus Z(h), g/h)_p = (W, g/h)_p$. The result follows.

4. Observe first that if R is a domain and $\mathfrak{p} \leq R$ is an prime ideal of R, then $(R/\mathfrak{p})_{\langle 0 \rangle}$ is isomorphic to $R_{\langle 0 \rangle}$. Now, by 3, we obtain that $k[X]_{\langle 0 \rangle} \cong (k[X]_{\mathfrak{m}_p})\langle 0 \rangle \cong (\mathcal{O}_{X,p})_{\langle 0 \rangle}$. The map

$$(\mathcal{O}_{X,p})_{\langle 0 \rangle} \longrightarrow K(X)$$
$$\frac{(U, f/g)_p}{(V, h/l)_p} \longmapsto [(U \cap V, fl/gh)]$$

can be seen to be a well-defined (use Lemma 1.5.2.4) isomorphism.

5. By Lemma 18.1.2.12, we have that $\bigcap_{\mathfrak{m} < k[X]} k[X]_{\mathfrak{m}} \cong k[X]$. By Remark 1.5.3.9, we have $\Gamma(\mathcal{O}_X, X) \hookrightarrow \mathcal{O}_{X,p}$ (in $K(X)$). We further have $k[X] \hookrightarrow \Gamma(\mathcal{O}_X, X)$. Consequently, we obtain via 3. the following

$$k[X] \hookrightarrow \Gamma(\mathcal{O}_X, X) \hookrightarrow \bigcap_{p \in X} \mathcal{O}_{X,p} \cong \bigcap_{p \in X} k[X]_{\mathfrak{m}_p} \hookrightarrow \bigcap_{\mathfrak{m} < k[X]} k[X]_{\mathfrak{m}} \cong k[X].$$

The result then follows.

6. We have $\dim X = \dim k[X]$ as any irreducible closed subset of X corresponds in a contravariant manner to a prime ideal of $k[X]$. By Theorem 18.7.2.1, we have $\dim k[X] = \operatorname{trdeg} K(X)/k$.

7. By 3, $\dim \mathcal{O}_{X,p} = \operatorname{ht} \mathfrak{m}_p$. By Theorem 18.7.2.2, we have $\operatorname{ht} \mathfrak{m}_p + \dim k[X]/\mathfrak{m}_p = \dim k[X]$. But since $k[X]/\mathfrak{m}_p \cong k$ by Nullstellensatz, therefore the above equation reduces to $\operatorname{ht} \mathfrak{m}_p = \dim k[X]$ and the right side is just $\dim X$.

$\square$

We next do the projective case. See Chapter 18, Section 18.1.2 for homogeneous localization of graded rings.

Lemma 1.5.3.11. *Let k be an algebraically closed field and X be a projective k-variety in $\mathbb{P}_k^n$. Let $U_i = \mathbb{P}_k^n \setminus Z(x_i)$ and $X_i := X \cap U_i$. Then*

$$\varphi_i : k[X_i]^a \cong k[X]^h_{(x_i)}$$

where $k[X_i]^a$ denotes the affine coordinate ring of $X_i \subseteq \mathbb{A}_k^n$ and $k[X]^h$ denotes the homogeneous coordinate ring of $X \subseteq \mathbb{P}_k^n$. Further the localization above is homogeneous.

Proof. Consider the map $k[y_1, \ldots, y_n] \to k[x_0, \ldots, x_n]$ mapping as $f(y_1, \ldots, y_n) \mapsto f\left(\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}\right)$. This can easily be seen to be a well-defined ring isomorphism mapping the ideal $I(X_i) \mapsto I(X_i)^h = I(X)^h_{(x_i)}$. The result follows by quotienting. $\square$

Proposition 1.5.3.12. *Let k be an algebraically closed field and X be a projective k-variety. Let $\mathfrak{m}_p = \langle \{f \in k[X] \mid f \text{ is homogeneous } \& f(p) = 0\} \rangle$ for any $p \in X$ and $k[X]$ be the homogeneous coordinate ring of X. Then,*

1. $\mathfrak{m}_p$ is a maximal ideal of $k[X]$ for every element $p \in X$,
2. $k[X]_{(\mathfrak{m}_p)} \cong \mathcal{O}_{X,p}$,
3. $k[X]_{(\langle 0 \rangle)} \cong K(X)$,
4. $\Gamma(\mathcal{O}_X, X) \cong k$.

Proof. Denote by $k[X]^h$ the homogeneous coordinate ring and $X_i := X \cap U_i$ where $U_i = \mathbb{P}_k^n \setminus Z(x_i)$. By Lemma **??**, $U_i \cong \mathbb{A}_k^n$ as varieties, therefore denote X_i^a to be the affine variety corresponding to $X_i \subseteq U_i$. We thus denote $k[X_i]^h$ for the homogeneous coordinate ring when $X_i \subseteq U_i$ and $k[X_i]^a$ to be the affine coordinate ring when $X_i \subseteq \mathbb{A}_k^n$. Let $R := k[X]^h$. The main idea of the last part is to use the theory of integral dependence together with algebraic closure of k.

1. Let $P \in X$, so $P \in X_i$ for some $i = 0, \ldots, n$. Thus, let $P^a \in X_i^a$ and by Lemma 1.5.3.11 and Proposition 1.5.3.10, we obtain that $\mathfrak{m}_{P^a}$ is a maximal ideal of $k[X_i]^a$. Thus, $\varphi_i(\mathfrak{m}_{P^a}) = \mathfrak{m}_P k[X]_{x_i}^h$ is a maximal ideal of $k[X]_{x_i}^h$.

2. We simply have the following for any $p \in X$ by irreducibility of X, by Lemma 1.5.3.11 and by Proposition 1.5.3.10:

$$\mathcal{O}_{X,p} \cong \mathcal{O}_{X_i,p} \cong \mathcal{O}_{X_i^a,p^a} \cong k[X_i]_{\mathfrak{m}_{p^a}}^a \cong \left(k[X]_{x_i}^h \right)_{\mathfrak{m}_{p^a}} \cong k[X]_{\mathfrak{m}_{p^a}}^h.$$

3. By irreducibility of X, by Lemma 1.5.3.11 and by Proposition 1.5.3.10, we have the following identifications

$$K(X) \cong K(X_i) \cong K(X_i^a) \cong k[X_i]_{\langle 0 \rangle}^a \cong \left(k[X]_{x_i}^h \right)_{\langle 0 \rangle} \cong k[X]_{\langle 0 \rangle}^h.$$

4. First note that $k \hookrightarrow \Gamma(\mathcal{O}_X, X)$. It would thus suffice to show that $\Gamma(\mathcal{O}_X, X) \hookrightarrow k$. Pick any $f \in \Gamma(\mathcal{O}_X, X)$. We wish to show that $f \in k$. Let $R = k[X]^h$. Note that we can embed $\Gamma(X, \mathcal{O}_X)$ inside the (non-homogeneous) fraction field $L = k[X]_{\langle 0 \rangle}^h$. Consequently, by algebraic closure of k, it would suffice to show that $f \in L$ satisfies a polynomial with coefficients in k. Since f is a regular function on each of the X_i, therefore $f \in k[X_i]^a \cong k[X]_{x_i}^h$. Consequently, $f = g_i / x_i^{n_i}$ in L where $\deg g_i = n_i$ and thus $x_i^{n_i} f \in R_{n_i}$ for each $i = 0, \ldots, n$. It thus follows that $\deg f = 0$ in L. Consequently, it would suffice to show that $f \in L$ is integral over R (as we can then obtain a polynomial in $k[x]$ whose zero is f by restricting to 0 degree coefficients). By Corollary 18.6.1.6, it would thus suffice to show that $R[f]$ is a finitely generated R-module.

   It would thus suffice if we show that $\exists M \in \mathbb{N}$ such that $\forall N \geq M$, $R_N f^m \subseteq R_N$ for all $m \geq 0$. Indeed, for $M = \sum_i n_i$, we see that $R_N f \subseteq R_N$ as for any $g \in R_N$, we have that each term of g will have to have one x_i whose power is $\geq n_i$. Repeatedly applying $R_N f \subseteq R_N$ yields $R_N f^m \subseteq R_N$ for all $m \geq 0$, as needed.

   $\square$

Remark 1.5.3.13. Note that in Proposition 1.5.3.12, 1, the maximal ideal $\mathfrak{m}_P$ does not contain all of non-constant polynomials in $k[X]$ because $\mathfrak{m}_p$ is generated by homogeneous polynomials vanishing at $p \in X$ and a polynomial with non-zero constant terms cannot be in such an ideal, thus such an $\mathfrak{m}_p$ will exactly be the ideal of all non-constant polynomials in $k[X]$, but then $p \in \bigcap_{f \in k[X], f(0)=0} Z(f) = \emptyset$.

We now show that affine varieties are completely determined by their coordinate rings in the following sense

Theorem 1.5.3.14. *Let k be a algebraically closed field. Then the following*

$$k[-] : \mathbf{AfVar}^{\mathrm{op}}{}_k \longrightarrow \mathbf{FGIAlg}_k$$
$$X \longmapsto k[X]$$
$$X \xrightarrow{\varphi} Y \longmapsto k[Y] \xrightarrow{k[\varphi]} k[X]$$

is a functor[18] *which induces an equivalence between the opposite category of affine varieties over k and finitely generated integral domains over k.*

Proof. **TODO.** □

We now show some examples of the machinery developed so far. We first show that any affine plane conic is isomorphic as a variety to either the parabola $y - x^2$ or the hyperbola $xy - 1$. Indeed, we use here the familiar high-school topic that one classifies conics on the basis of discriminant(!) This will further show that the usual substitutions that we so used to do in school days to reduce an algebraic equation into a simpler form can equivalently be stated in algebraic language as finding a correct automorphism of the corresponding ring in question.

Type up the sol tions from noteb Chapter 1.

Subvarieties

---

[18]Note that by Proposition 1.5.3.10, this is just the global sections functor.

1.5.4   Varieties as schemes

In this section we show how to realize a k-variety (see Definition 1.5.2.5) as a scheme. This will be essential as it fulfill all the reasons to work with schemes as they generalize the concept of varieties to just the right level where all algebro-geometric questions can be asked and be attempted to be solved.

We first show a fully-faithful functor which embeds the category of k-varieties into the category of k-schemes (that is, schemes over k). This will hence show how to obtain a scheme from a variety because, as the following construction of the relevant functor will show, it is not straightforward how should one begin defining it[19].

Definition 1.5.4.1. (Spectral space of X) For every topological space X, we can associate a topological space

$$t(X) := \{\text{All non-empty closed irreducible subsets of } X\}$$

where any closed set is given by $t(Y) \subseteq t(X)$ for a closed set $Y \subseteq X$. The following lemma shows that this indeed defines a topology on $t(X)$. We will call $t(X)$ the spectral space of X.

Lemma 1.5.4.2. *Let X be a space and $Y, Z, Y_i \subseteq X$ be closed subsets of X. Then,*
  1. $t(Y) \subseteq t(X)$,
  2. $t(Y \cup Z) = t(Y) \cup t(Z)$,
  3. $t\left(\bigcap_i Y_i\right) = \bigcap_i t(Y_i)$.

Proof. 1. Any closed irreducible subset of Y, where Y is closed in X, will again be closed and irreducible in X.
2. Any irreducible subset of $Y \cup Z$ cannot have non-empty intersection with both of them.
3. Follows from 1. $\qquad\square$

Indeed, our main idea is to show that for a variety V, the space $t(V)$ will eventually become a scheme. We have few observations about spectral spaces, before we realize that idea.

Lemma 1.5.4.3. *Let X, X_1, X_2 be spaces and $f : X_1 \to X_2$ be a continuous map. Then,*
  1. there is a one-to-one correspondence between closed subsets of X and closed subsets of $t(X)$,
  2. the following is a continuous map

$$t(f) : t(X_1) \longrightarrow t(X_2)$$
$$Y_1 \longmapsto \overline{f(Y_1)},$$

  3. the following is a functor

$$t : \mathbf{Top} \longrightarrow \mathbf{Top}$$
$$X \longmapsto t(X),$$

---

[19]However, one may take a hint (albeit quite vague) from Lemma 1.3.0.2 in the following construction.

4. the following is a continuous map

$$\alpha : X \longrightarrow t(X)$$
$$x \longmapsto \overline{\{x\}}.$$

Proof. 1. Follows from the definition of topology on the spectral space.

2. Let $Y_2 \subseteq X_2$ be closed so that $t(Y_2) \subseteq t(X_2)$ is closed. We wish to show that $(t(f))^{-1}(t(Y_2)) \subseteq t(X_1)$ is closed. This follows from the observation that for $Y_1 \in t(X_1)$, we have $\overline{f(Y_1)} \in t(Y_2) \iff Y_1 \in t(f^{-1}(Y_2))$.

3. Follows from 2.

4. Pick any closed $Y \subseteq X$ to thus obtain a closed $t(Y) \subseteq t(X)$. Then $\alpha^{-1}(t(Y)) = \{x \in X \mid \overline{\{x\}} \in t(Y)\} = \{x \in X \mid x \in Y\} = Y$. $\qquad\square$

We now give scheme structure to the space $t(X)$. But first, we need a small lemma.

Lemma 1.5.4.4. *Let $A = k[V]$ be the coordinate ring of an affine k-variety V over an algebraically closed field k. Then, for any open set $U \subseteq \operatorname{Spec}(A)$, the set of all closed points of U are dense in U.*

Proof. Since all closed points of $\operatorname{Spec}(A)$ are its maximal ideals by Nullstellensatz, thus, any closed point of U is a maximal ideal of A as well. Consequently, we may assume $U = D(f)$ is a basic open set for $f \in A$. But since $D(f) \cong \operatorname{Spec}(A_f)$ and closed points of any affine scheme are always dense, the result follows. $\qquad\square$

Theorem 1.5.4.5. *Let k be an algebraically closed field and $(V, \mathcal{O}_V)$ be a k-variety. Let $\alpha : V \to t(V)$ be the continuous map as defined in Lemma 1.5.4.3, 4. Then, $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme over k which admits an affine open cover by $\operatorname{Spec}(A)$ for $A = k[W]$ where W is an affine open subvariety of V.*

Proof. For better clarity of this important proof, we break it in multiple acts.

$$\textbf{Act 1}: \textit{We may assume } V \textit{ is an affine } k\textit{-variety.}$$

Since we wish to show that $t(V)$ is a scheme, hence we need to produce an open cover of $t(V)$ by affine schemes. Since V is covered by open affine k-varieties, thus if we can show that for an affine k-variety W, the space $t(W)$ is a scheme, then we would be done. Hence we may assume V is affine with coordinate ring $k[V] =: A$.

$$\textbf{Act 2}: t(V) \cong \operatorname{Spec}(A) \textit{ as topological spaces.}$$

Consider the usual maps that we know from our study of varieties:

$$t(V) \xrightleftharpoons[Z(-)]{I(-)} \operatorname{Spec}(A)$$

These are easily seen to be continuous inverses of each other by the correspondence between closed irreducible subsets of an affine variety and prime ideals of its coordinate ring (Lemma 1.2.1.1).

Act 3 : *The closed points of* $\operatorname{Spec}(A)$ *are points of* V.

We first construct the following map

$$\varphi : V \longrightarrow \operatorname{Spec}(A)$$
$$p \longmapsto \mathfrak{m}_p$$

where $\mathfrak{m}_p$ is defined together with some properties in Proposition 1.5.3.10. This is continuous by a small check on closed sets. Moreover, this is injective. Now, we claim that $\varphi(V) \subseteq \operatorname{Spec}(A)$ are all closed points of $\operatorname{Spec}(A)$. Indeed, this follows from the correspondence between closed points of $\operatorname{Spec}(A)$ and maximal prime ideals of A (Lemma 1.2.1.3). We will thus denote $\varphi(V)$ as the set of closed points of $\operatorname{Spec}(A)$.

Act 4 : *It is enough to show that* $\varphi_* \mathcal{O}_V \cong \mathcal{O}_{\operatorname{Spec}(A)}$.

Since we have the following commutative triangle

$$\begin{array}{ccc}
 & V & \\
{\scriptstyle \varphi} \swarrow & & \searrow {\scriptstyle \alpha} \\
\operatorname{Spec}(A) & \underset{\underset{I(-)}{\longleftarrow}}{\overset{\overset{Z(-)}{\longrightarrow}}{\cong}} & t(V)
\end{array} \quad ,$$

thus $\alpha_* \mathcal{O}_V \cong (Z \circ \varphi)_* \mathcal{O}_V = Z_* \varphi_* \mathcal{O}_V$. Since Z is an isomorphism, thus the reduction is justified.

Act 5 : $\varphi_* \mathcal{O}_V \cong \mathcal{O}_{\operatorname{Spec}(A)}$.

Let $U \subseteq \operatorname{Spec}(A)$ be an open set. We will construct an isomorphism between $\mathcal{O}_{\operatorname{Spec}(A)}(U)$ and $\mathcal{O}_V(\varphi^{-1}(U))$. Consider the map

$$\eta_U : \mathcal{O}_{\operatorname{Spec}(A)}(U) \longrightarrow \mathcal{O}_V(\varphi^{-1}(U))$$
$$s : U \to \amalg_{\mathfrak{p} \in U} A_{\mathfrak{p}} \longmapsto \eta_U(s) : \varphi^{-1}(U) \to k$$

where for any $q \in \varphi^{-1}(U)$, we define $\eta_U(s)(q) = s(\mathfrak{m}_q)(q)$. It clearly is a ring homomorphism which commutes with appropriate restriction maps. Thus, we need to show the following three statements in order to conclude.

1. $\eta_U(s)$ is regular,
2. η_U has zero kernel,
3. η_U is surjective.

In-fact, the above three statements are at the technical heart of the proof. The main driving force behind this is the density of closed points of open sets in $\operatorname{Spec}(A)$ (Lemma 1.5.4.4) and the identity principle of regular maps on a variety (Lemma 1.5.2.4).

Statement 1. is immediate as s is regular. For statement 2., suppose that $\eta_U(s) = 0$ over $\varphi^{-1}(U)$. Thus $s(\mathfrak{m}_q)(q) = f_q(q)/g_q(q) = 0$ for all $q \in \varphi^{-1}(U)$. Thus, $\eta_U(s)$ around q is represented by rational function f_q/g_q. By Lemma 1.5.2.4 on $\eta_U(s)$, we obtain that $f_q = 0$ for all $q \in \varphi^{-1}(U)$. Thus s is zero at all closed points of U, which are exactly $\varphi(\varphi^{-1}(U))$.

But since closed points of U are dense by Lemma 1.5.4.4 and s is a locally constant function, hence $s = 0$.

Finally, to see statement 3., pick any $f \in \mathcal{O}_V(\varphi^{-1}(U))$ and notice that $W := \varphi(\varphi^{-1}(U))$ is a dense subset of U (set of all closed points, Lemma 1.5.4.4). Thus, it is enough to define a locally constant function s over W whose extension $\tilde{s}$ over U is such that $\eta_U(\tilde{s}) = f$. Indeed, consider

$$s : W \longrightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$
$$\mathfrak{m}_q \longmapsto g_q/h_q$$

where g_q/h_q is the rational function representing f at the point $q \in V$. Clearly, the extension $\tilde{s}$ is in $\mathcal{O}_{\mathrm{Spec}(A)}(U)$ and it is mapped by η_U to f.

Act 6 : $(t(V), \alpha_* \mathcal{O}_V)$ *is a scheme over* k.

Now let V be a k-variety. We wish to show that $t(V)$ is a scheme over $\mathrm{Spec}\,(k)$. Thus we need to produce a map $t(V) \to \mathrm{Spec}\,(k)$, which is equivalent to a map $k \to \Gamma(\alpha_* \mathcal{O}_V, t(V))$ via the Theorem 1.3.0.5. Since $\Gamma(\alpha_* \mathcal{O}_V, t(V)) = \Gamma(\mathcal{O}_V, V) = A$ via Proposition 1.5.3.10, 5, the result follows.

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 1.5.4.6. Theorem 1.5.4.5 yields that the functor t restricts to the following

$$t : \mathbf{Var}_k \longrightarrow \mathbf{Sch}_k$$
$$(V, \mathcal{O}_V) \longmapsto (t(V), \alpha_* \mathcal{O}_V).$$

We will now show that this is a fully-faithful embedding. In other words, any map of $t(V_1) \to t(V_2)$ as of schemes over k is equivalent to a map $V_1 \to V_2$ of k-varieties.

Let us begin with some elementary properties of the residue fields of the k-scheme $t(V)$ attached to a k-variety V.

Lemma 1.5.4.7. *Let* k *be an algebraically closed field and let* V *be a* k-variety. A point $p \in t(V)$ *is closed if and only if* $\kappa(p) = k$.

Proof. (L $\Rightarrow$ R) Since $p \in t(V)$ is closed and closed points of $t(V)$ are exactly points of V, therefore $p \in V \subseteq t(V)$. Consequently, for an affine k-variety $X \subseteq V$ containing p, we obtain the following by Proposition 1.5.3.10, 3 and Nullstellensatz:

$$\kappa(p) = \mathcal{O}_{t(V),p}/\mathfrak{m}_{t(V),p} \cong \mathcal{O}_{V,p}/\mathfrak{m}_{V,p} \cong \mathcal{O}_{X,p}/\mathfrak{m}_{X,p} \cong k[X]_{\mathfrak{m}_p}/\mathfrak{m}_p k[X]_{\mathfrak{m}_p} \cong (k[X]/\mathfrak{m}_p k[X])_0 \cong k.$$

(R $\Rightarrow$ L) By Theorem 1.5.4.5, we have that for some open affine k-variety $X \subseteq V$, $p \in \mathrm{Spec}\,(k)[X]$. Consequently, $\kappa(p) = (k[X]/pk[X])_0 = k$ where p is treated as a prime ideal of $k[X]$. Consequently, we have that the domain $k[X]/pk[X] = k$ as we have inclusions $k \hookrightarrow k[X]/pk[X] \hookrightarrow (k[X]/pk[X])_0$. Thus $p \lneq k[X]$ is maximal. $\qquad\qquad$ $\square$

Proposition 1.5.4.8. *Let k be an algebraically closed field. Then there is a natural bijection*

$$\mathrm{Hom}_{\mathbf{Var}_k}(V_1, V_2) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(t(V_1), t(V_2)).$$

That is, the functor t is a fully-faithful embedding of k-varieties into schemes over k.

Proof. Exercise 2.15 of Hartshorne Chapter 2. $\square$

Complete the proof of embedding varieties into schemes, Chapter 1.

Let us now spell out all the properties that the scheme $t(V)$ satisfies for a k-variety V.

Proposition 1.5.4.9. *Let k be an algebraically closed field and V be a k-variety. Then, the scheme $t(V)$ over k is (for * properties, see Section 1.12)*

1. *integral,*
2. *noetherian,*
3. *finite type over k,*
4. *quasi-projective*,*
5. *separated*.*

Proof. 1. to 3. are immediate from the open covering by $\mathrm{Spec}\,(k[W])$ of $t(V)$ where $W \subseteq V$ is an open affine subvariety (Theorem 1.5.4.5). Consequently $t(V)$ is covered by spectrum of finite type k-algebras.

4. is also immediate as any k-variety is an open subset of an affine or a projective k-variety by definition. Since any affine k-variety can be seen as a projective k-variety, consequently, we have an open immersion of V into a closed subvariety of some projective space over k. This extends to an open immersion of $t(V)$ into a closed subscheme of $\mathbb{P}^n_k$.

5. Follows from 4. and Theorem 1.12.6.2. $\square$

We now state an important rectification result which precisely shows what type of schemes are those which are in the image of functor t as in Remark 1.5.4.6.

Corollary 1.5.4.10. *Let k be an algebraically closed field. Then, the functor of Remark 1.5.4.6*

$$t : \mathbf{Var}_k \longrightarrow \mathbf{QPISch}_k$$

establishes an equivalence between varieties over k and quasi-projective integral schemes over k. Further, the image of projective varieties under this functor is exactly the projective integral schemes over k.

Proof. By Proposition 1.5.4.8, we reduce to showing that t lands into quasi-projective schemes and is essentially surjective. Indeed, for a k-variety V, the scheme $t(V)$ is quasi-projective by Proposition 1.5.4.9, 4. Now, to show essential surjection, we first observe that open subschemes of $t(V)$ is in one-to-one bijection with open subsets of V. Consequently, it would suffice to show that any projective integral k-scheme X is in the essential image of t. Indeed, let V denote the closed points of X as a closed subscheme of some $\mathbb{P}^n_k$. Consequently, as closed points of a finite type k-scheme is dense (Lemma 1.12.2.6), therefore V is irreducible (note we are using irreducibility of X here), thus a projective variety in $\mathbb{P}^n_k$. Now, $t(V)$ and X have same underlying space. As a subspace of $\mathbb{P}^n_k$, $t(V)$ and X have both have the structure of a reduced scheme over the common underlying space. By uniqueness of reduced induced closed subscheme structure on a closed subset, we have that $t(V) \cong X$ (see Section 1.6.3). $\square$

We now redefine varieties as schemes and use them as such for the remainder of the sections.

Definition 1.5.4.11. (**Abstract and classical varieties**) Let k be an algebraically closed field. An *abstract variety* or simply a variety, is a separated, integral finite type k-scheme. Those varieties which are furthermore quasi-projective are exactly the varieties we defined earlier by Corollary 1.5.4.10. We will further call the notion of varieties we defined earlier in Definition 1.5.2.5 by referring to them as *classical varieties.*

1.6 Fundamental constructions on schemes

In this section, we would like to understand some of the basic constructions which one can perform with a collection of schemes.

1.6.1 Points of a scheme

Let X be a scheme. Pick any point $x \in X$. We then have the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. Hence we have a projection map

$$\mathcal{O}_{X,x} \to \kappa(x).$$

Consider now an open affine $x \in \operatorname{Spec}(A) \subseteq X$. Consequently, we have $\mathcal{O}_{X,x} \cong \mathcal{O}_{\operatorname{Spec}(A),x} \cong A_x$. Thus, denoting the inclusion $j_x : \operatorname{Spec}(A_x) \hookrightarrow \operatorname{Spec}(A)$, we obtain the following composition:

$$i_x : \operatorname{Spec}(\kappa(x)) \to \operatorname{Spec}(\mathcal{O}_{X,x}) = \operatorname{Spec}(A_x) \xrightarrow{j_x} \operatorname{Spec}(A) \hookrightarrow X.$$

Remember that $\operatorname{Spec}(A_x)$ can be interpreted as the affine subset in $\operatorname{Spec}(A)$ which is "very close" to $x \in \operatorname{Spec}(A)$. The map j_x takes the singleton point in $\operatorname{Spec}(\kappa(x))$ to $x \in X$. This map is usually called the *canonical map of point $x \in X$*. The map on stalks that i_x yields is the natural projection $\mathcal{O}_{X,x} \to \kappa(x)$. This map is quite unique as it is universal amongst all those maps $\operatorname{Spec}(K) \to X$ which maps to x. Indeed, we have the following.

Lemma 1.6.1.1. *Let X be a scheme and let $x \in X$ be a point. If K is a field and $f : \operatorname{Spec}(K) \to X$ is a map, then*
  1. If $f(\star) = x$, then $\kappa(x) \hookrightarrow K$.
  2. If $f(\star) = x$, then f factors via the canonical map i_x at point $x \in X$

$$
\begin{array}{ccc}
\operatorname{Spec}(\kappa(x)) & \xrightarrow{\ i_x\ } & X \\
\uparrow & \nearrow{\scriptstyle f} & \\
\operatorname{Spec}(K) & &
\end{array}
\quad .
$$

  3. $\operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec}(K), X) \cong \{x \in X \mid \kappa(x) \hookrightarrow K\}$.

Proof. 1. At the stalk, we have a local ring homomorphism $\varphi : \mathcal{O}_{X,x} \to K$. Consequently, $\operatorname{Ker}(\varphi) = \mathfrak{m}_{X,x}$. It then follows that $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \hookrightarrow K$.
2. Clearly f factors as above as a continuous map. To check the commutativity of sheaf maps, we need only check at stalks (Theorem 22.2.0.7). This is straightforward, as we get on stalks the following commutative diagram:

$$
\begin{array}{ccc}
\kappa(x) & \longleftarrow & \mathcal{O}_{X,x} \\
\downarrow & \swarrow{\scriptstyle f_\star^\sharp} & \\
K & &
\end{array}
\quad .
$$

3. It suffices to show that a morphism $f : \operatorname{Spec}(K) \to X$ is equivalent to the data of a point $x \in X$ such that $\kappa(x) \hookrightarrow K$. By 1, one side is immediate. Now consider a point $x \in X$

and a field extension $\kappa(x) \hookrightarrow K$. We wish to construct a map $f : \mathrm{Spec}\,(K) \to X$ such that the above data is obtained via the construction in 1 applied on f. Indeed, the map f on topological spaces is straightforward, $f(\star) = x$. On sheaves, it reduces to define a natural local ring homomorphism $\mathcal{O}_{X,x} \to K$. This is immediate, as we need only define this as $\mathcal{O}_{X,x} \to \kappa(x) \hookrightarrow K$. $\qquad\square$

The above lemma shows that defining a map $\mathrm{Spec}\,(K) \to X$ is equivalent to taking a point $x \in X$ such that K extends $\kappa(x)$. There is another similar important characterization of maps from $\mathrm{Spec}\left(\frac{k[x]}{x^2}\right)$ into X, which characterizes all rational points of X together with "direction" (that is, together with an element of the tangent space). We first define a rational point of a k-scheme. Recall that by Corollary 1.3.0.7, $\kappa(x)$ is a field extension of k. Further observe the definition of Zariski tangent space T_xX of a scheme as defined in Definition 1.11.1.10.

Definition 1.6.1.2. (**Rational points**) Let X be a k-scheme. Then a point $x \in X$ is said to be rational if $\kappa(x) = k$.

Let us denote $k[\epsilon] = k[x]/x^2$. The ring $k[\epsilon]$ is usually called the ring of *dual numbers*.

Proposition 1.6.1.3. [20] *Let X be a scheme over a field k. Then, we have a bijection*

$$\mathrm{Hom}_{\mathbf{Sch}/k}\left(\mathrm{Spec}\,(k[\epsilon]), X\right) \cong \{(x, \xi) \mid x \in X \text{ is a rational point } \& \ \xi \in T_xX\}$$

Proof. ($\Rightarrow$) Take a scheme homomorphism $f : \mathrm{Spec}\,(k[\epsilon]) \to X$. Note that we have a map

$$k[\epsilon] \to k[\epsilon]/\epsilon \cong k.$$

Consequently, we get a map $g : \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k[\epsilon])$ which by composing by f, we get

$$\mathrm{Spec}\,(k) \xrightarrow{g} \mathrm{Spec}\,(k[\epsilon]) \xrightarrow{f} X.$$

Observe that $\mathrm{Spec}\,(k[\epsilon])$ is a one point scheme, therefore $f(\mathrm{pt.}) = f \circ g(\mathrm{pt.}) =: x$. We wish to show that x is a rational point. By Lemma 1.6.1.1, 3, we have $\kappa(x) \hookrightarrow k$. But since X is a scheme over k, therefore $k \hookrightarrow \kappa(x)$. We further deduce from the fact that X is a k-scheme that we have a triangle

This shows that horizontal arrow above is an isomorphism. Thus, $\kappa(x) = k$. We now wish to obtain an element of T_xX.

At the point $x \in X$, we have a map $f : \mathrm{Spec}\,(k[\epsilon]) \to X$. This yields a map on stalks given by

$$\varphi : A \to k[\epsilon]$$

---

[20]Exercise II.2.8 of Hartshorne.

where $A = \mathcal{O}_{X,x}$ is the local ring at point $x \in X$ and φ is furthermore a local k-algebra homomorphism. Let $\mathfrak{m}$ be the maximal ideal of the local ring A. Then, $A/\mathfrak{m} = \kappa(x)$, which is equal to k as x is a rational point. Thus, A is a rational local k-algebra (Definition 18.1.2.20). It follows from Proposition 18.1.2.21 that φ is equivalent to an element of the tangent space $\xi \in TA$ and by definition, $TA = T_xX$. This completes the proof. $\qquad\square$

1.6.2 Gluing schemes

We now show how to obtain new schemes from old by the gluing construction. Indeed, the idea is simple, glue the underlying topological spaces of a certain collection of schemes and identifications and define a new structure sheaf over the resultant space which canonically makes it into a scheme. We will further see that there is a universal property that is satisfied by such a glue. We suggest that the reader make a diagram of *blobs* and draw the corresponding maps in order to see the naturality of the following.

Definition 1.6.2.1. (**Gluing datum**) A tuple of data $(I, \{X_i\}_{i \in I}, \{U_{ij}\}_{i,j \in I}, \{\varphi_{ij}\}_{i,j \in I})$ of an index set I, schemes X_i for each $i \in I$, open subschemes $U_{ij} \subseteq X_i$ for each $i, j \in I$ and scheme isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ for each $i, j \in I$ is a gluing datum if it satisfies the following:

1. $U_{ii} = X_i$ for all $i \in I$,
2. $\varphi_{ji} = \varphi_{ij}^{-1}$,
3. $\varphi_{ii} = \mathrm{id}_{U_{ii}} = \mathrm{id}_{X_i}$,
4. the cocycle condition,

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \text{ on } U_{ij} \cap U_{ik} \; \forall i, j, k \in I.$$

We then have that there is a unique glue of the above.

Proposition 1.6.2.2. *For a gluing datum* $(I, \{X_i\}_{i \in I}, \{U_{ij}\}_{i,j \in I}, \{\varphi_{ij}\}_{i,j \in I})$ *of schemes, there exists a unique scheme X with the following properties:*

1. there exists an open embedding of schemes

$$\phi_i : X_i \to X \text{ for each } i \in I,$$

2. $\phi_j \circ \varphi_{ij} = \phi_i$ on U_{ij} for all $i, j \in I$,
3. $X = \bigcup_{i \in I} \phi_i(X_i)$,
4. $\phi_i(X_i) \cap \phi_j(X_j) = \phi_i(U_{ij}) = \phi_j(U_{ji})$ for all $i, j \in I$.

Proof. The underlying space of X is obtained by gluing the underlying spaces of X_i in the usual manner;

$$X := \coprod_{i \in I} X_i / \sim$$

where $x_i \sim \varphi_{ij}(x_i)$ for all $x_i \in U_{ij}$ and $i, j \in I$. Let $\phi_i : X_i \to X$ be the canonical inclusion map. The topology is given on X via the quotient topology; $U \subseteq X$ is open if and only if $\phi_i^{-1}(U) \subseteq X_i$ is open for each $i \in I$. Then to define the sheaf $\mathcal{O}_X$, pick any open $U \subseteq X$ and define the sections over it as follows (let us write $\varphi_{ij} : \mathcal{O}_{U_{ij}} \xrightarrow{\cong} \mathcal{O}_{U_{ji}}$ as well):

$$\mathcal{O}_X(U) = \left\{ [(\phi_i^{-1}(U), s_i)] \mid \forall i, \; s_i \in \mathcal{O}_{X_i}(\phi_i^{-1}(U)) \text{ s.t. } \varphi_{ij}(\rho_{\phi_i^{-1}(U), \phi_i^{-1}(U) \cap U_{ij}}(s_i)) = \rho_{\phi_j^{-1}(U), \phi_j^{-1}(U) \cap U_{ji}}(s_j) \right\}.$$

By local nature, this is again a sheaf (also called the glued sheaf). Now, ϕ_i is an open embedding as for any open $U \subseteq X$, it follows that $\mathcal{O}_X(\varphi_i(X_i) \cap U) \cong \mathcal{O}_{X_i}(\phi_i^{-1}(U))$. Thus, X is a scheme as for each $x \in X$, $x \in \phi_i(X_i)$ which is a scheme. $\qquad\square$

The uniqueness of the glue follows from the fact the universal property that glued scheme satisfies. **TODO**. Gluing allows us to construct many non-affine schemes, like the projective n-scheme over k.

> Put some nice e
> amples of schem
> formed via gluin
> and do uniquene
> Chapter 1.

1.6.3   Reduced scheme of a scheme

For any scheme X, we can obtain a scheme with the same underlying space but with reduced structure sheaf. This procedure is called *reducing a scheme* to a reduced scheme.

Construction 1.6.3.1. Let X be a scheme. Consider the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)/\mathfrak{n}_U$ where $\mathfrak{n}_U$ is the nilradical of $\mathcal{O}_X(U)$ and denote this sheaf by $\mathcal{O}_X^{\mathrm{red}}$. The pair $(X, \mathcal{O}_X^{\mathrm{red}})$ will be called the *associated reduced scheme* of the scheme $(X, \mathcal{O}_X)$, usually denoted by X_{red}. Indeed, $(X, \mathcal{O}_X^{\mathrm{red}})$ is a scheme as the following result shows.

Lemma 1.6.3.2. [21] *Let X be a scheme. Then,*
  1. *the pair $(X, \mathcal{O}_X^{\mathrm{red}})$ is a scheme,*
  2. *there exists a map of schemes $\varphi : (X, \mathcal{O}_X^{\mathrm{red}}) \to (X, \mathcal{O}_X)$ which is a homeomorphism on the spaces.*

Proof. 1. Let $(\mathrm{Spec}\,(A), \mathcal{O}_{\mathrm{Spec}(A)})$ be an open affine of X. We shall show that $(\mathrm{Spec}\,(A), \mathcal{O}_{\mathrm{Spec}(A)}^{\mathrm{red}})$ is isomorphic to $(\mathrm{Spec}\,(A_{\mathrm{red}}), \mathcal{O}_{\mathrm{Spec}(A_{\mathrm{red}})})$. First, the isomorphism on spaces is straightforward as every prime ideal contains nilradical (nilradical is the intersection of all prime ideals, Lemma 18.1.2.9). We thus need to produce a sheaf morphism $\mathcal{O}_{\mathrm{Spec}(A)}^{\mathrm{red}} \to \mathcal{O}_{\mathrm{Spec}(A_{\mathrm{red}})}$ which is an isomorphism. Let us denote the presheaf $U \mapsto \mathcal{O}_X(U)/\mathfrak{n}_U$ by F. We first immediately reduce to showing the existence of a map $F \to \mathcal{O}_{\mathrm{Spec}(A_{\mathrm{red}})}$ which is an isomorphism on basic open sets, as we then obtain a map of sheaves $\mathcal{O}_{\mathrm{Spec}(A)}^{\mathrm{red}} \to \mathcal{O}_{\mathrm{Spec}(A_{\mathrm{red}})}$ by the universal property of sheafification (Theorem 22.1.0.1) which is an isomorphism on stalks (Theorem 22.2.0.6, 4).

Since sheaves and sheaf morphisms are uniquely determined by defining them on a basis, thus we further reduce to defining a presheaf map $F \to \mathcal{O}_{\mathrm{Spec}(A_{\mathrm{red}})}$ with above properties on a basis. Since $\mathrm{Spec}\,(A)$ has a canonical basis, namely, $\mathcal{B} = \{\mathrm{Spec}\,(A)_f\}_{f \in A}$, consequently one sees that isomorphism $A_f/\mathfrak{n}_f \cong (A/\mathfrak{n})_f$ can be naturally extended to a presheaf map $F \to \mathcal{O}_{\mathrm{Spec}(A_{\mathrm{red}})}$, which is an isomorphism on the basis $\mathcal{B}$.

2. Consider the map $f : (X, \mathcal{O}_X^{\mathrm{red}}) \to (X, \mathcal{O}_X)$ which is given by id_X on spaces but by the following quotient map $\mathcal{O}_X(U) \to \mathcal{O}_X(U)/\mathfrak{n}_U \to \mathcal{O}_X^{\mathrm{red}}(U)$. $\qquad\square$

There is a universal property of reduced schemes which says that a map out of a reduced scheme necessarily has to factor through the reduction of the codomain.

---

[21]Exercise II.2.3.b of Hartshorne.

Proposition 1.6.3.3. [22] *Let* $f : X \to Y$ *be a map of schemes with* X *being a reduced scheme. Then there exists a unique map of schemes* $g : X \to Y_{\mathrm{red}}$ *such that the triangle commutes:*

$$
\begin{array}{ccc}
Y & \xleftarrow{\;\varphi\;} & Y_{\mathrm{red}} \\
{\scriptstyle f}\big\uparrow & \nearrow{\scriptstyle g} & \\
X & &
\end{array}
\quad .
$$

Proof. The map g on spaces is immediate; it shoould be identical to f as φ is identity on spaces. The map $g^\flat$ on the other hand can be constructed as follows. First observe that if A and B are rings with B being reduced, then any ring map $\eta : A \to B$ extends to a unique map $\tilde{\eta} : A_{\mathrm{red}} \to B$ given by $a + \mathfrak{n} \mapsto \eta(a)$ which makes the triangle commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\;\eta\;} & B \\
\big\downarrow & \nearrow{\scriptstyle \tilde{\eta}} & \\
A_{\mathrm{red}} & &
\end{array}
\quad .
$$

In our case, we therefore get a unique map $\tilde{f_U^\flat}$ as below for any $U \subseteq Y$, which further gives us the required unique map $g_U^\flat : \mathcal{O}_Y^{\mathrm{red}}(U) \to \mathcal{O}_X(f^{-1}(U))$ which we need (by universality of sheafification, Theorem 22.1.0.1):

$$
\begin{array}{ccc}
& \xrightarrow{\makebox[3cm]{$f_U^\flat$}} & \\
\mathcal{O}_Y(U) \xrightarrow{\;\varphi_U\;} \mathcal{O}_Y^{\mathrm{red}}(U) & \xrightarrow{=:g_U^\flat} & \mathcal{O}_X(f^{-1}(U)) \\
\searrow \quad\quad \uparrow & \nearrow{\scriptstyle \tilde{f_U^\flat}} & \\
\mathcal{O}_Y(U)/\mathfrak{n}_U & &
\end{array}
\quad .
$$

One can then easily check that g as given above makes the triangle commute.

For each closed set $Y \subseteq X$ of a scheme, we construct a unique closed reduced subscheme structure over it.

Construction 1.6.3.4. (*Reduced induced subscheme*) **TODO**.

$\square$

1.6.4   Fiber product of schemes

One of the most important tool in scheme theory is that of fiber product of schemes. This is essential as this is exactly the right notion using which one can define intersection of subschemes, which is one of the fundamental goals of this book.

Existence of fiber products is equivalent to saying that the category of schemes **Sch** have all pullbacks. In particular, it is equivalent to saying that for any two S-schemes X

---

[22]Exercise II.2.3.c of Hartshorne.

and Y, their product in **Sch**$/S$ exists, called the *fiber product* denoted $X \times_S Y$.

However, we need to be more explicit than this abstract definition; we have to show that $X \times_S Y$ actually exists. Since we know how pushouts are constructed in the category of rings, their tensor products, therefore we can define it for affine schemes without much effort using the functor $\mathrm{Spec}\,(-) : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ of Theorem 1.3.0.5.

Definition 1.6.4.1. (**Fiber product of affine schemes**) Let the following be a coCartesian[23] diagram of rings (or of R-algebras)

$$
\begin{array}{ccc}
A \otimes_R B & \longleftarrow & B \\
\uparrow & & \uparrow g \\
A & \underset{f}{\longleftarrow} & R
\end{array}
$$

Since the $\mathrm{Spec}\,(-) : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ of Theorem 1.3.0.5 is right adjoint to global sections, therefore it preserves all limits of $\mathbf{Ring}^{\mathrm{op}}$, and thus, takes the above pushout diagram of R-algebras to a pullback diagram of affine schemes over $\mathrm{Spec}\,(R)$:

$$
\begin{array}{ccc}
\mathrm{Spec}\,(A \otimes_R B) & \longrightarrow & \mathrm{Spec}\,(B) \\
\downarrow & & \downarrow \mathrm{Spec}(g) \\
\mathrm{Spec}\,(A) & \underset{\mathrm{Spec}(f)}{\longrightarrow} & \mathrm{Spec}\,(R)
\end{array}
$$

We hence define $\mathrm{Spec}\,(A \otimes_R B)$ to be the fiber product of affine schemes $\mathrm{Spec}\,(A)$ and $\mathrm{Spec}\,(B)$ over $\mathrm{Spec}\,(R)$.

One defines the *fiber product of arbitrary S-schemes* as an S-scheme $X \times_S Y$ such that for any other S-scheme Z with map $f : Z \to X$ and $g : Z \to Y$ over S, there exists a unique map $u : Z \to X \times_S Y$ such that the diagram commutes

$$
\begin{array}{ccccc}
Z & & & & \\
& \searrow^{u} & \searrow^{g} & & \\
& & X \times_S Y & \longrightarrow & Y \\
& {}^{f}\downarrow & \downarrow & & \downarrow \\
& & X & \longrightarrow & S
\end{array}
$$

The most important part in this construction is the description of the structure sheaf of $X \times_S Y$. We shall explain that soon. We now show how to construct fiber products of arbitrary S-schemes.

Theorem 1.6.4.2. *Let X, Y be two S-schemes. The fiber product $X \times_S Y$ exists.*

Proof. This proof is rather an exercise in gluing and reduction techniques. Hence we do the proof as a reality check. □

---

[23]another name for pushout diagrams.

1.6.5   Applications of fiber product

We would now like to portray some of the applications of fiber products, especially in endowing the fibers of a morphism with a scheme structure.

examples of product of vari-Chapter 1.

Inverse image of a closed subscheme

Fibers of a map

Keep in mind the Lemma 22.4.0.3 and the surrounding remarks about stalks of sheaves for the remainder of this discussion. Let $f : X \to Y$ be a map and $y \in Y$ be a point. We endow $f^{-1}(y) \hookrightarrow X$ with a scheme structure. Define the *fiber* of f at y to be the following fiber product:

$$X_y := X \times_Y \mathrm{Spec}\,(\kappa(y)).$$

We at times denote it by $X \times_Y y$. Note that by natural projection map onto second factor, X_y is a scheme over $\kappa(y)$.

Lemma 1.6.5.1. *Fiber's space is homeomorphic to set-theoretic fiber.*

Example 1.6.5.2. We calculate explicit fibers of a map at every point of a familiar map. *Write solution of Exercise 3.10 of Hartshorne Chapter 2, written in notebook.*

The fibers of $\mathrm{Spec}\,(\mathbb{Z}[x]) \to \mathrm{Spec}\,(\mathbb{Z})$

We know that $\mathrm{Spec}\,(\mathbb{Z})$ is the final object in the category of schemes **Sch**. We also know that there is the canonical inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$. This induces a map

$$\varphi : \mathrm{Spec}\,(\mathbb{Z}[x]) \longrightarrow \mathrm{Spec}\,(\mathbb{Z}).$$

Understanding the fibers of this map will allow us to understand the affine arithmetic surface $\mathrm{Spec}\,(\mathbb{Z})$ (as $\mathbb{Z}[x]$ is a 2-dimensional ring). Note that we can already understand $\mathrm{Spec}\,(\mathbb{Z}[x])$ by the results surrounding Gauss' lemma as done in Theorem 18.1.4.3, but the following is a more geometric way of understadning this.

Proposition 1.6.5.3. *The prime ideals of $\mathbb{Z}[x]$ can be categorized into following three types.*
  1. *$\langle p \rangle$ where $p \in \mathbb{Z}$ is a prime,*
  2. *$\langle f(x) \rangle$ where $f(x) \in \mathbb{Z}[x]$ is an irreducible polynomial,*
  3. *$\langle p, f(x) \rangle$ where $p \in \mathbb{Z}$ is a prime and $f(x) \in \mathbb{Z}[x]$ irreducible in $\mathbb{Z}[x]$ which remains irreducible in $\mathbb{Z}/p\mathbb{Z}$,*

Proof. We will prove this by analyzing the fibers of $f : \mathrm{Spec}\,(\mathbb{Z}[x]) \to \mathrm{Spec}\,(\mathbb{Z})$. Pick a prime $p \in \mathbb{Z}$ and denote $X = \mathrm{Spec}\,(\mathbb{Z}[x])$. The fiber $X_p = \mathrm{Spec}\,(\mathbb{Z}[x]) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}\,(\kappa(p))$. As $\kappa(p) = \mathbb{F}_p$, finite field with p elements, therefore we have that $X_p = \mathrm{Spec}\,(\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{F}_p) = \mathrm{Spec}\,(\mathbb{F}_p[x])$. Note that for same reasons we have $X_{\mathfrak{o}} = \mathrm{Spec}\,(\mathbb{Q}[x])$.

As fibers of f covers the whole scheme, it follows that any point in $\mathbb{Z}[x]$ looks like one of the following:
  1. a prime ideal in $\mathbb{Q}[x]$,

2. a prime ideal in $\mathbb{F}_p[x]$.

Moreover, we have the following diagrams

$$
\begin{array}{ccc}
\mathrm{Spec}\,(\mathbb{F}_p[x]) \longrightarrow \mathrm{Spec}\,(\mathbb{Z}[x]) & \qquad & \mathbb{F}_p[x] \longleftarrow \mathbb{Z}[x] \\
\downarrow \qquad\quad\lrcorner\qquad \downarrow & & \uparrow \qquad\ulcorner\qquad \uparrow \\
\mathrm{Spec}\,(\mathbb{F}_p) \longrightarrow \mathrm{Spec}\,(\mathbb{Z}) & & \mathbb{F}_p \longleftarrow \mathbb{Z}
\end{array}
$$

.

$$
\begin{array}{ccc}
\mathrm{Spec}\,(\mathbb{Q}[x]) \longrightarrow \mathrm{Spec}\,(\mathbb{Z}[x]) & \qquad & \mathbb{Q}[x] \longleftarrow \mathbb{Z}[x] \\
\downarrow \qquad\quad\lrcorner\qquad \downarrow & & \uparrow \qquad\ulcorner\qquad \uparrow \\
\mathrm{Spec}\,(\mathbb{Q}) \longrightarrow \mathrm{Spec}\,(\mathbb{Z}) & & \mathbb{Q} \longleftarrow \mathbb{Z}
\end{array}
$$

Observe that $\mathbb{Z}[x] \to \mathbb{F}_p[x]$ is the mod-p map. Since every prime ideal of $\mathbb{Z}[x]$ now is a inverse image of a prime ideal by $\mathbb{Z}[x] \to \mathbb{F}_p[x]$ and $\mathbb{Z}[x] \to \mathbb{Q}[x]$, we get the desired result.   □

Geometric properties

1.6.6   Normal schemes and normalization

Do mainly Exercise 3.7, 3.8 of Chapter 2 of Hartshorne. Also do Exercise 3.17, 3.18 of Chapter 1 of Hartshorne.

1.7   Dimension of schemes

Do from Vakil, Hartshorne Exercise 3.20, 3.21, 3.22 of Chapter 2.
The notion of dimension of a geometric object serves as an essential tool for any attempt at its understanding. Schemes are no different and we have a notion of dimension for them. However, we also have a notion of dimension of rings. This section explores how these two interrelates and thus facilitate understanding of geometry of schemes.

1.7.1   General properties

Before moving to schemes that we will encounter the most, let us first give a general review of the notion of dimension of topological spaces and some general properties of dimension of schemes. Recall that the dimension of a topological space is the supremum of the length of the strictly decreasing chains of finite length of closed irreducible subsets of the space. Further for a space X and a closed irreducible subset $Z \subseteq X$, the codimension of Z in X is defined to be the supremum of the length of strictly increasing chains of closed irreducible subsets starting from Z. For an arbitrary closed subset $Y \subseteq X$, we define codim $(Y, X) = \inf_{Z \subseteq Y}$ codim (Z, X) where Z varies over all closed irreducible subsets of Y. For any closed set $Y \subseteq X$, if $\dim X < \infty$, we always have codim $(Y, X) \leq \dim X$.

Proposition 1.7.1.1. *Let X be a topological space. Then,*
  1. *If $Y \subseteq X$ is a subspace, then $\dim Y \leq \dim X$.*
  2. *If $\{U_i\}_{i \in I}$ is an open covering of X, then $\dim X = \sup_i \dim U_i$.*
  3. *Let $Y \subseteq X$ be a closed subspace and X be of finite dimension. If X is irreducible and $\dim Y = \dim X$, then $Y = X$.*

Proof. The main tool in all of them is just a clear understanding of the definition of dimension and of closed irreducible sets. We establish some terminologies to work with in this proof. For any space X a strictly decreasing chain of finite length of closed irreducible subsets will be called a *finite chain* of X and set of all finite chains will be denoted by $FC(X)$. We denote a chain by $Z_\bullet \in FC(X)$ and its length by $l(Z_\bullet)$. Consequently, $\dim X = \sup_{Z_\bullet \in FC(X)} l(Z_\bullet)$.
  1. First observe that if Y is closed then the result is immediate as any finite chain of Y will be a finite chain of X. Consequently, we reduce to showing that $\dim Y \leq \dim \bar{Y}$. In particular, we reduce to showing that if Y is dense in X, then $\dim Y \leq \dim X$. It further suffices to show existence of a length preserving map $FC(Y) \to FC(X)$. Indeed, for any $Z_\bullet \in FC(Y)$, one observes that $\mathrm{Cl}_X(Z_i)$ is a closed subset of X which is further irreducible in X. Consequently, $\mathrm{Cl}(Z_\bullet)$ is a finite chain of X of same length as of $Z_\bullet$.[24]
  2. By 1. we already have $\sup_i \dim U_i \leq \dim X$ so we need only show that $\dim X \leq \sup_i \dim U_i$. It suffices to show that for each $Z_\bullet \in FC(X)$, there exists $i \in I$ and $W_\bullet \in FC(U_i)$ such that $l(Z_\bullet) \leq l(W_\bullet)$. Let $r = l(Z_\bullet)$ and $i \in I$ be such that $U_i \cap Z_r \neq \emptyset$. Then, $W_\bullet = U_i \cap Z_\bullet$ forms a finite chain of U_i of same length as $Z_\bullet$. To see this, observe that if $U_i \cap Z_a = U_i \cap Z_b$ where we may assume $Z_a \supsetneq Z_b$, then the open set $U_i \cap Z_a$ of Z_a is contained in the closed set Z_b of Z_a, hence the closure of

---

[24]Actually we didn't needed the reduction to Y being dense in X.

$U_i \cap Z_a$ in Z_a is inside Z_b. But since Z_a is irreducible so $U_i \cap Z_a$ must be dense in Z_a, a contradiction.

3. Let $r = \dim X = \dim Y$. Suppose $Y \subsetneq X$. Let $Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ be a maximal finite chain of Y. Then the chain $X \supsetneq Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ is a finite chain in X as Y is closed. Thus $\dim X \geq 1 + r$, therefore $r = \dim Y \geq r + 1$, a contradiction.

$\square$

The following technical lemma was employed in proving the statement 2 of above, but is good to keep in handy.

Lemma 1.7.1.2. *Let X be a topological space and $Z_\bullet \in FC(X)$ a finite chain (in the terminology of Proposition 1.7.1.1) of length $l(Z_\bullet) = r$. If $U \subseteq X$ is an open set such that $U \cap Z_r \neq \emptyset$, then $U \cap Z_\bullet$ is a finite chain of length r in U.* $\square$

The following result gives a connection between all closed irreducible containing a given point and prime ideals of the local ring at that point.

Proposition 1.7.1.3. *Let X be a scheme and $x \in X$ be a point. We obtain an order reversing bijection*

$$\{Closed\ irreducibles\ Y\ of\ X\ containing\ x\} \cong \mathrm{Spec}\,(\mathcal{O}_{X,x}).$$

Proof. Denote the collection of all closed irreducibles of X containing x as I. Let $U = \mathrm{Spec}\,(A)$ be an affine open containing $x \in X$ so that $\mathcal{O}_{X,x} \cong A_x$. Consequently, we wish to show a bijection $I \cong \mathrm{Spec}\,(A_x)$, which is further equivalent to showing that I is bijective to all prime ideals of A contained in x. As all prime ideals of A contained in x is further bijective to all closed irreducible of U containing x by Lemma 1.2.1.1, we thus reduce to showing existence of a bijection between I and closed irreducibles of U containing x, denoted J.

Consider the following function

$$\varphi : I \longrightarrow J$$
$$Y \longmapsto Y \cap U.$$

Indeed, this map is well-defined as for any $Y \in I$, $\varphi(Y) = Y \cap U$ is first irreducible as any open subset of an irreducible set is irreducible. Further, it is closed in U as Y is closed. In order to show injectivity, we need only recall that any open subset of an irreducible set is dense. Finally, for surjectivity, take any $Z \in J$ so that Z is a closed irreducible in U containing x. Now let Y to be the closure of Z in X. We thus need only show that Y is irreducible in X. That follows immediately from the fact that closure of irreducible is again an irreducible, which in turn follows immediately from a simple observation on open subsets of the closure. $\square$

One then observes the following general result which will be used heavily in the future.

Lemma 1.7.1.4. *Let X be a scheme and Y be an irreducible closed subscheme of X with $\eta \in Y$ being its generic point. Then,*

$$\mathrm{codim}\,(Y, X) = \dim \mathcal{O}_{X,\eta}.$$

Proof. This is immediate from Proposition 1.7.1.3 as Y is the smallest closed irreducible containing η. $\square$

1.7.2   Dimension of finite type k-schemes

In this section, we prove various results surrounding the relationship between dimension of a given integral finite type k-scheme as a topological space and Krull dimension of various local rings.

Theorem 1.7.2.1. [25] *Let k be a field and X be a finite type integral k-scheme.*
  1. *If $U, V \subseteq X$ are two open affines which are spectra of finite type k-domains, then $\dim U = \dim V$.*
  2. *If $\{U_i\}_{i=1}^n$ is any finite open affine covering by spectra of finite type k-domains, then $\dim U_i = \dim X$ for all $i = 1, \ldots, n$.*
  3. *If $p \in X$ is a closed point, then $\dim X = \dim \mathcal{O}_{X,p}$.*
  4. *Let $K(X)$ be the function field of X. Then $\dim X = \operatorname{trdeg} K(X)/k$.*
  5. *If Y is a closed subset of X, then $\operatorname{codim}(Y, X) = \inf_{p \in Y} \dim \mathcal{O}_{X,p}$.*
  6. *If Y is a closed subset of X, then $\dim Y + \operatorname{codim}(Y, X) = \dim X$.*

Proof. The main tools are the Theorems 18.7.2.1 and 18.7.2.2.
  1. Observe that since X is irreducible, therefore U and V are dense open subsets of X, so $U \cap V \neq \emptyset$. Consequently, it will suffice to show that any dense affine open subset $W \subseteq U$ has same dimension as U. Indeed, U is spectra of finite type k-domain, so it is a separated finite type integral affine scheme, that is, an abstract affine variety. Consequently, by Proposition I.1.10 of cite[Hartshorne], $\dim W = \dim \overline{W} = \dim U$.
  2. Follows from Proposition 1.7.1.1, 2 and statement 1.
  3. As X is finite type, it admits a finite open affine covering by spectra of finite type k-domains. Let $U = \operatorname{Spec}(A)$ be one such open affine such that $p \in U$. Consequently, $p = \mathfrak{m} \in \operatorname{Spec}(A)$ represents a maximal ideal of R (Lemma 1.2.1.3). Thus, $\mathcal{O}_{X,p} \cong A_{\mathfrak{m}}$ and so $\dim \mathcal{O}_{X,p} = \dim A_{\mathfrak{m}}$. Note that A is a finite type k-algebra which is an integral domain. It thus follows by Theorem 18.7.2.2 that we have $\operatorname{ht} \mathfrak{m} + \dim A/\mathfrak{m} = \dim A$ and since $\dim A/\mathfrak{m} = 0$, therefore $\operatorname{ht} \mathfrak{m} = \dim A$. Further, since $\dim A_{\mathfrak{m}} = \operatorname{ht} \mathfrak{m}$, therefore we have $\dim \mathcal{O}_{X,p} = \dim A_{\mathfrak{m}} = \dim A = \dim U$. By statement 2, $\dim U = \dim X$ and the result follows.
  4. Function field is defined to be the local ring at the generic point of X, say $\eta \in X$ (Remark 1.4.2.5). Let $\eta \in \operatorname{Spec}(A)$ where $\operatorname{Spec}(A)$ is a member of an open affine cover of X by spectra of finite type k-domains. Observe that $\operatorname{Spec}(A)$ has η as its generic point as well. Consequently, $\dim \operatorname{Spec}(A) = \dim A = \operatorname{trdeg} K(A)/k$ and since $K(A) = \mathcal{O}_{\operatorname{Spec}(A),\eta} \cong \mathcal{O}_{X,\eta} = K(X)$, therefore $\dim \operatorname{Spec}(A) = \operatorname{trdeg} K(X)/k$. By statement 2, $\dim \operatorname{Spec}(A) = \dim X$ and the result follows.
  5. First observe that for any closed irreducible $Z \subseteq X$, we have $\operatorname{codim}(Z, X) \leq \dim X$. By statement 3, therefore, we have $\inf_{p \in Y} \dim \mathcal{O}_{X,p} = \inf_{p \in Y \text{ non-closed}} \mathcal{O}_{X,p}$. We will now show that for any closed irreducible subset $Z \subseteq X$ with $\eta \in Z$ its generic paint (schemes are *sober*[26]), we have $\dim \mathcal{O}_{X,\eta} = \operatorname{codim}(Z, X)$. By taking infimum, the result would then follow, so it would suffice to show the above claim. Let $\{\operatorname{Spec}(A_\alpha)\}$ be a finite open affine cover of X where A_α is a finite type k-domain. Observe that if $Z \cap \operatorname{Spec}(A_\alpha) \neq \emptyset$, then $\eta \in \operatorname{Spec}(A_\alpha)$. Now, $\eta \in \operatorname{Spec}(A_\alpha)$ is a

---

[25]Exercise II.3.20 of Hartshorne.
[26]a space where all closed irreducibles have a unique generic point.

point whose closure in $\mathrm{Spec}\,(A_\alpha)$ is $Z \cap \mathrm{Spec}\,(A_\alpha)$ so $Z \cap \mathrm{Spec}\,(A_\alpha)$ is a closed irreducible subspace of $\mathrm{Spec}\,(A_\alpha)$ whose generic point is η and thus $Z \cap \mathrm{Spec}\,(A_\alpha) \cong \mathrm{Spec}\,(A_\alpha/\eta)$, where we treat $\eta \lneq A_\alpha$ as a prime ideal of A_α. Consequently, $\dim \mathcal{O}_{X,\eta} = \dim \mathcal{O}_{\mathrm{Spec}(A_\alpha),\eta} = \dim(A_\alpha)_\eta = \mathrm{ht}\,\eta$. Since A_α is a finite type k-domain, therefore by Theorem 18.7.2.2, we obtain that $\mathrm{ht}\,\eta + \dim A_\alpha/\eta = \dim A_\alpha$, which thus yields $\mathrm{ht}\,\eta = \dim X - \dim A_\alpha/\eta$ by statement 2. It thus suffices to show that for some index α we get $\dim A_\alpha/\eta = \dim Z$ as then we would obtain $\dim \mathcal{O}_{X,\eta} = \dim X - \dim Z = \mathrm{codim}\,(Z, X)$.

Indeed, since $\{\mathrm{Spec}\,(A_\alpha/\eta)\}$ forms a finite open affine cover of Z, therefore by Proposition 1.7.1.1, 2 we get such an index α.

6. Observe that since $\mathrm{codim}\,(Y, X) < \infty$, therefore there exists a maximal closed irreducible $Z \subseteq Y$ such that $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(Z, X)$. Consequently, we have a finite chain of X, say $Z_\bullet$, ending at Z such that $l(Z_\bullet) = \mathrm{codim}\,(Y, X)$.

Let $U = \mathrm{Spec}\,(A)$ be an open affine where A is a finite type k-domain such that $U \cap Z \neq \emptyset$. Further, $\dim U \cap Y = \dim Y$. Consequently, by Lemma 1.7.1.2, we have $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(Z \cap U, U)$. Since $U \cap Y$ is a closed subscheme of U, therefore we may write $U \cap Y = \mathrm{Spec}\,(A/I)$ for an ideal $I \leq A$. Consequently, $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(\mathrm{Spec}\,(A/I), \mathrm{Spec}\,(A))$.

It is immediate from first definitions that

$$\mathrm{codim}\,(\mathrm{Spec}\,(A/I), \mathrm{Spec}\,(A)) = \inf_{\mathfrak{p} \supseteq I} \mathrm{codim}\,(\mathrm{Spec}\,(A/\mathfrak{p}), \mathrm{Spec}\,(A))$$

$$= \inf_{\mathfrak{p} \supseteq I} \mathrm{ht}\,\mathfrak{p}.$$

Now by Theorem 18.7.2.2 and above, we further obtain that

$$\mathrm{codim}\,(\mathrm{Spec}\,(A/I), \mathrm{Spec}\,(A)) = \inf_{\mathfrak{p} \supseteq I}(\dim A - \dim A/\mathfrak{p})$$

$$= \dim A - \sup_{\mathfrak{p} \supseteq I} \dim A/\mathfrak{p}$$

$$= \dim X - \dim U \cap Y$$

$$= \dim X - \dim Y$$

where $\dim A = \dim X$ because of statement 2.

$\square$

1.7.3 Dimension of fibers

In this section, we discuss the question of how the dimension of fibres of a morphism varies. We'll see that certain nice geometric situations are encoded in the maps for which the dimension of fibres is not too erratic.

1.8 Projective schemes

The most important type of examples that we will encounter in our study of algebraic geometry are subvarieties of projective space $\mathbb{P}^n_k$. Indeed, this is a construction which is fundamental because of the many nice properties enjoyed by realizing familiar constructions in it. One of them being this classical observation that any two straight lines are bound to intersect at atleast one point in the projective space. We shall see more equally nice results, not to mention the quadrics with which we wish to spend some considerable time as the main motivating example for us (Example 1.5.1.3) is itself realized as a quadric in projective space.

We recall that the notion of projective varieties, whose generalization we shall embark now on, has been covered in Section 1.5.

We first begin by defining the space $\mathrm{Proj}(S)$ of a graded ring $S = \bigoplus_{d \geq 0} S_d$.

Definition 1.8.0.1. (Projective spectrum of a graded ring) Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring and let $S_+ = \bigoplus_{d > 0} S_d$ be the ideal generated by non-zero degree elements. Denote

$$\mathrm{Proj}(S) := \{\mathfrak{p} \lneq S \mid \mathfrak{p} \text{ is homogeneous prime ideal } \& \ \mathfrak{p} \not\supseteq S_+\}.$$

The set $\mathrm{Proj}(S)$ is called the projective spectrum of the graded ring S.

Note that the latter condition is motivated by Remark 1.5.3.13. This is also used in a technical manner to show existence of a nice basis over $\mathrm{Proj}(S)$ in Lemma 1.8.1.3 and in other proofs as well. We now show that there is a natural topology over $\mathrm{Proj}(S)$, akin to the affine case.

Lemma 1.8.0.2. *Let S be a graded ring and denote for a homogeneous ideal $\mathfrak{a} \leq S$, the following subset of $\mathrm{Proj}(S)$:*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Then, for any homogeneous ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}_i$ of S, we obtain
  1. $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$,
  2. $\bigcap_i V(\mathfrak{a}_i) = V\left(\sum_i \mathfrak{a}_i\right)$.

Proof. Same as Lemma 1.2.0.1. $\square$

We thus obtain a topological space $\mathrm{Proj}(S)$ where a set is closed if and only if it is of the form $V(\mathfrak{a})$ for a homogeneous ideal $\mathfrak{a} \leq S$. This is called the Zariski topology over the $\mathrm{Proj}(S)$.

We now give some more topological properties of $\mathrm{Proj}(S)$.

1.8.1 Topological properties of $\mathrm{Proj}(S)$

The first obvious question is how does the inclusion $\mathrm{Proj}(S) \hookrightarrow \mathrm{Spec}\,(S)$ looks topologically?

Lemma 1.8.1.1. *Let S be a graded ring. The topology of $\mathrm{Proj}(S)$ is obtained by subspace topology of $\mathrm{Spec}\,(S)$. Thus, there is a continuous inclusion*

$$\mathrm{Proj}(S) \hookrightarrow \mathrm{Spec}\,(S).$$

Proof. Immediate from definitions. $\square$

We further note that for a graded ring S, the degree zero elements S_0 form a subring of S by the virtue of the fact that $S_d \cdot S_e \subseteq S_{d+e}$. Thus, we obtain a continuous map as the following shows.

Lemma 1.8.1.2. *Let S be a graded ring. Then the following is a continuous map*

$$\varphi : \mathrm{Proj}(S) \longrightarrow \mathrm{Spec}\,(S_0)$$
$$\mathfrak{p} \longmapsto \mathfrak{p} \cap S_0.$$

Proof. Pick any ideal $\mathfrak{a} \leq S_0$ and notice that it is already homogeneous in S. Consequently, $\varphi^{-1}(V(\mathfrak{a})_a) = V(\mathfrak{a})_h$ where $V(\mathfrak{a})_a \subseteq \mathrm{Spec}\,(S_0)$ and $V(\mathfrak{a})_h \subseteq \mathrm{Proj}(S)$. $\square$

We now find a collection of open sets which forms a basis for $\mathrm{Proj}(S)$. This is akin to Lemma 1.2.1.4.

Lemma 1.8.1.3. *Let S be a graded ring and $f, g \in S_d$ for some $d > 0$ be homogeneous elements. Denote*

$$D_+(f) := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid f \notin \mathfrak{p}\}.$$

Then,
   1. $D_+(f)$ is an open subset of $\mathrm{Proj}(S)$,
   2. $D_+(f) \cap D_+(g) = D_+(fg)$,
   3. $\{D_+(f)\}_{f \in S_d, d > 0}$ forms a basis of $\mathrm{Proj}(S)$.

Proof. 1. Since $D_+(f) = \mathrm{Proj}(S) \setminus V(f)$, thus $D_+(f)$ is open.
2. Straightforward.
3. Since for any $\mathfrak{p} \in \mathrm{Proj}(S)$, there exists $f \in S_d$ for some $d > 0$ such that $f \notin \mathfrak{p}$ as $\mathfrak{p}$ does not contain all of S_+, thus $\bigcup_{f \in S_d, d > 0} D_+(f) = \mathrm{Proj}(S)$. The rest follows by 2. $\square$

Remark 1.8.1.4. As tempting as it might be to think, but not all projective schemes are quasi-compact. An example is given by the graded ring $S = \mathbb{Z}[x_1, x_2, \dots]$, the polynomial ring over $\mathbb{Z}$ with countably infinitely many indeterminates. Then one observes that $\mathrm{Proj}(S) = \bigcup_{n=1}^{\infty} D_+(x_n)$. Moreover, as for any $\mathfrak{p} \in \mathrm{Proj}(S)$ can not contain S_+, therefore $\mathfrak{p}$ necessarily has to not contain some x_i, otherwise it contains S_+. Consequently, we cannot form a finite subcover of the above cover, showing that $\mathrm{Proj}(S)$ is not quasi-compact.

However, the following lemma might be helpful in checking when a projective scheme has a finite cover by basic open sets.

Lemma 1.8.1.5. *Let S be a graded ring and consider $X = \mathrm{Proj}(S)$. Let $f = f_0 + \cdots + f_n$ be a decomposition of $f \in S$ into homogeneous elements $f_d \in S_d$. Then,*

$$D(f) \cap X = (D(f_0) \cap X) \cup \bigcup_{d=1}^{n} D_+(f_d)$$

where we view $X \subseteq \mathrm{Spec}\,(S)$ and $D(f), D(f_0) \subseteq \mathrm{Spec}\,(S)$.

Proof. This is a rather straightforward proof. To show $(\subseteq)$, consider a point $\mathfrak{p} \in D(f) \cap X$ so that $f \notin \mathfrak{p}$. It follows from $f = f_0 + \cdots + f_n$ that for some $d = 0, \ldots, n$, $f_d \notin \mathfrak{p}$, which is in turn equivalent to stating that $\mathfrak{p} \in D_+(f_i)$ if $i \geq 1$ or $\mathfrak{p} \in D(f_0) \cap X$ if $d = 0$.

Conversely, pick $\mathfrak{p} \in (D(f_0) \cap X) \cup \bigcup_{d=1}^{n} D_+(f_d)$. We obtain that for some $d = 0, \ldots, n$, $f_d \notin \mathfrak{p}$. It follows from $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{p} \cap S_i$ that if $f \in \mathfrak{p}$, then we get by uniqueness of representatives of the direct sum that $f_d \in \mathfrak{p}$, a contradiction. $\qquad\square$

1.8.2   The structure sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$ and projective schemes

We have studied some basic properties of the topological space $\mathrm{Proj}(S)$ so far, we now construct a structure sheaf over it and make it, first, into a locally ringed space and, second, into a scheme. We first define the structure sheaf of projective spectrum, in which there is nothing new in comparison to projective varieties (see Definition 1.5.3.1).

Definition 1.8.2.1. (The structure sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$) Let S be a graded ring. Let $U \subseteq \mathrm{Proj}(S)$ be an open set of the projective spectrum of S. Define the following set

$$\mathcal{O}_{\mathrm{Proj}(S)}(U) := \Big\{ s : U \to \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \mid \forall \mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(\mathfrak{p})} \,\&\, \exists \text{ open } \mathfrak{p} \in V \subseteq U \,\&\, f, g \in S_d, d \geq 0 \text{ s.t. } \forall \mathfrak{q} \in V, g \notin \mathfrak{q} \,\&\, s(\mathfrak{q}) = f/g \Big\}.$$

From the fact that its elements are functions locally defined, one immediately obtains that $\mathcal{O}_{\mathrm{Proj}(S)}$ is a sheaf with obvious restriction maps. By appropriate restrictions on the domain, one further sees that under pointwise addition and multiplication, $\mathcal{O}_X(U)$ forms a commutative ring with 1.

Let us now show that $\mathrm{Proj}(S)$ is a scheme over $\mathrm{Spec}\,(S_0)$ in a natural manner.

Lemma 1.8.2.2. *Let S be a graded ring. Then $\mathrm{Proj}(S)$ is a scheme over $\mathrm{Spec}\,(S_0)$.*

Proof. We need only define a map $\mathrm{Proj}(S) \to \mathrm{Spec}\,(S_0)$. By Theorem 1.3.0.5, we need only construct a homomorphism $S_0 \to \Gamma(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$. This is straightforward, as we can interpret each $a \in S_0$ as a homogeneous regular function $s : \mathrm{Proj}(S) \to \coprod_{\mathfrak{p} \in \mathrm{Proj}(S)} S_{(\mathfrak{p})}$ mapping as $\mathfrak{p} \mapsto a/1$. $\qquad\square$

Thus, $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a ringed space. We now see that the stalk of this sheaf is isomorphic to the homogeneous localization. This will thus show that $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a locally ringed space (Lemma 18.1.2.13).

Lemma 1.8.2.3. *Let S be a graded ring and consider the ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$. For each $\mathfrak{p} \in \mathrm{Proj}(S)$, we have*

$$\mathcal{O}_{\mathrm{Proj}(S), \mathfrak{p}} \cong S_{(\mathfrak{p})}.$$

Proof. Consider the following map

$$\varphi : \mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}} \longrightarrow S_{(\mathfrak{p})}$$
$$(U, s)_{\mathfrak{p}} \longmapsto s(\mathfrak{p}).$$

It is straightforward to see that φ is a well-defined ring homomorphism. To see injectivity, suppose $(U, s)_{\mathfrak{p}} \mapsto 0$. Thus $s(\mathfrak{p}) = 0$. Consequently, for some open $V \subseteq U$ containing $\mathfrak{p}$ where s is given by f/g for $f, g \in S_d$, $d \geq 0$, we obtain $s(\mathfrak{q}) = f/g = 0$ for all $\mathfrak{q} \in V$. Thus $s = 0$ on V and hence $(U, s)_{\mathfrak{p}} = (V, \rho_{U,V}(s))_{\mathfrak{p}} = 0$. To see surjectivity, pick any $f/g \in S_{(\mathfrak{p})}$. Observe that $g \notin \mathfrak{p}$. Thus consider $(D_+(g), f/g)_{\mathfrak{p}} \in \mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}}$. $\square$

We now show that the locally ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a scheme. For this purpose we would need to show that $\mathrm{Proj}(S)$ is covered by affine opens. Indeed, we have the following lemma.

Lemma 1.8.2.4. *Let S be a graded ring and consider the locally ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$. For each $f \in S_d$, $d > 0$, we have the following isomorphism of locally ringed spaces*

$$\left(D_+(f), \mathcal{O}_{\mathrm{Proj}(S)|D_+(f)}\right) \cong \left(\mathrm{Spec}\left(S_{(f)}\right), \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right)}\right).$$

Proof. Consider the map

$$\varphi : D_+(f) \longrightarrow \mathrm{Spec}\left(S_{(f)}\right)$$
$$\mathfrak{p} \longmapsto (\mathfrak{p} \cdot S_f)_0.$$

By Lemma 18.1.2.16, it follows that φ is a bijection. To show that φ is an isomorphism it is sufficient to show that φ is a closed map. This is immediate as $\mathfrak{p} \supseteq \mathfrak{a}$ in $D_+(f)$ if and only if $(\mathfrak{p} \cdot S_f)_0 \supseteq (\mathfrak{a} \cdot S_f)_0$ in $\mathrm{Spec}\left(S_{(f)}\right)$.

We now wish to show isomorphism of corresponding sheaves. For this, we construct a map

$$\varphi^\flat : \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right)} \longrightarrow \varphi_* \mathcal{O}_{D_+(f)}$$

and show that this is an isomorphism. Indeed, we first observe a canonical isomorphism on stalks

$$\mathcal{O}_{D_+(f),\mathfrak{p}} \cong S_{(\mathfrak{p})} \xrightarrow{\eta_{\mathfrak{p}}} (S_{(f)})_{(\mathfrak{p}\cdot S_f)_0} \cong \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right),\varphi(\mathfrak{p})}.$$

Then one can construct the above isomorphism $\varphi^\flat$ by observing the following square for sections of the relevant sheaves over open $U \subseteq \mathrm{Spec}\left(S_{(f)}\right)$ and the corresponding $\varphi^{-1}(U) \subseteq D_+(f)$:

$$
\begin{array}{ccc}
U & \xleftarrow{\ \varphi\ \cong\ } & \varphi^{-1}(U) \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\
\coprod_{\mathfrak{p}\in\varphi^{-1}(U)}((S_{(f)})_{\varphi(\mathfrak{p})}) & \xleftarrow[\cong]{\ \Pi_{\mathfrak{p}}\eta_{\mathfrak{p}}\ } & \coprod_{\mathfrak{p}\in\varphi^{-1}(U)} S_{\mathfrak{p}}
\end{array}
\quad ,
$$

where $s \in \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right)}(U)$ and $t \in \mathcal{O}_{D_+(f)}(\varphi^{-1}(U))$ its image under $\varphi^\flat$ (which is defined by the above square). One can indeed check that $\varphi^\flat$ as defined is natural w.r.t restrictions. $\square$

Remark 1.8.2.5. Thus, for a graded ring S, we obtain a scheme $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$, which is called the projective scheme associated to a graded ring S.

We now give some more properties of $\mathrm{Proj}(S)$.

Proposition 1.8.2.6. [27] *Let S be a graded ring. Then,*
1. *$\mathrm{Proj}(S) = \emptyset$ if and only if $\forall s \in S_+$, s is a nilpotent element of S.*
2. *Let $\varphi : S \to T$ be a graded map of graded rings. Then $U = \{\mathfrak{q} \in \mathrm{Proj}(T) \mid \mathfrak{q} \not\supseteq \varphi(S_+)\}$ is an open set and the natural map*

$$f : U \longrightarrow \mathrm{Proj}(S)$$
$$\mathfrak{q} \longmapsto \varphi^{-1}(\mathfrak{q})$$

   defines a map of schemes.
3. *Let $\varphi : S \to T$ be a graded map of graded rings for which there exists $d_0 \in \mathbb{N}$ such that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$. Then, $U = \mathrm{Proj}(T)$ and $f : \mathrm{Proj}(T) \to \mathrm{Proj}(S)$ is an isomorphism.*

Proof. 1. The R $\implies$ L is immediate. Otherwise take an element $s \in S_d$. By Lemmas 1.8.1.3, 3 and 1.8.2.4, we obtain that $\mathrm{Spec}\,(S_{(s)}) = \emptyset$. Consequently, any prime ideal of $\mathrm{Spec}\,(S_s)$ has no zero degree terms, which can be seen to be not true. Consequently, $D(s) = \mathrm{Spec}\,(S_s) = \emptyset$. It follows from Lemma 1.2.2.7 that s is nilpotent.

2. The fact that U is open depends on φ being graded, i.e. $\varphi(S_d) \subseteq T_d$ for all $d \geq 0$. The continuity of f follows from the same observation. The map on sheaves is given by extending the natural map on stalks $\varphi_{(\mathfrak{q})} : S_{(\varphi^{-1}(\mathfrak{q}))} \to T_{(\mathfrak{q})}$, whose well-definedness, again, uses the fact that φ is graded.

3. The main trick here is to observe that if $s \in S_d$ for $d < d_0$, then raising some high enough power of s will make $s^n \in S_e$ where $\deg s^n \geq d_0$. For showing isomorphism on stalks $\varphi_{(\mathfrak{q})} : S_{(\varphi^{-1}(\mathfrak{q}))} \to T_{(\mathfrak{q})}$, it comes down to observing the following: let $s/t \in S_{(\varphi^{-1}(\mathfrak{q}))}$, then $s/t = st^n/t^{n+1}$ for any $n \in \mathbb{N}$. Then use the trick above. $\square$

Remark 1.8.2.7. The above Proposition 1.8.2.6 shows that the mapping $S \mapsto \mathrm{Proj}(S)$ is NOT functorial! However the statement 3. might give some hint how to fix this.

Next, we understand all closed subschemes of $\mathrm{Proj}(S)$ in the following two results (Corollary 1.8.2.10)

Proposition 1.8.2.8. *Let S, T be a graded rings.*
1. *If $\varphi : S \to T$ is a surjective graded map, then the open set $U = \mathrm{Proj}(T)$ and $f : \mathrm{Proj}(T) \to \mathrm{Proj}(S)$ is a closed immersion (see Proposition 1.8.2.6, 2).*
2. *Let $I \leq S$ be a homogeneous ideal and consider the ideal $I' = \bigoplus_{d \geq d_0} I_d$. Then, I, I' defines the same closed subscheme of $\mathrm{Proj}(S)$.*
3. *Let $I \leq S$ be a homogeneous ideal and let $\pi : S \to S/I$ be the natural projection. Then the closed subscheme $f : \mathrm{Proj}(S/I) \to \mathrm{Proj}(S)$ (as in 1.) has the ideal sheaf given by $\widetilde{I} \leq \mathcal{O}_{\mathrm{Proj}(S)}$.*

---

[27]Exercise II.2.14 of Hartshorne.

Proof. 1. $U = \operatorname{Proj}(T)$ because $\varphi(S_+) = T_+$. The fact that f is a topological immersion follows from the observations that $f(\operatorname{Proj}(T)) = V(\operatorname{Ker}(\varphi))$ where $\operatorname{Ker}(\varphi)$ is homogeneous and that for any ideal $\mathfrak{q} \leq T$, it follows from surjectivity that $\varphi(\varphi^{-1}(\mathfrak{q})) = \mathfrak{q}$. To show surjectivity of sheaves, it reduces to showing surjectivity of localization maps $S_{\varphi^{-1}(\mathfrak{q})} \xrightarrow{\varphi_{(\mathfrak{q})}} T_{(\mathfrak{q})}$, which is immediate from surjectivity of φ.

2. We wish to show an isomorphism as in the following commutative diagram:

$$\operatorname{Proj}(S/I) \xrightarrow{\;\;\cong\;\;} \operatorname{Proj}(S/I')$$
$$\searrow \qquad\qquad \swarrow$$
$$\operatorname{Proj}(S) \qquad\qquad .$$

Now since $(S/I)_d = S_d/I_d$ for all $d \geq 0$, therefore we have an isomorphism $\varphi : (S/I)_d \to (S/I')_d$ given by $s_d + I_d \mapsto s_d + I'_d$. The result follows from Proposition 1.8.2.6, 3.

3. We wish to show that $\operatorname{Ker}\left(f^{\flat} : \mathcal{O}_{\operatorname{Proj}(S)} \to f_*\mathcal{O}_{\operatorname{Proj}(S/I)}\right)$ is given by $\widetilde{I}$. It suffices to check this on basic open sets $D_+(g)$, $g \in S_d, d > 0$, by uniqueness of the sheaf defined on a basis. Indeed it follows that $f^{\flat}$ on $D_+(g)$ is given by the localisation map $S_{(g)} \to S_{(g)}/I_{(g)}$, whose kernel is $I_{(g)} = \widetilde{I}(D_+(g))$. $\qquad\square$

Proposition 1.8.2.9. *Let $S = A[x_0, \ldots, x_r]$ for a ring A and let $X = \operatorname{Proj}(S)$.*
1. *Let $I \leq S$ be a homogeneous ideal and denote $\bar{I} = \{s \in S \mid \forall i = 0, \ldots, r,\ \exists n_i\ s.t.\ x_i^{n_i} s \in I\}$ to be the saturation of I. Then, $\bar{I}$ is homogeneous.*
2. *Let $I, J \leq S$ be two homogeneous ideals. Then $\operatorname{Proj}(S/I) \cong \operatorname{Proj}(S/J)$ if and only if $\bar{I} = \bar{J}$.*
3. *Let $Y \hookrightarrow \operatorname{Proj}(S)$ be a closed subscheme. Then, $\Gamma_*(\mathcal{I}_Y)$ is a saturated ideal of S.*

Proof. 1. This follows from a simple consideration of the uniqueness of homogeneous decomposition of each element in a graded ring.

2. We may reduce to showing that I and $\bar{I}$ defines the same closed subscheme. We already have $I \hookrightarrow \bar{I}$ which translates to $V(\bar{I}) \hookrightarrow V(I)$. Conversely, pick $\mathfrak{p} \in V(I) \subseteq \operatorname{Proj}(S)$. We wish to show $\mathfrak{p} \supseteq \bar{I}$. Pick any $s \in \bar{I}$. Assume that $s \notin \mathfrak{p}$. It then follows that $\mathfrak{p} = \langle x_0, \ldots, x_r \rangle$ which is a prime ideal which contains S_+, thus $\mathfrak{p} \notin \operatorname{Proj}(S)$, a contradiction.

We then wish to show isomorphism of sheaves. Going to basic opens, this reduces to showing surjection is an injection:

$$(S/I)_{(f)} \longrightarrow (S/\bar{I})_{(f)}$$
$$\frac{s+I}{f^n} \longmapsto \frac{s+\bar{I}}{f^n}.$$

This follows from the fact that $\bar{I}$ is saturated[28].

3. Pick a homogeneous element $s \in S_d$ such that for each $i = 0, \ldots, r$, there exists $n_i \in \mathbb{N}$ such that $x_i^{n_i} s \in \Gamma(\mathcal{I}_Y(d + n_i), X)$. We wish to show that $s \in \Gamma_*(\mathcal{I}_Y)$. Note that $s \in \Gamma(\mathcal{O}_X(d), X)$. Cover X by $D_+(x_i)$ and consider the restrictions $x_i^{n_i} s \in \mathcal{I}_Y(d + n_i)(D_+(x_i))$. Multiplying (tensoring) $x_i^{n_i} s$ with $x_i^{-n_i} \in \mathcal{O}_X(-n_i)(D_+(x_i))$ yields $s \in \mathcal{I}_Y(d + n_i) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-n_i) \cong \mathcal{I}_Y(d)$ over $D_+(x_i)$. Thus, gluing these sections up from each $D_+(x_i)$, we get $s \in \Gamma(\mathcal{I}_Y(d), X) \subseteq \Gamma_*(\mathcal{I}_Y)$, as required. $\qquad\square$

---

[28]In-fact, this step shows exactly why the definition of saturation would've been made!

Using the above result, it is possible to find a characterization of closed subschemes of $\mathrm{Proj}(S)$ in terms of algebraic data.

Corollary 1.8.2.10. *Let $S = A[x_0, \ldots, x_r]$ be a graded ring for a ring A. Then there is a correspondence:*

$$\{\text{All closed subschemes of } \mathrm{Proj}(S)\} \quad \cong \quad \{\text{All saturated ideals of } S\}\ .$$

Proof. Follows from Proposition 1.8.2.9. □

Next, let us show how projective n-spaces over a ring changes with extension of scalars.

Definition 1.8.2.11. (Projective n-space over a ring) Let A be a ring. The projective n-space over A is defined to be $\mathbb{P}_A^n := \mathrm{Proj}(A[x_0, \ldots, n])$. By Lemma 1.8.2.2, $\mathbb{P}_A^n$ is a scheme over $\mathrm{Spec}\,(A)$.

We now see how $\mathbb{P}_A^n$ behaves under extension of scalars.

Lemma 1.8.2.12. *Let $A \to B$ be a map of rings and $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ be the corresponding map of affine schemes. Then,*

$$\mathbb{P}_B^n \cong \mathbb{P}_A^n \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(B).$$

Proof. Observe that $D_+(x_i) \subseteq \mathbb{P}_A^n$ for $i = 0, \ldots, n$ covers $\mathbb{P}_A^n$ as $A[x_0, \ldots, x_n]$ is finitely generated by x_i as an A-algebra. As filtered colimits commute with finite limits, thus together with Lemma 1.8.2.4 we obtain the following:

$$\mathbb{P}_A^n \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(B) = \left(\bigcup_{i=0}^{n} D_+(x_i) \right) \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(B)$$

$$\cong \bigcup_{i=0}^{n} D_+(x_i) \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(B)$$

$$\cong \bigcup_{i=0}^{n} \mathrm{Spec}\,\left(A[x_0, \ldots, x_n]_{(x_i)} \otimes_A B \right)$$

$$\cong \bigcup_{i=0}^{n} \mathrm{Spec}\,\left(A[x_0/_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i] \otimes_A B \right)$$

$$\cong \bigcup_{i=0}^{n} \mathrm{Spec}\,\left(B[x_0, \ldots, x_n]_{(x_i)} \right)$$

$$\cong \mathbb{P}_B^n.$$

□

Remark 1.8.2.13. Since any ring A is a $\mathbb{Z}$-algebra and $\mathbb{P}_A^n$ is naturally a $\mathbb{Z}$-scheme, therefore $\mathbb{P}_A^n \cong \mathbb{P}_{\mathbb{Z}}^n \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}\,(A)$, where the projection map $\mathbb{P}_A^n \to \mathrm{Spec}\,(A)$ is the usual structure map. This further motivates the construction of a projective space over any scheme.

Definition 1.8.2.14. (Projective n-space over a scheme) Let X be a scheme. The projective n-space over X is defined to be

$$\mathbb{P}_X^n := \mathbb{P}_{\mathbb{Z}}^n \times_{\mathrm{Spec}(\mathbb{Z})} X.$$

The natural projection map thus makes $\mathbb{P}_X^n$ a scheme over X.

1.8.3 Blowups

Do from Chapter 3 of Mumford and Hartshorne.

1.8.4 Projective schemes over a Noetherian ring

1.9 $\mathcal{O}_X$-modules

We will now cover certain types of important $\mathcal{O}_X$-modules that we will need in our study. Note that we defined $\mathcal{O}_X$-modules and various other algebraic constructions on them in Chapter 6, thus we assume the basic notion of $\mathcal{O}_X$-modules and its global algebra being known and we will thus specialize to the case of X being a scheme. The main goal is to define and study coherent and quasi-coherent modules over a scheme X. Its importance will manifest later in our study of projective schemes and their cohomology, the latter of which is an extremely powerful and versatile tool for doing geometry over schemes.

1.9.1 Coherent and quasi-coherent modules on schemes

Quasi-coherent sheaves form an integral part of the backbone of an attempt at doing geometry on schemes. Even though the definitions here makes sense in the setting of locally ringed spaces, but this theory is much more better behaved in the setting of schemes; for schemes, such sheaves have nice description on affine opens. This is the reason it is not included in Foundational Geometry, Chapter 6.

We first define the notion of quasicoherent modules on schemes.

Definition 1.9.1.1. (Quasicoherent and coherent $\mathcal{O}_X$-modules) Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is called quasicoherent if there exists an affine open cover $\{U_i := \operatorname{Spec}(R_i)\}_{i \in I}$ of X and $\{M_i\}_{i \in I}$ where M_i is an R_i-module such that $\mathcal{F}_{|U_i} \cong \widetilde{M_i}$ for all $i \in I$. Further, $\mathcal{F}$ is said to be a coherent module if each M_i is a finitely generated R_i-module for each $i \in I$.

Remark 1.9.1.2. There are five basic properties of quasi-coherent sheaves on a scheme, which we point out now.
1. Quasicoherence of a module can be checked locally.
2. The global sections functor of a quasicoherent module over an affine scheme is exact[29].
3. The image of the functor $\widetilde{(-)} : \mathbf{Mod}(R) \to \mathbf{Mod}(\mathcal{O}_{\operatorname{Spec}(R)})$ (see Definition 1.2.3.1 and remarks surrounding it) is precisely all quasicoherent modules over $\operatorname{Spec}(R)$.
4. Quasicoherence is preserved under inverse image. It is further preserved under direct image if domain is a Noetherian scheme or if the map is quasi-compact and separated.
5. The category of all quasicoherent modules

$$\mathbf{QCoh}(\mathcal{O}_X)$$

   is a Grothendieck-abelian category.

We will come to these results one by one. We first discuss some basic properties and examples.

Lemma 1.9.1.3. *Let $X = \operatorname{Spec}(A)$ be an affine scheme and $\mathfrak{a} \leq A$ be an ideal. Consider the corresponding closed immersion*

$$i : \operatorname{Spec}(A/\mathfrak{a}) = Y \hookrightarrow \operatorname{Spec}(A) = X.$$

---

[29]This in cohomological language means that the first cohomology group $H^1(X, \mathcal{F}) = 0$, as we shall see after few sections.

Then,

   1. $i_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module,
   2. $i_*\mathcal{O}_Y \cong \widetilde{A/\mathfrak{a}}$.

Proof. 1. Consider the following

$$\varphi : \mathcal{O}_X \times i_*\mathcal{O}_Y \longrightarrow i_*\mathcal{O}_Y$$

which on a basic open $D(f) \subseteq X$ for $f \in A$ is given by

$$\varphi_{D(f)} : A_f \times (A/\mathfrak{a})_{\bar{f}} \to (A/\mathfrak{a})_{\bar{f}}$$

as the usual A_f-module structure over $(A/\mathfrak{a})_{\bar{f}}$. Indeed, as the above maps are natural w.r.t. restrictions, this suffices by StacksProject Lemma 009O. Thus, $i_*\mathcal{O}_Y$ is an $\mathcal{O}_X$-module. Further, $i_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module as the open cover $\{D(f)\}_{f \in A}$ of X is such that $i_*\mathcal{O}_Y(D(f)) \cong (A/\mathfrak{a})_{\bar{f}}$ is an $\mathcal{O}_X(D(f)) \cong A_f$-module generated by $\bar{1}$ (in the case when $f \in A$, we have $i_*\mathcal{O}_Y(D(f)) = 0$ and so trivially a finitely generated A_f-module).

2. Again, by the use of above mentioned lemma, we may reduce to working over a basis of X. Choosing $\{D(f)\}_{f \in A}$ to be such a basis, we see that $i_*\mathcal{O}_Y(D(f)) \cong (A/\mathfrak{a})_f$ and $\widetilde{A/\mathfrak{a}}(D(f)) \cong (A/\mathfrak{a})_f$. Hence, we may define a map $i_*\mathcal{O}_Y \to \widetilde{A/\mathfrak{a}}$ which on basic opens is identity. Consequently, this map on stalks is identity. This by above used lemma again yields a unique sheaf morphism $\varphi : i_*\mathcal{O}_Y \to \widetilde{A/\mathfrak{a}}$ which is an isomorphism as at stalks it is an isomorphism. $\qquad\square$

Example 1.9.1.4. Let X be an integral noetherian scheme and let K be its function field. Let $\mathcal{K}$ be the constant sheaf of field K over X. Then $\mathcal{K}$ is a quasi-coherent $\mathcal{O}_X$-module.

As X is noetherian, therefore let $X = \bigcup_{i=1}^n \mathrm{Spec}\,(A_i)$ where A_i are noetherian rings. As X is integral, therefore by Lemma 1.4.2.2, each A_i is a noetherian domain. Thus we deduce that $K \cong Q(A_i)$ for each i, where $Q(A_i)$ is the fraction field of A_i. This is because $\mathrm{Spec}\,(A_i)$ are open and X irreducible. We now show that $\mathcal{K}$ is an $\mathcal{O}_X$-module.

Pick any open $U \subseteq X$. Recall from Chapter 22 that a section of $\mathcal{K}(U)$ is a continuous map $U \to K$ with K in discrete topology. For any point $p \in \mathrm{Spec}\,(A_i)$, as A_i is a domain, we see that $(A_i)_p \hookrightarrow Q(A_i) \cong K$ for each i. We thus deduce that K is an $\mathcal{O}_{X,x}$-algebra for each $x \in X$ in a natural way and $\mathcal{O}_{X,x} \subseteq K$. So we may now define

$$\mathcal{O}_X(U) \times \mathcal{K}(U) \to \mathcal{K}(U)$$
$$(c, s) \mapsto c \cdot s$$

where $c \cdot s : U \to K$ is defined by $c(x)s(x) \in K$, $c(x) \in \mathcal{O}_{X,x} \subseteq K$. This is continuous as each $c \in \mathcal{O}_X(U)$ is seen to be a continuous map $c : U \to \coprod_{x \in U} \mathcal{O}_{X,x} \subseteq K$ as it is locally constant (Remark 22.1.0.4 and that locally around each point we have an affine open inside every open). This is automatically compatible with restrictions. Consequently, $\mathcal{K}$ is an $\mathcal{O}_X$-module.

Next, to see this is quasi-coherent, we claim that the affine open cover $\{U_i = \mathrm{Spec}\,(A_i)\}_{i=1,\dots,n}$ is such that $\mathcal{K}_{|U_i}$ is isomorphic to $\widetilde{K}$. Consequently, we reduce to proving the following claim

: Let $X = \text{Spec}\,(A)$ be an affine scheme where A is a noetherian domain and let $K = Q(A)$ be its fraction field. Then, the constant sheaf $\mathcal{K}$ associated to K is isomorphic to the $\mathcal{O}_X$-module $\widetilde{K}$.

It suffices to construct a map $\varphi : \mathcal{K} \to \widetilde{K}$ defined only on a basis such that on basics it is an isomorphism. For this, we notice that since localization of a domain at an element is again a domain, therefore for each $g \in R$, the open $D(g) \subseteq X$ is connected. Hence, $\mathcal{K}(D(g)) = K$ and $\widetilde{K}(D(g)) \cong K_g = K$. Thus, we may define $\varphi_{D(g)} : K \to K$ to be identity which is then easily seen to be a sheaf morphism. Hence, these sheaves are isomorphic as $\mathcal{O}_X$-modules.

We now see an important lemma concerning quasicoherent modules over an affine scheme.

Lemma 1.9.1.5. *Let $\mathcal{F}$ be a quasicoherent module over an affine scheme $X = \text{Spec}\,(R)$. Then X admits a finite open affine cover $\{D(g_i)\}_{i=1}^n$ such that $\mathcal{F}_{|D(g_i)} \cong \widetilde{M_i}$ where M_i is an R_{g_i}-module.*

Proof. Since $\mathcal{F}$ is quasicoherent, therefore there exists an open affine cover $\{U_i = \text{Spec}\,(S_i)\}_i$ of X such that $\mathcal{F}_{|U_i} \cong \widetilde{M_i}$ where M_i is an S_i-module. Since subsets of the form $D(g)$ forms a basis of X therefore for $D(g) \subseteq U_i$ we obtain via Lemma 1.2.3.4, 2, that $\mathcal{F}_{|D(g)} \cong \widetilde{R_g \otimes_{S_i} M_i}$ as $D(g) \cong \text{Spec}\,(R_g)$. Since $N_i := R_g \otimes_{S_i} M_i$ is an R_g-module, so we have a cover of X by finitely many $D(g_i)$ by Lemma 1.2.1.6 such that $\mathcal{F}_{|D(g_i)} \cong \widetilde{N_i}$ where N_i is an R_{g_i}-module. $\square$

Using the above, we first show a technical lemma, which will be generalized later on, which will be used to show locality of quasi-coherent modules[30].

Lemma 1.9.1.6. *Let $X = \text{Spec}\,(A)$ be an affine scheme and $\mathcal{F} \in \textbf{QCoh}(X)$ be a quasi-coherent module. Let $D(f) \subseteq X$ be a basic open set for some $f \in A$.*

1. *If $s \in \Gamma(\mathcal{F}, X)$ is a global section of the module $\mathcal{F}$ such that s restricted on $D(f)$ is 0, then there exists $n > 0$ such that $f^n s = 0$ over X.*
2. *If $t \in \mathcal{F}(D(f))$, then there exists $n > 0$ such that $f^n t \in \mathcal{F}(D(f))$ extends to a global section of the module $\mathcal{F}$.*

Proof. 1. By Lemma 1.9.1.5, there exists a finite open cover $D(g_i)$ of X such that $\mathcal{F}_{|D(g_i)} \cong \widetilde{M_i}$. Denoting the restriction of s to $D(g_i)$ as $s_i \in M_i$, we see that the image of s_i is zero in $(M_i)_f$ when restricted to $D(fg_i) = D(f) \cap D(g_i)$. Consequently, for some $n_i > 0$, we have $f^{n_i} s_i = 0$ over $D(g_i)$. As g_i are finitely many, taking large enough n, we obtain $f^n s_i = 0$ over each $D(g_i)$. It follows that the global section $f^n s$ of the module $\mathcal{F}$ is such that it's restriction to each open set of an open cover of X is 0. By sheaf axioms, it follows that $f^n s = 0$ over X.

2. Fix the finite open affine cover $\{D(g_i)\}_{i=1}^n$ of X coming from Lemma 1.9.1.5. Consider all the finitely many intersections $D(g) \cap D(g_i) = D(fg_i)$. Restricting t from $D(f)$ to $D(fg_i)$, we obtain $t_i \in (M_i)_f$ for each i. Hence, for each i, there is some $n_i > 0$ such that $f^{n_i} t_i \in M_i = \mathcal{F}(D(g_i))$. By multiplying by large f^k to each $f^{n_i} t_i$ which are finitely many, we may arrange that $f^n t_i \in \mathcal{F}(D(g_i))$.

---

[30]The result is similar in flavour to Proposition 1.3.1.5.

We now form a matching family for the module $\mathcal{F}$ over the open cover $\{D(g_i)\}$ which would glue up to give the required global section. Indeed, fix two $D(g_i)$ and $D(g_j)$. Restrict $f^n t_i$ and $f^n t_j$ to $D(g_i) \cap D(g_j) = D(g_i g_j)$. Observe that over the even smaller open $D(fg_i g_j)$, the section $f^n t_i - f^n t_j$ is zero as $t_i = t_j = t$ over $D(fg_i g_j) \subseteq D(f)$. Hence by item 1 applied over $D(g_i g_j)$, there exists $m_{ij} > 0$ such that $f^{m_{ij}}(f^n t_i - f^n t_j) = 0$, hence $f^{n+m_{ij}}(t_i - t_j) = 0$ over $D(fg_i g_j)$. As i and j are finitely many, so taking m large enough, we obtain $f^{n+m} t_i = f^{n+m} t_j$ over $\mathcal{F}(D(g_i g_j))$ for each i and j. Thus, the family $\{f^{n+m} t_i\}$ is a matching family which glues up to give $s \in \Gamma(\mathcal{F}, X)$ such that its restriction over $D(f)$ is $f^{n+m} t$[31]. $\qquad\square$

---

[31]Note that we have implicitly used the fact the restriction maps of $\mathcal{F}$ preserves the respective module structures (see remarks surrounding Definition 6.5.0.1)

1.9.2   Further properties of quasi-coherent modules

We now discuss some more results which would culminate in the proofs of statements in Remark 1.9.1.2. In the first few results, we would set the background in order to be able to show that quasi-coherence is a local property. This would dispense the first statement of Remark 1.9.1.2.

Remark 1.9.2.1. Let $X = \mathrm{Spec}\,(R)$ be an affine scheme and $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module over X. Then, we obtain a map

$$\alpha : \widetilde{\Gamma(\mathcal{F}, X)} \longrightarrow \mathcal{F}$$

which on a basic open set $D(f) \subseteq X$, $f \in R$ is given by $\Gamma(\mathcal{F}, X)_f \to \mathcal{F}(D(f))$ mapping as $m/f^n \mapsto \rho_{X, D(f)}(m)/f^n$. Indeed, this is a $\mathcal{O}_X$-linear homomorphism which on stalks yields the $R_{\mathfrak{p}}$-linear map

$$\Gamma(\mathcal{F}, X)_{\mathfrak{p}} \longrightarrow \mathcal{F}_{\mathfrak{p}}$$

which is given by

$$\Gamma(\mathcal{F}, X) \otimes_R R_{\mathfrak{p}} \cong \Gamma(\mathcal{F}, X) \otimes_R \varinjlim_{f \notin \mathfrak{p}} R_f \cong \varinjlim_{f \notin \mathfrak{p}} \Gamma(\mathcal{F}, X) \otimes_R R_f = \varinjlim_{D(f) \ni \mathfrak{p}} \widetilde{\Gamma(\mathcal{F}, X)}(D(f)) \to \varinjlim_{D(f) \ni \mathfrak{p}} \mathcal{F}(D(f)).$$

We will see that this map α would become an isomorphism, especially due to the fact that quasicoherent modules behave very nicely on open affines of the form $D(f)$, as the Lemma 1.9.1.5 shows.

Th Lemma 1.9.1.5 is used to prove an important technical result, Lemma 5.3, Chapter 2 cite[Hartshorne]. We will prove a general version of that result later. Its importance is shown by the following corollary.

Corollary 1.9.2.2. *Let $\mathcal{F}$ be a quasicoherent sheaf over an affine scheme $X = \mathrm{Spec}\,(A)$. Then, there is a natural isomorphism of A_f-modules for each $f \in A$*

$$\Gamma(\mathcal{F}, X)_f \xrightarrow{\ \cong\ } \mathcal{F}(D(f))$$

given by $m/f^n \mapsto \rho(m)/f^n$ where ρ is the restriction map of $\mathcal{F}$ from X to $D(f)$

Proof. Follows from Lemma 5.3, Chapter 2 of cite[Hartshorne]. $\square$

Using the above, one proves the local nature of quasicoherence.

Proposition 1.9.2.3. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module over a scheme X. Then, $\mathcal{F}$ is quasicoherent if and only if for all open affine $U = \mathrm{Spec}\,(A) \subseteq X$ we have $\mathcal{F}_{|U} \cong \widetilde{M}$ where M is an A-module.*

Proof. We need to only show $R \Rightarrow L$. Let $\mathcal{F}$ be quasicoherent and $U = \mathrm{Spec}\,(A)$ open affine. We may assume $X = \mathrm{Spec}\,(A)$. Thus we need to show $\mathcal{F} \cong \widetilde{M}$ for an A-module M. By Lemma 1.9.1.5, we obtain an open affine cover $D(g_i)$ of X where $\mathcal{F}_{|D(g_i)} \cong \widetilde{M_i}$ for an A_{g_i}-module M_i. Let $M = \Gamma(\mathcal{F}, X)$, which is an A-module. By Corollary 1.9.2.2, we obtain a natural isomorphism $M_i \cong M_{g_i}$. Thus we have the required result. $\square$

A similar result is true for coherent modules.

Proposition 1.9.2.4. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module over a Noetherian scheme X. Then, $\mathcal{F}$ is coherent if and only if for all open affine $U = \operatorname{Spec}(A) \subseteq X$ we have $\mathcal{F}_{|U} \cong \widetilde{M}$ where M is a finitely generated A-module.*

Proof. See Proposition 5.4, Chapter 2 [Hartshorne]. $\qquad\square$

Corollary 1.9.2.5. *The image of the functor $\widetilde{(-)}$ of Definition 1.2.3.1 is exactly all quasicoherent modules over $\operatorname{Spec}(R)$. In other words, $\mathbf{Mod}(R) \equiv \widetilde{\mathbf{Mod}}(\mathcal{O}_{\operatorname{Spec}(R)}) = \mathbf{QCoh}(\mathcal{O}_{\operatorname{Spec}(R)})$. Further, this restricts to $\mathbf{Mod}(R)^{f.g.} \equiv \mathbf{Coh}(\mathcal{O}_{\operatorname{Spec}(R)})$.* $\qquad\square$

We next see the exactness of global sections functor.

Proposition 1.9.2.6. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ is quasicoherent, then*

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$

is exact.

Proof. Proposition 5.6, Chapter 2 [Hartshorne]. $\qquad\square$

The category of quasicoherent modules is further a Grothendieck-abelian category.

Theorem 1.9.2.7. *Let X be a scheme. The category $\mathbf{QCoh}(\mathcal{O}_X)$ is a Grothendieck-abelian category. Consequently, it is an abelian category which has all coproducts.*

Proof. Tag 077P, [Stacksproject]. $\qquad\square$

We now see stability of quasicoherence under inverse images

Lemma 1.9.2.8. *Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{G}$ be a quasicoherent $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a quasicoherent $\mathcal{O}_X$-module. If X, Y are Noetherian schemes and $\mathcal{G}$ is coherent, then $f^*\mathcal{G}$ is coherent.*

Proof. The first question is local in both X and Y by Proposition 1.9.2.3. Indeed, pick $x \in X$ and an open affine $V \ni f(x)$ in Y. Then by continuity there is an open affine $U \ni x$ in X such that $f(U) \subset V$. This shows that we may assume X and Y to be affine. The result is now immediate by Corollary 1.9.2.5 and Lemma 1.2.3.4, 2. The same technique works for coherent case. $\qquad\square$

As promised earlier, we state a general result about invertible modules and quasicoherent modules over schemes. This is a fundamental result and will be used to portray the simplicity of the techniques developed so far.

Lemma 1.9.2.9. *Let X be a scheme, $\mathcal{L} \in \operatorname{Pic}(X)$, $\mathcal{F} \in \mathbf{QCoh}(X)$, $f \in \Gamma(\mathcal{L}, X)$ and $s \in \Gamma(X, \mathcal{F})$. Denote by $X_f \subseteq X$ the open subset $X_f := \{x \in X \mid f_x \notin \mathfrak{m}_x \mathcal{L}_x\}$.*
   1. If X is quasicompact and s is such that $s|_{X_f} = 0$, then there exists $n \in \mathbb{N}$ such that $f^n s = 0$ in $\Gamma(\mathcal{L}^{\otimes n} \otimes \mathcal{F}, X)$.

2. *If X admits a finite affine open cover $\{U_i\}$ where $\mathcal{L}_{|U_i}$ is free (of rank 1) and $U_i \cap U_j$ is quasicompact, then for any $t \in \mathcal{F}(X_f)$, there exists $n \in \mathbb{N}$ such that $f^n t \in (\mathcal{L}^{\otimes n} \otimes \mathcal{F})(X_f)$ extends to a global section $s \in \Gamma(\mathcal{L}^{\otimes n} \otimes \mathcal{F}, X)$.*

Proof. 1. Cover X by finitely many affine open sets $U = \operatorname{Spec}(A)$ which satisfies $\varphi : \mathcal{L}_{|U} \cong \mathcal{O}_{X|U} \cong \mathcal{O}_{\operatorname{Spec}(A)}$. Further, denote $\mathcal{F}_{|U} = \widetilde{M}$ where M is an A-module (Corollary 1.9.2.5). By restricting f to U, we may write $g = \varphi_U(f) \in A$ and by restricting s to U, we may write $s \in M$. Since $s|_{X_f} = 0$ and $X_f \cap U = D(g)$, therefore $s/1 = 0$ in M_g. Consequently, there exists $n \in \mathbb{N}$ such that $g^n s = 0$ in $M = \Gamma(U, \mathcal{F}_{|U})$. We then observe the following isomorphisms:

$$\mathcal{L}_{|U}^{\otimes n} \otimes \mathcal{F} \cong \mathcal{O}_{X|U}^{\otimes n} \otimes \mathcal{F} \cong \mathcal{F}_{|U}$$

where the first one is $\varphi^{\otimes n} \otimes \operatorname{id}$. Consequently, we get isomorphisms in sections over U which implies that $f^{\otimes n} \otimes s \mapsto g^n s = 0$. hence $f^{\otimes n} \otimes s = 0$ in $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ over U. Since this happens for all finitely many Us, therefore taking large enough n, we observe that $f^n \otimes s = 0$ in $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ over X.

2.   □

These are some of the basic results on quasicoherent modules. We now do perhaps the most important application of $\mathcal{O}_X$-modules, that when X is a projective scheme.

1.9.3 $\mathcal{O}_{\operatorname{Proj}(S)}$-modules

It is extremely important to do all exercises from Chapter 2 of Hartshorne from 5.9 to 5.14, as they deal with geometry coming out of modules over projective schemes.

Let S be a graded ring and M a graded S-module. We attach a sheaf $\widetilde{M}$ to M over $\operatorname{Proj} S$.

Definition 1.9.3.1. $(\widetilde{M})$ Let S be a graded ring and M be a graded S-module. Then we define a sheaf $\widetilde{M}$ over $\operatorname{Proj}(S)$ given on an open set $U \subseteq \operatorname{Proj}(S)$ by

$$\widetilde{M}(U) := \left\{ s : U \to \coprod_{\mathfrak{p} \in U} M_{(\mathfrak{p})} \mid \forall \mathfrak{p} \in U, \, s(\mathfrak{p}) \in M_{(\mathfrak{p})} \, \& \, \exists \text{ open } \mathfrak{p} \in V \subseteq U \, \& \, m \in M_d, f \in S_d \text{ s.t. } f \notin \mathfrak{q} \, \& \, s(\mathfrak{q}) = m/f \, \forall \mathfrak{q} \in V \right\}.$$

The restrictions are the obvious ones. It is clear that if we treat S as a graded S-module, then $\widetilde{S} \cong \mathcal{O}_{\operatorname{Proj}(S)}$ where we treat $\mathcal{O}_{\operatorname{Proj}(S)}$ as an $\mathcal{O}_{\operatorname{Proj}(S)}$-module.

We have analogous results to the affine case on the behaviour of $\widetilde{M}$ on basis, on stalks and its quasicoherence.

Proposition 1.9.3.2. *Let S be a graded ring, M be a graded S-module, $X = \operatorname{Proj}(S)$ be the projective scheme over S and $\widetilde{M}$ to be the associated sheaf of M over X. Then,*
   1. for any $\mathfrak{p} \in X$,

$$(\widetilde{M})_{\mathfrak{p}} \cong M_{(\mathfrak{p})},$$

2. *for any $f \in S_d$, $d > 0$ and basic open $D_+(f)$,*

$$\widetilde{M}_{|D_+(f)} \cong \widetilde{M_{(f)}},$$

3. *the sheaf $\widetilde{M}$ is an $\mathcal{O}_X$-module which is furthermore quasicoherent,*
4. *if S is a noetherian ring and M is finitely generated, then $\widetilde{M}$ is coherent.*

Proof. 1. and 2. follows from repeating Lemma 1.8.2.4. Statement 3. follows from local property of quasicoherence (Proposition 1.9.2.3), the fact that sets of the form $D_+(f)$ for $f \in S_d$, $d > 0$ forms a basis of X (Lemma 1.8.1.3) and statement 2 above. Statement 4 follows from coherence being a local property for Noetherian schemes (Proposition 1.9.2.4) and statement 2 above. $\qquad\square$

Remark 1.9.3.3. The theory of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules is rich because of various constructions which interrelates the category $\mathbf{grMod}(S)$ of graded S-modules and graed maps and the category $\mathbf{Mod}(\mathcal{O}_{\mathrm{Proj}(S)})$. Indeed, these constructions is what we will study now, and these will be absolutely indispensable to do geometry in projective spaces $\mathrm{Proj}\, k[x_0, \ldots, x_n]/f$ for a homogeneous polynomial f.

Remark 1.9.3.4. The construction of $\mathcal{O}_X$-modules is functorial ($X = \mathrm{Proj}(S)$):

$$\widetilde{(-)} : \mathbf{grMod}(S) \longrightarrow \mathbf{QCoh}(\mathcal{O}_X)$$
$$M \longmapsto \widetilde{M}$$
$$M \overset{\varphi}{\to} N \longmapsto \widetilde{M} \overset{\eta}{\to} \widetilde{N}$$

where η on a basic open $D_+(f)$ is given by the localization maps $\eta_{D_+(f)} : M_{(f)} \to N_{(f)}$.

We first begin by twisting each $\mathcal{O}_{\mathrm{Proj}(S)}$-module.

Twists and Serre twists

Definition 1.9.3.5. (**Twists**) Let S be a graded ring and $X = \mathrm{Proj}(S)$. For each $n \in \mathbb{Z}$, we define the nth-*Serre twist* to be $\mathcal{O}_X(n)$ which is defined to be $\widetilde{S(n)}$, the sheaf associated to the n-th twisted graded S-module $S(n)$ (Definition 18.1.2.15). For each $\mathcal{O}_X$-module $\mathcal{F}$, we then define the nth-*twist of* $\mathcal{F}$ to be $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Remark 1.9.3.6. It is clear from definition that for each $d \geq 0$, any homogeneous element $s_d \in S_d$ of a graded ring S will define a global section $s_d \in \Gamma(\mathcal{O}_X(n), X)$ where $X = \mathrm{Proj}(S)$.

Some obvious questions are: what happens to n^{th}-twist of $\widetilde{M}$ for a graded S-module M, what is so special about $\mathcal{O}_X(n)$ in relation to $\mathcal{O}_X$? We answer these in the following result.

Proposition 1.9.3.7. *Let S be a graded ring generated by S_1 as an S_0-algebra and $X = \mathrm{Proj}(S)$. Then,*
1. *$\mathcal{O}_X(n)$ is an invertible module for all $n \in \mathbb{Z}$,*
2. *for any graded S-modules M, N and $n \in \mathbb{Z}$,*
    (a) *$\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$,*

   (b) $\widetilde{M}(n) \cong \widetilde{M(n)}$.
3. *Let T be a graded ring generated by T_1 as a T_0-algebra. Let M be a graded S-module and N a graded T-module. Let $\varphi : S \to T$ be a graded map and $f : U \to \mathrm{Proj}(S)$ be the corresponding map (Proposition 1.8.2.6). Then,*
   (a) $f_*(\widetilde{N}_{|U}) \cong \widetilde{{}_S N}$,
   (b) $f^*(\widetilde{M}) \cong \widetilde{(M \otimes_S T)}_{|U}$,
   (c) $f_*(\mathcal{O}_{\mathrm{Proj}(T)|U}) \cong \widetilde{T}$, where T is treated to be an S-module via φ.
4. *Let φ and f as in 3 and let $Y = \mathrm{Proj}(T)$. Then,*
   (a) $f_*(\mathcal{O}_Y(n)_{|U}) \cong f_*(\mathcal{O}_{Y|U})(n)$,
   (b) $f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)_{|U}$.
5. *For all $n, m \in \mathbb{Z}$,*

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(n + m).$$

Proof. 1. Cover X by basic open sets of the form $D_+(f)$ for $f \in S_1$. One then easily reduces to showing that $S(n)_{(f)}$ is a free $S_{(f)}$-module of rank 1. Indeed, one show that the following map is an $S_{(f)}$-linear map which is an isomorphism as $S_{(f)}$-modules: $S_{(f)} \to S(n)_{(f)}$ given by $s/f^k \mapsto sf^n/f^k$. One really needs f to be of degree 1 to be able to show that this is an isomorphism.

2. This reduces to finding natural isomorphism $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \cong (M \otimes_S N)_{(f)}$ which one can do constructing two sided inverses. One of these maps well-definedness will use the fact that degree of f is 1.

3. See Lemma 1.2.3.4 for $a)$ and $b)$ and observe that $f^{-1}(D_+(g)) = D_+(\varphi(g))$ for $g \in S$ homogeneous by a simple unravelling of definition of $U \subseteq \mathrm{Proj}(T)$. The statement $c)$ is immediate by looking the respective sections on a basic open set $D_+(g)$.

4. Statement $a)$ follows from $3.a)$ and $3.c)$ is immediate from $3.b)$.

5. Follows from $2.a)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 1.9.3.8. The twisting functor given by

$$\mathbf{Mod}(\mathcal{O}_X) \longrightarrow \mathbf{Mod}(\mathcal{O}_X)$$

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \longmapsto \mathcal{F}(n) \xrightarrow{f \otimes \mathrm{id}} \mathcal{G}(n)$$

is exact. This is immediate as $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ and thus localizing at a point x, we get $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_X(n)_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}$ where the latter isomorphism follows from Proposition 1.9.3.7, 1.

In general this tells us also that the stalks of all twisted sheaves $\mathcal{F}(n)$ is identical to that of $\mathcal{F}$.

Associated graded S-module

We now associate to each $\mathcal{O}_X$-module $\mathcal{F}$ a graded S-module, where $X = \mathrm{Proj}(S)$.

Definition 1.9.3.9. (Associated graded S-module) Let S be a graded ring, $X = \mathrm{Proj}(S)$ and $\mathcal{F}$ an $\mathcal{O}_X$-module. Define the associated graded S-module to be

$$\Gamma_*(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathcal{F}(n), X)$$

where the S-module structure is given as follows: we need only define the scalar multiplication for homogeneous elements, so let $s_d \in S_d$ and $t_n \in \Gamma(\mathcal{F}(n), X)$. Then define $s_d \cdot t_n$ to be the image of $s_d \otimes t_n \in \Gamma(\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(n), X)$ under the isomorphism $\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(n) \cong \mathcal{F}(n+d)$ via Proposition 1.9.3.7, 5, in order to obtain an element of $\Gamma(\mathcal{F}(n+d), X)$, as needed.

Remark 1.9.3.10. There are two main results about associated graded S-modules.

1. Let $S = A[x_0, \ldots, x_r]$ for a ring A and $r \geq 1$. Then $\Gamma_*(\mathcal{O}_X) \cong S$ for $X = \mathrm{Proj}(S)$. The relevance of this result is as follows. We know that the global sections of the structure sheaf over a projective scheme doesn't recover the homogeneous coordinate ring back. This result tells us that looking only at global sections of structure sheaf won't suffice (hopefully obvious by now), we need to instead look at global sections of all twists of the structure sheaf in order to recover the coordinate ring. For example, consider the quadric $xy - wz$ in $\mathbb{P}^5_k$. The corresponding coordinate ring is $S = k[w, x, y, z, a, b]/xy - wz \cong \frac{k[w,x,y,z]}{xy-wz}[a,b]$ and the corresponding scheme is $X = \mathrm{Proj}(S)$. Consequently, we can write $S = A[a, b]$ for $A = \frac{k[w,x,y,z]}{xy-wz}$ and thus this result would yield that S is isomorphic to $\Gamma_*(\mathcal{O}_X)$. Note that to use this result, we have to force ourselves to go 2 dimensions up.

2. Let S be a graded ring which is *finitely* generated by S_1 as an S_0-algebra and let $X = \mathrm{Proj}(S)$. Then, for any quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$, we obtain a natural isomorphism $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$. This result therefore tells us that the functor $M \mapsto \widetilde{M}$ of Remark 1.9.3.4 from graded S-modules to quasicoherent $\mathcal{O}_X$-modules is essentially surjective.

We'll later see that these results will allow us to obtain an equivalent criterion of when is a scheme over an affine scheme projective.

We now state these results and sketch their proofs.

Proposition 1.9.3.11. *Let $S = A[x_0, \ldots, x_r]$ for a ring A and $r \geq 1$ and denote $X = \mathrm{Proj}(S)$. Then $\Gamma_*(\mathcal{O}_X) \cong S$.*

Proof. The main idea is to keep reducing the problem to a problem about graded ring S. Since S is generated by x_is as an A-algebra, therefore $D_+(x_i)$ for $i = 0, \ldots, r$ covers X. An element in $\Gamma_*(\mathcal{O}_X)$ is given by a sum of elements $t_n \in \Gamma(\mathcal{O}_X(n), X)$. Let $t_n \in \Gamma(\mathcal{O}_X(n), X)$. The data of t_n is equivalently represented by the $(t_{n,0}, \ldots, t_{n,r})$ where $t_{n,i} \in \mathcal{O}_X(n)(D_+(x_i)) = S(n)_{(x_i)}$. Thus, $t_{n,i} \in S_{x_i}$ is a homogeneous element of degree n. Thus, $t = \sum_n t_n$ is equivalently represented by the tuple $(t_0, \ldots, t_r)$ where $t_i = \sum_n t_{n,i}$ such that the image of t_i under $S_{x_i} \to S_{x_i, x_j}$ is same as the image of t_j under $S_{x_j} \to S_{x_j, x_i}$ for all $i, j = 0, \ldots, r$. Note each of these S_{x_i, x_j} for varying i, j are contained in $R = S_{x_0, \ldots, x_r}$. Now, we have injective maps $S \to S_{x_i} \to S_{x_i, x_j} \to R$ and thus $t = (t_0, \ldots, t_r)$ as above is contained in $\bigcap_{i=0}^{r} S_{x_i} \hookrightarrow R$. In fact, any element of this intersection also corresponds to an element of $\Gamma_*(\mathcal{O}_X)$. Consequently, $\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^{r} S_{x_i}$. It is straightforward to see that this intersection is exactly S by writing a general homogeneous element of R and observing what it needs to satisfy to be in the intersection. $\square$

We now show the essential surjectivity of $\widetilde{(-)}$. The proof of this result nicely shows the elegance of the techniques developed so far.

Proposition 1.9.3.12. *Let S be a graded ring which is finitely generated by S_1 as an S_0-algebra and $X = \operatorname{Proj} S$.*

  1. For each $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural map

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}.$$

  2. For each quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$, the above map β is an isomorphism, that is,

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\ \cong\ } \mathcal{F}.$$

Proof. 1. Since $D_+(f)$ for $f \in S_1$ covers X, we may define β naturally only on $D_+(f)$. This is done as follows:

$$\Gamma_*(\mathcal{F})_{(f)} \xrightarrow{\ \beta_{D_+(f)}\ } \mathcal{F}(D_+(f))$$

$$\cong \Big\uparrow \varphi$$

$$\Gamma(\mathcal{F}(d)_{|D_+(f)} \otimes \mathcal{O}_X(-d), X)$$

where the diagonal map is given by

$$\frac{m}{f^d} \mapsto m \otimes \frac{1}{f^d}$$

and the isomorphism φ is given by restrictions.

2. In the above, we need to show that the diagonal map is an isomorphism. Suppose for some $m/f^d \in \Gamma_*(\mathcal{F})_{(f)}$, we have that $m \otimes 1/f^d = 0$ in $\Gamma(\mathcal{F}(d)_{|D_+(f)} \otimes \mathcal{O}_X(-d), X)$. Denote $\mathcal{G} = \mathcal{F}(d) \otimes \mathcal{O}_X(-d)$, which is quasicoherent. Note that $m \otimes 1/f^d = 0$ as an element in $\mathcal{G}(D_+(f))$ and also note that $D_+(f) = X_f$. Consequently from Lemma 1.9.2.9, 1, there exists $n \in \mathbb{N}$ such that $f^n \otimes m \otimes 1/f^d$ is zero as a global section of $\mathcal{O}_X(1)^{\otimes n} \otimes \mathcal{G} \cong \mathcal{F}(n)$. Hence, $f^{n-d}m = 0$ in $\Gamma(\mathcal{F}(n), X)$ and thus $\frac{m}{f^d} = \frac{f^{n-d}m}{f^n}$ is zero in $\Gamma_*(\mathcal{F})_{(f)}$. This shows injectivity. We now show surjectivity. Pick $t \in \mathcal{F}(D_+(f))$. By Lemma 1.9.2.9, 2, (which applies here as $D_+(f)$s are affine and finitely many whose intersection is again affine), we obtain a section $f^n t$ of $\mathcal{O}_X(1)^{\otimes n} \otimes \mathcal{F} \cong \mathcal{O}_X(n) \otimes \mathcal{F} \cong \mathcal{F}(n)$ over $D_+(f)$ which extends to a global section of $\mathcal{F}(n)$, say s. Consider s/f^n in $\Gamma_*(\mathcal{F})_{(f)}$, which maps to $s \otimes 1/f^n = t$ in $\mathcal{F}(D_+(f))$, as needed. $\qquad\square$

We can use these results to obtain a nice characterization of closed subschemes of projective schemes and an equivalent characterization of projective schemes over affine schemes. Denote by $\mathbb{P}^r_A = \operatorname{Proj}(A[x_0, \ldots, x_r])$ for a ring A.

Proposition 1.9.3.13. *Let $Y \hookrightarrow \mathbb{P}^r_A$ be a closed subscheme with ideal sheaf $\mathcal{I}_Y$ of the projective r-space over A. Then $I = \Gamma_*(\mathcal{I}_Y)$ is a homogeneous ideal of $A[x_0, \ldots, x_r]$ and we have*

$$Y \cong \operatorname{Proj}\left(A[x_0, \ldots, x_r]/I\right).$$

Proof. Let $S = A[x_0, \ldots, x_r]$. The fact that I is a homogeneous ideal of S follows from exactness twisting functor (Remark 1.9.3.8), left exactness of global sections and $\Gamma_*(\mathcal{O}_X) =$

S of Proposition 1.9.3.11. In order to show that $Y \cong \mathrm{Proj}(S/I)$, it is enough to show that they both define isomorphic ideal sheaves (Proposition 1.4.4.9, 3). The ideal sheaf of $\mathrm{Proj}(S/I)$ is $\widetilde{I}$ by Proposition 1.8.2.8, 3 and the ideal sheaf of Y is $\mathcal{I}_Y$. Since $I = \Gamma_*(\mathcal{I}_Y)$, therefore the result follows from Proposition 1.9.3.12, 2. $\qquad\square$

Proposition 1.9.3.14. *Let A be a ring. A scheme $Y \to \mathrm{Spec}\,(A)$ is projective if and only if $Y \cong \mathrm{Proj}(S)$ for a graded ring S with $S_0 = A$ and which is finitely generated by S_1 as an S_0-algebra.*

Proof. (L $\Rightarrow$ R) We have a closed immersion $Y \to \mathbb{P}^r_A$. From Proposition 1.9.3.13, it follows that $Y \cong \mathrm{Proj}(S)$ where $S = A[x_0, \ldots, x_r]/I$, but S_0 might not be A. By Proposition 1.8.2.8, 2, we may assume I to not have any degree 0 component. Thus, S as defined will satisfy the necessary criterion.
(R $\Rightarrow$ L) We have $S \cong A[x_0, \ldots, x_r]/I$, so by Proposition 1.8.2.8, 1, we have a closed immersion $\mathrm{Proj}(S) \to \mathbb{P}^r_A$. $\qquad\square$

Very ample invertible modules

We now study modules which determine when a scheme is projective.

Start the very a
invertible modu
Chapter 1.

Proposition 1.9.3.15. *Let Y be a Noetherian scheme. Then a scheme $f : X \to Y$ is projective if and only if f is proper and there exists a very ample invertible sheaf over X relative to Y.*

1.9.4 Vector bundles

Do Exercise 5.18 from Chapter 2.

1.10   Divisors

The notion of divisors is one of the central tools for understanding the geometrical properties of a given scheme. Indeed, in the special case of curves in projective plane, a (Weil) divisor is just a formal linear combination of points of the curve. From this data, one can in-fact recover the embedding of the curve in the projective plane. Hence the data that divisors of a scheme stores is rich in geometric information.

We will first define the notion of Weil divisors in those schemes in which the points lying on codimension 1 subset of the scheme are regular (see Lemma 1.7.1.4 for a motivation behind the definition).

Definition 1.10.0.1. (Regular in codimension 1) A scheme X is said to be regular in codimension 1 if the local rings $\mathcal{O}_{X,p}$ which are of dimension 1 are regular.

Remark 1.10.0.2. All non-singular abstract varieties are regular in codimension 1 as all local rings are regular. In this section, we will be working with schemes which are noetherian integral separated and regular in codimension 1. We call them *Weil schemes*. All non-singular abstract varieties are Weil schemes.

1.10.1   Weil divisors and divisor class group

We will define the notion of Weil divisors and divisor class group on Weil schemes.

Definition 1.10.1.1. (Weil divisors) Let X be a Weil scheme. A *prime divisor* is an integral closed subscheme of codimension 1. A *Weil divisor* is an element of the free abelian group generated by the set of all prime divisors, denoted Div X. A Weil divisor is denoted $\sum_{i=1}^{k} n_i Y_i \in$ Div X. A Weil divisor $\sum_i n_i Y_i$ is *effective* if $n_i \geq 0$ for all i.

We now look at a foundational result which will guide the further development. Its proof is important as it combines a lot of our previous knowledge.

Proposition 1.10.1.2. *Let X be a Weil scheme and $Y \subseteq X$ be a prime divisor with $\eta \in Y$ be its generic point. Then there is an injective map*

$$\mathrm{PDiv}(X) \to \mathrm{DVal}(K(X))$$

where PDiv *is the set of all prime divisors of X, $K(X)$ is the function field of X and* DVal$(K(X))$ *is the set of all discrete valuations over $K(X)$.*

Proof. Note that if Y is a prime divisor, then there is a generic point $\eta \in Y$. Since codim $(Y, X) = \dim \mathcal{O}_{X,\eta}$ by Lemma 1.7.1.4, therefore we obtain that $\mathcal{O}_{X,\eta}$ is a regular local ring of dimension 1. Now there is a special result for such rings, which in particular establishes equivalences of such rings with a lot of other type of rings. Indeed, by Theorem 18.9.0.8 we obtain that in our case $\mathcal{O}_{X,\eta}$ is a DVR. As the fraction field of $\mathcal{O}_{X,\eta}$ (it is a domain as X is integral) is $K(X)$, the function field of X (Lemma 1.3.1.3), therefore we have a valuation $v : K(X) \to \mathbb{Z}$ whose valuation ring is $\mathcal{O}_{X,\eta}$. By Lemma 1.12.4.7 (which holds for Y as Y is separated by Corollary 1.12.4.5, 2), the valuation v uniquely determines

the point $\eta \in X$ as v has center η because the valuation ring $\mathcal{O}_{X,\eta}$ of v dominates the local ring $\mathcal{O}_{X,\eta}$. As the information of point $\eta \in X$ yields the closed set $Y \subseteq X$, therefore the valuation $v : K(X) \to \mathbb{Z}$ uniquely determines the prime divisor Y. $\qquad \square$

Remark 1.10.1.3. As a consequence, we can study a prime divisor via the valuation that comes through the Proposition 1.10.1.2. Indeed, for a prime divisor $Y \subseteq X$ and the associated discrete valuation $v_Y : K(X) \to \mathbb{Z}$, we can think of the value $v_Y(f)$ for some $f/g \in K(X) \setminus \{0\}$ to be telling us the number of poles that f/g has along Y. We can justify this via the proof of Proposition 1.10.1.2 as follows. For a prime divisor $Y \subseteq X$ with generic point $\eta \in Y$, the corresponding valuation is obtained by the fact that $\mathcal{O}_{X,\eta}$ is a DVR in our case. For a DVR R with fraction field K, the corresponding discrete valuation $v : K \to \mathbb{Z}$ can be thought of as an abstraction of the idea that we want to know how many poles a fraction $f/g \in K$ has and v provides that data to us. In particular, we think that if $v(f/g)$ is positive, then that tells us f/g has that many zeros in Y and if $v(f/g)$ is negative then that many poles in Y. We mostly have only this idea in mind when dealing with valuations.

As we know from complex analysis that a meromorphic function can only have discretely many singularities, along the similar lines we have that an arbitrary $f \in K(X)$ can atmost have some poles or some zeros at only finitely many prime divisors.

Proposition 1.10.1.4. *TODO.*

1.10.2   Cartier divisors

1.10.3   Divisors and invertible modules

1.11 Smoothness & differential forms

In this section, we would like to understand the notion of smoothness in algebraic geometry. We will first begin by defining a non-singular point of a variety over an algebraically closed field, which would be an extrinsic definition. However, by a fundamental observation of Zariski, we can have an intrinsic definition of non-singular points, which would be in terms of regular local rings. The main thrust behind this latter definition will be the expectation that over non-signular points, the dimension of the tangent space is equal to the dimension of the variety (which is true in the case of, say manifolds). We would further see that for a variety over an algebraically closed field, the set of singular points is closed and proper.

We would then introduce the important notion of sheaf of differentials over a scheme. This will again allow us to characterize non-singular points of a variety, and much more.

1.11.1 Non-singular varieties

To start investigating the notion of non-singularity, we first investigate it in the setting of classical affine varieties (Definition 1.5.4.11). We will then proceed to abstract varieties.

Definition 1.11.1.1. (Non-singular points of a classical affine variety) Let k be an algebraically closed field and X be a classical affine k-variety with $I(X) = \langle f_1, \ldots, f_m \rangle \lneq k[x_1, \ldots, x_n]$. A point $p \in X$ is said to be non-singular if the $n \times m$ Jacobian matrix

$$[J_p]_{n \times m} = \left(\frac{\partial f_i}{\partial x_j}(p) \right)_{ij}$$

is of rank $n - \dim X$.

The first obvious question is whether the above definition is independent of the choice of the generators of prime ideal $I(X)$. The following lemma says yes.

Lemma 1.11.1.2. *Let k be an algebraically closed field and X be a classical affine k-variety. The definition of a non-singular point $p \in X$ is independent of the choice of the generating set of $I(X)$.*

Proof. Let $I(X) = \langle f_1, \ldots, f_m \rangle = \langle g_1, \ldots, g_l \rangle$. We wish to show that

$$\text{rank} \left[\frac{\partial f_i}{\partial x_j}(p) \right]_{ij} = \text{rank} \left[\frac{\partial g_i}{\partial x_j}(p) \right]_{ij}.$$

This follows immediately after writing $f_i = \sum_{a=1}^{l} c_{i_a} g_a$, $c_{i_a} \in k[x_1, \ldots, x_n]$, differentiating it and observing that $g_a(p) = 0$ for all $a = 1, \ldots, l$. $\qquad \square$

Remark 1.11.1.3. In geometry, one notes that at a smooth point, the dimension of tangent space equals the dimension of the manifold itself. We would like to do a similar construction here. Indeed, if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and 0 is regular for f, then we know by implicit function theorem that $M = Z(f) \subseteq \mathbb{R}^n$ is a smooth manifold with normal vector field $(\nabla f) : \mathbb{R}^n \to \mathbb{R}^n$. Consequently, one can define the tangent space $T_x M$ for $x \in M$ to be the set of all those vectors which are normal to $(\nabla f)_x$. We mimic this definition for classical affine varieties.

Definition 1.11.1.4. (Tangent space of a classical affine variety) Let k be an algebraically closed field and let X be a classical affine k-variety in $\mathbb{A}_k^n$ with $I(X) = \langle f_1, \ldots, f_m \rangle$. Denote for each $f \in k[x_1, \ldots, x_n]$ and $p \in \mathbb{A}_k^n$ the following linear functional

$$(df)_p : k^n \longrightarrow k$$

$$v \longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) v_i.$$

For a point $p \in X$, we define the tangent space $T_p X$ as the following k-vector space

$$T_p X := \{ v \in k^n \mid (df_i)_p(v) = 0 \ i = 1, \ldots, m \}$$
$$= \{ v \in k^n \mid (df)_p(v) = 0 \ \forall f \in I(X) \}.$$

We now show that this definition of tangent space is intrinsic. Indeed, we will show that the $T_p X = T\mathcal{O}_{X,p} := \operatorname{Hom}_k\left(\mathfrak{m}/\mathfrak{m}^2, k\right)$, where $(\mathcal{O}_{X,p}, \mathfrak{m})$ is the local ring at $p \in X$ and $\kappa(p) = k$ in this case (see Definition 18.1.2.17). Let us begin with a series of observations.

Lemma 1.11.1.5. *Let k be an algebraically closed field and $p \in \mathbb{A}_k^n$. Then the k-linear map*

$$\theta_p : k[x_1, \ldots, x_n] \longrightarrow k^n$$

$$f \longmapsto \left(\frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p) \right)$$

induces a k-linear isomorphism

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \cong k^n$$

where $\mathfrak{m}_p = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$ is the maximal ideal of $k[x_1, \ldots, x_n]$ corresponding to the point p.

Proof. Let $p = (p_1, \ldots, p_n)$. Observe that $\{\theta_p(x_i - p_i)\}_{i=1,\ldots,n}$ forms a basis of k^n. Consequently, θ_p restricts to a surjective k-linear map $\hat{\theta}_p : \mathfrak{m}_p \to k^n$. Now one observes that $f \in \operatorname{Ker}(\theta)_p$ if and only if $f \in \mathfrak{m}_p^2$. Thus we have by first isomorphism theorem that $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong k^n$. $\square$

Lemma 1.11.1.6. *Let k be an algebraically closed field and X be a classical affine k-variety in $\mathbb{A}_k^n$ with $p \in X$. Let $(\mathcal{O}_{X,p}, \mathfrak{m})$ denote the local ring of X at p and $I \leq k[x_1, \ldots, x_n]$ be the ideal of X. Then,*

$$\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}_p/(\mathfrak{m}_p^2 + I).$$

Proof. Let $A = k[x_1, \ldots, x_n]$. By Proposition 1.5.3.10, 3, we have $\mathfrak{m} = (\mathfrak{m}_p)_{\mathfrak{m}_p}/I_{\mathfrak{m}_p}$ and $\mathfrak{m}^2 = \left((\mathfrak{m}_p^2)_{\mathfrak{m}_p} + I_{\mathfrak{m}_p}\right)/I_{\mathfrak{m}_p}$. By quotienting, we obtain

$$\mathfrak{m}/\mathfrak{m}^2 \cong \frac{(\mathfrak{m}_p)_{\mathfrak{m}_p}}{(\mathfrak{m}_p^2 + I)_{\mathfrak{m}_p}}$$

$$\cong \left(\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + I} \right)_{\mathfrak{m}_p/(\mathfrak{m}_p^2 + I)}$$

$$\cong \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + I}.$$

$\square$

Recall the notion of regular local ring from Definition 18.1.2.19. We now see that non-singular points are classified by the local ring being regular.

Theorem 1.11.1.7. *Let k be an algebraically closed field and X be a classical affine k-variety and let $p \in X$. The following are equivalent:*

1. *The point $p \in X$ is non-singular.*
2. *The local ring $\mathcal{O}_{X,p}$ is regular.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{X,p}$. By definition, we have $\mathcal{O}_{X,p}$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}_{X,p}$. By Proposition 1.5.3.10, 7, we further have that $\mathcal{O}_{X,p}$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim X$. Whereas by Lemmas 1.11.1.6 and 1.11.1.5, we observe

$$
\begin{aligned}
\dim_k \mathfrak{m}/\mathfrak{m}^2 &= \dim_k \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + I} \\
&= \dim_k \left(\frac{\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}}{\frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2}} \right) \\
&= \dim_k \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2} - \dim_k \frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2} \\
&= n - \dim_k \frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2}.
\end{aligned}
$$

With these two observations, we thus reduce to proving that

$$
\dim_k \frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2} = \operatorname{rank} J_p
$$

where $J_p = \left[\frac{\partial f_i}{\partial x_j} \right]$ for $I = \langle f_1, \ldots, f_m \rangle$. This now follows by the following two rather straightforward observations; in the notations of Lemma 1.11.1.5 and its proof, one observes

1. $\hat{\theta}_p^{-1}(\theta_p(I))$ is isomorphic as k-vector space to $I + \mathfrak{m}_p^2$,
2. $\dim_k \theta_p(I) = \operatorname{rank} J_p$.

The result now follows. $\square$

With the above result, we formulate the following definition of non-singular abstract varieties.

Definition 1.11.1.8. (**Non-singular abstract variety**) Let k be an algebraically closed field. A variety X over k is said to be non-singular if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a regular local ring.

Remark 1.11.1.9. Note that in the definition of non-singular varieties, it is sufficient to demand that $\mathcal{O}_{X,x}$ is a regular local ring for all closed points $x \in X$ only. Indeed, by Lemma 1.3.1.2, local ring at a non-closed point is obtained by localizing the local ring at a closed point at a prime ideal. As the localization of a regular local ring at a prime ideal is again a regular local ring by Theorem **??**, the result follows.

We now define the Zariski (co)tangent space of a scheme at a point.

Definition 1.11.1.10. (**Zariski (co)tangent space**) Let X be a scheme and $x \in X$ be a point and let κ be the residue field at point x. Then

   1. the Zariski cotangent space at x is defined to be the κ-vector space

$$T_x^* X := \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2,$$

   2. the Zariski tangent space at x is defined to be the κ-vector space

$$T_x X := \mathrm{Hom}_\kappa \left(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, \kappa \right).$$

These are the analogues to the case in algebra (see Definition 18.1.2.17).

1.12 Morphism of schemes

The main use of schemes to answer geometric questions begin with defining various types of situations that one usually finds himself/herself in algebraic geometry. We discuss them here one by one by giving examples. In the first section we discuss some basic type of maps among schemes.

1.12.1 Locality, permanence and some basic maps

We first cover some basic type of maps between schemes.

Definition 1.12.1.1. (Quasi-compact maps) A map $f : X \to Y$ of schemes is said to be quasi-compact if there exists an affine open cover $\{V_i\}$ of Y such that the space $f^{-1}(V_i) \subseteq X$ is quasi-compact for each i.

Remark 1.12.1.2. Observe that a scheme X over k has a quasi-compact structure map $X \to \operatorname{Spec}(k)$ if and only if X is quasi-compact.

We now see that quasi-compact maps are local on target.

Proposition 1.12.1.3. [32] *A map $f : X \to Y$ is quasi-compact if and only if for each open affine $V \subseteq Y$, the space $f^{-1}(V) \subseteq X$ is quasi-compact.*

Proof. The ($\Leftarrow$) is immediate. For ($\Rightarrow$), pick any open affine $V \subseteq Y$. We wish to show that $f^{-1}(V)$ is quasi-compact. Let $V_i = \operatorname{Spec}(B_i)$ be the collection of open affines covering Y such that $f^{-1}(V_i)$ is quasi-compact. We now obtain a finite covering of V by affine opens which are affine open in V_i for some i as well. Indeed, by Lemma 1.4.4.3, we may cover $V \cap V_i$ by open affines which are basic open in both V and V_i. Doing this for each i, we obtain a cover of V by basic opens. As V is affine, so by Lemma 1.2.1.6 we obtain a finite collection of basic opens $\{D(g_i)\}_{i=1}^n$ where $g_i \in B_i$ such that $D(g_i)$ is a basic open in V as well.

We now have that $f^{-1}(V) = \bigcup_{i=1}^n f^{-1}(D(g_i))$. Hence it suffices to show that $f^{-1}(D(g_i))$ is a quasi-compact subspace. To this end, we first immediately reduce to assuming that X is quasicompact (by replacing X by $f^{-1}(V_i)$) and $Y = \operatorname{Spec}(B)$ is affine (by replacing Y by V_i). We now wish to prove that for any $g \in B$, $f^{-1}(D(g))$ is quasi-compact.

As X is quasi-compact, therefore there exists a finite affine open cover of X by $\operatorname{Spec}(A_i)$. It suffices to show that $\operatorname{Spec}(A_i) \cap f^{-1}(D(g))$ is a quasicompact space. Observe that $f|_{\operatorname{Spec}(A_i)} : \operatorname{Spec}(A_i) \to \operatorname{Spec}(B)$ is a morphism of affine schemes. It follows from Corollary 1.3.0.6 that $f|_{\operatorname{Spec}(A_i)}$ is induced from a ring map $\varphi_i : B \to A_i$. As $\operatorname{Spec}(A_i) \cap f^{-1}(D(g)) = (f|_{\operatorname{Spec}(A_i)})^{-1}(D(g)) = D(\varphi_i(g))$, which is an affine open, therefore by Lemma 1.2.1.6, we deduce that $\operatorname{Spec}(A_i) \cap f^{-1}(D(g))$ is quasi-compact, as required. $\square$

Definition 1.12.1.4. (Quasi-finite maps) A map $f : X \to Y$ of schemes is quasi-finite if for each $y \in Y$, the fiber X_y is a finite set.

---

[32]Exercise II.3.2 of Hartshorne.

Example 1.12.1.5. Let k be an algebraically closed field. Consider the map

$$f : X = \text{Spec}\left(\frac{k[x,y]}{y^2 - x^3}\right) \longrightarrow \mathbb{A}^1_k$$

obtained by the map $k[x] \to \frac{k[x,y]}{y^2 - x^3}$ given by $x \mapsto x + \langle y^2 - x^3 \rangle$. Take any point $\mathfrak{p} = \langle p(x) \rangle \in \mathbb{A}^1_k$. Hence the fiber is

$$X_{\mathfrak{p}} = \text{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k \kappa(\mathfrak{p})\right)$$

$$= \text{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k \frac{k[x]_{\mathfrak{p}}}{\mathfrak{p}k[x]_{\mathfrak{p}}}\right)$$

$$\cong \text{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k \left(\frac{k[x]}{\mathfrak{p}}\right)_{\mathfrak{p}}\right).$$

Let $\mathfrak{p} \neq \mathfrak{o}$. As $k[x]$ is a PID, therefore $\mathfrak{p}$ is a maximal ideal. Consequently, we have $\kappa(\mathfrak{p}) = k[x]/\mathfrak{p}$. Hence,

$$X_{\mathfrak{p}} \cong \text{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k \frac{k[x]}{p(x)}\right)$$

$$\cong \text{Spec}\left(\frac{k[x,y]}{y^2 - x^3, p(x)}\right).$$

As k is algebraically closed, therefore by weak Nullstellensatz, we obtain that $p(x) = x - a$ for some $a \in k$. Consequently, if we have $a \neq 0$ then

$$X_{\mathfrak{p}} \cong \text{Spec}\left(\frac{k[x,y]}{y^2 - x^3, x - a}\right)$$

$$\cong \text{Spec}\left(\frac{k[y]}{y^2 - a^3}\right)$$

$$\cong \text{Spec}\left(\frac{k[y]}{(y + a^{3/2})(y - a^{3/2})}\right)$$

$$\cong \text{Spec}\left(k \times k\right)$$

$$\cong \text{Spec}\left(k\right) \amalg \text{Spec}\left(k\right).$$

If $a = 0$, then

$$X_{\mathfrak{p}} \cong \text{Spec}\left(\frac{k[y]}{y^2}\right)$$

and we know that $k[y]/y^2$ has only one prime ideal, the one generated by $y + \langle y^2 \rangle \in k[y]/y^2$. Hence $X_{\mathfrak{p}}$ consists of two points at all non-zero closed points and of a single point at the origin, showing that f has finite fibers at all closed points. However, at the generic point

$\mathfrak{p} = \mathfrak{o}$, we have a more complicated story:

$$X_\mathfrak{o} \cong \mathrm{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k k(x)\right)$$

$$\cong \mathrm{Spec}\left(\frac{k(x)[x,y]}{y^2 - x^3}\right)$$

$$\cong \mathrm{Spec}\left(\frac{k(x)[y]}{y^2 - x^3}\right).$$

As $k(x)[y]$ is a PID, therefore points of $X_\mathfrak{o}$ are thus in bijective correspondence with prime ideals of $k(x)[y]$ containing $y^2 - x^3$, which in turn is in bijection with the set of irreducible factors of $y^2 - x^3$ in $k(x)[y]$. As $k(x)[y]$ is a UFD, therefore there can atmost be finitely many such irreducible factors. Hence, $X_\mathfrak{o}$ is finite.

Hence all fibres are finite, making f quasi-finite.

1.12.2 Finite type

We already considered one example of such maps in the case of schemes over a field in Section 1.4.3

Definition 1.12.2.1. (Locally finite type) Let $f : X \to Y$ be a map of schemes. Then f is said to be locally of finite type if there is an affine open cover $V_i = \mathrm{Spec}\,(B_i)$, $i \in I$ of Y such that for each $i \in I$, $f^{-1}(V_i)$ has an open affine cover $U_{ij} = \mathrm{Spec}\,(A_{ij})$, $j \in J$ such that for each j, the ring A_{ij} is finite type[33] B_i-algebra.

Definition 1.12.2.2. (Finite type) Let $f : X \to Y$ be a map of schemes. Then f is said to be of finite type if there is an open affine cover $V_i = \mathrm{Spec}\,(B_i)$, $i \in I$ of Y such that for each $i \in I$, $f^{-1}(V_i)$ has a finite open affine cover $U_{ij} = \mathrm{Spec}\,(A_{ij})$, $j = 1, \ldots, n$ such that for each j, A_{ij} is a finite type B_i-algebra.

It is an important observation that both the above definitions are local on target.

Proposition 1.12.2.3. [34] *A map $f : X \to Y$ is locally of finite type if and only if for all open affine $V = \mathrm{Spec}\,(B)$ in Y, there is an open affine cover $U_i = \mathrm{Spec}\,(A_i)$ of $f^{-1}(V)$ in X such that each A_i is a finite type B-algebra.*

Proof. The R $\Rightarrow$ L follows immediately. Let $V_i = \mathrm{Spec}\,(B_i)$ be an open affine cover of Y such that $f^{-1}(V_i)$ is covered by open affines $U_{ij} = \mathrm{Spec}\,(A_{ij})$ where each A_{ij} is a finite type B_i-algebra. Pick any affine open $V = \mathrm{Spec}\,(B)$ in Y and a point $x \in f^{-1}(V)$. We wish to find an open affine $x \in U = \mathrm{Spec}\,(A)$ inside $f^{-1}(V)$ such that A is a finite type B-algebra.

Consider $f(x) \in V$ and let $f(x) \in V \cap V_i$. Consequently, $x \in f^{-1}(V)$ will be contained in some U_{ij}, so $x \in f^{-1}(V) \cap U_{ij}$. By continuity of f, there exists a basic open $D(g) \subseteq V \cap V_i$ for some $g \in B_i$ which contains $f(x)$ such that $f^{-1}(D(g)) \subseteq f^{-1}(V) \cap U_{ij}$ is open. Restricting f to U_{ij}, we have $f : U_{ij} \to V_i$ which induces a map $\varphi : B_i \to A_{ij}$ which is of finite type. Denote $U = f^{-1}(D(g)) = D(\varphi(g)) = \mathrm{Spec}\left((A_{ij})_{\varphi(g)}\right) \subseteq f^{-1}(V) \cap U_{ij}$. We therefore get

---

[33]finite type algebra := finitely generated as an algebra.
[34]Exercise II.3.1 of Hartshorne.

that the restriction of f on U, which is given by $f : U \to D(g)$, induces the localization map on algebras $\varphi_g : (B_i)_g \to (A_{ij})_{\varphi(g)}$. As localization of algebras are finite type, therefore φ_g makes $(A_{ij})_{\varphi(g)}$ a finite type $(B_i)_g$-algebra.

By Lemma 1.4.4.3, we have an isomorphism $B_h \to (B_i)_g$. Thus, we have

$$B \to B_h \overset{\cong}{\to} (B_i)_g \to (A_{ij})_{\varphi(g)}$$

where each map is of finite type. Since composite of finite type maps is of finite type, therefore $(A_{ij})_{\varphi(g)}$ is a finite type B-algebra, as required.
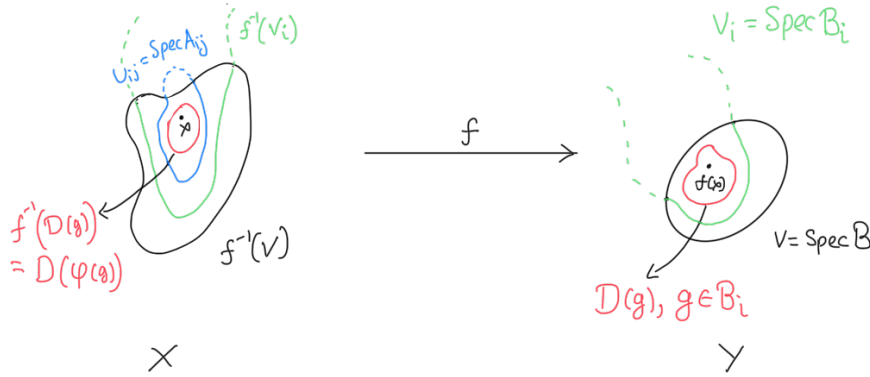
Figure 1.2: Sketch of the proof of Proposition 1.12.2.3

$\square$

We next see that finite type maps are also local on target and a nice property that they satisfy which says that finite type property descends to every open affine inside the inverse image of an open affine.

Theorem 1.12.2.4. [35] *Let $f : X \to Y$ be a map of schemes. Then,*
1. *f is of finite type if and only if f is locally of finite type and quasi-compact,*
2. *f is of finite type if and only if for every open affine $V = \operatorname{Spec}(B)$, the space $f^{-1}(V)$ can be covered by finitely many open affines $U_i = \operatorname{Spec}(A_i)$ where each A_i is a finite type B-algebra,*
3. *if f is of finite type, then for any open affine $V = \operatorname{Spec}(B) \subseteq Y$ and any open affine $U = \operatorname{Spec}(A) \subseteq f^{-1}(V)$, A is a finite type B-algebra.*

Proof. 1. $(\mathrm{L} \Rightarrow \mathrm{R})$ As f is of finite type, therefore there exists an open affine cover $\{V_i = \operatorname{Spec}(B_i)\}$ of Y such that $f^{-1}(V)$ can be covered by finitely many $U_{ij} = \operatorname{Spec}(A_{ij})$ where A_{ij} is a finite type B_i-algebra. Consequently, f is locally finite type. As each affine scheme is quasi-compact (Lemma 1.2.1.6) and finite union of quasi-compact spaces is quasi-compact, therefore we deduce that $f^{-1}(V)$ is quasi-compact.

$(\mathrm{R} \Rightarrow \mathrm{L})$ As f is locally of finite type, therefore there exists an open affine cover $\{V_i = \operatorname{Spec}(B_i)\}$ of Y such that $f^{-1}(V_i)$ is covered by open affines $U_{ij} = \operatorname{Spec}(A_{ij})$ where each A_{ij} is a finite type B_i-algebra. As f is quasicompact, therefore we have a finite subcover $U_{i1}, \ldots, U_{in}$ covering $f^{-1}(V)$, as required.

---

[35]Exercise II.3.3 of Hartshorne.
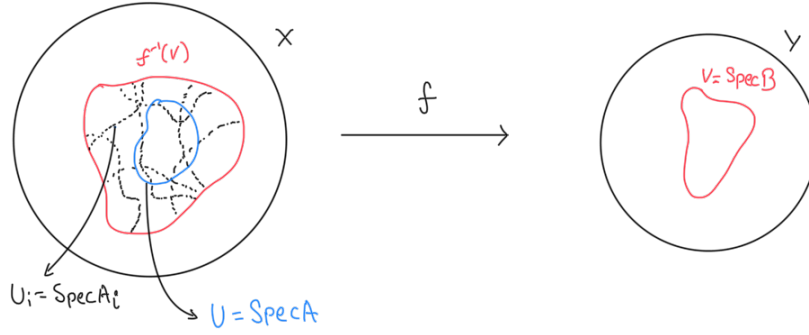
Figure 1.3: Sketch of the proof of Theorem 1.12.2.4, 3.

2. (R $\Rightarrow$ L) Immediate from definition.

(L $\Rightarrow$ R) Pick an open affine $V = \text{Spec}\,(B)$ in Y. We wish to show that $f^{-1}(V)$ is covered by finitely many open affines each of which is spectrum of a finite type B-algebra. Indeed, as f is quasi-compact by statement 1 above, therefore by Proposition 1.12.1.3, we see that $f^{-1}(V)$ is quasi-compact. Also by statement 1, f is of locally finite type. Hence by Proposition 1.12.2.3, $f^{-1}(V)$ is covered by spectra of finite type B-algebras. As $f^{-1}(V)$ is quasi-compact, we get a finite subcover, as required.

3. Pick any open affine $V = \text{Spec}\,(B)$ in Y and an open affine $U = \text{Spec}\,(A) \subseteq f^{-1}(V)$. As f is of finite type, therefore by statement 2 above, we obtain a finite collection $U_i = \text{Spec}\,(A_i)$ of open affines covering $f^{-1}(V)$. Observe that $U \cap U_i$ is an open set of U_i. By virtue of Lemma 1.4.4.3, we may cover $U \cap U_i$ by basic open sets of U_i which are basic open in U as well. Doing this for each i furnishes us with an open cover of U. As U is quasi-compact as it is affine (Lemma 1.2.1.6), consequently we get a finitely many elements $h_1, \ldots, h_n \in A$ such that $D(h_i) \subseteq U$ covers U and furthermore for each $i = 1, \ldots, n$, $D(h_i) \cong D(g_i)$ where $D(g_i) \subseteq U_i$ and $g_i \in A_i$. In particular, we have $A_{h_i} \cong (A_i)_{g_i}$. Now for each $i = 1, \ldots, n$, we have

$$B \to A_i \to (A_i)_{g_i} \cong A_{h_i}$$

where each of the arrows makes the codomain a finite type algebra over the domain. Hence, A_{h_i} is a finite type B-algebra. Consequently, we have $h_1, \ldots, h_n \in A$ such that $\cup_{i=1}^{n} D(h_i) = U$ (which is equivalent to saying that h_is generate the unit ideal of A by Lemma 1.2.1.5, 2) and A_{h_i} is a finite type B-algebra. It follows from Lemma 18.1.2.11 that A is a finite type B-algebra. This completes the proof. $\qquad\square$

We now list out some properties of finite type maps as we shall encounter them quite frequently.

Proposition 1.12.2.5. [36] *Properties of finite type maps.*

1. A closed immersion $X \to Y$ is of finite type.

---

[36]Exercise II.3.13 of Hartshorne.

2. *A quasicompact open immersion $X \to Y$ is of finite type.*
3. *Composition of finite type maps $X \to Y \to Z$ is of finite type.*
4. *Product of finite type schemes $X \to S$ and $Y \to S$ in $\mathbf{Sch}/S$ denoted $X \times_S Y \to S$ is of finite type.*
5. *Maps of finite type are stable under base extensions.*
6. *If $X \to Y$ is quasicompact and the composite $X \to Y \to Z$ is of finite type, then $X \to Y$ is of finite type.*
7. *If $X \to Y$ is of finite type and Y is noetherian, then X is noetherian.*

Proof. **TODO!**   □

The following is something we all expect, which indeed holds true for finite type schemes.

Lemma 1.12.2.6. *Let k be a field and X be a finite type k-scheme. The set of all closed points of X is dense in X.*

Proof. **TODO**.   □

Example 1.12.2.7. We give a number of examples of finite type maps.
1. Let k be a field. Consider the projection map $\pi : \mathbb{A}_k^2 \to \mathbb{A}_k^1$ defined by the k-algebra map $k[x] \to k[x,y]$ mapping as $x \mapsto x$. Note that π is a finite type map of schemes as the open covering of $\mathbb{A}_k^1$ as itself yields that $\pi^{-1}(\mathbb{A}_k^1) = \mathbb{A}_k^2$ and $\mathbb{A}_k^2$ is spectra of $k[x,y]$ which is a finite type $k[x]$ algebra via the above map. Indeed, $k[x,y]$ is generated by $\{y\}$ as a $k[x]$-algebra.
   We deduce that projection maps $\mathbb{A}_k^n \to \mathbb{A}_k^1$ are finite type maps for any $n \in \mathbb{N}$.
2. We next consider a family of curves parameterized by a parameter t. Consider the map
$$\mathbb{C}[t] \longrightarrow \mathbb{C}[t][x,y]\langle y^2 - x^3 - t\rangle.$$

This yields the following map at the level of schemes
$$X := \mathrm{Spec}\left(\frac{\mathbb{C}[t][x,y]}{\langle y^2 - x^3 - t\rangle}\right) \to \mathrm{Spec}\left(\mathbb{C}[t]\right).$$

Pick the closed point corresponding to $a \in \mathbb{C}$ in $\mathrm{Spec}\left(\mathbb{C}[t]\right)$. As $\mathbb{C}[t][x,y]/\langle y^2 - x^3 - t\rangle$ is a finite type $\mathbb{C}[t]$-algebra, therefore the above map of schemes is of finite type. Observe that the fiber of X at $a \in \mathrm{Spec}\left(\mathbb{C}[t]\right)$ (by abuse of notation) is given by
$$X_a = X \times_{\mathrm{Spec}(\mathbb{C}[t])} \mathrm{Spec}\left(\kappa(a)\right).$$

As $\kappa(a)$ is the fraction field of $\mathbb{C}[t]/\langle t - a\rangle$, which is $\mathbb{C}[a] = \mathbb{C}$, therefore we get the following
$$\begin{aligned}
X_a &= \mathrm{Spec}\left(\frac{\mathbb{C}[t][x,y]}{\langle y^2 - x^3 - t\rangle} \otimes_{\mathbb{C}[t]} \mathbb{C}[a]\right) \\
&\cong \mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{\langle y^2 - x^3 - a\rangle}\right).
\end{aligned}$$

Hence, we get the curve $y^2 - x^3 - a$ back as the fiber at the point $a \in \mathrm{Spec}\left(\mathbb{C}[t]\right)$.

3. Consider the map

$$k[t] \longrightarrow \frac{k[t][w, x, y, z]}{\langle (w - y)^2 + (x - z)^2 - t^2 \rangle}$$

which yields the map on geometric level as

$$X := \mathrm{Spec}\left(\frac{k[t][w, x, y, z]}{\langle (w - y)^2 + (x - z)^2 - t^2 \rangle} \right) \longrightarrow \mathrm{Spec}\left(k[t] \right).$$

Again, this is a finite type map and for a closed point $a \in k$ corresponding to the ideal $\langle t - a \rangle \lneq k[t]$, the fibre is

$$\begin{aligned}
X_a &\cong \mathrm{Spec}\left(\frac{k[t][w, x, y, z]}{\langle (w - y)^2 + (x - z)^2 - t^2 \rangle} \otimes_{k[t]} k[a] \right) \\
&\cong \mathrm{Spec}\left(\frac{k[w, x, y, z]}{\langle (w - y)^2 + (x - z)^2 - a^2 \rangle} \right)
\end{aligned}$$

on $\mathbb{A}^2_{\mathbb{R}}$).

4. Any projective variety $X \to \mathbb{P}^n_k$ will by definition be finite type over k (Theorem 1.12.6.2, 1). For example, the projective parabola $X = \mathrm{Proj}\left(\frac{k[x, y, z]}{y^2 - xz} \right)$ is a finite type scheme over k. Indeed, by Proposition 1.8.2.8, 1, we get that the natural map $X \to \mathrm{Proj}(k[x, y, z]) = \mathbb{P}^3_k$ coming from the quotient $k[x, y, z] \twoheadrightarrow \frac{k[x, y, z]}{y^2 - xz}$ (which is a graded map) is a closed immersion. Hence, it defines a closed subscheme of projective 3-space $\mathbb{P}^3_k$ over k.

5. **TODO**: *Add more examples from maps of projective schemes. This is important.*

1.12.3   Finite

This is a more stronger version of finite type maps discussed in previous section.

Definition 1.12.3.1. (Finite) Let $f : X \to Y$ be a map of schemes. Then f is said to be finite if there is an open affine covering $V_i = \mathrm{Spec}\left(B_i \right)$, $i \in I$ of Y such that $f^{-1}(V_i)$ is equal to an open affine $\mathrm{Spec}\left(A_i \right)$ where A_i is a finite B_i-algebra[37].

We see that finite maps are local on target.

Proposition 1.12.3.2. *A map $f : X \to Y$ of schemes is finite if and only if for each open affine $V = \mathrm{Spec}\left(B \right)$, we have $f^{-1}(V)$ is an open affine $\mathrm{Spec}\left(A \right)$ in X such that $B \to A$ makes A a finite B-algebra.*

Proof. (R $\Rightarrow$ L) is immediate from definitions.

(L $\Rightarrow$ R) Pick any open affine $V = \mathrm{Spec}\left(B \right)$ in Y. We first wish to show that $U = f^{-1}(V)$ is an affine scheme. We may employ criterion for affineness, Proposition 1.3.1.7, for this purpose. Hence for showing that U is affine, we reduce to finding $g_1, \ldots, g_n \in \Gamma(\mathcal{O}_{X|U}, U) = \mathcal{O}_X(U)$ such that U_{g_i} is affine and $\langle g_1, \ldots, g_n \rangle = \mathcal{O}_X(U)$.

As f is finite, therefore there exists an open affine covering $V_i = \mathrm{Spec}\left(B_i \right)$ of Y such that

---

[37]finite algebra := finitely generated as a module.

$f^{-1}(V_i) = \mathrm{Spec}\,(A_i) = U_i$ is affine and A_i is a finite B_i-algebra. Observe that $V \cap V_i$ forms an open covering of V. As V is affine, so it is quasi-compact (Lemma 1.2.1.6). Consequeently, we obtain a finite cover of V by V_is. Now cover each $V \cap V_i$ by basic opens which are basic in both V and V_i (Lemma 1.4.4.3). Doing this for each of the finitely many i, we obtain a cover of V by basic open sets. As V is quasi-compact (Lemma 1.2.1.6), therefore we have obtained a cover of V by finitely many basics $D(k_i)$ for $k_i \in B$ such that $D(k_i) \cong D(l_i)$ where $D(l_i) \subseteq V_i$ and $l_i \in B_i$ for $i = 1, \ldots, n$. Consequently by Lemma 1.2.1.5, the ideal generated by $k_1, \ldots, k_n$ in B is the unit ideal.

As we have

$$U = f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^{n} D(k_i)\right) = \bigcup_{i=1}^{n} f^{-1}(D(k_i)),$$

therefore by Lemma 1.3.1.4, we may write

$$U = \bigcup_{i=1}^{n} U_{\varphi(k_i)}$$

where $\varphi : B \to \mathcal{O}_X(U)$ is the map induced by the restricted map $f : U \to V$ on the global sections. Furthermore, as $\sum_{i=1}^{n} k_i B = B$, therefore $\sum_{i=1}^{n} \varphi(k_i)\mathcal{O}_X(U) = \mathcal{O}_X(U)$. Hence, it now suffices to show that each $U_{\varphi(k_i)}$ is affine.

We have $U_{\varphi(k_i)} = f^{-1}(D(k_i)) \cong f^{-1}(D(l_i)) = D(\varphi_i(l_i))$ where $\varphi_i : B_i \to A_i$ is the map on global sections obtained by the restriction $f : U_i \to V_i$. As $D(\varphi_i(l_i))$ is affine, thus, so is $U_{\varphi(k_i)}$. This shows that indeed, $f^{-1}(V)$ is an open affine.

We may now write $U = \mathrm{Spec}\,(A)$. We reduce now to showing that A is a finite B-algebra. For this observe that in the above, we obtained a finite open cover of U given by $U_{\varphi(k_i)} \cong D(\varphi_i(l_i))$ where $D(\varphi_i(l_i)) \subseteq U_i$. As $U = \mathrm{Spec}\,(A)$, therefore $\mathcal{O}_X(U) = A$, so we may let $\varphi(k_i) = g_i$ for $i = 1, \ldots, n$. Now, since $U_{\varphi(k_i)} = U_{g_i} = D(g_i) \cong D(\varphi_i(l_i))$, therefore we have $A_{g_i} \cong (A_i)_{\varphi_i(l_i)}$. As A_i is a finite B_i-algebra, therefore by Lemma 18.1.2.10, $(A_i)_{\varphi_i(l_i)}$ is a finite $(B_i)_{l_i}$-algebra. Further, as we saw in the beginning that $D(k_i) \cong D(l_i)$, hence we get $B_{k_i} \cong B_{l_i}$. We thus obtain a map $B_{k_i} \to A_{g_i}$ as in

$$
\begin{array}{ccc}
(A_i)_{\varphi_i(l_i)} & \xleftarrow{\;(\varphi_i)_{l_i}\;} & (B_i)_{l_i} \\
\| \wr & & \| \wr \\
A_{g_i} & \dashleftarrow & B_{k_i}
\end{array}
$$

which thus makes A_{g_i} a finite B_{k_i}-algebra, in particular, a finitely generated B_{k_i}-module. This is for each of the $i = 1, \ldots, n$, and since we have that $k_1, \ldots, k_n$ generates the unit ideal in B, hence by another application of Lemma 18.1.2.10, we deduce that A is a finite B-algebra, as required. $\square$

One important property of finite maps is that their fibres are finite.

Proposition 1.12.3.3. [38] *Let $f : X \to Y$ be a finite morphism. Then f is quasi-finite.*

---

[38]Exercise II.3.5, a).

Proof. Pick any $y \in Y$. We wish to show that $X_y = X \times_Y \operatorname{Spec}(\kappa(y))$ is a finite set. Let $V = \operatorname{Spec}(B)$ be an open affine containing y. Then since $f^{-1}(y) \subseteq f^{-1}(V)$, therefore we have $X_y = f^{-1}(V) \times_Y \operatorname{Spec}(\kappa(y))$. As f is finite, therefore $f^{-1}(V) = \operatorname{Spec}(A)$ for some finite B-algebra A. Hence we may replace X by $\operatorname{Spec}(A)$, Y by $\operatorname{Spec}(B)$ and point $y \in Y$ by a prime ideal $\mathfrak{q} \in \operatorname{Spec}(B)$. Consequently, we obtain

$$X_y = \operatorname{Spec}(A) \times_{\operatorname{Spec}(B)} \operatorname{Spec}(\kappa(\mathfrak{q}))$$
$$\cong \operatorname{Spec}\left(A \otimes_B \frac{B_\mathfrak{q}}{\mathfrak{q}B_\mathfrak{q}} \right).$$

As A is a finite B-algebra, therefore $A \cong B^n/I$ where $n \in \mathbb{N}$ and $I \leq B^n$ is an ideal. Further, any ideal of product of rings is product of ideals, therefore $A \cong B^n/I_1 \times \cdots \times I_n \cong B/I_1 \times \cdots \times B/I_n$ where $I_i \leq B$ is an ideal.

As tensor product is a filtered colimit and they commute over finite limits, therefore

$$A \otimes_B \frac{B_\mathfrak{q}}{\mathfrak{q}B_\mathfrak{q}} \cong \prod_{i=1}^n \frac{B}{I_i} \otimes_B \frac{B_\mathfrak{q}}{\mathfrak{q}B_\mathfrak{q}}$$
$$\cong \prod_{i=1}^n \frac{B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}}{I_i \cdot (B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q})}.$$

As $B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$ is a field, therefore $I_i \cdot (B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q})$ is either 0 or $B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} = \kappa(\mathfrak{q})$. Hence,

$$\operatorname{Spec}\left(A \otimes_B \frac{B_\mathfrak{q}}{\mathfrak{q}B_\mathfrak{q}} \right) \cong \coprod_{i=1}^n \operatorname{Spec}(R)$$

where R is either 0 or $\kappa(\mathfrak{q})$. Hence $\operatorname{Spec}(R)$ is either $\emptyset$ or $\{\text{pt.}\}$. Hence the above is atmost a finite set. $\square$

Another nice property enjoyed by finite maps is that they are closed.

Proposition 1.12.3.4. [39] *Let $f : X \to Y$ be a finite morphism. Then f is a closed map.*

Proof. Let $C \subseteq X$ be a closed set. We wish to show that $f(C)$ is a closed set. Recall that a set is closed if and only if its intersection with any open set is closed in that open set. Consequently, we reduce to showing that the intersection $f(C) \cap V$ where $V \subseteq Y$ is an open affine is closed in V. As $f^{-1}(f(C) \cap V) = C \cap f^{-1}(V)$ and f is finite, therefore we may write $f^{-1}(V) = U = \operatorname{Spec}(A)$ where A is a finite B-algebra. Consequently, we reduce to assuming $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ and $C \subseteq X$ given by $C = V(J)$ is a closed set where $J \leq A$ is an ideal. Denote by $\varphi : B \to A$ the corresponding map of rings.

As A is a finite B-algebra, therefore we may write $A \cong B/I_1 \times \cdots \times B/I_n$ where $I_i \leq B$ are ideals. Hence the ideal $J \leq A$ corresponds to $J = \bar{J}_1 \times \cdots \times \bar{J}_n \leq B/I_1 \times \cdots \times B/I_n$ where $J_i \leq B$ is an ideal containing I_i. Now, $C = V(J)$ is the set of prime ideals of A containing J. As $\operatorname{Spec}(A) \cong \coprod_{i=1}^n \operatorname{Spec}(B/I_i)$, hence C in $\operatorname{Spec}(A)$ is given by $\coprod_{i=1}^n V(\bar{J}_i)$. As the

---

[39] Exercise II.3.5, b).

$\varphi : B \to \prod_{i=1}^{n} B/I_i$ takes $b \mapsto (b + I_i)_{i=1,\dots,n}$ is the unique map obtained by projections $B \twoheadrightarrow B/I_i$, therefore

$$f(C) = f\left(\coprod_{i=1}^{n} V(\bar{J}_i)\right)$$
$$= \bigcup_{i=1}^{n} f(V(\bar{J}_i))$$
$$= \bigcup_{i=1}^{n} V(J_i)$$

where $V(J_i)$ is closed in Y. As finite union of closed sets is closed, hence we are through. $\quad\square$

Remark 1.12.3.5. [40] As tempting it might be to say that, but it is not true that a surjective, finite type, quasi-finite map is finite.

Indeed, let k be an algebraically closed field. Consider the map

$$f : \mathrm{Spec}\left(\frac{k[x,y]}{xy-1}\right) \amalg \mathrm{Spec}\,(k) \longrightarrow \mathbb{A}_k^1$$

induced by $k[x] \to \frac{k[x,y]}{xy-1} \times k$ given by $x \mapsto (x + \langle xy - 1\rangle, 0)$. As a nice exercise, one checks that (we write $a \in \mathbb{A}_k^1$ to mean $\langle x - a\rangle \in \mathbb{A}_k^1$)

1. $f^{-1}(0)$ is a singleton $(\mathrm{Spec}\,(k))$,
2. $f^{-1}(a)$ is a singleton, given by point (a, a^{-1}) (in particular, the point $\langle x - a, xy - 1\rangle$),
3. the generic fiber $f^{-1}(\mathfrak{o})$ is isomorphic to $\mathrm{Spec}\,(k(x))$ in $\mathrm{Spec}\left(\frac{k[x,y]}{xy-1}\right)$, hence a singleton.

Consequently, f is surjective, quasi-finite and furthermore of finite type. But still, $\frac{k[x,y]}{xy-1} \times k$ is not a finite $k[x]$-algebra.

Remark 1.12.3.6. Let k be a field. Observe that $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \cong \mathbb{A}_k^2$. However, the underlying set in $\mathbb{A}_k^2$ is not the product of underlying set of $\mathbb{A}_k^1$ with itself. Indeed, this is essentially due to the fact that every prime ideal of $k[x,y]$ is not of form $\mathfrak{p}_1 \times \mathfrak{p}_2$ where $\mathfrak{p}_1 \in k[x]$ and $\mathfrak{p}_2 \in k[y]$, as the prime ideal $xy - 1$ in $k[x,y]$ shows.

Example 1.12.3.7. We give some examples of finite maps.

1. Consider the canonical map $k[t] \to \frac{k[t,x]}{x^n-t}$ and the corresponding map $X = \mathrm{Spec}\left(\frac{k[t,x]}{x^n-t}\right) \to \mathrm{Spec}\,(k[t]) = \mathbb{A}_k^1$. As $\frac{k[t,x]}{x^n-t}$ is a finite $k[t]$-algebra of rank n, therefore $X \to \mathbb{A}_k^1$ is a finite map. Note that for each closed point $a \in \mathbb{A}_k^1$, the fiber $X_a \cong \mathrm{Spec}\left(\frac{k[x]}{x^n-a}\right)$, which has n closed points if k is algebraically closed and $a \neq 0$.
2. **TODO :** *Add more as you create or find.*

---

[40]Exercise II.3.5, c) of Hartshorne.

1.12.4 Separated

This notion corresponds to the Hausdorff property for topological spaces. Recall that a space X is Hausdorff if and only if the diagonal $\Delta : X \to X \times X$ is closed. We shall mimic this in the category of schemes.

Definition 1.12.4.1. (Separated) A map $f : X \to Y$ of schemes is said to be separated if the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. A scheme X is said to be separated if $X \to \mathrm{Spec}\,(\mathbb{Z})$ is separated.

It follows that any map of affine schemes is separated.

Lemma 1.12.4.2. *Let $f : \mathrm{Spec}\,(A) \to \mathrm{Spec}\,(B)$ be a map of affine schemes. Then f is separated.*

Proof. By Corollary 1.3.0.6, f corresponds to a map of rings $\varphi : B \to A$. Similarly, the diagonal map $\Delta : \mathrm{Spec}\,(A) \to \mathrm{Spec}\,(A) \times_{\mathrm{Spec}(B)} \mathrm{Spec}\,(A)$ corresponds to the B-algebra structure map over A, given by $m : A \otimes_B A \to A$, which is surjective. Consequently, by Corollary 1.4.4.10, Δ is a closed immersion. $\square$

Since any scheme locally is affine, we get a nice consequence of the above lemma.

Lemma 1.12.4.3. *Let $f : X \to Y$ be a map of schemes. Then the following are equivalent.*
  1. *f is separated.*
  2. *The diagonal $\Delta : X \to X \times_Y X$ has closed image.*

Proof. (1. $\Rightarrow$ 2.) Immediate.
(2. $\Rightarrow$ 1.) By the definition of diagonal, it is immediate that $\Delta : X \to X \times_Y X$ is a homeomorphism onto its image, which is further closed by the given hypothesis. Thus, we need only show that $\Delta^\flat : \mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is a surjective map. By Theorem 22.2.0.6, 3, this is a local property. Consequently, we further reduce to showing that for any point $x \in X$ there is an open set $f(x) \in V \subseteq X \times_Y X$ such that $\Delta^\flat_{|V} : \mathcal{O}_{V,f(x)} \to (\Delta_* \mathcal{O}_{\Delta^{-1}(V)})_{f(x)}$ is surjective. Now we may choose by continuity of f a small affine open $x \in U$ such that $f(U)$ is contained in an affine open V in Y. Consequently, $U \times_V U$ is an affine open subset of $X \times_Y X$ containing $f(x)$. We thus reduce to showing that $\mathcal{O}_{U \times_V U, f(x)} \to (\Delta_* \mathcal{O}_U)_x$ is surjective, which follows immediately from Lemma 1.12.4.2. $\square$

Next, we state an important characterization of separatedness which allows us to derive some very important and convenient results about it.

Theorem 1.12.4.4. *(Valuative criterion of separatedness) Let $f : X \to Y$ be a map of schemes where X is noetherian. Then the following are equivalent,*
  1. *f is separated.*
  2. *Pick any field K and any valuation ring R with fraction field K (see Section 18.9). Let $i : \mathrm{Spec}\,(K) \to \mathrm{Spec}\,(R)$ be the map corresponding to $R \hookrightarrow K$. For all $g : \mathrm{Spec}\,(R) \to Y$ and $h : \mathrm{Spec}\,(K) \to X$ such that the square commutes, there exists atmost one lift of g along f as to make the following diagram commute:*

$$
\begin{array}{ccc}
\mathrm{Spec}\,(K) & \xrightarrow{\ h\ } & X \\
{\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\
\mathrm{Spec}\,(R) & \xrightarrow{\ g\ } & Y
\end{array}
$$

Proof. See Theorem 4.3, Chapter 2 of cite[Hartshorne].   □

The following important corollaries can now easily be derived from this characterization.

Corollary 1.12.4.5. *Let us work in the category of noetherian schemes. Then,*
   1. *separated maps are stable under base extension,*
   2. *open and closed immersions are separated[41],*
   3. *composition of separated maps is separated,*
   4. *for a base scheme S, product of any two separated maps is separated in* **Sch**$/S$,
   5. *if the composite $X \to Y \to Z$ is separated, then $X \to Y$ is separated,*
   6. *a map $f : X \to Y$ is separated if and only if there is an open cover V_i of Y such that the restricted maps $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is separated.*

Proof. **TODO: From notebook.**   □

Separatedness of projective schemes

We next see that any projective scheme is separated.

Lemma 1.12.4.6. *Let S be a graded ring. Then,* $\mathrm{Proj}(S) \to \mathrm{Spec}\,(\mathbb{Z})$ *is separated.*

Proof. We need only check that the diagonal $\Delta : \mathrm{Proj}(S) \to \mathrm{Proj}(S) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Proj}(S)$ has closed image (Lemma 1.12.4.3). Since one can check a closed set locally and sets of the form $D_+(f) \times D_+(g)$ forms an open cover of $\mathrm{Proj}(S) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Proj}(S)$ for $f, g \in S_+$ homogeneous, therefore we reduce to checking that for $C = \Delta^{-1}(D_+(f) \times D_+(g))$, the restriction $\Delta|_C : C \to D_+(f) \times D_+(g)$ has closed image.

Since $C = D_+(fg) \cong \mathrm{Spec}\left(S_{(fg)}\right)$ and $D_+(f) \times D_+(g) \cong \mathrm{Spec}\left(S_{(f)} \otimes_{\mathbb{Z}} S_{(g)}\right)$, therefore we reduce to showing that the induced map $S_{(f)} \otimes_{\mathbb{Z}} S_{(g)} \to S_{(fg)}$ is surjective. This is clear, as for any $u/f^n g^n \in S_{(fg)}$ where let us denote $k = \deg f, l = \deg g$, for any m large enough such that all exponents in the below are positive, we obtain that

$$\frac{ug^{mk-n}}{f^{ml+n}} \otimes \frac{f^{ml}}{g^{mk}} \mapsto \frac{u}{f^n g^n}.$$

Thus, the image of Δ is closed[42].   □

Uniqueness of centers of valuations for varieties

We show a curious property for abstract varieties that any valuation defined over its function field has a unique *center*, if it exists[43]. See Definition 1.4.2.8 for definition of center points of a valuation over function field of an integral scheme.

Lemma 1.12.4.7. [44] *Let X be an integral scheme of finite type over k with function field K. If X is separated, then any valuation over K has a unique center if it exists.*

---

[41]in-fact, any topological immersion is separated, as is clear from the proof.

[42]in-fact we have also shown in the process that Δ is a closed immersion, thus we may not use Lemma 1.12.4.3

[43]It will always exist (and thus be unique) if the variety is proper, as is shown in the next section.

[44]Exericse II.4.5 a) of Hartshorne.

Proof. We will use the valuative criterion for this. Let $v : K \to G$ be a valuation over K with valuation ring $R \subseteq K$. Let $x, y \in X$ be two centers of v. As $K \subseteq K$, therefore by Lemma 1.6.1.1, 3, there exists a unique map $\text{Spec}\,(K) \to X$ mapping $\star \mapsto \eta$, where η is the generic point of X. It follows that we have the following commutative square

$$\begin{array}{ccc} \text{Spec}\,(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec}\,(R) & \longrightarrow & \text{Spec}\,(\mathbb{Z}) \end{array} \qquad (*)$$

As R is a local ring, therefore by [Lemma 01J6](#) of StacksProject, we have a bijection between maps $\text{Spec}\,(R) \to X$ and tuples (z, φ) where $z \in X$ and $\varphi : \mathcal{O}_{X,z} \to R$ is a local ring homomorphism. Consequently, as $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ are dominated by R, we obtain two tuples (x, ι_x) and (y, ι_y) where $\iota_x : \mathcal{O}_{X,x} \hookrightarrow R$ and $\iota_y : \mathcal{O}_{X,y} \hookrightarrow R$ are the two domination maps. Note that by the definition of domination, these two maps are local ring homomorphisms. Consequently, we get two maps $\text{Spec}\,(R) \to X$ which makes the $(*)$ commute. By the valuative crieterion of Theorem 1.12.4.4, the two maps $\text{Spec}\,(R) \to X$ are same, and thus so are the tuples (x, ι_x) and (y, ι_y), proving that $x = y$. $\qquad \square$

1.12.5 Proper

This and the next section brings us closer to detecting when a scheme is projective (i.e. is a subscheme of projective scheme). Proper maps corresponds roughly to the intuition that the scheme $X \to Y$ should not have any *missing points*.

Definition 1.12.5.1. (Universally closed and proper maps) A map $f : X \to Y$ is said to be universally closed if f is closed and for any base extension $Y' \to Y$, the base extension of X, denoted $f' : X' \to Y'$ is also closed as in the diagram below:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & \lrcorner & \downarrow f \\ Y' & \longrightarrow & Y \end{array} \cdot$$

Consequently, f is said to be proper if it is separated, finite type and is universally closed.

The main result here is again a valuative criterion which allows a lot of properties of such maps to be derived quite easily.

Theorem 1.12.5.2. *(Valuative criterion of properness) Let $f : X \to Y$ be a finite type map of schemes where X is noetherian. Then the following are equivalent.*
1. *f is proper.*
2. *Pick any field K and any valuation ring R with fraction field K (see Section 18.9). Let $i : \text{Spec}\,(K) \to \text{Spec}\,(R)$ be the map corresponding to $R \hookrightarrow K$. For all $g : \text{Spec}\,(R) \to Y$ and $h : \text{Spec}\,(K) \to X$ such that the square commutes, there exists a unique lift of*

g along f as to make the following diagram commute:

$$\begin{array}{ccc} \mathrm{Spec}\,(K) & \xrightarrow{\ h\ } & X \\ {\scriptstyle i}\downarrow & {\scriptstyle !}\ \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\,(R) & \xrightarrow{\ g\ } & Y \end{array}\ .$$

Note that whereas in Theorem 1.12.4.4 we had that there exists *atmost one* lift, here we have that there exists *unique* lift (it exists and there is only one).

Corollary 1.12.5.3. *Let us work in the category of noetherian schemes. Then,*

1. *if $X \to Y \to Z$ is proper and $Y \to Z$ is separated, then $X \to Y$ is proper,*
2. *closed immersion are proper,*
3. *proper maps are stable under base extensions,*
4. *composite of proper maps is proper,*
5. *for two proper schemes $X \to S, Y \to S$ in $\mathbf{Sch}/S$, their product $X \times_S Y \to S$ is proper,*
6. *a map $f : X \to Y$ is proper if and only if there exists an open cover V_i of Y such that the restriction $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is proper.*

Proof. **TODO : From notebook.** □

1.12.6   Projective

We now define maps of schemes which factors through a projective space over the target. This will be fundamental, as the most natural type of schemes we find in nature are projective varieties appearing as closed subschemes of the projective space over a field. Though we will work more generally, but this will pay off in some of the later discussions. See Definition 1.8.2.14 for projective spaces over a scheme.

Definition 1.12.6.1. (Projective and quasi-projective maps) Let $f : X \to Y$ be a map of schemes. We say f is projective if there exists an $n \in \mathbb{N}$ such that f factors as a closed immersion $X \to \mathbb{P}_Y^n$ followed by the struture map $\mathbb{P}_Y^n \to Y$ as in

$$\begin{array}{ccc} X & \xrightarrow{\text{cl. imm.}} & \mathbb{P}_Y^n \\ & {\scriptstyle f}\searrow & \downarrow \\ & & Y \end{array}\ .$$

Further, a map $f : X \to Y$ is said to be *quasi-projective* if f factors first into an open immersion $X \to X'$ and then a projective map $X' \to Y$ as in

$$\begin{array}{ccc} & X' & \xrightarrow{\text{cl. imm.}} & \mathbb{P}_Y^n \\ {\scriptstyle \text{op. imm.}}\uparrow & & & \downarrow \\ X & \xrightarrow{\ f\ } & Y & \end{array}\ .$$

Thus quasi-projective maps corresponds to the usual notion of quasi-projective varieties (open subsets of projective varieties in a projective n-space).

The important point to keep in mind about projective maps is that they are proper.

Theorem 1.12.6.2. *Let X and Y be noetherian schemes.*
  1. If $f : X \to Y$ is projective, then f is proper.
  2. If $f : X \to Y$ is quasi-projective, then f is finite type and separated.

Proof. 1. Since closed immersions are proper and proper maps are stable under base change (Corollary 1.12.5.3), we may reduce to showing that for each $n \in \mathbb{N}$, the scheme $\mathbb{P}_{\mathbb{Z}}^n \to \mathrm{Spec}\,(\mathbb{Z})$ is proper. It is clear that $\mathbb{P}_{\mathbb{Z}}^n$ is finite type $\mathbb{Z}$-scheme which is furthermore separated by Corollary 1.12.4.6.

In order to show that $\mathbb{P}_{\mathbb{Z}}^n$ is proper, we will proceed by induction over n. For $n = 0$, we have $\mathbb{P}_{\mathbb{Z}}^0 \cong \mathrm{Spec}\,(\mathbb{Z})$, which is trivially proper over $\mathrm{Spec}\,(\mathbb{Z})$. Now suppose $\mathbb{P}_{\mathbb{Z}}^{n-1}$ is proper over $\mathrm{Spec}\,(\mathbb{Z})$. We wish to show that $\mathbb{P}_{\mathbb{Z}}^n$ is proper. We will use valuative criterion for this (Theorem 1.12.5.2). Consider a valuation ring R with fraction field K such that we have maps g, h making the following commute:

$$\begin{array}{ccc} \mathrm{Spec}\,(K) & \xrightarrow{\;\;h\;\;} & \mathbb{P}_{\mathbb{Z}}^n \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,(R) & \xrightarrow{\;\;g\;\;} & \mathrm{Spec}\,(\mathbb{Z}) \end{array}.$$

Consequently, we wish to define a unique map $\mathrm{Spec}\,(R) \to \mathbb{P}_{\mathbb{Z}}^n$ which makes everything commute.

Denote $\mathrm{Spec}\,(K) = \{\star\}$ and $\xi = h(\star) \in \mathbb{P}_{\mathbb{Z}}^n$. We now observe that if $\xi \in V(x_{i_0})$ for any $i_0 = 0, \ldots, n$, then by the natural isomorphism $V(x_{i_0}) \cong \mathbb{P}_{\mathbb{Z}}^{n-1}$ and obvious restrictions, we get the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}\,(K) & \xrightarrow{\;\;h\;\;} & \mathbb{P}_{\mathbb{Z}}^{n-1} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,(R) & \xrightarrow{\;\;g\;\;} & \mathrm{Spec}\,(\mathbb{Z}) \end{array}.$$

Consequently, by inductive hypothesis, we have a unique lift $\mathrm{Spec}\,(R) \to \mathbb{P}_{\mathbb{Z}}^{n-1}$ and thus a map $\mathrm{Spec}\,(R) \to \mathbb{P}_{\mathbb{Z}}^n$ making the diagram commutative. This is sufficient by the fact that $\mathbb{P}_{\mathbb{Z}}^n$ is separated (Lemma 1.12.4.6) and by valuative criterion (Theorem 1.12.4.4).

We next need to cover the case when ξ is not in any hyperplane $V(x_i)$, that is, when $\xi \in \bigcap_{i=0}^n D_+(x_i)$. We will construct a map $\mathrm{Spec}\,(R) \to \mathbb{P}_{\mathbb{Z}}^n$ which makes everything commute and we will be done by separatedness of $\mathbb{P}_{\mathbb{Z}}^n$ (Lemma 1.12.4.6). In this case, we obtain that $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n, \xi} \cong \mathbb{Z}[x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i]$ for all $i = 0, \ldots, n$ as $D_+(x_i) \cong \mathrm{Spec}\,\big(\mathbb{Z}[x_0, \ldots, x_n]_{(x_i)}\big)$, (Lemma 1.8.2.4). Consequently, $x_i/x_j \in \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n, \xi}$ is invertible for all $i, j = 0, \ldots, n$, hence $x_i/x_j \notin \mathfrak{m}_{\mathbb{P}_{\mathbb{Z}}^n, \xi}$. Denote further $f_{ij} \in \kappa(\xi)$ to be the image of x_i/x_j under the map $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n, \xi} \to \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n, \xi}/\mathfrak{m}_{\mathbb{P}_{\mathbb{Z}}^n, \xi} = \kappa(\xi)$.

The map $h : \mathrm{Spec}\,(K) \to \mathbb{P}^n_{\mathbb{Z}}$ is equivalent to the data of the point $\xi \in \mathbb{P}^n_{\mathbb{Z}}$ and $\kappa(\xi) \hookrightarrow K$ (Lemma 1.6.1.1). Thus we have $f_{ij} \in K$ for all $i, j = 0, \ldots, n$. In order to define the map $j :$ $\mathrm{Spec}\,(R) \to \mathbb{P}^n_{\mathbb{Z}}$ in this case, it is sufficient to obtain a map $\mathbb{Z}[x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i] \to$ R such that the following commutes:

$$
\begin{array}{ccc}
K & \longleftarrow & Z[x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i] \\
\uparrow & & \uparrow \\
R & \longleftarrow & \mathbb{Z}
\end{array}
.
$$

We will now construct such a map. Let $v : K \to G$ be the valuation corresponding to the valuation ring R (so that R is the value ring of v), where G is a totally ordered abelian group. Consider the collection of elements $f_{10}, \ldots, f_{n0} \in K$ and denote $g_i = v(f_{i0}) \in G$. Let $g_m = \min_i g_i$. Consequently, for each $i = 0, \ldots, n$ we obtain $0 \leq g_i - g_m = v(f_{i0}) - v(f_{m0}) = v(f_{i0} f_{0m}) = v(f_{im})$. Thus, $f_{im} \in R$. Hence, we can construct the following map:

$$
Z[x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i] \longrightarrow R
$$
$$
\frac{x_i}{x_m} \longmapsto f_{im}.
$$

It is immediate that the above map makes the above diagram commute.

2. Since open immersions are separated (Corollary 1.12.4.5) and an open immersion $X \to X'$ where X is noetherian is immediately quasicompact, so by Proposition 1.12.2.5, 2, the result follows. $\qquad\square$

1.12.7   Flat

Look at the following MO post for more clarifications. Flat maps of schemes capture the notion of a "continuous family of schemes parameterized by points of a base scheme". However, the notion of flatness is very algebraic, as we shall soon see. We collect the properties of flat modules in the Special Topics, Chapter 18.

1.12.8   Smooth

1.12.9   Unramified

1.12.10   Étale

Étale maps is the place from where one enters the land of algebraic topology via algebraic geometry. Indeed, the fundamental goal here is to capture the notion of local isomorphism but in an algebraic context. The simplest place where one can understand them is a restricted version of this called finite étale maps. This is where we begin from as we shall need this in our discussion of Galois theory of schemes.

Finite étale

We refer to Algebra, Chapter 18 for background on separable algebras, in particular, to Definition 18.20.2.2 for free separable algebras.

We now define finite étale maps.

Definition 1.12.10.1. (**Finite étale scheme**) Let X be a base scheme. An X-scheme $p : Y \to X$ is said to be finite étale if there is an open affine covering of X given by $\{\mathrm{Spec}\,(A_i)\}_{i \in I}$ such that $p^{-1}(\mathrm{Spec}\,(A_i))$ is an open affine subscheme of Y given by $\mathrm{Spec}\,(B_i)$ such that the induced map $A_i \to B_i$ makes B_i a free separable A_i-algebra, for all $i \in I$. In such a situation, one calls Y a finite étale covering of X. Denote the category $\mathbf{\acute{E}t}_{\mathrm{fin}}(X)$ to be the full subcategory of $\mathbf{Sch}/X$ consisting of finite étale coverings of X.

Let us now give an example of finite étale scheme.

Example 1.12.10.2.

1.13   Coherent and quasi-coherent sheaf cohomology

All schemes in this section are Noetherian. We now start some actual geometry. We would need the cohomology of abelian sheaves over a space (Chapter 22) and the notion of Noetherian schemes (Section 1.4) in this section. *Apart from Mumford, you may like to give Hida a visit.*

We refer to Topics in Sheaf Theory, Chapter 22 for classical Čech cohomology on topological spaces.

1.13.1   Application : Serre-Grothendieck duality

Do from Hida and Hartshorne .

1.13.2   Application : Riemann-Roch theorem for curves

Do from Hida and Hartshorne.

Chapter 2

Elliptic Curves

Start writing from Hida's Geometric modular forms and elliptic curves. Need to complete divisors, Riemann-Roch, . Main topics are : definition and basic properties, GMF of level 1, A bit of classical Weierstrass theory of elliptic curves over $\mathbb{C}$, Put background on p-adic fields and then you may cover some bit of elliptic curves over them. The important part is level structure of elliptic curves, L-functions and moduli problems. For deformation, Drinfeld's theorem might do for now.

Chapter 3

Galois Theory

Contents

After all this time, we finally do Galois theory of schemes...

This chapter has the humble goal of proving the following theorem.

Theorem 3.0.0.1. *Let X be a connected scheme and let $\acute{\textbf{E}}\textbf{t}_{\text{fin}}(X)$ be the category of finite étale coverings of X and $\pi(X)$-$\textbf{Set}$ be the category of sets with $\pi(X)$-action where $\pi(X)$ is the étale fundamental group of X. Then,*

$$\acute{\textbf{E}}\textbf{t}_{\text{fin}}(X) \equiv \pi(X)\text{-}\textbf{Set}.$$

This is a statement about general (connected) schemes X. When we instantiate X into some special cases, we will, to our surprise, recover the classical Galois theories of fields, of commutative algebras and of infinite dimensional commutative algebras. Hence, the importance of this theorem is paramount.

We will use the results and definitions of Chapter 1 quite freely. The final goal of this chapter is to see what the the Theorem 3.0.0.1 says about an elliptic curve. Hence we would be computing the étale fundamental group of an elliptic curve as an example.

3.1   The case of elliptic curves

Chapter 4

Deformation Theory

In this chapter, we work out the classical deformation theory of schemes.

Chapter 5

Algebraic Geometry

Contents

We would like to discuss and give an overview of some distinguished topics of interest in contemporary algebraic geometry.

5.1 Functor of points

Cover in french from Demazure-Gabriel.

5.2 Analytification and GAGA

Cover from Neeman.

5.3 Intersection theory

Let X be an algebraic variety over an algebraically closed field k. Suppose $U, V \subseteq X$ are irreducible subvarieties of X. Suppose that, set theoretically, $U \cap V \neq \emptyset$. Then, for an element $x \in U \cap V$, one can ask whether the varieties U and V intersect at x only once, or twice, or thrice or.... This is a very intuition specific question, for example the tangent to parabola $y - x^2$ at $(0,0)$ might seem to meet it only at one point, but as soon we perturb the

tangent line, we see that the single point suddenly devolves into two points on the parabola. One may then intuitively say then that $(0,0) \in (y - x^2) \cap (y)$ has *intersection multiplicity* 2. The goal of this section is to illuminate a definition of intersection multiplicity that will be helpful in elucidating the above intuition.

5.3.1   Intersection in projective space

Do from Hartshorne Chapter 1, Section 7.

5.4 Overview of K-theory of schemes

Do several exercises from Chapter 2 and Chapter 3 of Hartshorne on Grothendieck group K_0 and K_1.

5.4.1 Overview

5.5 The Riemann-Hilbert correspondence

5.6 Serre's intersection formula

5.7 Étale topology of Sch

See Bhatt's notes on the topic and Stacksproject.

Part II

The Differential Viewpoint

Goals:

1. Define real and complex manifolds from locally ringed spaces, examples (Wedhorn).
2. Basic constructions like linearization, product, fiber products, submanifolds, quotients (Wedhorn for theory and Bredon for applications).
3. $\mathcal{O}_X$-modules and global algebra.
4. Lie groups (Wedhorn and Taubes).
5. Torsors and 1st-Cech cohomology group (Wedhorn and Mumford's chapter on cohomology of sheaves).
6. Bundles and applications (Taubes Ch.3,4,5,6,7,10, Wedhorn and Bredon both for theory and their exercises for applications).
7. Singular homology and cohomology as ES-axioms. Properties, applications and results (Bredon and May). Singular cohomology as sheaf cohomology (Wedhorn Chapter 11).
8. Fundamental group and covering maps (classification) as etale spaces of certain sheaves (Bredon Chapter 3 and my algebraic topology notes).
9. Differential forms and de-Rham cohomology (Wedhorn's Section 8.6 and Bredon's Chapter 5, with examples and exercises).
10. ($\star$ **Geometric milestone**) *Covariant derivative, connections, classes and curvature* (Taubes Ch. 11,12,13,14,15,16).
11. ($\star$ **Algebraic milestone**) *Cohomological methods in geometry* (Bredon's Chapter 6 full).
12. ($\star$ **Homotopical milestone**) *Homotopical methods* (Bredon's Chapter 7 and May).

I have to rearrange the following chapters to suit the above outline.

These chapters need not be filled with unwarranted details. They should provide the point of the construction clearly and all minute details can be safely skipped over after understanding them.

Chapter 6

Foundational Geometry

Contents

Complete this chapter from Wedhorn's manifolds, sheaves and cohomology, and by Bredon's topology and geometry.

We will define the notion of a real and complex manifold. Some foundational constructions are made on them. We will take a rather modern viewpoint on the matter. We will further discuss ....

6.1 Locally ringed spaces and manifolds

We will make very fluid use of sheaves (see Chapter 22). Let us begin by the foundational structure in all of geometry, a (locally)ringed space.

Definition 6.1.0.1. (**Ringed and locally ringed spaces**) A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative R-algebras.

The space $(X, \mathcal{O}_X)$ is locally ringed if the stalk $\mathcal{O}_{X,x}$ at each point $x \in X$ is a local ring. The sheaf $\mathcal{O}_X$ is called the structure sheaf of X.

In order to understand the relation between two such spaces, we next have to understand the morphism of (locally)ringed spaces. For a motivation, see Example 1.2.2.1.

Definition 6.1.0.2. (**Morphism of ringed and locally ringed spaces**) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. A morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and a map of sheaves over X denoted $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed, then for $(f, f^\sharp)$ to be morphism of locally ringed spaces has to satisfy an additional condition that the induced map on stalks is a map of local rings. That is, for each $x \in X$, the induced map on stalks

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is such that $(f_x^\sharp)^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,f(x)}$ (see Special Topics, Remark 22.4.0.6). We call this map the comorphism at $x \in X$. In particular, this map is given by the unique map obtained by universality of direct limits under question: consider any open $V \ni f(x)$ in Y, we then obtain the following diagram:

$$
\begin{array}{ccc}
\mathcal{O}_{X,x} & \xleftarrow{\quad =: f_x^\sharp \quad} & \mathcal{O}_{Y,f(x)} \\
{\scriptstyle \iota_{f^{-1}(V)}}\Big\uparrow \quad {\scriptstyle \iota_{f^{-1}(V)} \circ f_V^\flat} & & \Big\uparrow{\scriptstyle \iota_V} \\
\mathcal{O}_X(f^{-1}(V)) & \xleftarrow[\quad f_V^\flat \quad]{} & \mathcal{O}_Y(V)
\end{array}
$$

In most of our purposes, the map $f^\flat$ will be given on sections by composing with f. In such situations, the map on stalks being local corresponds to the geometric intuition that all non-invertible functions around some open subset of $f(x)$ comes from non-invertible maps around x. This in some sense makes sure that the local data around $f(x)$ is completely available via f.

Definition 6.1.0.3. (**Composition**) Composition of two maps of locally ringed spaces is defined in the obvious manner. For $X \xrightarrow{g} Y \xrightarrow{f} Z$, we get maps $g^\flat : \mathcal{O}_Y \to g_*\mathcal{O}_X$ and $f^\sharp : f^{-1}\mathcal{O}_Z \to \mathcal{O}_Y$. Then, the map $f \circ g : X \to Z$ is defined on space level by just the composite $f \circ g$ of the continuous maps and on the sheaf level as the corresponding flat and sharp maps of $f \circ g : X \to Z$:

$$h^\flat : \mathcal{O}_Z \longrightarrow (f \circ g)_*\mathcal{O}_X$$
$$h^\sharp : (f \circ g)^{-1}\mathcal{O}_Z \longrightarrow \mathcal{O}_X.$$

In particular, for an open set $U \subseteq Z$, the corresponding map on local sections $h_U^\flat$ is given by the following composite:

$$
\begin{array}{ccc}
\mathcal{O}_Z(U) & \xdashrightarrow{\quad h_U^\flat \quad} (f_*g_*\mathcal{O}_X)(U) = & \mathcal{O}_X(g^{-1}f^{-1}(U)) \\
\Big\downarrow{\scriptstyle f_U^\flat} & & \Big\uparrow{\scriptstyle g_{f^{-1}(U)}^\flat} \\
(f_*\mathcal{O}_Y)(U) & = & \mathcal{O}_Y(f^{-1}(U))
\end{array}
$$

Similarly, the corresponding morphism of stalks given by $h_x^\sharp$ is given by the usual

$$h_x^\sharp : (g^{-1}f^{-1}\mathcal{O}_Z)_x \cong \mathcal{O}_{Z,f(g(x))} \longrightarrow \mathcal{O}_{X,x}$$

which is the composite

$$
\begin{array}{ccc}
\mathcal{O}_{Z,h(x)} & \longleftarrow & \mathcal{O}_Z(W) \\
{\scriptstyle f_{g(x)}^\sharp}\downarrow & & \downarrow{\scriptstyle f_W^\flat} \\
\mathcal{O}_{Y,g(x)} & \longleftarrow & \mathcal{O}_Y(f^{-1}(W)) \\
{\scriptstyle g_x^\sharp}\downarrow & & \downarrow{\scriptstyle g_{f^{-1}(W)}^\flat} \\
\mathcal{O}_{X,x} & \longleftarrow & \mathcal{O}_X(g^{-1}(f^{-1}(W)))
\end{array} \quad .
$$

Lemma 6.1.0.4. *Let $h : X \xrightarrow{g} Y \xrightarrow{f} Z$ be a morphism of ringed spaces. Consider the base change functors corresponding to maps g and f:*

$$g^{-1} : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(X)$$
$$f_* : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(Z).$$

and consider the following composite in $\mathbf{Sh}(Y)$

$$f^{-1}\mathcal{O}_Z \xrightarrow{\ f^\sharp\ } \mathcal{O}_Y \xrightarrow{\ g^\flat\ } g_*\mathcal{O}_X \ .$$

Then,
1. *$g^{-1}(g^\flat \circ f^\sharp) \cong h^\sharp$,*
2. *$f_*(g^\flat \circ f^\sharp) \cong h^\flat$.*

Proof. These are cumbersome but straightforward identities. For example, one has to observe that $f_*(f^\sharp) \cong f^\flat$ and that for an open set $U \subseteq Z$, we have $(f_*(g^\flat))_U = g_{f^{-1}(U)}^\flat$. $\square$

We have a simple lemma for isomorphism of ringed spaces.

Lemma 6.1.0.5. *Let $f : X \to Y$ be a morphism of ringed spaces. Then, f is an isomorphism if and only if $f : X \to Y$ is a homeomorphism and $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

Proof. (L $\Rightarrow$ R) Use Theorem 22.2.0.6, 3 and 4.
(R $\Rightarrow$ L) One can explicitly construct a map of sheaves in the other direction in a straightforward manner. $\square$

An open subspace of a ringed space also inherits the structure of a ringed space.

Definition 6.1.0.6. (Open subspace and embedding) Let $(X, \mathcal{O}_X)$ be a (locally) ringed space. An open subspace of $(X, \mathcal{O}_X)$ is an open subset $i : U \hookrightarrow X$ together with the inverse image sheaf $i^{-1}\mathcal{O}_X = \mathcal{O}_{X|U}$[1]. The pair $(U, \mathcal{O}_{X|U})$ is called an open subspace, $(U, \mathcal{O}_{X|U}) \hookrightarrow (X, \mathcal{O}_X)$. A map $(j, j^\sharp) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is an open embedding if $U := j(Z) \hookrightarrow X$ is open and $(j, j^\sharp) : (Z, \mathcal{O}_Z) \to (U, \mathcal{O}_{X|U})$ is an isomorphism of ringed spaces.

---

[1]It's a trivial matter to observe that inverse image of a sheaf to an open inclusion will be the restriction sheaf (see Lemma 22.4.0.3).

An important concept is of local isomorphism of ringed spaces, which will prove it's worth while defining manifolds.

Definition 6.1.0.7. (**Local isomorphism**) Let $f : X \to Y$ be a morphism of ringed spaces. One calls f to be a local isomorphism if there exists an open cover $\{U_i\}_{i \in I}$ of X such that $f|_{U_i} : U_i \to Y$ is an open embedding for all $i \in I$.

6.1.1 Local models and manifolds

Before we proceed further, we have to clearly state some of our local model spaces that we are going to use while defining the manifolds. Therefore the following example of ringed spaces are foundational.

Example 6.1.1.1. (*Sheaf of C^α-maps*) Let $X \subseteq \mathbb{R}^n$ be an open set and $\alpha \in \mathbb{N}^\infty$. One defines the following presheaf

$$\mathcal{C}^\alpha_{X;\mathbb{R}^m} := \{f : X \to \mathbb{R}^m \mid f \text{ is } C^\alpha\}$$

where the restriction maps are usual functional restrictions. Then, $\mathcal{C}^\alpha_{X;\mathbb{R}^m}$ forms a sheaf, called the sheaf of C^α maps on X. This sheaf has stalks as local rings which can be seen quite easily (set of all functions defined in *some* neighborhood of $x \in X$ has a ring structure with maximal ideal being all those functions taking value 0 at x). Hence, $(X, \mathcal{C}^\alpha_{\mathbb{R}^m})$ is a locally ringed space, where we dropped the subscript X for notational convenience.

Example 6.1.1.2. (*Sheaf of holomorphic maps*) Let $X \subseteq \mathbb{C}^n$ be an open set. One defines the following presheaf

$$\mathcal{C}^{\text{hol}}_{X;\mathbb{C}^m} := \{f : X \to \mathbb{C}^m \mid f \text{ is holomorphic}\}$$

where the restriction maps are the usual functional restriction. This is easily seen to be a sheaf, called the sheaf of holomorphic functions over X. This endows $(X, \mathcal{C}^{\text{hol}}_{\mathbb{C}^m})$ with the structure of a locally ringed space.

With these two examples, we can come to the notion of real and complex manifolds as follows.

Definition 6.1.1.3. (**Real and complex manifolds**) Let X be a Hausdorff and second-countable topological space. Then,

1. A locally $\mathbb{R}$-ringed space $(X, \mathcal{O}_X)$ is a real C^α-manifold if there exists an open covering $\{U_i\}_{i \in I}$ of X and for each $i \in I$, there exists a positive integer $n_i \in \mathbb{N}$ and an isomorphism of locally $\mathbb{R}$-ringed spaces $\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^\alpha_{\mathbb{R}})$ for some open $Y_i \subseteq \mathbb{R}^{n_i}$. Hence a real C^α-manifold structure on X is the following tuple of data:

$$\left(X, \mathcal{O}_X, \{U_i\}_{i \in I}, \{Y_i \subseteq \mathbb{R}^{n_i}\}_{i \in I}, \{\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^\alpha_{\mathbb{R}})\}_{i \in I}\right)$$

2. A locally $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ is a complex manifold if there exists an open covering $\{U_i\}_{i \in I}$ of X and for each $i \in I$ there exists $n_i \in \mathbb{N}$ and an isomorphism of locally $\mathbb{C}$-ringed spaces $\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^{\text{hol}}_{\mathbb{C}})$ for some open $Y_i \subseteq \mathbb{C}^{n_i}$. Hence a complex manifold structure on X is the following tuple of data:

$$\left(X, \mathcal{O}_X, \{U_i\}_{i \in I}, \{Y_i \subseteq \mathbb{C}^{n_i}\}_{i \in I}, \{\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^{\text{hol}}_{\mathbb{C}})\}_{i \in I}\right)$$

In both of these, the isomorphisms $\{\varphi_i\}$ are called *charts* of the manifold. Also, we can rather consider $\{\varphi_i\}_{i \in I}$ to be open embeddings. A map of manifolds is just defined to be a map of locally ringed spaces. Let $\mathbf{Mfd}_\alpha^\mathbb{R}$ and $\mathbf{Mfd}^\mathbb{C}$ denote the category of real C^α and complex manifolds respectively. A map of manifolds are just locally ringed maps between them. Isomorphisms in them are called C^α-diffeomorphism and biholomorphic maps respectively.

Let us now dwell into some of the immediate observations and remarks coming out of this definition. Let us first ease some notations. Let $(X, \mathcal{O}_X)$ be a real or complex manifold. The local chart (U_i, φ_i) is usually denoted by (U_i, x) where $x : U_i \to \mathbb{R}^n$ is a local embedding of locally ($\mathbb{R}$ or $\mathbb{C}$)-ringed spaces, where n depends on U_i. We usually suppress all the sheaves and their morphisms unless necessary (we will soon see why that's the case). For a local chart (U_i, x), the n component maps $\pi_j \circ x : U_i \to \mathbb{R}$ are denoted by x^j. Moreover, since $x : U \to x(U)$ is an isomorphism, therefore we denote $x^{-1} : x(U) \to U$ to be its inverse. All this will come in handy when we will start doing geometry over $(X, \mathcal{O}_X)$.

Let $(X, \mathcal{O}_X)$ be a real or complex manifold. We call an open subspace $(U, \mathcal{O}_{X|U}) \hookrightarrow (X, \mathcal{O}_X)$ an open submanifold.

One now sees that any morphism of manifolds as locally ringed spaces is completely determined by what happens at the level of points. In-fact, the sheaf allowed on X is also restricted if its a manifold. This is why we usually completely suppress the map of sheaves from our notation as that will be vacuous as long as we are working with map of manifolds. Let $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$ be two manifolds ($\mathbb{R}$ or $\mathbb{C}$, but both of same type). We can define a sheaf $\mathcal{O}_{M;N}$ on M given by following sections: for some open $U \subseteq M$, we have a sheaf

$$\mathcal{O}_{M;N}(U) := \{f : (U, \mathcal{O}_{X|U}) \to (N, \mathcal{O}_N) \mid f \text{ is a map of manifolds}\}.$$

Now we show a foundational result which says that the notion of morphism of locally ringed spaces are nothing new in the classical world of $\mathbb{R}^n$ or $\mathbb{C}^n$. We place high importance on the following result as it becomes our point of departure (and thus a point of motivation) as to why the notion of a morphism of locally ringed spaces is defined as what it is; because it is the right notion of a "geometric map" in more abstract situations.

Theorem 6.1.1.4. *Let K be either $\mathbb{R}$ or $\mathbb{C}$, $X \subseteq K^n$ and $Y \subseteq K^m$ be two open subsets of the standard spaces. If $f : (X, \mathcal{C}_X^\alpha) \to (Y, \mathcal{C}_Y^\alpha)$ is a map of locally ringed spaces, then*
1. $f^\flat : \mathcal{C}_Y^\alpha \to f_* \mathcal{C}_X^\alpha$ is given on an open set $V \subseteq Y$ by the standard composition map

$$f_V^\flat : \mathcal{C}_Y^\alpha(V) \longrightarrow \mathcal{C}_X^\alpha(f^{-1}(V))$$
$$V \xrightarrow{t} K \longmapsto f^{-1}(V) \xrightarrow{f} V \xrightarrow{t} K,$$

2. f is a C^α-map.

Remark 6.1.1.5. As a slogan, we may remember the above theorem as the following principle:

In $\mathbb{R}^n$ or $\mathbb{C}^n$, locally ringed maps are exactly real C^α or holomorphic maps.

As a consequence of this, whenever we would like to consider C^α maps from, say $\mathbb{R}^n$ to $\mathbb{R}^m$, we might as well ask to produce a map of locally ringed spaces $(\mathbb{R}^n, \mathcal{C}^\alpha_{\mathbb{R}^n})$ to $(\mathbb{R}^m, \mathcal{C}^\alpha_{\mathbb{R}^m})$, which again shows how much geometric information is hidden in the notion of sheaves.

Proof of Theorem 6.1.1.4. [2] Pick any open $V \subseteq Y$ and any $t \in \mathcal{C}^\alpha_Y(V)$. We wish to show that $f^\flat_V(t) = t \circ f$ as a map $f^{-1}(V) \to K$. Consequently, pick any point $p \in f^{-1}(V)$. We wish to show that $f^\flat_V(t)(p) = t(f(p))$. Since $\mathcal{C}^\alpha_Y$ and $\mathcal{C}^\alpha_X$ are sheaves, therefore it is enough to show that $f^\flat_V(t)$ and $t \circ f$ have same image in the stalk. Since we have the following commutative square

$$
\begin{array}{ccc}
\mathcal{C}^\alpha_Y(V) & \xrightarrow{\;f^\flat_V\;} & \mathcal{C}^\alpha_X(f^{-1}(V)) \\
\downarrow & & \downarrow \\
\mathcal{C}^\alpha_{Y,f(p)} & \xrightarrow{\;f^\sharp_p\;} & \mathcal{C}^\alpha_{X,p}
\end{array}
\quad ,
$$

therefore we reduce to showing that $f^\sharp_p((V,t)_{f(p)}) = (f^{-1}(V), f^\flat_V(t))_p$. In other words, we wish to show that the following triangle commutes

$$
\begin{array}{ccc}
\mathcal{C}^\alpha_{Y,f(p)} & \xrightarrow{\quad f^\sharp_p \quad} & \mathcal{C}^\alpha_{X,p} \\
& \searrow^{\mathrm{ev}_{f(p)}} \quad \swarrow^{\mathrm{ev}_p} & \\
& K &
\end{array}
\quad .
$$

But this is immediate from the fact that the K-algebra homomorphism $f^\sharp_p$ is a local ring homomorphism and the kernels of the evaluation maps are exactly the corresponding unique maximal ideals, so by quotienting by the maximal ideals, we obtain a K-algebra homomorphism $K \to K$ which necessarily is identity. Hence the triangle indeed commutes.
In order to show that the map f is a C^α-map, we need only show that the m projection maps $\pi_i : K^m \to K$ when composed with f yields C^α maps given by $X \to K$, but that is immediate from 1. $\qquad\square$

Using the above result, one can show that any manifold essentially has a unique structure sheaf of the form $\mathcal{O}_{X;\mathbb{R}}$ or $\mathcal{O}_{X;\mathbb{C}}$.

Proposition 6.1.1.6. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. If $(X, \mathcal{O}_X)$ is a real or complex manifold, then $\mathcal{O}_X \cong \mathcal{O}_{X;\mathbb{R}}$ or $\mathcal{O}_X \cong \mathcal{O}_{X;\mathbb{C}}$.*

Proof. We wish to show that there is an isomorphism of sheaves $\varphi : \mathcal{O}_{X;\mathbb{R}} \to \mathcal{O}_X$. For an open set $U \subseteq X$, we define φ_U as follows:

$$
\varphi_U : \mathcal{O}_{X;\mathbb{R}}(U) \longrightarrow \mathcal{O}_X(U)
$$
$$
t : (U, \mathcal{O}_{X|U}) \to (\mathbb{R}, \mathcal{C}^\alpha_\mathbb{R}) \longmapsto t^\flat_\mathbb{R}(\mathrm{id}_\mathbb{R}).
$$

We claim that this map of sheaves is an isomorphism. We need only show that the map on stalks $\varphi_x : \mathcal{O}_{X;\mathbb{R},x} \to \mathcal{O}_{X,x}$ is an isomorphism. So we may assume that X has a global

---
[2]First proof in my new creator of meaning!

chart $\eta : (X, \mathcal{O}_X) \cong (W, \mathcal{C}^\alpha_{W;\mathbb{R}})$ where $W \subseteq \mathbb{R}^n$ is an open subset. Consequently, we have $\eta^\sharp_x : \mathcal{C}^\alpha_{W;\mathbb{R},\eta(x)} \cong \mathcal{O}_{X,x}$. Furthermore, $\mathcal{O}_{X;\mathbb{R},x} \cong \mathcal{C}^\alpha_{W;\mathbb{R},\eta(x)}$. Consequently, we wish to show that $\varphi_x : \mathcal{C}^\alpha_{W;\mathbb{R},\eta(x)} \to \mathcal{C}^\alpha_{W;\mathbb{R},\eta(x)}$ given by $(W, t : W \to \mathbb{R})_{\eta(x)} \mapsto (W, t^\flat_\mathbb{R}(\mathrm{id}_\mathbb{R}))_{\eta(x)}$ is an isomorphism. Since by Theorem 6.1.1.4, 1, the map $t^\flat_\mathbb{R}$ is given by precomposition by t, therefore $t^\flat_\mathbb{R}(\mathrm{id}_\mathbb{R})$ is just t. Consequently, φ_x is identity, which proves the result. $\qquad\square$

6.2 Linearization

till fiber prod-
Chapter 6.

6.3 Constructions on manifolds

6.4 Lie groups

6.5 Global algebra

Let $(X, \mathcal{O}_X)$ be a locally ringed space. We will discuss here the operations on and properties of $\mathbf{Mod}(\mathcal{O}_X)$, the category of $\mathcal{O}_X$-modules [3]. An $\mathcal{O}_X$-module is a sheaf $\mathcal{M}$ on X such that $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module and the restriction maps of $\mathcal{M}$ are given as module homomorphism w.r.t the corresponding restriction map of $\mathcal{O}_X$ (more precisely below). There are several important constructions and properties that one can make with these. In-fact, just like one understands a ring R by understanding R-modules, one can understand $\mathcal{O}_X$ by understanding $\mathcal{O}_X$-modules. The similarity runs deeper as we can also define in certain cases the very same constructions we do in module, but in the case of $\mathcal{O}_X$-modules, and these constructions and operations becomes indispensable in doing geometry over locally ringed spaces of special kind, like schemes. A lot of such phenomenon is merely due to the fact that $\mathbf{Mod}(\mathcal{O}_X)$ is an abelian category. In-fact, notice that for each singleton space $X = \{\text{pt.}\}$, a ring R can be seen as the structure sheaf $\mathcal{O}_X$ over X and any R-module as a $\mathcal{O}_X$-module. Hence one may also think of the concept of $\mathcal{O}_X$-modules as the global version of classical commutative algebra.

Needless to say, this is an indispensable section for the purposes of geometry in general.

Let us first observe that over any topological space X, the product of two sheaves $\mathcal{F}, \mathcal{G}$ over X defined by $(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$ is indeed a sheaf with restriction maps as products of the restrictions. This allows us to define $\mathcal{O}_X$-modules very naturally.

For the rest of this section, we fix a ringed space $(X, \mathcal{O}_X)$.

Definition 6.5.0.1. ($\mathcal{O}_X$-**modules**) An abelian sheaf $\mathcal{F}$ over X is an $\mathcal{O}_X$-module if there is a map of sheaves

$$\mathcal{O}_X \times \mathcal{F} \longrightarrow \mathcal{F}$$
$$(c, s) \longmapsto cs$$

where $c \in \mathcal{O}_X(U), s \in \mathcal{F}(U)$ for all open $U \subseteq X$ which endows $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module structure.

An $\mathcal{O}_X$-linear map of $\mathcal{O}_X$-modules is defined as a sheaf map $\varphi : \mathcal{F} \to \mathcal{G}$ between $\mathcal{O}_X$-modules such that for each open $U \subseteq X$, the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-linear map and that the restrictions preserves the respective module structures.

The above definition, when unravelled, yields that the scalar multiplication of each $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ commutes with restrictions; for $c \in \mathcal{O}_X(U)$, $s \in \mathcal{F}(U)$ and an open subset $V \subseteq U$, we have $(c \cdot s)|_V = c|_V \cdot s|_V$.

Remark 6.5.0.2. For an $\mathcal{O}_X$-module $\mathcal{F}$ we have the following easy observations:
1. $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module for all $x \in X$. Indeed, this follows from the universal property of direct limits and the fact that direct limits commutes with product; we have the

---

[3]we will give some general constructions for arbitrary sheaves over a topological case at times, before specializing to $\mathcal{O}_X$-module case.

following diagram

$$
\begin{array}{ccc}
\mathcal{O}_{X,x} \times \mathcal{F}_x & & \\
\cong \uparrow & \diagdown & \\
\varinjlim_{x \in V} \mathcal{O}_X(V) \times \mathcal{F}(V) & \overset{\tilde{m}}{\dashrightarrow} & \\
\uparrow & & \\
\mathcal{O}_X(U) \times \mathcal{F}(U) \overset{m_U}{\longrightarrow} \mathcal{F}(U) \longrightarrow & \mathcal{F}_x
\end{array}
\quad .
$$

Explicitly, the $\mathcal{O}_{X,x}$-module structure on $\mathcal{F}_x$ is given by

$$
\mathcal{O}_{X,x} \times \mathcal{F}_x \longrightarrow \mathcal{F}_x
$$
$$
((U,c)_x, (U,s)_x) \longmapsto (U, c \cdot s)_x
$$

where we may assume c and s are defined on same open neighborhood of x by appropriately restricting.

2. For a homomorphism $f : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules, we get a $\mathcal{O}_{X,x}$-module homomorphism $f_x : \mathcal{F}_x \to \mathcal{G}_x$ mapping as $(U,s)_x \mapsto (U, f_U(s))_x$ for each $x \in X$,

3. Let X be locally ringed space. Then, $\mathcal{F}_x / \mathfrak{m}_{X,x} \mathcal{F}_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x} / \mathfrak{m}_{X,x} \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a $\kappa(x)$-vector space. This is called the *fiber of module $\mathcal{F}$ over x*, denoted by $\mathcal{F}(x)$. Recall this is how the fiber of a module over a prime ideal of the ring is defined.

We first give few basic constructions, which is useful to keep in mind.

Definition 6.5.0.3. (Support of a sheaf) Let X be a topological space and $\mathcal{F}$ be an abelian sheaf over X. Let $U \subseteq X$ be an open set. For $s \in \mathcal{F}(U)$, we define the support of s as the subset

$$
\mathrm{Supp}\,(s) := \{x \in U \mid (U,s)_x \neq 0 \text{ in } \mathcal{F}_x\}.
$$

We further define the support of the sheaf as

$$
\mathrm{Supp}\,(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.
$$

Support of a section is always a closed subset, but the support of a sheaf may not be closed.

Lemma 6.5.0.4. [4] *Let X be a space and $\mathcal{F}$ be a sheaf over X with $s \in \mathcal{F}(U)$ for an open set $U \subseteq X$. Then $\mathrm{Supp}\,(s) \subseteq U$ is a closed subset of U.*

Proof. Take any point $y \in U \setminus \mathrm{Supp}\,(s)$. We will find an open set $W \subseteq U \setminus \mathrm{Supp}\,(s)$ with $W \ni y$. Indeed, as $(U,s)_y = 0$, therefore we get a $W \subseteq U$ with $s|_W = 0$. For any $z \in W$, one further checks that $(U,s)_z = (W, s|_W)_z = 0$. Thus, $z \notin \mathrm{Supp}\,(s)$ and consequently, $W \subseteq U \setminus \mathrm{Supp}\,(s)$. □

Do skyscraper and subsheaf with support (Exercises 1.17 and 1.20 in Hartshorne.)

---

[4]Exercise II.1.14 of Hartshorne.

6.5.1 Global algebra : The algebra of $\mathcal{O}_X$-modules

In our quest to do geometry over schemes, we will make heavy use of the algebra of sheaves, especially that of exact sequences, so we give a lot of constructions that we may have to make out in the wild. We will make heavy use of sheafification (Theorem 22.1.0.1) in the sequel. An important question that arises is whether sheafification of an algebraic construction over collection of $\mathcal{O}_X$-modules actually is again an $\mathcal{O}_X$-module or not? The answer is yes, as can be easily checked by explicitly looking at sections of sheafification directly (see Remark 22.1.0.4 to observe that its not difficult, anyways we will show the explicit checks consistently).

Caution 6.5.1.1. The following pages might seem to be filled with *unnecessary details* about checking whether a given construction on $\mathcal{O}_X$-modules results in an $\mathcal{O}_X$-module or not. While for some this might be unnecessary, but working this out in experience has been satisfying and tends to give a deeper understanding of the various module structures (*algebraic* modules) that gets associated with an $\mathcal{O}_X$-module $\mathcal{F}$ and how they interrelate. Indeed, we will see that with more elaborate constructions, we get more and more module structures to handle with. Thus it is necessary to work some details out of this. At any rate, we will be using notions presented in the sequel quite frequently in algebraic geometry and in particular while doing cohomology (Cĕch cohomology in particular!) so we need a good knowledge of the $\mathcal{O}_X$-modules and their internal technicalities.

Remark 6.5.1.2. Since there are a lot of constructions in the sequel, so to have a sense of mental clarity, let us list them here:

- Submodules and ideals of $\mathcal{O}_X$.✓
- Quotient of modules.✓
- Image and kernel modules.✓
- Exact sequences of modules.✓
- The $\Gamma(\mathcal{O}_X, X)$-module $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.✓
- $\mathcal{H}om_{\mathcal{O}_X}$ module.✓
- Direct sum of modules.✓
- Direct product of modules.✓
- Tensor product of modules.✓
- Free, locally free & finite locally free $\mathcal{O}_X$-modules.✓
- Invertible modules and the Picard group.✓
- Direct and inverse image modules.✓
- Sums & intersections of submodules.
- Modules generated by sections.
- Inverse limit.
- Direct limit.
- Symmetric & exterior algebras.
- $\mathcal{E}xt$ module.
- $\mathcal{T}or$ module.
- $\otimes$-$\mathcal{H}om$ adjunction.

Submodules and ideals of $\mathcal{O}_X$

Definition 6.5.1.3. (Submodules and ideals) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. A *submodule* of $\mathcal{F}$ is an $\mathcal{O}_X$-module which is a subsheaf $\mathcal{G} \subseteq \mathcal{F}$ such that for all open $U \subseteq X$, the inclusion

$$\mathcal{G}(U) \hookrightarrow \mathcal{F}(U)$$

is an $\mathcal{O}_X(U)$-module homomorphism. Since $\mathcal{O}_X$ is an $\mathcal{O}_X$-module, thus, to be in line with usual terminology, we define submodules of $\mathcal{O}_X$ as *ideals* of $\mathcal{O}_X$.

Remark 6.5.1.4. Note that for any $\mathcal{O}_X$ submodule $\mathcal{G} \subseteq \mathcal{F}$, we get a submodule $\mathcal{G}_x \subseteq \mathcal{F}_x$ of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.

Quotient of modules

Definition 6.5.1.5. (Quotient modules) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{G}$ be a submodule of $\mathcal{F}$. The *quotient module* is the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$, denoted by $\mathcal{F}/\mathcal{G}$ (see Definition 22.2.0.4). Indeed, $\mathcal{F}/\mathcal{G}$ is an $\mathcal{O}_X$-module by the following lemma.

Lemma 6.5.1.6. $\mathcal{F}/\mathcal{G}$ *is an $\mathcal{O}_X$-module.*

Proof. We will use the definition of sheafification as given in Remark 22.1.0.4. For each open set $U \subseteq X$, consider the following map:

$$\eta_U : \mathcal{O}_X(U) \times (\mathcal{F}/\mathcal{G})(U) \longrightarrow (\mathcal{F}/\mathcal{G})(U)$$
$$(c,s) \longmapsto \eta_U(c,s) : U \to \amalg_{x \in U} \mathcal{F}_x/\mathcal{G}_x$$

where $\eta_U(c,s)(x) := c_x \cdot s(x)$ where $c_x \in \mathcal{O}_{X,x}$ and $s(x) \in \mathcal{F}_x/\mathcal{G}_x$ and the multiplication $c_x \cdot s(x)$ is coming from the $\mathcal{O}_{X,x}$-module structure that $\mathcal{F}_x/\mathcal{G}_x$ has. We now need to show following two statements:

1. $\eta_U(c,s)$ is indeed in $(\mathcal{F}/\mathcal{G})(U)$,
2. $\eta : \mathcal{O}_X \times \mathcal{F}/\mathcal{G} \to \mathcal{F}/\mathcal{G}$ is a sheaf map.

For statement 1, we need to show that for each $x \in U$, there exists an open set $x \in V \subseteq U$ and there exists $r \in \mathcal{F}(U)/\mathcal{G}(U)$ such that for all $y \in V$ we have the equality $c_y \cdot s(y) = r_y$ in $\mathcal{F}_y/\mathcal{G}_y$. Indeed, this can easily be seen via the fact that $s \in (\mathcal{F}/\mathcal{G})(U)$. Statement 2 is immediate after drawing the relevant square whose commutativity is under investigation. $\square$

Remark 6.5.1.7. Note further that we get a natural map

$$\mathcal{F} \to \mathcal{F}/\mathcal{G}$$

which factors through the inclusion of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$ into the sheaf $\mathcal{F}/\mathcal{G}$.

Image and kernel modules

Definition 6.5.1.8. (Image and kernel modules) Let $f : \mathcal{F} \to \mathcal{G}$ be a $\mathcal{O}_X$-module homomorphism. We then get the image sheaf $\mathrm{Im}\,(f)$ and the kernel sheaf $\mathrm{Ker}\,(f)$ by Definition 22.2.0.5. Indeed, both of these are $\mathcal{O}_X$-modules as the following lemma shows.

Lemma 6.5.1.9. $\mathrm{Im}\,(f)$ *and* $\mathrm{Ker}\,(f)$ *are* $\mathcal{O}_X$*-modules.*

Proof. $\mathrm{Ker}\,(f)$ is straightforward. For $\mathrm{Im}\,(f)$, we first observe that if we denote $\mathrm{Im}\,(f) = (\mathrm{im}\,(f))^{++}$, then $(\mathrm{im}\,(f))_x = f_x(\mathcal{F}_x)$. We thus define the $\mathcal{O}_X$-module structure on $\mathrm{Im}\,(f)$ as follows:

$$\eta_U : \mathcal{O}_X(U) \times \mathrm{Im}\,(f)(U) \longrightarrow \mathrm{Im}\,(f)(U)$$
$$(c, s : U \to \amalg_{x \in U} f_x(\mathcal{F}_x)) \longmapsto \eta_U(c, s)$$

where $\eta_U(c, s)(x) = c_x \cdot s(x)$ where $s(x) \in f_x(\mathcal{F}_x) \subseteq \mathcal{G}_x$. One checks like for quotient modules that this defines an $\mathcal{O}_X$-module structure on $\mathrm{Im}\,(f)$. Further, it is clear that $\mathrm{Im}\,(f) \subseteq \mathcal{G}$. $\square$

Corollary 6.5.1.10. *For a* $\mathcal{O}_X$*-module homomorphism* $f : \mathcal{F} \to \mathcal{G}$*, we have* $\mathrm{Ker}\,(f) \leq \mathcal{F}$ *and* $\mathrm{Im}\,(f) \leq \mathcal{G}$ *are submodules.*

Proof. Use Remark 22.1.0.4 to get this immediately. $\square$

We have a "first isomorphism theorem" for modules then.

Lemma 6.5.1.11. *For a map* $f : \mathcal{F} \to \mathcal{G}$ *of* $\mathcal{O}_X$*-modules, we obtain an isomorphism*

$$\mathcal{F}/\mathrm{Ker}\,(f) \cong \mathrm{Im}\,(f).$$

Proof. For each $x \in X$ let $\varphi_x : \mathcal{F}_x/\ker f_x \xrightarrow{\cong} \mathrm{im}\,(f_x)$. Then we define the following for any $U \subseteq X$ open

$$(\mathcal{F}/\mathrm{Ker}\,(f))(U) \longrightarrow \mathrm{Im}\,(f)(U)$$
$$s : U \to \amalg_{x \in U} \mathcal{F}_x/\ker f_x \mapsto \varphi \circ s$$

where $(\varphi \circ s)(x) = \varphi_x(s(x))$. This is immediately an isomorphism by going to stalks (Theorem 22.2.0.6, 3). $\square$

Exact sequences of modules

Definition 6.5.1.12. (**Exact sequences**) A sequence of $\mathcal{O}_X$-modules

$$\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}''$$

is said to be *exact* if $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$.

Remark 6.5.1.13. By Lemma 22.2.0.8, $\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}''$ is exact if and only if $\mathrm{Ker}\,(g_x) = \mathrm{Im}\,(f_x)$ at all points $x \in X$.

Do Hartshorne E
ercise 1.6, 1.7, 1
1.16, 1.19 for ex
sequences, Chap

The $\Gamma(\mathcal{O}_X, X)$-module $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$

We now consider the set of all $\mathcal{O}_X$-module homomorphisms $f : \mathcal{F} \to \mathcal{G}$ and observe very easily that it has a $\Gamma(\mathcal{O}_X, X)$-module structure. This generalizes the fact that under pointwise addition and scalar multiplication, the set $\mathrm{Hom}_R(M, N)$ for two R-modules M, N is again an R-module.

Definition 6.5.1.14. ($\Gamma(\mathcal{O}_X, X)$**-module** $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$) Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. Then the collection of all $\mathcal{O}_X$-module homomorphisms $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a $\Gamma(X, \mathcal{O}_X)$-module. Indeed, for two $f, g \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $c \in \Gamma(\mathcal{O}_X, X)$, we define $f + g : \mathcal{F} \to \mathcal{G}$ by $s \mapsto f(s) + g(s)$ and we define $c \cdot f : \mathcal{F} \to \mathcal{G}$ by $s \mapsto \rho_{X,U}(s) \cdot f(s)$ for any open set $U \subseteq X$ and $s \in \mathcal{F}(U)$.

We will now globalize the construction of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to obtain an $\mathcal{O}_X$-module out of it.

$\mathcal{H}om_{\mathcal{O}_X}$ module

Definition 6.5.1.15. (**Hom module** $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$) Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. Then the following presheaf

$$U \mapsto \mathcal{H}om_{\mathcal{O}_{X|U}}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$$

with restriction given by restriction of sheaf maps, is an $\mathcal{O}_X$-module denoted by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, as the following lemma shows.

Lemma 6.5.1.16. $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *is an $\mathcal{O}_X$-module*

Proof. The fact that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf can be seen immediately. The $\mathcal{O}_X$-module structure is defined as follows: pick any open $U \subseteq X$

$$\eta_U : \mathcal{O}_X(U) \times \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$$
$$(c, f) \longmapsto cf$$

where $cf : \mathcal{F}_{|U} \to \mathcal{G}_{|U}$ is given on an open set $V \subseteq U$ by

$$(cf)_V : \mathcal{F}(V) \longmapsto \mathcal{G}(V)$$
$$s \longmapsto \rho_{U,V}(c) \cdot f_V(s).$$

One easily check that η is a well-defined natural map of sheaves, thus making $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ into an $\mathcal{O}_X$-module. $\qquad\square$

Do Exercise 1.15 Chapter 2 of Hartshorne as well.

Remark 6.5.1.17. It is in general not true that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$.

We now define the dual of a module in the obvious manner.

Definition 6.5.1.18. (**Dual module**) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The dual of $\mathcal{F}$ is defined to be the module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. We denote the dual by $\mathcal{F}^\vee$.

There are some isomorphisms regarding $\mathcal{H}om$ that is akin to their usual algebraic counterparts. We outline them in the following lemma.

Lemma 6.5.1.19. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then,*
1. *$\mathcal{H}om(\mathcal{O}_X^n, \mathcal{F}) \cong \mathcal{H}om(\mathcal{O}_X, \mathcal{F})^n$,*
2. *$\mathcal{H}om(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$.*

Proof. In both cases we construct a map and its inverses and it is straightforward to see that they are well-defined, natural and indeed inverses of each other.
1. Consider the map

$$\mathcal{H}om(\mathcal{O}_X^n, \mathcal{F}) \longrightarrow \mathcal{H}om(\mathcal{O}_X, \mathcal{F})^n$$

which on an open set $U \subseteq X$ maps as

$$\mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}^n, \mathcal{F}_{|U}\right) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right)^n$$
$$f : \mathcal{O}_{X|U}^n \to \mathcal{F}_{|U} \longmapsto (f_i)_{i=1,\ldots,n}$$

where for $V \subseteq U$, we have that $f_{i,V} : \mathcal{O}_X(V) \to \mathcal{F}(V)$ maps as $s \mapsto s \cdot f_V(e_i) = f_V(s \cdot e_i)$ where e_i is i^{th} standard vector in $\mathcal{O}_X(V)^n$. Conversely, define the map

$$\mathcal{H}om(\mathcal{O}_X, \mathcal{F})^n \longrightarrow \mathcal{H}om(\mathcal{O}_X^n, \mathcal{F})$$

which on $U \subseteq X$ open maps as

$$\left(g_i : \mathcal{O}_{X|U} \to \mathcal{F}_{|U}\right)_{i=1,\ldots,n} \longmapsto g : \mathcal{O}_{X|U}^n \to \mathcal{F}_{|U}$$

where on $V \subseteq U$ open, we define $g_V : \mathcal{O}_X(V)^n \to \mathcal{F}(V)$ as $(s_1, \ldots, s_n) \mapsto \sum_{i=1}^n g_{i,V}(s_i) = \sum_{i=1}^n s_i \cdot g_{i,V}(e_i)$.
2. Define the map

$$\mathcal{H}om(\mathcal{O}_X, \mathcal{F}) \longrightarrow \mathcal{F}$$

on open $U \subseteq X$ by

$$\mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right) \longrightarrow \mathcal{F}(U)$$
$$f : \mathcal{O}_{X|U} \to \mathcal{F}_{|U} \longmapsto f_U(1).$$

Define the inverse

$$\mathcal{F} \longrightarrow \mathcal{H}om(\mathcal{O}_X, \mathcal{F})$$

on an open set $U \subseteq X$ by

$$\mathcal{F}(U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right)$$
$$s \longmapsto f : \mathcal{O}_{X|U} \to \mathcal{F}_{|U}$$

where for an open set $V \subseteq U$, we define $f_V(t) = f_V(t \cdot 1) := t \cdot s$. $\qquad\square$

Direct sum of modules

Definition 6.5.1.20. (Direct sum of modules) Let $\{\mathcal{F}_i\}_{i\in I}$ be a family of $\mathcal{O}_X$-modules. The direct sum of $\mathcal{F}_i$ is the sheafification of the presheaf

$$U \mapsto \bigoplus_{i\in I} \mathcal{F}_i(U)$$

whose restriction is the direct sum of the corresponding restrictions. We denote this sheaf by $\bigoplus_{i\in I} \mathcal{F}_i$ and it is an $\mathcal{O}_X$-module by the following lemma. If for all $i \in I$, we have $\mathcal{F}_i = \mathcal{F}$, then we write

$$\bigoplus_{i\in I} \mathcal{F} = \mathcal{F}^{\oplus I} = \mathcal{F}^{(I)}$$

as usually is done in algebra.

Lemma 6.5.1.21. $\bigoplus_{i\in I} \mathcal{F}_i$ *is an $\mathcal{O}_X$-module and $\left(\bigoplus_{i\in I} \mathcal{F}_i\right)_x \cong \bigoplus_{i\in I} \mathcal{F}_{i,x}$ for all $x \in X$.*

Proof. Since stalks functor is left adjoint (to skyscraper, we didn't covered this but this is a basic known fact), therefore it preserves all colimits and thus $\left(\bigoplus_{i\in I} \mathcal{F}_i\right)_x \cong \bigoplus_{i\in I} \mathcal{F}_{i,x}$. Now, the $\mathcal{O}_X$-module structure over $\bigoplus_{i\in I} \mathcal{F}_i$ is obtained as follows: pick any $U \subseteq X$ open and consider the map

$$\eta_U : \mathcal{O}_X(U) \times \left(\bigoplus_{i\in I} \mathcal{F}_i\right)(U) \longrightarrow \left(\bigoplus_{i\in I} \mathcal{F}_i\right)(U)$$

$$(c, s : U \to \amalg_{x\in U} \oplus_{i\in I} \mathcal{F}_{i,x}) \longmapsto cs$$

where $cs(x) = c_x \cdot s(x)$ where $s(x) \in \bigoplus_{i\in I} \mathcal{F}_{i,x}$ and $\bigoplus_{i\in I} \mathcal{F}_{i,x}$ is an $\mathcal{O}_{X,x}$-module. By exactly same techniques employed in proving them in earlier cases, it can be observed that the above defines a map $\eta : \mathcal{O}_X \times \oplus_{i\in I}\mathcal{F}_i \to \oplus_{i\in I}\mathcal{F}_i$ which is a sheaf map. $\square$

We now cover the other construction we know from algebra.

Direct product of modules

Definition 6.5.1.22. (Direct product of modules) Let $\{\mathcal{F}\}_{i\in I}$ be a family of $\mathcal{O}_X$-modules. The direct product of them is defined to be the sheaf

$$U \mapsto \prod_{i\in I} \mathcal{F}_i(U)$$

with product of restrictions as its restriction. Indeed, it is immediate it is a sheaf and that the canonical map $\eta_U : \mathcal{O}_X(U) \times \prod_{i\in I} \mathcal{F}_i(U) \to \prod_{i\in I} \mathcal{F}_i(U)$ mapping as $(c, (s_i)_{i\in I}) \mapsto (c \cdot s_i)_{i\in I}$ makes $\prod_{i\in I} \mathcal{F}_i$ an $\mathcal{O}_X$-module. If $\mathcal{F}_i = \mathcal{F}$ for all $i \in I$, then we denote

$$\prod_{i\in I} \mathcal{F} = \mathcal{F}^{\prod I} = \mathcal{F}^I$$

as is usually done in algebra.

We now define tensor product of two $\mathcal{O}_X$-modules.

Tensor product of modules

Definition 6.5.1.23. (Tensor product of modules) Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. The tensor product of $\mathcal{F}$ and $\mathcal{G}$ is given by the sheafification of the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, as the following lemma shows.

Lemma 6.5.1.24. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is an $\mathcal{O}_X$-module and $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$ for each $x \in X$.

Proof. The second statement is immediate from Lemma 18.4.1.1. The $\mathcal{O}_X$-module structure is the obvious one: pick any open $U \subseteq X$ and then consider the map

$$\eta_U : \mathcal{O}_X(U) \times (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) \longrightarrow (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U)$$
$$(a, s : U \to \amalg_{x \in U} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x) \longmapsto as$$

where $as(x) = a_x s(x)$. One easily checks that this defines a well-defined natural sheaf map. $\qquad\square$

A simple observation also yields the usual identity we know from modules.

Lemma 6.5.1.25. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then,

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{F}.$$

Proof. Consider the map

$$\eta : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \longrightarrow \mathcal{F}$$

given on an open $U \subseteq X$ by the map corresponding to the following natural isomorphism (Theorem 22.1.0.1)

$$\eta_U : \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \overset{\cong}{\to} \mathcal{F}(U).$$

This yields the similar isomorphic map on stalks via Lemma 6.5.1.24 to yield the result via Theorem 22.2.0.6, 3. $\qquad\square$

Tensor product of modules is obviously commutative.

Lemma 6.5.1.26. Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. Then, $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{G} \otimes \mathcal{F}$.

Proof. Construct the map $\tilde{\eta} : \mathcal{F} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{F}$ as the unique map corresponding to the following

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \xrightarrow[\cong]{\eta_U} \mathcal{G}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U)$$
$$\searrow{\scriptstyle j_U \eta_U} \qquad \downarrow{\scriptstyle j_U} \qquad .$$
$$(\mathcal{G} \otimes \mathcal{F})(U)$$

This map on the stalks gives the usual twist isomorphism $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \cong \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$. $\qquad\square$

Free, locally free & finite locally free $\mathcal{O}_X$-modules

Definition 6.5.1.27. (Free, locally free and finite locally free modules) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then,

1. $\mathcal{F}$ is called *free* if $\mathcal{F} \cong \mathcal{O}_X^{(I)}$ for some index set I,
2. $\mathcal{F}$ is called *locally free* if for all $x \in X$, there exists open $U \ni x$ such that $\mathcal{F}_{|U} \cong \mathcal{O}_{X|U}^{(I_x)}$ where I_x is an indexing set depending on x,
3. $\mathcal{F}$ is called *finite locally free* if $\mathcal{F}$ is locally free and the indexing set I_x is finite for each $x \in X$. If $I_x = I$ and I has size n, then we say that $\mathcal{F}$ is *locally free of rank n*.

We now observe that the hom sheaf of two locally free modules of finite rank is again locally free of finite rank.

Lemma 6.5.1.28. *Let $\mathcal{F}, \mathcal{E}$ be two locally free $\mathcal{O}_X$-modules of ranks n and m respectively. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})$ is a locally free module of rank nm.*

Proof. For each $x \in X$, there exists an open set $U \ni x$ such that $\mathcal{F}_{|U} \cong \mathcal{O}_{X|U}^n$ and $\mathcal{E}_{|U} \cong \mathcal{O}_{X|U}^m$. We then observe the following

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})(U) = \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{F}_{|U}, \mathcal{E}_{|U}\right) \cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}^n, \mathcal{O}_{X|U}^m\right) \cong \mathcal{O}_{X|U}^{nm}$$

where the last isomorphism can be established easily by reducing to the usual module case $(\mathrm{Hom}_R(R^n, R^m) \cong R^{nm})$. $\square$

An important corollary of the above lemma is as follows.

Corollary 6.5.1.29. *Let $\mathcal{F}$ be be a locally free module of rank n. Then the dual $\mathcal{F}^\vee$ is locally free of rank n.*

Proof. By Lemma 6.5.1.28, $\mathcal{F}^\vee$ is locally free of rank n. $\square$

Invertible modules and the Picard group

Definition 6.5.1.30. (Invertible modules) An $\mathcal{O}_X$-module $\mathcal{L}$ is said to be invertible if it is locally free of rank 1.

The name is justified by the fact that the set of all invertible modules upto isomorphism forms a group under tensor product and is one of the important invariants of a (ringed) space amongst many others. We now show that indeed this forms a group. We will drop the subscript $\mathcal{O}_X$ from the tensor product, for clarity, in the following.

Proposition 6.5.1.31. *Let $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ be invertible $\mathcal{O}_X$-modules. Then,*

1. *$\mathcal{L}_1 \otimes \mathcal{L}_2$ is invertible,*
2. *$(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \mathcal{L}_3 \cong \mathcal{L}_1 \otimes (\mathcal{L}_2 \otimes \mathcal{L}_3)$,*
3. *$\mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{O}_X$.*

Proof. 1. This is a local question, so pick $x \in X$ and an open set $U \ni x$ such that $\mathcal{L}_{1|U} \cong \mathcal{O}_{X|U} \cong \mathcal{L}_{2|U}$. We wish to construct a natural map $(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2)(U) \to \mathcal{O}_X(U)$ which is an isomorphism. By Theorem 22.1.0.1, it suffices to show a natural isomorphism

$\mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U) \to \mathcal{O}_X(U)$. This is constructed quite easily as $\mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U) \cong \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \cong \mathcal{O}_X(U)$. Thus we just need to consider $\mathrm{id}_{\mathcal{O}_X(U)}$.

2. This is again a local question, which can be answered by establishing an isomorphism (by using Theorem 22.1.0.1)

$$(\mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_3(U) \cong \mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} (\mathcal{L}_2(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_3(U))$$

for any open $U \subseteq X$, but that is an immediate observation from algebra.

3. By Corollary 6.5.1.29, we have that $\mathcal{L}^\vee$ is invertible. By Theorem 22.2.0.6, 3., the result would follow if we can show that there is a natural $\mathcal{O}_X$-linear map $\varphi : \mathcal{L}^\vee \otimes \mathcal{L} \to \mathcal{O}_X$ such that for each point $x \in X$ there exists an open set $x \in U \subseteq X$ such that on U, φ yields an $\mathcal{O}_X(U)$-linear isomorphism $(\mathcal{L}^\vee \otimes \mathcal{L})(U) \cong \mathcal{O}_X(U)$. We may take U small enough so that $\mathcal{L}^\vee_{|U} \cong \mathcal{O}_{X|U} \cong \mathcal{L}_{|U}$. Thus, after replacing X by U, we may assume $\mathcal{L} = \mathcal{O}_X = \mathcal{L}^\vee$. By Lemmas 6.5.1.19 and 6.5.1.25, we obtain the following isomorphisms

$$\mathcal{L}^\vee \otimes \mathcal{L} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \otimes \mathcal{O}_X \cong \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes \mathcal{O}_X \cong \mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X.$$

This can easily be promoted to a sheaf map. $\square$

Definition 6.5.1.32. (Picard group of X) The Picard group of X is defined to be the set of all isomorphism classes of invertible modules with the operation of tensor product. We denote this by

$$\mathrm{Pic}(X)$$

The Proposition 6.5.1.31 and Lemma 6.5.1.26 shows that $\mathrm{Pic}(X)$ is indeed an abelian group.

Direct and inverse image modules

In this and the next sections, we show how the modules behave under map of ringed spaces.

Definition 6.5.1.33. (Direct image) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the direct image of $\mathcal{F}$ under f is the direct image sheaf $f_*\mathcal{F}$ which is again an $\mathcal{O}_Y$-module given by the following composition

$$\mathcal{O}_Y \times f_*\mathcal{F} \xrightarrow{f^\flat \times \mathrm{id}} f_*\mathcal{O}_X \times f_*\mathcal{F} \xrightarrow{f_*m} f_*\mathcal{F}$$

where $m : \mathcal{O}_X \times \mathcal{F} \longrightarrow \mathcal{F}$ is the $\mathcal{O}_X$-module structure on $\mathcal{F}$. Note that f_* commutes with products as f_* is a right-adjoint.

The inverse image of a module, on the other hand, is an involved construction.

Definition 6.5.1.34. (Inverse image) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. The inverse image of $\mathcal{G}$ is defined to be the map

$$f^*\mathcal{G} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$

which is indeed an $\mathcal{O}_X$-module as the following lemma shows.

Lemma 6.5.1.35. *The sheaf $f^*\mathcal{G}$ is an $\mathcal{O}_Y$-module.*

Proof. We need to show three statements:

1. $\mathcal{O}_X$ is an $f^{-1}\mathcal{O}_Y$-module.
2. $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module.
3. $f^*\mathcal{G}$ is an $\mathcal{O}_X$-module.

Statement 1 follows from the following composition

$$f^{-1}\mathcal{O}_Y \times \mathcal{O}_X \xrightarrow{f^\sharp \times \mathrm{id}} \mathcal{O}_X \times \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

where the latter is just the multiplication structure on $\mathcal{O}_X$. Statement 2 follows from $\mathcal{O}_Y$-module structure on $\mathcal{G}$ and the fact that $f^{-1}(\mathcal{G} \times \mathcal{G}') = f^{-1}\mathcal{G} \times f^{-1}\mathcal{G}'$ for two sheaves $\mathcal{G}, \mathcal{G}'$ over Y. Indeed, the latter follows from the fact that $f^+(\mathcal{G} \times \mathcal{G}') = f^+\mathcal{G} \times f^+\mathcal{G}'$, which in turn follows from the fact that filtered colimit commutes with finite limits. Statement 3 now follows immediately. $\qquad\square$

We now state an important result, that is $f_* \vdash f^*$.

Proposition 6.5.1.36. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces. Then,*

$$\mathbf{Mod}(\mathcal{O}_Y) \underset{f_*}{\overset{f^*}{\underset{\perp}{\rightleftarrows}}} \mathbf{Mod}(\mathcal{O}_X) \ .$$

In other words, we have a natural isomorphism of groups

$$\mathrm{Hom}_{\mathcal{O}_X}\left(f^*\mathcal{G}, \mathcal{F}\right) \cong \mathrm{Hom}_{\mathcal{O}_Y}\left(\mathcal{G}, f_*\mathcal{F}\right).$$

Proof. Omitted. $\qquad\square$

Sums & intersections of submodules

Modules generated by sections

Inverse limit

Do Hartshorne Exercise 1.12 as well.

Direct limit

Do Hartshorne Exercise 1.11 as well.

Symmetric & exterior powers

Do from Section 17.21 of Stacksproject and Exercise 5.16 of Hartshorne.

$\mathcal{E}xt$ **module**

$\mathcal{T}or$ **module**

$\otimes$-$\mathcal{H}om$ **adjunction**

6.5.2 The abelian category of $\mathcal{O}_X$-modules

We now show an important result that category of $\mathcal{O}_X$-modules over any ringed space is an abelian category (thus we can do whole of homological algebra over it!). We have essentially done everything, but we write it here for clear reference.

Theorem 6.5.2.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $\mathbf{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules is an abelian category.*

Proof. ⬚

Theorem 6.5.2.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the abelian category $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives.*

Write the short
of $\mathbf{Mod}(\mathcal{O}_X)$ be
abelian, Chapter

6.6 Torsors and 1$^{\text{st}}$-Cech cohomology group

Once we have understood the constructions of the last section, we can now start doing some serious geometry over our manifolds. Indeed, this is what we start laying out in this section.

6.7 Bundles

We give here the general theory of fiber, principal and vector bundles. When the need arises, we will instantiate this into different areas (like in the chapter on differential geometry). The material in previous chapter will allow a very united way of looking at the notion of bundles, and will start portraying the intimate connection that bundles and cohomology has.

6.8   Homology, cohomology and properties

Give a remark of viewing singular cohomology as ES-axioms and also for manifolds as sheaf cohomology.

We will begin by introducing (co)homology from an axiomatic point of view and will derive few properties off of it. This will come in handy for discussing the main properties of differential manifolds in (co)homological language, especially characteristic classes and orientations and what not. The main thing that we wish to do is the Hurewicz theorem, which will allow us to connect homotopy groups and homology groups on the one hand, and will allow us to prove the uniqueness of homology theories for CW complexes on the other hand.

All spaces X are assumed to be compactly generated, that is, X is a weak Hausdorff (compact subspaces are closed) k-space (compactly closed subspaces are closed).

We will use the theory of cofibrations and fibrations as developed in Chapter 8 quite freely.

6.8.1   Axioms of a homology theory

Consider the category $\mathbf{Top}_2$ of pairs (X, A) of spaces where $A \hookrightarrow X$ and maps $(X, A) \to (Y, B)$ which consists of the pair $f : X \to Y$ and $g : A \to B$ such that the resulting square commutes.

Definition 6.8.1.1. (Homology theory) A homology theory for a group π is a sequence of functors

$$H_q(-, -; \pi) : \mathbf{Top}_2 \longrightarrow \mathbf{AbGrp}$$

for $q \in \mathbb{Z}$ equipped with natural transformations

$$\partial : H_q(-, -; \pi) \longrightarrow H_{q-1}(-, -; \pi)$$

whose component at (X, A) is given by $\partial : H_q(X, A; \pi) \to H_{q-1}(A, \emptyset; \pi)$. Denote $H_q(X; \pi) := H_q(X, \emptyset; \pi)$. This data must satisfy the following axioms:

1. *(Homology of a point)* : If $X = \{\text{pt.}\}$, then homology must be concentrated at degree 0:

$$H_q(\{\text{pt.}\}; \pi) = \begin{cases} \pi & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

2. *(Homology long exact sequence)* : The trivial inclusions $A \hookrightarrow X$ and $(X, \emptyset) \hookrightarrow (X, A)$

induces the following long exact sequence:

$$
\begin{array}{ccc}
H_q(A;\pi) \xleftarrow{\hspace{1cm}} H_q(X;\pi) \longrightarrow H_q(X,A;\pi) \\
H_{q-1}(A;\pi) \xleftarrow{\hspace{1cm}} H_{q-1}(X;\pi) \longrightarrow H_{q-1}(X,A;\pi)
\end{array}
$$

3. *(Excision invariance)* : For an excisive triple (X, A, B), that is $A, B \hookrightarrow X$ and $X = A^\circ \cup B^\circ$, the inclusion $(X, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism at all degree $q \in \mathbb{Z}$:
$$
H_q(X, A \cap B; \pi) \xrightarrow{\cong} H_q(X, B; \pi) \ .
$$

4. *(Coproduct preserving)* : If (X_i, A_i) is an arbitrary collection of objects in $\mathbf{Top}_2$, then the homology in any degree of their disjoint union is the sum of the corresponding homology groups:
$$
H_q\left(\coprod_i (X_i, A_i); \pi\right) \xrightarrow{\cong} \bigoplus_i H_q(X_i, A_i; \pi).
$$

5. *(π_*-insensitivity)* : If $f : (X, A) \longrightarrow (Y, B)$ is a weak equivalence, then in all degrees the corresponding homology groups are isomorphic:
$$
f_p : H_q(X, A; \pi) \xrightarrow{\cong} H_q(Y, B; \pi).
$$

Remark 6.8.1.2. In nature, there are some homology theories which satisfy all of the above axioms except the dimension axiom, that is, the group that they assign to a point is not concentrated in degree 0 (axiom 1. above). A famous example of this is K-theory via the Bott-periodicity theorem. One calls such a homology theory to be a *generalized homology theory*. All results that we will derive here will hold true for a generalized homology theory E_q.

6.8.2 Reduced homology

For each homology theory $E_q(-, -)$, we can construct a based version of the theory denoted $\tilde{E}_q(-, \text{pt.})$. For a based space $(X, \text{pt.})$, define the following
$$
\tilde{E}_q(X) := E_q(X, \text{pt.}).
$$

Let us spell out some basic relations of this reduced homology $\tilde{E}_q$ to that of original homology E_q.

Lemma 6.8.2.1. *Let π be a group and E_q be a homology theory. Let $(X, \text{pt.})$ be a based space and $(A, \text{pt.}) \hookrightarrow (X, \text{pt.})$ be a based subspace. Then the reduced homology theory $\tilde{E}_q$ satisfies the following:*

1. $E_q(X, X) \cong E_q(X)$.
2. If $A \hookrightarrow X \xrightarrow{r} A$ is a retract of X, then the map $\iota_* : E_q(A) \to E_q(X)$ is injective and $r_* : E_q(X) \to E_q(A)$ is surjective.
3. If $A \hookrightarrow X \xrightarrow{r} A$ is a retract of X, then the following natural maps form a short-exact sequence of E-homology groups:

$$0 \to E_q(A) \to E_q(X) \to E_q(X, A) \to 0.$$

4. $E_q(X) = \tilde{E}_q(X) \oplus E_q(\mathrm{pt.})$.
5. There is a long exact sequence

Proof. 1. The pair $(X, X, \emptyset)$ forms an excisive triad.

2. This is a set theoretic observation; any factorization of identity is a monic followed by an epic.

3. By homology long-exact sequence, we reduce to showing the injectivity of $E_q(A) \to E_q(X)$ and surjectivity of $E_q(X) \to E_q(X, A)$. The former follows from the above. For the latter, we have $E_q(X, X) \cong E_q(X) \to E_q(X, A)$ and then we conclude by an application of 2.

4. This follows from the splitting lemma of groups. Note that the left map in the short-exact sequence of groups obtained in 3. is split by the retraction $r_* : E_q(X) \to E_q(A)$.

5. Consider $i : A \hookrightarrow X$. Then $E_q(A) \to E_q(X)$ takes $E_q(\mathrm{pt.})$ to $E_q(\mathrm{pt.})$ isomorphically. Hence we may quotient it out under the exactness to get the desired sequence. $\square$

6.9   Differential forms and de-Rham cohomology

Do this from Section 8.6 and Section 10.4 of Wedhorn, via sheaf cohomology. Add motivation from courses.

6.9.1   Differential forms on $\mathbb{R}^n$

We first discuss differential forms on $\mathbb{R}^n$ as this provides clear and sufficient motivation for the abstract treatment of differential forms in all other places where it is used. We begin by defining the main ingredients. The material of Section 18.4 is used in the following.

Definition 6.9.1.1. (Coordinate forms on $\mathbb{R}^n$) Fix $n \in \mathbb{N}$. Let $V = \mathbb{R}^n$ be the n-dimensional $\mathbb{R}$-module. The functional

$$dx_i : V \longrightarrow \mathbb{R}$$
$$(x_1, \ldots, x_n) \longmapsto x_i$$

is called the ith-coordinate form on V, for each $i = 1, \ldots, n$. Note that dx_i is a 1-form/1-tensor, i.e. $dx_i \in M^1(V) = V^*$. Observe that dx_i is the dual basis of V^* corresponding to standard basis e_i of V.

Next, we define a multilinear map which for each choices of axes, gives the volume of the parallelopiped obtained by the projection along those axes, given a parallelopiped spanned by some vectors.

Definition 6.9.1.2. (Projection forms on $\mathbb{R}^n$) Fix $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Let $V = \mathbb{R}^n$ be the n-dimensional $\mathbb{R}$-module. Let $I = (i_1, \ldots, i_k)$ be an ordered k-tuple where $1 \leq i_j \leq n$ for each $j = 1, \ldots, k$. Then, we define the I-projection form as

$$dx_I := \pi_k(dx_{i_1} \otimes \ldots \otimes dx_{i_k}) = D_I$$

which is an alternating k-form on V, that is $dx_I \in \Lambda^k(V)$ (see Example 18.4.3.11). More explicitly, it is given by the following k-linear form on V

$$dx_I : V \times \cdots \times V \longrightarrow \mathbb{R}$$
$$(v_1, \ldots, v_k) \longmapsto \det \begin{bmatrix} dx_{i_1}(v_1) & dx_{i_2}(v_1) & \ldots & dx_{i_k}(v_1) \\ dx_{i_1}(v_2) & dx_{i_2}(v_2) & \ldots & dx_{i_k}(v_2) \\ \vdots & \vdots & \ldots & \vdots \\ dx_{i_1}(v_k) & dx_{i_2}(v_k) & \ldots & dx_{i_k}(v_k) \end{bmatrix}.$$

Remark 6.9.1.3. Recall from Theorem 18.4.3.14 that $\Lambda^k(V)$ has basis given by dx_I for distinct increasing k-tuples from $1, \ldots, n$. Thus, $\{dx_I\}_I$ forms an $\mathbb{R}$-basis of $\Lambda^k(V)$ of size ^nC_k.

Remark 6.9.1.4. Recall that wedge product of forms is given by the following (where one defines them only on the basis elements)

$$\Lambda^k(V) \times \Lambda^l(V) \longrightarrow \Lambda^{k+l}(V)$$
$$(dx_I, dx_J) \longmapsto dx_I \wedge dx_J := dx_{(I,J)}$$

where recall that $dx_{(I,J)}$ will be zero if there is any index common in I and J (see Definition 18.4.4.1), where I, J are increasing tuples of indices from $\{1, \ldots, n\}$ of lengths k and l respectively. From the above, we see that for any alternating k-form $\omega = \sum_I a_I dx_I$ and alternating l-form $\eta = \sum_J b_j dx_J$, their wedge product is defined as

$$\omega \wedge \eta = \sum_J \sum_I a_I b_J (dx_I \wedge dx_J).$$

Remark 6.9.1.5. Let $U \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. Observe that $\mathcal{C}^\infty(U)$, the ring of smooth $\mathbb{R}$-valued functions on U, is an $\mathbb{R}$-algebra. In the same vein, we know that alternating k-forms $\Lambda^k(\mathbb{R}^n)$ forms an $\mathbb{R}$-vector space of dimension ${}^nC_k$ (see Theorem 18.4.3.14).

Definition 6.9.1.6. (**Differential k-forms**) Let $U \subseteq \mathbb{R}^n$ be an open set and $0 \leq k \leq n$. The module of differential k-forms is defined to be the following $\mathbb{R}$-vector space

$$\Omega_U^k = \Lambda^k(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathcal{C}^\infty(U).$$

As $\Lambda^k(\mathbb{R}^n)$ is a free $\mathbb{R}$-module with rank ${}^nC_k$, therefore Ω_U^k is a free $\mathcal{C}^\infty(U)$-module of rank ${}^nC_k$.

Remark 6.9.1.7. Observe that $\{\Omega_U^k\}$ obtains the wedge product structure from the wedge product on $\{\Lambda^k(\mathbb{R}^n)\}$ as we may define for $\omega = \sum_I f_I dx_I \in \Lambda^k(\mathbb{R}^n)$ and $\eta = \sum_J g_J dx_J$ the following

$$\omega \wedge \eta := \left(\sum_I f_I dx_I \right) \wedge \left(\sum_J g_J dx_J \right)$$
$$= \sum_I \sum_J f_I g_J dx_I \wedge dx_J.$$

Thus, $\bigoplus_{k \geq 0} \Omega_U^k$ forms a graded $\mathcal{C}^\infty(U)$-algebra.

Remark 6.9.1.8. An arbitrary element $\omega \in \Omega_U^k$ is called a differential k-form over U and is written as

$$\omega = \sum_{I \in X_k} f_I(x_1, \ldots, x_n) dx_I$$

where X_k is the set of size ${}^nC_k$ of all k-combinations in increasing order of $\{1, \ldots, n\}$ and $f_I \in \mathcal{C}^\infty(U)$ is a smooth function. Observe that $\Omega_U^0 = \mathcal{C}^\infty(U)$.

We now construct the exterior derivative which will be a differential over the chain complex Ω_U^k, as we will see soon.

Definition 6.9.1.9. (**Exterior derivative**) Let $U \subseteq \mathbb{R}^n$ be an open subset and $\{\Omega_U^k\}_{k \in \mathbb{N}}$ be the modules of differential k-forms. For each $k \in \mathbb{N} \cup \{0\}$, we define a map $d : \Omega_U^k \to \Omega^{k+1}{}_U$ as follows. Define for $k = 0$ the following

$$d : \Omega_U^0 \longrightarrow \Omega_U^1$$
$$f \longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

where since $f \in \mathcal{C}^\infty(U)$ is smooth, therefore so is $\partial f / \partial x_i$. Further, since $dx_i \in \Lambda^1(\mathbb{R}^n)$, therefore the above is well-defined. For $k \geq 1$, we define d as follows

$$d : \Omega_U^k \longrightarrow \Omega_U^{k+1}$$
$$\omega = \sum_{I \in X_k} f_I dx_I \longmapsto d\omega = \sum_{I \in X_k} df_I \wedge dx_I$$

where $dx_I \in \lambda^k(\mathbb{R}^n)$. Observe that $df_I \in \Omega_U^1$, thus indeed $df_I \wedge dx_I \in \Omega_U^{k+1}$. This map d is called the exterior derivative of differential forms.

The following are immediate but important properties of exterior derivative. **TODO**.

Chapter 7

Foundational Differential Geometry

Contents

The material of previous chapter has already gotten us very close to topics in geometry over C^∞-manifolds.

7.1   Bundles in differential geometry and applications

7.2   Fundamental group and covering maps

The following is an important example of a covering map.

Lemma 7.2.0.1. *The map $\varphi : S^1 \to S^1$ given by $z \mapsto z^n$ is a covering map.*

Proof. Pick any $z_0 = e^{i\theta_0} \in S^1$. We wish to show that there exists an open set $U_0 \ni z_0$ in S^1 such that

$$\varphi^{-1}(U_0) = \coprod_{k=0}^{n-1} V_k$$

where V_k are open in S^1 and $\varphi|_{V_k} : V_k \to U_0$ is a homeomorphism.

Denote by $\gamma : \mathbb{R} \to S^1$ the continuous surjective map given by $t \mapsto e^{it}$. Thus, $z_0 = \gamma(\theta_0)$. Consider the interval $I_0 = \left(\theta_0 - \frac{\pi}{n}, \theta_0 + \frac{\pi}{n}\right)$. As the map $\gamma : \mathbb{R} \to S^1$ is an open map, therefore we have $U_0 = \gamma(I_0)$ which is an open set of S^1 containing z_0. We claim that U_0 is

an evenly covered neighborhood for z_0. Indeed, we see that

$$
\begin{aligned}
\varphi^{-1}(U_0) &= \{z \in S^1 \mid z^n \in U_0\} \\
&= \left\{e^{i\theta} \in S^1 \mid e^{ni\theta} \in \gamma(I_0)\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \exists \kappa \in I_0 \text{ s.t. } \gamma(\kappa) = e^{i\kappa} = e^{ni\theta}\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \exists \kappa \in I_0 \text{ s.t. } n\theta = \kappa + 2k\pi, \text{ for some } k \in \mathbb{Z}\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \exists \kappa \in I_0 \text{ s.t. } \theta = \frac{\kappa}{n} + \frac{2\pi k}{n}, \text{ for some } k \in \mathbb{Z}\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \theta \in \coprod_{k \in \mathbb{Z}} \left(\frac{\theta_0}{n} - \frac{\pi}{n^2} + \frac{2\pi k}{n}, \frac{\theta_0}{n} + \frac{\pi}{n^2} + \frac{2\pi k}{n}\right)\right\} \\
&= \coprod_{k=0}^{n-1} \gamma\left(\left(\frac{\theta_0}{n} - \frac{\pi}{n^2} + \frac{2\pi k}{n}, \frac{\theta_0}{n} + \frac{\pi}{n^2} + \frac{2\pi k}{n}\right)\right).
\end{aligned}
$$

This completes the proof. $\qquad\square$

7.3 Cohomological methods

7.4 Covariant derivative, connections, classes and curvatures

Chapter 8

Foundational Homotopy Theory

Contents

In this chapter, we will work only in the category of compactly generated spaces, unless stated otherwise.

Do this chapter from May (most importantly some other sections on spectra, EM-spaces, cobordism...) and Bredon's last chapter.

8.1 Cofibrations and cofiber sequences

Most of the long exact sequences appearing in algebraic topology are derived from the topics that we will cover in this chapter. These should rather be seen as an important conceptual tool in order to do computations. We will begin with cofibrations, closed subspaces from whose homotopies can be extended to the whole space, and then fibrations, which can be

thought of as generalizations of covering spaces (more generally, fiber bundles) which one studies in a first course in algebraic topology.

Cofibrations can be treated as an intermediary tool for developing more sophisticated concepts in algebraic topology. In particular, we will be using this to derive an exact sequence of groups out of a map of based spaces.

Definition 8.1.0.1. (**Cofibrations**) A map $i : A \to X$ is a cofibration if it satisfies the homotopy extensions property, that is, if $f : X \to Y$ is a continuous map such that there is a homotopy $h : A \times I \to Y$ where $h(-, 0) = f \circ i$, then that homotopy can be lifted to $\tilde{h} : X \times I \to Y$ where $\tilde{h}(-, 0) = f$. More abstractly, if $h \circ i_0 = f \circ i$ in the following diagram, then there exists $\tilde{h}$ such that the following commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\quad i_0 \quad} & A \times I \\
{\scriptstyle i}\downarrow & {\scriptstyle i \times \text{id}}\searrow \quad X \times I & \downarrow{\scriptstyle h} \\
& {\scriptstyle i_0}\nearrow \quad\quad \dashrightarrow{\scriptstyle \tilde{h}} & \\
X & \xrightarrow[\quad f \quad]{} & Y
\end{array}
\quad .
$$

One sees that pushout of a cofibration along any map is a cofibration.

Lemma 8.1.0.2. *Let $i : A \to X$ be a cofibration and $f : A \to B$ be any other map. Then, the pushout $j : B \to B \cup_f X$ is a cofibration.*

Proof. Take any map $g : B \cup_f X \to Y$ and a homotopy $h : B \times I \to Y$ where $h \circ i_0 = g \circ j$. We have the following diagram:

$$
\begin{array}{ccccc}
A & & & & \\
 & \searrow{\scriptstyle f} & & & \\
 & & B & \xrightarrow{\quad i_0 \quad} & B \times I \\
 & & & {\scriptstyle j \times \text{id}}\searrow & \\
{\scriptstyle i \text{ (cofibration)}}\downarrow & \text{p.o.} \quad {\scriptstyle j}\downarrow & (B \cup_f X) \times I & & \downarrow{\scriptstyle h} \\
 & & {\scriptstyle i_0}\nearrow & & \\
 & B \cup_f X & \xrightarrow[\quad g \quad]{} & & Y \\
 & \nearrow & & & \\
X & & & &
\end{array}
\quad .
$$

We wish to show that there is a map $\tilde{h} : (B \cup_f X) \times I \to Y$ which commutes with the diagram shown above. Since we have the following pushout square:

$$
\begin{array}{ccc}
B \cup_f X & \longleftarrow & X \\
{\scriptstyle j}\uparrow & \text{p.o.} & \uparrow{\scriptstyle i} \\
B & \xleftarrow[\ f\ ]{} & A
\end{array}
\quad ,
$$

therefore after applying functor $- \times I$, which has a right adjoint, so is colimit preserving (we are working in the category of compactly generated spaces which is cartesian closed), we get the following pushout square which is closer to what we have in the first diagram:

$$
\begin{array}{ccc}
(B \cup_f X) \times I & \longleftarrow & X \times I \\
{\scriptstyle j \times \mathrm{id}} \uparrow & \text{p.o.} & \uparrow {\scriptstyle i \times \mathrm{id}} \cdot \\
B \times I & \underleftarrow{\ f \times \mathrm{id}\ } & A \times I
\end{array}
$$

Now, we get a map h' as below by the virtue of i being a cofibration:

Next, by the universal property of pushout $(B \cup_f X) \times I$, we get a map $\tilde{h}$

which satisfies the required commutativity. □

8.1.1 Reduction to mapping cylinder

To check that a map $i : A \to X$ is a cofibration, we can reduce to checking the homotopy extension property of maps of the form $f : X \to Mi$ where Mi is the *mapping cylinder*,

defined as the pushout $Mi := X \cup_i (A \times I)$:

$$
\begin{array}{ccc}
Mi & \longleftarrow & A \times I \\
\uparrow & \llcorner & \uparrow{\scriptstyle i_0} \\
X & \longleftarrow_{\, i} & A
\end{array}
\quad .
$$

More pictorially, Mi is formed by gluing cylinder $A \times I$ to X along A. In mind, one pictures a cylinder "popping out" of X from where A lived in X. Indeed, we have the following lemma:

Lemma 8.1.1.1. *Let $i : A \to X$ be a map. Then the following are equivalent:*
  1. *i is a cofibration.*
  2. *i satisfies homotopy extension property for any $f : X \to Y$ and for any Y.*
  3. *i satisfies homotopy extension property for any $f : X \to Mi$.*

Proof. The only non-trivial part is to show $3 \Rightarrow 2$. Take any map $f : X \to Y$ and any homotopy $h : A \times I \to Y$. Consider

The map h_1 is formed by homotopy extension property of i for $X \to Mi$ and g is formed by universal property of pushout which is Mi. The map $gh_1 : X \times I \to Y$ follows the required commutativity relations. $\qquad\square$

Construction 8.1.1.2. (*Suspension-loop space adjunction*) Consider the category of compactly generated spaces based $\mathbf{Top}_*$. Denote by ΣX the *reduced suspension* as the smash product $X \wedge S^1$ and ΩX as the *loop space* $\mathrm{Hom}_{\mathbf{Top}_*}\left(S^1, X\right)$. Consider the map

$$
\mathrm{Hom}_{\mathbf{Top}_*}\left(\Sigma X, Y\right) \longrightarrow \mathrm{Hom}_{\mathbf{Top}_*}\left(X, \Omega Y\right)
$$
$$
f : \Sigma X \to Y \longmapsto \tilde{f} : x \mapsto (t \mapsto f([(x,t)])).
$$

This has the inverse given by $\tilde{f} \mapsto f$ where f maps as $[(x,t)] \mapsto \tilde{f}(x)(t)$. One can check that these are well-defined and are indeed inverses of each other. This is the famous and important $\Sigma - \Omega$ adjunction.

Now, we notice that the $[\Sigma X, Y] := \pi_0(\mathrm{Hom}_{\mathbf{Top}_*}\left(\Sigma X, Y\right))$, which is just the set of path-components of $\mathrm{Hom}_{\mathbf{Top}_*}\left(\Sigma X, Y\right)$, has a group structure obtained by the above isomorphism defined as follows: for $f, g \in \mathrm{Hom}_{\mathbf{Top}_*}\left(\Sigma X, Y\right) \cong \mathrm{Hom}_{\mathbf{Top}_*}\left(X, \Omega Y\right)$, define $f + g$ as

$$
(f + g)([(x,t)]) := \begin{cases} f([(x, 2t)]) & \text{if } 0 \leq t \leq 1/2 \\ g([x, 2t - 1]) & \text{if } 1/2 \leq t \leq 1. \end{cases}
$$

This tells us that $[\Sigma X, Y] \cong [X, \Omega Y]$ is a group.

8.1.2 Replacement by cofibration

Let $f : X \to Y$ be an arbitrary map of spaces. We can replace f by a cofibration followed by a weak-equivalence.

Construction 8.1.2.1. Let $f : X \to Y$ be a map of spaces. Consider the following commutative triangle:

$$
\begin{array}{ccc}
 & Mf & \\
{}^{j}\nearrow & & \searrow^{r} \\
X & \xrightarrow{\ \ \ f\ \ \ } & Y
\end{array}
$$

where $Mf = Y \cup_f (X \times I)$ is the mapping cylinder and the other two maps are given by $j(x) = (x, 1)$ and $r|_Y = \mathrm{id}_Y$ and $r|_{X \times I}(x, t) = f(x)$ for $t > 0$. We first see that $j : X \to Mf$ is a cofibration. Indeed, if $g : Mf \to Z$ is any map and we have a diagram as in Definition 8.1.0.1, then we can form the required homotopy $\tilde{h} : Mf \times I \to Z$ by defining

$$
\tilde{h}([(x, s)], t) := \begin{cases} x & \text{if } x \in Y \\ h(x, st) & \text{if } [(x, s)] \in X \times I. \end{cases}
$$

We then see that $\tilde{h}(j \times \mathrm{id})(x, t) = \tilde{h}([(x, 1)], t) = h(x, t)$ and that $\tilde{h}i_0([(x, s)]) = \tilde{h}([(x, s)], 0) = h(x, 0) = g(x)$. So we have the required extension and hence $j : X \to Mf$ is a cofibration. Next, we wish to see that $r : Mf \to Y$ is a homotopy equivalence. For this, we have a map $i : Y \to Mf$ taking $y \mapsto [y]$. We then see that $ri = \mathrm{id}_Y$ and $ir \simeq \mathrm{id}_{Mf}$. The former is simple and the latter is established by the following homotopy $h : Mf \times I \to Mf$ mapping as $([(x, s)], t) \mapsto [(x, st)]$. This is indeed a homotopy from ir to id_{Mf}. Thus, $r : Mf \to Y$ establishes that Y is a deformation retract of the mapping cylinder Mf.

Hence, one can replace a map of spaces $f : X \to Y$ by a cofibration $j : X \to Mf$ followed by a homotopy equivalence $r : Mf \to Y$.

8.1.3 Based cofiber sequences

The main point of cofiber sequences is to obtain an exact sequence of groups, which will prove to be helpful later.

Definition 8.1.3.1. (Homotopy cofiber/Mapping cone) Let $f : X \to Y$ be a based map and let $j : X \to Mf$ be it's cofibrant replacement. The homotopy cofiber Cf of f is defined to be the quotient of the mapping cylinder Mf of f by the image of the map j taking $x \mapsto (x, 1)$. That is,

$$
Cf := Mf/j(X).
$$

Pictorially, one sees that the mapping cone Cf of $f : X \to Y$ is obtained by gluing Y to the cone of X at it's base. We are now ready to construct cofiber sequence of a based map $f : X \to Y$.

Construction 8.1.3.2. (*Cofiber sequence*) Let $f : X \to Y$ be a based map and denote Cf to be the mapping cone of f. We have a natural map $i : Y \to Cf$ which is the

inclusion of Y into the mapping cone. This is a cofibration because it is the pushout (Lemma 8.1.0.2) of the inclusion $X \to CX$ of X into the 0-th level of the cone CX and this inclusion is a cofibration. Consider also the map $-\Sigma f : \Sigma X \to \Sigma Y$ which maps $[(x,t)] \mapsto [(f(x), 1-t)]$. We have another natural map from the mapping cone to its quotient by Y given by $\pi : Cf \to Cf/Y \cong \Sigma X$. We then get the following sequence of spaces, called the cofiber sequence of map f:

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma Cf \xrightarrow{-\Sigma \pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 i} \Sigma^2 Cf \to \cdots .$$

8.2   Fibrations and fiber sequences

8.3   Higher homotopy groups

8.4   CW-complexes : Basic theory

One of the important properties of compactly generated spaces is that any such space can be approximated upto homotopy by a class of spaces constructed in a rather simple manner. These are precisely the CW complexes. Once the above approximation theorems are set up, we can safely reduce a lot of computation in homology to such a CW-approximation. Moreover, the reductions run so deep that in-fact any homology theory E_q on general compactly generated spaces necessarily induces and comes from the restriction of E_q to CW-complexes. An application of Hurewicz theorem will then tell us that upto natural isomorphism, there is a unique homology theory over CW-complexes.

8.5   CW-complexes : Approximation theorems

8.6   CW-complexes : Homotopy theory

8.7   Homotopy groups of Lie groups

8.8   Homotopy and homology : Hurewicz's theorem

8.9   Homotopy and homology : Whitehead's theorem

8.10   Eilenberg-Maclane spaces

8.11   Spectra

Spectra are objects which generalizes both the notion of cohomology theories and spaces, in that there are mappings from cohomology theories and spaces into the homotopy category of spectra. Thus, one needs to construct a good category of spectra, give a model structure on it and thus by Quillen's theory obtain this absolutely wonderful homotopy category of spectra, which unites the viewpoint of cohomology and spaces. However, we are getting

ahead of ourselves, as finding the right homotopy category and giving a construction of category of spectra is easier said than done. We will meet this topic later in our discussion of ∞-categories (they will form a prototypical example of stable ∞-categories).

8.12 Lifting problems and obstructions

Chapter 9

Classical Ordinary Differential Equations

Contents

We will prove some basic existence/uniqueness results about ODEs here, with a classical/analytic viewpoint in mind. Let us first begin by stating what is meant by an *initial value problem* and what is meant by *solving an initial value problem*. A main focus will be on doing analytical proofs, which is always extremely helpful. In particular, we will see how much weird and pathological behaviors can emerge after *passing to limit*, thus justifying why commuting with limits is a sought after property in all over analysis.

9.1   Initial value problems

Let us begin by understanding what is meant by a differential equation. Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set. Consider a continuous function $f : D \to \mathbb{R}^n$ mapping as $(t, x) \mapsto f(t, x)$ where $t \in \mathbb{R}, x \in \mathbb{R}^n$. A fundamental goal that one wishes to achieve is to find a "nice" function $x : I \subseteq \mathbb{R} \to D$ such that the function f can be known upto first derivatives, that is, we want to construct such a function $x : I \to \mathbb{R}^n$ such that it can tell us the following about f:

1. (*Correct domain*)$\forall t \in I$, we shall have $(t, x(t)) \in D$,
2. (*Differential equation*)$\frac{dx}{dt}(t_0) = f(t_0, x(t_0)), \forall t_0 \in I$. That is, the first derivative of x can give us exactly the values that f takes on D.

To find such a function x, the main difficulty is the condition 2 above, for this requires $x : I \to \mathbb{R}^n$ to be continuously differentiable (so of class C^1) and that we necessarily have to construct a function x by the knowledge only of it's first derivative (which is $f(t, x)$).

This problem of constructing a C^1 map $x : I \subseteq \mathbb{R} \to \mathbb{R}^n$ from only the data of it's continuous first derivative is called the process of solving a differential equation.

Clearly, many C^1 maps can have same first derivative (we need only add a scalar in front). So the uniqueness of the above problem is hopeless. However, one can add an extra data to the problem above that x shall satisfy and then we do get uniqueness at times. In particular, we demand the following from x:

3. (*Initial value*) for some fixed $s_0 \in I$ and $x_0 \in \mathbb{R}^n$, we require $x(s_0) = x_0$.

We then define an initial value problem (IVP) as follows:

Definition 9.1.0.1. (IVP & solutions) Let $f : D \to \mathbb{R}^n$ be a continuous map on an open set $D \subseteq \mathbb{R} \times \mathbb{R}^n$. An IVP is a construction problem where from the tuple of data $(f, (t_0, x_0))$ for some $(t_0, x_0) \in D$, we have to construct the following:

1. an interval $I \subseteq \mathbb{R}$ containing t_0,
2. a function $x : I \to \mathbb{R}^n$.

This function x should then satisfy the following:

1. $(t, x(t)) \in D \; \forall t \in I$,
2. $\frac{dx}{dt}(t) = f(t, x) \; \forall t \in I$,
3. $x(t_0) = x_0$.

We identify the above IVP with the tuple $(f, (t_0, x_0))$. If such a function $x : I \to \mathbb{R}^n$ exists, then it is called a *solution to the IVP* $(f, (t_0, x_0))$.

9.1.1   Existence: Peano's theorem

We have an elementary result which tells us that, if the solution exists, then what should be its form.

Lemma 9.1.1.1. *Let $f : D \to \mathbb{R}^n$ be a continuous map and $(f, (t_0, x_0))$ be an IVP. Then, a continuous map $x : I \to \mathbb{R}^n$ is a solution to the IVP $(f, (t_0, x_0))$ if and only if $\forall t \in I$, $x(t)$ is the following line integral*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Proof. (L $\Rightarrow$ R) Since x is a solution, therefore $\frac{dx}{dt}(t) = f(t, x(t))\ \forall t \in I$ and $t_0 \in I$. Then use fundamental theorem of calculus to calculate the line integral of the vector field ∇x along the line joining t_0 and t.

(R $\Rightarrow$ L) By continuity of x, we get that $t \mapsto f(t, x(t))$ is continuous. Since $x(t_0) = x_0$, therefore by continuity of $t \mapsto (t, x(t))$, there exists an open interval $I \ni t_0$ of $\mathbb{R}$ such that $(t, x(t)) \in D$ for all $t \in I$. It then follows by an application of fundamental theorem of calculus that $\frac{dx}{dt}(t) = f(t, x(t))$ for each $t \in I$. $\qquad\qquad\qquad\qquad\qquad$ $\square$

We next do Peano's theorem, which tells us that indeed solutions exists. This when combined with above tells us that solutions to IVP $(f, (t_0, x_0))$ exists and is of same "form". However, it will require a classic result in analysis called Arzela-Ascoli theorem. Let us do that first.

Theorem 9.1.1.2. *(Arzela-Ascoli theorem) Let $x_n : [0, 1] \to \mathbb{R}^n$ be a sequence of continuous functions such that $\{x_n\}$ is a uniformly bounded and equicontinuous family of maps. Then there exists a subsequence of $\{x_n\}$ which is uniformly convergent.*

Proof. We need to define a function on whole of $[0, 1]$. Since $\mathbb{Q} \cap [0, 1]$ is dense, so we might as well define it only for the rational points and then pass to the limit to get a function over whole of $[0, 1]$. This will be the main idea of the proof. $\qquad\qquad$ $\square$

We can now approach the existence result.

Theorem 9.1.1.3. *(Peano's theorem) Let $f : D \to \mathbb{R}^n$ be a continuous map where $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is open and let $(t_0, x_0) \in D$ so that the tuple $(f, (t_0, x_0))$ forms an IVP. Choose $r > 0$ and $c > 0$ such that $[t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \subseteq D$[1]. Then, denoting $M := \max_{x \in [t_0 - c, t_0 + c] \times \overline{B}_r(x_0)} f(x)$ and $h := \min\{c, \frac{r}{M}\}$, there exists a solution to the IVP $(f, (t_0, x_0))$ given by*

$$x : [t_0 - h, t_0 + h] \longrightarrow \overline{B}_r(x_0).$$

Proof. We will construct the solution x in a limiting manner. First, we may replace t_0 by 0 as we can shift the solution to t_0 thus obtained. Second, we may define x on $[0, h]$ as we may translate and scale the solution as desired. Now, consider the sequence of functions defined as follows:

$$x_n(t) : [0, h] \longrightarrow \mathbb{R}^n$$

$$t \longmapsto \begin{cases} x_0 \text{ if } t \in [0, \frac{h}{n}], \\ x_0 + \int_0^{t - h/n} f(s, x_n(s))ds & \text{if } t \in [\frac{h}{n}, h]. \end{cases}$$

So we obtain a sequence of functions $\{x_n\}$ defined over $[0, h]$. Now, in the limiting case, we will have a function exactly of the form required by Lemma 9.1.1.1, so we reduce to showing that a subsequence of the above converges and converges to a continuous function. We will use the Arzela-Ascoli (Theorem 9.1.1.2) for showing this. We thus need only show that the

---

[1]That is, choose a basic closed set around (t_0, x_0) in D.

sequence $\{x_n\}$ is uniformly bounded and equicontinuous. For uniform boundedness, we will simply show that $x_n(t) \in \overline{B}_r(x_0) \ \forall t \in [0, h]$. This follows from the following:

$$
\begin{aligned}
|x_n(t) - x_0| &\leq \left| \int_0^{t-h/n} f(s, x_n(s))ds \right| \\
&\leq \int_0^{t-h/n} |f(s, x_n(s))| \, ds \\
&\leq M\left(t - \frac{h}{n}\right) \\
&\leq Mh \\
&\leq r.
\end{aligned}
$$

Finally, to see equicontinuity, we may simply observe that for any $\epsilon > 0$ and for any $n \in \mathbb{N}$,

$$
\begin{aligned}
|x_n(s) - x_n(t)| &\leq \left| \int_{t-h/n}^{s-h/n} f(u, x_n(u)) \right| du \\
&\leq \int_{t-h/n}^{s-h/n} |f(u, x_n(u))| \, du \\
&\leq M(s - t).
\end{aligned}
$$

This shows equicontinuity. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 9.1.1.4. (*Comments on proof of Theorem 9.1.1.3*) The main idea of the proof was to *find the required function through a limiting procedure*, where to make sure that we do get the limit, we used Arzela-Ascoli. One of the foremost things we did as well was to *reduce to the nicest possible setting*, which will be very necessary to clear things around.

9.1.2   Uniqueness: Picard-Lindelöf theorem

We will now show that for an IVP $(f, (t_0, x_0))$, we may get unique solutions provided some hypotheses on f. In order to understand what this hypothesis on f is, we need to review Lipschitz and contractive functions.

Definition 9.1.2.1. ((locally)**Lipschitz functions**) A map $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function if $\exists L > 0$ such that $\forall \, x, y \in R$, we have

$$
\|f(x) - f(y)\| < L\|x - y\|.
$$

The function f is called locally Lipschitz if $\forall x \in E$, there exists $r > 0$ such that $f|_{B_r(x)}$ is a Lipschitz map.

Example 9.1.2.2. The map $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^{1/3}$ is not locally Lipschitz at $x = 0$. This is because if it is so, then $\exists \epsilon > 0$ such that on $B_\epsilon(0)$ the map f is Lipschitz. But for $x, y \in B_\epsilon(0)$ we get

$$
\begin{aligned}
|x - y| &= \left| (x^{1/3})^3 - (y^{1/3})^3 \right| \\
&= \left| (x^{1/3} - y^{1/3})(x^{2/3} + y^{2/3} + (xy)^{1/3}) \right| \\
&\leq 2\epsilon.
\end{aligned}
$$

Thus,

$$\left|x^{1/3} - y^{1/3}\right| \leq \frac{2\epsilon}{\left|x^{2/3} + y^{2/3} + (xy)^{1/3}\right|},$$

which shows that f can not be Lipschitz on $B_\epsilon(0)$.

We have that all continuously differentiable maps are locally Lipschitz.

Lemma 9.1.2.3. *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a C^1-map on an open set E, then f is locally Lipschitz.*

Proof. Take $a \in E$. By Theorem **??**, 2, we reduce to showing that there exists a $\epsilon > 0$ such that $\overline{B}_\epsilon(0) \subset E$ so that the continuous map $Df : E \to L(\mathbb{R}^n, \mathbb{R}^m)$ achieves maxima on the compact set. This follows from the fact that E is open. $\qquad\square$

One definition that we will need is that of uniform Lipschitz.

Definition 9.1.2.4. (**Uniform Lipschitz**) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Then f is called uniformly Lipschitz w.r.t. x if there exists $L > 0$ such that

$$\|f(t, x) - f(t, y)\| < L\|x - y\|$$

for all $(t, x), (t, y) \in D$.

A contraction is defined in an obvious manner.

Definition 9.1.2.5. (**Contractive mappings**) Let $f : X \to X$ be a continuous map of metric spaces. Then f is said to be contractive if there exists $0 < \lambda < 1$ such that

$$d(f(x), f(y)) < \lambda d(x, y)$$

for all $x, y \in X$.

Our goal is to find the conditions that one must impose on f for the IVP $(f, (t_0, x_0))$ to have a unique solution. This means we need to find a solution $x : I \to \mathbb{R}^n$ in such a manner that x is the unique solution possible on that interval I. Now a place uniqueness comes into the picture is Banach fixed point theorem. Indeed, we will use it to find such an interval I and map x so that it would be unique for the said IVP.

Theorem 9.1.2.6. *(Banach fixed point theorem) Let X be a complete metric space and $f : X \to X$ be a contractive mapping. Then, f has a unique fixed point.*

Proof. We will first show the existence of such a fixed point. There is an obvious process of doing so. Take any point $x_0 \in X$. We then form the sequence $\{x_n\}$ in X where $x_n = f^n(x_0)$.

We claim that $\{x_n\}$ is Cauchy. Indeed, we have that for any $\epsilon > 0$ (we may take $n \geq m$):

$$
\begin{aligned}
d(x_n, x_m) &= d(f^n(x_0), f^m(x_0)) \\
&< \lambda^m d(f^{n-m}(x_0), x_0) \\
&< \lambda^m \left(d(f^{n-m}(x_0), f(x_0)) + d(f(x_0), x_0) \right) \\
&< \lambda^m \left(\lambda d(f^{n-m-1}(x_0), x_0) + d(x_1, x_0) \right) \\
&= \lambda^{m+1} d(f^{n-m-1}(x_0), x_0) + \lambda^m d(x_1, x_0) \\
&< d(x_1, x_0) \left(\lambda^m + \cdots + \lambda^n \right) \\
&= \lambda^m \frac{1 - \lambda^{n-m}}{1 - \lambda} d(x_1, x_0) \\
&< \frac{\lambda^m}{1 - \lambda} d(x_1, x_0).
\end{aligned}
$$

Next, by completeness of X, we have that there exists $x = \varinjlim_n x_n$ in X. Now, $f(x) = f(\varinjlim_n x_n) = \varinjlim_n f(x_n)$ by continuity and $\varinjlim_n f(x_n) = x$ by definition of x_n. The uniqueness is simple by contractive property of f. $\square$

We now come to the main result, the uniqueness of solutions of IVP. Before stating it, let us state how we will be proving it, using the following bijection between solutions of $(f, (t_0, x_0))$ and fixed points of certain mapping.

Construction 9.1.2.7. Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping where D is open and let $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Now consider the following space for some $c > 0$

$$
X := C^1 \left[[t_0 - c, t_0 + c], \mathbb{R}^n \right]
$$

and consider the following map

$$
T : X \longrightarrow X
$$

$$
x(t) \longmapsto T(x)(t) := x_0 + \int_{t_0}^t f(s, x(s)) ds.
$$

Then, by Lemma 9.1.1.1, we see that $x(t) \in X$ is a solution of $(f, (t_0, x_0))$ if and only if $T(x(t)) = x(t)$. Hence

$$
\{Solutions \ of \ IVP \ (f, (t_0, x_0))\} \cong \{Fixed \ points \ of \ T : X \to X\}.
$$

Theorem 9.1.2.8. *(Weak Picard-Lindelöf) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ such that $(f, (t_0, x_0))$ forms an IVP. Choose $c > 0$ and $r > 0$ such that $[t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \subseteq D$. Denote $M := \max_{x \in [t_0-c, t_0+c] \times \overline{B}_r(x_0)} f(x)$. If the map*

$$
f : [t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \longrightarrow \mathbb{R}^n
$$

is uniformly Lipschitz w.r.t. x and Lipschitz constant being L, then, denoting $h := \min\{c, \frac{r}{M}, \frac{1}{L}\}$, there exists a unique solution of IVP $(f, (t_0, x_0))$ given by

$$
x : [t_0 - h, t_0 + h] \longrightarrow \overline{B}_r(x_0).
$$

Proof. The main part of the proof will be the idea in Construction 9.1.2.7 and Banach fixed point theorem. Let X denote the following space

$$X := \left\{ y \in C^0\left[[t_0 - h, t_0 + h], \mathbb{R}^n\right] \mid y(t_0) = x_0 \ \& \ \sup_{x \in [t_0 - h, t_0 + h]} \|x_0 - y(t_0)\| \leq hM \right\}.$$

Consider the following function on X

$$T : X \longrightarrow X$$

$$y \longmapsto x_0 + \int_{t_0}^{t} f(s, y(s))ds.$$

By Theorem 9.1.2.6, we reduce to showing that function X is complete and T is a contraction mapping. Let us first show completeness of X. We will show that $X \hookrightarrow C\left[[t_0 - h, t_0 + h], \mathbb{R}^n\right]$ is a closed subspace and it will suffice since $C\left[[t_0 - h, t_0 + h], \mathbb{R}^n\right]$ is complete and closed subspaces of complete spaces are complete. . $\qquad\qquad\qquad\qquad\qquad$ □ | Complete the proof of Theorem 9.1. Chapter 9.

We will now prove Picard-Lindelöf again but with a weakening of hypotheses as compared to Theorem 9.1.2.8. This is important because most of the time one doesn't has the information of Lipschitz constant L as is required in Theorem 9.1.2.8 while constructing the interval of the solution.

Lemma 9.1.2.9. *Something about Picard iterates: If f is Lipschitz with constant $L > 0$, then the Picard iterates $\{x_n(t)\}$ satisfies*

$$\|x_{n+1}(t) - x_n(t)\| \leq \frac{ML^n(t - t_0)^{n+1}}{(n+1)!}.$$

Theorem 9.1.2.10. *(Strong Picard-Lindelöf) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map on an open set D and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Choose $c > 0$ and $r > 0$ such that $[t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \subseteq D$. Denote $M := \max_{x \in [t_0 - c, t_0 + c] \times \overline{B}_r(x_0)} f(x)$. If the map*

$$f : [t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \longrightarrow \mathbb{R}^n$$

is uniformly Lipschitz w.r.t. x, then, for any $h < \min\{c, \frac{r}{M}\}$, there exists a unique solution of IVP $(f, (t_0, x_0))$ given by

$$x : [t_0 - h, t_0 + h] \longrightarrow \overline{B}_r(x_0).$$

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □ | Start the proof Theorem 9.1.2.1 Chapter 9.

The following corollary tells us an alternate sufficient condition on f for the existence of unique solution to an IVP on f.

Corollary 9.1.2.11. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous map where D is open. If $\frac{\partial f_i}{\partial x_j} : D \to \mathbb{R}$ are continuous maps for all $1 \leq i, j \leq n$, then for each $(t_0, x_0) \in D$ there exists an open neighborhood around $(t_0, x_0) \in D$ in which there is a unique solution to IVP $(f, (t_0, x_0))$.*

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □ | Start the proof Corollary 9.1.2.1 Chapter 9.

Remark 9.1.2.12. In practice, to reduce to an open neighborhood where the solution is unique, the above corollary will be useful.

9.1.3   Continuation of solutions

Consider the map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(t, x) \mapsto x^2$ and $(0, 1) \in \mathbb{R} \times \mathbb{R}$. One sees that the IVP $(f, (0, 1))$ has a solution given by

$$x : (-1, 1) \longrightarrow \mathbb{R}$$

$$t \longmapsto \frac{1}{1 - t}.$$

However, this solution can be "extended"/"continued" to the following solution of the said IVP

$$y : (-\infty, 1) \longrightarrow \mathbb{R}$$

$$t \longmapsto \frac{1}{1 - t}.$$

These two are different solutions but the domain of one is inside the domain of the other. This concept of solutions extending from one domain to a larger domain will be investigated in this section.

The following definition is obvious.

Definition 9.1.3.1. (Continuation of solutions) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : I \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$. Then the solution x is said to be continuable if there exists a solution y of $(f, (t_0, x_0))$ given by $y : J \to \mathbb{R}^n$ where $J \supseteq I$ and $y|_I = x$.

The following theorem tells us a sufficient criterion on the solution which would make it continuable to some larger interval.

Theorem 9.1.3.2. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : (a, b) \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$.*
  1. *If $\overrightarrow{\lim}_{t \to b^-} x(t)$ exists and $(b, \overrightarrow{\lim}_{t \to b^-} x(t)) \in D$, then there exists $\epsilon > 0$ such that x can be continued to a solution $\tilde{x} : (a, b + \epsilon) \to \mathbb{R}^n$.*
  2. *If $\overrightarrow{\lim}_{t \to a^+} x(t)$ exists and $(a, \overrightarrow{\lim}_{t \to a^+} x(t)) \in D$, then there exists $\epsilon > 0$ such that x can be continued to a solution $\tilde{x} : (a - \epsilon, b) \to \mathbb{R}^n$.*

Proof.   □

The following lemma states that for mild conditions on f, the boundary limits might exist for a solution.

Lemma 9.1.3.3. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. If f is bounded, then for any solution $x : (a, b) \to \mathbb{R}^n$, the limits*

$$\overrightarrow{\lim_{t \to b^-}} x(t) \ \& \ \overrightarrow{\lim_{t \to a^+}} x(t) \ exist.$$

Proof. Use Lemma 9.1.1.1 to get that x is uniformly continuous over (a, b), so it has unique extension to its boundary.   □

9.1.4 Maximal interval of solutions

Let $(f, (t_0, x_0))$ be an IVP and let $x : I \to \mathbb{R}^n$ be a solution. A natural question is whether there is a "maximal continuation" of x in the sense of Definition 9.1.3.1. This is what we investigate here. The following definition is clear.

Definition 9.1.4.1. (Maximal interval of solution) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. The maximal interval of solution x is an interval $J \subseteq \mathbb{R}$ such that there exists a continuation of x on J and there is no continuation of $z : L \to \mathbb{R}^n$ of y where $L \supsetneq J$.

Lemma 9.1.4.2. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. If $x : I \to \mathbb{R}^n$ is a solution of $(f, (t_0, x_0))$, then there exists a maximal interval of solution x.*

Proof. This is a simple application of Zorn's lemma on the poset

$$P = \{y : J \to \mathbb{R}^n \mid y \text{ is a continuation of } x\}$$

where $y \leq z$ iff z is a continuation of x. $\qquad\square$

We wish to now find a characterization of maximal intervals of a solution. That is, we wish to know when can we say that a given solution is maximal.

Proposition 9.1.4.3. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : (a, b) \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$. Then,*
1. *The interval $[t_0, b)$ is a right maximal interval of solution x if and only if for any compact subset $K \subseteq D$, there exists $t \in [t_0, b)$ such that $(t, x(t)) \notin K$.*
2. *The interval $(a, t_0]$ is a left maximal interval of solution x if and only if for any compact subset $K \subseteq D$, there exists $t \in (a, t_0]$ such that $(t, x(t)) \notin K$.*

Proof. By symmetry, we reduce to showing 1. The main idea is to use the maximality and the results of previous section. $\qquad\square$

Complete the proof of Proposition 9.1.4.3, Chapter

9.1.5 Solution on boundary

In this section, we investigate the limiting cases of solutions of ODEs on a maximal interval (see Lemma 9.1.4.2). We see that if the one-sided limit of a maximal solution exists, then it's graph has to lie on the boundary of the domain.

Theorem 9.1.5.1. *Let $f : D \to \mathbb{R}^n$ be a continuous map where $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is open and let $(t_0, x_0) \in D$ so to make $(f, (t_0, x_0))$ an IVP. If $x : I \to \mathbb{R}^n$ is a solution to $(f, (t_0, x_0))$ and $I = (a, b)$ is a maximal interval of solution, then*
1. *If $\partial D \neq \emptyset$, $b < \infty$ and $\varinjlim_{t \to b^-} x(t)$ exists, then*

$$\left(b, \varinjlim_{t \to b^-} x(t) \right) \in \partial D.$$

2. If $\partial D = \emptyset$, $b < \infty$ then

$$\limsup_{t \to b^-} x(t) = \infty.$$

A similar statement holds for left sided limit towards a.

Proof. 1. Suppose not. Then $\left(b, \varinjlim_{t \to b^-} x(t)\right) \in D$ as D is open. It follows from Lemma 9.1.3.2 that $[t_0, b)$ is not maximal.

2. Suppose not. Then $\limsup_{t \to b^-} x(t) \neq \infty$. Hence, there exists $M > 0$ such that $\|x(t)\| < M$ for all $t \in [t_0, b)$. Now, construct $K = [t_0, b] \times C$ where C is a compact disc such that $\forall t \in [t_0, b)$, $x(t) \in C$, which can be chosen as an appropriate disc in $B_M(x_0)$. Since $K \subseteq D$, therefore by Proposition 9.1.4.3 we get a contradiction to maximality of $[t_0, b)$. $\qquad\square$

That's all we have to say here, so far.

9.1.6   Global solutions

So far we have studied solutions $x(t)$ to IVP defined only on some small enough intervals I such that $(t, x(t)) \in D$. However, we defined $D \subseteq \mathbb{R} \times \mathbb{R}^n$ as an arbitrary open set. In this section we would restrict to certain type of domains D, namely of the form $D = I \times \mathbb{R}^n$ and will try to study whether we can obtain a solution $x(t) : I \to \mathbb{R}^n$ to an IVP $(f, (t_0, x_0))$. If they exists, we call such a solution to be a *global solution* of the IVP $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ with initial values $(t_0, x_0) \in I \times \mathbb{R}^n$.

Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where $I \subseteq \mathbb{R}$ is an open interval and choose $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : J \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$. The main result of this section says that every such solution $x(t)$ can be extended to a global solution on I given some regularity conditions of values of f.

Theorem 9.1.6.1. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where $I \subseteq \mathbb{R}$ is an open interval and choose $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Suppose*

$$\|f(t, x)\| \leq M(t) + \|x\| N(t)$$

where $M, N : I \to \mathbb{R}$ are non-negative continuous maps, $\forall (t, x) \in I \times \mathbb{R}^n$. Then any solution $x : J \to \mathbb{R}^n$ of $(f, (t_0, x_0))$ can be continued to a solution $\tilde{x} : I \to \mathbb{R}^n$.

Proof. Omitted due to unnecessary technicality. See [[**??**] Kelley-Peterson?]. $\qquad\square$

We are more interested in the applications of the above theorem, which we now present.

Corollary 9.1.6.2. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be uniformly Lipschitz w.r.t. x. Then, there exists a unique global solution $x : I \to \mathbb{R}^n$ of the IVP $(f, (t_0, x_0))$.*

Proof. We get that $\exists L > 0$ such that

$$\|f(t,x) - f(t,y)\| < L\|x - y\|$$

for all $t \in I$ and $x, y \in \mathbb{R}^n$. In particular for $y = 0$, we get

$$\|f(t,x)\| \leq \|f(t,x) - f(t,0)\| + \|f(t,0)\|$$
$$\leq L\|x\| + \|f(t,0)\|$$

where $N(t) = L$ and $M(t) = \|f(t,0)\|$ in the notation of Theorem 9.1.6.1. Hence, by the same theorem, if there exists a solution of $(f, (t_0, x_0))$, say x on $J \subseteq I$, then it extends to a solution on I. Now by Strong Picard-Lindelöf (Theorem 9.1.2.10), we conclude that there is a unique solution on I; if there are two solutions on I, then by restriction on the interval obtained from Picard-Lindelöf, we would get a contradiction to it's uniqueness. $\square$

For a system of equations linear in x, for $x \in \mathbb{R}^n$, we have the following result.

Corollary 9.1.6.3. *Let $f(t,x) = A(t)x + b(t)$ be a map from $I \times \mathbb{R}^n$ to $\mathbb{R}^n$ where $A(t) \in C(I, \mathbb{R}^{n \times n})$ and $b \in C(I, \mathbb{R}^n)$ for an open interval $I \subseteq \mathbb{R}$ and $x = (x_1, \ldots, x_n)$. For $(t_0, x_0) \in I \times \mathbb{R}^n$, consider the IVP $(f, (t_0, x_0))$. Then there exists a unique solution*

$$x : I \times \mathbb{R}^n \to \mathbb{R}^n.$$

Proof. Using triangle inequality, we obtain

$$\|f(t,x)\| \leq \|A(t)\|\|x\| + \|b\|.$$

The result follows by an application of Theorem 9.1.6.1 and Corollary 9.1.6.2. $\square$

9.2 Linear systems

So far, we covered solutions of ODE of the form

$$\frac{dx}{dt} = f(t, x(t))$$

where $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $x : I \to \mathbb{R}^n$. In particular, $\frac{dx}{dt}$ is given as

$$\frac{dx}{dt}(t) = \begin{bmatrix} \frac{dx_1}{dt} & \frac{dx_2}{dt} & \cdots & \frac{dx_n}{dt} \end{bmatrix}$$

where each $x_i : I \to \mathbb{R}$. On the other hand, the right side consists of $f(t, x)$, which is a continuous function from a subset of $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$.

In this section, we would now study in detail a particular type of IVP in which the aforementioned function $f(t, x)$ is a linear map. In particular, the mapping f is given by

$$f : D \subseteq \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(t, x) \longmapsto Ax$$

for a real matrix A.

Remark 9.2.0.1. One should keep in mind that these are not new ODEs; a linear system is same as $\frac{dx}{dt} = f(t,x)$ where $f(t,x) = Ax$, so they are special cases of general ODEs and have special properties like uniqueness of solutions. In particular, all the results of the previous section on general ODEs will obviously hold in the linear case.

Remark 9.2.0.2. By Lemma 9.1.1.1, we know that a solution of $\frac{dx}{dt} = Ax$ is necessarily of the form

$$x(t) = x_0 + A \int_0^t x(s)ds.$$

9.2.1   Some properties of matrices

Let us begin by stating some of the properties of matrix algebra, especially of exponential of matrices as it will be used in Theorem 9.2.2.1. Since these are not fancy results so we omit the proof of all except the main observations required in each.

Theorem 9.2.1.1. *Let* $A, B \in M_n(\mathbb{R})$. *Then,*
  1. $\|A + B\| \leq \|A\| + \|B\|$.
  2. $\|AB\| \leq \|A\|\|B\|$.
  3. *The series* e^X *defined by*

$$e^X := \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

  converges for all $X \in M_n(\mathbb{R})$.
  4. $e^0 = I$.
  5. $(e^A)^T = e^{A^T}$.
  6. e^X *is invertible and* $(e^X)^{-1} = e^{-X}$ *for all* $X \in M_n(\mathbb{R})$.
  7. *If* $AB = BA$, *then* $e^{A+B} = e^A e^B = e^B e^A$.
  8. *If* $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, *then* $e^A = \operatorname{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.
  9. *If* P *is invertible, then* $e^{PAP^{-1}} = Pe^A P^{-1}$.

Proof. We omit the proof of all but the 3. To show that the series converges, by M-test, we reduce to showing that $\sum_n \frac{\|X\|^n}{n!}$ converges as

$$\|\frac{X^n}{n!}\| \leq \frac{\|X\|^n}{n!}.$$

Indeed, it converges to $e^{\|X\|}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Out of the above, perhaps the most important is the last one, as it tells us that if we have a diagonalizable matrix $A = PDP^{-1}$, then knowing its eigenvalues (that is, knowing D) and the matrix P is enough for us to calculate the e^A. Indeed, one should note that the exponent of a matrix is not easy to compute all the time!

We now give the lemma which will be quite useful for our goals, that the derivative of exponential of matrices is the obvious one.

Lemma 9.2.1.2. *Let* $X \in M_n(\mathbb{R})$*. Then,*

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Proof. One would need to interchange two limits at one point, which could only be done if the convergences are uniform. This could be shown by M-test. $\square$

9.2.2 Fundamental theorem of linear systems

The most important theorem for linear systems of the form $\frac{dx}{dt} = Ax$ is that that they have a unique solution.

Theorem 9.2.2.1. *Let* $A \in M_n(\mathbb{R})$*. Then for any* $x_0 \in \mathbb{R}^n$*, the IVP*

$$\frac{dx}{dt} = Ax(t)$$

with $x(0) = x_0$ *has a unique solution given by*

$$x(t) = e^{At}x_0.$$

Proof. Suppose $y(t)$ is another solution. Then, define $z(t) = e^{-At}y(t)$. Differentiating this, we get

$$\frac{d}{dt}z(t) = -Ae^{-At}y(t) + e^{-At}\frac{dy}{dt}(t).$$

Since $\frac{dy}{dt} = Ay$, thus the above equation gives $\frac{d}{dt}z(t) = -Ae^{-At}y + e^{-At}Ay = 0$. Hence $z(t) = c$ is constant, therefore $y(t) = ce^{At}$. Since $y(0) = x_0 = c$, therefore $y = x$. $\square$

Non-homogeneous linear systems

A non-homogeneous linear system is a linear IVP with an *offset*; they are of the form:

$$\frac{dx}{dt} = Ax(t) + b(t)$$

with $x(0) = x_0$. Their solution have a peculiar form.

Lemma 9.2.2.2. *Let* $\frac{dx}{dt} = Ax(t) + b(t)$ *with* $x(0) = x_0$ *be a non-homogeneous IVP for* $A \in M_n(\mathbb{R})$*. Then* x *is a solution if and only if*

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}b(s)ds.$$

Proof. We can multiply the IVP by e^{-At} to obtain

$$e^{-At}\frac{dx}{dt} = Ae^{-At}x + e^{-At}b(t)$$

$$e^{-At}\frac{dx}{dt} - Ae^{-At}x = e^{-At}b(t)$$

$$\frac{d}{dt}[e^{-At}x] = e^{-At}b(t)$$

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-As}b(s)ds.$$

One can easily check that the given form satisfies the IVP, by an application of fundamental theorem of calculus.   □

Finding e^A explicitly?

We will show by example how to find e^A by Jordan decomposition $A = PBP^{-1}$ where B is a matrix in Jordan form.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}.$$

One first calculates the eigenvalues of the matrix A. We get that these are $2, 2, 2$, that is there is only on eigenvalue with algebraic multiplicity 3. In order to find the Jordan form B, we need only now find the geometric multiplicity of

lete the method
Jordan form
atrix, Chapter

9.3   Stability of linear systems in $\mathbb{R}^2$

Consider the linear IVP given by

$$\frac{dx}{dt} = Ax(t)$$

with $x(0) = x_0$ where $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$ and $A \in M_2(\mathbb{R})$. From the fundamental theorem, we know that the solution is of the form $x(t) = e^{At}x_0$. By Jordan form, we know that there exists base change matrix $P \in GL_2(\mathbb{R})$ such that $A = P^{-1}BP$ where B is in Jordan form and hence it is of either of the three forms:

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

By Theorem 9.2.1.1, 9, we get

$$x(t) = e^{At}x_0 = e^{P^{-1}BPt}x_0 = P^{-1}e^{Bt}Px_0$$

so we reduce to understanding the plots of $e^{Bt}x_0$ for the aforementioned three cases, in order to understand the plot of $e^{At}x_0$ as both are related by coordinate transformation by P.

A phase portrait of a linear system

$$\frac{dx}{dt} = Ax(t)$$

is a plot of $x_1(t)$ vs $x_2(t)$ for various choices of initial points. Indeed, the choice of initial points is paramount if one ought to find the behavior of solutions. On the basis of the analysis of the three cases for B, we make the following definitions.

Definition 9.3.0.1. Let $\frac{dx}{dt} = Ax$ be a linear system where $\det A \neq 0$ and $A \in M_2(\mathbb{R})$. Then, the system is said to have

1. **saddle at origin** if $A \sim \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ where $\lambda < 0 < \mu$,

2. **node at origin** if
   (a) $A \sim \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ where λ, μ have same sign,
   (b) $A \sim \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$,

3. **focus at origin** if $A \sim \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$,

4. **center at origin** if $A \sim \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$.

9.4 Autonomous systems

An IVP is said to be *autonomous* if the governing equation

$$\frac{dx}{dt} = f(x(t))$$

is such that the continuous map $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is independent of time parameter t and we further assume that $f \in C^1$. In such a case we write $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and for a fixed initial datum, the maximal interval of existence is unique as well (see Corollary 9.1.2.11)

One calls a point $x_0 \in D$ to be an *equilibrium point* of the $\frac{dx}{dt} = f(x(t))$ if $f(x_0) = 0$.

9.4.1 Flows and Liapunov stability theorem

In our attempt at a better understanding of the autonomous system's dependence on initial point, we develop a basic machinery to handle it. The phase plots were a tool only available for linear systems, but we are not dealing with then in this section. Note however that a linear system is also autonomous.

The first tool we want to make is the notion of flows.

Definition 9.4.1.1. (**Flows**) Consider the following autonomous ODE

$$\frac{dx}{dt} = f(x(t))$$

where $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map. Denote by $\varphi(-, y) : I_y \to E$ to be a solution of the IVP $(f, (0, y))$ defined on **the** maximal interval of existence I_y for $\varphi(-, y)$ (Lemma 9.1.4.2). The map

$$\varphi : I \times E \longrightarrow E$$
$$(t, y) \longmapsto \varphi(t, y)$$

is called the flow of the system and the map $\varphi(t, -) : E \to E$ is called the *flow of the system at time t.* As we argued in the beginning, there is only one maximal interval of existence for each initial datum.

Remark 9.4.1.2. For a pair $(t, y) \in I \times E$, the value of the flow $\varphi(t, y) \in E$ tells us where the solution $\varphi(-, y)$ takes the initial point y at time t.

We have some obvious observations.

Lemma 9.4.1.3. *Consider the following autonomous ODE*

$$\frac{dx}{dt} = f(x(t)).$$

Let $\varphi : I \times E \to E$ be the flow of the system. Then,
    1. $\varphi(0, y) = y$.
    2. $\varphi(s, \varphi(t, y)) = \varphi(s + t, y)$.
    3. $\varphi(-t, \varphi(t, y)) = y$.

Proof. Trivial.   □

We now define the important notions surrounding stability.

Definition 9.4.1.4. (**Stability**) Consider the following autonomous ODE

$$\frac{dx}{dt} = f(x(t)).$$

Let $\varphi : I \times E \to E$ be the flow of the system.
    1. An equilibrium point $x_0 \in E$ is said to be *(Liapunov)stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x \in B_\delta(x_0) \implies \varphi(t, x) \in B_\epsilon(x_0) \; \forall t \geq 0$.
    2. An equilibrium point $x_0 \in E$ is said to be *unstable* if it is not stable.
    3. An equilibrium point $x_0 \in E$ is said to be *asymptotically stable* if it is stable and $\exists r > 0$ such that

$$x \in B_r(x_0) \implies \lim_{\overrightarrow{t \to \infty}} \varphi(t, x) = x_0.$$

We are now ready to state one of the most important results in stability theory, the Liapunov stability theorem. This result gives a sufficient condition for stability of a given point in the domain of $f : E \to \mathbb{R}^n$ of an autonomous system.

Theorem 9.4.1.5. *(Liapunov stability theorem) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and*

$$\frac{dx}{dt} = f(x(t))$$

be a given autonomous system with $x_0 \in E$ being an equilibrium point. If there exists a map of class C^1

$$V : E \to \mathbb{R}$$

such that $V(x_0) = 0$ and $V(x) > 0$ for all $x \in E \setminus \{x_0\}$, then

1. *if $V'(x) \leq 0$ for all $x \in E \setminus \{x_0\}$, then x_0 is stable,*
2. *if $V'(x) < 0$ for all $x \in E \setminus \{x_0\}$, then x_0 is asymptotically stable,*
3. *if $V'(x) > 0$ for all $x \in E \setminus \{x_0\}$, then x_0 is unstable.*

Remark 9.4.1.6. It is important to note that for most of the autonomous systems in nature, the function V as above which will do the job will be the energy functional of the physical system, that is, sum of kinetic and potential energy.

Proof of Theorem 9.4.1.5. _____ $\square$

Start the proof
Theorem 9.4.1.5
Chapter 9.

9.5 Linearization and flow analysis

Consider the following *system*:

$$x' = f(x)$$

where $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map and E is an open set. In the terminology of what we have covered so far, we have an autonomous system. In general, the above system may not be linear, as we studied previously. However, we can *linearize* the system at an equilibrium point x_0, as we shall show below. Indeed, this allows us to analyze the general autonomous system around each point as if it were linear.

Construction 9.5.0.1. (*Linearization of system at a point*) Let $E \subseteq \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}^n$ be a C^1 map. Let $x_0 \in E$ be an equilibrium point. For any $x \in E$, by Taylor's theorem, we get

$$\begin{aligned} f(x) &= f(x_0) + Df(x_0)(x - x_0) + \text{higher order terms} \\ &= Df(x_0)(x - x_0) + \text{higher order terms} \\ &= A(x - x_0) + \text{higher order terms}. \end{aligned}$$

We thus call the $x' = Df(x_0)x$ to be the *linearization of the system f at point x_0*.

Few definitions are in order.

Definition 9.5.0.2. (Hyperbolic, sink, source & saddle points) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. An equilibrium point $x_0 \in E$ is said to be:
1. *hyperbolic* if all eigenvalues of $Df(x_0)$ has non-zero real part,
2. *sink* if all eigenvalues of $Df(x_0)$ has negative real part,
3. *source* if all eigenvalues of $Df(x_0)$ has positive real part,
4. *saddle* if there exists eigenvalues λ, μ of $Df(x_0)$ such that real part of λ is > 0 and real part of μ is < 0.

9.5.1 Stable manifold theorem

"For a non-linear system, there are stable and unstable submanifolds, so that once you are in either of them, the flow will constrain you to remain there."

We will do an important theorem in the theory of linearization of autonomous systems. We shall avoid the proof this theorem. A reference is pp 107, [cite Perko]. Let us first define three important subspaces corresponding to a linear system.

Definition 9.5.1.1. (Stable, unstable & center subspaces) Let

$$x' = Ax$$

be a linear system where $A \in M_n(\mathbb{R})$. Let $\lambda_j = a_j + ib_j$ be eigenvalues of A and $w_j = u_j + iv_j$ be a generalized eigenvector of λ_j. Then,

1. the stable subspace E^s is defined to be the span of all u_j, v_j in $\mathbb{R}^n$ for those $j = 1, \ldots, n$ such that $a_j < 0$,
2. the unstable subspace E^u is defined to be the span of all u_j, v_j in $\mathbb{R}^n$ for those $j = 1, \ldots, n$ such that $a_j > 0$,
3. the center subspace E^c is defined to be the span of all u_j, v_j in $\mathbb{R}^n$ for those $j = 1, \ldots, n$ such that $a_j = 0$.

Lemma 9.5.1.2. *Let $x' = Ax$ be a linear system for $A \in M_n(\mathbb{R})$. Then,*
1. *$\mathbb{R}^n = E^s \oplus E^u \oplus E^c$,*
2. *E^s, E^u and E^c are invariant under the flow $\varphi(t, x)$ of the linear system, which as we know is given by $e^{At}x$.*

Proof. 1. This is easy, as generalized eigenvectors always span the whole space.
2. We need only show that for a generalized eigenvector w_j corresponding to $\lambda_j = a_j + ib_j$ with $a_j < 0$, the vector $A^k w_j$ is again a genralized eigenvector. Indeed, this follows from definition of a generalized eigenvector as $(A - \lambda_j I)w_j$ is again a generalized eigenvector. $\square$

We now come to the real deal.

Theorem 9.5.1.3. *(Stable manifold theorem) Let $E \subseteq \mathbb{R}^n$ be an open subset with $0 \in E$, consider $f : E \to \mathbb{R}^n$ to be a C^1-map and consider the system that it defines. Denote E^s and E^u to be the stable and unstable subspaces of the system $x' = Df(0)x$. If,*
- *$f(0) = 0$,*
- *$Df(0) : \mathbb{R}^n \to \mathbb{R}^n$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part,*

then:
1. *There exists a k-dimensional differentiable manifold S inside E such that*
   (a) $T_0 S = E^s$,
   (b) for all $t \geq 0$ and for all $x \in S$, we have

$$\varphi(t, x) \in S,$$

   (c) for all $x \in S$, we have

$$\lim_{t \to \infty} \varphi(t, x) = 0.$$

2. *There exists an $n - k$-dimensional differentiable manifold inside E such that*
   (a) $T_0 U = E^u$,
   (b) for all $t \leq 0$ and for all $x \in U$, we have

$$\varphi(t, x) \in S,$$

(c) for all $x \in U$, we have

$$\lim_{t \to -\infty} \varphi(t, x) = 0.$$

Let us explain via an example

Example 9.5.1.4. Consider the system

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{bmatrix}.$$

This is not a linear system as for $f((x_1, x_2, x_3)) = (-x_1, -x_2 + x_1^2, x_3 + x_1^2)$, the above system is given by

$$x' = f(x) \tag{9.1}$$

and $f(x)$ is clearly not linear in x. However, note that $f(0) = 0$. Thus, linearizing the system (9.1) at 0, we obtain the linear system

$$x' = Ax \tag{9.2}$$

where

$$A := Df(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So A has two eigenvalues with negative real part, namely -1 and -1 and one eigenvalue with positive real part, namely 1. In particular A is diagonalizable, hence E^s and E^u are just span of the eigenvectors (as all generalized eigenvectors in this case are just your regular eigenvectors). Hence we see

$$E^s = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$E^u = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence $E^s = x$-y plane and E^u is the z-axis of $\mathbb{R}^3$.

By an application of stable manifold theorem on this system, the stable manifold S is of dimension 2 and unstable manifold U is of dimension 1. Now, by elementary calculations, we can actually solve the linear system (9.2) and we thus obtain the following solution

$$x_1(t) = c_1 e^{-t}$$
$$x_2(t) = c_2 e^{-t} + c_1^2 (e^{-t} - e^{-2t})$$
$$x_3(t) = c_3 e^t + \frac{c_1^2}{3} (e^t - e^{-2t}).$$

Hence, the flow of the system is given by

$$\varphi : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(t, (c_1, c_2, c_3)) \longmapsto \begin{pmatrix} cc_1 e^{-t} \\ c_2 e^{-t} + c_1^2 (e^{-t} - e^{-2t}) \\ c_3 e^t + \frac{c_1^2}{3}(e^t - e^{-2t}) \end{pmatrix}.$$

Now, notice the following for any $c = (c_1, c_2, c_3) \in \mathbb{R}^3$

$$\lim_{t \to \infty} \varphi(t, c) = 0 \iff c_3 + \frac{c_1^2}{3} = 0$$

$$\lim_{t \to -\infty} \varphi(t, c) = 0 \iff c_1 = c_2 = 0.$$

Notice the fact that the above equivalence is very particular to this example. But this leads us to the following conclusions

$$S = \{(c_1, c_2, c_3) \in \mathbb{R}^3 \mid c_3 + c_1^2/3\}$$

$$U = \{(c_1, c_2, c_3) \in \mathbb{R}^3 \mid c_1 = c_2 = 0\} \cong z\text{-axis}.$$

Note that it is indeed true that for all $c \in S$ and any $t \geq 0$, $\varphi(t, c) \in S$. Similarly for U. Finally, one can check that $T_0 S = E^s$ and $T_0 U = E^u$, where the latter is immediate.

9.5.2   Poincaré-Bendixon theorem

So far, for a system we have defined its flow. Flow or integral curves of the system holds important information about the system at hand. However, we have not done any serious analysis with them. We shall begin the analysis of flows of a system now and prove the aforementioned theorem. It's use is predominantly to find closed trajectories of a system, which most of the times appears as a boundary of two differing phenomenon of the system, hence the importance of closed trajectories and of the theorem.

We first set up the terminology to be used in order to define basic objects of analysis of flow of a system.

Definition 9.5.2.1. (ω & α -limit set) Let $E \subseteq \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}^n$ be a C^1 map. Let $\varphi : \mathbb{R} \times E \to \mathbb{R}^n$ be the flow of the system. Then,
  1. a point $y \in E$ is said to be a ω-limit point of $x \in E$ if there exists a sequence $t_1 < t_2 < \cdots < t_n < \ldots$ in $\mathbb{R}$ such that $\lim_{n \to \infty} t_n = \infty$ and $\lim_{n \to \infty} \varphi(t_n, x) = y$.
  2. a point $y \in E$ is said to be an α-limit point of $x \in E$ if there exists a sequence $t_1 > t_2 > \cdots > t_n > \ldots$ in $\mathbb{R}$ such that $\lim_{n \to \infty} t_n = -\infty$ and $\lim_{n \to \infty} \varphi(t_n, x) = y$.
Let $x \in E$, the set of all ω and α limit points of x are denoted $L_\omega(x)$ and $L_\alpha(x)$ respectively.

The following are some simple observations from the definition

Lemma 9.5.2.2. *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system given by it.*

1. *If $y \in L_\omega(x)$ and $z \in L_\omega(y)$ then $z \in L_\omega(x)$.*
2. *If $y \in L_\omega(x)$ and $z \in L_\alpha(y)$ then $z \in L_\omega(x)$.*
3. *For any $x \in E$, the limit sets $L_\omega(x)$ and $L_\alpha(x)$ are closed in E.*

Using the concept of limit points, we can define certain nice subspaces of E conducive to them.

Definition 9.5.2.3. (Positively invariant set) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it. A region $D \subseteq E$ is said to be positively invariant if for all $x \in D$, $\varphi(t, x) \in D$ for all $t \geq 0$ where $\varphi : \mathbb{R} \times E \to E$ is the flow.

We then have the following simple result.

Lemma 9.5.2.4. *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it.*
1. *If x, z are on same flow line/trajectory, then $L_\omega(x) = L_\omega(z)$.*
2. *For any $x \in E$, the limit set $L_\omega(x)$ is positively invariant.*
3. *If $D \subseteq E$ is a closed positively invariant set, then for all $x \in D$, $L_\omega(x) \subseteq D$.*

We now define another set of tools helpful in doing flow analysis. First is a notion which will come in handy while trying to discuss both the topology of underlying space and the flow together. A hyperplane in $\mathbb{R}^n$ is a codimension 1 linear subspace.

Definition 9.5.2.5. (Local sections) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it. Let $0 \in E$. A local section S of f is an open connected subset of a linear hyperplane $H \subseteq \mathbb{R}^n$ such that $0 \in S$ and H is transverse to f, that is, $f(x) \notin H$ for all $x \in S$.

The next tool helps to "straighten" out flow around a local section.

Definition 9.5.2.6. (Flow box) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it. Let $0 \in E$ and S be a local section of f. A flow box around S is a diffeomorphism Φ between $(-\epsilon, \epsilon) \times S \subseteq \mathbb{R} \times E$ and $V_\epsilon \subseteq E$ given by $V_\epsilon := \{\varphi(t, x) \mid t \in (-\epsilon, \epsilon), x \in S\}$:

$$\Phi : (-\epsilon, \epsilon) \times S \longrightarrow V_\epsilon$$
$$(t, x) \longmapsto \varphi(t, x).$$

We identify $(-\epsilon, \epsilon) \times S$ as the flow box around S.

For a flow box, the diffeomorphism is important as it tells us that we can assume WLOG in a flow box that flow line are identical to the orthogonal coordinate system of $(-\epsilon, \epsilon) \times S \subseteq \mathbb{R}^{n+1}$.

We would now like to do flow analysis for the special case of planar systems. Indeed, the main theorem of this section is about the behaviour of certain limit sets of planar systems.

Let us first observe that for a planar system, any local section intersects a flow line at only discretly many points.

Lemma 9.5.2.7. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $x \in E$ and consider a local section S around x. Let*

$$\Sigma := \{\varphi(t, x) \in E \mid t \in [-l, l]\}.$$

Then $\Sigma \cap S$ is discrete.

Next, we see that if a sequence of points in a local section S of a planar system is monotonous in S and those same points appear in a trajectory, then it is monotonous in that trajectory as well. Indeed, a sequence of points $\{\varphi(t_n, x)\}$ along a trajectory is said to be *monotonous* if $\varinjlim_{n \to \infty} t_n = \infty$. Note that for a planar system, a codimension 1 linear subspace is a line, hence it has an inherent order and thus we can talk about monotonous sequences in a local section.

Proposition 9.5.2.8. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let S be a local section of the system. If $x_n = \varphi(t_n, x)$ is a sequence of points monotonous along the trajectory and $x_n \in S$, then $\{x_n\}$ are monotonous in S as well.*

One can use the above proposition to deduce some "eventual" properties of points in a local section by observing their intersection points with a flow line (which are discrete). Further it can be used for replacing a sequence along a trajectory to a sequence along a local section, which might be easier to analyze (as it's behaviour will just be that of monotonous sequences in $\mathbb{R}$).

Next we see an important observation, that trajectories of some special points cannot intersect a local section at more than one point(!)

Lemma 9.5.2.9. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. For some $x \in E$, let $y \in L_\omega(x) \cup L_\alpha(x)$. Then the trajectory of y intersects any local section at not more than single point.*

The next result is interesting, for it says that if the trajectory of a point intersects a local section, then there is a whole neighborhood worth of point around it, each of whose trajectories will intersect the local section(!) In some sense, this corresponds to the continuity of flow.

Proposition 9.5.2.10. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $\varphi : \mathbb{R} \times E \to \mathbb{R}^2$ denote the flow of the system. Let S be a local section around $y \in E$. If there exists $z_0 \in E$ such that for some $t_0 > 0$ we have $\varphi(t_0, z_0) = y$, then*
  1. there exists an open set $U \ni z_0$,
  2. there exists a unique C^1-map $\tau : U \to \mathbb{R}$,
where τ has the property that $\tau(z_0) = t_0$ and

$$\varphi(\tau(z), z) \in S \; \forall z \in U.$$

With this, we define the main object of study, a closed orbit.

Definition 9.5.2.11. (**Closed orbits**) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. A closed orbit is a periodic trajectory which doesn't contain an equilibrium point.

Note that if a trajectory contains an equilibrium point, then it will terminate after some finite time, hence the above requirement.

We now come to the main theorem of this section, which tells us a sufficient condition to find a closed orbits of a planar system.

Theorem 9.5.2.12. *(Poincaré-Bendixon theorem) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $x \in E$ be such that $L_\omega(x)$ $(L_\alpha(x))$ is a non-empty compact limit set which doesn't contain an equilibrium point. Then $L_\omega(x)$ $(L_\alpha(x))$ is a closed orbit.*

Let us now give some applications of the above theorem. First, we can classify limit sets $L_\omega(x)$ completely.

Theorem 9.5.2.13. *(Classification of limit sets) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $x \in E$ be such that $L_\omega(x)$*
- *is connected,*
- *is compact,*
- *has finitely many equilibrium points.*

Then one of the following holds
1. *$L_\omega(x)$ is a singleton.*
2. *$L_\omega(x)$ is periodic trajectory with no equilibrium points.*
3. *$L_\omega(x)$ consists of equilibrium points $\{x_j\}$ and a set of non-periodic trajectories $\{\gamma_i\}$ such that for all i, the trajectory γ_i tends to some x_j as $t \to \pm\infty$.*

The main use of Poincaré-Bendixon is to find limit cycles.

Definition 9.5.2.14. (**Limit cycles**) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. A limit cycle is a periodic trajectory γ such that there exists $x \in E$ for which $\gamma \subseteq L_\omega(x)$ or $\gamma \subseteq L_\alpha(x)$.

We now state the corollary of Poincaré-Bendixon which allows us to find the existence of limit cycles.

Corollary 9.5.2.15. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. If there exists a subseteq $D \subseteq E$ such that D*
1. *is compact,*
2. *is positively invariant,*
3. *has no equilibrium points,*

then there exists a limit cycle in D.

Proof. By Poincaré-Bendixon, we need only find $x \in D$ such that $L_\omega(x)$ is compact, as then $L_\omega(x)$ itself will be the limit cycle. This is straightforward, as D is positively invariant and compact, so $L_\omega(x)$ is inside D and is closed (hence compact). $\square$

9.6   Second order ODE

We now discuss some basic theory of second order ordinary differential equations.

Definition 9.6.0.1. (Second order system and solutions) Let $I \subseteq \mathbb{R}$ be an interval of $\mathbb{R}$ and consider $a_0, a_1, a_2, g \in C(I)$ to be four continuous maps $I \to \mathbb{R}$ such that $a_0(x) > 0 \ \forall x \in I$. Then, a second order system with parameters a_0, a_1, a_2, g is given by

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = g(x). \tag{9.3}$$

Note that $y' := \frac{dy}{dx}$. A *solution* of a second order system (q, r, f) is a $C^2(I)$ map $y(x)$ such that it satisfies (9.3).

Remark 9.6.0.2. A second order ODE can be written in the form

$$y'' + q(x)y' + r(x)y = f(x)$$

where $q, r, f \in C(I)$. This form is the one that we shall use and will identify a second order system by the tuple (q, r, f).

Remark 9.6.0.3. On the $\mathbb{R}$-vector space $C^2(I)$ of twice continuously differentiable functions, every 2nd order system (q, r, f) defines a linear transformation

$$L : C^2(I) \longrightarrow C(I)$$
$$y(x) \longmapsto (D^2 + q(x)D + r(x))y$$

where $D : C^2(I) \to C(I)$ is the derivative transformation $y \mapsto y'$, which is evidently linear. In this notation, we can write a second order system (q, r, f) as

$$Ly = f$$

where $L = D^2 + qD + r$. We call this linear transformation L the *transform associated to* (q, r, f).

Definition 9.6.0.4. (Solution space) Let (q, r, f) be a 2nd order system and $L : C^2(I) \to C(I)$ be the associated transform. The solution space of (q, r, f) is defined as the $\text{Ker}(L) \subseteq C^2(I)$. Note that the set of all solutions of (q, r, f) in $C^2(I)$ is given by $L^{-1}(f) \subseteq C^2(I)$.

Lemma 9.6.0.5. *Let (q, r, f) be a 2nd order system and L be the associated transform. Then* $\dim_{\mathbb{R}}(\text{Ker}(L)) = 2$.

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We now observe that one can obtain all solutions of the 2nd order system $S := (q, r, f)$ by obtaining a basis of the solution space of S and one solution of S.

Lemma 9.6.0.6. *Let $S = (q, r, f)$ be a 2nd order system and L be the associated transform. Then, for any $y_p \in L^{-1}(f)$*

$$L^{-1}(f) = y_p + \text{Ker}(L).$$

Proof. Observe that $y - y_p \in \mathrm{Ker}\,(L)$ and a linear transformation has all fibers of same size. $\qquad\square$

We define a tool which helps in distinguishing independent or dependent solutions of a homogeneous system.

Definition 9.6.0.7. (Wronskian) Let $f, g \in C^1(I)$. The Wronskian of f and g is given by

$$W(f,g) : I \to \mathbb{R}$$

where for any $x \in I$, we have

$$
\begin{aligned}
W(f,g)(x) :&= \det \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} \\
&= f(x)g'(x) - g(x)f'(x).
\end{aligned}
$$

Lemma 9.6.0.8. *Let $(q, r, 0)$ be a homogeneous system and let $y_1, y_2 \in C^2(I)$ be two solutions. Then,*
1. *$W(y_1, y_2)$ is either constant 0 for all $x \in I$ or $W(y_1, y_2)(x) \neq 0$ for all $x \in I$.*
2. *y_1, y_2 are linearly independent if and only if $W(y_1, y_2) \neq 0 \; \forall x \in I$.*

Proof. $\qquad\square$

9.6.1 Zero set of homogeneous systems

Let (q, r, f) be a 2nd order system and let y be a solution. There are some peculiar properties of the zero set $Z(y) := \{x \in I \mid y(x) = 0\} \subseteq \mathbb{R}$. We first show that the set $Z(y)$ is discrete if the system is homogeneous.

Lemma 9.6.1.1. *Let $(q, r, 0)$ be a 2nd order homogeneous system and let y be a solution. The zeroes of $y(x)$ are isolated, that is, $Z(y)$ is discrete.*

Proof. $\qquad\square$

Strum separation and comparison theorems

These theorems are at the heart of the analysis of zeros of homogeneous systems.

Theorem 9.6.1.2. *(Strum separation theorem) Let $(q, r, 0)$ be a 2nd order homogeneous system. Let y_1, y_2 be two distinct linearly independent solutions of the system. Then,*
1. *$Z(y_1)$ and $Z(y_2)$ are disjoint.*
2. *$Z(y_1)$ and $Z(y_2)$ are braided, that is, for any two x_1^1 and x_2^1 in $Z(y_1)$, there exists $x_1^2 \in Z(y_2)$ between them, and vice versa.*

Theorem 9.6.1.3. *(Strum comparison test) Consider two homogeneous 2nd order systems $(0, r_1, 0)$ and $(0, r_2, 0)$. Let y be a solution of $(0, r_1, 0)$ and u be a solution of $(0, r_2, 0)$, both non-trivial. Let $x_1, x_2 \in Z(u)$ such that*
1. *$r_1(x) \geq r_2(x)$ for all $x \in (x_1, x_2)$,*
2. *$\exists\, x_k \in (x_1, x_2)$ such that $r_1(x_k) > r_2(x_k)$.*
Then, there exists $z \in Z(y)$ such that $z \in (x_1, x_2)$.

9.6.2   Boundary value problems

A *boundary value problem* (BVP) is a second order system on an interval $I = [a, b]$ given by

$$y'' + qy' + ry = f$$

for $q, r, f \in C(I)$ such that its solutions has to satisfy certain conditions on the boundary given by

$$B_a(y) := \alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$B_b(y) := \alpha_2 y(b) + \beta_2 y'(b) = 0$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$. This is clearly a different problem than that of IVP. However, with some construction, we can convert this problem into a pair of 2nd order IVPs. It will turn out that the solution of this pair has important consequences for the original IVP at hand.

Reduction to a pair of 2nd order IVPs and criterion for uniqueness of BVP solution

Theorem 9.6.2.1. *Let $I = [a, b]$ and $q, r, f \in C(I)$. Consider the 2nd order system (q, r, f) and denote the associated transform as $L : C^2(I) \to C^2(I)$. From the system (q, r, f) consider the BVP given explicitly by*

$$Ly := y'' + qy' + ry = f \tag{9.4}$$
$$B_a(y) := \alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$B_b(y) := \alpha_2 y(b) + \beta_2 y'(b) = 0$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$. Construct the following two 2nd order IVPs

$$Ly := y'' + qy' + ry = 0 \tag{9.5}$$
$$y(a) = \beta_1$$
$$y'(a) = -\alpha_1$$

and

$$Ly := y'' + qy' + ry = 0 \tag{9.6}$$
$$y(b) = \beta_2$$
$$y'(b) = -\alpha_2.$$

Then the following are equivalent
   1. Let y_1 be a solution of (9.5) and y_2 be a solution of (9.6). Then y_1 and y_2 are linearly independent in the solution space $\mathrm{Ker}\,(L)$.
   2. The homogeneous BVP

$$Ly := y'' + qy' + ry = 0 \tag{9.7}$$
$$B_a(y) = 0$$
$$B_b(y) = 0$$

   has only 0 as solution.
   3. The BVP (9.4) has a unique solution.

Variation of parameters

Variation of parameters can give us a general form of a particular solution of $Ly = f$, in terms of the solutions of IVPs (9.5) and (9.6). Indeed, we have the following theorem.

Theorem 9.6.2.2. *Let y_1 be a solution of (9.5) and y_2 be a solution of (9.6). Let*

$$c_1(x) = \int_a^x \frac{-f(s)y_2(s)}{W(y_1, y_2)(s)} ds \tag{9.8}$$

$$c_2(x) = \int_a^x \frac{f(s)y_1(s)}{W(y_1, y_2)(s)} ds.$$

Then,

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) \tag{9.9}$$

is a particular solution of $Ly = f$ with $y_p(a) = 0$.

Further, we obtain a general form of solution of BVP (9.4).

Theorem 9.6.2.3. *Consider the notations of Theorems 9.6.2.1 and 9.6.2.2.*
  1. Any solution y of BVP (9.4) is

$$y = y_p - c_1(b)y_1$$

  where y_1 is a solution of (9.5) and $c_1(x)$ is defined in (9.8).
  2. Any solution of the BVP (9.4) is given by the integral

$$y(x) = \int_a^b G(x, s)f(s)ds \tag{9.10}$$

  for all $x \in I$, where

$$G(x, s) = \begin{cases} \frac{y_1(x)y_2(s)}{W(y_1, y_2)(s)} & \text{if } x \leq s \leq b \\ \frac{y_1(s)y_2(x)}{W(y_1, y_2)(s)} & \text{if } a \leq s \leq x. \end{cases} \tag{9.11}$$

  This map G is called the Green's function for the transformation $L : C^2(I) \to C(I)$.

Strum-Liouville system

Let $p, q \in C^2(I)$ and $f \in C(I)$ with $p > 0$. Define the 2nd order system

$$py'' + p'y' + qy = f.$$

We can write it in neater terms as follows

$$(py')' + qy = f. \tag{9.12}$$

We will call this the *Strum-Liouville system*, denoted by (p, q, f), and the associated transform as $L : C^2(I) \to C(I)$ mapping $y \mapsto (py')' + qy$. Consequently, (9.12) can be written as

$$Ly := (py')' + qy = f.$$

We have some basic results about the associated transform L.

Lemma 9.6.2.4. *Let (p, q, f) be a Strum-Liouville system and L be the associated transform.*

1. *(Lagrange's identity) If $y_1, y_2 \in C^2(I)$, then*

$$y_1 L y_2 - y_2 L y_1 = (pW(y_1, y_2))'.$$

2. *(Abel's formula) If y_1, y_2 are solutions of $Ly = 0$, that is, they are solutions of the Strum-Liouville system defined by $(p, q, 0)$, then*

$$W(y_1, y_2) = c/p$$

for some constant $c \in \mathbb{R}$.

Strum-Liouville Boundary Value Problems (SL-BVPs)

Consider a homogeneous Strum-Liouville system $(p, q, 0)$ and let L be the associated transform. Consider $r \in C(I)$ and $\lambda \in \mathbb{C}$. Then, a *Strum-Liouville boundary value problem* is a following type of 2nd order BVP

$$Ly + \lambda ry = 0 \tag{9.13}$$

with

$$B_a(y) = 0$$
$$B_b(y) = 0.$$

Strum-Liouville EigenValue Problems (SL-EVPs)

An SL-EVP consists of an SL-BVP (9.13) and the following question: find $\lambda \in \mathbb{C}$ such that the SL-BVP (9.13) admits a non-zero solution $y_\lambda \in C^2(I)$. In such a case λ is called the *eigenvalue* and y_λ the *eigenfunction* of the corresponding SL-EVP. We then call the tuple (p, q, r) as the SL-EVP.

Types of SL-EVPs

We further classify an SL-EVP (p, q, r) based on the properties of the underlying functions.
1. **regular** if $p > 0$ and $r > 0$ on $[a, b]$,
2. **singular** if $p > 0$ on (a, b), $p(a) = 0 = p(b)$ and $r \geq 0$ on $[a, b]$,
3. **periodic** if $p > 0$ on $[a, b]$, $p(a) = p(b)$ and $r > 0$ on $[a, b]$.
We next see that any eigenvalue of SL-EVP is always real.

Lemma 9.6.2.5. *Let (p, q, r) be a regular SL-EVP. Then all eigenvalues of (p, q, r) are real.*

Strum-Liouville theorem for regular SL-EVP

Properties of regular SL-EVPs

Examples

Chapter 10

K-Theory of Vector Bundles

Chapter 11

Jet Bundles

It is through the concepts explained in this chapter that we shall begin doing some geometry over our base manifolds. A classical use of differential equations is elaborated in Chapter 9, whereas here we shall be more conceptual in order to elucidate the underlying structure of the notion of "differential equations".

Part III

The Complex Analytic Viewpoint

Chapter 12

Analysis on Complex Plane

Contents

We will here review some of the classical results of complex function theory of one variable, namely the following four topics:

- Analytic functions; Cauchy-Riemann equations, harmonic functions.
- Complex integration; Zeroes of analytic functions, winding numbers, Cauchy's formula and theorem, Liouville's theorem, Morera's theorem, open-mapping theorem, Schwarz's lemma.
- Singularities; Classification, Laurent series, Casorati-Weierstrass theorem, residues and applications, meromorphic maps, Rouché's theorem.

- Conformal maps; Möbius transformations, normality and compactness, Riemann mapping theorem.

All this is important as it will build one's intuition of geometry in complex case, which is something we don't learn as early in our studies as, say, real geometry. Of-course this would be of immense use in complex algebraic geometry, some if which we shall cover in later chapters. Moreover, a complex manifold by definition locally looks like $\mathbb{C}^n$, hence it is imperative that we understand the geometry and analysis of complex plane and make it as second nature as the usual geometry over $\mathbb{R}^2$ is to us.

12.1   Holomorphic functions

Let $\Omega \subseteq \mathbb{C}$ denote an open subset of the complex plane $\mathbb{C}$ for the rest of this chapter. Consider a function $f : \Omega \to \mathbb{C}$. Motivated by the classical case of real differentiability in one variable, we can define a notion of differentiation for f at $a \in \Omega$.

Definition 12.1.0.1. (**$\mathbb{C}$-differentiable/holomorphic functions**) A function $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable or holomorphic at $a \in \Omega$ if the following limit exists:

$$\lim_{\substack{\longrightarrow \\ z \to 0}} \frac{f(a+z) - f(a)}{z}$$

in which case it's value is said to be the derivative of f at a and is denoted by $\frac{df}{dz}(a) = f'(a) \in \mathbb{C}$.

Remark 12.1.0.2. As we shall soon see, this seemingly innocuous definition for some surprising reason gives the following fantastic results:

1. Theorems 12.1.1.2 and **??** tells us:

$$\{All\ \mathbb{C}\text{-}differentiable\ maps\ f : \Omega \to \mathbb{C}\}$$

$$\|\wr\|$$

$$\{All\ pairs\ of\ differentiable\ maps\ u, v : \Omega \to \mathbb{R},\ related\ by\ CR\text{-}equations\}$$

2. Corollary 12.1.2.2 and Theorem **??** tells us:

$$\mathbb{C}\text{-}differentiable\ maps\ are\ conformal.$$

3. Theorem **??** tells us:

$$\mathbb{C}\text{-}differentiable\ functions\ are\ harmonic.$$

Moreover, Theorem **??** tells us that if Ω is simply connected, then

$$\{Harmonic\ functions\ \Omega \subseteq \mathbb{R}^2 \cong \mathbb{C}\}$$

$$\|\wr\|$$

$$\{\mathbb{C}\text{-}differentiable\ functions\ on\ \Omega \subseteq \mathbb{C}\}$$

4. Theorem **??** tells us:

$$\textit{Contour integral of a } \mathbb{C}\textit{-differentiable map around a loop is 0}.$$

5. Theorem **??** tells us:

> *A $\mathbb{C}$-differentiable function inside a disc is determined by its values on the disc's boundary.*

6. Corollary 12.2.3.5 tells us:

$$\{\mathbb{C}\textit{-differentiable maps } f : \Omega \to \mathbb{C}\}$$

$$\|\mathbb{R}$$

$$\{\textit{Analytic maps } f : \Omega \to \mathbb{C}\}$$

This shows the sheer importance of the notion of $\mathbb{C}$-differentiability, which we will explore later in a more *local* setting. Our goal in the rest of this chapter is to provide rather quick proofs to these results while portraying the main ideas employed in them.

Let us start by analyzing some elementary properties of holomorphic maps.

12.1.1   Cauchy-Riemann equations

Let $f : \Omega \to \mathbb{C}$ be a holomorphic map on an open subset $\Omega \subseteq \mathbb{C}$. Now, there is a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{C}$ given by $(x, y) \mapsto x + iy$. Composing f with this map, we get that f can equivalently be stated as the data of two real valued maps $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x, y) = \Re f(\varphi(x, y))$ and $v(x, y) = \Im f(\varphi(x, y))$.

Like in the case of $\mathbb{R}$-differentiability, in our case we can also define partial differential operators of f w.r.t. x, y and z.

Definition 12.1.1.1. (Partial differential operators on f) Let $f : \Omega \to \mathbb{C}$ be a holomorphic map on an open subset Ω of $\mathbb{C}$. Then, we define the following quantities in an obvious manner:
1. $\frac{\partial f}{\partial x} := \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.
2. $\frac{\partial f}{\partial y} := \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$.
3. $\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$.
4. $\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$.

Then the fact that f is holomorphic can be equivalently stated in terms of real differentiability of the maps u and v as the following theorem states.

Theorem 12.1.1.2. *Suppose $f : \Omega \to \mathbb{C}$ is any $\mathbb{C}$-valued function on an open set Ω of $\mathbb{C}$. Then write $f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \rightrightarrows \mathbb{R}$.*
1. *$f : \Omega \to \mathbb{C}$ is holomorphic at $z_0 \in \Omega$ if and only if u, v are real differentiable and satisfy any of the following equivalent PDEs at z_0:*
   (a) *$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.*
   (b) *$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$.*

(c) $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}$.

(d) $\frac{\partial f}{\partial \bar{z}} = 0$.

2. If $u, v : \Omega \to \mathbb{R}$ is a pair of C^1-maps satisfying the CR-equations, then $f := u + iv$ is a holomorphic map.

Proof. Equivalence of the four PDEs is straightforward. Now let $f : \Omega \to \mathbb{C}$ be a holomorphic map. This means that for any $a \in \Omega$, we have

$$\frac{\partial f}{\partial z}(a) = \lim_{z \to 0} \frac{f(a+z) - f(a)}{z}.$$

The required PDEs for u and v follows by letting z approach 0 first from real axis and then from imaginary axis and deeming them equal.

Next, we may write $R(z) = f(a+z) - f(a) - cz$ for some $c = c_1 + ic_2$ and then $R(z) = R_u(z) + iR_v(z)$ where $R_u(z) = u(a+z) - u(a) - c_1 x + c_2 y$ and $R_v(z) = v(a+z) - v(a) - c_2 x - c_1 y$. Then, f is holomorphic at a with $\frac{df}{dz}(a) = c$ if and only if $\lim_{z \to 0} \frac{R(z)}{z} = 0$. But the latter happens if and only if $\lim_{z \to 0} \frac{R_u(z)}{z} = 0 = \lim_{z \to 0} \frac{R_v(z)}{z}$. Now $\frac{R_u(z)}{z} = 0$ if and only if $c_1 = \frac{\partial u}{\partial x}(a)$ and $c_2 = -\frac{\partial u}{\partial y}(a)$. Similarly, $\lim_{z \to 0} \frac{R_v(z)}{z} = 0$ if and only if $c_2 = \frac{\partial v}{\partial x}(a)$ and $c_1 = \frac{\partial v}{\partial y}(a)$. $\qquad\square$

12.1.2   Conformal maps

We will now show that holomorphic maps "preserves angles". The meaning of angle is not well-defined a-priori on the complex plane, so we will have to develop that first.

A *curve* in $\mathbb{C}$ is a continuous map $\gamma : I \to \mathbb{C}$. It is said to be *differentiable* if $\Re\gamma : I \to \mathbb{R}$ and $\Im\gamma : I \to \mathbb{R}$ are differentiable $\mathbb{R}$-valued functions. It is said to be regular at $t \in I$ if $\gamma'(t) \neq 0 \in \mathbb{C}$. Now, let $\gamma_1, \gamma_2 : I \to \mathbb{C}$ be two curves which intersect at $\gamma_1(t_1) = \gamma_2(t_2)$ for $t_1, t_2 \in I$ such that γ_i is regular at t_i, $i = 1, 2$. Such an intersection is said to be *regular*. Then, the angle of intersection of γ_1 and γ_2 at a regular point is defined to be:

$$\angle \gamma_1(t_1), \gamma_2(t_2) := \arg \gamma_2'(t_2) - \arg \gamma_1'(t_1).$$

A function $f : \Omega \to \mathbb{C}$ is *conformal* at $z_0 \in \Omega$ if f preserves angles of all regular intersections of two curves at z_0.

It is now an easy observation that holomorphic maps will be conformal.

Lemma 12.1.2.1. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic map on an open set Ω of $\mathbb{C}$. If $z_0 \in \Omega$ such that $f'(z_0) \neq 0$, then for any two curves γ_1, γ_2 such that $\gamma_1(t_1) = z_0 = \gamma_2(t_2)$ and γ_1, γ_2 are regular at t_1, t_2 respectively, then*

$$\angle \gamma_1(t_1), \gamma_2(t_2) = \angle f \circ \gamma_1(t_1), f \circ \gamma_2(t_2).$$

Proof. The result follows from chain rule and the fact that $\arg wz = \arg w + \arg z$. $\qquad\square$

A map $f : \Omega \to \mathbb{C}$ is called *conformal* if it preserves angles of all regularly intersecting curves. Thus,

Corollary 12.1.2.2. *All holomorphic functions are conformal except at those points at which derivative is zero.* □

We will now show that even an arbitrary conformal map $f : \Omega \to \mathbb{C}$ is also holomorphic.

Theorem 12.1.2.3. *Let $f : \Omega \to \mathbb{C}$ be a conformal map such that $\Re f$ and $\Im f$ are of class C^1. Then,*
1. *f is holomorphic.*
2. *$f'(z) \neq 0$ for all $z \in \Omega$.*

Proof. **TODO.** □

12.1.3 Harmonic maps

A function $f : \Omega \to \mathbb{C}$ is said to be harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

Lemma 12.1.3.1. *Let $f = u + iv : \Omega \to \mathbb{C}$ be a function where $u, v : \Omega \rightrightarrows \mathbb{R}$. Then, f is harmonic if and only if u and v are harmonic (in $\mathbb{R}$-sense).*

Proof. A straightforward application of Cauchy-Riemann. □

Lemma 12.1.3.2. *All holomorphic maps are harmonic.*

Proof. **TODO.** □

Lemma 12.1.3.3. *All conformal maps are harmonic.*

Proof. **TODO.** □

12.1.4 Stereographic projection and the Riemann sphere

TODO.

12.1.5 Linear fractional transformations

A linear fractional transformation is a map

$$\varphi : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$. These are important as they provide a class of workable examples of rational functions, which are pretty much the bread and butter of algebraic geometry. Moreover, these maps arrange themselves in a group and it then follows that it contains as a subgroup the biholomorphic automorphism group of lots of geometric objects of interest (see Lemmas 12.1.5.2, 12.1.5.3). However, these maps makes the most sense on the complex projective line, $\mathbb{C}P^1$, the quotient of $\mathbb{C}^2$ by all lines passing through origin, which is the

usual Riemann sphere $\bar{\mathbb{C}}$ (see Lemma **??**).

Let us work out this connection in detail. We have the following maps:

$$\alpha : \mathbb{C}^2 \longrightarrow \mathbb{C}P^1 \xrightarrow{\cong} \bar{\mathbb{C}}$$

$$(w, z) \longmapsto [w, z] \longmapsto \tfrac{w}{z}$$

Notice that $L_f(\bar{\mathbb{C}})$, the collection of all linear fractional transforms on $\bar{\mathbb{C}}$ forms a group where the identity is given when $a = 0 = c$. The multiplication of two fractional transforms is again a fractional transform, as can be checked easily. Hence, it follows that $L_f(\bar{\mathbb{C}})$ is a subgroup of all biholomorphic maps of $\bar{\mathbb{C}}$, the Aut $\left(\bar{C}\right)$. Hence we have a hold on one type of global biholomorphic maps of the Riemann sphere(!)

We then have the following result.

Lemma 12.1.5.1. *Let $\bar{\mathbb{C}}$ denote the Riemann sphere. Then,*

$$L_f(\bar{\mathbb{C}}) \cong GL_2(\mathbb{C})/\mathbb{C}^\times I_2$$

Proof. There's a natural map

$$\kappa : GL_2(\mathbb{C}) \longrightarrow L_f(\bar{\mathbb{C}})$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \frac{az + b}{cz + d}.$$

This is a group homomorphism, as can be checked easily. The kernel of this homomorphism consists of matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $\frac{az+b}{cz+d} = z$. Unravelling, we get $c = 0 = b$ and $a = d \neq 0$. $\square$

This group is also known by *projective general linear group*, $PGL_2(\mathbb{C}) := L_f(\bar{\mathbb{C}})$. The group $L_f(\bar{\mathbb{C}})$ also has some special subgroups. For example, it consists of all biholomorphic maps of $D^\circ := \{z \in \mathbb{C} \mid |z| < 1\}$.

Lemma 12.1.5.2. *For the open unit ball D°, we have*

$$\text{Aut}\,(D^\circ) \cong \left\{ \frac{t(z - a)}{1 - \bar{a}z} \mid |t| = 1 \ \& \ a \in D^\circ \right\}.$$

Proof. **TODO.** $\square$

Similarly, it also contains an isomorphic copy of all biholomorphic maps of upper half plane $\mathbb{H}$.

Lemma 12.1.5.3. *For the upper half plane $\mathbb{H} \subset \mathbb{C}$, we have*

$$\text{Aut}\,(\mathbb{H}) \cong SL_2(\mathbb{R}) \subset GL_2(\mathbb{C}).$$

Proof. **TODO.** $\square$

Properties

Let us now state some basic properties of fractional transforms.

Lemma 12.1.5.4. *If $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a non-identity fractional transform, then either it has one or two fixed points, but not zero.*

Proof. A non-identity fractional transform $\varphi(z) = \frac{az+b}{cz+d}$ follows that either $b, d \neq 0$ or $a \neq c$. Suppose the former is not the case. Now if $\varphi(z) = z$, then it follows that $cz^2 + (d-a)z - b = 0$ where $b = d = 0$. Thus we obtain $z(cz - a) = 0$, which gives atleast one and atmost two solutions. Similarly, if $a = c$, then $b, d \neq 0$. It then follows that the above quadratic has either one or two solutions. □

Another property of fractional transforms is that they are uniquely determined by how they map on three points.

Lemma 12.1.5.5. *If z_1, z_2, z_3 and w_1, w_2, w_3 are two pair of distinct points in $\bar{\mathbb{C}}$, then there exists a unique fractional transform $\varphi \in L_f(\bar{\mathbb{C}})$ such that*

$$f(z_i) = w_i \ \forall i = 1, 2, 3.$$

Proof. Uniqueness follows from the fact that if $\varphi, \varpi : \bar{\mathbb{C}} \rightrightarrows \bar{\mathbb{C}}$ are two fractional transforms taking $z_i \mapsto w_i$, then the fractional transform $\varphi \circ \varpi^{-1}$ has 3 fixed points. It follows from Lemma 12.1.5.4 that $\varphi \circ \varpi^{-1} = \text{id}$.

To show existence, take any arbitrary triple $v_1, v_2, v_3 \in \bar{\mathbb{C}}$. We will show that one can construct a fractional transform depending on v_i mapping as $z_i \mapsto v_i$. Denote then the map φ, $z_i \mapsto v_i$ and ϖ, $w_i \mapsto v_i$. Then $\varpi^{-1} \circ \varphi$ would be the required map. Since v_i can be arbitrary, therefore we choose it as per our convenience. It is perhaps easier to write it for $\infty, 0, 1$. □

One last basic property that may be observed for fractional transforms is that they are conformal.

Lemma 12.1.5.6. *All fractional transforms $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ are conformal.*

Proof. Since fractional transforms are holomorphic, therefore by Corollary 12.1.2.2, we reduce to showing that $\varphi'(z) \neq 0$ for all $z \in \bar{\mathbb{C}}$. Indeed, we have

$$\phi'(z) = \frac{ad - bc}{(cz + d)^2},$$

where since $ad - bc \neq 0$ by definition, therefore $\phi'(z) \neq 0$. □

Example : The Cayley transform

We will discuss here the properties of the following fractional transform, known by Cayley's name:

$$\varphi : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$$
$$z \longmapsto \frac{z + i}{z - i}.$$

12.1.6 e^z**,** $\log z$**,** $\arg z$

12.1.7 Branches of $\log z$ **and** $\arg z$

12.2 La théorie des cartes holomorphes

The theory of holomorphic maps. We now begin another part of complex function theory which is at the heart of a lot of sources of interest in it. We first consider the line integrals.

12.2.1 Line integrals

Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous function. Suppose $G \subseteq \mathbb{C}$ is an open subset containing γ and it's interior and let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map $f : G \to \mathbb{C}$. Then, the *line integral* of f along γ is defined as

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$$

where definite integral of a complex valued function $g : [a, b] \to \mathbb{C}$ where $g = u + iv$ is given simply as the Riemann integral on each of the real and imaginary parts:

$$\int_a^b g(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt.$$

A continuous map $\gamma : [a, b] \to \mathbb{C}$ is called piecewise C^1 if γ is C^1 at all but finitely many points and where it isn't differentiable, one sided derivative exists.

Few properties of line integrals are in order.

Theorem 12.2.1.1. *Let $\gamma : [a, b] \to \mathbb{C}$ be a curve in $\mathbb{C}$ and let $G \subseteq \mathbb{C}$ be an open subset containing γ. Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map over G. Then,*
 1. (FTOC) If γ is piecewise C^1, then

$$\int_a^b \gamma'(t)dt = \gamma(b) - \gamma(a).$$

 2. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$ where G contains γ, then

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a)).$$

 So if γ is a closed loop, then integral of f' along it is 0.
 3. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$ and $\tilde{\gamma}$ is a reparametrization of γ, then $\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$.
 4. (Estimate) If $f \in \mathcal{C}^{\mathrm{hol}}(G)$ and $M = \sup_{t \in [a,b]} |f(\gamma(t))|$, then

$$\left| \int_\gamma f(z)dz \right| \le ML(\gamma)$$

 where $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$ is the arc-length.

Proof. Assuming 1 by FTOC on each piece, all results follows from basic analysis. $\square$

12.2.2   Cauchy's theorem - I

We will now state the Cauchy's theorems on holomorphic maps and integrals. This will be a special case of the general version, which we shall do later, for we will find almost all of the traditional applications without needing that generality. We will begin with it's infantile version, which is quite simple to state now with line integrals in our pouch.

Theorem 12.2.2.1. *(Cauchy's theorem) Let $\gamma : [a, b] \to \mathbb{C}$ be a closed piecewise C^1 loop in $\mathbb{C}$ and let $G \subseteq \mathbb{C}$ be a convex open set containing γ and it's interior $\mathrm{Int}(\gamma)$. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$, then*

$$\int_\gamma f(z)dz = 0.$$

Then there is the Cauchy integral formula.

Theorem 12.2.2.2. *(Cauchy's integral formula) Let C be a circle oriented in the counterclockwise manner and let $G \subseteq \mathbb{C}$ be an open set containing C and its interior $\mathrm{Int}(C)$. Then,*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

for all $z \in \mathrm{Int}(C)$.

Remark 12.2.2.3. Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on open $G \subseteq \mathbb{C}$. The integral formula tells us that the value of f at $z \in G$ can be given in terms of line integral of f around a small enough circle C in the CCW orientation centered at z so that $C \subseteq G$. Hence the integral formula tells us that holomorphic maps are pretty much completely determined by taking their line integrals around circles.

We will provide some results which can be derived from them. In particular, using these, we would be able to show that a holomorphic map is analytic (Corollary 12.2.3.5).

Proof of Cauchy's theorem : Holomorphic maps have primitives

A *primitive* of a holomorphic map f is a holomorphic map g such that $g' = f$. We first state the following theorem without proof using which we will prove the Cauchy's theorem.

Theorem 12.2.2.4. *(Cauchy's triangle theorem) Let T be a triangle in $\mathbb{C}$ and $G \subseteq \mathbb{C}$ be an open set containing T and $\mathrm{Int}(T)$. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$, then*

$$\int_T f(z)dz = 0.$$

Proof. [**??**] [Sarason]. □

Now, we will prove the following lemma using the above triangle theorem.

Lemma 12.2.2.5. *Let $G \subseteq \mathbb{C}$ be a convex open set and $f \in \mathcal{C}^{\mathrm{hol}}(G)$. Then there exists a map $g \in \mathcal{C}^{\mathrm{hol}}(G)$ such that $g' = f$.*

Proof. For a fixed $z_0 \in G$, define

$$g : G \longrightarrow \mathbb{C}$$

$$z \longmapsto \int_{[z_0,z]} f(z)dz$$

where $[z_0, z]$ denotes the path formed by line joining z_0 and z in G. We claim that for all $z \in G$, $g'(z) = f(z)$. Indeed, pick any $z_1 \in G$ to form triangle $T = (z_0, z_1, z)$ inside G (G is convex). Then, by Theorem 12.2.2.4, we get the following

$$0 = \int_T f(w)dw$$

$$= \int_{[z_0,z_1]} f(w)dw + \int_{[z_1,z]} f(w)dw + \int_{[z,z_0]} f(w)dw$$

$$g(z) - g(z_1) = \int_{[z_1,z]} f(w)dw.$$

We wish to estimate

$$\left| \frac{g(z) - g(z_1)}{z - z_1} - f(z_1) \right| = \left| \frac{1}{z - z_1} \int_{[z_1,z]} f(w)dw - f(z_1) \right|$$

$$= \left| \frac{1}{z - z_1} \int_{[z_1,z]} (f(w) - f(z_1))dw \right|$$

$$\leq \frac{1}{|z - z_1|} \int_{[z_1,z]} |f(w) - f(z_1)| \, dw.$$

Since f is continuous, therefore for any $\epsilon > 0$, there is a $\delta > 0$ such that $|w - z_1| < \delta$ implies $|f(w) - f(z_1)| < \epsilon$. Hence, for $|w - z_1| < \delta$, we get

$$\leq \frac{1}{|z - z_1|} \int_{[z_1,z]} \epsilon dw$$

$$= \epsilon.$$

Hence as $z \to z_1$, the above difference $\to 0$. $\square$

Proof of Theorem 12.2.2.1. Since $f \in \mathcal{C}^{\mathrm{hol}}(G)$, therefore by Lemma 12.2.2.5, there exists $g \in \mathcal{C}^{\mathrm{hol}}(G)$ such that $g' = f$. Hence the result follows by Theorem 12.2.1.1, 2. $\square$

Proof of Cauchy's integral formula : Cauchy integrals

We would like to present the proof of Cauchy integral formula as it portrays how to use the fact that integral of holomorphic maps around closed loops are zero (Theorem 12.2.2.1).

Proof of Theorem 12.2.2.2. Pick any $z_0 \in \mathrm{Int}(C)$. We shall show the result for this chosen z_0. We shall use the Cauchy's theorem 12.2.2.1 in an essential manner. Indeed, consider the following figure on the complex plane inside G: Integrating the holomorphic map $\frac{f(w)}{w-z_0}$
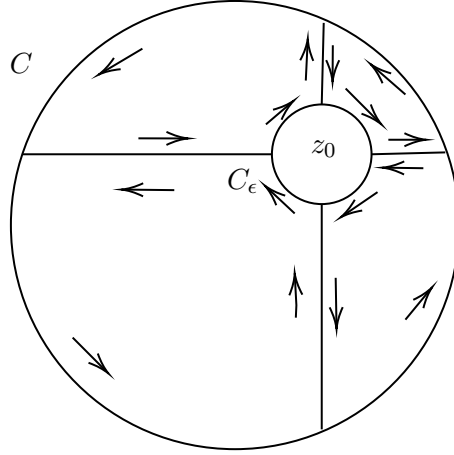
Figure 12.1: Contour over which to integrate $\frac{f(w)}{w-z_0}$.

over the each of the four regions will give zero by Theorem 12.2.2.1. However, summing them up, one can see that we get the difference $\int_C \frac{f(w)}{w-z_0}dw - \int_{C_\epsilon} \frac{f(w)}{w-z_0}dw$, which should thus be zero, yielding us $\int_C \frac{f(w)}{w-z_0}dw = \int_{C_\epsilon} \frac{f(w)}{w-z_0}dw$. Note this is true for all $\epsilon < d(z_0, C)$.

Now recall that $\int_C \frac{1}{z}dz = 2\pi i$. Hence, we get the following estimate for any chosen $\epsilon < d(z_0, C)$

$$\left| \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z_0}dw - f(z_0) \right| = \left| \frac{1}{2\pi i} \int_{C_\epsilon} \frac{f(w)}{w-z_0}dw - f(z_0) \right|$$
$$= \left| \frac{1}{2\pi i} \int_{C_\epsilon} \frac{f(w) - f(z_0)}{w-z_0}dw \right|$$

Now, by Theorem 12.2.1.1, 4, let $M_\epsilon = \sup_{w \in C_\epsilon} \left| \frac{f(w)-f(z_0)}{w-z_0} \right|$ to obtain the following inequality

$$\leq \frac{M_\epsilon}{2\pi} L(C_\epsilon)$$
$$= \frac{M_\epsilon}{2\pi} 2\pi\epsilon$$
$$= \epsilon M_\epsilon.$$

Since f is holomorphic, therefore $\varinjlim_{\epsilon \to 0} M_\epsilon = |f'(z_0)|$. Hence, $\varinjlim_{\epsilon \to 0} \epsilon M_\epsilon = 0$, which gives the desired result. $\square$

12.2.3   Theory of holomorphic maps

We now present applications of the two highly useful results of Cauchy (Theorems 12.2.2.1, 12.2.2.2). The results covered here are as follows:

- Mean value property.

- Power series representation of Cauchy integrals.
- Morera's theorem.
- Derivatives.
- Liouville's theorem.
- Identity theorem.
- Maximum modulus theorem.
- Schwarz's lemma.
- Classification of bijective holomorphic maps of open unit ball.
- Open mapping theorem.
- Fundamental theorem of algebra.
- Inverse function theorem.
- Local m^{th} power property.
- Harmonic conjugates.

These results lie at the heart of complex analysis.

Let us begin by understanding the behavior of a holomorphic map around a circle centered at a point.

Mean value property of holomorphic maps

Proposition 12.2.3.1. *Let $G \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{C}^{\text{hol}}(G)$. Then, for all $z_0 \in G$ and C_r a circle of radius r centered at z_0 contained inside G together with its interior $\text{Int}(C)$, we have*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt.$$

Proof. Using the integral formula (Theorem 12.2.2.2) and using $\gamma(t) = z_0 + re^{it}$ as a parameterization of C_r, the result follows. $\square$

Power series representation of Cauchy integrals

We will in this section show that functions defined by Cauchy integrals are analytic. Since holomorphic maps are given by Cauchy integrals, thus we would be able to show that holomorphic maps are analytic.

Definition 12.2.3.2. (Maps given by Cauchy integral) Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 curve in $\mathbb{C}$ and $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on an open subset $G \subseteq \mathbb{C}$ where G contains $\text{Im}(\gamma)$. Define the following map

$$\tilde{f} : \mathbb{C} \setminus \text{Im}(\gamma) \longrightarrow \mathbb{C}$$
$$z \longmapsto \int_\gamma \frac{f(w)}{w - z} dw.$$

Then $\tilde{f}$ is called the Cauchy integral associated to $f \in \mathcal{C}^{\text{hol}}(G)$ and $\gamma : [a, b] \to G$.

We first show that holomorphic maps are given by Cauchy integrals.

Lemma 12.2.3.3. *Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on an open set $G \subseteq \mathbb{C}$. Then f is locally given by a Cauchy integral.*

Proof. Indeed, by Theorem 12.2.2.2, we see that for all $z \in G$, choosing a small circle C_z around z and such that C_z and $\mathrm{Int}(C_z)$ are inside G, we can write

$$f(z) = \frac{1}{2\pi i} \int_{C_z} \frac{f(w)}{w - z} dw.$$

Hence locally f looks like a Cauchy integral. $\qquad\square$

We now show that Cauchy integrals are analytic, making holomorphic maps analytic by above lemma.

Proposition 12.2.3.4. *Maps defined by Cauchy integrals are analytic.*

Proof. Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ where G is open and let $\gamma : [a, b] \to G$ be a piecewise C^1 curve. We wish to show that $\tilde{f}$ defined on $\mathbb{C} \setminus \mathrm{Im}\,(\gamma)$ is given locally by power series. Indeed, pick any $z \in \mathbb{C} \setminus \mathrm{Im}\,(\gamma)$. Since $\mathrm{Im}\,(\gamma)$ is closed, therefore there exists a ball of radius r, B_r, such that $B_r \subseteq \mathbb{C} \setminus \mathrm{Im}\,(\gamma)$. In order to expand $\tilde{f}(z) = \int_\gamma \frac{f(w)}{w-z} dw$ as a power series, we first focus on $\frac{1}{w-z}$, where $w \in \mathrm{Im}\,(\gamma)$ and z is as above. Indeed, for any $z_0 \in B_r$, we have $|z - z_0| < r$ and $|w - z_0| > r$, thus yielding that $\left|\frac{z-z_0}{w-z_0}\right| < 1$ and hence we can write

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{(w - z_0)(1 - \frac{z-z_0}{w-z_0})}$$

$$= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}.$$

Moreover the convergence is uniform as we are within the radius of convergence. Now, $f(w) \leq M$ for all $w \in \mathrm{Im}\,(\gamma)$ as $\mathrm{Im}\,(\gamma)$ is compact and f is continuous over it. Hence we get that that following holds for all $w \in \mathrm{Im}\,(\gamma)$

$$\frac{f(w)}{w - z} = \sum_{n=0}^{\infty} \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}}.$$

Taking integral both sides, it thus follows from uniform convergence of above series that

$$\tilde{f}(z) = \int_\gamma \frac{f(w)}{w - z} dw = \int_\gamma \sum_{n=0}^{\infty} \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}} dw$$

$$= \sum_{n=0}^{\infty} \int_\gamma \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}} dw$$

$$= \sum_{n=0}^{\infty} \left(\int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} dw\right) (z - z_0)^n.$$

Hence locally $\tilde{f}$ looks like a power series, i.e. it is analytic. $\qquad\square$

Corollary 12.2.3.5. *Holomorphic maps are analytic.*

Proof. By Lemma 12.2.3.3, holomorphic maps are given by Cauchy integrals. By Proposition 12.2.3.4, maps given by Cauchy integrals are analytic. $\qquad\square$

Morera's theorem : Converse of Cauchy's triangle theorem

Proposition 12.2.3.6. *If $f : G \to \mathbb{C}$ is a continuous map on an open set $G \subseteq \mathbb{C}$ such that for all triangles $T \subseteq G$ where $\mathrm{Int}(T) \subseteq G$ as well we get*

$$\int_T f(z)dz = 0,$$

then f is holomorphic.

Proof. $\qquad\square$

Derivatives of a holomorphic map

Proposition 12.2.3.7. *Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on an open set $G \subseteq \mathbb{C}$. Then, f is differentiable to all orders and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(w)}{(w-z)^{n+1}} dw$$

where C_r is a circle in CCW orientation of radius r such that $C_r \subseteq G$ and $\mathrm{Int}(C_r) \subseteq G$. Moreover, for all $z \in \mathrm{Int}(C_r)$ with z_0 as center, we have that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \left(\int_{C_r} \frac{f(w)}{(w-z)^{n+1}} dw \right) (z-z_0)^n$$

Proof. $\qquad\square$

Liouville's theorem

A holomorphic map f on the entire complex plane, that is $f \in \mathcal{C}^{\mathrm{hol}}(\mathbb{C})$, is said to be *entire*.

Proposition 12.2.3.8. *Any entire bounded function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

Proof. $\qquad\square$

Zeroes of holomorphic maps

Proposition 12.2.3.9. *Let $G \subseteq \mathbb{C}$ be an open connected subset of $\mathbb{C}$. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$ is a holomorphic map on G, then the zero set $V(f) = \{z \in G \mid f(z) = 0\}$ has no limit point in G i.e. either $V(f) = G$ or $V(f)$ is discrete.*

Proof. $\qquad\square$

Identity theorem

Proposition 12.2.3.10. *Let $f, g \in \mathcal{C}^{\mathrm{hol}}(G)$ be two holomorphic maps defined on an open connected set $G \subseteq \mathbb{C}$. Then $f = g$ on G if and only if there exists a set $A \subseteq G$ which has a limit point in G such that $f|_A = g|_A$.*

Proof. □

Corollary 12.2.3.11. *Let f, g be two holomorphic maps on open connected subset $G \subseteq \mathbb{C}$ such that there exists an open set $U \subsetneq G$ contained inside of G such that $\partial U \neq \emptyset$ and $\bar{U} \subseteq G$ and $f|_U = g|_U$. Then $f = g$ on G.*

Proof. Indeed, since any element in ∂U is a limit point of U in G and $f|_U = g|_U$, then the result follows by Proposition 12.2.3.10. □

Corollary 12.2.3.12. *Let f, g be two holomorphic maps on open connected subset $G \subseteq \mathbb{C}$ such that there exists a closed ball $B \subset G$ on which $f|_B = g|_B$, then $f = g$ on G.*

Proof. A closed ball has non-empty interior. The result follows by Corollary 12.2.3.11. □

Maximum modulus principle

Proposition 12.2.3.13. *Let $G \subseteq \mathbb{C}$ be an open connected set and $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on G. Then $|f|$ doesn't achieves local maxima in G.*

Proof. □

Schwarz's lemma

Lemma 12.2.3.14. *Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc. If $f \in \mathcal{C}^{\mathrm{hol}}(D)$ is a holomorphic map $f : D \to D$ such that $f(0) = 0$, then*
  1. $|f(z)| \leq |z|$ *for all $z \in D$.*
  2. $|f'(0)| \leq 1$.
  3. *If f is not of the form λz for $\lambda \in S^1$, then the inequality in 1. & 2. is strict at all points $z \in D \setminus \{0\}$. In particular, if there exists $z_0 \in D \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, then $f(z) = \lambda z$ for $|\lambda| = 1$ and $\lambda = f'(0)$.*

Proof. Consider the map defined by

$$g : D \longrightarrow \mathbb{C}$$

$$z \longmapsto \begin{cases} \frac{f(z)}{z} & \text{if } z \in D \setminus \{0\} \\ f'(0) & \text{if } z = 0. \end{cases}$$

Clearly g is holomorphic. Now, for any $r \in (0,1)$, for $C_r \subset D$, by maximum modulus, Proposition 12.2.3.13, we have

$$|g(z)| < \frac{1}{r}$$

for all $z \in \text{Int}(C_r)$. Taking limit as $r \to 1$, we obtain $|g(z)| \leq 1$ for all $z \in D$. Now, if $\exists w \in D$ such that $|f(w)| = |w|$, then $|g(w)| = 1$. Since $|g(z)| < 1$ for all $z \in D$ as shown above, therefore by another use of maximum modulus, Proposition 12.2.3.13, it follows that $g(z) = \lambda$ is a constant where $|\lambda| = 1$. Thus $f(z) = \lambda z$. $\square$

Corollary 12.2.3.15. *(Pick's lemma) Let $f : D \to D$ be a holomorphic map where $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Then, for any two points $z, w \in D$*

$$\left| \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \right| \leq \left| \frac{z - w}{1 - z\bar{w}} \right|$$

except if f is a linear fractional transform mapping disc onto itself.

Proof. Define for each $w \in D$ the following fractional transform

$$g_w : D \longrightarrow D$$
$$z \longmapsto \frac{z - w}{1 - z\bar{w}}.$$

Then apply Schwarz's lemma (Lemma 12.2.3.14) on $g_{f(w)} \circ f \circ g_w^{-1} : D \to D$ as fractional transforms are biholomorphic. $\square$

Classification of bijective holomorphic maps of open unit ball

We shall classify all bijective holomorphic maps $f : D \to D$ for $D := \{z \in \mathbb{C} \mid |z| < 1\}$ and see that in the process that they are biholomorphic as well. For this, we first define the following important map which we encountered in Pick's lemma (Corollary 12.2.3.15). Define the following map for each $\alpha \in D$:

$$\varphi_\alpha : \bar{D} \longrightarrow \bar{D}$$
$$z \longmapsto \frac{z - \alpha}{1 - \bar{\alpha}z}.$$

This is indeed a holomorphic map over $\bar{D}$. We now see that this is biholomorphic.

Theorem 12.2.3.16. *For any $\alpha \in D$, the map $\varphi_\alpha : \bar{D} \to \bar{D}$ is such that*
   1. *φ_α takes D to D,*
   2. *φ_α takes ∂D to ∂D,*
   3. *φ_α is injective,*
   4. *φ_α is surjective,*
   5. *φ_α has a holomorphic inverse given by $\varphi_{-\alpha}$.*

Proof. Fix an $\alpha \in D$. We first show 2. For any $z \in \partial D$, we can write $z = e^{it}$ for $t \in \mathbb{R}$.

Thus we have

$$\left|\varphi_\alpha(e^{it})\right| = \left|\frac{e^{it} - \alpha}{1 - \bar{\alpha}e^{it}}\right|$$

$$= \left|\frac{e^{it} - \alpha}{1 - \bar{\bar{\alpha}}e^{\bar{i}t}}\right|$$

$$= \left|\frac{e^{it} - \alpha}{1 - \alpha e^{-it}}\right|$$

$$= \left|\frac{e^{it} - \alpha}{e^{it} - \alpha}\right|$$

$$= 1.$$

Thus, $\varphi_\alpha(e^{it}) \in \partial D$. This shows 2. Now we show 1. For this, by maximum modulus (Proposition 12.2.3.13), we have that $|\varphi_\alpha|$ achieves maxima on ∂D, and by 1., that maxima is 1, hence at every point of ∂D does $|\varphi_\alpha|$ achieves maxima. Hence $\varphi_\alpha(D) \subseteq D$. This shows 1. Next, it is a matter of simple calculation to see that $\varphi_\alpha \circ \varphi_{-\alpha} = \mathrm{id}_{\bar{D}}$ and thus by symmetry $\mathrm{id}_{\bar{D}} = \varphi_{-\alpha} \circ \varphi_\alpha$. Hence, φ_α is a biholomorphic map taking D onto D and ∂D onto ∂D. $\qquad\square$

We would now like to see that all biholomorphic maps of open unit ball are given by some unit modulus scalar multiples of φ_α. However, we need an idea to do so, which is provided by the following result.

Proposition 12.2.3.17. *(Extremality) For fixed $\alpha, \beta \in D$, denote $\mathcal{C}_{\alpha,\beta}$ to be the class of holomorphic maps into the unit disc $f : D \to D$ such that $f(\alpha) = \beta$. Then,*

1. we have

$$\sup_{f \in \mathcal{C}_{\alpha,\beta}} \left|f'(\alpha)\right| = \frac{1 - |\beta|^2}{1 - |\alpha|^2}.$$

2. The map $f \in \mathcal{C}_{\alpha,\beta}$ achieving the suprema is given by the following rational map

$$f = \varphi_{-\beta} \circ \lambda \circ \varphi_\alpha$$

   where $\lambda \in \partial D$ is a scalar.

Proof. 1. We need only show that for each $f \in \mathcal{C}_{\alpha,\beta}$, we get

$$\left|f'(\alpha)\right| \leq \frac{1 - |\beta|^2}{1 - |\alpha|^2}.$$

Indeed, this simply follows from a similar idea as used in the proof Pick's lemma (Corollary 12.2.3.15) above; consider the map $g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$ and use Schwarz's lemma (Lemma 12.2.3.14) on it to get the bound $|g'(0)| \leq 1$. Now use chain rule while keeping in mind that $\varphi'(0) = 1 - |\alpha|^2$ and $\varphi'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}$.

2. From proof of 1, it follows that the equality is achieved if and only if $|g'(0)| = 1$. By Schwarz's lemma (Lemma 12.2.3.14) this happens only if $g(z) = \lambda z$ for $\lambda \in \partial D$. Rest follows by composing with inverses of φ_β and $\varphi_{-\alpha}$ which we know from Theorem 12.2.3.21, 5. $\qquad\square$

We now come to the real deal. The following shows that all bijective holomorphic maps $D \to D$ are biholomorphic and are given by unit modulus scalar multiples of φ_α for some $\alpha \in D$. However we shall need a topic which we will cover in the next few sections, namely the inverse function theorem for one complex variable (see Section **??**, Theorem **??**). Moreover we shall also need another result which we do only in a further section called Rouché's theorem (Section **??**, Theorem **??**).

Theorem 12.2.3.18. *(Bijective holomorphic maps $D \to D$) Let $f : D \to D$ be a bijective holomorphic map. Denote $\alpha \in D$ to be the unique element such that $f(\alpha) = 0$. Then, there exists $\lambda \in \partial D$ such that*

$$f = \lambda \varphi_\alpha.$$

Proof. Consider the set-theoretic inverse of f, denoted $g : D \to D$. By Rouché's theorem (Theorem **??**) and by inverse function theorem (Theorem **??**), we obtain that $g \in \mathcal{C}^{\mathrm{hol}}(D)$. Now by chain rule we obtain $g'(f(\alpha))f'(\alpha) = 1$, that is, $g'(0) = 1/f'(\alpha)$. Now by Proposition 12.2.3.17, we obtain the following inequality for f and g where $f(\alpha) = 0$ and $g(0) = \alpha$:

$$|f(z)| \leq \frac{1}{1 - |\alpha|^2}$$
$$|g(z)| \leq 1 - |\alpha|^2.$$

In particular, we obtain that $1 - |\alpha|^2 \geq g'(0) = 1/f'(\alpha) \geq 1 - |\alpha|^2$, thus $g'(0) = 1 - |\alpha|^2$. Similarly, $|f'(\alpha)| = \frac{1}{1-|\alpha|^2}$. Hence f achieves the suprema of Proposition 12.2.3.17, 1. By Proposition 12.2.3.17, the result follows. $\square$

Corollary 12.2.3.19. *There is a bijection*

$$\left\{ \textit{Bijective holomorphic maps } f : D \to D \right\}$$

$$\cong$$

$$\left\{ \textit{Rational functions of the form } \lambda \tfrac{z-\alpha}{1-\bar{\alpha}z}, \ \alpha \in D, \ \lambda \in \partial D \right\}.$$

$\square$

Using this and Schwarz's lemma, we can show that a holomorphic map $f : D \to D$ can have atmost one fixed point.

Corollary 12.2.3.20. *Let $f : D \to D$ be a holomorphic map. Then f has atmost one fixed point.*

Proof. The idea is quite simple and we have used it already in the proof of Pick's lemma (Corollary 12.2.3.15). Indeed, we will construct $\varphi_\alpha : D \to D$ in such a manner that Schwarz's lemma can be applied to $\varphi \circ f \circ \varphi^{-1}$ and will use the results about the function φ_α (Theorem 12.2.3.21).

Suppose $z_1 \neq z_2 \in D$ are two fixed points of f. Consider the map $\varphi_{z_1}(z) := \frac{z-z_1}{1-\bar{z_1}z}$. This is a biholomorphic mapping $\varphi_{-z_1} : D \to D$. Consider

$$h = \varphi_{z_1} \circ f \circ \varphi_{-z_1}.$$

Then $h : D \to D$ is a holomorphic map and $h(0) = 0$. Applying Schwarz's lemma (Lemma 12.2.3.14), we obtain that $|h(z)| \leq |z|$. But notice that $h(\varphi_{z_1}(z_2)) = \varphi_{z_1}(z_2)$. Thus $\varphi_{z_1}(z_2)$ is a fixed point of h. Moreover, $\varphi_{z_1}(z_2) \neq 0$ as other wise $z_2 = z_1$, a contradiction. Thus, there exists $w \in D$ such that $|h(w)| = |w|$ (in particular, for $w = \varphi_{z_1}(z_2)$). Hence by contrapositive of Lemma 12.2.3.14, 3, we obtain that $h(z) = \lambda z$. Since $h(w) = w = \lambda w$, we obtain that $\lambda = 1$. Hence $h = \mathrm{id}$, thus $f = \mathrm{id}$. $\qquad \square$

Open mapping theorem

This theorem is quite an important result in the theory of holomorphic maps. It says a very simple thing, *all holomorphic maps on open connected sets are open maps*(!)

Theorem 12.2.3.21. *Let $G \subseteq \mathbb{C}$ be an open connected subset and let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a non-constant holomorphic map $f : G \to \mathbb{C}$. Then f is an open map.*

Proof. $\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \square$

Fundamental theorem of algebra

Proposition 12.2.3.22. *Every non-constant polynomial $f(x) \in \mathbb{C}[x]$ can be factored into linear factors.*

Proof. Suppose $f(x) \in \mathbb{C}[x]$ is a non-constant polynomial given by

$$f(x) = a_n x^n + \cdots + a_1 x + a_0.$$

Suppose to the contrary that f has no zeros in $\mathbb{C}$. Then $g(x) = \frac{1}{f(x)} : \mathbb{C} \to \mathbb{C}$ is an entire map. We wish to use Liouville's theorem (Proposition 12.2.3.8) on $g(x)$ in order to obtain a contradiction. Indeed, to get an upper bound for $|g(x)|$, we need a lower bound for $|f(x)|$. To this end we have

$$\begin{aligned}
|f(x)| &\geq |a_n x^n + \cdots + a_1 x + a_0| \\
&\geq |a_n x^n| \left| \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} \right) \right| \\
&\geq |a_n x^n| \left(1 - \left| \frac{a_{n-1}}{a_n x} \right| - \cdots - \left| \frac{a_1}{a_n x^{n-1}} \right| - \left| \frac{a_0}{a_n x^n} \right| \right)
\end{aligned}$$

where the last inequality comes from triangle inequality. Now write $h(x) = 1 - \left| \frac{a_{n-1}}{a_n x} \right| - \cdots - \left| \frac{a_1}{a_n x^{n-1}} \right| - \left| \frac{a_0}{a_n x^n} \right|$. In order to get a further lower bound for $|f(x)|$, we need to get an upper bound for $h(x)$. Since $h(x) \to 0$ as $x \to \infty$, therefore for some $R > 0$, we shall have $h(x) \leq \frac{1}{3}$ for $|x| > R$. Thus, we get

$$|f(x)| \geq |a_n R^n| \frac{2}{3}$$

for $|x| > R$. Now on $|x| \leq R$, by continuity of $|f|$ on a compact domain, we get that it achieves a minima, and hence $|f|$ is a lower bounded map and hence $g(x)$ is an upper bounded map. $\qquad\square$

Inverse function theorem

Remember that for a differentiable map $f : \mathbb{R}^n \to \mathbb{R}^n$, if $x_0 \in \mathbb{R}^n$ is a point such that Df_{x_0} is invertible, the inverse function theorem tells us that f is a diffeomorphism in some neighborhood around x_0. A similar statement is true for holomorphic maps $f : G \subseteq \mathbb{C} \to \mathbb{C}$.

Theorem 12.2.3.23. *(Inverse function theorem) Let $G \subseteq \mathbb{C}$ be an open connected set and $\varphi \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on G. If for $z_0 \in G$ we have that $f'(z_0) \neq 0$, then there exists a neighborhood $z_0 \in V \subseteq G$ such that*

1. *$\varphi|_V : V \to \varphi(V)$ is bijective,*
2. *$\varphi(V) \subseteq G$ is open,*
3. *the map $\psi : \varphi(V) \to V$ given by $\varphi(z) \mapsto z$ is in $\mathcal{C}^{\mathrm{hol}}(\varphi(V))$,*
4. *the map $\varphi|_V : V \to \varphi(V)$ is biholomorphic.*

We will now prove it. Let us begin with the following simple lemma.

Lemma 12.2.3.24. *Let $G \subseteq \mathbb{C}$ be an open connected set. If $f : G \to \mathbb{C}$ is a holomorphic map, then the map defined by*

$$g : G \times G \longrightarrow \mathbb{C}$$
$$(z, w) \longmapsto \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w, \\ f'(z) & \text{if } z = w \end{cases}$$

is continuous.

Proof. Clearly g is continuous for all (z, w) with $z \neq w$. Pick any $a \in G$. We will show that g is continuous at (a, a). For that, we wish to estimate $|g(z, w) - g(a, a)|$. For this, note that we can write $g(z, w)$ as follows where γ is the straight path $\gamma(t) = (1 - t)z + tw$:

$$
\begin{aligned}
g(z, w) &= \frac{f(z) - f(w)}{z - w} \\
&= \frac{f(\gamma(0)) - f(\gamma(1))}{\gamma(0) - \gamma(1)} \\
&= \frac{1}{w - z} \int_\gamma f'(z) dz \\
&= \int_0^1 f'(\gamma(t)) dt
\end{aligned}
$$

where the third equality follows from Theorem 12.2.1.1, 2. Thus we can write

$$\begin{aligned}
|g(z,w) - g(a,a)| &= \left| \int_0^1 f'(\gamma(t))dt - f'(a) \right| \\
&= \left| \int_0^1 f'(\gamma(t)) - f'(a)dt \right| \\
&\leq \int_0^1 \left| f'(\gamma(t)) - f'(a) \right| dt.
\end{aligned}$$

Now by continuity of f', the estimate follows. □

We can now prove the inverse function theorem.

Proof of Theorem 12.2.3.23. 1. The surjectivity is clear. For injectivity, we will show that for two $z_1 \neq z_2 \in V$, $|\varphi(z_1) - \varphi(z_2)| \geq M$ for some $M > 0$ using the lemma just proved. Indeed, using Lemma 12.2.3.24 and triangle inequality, we obtain for $\epsilon = \frac{1}{2}|\varphi'(z_0)|$ an open set $\tilde{V}$ containing (z_0, z_0) such that for all $(z_1, z_2) \in \tilde{V}$ with $z_1 \neq z_2$ we get the following

$$\left| \left| \frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2} \right| - |\varphi'(z_0)| \right| \leq \left| \frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2} - \varphi'(z_0) \right| < \epsilon = \frac{1}{2}|\varphi'(z_0)|.$$

Using this, we obtain that

$$|\varphi(z_1) - \varphi(z_2)| \geq \frac{1}{2}|\varphi'(z_0)| |z_1 - z_2|.$$

Thus for $z_1, z_2 \in V \subseteq G$ where V is obtained by projecting a small open set inside $\tilde{V}$ back to G, we see that on that V φ is injective.

2. This is just open mapping theorem, Theorem 12.2.3.21.

3. Shrink V enough so that $\varphi'(z) \neq 0$ for all $z \in V$. Then everything is straightforward using

$$|\varphi(z_1) - \varphi(z_2)| \geq \frac{1}{2}|\varphi'(z_0)| |z_1 - z_2|$$

which we obtained in 1. □

Local m^{th} power property

Any holomorphic map around a point can be represented by the mth power of some other special holomorphic map. Indeed, this is what the following theorem tells us.

Theorem 12.2.3.25. *Let $G \subseteq \mathbb{C}$ be an open-connected subset of $\mathbb{C}$ and let $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on G. Let $z_0 \in G$ and denote $w_0 = f(z_0)$. Let m be the order of zero that $f - w_0$ has at z_0. Then, there exists an open set $z_0 \in V \subseteq G$ and a holomorphic map*

$$\varphi : V \to \mathbb{C}$$

in $\mathcal{C}^{\text{hol}}(G)$ such that

1. $f(z) = w_0 + (\varphi(z))^m$ for all $z \in V$,
2. φ' is nowhere vanishing in V, i.e. has no zero in V,
3. there exists $r > 0$ such that φ is biholomorphic onto $D_r(0)$, the open disc of radius r around 0. Thus, $\varphi : V \to D_r(0)$ is bijective.

Proof. The main point of the proof is to try to represent the desired φ as $\exp \frac{??}{m}$. We just need to fill ?? correctly. Since $f - w_0$ has zero of order m at z_0, therefore there exists $g \in \mathcal{C}^{\mathrm{hol}}(G)$ such that

$$f(z) - w_0 = (z - z_0)^m g(z).$$

Now, by appropriately shrinking G away from zeros of g, we may assume $g \neq 0 \forall z \in G\backslash\{z_0\}^1$. Thus we have that $\frac{g'}{g}$ is holomorphic on G (this is our V). By Lemma 12.2.2.5, we get $h \in \mathcal{C}^{\mathrm{hol}}(G)$ such that $h' = \frac{g'}{g}$. We now claim that $g = \exp h$. Indeed, it is a simple matter to see that the derivative of $g \exp -h$ is zero. Thus, by using surjectivity of $\exp$, we can absorb the additive constant into h to obtain the above claim. One then sees that

$$\varphi(z) = (z - z_0) \exp \frac{h(z)}{m}$$

does the job for 1. The rest is straightforward. □

Harmonic conjugates

We will now show that any real valued harmonic map $u : G \subseteq \mathbb{R}^2 \to \mathbb{R}$ defines a unique (upto some constant) holomorphic map $g : G \to \mathbb{C}$ whose real part is u.

Theorem 12.2.3.26. *Let $G \subseteq \mathbb{C}$ be a convex open connected set. Let $u : G \to \mathbb{R}$ be a harmonic real valued function. Then, there exists holomorphic map $g : G \to \mathbb{C}$ unique upto an additive constant such that*

$$\Re g = u.$$

Proof. The main idea is to construct a holomorphic map f on G via the data of partial derivatives of u, and then use the Lemma 12.2.2.5, to get a primitive g, which will do the job. Indeed, we can make f via the following observation: u is harmonic real valued function if and only if $\frac{\partial^2}{\partial \bar{z} \partial z} = 0$. Using this, just define $f = u_x - i u_y$ and to show that f is holomorphic, observe that $\frac{\partial f}{\partial \bar{z}} = 0$. □

In combination with Lemma 12.1.3.2, we get that

Corollary 12.2.3.27. *Let $G \subseteq \mathbb{C}$ be open connected. Then,*

$$\{g : G \to \mathbb{C} \text{ is holomorphic}\} \cong \{u : G \to \mathbb{R} \text{ is harmonic}\}$$

where we identify functions upto additive constant. □

---

[1] We are implicitly using the isolated zeros theorem (Theorem **??**) which we shall do later.

12.3   Singularities

Consider the map $f(z) = 1/z$ on $\mathbb{C}^\times$. It is holomorphic. However, at $z = 0$, it is not holomorphic. Such points are called singularities of f, as we shall define more clearly later. Our goal is to study this phenomenon more carefully in this section. For this, we first need to develop a tool for local analysis of such "bad" points (some may also call it "the" points).

12.3.1   Laurent series

Definition 12.3.1.1. (Laurent series) A Laurent series centered at $z_0 \in \mathbb{C}$, denoted by $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, is a series of functions defined on some annulus $A_{z_0}(R_1, R_2) := \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$ centered at z_0 for $0 \leq R_1 < R_2$ such that the series converges at all points $z \in A_{z_0}(R_1, R_2)$. That is, the sequence of holomorphic maps $\{\sum_{n=-N}^{n=N} a_n(z - z_0)^n\}_N$ on $A_{z_0}(R_1, R_2)$ converges uniformly and absolutely to a holomorphic function $f : A_{z_0}(R_1, R_2) \to \mathbb{C}$ (by Weierstrass theorem).

For a Laurent series, we can find the coefficents in terms of Cauchy integral of the function it represents.

Lemma 12.3.1.2. *Let $f(z) = \sum_{n=-\infty}^{n=\infty} a_n(z - z_0)^n$ be a Laurent series around $z_0 \in \mathbb{C}$ in an annulus. Then for all $n \in \mathbb{Z}$*

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

where $R_1 < r < R_2$.

Proof. Use the uniform convergence of the Laurent series on $\frac{f(z)}{(z-z_0)^{n+1}}$ (so to limits out of integrals) and the fact that $\int_{C_r} (z - z_0)^n dz = 2\pi i$. $\square$

By Cauchy-Hadamard theorem for calculation of radius of convergence we also get the parameters for the maximum annulus on which a Laurent series can exist.

Lemma 12.3.1.3. *For a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, the smallest value of R_1 and largest value of R_2 such that $f(z)$ converges on $A_{z_0}(R_1, R_2)$ is given by*

1. $R_1 = \limsup_{n \to \infty} |a_{-n}|^{\frac{1}{n}}$
2. $R_2 = \frac{1}{\limsup_{n \to \infty} |a_n|^{\frac{1}{n}}}$

Proof. Straightforward use of Cauchy-Hadamard. $\square$

Our goal now is to show the following theorem.

Theorem 12.3.1.4. *Consider any $0 < R_1 < R_2$ and any $z_0 \in \mathbb{C}$. If $f : A_{z_0}(R_1, R_2) \to \mathbb{C}$ is holomorphic, then it is represented by a Laurent series.*

We shall see this in the following subsections.

Cauchy integral in a neighborhood of ∞

Cauchy formula for an annulus

Existence of Laurent series

12.3.2   Isolated singularities : Removable, poles and essential

We now come to the main matter of the present study, the notion of singularities. A holomorphic function $f : G \to \mathbb{C}$ is said to have an *isolated singularity* at $z_0 \notin G$ if there exists a punctured disc $A_{z_0}(0, r) \hookrightarrow G$. Consequently, by Theorem 12.3.1.4, we obtain a Laurent series expansion of f in $A_{z_0}(0, r)$. Let us denote it by

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n z^n.$$

We can then classify the isolated singularity z_0 into three types:
1. z_0 is a *removable singularity* if $a_n = 0$ for all $n < 0$,
2. z_0 is a *pole of order m* if $\min\{n < 0 \mid a_n \neq 0\} = m$,
3. z_0 is an *essential singularity* if $\min\{n < 0 \mid a_n \neq 0\} = -\infty$ or unbounded.

There are three characterizing theorems of each of the three kinds of singularities.

Theorem 12.3.2.1. *(Riemann's extension theorem) Let $f : G \to \mathbb{C}$ be a holomorphic map. Then the following are equivalent.*
1. *The point $z_0 \in \mathbb{C} \setminus G$ is a removable singularity of f.*
2. *There exists a punctured disc $A_{z_0}(0, r) \hookrightarrow G$ such that f is bounded on it.*

Theorem 12.3.2.2. *(Criterion for a pole) Let $f : G \to \mathbb{C}$ be a holomorphic map. Then the following are equivalent.*
1. *The point $z_0 \in \mathbb{C} \setminus G$ is a pole of f of some order.*
2. *We have*

$$\varlimsup_{z \to z_0} |f(z)| = \infty.$$

Theorem 12.3.2.3. *(Casorati-Weierstrauss theorem) Let $f : G \to \mathbb{C}$ be a holomorphic map. If the point $z_0 \in \mathbb{C} \setminus G$ is an essential singularity of f, then there exists a punctured disc $A_{z_0}(0, r) \hookrightarrow G$ such that $f(A_{z_0}(0, r))$ is dense in $\mathbb{C}$.*

The last theorem in particular shows the chaotic behaviour of essential singularities. We shall prove them in the remaining part of this section.

Complete the pr
of characterizati
Chapter 12.

12.4   Cauchy's theorem - II

Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1 curve in $\mathbb{C}$ and let $\Omega = \mathbb{C}^\times \setminus \mathrm{Im}(\gamma)$. We define the *index of γ* to be the following map over Ω:

$$\mathrm{Ind}_\gamma(z) : \Omega \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} dw.$$

Lemma 12.4.0.1. *Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1 curve in $\mathbb{C}$ and let $\Omega = \mathbb{C}^\times \setminus$ $\text{Im}(\gamma)$. Then $\text{Ind}_\gamma(z)$ is a holomorphic map on Ω.*

Proof. This follows from Proposition 12.2.3.4, as $\text{Ind}_\gamma(z)$ is the Cauchy integral of the constant function 1. $\square$

The following is the main theorem that we shall use.

Theorem 12.4.0.2. *Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1 curve in $\mathbb{C}$ and let $\Omega = \mathbb{C}^\times \setminus \text{Im}(\gamma)$. Then,*
1. *$\text{Ind}_\gamma(z)$ is an integer valued map,*
2. *$\text{Ind}_\gamma(z)$ is constant on each connected component of Ω,*
3. *$\text{Ind}_\gamma(z)$ is 0 on unbounded component of Ω.*

We now introduce the main Cauchy's theorem.

12.4.1   General Cauchy's theorem

To state the Cauchy's theorem in full generality, we first need to build the small language of chains, which is just a slight generalization of curves. Let $\{\gamma_i : I_i \to \mathbb{C}\}_{i=1}^n$ be a finite collection of piecewise C^1 curves over $\mathbb{C}$. A *chain generated by* $\{\gamma_i\}$ is a formal sum of the form

$$\Gamma = \gamma_1 + \cdots + \gamma_n.$$

One can be more precise here by treating Γ as an element of the free abelian group of all singular 1-chains, but we don't need that technology right now. We denote

$$\text{Im}(\Gamma) := \bigcup_{i=1}^n \text{Im}(\gamma_i).$$

Moreover, for a continuous map $f : \text{Im}(\Gamma) \to \mathbb{C}$, we further denote

$$\int_\Gamma f(z)dz := \sum_{i=1}^n \int_{\gamma_i} f(z)dz.$$

We can further define the *index of a chain* Γ as simply the sum of indices of individual curves:

$$\text{Ind}_\Gamma(z) := \sum_{i=1}^n \text{Ind}_{\gamma_i}(z)$$

for all $z \in \Omega$, where $\Omega = \mathbb{C}^\times \setminus \text{Im}(\Gamma)$. Note that the set Ω here will have multiple components if each element of the cycle is a distinct loop. Indeed, if $\Gamma = \gamma_1 + \cdots + \gamma_n$ is a cycle where each γ_i is a closed loop, then we call Γ a *cycle*. The general Cauchy's theorem is then a statement about integral over cycles.

Theorem 12.4.1.1. *(Cauchy's theorem) Let $\Omega \subseteq \mathbb{C}$ be an open set and $\Gamma \hookrightarrow \Omega$ be a cycle such that*

$$\mathrm{Ind}_\Gamma(z) = 0 \ \ \forall z \in \mathbb{C}^\times \setminus \Omega.$$

Let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then,
   1. *(Integral formula)*

$$\mathrm{Ind}_\Gamma(z)f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw$$

   for all $z \in \Omega \setminus \mathrm{Im}\,(\Gamma)$.
   2. *(Integral theorem)*

$$\int_\Gamma f(z)dz = 0,$$

   3. *if $\Gamma_0, \Gamma_1 \hookrightarrow \Omega$ are two cycles such that $\mathrm{Ind}_{\Gamma_0}(z) = \mathrm{Ind}_{\Gamma_1}(z)$ for all $z \notin \Omega$, then*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

The most important of the above triad of conclusions is the first one, which clearly generalizes the known integral formula.

12.4.2   Homotopy & Cauchy's theorem

Theorem 12.4.2.1. *Let $\Omega \subseteq \mathbb{C}$ be an open-connected set. If $\gamma_0, \gamma_1 \hookrightarrow \Omega$ are two piecewise C^1 closed loops in Ω such that they are homotopic in Ω, then*

$$\mathrm{Ind}_{\gamma_0}(z) = \mathrm{Ind}_{\gamma_1}(z) \ \ \forall z \notin \Omega.$$

This has some major corollaries in combination with Theorem 12.4.1.1.

Corollary 12.4.2.2. *Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. If $\gamma_0, \gamma_1 \hookrightarrow \Omega$ are two piecewise C^1 closed loops in Ω such that they are homotopic in Ω, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Corollary 12.4.2.3. *Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. If $\gamma \hookrightarrow \Omega$ is a piecewise C^1 closed loop in Ω and Ω is simply connected, then*

$$\int_\gamma f(z)dz = 0.$$

12.5   Residues and meromorphic maps

Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and $f : \Omega \to \mathbb{C}$ be holomorphic. Let $z_0 \notin \Omega$ be a point of isolated singularity of f. The *residue of f at z_0* is then defined to be the coefficient a_{-1} of the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

of the map f around z_0. We denote residue of f at z_0 by $\mathrm{res}_{z_0}(f) := a_{-1}$. For example, consider the following integral where C_r is a circle of radius r centered at z_0

$$\int_{C_r} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Since all terms $a_n(z - z_0)^n$ are $n \neq -1$ contributes zero integral as the positive parts of holomorphic in the interior of the loop and the negative parts are derivatives of constant 1, which is zero, therefore the only non-zero term is contributed by $n = -1$. Consequently, we have

$$\int_{C_r} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = 2\pi i a_{-1}$$
$$= 2\pi i \, \mathrm{res}_{z_0}(f)$$

where $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

We now define a class of holomorphic maps which one encounters often in complex analysis.

Definition 12.5.0.1. (Meromorphic maps) Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and $f : \Omega \to \mathbb{C}$ be any function. We say that f is meromorphic if
   1. there exists a set $A \subset \Omega$ which has no limit points in Ω,
   2. $f : \Omega \setminus A \to \mathbb{C}$ is holomorphic,
   3. every point of A is a pole of f.
One often calls the set A as the *set of poles of f*.

There are some observations to be made.

Lemma 12.5.0.2. *Let $f : \Omega \to \mathbb{C}$ be a meromorphic map on an open-connected set Ω. Then, the set of poles of f is atmost countable.*

Proof. Let $A \subset \Omega$ be the set of poles of f. Covering Ω by countably many compact sets $\{K_i\}$, we observe that intersection of each of $K_i \cap A$ has to be atmost finite, otherwise there exists a sequence in $K_i \cap A$, which consequently admits a convergent subsequence, that is, a limit point in Ω. Consequently, A is a countable union of finite sets. $\qquad\square$

Remark 12.5.0.3. For the purposes of residue of f at $a \in A$, one can replace analysis of f with analysis of f by the analysis of $Q = \sum_{n=-m}^{-1} a_n (z-a)^n$, called the principal part of f at a where m is the order of pole of f at $a \in A$. Clearly, $\text{res}_a Q = \text{res}_a f$. Moreover, one sees that

$$\text{res}_a(f)\text{Ind}_\gamma(a) = \frac{1}{2\pi i} \int_\gamma Q(z)dz$$

where γ is a piecewise C^1-loop centered at a, in $\Omega \setminus A$. This is again a consequence of the fact that all terms inside the integral are zero except the one corresponding to a_{-1}. Indeed, this hints at a general phenomenon, which is clarified by the following theorem.

Theorem 12.5.0.4. *(The residue theorem) Let $\Omega \subseteq \mathbb{C}$ be an open-connected set. If $f : \Omega \to \mathbb{C}$ is a meromorphic map with $A \subseteq \Omega$ its set of poles and Γ a cycle in $\Omega \setminus A$ such that*

$$\text{Ind}_\Gamma(z) = 0 \ \ \forall z \notin \Omega,$$

then

$$\frac{1}{2\pi i} \int_\Gamma f(z)dz = \sum_{a \in A} \text{res}_a(f)\text{Ind}_\Gamma(a).$$

We now an important result, which gives us information of zeroes of holomorphic maps on certain subsets.

Theorem 12.5.0.5. *Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1-loop in an open-connected set $\Omega \subseteq \mathbb{C}$ such that*
1. $\text{Ind}_\gamma(z) = 0$ for all $z \notin \Omega$,
2. $\text{Ind}_\gamma(z) = 0$ or 1 for all $z \in \Omega \setminus \text{Im}(\gamma)$.
Then we have that for any holomorphic maps $f, g : \Omega \to \mathbb{C}$, denoting $\Omega_1 := \{z \in \Omega \setminus \text{Im}(\gamma) \mid \text{Ind}_\gamma(z) = 1\}$ and $N_f = \#Z(f) \cap \Omega_1$, we get that
1. if f has no zeros on $\text{Im}(\gamma) \subseteq \Omega$, then

$$N_f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}dz = \text{Ind}_{f \circ \gamma}(0).$$

2. (Rouché's theorem) if

$$|f(z) - g(z)| < |g(z)| \ \ \forall z \in \text{Im}(\gamma),$$

then $N_g = N_f$.

Part IV

The Categorical Viewpoint

Out of the four, this is the most foundational and the deepest one of them all. It has to be, as the main motive here is to understand some of the foundational notions of geometry, like *intersection* and *deformation*, and to act as their natural mathematical residential address. However, we would need to cover a lot of ground before we start doing geometry in this new world, most of it is due to a fundamental different way of thinking than what is done classically (more categorical than set theoretic, the latter is abound in some of the previous chapters of this book). But the rewards are high, for it will provide us a deeper understanding of fundamental questions raised throughout this book, one of them being the question of *a concrete, robust and complete theory of intersections of manifolds and schemes.*

Chapter 13

Classical Topoi

We give a *general review*[1] of classical topos theory, the main references being the cite[MacMoe], cite[Moe] and cite[Elephant]. *You can add material from your past notes.*

13.1 Topoi, maps of topoi and examples

13.2 Topoi as generalized spaces

13.3 Universal constructions on topoi

13.4 Cohomology and homotopy of topoi

_____

[1]Its important to highlight that this is supposed to be an overview not an encyclopedic reference!

Chapter 14

Language of ∞-Categories

Contents

In this chapter we give an overview of the language of ∞-categories. The ultimate goal of this chapter is to define the ∞-category of ∞-groupoids and state the Yoneda lemma in ∞-categorical setting. Therefore, if you believe that ∞-groupoids are spaces (homotopy hypothesis), then we would construct the ∞-category of spaces by the end of this chapter. In the process, we will learn about the techniques invovled in manipulating ∞-categories.

14.1 Simplicial sets

The goal of simplicial sets is to obtain a combinatorial approximation of topological spaces. We describe simplicial sets as a presheaf over simplex category. One can motivate themselves why this definition is the correct definition for combinatorially handling topological spaces by reading the review paper cite[Bergner's simplicial set paper]. We give some basic properties of such objects. We also give a general important result about presheaves, "every presheaf is a colimit of representable presheaves". This is of fundamental importance in the development of ∞-categories.

Remark 14.1.0.1. (*Notations*)

1. As we will be frequently constructing and dealing with presheaf category over a category $\mathbf{C}$, therefore instead of only denoting it by $\mathbf{PSh}(\mathbf{C})$, we will also be denoting it by $\widehat{\mathbf{C}}$, depending on the convenience of the given situation.
2. We will denote objects a category $\mathbf{C}$ by lowercase alphabets like $a, b, c, .., x, y, z$, morphisms in $\mathbf{C}$ by lowercase letters like $f, g, h, \ldots$ and functors $\mathbf{C} \to \mathbf{D}$ by uppercase alphabets like $A, B, C, ..., X, Y, Z$ and also by lowercase alphabets like $f, g, h, \ldots$.
3. Let $\mathbf{C}$ be a category and $c \in \mathbf{C}$ be an object. We denote by $h_c : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ the contravariant hom-functor given by $a \mapsto \mathrm{Hom}_{\mathbf{C}}(a, c)$.
4. We will denote ∞-categories by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$.

Let us also as a reminder put the usual Yoneda lemma.

Lemma 14.1.0.2. *(Yoneda lemma) Let* $\mathbf{A}$ *be a category,* $a \in \mathbf{A}$ *be an object and* F *be a presheaf over* $\mathbf{A}$. *Then, there is a natural isomorphism*

$$\mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, F) \overset{\cong}{\longrightarrow} F(a).$$

Proof. Consider the following maps

$$\varphi : \mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, F) \longrightarrow F(a)$$
$$\alpha \longmapsto \alpha_a(\mathrm{id}_a).$$

and

$$\xi : F(a) \longrightarrow \mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, F)$$
$$x \longmapsto \beta : h_a \to F$$

where β is defined as follows. Consider any $a' \in \mathbf{A}$ and any $f \in h_a(a')$. Define $\beta_{a'} : h_a(a') \to F(a')$ by mapping as $f \mapsto F(f)(x)$.

One then sees that $\xi \circ \varphi = \mathrm{id}$ by naturality of α and that $\varphi \circ \xi = \mathrm{id}$ by funtoriality of F. $\qquad\square$

Corollary 14.1.0.3. *Let* $\mathbf{A}$ *be a category and define* $\mathrm{Yon} : A \to \widehat{\mathbf{A}}$ *to be the Yoneda functor given by* $a \mapsto h_a$ *and* $a \to b \mapsto h_a \to h_b$. *Then,* Yon *is fully-faithful.*

Proof. We have by Yoneda lemma (Lemma 14.1.0.2) that $\mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, h_b) \cong h_b(a) = \mathrm{Hom}_{\mathbf{A}}(a, b)$. $\qquad\square$

14.1.1   Extension of functor by colimits-I

Consider the category of all finite sets $[n]$ with $n + 1$ elements with linear order $0 < 1 < \cdots < n$ and mappings being the order preserving map. Denote this category by $\boldsymbol{\Delta}$ and call it the *simplex category*. A *simplicial set* is then a presheaf over $\boldsymbol{\Delta}$. Let the category of all simplicial sets be denoted $\mathbf{sSet}$.

The following is a fundamental result in category theory.

Theorem 14.1.1.1. *Let* $\mathbf{A}$ *be a small category and* $\mathbf{C}$ *be a locally small category with all small colimits. Then, for any functor*

$$f : \mathbf{A} \to \mathbf{C}$$

we get two functors

$$f_! : \widehat{\mathbf{A}} \longrightarrow \mathbf{C}$$

and

$$f^* : \mathbf{C} \longrightarrow \widehat{\mathbf{A}}$$
$$c \longmapsto \mathrm{Hom}_{\mathbf{C}} \left(f(-), c \right)$$

such that

1. $f_!$ *is left adjoint of* f^*

$$\widehat{\mathbf{A}} \; \underset{f^*}{\overset{f_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \; \mathbf{C} \; .$$

   The functor $f_!$ *is called the extension of* f *by colimits.*

2. *Denoting* $\mathrm{Yon} : \mathbf{A} \hookrightarrow \widehat{\mathbf{A}}$ *to be the Yoneda embedding, we get that the following commutes upto a unique natural isomorphism*

$$
\begin{array}{ccc}
\widehat{\mathbf{A}} & \xrightarrow{\;f_!\;} & \mathbf{C} \\[4pt]
\mathrm{Yon} \Big\uparrow & \nearrow & \\[2pt]
\mathbf{A} & f &
\end{array}
\; .
$$

   That is, $f_!(h_a) \cong f(a)$ *where* $h_a = \mathrm{Hom}_{\mathbf{A}}(-, a)$.

Proof. The main heart of the proof is the fact that every presheaf is a colimit of representables (Proposition 1.1.8 of cite[Cis]).

1. We define $f_! : \widehat{\mathbf{A}} \to \mathbf{C}$ as follows. Pick any $X \in \widehat{\mathbf{A}}$. Consider the functor $\varphi_X : \mathbf{A}/X \to \widehat{\mathbf{A}}$ from the category of elements of X given by $(a, s) \mapsto h_a$ and on maps by $f : (a, s) \to (b, t)$ by $h_f : h_a \to h_b$. By Proposition 1.1.8 of cite[Cis], we have $\varinjlim_{(a,s)} h_a = X$. Define $f_!(X) = \varinjlim_{(a,s)} f(a)$ which exists in $\mathbf{C}$ as $\mathbf{C}$ has all small colimits. Consequently, we obtain the following natural isomorphisms by Yoneda lemma (Lemma 14.1.0.2) and limit preserving properties of contravariant homs

$$\mathrm{Hom}_{\mathbf{C}} \left(f_!(X), c \right) \cong \mathrm{Hom}_{\mathbf{C}} \left(\varinjlim_{(a,s)} f(a), c \right) \cong \varprojlim_{(a,s)} \mathrm{Hom}_{\mathbf{C}} \left(f(a), c \right)$$

$$\cong \varprojlim_{(a,s)} \mathrm{Hom}_{\widehat{\mathbf{A}}} \left(h_a, f^*(c) \right) \cong \mathrm{Hom}_{\widehat{\mathbf{A}}} \left(\varinjlim_{(a,s)} h_a, f^*(c) \right)$$

$$\cong \mathrm{Hom}_{\widehat{\mathbf{A}}} \left(X, f^*(c) \right).$$

2. Since $\mathrm{Hom}_{\mathbf{C}} \left(f_!(h_a), c \right) \cong \mathrm{Hom}_{\widehat{\mathbf{A}}} \left(h_a, f^*c \right) \cong f^*(c)(a) = \mathrm{Hom}_{\mathbf{C}} \left(f(a), c \right)$ for all objects $c \in \mathbf{C}$, therefore the result follows by Corollary 14.1.0.3. $\qquad \square$

The above theorem will take a central place in constructions of this chapter. Indeed, let us point the following applications of this theorem before stating its proof.

Corollary 14.1.1.2. *Let* $\mathbf{A}$ *be a small category and* $\mathbf{C}$ *be a category with small colimits. If* $F : \widehat{\mathbf{A}} \to \mathbf{C}$ *is a colimit preserving functor. Then*
  1. there exists $f : \mathbf{A} \to \mathbf{C}$ *such that* $f_! \cong F$ *naturally,*
  2. F *has a right adjoint.*

Proof. We first define the required f. For any object $a \in \mathbf{A}$, define $f(a) := F(h_a)$. Then by Theorem 14.1.1.1, 1, it follows that $f_!(h_a) \cong f(a) = F(h_a)$. Since we know that every presheaf is presented as a colimit of representable functors indexed by its category of elements, thus we get that for any $X \in \widehat{\mathbf{A}}$, $F(X) \cong f_!(X)$ naturally. The second conclusion also follows from Theorem 14.1.1.1, 2. $\square$

The following result will be important in order to define simplicial mapping spaces.

Proposition 14.1.1.3. *For any small category* $\mathbf{A}$*, the presheaf category* $\widehat{\mathbf{A}}$ *is Cartesian closed where the internal hom object is defined by*

$$\underline{\mathrm{Hom}}\,(X, Y)(a) := \mathrm{Hom}_{\widehat{\mathbf{A}}}\,(X \times h_a, Y)$$

and on morphisms as

$$\underline{\mathrm{Hom}}\,(X, Y)(f) := \mathrm{Hom}_{\widehat{\mathbf{A}}}\,(X \times h_f, Y).$$

Proof. We wish to show that $\underline{\mathrm{Hom}}\,(-, -)$ acts as internal hom object in $\widehat{\mathbf{A}}$. This can be seen by establishing following natural bijections

$$\mathrm{Hom}_{\widehat{\mathbf{A}}}\,(T \times X, Y) \cong \mathrm{Hom}_{\widehat{\mathbf{A}}}\,(T, \underline{\mathrm{Hom}}\,(X, Y))$$

This follows from contemplating the functor $f_X : \mathbf{A} \to \widehat{\mathbf{A}}$ for all $X \in \widehat{\mathbf{A}}$ given by $a \mapsto X \times \mathrm{Yon}(-)$, together with Theorem 14.1.1.1. Indeed, first observe that $\widehat{\mathbf{A}}$ is locally small with all small colimits. Second, observe from the proof of Theorem 14.1.1.1 that for any $Z \in \widehat{\mathbf{A}}$, we have that $f_{X!} : \widehat{\mathbf{A}} \to \widehat{\mathbf{A}}$ takes any $Z \in \widehat{\mathbf{A}}$ and maps it to $f_{X!}(Z) = \varinjlim_{(a,s)} f_X(a)$ where (a, s) varies over $\mathbf{A}/Z$, the category of elements of Z. Since $f_X(a) = X \times h_a$ and filtered colimits commute with finite limits, therefore we get a natural isomorphism $f_{X!}(Z) \cong X \times Z$. Consequently, the adjunction of Theorem 14.1.1.1 completes the proof. $\square$

14.1.2   Eilenberg-Zilber categories

This is an abstraction of the type of combinatorial proofs that we would like to make in the simplex category $\mathbf{\Delta}$. Indeed, as we would have to work with surjections and injections in $\mathbf{\Delta}$ primarily, which interacts with the the size of $[n]$ (which we would define to be n in a minute) as an injection would only increase the size and a surjection would only decrease it, therefore we need a systematic toolset to work with these things. In particular, if we denote $\mathbf{\Delta}_+$ to be the subcategory of all injective maps, $\mathbf{\Delta}_-$ to be the subcategory of all surjective maps in $\mathbf{\Delta}$ and $d : \mathrm{Ob}(\mathbf{\Delta}) \to \mathbb{N}$ the size map, then we have the following properties about the tuple $(\mathbf{\Delta}, \mathbf{\Delta}_+, \mathbf{\Delta}_-, d)$:

1. All bijections are in both $\mathbf{\Delta}_+$ and $\mathbf{\Delta}_-$.
2. The dimension map d takes bijective cardinals to the same natural.
3. Let $\sigma : [n] \to [m]$ in $\mathbf{\Delta}$ which is not bijective. If σ is injective (i.e. in $\mathbf{\Delta}_+$), then $d(a) < d(b)$ and if σ is surjective (i.e. in $\mathbf{\Delta}_-$), then $d(a) > d(b)$.
4. Any map $\sigma : [n] \to [m]$ in $\mathbf{\Delta}$ factors as a surjection followed by an injection.
5. For any surjective map $\sigma : [n] \to [m]$ in $\mathbf{\Delta}$, there exists a section $\pi : [m] \to [n]$, i.e. such that $\pi\sigma = \mathrm{id}_{[n]}$.
6. If $\sigma : [n] \to [m]$ is surjective, then the set of sections of σ uniquely determines the map σ.

Remark 14.1.2.1. We need this abstraction of properties of $\mathbf{\Delta}$ so that with the same techniques, we can work with bisimplicial sets, which is important if one wishes to consider simplicial objects in **sSet**.

These considerations about $\mathbf{\Delta}$ motivates the following definition.

Definition 14.1.2.2. (**Eilenberg-Zilber categories**) A category $\mathbf{A}$ is said to be an Eilenberg-Zilber category (or much simply, *EZ cateogry*), if there exists subcategories $\mathbf{A}_+, \mathbf{A}_-$ and a function $d : \mathrm{Ob}(\mathbf{A}) \to \mathbb{N}$ which satisfies the following axioms:

1. All isomorphisms of $\mathbf{A}$ are in both $\mathbf{A}_+$ and $\mathbf{A}_-$.
2. If a, a' are isomorphic objects in $\mathbf{A}$, then $d(a) = d(a')$.
3. Let $\sigma : a \to a'$ not be an isomorphism. If σ is in $\mathbf{A}_+$, then $d(a) < d(a')$. If σ is in $\mathbf{A}_-$, then $d(a) > d(a')$.
4. For any map $\sigma : a \to a'$ in $\mathbf{A}$, there exists unique factorization of σ into maps $p : a \to c$ in $\mathbf{A}_-$ and $i : c \to a'$ in $\mathbf{A}_+$

$$
\begin{array}{ccc}
 & c & \\
{}^{p}\nearrow & & \searrow^{i} \\
a & \xrightarrow[\sigma]{} & a'
\end{array}
\quad .
$$

5. If $\sigma : a \to a'$ is a map in $\mathbf{A}_-$, then there exists a section $\pi : a' \to a$, i.e. $\pi\sigma = \mathrm{id}_a$.
6. If σ, σ' are two maps in $\mathbf{A}_-$ such both of them has the same collection of sections, then $\sigma = \sigma'$.

The main thrust of this section is to discuss presheaves over an EZ category $\mathbf{A}$, keeping in mind the prototypical case of $\mathbf{A} = \mathbf{\Delta}$. Indeed, the main result and its corollaries will serve first as a practice for the type of arguments we shall need later and also as a tool to be consistently used later in constructions with simplicial sets (which are, presheaves over $\mathbf{\Delta}$).

Example 14.1.2.3. By previous discussion, it is clear that the simplex category $\mathbf{\Delta}$ is an EZ category.

Let $\mathbf{A}$ be an EZ category and X a presheaf over $\mathbf{A}$. Then the category of elements of X, $\mathbf{A}/X$, is an EZ category again. Indeed, define $(\mathbf{A}/X)_+$ exactly as those pairs (a, s) where $a \in \mathbf{A}_+$ and $(\mathbf{A}/X)_-$ exactly as those pairs (a, s) where $a \in \mathbf{A}_-$. Further for an object (a, s), define $d(a, s) = d(a)$, where the latter d is coming from the EZ structure on $\mathbf{A}$.

We begin our discussion by first defining some important terminologies. Keep in mind that the prototypical example of $\boldsymbol{\Delta}$ as an EZ category acts as a motivation for the following constructions

Definition 14.1.2.4. (**Decomposition of a section, degenerate and non-degenerate sections**) .**TODO**. *Do degenerate sections, decomposition of a section, non-degenerate sections, n-skeleton of a presheaf, Lemma 1.3.6 as a practice and Corollary 1.3.10.*

14.1.3 Categories as simplicial sets

14.1.4 ∞-categories

14.1.5 Theorem of Boardman-Vogt

14.2 Classical homotopical algebra

14.2.1 Factorisation systems and model categories

14.2.2 Basic constructions

14.2.3 Construction of model structures on PSh(C)

14.2.4 From one model structure to many

Chapter 15

Homotopical Algebra

Chapter 16

Stable ∞-Categories

Chapter 17

Algèbre Commutative Dérivée

Part V

Special Topics

Chapter 18

Commutative Algebra

Contents

In this chapter, we collect topics from contemporary commutative algebra. The most need of all this material comes from algebraic goemetry. In particular, in the following, we list out the topics that we would need for our treatment of basic algebraic geometry.

1. Dimension theory : For dimension of schemes, Hauptidealsatz, local complete intersection, etc.
2. Integral dependence : For proper maps between affine varieties, normalization, finiteness of integral closure, certain DVRs of dimension 1, etc.
3. Field theory : For birational classification of varieties, primitive element theorem, basic algebra in general, etc.
4. Completions : Local analysis of singularities, formal schemes, complete local rings, Cohen structure theorem, Krull's theorem, etc.
5. Valuation rings : For curves and their non-singular points (DVRs) and various equivalences, Dedekind domains, etc.
6. Multiplicities : For intersections in projective spaces, intersection multiplicity, Hilbert polynomials, flat families, etc.
7. Kähler differentials : For differential forms on schemes, this will be used consistently in further topics.
8. Depth and Cohen-Macaulay : For local complete intersections, blowing up, etc.
9. Tor and Ext functors : They are tools for other algebraic notions, generizable to global algebra, tor dimension, etc.
10. Projective modules : For vector bundles, projective dimension and Ext, pd + depth = dim for regular local rings, etc.
11. Flatness : Family of schemes varying continuously, smooth and étalé maps, etc.
12. Lifting properties - Étale, unramified and smooth morphisms : These are used heavily for the corresponding scheme maps, and beyond.

Notation 18.0.0.1. Let R be a ring and $f(x) \in R[x]$ be a polynomial. We will denote $c_n(f) \in R$ to be the coefficient of x^n in $f(x)$. If $f(x,y) \in R[x,y]$, then we will denote $c_{n,m}(f) \in R$ to be the coefficient of $x^n y^m$ in $f(x,y)$. We may also write $c_{x^n}(f)$ for $c_n(f)$ and $c_{x^n y^m}(f)$ for $c_{n,m}(f)$ if it makes statements more clear.

18.1 General algebra

We discus here general results about prime ideals, modules and algebras.

18.1.1 Jacobson radical and Nakayama lemma

Let R be a ring. Denote the set of all units of R as $R^\times$. The *Jacobson radical* is an ideal $\mathfrak{r}$ of R formed by the intersection of all maximal ideals of R. A *finitely generated R-module* M is a module which has a finite collection of elements $\{x_1, \ldots, x_n\} \subset M$ such that for any $z \in M$, there are $r_1, \ldots, r_n \in R$ so that $z = r_1 x_1 + \cdots + r_n x_n$. More concisely, if there is a surjection R-module homomorphism $R^n \twoheadrightarrow M$. We then have the following results about $\mathfrak{r}$.

Proposition 18.1.1.1. *Let R be a ring and let $\mathfrak{r}$ denotes it Jacobson radical. Then,*
1. *$x \in \mathfrak{r}$ if and only if $1 - xy \in R^\times$ for any $y \in R$.*
2. *(Nakayama lemma) Let M be a finitely generated R-module. If $\mathfrak{q} \subseteq \mathfrak{r}$ is an ideal of R such that $\mathfrak{q}M = M$, then $M = 0$.*
3. *Let M be a finitely generated module and $\mathfrak{q} \subseteq \mathfrak{r}$. Let $N \leq M$ be a submodule of M such that $M = N + \mathfrak{q}M$, then $M = N$.*
4. *If R is a local ring and M, N are two finitely generated modules, then*

$$M \otimes_R N = 0 \iff M = 0 \text{ or } N = 0.$$

Proof. 1. (L $\Rightarrow$ R) Suppose there is $y \in R$ such that $1 - xy \notin R^\times$. Since each non-unit element is contained in a maximal ideal by Zorn's lemma, therefore $1 - xy \in \mathfrak{m}$ for some maximal ideal. Since $x \in \mathfrak{r}$, therefore $x \in \mathfrak{m}$. Hence $xy, 1 - xy \in \mathfrak{m}$, which means that $1 \in \mathfrak{m}$, a contradiction.
(R $\Rightarrow$ L) Suppose $1 - xy \in R^\times$ for all $y \in R$ and $x \notin \mathfrak{r}$. Then, again by Zorn's lemma we have $x \in R^\times$. Hence let $y = x^{-1}$ to get that $1 - xy = 1 - 1 = 0 \in R^\times$, a contradiction.

2. Suppose $M \neq 0$. Since M is finitely generated, therefore there is a submodule $N \subset M$ such that M/N is simple (has no proper non-trivial submodule). Simple R-modules are isomorphic to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R via the map $M' \mapsto R/\mathrm{Ann}(x)$ where $x \neq 0$ in M. Therefore $M/N \cong R/\mathfrak{m}$. Then, $\mathfrak{m}R \neq R$ which is same as $\mathfrak{m}M \neq M$. Since $\mathfrak{q} \subseteq \mathfrak{r} \subseteq \mathfrak{m}$, hence $\mathfrak{q}M \neq M$, a contradiction.

3. Apply 2. on M/N.

4. The only non-trivial part is L $\Rightarrow$ R. Since $(M \otimes_R N)/\mathfrak{m}(M \otimes_R N) = M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} N/\mathfrak{m}N$, therefore we have $M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} N/\mathfrak{m}N = 0$. Since $R/\mathfrak{m}$ is a field therefore $M/\mathfrak{m}M = 0$ WLOG. Hence, $M = \mathfrak{m}M$ and since R is local, therefore $\mathfrak{r} = \mathfrak{m}$. We conclude by Nakayama. $\qquad\square$

18.1.2 Localization

We next consider localization of rings and R-modules. Take any multiplicative set $S \subset R$ which contains 1. Then, localizing an R-module M on S is defined as

$$S^{-1}M := \{m/s \mid m \in M, s \in S\}.$$

where $m/s = n/t$ if and only if $\exists u \in S$ such that $u(mt - ns) = 0$. We have that $S^{-1}M$ is an R-module where addition $m/s + n/t = (mt + ns)/st$. In the case when $M = R$, we get a ring structure on $S^{-1}R$ as well where multiplication is given by $m/s \cdot n/t := mn/st$. There is a natural map $M \to S^{-1}M$ which maps $m \mapsto m/1$ and it may not be an injection if $\exists m \in M$ and $s \in S$ such that $s \cdot M = 0$.

Lemma 18.1.2.1. *Let $S \subset R$ be a multiplicative set in a ring R and M be an R-module. Then,*

$$S^{-1}M \cong S^{-1}R \otimes_R M.$$

Proof. One can do this by directly checking the universal property of tensor product of $S^{-1}R$ and M over R for $S^{-1}M$. We have the map $\varphi : S^{-1}R \times M \to S^{-1}M$ given by $(r/s, m) \mapsto rm/s$. Now for any bilinear map $f : S^{-1}R \times M \to N$, we can define the map $\tilde{f} : S^{-1}M \to N$ given by $\tilde{f}(m/s) := f(1/s, m)$. Clearly, $\tilde{f}$ is well-defined and $\tilde{f}\varphi = f$. Moreover, if $g : S^{-1}M \to N$ is such that $g\varphi = f$, then $g(m/s) = f(1/s, m) = \tilde{f}(m/s)$. Hence $\tilde{f}$ is unique with this property. $\square$

Lemma 18.1.2.2. *Localization w.r.t a multiplicative set $S \subset R$ is an exact functor on $\mathbf{Mod}(R)$.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Then we have the localized sequence $S^{-1}M' \to S^{-1}M \to S^{-1}M''$. Since $S^{-1}0 = 0$, therefore this is left exact. Exactness at middle follows from exactness at middle of the first sequence. The right exactness can be seen by right exactness of tensor product functor $S^{-1}R \otimes_R -$ and by Lemma 18.1.2.1. $\square$

Lemma 18.1.2.3. *Let R be a ring and $S \subset R$ be a multiplicative set. Then*

$$\{prime\ ideals\ of\ R\ not\ intersecting\ S\} \overset{\cong}{\longrightarrow} \{prime\ ideals\ of\ S^{-1}R\}$$
$$\mathfrak{p} \longmapsto S^{-1}\mathfrak{p}$$

Proof. Trivial. $\square$

Next we see an important property of modules, that is their "local characteristic". This means that one can check whether an element of a module is in a submodule by checking it locally at each prime, as the following lemma suggests. This has geometric significance in algebraic geometry (M induces and is induced by a quasi-coherent sheaf over $\mathrm{Spec}\,(R)$, see **??**).

Lemma 18.1.2.4. *Let M be an R-module. Then,*
   1. *$M \neq 0$ if and only if there exists a point $\mathfrak{p} \in \mathrm{Spec}\,(R)$ such that $M_{\mathfrak{p}} \neq 0$.*
   2. *If $N \subset M$ is a submodule and $0 \neq x \in M$, then $x \in N$ if and only if $x \in N_{\mathfrak{p}} \subseteq M_{\mathfrak{p}}$ for each point $\mathfrak{p} \in \mathrm{Spec}\,(R)$.*

Proof. 1. (L $\Rightarrow$ R) Since $\exists x \in M$ which is non-zero, therefore consider the annihilator ideal $\mathrm{Ann}(x) = \{r \in R \mid rx = 0\}$ of R. Then, this ideal is contained in a maximal ideal $\mathfrak{m}$ of R by Zorn's lemma. Hence consider $M_{\mathfrak{m}}$, which contains $x/1$. Now if there exists $r \in R \setminus \mathfrak{m}$

such that $rx = 0$, then $r \in \mathrm{Ann}(x)$, but since $\mathfrak{m} \supseteq \mathrm{Ann}(x)$, hence we have a contradiction. (R $\Rightarrow$ L) Let $\mathfrak{p} \in \mathrm{Spec}\,(R)$ be such that $x/r \in M_\mathfrak{p}$ and $x/r \neq 0$. Since $M_\mathfrak{p}$ is an R-module, therefore $r \cdot (x/r)$ is well-defined in $M_\mathfrak{p}$. Hence $(rx)/r = x/1 \in M_\mathfrak{p}$. If $x/1 = 0$ in $M_\mathfrak{p}$, therefore $\varphi_\mathfrak{p}(x) = 0$ and hence $x = 0$ as $\varphi_\mathfrak{p}$ is injective. Thus, $x/r = 0$ in $M_\mathfrak{p}$, a contradiction. Therefore $x/1 \neq 0$ and hence $x \neq 0$ in M.

2. This follows from using 1. on the module $(N + Rx)/N$. We do this by observing the following chain of equivalences, whose key steps are explained below:

$x \in N \iff N + Rx = N \iff (N + Rx)/N = 0 \iff ((N + Rx)/N)_\mathfrak{p} \ \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff$
$(N + Rx)_\mathfrak{p}/N_\mathfrak{p} = 0 \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff (N + Rx)_\mathfrak{p} = N_\mathfrak{p} \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff$
$N_\mathfrak{p} + (Rx)_\mathfrak{p} = N_\mathfrak{p} \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff (Rx)_\mathfrak{p} \subseteq N_\mathfrak{p} \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff \varphi_\mathfrak{p}(x) = x/1 \in N_\mathfrak{p} \forall \mathfrak{p} \in \mathrm{Spec}\,(R).$

For two submodules $N, K, L \subset M$ where $L \subseteq N$ and $\mathfrak{p} \in \mathrm{Spec}\,(R)$, we get $(N/L)_\mathfrak{p} = N_\mathfrak{p}/L_\mathfrak{p}$ by exactness of localization (Lemma 18.1.2.2) on the exact sequence

$$0 \to L \to N \to N/L \to 0.$$

Finally $(N + K)_\mathfrak{p} = N_\mathfrak{p} + K_\mathfrak{p}$ in $M_\mathfrak{p}$ is true by direct checking and where we use the primality of $\mathfrak{p}$. $\qquad\square$

Remark 18.1.2.5. (*Few life hacks*) The above proof tells us few ways how one can approach the problems in ring theory. Note especially that $x \in N$ if and only if $N + Rx = N$, which quickly turns a set-theoretic relation into an algebraic one, where we can now use various constructions as we did, like localization.

The following is the universal property for localization.

Proposition 18.1.2.6. *Let R be a ring and S be a multiplicative set. If $\varphi : R \to T$ is a ring homomorphism such that $\varphi(S) \subseteq T^\times$ where $T^\times$ is the unit group of T, then there exists a unique map $\tilde\varphi : S^{-1}R \to T$ such that the following commutes*

$$
\begin{array}{ccc}
R & \xrightarrow{\ \varphi\ } & T \\
{\scriptstyle i}\downarrow & \nearrow{\scriptstyle \tilde\varphi} & \\
S^{-1}R & &
\end{array}
\quad.
$$

Proof. Pick any ring map $\varphi : R \to T$. Take any map $f : S^{-1}R \to T$ which makes the above commute. We claim that $f(r/s) = \varphi(r)\varphi(s)^{-1}$. Indeed, we have that $f(r/1) = \varphi(r)$ for all $r \in R$. Further, for any $s \in S$, we have $f(1/s) = 1/f(s/1) = 1/\varphi(s) = \varphi(s)^{-1}$. Consequently, we get for any $r/s \in S^{-1}R$ the following

$$f\left(\frac{r}{s}\right) = f\left(\frac{r}{1} \cdot \frac{1}{s}\right) = f\left(\frac{r}{1}\right) \cdot f\left(\frac{1}{s}\right) = \varphi(r)\varphi(s)^{-1}.$$

This proves uniqueness. Clearly, this is a ring homomorphism. This completes the proof. $\quad\square$

Remark 18.1.2.7. As Proposition 18.1.2.6 is the universal property of localization, therefore the construction $S^{-1}R$ is irrelevant; the property above completely characterizes localization upto a unique isomorphism.

Lemma 18.1.2.8. *Let R be a ring and $f \in R \setminus \{0\}$. Then,*

$$R_f \cong \frac{R[x]}{\langle fx - 1 \rangle}.$$

In particular, R_f is a finite type R-algebra.

Proof. We shall use Proposition 18.1.2.6. We need only show that $R[x]/\langle fx - 1 \rangle$ satisfies the same universal property as stated in Proposition 18.1.2.6. Indeed, we first have the map $i : R \to R[x]/\langle fx - 1 \rangle$ given by $r \mapsto r + \langle fx - 1 \rangle$. Let $\varphi : R \to T$ be any map such that $\varphi(f) \in T^\times$. We claim that there exists a unique map $\tilde{\varphi} : R[x]/\langle fx - 1 \rangle \to T$ such that $\tilde{\varphi} \circ i = \varphi$. Indeed, take any map $g : R[x]/\langle fx - 1 \rangle \to T$ such that $g \circ i = \varphi$. Thus, for all $r \in R$, we have $g(r + \langle fx - 1 \rangle) = \varphi(r)$. As $fx + \langle fx - 1 \rangle = 1 + \langle fx - 1 \rangle$, therefore we obtain that $g(f + \langle fx - 1 \rangle) \cdot g(x + \langle fx - 1 \rangle) = \varphi(f) \cdot g(x + \langle fx - 1 \rangle) = 1$. Hence, we see that $g(x + \langle fx - 1 \rangle) = \varphi(f)^{-1}$. Hence for any element $p(x) + \langle fx - 1 \rangle$, we see that $f(p(x) + \langle fx - 1 \rangle) = p(\varphi(f)^{-1})$. This makes g unique well-defined ring homomiorphism. This completes the proof. $\square$

The following is a simple but important application of technique of localization.

Lemma 18.1.2.9. *Let R be a ring. Then the nilradical of R, $\mathfrak{n}$, the ideal consisting of nilpotent elements is equal to the intersection of all prime ideals of R:*

$$\mathfrak{n} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}.$$

Proof. Take $x \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$. We then have $x \in \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Spec}(R)$. Hence if for each $n \in \mathbb{N}$ we have that $x^n \neq 0$, then we get that $S = \{1, x, x^2, \dots\}$ forms a multiplicative system. Considering the localization $S^{-1}R$, we see that it is non-zero. Therefore $S^{-1}R$ has a prime ideal, which corresponds to a prime ideal $\mathfrak{p}$ of R which does not intersects S, by Lemma 18.1.2.3. But this is a contradiction as x is in every prime ideal.
Conversely, take any $x \in \mathfrak{n}$ and any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$. Since $x^n = 0$ for some $n \in \mathbb{N}$, therefore $x^n \in \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Spec}(R)$. Hence it follows from primality of each $\mathfrak{p}$ that $x \in \mathfrak{p}$. $\square$

We next give two results which are of prominent use in algebraic geometry. The first result says that finite generation of a module can be checked locally.

Lemma 18.1.2.10. *Let M be an R-module and suppose $f_i \in R$ are elements such that $\sum_{i=1}^n Rf_i = R$. Then, the following are equivalent:*
   1. M is a finitely generated R-module.
   2. M_{f_i} is a finitely generated R_{f_i}-module for all $i = 1, \dots, n$.

Proof. (1. $\Rightarrow$ 2.) This is simple, as finite generation is preserved under localization.
(2. $\Rightarrow$ 1.) Let M_{f_i} be generated by $m_{ij}/(f_i)^{n_{ij}}$ for $j = 1, \dots, n_i$. Let $N \leq M$ be a submodule generated by m_{ij} for each $j = 1, \dots, n_i$ and for each $i = 1, \dots, n$. Clearly, N is a finitely generated R-module. Moreover, N_{f_i} for each $i = 1, \dots, n$ is equal to M_{f_i}. Since localization at a prime ideal $\mathfrak{p} \lneq R$ is given by direct limit of all localization of elements not

in $\mathfrak{p}$, therefore $(M/N)_{\mathfrak{p}} \cong \varinjlim_{i=1,\ldots,n} (M/N)_{f_i}$ and since $M_{f_i} = N_{f_i}$, therefore $(M/N)_{f_i} = 0$. It follows that $(M/N)_{\mathfrak{p}} = 0$ for all primes $\mathfrak{p}$ and hence $M/N = 0$ by Lemma 18.1.2.4, 1, hence $M = N$ and M is finitely generated. $\qquad\square$

The second result gives a partial analogous result as to Lemma 18.1.2.10 did, but for algebras. This is again an important technical tool used often in algebraic geometry.

Lemma 18.1.2.11. *Let A be a ring and B be an A-algebra. Suppose $f_1, \ldots, f_n \in B$ are such that $\sum_{i=1}^n B f_i = B$. If for all $i = 1, \ldots, n$, B_{f_i} is a finitely generated A-algebra, then B is a finitely generated A-algebra.*

Proof. Let B_{f_i} be generated by

$$\left\{ \frac{b_{ij}}{f_i^{n_j}} \right\}_{j=1,\ldots,M_i}$$

as an A-algebra, for each $i = 1, \ldots, n$. Further, we have $c_1, \ldots, c_n \in B$ such that $c_1 f_1 + \cdots + c_n f_n = 1$. We claim that $S = \{b_{ij}, f_i, c_i\}_{i,j}$ is a finite generating set for B.

Let C be the sub-algebra of B generated by S. Pick any $b \in B$. We wish to show that $b \in C$. Fix an $i = 1, \ldots, n$. Observe that the image of b in the localized ring B_{f_i} is generated by some polynomial with coefficients in A and indeterminates replaced by

$$\left\{ \frac{b_{ij}}{f_i^{n_j}} \right\}_{j=1,\ldots,M_i}.$$

We may multiply b by $f_i^{N_i}$ for N_i large enough so that $f_i^{N_i} b$ is then represented by a polynomial with coefficients in A evaluated in f_i and b_{ij} for $j = 1, \ldots, M_i$. Consequently, $f_i^{N_i} b \in C$, for each $i = 1, \ldots, n$. Observe that $f_1, \ldots, f_n$ in C generates the unit ideal in C. By Lemma 18.21.0.2, 2, we see that $f_1^{N_1}, \ldots, f_n^{N_n}$ also generates the unit ideal in C. Hence, we have $d_1, \ldots, d_n \in C$ such that $1 = d_1 f_1^{N_1} + \cdots + d_n f_n^{N_n}$. Multiplying by b, we obtain $b = d_1 f_1^{N_1} b + \cdots + d_n f_n^{N_n} b$ where by above, we now know that each term is in C. This completes the proof. $\qquad\square$

An observation which is of importance in the study of varieties is the following

Lemma 18.1.2.12. *Let R be an integral domain. Then*

$$\bigcap_{\mathfrak{m} < R} R_{\mathfrak{m}} \cong R$$

where the intersection runs over all maximal ideals $\mathfrak{m}$ of R and the intersection is carried out in the fraction field $R_{\langle 0 \rangle}$.

Proof. We already have that

$$R \hookrightarrow R_{\mathfrak{m}}$$

for any maximal ideal $\mathfrak{m} < R$. Thus,

$$R \hookrightarrow \bigcap_{\mathfrak{m} < R} R_{\mathfrak{m}}.$$

Thus it would suffice to show that $\bigcap_{\mathfrak{m} < R} R_{\mathfrak{m}} \hookrightarrow R$. Indeed, consider the following map

$$\bigcap_{\mathfrak{m} < R} R_{\mathfrak{m}} \longrightarrow R$$

$$[f_{\mathfrak{m}}/g_{\mathfrak{m}}] \longmapsto f_{\mathfrak{m}} g_{\mathfrak{m}'}$$

where $f_{\mathfrak{m}}/g_{\mathfrak{m}} = f_{\mathfrak{m}'}/g_{\mathfrak{m}'}$ for two maximal ideals $\mathfrak{m}, \mathfrak{m}'$ in R. Thus, $f_{\mathfrak{m}} g_{\mathfrak{m}'} = f_{\mathfrak{m}'} g_{\mathfrak{m}}$. Hence the above map is well-defined and is injective as $f_{\mathfrak{m}} g_{\mathfrak{m}'} = 0$ implies $f_{\mathfrak{m}} = 0$ as $g_{\mathfrak{m}'} \neq 0$. The result follows. $\qquad\square$

Homogeneous localization

The following is a discussion on localization of a graded ring S at a homogeneous prime ideal $\mathfrak{p}$. Let T denote the multiplicative subset of S consisting of all homogeneous elements not contained in $\mathfrak{p}$. Consequently, we define

$$S_{(\mathfrak{p})} := (T^{-1}S)_0$$

where $(T^{-1}S)_0$ is the degree 0 elements in the localization $T^{-1}S$. We call this the *homogeneous localization* of the graded ring S at the homogeneous prime ideal $\mathfrak{p}$. Thus $S_{(\mathfrak{p})} = (S_{\mathfrak{p}})_0$, i.e. homogeneous localization just picks out degree 0 elements from the usual localization. Note that the usual localization $T^{-1}S$ is a graded ring where grading is given by subtracting the degree of numerator by grading of denominator.

Lemma 18.1.2.13. *Let S be a graded ring and $\mathfrak{p}$ be a homogeneous prime ideal of S. Then, the homogeneous localization $S_{\mathfrak{p}}$ is a local ring.*

Proof. Consider the set $\mathfrak{m} := (\mathfrak{p} \cdot T^{-1}S) \cap S_{\mathfrak{p}}$. Then, $\mathfrak{m}$ is a maximal ideal of $S_{\mathfrak{p}}$ as any element not in $\mathfrak{m}$ in $S_{\mathfrak{p}}$ is a fraction f/g where $\deg f = \deg g$ and $f \notin \mathfrak{p}$ and thus it is invertible. Consequently, $S_{\mathfrak{p}}$ is local. $\qquad\square$

Remark 18.1.2.14. Note that if S is a graded domain, then $S_{(\langle 0 \rangle)}$ yields a field whose elements are of the form f/g where $\deg f = \deg g$ and f, g g is a non-zero homogeneous element of S. This field is called the *homogeneous fraction field* of graded domain S. This is a subfield of usual fraction field $S_{\langle 0 \rangle}$.

Let S be a graded ring and $g \in S$ be a homogeneous element. The *homogeneous localization of S at g* is defined to be the following subring of S_g:

$$S_{(g)} := \{f/g^n \in S_g \mid f \text{ is homogeneous with } \deg f = n \deg g, \ n \in \mathbb{N}\} \leq S_g.$$

Let S be a graded ring. Then an S-module M is said to be *graded S-module* if $M = \bigoplus_{d \in \mathbb{Z}} M_d$ where $M_d \leq M$ is a subgroup of M such that $S_d \cdot M_e \subseteq M_{d+e}$. Then, for a homogeneous element $g \in S$, we denote by $M_{(g)}$ the following submodule of M_g:

$$M_{(g)} := \{m/g^n \mid m \text{ is homogeneous with } \deg m = n \deg g, \ n \in \mathbb{N}\} \leq M_g.$$

For each graded S-module M, one can attach a sequence of graded modules.

Definition 18.1.2.15. (**Twisted modules**) Let S be a graded ring and M a graded S-module. Then, define

$$M(l) := \bigoplus_{d \in \mathbb{Z}} M_{d+l}$$

to be the *l-twisted graded module of M*.

An important lemma with regards to localization of a graded ring at a positive degree element is as follows, it will prove its worth in showing that projective spectrum of a graded ring is a scheme (see Lemma **??**).

Lemma 18.1.2.16. *Let S be a graded ring and $f \in S_d$, $d > 0$. Then we have a bijection*

$$D_+(f) \cong \mathrm{Spec}\left(S_{(f)}\right)$$

where $D_+(f) \subseteq \mathrm{Spec}\,(S)$ is the set of all homogeneous prime ideals of S which does not contain f and does not contain S_+.

Proof. Consider the following map

$$\varphi : D_+(f) \longrightarrow \mathrm{Spec}\left(S_{(f)}\right)$$
$$\mathfrak{p} \longmapsto (\mathfrak{p} \cdot S_f)_0,$$

that is, the degree zero elements of the prime ideal $\mathfrak{p} \cdot S_f$ of S_f. Indeed, $\varphi(\mathfrak{p})$ is a prime ideal of $S_{(f)}$. Further, if $(\mathfrak{p} \cdot S_f)_0 = (\mathfrak{q} \cdot S_f)_0$ for $\mathfrak{p}, \mathfrak{q} \in D_+(f)$, then for any $g \in \mathfrak{p}$, one observes via above equality that $g \in \mathfrak{q}$. Consequently, $\mathfrak{p} = \mathfrak{q}$. Thus φ is injective. For surjectivity, pick any prime ideal $\mathfrak{p} \in \mathrm{Spec}\left(S_{(f)}\right)$. We will construct a prime ideal $\mathfrak{q} \in D_+(f)$ such that $\varphi(\mathfrak{q}) = \mathfrak{p}$. Indeed, let $K = \{g \in S \mid g \text{ is homogeneous } \& \exists n > 0 \text{ s.t. } g/f^n \in \mathfrak{p}\}$ and consider the ideal

$$\mathfrak{q} = \langle K \rangle.$$

We thus need to check the following statements to complete the bijection:
1. $\mathfrak{q}$ is not the unit ideal of S,
2. $\mathfrak{q}$ is homogeneous in S,
3. $\mathfrak{q}$ is prime in S,
4. $\mathfrak{q}$ doesn't contain f,
5. $(\mathfrak{q} \cdot S_f)_0 = \mathfrak{p}$.

Statement 4 tells us that $\mathfrak{q}$ doesn't contain S_+. Statement 1 follows from a degree argument; if $1 \in \mathfrak{q}$, then $1 = a_1 g_1 + \cdots + a_m g_m$ for $g_i \in K$ and $a_i \in S$, but 1 is a degree 0 element whereas the minimum degree of the right is atleast > 0. Statement 2 is immediate as $\mathfrak{q}$ is generated by homogeneous elements. For statement 3, it is enough to check for homogeneous elements $h, k \in S$ that $hk \in \mathfrak{q} \implies h \in \mathfrak{q}$ or $k \in \mathfrak{q}$. This is immediate, after observing that any homogeneous element of $\mathfrak{q}$ is in K because K is the set of all homogeneous elements of S of positive degree which is not a power of f. Statements 4 and 5 are immediate checks. $\square$

Local rings

A ring R is said to be *local* if there is a unique maximal ideal of R. In such a case we denote it by $(R, \mathfrak{m})$.

Definition 18.1.2.17. (Zariski (co)tangent space) Let $(R, \mathfrak{m})$ be a local ring. Then, we define the Zariski *cotangent space* of $(R, \mathfrak{m})$ to be $T^*R = \mathfrak{m}/\mathfrak{m}^2$ and the Zariski *tangent space* to be its dual $TR = \operatorname{Hom}_k\left(\mathfrak{m}/\mathfrak{m}^2, k\right)$.

Remark 18.1.2.18. The Zariski cotangent space T^*R is a κ-vector space where $\kappa = R/\mathfrak{m}$ is the residue field. Indeed, the scalar multiplication is given by

$$\kappa \times T^*R \longrightarrow T^*R$$
$$(c + \mathfrak{m}, x + \mathfrak{m}^2) \longmapsto cx + \mathfrak{m}^2$$

where $c \in R$ and $x \in \mathfrak{m}$. Indeed, this is well-defined as can be seen by a simple check. Consequently, the tangent space $TR = \operatorname{Hom}_k\left(\mathfrak{m}/\mathfrak{m}^2, k\right)$ is also a κ-vector space.

Definition 18.1.2.19. (Regular local ring) Let $(A, \mathfrak{m})$ be a local ring with $k = A/\mathfrak{m}$ being the residue field. Then A is said to be regular if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

There is an important geometric lemma that one should keep in mind about certain local rings.

Definition 18.1.2.20. (Rational local k-algebras) Let k be a field. A local k-algebra $(R, \mathfrak{m})$ is said to be rational if its residue field $\kappa = R/\mathfrak{m}$ is isomorphic to the field k.

Rational local k-algebras have a rather simple tangent space.

Proposition 18.1.2.21. *Let $(A, \mathfrak{m}_A)$ be a rational local k-algebra. Then,*

$$TA \cong \operatorname{Hom}_{k,\mathrm{loc}}(A, k[\epsilon])$$

where $k[\epsilon] := k[x]/x^2$ is the ring of dual numbers and $\operatorname{Hom}_{k,\mathrm{loc}}(A, k[\epsilon])$ denotes the set of all local k-algebra homomorphisms.

Proof. Pick any k-algebra homomorphism $\varphi : A \to k[\epsilon]$. Denote by $\mathfrak{m}_\epsilon = \langle \epsilon \rangle \lneq k[\epsilon]$ the unique maximal ideal of $k[\epsilon]$. Since

$$k[\epsilon]/\mathfrak{m}_\epsilon \cong k,$$

therefore $k[\epsilon]$ is a rational local k-algebra as well. By Lemma 18.21.0.7, we may write $A = k \oplus \mathfrak{m}_A$ and $k[\epsilon] = k \oplus \mathfrak{m}_\epsilon$. We now claim that the datum of a local k-algebra homomorphism $\varphi : A \to k[\epsilon]$ is equivalent to datum of a k-linear map of k-modules $\theta : \mathfrak{m}_A/\mathfrak{m}_A^2 \to k$.

Indeed, we first observe that for any $\varphi : A \to k[\epsilon]$ as above, we have $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_\epsilon$. Thus, $\varphi(\mathfrak{m}_A^2) \subseteq \mathfrak{m}_\epsilon^2 = 0$. Thus, we deduce that for any such φ, $\operatorname{Ker}(\varphi) \supseteq \mathfrak{m}_A^2$. It follows from universal property of quotients that any such φ is in one-to-one correspondence with k-algebra homomorphisms

$$\tilde{\varphi} : A/\mathfrak{m}_A^2 \cong k \oplus (\mathfrak{m}_A/\mathfrak{m}_A^2) \longrightarrow k[\epsilon].$$

As $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_\epsilon$, therefore $\tilde{\varphi}(\mathfrak{m}_A/\mathfrak{m}_A^2) \subseteq \mathfrak{m}_\epsilon$. Thus, we obtain a k-linear map of k-modules

$$\theta : \mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow k \cong \mathfrak{m}_\epsilon$$

where $\mathfrak{m}_\epsilon \cong k$ as k-modules. It suffices to now show that from any such θ, one can obtain a unique k-algebra map $\tilde{\varphi} : k \oplus (\mathfrak{m}_A/\mathfrak{m}_A^2) \to k[\epsilon]$, which furthermore sets up a bijection between all such $\tilde{\varphi}$ and θ.

Indeed, from k-linear map θ, we may construct the following k-algebra map

$$\tilde{\varphi} : k \oplus (\mathfrak{m}_A/\mathfrak{m}_A^2) \longrightarrow k[\epsilon]$$
$$(k + \bar{m}) \longmapsto k + \theta(\bar{m})\epsilon.$$

Then we observe that $\tilde{\varphi}$ is a k-algebra homomorphism as

$$\begin{aligned}
\tilde{\varphi}((k_1 + \bar{m}_1)(k_2 + \bar{m}_2)) &= \tilde{\varphi}(k_1 k_2 + k_1 \bar{m}_2 + k_2 \bar{m}_1 + \bar{m}_1 \bar{m}_2) \\
&= k_1 k_2 + k_1 \theta(\bar{m}_2)\epsilon + k_2 \theta(\bar{m}_1)\epsilon + \theta(\bar{m}_1 \bar{m}_2)\epsilon \\
&= k_1 k_2 + k_1 \theta(\bar{m}_2)\epsilon + k_2 \theta(\bar{m}_1)\epsilon \\
&= (k_1 + \theta(\bar{m}_1)\epsilon) \cdot (k_2 + \theta(\bar{m}_2)\epsilon) \\
&= \tilde{\varphi}(k_1 + \bar{m}_1) \cdot \tilde{\varphi}(k_2 + \bar{m}_2).
\end{aligned}$$

Hence, from θ one obtain $\tilde{\varphi}$ back, thus setting up a bijection and completing the proof. $\square$

18.1.3   Structure theorem

Let M be a finitely generated R-module. We can understand the structure of such modules completely in terms of the ring R, when R is a PID (so that it's UFD). This is the content of the structure theorem. We first give the following few propositions which is used in the proof of the structure theorem but is of independent interest as well, in order to derive a usable variant of structure theorem. The following theorem tells us a direct sum decomposition exists for any finitely free torsion module over a PID.

Proposition 18.1.3.1. *Let M be a finitely generated torsion module over a PID R. If* $\mathrm{Ann}(M) = \langle c \rangle$ *where $c = p_1^{k_1} \ldots p_r^{k_r}$ and $p_i \in R$ are prime elements, then*

$$M \cong M_1 \oplus \cdots \oplus M_r$$

where $M_i = \{x \in M \mid p_i^{r_i} x = 0\} \leq M$ for all $i = 1, \ldots, r$, that is, where $\mathrm{Ann}(M_i) = \langle p_i^{r_i} \rangle$ for all $i = 1, \ldots, r$.

The next result tells us that we can further write each of the above M_is as a direct sum decomposition of a special kind.

Proposition 18.1.3.2. *Let M be a finitely generated torsion module over a PID R. If* $\mathrm{Ann}M = \langle p^r \rangle$ *where $p \in R$ is a prime element, then there exists $r_1 \geq r_2 \geq \cdots \geq r_k \geq 1$ such that*

$$M \cong R/\langle p^{r_1} \rangle \oplus \cdots \oplus R/\langle p^{r_k} \rangle.$$

The structure theorem is as follows.

Theorem 18.1.3.3. *(Structure theorem) Let R be a PID and M be a finitely generated R-module. Then there exists an unique $n \in \mathbb{N} \cup \{0\}$ and $q_1, \ldots q_r \in R$ unique upto units such that $q_{i-1} | q_i$ for all $i = 2, \ldots, r$ and*

$$M \cong R^n \oplus R/\langle q_1 \rangle \oplus \cdots \oplus R/\langle q_r \rangle.$$

The most useful version of this is the following:

Corollary 18.1.3.4. *Let M be a finitely generated torsion module over a PID R. Then, there exists k-many prime elements $p_1, \ldots, p_k \in R$, $n_j \in \mathbb{N}$ for each $j = 1, \ldots, k$ and $1 \leq r_{1j} \leq \cdots \leq r_{n_j j} \in \mathbb{N}$ for each $j = 1, \ldots, k$ such that*

$$M \cong \bigoplus_{j=1}^{k} \left(R/\langle p_j^{r_{1j}} \rangle \oplus \cdots \oplus R/\langle p_j^{n_j j} \rangle \right).$$

Proof. This is a consequence of Propositions 18.1.3.1 and 18.1.3.2. $\square$

This is the famous structure theorem for finitely generated modules over a PID. Note that the ring $\mathbb{Z}$ is PID and any abelian group is a $\mathbb{Z}$-module. Thus, we can classify finitely generated abelian groups using the structure theorem.

Example 18.1.3.5. An example of a module which is not finitely generated is the polynomial module $R[x]$ over a ring R. Indeed, the collection $\{1, x, x^2, \ldots\}$ will make it free but not finitely generated.

Example 18.1.3.6. Classification of all abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$, for example, can be achieved via structure theorem. Indeed using Corollary 18.1.3.4, we will get that there are 6 total such abelian groups given by
- $\left(\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}} \right)$
- $\left(\frac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}} \right)$
- $\left(\frac{\mathbb{Z}}{2^3\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}} \right)$
- $\left(\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{3^2\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}} \right)$
- $\left(\frac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{3^2\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}} \right)$
- $\left(\frac{\mathbb{Z}}{2^3\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{3^2\mathbb{Z}} \right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}} \right)$

18.1.4   UFDs

Add results surrounding primitive polynomials and Gauss' lemma here from notebook.

Spectra of polynomial rings over UFDs

We now calculate the prime spectra of polynomial rings over UFDs. For that, we need the following two lemmas.

Lemma 18.1.4.1. *Let R be a UFD and $I \leq R[x]$ be an ideal containing two elements with no common factors. Then I contains a non-zero constant from R.*

Proof. Indeed, let $f, g \in R[x]$ be two elements with no common factors. Let Q denote the fraction field of R. We first claim that $f, g \in Q[x]$ have no common factor as well. Indeed, suppose $h(x) \in Q[x]$ is a common factor of $f(x)$ and $g(x)$. It follows from the result on primitive polynomials that we can write $h(x) = ch_0(x)$ where $c \in Q$ and $h_0(x) \in R[x]$ is primitive. Hence, we see that $h_0(x) \in R[x]$ is a polynomial such that $h_0 \div f$ and $h_0 \div g$ in $Q[x]$. Again, by general results in UFD, we then conclude that $h_0 \div f$ and $h_0 \div g$ in $R[x]$. As f and g have no common factor, therefore $h_0(x) \in R[x]$ is a unit. Hence $h(x) \in Q[x]$ is a unit. Thus, there is no common factor of $f(x)$ and $g(x)$ in $Q[x]$ if there is none in $R[x]$.

Hence, $f(x), g(x)$ in $Q[x]$ have gcd 1, where $Q[x]$ is a PID. Consequently, $f(x)$ and $g(x)$ generates the unit ideal in $Q[x]$. It follows that there exists $p(x), q(x) \in Q[x]$ such that

$$1 = p(x)f(x) + q(x)g(x).$$

By theorem on primitive polynomials, we may write $p(x) = \frac{a}{b}p_0(x)$ and $q(x) = \frac{c}{d}q_0(x)$ where $a/b, c/d \in Q$ and $p_0(x), q_0(x) \in R[x]$ are primitive. The above equation hence becomes

$$1 = \frac{a}{b}p_0(x)f(x) + \frac{c}{d}q_0(x)g(x)$$
$$= \frac{adp_0(x)f(x) + bcq_0(x)g(x)}{bd},$$

which thus yields

$$bd = adp_0(x)f(x) + bcq_0(x)g(x)$$

where RHS is in $I \leq R[x]$ because $ad, p_0, bc, q_0 \in R[x]$ and $f, g \in I$ and LHS is in R. Hence $I \cap R$ is not zero. $\qquad \square$

Lemma 18.1.4.2. *Let R be a PID and $f, g \in R[x]$ be non-zero polynomials such that f and g have no common factors. Then,*

1. *any prime ideal $\mathfrak{p} \lneq R[x]$ containing f and g is maximal,*
2. *any maximal ideal $\mathfrak{m} \lneq R[x]$ containing f and g is of the form $\langle p, h(x) \rangle$ where $p \in R$ is prime and $h(x)$ is prime modulo p,*
3. *there are only finitely many maximal ideals of $R[x]$ containing f and g.*

Proof. 1. : Let $\mathfrak{p} \lneq R[x]$ be a prime ideal containing f and g. Observe by Lemma 18.1.4.1 that there exists $b \in R \setminus 0$ such that $b \in \mathfrak{p} \cap R$, that is, $\mathfrak{p} \cap R \neq 0$. As R is a PID and $\mathfrak{p} \cap R$ is a prime ideal of R, therefore $\mathfrak{p} \cap R = pR$ for some prime element $p \in \mathfrak{p} \cap R$. We wish to show that $R[x]/\mathfrak{p}$ is a field. Indeed, we see that (note $\langle p, \mathfrak{p} \rangle = \mathfrak{p}$ as $p \in \mathfrak{p}$)

$$\frac{R[x]}{\mathfrak{p}} \cong \frac{\frac{R[x]}{pR[x]}}{\frac{\langle p, \mathfrak{p} \rangle}{pR[x]}} = \frac{\frac{R[x]}{pR[x]}}{\frac{\mathfrak{p}}{pR[x]}}$$
$$\cong \frac{\frac{R}{pR}[x]}{\overline{\mathfrak{p}}}$$

where $\overline{\mathfrak{p}} = \pi(\mathfrak{p})$ where $\pi : R[x] \twoheadrightarrow \frac{R}{pR}[x]$ is the quotient map. As R is a PID and pR is a non-zero prime ideal, therefore it is maximal. Consequently, R/pR is a field and hence

$\frac{R}{pR}[x]$ is a PID. Suppose $\overline{\mathfrak{p}} = 0$, then f and g have a common factor given by $p \in R$, which is not possible. Consequently, $\overline{\mathfrak{p}}$ is a proper prime ideal of $\frac{R}{pR}[x]$ by correspondence theorem. But in PIDs, non-zero prime ideals are maximal ideals, hence we obtain that $\frac{R}{pR}[x]/\overline{\mathfrak{p}}$ is a field, as required.

2. : Let $\mathfrak{m} \lneq R[x]$ be a maximal ideal of $R[x]$ containing f and g. Hence, from Lemma 18.1.4.1 and R being a PID, there exists $p \in R$ a prime such that $\mathfrak{m} \cap R = pR$. Hence R/pR is a field as R is a PID and pR a non-zero prime ideal (so maximal). Consequently, we have a quotient map

$$\pi : R[x] \twoheadrightarrow \frac{R[x]}{pR[x]} \cong \frac{R}{pR}[x]$$

As $p \in \mathfrak{m}$, therefore by correspondence thereom $\pi(\mathfrak{m}) = \overline{\mathfrak{m}}$ is a maximal ideal of $\frac{R}{pR}[x]$. As R/pR is a field, therefore $\frac{R}{pR}[x]$ is a PID. Hence, $\overline{\mathfrak{m}} = \langle \overline{h(x)} \rangle$ for some $h(x) \in R[x]$ such that $\overline{h(x)}$ is irreducible (so it generates a maximal ideal). Again, by correspondence theorem we have $\pi^{-1}(\overline{\mathfrak{m}}) = \mathfrak{m} = h(x)R[x] + pR[x] = \langle p, h(x) \rangle$, as required.

3. : We will use notations of proof of 2. above. Take any maximal ideal $\mathfrak{m} = \langle p, h(x) \rangle \lneq R[x]$ which contains $f(x)$ and $g(x)$, $p \in R$ is prime and $h(x)$ is irreducible modulo p. As R is a PID, so it is a UFD, hence $R[x]$ is a UFD by Gauss' lemma. Hence, writing $f(x)$ and $g(x)$ as product of prime factors in $R[x]$, we observe that there exists distinct primes $p(x), q(x) \in R[x]$ such that $p(x), q(x) \in \mathfrak{m}$. Replacing f by p and g by q, we may assume f and g are irreducible (or prime) in $R[x]$.

By Lemma 18.1.4.1, there exists $b \in R \setminus 0$ such that $b \in \mathfrak{m} \cap R$. As the proof of 2. above shows, $p \div b$ in R. As R is a PID, so it is a UFD, hence there are only finitely many choices for p.

Now, going modulo prime p, we see that $\overline{f(x)}, \overline{g(x)} \in \overline{\mathfrak{m}} \lneq \frac{R}{pR}[x]$ has a common factor in $\frac{R}{pR}[x]$, given by $\overline{h(x)}$ as $\overline{\mathfrak{m}} = \langle \overline{h(x)} \rangle$ (by proof of 2.). As $\overline{h(x)}$ generates a maximal ideal in $\frac{R}{pR}[x]$, therefore $\overline{h(x)}$ is a prime element of $\frac{R}{pR}[x]$, which has to divide $\overline{f(x)}$ and $\overline{g(x)}$. As $\frac{R}{pR}[x]$ is a PID, therefore there are only finitely many choices for $\overline{h(x)}$, and since $\mathfrak{m} = \pi^{-1}(\langle \overline{h(x)} \rangle)$, therefore every choice of p as above, yields finitely many choices for $\mathfrak{m}$.

Consequently, there are finitely many choices for p and once p is fixed, there are only finitely many choices for the ideal $\overline{\mathfrak{m}}$. As $\mathfrak{m} = \pi^{-1}(\overline{m})$, therefore there are finitely many maximal ideals containing f and g. $\square$

We now classify $\mathrm{Spec}\,(R[x])$ for a UFD R.

Theorem 18.1.4.3. *Let R be a PID. Any prime ideal $\mathfrak{p} \lneq R[x]$ is of one of the following forms*

1. *$\mathfrak{p} = \mathfrak{o}$,*
2. *$\mathfrak{p} = \langle f(x) \rangle$ for some irreducible $f(x) \in R[x]$,*
3. *$\mathfrak{p} = \langle p, h(x) \rangle$ for some prime $p \in R$ and $h(x) \in R[x]$ irreducible modulo p and this is also a maximal ideal.*

Proof. Indeed, pick any prime ideal $\mathfrak{p} \lneq R[x]$. If $\mathfrak{p}$ is 0, then it is prime as $R[x]$ is a domain. We now have two cases. If $\mathfrak{p}$ is principal, then $\mathfrak{p} = \langle f(x) \rangle$ for some $f(x) \in R[x]$. As $\langle f(x) \rangle$

is prime therefore $f(x)$ is a prime element. As $R[x]$ is a UFD by Gauss' lemma, therefore $f(x)$ is also irreducible. Consequently, $\mathfrak{p} = \langle f(x) \rangle$ where $f(x)$ is irreducible.

On the other hand if $\mathfrak{p}$ is not principal, there exists $f(x), g(x) \in \mathfrak{p}$ such that $f(x) \nmid g(x)$ and $g(x) \nmid f(x)$. As $R[x]$ is a UFD and $\mathfrak{p}$ is prime, therefore there exists prime factors of f and g which are in $\mathfrak{p}$. Replacing f and g by these prime factors, we may assume f and g are distinct irreducibles in $\mathfrak{p}$. Consequently, by Lemma 18.1.4.2, we see that $\mathfrak{p} = \langle p, h(x) \rangle$ for some prime $p \in R$ and $h(x)$ irreducible modulo p. Moreover by Lemma 18.1.4.2 we know that $\mathfrak{p}$ in this case is maximal. $\qquad \square$

We now portray their use in the following.

Lemma 18.1.4.4. *Let F be an algebraically closed field. Then,*
1. *every non-constant polynomial $f(x,y) \in F[x,y]$ has at least one zero in F^2,*
2. *every maximal ideal of $F[x,y]$ is of the form $\mathfrak{m} = \langle x - a, y - b \rangle$ for some $a, b \in F$.*

Proof. 1. : Take any polynomial $f(x,y) \in F[x,y]$. Going modulo y, we see that $\overline{f(x,y)} \in F[x,y]/\langle y \rangle = F[x]$. If $\overline{f(x,y)} = 0$, then $(a,0)$ is a root of $f(x,y)$ for any $a \in F$. if $\overline{f(x,y)} \neq 0$, then since F is algebraically closed, therefore we may write $\overline{f(x,y)} = (x - a_1)\dots(x - a_n)$. Consequently, any $(a_i, 0)$ is a zero of $f(x,y)$. Hence, in any case, $f(x,y)$ has a root in F^2.
2. : Let $R = F[x]$. We know that R is a PID. Take any maximal ideal $\mathfrak{m} \leq R[y] = F[x,y]$. Then by Theorem 18.1.4.3, we have that either $\mathfrak{m} = \langle f(x,y) \rangle$ where $f(x,y)$ is irreducible or $\mathfrak{m} = \langle p(x), h(x,y) \rangle$ where $p(x) \in R$ is prime and $h(x,y)$ is irreducible modulo $p(x)$.

In the former, we claim that $\langle f(x,y) \rangle$ is not maximal. Indeed, by item 1, we have that $f(x,y)$ has a zero in F^2, say (a,b). Dividing $f(x,y)$ by $y - b$ in $R[y]$, we obtain $f(x,y) = h(x,y)(y - b) + k(x)$, where $k(x) \in R$. Consequently, $k(a) = 0$. Hence, $k(x) = (x - a)l(x)$. Thus, we have $f(x,y) = h(x,y)(y - b) + (x - a)l(x)$, showing $f(x,y) \in \langle x - a, y - b \rangle$. By Theorem 18.1.4.3 above, we know that $\langle x - a, y - b \rangle \in R[y]$ is a maximal ideal and we also know that it contains $f(x,y)$. We hence need only show that $\langle f(x,y) \rangle \subsetneq \langle x - a, y - b \rangle$. Indeed, observe that $x - a \notin \langle f(x,y) \rangle$ as if it is, then $f(x,y) \div x - a$. But then $f(x,y)$ is in R, hence $y - b \notin \langle f(x,y) \rangle$. So in either case, $\langle f(x,y) \rangle$ is properly contained in $\langle x - a, y - b \rangle$, showing that $\langle f(x,y) \rangle$ cannot be maximal. Thus, no maximal ideal of $R[y]$ can be of the form $\langle f(x,y) \rangle$.

In the latter, where $\mathfrak{m} = \langle p(x), h(x,y) \rangle$ where $p(x) \in R$ is prime and $h(x,y)$ is irreducible modulo $p(x)$, we first see that $p(x) = x - a$ for some $a \in F$ as $R = F[x]$ and only primes of $F[x]$ are of this type. Let $\pi : R \twoheadrightarrow \frac{R}{p(x)R}[y] \cong \frac{R}{\langle x-a \rangle}[y] \cong F[y]$ be the quotient map by the ideal $p(x)R[y]$. Then we see that by correspondence theorem, $\pi(\mathfrak{m}) = \overline{\mathfrak{m}} = \langle \overline{h(x,y)} \rangle$ is a prime ideal of $F[y]$. Hence, $\overline{\mathfrak{m}} = \langle k(y) \rangle$ for some $k(y) \in F[y]$. Further, since $\overline{\mathfrak{m}}$ is prime and F algebraically closed, therefore $k(y) = y - b$. Thus, we see that modulo $p(x)$ we have $h(x,y) = k(y) = y - b$. We then see that $\mathfrak{m} = \pi^{-1}(\overline{\mathfrak{m}}) = \pi^{-1}(\langle \overline{y - b} \rangle) = \langle p(x), y - b \rangle = \langle x - a, y - b \rangle$, as required. $\qquad \square$

Another example gives us finiteness of intersection of two algebraic curves over an algebraically closed field.

Proposition 18.1.4.5. *Let F be an algebraically closed field and $f, g \in F[x,y]$ be two polynomials with no common factors. Then, $Z(f) \cap Z(g)$ is a finite set, that is, f and g intersects at finitely many points in $\mathbb{A}_F^2$.*

Proof. We first show that for any $h(x, y) \in F[x, y]$, $h(a, b) = 0$ for some $(a, b) \in F^2$ if and only if $h \in \langle x - a, y - b \rangle$. Clearly, $(\Leftarrow)$ is immediate. For $(\Rightarrow)$, we proceed as follows. Going modulo $y - b$ in $F[x, y]$, we obtain $\overline{h(x, y)} \in F[x, y]/\langle y - b \rangle \cong F[x]$. Observe that $\langle y - b \rangle$ is the kernel of the map $F[x, y] \to F[x]$ taking $y \mapsto b$, hence $\overline{h(x, y)} = \overline{h(x, b)}$. As F is algebraically closed, therefore we may write

$$\overline{h(x, b)} = \overline{h(x, y)} = (x - c_1) \ldots (x - c_n)$$

for $c_i \in F$. As, $h(a, b) = 0$, therefore $(x - a) \div \overline{h(x, b)}$. Hence, for some i, we must have $c_i = a$. This allows us to write

$$h(x, y) - (x - a)k(x) \in \langle y - b \rangle$$

for some $k(x) \in F[x]$. It follows that for some $q(x, y) \in F[x, y]$ we have

$$h(x, y) - (x - a)k(x) = (y - b)q(x, y)$$

Thus, $h(x, y) \in \langle x - a, y - b \rangle$. This completes the proof of the claim above.

Now, using above claim $f(a, b) = 0 = g(a, b)$ if and only if $f, g \in \langle x - a, y - b \rangle$. By Lemma 18.1.4.2, as f and g have no common factors, therefore there are finitely many maximal ideals containing f and g. Further, by Lemma 18.1.4.4, we know that each such maximal ideal is of the form $\langle x - a, y - b \rangle$. Hence, there are only finitely many maximal ideals containing f and g, each of which looks like $\langle x - a, y - b \rangle$. Hence, by above claim, there are finitely many points $(a, b) \in F^2$ such that $f(a, b) = 0 = g(a, b)$. $\qquad \square$

18.1.5   Finite type k-algebras

We discuss basic theory of finite type k-algebras, that is, algebras of form $k[x_1, \ldots, x_n]/I$.

Recall that for a field k, we denote by $k[x]$ the polynomial ring in one variable and we denote the rational function field $k(x)$ to be the field obtained by localizing at prime $\mathfrak{o}$. Further if K/k is a field extension and $\alpha \in K$, then $k[\alpha]$ is a subring of K generated by $\alpha \in K$ and it contains k. Whereas, $k(\alpha)$ is a field extension $k \hookrightarrow k(\alpha) \hookrightarrow K$. The following lemma shows that if K is algebraic, then $k(\alpha) = k[\alpha]$.

Lemma 18.1.5.1. *Let k be a field and K/k be an algebraic extension. If $\alpha_1, \ldots, \alpha_n \in K$, then $k[\alpha_1, \ldots, \alpha_n] = k(\alpha_1, \ldots, \alpha_n)$.*

Proof. The proof uses a standard observation in field theory. First, let $f_1(x) \in k[x]$ be the minimal polynomial of α_1. Consequently, by a standard result in field theory, $k[\alpha_1] = k[x]/f_1(x)$ is a field. Thus $k[\alpha_1] = k(\alpha_1)$. Now observe that $K/k(\alpha_1)$ is an algebraic extension. Consequently, the same argument will yield $k(\alpha_1)[\alpha_2]$ to be a field. By above, we thus obtain $k(\alpha_1)[\alpha_2] = k[\alpha_1][\alpha_2] = k[\alpha_1, \alpha_2]$ to be a field. Consequently, $k[\alpha_1, \alpha_2] = k(\alpha_1, \alpha_2)$. One completes the proof now by induction. $\qquad \square$

Lemma 18.1.5.2. *Let k be a field and K/k be an algebraic extension. Then the homomorphism*

$$k[x_1, \ldots, x_n] \longrightarrow k(\alpha_1, \ldots, \alpha_n)$$
$$x_i \longmapsto \alpha_i$$

has kernel which is a maximal ideal generated by n elements.

Proof. (*Sketch*) Use the proof of Lemma 18.1.5.1 to obtain that for each $1 \leq i \leq n$, we have that $k(\alpha_1, \ldots, \alpha_{i-1})[\alpha_i] \cong k(\alpha_1, \ldots, \alpha_{i-1})[x_i]/p_i(\alpha_1, \ldots, \alpha_{i-1}, x_i)$ and divide an element $p \in k[x_1, \ldots, x_n]$ in the kernel inductively by p_i and replacing p_i by remainder, starting at $i = n$. $\qquad\square$

Let us now observe a basic fact. Next, we observe that residue fields of any point in an affine n-space over k is an algebraic extension of k. **TODO : Till Jacobson rings from Matsumura.**

18.2   Noetherian modules

Let R be a ring. An R-module M is said to be *noetherian* if it satisfies either of the following equivalent properties:

1. Every increasing chain of submodules of M eventually stabilizes.
2. Every non-empty family of submodules of M has a maximal element.
3. Every submodule is finitely generated.

We prove the equivalence of 1 and 3 as in Proposition 18.2.0.3. But before, let us see that noetherian hypothesis descends to submodules and to quotients:

Lemma 18.2.0.1. *Let R be a ring and M be a noetherian R-module.*
1. *If N is a submodule of M, then N is noetherian.*
2. *If M/N is a quotient of M, then M/N is noetherian.*

Proof. 1. Take any submodule of M which is in N, then it is a submodule of N which is finitely generated.

2. Take any submodule of M/N, which is of the form K/N where $K \subseteq M$ is a submodule of M containing N. Hence K is finitely generated and so is N. Thus K/N is finitely generated. $\square$

We also have that a finitely generated module over noetherian ring necessarily has to be noetherian, so every submodule is also finitely generated, which is not usually the case. This is another hint why having noetherian hypothesis can greatly ease calculations.

Lemma 18.2.0.2. *Let R be a noetherian ring and let M be an R-module. Then M is a noetherian module if and only if M is finitely generated.*

Proof. The only non-trivial side is R $\Rightarrow$ L. Since M is finitely generated, therefore there is a surjection $f : R^n \twoheadrightarrow M$ where R^n is noetherian as R is noetherian (you may like to see it as a consequence of Corollary 18.2.0.5). Now take an increasing chain of submodules $N_0 \subseteq N_1 \subseteq \ldots$ of M. This yields an increasing chain of ideals $f^{-1}(N_0) \subseteq f^{-1}(N_1) \subseteq \ldots$, which stabilizes as R is noetherian. Applying f to the chain again we get that $N_0 \subseteq N_1 \subseteq \ldots$ stabilizes. $\square$

Here's the proof of equivalence as promised.

Proposition 18.2.0.3. *Let R be a ring. An R-module M is noetherian if and only if every submodule of M is finitely generated.*

Proof. (L $\implies$ R) Suppose R-module M is noetherian and let $S \subseteq M$ be a submodule of M. Note S is also noetherian. This means that any subcollection of submodules of S has a maximal element. Let such a subcollection be the collection of all finitely generated submodules of S, which clearly isn't empty as $\{0\}$ is there. This would have a maximal element, say N. If $N = S$, we are done. If not, then take $x \in S \setminus N$ and look at $N + Rx \subset S$. Clearly this is a submodule of S strictly containing N and is also finitely generated as N is too. This contradicts the maximality of N. Hence every submodule of M is finitely generated.

(R $\implies$ L) Let every submodule of M be finitely generated. We wish to show that this

makes M into a noetherian module. So take any ascending chain of submodules $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$. Consider the union $S = \cup_{i=0}^{\infty} S_i$. S is also a submodule because for any $x, y \in S$, since $\{S_i\}$ is an ascending chain, there exists S_i such that $x, y \in S_i$, and so $x + y \in S_i \subseteq S$. By hypothesis, $S = \langle x_1, \ldots, x_k \rangle$. Let S_{n_i} be the smallest submodule containing x_i. Then $S_{\max n_i}$ is a member of the chain which contains each of the x_is, which thus means that the $S_{\max n_i}$ is generated by x_is because if it didn't then S would have either a smaller or a larger generating set, contradicting the generation by $x_1, \ldots, x_k$. Hence the chain stabilizes after $S_{\max n_i}$. $\square$

The reason one dwells with the noetherian hypothesis is reflected in the following properties enjoyed by it. Given a short exact sequence of modules, it is possible to figure out whether the middle module is noetherian or not by checking the same for the other two:

Proposition 18.2.0.4. *Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of R-modules. Then, the module M is noetherian if and only if M' and M'' are noetherian.*

Proof. (L $\implies$ R) Let M be noetherian. Then if we consider any ascending chain in M' or M'', then we get an ascending chain in M because of the maps f and g. Remember inverse image of an injective and direct image of a surjective module homomorphism of a submodule is also a submodule.

(R $\implies$ L) Consider an ascending chain of submodules $S_0 \subseteq S_1 \subseteq \ldots$ in M. We then have two more ascending chains $\{(f)^{-1}(S_i)\}$ and $\{g(S_i)\}$ in M' and M'' respectively. Since these are noetherian, therefore for both of them $\exists k \in \mathbb{N}$ such that these two chains stabilizes after k. Now, we wish to show that $\{S_i\}$ also stabilizes after k. For this, we just need to show that $S_{k+1} \subseteq S_k$. Hence take any $m \in S_{k+1}$. We have $g(m) \in g(S_k)$, therefore $\exists s \in S_k$ such that $g(m) = g(s) \implies g(m-s) = 0$ in M''. Since the sequence is exact, therefore $\exists m' \in M'$ such that $f(m') = m - s$, or, $m - s \in \text{im}(f)$. Since $m \in S_{k+1}$ and $s \in S_k \subseteq S_{k+1}$, therefore $m - s \in S_{k+1}$. Hence $m - s \in \text{im}(f) \cap S_{k+1}$ and since $\text{im}(f) \cap S_{k+1} = \text{im}(f) \cap S_k$, therefore $m - s \in S_k$ and thus $m \in S_k$. This proves $S_{k+1} \subseteq S_k$, proving $S_k = S_{k+1} = \ldots$. $\square$

An easy consequence of the above is that direct sum of finitely many noetherian modules is again noetherian:

Corollary 18.2.0.5. *Suppose $\{M_i\}_{i=1}^{n}$ be a collection of noetherian R-modules. Then $\bigoplus_{i=1}^{n} M_i$ is also a noetherian R-module.*

Proof. Since the sum $\bigoplus_{i=1}^{n} M_i$ sits at the middle of the following short exact sequence:

$$0 \longrightarrow M_1 \xrightarrow{f} \bigoplus_{i=1}^{n} M_i \xrightarrow{g} \bigoplus_{i=2}^{n} M_i \longrightarrow 0$$

where f is given by $m \longmapsto (m, 0, \ldots, 0)$ and g is given by $(m_1, \ldots, m_n) \longmapsto (m_2, \ldots, m_n)$. The fact that this is indeed exact is simple to see. One can next use induction to complete the proof. $\square$

An important result in the theory of noetherian modules is the following, which gives us few more (but highly important) examples of noetherian rings in nature. In particular it tells us that the one of the major class of rings which are studied in algebraic geometry, polynomial rings over algebraically closed fields, are noetherian.

Theorem 18.2.0.6. *(Hilbert basis theorem) Let R be a ring. If R is noetherian, then*

1. *$R[x_1, \ldots, x_n]$ is noetherian,*
2. *$R[[x_1, \ldots, x_n]]$ is noetherian.*

Proof. 1. We need only show that if R is noetherian then so is $R[x]$. Pick any ideal $I \leq R[x]$. We wish to show it is finitely generated. We go by contradiction, let I not be finitely generated.

Let $f_1 \in I$ be the smallest degree non-constant polynomial[1] and denote $I_1 = \langle f_1 \rangle$. Let $f_2 \in I \setminus I_1$ be the smallest degree non-constant polynomial and denote $I_2 = \langle f_1, f_2 \rangle$. Inductively, we define $I_n = \langle f_1, \ldots, f_n \rangle$ where $f_n \in I \setminus I_{n-1}$ is of least degree non-constant. As I is not finitely generated, therefore for all $n \in \mathbb{N}$, $I_n \lneq I$. Let $f_n(x) = a_n x^m +$ other terms for each $n \in \mathbb{N}$ so that $a_n \in R$ represents the coefficient of the leading term of $f_n(x)$. Consequently, we obtain a sequence $\{a_n\} \subseteq R$. Let $J = \langle a_1, \ldots, a_n, \ldots \rangle$. As R is noetherian, therefore there exists $n \in \mathbb{N}$ such that $J = \langle a_1, \ldots, a_n \rangle$. It follows that for some $r_1, \ldots, r_n \in R$ we have

$$a_{n+1} = r_1 a_1 + \cdots + r_n a_n.$$

We claim that $I = \langle f_1, \ldots, f_n \rangle =: I_n$.

If not then $f_{n+1} \in I \setminus I_n$ is of least degree non-constant. We will now show that $f_{n+1} \in I_n$, thus obtaining a contradiction. Indeed, we have by the way of choice of f_{n+1} that $\deg f_{n+1} \geq \deg f_i$ for each $i = 1, \ldots, n$. Consequently the polynomial

$$g = \sum_{i=1}^{n} r_i f_i \cdot x^{\deg f_{n+1} - \deg f_i}$$

has the property that its degree is equal to $\deg f_{n+1}$ and the coefficient of its leading term is equal to f_{n+1}. It follows that the polynomial $g - f_{n+1} \in I$ has degree strictly less than that of f_{n+1}. By minimality of f_{n+1}, it follows that $g - f_{n+1} \in I_n$. Note that by construction $g \in I_n$. Hence $f_{n+1} \in I_n$, as required.

2. **TODO** : *Write it from your exercise notebook.* □

18.3 $\mathrm{Supp}\,(M)$, $\mathrm{Ass}\,(M)$ and primary decomposition

Let R be a ring and M be a finitely generated R-module. In the classical case when R is a field and M is then a finite dimensional R-vector space, if $x \in M$ then if even a single element of R annihilate x, then all elements of R annihilate x. This luxury is not enjoyed when R is a ring because not all elements of R may be invertible. What one does then is to study the associated annihilating ideals corresponding to each element of M. The global version of this idea is exactly the concept of *annihilator ideal of M*, i.e. $\mathfrak{a}_M := \{r \in R \mid rM = 0\}$.

Now, if we have an R-module M, then we get an ideal of R. This gives us a closed subset of $\mathrm{Spec}\,(R)$ (see Section 1.2). A basic question that then arises is what is the relationship between the module M and the closed set $V(\mathfrak{a}_M) \hookrightarrow \mathrm{Spec}\,(R)$. The following answers that.

---

[1]this exists by well-ordering by degree.

Lemma 18.3.0.1. *Let R be a ring and M be a finitely generated R-module. If $\mathfrak{p} \in \mathrm{Spec}\,(R)$ and $\mathfrak{a}_M = \mathrm{Ann}(M)$ be the annihilator ideal, then the following are equivalent:*

1. $M_\mathfrak{p} \neq 0$.
2. $\mathfrak{p} \in V(\mathfrak{a}_M)$.

Proof. If we can show that $\mathrm{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p}) = (\mathfrak{a}_M)_\mathfrak{p}$, then we have the following equivalence

$$M_\mathfrak{p} \neq 0 \iff \mathrm{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq R_\mathfrak{p} \iff (\mathfrak{a}_M)_\mathfrak{p} \lneq R_\mathfrak{p} \iff \mathfrak{a}_M \subseteq \mathfrak{p}$$

where last equivalence follows from a modified version of Lemma 18.1.2.3. Hence we reduce to showing that $\mathrm{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p}) = (\mathfrak{a}_M)_\mathfrak{p}$. It is easy to see that $\mathrm{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p}) \supseteq (\mathfrak{a}_M)_\mathfrak{p}$. Let $r/s \in \mathrm{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p})$. We wish to show that $r/s \in (\mathfrak{a}_M)_\mathfrak{p}$. Since M is finitely generated, therefore let $\{m_1, \ldots, m_n\}$ be a generating set of M. We thus reduce to showing that $r/s \cdot m_i/1 = 0$ for each $i = 1, \ldots, n$. This is exactly the data provided by the fact that $r/s \in \mathrm{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p})$. $\square$

The above lemma hence gives us a closed subset of $\mathrm{Spec}\,(R)$ attached to each finitely generated R-module M. This has a name.

Definition 18.3.0.2. (**Support of a module**) Let R be a ring and M be a finitely generated R-module. Let $\mathfrak{a}_M$ be the annihilator ideal of M. Then, the support of the module M is defined to be the closed set $\mathrm{Supp}\,(M) := V(\mathfrak{a}_M) \hookrightarrow \mathrm{Spec}\,(R)$. By Lemma 18.3.0.1, it is equivalently given by the set of all those points $\mathfrak{p} \in \mathrm{Spec}\,(R)$ such that $M_\mathfrak{p} \neq 0$.

We then define prime ideals associated to an R-module.

Definition 18.3.0.3. (**Associated prime ideals**) Let R be a noetherian ring and M be an R-module. A prime ideal $\mathfrak{p} \in \mathrm{Spec}\,(R)$ is said to be associated to M if there exits $m \in M$ such that

$$\mathfrak{p} = \{r \in R \mid rm = 0\}.$$

The subspace of $\mathrm{Spec}\,(R)$ of all prime ideals associated to M is denoted $\mathrm{Ass}\,(M) \hookrightarrow \mathrm{Spec}\,(R)$.

One can have the following alternate definition of an associated prime ideal.

Lemma 18.3.0.4. *Let R be a noetherian ring and M be an R-module. Then,*

$$\mathfrak{p} \in \mathrm{Ass}\,(M) \iff \exists N \leq M \text{ such that } N \cong R/\mathfrak{p}.$$

Proof. L $\Rightarrow$ R is easy, just consider the map $R \to M$ given by $r \mapsto rm$ where $m \in M$ corresponds to $\mathfrak{p}$. Conversely, take any $0 \neq n \in N$. Then $\mathfrak{p} = \{r \in R \mid rn = 0\}$ as if $r \in R$ is such that $rn = 0$ and $n = s + \mathfrak{p}$, then $rn = rs + \mathfrak{p} = \mathfrak{p}$, that is $rs \in \mathfrak{p}$ and since $s \notin \mathfrak{p}$, therefore $r \in \mathfrak{p}$. Conversely, if $r \in \mathfrak{p}$ then for all $n \in N$, $rn = 0$. $\square$

So, for an R-module M, we get two subspaces of $\mathrm{Spec}\,(R)$, one is the closed subspace called support $\mathrm{Supp}\,(M)$ and the other is $\mathrm{Ass}\,(M)$. Support will be used later, but the concept of associated prime ideals of M have a deeper connection with the ring R. They are not unrelated.

Lemma 18.3.0.5. *Let M be an R-module. Then $\text{Ass}(M) \hookrightarrow \text{Supp}(M) \hookrightarrow \text{Spec}(R)$.*

Proof. For $\mathfrak{p} \in \text{Ass}(M)$ let $m \in M$ such that its annihilator is $\mathfrak{p}$. Then, for any $r \in \mathfrak{a}_M$, $rm = 0$ and hence $r \in \mathfrak{p}$. Thus $\mathfrak{p} \in V(\mathfrak{a}_M) = \text{Supp}(M)$. $\square$

We wish to show the following result from which primary decomposition follows.

Theorem 18.3.0.6. *Let R be a noetherian ring and M be a finitely generated R-module. Then there exists an injective map*

$$M \longrightarrow \prod_{\mathfrak{p} \in \text{Ass}(M)} E_{\mathfrak{p}}$$

where for each $\mathfrak{p} \in \text{Ass}(M)$, $E_{\mathfrak{p}}$ is an R-module where $\text{Ass}(E_{\mathfrak{p}}) \hookrightarrow \text{Spec}(R)$ is a singleton given by $\{\mathfrak{p}\}$.

This result clearly tells us that points of $\text{Ass}(M)$ are somewhat special. Let us investigate[2].

Lemma 18.3.0.7. *Let R be a noetherian ring and M be a finite R-module[3]. Then,*
  1. *If $N \subseteq M$ is a submodule, then $\text{Ass}(N) \subseteq \text{Ass}(M)$.*
  2. *If $N \subseteq M$ is a submodule, then $\text{Supp}(N) \subseteq \text{Supp}(M)$.*
  3. *If $N \subseteq M$ is a submodule, then $\text{Ass}(N) \subseteq \text{Ass}(M) \subseteq \text{Ass}(N) \cup \text{Ass}(M/N)$.*
  4. *For any point $\mathfrak{p} \in \text{Spec}(R)$, we have $\mathfrak{a}_{R/\mathfrak{p}} := \text{Ann}(R/\mathfrak{p}) = \mathfrak{p}$. Thus, $\text{Supp}(R/\mathfrak{p}) = V(\mathfrak{p})$ is an irreducible closed subset of $\text{Spec}(R)$.*
  5. *For any point $\mathfrak{p} \in \text{Spec}(R)$, we have $\text{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$. Thus, $\text{Ass}(R/\mathfrak{p})$ is exactly the generic point of $\text{Supp}(R/\mathfrak{p})$.*
  6. *For all $\mathfrak{p} \in \text{Spec}(R)$, there exists a maximal submodule $N \subseteq M$ such that $\mathfrak{p} \notin \text{Ass}(N)$.*
  7. *For all $\mathfrak{p} \in \text{Ass}(M)$, there exists a maximal submodule $N \subsetneq M$ such that $\mathfrak{p} \notin \text{Ass}(N)$ and none of these maximal submodules are isomorphic to $R/\mathfrak{p}$.*

Proof. Note that by Lemma 18.2.0.2, M is a Noetherian module.
  1. If $\mathfrak{p} \in \text{Ass}(N)$, then for some $n \in N$, $\mathfrak{p} = \{r \in R \mid rn = 0\}$. Result follows as $n \in M$.
  2. If $\mathfrak{p} \in \text{Supp}(N)$, then $\mathfrak{p} \supseteq \mathfrak{a}_N$. Result follows as $\mathfrak{a}_N \supseteq \mathfrak{a}_M$.
  3. By 1, we need only show $\text{Ass}(M) \subseteq \text{Ass}(N) \cup \text{Ass}(M/N)$. Pick $\mathfrak{p} \in \text{Ass}(M)$. By the Lemma 18.3.0.4 and it's proof, the submodule E of M containing of all elements of M who have annihilator as $\mathfrak{p}$ is isomorphic to $R/\mathfrak{p}$. If $E \cap N =$, then M/N has a submodule isomorphic to $R/\mathfrak{p}$ and hence $\mathfrak{p} \in \text{Ass}(M/N)$. Otherwise if $E \cap N \neq \emptyset$, then pick $x \in E \cap N$. Since $x \in E$, so annihilator of x is $\mathfrak{p}$ and thus $\mathfrak{p} \in \text{Ass}(E \cap N)$. By another use of Lemma 18.3.0.4, there is a submodule $F \subseteq E \cap N$ which is isomorphic to $R/\mathfrak{p}$. It follows that N has a submodule isomorphic to $R/\mathfrak{p}$. By a final use of Lemma 18.3.0.4, we conclude that $\mathfrak{p} \in \text{Ass}(N)$.
  4. $\text{Ann}(R/\mathfrak{p}) = \{r \in R \mid r(R/\mathfrak{p}) = \mathfrak{p}\}$. It follows from primality of $\mathfrak{p}$ that $\text{Ann}(R/\mathfrak{p}) = \mathfrak{p}$.
  5. As above, this reduces to primality of $\mathfrak{p}$.

---

[2]The following was a personal investigation of the author, who, in the process of overcoming his inexperience as a true researcher, would like to apologize for the mess that is the Lemma 18.3.0.7.

[3]this is just another name for finitely generated R-modules.

6. The set of all submodules N of M satisfying Ass $(N) \notin p$ has a maximal element as M is a noetherian module.

7. If $\mathfrak{p} \in$ Ass (M), then the maximal N obtained from 5 cannot be M. The other fact follows from 4.

$\square$

With the above investigation, we are now ready to prove Theorem 18.3.0.6.

Proof of Theorem 18.3.0.6. **TODO**. $\square$

The primary decomposition now is a corollary.

Corollary 18.3.0.8. *(Primary decomposition theorem[4])* _____

> Complete this fr
> Local Algebra, C
> ter 18.

---

[4]for finitely generated modules over a Noetherian ring.

18.4   Tensor, symmetric & exterior algebras

18.4.1   Results on tensor products

We collect some important results on tensor products in this section which are used all over the text. We first show a helpful lemma showing that tensor product commutes with direct limits in all positions.

Lemma 18.4.1.1. *Let M_i, N_i bet R_i-modules where I is directed set and $\{M_i\}, \{N_i\}$ and $\{R_i\}$ are directed systems of modules and rings. Let $M := \varinjlim_{i \in I} M_i$, $N := \varinjlim_{i \in I} N_i$ and $R := \varinjlim_{i \in I} R_i$. Then,*

$$\varinjlim_{i \in I}(M_i \otimes_{R_i} N_i) \cong M \otimes_R N$$

as R-modules.

Proof. We will construct R-linear maps $f : \varinjlim_{i \in I}(M_i \otimes_{R_i} N_i) \longleftrightarrow M \otimes_R N : g$ which will be inverses to each other. We first construct f as follows. For each $i \in I$, we have

$$f_i : M_i \otimes_{R_i} N_i \to M \otimes_{R_i} N \to M \otimes_R N$$

given by $(m_i \otimes n_i) \mapsto ((m_i) \otimes (n_i)) \mapsto ((m_i) \otimes (n_i))$. Note that M, N are R_i-modules canonically. By universal property of $\varinjlim_{i \in I}$, we obtain f as above. To construct g, we need only construct an R-bilinear map

$$M \times N \longrightarrow \varinjlim_{i \in I}(M_i \otimes_{R_i} N_i)$$

$$((m_i)_{i \in I}, (n_i)_{i \in I}) \longmapsto ((m_i \otimes n_i)_{i \in I}).$$

This can be said to be R-bilinear, thus yielding a map g as required. It is straightforward to see they are inverses to each other. $\square$

We now give some highly useful lemmas on tensor product which is of important use in calculations in algebraic geometry.
Add tensor product formulas, anything that you can find or create.
Add fibre of a prime ideal!
Do from Section 10.13 of Stacksproject and from Appendix 2.3 of Eisenbud.

18.4.2   Determinants

Fix a commutative ring R with unity for the remainder of this section. We shall show in this section that there exists a unique determinant map over $M_n(R)$. This will motivate further notions discussed in later sections.
    We begin by defining a multilinear map over $M_n(R)$.

Definition 18.4.2.1. (Multilinear map over $M_n(R)$) Let $n \in \mathbb{N}$ and consider $M_n(R)$. An n-linear map over $M_n(R)$ is a function

$$D : M_n(R) \longrightarrow R$$

which is linear in each row. That is, if A_i denotes the ith-row of matrix A and $c \in R$, then for each $i = 1, \ldots, n$, we have

$$D(A_1, \ldots, A_{i-1}, cA_i + B_i, A_{i+1}, \ldots, A_n) = cD(A_1, \ldots, A_{i-1}, A_i, A_{i+1}, \ldots, A_n)$$
$$+ D(A_1, \ldots, A_{i-1}, B_i, A_{i+1}, \ldots, A_n).$$

We may abbreviate the above by simply writing $D(cA_i + B_i) = cD(A_i) = D(B_i)$.

Example 18.4.2.2. The map

$$D : M_n(R) \longrightarrow R$$
$$A \longmapsto cA_{1k_1} A_{2k_2} \ldots A_{nk_n}$$

is an n-linear map where $c \in R$ is a constant and $1 \leq k_i \leq n$ are n integers.

We first see that linear combination of n-linear maps is again n-linear.

Lemma 18.4.2.3. *Let $D_1, \ldots, D_r$ be n-linear maps and $c_1, \ldots, c_r \in R$. Then $c_1 D_1 + \cdots + c_r D_r$ is an n-linear map.*

Proof. By induction, we may assume $r = 2$. Now this is a straightforward check. $\square$

We now come more closer to determinants by defining the following type of n-linear maps.

Definition 18.4.2.4. (Alternating & determinant maps) An n-linear map $D : M_n(R) \to R$ is said to be alternating if
   1. $D(A) = 0$ if $A_i = A_j$ for any $i \neq j$,
   2. $D(\sigma_{ij}(A)) = -D(A)$ where σ_{ij} swaps rows A_i and A_j.
An alternating n-linear map $D : M_n(R) \to R$ is said to be determinant if $D(I_n) = 1$.

Proposition 18.4.2.5. *If $D : M_n(R) \to R$ is an n-linear map such that $D(A) = 0$ whenever $A_i = A_{i+1}$ for some $1 \leq i \leq n$, then D is alternating.*

Proof. Let $A \in M_n(R)$ and $1 \leq i \neq j \leq n$ be such that $A_i = A_j$. We first wish to show that $D(\sigma_{ij}(A)) = -D(A)$. We may assume $j > i$. We go by strong induction over $j - i$. We first show this for $j = i+1$. Indeed, we then have $D(\sigma_{i,i+1}(A)) = D(A_{i+1}, A_i)$. Writing $0 = D(A_{i+1} + A_i, A_i + A_{i+1}) = D(A_{i+1}, A_i) + D(A_i, A_{i+1})$. Thus we get $D(A_{i+1}, A_i) = -D(A_i, A_{i+1})$.

In the inductive case, suppose $D(\sigma_{ij}(A)) = -D(A)$ for all $j - i \leq k$. We wish to show that if $j - i = k+1$, then the same holds. As $\sigma_{i,i+k+1}(A) = \sigma_{i+k,i+k+1} \circ \sigma_{i,i+k} \circ \sigma_{i+k,i+k+1}(A)$, therefore we are done.

To get that $D(A) = 0$ for A such that $A_i = A_j$ for some $j > i$, we may simply swap rows till they are adjacent, which will be zero by our hypothesis. $\square$

We now define the main candidate for the determinant function over $M_n(R)$.

Definition 18.4.2.6. (E_j) Let $D : M_{n-1}(R) \to R$ be an $n-1$-linear map. For each $1 \leq j \leq n$, define the following map

$$E_j : M_n(R) \longrightarrow R$$
$$A \longmapsto \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D(A[i|j]).$$

Further denote $D_{ij}(A) := D(A[i|j])$.

Theorem 18.4.2.7. *Let $n \in \mathbb{N}$ and $D : M_{n-1}(R) \to R$ be an alternating $n-1$-linear map. For each $1 \leq j \leq n$, the map $E_j : M_n(R) \to R$ defined as above is an alternating n-linear map. If moreover D is a determinant map, then so is each E_j.*

Proof. Fix $1 \leq j \leq n$. We first wish to show that E_j is n-linear. As $D_{ij} : M_n(R) \to R$ is linear in every row except i. Thus $A \mapsto A_{ij} D_{ij}(A)$ is n-linear. It follows from Lemma 18.4.2.3 that E_j is n-linear.

To show that E_j is alternating, it would suffice from Proposition 18.4.2.5 to show that $E_j(A) = 0$ if A has any two adjacent rows equal, say $A_k = A_{k+1}$. This one checks directly by the definition of E_j.

To see that E_j is determinant if D is determinant is also easy to see. $\square$

We now show the uniqueness of determinants and alternating n-linear maps (upto the value on I_n).

Theorem 18.4.2.8. *Let $D : M_n(R) \to R$ be an alternating n-linear map over $M_n(R)$. Then,*

1. D is given explicitly on $A \in M_n(R)$ by

$$D(A) = \left(\sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{1\sigma(1)} \ldots A_{n\sigma(n)} \right) D(I),$$

hence D is unique upto its value over I,

2. if D is determinant map, then it is uniquely given by

$$D(A) = \det A := \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{1\sigma(1)} \ldots A_{n\sigma(n)},$$

3. any alternating map D on $M_n(R)$ is thus uniquely determined by its value on I as

$$D(A) = (\det A) \cdot D(I).$$

Proof. The proof is straightforward but tedious. **TODO**. $\square$

Corollary 18.4.2.9. *Let $n \in \mathbb{N}$.*
1. If $A, B \in M_n(R)$, then $\det(AB) = \det(A) \cdot \det(B)$.
2. If $B \in M_n(R)$ is obtained by $B_j = A_j + cA_i$ for some fixed $1 \leq i, j \leq n$ and rest of the rows of B are identical to A, then $\det(B) = \det(A)$.

3. *If $M \in M_{r+s}(R)$ is given by*

$$M = \begin{bmatrix} A_{r \times r} & B_{r \times s} \\ 0 & C_{s \times s} \end{bmatrix}$$

   then $\det(M) = \det(A) \cdot \det(C)$.
4. *For each $1 \leq j \leq n$, we have*

$$\det(A) = E_j(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(A[i|j]).$$

Proof. (*Sketch*) For 1. we can contemplate

$$D : M_n(R) \longrightarrow R$$
$$A \mapsto \det(AB).$$

One claims that D is an n-linear alternating map. Then apply Theorem 18.4.2.8, 3.

2. Follows by multilinearity of det.

3. As elementary row operations only change determinant upto sign and restricting an $r + s$-linear alternating map to first r or last s entries keeps it r-linear and s-linear alternating respectively, therefore the result follows.

4. Follows from Theorem 18.4.2.7 and Theorem 18.4.2.8. $\qquad \square$

Construction 18.4.2.10. (*Adjoint of a matrix*) Let $A \in M_n(R)$ be a square matrix. By Corollary 18.4.2.9, the sum $E_j(A) = \det(A)$ for each $1 \leq j \leq n$

$$\det(A) = \sum_{i=1}^{n} A_{ij} (-1)^{i+j} \det(A[i|j]).$$

Hence, let us define $C_{ij} := (-1)^{i+j} \det(A[i|j])$ as the ij^{th}-*cofactor of A*. Consequently, we get a matrix $(\text{Adj}A)_{ij} = C_{ji}$, called the *adjoint matrix*. Hence, we may rewrite the determinant as

$$\det(A) = \sum_{i=1}^{n} A_{ij} C_{ij}$$
$$= \sum_{i=1}^{n} (\text{Adj}A)_{ji} A_{ij}.$$

Thus,

$$\det(A)I = \text{Adj}(A) \cdot A.$$

This also allows us to write that in the case when A is invertible, we have

$$A^{-1} = \frac{1}{\det A} \text{Adj}(A).$$

As similar matrices have same determinant, therefore each linear operator on a finite dimensional vector space has a unique determinant. Thus determinants are invariants of linear operators upto similarity.

18.4.3   Multilinear maps

We now put the previous discussion in a more abstract framework where we work with modules over a commutative ring with 1. We first recall that the rank of a finitely generated module is the size of the smallest generating set. Further recall that a finitely generated free R-module V has a well-defined rank and the smallest generating set is moreover a basis of V (i.e. linearly independent set of generators).

   For this section, we would hence fix a commutative ring R with 1.

Definition 18.4.3.1. (*r-linear forms over a module*) Let V be an R-module. An r-linear form L over V is a function

$$L : V^r = V \times \cdots \times V \longrightarrow R$$

such that for any $c \in R$, $\beta_i \in V$ and $(\alpha_1, \ldots, \alpha_r) \in V^r$, we have

$$L(\alpha_1, \ldots, c\alpha_i + \beta_i, \ldots, \alpha_n) = cL(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) + L(\alpha_1, \ldots, \beta_i, \ldots, \alpha_n)$$

for any $1 \leq i \leq r$. An r-linear form is usually also called an r-tensor. A 2-linear form/tensor is also usually called a bilinear form. Note that an r-linear form may not be linear. Denote the R-module of all r-linear forms by $M^r(V)$.

Remark 18.4.3.2. Let $f_1, \ldots, f_r \in V^* = \mathrm{Hom}_R(V, R) = M^1(V)$ be a collection of linear functionals. We then obtain $L \in M^r(V)$ given by

$$L(\alpha_1, \ldots, \alpha_r) = f_1(\alpha_1) \cdot \cdots \cdot f_r(\alpha_r).$$

Example 18.4.3.3. We give some examples.
   1. Let $V = R^n$ be a free R-module of rank n. Then for a fixed matrix $A \in M_n(R)$, the map

$$V \times V \longrightarrow R$$
$$(x, y) \longmapsto x^t A y$$

   is a bilinear form over V.
   2. Let $V = R^n$ be a free R-module of rank n. Then we obtain the following n-linear form

$$\det : V^n \longrightarrow R$$
$$(\alpha_1, \ldots, \alpha_n) \longmapsto \det(A)$$

   where $A \in M_n(R)$ whose i^{th}-row is α_i. Hence, determinant is an n-tensor/n-linear form over V.

Remark 18.4.3.4. (General expression of an r-linear form) Let $L \in M^r(V)$ be an r-form over an R-module V where V is a free module of rank n. Further denote $e_1, \ldots, e_n$ be a

basis of V. For any $(\alpha_1, \ldots, \alpha_r) \in V^r$, we may write $\alpha_i = \sum_{j=1}^n A_{ij} e_j$. Hence we have $A \in M_{r \times n}(R)$. This yields by n-linearity of L that

$$
\begin{aligned}
L(\alpha_1, \ldots, \alpha_r) &= \sum_{j_r=1}^n \cdots \sum_{j_1=1}^n A_{1j_1} \ldots A_{rj_r} L(e_{j_1}, \ldots, e_{j_r}) \\
&= \sum_{J=\{j_1,\ldots,j_r\}} A_J L(e_J)
\end{aligned}
$$

where $J \in X$ where X is the set of all r-tuples with entries in $\{1, \ldots, n\}$. There are therefore n^r terms in the above sum.

Definition 18.4.3.5. (**Tensor product of linear forms**) Let M be an R-module. We then define

$$
\begin{aligned}
- \otimes - : M^r(V) \times M^s(V) &\longrightarrow M^{r+s}(V) \\
(L, M) &\longmapsto L \otimes M
\end{aligned}
$$

where $L \otimes M : V^{r+s} \to R$ is given by $(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) \mapsto L(\alpha_1, \ldots, \alpha_r) M(\beta_1, \ldots, \beta_s)$.

Remark 18.4.3.6. We have following observations about tensor of forms:
1. $L \otimes (T + S) = L \otimes T + L \otimes S$,
2. $(L \otimes T) \otimes N = L \otimes (T \otimes N)$,
3. $c(L + T) \otimes S = cL \otimes S + cT \otimes S$,
4. $L \otimes T \neq T \otimes L$.

We now come to an important theorem about $M^r(V)$

Theorem 18.4.3.7. *Let V be a free R-module of rank n and $B = \{e_1, \ldots, e_n\} \subseteq V$ be a basis of V. Let X denote the set of all r-tuples with entries in $\{1, \ldots, n\}$. Then,*
1. *the R-module $M^r(V)$ is free of rank n^r,*
2. *a basis of $M^r(V)$ is given by $f_J = f_{j_1} \otimes \ldots \otimes f_{j_r}$ where $B^* = \{f_1, \ldots, f_n\} \subseteq V^* = M^1(V)$ is the dual basis of B, where $J = \{j_1, \ldots, j_r\}$ varies over all elements of X.*

Proof. (*Sketch*) We claim that $\{f_J\}_{J \subseteq X}$ forms a basis of $M^r(V)$. Pick any $(\alpha_1, \ldots, \alpha_r) \in V^r$, then by Remark 18.4.3.4, we first have $\alpha_i = \sum_{j=1}^n f_j(\alpha_i) e_j$. Consequently,

$$
\begin{aligned}
L(\alpha_1, \ldots, \alpha_r) &= \sum_{J=\{j_1,\ldots,j_r\}} L(e_{j_1}, \ldots, e_{j_r}) \cdot f_{j_1} \otimes \ldots \otimes f_{j_r}(\alpha_1, \ldots, \alpha_r) \\
&= \sum_{J=\{j_1,\ldots,j_r\}} L(e_J) f_{j_1} \otimes \ldots \otimes f_{j_r}(\alpha_1, \ldots, \alpha_r).
\end{aligned}
$$

Thus, $\{f_J\}_{J \subseteq X}$ spans $M^r(V)$. For linear independence, take any combination

$$
\sum_{J \subseteq X} c_J f_J = 0.
$$

On the LHS, apply e_I to get $c_I = 0$ for each $I \subseteq X$. $\qquad \square$

Definition 18.4.3.8. (**Alternating r-linear forms**) Let V be an R-module. An r-linear form $L \in M^r(V)$ is said to be alternating if

1. $L(\alpha_1, \ldots, \alpha_r) = 0$ if $\alpha_i = \alpha_j$ for $i \neq j$,
2. $L(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)}) = \operatorname{sgn}(\sigma) L(\alpha_1, \ldots, \alpha_r)$ for all $\sigma \in S_r$.

The collection of all alternating r-linear forms is denoted by $\Lambda^r(V)$ and its a submodule of $M^r(V)$. Note that the second axiom follows from 1, but is important to keep it in mind.

Observe that $\Lambda^1(V) = M^1(V) = V^*$.

Remark 18.4.3.9. Consider $V = R^n$, a free R-module of rank n. We saw earlier that $\det \in M^n(V)$ is an n-linear form.. Theorem 18.4.2.8 shows that det is moreover an unique alternating form with $\det(e_1, \ldots, e_n) = 1$. Thus, $\det \in \Lambda^n(V) \subseteq M^n(V)$ is the unique alternating n-linear form over V such that $\det(e_1, \ldots, e_n) = 1$, i.e. $\Lambda^n(V)$ is a free R-module of rank 1.

Construction 18.4.3.10. Let V be an R-module. We now construct an R-linear map $\pi_r : M^r(V) \to \Lambda^r(V)$. For each $L \in M^r(V)$, define $L_\sigma \in M^r(V)$ given by $L_\sigma(\alpha_1, \ldots, \alpha_r) = L(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)})$ for $(\alpha_1, \ldots, \alpha_r) \in V^r$. Consequently, we claim that the following map is well-defined:

$$\pi_r : M^r(V) \longrightarrow \Lambda^r(V)$$
$$L \longmapsto \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) L_\sigma.$$

Indeed, we have to show that $\pi_r L$ is an alternating form. Let $(\alpha_1, \ldots, \alpha_r) \in V^r$ be such that $\alpha_i = \alpha_j$ for $i \neq j$. We wish to show that $\pi_r L(\alpha_1, \ldots, \alpha_r) = 0$. Let $\tau = (ij)$ be the transposition swapping i and j. First observe that the map $S_r \to S_r$ given by $\sigma \mapsto \tau\sigma$ is a bijection. Consequently, if we let $\sigma_1, \ldots, \sigma_{\frac{n!}{2}}$ to be any $\frac{n!}{2}$ elements of S_r, then the rest $\frac{n!}{2}$ are given by $\tau\sigma_i$, $i = 1, \ldots, n!/2$. Consequently,

$$\pi_r L(\alpha_1, \ldots, \alpha_r) = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) L(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)})$$
$$= \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) L(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)})$$
$$= \sum_{i=1}^{\frac{n!}{2}} \operatorname{sgn}(\sigma_i) L(\alpha_{\sigma_i(1)}, \ldots, \alpha_{\sigma_i(r)}) + \sum_{i=1}^{\frac{n!}{2}} \operatorname{sgn}(\tau\sigma_i) L(\alpha_{\tau\sigma_i(1)}, \ldots, \alpha_{\tau\sigma_i(r)})$$
$$= \sum_{i=1}^{\frac{n!}{2}} \operatorname{sgn}(\sigma_i) L(\alpha_{\sigma_i(1)}, \ldots, \alpha_{\sigma_i(r)}) + \sum_{i=1}^{\frac{n!}{2}} -\operatorname{sgn}(\sigma_i) L(\alpha_{\sigma_i(1)}, \ldots, \alpha_{\sigma_i(r)})$$
$$= 0.$$

Hence, π_r is indeed an R-linear map from $M^r(V)$ into $\Lambda^r(V)$.

Finally note that if $L \in \Lambda^r(V)$, then $\pi_r L = r! L$ as $L_\sigma = \operatorname{sgn}(\sigma) L$ for any $\sigma \in S_r$.

Example 18.4.3.11. Let $V = R^n$ be the free R-module of rank n. Let $e_1, \ldots, e_n \in V$ be the standard R-basis of V. Further, let $f_1, \ldots, f_n \in M^1(V)$ be the associated dual basis. Note that for any $\alpha \in V$, we have $\alpha = f_1(\alpha)e_1 + \ldots f_n(\alpha)e_n$. Then, we get an n-form

$$L = f_1 \otimes \ldots \otimes f_n \in M^n(V).$$

Consequently we obtain an alternating n-form given by

$$\pi_r L = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}).$$

Observe that for any $(\alpha_1, \ldots, \alpha_n) \in V^n$, we obtain

$$\pi_r L(\alpha_1, \ldots, \alpha_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)} \right)(\alpha_1, \ldots, \alpha_n)$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(f_{\sigma(1)}(\alpha_1) \cdot \ldots \cdot f_{\sigma(n)}(\alpha_n) \right).$$

This is exactly the determinant of the $n \times n$ matrix over R given by $A = (f_j(\alpha_i))$. That is, $\pi_r L = \det$.

The following properties of π_r will become important later on.

Proposition 18.4.3.12. *Let V be an R-module and $L \in M^r(V)$ and $M \in M^s(V)$ be r and s-forms over V respectively. Then,*

$$\pi_{r+s}(\pi_r(L) \otimes \pi_s(M)) = r!s!\pi_{r+s}(L \otimes M).$$

Proof. **TODO** *: Magnum tedium.* $\qquad\square$

The above has a very nice and useful corollary.

Corollary 18.4.3.13. *Let V be a free R-module of rank n with $f_1, \ldots, f_n \in V^*$ be a dual basis of V^*. Let $I \in X_r$ and $J \in X_s$ where X_r and X_s are the sets of r and s combinations of $\{1, \ldots, n\}$, respectively, such that I and J are disjoint ($i_k \neq j_l$ for any $1 \leq k \leq r$, $1 \leq l \leq s$). Denote $D_I = \pi_r(f_I)$ and $D_J = \pi_s(f_J)$ where $f_I = f_{i_1} \otimes \ldots \otimes f_{i_r} \in M^r(V)$ and $f_J = f_{j_1} \otimes \ldots \otimes f_{j_s} \in M^s(V)$. Then,*

$$\pi_{r+s}(D_I \otimes D_J) = r!s!D_{I \amalg J}.$$

Proof. Follows immediately from Proposition 18.4.3.12 $\qquad\square$

We now come to the main result about alternating forms.

Theorem 18.4.3.14. *Let V be a free module of rank n over R.*
  1. If $r > n$, then $\Lambda^r(V) = 0$.
  2. If $0 \leq r \leq n$, then rank of $\Lambda^r(V)$ is ${}^nC_r$.

Proof. (*Sketch*) Using Remark 18.4.3.4, statement 1 is straightforward. For 2, observe that we can write for $(\alpha_1, \ldots, \alpha_r) \in V^r$, $r \leq n$ as follows

$$L(\alpha_1, \ldots, \alpha_r) = \sum_{J=\{j_1, \ldots, j_r\} \in X} L(e_J)(f_{j_1} \otimes \ldots \otimes f_{j_r})(\alpha_1, \ldots, \alpha_r)$$

where X is the set of all r-permutations of $\{1, \ldots, n\}$ (as for any repeatitions, the corresponding term is 0). Now, partitioning the set X into classes in which permutations represent the same combination, we obtain an indexing set $\hat{X}$ of size ${}^nC_r$. Again, by the fact that L is alternating, we observe $\text{sgn}(\sigma)L(e_{j_1}, \ldots, e_{j_r}) = L(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(r)}})$. Consequntly we may write the above sum as

$$L(\alpha_1, \ldots, \alpha_r) = \sum_{J=\{j_1, \ldots, j_r\} \in \hat{X}} L(e_{j_1}, \ldots, e_{j_r}) \sum_{\sigma \in S_r} \text{sgn}(\sigma) \left(f_{j_{\sigma(1)}} \otimes \ldots \otimes f_{j_{\sigma(r)}} \right)(\alpha_1, \ldots, \alpha_r).$$

Therefore denote for each $J \in \hat{X}$ the following

$$D_J = \sum_{\sigma \in S_r} \text{sgn}(\sigma) \left(f_{j_{\sigma(1)}} \otimes \ldots \otimes f_{j_{\sigma(r)}} \right).$$

One can observe that the D_J for each $J \in \hat{X}$ can alternatively be written as

$$D_J = \pi_r(f_{j_1} \otimes \ldots \otimes f_{j_r}).$$

The above shows that D_J is in $\Lambda^r(V)$ and that it spans $\Lambda_r(V)$. The claim now is that these are also linearly independent. Indeed, that follows immediately by using the fact that f_js are dual basis of e_js. $\qquad \square$

We can now abstractly obtain the determinant of a linear operator $T : V \to V$ on a free R-module V of rank n.

Corollary 18.4.3.15. *Let V be a free R-module of rank n and $T : V \to V$ be an R-linear operator. Then,*
   1. rank of $\Lambda^n(V) = 1$,
   2. there exists a unique $c_T \in R$ such that for all $L \in \Lambda^n(V)$,

$$L \circ T = c_T L.$$

This c_T is defined to be the determinant of the operator T.

Proof. Statement 1 follows from Theorem 18.4.3.14. For statement 2, one need only observe that $L \circ T$ is again an alternating n-tensor and then use statement 1. $\qquad \square$

18.4.4   Exterior algebra

Let us first make the exterior algebra over characteristic 0 fields, before moving to arbitrary ring.

Definition 18.4.4.1. (**Wedge product**) Let K be a field of characteristic 0 and V be an R-vector space. For any $r, s \in \mathbb{N}$, define

$$\Lambda_r(V) \times \Lambda^s(V) \longrightarrow \Lambda^{r+s}(V)$$

$$(L, M) \longmapsto L \wedge M := \frac{1}{r!s!}\pi_{r+s}(L \otimes M).$$

Observe that $D_I \wedge D_J = \frac{1}{r!s!}\pi_{r+s}(\pi_r(f_I) \otimes \pi_s(f_J)) = \frac{r!s!}{r!s!}\pi_{r+s}(f_I \otimes f_J)$ and the latter is either 0 if I and J have a common index or $D_{I \amalg J}$ if they are distinct. This follows from Proposition 18.4.3.12.

In the following result, we see that wedge product is a anti-commutative, distributive and associative operation.

Proposition 18.4.4.2. *Let V be a K-vector space over a field K of characteristic 0.*
  1. *Let $\omega, \eta \in \Lambda^k(V), \phi \in \Lambda^l(V)$. Then, wedge product is distributive as*

$$(\omega + \eta) \wedge \phi = \omega \wedge \phi + \eta \wedge \phi,$$

  2. *Let $\omega \in \Lambda^k(V), \eta \in \Lambda^l(V)$. Then, wedge product is anti-commutative as*

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega,$$

  3. *Let $\omega \in \Lambda^k(V), \eta \in \Lambda^l(V), \phi \in \Lambda^m(V)$. Then, wedge product is associative as*

$$(\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi).$$

Proof. We need only check these identities on the basis elements $\{D_I\}$ of each $\Lambda^r(V)$.
  1. Let $\omega = D_I, \eta = D_J$ and $\varphi = D_M$. Then,

$$(D_I + D_J) \wedge D_M = \pi_{k+l}((D_I + D_J) \otimes D_M) = \pi_{k+l}(D_I \otimes D_M + D_J \otimes D_M)$$
$$= \pi_{k+l}(D_I \otimes D_M) + \pi_{k+l}(D_J \otimes D_M) = D_I \wedge D_M + D_J \wedge D_M$$

as required.
  2. **TODO**. $\qquad\qquad\square$

Using above, we come to the following definition.

Definition 18.4.4.3. (**Exterior algebra**) Let V be a K-vector space where K is a field of characteristic 0. Then the exterior algebra over V is

$$\Lambda(V) = K \oplus \Lambda^1(V) \oplus \Lambda^2(V) \ldots$$
$$= K \oplus \bigoplus_{k \geq 1} \Lambda^k(V)$$

where the product is given by wedge product which by Proposition 18.4.4.2 is associative, unital, distributive but non-commutative. This is also sometimes called the Grassmann algebra over V.

Remark 18.4.4.4. Observe that if V is of dimension n, then

$$\Lambda(V) = K \oplus \bigoplus_{k=1}^{n} \Lambda^k(V)$$

as all the higher forms are automatically 0. Consequently, the dimension of $\Lambda(V)$ by Theorem 18.4.3.14 is seen to be

$$\dim_K \Lambda(V) = 1 + \sum_{k=1}^{n} {}^nC_k$$
$$= \sum_{k=0}^{n} {}^nC_k$$
$$= 2^n.$$

Remark 18.4.4.5. Let V be a K-vector space of dimension n, where K is of characteristic 0. The exterior algebra $\Lambda(V)$ is a graded K-algebra of dimension 2^n over K. Indeed, the grading is correct as if $\omega \in \Lambda^k(V), \eta \in \Lambda^l(V)$, then $\omega \wedge \eta \in \Lambda^{k+l}(V)$.

18.5 Field theory

Add first chapter of your bachelor's thesis in four sections : algebraic, separable, normal and Galois extensions.

18.5.1 Transcendence degree

Definition 18.5.1.1. (**Transcendence**) Let K/k be a field extension.
1. A collection of elements $\{\alpha_i\}_{i \in I}$ of K is said to be *algebraically independent* if the map

$$k[x_i \mid i \in I] \longrightarrow K$$
$$x_i \longmapsto \alpha_i$$

is injective.
2. A *transcendence basis* of K/k is defined to be an algebraically independent set $\{\alpha_i \mid i \in I\}$ of K/k such that $K/k(\alpha_i \mid i \in I)$ is an algebraic extension.
3. The extension K/k is said to be *purely transcendental* if $K \cong k(x_i \mid i \in I)$ for some indexing set I.

Lemma 18.5.1.2. *Let K/k be a field extension. Then, $\{\alpha_i\}_{i \in I}$ is a transcendence basis of K/k if and only if $\{\alpha_i\}_{i \in I}$ is a maximal algebraically independent set of K/k.*

Proof. (L $\Rightarrow$ R) If $\{\alpha_i\}_{i \in I}$ is not maximal, then there exists $S \subset K$ containing $\{\alpha_i\}_{i \in I}$ such that S is algebraically independent. Let $\beta \in S \setminus \{\alpha_i\}_{i \in I}$. But since $K/k(\{\alpha_i\}_{i \in I})$ is an algebraic extension and $\beta \notin k(\{\alpha_i\}_{i \in I})$ by algebraic independence of S, therefore we have a contradiction to algebraic nature of the extension $K/k(\{\alpha_i\}_{i \in I})$.
(R $\Rightarrow$ L) Suppose $K/k(\{\alpha_i\}_{i \in I})$ is not algebraic. Then there exists $\beta \in K$ which is transcendental over $k(\{\alpha_i\}_{i \in I})$. Thus the set $\{\alpha_i\}_{i \in I} \cup \{\beta\}$ is a larger algebraically independent set, contradicting the maximality. $\square$

Lemma 18.5.1.3. *Let K/k be a field extension. Then any two transcendence basis have the same cardinality.*

Proof. See Tag 030F of cite[Stacksproject]. $\square$

Definition 18.5.1.4. (**Transcendence degree**) Let K/k be a field extension. The cardinality of any transcendence basis is said to be the transcendence degree, denoted trdeg K/k. Furthermore, if A is a domain containing k, then we define trdeg A/k to be the transcendence degree of A_0, the field of fractions of A, over k.

Remark 18.5.1.5. Let K/k be a field extension. If trdeg $K/k = 1$, then there exists $\alpha \in K$ such that α is not an algebraic element over k but $K/k(\alpha)$ is algebraic. In particular, for any transcendental element $\alpha \in K$ over k, the set $\{\alpha\}$ is algebraically independent over k. Precisely, there is a one-to-one bijection between the set of all singletons which are algebraically independent and all transcendental elements of K/k.

Example 18.5.1.6. There are some basic examples which reader might have encountered. For example, one knows that $\mathbb{Q}(\pi)/\mathbb{Q}$ is transcendental as $\pi \in \mathbb{Q}(\pi)$ is not algebraic over $\mathbb{Q}$. Consequently, trdeg $\mathbb{Q}(\pi)/\mathbb{Q}$ is 1, as $\mathbb{Q}(\pi)/\mathbb{Q}(\pi)$ is algebraic.

For another example, consider the next obvious situation of $\mathbb{Q}(e, \pi)/\mathbb{Q}$. Since $\{e\}$ and $\{\pi\}$ are algebraically independent sets over $\mathbb{Q}$, therefore trdeg in this case is ≥ 1. But it is an unknown problem whether $\{e, \pi\}$ forms an algebraically independent set over $\mathbb{Q}(!)$ Consequently, if they do, then trdeg $\mathbb{Q}(e, \pi)/\mathbb{Q} = 2$ and if they don't, then the best we can say is trdeg $\mathbb{Q}(e, \pi)/\mathbb{Q} \geq 1$.

Example 18.5.1.7. We have trdeg $k(x_1, \ldots, x_n)/k = n$ as $\{x_1, \ldots, x_n\}$ forms a maximal algebraically independent set.

We observe some basic first properties of transcendence degree.

Lemma 18.5.1.8. *Let $A = k[\alpha_1, \ldots, \alpha_n]$ be an integral domain where $\alpha_i \in K$ for some field extension K/k. If* trdeg $A/k = r > 0$, *then there exists $\alpha_{i_1}, \ldots, \alpha_{i_r}$ which are transcendental over k.*

18.6   Integral dependence

The main topic of interest of study in this section is the following question: "*let R be a ring and S be an R-algebra. How do all those elements of S behave like which satisfy a polynomial with coefficients in R?*".

18.6.1   Definitions and basic theory

In order to investigate this further, let us bring some definitions.

Definition 18.6.1.1. (**Integral elements and integral algebra**) Let R be a ring and S be an R-algebra. An element $s \in S$ for which there exists $p(x) \in R[x]$ such that $p(s) = 0$ in S is said to be an *integral element* over R. Further, S is said to be *integral over R* if every element of S is integral over R.

To begin deriving properties, we would need a fundamental result about endomorphisms of finitely generated modules.

Theorem 18.6.1.2. *(Cayley-Hamilton theorem) Let R be a ring, M be a finitely generated R-module generated by n elements and $I \leq R$ be an ideal. If $\varphi : M \to M$ is an R-linear map such that*

$$\varphi(M) \subseteq IM,$$

then there exists a monic polynomial

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$

in $R[x]$ such that $p(\varphi) = 0$.

Proof. Theorem 4.3, pp 120, [cite Eisenbud]. $\qquad\square$

There are two important corollaries of Cayley-Hamilton which will remind the reader of finite-dimensional vector space case.

Corollary 18.6.1.3. *Let R be a ring and M be a finitely generated R-module. If $\phi : M \to M$ is a surjective R-module homomorphism, then ϕ is an isomorphism.*

Proof. $\qquad\square$

Corollary 18.6.1.4. *Let R be a ring and M be a finitely generated R-module. If M is free of finite rank, then any generating set of n elements of M is linearly independent.*

Proof. $\qquad\square$

Investigation into the question posed in the beginning first yields us the following result, which tells in particular that adjoining a root of any polynomial $p(x) \in R[x]$ to R yields a finitely generated R-module.

Proposition 18.6.1.5. *Let R be a ring and $I \leq R$ be an ideal. Denote $S = R[x]/I$ and $s \in S$ be the image of x under natural projection of $x \in R[x]$ to S. Then,*

1. S is generated by $\leq n$ elements as an R-module if and only if I contains a monic polynomial of degree $\leq n$.

2. S is finitely generated free R-module if and only if I is generated by a monic polynomial, a free basis in which case is given by $\{1, s, s^2, \ldots, s^{n-1}\}$.

Proof. 1.   □

A simple corollary of this gives us a way to check integrality of an element.

Corollary 18.6.1.6. *Let R be a ring and S be an R-algebra. Then $z \in S$ is integral over R if and only if $R[z] \leq S$ is a finitely generated R-module.*

One can obtain a further corollary of Proposition 18.6.1.5 that finite generation of an algebra by integral elements as an algebra is equivalent to finite generation as an R-module.

Corollary 18.6.1.7. *Let R be a ring and S be an R-algebra. Then S is a finite R-algebra if and only if S is generated by finitely many integral elements as an R-algebra.*

The following result show that all integral elements form a subring of S.

Proposition 18.6.1.8. *Let R be a ring and S be an R-algebra. The set of all elements of S integral over R forms a subalgebra of S.*

With this, a natural situation is when every element of S is integral over R.

Definition 18.6.1.9. (**Normalization & integral extension**) Let R be a ring and S be an R-algebra. The subalgebra A of all integral elements of S over R is said to be the *integral closure of S over R*. One also calls A the *normalization of R in S*. Further, if $R \hookrightarrow S$ is a ring extension and every element of S is integral over R, then S is said to be an *integral extension of R*.

A special situation in above is when R is a domain and S is its fraction field.

Definition 18.6.1.10. (**Normal domain**) Let R be a domain and S be its fraction field. If the normalization of R in S is R itself, then R is said to be a normal domain.

18.6.2   Normal domains & Noether normalization

This geometrically connects to normalization of a variety.

Further investigation into normal domains lets us identify all UFDs as normal domains.

Proposition 18.6.2.1. *All unique factorization domains are normal domains.*

Consequently, $\mathbb{Z}, k[x_1, \ldots, x_n]$ and $\mathbb{Z}[x_1, \ldots, x_n]$ are normal as well. In-fact we have that a ring is normal if and only if the polynomial ring in one variable over it is normal.

Lemma 18.6.2.2. *A ring R is normal if and only if $R[x]$ is normal.*

Further, we can obtain a generalization of the fact that a monic irreducible in $\mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$.

Proposition 18.6.2.3. *Let $R \hookrightarrow S$ be a ring extension and let $f \in R[x]$ be a monic polynomial. If $f = gh$ in $S[x]$ where g and h are monic, then the coefficients of g and h are integral over R.*

We also obtain that any monic irreducible in the polynomial ring in one variable over a normal domain is prime.

Lemma 18.6.2.4. *Let R be a ring and $f(x) \in R[x]$ be a monic irreducible. If R is a normal domain, then $f(x)$ is a prime element.*

Thus, for normal domains R, monic irreducible and monic prime polynomials are equivalent concepts.

Finally, as a big use of normalization in geometry, we obtain a version of Noether normalization theorem, which says that finitely generated domains have normalization which is again a finitely generated domain.

Theorem 18.6.2.5. *Do Noether normalization from Eisenbud.*

18.6.3   Integral extensions

This geometrically connects to proper maps between affine varieties. We will cover Cohen-Seidenberg theorems about primes in an integral extension.

18.7   Dimension theory

We will discuss the notion of dimension of rings and how that notion corresponds to dimension of the corresponding affine scheme. Further, the notion of dimension applied to algebraic geometry will garnish us with a concrete geometric intuition to situations which otherwise may feel completely sterile.

18.7.1   Dimension, height & coheight

As usual, all rings are commutative with 1.

Definition 18.7.1.1. (Dimension of a ring) Let R be a ring. Then $\dim R$ is defined as follows

$$\dim R := \sup_r \{\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_r \mid \mathfrak{p}_i \text{ are prime ideals of } R\}.$$

Definition 18.7.1.2. (Height/coheight of a prime ideal) Let R be a ring and $\mathfrak{p} \lneq R$ be a prime ideal. Then height of $\mathfrak{p}$ is defined as follows:

$$\operatorname{ht} \mathfrak{p} := \sup_r \{\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_r \mid \mathfrak{p}_i \text{ are prime ideals of } R\}.$$

Similarly, the coheight of $\mathfrak{p}$ is defined by

$$\operatorname{coht} \mathfrak{p} := \sup_r \{\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r \mid \mathfrak{p}_i \text{ are prime ideals of } R\}$$

Remark 18.7.1.3. Note that the dimension of a prime ideal $\mathfrak{p}$ as a ring may not be same as its height in R, as there might be many more primes in $\mathfrak{p}$ which may fail to be primes in the ring R. But clearly, $\dim \mathfrak{p} \geq \operatorname{ht} \mathfrak{p}$.

Recall that the *dimension of a topological space X* is defined as

$$\dim X = \sup_r \{Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r \mid Z_i \text{ are irreducible closed subsets of } X\}.$$

We now have some immediate observations about height, coheight and dimension.

Lemma 18.7.1.4. *Let R be a ring. Then,*
   1. $\operatorname{ht} \mathfrak{p} = \dim R_{\mathfrak{p}},$
   2. $\operatorname{coht} \mathfrak{p} = \dim R/\mathfrak{p},$
   3. $\operatorname{ht} \mathfrak{p} + \operatorname{coht} \mathfrak{p} \leq \dim R.$

Proof. Prime ideals of $R/\mathfrak{p}$ are in one-to-one order preserving bijection with prime ideals of R containing $\mathfrak{p}$. Prime ideals of $R_{\mathfrak{p}}$ are in one-to-one order preserving bijection with prime ideals of R contained in $\mathfrak{p}$. Let Y denote the length of all chains of prime ideals of R passing through $\mathfrak{p}$. Consequently, $\sup Y \leq \dim X$. But $\sup Y = \operatorname{ht} \mathfrak{p} + \operatorname{coht} \mathfrak{p}$. $\square$

Lemma 18.7.1.5. *Let R be a PID. Then, $\dim R = 1$. Consequently, $\mathbb{Z}$ and $k[x]$ are one dimensional rings for any field k[5].*

---
[5]as the intuition agrees!

Proof. Any chain is either of the form $\langle 0 \rangle$ or $\langle x \rangle \supsetneq \langle 0 \rangle$. $\qquad\square$

Further, by Theorem 18.1.4.3 we see the following.

Lemma 18.7.1.6. *If R is a PID which is not a field, then $\dim R[x] = 2$.*

Proof. Indeed, by Theorem 18.1.4.3, the longest chain of prime ideals of the form $\mathfrak{o} \lneq \langle f(x) \rangle \lneq \langle p, h(x) \rangle$ where $f(x)$ is irreducible and $h(x)$ is irreducible modulo prime $p \in R$, as one can see immediately. $\qquad\square$

The following is also a simple assertion, which basically is why one introduces dimension of a ring.

Lemma 18.7.1.7. *Let R be a ring. Then,*

$$\dim \operatorname{Spec}(A) = \dim A.$$

Proof. Immediate from definitions and Lemma 1.2.1.1. $\qquad\square$

Let us now give some more helpful notions, especially the dimension of an R-module.

Definition 18.7.1.8. (Dimension of a module and height of ideals) Let M be an R-module. Then the dimension of M is defined as

$$\dim M := \dim R/\operatorname{Ann}(M).$$

Further, for an ideal $I \leq R$, we define the height of I as the infimum of heights of all prime ideals above I:

$$\operatorname{ht} I := \inf\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \supseteq I, \ \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

We have the corresponding topological result.

Lemma 18.7.1.9. *Let R be a ring and M be a finitely generated R-module. Then,*

$$\dim M = \dim \operatorname{Supp}(M)$$

where $\operatorname{Supp}(M) \subseteq \operatorname{Spec}(R)$ is the support of the module M.

Proof. The result follows as $\operatorname{Supp}(M)$ is the closed subset $V(\operatorname{Ann} M)$ so that any irreducible closed set in $\operatorname{Supp}(M)$ will be irreducible closed in $\operatorname{Spec}(R)$ and then we can use Lemma 1.2.1.1. $\qquad\square$

18.7.2 Dimension of finite type k-algebras

In algebraic geometry, one is principally interested in finite type algebras over a field. Thus it is natural to engage in the study of their dimensions. We discuss some elementary results in this direction in this section. See Section 18.1.5 for basics of finite type k-algebras.

The main results are as follows.

Theorem 18.7.2.1. *Let k be a field and A be a finite type k-algebra which is a domain[6].
Then,*

$$\dim A = \operatorname{trdeg} A/k.$$

Theorem 18.7.2.2. *Let k be a field and A be a finite type k-algebra which is a domain and
let $\mathfrak{p} \lneq A$ be a prime ideal. Then,*

$$\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} = \dim A.$$

We now prove these results.

---

[6]note that such algebras are exactly the ones which correspond to affine algebraic varieties.

18.8 Completions

Do from Chapter 7 of Eisenbud

18.9   Valuation rings

Definition 18.9.0.1. (Valuation on a field) Let K be a field and G be an abelian group. A function $v : K \to G \cup \{\infty\}$ is said to be a valuation of K with values in G if v satisfies

1. $v(xy) = v(x) + v(y)$,
2. $v(x + y) \geq \min\{v(x), v(y)\}$,
3. $v(x) = \infty$ if and only if $x = 0$.

Let $\mathrm{Val}(K, G)$ denote the set of all valuations over K with values in G.

Few immediate observations are in order.

Lemma 18.9.0.2. *Let K be a field, G be an abelian group and $v \in \mathrm{Val}(K, G)$ be a valuation. Then,*

1. *$R = \{x \in K \mid v(x) \geq 0\} \cup \{0\}$ is a subring of K,*
2. *$\mathfrak{m} = \{x \in K \mid v(x) > 0\} \cup \{0\}$ is a maximal ideal of R,*
3. *$(R, \mathfrak{m})$ is a local ring,*
4. *R is an integral domain,*
5. *$R_{\langle 0 \rangle} = K$,*
6. *$\forall x \in K$, $x \in R$ or $x^{-1} \in R$.*

Proof. Items 1 and 4 are immediate from the axioms of valuations. Items 2 and 3 are immediate from the observation that $\{x \in K \mid v(x) = 0\} \cup \{0\}$ is a field in R. For items 5 and 6, we need to observe that $v(1) = 0$ and for any $x \in K^{\times}$, $v(x^{-1}) = -v(x)$. $\qquad\square$

Remark 18.9.0.3. We call the subring $R \subset K$ above corresponding to a valuation v over K to be the *value ring of v*.

Definition 18.9.0.4. (Valuation rings) Let R be an integral domain. Then R is said to be a valuation ring if it is the value ring of some valuation over $K = R_{\langle 0 \rangle}$.

Definition 18.9.0.5. (Domination) Let K be a field and $A, B \subset K$ be two local rings in K. Then B is said to dominate A if $B \supseteq A$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$.

There is an important characterization of valuation rings inside a field K with respect to all local rings in K.

Theorem 18.9.0.6. *Let K be a field and $R \subset K$ be a local ring. Denote $\mathrm{Loc}(K)$ to be the set of all local rings in K together with the partial order of domination. Then, the following are equivalent,*

1. *R is a valuation ring.*
2. *R is a maximal element of the poset $\mathrm{Loc}(K)$.*

Furthermore, for every local ring $S \in \mathrm{Loc}(K)$, there exists a valuation ring $R \in \mathrm{Loc}(K)$ which dominates S.

Proof. See Tag 00I8 of cite[Stacksproject]. $\qquad\square$

An important type of valuation rings are where the value group is the integers.

Definition 18.9.0.7. (Discrete valuation rings) Let R be a domain. Then R is said to be a discrete valuation ring (DVR) if the value group of R is the integers $\mathbb{Z}$.

It turns out that noetherian local domains of dimension 1 have some important characterizations.

Theorem 18.9.0.8. *Let A be a noetherian local domain of dimension 1. Then the following are equivalent*

1. *A is a DVR,*
2. *A is a normal domain,*
3. *A is a regular local ring,*
4. *the maximal ideal of A is principal.*

Proof. Do it from Atiyah-Macdonald page 94. □

18.10   Tor and Ext functors

Start doing from Appendix 3 of Eisenbud.

18.11   Projective modules

In this section we define an important object in the study of algebraic K-theory, projective modules. These generalize finitely generated free R-modules. This notion is further used in a very important geometric concept called depth and Cohen-Macaulay condition. In order to reach there, we would need a concept called projective dimension, which we cover here.

18.12   Multiplicities

18.13 Kähler differentials

18.14   Depth and Cohen-Macaulay

18.15 Filtrations

Do from Chapter 5 of Eisenbud

18.16 Flatness

Complete this from Eisenbud Chapter 7 and appendix of Sernesi on Flatness, especially Proposition A.2.

This is one of the important parts of commutative algebra, as this notion corresponds to the idea of a continuous family of schemes, in some sense, as is discussed in the respective part above.

Definition 18.16.0.1. (**Flat modules and flat map of rings**) Let R be a ring. An R-module M is said to be flat if for any short exact sequence of R-modules

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

the following sequence is exact

$$0 \longrightarrow M \otimes_R N_1 \longrightarrow M \otimes_R N_2 \longrightarrow M \otimes_R N_3 \longrightarrow 0 .$$

A map $\varphi : A \to B$ is a flat map if B is a flat A-module. In this case one also calls B to be a flat A-algebra.

Remark 18.16.0.2. 1. By right exactness of tensor products, it is sufficient to check that the s.e.s. $0 \to N_1 \to N_2$ is taken to s.e.s $0 \to M \otimes_R N_1 \to M \otimes_R N_2$.

2. Since localisation is an exact functor (Lemma 18.1.2.2), thus the natural map $A \to S^{-1}A$ is a flat map for any multiplicative set $S \subseteq A$.

18.17   Lifting properties : Étale maps

18.18 Lifting properties : Unramified maps

18.19   Lifting properties : Smooth maps

18.20 Simple, semisimple and separable algebras

These algebras are at the heart of the Galois phenomenology, i.e. all things related to polynomials splitting in a bigger field or not. Our study of these objects will thus motivate the study of the corresponding geometrical picture.

Find and write about these alge Chapter 18.

18.20.1 Semisimple algebras

Definition 18.20.1.1. (Semisimple algebras over a field k) Let A be a k-algebra. Then A is a semisimple k-algebra if the Jacobson radical of A is 0.

18.20.2 Separable algebras

We will first study a rather special type of separable algebras, which are finitely generated and free as modules. Let us first give an example of such an algebra which is motivating our definition given later.

Example 18.20.2.1. Consider a ring A and the A-algebra A^n. There is something special about A^n; it is "separated" into finitely pieces which looks like A. This can be formalized. Indeed, we have the most obvious fact about such algebras that the obvious map

$$\varphi : A^n \longrightarrow \operatorname{Hom}_A\left(A^n, A\right)$$
$$(a_1, \ldots, a_n) \longmapsto e_i \mapsto a_i$$

is an isomorphism of A-algebras. More specifically, the map φ takes $(a_i) = (a_1, \ldots, a_n)$ to the following mapping

$$\varphi((a_i)) : A^n \longrightarrow A$$
$$(b_1, \ldots, b_n) \longmapsto a_1 b_1 + \cdots + a_n b_n.$$

We now wish to generalize this. That is to say, taking above phenomenon as a definition we want to generalize when an A-algebra B "separates" into simple pieces. For this to work, we need to find an alternate characterization of the above phenomenon. For this, a little bit of thought shows that the above map is obtained as the dual map of the $\phi \in \operatorname{Hom}_A\left(A^n, \operatorname{Hom}_A\left(A^n, A\right)\right)$ under the $\otimes$-Hom adjunction

$$\operatorname{Hom}_A\left(A^n \times A^n, A\right) \cong \operatorname{Hom}_A\left(A^n, \operatorname{Hom}_A\left(A^n, A\right)\right)$$

where the isomorphism is given by

$$(A^n \times A^n \xrightarrow{f} A) \longmapsto ((a_i) \mapsto ((b_i) \mapsto f((a_i), (b_i)))).$$

Now, consider the map

$$\tilde{\phi} : A^n \times A^n \longrightarrow A$$
$$((a_i), (b_i)) \longmapsto \sum_{i=1}^{n} a_i b_i.$$

The tensor-hom isomorphism tells us that $\tilde{\phi}$ is the dual map of ϕ above. Now notice that this dual map $\tilde{\phi}$ has a very simple description; it is given by the following commutative diagram:

$$
\begin{array}{ccc}
A^n \times A^n & \xrightarrow{\ \tilde{\phi}\ } & A \\
\downarrow & \nearrow{\scriptstyle \text{Tr}} & \\
\operatorname{Hom}_A(A^n, A^n) & &
\end{array}
$$

It is this dual map that we shall generalize to the setting of arbitrary A-algebra B which is finitely generated and free of rank n. Indeed, for any A-algebra B and chose any generating set of B as an A-module, so that for any element $b \in B$, we can write $b = (b_1, \ldots, b_n) \in A^n$. We thus get a natural map $\tilde{\phi}$ as in the diagram below

$$
\begin{array}{ccccc}
(b, c) & & B \times B & \xrightarrow{\ \tilde{\phi}\ } & A \\
\downarrow & & {\scriptstyle \kappa}\downarrow & \nearrow{\scriptstyle \text{Tr}} & \\
(b_i c_j)_{1 \le i, j \le n} & & \operatorname{Hom}_A(B, B) & &
\end{array}
$$

Now, consider the tensor-hom dual of $\tilde{\phi}$ to obtain

$$
\phi : B \longrightarrow \operatorname{Hom}_A(B, A)
$$
$$
b \longmapsto \left(c \mapsto \tilde{\phi}(b, c) \right).
$$

In order to mimic the case of A^n, we would require the map ϕ to be an isomorphism. Indeed, this is what we do in the definition given below.

Before defining a nice class of separable algebras, let us define an A-algebra B to be *finitely free* if B is finitely generated and free as an A-module.

Definition 18.20.2.2. (Free separable algebras) Let A be a ring and B be a finitely free A-algebra of rank n and chose a generating set of B, so for $b \in B$, we can write $b = (b_1, \ldots, b_n)$ for $b_i \in A$. Define $\tilde{\varphi}$ to be the following map

$$
\begin{array}{ccccc}
(b, c) & & B \times B & \xrightarrow{\ \tilde{\varphi}\ } & A \\
\downarrow & & {\scriptstyle \kappa}\downarrow & \nearrow{\scriptstyle \text{Tr}} & \\
(b_i c_j)_{1 \le i, j \le n} & & \operatorname{Hom}_A(B, B) & &
\end{array}
$$

Then B is said to be a separable A-algebra if the tensor-hom dual map $\varphi : B \to \operatorname{Hom}_A(B, A)$ is an isomorphism of A-algebras.

We would now like to show how separable algebras become familiar in the case of algebras over a field.

Proposition 18.20.2.3. *Let k be a field and A be an k-algebra. Then, the following are equivalent*

1. A is a free separable k-algebra.

2. $A = \prod_{i=1}^{n} K_i$ where K_i are finite separable extensions of field k.

Proof. □

Another characterization of separable algebras is as follows.

Lemma 18.20.2.4. *Let A be a ring and B be a finitely free A-algebra. Then the following are equivalent.*
1. *B is a separable A-algebra.*
2. *For all $\{w_1, \ldots, w_n\}$ in B which is a generating set of free A-module B, we have*

$$\det\left(\mathrm{Tr}(w_i w_j)_{1 \leq i,j \leq n}\right) \in A^{\times}.$$

Proof. □

18.21   Miscellaneous

We collect in this section results which doesn't fit in any other prior section. Perhaps this means our arrangement of material is not optimal.

The following result is a generalization of Lagrange interpolation formula.

Lemma 18.21.0.1. *Let K/F be an algebraic field extension. Then for any $\alpha_1, \ldots, \alpha_n \in K$, such that α_i is not equal to any α_j nor any of its conjugate, and for any choice $\beta_1, \ldots, \beta_n \in K$, there exists a polynomial $f(x) \in F[x]$ such that $f(\alpha_i) = \beta_i$ for all $i = 1, \ldots, n$.*

Proof. Let $\alpha_1, \ldots, \alpha_n \in K$ be such that α_j is not equal to α_i nor any of its conjugates for any $j \neq i$. Let $\beta_1, \ldots, \beta_n \in K[\alpha_i]$. We wish to find a polynomial $f(x) \in F[x]$ such that $f(\alpha_i) = \beta_i$ for each $i = 1, \ldots, n$.

We first observe that as K is an algebraic extension of F, therefore there exists $p_i(x) \in F[x]$ which is the minimal polynomial of $\alpha_i \in K$. This polynomial is obtained by looking at the kernel of evaluation at α_i, $\varphi_i : F[x] \to K$ where $x \mapsto \alpha_i$. Consequently, $p_i(x)$ is a monic irreducible polynomial of least degree in $F[x]$ such that $p_i(\alpha_i) = 0$, for each $i = 1, \ldots, n$.

As $\mathfrak{m}_i := \langle p_i(x) \rangle \leq F[x]$ are maximal ideals and $p_i(x) \neq p_j(x)$ because $\alpha_i \neq \alpha_j, \overline{\alpha_j}$[7], therefore $\mathfrak{m}_i + \mathfrak{m}_j = F[x]$ for all $i \neq j$. Hence $\mathfrak{m}_i$ are comaximal. Consequently, we obtain by Chinese remainder theorem that

$$F[x] \twoheadrightarrow \frac{F[x]}{\mathfrak{m}_1 \ldots \mathfrak{m}_n} \xrightarrow{\cong} \frac{F[x]}{\mathfrak{m}_1} \times \cdots \times \frac{F[x]}{\mathfrak{m}_n} \xrightarrow{\cong} F[\alpha_1] \times \cdots \times F[\alpha_n]$$
.

$$f(x) \longmapsto f(x) + \mathfrak{m}_1 \ldots \mathfrak{m}_n \longmapsto (f(x) + \mathfrak{m}_i)_i \longmapsto (f(\alpha_1), \ldots, f(\alpha_n))$$

Consequently, by above diagram, for the elements $(\beta_1 \ldots, \beta_n) \in F[\alpha_1] \times \cdots \times F[\alpha_n]$, there exists a polynomial $f(x) \in F[x]$ such that $(f(\alpha_1), \ldots, f(\alpha_n)) = (\beta_1, \ldots, \beta_n)$. Hence $f(\alpha_i) = \beta_i$ for each $i = 1, \ldots, n$. This completes the proof. □

The following is a general exercise in basic ideal theory.

_____
[7]because conjugates have same minimal polynomials.

Lemma 18.21.0.2. *Let R be a commutative ring with unity. Let $\mathfrak{p} \lneq R$ be a prime ideal and $I, J \leq R$ be ideals. Then,*

1. *$I^k \subseteq \mathfrak{p}$ for some $k \geq 0$ implies $I \subseteq \mathfrak{p}$,*
2. *the following are equivalent:*
   (a) $\sqrt{I} + \sqrt{J} = R$,
   (b) $I + J = R$,
   (c) $I^k + J^l = R$ for all $k, l > 0$.

Proof. 1. Let $I \leq R$ be an ideal and $\mathfrak{p} \lneq \mathbb{R}$ be a prime ideal. Then, we wish to show that $I^k \subseteq \mathfrak{p} \implies I \subseteq \mathfrak{p}$ for any $k \in \mathbb{N}$.

Indeed, pick any $x \in I$. As $x^k \in I$, therefore $x^k \in \mathfrak{p}$. As $x^k = x \cdot x^{k-1} \in \mathfrak{p}$, therefore either $x \in \mathfrak{p}$ of $x^{k-1} \in \mathfrak{p}$. If the former, then we are done. If the latter, then we have $x^{k-1} = x \cdot x^{k-2} \in \mathfrak{p}$. Continuing in this manner, we eventually reach to the conclusion that $x \in \mathfrak{p}$.

2. $((a) \Rightarrow (b))$: As we have $x \in \sqrt{I}$ and $y \in \sqrt{J}$ such that $x + y = 1$, therefore for some $n, m \in \mathbb{N}$ we have $x^n \in I$ and $y^m \in J$. Now, observe that

$$1 = 1^{n+m} = (x+y)^{n+m} = \sum_{r=0}^{n+m} {}^{n+m}C_r x^r y^{n+m-r}$$

$$= \sum_{r=0}^{n} {}^{n+m}C_r x^r y^{n+m-r} + \sum_{r=n+1}^{n+m} {}^{n+m}C_r x^r y^{n+m-r}.$$

If $0 \leq r \leq n$, then $y^{n+m-r} \in J$ and if $n+1 \leq r \leq n+m$, then $x^r \in I$. Hence $\sum_{r=0}^{n} {}^{n+m}C_r x^r y^{n+m-r} \in J$ and $\sum_{r=n+1}^{n+m} {}^{n+m}C_r x^r y^{n+m-r} \in I$. This shows that there exists $a \in I$ and $b \in J$ then $a + b = 1$.

$((b) \Rightarrow (c))$: As we have $x \in I$ and $y \in J$ such that $x + y = 1$, thus writing $1 = 1^{k+l}$ again, we see

$$1 = 1^{k+l} = (x+y)^{k+l}$$

$$= \sum_{r=0}^{k+l} {}^{k+l}C_r x^r y^{k+l-r}$$

$$= \sum_{r=0}^{k} {}^{k+l}C_r x^r y^{k+l-r} + \sum_{r=k+1}^{k+l} {}^{k+l}C_r x^r y^{k+l-r}.$$

If $0 \leq r \leq k$, then $y^{k+l-r} \in J^l$ and if $k+1 \leq r \leq k+l$, then $x^r \in I^k$. Consequently, we have $\sum_{r=0}^{k} {}^{k+l}C_r x^r y^{k+l-r} \in J^l$ and $\sum_{r=k+1}^{k+l} {}^{k+l}C_r x^r y^{k+l-r} \in I^k$. Hence there exists $a \in I^k$ and $b \in J^l$ such that $a + b = 1$.

$((c) \Rightarrow (a))$: Setting $k = l = 1$, we have that there exists $x \in I$ and $y \in J$ such that $x + y = 1$. As $\sqrt{I} \supseteq I$ and $\sqrt{J} \supseteq J$, therefore $x \in \sqrt{I}$ and $y \in \sqrt{J}$ such that $x + y = 1$. Hence $\sqrt{I} + \sqrt{J} = R$. This completes the proof. $\qquad \square$

The following is a counterexample to the claim that a sub-algebra of a finite type algebra is a finite type algebra.

Lemma 18.21.0.3. *Let R be a ring. The ring $R[t, tx, tx^2, \ldots, tx^i, \ldots]$ is neither a finite type R-algebra nor a finite type $R[t]$-algebra.*

Proof. Let $S = R[t, tx, tx^2, tx^3, \ldots]$. We wish to show that S is not a finitely generated R or $R[t]$ algebra.

a) We first show that S is not finitely generated R-algebra. Indeed, let $p_1, \ldots, p_n \in S$ be generators of S as an R-algebra. Then, we have that $p_i \in R[t, tx, \ldots, tx^{m_i}]$ as a polynomial can atmost be in finitely many indeterminates. Hence, letting $M = \max_i m_i$, we obtain that $p_1, \ldots, p_n \in R[t, tx, \ldots, tx^M]$. It then follows that the R-algebra generated by $p_1, \ldots, p_n$ will only be inside $R[t, tx, \ldots, tx^M]$. We consequently reduce to showing that $R[t, tx, \ldots, tx^M] \neq S$.

Let $tx^{M+1} \in S$. We claim that $tx^{M+1} \notin R[t, tx, \ldots, tx^M]$. Assuming to the contrary, we have that for some $a_{k_0, \ldots, k_M} \in R$

$$
tx^{M+1} = \sum_{k_0, \ldots, k_M} a_{k_0, \ldots, k_M} t^{k_0} \ldots (tx^M)^{k_M}
$$
$$
= \sum_{k_0, \ldots, k_M} a_{k_0, \ldots, k_M} t^{k_0 + \cdots + k_M} \cdot x^{k_1 + 2k_2 + \cdots + Mk_M}.
$$

We thus deduce that $a_{k_0, \ldots, k_M} \neq 0$ if and only if $k_0 + \cdots + k_M = 1$. As $k_i \in \mathbb{Z}_{\geq 0}$, we further deduce that the only non-zero coefficients are $a_{1,0,\ldots,0}, a_{0,1,\ldots,0}, \ldots, a_{0,0,\ldots,1}$. Hence, the above equation reduces to

$$
tx^{M+1} = a_{1,0,\ldots,0} t + a_{0,1,\ldots,0} tx + \cdots + a_{0,0,\ldots,1} tx^M.
$$

Clearly, for no choice of coefficients $a_{1,0,\ldots,0}, a_{0,1,\ldots,0}, \ldots, a_{0,0,\ldots,1}$ in R can we make both sides equal in $R[t, x]$. This is a contradiction.

b) We now wish to show that S is not finitely generated as an $R[t]$-algebra. Assuming to the contrary, there exists $p_1, \ldots, p_n \in S$ such that S is generated by them as an $R[t]$-algebra. Again for the same reason as in a), we see that $p_1, \ldots, p_n \in R[t, tx, \ldots, tx^M]$ for some $M \in \mathbb{Z}_{>0}$. Now, as $R[t, tx, \ldots, tx^M] = R[t][tx, tx^2, \ldots, tx^M]$, therefore the $R[t]$-algebra generated by $p_1, \ldots, p_n$ will only be inside $R[t][tx, tx^2, \ldots, tx^M]$. Hence, we reduce to showing that $R[t][tx, tx^2, \ldots, tx^M] \neq S$. To this end, the exact same technique as in part a) works verbatim, as we need only show that $tx^{M+1} \notin R[t][tx, tx^2, \ldots, tx^M] = R[t, tx, \ldots, tx^M]$.

This completes the proof. $\qquad\square$

The following result characterizes all ideals of $F[[x]]$ and tells us that localization of $F[[x]]$ at $1/x$ yields the Laurent series ring.

Proposition 18.21.0.4. *Let F be a field and $R = F[[x]]$.*
1. *An element in $a = a_0 + a_1 x + \cdots \in R$ is a unit if and only if $a_0 \neq 0$.*
2. *Every non-zero ideal of R is of the form $x^k R$.*
3. *$R[x^{-1}] = Q(R) = F((x))$.*

Proof. 1. ($\Rightarrow$) Since $\sum_{i \geq 0} a_i x^i$ is a unit in $F[[x]]$, therefore there exists $\sum_{i \geq 0} b_i x^i$ which is an inverse of $\sum_{i \geq 0} a_i x^i$. Consequently, we have

$$
(a_0 + a_1 x + \ldots) \cdot (b_0 + b_1 x + \ldots) = 1
$$
$$
(a_0 b_0 + (a_1 b_0 + a_0 b_1) x + \ldots) = 1.
$$

Comparing the degree 0 term both sides, we obtain $a_0 b_0 = 1$. Therefore, if $a_0 = 0$, then $a_0 b_0 = 0$ and we would thus obtain a contradiction.

($\Leftarrow$) Suppose $a_0 \neq 0$. We wish to find $\sum_{i \geq 0} b_i x^i$ such that $\left(\sum_{i \geq 0} a_i x^i \right) \cdot \left(\sum_{j \geq 0} b_j x^j \right) = 1$. We can calculate what b_is should be by observing the following:

$$\left(\sum_{i \geq 0} a_i x^i \right) \cdot \left(\sum_{j \geq 0} b_j x^j \right) = \sum_{k \geq 0} c_k x^k$$

where $c_k = \sum_{i+j=k} a_i b_j$. We now claim that there exists a unique solution for each b_i in the equations given by setting $c_0 = 1$ and $c_k = 0$ for all $k \geq 1$. We show this by strong induction. Indeed, for $c_0 = a_0 b_0 = 1$ yields that $b_0 = a_0^{-1}$. For $k = 1$, we have $c_1 = a_1 b_0 + a_0 b_1 = 0$ which thus yields $b_1 = -a_0^{-1} a_1 b_0$. We now wish to show that if b_l has a unique solution for all $l = 0, \ldots k - 1$, then b_k has a unique solution as well. Indeed, b_k satisfies the following equation coming from $c_k = 0$:

$$0 = \sum_{i+j=k} a_i b_j$$

$$= a_0 b_k + \sum_{i+j=k, j < k} a_i b_j.$$

By inductive hypothesis, for all $0 \leq j < k$, b_j has a unique solution. Consequently by the above, b_k has a unique solution as well. This completes the induction which yields the required formal power series.

$\sum_{j \geq 0} b_j x^j$ which acts as the inverse of $\sum_{i \geq 0} a_i x^i$. 2. We wish to show that any non-zero ideal $I \leq R$ is of the form $I = x^k R$ where $k \in \mathbb{N}$. Pick any ideal $I \leq R$. For any power series $p(x) = c_n x^n + c_{n+1} x^{n+1} + \ldots$ where $c_n \neq 0$, we define n to be the **co-degree** of $p(x)$. Then, let $p(x) = c_k x^k + c_{k+1} x^{k+1} + \ldots$ be the element of I with least co-degree (such an element exists by virtue of well-ordering of $\mathbb{N}$). Consequently, we obtain $p(x) = x^k (c_k + c_{k+1} x + \ldots)$.

We thus claim that $I = x^k R$. Indeed, pick any $f(x) \in I$. Then, $f(x) = d_n x^n + d_{n+1} x^{n+1} + \ldots$ where $d_n \neq 0$. Hence, we may write $f(x) = x^n (d_n + d_{n+1} x + \ldots)$. By item 1, we know that $d_n + d_{n+1} x + \ldots$ is a unit in R, so that we may write $f(x) = x^n u$, $u \in R$ is a unit. Now, as $f(x) \in I$, thus co-degree of f is atleast k as $p(x) \in I$ with co-degree k is the least co-degree element. Consequently, we may write $f(x) = x^k x^{n-k} u$. Hence $f(x) \in x^k R$. Conversely, pick any $x^k g(x) \in x^k R$. Since $p(x) = x^k (c_k + c_{k+1} x + \ldots)$ where $c_k \neq 0$, therefore $c_k + c_{k+1} x + \ldots$ is a unit, hence $p(x) = x^k v$ for some unit $v \in R$. Thus, $x^k \in I$ and hence $x^k g(x) \in I$. This completes the proof.

3. We wish to show that $R[\frac{1}{x}] = Q(R)$, the fraction field of R, i.e. $F((x))$. Indeed, as $x \in R$ is a non-zero element, therefore $1/x \in Q(R)$ and consequently, $R[\frac{1}{x}] \subseteq Q(R)$. We now wish to show that converse also holds.

Pick any $\frac{f(x)}{g(x)} \in Q(R)$ where $f(x), g(x) \in R$ are power series. Let $f(x)$ have co-degree n and $g(x)$ have co-degree m. We may then write

$$\frac{f(x)}{g(x)} = \frac{c_n x^n + c_{n+1} x^{n+1} + \ldots}{d_m x^m + d_{m+1} x^{m+1} + \ldots}$$

where $c_n, d_m \neq 0$. We may further write above as

$$\frac{f(x)}{g(x)} = \frac{x^n u}{x^m v}$$

for units $u = c_n + c_{n+1}x + \ldots, v = d_m + d_{m+1}x + \cdots \in R$ (by item 1).

If $n > m$, then $\frac{f(x)}{g(x)} = \frac{x^{n-m}w}{1}$ for some unit $w \in R$ and we know that $\frac{x^{n-m}}{1} \in R[\frac{1}{x}]$. If $n < m$, then $\frac{f(x)}{g(x)} = \frac{w}{x^{m-n}}$ for some unit $w \in R$ and we know that $\frac{1}{x^{m-n}} \in R[\frac{1}{x}]$. Finally if $n = m$, then $\frac{f(x)}{g(x)}$ is a unit of R and hence of $R[\frac{1}{x}]$.

Hence in all cases, $\frac{f(x)}{g(x)} \in R[\frac{1}{x}]$. We thus conclude $Q(R) \subseteq R[\frac{1}{x}]$, completing the proof. $\square$

In the following theorem, we show some important properties of the ring $\mathbb{Z}[\omega]$, where ω is a third root of unity.

Theorem 18.21.0.5. *Let $R = \mathbb{Z}[\omega]$ where $\omega = e^{\frac{2\pi i}{3}}$ is a cube root of unity.*
1. *R is a Euclidean domain.*
2. *The function given by*

$$f : \mathrm{Spec}\,(\mathbb{Z}[\omega]) \longrightarrow \mathrm{Spec}\,(\mathbb{Z})$$

$$\pi \longmapsto \begin{cases} p & \text{if } \pi = p \text{ upto associates}, \\ \pi\bar{\pi} & \text{else}. \end{cases}$$

   is surjective such that $f^{-1}(p)$ is either $\{\pi, \bar{\pi}\}$ or $\{p\}$ (upto associates) for any prime $p \in \mathrm{Spec}\,(\mathbb{Z})$.
3. *Let $p \in \mathbb{Z}$ be a prime. The following are equivalent*
   (i) *p splits in $\mathbb{Z}[\omega]$, that is $p = \alpha\bar{\alpha}$ for some $\alpha \in \mathbb{Z}[\omega]$,*
   (ii) *$x^2 \pm x + 1$ has a root in $\mathbb{F}_p$, that is, $\exists a \in \mathbb{F}_p$ such that $a \neq 1$ and $a^3 = \pm 1$,*
   (iii) *either $p = 3$ or $p = 1 \mod 3$.*
4. *Take any $n \in \mathbb{Z}$. The following are equivalent*
   (i) *$n = a^2 \pm ab + b^2$ for some $a, b \in \mathbb{Z}$,*
   (ii) *primes $2 \mod 3$ occurs evenly many times in the prime factorization of n.*

Proof. 1. We first wish to show that R is a Euclidean domain. We claim that the following function

$$d : R \setminus \{0\} \longrightarrow \mathbb{N} \cup \{0\}$$
$$\alpha = a + b\omega \longmapsto \alpha\bar{\alpha} = a^2 + b^2 - ab$$

satisfies the axiom of size function for R. Indeed, pick any $\alpha, \beta \in R$ where $\beta \neq 0$. We may then write

$$\frac{\alpha}{\beta} = \frac{\alpha\bar{\beta}}{\beta\bar{\beta}} = \frac{\alpha\bar{\beta}}{c} = a + ib$$

where $a, b \in \mathbb{Q}$. As any rational $x \in \mathbb{Q}$ can be written as $x = n + q$ where $n \in \mathbb{Z}$ and $0 \le q \le 1/2$, therefore we may write

$$\frac{\alpha}{\beta} = a + ib = (n_1 + r_1) + \omega(n_2 + r_2)$$

where $n_1, n_2 \in \mathbb{Z}$ and $0 \le r_1, r_2 \le 1/2$. Thus,

$$\alpha = \beta(n_1 + \omega n_2) + \beta(r_1 + \omega r_2) \tag{1.1}$$

As $\alpha, \beta(n_1 + \omega n_2) \in R$, therefore by (1.1) we deduce that $\beta(r_1 + \omega r_2) \in R$.
Note that since the size function d is the norm map, which is actually a multiplicative map defined on whole of $\mathbb{C}$ as

$$\mathbb{C} \longrightarrow \mathbb{R}$$
$$z \longmapsto z\bar{z},$$

hence, we see that

$$d(\beta(r_1 + \omega r_2)) = \beta\bar{\beta}(r_1^2 + r_2^2 - r_1 r_2)$$
$$\le \beta\bar{\beta}\left(\frac{1}{2^2} + \frac{1}{2^2}\right)$$
$$= \frac{\beta\bar{\beta}}{2}$$
$$< \beta\bar{\beta}$$
$$= d(\beta).$$

Thus, Eq. (1.1) is the required division of α by β. This proves that R is a Euclidean domain.

2. Let R be an arbitrary Euclidean domain and let $\operatorname{Spec}(R)$ denote the set of all prime ideals of R. As R is a Euclidean domain, therefore it is a PID. Consequently, $\operatorname{Spec}(R)$ is in one-to-one bijection with prime/irreducible elements of R together with 0. Hence, we write $p \in \operatorname{Spec}(R)$ to mean a prime element of R. We know that $\mathbb{Z}[\omega]$ and $\mathbb{Z}$ are Euclidean domains.
We wish to show that there is a surjective map

$$f : \operatorname{Spec}(\mathbb{Z}[\omega]) \longrightarrow \operatorname{Spec}(\mathbb{Z})$$
$$\pi \longmapsto \begin{cases} p & \text{if } \pi = p \text{ upto associates,} \\ \pi\bar{\pi} & \text{else.} \end{cases}$$

such that $f^{-1}(p)$ is either $\{\pi, \bar{\pi}\}$ or $\{p\}$ (upto associates) for any prime $p \in \operatorname{Spec}(\mathbb{Z})$ where $\pi \in \operatorname{Spec}(\mathbb{Z}[\omega])$ is a prime element.
We first observe that $\mathbb{Z}[\omega]$ has a non-trivial automorphism given by $\alpha = a + b\omega \mapsto \bar{\alpha} = a + b\omega^2$. Pick $\pi \in \operatorname{Spec}(\mathbb{Z}[\omega])$ a non-zero prime element. Observe that automorphisms takes a prime element to a prime element. As $\mathbb{Z}$ is a UFD, therefore for $p_1, \ldots, p_l \in$

Spec $(\mathbb{Z})$ non-zero primes, and $\pi_1, \ldots, \pi_k \in$ Spec $(\mathbb{Z}[\omega])$ non-zero primes, we may write

$$\pi\bar{\pi} = a^2 + b^2 - ab$$
$$= p_1 \ldots p_l$$
$$= \pi_1 \ldots \pi_k$$

where the last equality comes from writing prime factorization of each p_i in $\mathbb{Z}[\omega]$. Now, as $\mathbb{Z}[\omega]$ is a UFD, therefore $k = 2$ and hence $l \leq 2$. We now have two cases

(i) If $l = 2$, then $\pi\bar{\pi} = p_1 p_2$. Expanding each p_i into product of primes in $\mathbb{Z}[\omega]$, we immediately deduce by unique factorization in $\mathbb{Z}[\omega]$ that $p_1 = \pi$ and $p_2 = \bar{\pi}$ upto associates (wlog). Hence, $\bar{\pi} = p_2 = p_1$. That is,

$$\pi\bar{\pi} = p^2.$$

(ii) If $l = 1$, then

$$\pi\bar{\pi} = p$$

for some non-zero prime $p \in$ Spec $(\mathbb{Z})$.

This defines the function $f : $ Spec $(\mathbb{Z}[\omega]) \to$ Spec $(\mathbb{Z})$. Next, we wish to show that this is surjective. Indeed, pick any non-zero $p \in$ Spec $(\mathbb{Z})$. Using prime factorization in $\mathbb{Z}[\omega]$, we obtain primes $\pi_1, \ldots, \pi_k$ in $\mathbb{Z}[\omega]$ such that

$$p = \pi_1 \ldots \pi_k.$$

Again using the conjugation automorphism yields us

$$p^2 = (\pi_1 \bar{\pi}_1) \ldots (\pi_k \bar{\pi}_k).$$

Note $\pi_i \bar{\pi}_i \in \mathbb{Z}$. Hence, by unique factorization of $\mathbb{Z}$, we obtain $k \leq 2$. We now have two cases

(i) If $k = 2$, then $p^2 = (\pi_1 \bar{\pi}_1)(\pi_2 \bar{\pi}_2)$. As π_i are not units, we deduce that $p = \pi_1 \bar{\pi}_1$ and $p = \pi_2 \bar{\pi}_2$. Consequently, we have $\pi_1 \bar{\pi}_1 = \pi_2 \bar{\pi}_2$. Thus, by unique factorization of $\mathbb{Z}[\omega]$, we further deduce that $\pi_1 = \pi_2$ or $\bar{\pi}_2$. Hence, $p = \pi\bar{\pi}$ for a unique $\pi \in$ Spec $(\mathbb{Z}[\omega])$.

(ii) If $k = 1$, then

$$p^2 = \pi\bar{\pi}$$

for some $\pi \in$ Spec $(\mathbb{Z}[\omega])$. Writing p as a product of primes in $\mathbb{Z}[\omega]$, we immediately deduce of unique factorization of $\mathbb{Z}[\omega]$ that $p = \pi'$ upto units for some non-zero prime $\pi' \in$ Spec $(\mathbb{Z}[\omega])$. Consequently, $p^2 = \pi'\bar{\pi'} = \pi\bar{\pi}$. Again by unique factorization of $\mathbb{Z}[\omega]$, we immediately deduce that $\pi = \pi'$ upto units.

This shows the surjectivity of the map f.

3. (i) $\iff$ (ii) : By part b), p splits in $\mathbb{Z}[\omega]$ iff p is not prime in $\mathbb{Z}[\omega]$. This happens iff $\mathbb{Z}[\omega]/p$ is not a domain. We now observe

$$
\begin{aligned}
\frac{\mathbb{Z}[\omega]}{p\mathbb{Z}[\omega]} &\cong \frac{\frac{\mathbb{Z}[x]}{\langle x^2+x+1\rangle}}{\frac{\langle p,x^2+x+1\rangle}{\langle x^2+x+1\rangle}} \\
&\cong \frac{\mathbb{Z}[x]}{\langle p, x^2+x+1\rangle} \\
&\cong \frac{\frac{\mathbb{Z}[x]}{p\mathbb{Z}[x]}}{\frac{\langle p,x^2+x+1\rangle}{p\mathbb{Z}[x]}} \\
&\cong \frac{\mathbb{F}_p[x]}{\langle x^2+x+1\rangle}.
\end{aligned}
$$

Hence, p is not prime in $\mathbb{Z}[\omega]$ iff $x^2 + x + 1$ is reducible in $\mathbb{F}_p[x]$. As a polynomial of degree 2 or 3 over a field is reducible iff it has a root in the field, therefore p is not prime in $\mathbb{Z}[\omega]$ iff $x^2 + x + 1$ has a root in $\mathbb{F}_p$. Similarly, since ω^2 has minimal polynomial $x^2 - x + 1$ and $\mathbb{Z}[\omega] = \mathbb{Z}[\omega^2]$, hence repeating the above yields p is not prime in $\mathbb{Z}[\omega]$ iff $x^2 - x + 1$ has a root in $\mathbb{F}_p[x]$.

(ii) $\Rightarrow$ (iii) : If $p = 2$, then $x^2 \pm x + 1$ has no roots in $\mathbb{F}_2$. Consequently, let $p \neq 2, 3$. We then wish to show that $p = 1 \mod 3$. Let $a \in \mathbb{F}_p$ be the root of $f(x) = x^2 \pm x + 1$. Thus, $a^3 = \pm 1$. Observe that $a \neq \pm 1$ as if $a = 1$, then $f(1)$ and $f(-1)$ are either 1 or 3 and since $p \neq 3$, therefore $f(1), f(-1) \neq 0$, a contradiction.
As $a^3 = \pm 1$ and $a \neq \pm 1$, therefore the order of $a \in \mathbb{F}_p^*$ is either 3 or 6. In either case, as $|\mathbb{F}_p^*| = p - 1$, therefore by Lagrange's theorem, $3 \div p - 1$ or $6 \div p - 1$. But in both cases, we have $p = 1 \mod 3$.

(iii) $\Rightarrow$ (ii) : If $p = 3$, then $1 \in \mathbb{F}_3$ is root of $x^2 + x + 1$ and 2 is the root of $x^2 - x + 1$. If $p = 1 \mod 3$, then we proceed as follows. As $\mathbb{F}_p^*$ is a cyclic group of order $p - 1$ and since $p - 1 = 3k$ for some $k \in \mathbb{Z}$, hence there exists an element $a \in \mathbb{F}_p$ of order 3. Consequently, we have $a^3 = 1$ and thus $x^3 - 1$ in $\mathbb{F}_p[x]$ has a root. As $x^3 - 1 = (x - 1)(x^2 + x + 1)$ and $a \neq 1$, hence a is a root of $x^2 + x + 1$.
Now since

$$
\frac{\mathbb{F}_p[x]}{\langle x^2+x+1\rangle} \cong \frac{\mathbb{F}_p[x-1]}{\langle (x-1)^2+(x-1)+1\rangle} = \frac{\mathbb{F}_p[x]}{\langle x^2-x+1\rangle}
$$

therefore if $x^2 + x + 1$ has a root in $\mathbb{F}_p$, then so does $x^2 - x + 1$.

4. (i) $\Rightarrow$ (ii) : Write the prime factorization of n in $\mathbb{Z}[\omega]$ as follows

$$
\begin{aligned}
n &= (a + b\omega)(a + b\omega^2) \\
&= (\pi_1 \ldots \pi_k)(\bar{\pi}_1 \ldots \bar{\pi}_k) \\
&= (\pi_1 \bar{\pi}_1) \ldots (\pi_k \bar{\pi}_k).
\end{aligned}
$$

From parts b) and c), we know that for any prime element $\pi \in \mathbb{Z}[\omega]$, we have $\pi\bar{\pi} = p$ iff $p = 3$ or $1 \mod 3$ and $\pi\bar{\pi} = p^2$ iff $p = 2 \mod 3$. Consequently, we have

$$
n = (p_1 \ldots p_m)(p_{m+1}^2 \ldots p_k^2)
$$

where we call primes $p_1, \ldots, p_m$ which are either 3 or 1 mod 3 of **split type**. Similarly, we call the primes $p_{m+1}, \ldots, p_k$ which are 2 mod 3 of **unsplit type**. From above it is clear that unsplit type primes appear evenly many times (they appear in squares) in the prime factorization of n.

(ii) $\Rightarrow$ (i) : Let $n \in \mathbb{Z}$ be such that its prime factorization in $\mathbb{Z}$ is as follows

$$n = (p_1 \ldots p_m)(q_1^{2k_1} \ldots q_n^{2k_n})$$

where q_i are primes of unsplit type, that is, $q_i = 2$ mod 3 and p_i are of split type, that is, 3 or 1 mod 3. Now, by part b), we may write $p_i = \pi_i \bar{\pi}_i$ as they split in $\mathbb{Z}[\omega]$ and $q_i = \xi_i$, where ξ_i, π_i are primes in $\mathbb{Z}[\omega]$.
It follows that we may write

$$\begin{aligned}
n &= (\pi_1 \bar{\pi}_1 \ldots \pi_m \bar{\pi}_m) \left(\xi_1^{2k_1} \ldots \xi_n^{2k_n} \right) \\
&= (\xi_1^{k_1} \ldots \xi_n^{k_n})(\pi_1 \ldots \pi_m) \cdot (\xi_1^{k_1} \ldots \xi_n^{k_n})(\bar{\pi}_1 \ldots \bar{\pi}_m) \\
&= \alpha \bar{\alpha}
\end{aligned}$$

where $\alpha = (\xi_1^{k_1} \ldots \xi_n^{k_n})(\pi_1 \ldots \pi_m) = a + b\omega$, as required.
This completes the proof. $\qquad \square$

Example 18.21.0.6. As an example use of above we may now find all ordered tuples $(a, b) \in \mathbb{Z}^2$ such that $2100 = a^2 - ab + b^2$.
Observe that

$$\begin{aligned}
2100 &= 2^2 \cdot 3 \cdot 5^2 \cdot 7 \\
&= 2^2 \cdot 5^2 \cdot (2 + \omega)(2 + \omega^2)(3 + \omega)(3 + \omega^2).
\end{aligned}$$

We now wish to find the distinct $\alpha \in \mathbb{Z}[\omega]$ such that $2100 = \alpha \bar{\alpha}$. For this, we first need to find all units of $\mathbb{Z}[\omega]$.
Indeed, we claim that the units of $\mathbb{Z}[\omega]$ are $1, -1, \omega, -\omega, 1 + \omega, -1 - \omega$. We give a terse proof of this fact as follows. Let $a + b\omega \in \mathbb{Z}[\omega]$ be a unit, so that there exists $c + d\omega$ such that $(a + b\omega)(c + d\omega) = 1$. Then, the multiplicative map

$$\mathbb{Z}[\omega] \to \mathbb{Z}$$
$$\alpha \mapsto \alpha \bar{\alpha}$$

yields in $\mathbb{Z}$ that $(a^2 + b^2 - ab)(c^2 + d^2 - cd) = 1$. This forces $a^2 + b^2 - ab = 1 = c^2 + d^2 - cd$. From these equations one can deduce that $c + d\omega = (a - b) - b\omega$. Hence, $a + b\omega$ is a unit iff $a^2 + b^2 - ab = 1$. It follows by AM-GM inequality on a^2 and b^2 that $ab \leq 1$. Hence, we deduce that $a = 1, b = 1$ or $a = -1, b = -1$ or $a = 0$ or $b = 0$. Correspondingly, we get the six units of $\mathbb{Z}[\omega]$ as mentioned above.
In order to count the number of distinct pairs $(a, b) \in \mathbb{Z}^2$ such that $n = a^2 + b^2 - ab = (a + b\omega)(a + b\omega^2)$ properly, let us bring some notations. Let $X_n = \{(a + b\omega) \mid (a + b\omega)(a + b\omega^2) = n\} \subseteq \mathbb{Z}[\omega]$. Denote $f : \mathbb{Z}[\omega] \to \mathbb{Z}$ to be the multiplicative map $\alpha \mapsto \alpha \bar{\alpha}$. We thus have $X_n = f^{-1}(n)$. Now observe that

1. for each $a + b\omega \in X_n$, we have $b + a\omega \in X_n$,
2. for each $a + b\omega \in X_n$, we have $a + b\omega^2 \in X_n$,
3. for each $a + b\omega \in X_n$ and $u \in \mathbb{Z}[\omega]$ a unit, we have $u(a + b\omega) \in X_n$. This is because in $\mathbb{Z}[\omega]$, inverse of a unit is its conjugate.

Our goal is to count ordered tuples $(a, b) \in \mathbb{Z}^2$ such that $n = a^2 + b^2 - ab$. Immediately, we see that such ordered tuples are in bijection with X_n. Hence, we reduce to counting X_n.

From the above discussion, we see the elements in X_n obtained by multiplying by units are

- $2 \cdot 5 \cdot 1 \cdot (2 + \omega)(3 + \omega) = 50 + 40\omega$,
- $2 \cdot 5 \cdot -1 \cdot (2 + \omega)(3 + \omega) = -50 - 40\omega$,
- $2 \cdot 5 \cdot \omega \cdot (2 + \omega)(3 + \omega) = -40 + 10\omega$,
- $2 \cdot 5 \cdot (-\omega) \cdot (2 + \omega)(3 + \omega) = 40 - 10\omega$,
- $2 \cdot 5 \cdot (1 + \omega) \cdot (2 + \omega)(3 + \omega) = 10 + 50\omega$,
- $2 \cdot 5 \cdot (-1 - \omega) \cdot (2 + \omega)(3 + \omega) = -10 - 50\omega$,
- $2 \cdot 5 \cdot 1 \cdot (2 + \omega^2)(3 + \omega) = 40 - 10\omega$.
- $2 \cdot 5 \cdot -1 \cdot (2 + \omega^2)(3 + \omega) = -40 + 10\omega$,
- $2 \cdot 5 \cdot \omega \cdot (2 + \omega^2)(3 + \omega) = 10 + 50\omega$,
- $2 \cdot 5 \cdot (-\omega) \cdot (2 + \omega^2)(3 + \omega) = -10 - 50\omega$,
- $2 \cdot 5 \cdot (1 + \omega) \cdot (2 + \omega^2)(3 + \omega) = 50 + 40\omega$,
- $2 \cdot 5 \cdot (-1 - \omega) \cdot (2 + \omega^2(3 + \omega) = -50 - 40\omega$.

Similarly, those obtained by swapping are

- $40 + 50\omega$,
- $-40 - 50\omega$,
- $10 - 40\omega$,
- $50 + 10\omega$,
- $-50 - 10\omega$.

Hence, there are 12 such ordered tuples $(a, b) \in \mathbb{Z}^2$ given by $(40, 50), (-40, -50), (10, -40),$ $(50, 10), (-50, 10), (50, 40), (-50, -40), (-40, 10), (10, 50), (-10, -50).$

The following is a simple but powerful lemma about certain type of k-algebras.

Lemma 18.21.0.7. *Let k be a field and A be a k-algebra such that there is a maximal ideal $\mathfrak{m} \lneq A$ for which $A/\mathfrak{m} \cong k$. Then,*

$$A \cong k \oplus \mathfrak{m}$$

where $k \oplus \mathfrak{m}$ obtains the k-algebra structure from A.

Proof. Consider the triangle

Pick any $a \in A$. We have $\pi(a) \in A/\mathfrak{m} \cong k$, so let $\pi(a) \in k$ by identifying under that isomorphism. Consequently, we may write $a = \pi(a) + (a - \pi(a))$. Note since $\pi(a - \pi(a)) = \pi(a) - \pi(\pi(a)) = \pi(a) - \pi(a) = 0$ by the commutativity of the above, therefore $a \in \mathfrak{m}$.

Furthermore $\mathfrak{m} \cap k = 0$ is immediate as $\mathfrak{m}$ is a proper ideal. It follows that $A = k \oplus \mathfrak{m}$ as k-linear subspaces, and thus $k \oplus \mathfrak{m}$ is a k-algebra as well, isomorphic to A, where, since $(k_1 + m_1) \cdot (k_2 + m_2) = k_1 k_2 + k_1 m_2 + k_2 m_1 + m_1 m_2$ inside of A, hence we may define the k-algebra structure on $k \oplus \mathfrak{m}$ as

$$(k_1, m_1) \cdot (k_2, m_2) = (k_1 k_2, k_1 m_2 + k_2 m_1 + m_1 m_2)$$

for $(k_i, m_i) \in k \oplus \mathfrak{m}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Chapter 19

K-Theory of Rings

Contents

Author's note : This chapter will be added around March, during the talks on K-theory by GST IISER Mohali. The goal is to understand K_0, K_1, K_2 of rings concretely and then define higher K-groups categorically.

A lot of the subsections made below are not permanent as they will be absorbed into the sections in Special Topics, like projective and free modules should go into Algebra (only the commutative case, the non-commutative case shall be given in a small remark). A lot others will be given a theorem or a lemma.

19.1 Lower K-theory of rings

Complete this section, Chapter 19

We shall study the basics of K-theory here, atleast the algebraic picture which was classically the first one to be developed. Algebraic K-theory, or atleast the idea thereof, was

first developed by Grothendieck in his far reaching generalization of Riemann-Roch theorem while studying intersection theory of varieties.

The lower K-theory of rings was the first view of algebraic K-theory that people found. Indeed, $K_0(R)$ for a ring R with unity was defined and used by Grothendieck whereas later $K_1(R), K_2(R)$ was defined later by Hymann Bass and John Milnor. The existence of a nine term long-exact sequence then raised a question whether one can define higher K-groups. Indeed, this was done by Daniel Gray Quillen who won Fields medal for this work. The general procedure in which he defined $K_n(R)$ was to first obtain a topological space from a ring R in such a manner so to make its first 3 homotopy groups identical to K_0, K_1, K_2 and then defined higher K-groups by the higher homotopy groups of that space.

Let us now see the various places that we will find ourself in during the process of finding the right notions of K-groups.

19.1.1   The grand schema

The following is the picture which we shall traverse througout these notes.
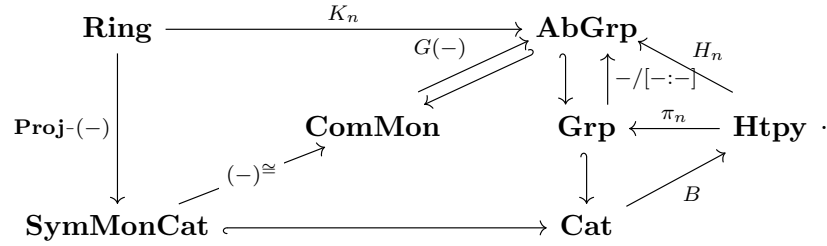
We shall see.

19.1.2   Group completion

For a commutative semigroup $(S, \star)$, we can construct an equivalence relation on it which mimics the way on obtains the group of integers out of the semigroup of naturals. Indeed, define the following on $S \times S$:

$$(a, b) \sim (c, d) \iff \exists t \in S \text{ s.t. } a \star d \star t = b \star c \star t.$$

Define $G(S) = (S \times S)/\sim$ to be the *Grothendieck group* of S, also called group completion of S. The operation on the elements is given as follows

$$[(a, b)] + [(c, d)] := [(a \star c, b \star d)].$$

Indeed since $\star$ is commutative, therefore $+$ is commutative. Moreover, identity is given by $[(a, a)]$ for any $a \in S$. The inverse of $[(a, b)]$ is obviously given by $[(b, a)]$. Examples are as follows:
  1. $G(\mathbb{N}, +) = (\mathbb{Z}, +)$,
  2. $G(\mathbb{N}, \cdot) = 0$ as there is an absorbing element in $(\mathbb{N}, \cdot)$,

3. for a linear order $(L, \leq)$, the operation of taking minimums is associative, hence $(L, \min)$ is a semigroup. Its Grothendieck group is 0 as for $\min(a, d)$ and $\min(b, c)$, because of linearity, there is an element t which is below both of them.

We now see that group completion has a right adjoint.

Theorem 19.1.2.1. *The following is an adjunction*

$$\mathbf{ComSGrp} \; \underset{\longleftarrow}{\overset{\overset{G}{\longrightarrow}}{\perp}} \; \mathbf{AbGrp} \; .$$

Proof. The unit of this adjunction is given as follows. For any semigroup S, define the following map:

$$i_S : S \longrightarrow G(S)$$
$$s \longmapsto [(s \star s, s)].$$

Indeed, this makes the following triangle commutative for any abelian group A and a map of semigroups $f : S \to A$:

$$S \xrightarrow{\;i_S\;} G(S)$$

$$f \searrow \quad \downarrow \tilde{f}$$

$$A$$

where $\tilde{f}([(a, b)]) = f(a) - f(b)$. This is clearly a map of semigroups. $\qquad \square$

Call the element $[(s \star s, s)] \in G(S)$ for an $s \in S$ to be the *klasse* of $s \in S$ and is denoted by $[s] \in G(S)$. This is where the subject K-theory got its name.

19.1.3 Cofinality

An important aspect of the Grothendieck construction is that of cofinality. This in form of a prose means *"Going to end together"*. Indeed, for a semigroup S, a sub-semigroup $T \leq S$ is said to be *cofinal* in S if for all $s \in S$, there exists $s' \in S$ such that $s \star s' \in T$. Indeed, for example, the subsemigroup of all multiples of 3 in $(\mathbb{N}, +)$ forms a subsemigroup which is cofinal in $\mathbb{N}$.

The main thrust behind this notion is because of the following theorem, which in particular tells us that cofinal sub-semigroups becomes subgroups of the Grothendieck group.

Theorem 19.1.3.1. *(Cofinality theorem) Let $(S, \star)$ be a semigroup and T be a cofinal sub-semigroup of S. Then,*
  1. *$G(T) \leq G(S)$,*
  2. *if $m, m' \in S$ are such that $[m] = [m'] \in G(S)$, then there exists $n \in T$ such that $m \star n = m' \star n$,*
  3. *for all $x \in G(S)$, there exists $m \in S$ and $n \in T$ such that $x = [m] - [n]$.*

Proof. **TODO.** $\qquad \square$

19.1.4   Source of examples : SymMonCat

A good source of examples of interesting semigroups (and thus their group completions) is provided by the categorification of the notion of semigroups in the name of *symmetric monoidal categories*. We define symmetric monoidal categories as follows.

Definition 19.1.4.1. (**Symmetric monoidal categories**) A category $\mathbf{C}$ is said to be symmetric monoidal if there exists following additional structures on top of $\mathbf{C}$
  1. $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, the tensor bifunctor,
  2. $e \in \mathbf{C}$, the unit object,
  3. **TODO.**

SymMonCat $\to$ ComMon

We can obtain a commutative monoid by identifying all isomorphic objects in a symmetric monoidal category.

Construction 19.1.4.2. (*Commutative monoid from symmetric monoidal categories*) **TODO**.

19.1.5 $K_0(R)$

We now construct K_0 of a ring with unity.

Construction 19.1.5.1. Let R be a ring with 1. Consider the category of all right projective R-modules, denoted **Proj**-R. Note that it forms a symmetric monoidal category, where the tensor is the direct sum and the unit object is the zero module:

$$(\mathbf{Proj}\text{-}R, \oplus, 0) .$$

By Construction 19.1.4.2, we obtain a commutative monoid

$$(\mathbf{Proj}\text{-}R, \oplus, 0)^{\cong} .$$

Now, commutative monoids are commutative semi-groups, therefore we have

$$\mathbf{ComMon} \hookrightarrow \mathbf{ComSGrp}.$$

Using the group completion functor of Theorem 19.1.2.1, we obtain an abelian group, which is defined to be the K_0 *of the ring* R:

$$K_0(R) := G\left(\mathbf{Proj}\text{-}R, \oplus, 0\right)^{\cong} .$$

Thus K_0 can be treated as a composition of functors:

$$
\begin{array}{ccc}
\mathbf{SymMonCat} & \xrightarrow{(-)^{\cong}} & \mathbf{ComSGrp} \\
{\scriptstyle \mathbf{Proj}\text{-}(-)} \Big\uparrow & & \Big\downarrow {\scriptstyle G(-)} \\
\mathbf{Ring} & \dashrightarrow[K_0(-)] & \mathbf{AbGrp}
\end{array} \quad .
$$

Examples

Discussion 19.1.5.2. Let us discuss what we are about to do. Continuing from what we have done, we will now look at some more exotic examples of Grothendieck groups, notably the connection between class groups in algebraic number theory and reduced Grothendieck group for Dedekind domains. The functor K_0 behaves well with respect to finite direct sums and direct limits. Reinterpreting finitely generated projective modules as infinite idempotent matrices, one gets an alternate presentation of K_0, which directly shows that it is a Morita invariant.

K-theory behaves a lot like a homology theory (Definition 6.8.1.1), thus facilitating computations. In fact, given a two-sided ideal I of a ring R, there is an exact sequence relating the K-groups of the rings R, R/I and a relative K-group of the pair (R, I). Moreover, the relative K-group $K_0(R, I)$ only depends on the difference between R and R/I, which is the ideal I thought of as a ring without unity—an analogue of the excision theorem in homology.

19.1.6 K_0 of Dedekind domains

19.1.7   Idempotent matrices and K_0

19.1.8   Reduced and relative K_0

19.1.9   Excision and relative K_0-sequence

Discussion 19.1.9.1. Let us discuss what we are next going to do. It is said that the Grothendieck group $K_0(R)$ is orthogonal to all higher K-groups in the sense that the former looks at the spread of projective modules while the rest only look at the eventual behaviour as the size of those modules grows.

Hyman Bass's group $K_1(R)$ is the intended "value group for the determinant" of an invertible matrix over R and can be defined as the abelianization of the direct limit of automorphism groups of finitely generated projective modules. We will look at two different, yet equivalent, approaches to this group. Thanks to Whitehead, we will be dealing with invertible matrices without worrying about their sizes. Gaussian elimination and Dieudonne's determinant map will help us to compute $K_1(R)$ in familiar examples. Finally we will extend the exact sequence from last few sections to a six term long exact sequence.

19.1.10 $K_1(R)$

19.1.11   **Elementary matrices and K_1**

19.1.12   **Examples**

19.1.13   **Relative K_1 and relative K_1-sequence**

19.1.14   **Clutching and K_0-K_1-sequence**

Discussion 19.1.14.1. Let us discuss what we are about to do. Motivated by topological K-theory, the ring $R[t]$ of polynomials and the ring $R[t, t^{-1}]$ of Laurent polynomials can be viewed as "algebraic version" of cylinder and desuspension for the ring R respectively. The study of K-groups for these rings by Grothendieck and Bass-Heller-Swan resulted in the definition of negative K-groups for R, which extend the 6-term long exact sequence on the right.
Positive K-theory of a ring R is supposed to measure abelian invariants of the highly non-abelian group $GL(R)$. At the end we will briefly study universal central extension of the group $E(R)$ of elementary matrices to define $K_2(R)$ and see how to extend the 6-term exact sequence on the left.

$\vdots$

19.2   Higher K-theory of rings

Our goal in this section is to construct the K-theory space. For that we would need ideas coming from the world of simplicial sets, which is present in Chapter 14, Language of ∞-Categories.

Chapter 20

Abstract Analysis

Contents

We will discuss some topics from integration theory and functional analysis (topological vector spaces).

20.1 Integration theory

We would like to state and portray the uses of some of the important and highly usable results of integration theory, elucidating in the process the analytical thought which is of

paramount importance in any route of exploration in this field[1]. We give bare-bone proofs as all this is standard, but we will highlight the main part of the proof by $\heartsuit$ or if there are many main parts, then by $\heartsuit\heartsuit\ldots$(!) Let us first begin with some motivation behind modern measure theory.

We know that the class of all Riemann integrable functions on $[a, b]$, denoted $R([a, b])$, is not complete under pointwise limit (a sequential approximation of Dirichlet's function shows that). Further, motivated by Weierstrass approximation, one would like to have commutability results between $\varinjlim$ and $\int$, which again $R([a, b])$ lacks. Consequently, one is motivated to find a larger class of "integrable" functions for which these defects would be rectified.

The idea that H. Lebesgue had was quite simple. He continued the idea of Riemann (that is, of partitions) but made sure that the function under investigation is much more intertwined in with it. Indeed, for a bounded function $f : [a, b] \to \mathbb{R}$, we contain the image $\text{Im}\,(f) \subseteq [\alpha, \beta]$ and then consider a partition $\mathcal{P} = \{I_i\}_{i=1}^n$ where I_i is an interval. Now choose $\xi_i \in f^{-1}(I_i) =: J_i$ for each i. Consequently, we may naturally define *Lebesgue sum of f w.r.t. $\mathcal{P}$* as follows

$$L(f, \mathcal{P}) := \sum_{i=1}^n f(\xi_i) m(J_i),$$

where $m(J_i)$ is supposed to be some sort of measure of J_i. Note that J_i in general might be very bad (may not even be an interval!). To complete this idea of "integration", we are naturally led to considering more general notions of measures. Indeed, this is what we will pursue in this course.

Remark 20.1.0.1. (*Pseudo-definition of measure*) First, what do we expect from a notion of measure on $\mathbb{R}$? Perhaps the following is the minimum conditions we would require to call a function "measure": A function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is said to be a *pseudo-measure* if it satisfies the following

1. (*measure of intervals*) for any interval I, the measure $\mu(I) = l(I)$ where l is the length function,

2. (*measure of disjoint unions*) for any disjoint sequence of subsets $\{A_n\}$, $\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$,

3. (*translation invariance*) for any subset A and $x \in \mathbb{R}$, we have $\mu(A + x) = \mu(A)$.

We will call such a function a *pseudo-measure on $\mathbb{R}$*. Observe that for $A \subseteq B$, we obtain $\mu(A) \leq \mu(B)$ by breaking $B = A \cup B \setminus A$. We call μ a pseudo-measure because it does not exists!

Theorem 20.1.0.2. *(Vitali set) There exists no pseudo-measure on $\mathbb{R}$. In paritcular, there exists a set $V \subseteq \mathbb{R}$ such that for a pseudo-measure μ, $\mu(V) \notin [0, \infty]$.*

Proof. We will construct such a set V. Begin with the closed interval $J = [0, 1]$. Define an equivalence relation $\sim$ on J given as follows:

$$x \sim y \iff x - y \in \mathbb{Q}.$$

---

[1]One may argue, instead, in whole of mathematics.

This can easily be seen to be an equivalence relation on J. We have first some observations to make about this equivalence relation and the consequent partition of J that it entails.

1. Observe that the class of any rational r in J under $\sim$ is simply $[0]$, as $r - 0 \in \mathbb{Q}$.
2. Every equivalence class is countable in size. Indeed, for any $x \in J$, the class $[x]$ is just translate of x by rationals, which is countable.
3. There are uncountably many equivalence classes. Indeed, if there were atmost countably many equivalence classes, then by statement 2 above, it would follow there are atmost countably many elements in J, which is a contradiction.

Consequently, this equivalence relation partitions J into following classes:

$$J = \bigcup_{\alpha \in \mathcal{I}} [\alpha]$$

where $\mathcal{I}$ is an uncountable set.

We would now construct the set V as follows. First, let us assume axiom of choice, so that for each class $[\alpha]$, we may pick an element $r_\alpha \in [\alpha]$ and would thus obtain a subset of J, denoted $V = \{r_\alpha \mid \alpha \in I\}$. We call this the Vitali set.

Consider the set $Q = [-1, 1] \cap \mathbb{Q}$. Since it is countable so consider an enumeration $Q = \{q_n\}$. Now consider the translates $V + q_n$ for all $n \in \mathbb{N}$ and their union $X = \bigcup_n V + q_n$. We now observe the following two facts about X.

1. If $n \neq m$, then $(V + q_n) \cap (V + q_m) = \emptyset$. Indeed, if $x \in (V + q_n) \cap (V + q_m)$, then $x = r_a + q_n = r_b + q_m$. Consequently, $r_a - r_b \in \mathbb{Q}$ and hence $[a] = [b]$. But by single choice of r_c for each $c \in \mathcal{I}$, we get $r_a = r_b$ and thus $q_n = q_m$ from above, which is a contradiction.
2. $J = [0, 1] \subseteq X$. Indeed, for any $x \in [0, 1]$, consider the class $[a]$ in which x is present. Consequently we have a unique $r_a \in V$ corresponding to x which satisfies $x \in [r_a]$. Thus, $x = r_a + t$ where $t \in \mathbb{Q}$. We may write $t = q_n$ to obtain that $x \in V + q_n$, as desired.
3. $X \subseteq [-1, 2]$. Indeed, this follows immediately since $X = \bigcup_n V + q_n$ where q_ns are rationals in $[-1, 1]$ and $V \subseteq [0, 1]$.

With the above three observations, we obtain the following inclusions:

$$[0, 1] \subseteq \bigcup_n V + q_n \subseteq [-1, 2].$$

Now, if we apply the pseudo-measure μ on the above inclusions, we will obtain the following:

$$1 \leq \sum_n \mu(V) \leq 3.$$

If $\mu(V) = 0, \infty$, then we have an immediate contradiction. Else if $0 < \mu(V) < \infty$, then $\sum_n \mu(V) = \infty$ and we again have a contradiction. Thus, $\mu(V) \notin [0, \infty]$, a contradiction. $\square$

Remark 20.1.0.3. The main issue in pseudo-measures is that we trying to get a measure on *all* of the subsets of $\mathbb{R}$. By Theorem 20.1.0.2, this is hopeless. What we shall now do instead is to obtain a measure not on all of the subsets of $\mathbb{R}$, but rather on only a

subcollection of subsets of $\mathbb{R}$, and we shall choose this subcollection in a manner so that we don't allow sets like Vitali sets. Indeed, this becomes our point of departure for the abstract definition of σ-algebras and measure/measurable spaces, the need for the right domain of a measure function.

20.1.1 σ-algebras and measures

We first define an algebra and a σ-algebra over a set X.

Definition 20.1.1.1. (Algebra and σ-algebra) Let X be a set. An algebra is a subset of the power set $\mathcal{A} \subseteq P(X)$ such that

1. $X \in \mathcal{A}$,
2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,
3. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_{i=1}^{n} A_i \in \mathcal{A}$.

If $\mathcal{A}$ further satisfies that

4. $\{A_n\}$ is a sequence in $\mathcal{A}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$,

then $\mathcal{A}$ is said to be a σ-algebra.

Remark 20.1.1.2. The name "algebra" is indeed coming from algebra(!) Consider the tuple $(\mathcal{P}(X), \Delta, \cap)$ where $A \Delta B$ is the symmetric difference. One observe that then forms a commutative ring with unity with addition being symmetric difference and multiplication being intersection. One now sees that an algebra in the sense of Definition 20.1.1.1 indeed is closed under symmetric differences, has X and is closed under intersections. Consequently, an algebra $\mathcal{A}$ (in the sense of Definition 20.1.1.1) is a subalgebra of the algebra $\mathcal{P}(X)$ (in the usual sense). This therefore motivates the name.

For fun, we also define the following notion.

Definition 20.1.1.3. (Constructible algebra and Borel σ-algebra) Let (X, τ) be a topological space. Then a constructible algebra $\mathcal{C}$ is defined to be the smallest algebra containing the topology τ and the Borel σ-algebra is defined to be the smallest σ-algebra containing τ.

We may later dwell on the former for Zariski spaces (also known as spectral spaces) and their importance in algebraic geometry.

Definition 20.1.1.4. (σ-algebra generated by a class) Let X be a set and $S \subseteq P(X)$ be a class of sets of X. Then the σ-algebra generated by S is the smallest σ-algebra containing S, denoted $\sigma(S)$.

Consequently, one defines the Borel σ-algebra to be the σ-algebra generated by τ.

Lemma 20.1.1.5. *Let X be a set and $S \subseteq P(X)$ a class of subsets of X. Let $A \subseteq X$ be a subset. Denote by $S \cap A = \{B \cap A \mid B \in S\}$. Then,*

$$\sigma_A(S \cap A) = \sigma(S) \cap A.$$

where $\sigma_A(S \cap A)$ denotes the smallest σ-algebra over A generated by the class $S \cap A \subseteq P(A)$.

Proof. It is easy to see that $\sigma_A(S \cap A) \hookrightarrow \sigma(S) \cap A$ by considering that $S \cap A \subseteq \sigma(S) \cap A$. Conversely, we use the generating set principle. That is, since we wish to show that for any $B \in \sigma(S)$, we have $B \cap A \in \sigma_A(S \cap A)$, therefore we define

$$\mathcal{S} := \{B \in \sigma(S) \mid B \cap A \in \sigma_A(S \cap A)\}$$

and then observe quite easily that $\mathcal{S}$ is a σ-algebra over X inside $\sigma(S)$ containing S. Thus $\mathcal{S} = \sigma(S)$, as needed. $\square$

We now define a measure.

Definition 20.1.1.6. (Measure and measure spaces) Let X be a non-empty set together with a σ-algebra $\mathcal{A}$ of X. The tuple $(X, \mathcal{A})$ is defined to be a measurable space. A measure on $(X, \mathcal{A})$ is a set function

$$\mu : \mathcal{A} \longrightarrow [0, \infty]$$

such that $\mu(\emptyset) = 0$ and for a disjoint sequence of sets $\{A_n\} \subseteq \mathcal{A}$, we have $\mu(\cup_n A_n) = \sum_n \mu(A_n)$. The triple $(X, \mathcal{A}, \mu)$ is said to be a measure space.

The following are some finiteness conditions we would like to have on measure spaces.

Definition 20.1.1.7. (Finiteness conditions) Let $(X, \mathcal{A}, \mu)$ be a measure space. Then,
1. X is said to be *finite* if $\mu(X) < \infty$,
2. X is said to be *σ-finite* if there exists $\{A_n\} \subseteq \mathcal{A}$ such that $\bigcup_n A_n = X$ and $\mu(A_n) < \infty$,
3. X is said to be *semi-finite* if for all $A \in \mathcal{A}$ such that $\mu(A) = \infty$, there exists $B \subseteq A$ such that $B \in \mathcal{A}$ and $\mu(B) < \infty$.

X-indexed $\mathbb{R}$-series

We would now like to make sense of the sum $\sum_{x \in X} f(x)$ where $f : X \to [0, \infty]$ is an arbitrary function.

Definition 20.1.1.8. (X-indexed $\mathbb{R}$-series) Let $f : X \to [0, \infty]$ be a function where X is a set. We define the series $\sum_{x \in X} f(x)$ as follows:

$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) \mid F \subseteq X \text{ is finite} \right\}.$$

The following are some basic properties of X-indexed $\mathbb{R}$-series.

Proposition 20.1.1.9. *Let X be a set and $f : X \to [0, \infty]$ be a function. Denote $S = \{x \in X \mid f(x) > 0\}$.*
1. *If S is uncountable then $\sum_{x \in X} f(x) = \infty$.*
2. *If S is countably finite then for any bijection $\varphi : \mathbb{N} \to S$, we have*

$$\sum_{x \in X} f(x) = \sum_{n \in \mathbb{N}} f(\varphi(n)).$$

Proof. 1. Write $S = \bigcup_n S_n$ where $S_n = \{f(x) > 1/n\}$. Note that S_n forms an increasing sequence of sets. As S is uncountable, there exists $N \in \mathbb{N}$ such that S_N is uncountable. Consequently, for any finite set $F \subseteq S_N$, we have $\sum_{x \in F} f(x) \geq \frac{|F|}{N}$. As $\sum_{x \in F} f(x) \leq \sum_{x \in X} f(x)$, therefore

$$\frac{|F|}{N} \leq \sum_{x \in X} f(x). \tag{$\heartsuit$}$$

As $F \subseteq S_N$ is arbitrary finite set and S_N is uncountable, therefore we get the desired result.

2. Pick any bijection $\varphi : \mathbb{N} \to S$ and pick a finite set $F \subseteq X$. We have $\sum_{x \in F} f(x) = \sum_{x \in F \cap S} f(x)$, so replace $F \subseteq X$ by a finite set $F \subseteq S$. Let $n \in \mathbb{N}$ be large enough so that $\varphi(\{1, \ldots, n\}) \supseteq F$. Consequently, we have

$$\sum_{x \in F} f(x) \leq \sum_{k=1}^{n} f(\varphi(k)) \leq \sum_{x \in X} f(x). \tag{$\heartsuit$}$$

Take $n \to \infty$ in the above inequality to obtain

$$\sum_{x \in F} f(x) \leq \sum_{k=1}^{\infty} f(\varphi(k)) \leq \sum_{x \in X} f(x).$$

Take sup over all finite subsets F of X in the above inequality to obtain

$$\sum_{x \in X} f(x) \leq \sum_{n \in \mathbb{N}} f(\varphi(n)) \leq \sum_{x \in X} f(x),$$

which yields the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

20.1.2   Basic results on measure spaces

We have the following first result.

Proposition 20.1.2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*
   1. *If $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*
   2. *If $A, B \in \mathcal{A}$ and $A \subseteq B$ where $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.*
   3. *For any sequence $\{A_n\} \subseteq \mathcal{A}$, we have*

$$\mu\left(\bigcup_n A_n\right) \leq \sum_n \mu(A_n).$$

   4. *If $\{A_n\} \subseteq \mathcal{A}$ is an increasing sequence of measurable sets, then*

$$\mu\left(\bigcup_n A_n\right) = \varinjlim_n \mu(A_n).$$

   5. *If $\{A_n\} \subseteq \mathcal{A}$ is a decreasing sequence of measurable sets where $\mu(A_1) < \infty$, then*

$$\mu\left(\bigcap_n A_n\right) = \varinjlim_n \mu(A_n).$$

6. *If X is σ-finite, then X is semi-finite.*

Proof. Statements 1. and 2. are immediate from the disjoint decomposition $B = A \amalg (B \setminus A)$. For 3. note that for any $\{A_n\} \subseteq \mathcal{A}$, we can form a disjoint sequence $\{B_n\} \subseteq \mathcal{A}$ such that $\bigcup_n A_n = \coprod_n B_n$ and $\mu(B_n) \leq \mu(A_n)$. Statement 4. also follows from similar reasons, where we can now let $B_n = A_n \setminus A_{n-1}$. Let us do statement 5. in some detail.

Observe that the sequence $C_1 = \emptyset$ and $C_n = A_1 \setminus A_n$ is an increasing sequence of sets. Thus, we have by statement 4. that

$$\mu\left(\bigcup_n C_n\right) = \varinjlim_n \mu\left(C_n\right). \tag{$\heartsuit$}$$

We can write $A_1 = (A_1 \setminus A_n) \amalg A_n$. Using statement 2. we obtain that

$$\mu(A_1) = \mu(C_n) + \mu(A_n)$$
$$\mu(A_1) - \mu(A_n) = \mu(C_n). \tag{$\heartsuit\heartsuit$}$$

We now claim that $\bigcap_n A_n = A_1 \setminus \bigcup_n C_n$. Indeed, for $x \in \bigcap_n A_n$, $x \in A_n \subseteq A_1$ for all n and thus $x \in A_1$. But if $x \in C_n$ for some n, then $x \notin A_n$, consequently a contradiction. Hence $x \in A_1 \setminus \bigcup_n C_n$. Conversely, for $x \in A_1 \setminus \bigcup_n C_n$ and any $n \in \mathbb{N}$, we have that if $x \notin A_n$, then $x \in A_1 \setminus A_n = C_n$, a contradiction. Hence the claim is proved.

As each $C_n \subseteq A_1$, thus $\bigcup_n C_n \subseteq A_1$. Consequently, by statement 2. and above claim we obtain that

$$\mu\left(\bigcap A_n\right) = \mu(A_1) - \mu\left(\bigcup_n C_n\right)$$
$$= \mu(A_1) - \varinjlim_n \mu(C_n)$$
$$= \mu(A_1) - \varinjlim_n \left(\mu(A_1) - \mu(A_n)\right)$$
$$= \varinjlim_n \mu(A_n).$$

This proves statement 5.

For statement 6. pick any $A \in \mathcal{A}$ with $\mu(A) = \infty$. We wish to construct a subset $B \subseteq A$ with $B \in \mathcal{A}$ and $0 < \mu(B) < \infty$. Let $\{D_n\} \subseteq \mathcal{A}$ be a collection of finite measure sets such that $\bigcup_n D_n = X$. Note that we can assume D_n are disjoint by suitably replacing D_n by $D_n \setminus D_1 \cup \cdots \cup D_{n-1}$. Assume to the contrary, so that for each $B \subseteq A$ with $B \in \mathcal{A}$, either $\mu(B) = 0$ or $\mu(B) = \infty$. Let $D_n \cap A$ be such that $D_n \cap A \neq \emptyset$. Consequently, $\mu(D_n \cap A) = 0$ or ∞. The latter isn't possible, therefore $\mu(D_n \cap A) = 0$ for all $n \in \mathbb{N}$.

Since we have $A = \coprod_n D_n \cap A$, therefore $\mu(A) = \sum_n \mu(D_n \cap A) = 0$, a contradiction to the fact that $\mu(A) = \infty$. $\qquad\square$

We now cover an important example of a measure.

Construction 20.1.2.2. (*Measures from a positive function*) Let $(X, \mathcal{A})$ be a measurable space and $f : X \to [0, \infty]$ be a function. We construct the following map

$$\{\text{All functions } X \to [0, \infty]\} \longrightarrow \{\text{Measures on } (X, \mathcal{A})\}.$$

Indeed, define

$$\mu_f : \mathcal{A} \longrightarrow [0, \infty]$$
$$A \longmapsto \sum_{x \in A} f(x).$$

We claim that μ_f forms a measure.

It is clear that $\mu_f(\emptyset) = 0$. Consequently we need to show that for a disjoint collection $\{A_n\} \subseteq \mathcal{A}$, we have

$$\mu_f \left(\coprod_n A_n \right) = \sum_n \mu_f(A_n).$$

We first have that

$$\mu_f \left(\coprod_n A_n \right) = \sup \left\{ \sum_{x \in F} f(x) \mid F \subseteq \coprod_n A_n \text{ is finite} \right\} \tag{1}$$

and

$$\sum_n \mu_f(A_n) = \sum_n \sup \left\{ \sum_{x \in G} f(x) \mid G \subseteq A_n \text{ is finite} \right\}. \tag{2}$$

We first show that (1) $\leq$ (2). We need only show that for a finite set $F \subseteq \amalg_n A_n$, we have $\sum_{x \in F} f(x) \leq (2)$. Indeed, as $F_n := F \cap A_n$ is a collection of disjoint finite set where $F_n \subseteq A_n$ and only for finitely many n is F_n non-empty, therefore $\sum_{x \in F} f(x) = \sum_n \sum_{x \in F_n} f(x) \leq (2)$.

Conversely, we now wish to show that (2) $\leq$ (1). We use a standard technique for this. Pick any $\epsilon > 0$. For each $n \in \mathbb{N}$, we obtain a finite set $G_n \subseteq A_n$ such that

$$\mu_f(A_n) - \frac{\epsilon}{2^n} \leq \sum_{x \in G_n} f(x). \tag{$\heartsuit$}$$

Summing this till $N \in \mathbb{N}$, we obtain

$$\sum_{n=1}^N \left(\mu_f(A_n) - \frac{\epsilon}{2^n} \right) \leq \sum_{n=1}^N \sum_{x \in G_n} f(x) = \sum_{x \in \amalg_{n=1}^N G_n} f(x) \leq (1).$$

Now take $N \to \infty$ and $\epsilon \to 0$ to obtain the result[2].

Observe that the map defined above in Construction 20.1.2.2 is neither injective nor surjective, and that's good, otherwise measure theory would have been redundant. We now study completions of a measure space.

Remark 20.1.2.3. The goal of next few sections is to establish a good measure on $\mathbb{R}^n$ through which we can proceed to a theory of integration of measurable functions. Indeed, this goal was achieved by Lebesgue and he constructed what will be called the Lebesgue measure on $\mathbb{R}^n$. Hence, one should view the goal of the next few sections as to construct this measure space $(\mathbb{R}^n, \mathcal{M}, m)$, which is highly usable (as we will see in the integration theory) and is the gold standard of modern analysis.

---

[2] We call this the ϵ-*wiggle around* inf *and* sup *technique*.

20.1.3 Completion of a measure space

Definition 20.1.3.1. (**Null sets and complete measure spaces**) Let $(X, \mathcal{A}, \mu)$ be a measure space. A null set is an element $A \in \mathcal{A}$ such that $\mu(A) = 0$. The collection of all null sets is written as $\mathrm{Null}(\mathcal{A}) \subseteq \mathcal{A}$. A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if for all $A \in \mathrm{Null}(\mathcal{A})$, $\mathcal{P}(A) \subseteq \mathcal{A}$.

Remark 20.1.3.2. Note that for a measure space $(X, \mathcal{A}, \mu)$, the collection of all null sets $\mathrm{Null}(\mathcal{A})$ contains $\emptyset$ and is closed under countable union. Indeed, for $\{A_n\} \subseteq \mathrm{Null}(\mathcal{A})$, we have $\mu(\cup_n A_n) \leq \sum_n \mu(A_n) = 0$ by Proposition 20.1.2.1, 3.

Definition 20.1.3.3. (**Extension of measure spaces**) Let $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}', \mu')$ be two measure spaces. Then we say that $(X, \mathcal{A}', \mu')$ is an extension of $(X, \mathcal{A}, \mu)$ if $\mathcal{A}' \supseteq \mathcal{A}$ and $\mu'|_{\mathcal{A}} = \mu$.

We will now for each measure space $(X, \mathcal{A}, \mu)$ will construct an extension of it which will be complete.

Construction 20.1.3.4. Let $(X, \mathcal{A}, \mu)$ be a measure space. Consider the following collection

$$\hat{\mathcal{A}} := \{A \cup B \mid A \in \mathcal{A}, B \subseteq N, N \in \mathrm{Null}(\mathcal{A})\}.$$

Define $\hat{\mu} : \hat{\mathcal{A}} \to [0, \infty]$ as $A \cup B \mapsto \mu(A)$.

Theorem 20.1.3.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then, $(X, \hat{\mathcal{A}}, \hat{\mu})$ is a complete measure space extending $(X, \mathcal{A}, \mu)$. We call it the completion of $(X, \mathcal{A}, \mu)$.*

Proof. We need to show the following things.
  1. $\hat{\mathcal{A}}$ is a σ-algebra,
  2. $\hat{\mu}$ is a measure,
  3. $\hat{\mu}|_{\mathcal{A}} = \mu$,
  4. $(X, \hat{\mathcal{A}}, \hat{\mu})$ is complete.
The first three are straightforward. We show 4. in some detail.

Pick $A \cup B \in \hat{\mathcal{A}}$ such that $\hat{\mu}(A \cup B) = \mu(A) = 0$. Then $A \in \mathrm{Null}(\mathcal{A})$. Further, $B \subseteq N$ where $N \in \mathrm{Null}(\mathcal{A})$. Let $C \subseteq A \cup B$. Then $C = (C \cap A) \cup (C \cap B)$. Since $C \cap A \subseteq A$ and $C \cap B \subseteq N$, therefore $C \subseteq A \cup N$ where $A \cup N \in \mathrm{Null}(\mathcal{A})$. Consequently, we may write $C = \emptyset \cup C$ where C is a subset of a null set. Hence $C \in \hat{\mathcal{A}}$. $\qquad\square$

Example 20.1.3.6. Let $X = \{1, 2, 3\}$ and $\mathcal{A} = \{\emptyset, X, \{1\}, \{2, 3\}\}$. Define $\mu : \mathcal{A} \to [0, \infty]$ by $\mu(\emptyset) = 0 = \mu(\{2, 3\})$ and $\mu(\{1\}) = \mu(X)$. Clearly, $(X, \mathcal{A}, \mu)$ is a measure space which is not complete. We calculate its completion $(X, \hat{\mathcal{A}}, \hat{\mu})$. By Construction 20.1.3.4, as the only null set is $\{2, 3\}$, we have

$$\hat{\mathcal{A}} = \{\emptyset, X, \{1\}, \{2, 3\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\}.$$

Hence $\hat{\mathcal{A}} = \mathcal{P}(X)$. Similarly, $\hat{\mu}$ is easy to find by the definition in Construction 20.1.3.4.

20.1.4   Outer and Lebesgue measures

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We wish to understand the type of sets of $\mathbb{R}$ we obtain by looking at the *area below f*. That is the set $A = \{(x, y) \in [a, b] \times \mathbb{R} \mid f(x) < y\}$. The high-school Newtonian integration proceeds by approximating the set A from the *outside* in the sense that we take bigger rectangles E containing A and then try to make it small so that in the limit we obtain A. We would now like to formalize this idea and this is the main purpose of outer measures. Indeed, note that in the above example, we have that the target set A has the following property with respect to all $E \subseteq \mathbb{R}^2$: the area of E is recovered by taking area of $E \cap A$ and adding it to area of $E \cap A^c$. This will be the main force behind our definition of outer measure.

Definition 20.1.4.1. (Outer measures and measurable sets) Let X be a non-empty set. A function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is said to be an outer measure if it satisfies the following properties:
   1. $\mu^*(\emptyset) = 0$,
   2. $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$,
   3. $\mu^* \left(\bigcup_n A_n \right) \leq \sum_n \mu^*(A_n)$.
A subset $A \subseteq X$ is said to be μ^*-measurable if for all $E \subseteq X$ we have that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We will denote the collection of all μ^*-measurable sets by $\mathcal{M}$.

Remark 20.1.4.2. Let X be a set with an outer measure μ^*. By subadditivity (the 3rd axiom of outer measure), we obtain that $A \subseteq X$ is μ^*-measurable if and only if for all $E \subseteq X$ we have the inequality

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We now construct an important outer measure on $\mathbb{R}^n$. This is the precursor to the measure that we finally aim for (Remark 20.1.2.3). This also portrays the use of the idea we discussed in the beginning.

Construction 20.1.4.3. (*Lebesgue outer measure on $\mathbb{R}^n$*) Consider $\mathbb{R}^m$ and for an box $I \subseteq \mathbb{R}^n$, by which we mean a product of interval $I = I_1 \times \cdots \times I_m$ for $I_i \subseteq \mathbb{R}$, denote $v(I)$ to be its volume; $v(I) = \prod_{i=1}^m l(I_i)$. For any $A \subseteq \mathbb{R}^n$, we define

$$\mu^*(A) = \inf \left\{ \sum_n v(I_n) \mid \bigcup_n I_n \supseteq A, \ I_n \text{ are boxes} \right\}.$$

We claim that μ^* forms an outer measure on $\mathbb{R}^n$.

   Indeed, $\mu^*(\emptyset) = 0$ as $\emptyset \subseteq (-1/k, 1/k)^m$ for all $n \in \mathbb{N}$ so we have $\mu^*(A) \leq 2^m/n^m$. Taking $n \to \infty$ does the job.

   Let $A \subseteq B$ in $\mathbb{R}^m$. Observe that to show $\mu^*(A) \leq \mu^*(B)$ we need only show that $\{\sum_n v(I_n) \mid \bigcup_n I_n \supseteq A, \ I_n \text{ are boxes}\} \supseteq \{\sum_n v(I_n) \mid \bigcup_n I_n \supseteq B, \ I_n \text{ are boxes}\}$. But this is trivial as and sequence of boxes $\{I_n\}$ covering B also covers A.

Finally we wish to show countable subadditivity. Pick $\{A_n\} \subseteq \mathcal{P}(\mathbb{R}^m)$. We wish to show that

$$\mu^* \left(\bigcup_n A_n \right) \leq \sum_n \mu^*(A_n).$$

We use the ϵ-wiggle around sup and inf technique to show this, as discussed earlier in Construction 20.1.2.2. Pick any $\epsilon > 0$ and observe that we have a sequence of boxes $\{I_{n,k}\}_k$ for each $n \in \mathbb{N}$ such that $\bigcup_k I_{n,k} \supseteq A_n$ and

$$\mu^*(A_n) + \frac{\epsilon}{2^n} \geq \sum_k v(I_{n,k}). \qquad (\heartsuit)$$

Observe further that $\bigcup_n \bigcup_k I_{n,k} \supseteq \bigcup_n A_n$. Consequently, we have $\sum_n \sum_k v(I_{n,k}) \geq \mu^*(\bigcup_n A_n)$. Hence,

$$\sum_n \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) \geq \sum_n \sum_k v(I_{n,k}) \geq \mu^* \left(\bigcup_n A_n \right).$$

Hence μ^* is an outer measure on $\mathbb{R}^n$.

Note that the only place we required knowledge about boxes explicitly was only to show that $\mu^*(\emptyset) = 0$. This motivates the following simple result

Theorem 20.1.4.4. *Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ be a collection of sets containing $\emptyset$ and X. Let $l : \mathcal{S} \to [0, \infty]$ be a function such that $l(\emptyset) = 0$. Then μ^* defined by*

$$\mu^* : \mathcal{P}(X) \longrightarrow [0, \infty]$$
$$A \longmapsto \inf \left\{ \sum_n l(I_n) \mid \bigcup_n I_n \supseteq A, \ I_n \in \mathcal{S} \right\}$$

is an outer measure on X.

Proof. Verbatim to Construction 20.1.4.3, except that $\mu^*(\emptyset) = 0$ follows now by the assumption that $l(\emptyset) = 0$ and $\emptyset \in \mathcal{P}(X)$ so that $\emptyset$ forms its own covering. $\square$

20.1.5 Extension theorems-I : Carathéodory's theorem

Let X be a set and μ^* be an outer measure on X. We now show that one can construct a complete measure space on X out of this data. This means that we get a measure on $\mathbb{R}$ corresponding to Lebesguw outer measure on $\mathbb{R}$. However there will still be some issue which we will fix in the next section. This is attributed to Constantin Carathéodory.

Theorem 20.1.5.1. *Let X be a set and μ^* be an outer measure on X. Denote the collection of all μ^*-measurable sets by $\mathcal{M}$. Then $(X, \mathcal{M}, \mu)$ is a complete measure space where $\mu = \mu^*|_{\mathcal{M}}$.*

Proof. We begin by showing that the collection $\mathcal{M}$ is a σ-algebra. Note that $A \in \mathcal{M}$ if and only if for each $E \subseteq X$, we have $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Clearly $A = \emptyset$ trivially satisfies this. Similarly it is clear that if $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$. So we need only show that $\mathcal{M}$ is closed under countable union. To this end, we first show that $\mathcal{M}$ is closed under finite union.

Consider $A_1, A_2 \in \mathcal{M}$. We wish to show that $A_1 \cup A_2 \in \mathcal{M}$. Indeed, we have the following by the fact that $A_i \in \mathcal{M}$

$$
\begin{aligned}
\mu^*(E) &\geq \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c) \\
&\geq \mu^*(E \cap A_1 \cap A_2) + \mu^*(E \cap A_1 \cap A_2^c) + \mu^*(E \cap A_1^c) \\
&\geq \mu^*(E \cap A_1 \cap A_2) + \mu^*(E \cap A_1 \cap A_2^c) + \mu^*(E \cap A_1^c \cap A_2) + \mu^*(E \cap A_1^c \cap A_2^c).
\end{aligned}
$$

By countable subadditivity of outer measures and the fact that $E \cap (A_1 \cup A_2) = (E \cap (A_1 \cap A_2^c)) \cup (E \cap (A_2 \cap A_1^c)) \cup (E \cap A_1 \cap A_2)$, we further obtain that

$$
\mu^*(E) \geq \mu^*(E \cap (A_1 \cup A_2)) + \mu^*(E \cap (A_1 \cup A_2)^c).
$$

We will now show that $\mathcal{M}$ is closed under countable union by using above. Let $\{A_n\} \subseteq \mathcal{A}$. We can always assume that $\{A_n\}$ are disjoint by replacing A_n by $A_n \setminus A_1 \cup \cdots \cup A_{n-1}$ as this does not changes the union. Let $A = \coprod_n A_n$. We wish to show that $A \in \mathcal{M}$, that is, for any $E \subseteq X$ we wish to show that

$$
\mu^*(E) \geq m^*(E \cap A) + \mu^*(E \cap A^c).
$$

Denote $B_n = \coprod_{k=1}^n A_k$. By the above, we have that $B_n \in \mathcal{M}$. Consequently, we have

$$
\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c). \tag{$*$}
$$

We first observe that by the fact that $A_n \in \mathcal{M}$, we obtain the following recursion relation for $\mu^*(E \cap B_n)$

$$
\begin{aligned}
\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \hspace{2cm} (\heartsuit) \\
&= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\
&\ \ \vdots \\
&= \sum_{k=1}^n \mu^*(E \cap A_k).
\end{aligned}
$$

Using the above on $(*)$ and the fact that $B_n^c \supseteq A^c$ together with continuity of measure, we obtain

$$
\begin{aligned}
\mu^*(E) &= \sum_{k=1}^n \mu^*(E \cap A_k) + \mu^*(E \cap B_n^c) \\
&\geq \sum_{k=1}^n \mu^*(E \cap A_k) + \mu^*(E \cap A^c).
\end{aligned}
$$

Taking $n \to \infty$, we obtain

$$\mu^*(E) \geq \sum_n \mu^*(E \cap A_n) + \mu^*(E \cap A^c). \qquad (**)$$

By countable subadditivity of μ^*, we further obtain

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

This shows that $A \in \mathcal{M}$ and hence this completes the proof that $\mathcal{M}$ is a σ-algebra.

We next wish to show that μ is a measure on $(X, \mathcal{M})$. As $\mu^*(\emptyset) = 0$, therefore $\mu(\emptyset) = 0$. Now pick a disjoint collection $\{A_n\} \subseteq \mathcal{M}$. We wish to show that $\mu^*\left(\coprod_n A_n\right) = \sum_n \mu^*(A_n)$. Indeed, by the countable subadditivity of outer measure, we already have that $\mu^*\left(\coprod_n A_n\right) \leq \sum_n \mu^*(A_n)$. Hence we wish to show that $\mu^*\left(\coprod_n A_n\right) \geq \sum_n \mu^*(A_n)$. This follows at once by letting $E = \coprod_n A_n$ in $(**)$ above.

We finally wish to show that μ is a complete measure. Pick any $N \in \mathcal{M}$ such that $\mu(N) = 0$ and then pick any $A \subseteq N$. We wish to show that $A \in \mathcal{M}$. Pick any $E \subseteq X$. We wish to show that

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Since $\mu^*(E \cap A) \leq \mu^*(A)\mu^*(N) = 0$, so we reduce to showing that

$$\mu^*(E) \geq \mu^*(E \cap A^c).$$

As $N \in \mathcal{M}$ and $\mu^*(E \cap N) = 0$ for similar reasons as above, therefore we have

$$\mu^*(E) = \mu^*(E \cap N^c)$$
$$\geq \mu^*(E \cap A^c)$$

as $A \subseteq N$. Hence $A \in \mathcal{M}$ and this completes the proof. $\square$

20.1.6 Extension theorems-II : Premeasures and Carathéodory-Hahn theorem

In the previous section, we constructed a complete measure by the data of an outer measure. By Theorem 20.1.4.4, we also saw that one can obtain an outer measure on any set through a "length function" defined on a collection of "good sets". We then in Theorem 20.1.5.1 constructed a complete measure space out of it. Unfortunately, the notion of outer measures as constructed in Theorem 20.1.4.4 is to broad for applications. Moreover, the aforementioned "length function" and "good sets" are not actually good enough to see some light of applications; these are too structureless. For this reason, we would jazz up our notion of "length function" by considering a *premeasure* and "good sets" by considering a collection of sets which should form an *algebra*. We would then prove an analogue of Carathéodory's theorem in this setup and as an application, we would see in the next section how to make sense of the traditionally difficult to precisely understand notion of a distribution.

20.1.7   Extension theorems-III : Distribution functions

20.1.8   Lebesgue-Stieltjes measure

20.1.9 Applications-I : Measure spaces and measurable functions

We now present applications of the above theory. This is, in particular, to showcase the true strength of abstract analysis. This can also be used to strengthen one's intuition about the topic.

σ-algebras and measure spaces

Lemma 20.1.9.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that μ is σ-finite if and only if there exists a countable disjoint family of measurable sets $\{A_n\}$ such that $X = \coprod_n A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.*

Proof. Note that R $\implies$ L is immediate from definition. Let μ be σ-finite. Then there exists $\{B_n\} \subseteq \mathcal{A}$ such that $\mu(B_n) < \infty$ and $\bigcup_n B_n = X$. Define $A_1 = B_1$ and $A_n = B_n \setminus B_1 \cup \cdots \cup B_{n-1}$. As $\mathcal{A}$ is a σ-algebra, so $\{A_n\} \subseteq \mathcal{A}$. Moreover, $A_n \cap A_m = \emptyset$ for all $n \neq m$ because if $m > n^3$ and $x \in A_m \cap A_n$, then $x \in B_m \setminus B_1 \cup \cdots \cup B_n \cup \dots B_{m-1}$ and $x \in B_n \setminus B_1 \cup \dots B_{n-1}$, a contradiction. As $A_n \subseteq B_n$, therefore $\mu(A_n) \leq \mu(B_n) < \infty$. To complete the proof, we need only show that $\bigcup_n A_n = \bigcup_n B_n$.

Pick any $x \in \bigcup_n A_n$. Then $x \in B_n \setminus B_1 \cup \cdots \cup B_{n-1}$ for some $n \in \mathbb{N}$. Thus, $x \in B_n$ and hence $x \in \bigcup_n B_n$. Conversely, pick $x \in \bigcup_n B_n$. Then $x \in B_n$ for some $n \in \mathbb{N}$. Now, either $x \in B_n \setminus B_1 \cup \cdots \cup B_{n-1}$ or $x \in B_n \cap (B_1 \cup \cdots \cup B_{n-1})$. If the former is true, then $x \in A_n$ and we are done. If the latter is true, then we may assume $x \in B_{n-1} \cap B_n$. Now again either $x \in B_{n-1} \setminus B_1 \cup \cdots \cup B_{n-2}$ or $x \in B_n \cap B_{n-1} \cap (B_1 \cup \cdots \cup B_{n-2})$. Repeating this process inductively, we will end up in either of the following cases:

1. $x \in A_k$ for some $1 \leq k \leq n$,
2. $x \in B_1 \cap \cdots \cap B_n$.

As $B_1 = A_1$ by construction, therefore in either case we are done. $\qquad\square$

Lemma 20.1.9.2. *Given $\mathcal{S} \subseteq \mathcal{P}(X)$, denote by $\mathcal{A}(\mathcal{S})$ the σ-algebra generated by $\mathcal{S}$. Then,*

$$\mathcal{A}(\mathcal{S}) = \mathcal{A}(\mathcal{A}(\mathcal{S})).$$

Proof. Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ be an arbitrary collection of subsets of X. If X is empty then the statement is vacuously true, so let X be non-empty. Since the σ-algebra generated by $\mathcal{A}(\mathcal{S})$ is the intersection of all σ-algebras containing $\mathcal{A}(\mathcal{S})$, therefore we have that $\mathcal{A}(\mathcal{A}(\mathcal{S})) = \bigcap_{\mathcal{C} \supseteq \mathcal{A}(\mathcal{S})} \mathcal{C}$. Since $\mathcal{A}(S)$ is a σ-algebra containing $\mathcal{A}(\mathcal{S})$, therefore $\mathcal{A}(\mathcal{A}(\mathcal{S})) \subseteq \mathcal{A}(\mathcal{S})$. Since $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{C}$ for all σ-algebras $\mathcal{C}$ containing $\mathcal{A}(\mathcal{S})$, therefore $\mathcal{A}(\mathcal{A}(\mathcal{S})) \supseteq \mathcal{A}(\mathcal{S})$. $\qquad\square$

Lemma 20.1.9.3. *Let $\mathcal{A}(\mathcal{S})$ be the σ-algebra generated by a set $\mathcal{S} \subseteq \mathcal{P}(X)$. Then, $\mathcal{A}(\mathcal{S})$ is the union of the σ-algebras generated by Y as Y ranges over all countable subsets of $\mathcal{S}$.*

Proof. Let X be a non-empty set and $\mathcal{S} \subseteq \mathcal{P}(X)$. We wish to show that

$$\mathcal{A}(\mathcal{S}) = \bigcup_{\mathcal{Y} \subseteq \mathcal{S}, \text{ countable}} \mathcal{A}(\mathcal{Y}).$$

---

[3]which we may assume wlog.

Let $\mathcal{Y} \subseteq \mathcal{S}$ be a countable subcollection. Then, $\mathcal{A}(\mathcal{Y}) \subseteq \mathcal{A}(\mathcal{S})$. Consequently, $\bigcup_{\mathcal{Y} \subseteq \mathcal{S}, \text{ countable}} \mathcal{A}(\mathcal{Y}) \subseteq \mathcal{A}(\mathcal{S})$. Conversely, we need to show that

$$\mathcal{A}(\mathcal{S}) \subseteq \bigcup_{\mathcal{Y} \subseteq \mathcal{S}, \text{ countable}} \mathcal{A}(\mathcal{Y}).$$

We claim that $\bigcup_{\mathcal{Y} \subseteq \mathcal{S}, \text{ countable}} \mathcal{A}(\mathcal{Y})$ is a σ-algebra containing $\mathcal{S}$. This would complete the proof as $\mathcal{A}(\mathcal{S})$ is the smallest σ-algebra containing $\mathcal{S}$.

Denote $\mathcal{Z} = \bigcup_{\mathcal{Y} \subseteq \mathcal{S}, \text{ countable}} \mathcal{A}(\mathcal{Y})$. As $\mathcal{A}(\mathcal{Y})$s are σ-algebras, therefore $\mathcal{Z}$ contains X and $\emptyset$. Let $A \in \mathcal{Z}$. Then $A \in \mathcal{A}(\mathcal{Y})$ for some $\mathcal{Y} \subseteq \mathcal{S}$ countable. Consequently, $A^c \in \mathcal{A}(\mathcal{Y})$ and thus $A^c \in \mathcal{Z}$. Let $\{A_n\} \subseteq \mathcal{Z}$ be a countable collection of sets. Then $A_n \in \mathcal{A}(\mathcal{Y}_n)$ for all $n \in \mathbb{N}$. Further, we have that $\mathcal{Y}_k \subseteq \mathcal{A}(\bigcup_n \mathcal{Y}_n)$ for all $k \in \mathbb{N}$ as $\mathcal{Y}_k \subseteq \bigcup_n \mathcal{Y}_n$. As $\mathcal{Y}_k$ are countable and countable union of countable sets is countable, therefore $\bigcup_n \mathcal{Y}_n$ is countable. Thus, we have

$$A_k \in \mathcal{A}(\mathcal{Y}_k) \subseteq \mathcal{A}\left(\bigcup_n \mathcal{Y}_n\right) \subseteq \mathcal{Z} \; \forall k \in \mathbb{N}.$$

Thus from above, we obtain that

$$\bigcup_k A_k \in \mathcal{A}\left(\bigcup_n \mathcal{Y}_n\right) \subseteq \mathcal{Z}.$$

Hence, $\mathcal{Z}$ is a σ-algebra. To complete the proof, we need only show that $\mathcal{Z}$ contains $\mathcal{S}$.

Let $A \in \mathcal{S}$. Then since $\{A\}$ is a countable subset of $\mathcal{S}$, therefore $\mathcal{A}(\{A\})$ is contained in $\mathcal{Z}$ and thus $A \in \mathcal{Z}$. $\qquad \square$

Lemma 20.1.9.4. *The σ-algebra generated by*
    1. $\mathcal{S} = \{(a, b] \mid a < b \in \mathbb{Q}\}$,
    2. $\mathcal{S} = \{(a, n] \mid a \in \mathbb{Q}, n \in \mathbb{Z}\}$,
is the Borel σ-algebra on $\mathbb{R}$.

Proof. 1. Let $\mathcal{S} = \{(a, b] \mid a, b \in \mathbb{Q}\}$. We wish to show that $\mathcal{A}(\mathcal{S}) = \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$. Since $(a, b]$ for $a, b \in \mathbb{Q}$ is contained in $\mathcal{B}$ as $(a, b] = (a, b) \cup \bigcap_{n \in \mathbb{N}} (b - 1/n, b + 1/n)$, therefore $\mathcal{S} \subseteq \mathcal{B}$. Consequently, $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{B}$ as $\mathcal{B}$ is the smallest σ-algebra containing open intervals.

Since we also know that $\mathcal{B}$ is generated by the collection of all closed intervals $[a, b]$ in $\mathbb{R}$, therefore to show that $\mathcal{B} \subseteq \mathcal{A}(\mathcal{S})$, it would suffice to show $[a, b] \in \mathcal{A}(\mathcal{S})$ where $a < b$ in $\mathbb{R}$. Pick $a < b$ in $\mathbb{R}$. By density of $\mathbb{Q}$, we may pick $\{a_n\}$ to be an increasing sequence such that $a_n \in \mathbb{Q}$, $a_n < a$ and $\lim_{n \to \infty} a_n = a$. Similarly, we may pick a decreasing sequence $\{b_n\}$ such that $b_n \in \mathbb{Q}$, $b_n > b$ and $\lim_{n \to \infty} b_n = b$. Consequently, we claim that

$$[a, b] = \bigcap_n (a_n, b_n]$$

where $(a_n, b_n] \in \mathcal{S}$. Indeed, $(\subseteq)$ is clear. For $(\supseteq)$, take $x \in \bigcap_n (a_n, b_n]$. Hence $a_n < x \leq b_n$. Taking $n \to \infty$, we get $a \leq x \leq b$ as desired. Thus, $[a, b] \in \mathcal{A}(\mathcal{S})$.

2. Let $\mathcal{S} = \{(a, n] \mid a \in \mathbb{Q}, \ n \in \mathbb{N}\}$. We wish to show that $\mathcal{A}(\mathcal{S}) = \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$. Since $(a, n]$ for $a \in \mathbb{Q}$ and $n \in \mathbb{N}$ is contained in $\mathcal{B}$ as $(a, n] = (a, n) \cup \bigcap_{k \in \mathbb{N}}(n - 1/k, n + 1/k)$, therefore $\mathcal{S} \subseteq \mathcal{B}$. Consequently, $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{B}$.

Since we also know that $\mathcal{B}$ is generated by the collection of all open intervals of the form (a, ∞), $a \in \mathbb{R}$, therefore to show that $\mathcal{B} \subseteq \mathcal{A}(\mathcal{S})$, it would suffice to show $(a, \infty) \in \mathcal{A}(\mathcal{S})$ for all $a \in \mathbb{R}$. Pick (a, ∞) for some $a \in \mathbb{R}$. By density of $\mathbb{Q}$, there exists a decreasing sequence $\{a_n\}$ in $\mathbb{R}$ such that $a_n \in \mathbb{Q}$, $a_n > a$ and $\lim_{n \to \infty} a_n = a$. Consequently, we claim that

$$(a, \infty) = \bigcup_n (a_n, n]$$

where $(a_n, n] \in \mathcal{S}$. Indeed, for $(\subseteq)$, take $x \in (a, \infty)$. We therefore have $a < x < \infty$. As $\lim_{n \to \infty} a_n = a$ and $a_n > a$ for all $n \in \mathbb{N}$, therefore there exists $N \in \mathbb{N}$ such that $a < a_n \leq a_N < x$ for all $n \geq N$. Consequently, for some large $n \in \mathbb{N}$ greater than N such that $x \leq n$, we obtain $a_n < x \leq n$ and hence $x \in (a_n, n]$. For $(\supseteq)$, take $x \in \bigcup_n (a_n, n]$ and thus we get $a < a_n < x \leq n < \infty$. Thus, $(a, \infty) \in \mathcal{A}(\mathcal{S})$. $\qquad \square$

Lemma 20.1.9.5. *The Borel σ-algebra on $\mathbb{R}^2$ is generated by*

$$\{(I \times \mathbb{R}) \cup (\mathbb{R} \times J) \mid I, J \subseteq \mathbb{R}, \ open \ intervals\}.$$

Proof. Let $\mathcal{S} = \{(I \times \mathbb{R}) \cup (\mathbb{R} \times J) \mid I, J \subseteq \mathbb{R} \text{ is open}\}$. We wish to show that $\mathcal{A}(\mathcal{S}) = \mathcal{B}$ where $\mathcal{B}$ is the σ-algebra of $\mathbb{R}^2$.

As $\mathcal{S}$ is a collection of open sets of $\mathbb{R}^2$ and $\mathcal{B}$ is generated by all open sets of $\mathbb{R}^2$, therefore $\mathcal{S} \subseteq \mathcal{B}$ and thus $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{B}$.

We now wish to show that $\mathcal{B} \subseteq \mathcal{A}(\mathcal{S})$. It would suffice to show that any open set $U \subseteq \mathbb{R}^2$ is in $\mathcal{A}(\mathcal{S})$. Note that $\mathcal{A}(\mathcal{S})$ consists of all open rectanlges $I \times J = (I \times \mathbb{R}) \cap (\mathbb{R} \times J)$. Thus, it would suffice to show that U can be written as countable union of open rectangles. Recall that open rectangles forms a basis for the usual topology on $\mathbb{R}^2$. Consider the collection of all open rectangles K inside U whose vertices have both rational coordinates. We claim that the union of such open rectangles is equal to U. Indeed, their union is inside U and for any $x \in U$, there exists an open ball $x \in B \subseteq U$, so there exists an open rectangle K inside B which contains x and has vertices which have both rational coordinates. Thus U is equal to the union of all such rectangles. Since there are only countably many such open rectangles as they are parameterized by choice of 4 points in $\mathbb{Q}^2 \cap U$ which is atmost countably many, therefore we have obtained a countable cover of U by open rectangles. This completes the proof. $\qquad \square$

Lemma 20.1.9.6. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $A, B \in \mathcal{A}$. Then,*

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

Proof. Observe that we can write

$$A \cup B = (A \setminus (A \cap B)) \cup B$$

where the right side is a disjoint union. Consequently, we have

$$\mu(A \cup B) = \mu(A \setminus A \cap B) + \mu(B). \tag{6.1}$$

We now have two cases. If $\mu(A \cap B) = \infty$, then since $\mu(A \cap B) \leq \mu(A), \mu(B)$ and $\mu(A) \leq \mu(A \cup B)$, therefore we get $\mu(A \cup B) = \mu(A \cap B) = \mu(A) = \mu(B) = \infty$, so that the statement to be proven is a tautology. Else if $\mu(A \cap B) < \infty$, then we can write

$$\mu(A \setminus A \cap B) = \mu(A) - \mu(A \cap B).$$

Consequently, by Eq. (6.1) and the fact that $\mu(A \cap B) < \infty$, we have

$$\mu(A \cup B) = \mu(A) - \mu(A \cap B) + \mu(B)$$
$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

This completes the proof. $\qquad \square$

Lemma 20.1.9.7. *Let $x \in \mathbb{R}$ and let B be a Borel subset of $\mathbb{R}$. Then, $x + B$ and xB are Borel subsets of $\mathbb{R}$ (that is, Borel subsets of $\mathbb{R}$ are translation and dilation invariant).*

Proof. 1. Let $x \in \mathbb{R}$ and $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. We wish to show that for all $B \in \mathcal{B}$, the translate $x + B \in \mathcal{B}$. Consequently, we wish to show

$$x + \mathcal{B} \subseteq \mathcal{B}$$

where $x + \mathcal{B} = \{x + B \mid B \in \mathcal{B}\}$. We use the following standard technique to show this.

Consider the following collection

$$\mathcal{C} = \{B \in \mathcal{B} \mid x + B \in \mathcal{B}\}.$$

Our goal is to show that $\mathcal{C} = \mathcal{B}$. Note that $\mathcal{C} \subseteq \mathcal{B}$. Conversely, we wish to show that $\mathcal{B} \subseteq \mathcal{C}$. This would follow immediately if we show that $\mathcal{C}$ is a σ-algebra containing all open intervals, as $\mathcal{B}$ is the σ-algebra generated by all open intervals.

We now establish that $\mathcal{C}$ is a σ-algebra. Since $x + \mathbb{R} = \mathbb{R}$ and $x + \emptyset = \emptyset$, therefore $\mathbb{R}, \emptyset \in \mathcal{C}$. Let $A \in \mathcal{C}$. We wish to show that $A^c \in \mathcal{C}$. Since $x + A \in \mathcal{B}$, therefore $(x + A)^c \in \mathcal{B}$. Thus it suffices to show that $(x + A)^c = x + A^c$. Indeed, we have the following equalities

$$(x + A)^c = \{y \in \mathbb{R} \mid y \notin x + A\}$$
$$= \{y \in \mathbb{R} \mid y - x \notin A\}$$
$$= \{y \in \mathbb{R} \mid y - x \in A^c\}$$
$$= \{y \in \mathbb{R} \mid y \in x + A^c\}$$
$$= x + A^c.$$

Let $\{A_n\} \subseteq \mathcal{C}$. We wish to show that $\bigcup_n A_n \in \mathcal{C}$. We have that for each $n \in \mathbb{N}$, $x + A_n \in \mathcal{B}$. It would thus suffice to show that

$$x + \bigcup_n A_n = \bigcup_n (x + A_n).$$

Indeed, take $x + a \in x + \bigcup_n A_n$. Hence $a \in A_n$ for some $n \in \mathbb{N}$. Consequently, $x + a \in x + A_n$. Thus $x + a \in \bigcup_n (x + A_n)$. Conversely, let $z \in \bigcup_n (x + A_n)$. Then $z = x + y_n$ for $y_n \in A_n$. Consequently, $z \in x + \bigcup_n A_n$. This show that $\mathcal{C}$ is a σ-algebra.

To complete the proof, we now need only show that $\mathcal{C}$ has all open intervals. This is immediate, as we show now. Take any $(a, b) \subseteq \mathbb{R}$. Since $x + (a, b) = (x + a, x + b) \in \mathcal{B}$, therefore $(a, b) \in \mathcal{C}$.

2. Let $x \in \mathbb{R}$ and $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. We wish to show that for all $B \in \mathcal{B}$, the dilate $x \cdot B \in \mathcal{B}$. Note that $x \cdot B = \{xb \mid b \in B\}$. Consequently, we wish to show

$$x \cdot \mathcal{B} \subseteq \mathcal{B}$$

where $x \cdot \mathcal{B} = \{x \cdot B \mid B \in \mathcal{B}\}$. If $x = 0$, then $x \cdot \mathcal{B} = \{0\}$ and that is trivially inside $\mathcal{B}$ as $\{0\} = \bigcap_n (-1/n, 1/n)$. Thus we now assume that $x \neq 0$. We use the following standard technique to show the above inclusion.

Consider the following collection

$$\mathcal{C} = \{B \in \mathcal{B} \mid x \cdot B \in \mathcal{B}\}.$$

Our goal is to show that $\mathcal{C} = \mathcal{B}$. Note that $\mathcal{C} \subseteq \mathcal{B}$. Conversely, we wish to show that $\mathcal{B} \subseteq \mathcal{C}$. This would follow immediately if we show that $\mathcal{C}$ is a σ-algebra containing all open intervals, as $\mathcal{B}$ is the σ-algebra generated by all open intervals.

We now establish that $\mathcal{C}$ is a σ-algebra. Observe that $x \cdot \mathbb{R} = \mathbb{R}$. Indeed, as $x \cdot \mathbb{R} \subseteq \mathbb{R}$ is clear, we can also write any $a \in \mathbb{R}$ as $x \cdot x^{-1} a$. We also have $x \cdot \emptyset = \emptyset$. Therefore $\mathbb{R}, \emptyset \in \mathcal{C}$. Let $A \in \mathcal{C}$. We wish to show that $A^c \in \mathcal{C}$. Since $x \cdot A \in \mathcal{B}$, therefore $(x \cdot A)^c \in \mathcal{B}$. Thus it suffices to show that $(x \cdot A)^c = x \cdot A^c$. Indeed, we have the following equalities

$$\begin{aligned}
(x \cdot A)^c &= \{y \in \mathbb{R} \mid y \notin x \cdot A\} \\
&= \{y \in \mathbb{R} \mid x^{-1} y \notin A\} \\
&= \{y \in \mathbb{R} \mid x^{-1} y \in A^c\} \\
&= \{y \in \mathbb{R} \mid y \in x \cdot A^c\} \\
&= x \cdot A^c.
\end{aligned}$$

Let $\{A_n\} \subseteq \mathcal{C}$. We wish to show that $\bigcup_n A_n \in \mathcal{C}$. We have that for each $n \in \mathbb{N}$, $x \cdot A_n \in \mathcal{B}$. It would thus suffice to show that

$$x \cdot \bigcup_n A_n = \bigcup_n (x \cdot A_n).$$

Indeed, take $x \cdot a \in x \cdot \bigcup_n A_n$. Hence $a \in A_n$ for some $n \in \mathbb{N}$. Consequently, $x \cdot a \in x \cdot A_n$. Thus $x \cdot a \in \bigcup_n (x \cdot A_n)$. Conversely, let $z \in \bigcup_n (x \cdot A_n)$. Then $z = x \cdot y_n$ for $y_n \in A_n$. Consequently, $z \in x \cdot \bigcup_n A_n$. This show that $\mathcal{C}$ is a σ-algebra.

To complete the proof, we now need only show that $\mathcal{C}$ has all open intervals. This is immediate, as we show now. Take any $(a, b) \subseteq \mathbb{R}$. If $x > 0$, then we have $x \cdot (a, b) = (x \cdot a, x \cdot b) \in \mathcal{B}$, therefore $(a, b) \in \mathcal{C}$. If $x < 0$, then we have $x \cdot (a, b) = (x \cdot b, x \cdot a) \in \mathcal{B}$, therefore $(a, b) \in \mathcal{C}$. $\qquad \square$

Lemma 20.1.9.8. *Let $(X, \mathcal{A})$ be a measurable space and let $\{\mu_i\}_{i=1}^n$ be a finite collection of measures on $(X, \mathcal{A})$. If $r_1, \ldots, r_n \in \mathbb{R}_{\geq 0}$, then $\sum_i r_i \mu_i$ is a measure on $(X, \mathcal{A})$ (that is, positive linear combination of measures is a measure).*

Proof. Let $(X, \mathcal{A})$ be a measurable space and $\{\mu_i\}_{i=1}^n$ be a collection of measures on it. Let $\{r_i\}_{i=1}^n \subseteq \mathbb{R}_{\geq 0}$. We wish to show that $\mu = \sum_{i=1}^n r_i \mu_i$ is a measure on $(X, \mathcal{A})$. First we may assume that each $r_i > 0$ as if any $r_j = 0$, then $\mu(A) = \sum_{i=1}^n r_i \mu_i(A) = \sum_{i \neq j} r_i \mu_i(A) + r_j \mu_j(A)$, therefore if $\mu_j(A) < \infty$, then $r_j \mu_j(A) = 0$ and if $\mu_j(A) = \infty$, then since $0 \cdot \infty = 0$, therefore still $r_j \mu_j(A) = 0$. Further, if all $r_i = 0$, then $\mu = 0$, which is the trivial measure. Consequently, we assume that $r_i > 0$ for all $i = 1, \ldots, n$.

We now show that μ is a measure on $(X, \mathcal{A})$. We have $\mu(\emptyset) = \sum_{i=1}^n r_i \mu_i(\emptyset) = \sum_{i=1}^n r_i \cdot 0 = 0$. Let $\{A_n\} \subseteq \mathcal{A}$ be a collection of disjoint measurable sets. We wish to show that

$$\mu\left(\coprod_k A_k\right) = \sum_k \mu(A_k).$$

We have

$$\mu\left(\coprod_k A_k\right) = \sum_{i=1}^n r_i \mu_i\left(\coprod_k A_k\right)$$

$$= \sum_{i=1}^n r_i \sum_{k=1}^\infty \mu_i(A_k).$$

We now claim that

$$\sum_{i=1}^n r_i \sum_{k=1}^\infty \mu_i(A_k) = \sum_{k=1}^\infty \sum_{i=1}^n r_i \mu_i(A_k) \tag{8.1}$$

and showing this will complete the proof as

$$\sum_{k=1}^\infty \sum_{i=1}^n r_i \mu_i(A_k) = \sum_{k=1}^\infty \mu(A_k).$$

We have few cases for establishing Eq. (8.1).

1. *If for all $i = 1, \ldots, n$, the series $\sum_{k=1}^\infty \mu_i(A_k)$ is finite.* Then, $\sum_{i=1}^n r_i \sum_{k=1}^\infty \mu_i(A_k) = \sum_{i=1}^n \sum_{k=1}^\infty r_i \mu_i(A_k)$. Now, if $\sum_n x_n, \sum_n y_n$ are two convergent positive series, then their linear combination $c \sum_n x_n + d \sum_n y_n$ is equal to $\sum_n c x_n + d y_n$, where $c, d \in \mathbb{R}_{\geq 0}$. Indeed, this follows at once from the equality $c \varprojlim_{n \to \infty} \sum_{k=1}^n x_k + d \varprojlim_{n \to \infty} \sum_{k=1}^n y_k = \varprojlim_{n \to \infty} \sum_{k=1}^n c x_k + d y_k$, which follows from the fact that both the limit exists and $c, d \in \mathbb{R}$. Consequently, we have

$$\sum_{i=1}^n \sum_{k=1}^\infty r_i \mu_i(A_k) = \sum_{k=1}^\infty \sum_{i=1}^n r_i \mu_i(A_k),$$

which is what we needed.

2. *If there exists $i_0 = 1, \ldots, n$ such that the series $\sum_{k=1}^{\infty} \mu_{i_0}(A_k) = \infty$. In this case, in the Eq. (8.1), the left side is ∞. The right side is also infinity as shown below:*

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n} r_i \mu_i(A_k) \geq \sum_{k=1}^{\infty} r_{i_0} \mu_{i_0}(A_k)$$
$$= \infty$$

where the first inequality follows from $r_i > 0$ for all $i = 1, \ldots, n$ and measure being positive by definition. Consequently, Eq. (8.1) follows in this case as well.

This completes the proof. □

Lemma 20.1.9.9. *For any set X and a subset $S \subseteq X$, the collection*

$$\mathcal{A}_S = \{A \subseteq X \mid A \subseteq S \text{ or } A^c \subseteq S\}$$

is a σ-algebra on X.

Proof. Let X be a non-empty set, $S \subseteq X$ and define

$$\mathcal{A}_S := \{A \subseteq X \mid A \subseteq S \text{ or } A^c \subseteq S\}.$$

We claim that this forms a σ-algebra on X. As $X^c = \emptyset \subseteq S$, therefore $X \in \mathcal{A}_S$ and $\emptyset \in \mathcal{A}_S$. Let $A \in \mathcal{A}_S$. If $A \subseteq S$, then A^c is such that $(A^c)^c = A \subseteq S$, so $A^c \in \mathcal{A}_S$. If $A^c \subseteq S$, then A^c is such that $A^c \subseteq S$, so $A^c \in \mathcal{A}_S$. So in both cases $\mathcal{A}_S$ is closed uncer complements.

Let $\{A_n\} \subseteq \mathcal{A}_S$ be a collection of subsets. We wish to show that $\bigcup_n A_n \in \mathcal{A}_S$. We have three cases.

C1. $A_n \subseteq S$ for all $n \in \mathbb{N}$. Then $\bigcup_n A_n \subseteq S$ and thus $\bigcup_n A_n \in \mathcal{A}_S$.
C2. $\exists A_m$ such that $A_m \not\subseteq S$. Then $A_m^c \subseteq S$. We then observe by De-Morgan's law that

$$\left(\bigcup_n A_n \right)^c = \bigcap_n A_n^c \subseteq A_m^c \subseteq S.$$

Consequently, $\bigcup_n A_n \in \mathcal{A}_S$.
C3. $A_n \not\subseteq S$ for all $n \in \mathbb{N}$. Then $A_n^c \subseteq S$ for all $n \in \mathbb{N}$. We again observe by De-Morgan's law that

$$\left(\bigcup_n A_n \right)^c = \bigcap_n A_n^c \subseteq A_m^c \subseteq S \; \forall m \in \mathbb{N}.$$

Consequently, $\bigcup_n A_n \in \mathcal{A}_S$.
In all three cases, $\bigcup_n A_n \in \mathcal{A}_S$. Hence $\mathcal{A}_S$ is a σ-algebra. □

Lemma 20.1.9.10. *Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space, and let $\mu(A) = \infty$ for some $A \in \mathcal{A}$. If $M > 0$, then there exists $B \subseteq A$ such that $M < \mu(B) < \infty$.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a semi-finite measure space and $A \in \mathcal{A}$ such that $\mu(A) = \infty$. We wish to show that for all $M > 0$, there exists a subset $B \subseteq A$ such that $B \in \mathcal{A}$ and $M < \mu(B) < \infty$.

We wish to show that there exists measurable subsets of A of arbitrarily large size. Therefore, consider the collection

$$S = \{\mu(B) \mid B \subseteq A, B \in \mathcal{A}, \mu(B) < \infty\}.$$

Denote $l = \sup S$. We wish to show that $l = \infty$. Pick a sequence $\{B_n\} \subseteq S$ such that $\lim_{n \to \infty} \mu(B_n) = l$. We first claim that

$$\mu\left(\bigcup_n B_n\right) = l \tag{10.1}$$

Clearly, $\bigcup_n B_n \in \mathcal{A}$. Observe that since

$$\mu(B_k) \leq \mu\left(\bigcup_n B_n\right)$$

for all $k \in \mathbb{N}$, therefore taking $k \to \infty$, we easily obtain

$$l \leq \mu\left(\bigcup_n B_n\right).$$

Conversely, we wish to show that

$$\mu\left(\bigcup_n B_n\right) \leq l.$$

Let $D_1 = B_1$, $D_2 = B_1 \cup B_2$ and in general $D_n = B_1 \cup \cdots \cup B_n$. Then we observe that $\{D_n\} \subseteq \mathcal{A}$ forms an increasing sequence of sets with $\bigcup_n D_n = \bigcup_n B_n$. Consequently,

$$\mu\left(\bigcup_n B_n\right) = \mu\left(\bigcup_k D_k\right) = \lim_{k \to \infty} \mu(D_k).$$

Since $D_k \subseteq A$ is such that $\mu(D_k) \leq \sum_{i=1}^k \mu(B_i) < \infty$ (by subadditivity), therefore $\mu(D_k) \in S$ for all $k \in \mathbb{N}$. Consequently,

$$\lim_{k \to \infty} \mu(D_k) \leq l.$$

Therefore we obtain $\mu\left(\bigcup_n B_n\right) \leq l$. Hence this completes the proof of Eq. (10.1).

Since we wish to show that $l = \infty$, so assume to the contrary that $l < \infty$. It follows from Eq. (10.1) that $\mu\left(\bigcup_n B_n\right) < \infty$ and therefore $\bigcup_n B_n \in S$. Let $C = \bigcup_n B_n$. Then consider $A_1 = A \setminus C$. Since $\mu(A_1) = \mu(A) - \mu(C)$ as $\mu(C) < \infty$, therefore we have $\mu(A_1) = \infty - \mu(C) = \infty$. It follows from semifiniteness that there exists $C_1 \subseteq A_1$ such that $C_1 \in \mathcal{A}$ and $0 < \mu(C_1) < \infty$. Note that C_1 and C are disjoint. It follows that the disjoint union $C_1 \cup C \subseteq A$ is such that $\mu(C \cup C_1) \in S$. But since $\mu(C_1 \cup C) = \mu(C_1) + \mu(C) > \mu(C) = l$, therefore S contains an element which is strictly larger than its supremum, a contradiction. Hence $l = \infty$ and this completes the proof. $\square$

Lebesgue measure on $\mathbb{R}$

In this section $(\mathbb{R}, \mathcal{M}, m)$ denotes the Lebesgue measure space on $\mathbb{R}$ and m^* denotes the Lebesgue outer measure on $\mathbb{R}$.

Lemma 20.1.9.11. *Every Borel subset of $\mathbb{R}$ is Lebesgue measurable.*

Proof. Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space over $\mathbb{R}$. We wish to show that the σ-algebra of Borel sets denoted $\mathcal{B}$ is in $\mathcal{M}$. Denote by $\mathcal{A}$ the following:

$$\mathcal{A} = \{\text{disjoint finite union of intervals of form } (-\infty, a], (b, \infty), (a, b] \text{ for } a < b \in \mathbb{R}\}. \quad (1.1)$$

By construction of Lebesgue measure, we know that $\mathcal{A} \subseteq \mathcal{M}$. We thus claim that the σ-algebra generated by $\mathcal{A}$ contains $\mathcal{B}$, that is, $\langle \mathcal{A} \rangle \supseteq \mathcal{B}$. This will conclude the proof.

Indeed, as we know that $\mathcal{B}$ is generated by all closed intervals of the form $(-\infty, a]$ for all $a \in \mathbb{R}$, therefore it suffices to show that $(-\infty, a] \in \langle \mathcal{A} \rangle$, but that is a tautology as $(-\infty, a]$ is in $\mathcal{A}$. Hence $\mathcal{B} \subseteq \langle A \rangle$. $\square$

Lemma 20.1.9.12. *Let A be a subset of $\mathbb{R}$ and $c \in \mathbb{R}$. Then,*
1. *$m^*(A + c) = m^*(A)$,*
2. *$A \in \mathcal{M}$ if and only if $A + c \in \mathcal{M}$,*
3. *if $A \in \mathcal{M}$, then $m(A + c) = m(A)$.*

Proof. Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$. Take $A \subseteq \mathbb{R}$ and for $c \in \mathbb{R}$ define $A + c = \{a + c \in \mathbb{R} \mid a \in A\}$. Let us set up some notation. For any $E \subseteq \mathbb{R}$, we denote

$$C(E) = \left\{ \{I_n\} \mid \bigcup_n I_n \supseteq A, \ I_n = (a_n, b_n] \in \mathcal{A} \right\} \quad (*)$$

where $\mathcal{A}$ is the algebra defined in Eq. (1.1). Further, let us denote

$$\Sigma C(E) = \left\{ \sum_n l(I_n) \in [0, \infty] \mid \{I_n\} \in C(E) \right\} \quad (**)$$

where $l((a, b]) = b - a$ is the length function. By definition, we have $m^*(E) := \inf \Sigma C(E)$.

(i) : We first wish to show that the Lebesgue outer measure m^* is translation invariant. That is, $m^*(A + c) = m^*(A)$. We first show $m^*(A + c) \geq m^*(A)$. Pick any $\{I_n\} \in C(A)$. Then we claim that $\{I_n + c\}$ is an element of $C(A + c)$. Indeed, denoting $I_n = (a_n, b_n]$, we immediately get $I_n + c = (a_n + c, b_n + c]$. Now to see that $\bigcup_n (I_n + c) \supseteq A + c$, pick any $a + c \in A + c$ where $a \in A$. Then, as $\bigcup_n I_n \supseteq A$, therefore $a \in I_n$ for some n and thus $a + c \in I_n + c$. It follows that $\{I_n + c\} \in C(A + c)$. Further note that $l(I_n) = l(I_n + c)$ by definition. Consequently, we have

$$\Sigma C(A) \subseteq \Sigma C(A + c).$$

Taking infima, we yield $m^*(A) = \inf \Sigma C(A) \leq \inf \Sigma C(A + c) = m^*(A + c)$, that is $m^*(A) \leq m^*(A + c)$.

Conversely, we wish to show that $m^*(A) \geq m^*(A + c)$. For this, we use the standard technique of ϵ-*wiggle around* inf. Fix $\epsilon > 0$. By definition of $m^*(A)$, there exists $\{I_n\} \in C(A)$ where $I_n = (a_n, b_n]$ such that

$$m^*(A) + \epsilon > \sum_n b_n - a_n. \tag{2.1}$$

Note that we can write the above as

$$m^*(A) + \epsilon > \sum_n (b_n + c) - (a_n + c)$$
$$= \sum_n l((a_n + c, b_n + c])$$
$$= \sum_n l(I_n + c).$$

We have $\{I_n + c\} \in C(A + c)$ as shown previously, therefore we obtain

$$m^*(A) + \epsilon > \sum_n l(I_n + c) \geq \inf \Sigma C(A + c) = m^*(A + c).$$

Hence we have $m^*(A) + \epsilon > m^*(A + c)$. Taking $\epsilon \to 0$, we obtain $m^*(A) \geq m^*(A + c)$. This completes the proof.

(ii) : We next wish to show that $A + c \in \mathcal{M}$ if and only if $A \in \mathcal{M}$. Observe that it suffices to show that $A \in \mathcal{M} \implies A + c \in \mathcal{M}$. Indeed, for the converse, take $B = A + c \in \mathcal{M}$. To show that $A \in \mathcal{M}$, it would suffice to show that $B - c \in \mathcal{M}$, which would follow at once by previous. Hence, we may only show that $A \in \mathcal{M} \implies A + c \in \mathcal{M}$.

Pick $A \in \mathcal{M}$. Fix $\epsilon > 0$. By regularity theorems, there exists open $U \supseteq A$ such that $m^*(U \setminus A) < \epsilon$. We now claim the following three statements:

1. $U + c$ *is open* : Indeed, pick any $x + c \in U + c$ where $x \in U$. As U is open, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq U$. Consequently, $(x - \delta + c, x + \delta + c) \subseteq U + c$, hence $U + c$ is open.

2. $U + c$ *contains* $A + c$: Pick any $a + c \in A + c$ where $a \in A$. As $U \supseteq A$, therefore $a + c \in U + c$.

3. $(U + c) \setminus (A + c)$ *equals* $(U \setminus A) + c$: We first show $(U + c) \setminus (A + c) \subseteq (U \setminus A) + c$. Pick any $x + c \in (U + c) \setminus (A + c)$. Then $x + c \in U + c$ and $x + c \notin A + c$. Thus, $x \in U$ and $x \notin A$. Hence $x \in U \setminus A$ and thus $x + c \in U \setminus A + c$.
   Conversely, pick $x + c \in (U \setminus A) + c$. Then $x \in U \setminus A$ and thus $x + c \in U + c$ and $x + c \notin A + c$. Thus $x + c \in (U + c) \setminus (A + c)$. This completes the proof of this claim.

By above three claims, we conclude that $U + c$ is an open set containing $A + c$ such that

$$m^*(U + c \setminus A + c) = m^*((U \setminus A) + c) \overset{(ii)}{=} m^*(U \setminus A) < \epsilon.$$

By regularity theorems, we conclude the proof.

(iii) : We wish to show that if $A \in \mathcal{M}$, then $m(A + c) = m(A)$. This is immediate from *(i)* as $m = m^*|_{\mathcal{M}}$. $\qquad\square$

Lemma 20.1.9.13. *Let A be a subset of $\mathbb{R}$ and $c \in \mathbb{R}$. Then,*

1. $m^*(cA) = |c|\, m^*(A)$,

2. for $c \neq 0$, $A \in \mathcal{M}$ if and only if $cA \in \mathcal{M}$,

3. if $A \in \mathcal{M}$, then $m(cA) = |c|\, m(A)$.

Proof. Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space. Take any $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$.

(i) : We first wish to show that $m^*(cA) = |c|\, m^*(A)$. If $c = 0$, then the equality is immediate as $cA = \{0\}$ and we know that $m^*(\{0\}) = 0$ as $0 \in (-1/n, 1/n]$ for all $n \in \mathbb{N}$ so that $m^*(\{0\}) \leq 2/n$. Taking $n \to \infty$, we get that $m^*(\{0\}) = 0$. So we assume from now on that $c \neq 0$. We first immediately reduce to showing either one of

$$m^*(cA) \geq |c|\, m^*(A) \text{ or } m^*(cA) \leq |c|\, m^*(A)$$

Indeed, the other side follows by replacing A by cA and replacing c by $1/c$ in either of the above. We now have two cases based on c being positive or negative.

If $c > 0$, then we proceed as follows. We follow the convention of Eqns $(*)$ and $(**)$ as set up in Q2. We use the standard technique of ϵ-wiggle around inf. Fix $\epsilon > 0$. By definition of outer measure, there exists $\{I_n\} \in C(cA)$ where $I_n = (a_n, b_n]$ such that

$$m^*(cA) + \epsilon > \sum_n l(I_n). \tag{3.1}$$

As $\bigcup_n I_n \supseteq cA$ and $c > 0$, therefore we claim that $\bigcup_n (\frac{1}{c} I_n) \supseteq A$. Indeed, for any $a \in A$, $cA \in I_n$. Thus $ca \in (a_n, b_n]$. Consequently, $a \in (a_n/c, b_n/c] = (\frac{1}{c} I_n)$. Thus, $\{\frac{1}{c} I_n\} \in C(A)$. Consequently, we have

$$\sum_n l\left(\frac{1}{c} I_n\right) = \sum_n \frac{1}{c} l(I_n) \geq m^*(A).$$

Consequently, $\sum_n l(I_n) \geq c\, m^*(A)$. Using this in Eq. (3.1), we thus obtain

$$m^*(cA) + \epsilon > \sum_n l(I_n) \geq c\, m^*(A).$$

Taking $\epsilon \to 0$, we obtain $m^*(cA) \geq c\, m^*(A)$, as required.

If $c < 0$, then we begin similarly to the previous case. Fix $\epsilon > 0$. There exists $\{I_n\} \in C(A)$ where $I_n = (a_n, b_n]$ such that

$$m^*(A) + \epsilon > \sum_n l(I_n). \tag{3.2}$$

Note that $cI_n = c(a_n, b_n] = [cb_n, ca_n)$ as $c < 0$ and this type of set is not half-open and is thus not in $\mathcal{A}$, the algebra of half-opens of Eq. (1.1). Consequently, we have to use ϵ-wiggle to find a new collection of intervals obtained via cI_n which are half open but their sum of lengths in only in ϵ-neighborhood of those $\{cI_n\}$. Indeed, for each $n \in \mathbb{N}$, we may construct

$$J_n = \left(cb_n - \frac{\epsilon}{2^{n+1}}, ca_n + \frac{\epsilon}{2^{n+1}}\right].$$

Note that $J_n \supseteq cI_n$. As $\bigcup_n cI_n \supseteq cA$, therefore $\bigcup_n J_n \supseteq cA$. Thus $\{J_n\} \in C(cA)$. Consequently,

$$
\begin{aligned}
m^*(cA) &\leq \sum_n l(J_n) \\
&= \sum_n c(a_n - b_n) + \frac{2\epsilon}{2^{n+1}} \\
&= \sum_n -c(b_n - a_n) + \sum_n \frac{\epsilon}{2^n} \\
&= -c\sum_n (b_n - a_n) + \epsilon \\
&= -c\sum_n l(I_n) + \epsilon
\end{aligned}
$$

where the third line follows from the series being positive and thus we can rearrange such a series. It thus follows by Eq. (3.2) and above that

$$
\begin{aligned}
m^*(cA) &< -c(m^*(A) + \epsilon) + \epsilon \\
&= -cm^*(A) + \epsilon(1 - c).
\end{aligned}
$$

Taking $\epsilon \to 0$, we obtain ($-c = |c|$ as $c < 0$)

$$
m^*(cA) \leq |c|\, m^*(A)
$$

as required. This completes the proof.

(ii) : We now wish to show that for $c \neq 0$, $A \in \mathcal{M}$ if and only if $cA \in \mathcal{M}$. Note that this is not true for $c = 0$ as if we take a non-measurable set $V \subseteq \mathbb{R}$, then $cV = \{0\}$ is measurable but V is not.

   Pick $c \neq 0$. We first note that showing only $A \in \mathcal{M} \implies cA \in \mathcal{M}$ is sufficient. Indeed, the other side follows by replacing c by $1/c$ in the above. So we reduce to showing $A \in \mathcal{M} \implies cA \in \mathcal{M}$.

   Pick $A \in \mathcal{M}$ and $c \neq 0$ in $\mathbb{R}$. Fix $\epsilon > 0$. By regularity theorems, there exists open $U \setminus A$ such that $m^*(U \setminus A) < \frac{\epsilon}{|c|}$. We now claim the following statements:

1. *cU is open* : Pick $cx \in cU$ where $x \in U$. As U is open therefore there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq U$. Consequently, $c(x - \delta, x + \delta) = (c(x + \delta), c(x - \delta)) \subseteq cU$ and contain cx. Hence cU is open.
2. *cU contains cA* : Pick any cx in cA. Then $x \in A$. As $U \subseteq A$, therefore $x \in U$ and hence $cx \in cU$.
3. *$cU \setminus cA$ equals $c(U \setminus A)$* : For ($\subseteq$), pick any $cx \in cU \setminus cA$. Then $cx \in cU$ and $cx \notin cA$. Thus, $x \in U$ and $x \notin A$, that is $\in U \setminus A$ and thus $cx \in c(U \setminus A)$. Conversely to show ($\supseteq$), pick any $cx \in c(U \setminus A)$ where $x \in U \setminus A$. Thus, $x \in U$ and $x \notin A$. Thus $cx \in cU$ and $cx \notin cA$. Thus $cx \in cU \setminus cA$.

Following the above three lemmas, we conclude that cU is an open set containing cA such that

$$
m^*(cU \setminus cA) = m^*(c(U \setminus A)) \overset{(i)}{=} |c|\, m^*(U \setminus A) < |c|\,\frac{\epsilon}{|c|} = \epsilon.
$$

Thus by regularity theorems, $cA \in \mathcal{M}$ as well.

(iii) : We wish to show that if $A \in \mathcal{M}$, then $m(cA) = |c| \, m(A)$. But this is immediate from *(i)* as $m = m^*|_{\mathcal{M}}$. This completes the whole proof. $\qquad\square$

Lemma 20.1.9.14. *For each subseteq $A \subseteq \mathbb{R}$, there exists a Borel subset $B \supseteq A$ such that*

$$m^*(A) = m(B).$$

Proof. We wish to show that for each $A \subseteq \mathbb{R}$, there exists a Borel set $B \supseteq A$ such that $m(B) = m^*(A)$. We divide into two cases based on outer measure of A. We will follow the notations of Eq. $(*)$ and $(**)$.

If $m^*(A) = \infty$. In this case, we claim that $B = \mathbb{R}$ will work. Indeed $\mathbb{R}$ is open and thus Borel. We thus claim that $m(\mathbb{R}) = \infty$. Indeed, for $I_n = (n, n+1]$, $n \in \mathbb{Z}$, we have that $\{I_n\}$ are disjoint and $\coprod_n I_n = \mathbb{R}$. As m is a measure and I_n are measurable, therefore

$$m(\mathbb{R}) = \sum_n m(I_n) = \sum_n 1 = \infty.$$

Hence $B = \mathbb{R}$ will work.

If $m^*(A) < \infty$, then we proceed as follows. For each $N \in \mathbb{N}$, there exists $\{I_n^N\} \in C(A)$ such that

$$m^*(A) + \frac{1}{N} > \sum_n l(I_n^N).$$

Define $U_N = \bigcup_n I_n^N$. As each half open interval $(a, b] = \bigcap_{n \in \mathbb{N}} (a, b + 1/n)$ is a Borel set, therefore U_N is a Borel set. Observe that

$$m(U_N) \leq \sum_n m^*(I_n^N) = \sum_n l(I_n^N) < m^*(A) + \frac{1}{N}.$$

Note that in the above we have used the fact that Lebesgue measure restricted to half opens is exactly the length function. We thus have for each $N \in \mathbb{N}$ a Borel set U_N containing A such that

$$m(U_N) < m^*(A) + \frac{1}{N}. \tag{4.1}$$

Denote $B_K = \bigcap_{N=1}^{K} U_N$. Then each B_K is Borel and $\{B_K\}$ is a decreasing sequence of sets. Furthermore, $\bigcap_{K=1}^{\infty} B_K = \bigcap_{N=1}^{\infty} U_N$. Denote $B = \bigcap_{K=1}^{\infty} B_K$. Observe that $B \supseteq A$ as $B_K \supseteq A$ for each $K \in \mathbb{N}$. Consequently, by continuity of m^* we have

$$m(B) \geq m^*(A).$$

For the converse, first note that by Eq. (4.1), $m(U_1) < \infty$. Thus by monotone convergence property of measures, we obtain that $\lim_{K \to \infty} m(B_K) = m\left(\bigcap_{K=1}^{\infty} B_K\right)$. It follows from

above, $B_K \subseteq U_K$ and Eq. (4.1) that

$$
\begin{aligned}
m(B) &= m\left(\bigcap_{K=1}^{\infty} B_K\right)\\
&= \lim_{K\to\infty} m(B_K)\\
&\leq \lim_{K\to\infty} m(U_K)\\
&\overset{(4.1)}{<} \lim_{K\to\infty}\left(m^*(A) + \frac{1}{K}\right)\\
&\leq m^*(A).
\end{aligned}
$$

Thus $m(B) \leq m^*(A)$ and we are done. $\qquad\square$

Lemma 20.1.9.15. *TODO : Type up Q5-Q10 of HW#2.*

Lemma 20.1.9.16. *A bounded set $E \subseteq \mathbb{R}$ is measurable if and only if $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$ for all bounded subsets $A \subseteq \mathbb{R}$.*

Proof. Let E be a bounded set of $\mathbb{R}$. We wish to show that E is measurable if and only if for all bounded sets $A \subseteq \mathbb{R}$, we get $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$.

The ($\Rightarrow$) is immediate from definitions. For ($\Leftarrow$), we proceed as follows. We wish to show that for any $F \subseteq \mathbb{R}$, we have

$$
m^*(F) \geq m^*(F \cap E) + m^*(F \cap E^c).
$$

Indeed, if $m^*(F) = \infty$, then there is nothing to show. So we assume $m^*(F) < \infty$. Observe then that $m^*(F \cap E), m^*(F \cap E^c) \leq m^*(F) < \infty$. Fix $\epsilon > 0$. There exists a sequence $\{I_n\}$ of half-opens such that $\bigcup_n I_n \supseteq F$ and

$$
m^*(F) + \epsilon > \sum_n m^*(I_n)
$$

where we are using the fact that measure of a half-open interval is its length. Observe that for each $n \in \mathbb{N}$, we have $m^*(F) + \epsilon > m^*(I_n)$, thus each I_n is a half-open interval with bounded length, hence I_n is bounded as a set. Consequently, we have

$$
\begin{aligned}
m^*(F) + \epsilon &> \sum_n m^*(I_n)\\
\text{(by hypothesis)}\quad &\geq \sum_n m^*(I_n \cap E) + m^*(I_n \cap E^c)\\
\text{(by rearrangement of +ve series)}\quad &= \sum_n m^*(I_n \cap E) + \sum_n m^*(I_n \cap E^c)\\
\text{(by subadditivity)}\quad &\geq m^*\left(\bigcup_n I_n \cap E\right) + m^*\left(\bigcup_n I_n \cap E^c\right)\\
\text{(by } \cup_n I_n \supseteq F)\quad &\geq m^*(F \cap E) + m^*(F \cap E^c).
\end{aligned}
$$

This completes the proof. $\qquad\square$

Measurable functions

Notation 20.1.9.17. At times, we will write a subset of X as follows:

$$\{x \in X \mid \mathcal{P}_x \text{ is true}\} = \{\mathcal{P}_x \text{ is true}\}.$$

This makes some constructions much more clearer to see and interpret.

Lemma 20.1.9.18. *Let $f : X \to Y$ be a function and $\mathcal{A}$ be an algebra on Y. Then,*

$$\langle f^{-1}(\mathcal{A}) \rangle = f^{-1}(\langle \mathcal{A} \rangle).$$

Proof. Let $f : X \to Y$ be a function and $\mathcal{A}$ be an algebra over Y. We wish to show that

$$\langle f^{-1}(\mathcal{A}) \rangle = f^{-1}(\langle \mathcal{A} \rangle). \tag{2.1}$$

We first claim that $f^{-1}(\langle \mathcal{A} \rangle)$ is a σ-algebra over X. Indeed, as $Y, \emptyset \in \langle \mathcal{A} \rangle$, we have $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$. Further, if $B \in f^{-1}(\langle \mathcal{A} \rangle)$, then $B = f^{-1}(A)$ for some $A \in \langle \mathcal{A} \rangle$. Hence $B^c = f^{-1}(A)^c = f^{-1}(A^c)$ and $A^c \in \langle A \rangle$ as $\langle A \rangle$ is a σ-algebra. Finally, pick $\{B_n\} \subseteq f^{-1}(\langle A \rangle)$. Then $B_n = f^{-1}(A_n)$ for $A_n \in \langle A \rangle$. Consequently, $\bigcup_n B_n = \bigcup_n f^{-1}(A_n) = f^{-1}\left(\bigcup_n A_n\right)$ and since $\bigcup_n A_n \in \langle A \rangle$, hence this proves that $f^{-1}(\langle A \rangle)$ is a σ-algebra.

We now show ($\subseteq$) part of Eq. (2.1). Indeed, by above, it would suffice to show that $f^{-1}(\mathcal{A})$ is contained in the σ-algebra $f^{-1}(\langle A \rangle)$. Pick any $B \in f^{-1}(\mathcal{A})$, so that $B = f^{-1}(A)$ where $A \in \mathcal{A}$. As $\mathcal{A} \subseteq \langle A \rangle$, therefore $A \in \langle A \rangle$. It follows that $B = f^{-1}(A) \in f^{-1}(\langle A \rangle)$. This shows that $\langle f^{-1}(\mathcal{A}) \rangle \subseteq f^{-1}(\langle A \rangle)$.

We now show ($\supseteq$) part of Eq. (2.1). We will use the standard technique of *good sets* for this. Consider

$$\mathcal{C} := \{A \in \langle \mathcal{A} \rangle \mid f^{-1}(A) \in \langle f^{-1}(\mathcal{A}) \rangle\} \subseteq \langle \mathcal{A} \rangle$$

We now claim the following two statements:
1. $\mathcal{C}$ *is a σ-algebra on Y* : Indeed, $Y = f^{-1}(X)$ and $\emptyset = f^{-1}(\emptyset)$ where $X, \emptyset \in \langle \mathcal{A} \rangle$ and $X, \emptyset \in \langle f^{-1}(\mathcal{A}) \rangle$. Further, for $A \in \mathcal{C}$, we have $f^{-1}(A) \in \langle f^{-1}(\mathcal{A}) \rangle$ and thus $(f^{-1}(A))^c = f^{-1}(A^c) \in \langle f^{-1}(\mathcal{A}) \rangle$. Thus $A^c \in \mathcal{C}$. Finally, pick $\{A_n\} \subseteq \mathcal{C}$. Then $f^{-1}(A_n) \in \langle f^{-1}(\mathcal{A}) \rangle$ for each $n \in \mathbb{N}$. Thus, $\bigcup_n f^{-1}(A_n) = f^{-1}\left(\bigcup_n A_n\right) \in \langle f^{-1}(\mathcal{A}) \rangle$. It then follows that $\bigcup_n A_n \in \mathcal{C}$. This shows that $\mathcal{C}$ is a σ-algebra.
2. $\mathcal{C} \supseteq \mathcal{A}$: Pick any $A \in \mathcal{A}$. As $\langle f^{-1}(\mathcal{A}) \rangle$ contains $f^{-1}(\mathcal{A})$, so $f^{-1}(A) \in \langle f^{-1}(\mathcal{A}) \rangle$.

We now conclude the proof. As $\mathcal{C}$ is a σ-algebra containing $\mathcal{A}$ and inside $\langle \mathcal{A} \rangle$, therefore $\mathcal{C} = \langle \mathcal{A} \rangle$. It follows that for each $A \in \langle \mathcal{A} \rangle$, we have $f^{-1}(A) \in \langle f^{-1}(\mathcal{A}) \rangle$, that is $f^{-1}(\langle \mathcal{A} \rangle) \subseteq \langle f^{-1}(\mathcal{A}) \rangle$, as required. This completes the proof. □

Lemma 20.1.9.19. *Let $(X, \mathcal{M}, m)$ be the Lebesgue measure space. Let $A \in \mathcal{M}$ be a bounded set such that $0 < m(A) < \infty$. For each $0 < M < m(A)$, there exists a $B \subsetneq A$ such that $B \in \mathcal{M}$ and $m(B) = M$.*

Proof. There are two proofs that we wish to present, one uses Lemma 20.1.9.20 and other is independent. The latter uses a nice technique which we would like to write down concretely.

Method 1 : *(Using Lemma 20.1.9.20)* Consider the map

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto m(A \cap (-\infty, x]).$$

As A is a bounded set, therefore $m(A) < \infty$ as there exists a bounded interval $I \supseteq A$ where $I = [c, d]$. By Lemma 20.1.9.20, the map f is a continuous map. Let $a \in \mathbb{R}$ be such that $a < c$. Then $f(a) = m(A \cap (-\infty, a]) = m(\emptyset) = 0$. Let $b \in \mathbb{R}$ such that $b > d$. Then, $f(b) = m(A \cap (-\infty, b]) = m(A)$. On the interval $J = [a, b]$ we have $f(a) = 0$ and $f(b) = m(A)$. By intermediate value property of f, there exists $c \in J$ such that $f(c) = M$. Consequently, $A \cap (-\infty, c]$ is a measurable subset of A whose measure is M.

Method 2 : *(Exponential subdivision technique)* We shall explicitly construct $B \subsetneq A$ such that $m(B) = M$. First, we observe that the question is invariant under translation and dilation. Hence we may, after suitable dilation and translation, assume that $A \subseteq [0, 1)$. For each $n \in \mathbb{N}$, consider the following partition of $[0, 1)$

$$P_n : 0 < x_1 = \frac{1}{2^n} < x_2 = 2 \cdot \frac{1}{2^n} < \cdots < x_{2^n - 1} = (2^n - 1) \cdot \frac{1}{2^n} < 1.$$

Denote $I_{n,j} = [\frac{j-1}{2^n}, \frac{j}{2^n})$ for each $j = 1, \ldots, 2^n$. Observe that $I_{n,j}$ are disjoint and, denoting $A_{n,j} = A \cap I_{n,j}$, we further have a disjoint collection $\{A_{n,j}\}$ of measurable subsets[4] of A such that

$$\coprod_{j=1}^{2^n} A_{n,j} = A.$$

Further, we have that

$$\sum_{j=1}^{2^n} m(A_{n,j}) = m\left(\coprod_{j=1}^{2^n} A_{n,j}\right)$$
$$= m(A)$$

and that

$$m(A_{n,j}) \leq m(I_{n,j}) = \frac{1}{2^n}.$$

Now, for each $n \in \mathbb{N}$, let N_n be the largest index such that

$$\sum_{j=1}^{N_n} m(A_{n,j}) \leq M.$$

---

[4]measurable because A and $I_{n,j}$ are measurable

By the choice of index N_n, we observe that

$$M < \sum_{j=1}^{N_n+1} m(A_{n,j})$$

$$= \sum_{j=1}^{N_n} m(A_{n,j}) + m(A_{n,N_n+1})$$

$$\leq \sum_{j=1}^{N_n} m(A_{n,j}) + \frac{1}{2^n}.$$

Denoting $C_n = \amalg_{j=1}^{N_n} A_{n,j}$, we obtain,

$$M - \frac{1}{2^n} < \sum_{j=1}^{N_n} m(A_{n,j}) = m(C_n) \leq M. \tag{3.1}$$

We now claim that $\{C_n\}$ is an increasing sequence of measurable subsets of A. First observe that for each $n \in \mathbb{N}$, we have that N_{n+1} is either $2N_n - 1$ or $2N_n$. Indeed, pick any $x \in C_n$. Then $x \in A_{n,j}$ where $j = 1, \ldots, N_n$. Expanding this, we have

$$x \in A_{n,j}$$

$$= A \cap \left[\frac{j-1}{2^n}, \frac{j}{2^n} \right)$$

$$= A \cap \left(\left[\frac{2(j-1)}{2^{n+1}}, \frac{2j-1}{2^{n+1}} \right) \amalg \left[\frac{2j-1}{2^{n+1}}, \frac{2j}{2^{n+1}} \right) \right)$$

$$= A_{n+1,2j-1} \amalg A_{n+1,2j}. \tag{3.2}$$

As $N_{n+1} = 2N_n - 1$ or $2N_n$, therefore for $j = 1, \ldots, N_n$, $2j = 2, \ldots, 2N_n$, hence in Eq. (3.2), we obtain that $x \in A_{n+1,2j-1}$ or $x \in A_{n+1,2j}$ and as $2j \leq 2N_n$, hence $x \in C_{n+1}$. This shows that $C_n \subseteq C_{n+1}$.

Applying $\lim_{n \to \infty}$ on Eq. (3.1), we thus obtain

$$M \leq \lim_{n \to \infty} m(C_n) \leq M.$$

Thus, by monotone convergence of measures, we conclude

$$M = \lim_{n \to \infty} m(C_n)$$

$$= m \left(\bigcup_n C_n \right).$$

As $C_n \subseteq A$ for each $n \in \mathbb{N}$, therefore $\bigcup_n C_n \subseteq A$. Consequently we have obtained a subset of A whose measure is M. $\qquad \square$

Lemma 20.1.9.20. *Let $(X, \mathcal{M}, \mu)$ be the Lebesgue measure space and $A \in \mathcal{M}$ be a bounded set. Then the function*

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \longmapsto m(A \cap (-\infty, x])$$

is continuous.

Proof. Let $A \in \mathcal{M}$ which has finite measure. We wish to show that

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto m(A \cap (-\infty, x])$$

is continuous. Pick any $a \in \mathbb{R}$ and any $\epsilon > 0$. We wish to find a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. We now have two cases with respect to the position of x and a in $\mathbb{R}$.

1. *If $a \le x$:* then $f(x) - f(a)$ can be rewritten as follows:

$$\begin{aligned}
|f(x) - f(a)| &= m(A \cap (-\infty, x]) - m(A \cap (-\infty, a]) \\
&= m(A \cap (-\infty, x] \setminus A \cap (-\infty, a]) \\
&= m(A \cap (a, x]) \\
&\le m((a, x]) \\
&= x - a.
\end{aligned}$$

Therefore taking $\delta = \epsilon$, we would be done.

2. *If $a > x$:* then $|f(x) - f(a)|$ can be written as

$$\begin{aligned}
|f(x) - f(a)| &= |f(a) - f(x)| \\
&= m(A \cap (-\infty, a]) - m(A \cap (-\infty, x]) \\
&= m(A \cap (x, a]) \\
&\le m((x, a]) \\
&= a - x.
\end{aligned}$$

Thus, again, taking $\delta = \epsilon$ would do the job.
This completes the proof. $\qquad\square$

Lemma 20.1.9.21. *Let X be a measurable space and let $f : X \to \mathbb{R}$ be a function. Suppose $\{x \in X \mid a \le f(x) < b\}$ is measurable for all $a < b$. Then f is a measurable function.*

Proof. As the Borel σ-algebra on $\mathbb{R}$ is generated by sets of the form $[a, \infty)$ for $a \in \mathbb{R}$, therefore for a fixed $a \in \mathbb{R}$ we need only show that $f^{-1}([a, \infty))$ is measurable in X.

We can write

$$\begin{aligned}
f^{-1}([a, \infty)) &= \{a \le f(x)\} \\
&= \bigcup_{n > a \text{ in } \mathbb{N}} \{a \le f(x) < n\}.
\end{aligned}$$

As we are given that $\{a \le f(x) < b\}$ are measurable for all $a < b \in \mathbb{R}$ and countable union of measurable sets is measurable, therefore $f^{-1}([a, \infty))$ is measurable. $\qquad\square$

Lemma 20.1.9.22. *All monotone functions $f : \mathbb{R} \to \mathbb{R}$ are measurable.*

Proof. We wish to show that all monotone functions $f : \mathbb{R} \to \mathbb{R}$ are measurable. Note that we may first reduce to assuming that f is non-decreasing as if f is non-increasing, then $-f$

will be non-decreasing.

Hence let $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing. As Borel σ-algebra of $\mathbb{R}$ is generated by intervals of the form $[a, \infty)$, $a \in \mathbb{R}$, therefore it suffices to check that $f^{-1}([a, \infty))$ is measurable in $\mathbb{R}$. Observe

$$f^{-1}([a, \infty)) = \{a \leq f(x)\}.$$

We now have two cases to handle.

1. *If $a \in f(\mathbb{R})$:* Then there exists $b \in \mathbb{R}$ such that $f(b) = a$. We may write

$$\{a \leq f(x)\} = \{a < f(x)\} \amalg \{a = f(x)\}.$$

Now since f is non-decreasing, therefore $f(x) > f(y)$ implies $x > y$. Further, we have that $f^{-1}(a) = \{a = f(x)\}$ is measurable as singletons are Borel. Consequently, we have

$$\begin{aligned}\{a \leq f(x)\} &= \{f(b) < f(x)\} \amalg \{a = f(x)\} \\ &= (b, \infty) \amalg f^{-1}(a).\end{aligned}$$

Hence $f^{-1}([a, \infty))$ is measurable.

2. *If $a \notin f(\mathbb{R})$:* We further have two cases.

   (a) *If there exists $b \in \mathbb{R}$ such that $b \notin \{a \leq f(x)\}$:* Observe first that in this case $f(b) < a$. We claim that in this case $\{a \leq f(x)\}$ is lower bounded by b. Indeed, suppose not. Then there exists $y < b$ such that $y \in \{a \leq f(x)\}$. Then $a \leq f(y) \leq f(b) < a$, a contradiction. Hence $\{a \leq f(x)\}$ is bounded below.
   Let $c = \inf\{a \leq f(x)\}$, which now exists. Consequently, we have two more cases:
   - *If $f(c) \geq a$:* That is, if $c \in \{a \leq f(x)\}$. Then we claim

   $$\{a \leq f(x)\} = [c, \infty).$$

   which is clearly a measurable. Indeed, for some $x \in \mathbb{R}$ such that $f(x) \geq a$, then $x \geq c$. Conversely, if $b \geq c$ in $\mathbb{R}$, then $f(b) \geq f(c) \geq a$, so $b \in \{a \leq f(x)\}$. This proves the claim.
   - *If $f(c) < a$:* That is, if $c \notin \{a \leq f(x)\}$. Then we claim

   $$\{a \leq f(x)\} = (c, \infty)$$

   which is clearly a measurbale set. Indeed, for $x \in \mathbb{R}$ such that $f(x) \geq a$, $x > c$. Further $x \neq c$ as otherwise $f(x) < a$. Conversely, if $b > c$, then there exists $d \in \{a \leq f(x)\}$ such that $c < d < b$ as c is the infimum. Consequently, $a \leq f(d) \leq f(b)$. Hence $b \in \{a \leq f(x)\}$. This proves the claim.

   (b) *If there doesn't exists any $b \in \mathbb{R}$ such that $f(b) < a$:* Then for all $b \in \mathbb{R}$ we have $f(b) \geq a$. Consequently, $f^{-1}([a, \infty)) = \{a \leq f(x)\} = \mathbb{R}$, which is measurable.

Hence in all cases $f^{-1}([a, \infty))$ is a measurable set. This completes the proof. $\qquad\square$

Lemma 20.1.9.23. *Let $f : X \to \mathbb{C}$ be a complex measurable function on a measurable space X. Then, there exists a complex measurable function $g : X \to \mathbb{C}$ such that $|g| = 1$ and $f = g\,|f|$.*

Proof. Let $f : X \to \mathbb{C}$ be a measurable function. We wish to find a measurable function $g : X \to \mathbb{C}$ such that $|g| = 1$ and $f = g\,|f|$.

As $|f| = f\chi_{\{f(x) \geq 0\}} - f\chi_{\{f(x) < 0\}}$, therefore $|f|$ is a measurable function. Denote $E = \{|f(x)| = 0\}$. Consequently, we define g as follows:

$$g(x) = \begin{cases} \frac{f(x)}{|f(x)|} & \text{if } x \in E^c \\ 1 & \text{if } x \in E. \end{cases}$$

We first wish to show that g is measurable. For this, we shall use the fact that measurability of g can be checked on a cover $\{E_\alpha\}$ of X such that $g|_{E_\alpha}$ is measurable. Thus in our case, we need only show that $g|_E$ and $g|_{E^c}$ are measurable. On E, g is a constant, hence measurable. On E^c, g is $f/|f|$. As $|f|$ is not zero on E^c, therefore by Lemma 20.1.9.25, $f/|f|$ is measurable. Hence, g is measurable.

We now see that $|g|\,(x) = \left| \frac{f(x)}{|f(x)|} \right| = 1$ on E^c and $|g(x)| = 1$ on E. Thus $|g| = 1$ on X. Further, if $x \in E$, then $f(x) = 0 = g(x)\,|f|\,(x)$. If $x \in E^c$, then $g(x) = \frac{f(x)}{|f(x)|}$ which implies $|f(x)|\,g(x) = f(x)$. This shows that in all cases, $f = g\,|f|$. $\qquad\square$

Example 20.1.9.24. It is not true that if $f : [0,1] \to \mathbb{R}$ is a function whose each fibre is measurable, then f is measurable.

Consider the following function

$$f : [0,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} x & \text{if } x \in V^c \\ x + N & \text{if } x \in V \end{cases}$$

where $V \subseteq [0,1]$ denotes the Vitali set and $N = 3$. Then, for each $y \in \mathbb{R}$, we have that $f^{-1}(y)$ is atmost a singleton, which is measurable in $[0,1]$. However, for any $1 < b < N$, we see that $f^{-1}((b,\infty)) = V$, which is not measurable. Hence f is a non-measurable function whose fibres are measurable.

Lemma 20.1.9.25. *Let $f, g : X \to \mathbb{C}$ be a measurable function such that $\{g(x) \neq 0\} = X$. Then f/g is measurable.*

Proof. Let $f, g : X \to \mathbb{C}$ be a measurable function such that $\{g(x) \neq 0\} = X$. Then we wish to show that f/g is measurable.

We first have that $(f,g) : X \to \mathbb{R}^2$ given by $x \mapsto (f(x), g(x))$ is measurable. Consequently, we consider the composite

$$X \xrightarrow{(f,g)} \mathbb{R}^2 \setminus \{y = 0\} \xrightarrow{\Phi} \mathbb{R}$$

where $\Phi(x,y) = \frac{x}{y}$. As Φ is continuous, therefore the composite $\Phi \circ (f,g)$ is measurable. Consequently, we obtain that the map $x \mapsto \frac{f(x)}{g(x)}$ is measurable, but this is exactly f/g over X. This completes the proof. $\qquad\square$

Lemma 20.1.9.26. *Let $f, g : X \to \overline{\mathbb{R}}$ be measurable functions and pick any $r_0 \in \overline{\mathbb{R}}$. Then the map*

$$h : X \longrightarrow \overline{\mathbb{R}}$$

$$x \longmapsto \begin{cases} r_0 & \text{if } f(x) = -g(x) = \pm\infty \\ f(x) + g(x) & \text{else} \end{cases}$$

is measurable[5].

Proof. Let $f, g : X \to \overline{\mathbb{R}}$ be measurable functions and pick any $r_0 \in \overline{\mathbb{R}}$. Then we wish to show that the map

$$h : X \longrightarrow \overline{\mathbb{R}}$$

$$x \longmapsto \begin{cases} r_0 & \text{if } f(x) = -g(x) = \pm\infty \\ f(x) + g(x) & \text{else} \end{cases}$$

is measurable.

Define the following sets

$$E = \{f(x) = \infty = -g(x)\}$$
$$F = \{f(x) = -\infty = -g(x)\}.$$

As $E = f^{-1}(\infty) \cap g^{-1}(-\infty)$ and $F = f^{-1}(-\infty) \cap g^{-1}(\infty)$, therefore they are measurable. Observe that E and F are disjoint. We thus need only show that h restricted to E, F and $X \setminus (E \amalg F)$ is measurable.

1. On E : As $h|_E$ is constant r_0, therefore $h|_E$ is measurable.
2. On F : As $h|_F$ is again constant r_0, therefore $h|_F$ is measurable.
3. On $X \setminus (E \amalg F)$: We first deduce that

$$X \setminus (E \amalg F) = X \cap E^c \cap F^c$$
$$= E^c \cap F^c$$
$$= (\{f(x) \neq \infty\} \cup \{g(x) \neq -\infty\}) \bigcap (\{f(x) \neq -\infty\} \cup \{g(x) \neq \infty\})$$

Let $G = \{f(x) \in \mathbb{R}\}$ and $H = \{g(x) \in \mathbb{R}\}$. Then we may write $X \setminus (E \amalg F)$ as

$$X \setminus (E \amalg F) = (G \cup H \cup \{f(x) = -\infty\} \cup \{g(x) = \infty\}) \bigcap (G \cup H \cup \{f(x) = \infty\} \cup \{g(x) = -\infty\})$$
$$= (G \cup H) \cup \left((\{f(x) = -\infty\} \cup \{g(x) = \infty\}) \bigcap (\{f(x) = \infty\} \cup \{g(x) = -\infty\}) \right)$$
$$= (G \cup H) \cup \underbrace{\{f(x) = -\infty = g(x)\}}_{=:A} \cup \underbrace{\{f(x) = \infty = g(x)\}}_{=:B}.$$

As $h|_{G \cup H}$ is $(f + g)|_{G \cup H}$ and on $G \cup H$, $f + g : G \cup H \to \mathbb{R}$, therefore h is measurable. We thus reduce to checking that $h|_A$ and $h|_B$ are measurable. On both of them, one immediately observes that h is constant $-\infty$ and ∞ respectively. Hence, $h|_A$ and $h|_B$ are measurable. As h restricted to $G \cup H$, A and B is measurable therefore h restricted to $X \setminus (E \amalg F)$ is measurable.

---

[5]This question in particular shows that modifying a measurable function at a single point doesn't affect measurability at all.

This completes the proof. □

Example 20.1.9.27. It is not true in general that if for a function $f : X \to \mathbb{R}$, the $|f| : X \to [0, \infty]$ is measurable then f is measurable.

Indeed, consider the following function where $V \subseteq [0, 1]$ denotes the Vitali set:

$$f : [0, 1] \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} -x & \text{if } x \in V \\ x & \text{if } x \in V^c. \end{cases}$$

Then, $|f| = \text{id}_{[0,1]}$ which is measurable whereas f is not measurable as $f^{-1}((-\infty, 0)) = V$, which is not a measurable set.

Lemma 20.1.9.28. *Let* $(X_1, \mathcal{A}_{X_1}, \mu_1)$ *be a measure space,* $(X_2, \mathcal{A}_{X_2})$ *be a measurable space and* $f : X_1 \to X_2$ *be a measurable function. Then*

$$\mu_2 : \mathcal{A}_{X_2} \longrightarrow [0, \infty]$$
$$B \longmapsto \mu_1(f^{-1}(B))$$

is a measure on $(X_2, \mathcal{A}_{X_2})$.

Proof. We first immediately observe that $\mu_2(\emptyset) = \mu_1(f^{-1}(\emptyset)) = \mu_1(\emptyset) = 0$. We thus reduce to showing that for any disjoint collection $\{B_n\} \subseteq \mathcal{A}_{X_2}$, we have $\mu_2(\coprod_n B_n) = \sum_n \mu_2(B_n)$. To this end, observe that

$$\mu_2\left(\coprod_n B_n\right) = \mu_1\left(f^{-1}\left(\coprod_n B_n\right)\right)$$
$$= \mu_1\left(\coprod_n f^{-1}(B_n)\right)$$
$$= \sum_n \mu_1(f^{-1}(B_n))$$
$$= \sum_n \mu_2(B_n).$$

This completes the proof. □

Lemma 20.1.9.29. *Let* $(X, \mathcal{A}, \mu)$ *be a measure space and* $f : X \to \mathbb{R}$ *be a measurable function such that* $\mu(\{|f(x)| \geq \epsilon\}) = 0$ *for all* $\epsilon > 0$. *Then* $f = 0$ *almost everywhere.*

Proof. We first claim that it suffices to show that $\{|f(x)| > 0\}$ is a null set. Indeed, this is because $\{f(x) \neq 0\} = \{|f(x)| > 0\}$. Hence it suffices to show that $|f| = 0$ a.e.

Define for each $n \in \mathbb{N}$ the following subset of X

$$E_n = \{|f(x)| > 1/n\}.$$

We claim that

$$\{|f(x)| > 0\} = \bigcup_{n \in \mathbb{N}} E_n.$$

Indeed, for ($\subseteq$), pick $x \in X$ such that $|f(x)| > 0$. Then there exists $n \in \mathbb{N}$ such that $|f(x)| > 1/n$. Hence $x \in E_n$. Conversely pick $x \in E_n$, then by way of construction of E_n, we have $|f(x)| > 1/n > 0$.

Observe that $\{E_n\}$ is an increasing sequence of sets as if $x \in E_n$ then $|f(x)| > \frac{1}{n} > \frac{1}{n+1}$, so $x \in E_{n+1}$. It then follows by monotone convergence property of measures that

$$\mu(\{|f(x) > 0|\}) = \mu\left(\bigcup_n E_n\right) = \varprojlim_{n \to \infty} \mu(E_n) = \varprojlim_{n \to \infty} 0 = 0.$$

This completes the proof. $\qquad\qquad\square$

Example 20.1.9.30. The statement of Egoroff's theorem depends crucially on the fact that each function in the sequence $\{f_n\}$ is measurable. Indeed, we show by the way of an example that the conclusion of Egoroff's theorem is not true when f_n's are not measurable.

We wish to show that the statement of Egoroff's theorem fails if we drop the condition that functions be measurable.

Consider the measure space $(\mathbb{Z}, \mathcal{A}, \mu)$ where $\mathcal{A} = \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, \mathbb{Z}\backslash 2\mathbb{Z}\}$ and $\mu(\emptyset) = 0 = \mu(2\mathbb{Z})$, $\mu(\mathbb{Z}) = 1 = \mu(\mathbb{Z} \setminus 2\mathbb{Z})$. Consider the functions $f_n : (\mathbb{Z}, \mathcal{A}, \mu) \to \mathbb{R}$ where $\mathbb{R}$ has the Borel measure, given by

$$f_n(k) = \frac{k}{n}$$

for all $k \in \mathbb{Z}$. Observe that $\{f_n\}$ pointwise converges to the constant 0 function at all points of $\mathbb{Z}$. Further note that f_n is not measurable as $f_n^{-1}(\{k/n\}) = \{k\}$ is not a measurable set in $\mathcal{A}$ but $\{k/n\}$ is Borel measurable.

To show that this is a counterexample, it would suffice to show that there exists an $\epsilon_0 > 0$ such that for all measurable sets $E \in \mathcal{A}$, either $\mu(E^c) \geq \epsilon_0$ or f_n does not converges uniformly to 0 on E. We claim that in our situation, $\epsilon_0 = 1/2$ works. Indeed, for $E = \emptyset, 2\mathbb{Z}$, we have $\mu(E^c) = 1 > 1/2$. Thus we reduce to showing that f_n does not converges uniformly on $\mathbb{Z}$ and $\mathbb{Z} \setminus 2\mathbb{Z}$. Indeed, observe that $\sup_{k \in \mathbb{Z}} |f_n(k)| = \sup_{k \in \mathbb{Z}} k/n = \infty$ for each $n \in \mathbb{N}$. As f_n converges uniformly if and only if $\sup_{k \in \mathbb{Z}} |f_n(k)| \to 0$ as $n \to \infty$, therefore we deduce that f_n does not converge uniformly over $\mathbb{Z}$. Similarly, it doesn't converge uniformly over $\mathbb{Z} \setminus 2\mathbb{Z}$.

Lemma 20.1.9.31. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. If $f = g$ almost everywhere, then $f = g$.*

Proof. Indeed, consider $h = f - g$. Suppose $h \neq 0$, therefore there exists $x_0 \in \mathbb{R}$ such that $h(x_0) \neq 0$. By continuity of h, there exists $\epsilon > 0$ such that $(x_0 - \epsilon, x_0 + \epsilon) \subseteq \{h(x) \neq 0\}$. Hence, $2\epsilon < m(\{h(x) \neq 0\}) = 0$, which yields $0 < 2\epsilon \leq 0$, a contradiction. $\qquad\square$

Lemma 20.1.9.32. *Let $(X, \mathcal{S}, \mu)$ be a measure space and $f_n, f : X \to \overline{\mathbb{R}}$ be measurable functions such that $f_n \to f$ pointwise almost everywhere. Then, there exists measurable functions $g_n : X \to \overline{\mathbb{R}}$ such that $f_n = g_n$ almost everywhere and $g_n \to f$ pointwise.*

Proof. Indeed, as f_n converges pointwise to f almost everywhere, therefore the set $E = \{\varprojlim_{n \to \infty} f_n(x) \neq f(x)\}$ is a zero measure set. Consequently, we may define

$$g_n : X \longrightarrow \overline{\mathbb{R}}$$

$$x \longmapsto \begin{cases} f_n(x) & \text{if } x \notin E \\ f(x) & \text{if } x \in E. \end{cases}$$

We then observe that $\{g_n(x) \neq f_n(x)\} = E$, which is of measure zero, hence $g_n = f_n$ almost everywhere. Furthermore, we see that for any $x \in X$,

$$\varprojlim_{n \to \infty} g_n(x) = \begin{cases} \varprojlim_{n \to \infty} f_n(x) = f(x) & \text{if } x \notin E \\ \varprojlim_{n \to \infty} f(x) = f(x) & \text{if } x \in E. \end{cases}$$

Thus, $\varprojlim_{n \to \infty} g_n = f$ pointwise. This completes the proof. $\square$

Example 20.1.9.33. We wish to show that there exists continuous function $f : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f : \mathbb{R} \to \mathbb{R}$ is not Lebesgue measurable.

While learning about the existence of a non-Borel measurable set, one learns about the existence of a homeomorphism $\varphi : [0,1] \to [0,2]$ such that $m(\varphi(C)) = 1 > 0$ where $C \subseteq [0,1]$ is the Cantor set. Indeed, if $\mathcal{C} : [0,1] \to [0,1]$ denotes the Cantor function, then φ is constructed by defining $\varphi(x) = \mathcal{C}(x) + x$. As, $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = 1$, therefore $\varphi(0) = 0$ and $\varphi(1) = 2$. Consequently, we may define a continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x - 1 & \text{if } x > 2 \\ \varphi^{-1}(x) & \text{if } x \in [0,2] \\ x & \text{if } x < 0. \end{cases}$$

Observe that f is continuous as f is obtained by gluing three continuous functions at points where they agree.

As $m(\varphi(C)) = 1 > 0$ for Cantor set C, therefore there exists a non-measurable set $V \subseteq \varphi(C) \subseteq [0,2]$. But since $f(V) = \varphi^{-1}(V) \subseteq \varphi^{-1}(\varphi(C)) = C$ and C is a null set, therefore by completeness of Lebesgue measure, it follows that $f(V)$ is a Lebesgue measurable set. Consequently, we may define $g = \chi_{f(V)} : \mathbb{R} \to \mathbb{R}$, which is Lebesgue measurable as $f(V)$ is Lebesgue measurable. We thus have

$$\mathbb{R} \xrightarrow[\text{cont.}]{f} \mathbb{R} \xrightarrow[\text{Leb. msble}]{g} \mathbb{R} \ .$$

We claim that $h := g \circ f$ is not Lebesgue measurable. Indeed, observe that $h^{-1}(\{1\}) = (g \circ f)^{-1}(\{1\}) = f^{-1}(g^{-1}(\{1\})) = f^{-1}(f(V))$. But as f restricted to $[0,2]$ is a homeomorphism from $[0,2]$ to $[0,1]$ because on $[0,2]$, f is equal to φ^{-1}, hence $f^{-1}(f(V)) = V$. Hence $h^{-1}(\{1\}) = V$, where $\{1\}$ is measurable but $V \subseteq [0,2]$ is non-measurable. This shows that h is not measurable. This completes the proof.

20.1.10 Main results : Commutation of $\int$ and $\varinjlim$

We begin with the simple approximation theorem of positive measurable functions.

Theorem 20.1.10.1. *Let (X, M, μ) be a measure space and let $f : X \to \mathbb{R}$ be a positive measurable map. Then, there exists a sequence of simple functions $\{s_n\}_{n \in \mathbb{N}}$ such that*
  1. *$0 \leq s_n(x) \leq s_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$,*
  2. *$\varinjlim_{n \to \infty} s_n(x) = f(x)$ for all $x \in X$.*

We have the famous monotone convergence theorem.

Theorem 20.1.10.2. *(MCT) Let (X, M, μ) be a measure space and let $f_n : X \to \mathbb{R}$ be a sequence of positive measurable maps. Suppose*
  1. *$\varinjlim_n f_n(x)$ exists and is equal to $f(x)$ for some measurable $f : X \to \mathbb{R}$,*
  2. *$f_n(x) \leq f_{n+1}(x)$ for all $x \in X$ and $n \in \mathbb{N}$.*
Then,

$$\varinjlim_{n \to \infty} \int_X f_n d\mu = \int_X \varinjlim_{n \to \infty} f_n d\mu.$$

We have the Fatou's lemma, which is the best thing we got without any hypotheses on the sequence of functions under consideration.

Theorem 20.1.10.3. *(Fatou's lemma) Let (X, M, μ) be a measure space and let $f_n : X \to \mathbb{R}$ be a sequence of positive measurable maps. Then,*

$$\int_X \liminf_{n \to \infty} f_n d\mu \leq \liminf_{n \to \infty} \int_X f_n d\mu.$$

Similarly, we have the famous dominated convergence theorem.

Theorem 20.1.10.4. *(DCT) Let (X, M, μ) be a measure space and let $f_n : X \to \mathbb{C}$ be a sequence of measurable maps. Suppose*
  1. *$\varinjlim_n f_n(x)$ exists and is equal to $f(x)$ for some measurable $f : X \to \mathbb{C}$,*
  2. *there exists a measurable $g : X \to \mathbb{R}$ of class L^1 such that $|f_n(x)| \leq g(x)$ for all $x \in X$ and $n \in \mathbb{N}$.*
Then,

$$\varinjlim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0$$

and

$$\varinjlim_{n \to \infty} \int_X f_n d\mu = \int_X \varinjlim_{n \to \infty} f_n d\mu.$$

A simple corollary of the MCT tells us an equivalent story for decreasing sequence of maps where first term is L^1, as compared to the statement of MCT.

Corollary 20.1.10.5. *Let (X, M, μ) be a measure space and let $f_n : X \to \mathbb{R}$ be a sequence of positive measurable maps. Suppose*

1. $\lim_{\to n} f_n(x)$ exists and is equal to $f(x)$ for some measurable $f : X \to \mathbb{R}$,
2. $f_n(x) \geq f_{n+1}(x)$ for all $x \in X$ and $n \in \mathbb{N}$,
3. $f_1(x) \in L^1$.

Then,

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X \lim_{n \to \infty} f_n d\mu.$$

Proof. Since $f \leq f_n \leq f_1$, therefore $f \in L^1$. Now, consider the (not necessarily positive!) measurable sequence $g_n = f - f_n$. Since f_n decreases, therefore g_n increases. Now, $\lim_{\to n} g_n = 0$ as $\lim_{\to n} f_n = f$. Since $0 \in L^1$, therefore Hence, by MCT Theorem 20.1.10.2, we get that $\lim_{\to n} \int_X g_n dm = \int_X \lim_{\to n} g_n dm$. Expanding it and using the fact that f is in L^1 (so you can cancel $\int_X f dm$ both sides!) gives the desired result. $\qquad\square$

Another important result which is of tremendous usability is the fact that Riemann and Lebesgue agree on compact domains(!)

Theorem 20.1.10.6. *(Riemann = Lebesgue on $[a, b]$) Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval and $f : [a, b] \to \mathbb{R}$ be a bounded map. Then, the Riemann integral and Lebesgue integral of f agrees on $[a, b]$. That is,*

$$\int_a^b f(x) dx = \int_{[a,b]} f dm$$

where m is the Lebesgue measure of $\mathbb{R}$.

20.1.11   Properties of L^1 maps

We would in this section quickly portray some of the easy properties of L^1-maps which are good to keep in mind. The first tells us that a high schooler's dream of claiming a map to be zero if integral is zero is *almost* true for L^1 maps.

Lemma 20.1.11.1. *Let $f : X \to \mathbb{C}$ be a measurable map where (X, M, m) is a measure space. Suppose $f \in L^1$. Then, $\int_F f dm = 0$ for all $F \in M$ if and only if $f = 0$ almost everywhere.*

Proof. One side is trivial. For the other, we may reduce to the case when f is real valued. Let $A = \{x \in X \mid f^-(x) > 0\}$. As f^- is measurable, therefore $A \in M$. Since $\int_A f dm = 0$, therefore $\int_A f^+ - f^- dm = \int_A f^+ = \int_A f^- dm$. If $x \in A$, then $f^-(x) > 0$, and hence $f^+(x) = 0$. Hence $\int_A f^+ dm = 0$ and hence $\int_A f^- dm = 0$. Since $f^- \geq 0$, therefore $f^- = 0$ almost everywhere. We thus have $\int_X f dm = \int_X f^+ dm = 0$ as $X \in M$. Since $f^+ \geq 0$, therefore $f^+ = 0$ almost everywhere. $\qquad\square$

Lemma 20.1.11.2. *Let (X, M, m) be a measure space and $f : X \to \mathbb{R}$ be a measurable map with $f \geq 0$. Then,*

$$m(\{x \in X \mid f(x) = \infty\}) = 0.$$

Proof. This again uses the standard idea of breaking the set which we wish to measure into sets whose bounds on measure is known. Indeed, observe that

$$E := \{f(x) = \infty\} = \bigcap_{n \in \mathbb{N}} \{f(x) > n\} =: \bigcap_{n \in \mathbb{N}} E_n.$$

Moreover, $\{E_n\}$ is decreasing. Thus,

$$m(E) = \varinjlim_{n \to \infty} m(E_n).$$

Now we obtain bound on $m(E_n)$. Indeed,

$$nm(E_n) = \int_{E_n} n\, dm \le \int_{E_n} f(x)\, dm \le \int_X f(x)\, dm =: I < \infty.$$

Thus $m(E_n) \le I/n$. Hence $\varinjlim_{n \to \infty} m(E_n) = 0$. $\qquad\square$

20.1.12 Applications-II : Integration

TODO : Add HW# 5 material here.

We present important applications of the above results, showcasing the power of their usage. At parts here, we are proving results from Folland's exercises ([**??**]).

Lemma 20.1.12.1. *The Lebesgue integral*

$$\int_0^1 \frac{x^p - 1}{\log x}\, dx$$

exists for $p > -1$.

Proof. The first idea is to break p into cases. In some cases, it is obvious why the above integral exists, in others, we have to work. Denote $f_p(x) = \frac{x^p - 1}{\log x}$.

$$\textbf{Act 1}: p > 0$$

In this regime, we can bound the $\int_0^1 f_p(x)\, dx$ by a fixed quantity. Indeed, since $f_p(x)$ is positive, it will suffice. Observe that

$$\frac{x^p - 1}{\log x} = \frac{1 - x^p}{-\log x} \le \frac{1}{-\log x}.$$

Now, $-\log x$ can be lower bounded by $1 - ax$ for some $0 < a < 1$ by an easy graphical observation. Hence, continuing above, we get

$$\frac{x^p - 1}{\log x} \le \frac{1}{1 - ax}.$$

The integral then translates to

$$\int_0^1 \frac{x^p - 1}{\log x}\, dx \le -\int_0^1 \frac{1}{1 - ax}\, dx = -\frac{\log(1 - a)}{a} < \infty.$$

$$\textbf{Act 2}: -1 < p < 0$$

This is the regime in which we got to work a bit. First, from some graphical observations about $x^p - 1$ and $\log x$, we conclude the following:

1. $x^p - 1$ is positive and $\log x$ is negative, so that $\frac{x^p-1}{\log x}$ is negative.
2. Viewing $1/\log x$ as an **attenuating factor**[6], we see that $0 < 1/\log x < -1$ for $0 < x < 1/e$ and $1/\log x \leq -1$ for $1/e \leq x < 1$.
3. On $1/e < x < 1$, $\log x > 1 - x$. Hence $1/\log x < 1/1 - x$.

With this, we write our integral as

$$\int_0^1 \frac{x^p - 1}{\log x} dx = \int_0^{1/e} \frac{x^p - 1}{\log x} dx + \int_{1/e}^1 \frac{x^p - 1}{\log x} dx$$

$$< \int_0^{1/e} (1 - x^p) dx + \int_0^1 \frac{x^p - 1}{x - 1} dx$$

Now the first integral is bounded while the second is bounded as the derivative of x^p exists at $x = 1$.

$\square$

Lemma 20.1.12.2. *Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty, -\infty\}$ be a measurable map with $(\mathbb{R}, M, m)$ be a measure structure on $\mathbb{R}$. If there exists $M > 0$ such that for all $E \in M$ such that $0 < m(E) < \infty$ we have that*

$$\left| \frac{1}{m(E)} \int_E f \, dm \right| < M,$$

then

$$|f(x)| \leq M \ a.e..$$

Proof. Let $A = \{x \in \mathbb{R} \mid |f(x)| > M\}$. We can write it as $A = A_+ \cup A_-$ where $A_+ = \{x \in \mathbb{R} \mid f(x) > M\}$ and $A_- = \{x \in \mathbb{R} \mid f(x) < -M\}$. Clearly these are disjoint and covers A. Hence, we wish to show

$$m(A) = m(A_+) + m(A_-) = 0$$

which is equivalent to showing that $m(A_+) = m(A_-) = 0$ as measures are always positive.

$$\textbf{Act 1}: m(A_+) = 0.$$

The way A_+ and A_- are defined, it is natural for the next step to be a consideration of integral of f over these. Indeed, we observe that, due to the fact that $f \in L^1$ and $A_+ \subseteq \mathbb{R}$

$$Mm(A_+) = \int_{A_+} M \leq \int_{A_+} |f| \leq \int_{\mathbb{R}} |f| \, dm < \infty.$$

---

[6]we view $1/\log x$ as an attenuating factor instead of $x^p - 1$ as if we remove $1/\log x$, then we would be left with $x^p - 1$, whose integral is easy to find.

Thus, $\infty > \int_{A_+} f\,dm \geq Mm(A_+)$. Note we dropped the absolute sign as f is positive on A_+. Hence $m(A_+) \neq \infty$.

Now suppose $0 < m(A_+) < \infty$. Then by hypothesis, we can write

$$\int_{A_+} f\,dm < Mm(A_+),$$

which is a contradiction. Hence $m(A_+) = 0$.

Act 2 : $m(A_-) = 0$.

Again using $f \in L^1$ and $A_- \subseteq \mathbb{R}$, we get

$$\int_{A_-} |f|\,dm \leq \int_{\mathbb{R}} |f|\,dm < \infty.$$

Since $\left| \int_{A_-} f\,dm \right| \leq \int_{A_-} |f|\,dm$ and since $\int_{A_-} f\,dm < \int_{A_-} -M\,dm = -Mm(A_-)$ so that $\left| \int_{A_-} f\,dm \right| > Mm(A_-)$, therefore we get

$$Mm(A_-) < \left| \int_{A_-} f\,dm \right| \leq \int_{A_-} |f|\,dm < \infty.$$

Hence $m(A_-) \neq \infty$. Now with this, if we assume $\infty > m(A_-) > 0$, then by hypothesis, we obtain

$$\left| \int_{A_-} f\,dm \right| \leq m(A_-)M,$$

which contradicts the above inequality. Hence $m(A_-) = 0$.

$\square$

Lemma 20.1.12.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable map where the domain $\mathbb{R}$ has a measure structure $(\mathbb{R}, M, m)$. If $f \in L^1$ and $f \geq 0$, then for all $E \in M$*

$$\lim_{n \to \infty} \int_E f^{\frac{1}{n}}\,dm = m(E).$$

Proof. The fundamental observation that one has to make here is that if $y \in [0, \infty)$, then $y^{1/n}$ increases to 1 on $(0, 1]$ and $y^{1/n}$ decreases to 1 on $(1, \infty)$. Indeed, pick any $E \in M$ and define

$$E_\leq := E \cap \{x \in \mathbb{R} \mid f(x) \leq 1\}$$
$$E_> := E \cap \{x \in \mathbb{R} \mid f(x) > 1\}.$$

We thus have a disjoint measurable cover of E and hence $m(E) = m(E_\leq) + m(E_>)$. Hence we get that

$$\lim_{n \to \infty} \int_E f^{\frac{1}{n}}\,dm = \lim_{n \to \infty} \int_{E_\leq} f^{\frac{1}{n}}\,dm + \lim_{n \to \infty} \int_{E_>} f^{\frac{1}{n}}\,dm.$$

Now, we have two integrals to consider.

$$\mathbf{Act\ 1} : \varinjlim_{n\to\infty} \int_{E_\le} f^{\frac{1}{n}} dm = m(E_\le).$$

Since $f^{\frac{1}{n}}$ is a sequence of positive measurable maps increasing to 1, therefore by MCT (Theorem 20.1.10.2), we get that

$$\varinjlim_{n\to\infty} \int_{E_\le} f^{\frac{1}{n}} dm = \int_{E_\le} \varinjlim_{n\to\infty} f^{\frac{1}{n}} dm$$
$$= \int_{E_\le} 1 dm$$
$$= m(E_\le).$$

$$\mathbf{Act\ 2} : \varinjlim_{n\to\infty} \int_{E_>} f^{\frac{1}{n}} dm = m(E_>).$$

It is this place where we will have to use the fact that $f \in L^1$. Since $f^{\frac{1}{n}}$ is a sequence of positive measurable maps decreasing to 1 where f is L^1. Hence, by Corollary 20.1.10.5 of MCT, we get that

$$\varinjlim_{n\to\infty} \int_{E_>} f^{\frac{1}{n}} dm = \int_{E_>} \varinjlim_{n\to\infty} f^{\frac{1}{n}} dm$$
$$= \int_{E_>} 1 dm$$
$$= m(E_>).$$

This completes the proof, as we have showed $\varinjlim_{n\to\infty} \int_E f^{\frac{1}{n}} dm = \varinjlim_{n\to\infty} \int_{E_\le} f^{\frac{1}{n}} dm + \varinjlim_{n\to\infty} \int_{E_>} f^{\frac{1}{n}} dm = m(E_\le) + m(E_>) = m(E).$ $\qquad\square$

Lemma 20.1.12.4. *Let $(\mathbb{R}, M, m)$ be a measure space with $E \in M$ and $f_n : E \to \mathbb{R} \cup \{\pm\infty\}$ be a sequence of measurable maps. If there exists a measurable map $f : E \to \mathbb{R} \cup \{\pm\infty\}$ such that*

$$|f_n(x)| \le |f(x)| \quad a.e. \ \forall n \in \mathbb{N},$$

then

$$\int_E \limsup_n f_n dm \le \limsup_n \int_E f_n dm \le \liminf_n \int_E f_n dm \le \int_E \liminf f_n dm.$$

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 20.1.12.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be an L^1 map where domain has a measure structure given by $(\mathbb{R}, M, m)$. Pick $E \in M$ and let $E_n := \{x \in E \mid f(x) \ge n\}$. Then,*
   1. $m(E_n) \le \frac{1}{n} \int_{E_n} |f| \, dm$,
   2. $\varinjlim_{n\to\infty} \int_{E_n} |f| \, dm$,
   3. $\varinjlim_{n\to\infty} n m(E_n) = 0$.

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 20.1.12.6. *Let E be a non-measurable set and $f : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$f(x) = \begin{cases} e^x & \text{if } x \in E \\ -e^x & \text{if } x \notin E \end{cases}.$$

Then

1. *$\{x \in \mathbb{R} \mid f(x) = c\}$ is measurable for all $x \in \mathbb{R}$.*
2. *f is not a measurable map.*

Proof. ☐

Lemma 20.1.12.7. *Let (X, M, m) be a measure space and $f : X \times [a, b] \to \mathbb{C}$ be a function such that $f(x, t) : X \to \mathbb{C}$ is measurable for all $t \in [a, b]$. Let $F(t) := \int_X f(x, t)dm$. Suppose there exists $g \in L^1$ such that*

$$|f(x, t)| \le |g(x)| \, \forall x \in X$$

for every $t \in [a, b]$. If $\varinjlim_{t \to t_0} f(x, t) = f(x, t_0)$ for every $x \in X$, then

$$\varinjlim_{t \to t_0} F(t) = F(t_0).$$

Proof. Clearly we should use DCT. However, we first need to get a sequence of functions for it. Indeed, since we know that $\varinjlim_{t_n \to t_0} f(x, t) = f(x, t_0)$, thus for any sequence $t_n \to t_0$, we have $\varinjlim_{n \to \infty} f(x, t_n) = f(x, t_0)$. Hence we may define $f_n(x) = f(x, t_n)$ which are by definition measurable. Moreover, we have $|f_n(x)| \le |g(x)|$ for all $x \in X$ where $g \in L^1$. Hence, by DCT, we obtain

$$\begin{aligned} \varinjlim_{n \to \infty} F(t_n) = \varinjlim_{n \to \infty} \int_X f_n(x)dm &= \int_X \varinjlim_{n \to \infty} f_n(x)dm \\ &= \int_X f(x, t_0)dm \\ &= F(t_0). \end{aligned}$$

Since $t_n \to t_0$ is arbitrary, therefore $\varinjlim_{t \to t_0} F(t) = F(t_0)$. ☐

20.1.13 Product measures

20.1.14 Fubini-Tonelli theorem

20.1.15 Applications-III : Product and Fubini-Tonelli

20.1.16 Signed measures

20.1.17   Elements of L^p spaces

Let (X, M, m) be a measure space. The $L^p(X)$ is the set of all measurable maps $f : X \to \mathbb{C}$ such that $(\int_X |f|^p \, dm)^{1/p} < \infty$. This is a complete normed linear space for $1 \leq p \leq \infty$, where the triangle inequality is also sometimes called Minkowski's inequality, which follows from Hölder's inequality, which say that if $f \in L^p, g \in L^{p^*}$, then $fg \in L^1$ and $\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p^*}}$ where $1/p + 1/p^* = 1$. Egoroff's theorem tells us that a pointwise convergence of a sequence of measurable functions in a finite measure space is almost uniformly convergent, except on a set of finite measure and this set's measure can be chosen as small as we want. Further, convergence in L^p norm implies that there is a subsequence which converges pointwise. The L^∞ norm is also defined. It is given by the essential supremum of the measurable function f:

$$\|f\|_{L^\infty} := \inf\{M > 0 \mid m(\{|f(x)| \geq M\}) = 0\}.$$

On any finite measure space X, we further have that $L^q(X) \hookrightarrow L^p(X)$ for all $\infty \geq q \geq p \geq 1$.

Chapter 21

Homological Methods

Contents

Methods employed in homological algebra comes in handy to attack certain type of local-global problems in geometry. We would like to discuss some foundational homological algebra in this chapter in the setting of additive and abelian categories. The main goal is not to illuminate foundations but to quickly get to the working theory which can allow us to develop deeper results elsewhere in this notebook. Using the Freyd-Mitchell embedding theorem, we can always assume that any abelian category $\mathbf{A}$ is a subcategory of $\mathbf{AbGrp}$. Thus we will freely do the technique of diagram chasing in the following, implicitly assuming $\mathbf{A}$ to be embedded in $\mathbf{AbGrp}$. The main example to keep in mind throughout this chapter is of-course the category of R-modules, $\mathbf{Mod}(R)$.

21.1 The setup : abelian categories

Let us begin with the basic definitions. Let $\mathbf{A}$ be a category. Then $\mathbf{A}$ is said to be *preadditive* if for any $x, y \in \mathbf{A}$, the homset $\mathrm{Hom}\,(x, y)$ is a group and the composition $\mathrm{Hom}\,(x, y) \times \mathrm{Hom}\,(y, z) \to \mathrm{Hom}\,(x, z)$ is a group homomorphism. For two preadditive categories $\mathbf{A}, \mathbf{B}$ a functor $F : \mathbf{A} \to \mathbf{B}$ is called *additive* if for all $x, y \in \mathbf{A}$, the function $\mathrm{Hom}_{\mathbf{A}}\,(x, y) \to \mathrm{Hom}_{\mathbf{B}}\,(Fx, Fy)$ is a group homomorphism.

Let $\mathbf{A}$ be a preadditive category and $f : x \to y$ be an arrow. This mean for any two object $w, z \in \mathbf{A}$, there is a zero arrow $0 \in \mathrm{Hom}\,(w, z)$. Then, we can define the usual notions of algebra as follows.

1. $i : \mathrm{Ker}\,(f) \to x$ is defined by the following universal property w.r.t. $fi = 0$:

$$
\begin{array}{c}
x' \\
\exists! \Big\downarrow \qquad \searrow {\scriptstyle \forall g \text{ s.t. } fg = 0} \\
\mathrm{Ker}\,(f) \xrightarrow{\;\;i\;\;} x \xrightarrow{\;\;f\;\;} y
\end{array}
$$

2. $j : y \to \mathrm{CoKer}\,(f)$ defined by the following universal property w.r.t. $jf = 0$:

$$
\begin{array}{c}
{\scriptstyle \forall g \text{ s.t. } gf = 0} \nearrow \quad y' \\
\qquad\qquad\qquad \Big\uparrow {\scriptstyle \exists!} \qquad . \\
x \xrightarrow{\;\;f\;\;} y \xrightarrow{\;\;j\;\;} \mathrm{CoKer}\,(f)
\end{array}
$$

3. $k : x \to \mathrm{CoIm}\,(f)$ is defined to be the cokernel of the kernel map $i : \mathrm{Ker}\,(f) \to x$.

4. $l : \mathrm{Im}\,(f) \to y$ is defined to be the kernel of the cokernel map $j : y \to \mathrm{CoKer}\,(f)$.

Hence, for each $f : x \to y$ in a preadditive category $\mathbf{A}$, we can contemplate the following four type of maps:

$$
\begin{array}{ccc}
\mathrm{Ker}\,(f) & & \mathrm{CoKer}\,(f) \\
\downarrow & & \uparrow \\
x & \xrightarrow{\;\;f\;\;} & y \\
\downarrow & & \uparrow \\
\mathrm{CoIm}\,(f) & & \mathrm{Im}\,(f)
\end{array}
$$

Lemma 21.1.0.1. *In a preadditive category, if a coproduct $x \oplus y$ exists, then so does the product $x \times y$ and vice versa. In such a case, $x \oplus y \cong x \times y$.* $\qquad\square$

A preadditive category $\mathbf{A}$ is said to be *additive* if it contains all finite products, including the empty ones. By the above lemma, we require zero objects and sums of objects to exist.

An additive category $\mathbf{A}$ is said to be *abelian* if all kernels and cokernels exist and the natural map for each $f : x \to y$ in $\mathbf{A}$

$$
\mathrm{CoIm}\,(f) \to \mathrm{Im}\,(f)
$$

is an isomorphism. This intuitively means that the first isomorphism theorem holds in abelian categories by definition.

21.2 (co)Homology, resolutions and derived functors

In this section, we shall discuss basic topics of homological algebra in abelian categories, which we shall need to setup the sheaf cohomology in geometry and Lie group cohomology in algebra and etcetera, etcetera.

21.2.1 (co)Homology

We define cochain complexes and maps, cohomology and homotopy of such. Since this section is mostly filled with *trivial matters*, therefore we shall allow ourselves to be a bit sketchy with proofs.

Complete the homology part and projective resolution Chapter 18.

Definition 21.2.1.1. (Cochain complexes, maps and cohomology) A cochain complex $A^\bullet$ is a sequence of object $\{A^i\}_{i\in\mathbb{Z}}$ with a map $d^i : A^i \to A^{i+1}$ called the coboundary maps which satisfies $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. A map $f : A^\bullet \to B^\bullet$ of cochain complexes is defined as a collection of maps $f^i : A^i \to B^i$ such that the following commutes

$$
\begin{array}{ccc}
A^i & \xrightarrow{d^i} & A^{i+1} \\
{\scriptstyle f^i}\downarrow & & \downarrow{\scriptstyle f^{i+1}} \\
B^i & \xrightarrow{d^i} & B^{i+1}
\end{array} \; .
$$

That is, $d^i f^i = f^{i+1} d^i$ for each $i \in \mathbb{Z}$. For a cochain complex $A^\bullet$, we define the ith cohomology object as the quotient

$$h^i(A^\bullet) := \text{Ker}\left(d^i\right)/\text{Im}\left(d^{i-1}\right).$$

With the obvious notion of composition, we thus obtain a category of cochain complexes $\text{coCh}(\mathbf{A})$ over the abelian category $\mathbf{A}$.

We now show that h^i forms a functor over $\text{coCh}(\mathbf{A})$.

Lemma 21.2.1.2. *Let $\mathbf{A}$ be an abelian category. The ith-cohomology assignment is a functor*

$$
\begin{aligned}
h^i : \text{coCh}(\mathbf{A}) &\longrightarrow \mathbf{A} \\
A^\bullet &\longmapsto h^i(A^\bullet).
\end{aligned}
$$

Proof. For a map of complexes $f : A^\bullet \to B^\bullet$, we first define the map $h^i(f)$

$$
\begin{aligned}
h^i(f) : h^i(A^\bullet) &\longrightarrow h^i(B^\bullet) \\
a + \text{Im}\left(d^{i-1}\right) &\longmapsto f^i(a) + \text{Im}\left(d^{i-1}\right).
\end{aligned}
$$

This is well defined group homomorphism. Further, it is clear that this is functorial. $\qquad\square$

With this, we obtain the cohomology long-exact sequence.

Lemma 21.2.1.3. *Let* $\mathbf{A}$ *be an abelian category and* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *a short-exact sequence in* $\mathrm{coCh}(\mathbf{A})$. *Then there is a map* $\delta^i : h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ *for each* $i \in \mathbb{Z}$ *such that the following is a long exact sequence*

$$
\begin{array}{ccccc}
h^i(A^\bullet) & \longrightarrow & h^i(B^\bullet) & \longrightarrow & h^i(C^\bullet) \\
 & & \delta^i & & \\
h^{i+1}(A^\bullet) & \longrightarrow & h^{i+1}(B^\bullet) & \longrightarrow & h^{i+1}(C^\bullet)
\end{array}
$$

Proof. (*Sketch*) The proof relies on chasing an element $c \in \mathrm{Ker}(d)$ of C^i till we obtain an element $a \in A^{i+1}$ in $\mathrm{Ker}(d)$, in the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A^{i-1} & \longrightarrow & B^{i-1} & \longrightarrow & C^{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & A^{i} & \longrightarrow & B^{i} & \longrightarrow & C^{i} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & A^{i+1} & \longrightarrow & B^{i+1} & \longrightarrow & C^{i+1} & \longrightarrow & 0
\end{array}
$$

The chase is straightforward and is thus omitted. The resultant map is indeed a well-defined group homomorphism. $\qquad\square$

We now define homotopy of maps of complexes

Definition 21.2.1.4. (**Homotopy between maps**) Let $\mathbf{A}$ be an abelian category and $f, g : A^\bullet \to B^\bullet$ be two maps of cochain complexes. Then a homotopy between f and g is defined to be a collection of maps $k := \{k^i : A^i \to B^{i-1}\}_{i \in \mathbb{Z}}$ such that $f^i - g^i = dk^i + k^{i+1}d$ for each $i \in \mathbb{Z}$:

$$
\begin{array}{ccc}
 & A^i \xrightarrow{\ d\ } A^{i+1} & \\
{\scriptstyle k^i}\swarrow \ {\scriptstyle g^i}\downarrow\ {\scriptstyle f^i}\downarrow & & \swarrow{\scriptstyle k^{i+1}} \\
B^{i-1} \xrightarrow[\ d\ ]{} B^i & &
\end{array}
$$

As one might expect, homotopic maps induces *same* (not isomorphic, but actually same) maps on cohomology.

Lemma 21.2.1.5. *Let* $\mathbf{A}$ *be an abelian category and* $f, g : A^\bullet \to B^\bullet$ *be two maps of cochain complexes. If* $k : f \sim g$ *is a homotopy between* f *and* g, *then* $h^i(f) = h^i(g)$ *as maps* $h^i(A^\bullet) \to h^i(B^\bullet)$ *for all* $i \in \mathbb{Z}$.

Proof. (*Sketch*) Pick any $a \in \mathrm{Ker}(d)$ in A^i. We wish to show that $f^i(a) - g^i(a) \in \mathrm{Im}(d)$. This follows from unravelling the definition of homotopy $k : f \sim g$. $\qquad\square$

We now define the notion of exact functors between two abelian categories.

Definition 21.2.1.6. (**Exactness of functors**) Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories. A functor $F : \mathbf{A} \to \mathbf{B}$ is said to be

1. *additive* if the map $\mathrm{Hom}_{\mathbf{A}}(A, B) \to \mathrm{Hom}_{\mathbf{B}}(FA, FB)$ is a group homomorphism,
2. *left exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $0 \to FA' \to FA \to FA''$ is exact,
3. *right exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $FA' \to FA \to FA'' \to 0$ is exact,
4. *exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $0 \to FA' \to FA \to FA'' \to 0$ is exact,
5. *exact at middle* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $FA' \to FA \to FA''$ is exact.

Remark 21.2.1.7. It is important to keep in mind that all the above definitions are made for *short* exact sequences; a left exact A functor may not map a long exact sequence $0 \to A_1 \to \dots$ to a long exact sequence $0 \to FA_1 \to \dots$.

There are two prototypical examples of such functors in the category of R-modules.

Example 21.2.1.8. $(- \otimes_R M$ and $\mathrm{Hom}_R(M, -))$ Let R be a commutative ring and M be an R-module It is a trivial matter to see that the functor $- \otimes_R M : \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ is right exact but not left exact as applying $- \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ on the following shows where $\gcd(n, m) = 1$:

$$0 \to n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Indeed, $n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is not injective as the former is an infinite ring whereas the latter is finite.

Consider the covariant hom-functor $\mathrm{Hom}_R(M, -) : \mathbf{Mod}(R) \to \mathbf{Mod}(R)$. This can easily be seen to be left exact. This is not right exact as applying $- \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ to the above exact sequence would yield (note that $\mathrm{Hom}_R(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$).

We now study the dual notion of homology, without much change. **TODO.**

21.2.2 Resolutions

We begin with injective objects, resolutions and having enough injectives.

Definition 21.2.2.1. (**Injective objects and resolutions**) Let $\mathbf{A}$ be an abelian category. An object $I \in \mathbf{A}$ is said to be injective if the functor thus represented, $\mathrm{Hom}_{\mathbf{A}}(-, I) : \mathbf{A}^{\mathrm{op}} \to \mathbf{AbGrp}$ is exact. An injective resolution of an object $A \in \mathbf{A}$ is an exact cochain complex

$$A \xrightarrow{\epsilon} I^0 \to I^1 \to \dots$$

where each I^i is an injective object. We denote an injective resolution of A by $\epsilon : A \to I^{\bullet}$.

We see that any two injective resolutions of an object are homotopy equivalent.

Lemma 21.2.2.2. *Let* **A** *be an abelian category and* $A \in \mathbf{A}$ *be an object with two injective resolutions* $\epsilon : A \to I^\bullet$ *and* $\eta : A \to J^\bullet$. *Then there exists a homotopy* $k : \epsilon \sim \eta$.

Proof. Comparison Theorem 2.3.7, pp 40, [cite Weibel Homological Algebra]. □

We then define when an abelian category has enough injectives.

Definition 21.2.2.3. (**Enough injectives**) An abelian category **A** is said to have enough injectives if for each object $A \in \mathbf{A}$, there is an injective object $I \in \mathbf{A}$ such that A is a subobject of I, $A \leq I$.

In such abelian categories, all objects have injective resolutions.

Lemma 21.2.2.4. *Let* **A** *be an abelian category with enough injectives. Then all objects* $A \in \mathbf{A}$ *admit injective resolutions* $\epsilon : A \to I^\bullet$.

Proof. Exercise 2.3.5 and Corollary 2.3.11, pp 42, [cite Weibel Homological Algebra]. □

We next do projective objects, resolutions and having enough projectives. **TODO**.

21.2.3   Derived functors

For each covariant left exact functor $F : \mathbf{A} \to \mathbf{B}$ between abelian categories, we will produce a sequence of functors $R^i F$ for each $i \geq 0$.

Definition 21.2.3.1. (**Right derived functors of a left-exact functor**) Let $F : \mathbf{A} \to \mathbf{B}$ be a left exact functor of abelian categories where **A** has enough injectives. Then, define for each $i \geq 0$ the following

$$R^i F : \mathbf{A} \longrightarrow \mathbf{B}$$
$$A \longmapsto h^i(F(I^\bullet))$$

where $\epsilon : A \to I^\bullet$ is any injective resolution of A. We call $R^i F$ the ith right derived functor of the left exact functor F.

Remark 21.2.3.2. Indeed the above definition is well-defined, by Lemmas 21.2.1.5 and 21.2.2.2. Further, keep in mind the Remark 21.2.1.7.

The most important fact about right derived functors is that they are equivalent to the datum of what will be defined as universal δ-functors. In the rest of this section we setup the definitions and only state the result.

We collect what is most important about $R^i F$ in the following result.

the remaining
n universal δ-
rs, Chapter 21.

21.3   Filtrations

21.4   Gradings

21.5 Spectral sequences

Do from Weibel Chapter 5, Eisenbud Appendix 3 and Stacksproject.

Chapter 22

Foundational Sheaf Theory

Contents

The notion of sheaves plays perhaps the most important role in modern viewpoint of geometry. It is thus important to understand the various constructions that one can make on them. We assume the reader knows the definition of a sheaf on a space X and morphism of sheaves. We begin with some recollections.

Recall that a map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on X defines for each point $x \in X$ a map of stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ given by $s_x \mapsto \varphi_U(s)_x$ where s is a section of $\mathcal{F}$ over $U \subseteq X$. One can check quite easily that this is well-defined and that this map φ_x is in-fact the unique map given by the universal property of the colimit in the diagram below:

$$
\begin{array}{ccc}
\mathcal{G}(U) & \longrightarrow & \varinjlim_{V \ni x} \mathcal{G}(V) \\
{\scriptstyle \varphi_U} \uparrow & & \uparrow {\scriptstyle \varphi_x} \\
\mathcal{F}(U) & \longrightarrow & \varinjlim_{V \ni x} \mathcal{F}(V)
\end{array} \quad .
$$

Hence φ_x is the unique map which makes the above diagram commute.

Note that $\mathcal{F} \hookrightarrow \mathcal{G}$ is a subsheaf if $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ such that for $U \hookrightarrow V$, the restriction map $\rho_{V,U} : \mathcal{G}(V) \to \mathcal{G}(U)$ restricts to $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$.

For an abelian group A and a space X, one defines the constant sheaf $\mathcal{A}$ as the sheaf which for each open set $U \subseteq X$ assigns $\mathcal{A}(U) = \{s : U \to A \mid s \text{ is continuous}\}$, where A is

given the discrete topology. One sees instantly that this is a sheaf. Further one observes that if $U = U_1 \amalg \cdots \amalg U_k$ where U_i are components of open set U, then $\mathcal{A}(U) \cong A \oplus \cdots \oplus A$ k-times.

We now begin by showing how to construct a sheaf out of a presheaf over X.

22.1   The sheafification functor

Let X be a topological space, denote the category of presheaves on X by $\mathbf{PSh}(X)$ and denote the category of sheaves over X by $\mathbf{Sh}(X)$. We have a canonical inclusion functor $i : \mathbf{PSh}(X) \hookrightarrow \mathbf{Sh}(X)$. We construct it's left adjoint commonly known as the process of sheafifying a presheaf.

Theorem 22.1.0.1. *(Sheafification) Let X be a topological space and let F be a presheaf over X. Then there exists a pair $(\mathcal{F}, i_F)$ of a sheaf $\mathcal{F}$ and a map $i_F : F \to \mathcal{F}$ such that for any sheaf $\mathcal{G}$ and a morphism of presheaves $\varphi : F \to \mathcal{G}$, there exists a unique morphism of sheaves $\tilde{\varphi} : \mathcal{F} \to \mathcal{G}$ such that the following commutes*

$$
\begin{array}{ccc}
\mathcal{F} & \dashrightarrow^{\tilde{\varphi}} & \mathcal{G} \\
{\scriptstyle i_F}\uparrow & \nearrow_{\varphi} & \\
F & &
\end{array}
\quad ,
$$

that is, we have a natural bijection

$$
\mathrm{Hom}_{\mathbf{PSh}(X)}(F, \mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, \mathcal{G}).
$$

Moreover:

1. *$(\mathcal{F}, i_F)$ is unique upto unique isomorphism.*
2. *For every $x \in X$, the map on stalks $i_{F,x} : F_x \to \mathcal{F}_x$ is bijective.*
3. *For any map of presheaves $\varphi : F \to G$, we get a map of sheaves $\tilde{\varphi} : \mathcal{F} \to \mathcal{G}$ such that the natural square commutes:*

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\tilde{\varphi}} & \mathcal{G} \\
{\scriptstyle i_F}\uparrow & & \uparrow{\scriptstyle i_G} \\
F & \xrightarrow{\varphi} & G
\end{array}
\quad .
$$

Hence we have a functor

$$
(-)^{++} : \mathbf{PSh}(X) \longrightarrow \mathbf{Sh}(X)
$$
$$
F \longmapsto F^{++} := \mathcal{F}.
$$

Proof. We explicitly construct the sheaf $\mathcal{F}$ out of F. We define the local sections of $\mathcal{F}$ by using germs and turning the gluing condition of sheaf definition onto itself. In particular, define

$$
\mathcal{F}(U) := \left\{ ((s_{i_x})_x) \in \prod_{x \in U} F_x \;\middle|\; \forall x \in U,\ \exists \text{ open } W \ni x \ \&\ t \in F(W) \text{ s.t. } \forall p \in W,\ t_p = (s_{i_p})_p \right\}.
$$

The restriction map for $U \hookrightarrow V$ of $\mathcal{F}$ is given by $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, $((s_{i_x})_x) \mapsto ((s_{i_x})_x)_{x \in U}$, that is, $\rho_{V,U}$ is just the projection map. Next, we show that $\mathcal{F}$ satisfies the gluing criterion and that is where we will see how the above definition of sections of $\mathcal{F}$ came about. Take an open set $U \subseteq X$ and an open cover $U = \bigcup_{i \in I} U_i$. Let $s_i \in \mathcal{F}(U_i)$ be a corresponding collection of sections such that for all $i, j \in I$, we have $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$. We wish to thus construct a section $t \in \mathcal{F}(U)$ such that $\rho_{U, U_i}(t) = s_i$ for all $i \in I$. Indeed let $((t_{i_x})_x) \in \prod_{x \in U} F_x$ where $t := (t_{i_x})_x = (s_i)_x$ if $x \in U_i$. Then since for any $x \in U$, there exists $U \supseteq U_i \ni x$ and $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U, U_i}(t) = ((t_{i_x})_x)_{x \in U_i} = ((s_i)_x)_{x \in U_i}$, we thus conclude that $t \in \mathcal{F}(U)$. So $\mathcal{F}$ satisfies the gluing condition. The locality is quite simple. Next the map i_F is given as follows on sections:

$$i_{F,U} : F(U) \longrightarrow \mathcal{F}(U)$$
$$s \longmapsto (s_x).$$

Now, it can be seen by definition of colimits that $\mathcal{F}_x = F_x$. Finally, let $\mathcal{G}$ be a sheaf and let $\varphi : F \to \mathcal{G}$ be a morphism of presheaves, then we can define $\tilde{\varphi}$ by gluing the germs as follows:

$$\tilde{\varphi}_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$
$$((s_{i_x})_x) \longmapsto [\varphi_{W_x}(s_{i_x})]$$

where $[\varphi_{W_x}(s_{i_x})]$ denotes the unique section in $\mathcal{G}(U)$ that one gets by considering the open cover $\bigcup_{x \in U} W_x$ where $s_{i_x} \in \mathcal{F}(W_x)$ and considering the gluing of corresponding sections $\varphi_{W_x}(s_{i_x}) \in \mathcal{G}(W_x)$. These sections agree on intersections because φ is a natural transformation and (s_{i_x}) agree on intersections as sections of $\mathcal{F}(U)$. Hence we have the unique map $\tilde{\varphi}$. Moreover, it is clear that $\tilde{\varphi} \circ i_F = \varphi$. $\square$

Corollary 22.1.0.2. *Let F be a presheaf over a topological space X, then for all $x \in X$, $F_x = (F^{++})_x$.*

Proof. By construction of F^{++}. $\square$

Corollary 22.1.0.3. *If $\mathcal{F}$ is a sheaf over a topological space X, then $\mathcal{F}^{++} = \mathcal{F}$.*

Proof. Follows immediately from the universal property of the sheafification, Theorem 22.1.0.1. $\square$

Remark 22.1.0.4. The sections of sheaf $\mathcal{F}$ in an open set U containing x is defined in such a manner so that $f \in \mathcal{F}(U)$ can be constructed locally out of sections of F. In particular, we can write $\mathcal{F}(U)$ more clearly as follows

$$\mathcal{F}(U) = \left\{ s : U \to \coprod_{x \in U} F_x \mid \forall x \in U,\ s(x) \in F_x\ \&\ \exists\ \text{open } x \in V \subseteq U\ \&\ \exists t \in F(V)\ \text{s.t.}\ s(y) = t_y\ \forall y \in V \right\}.$$

Note that this is exactly the realization that $\mathcal{F}(U)$ is the set of section of the étale space of the sheaf $\mathcal{F}$ (see Section 22.3). Most of the time in practice, we would work with the universal property of $\mathcal{F}$ in Theorem 22.1.0.1 as it is much more amenable, but the above must be kept in mind as it is used, for example, to make sure that certain algebraic constructions of $\mathcal{O}_X$-modules remains $\mathcal{O}_X$-modules (no matter how trivial they may sound).

22.2   Morphisms of sheaves

All sheaves are abelian sheaves in this section. One of the most important aspects of using sheaves is that the injectivity and bijectivity of φ_x can be checked on sections. We first show that taking stalks is functorial

Lemma 22.2.0.1. *Let X be a topological space, $\mathcal{F}, \mathcal{G}$ be two sheaves over X and $x \in X$ be a point. Then the following mapping is functorial:*

$$\mathbf{Sh}(X) \longrightarrow \mathbf{AbGrp}$$
$$\mathcal{F} \longmapsto \mathcal{F}_x$$
$$\mathcal{F} \xrightarrow{f} \mathcal{G} \longmapsto \mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x.$$

Proof. Immediate, just remember how composition of two natural transforms is defined. $\square$

Another simple lemma about sheaves and stalks is that equality of two sections can be checked at the stalk level.

Lemma 22.2.0.2. *Let X be a topological space and $\mathcal{F}$ be a sheaf over X. If $s, t \in \mathcal{F}(U)$ for some open $U \subseteq X$ such that $(U, s)_x = (U, t)_x \ \forall x \in U$, then $s = t$ in $\mathcal{F}(U)$.*

Proof. By equality on stalks, it follows that we have an open set $W_x \ni x$ in U for all $x \in U$ such that $\rho_{U,W_x}(s) = \rho_{U,W_x}(t)$. The result follows from the unique gluing property of sheaf $\mathcal{F}$. $\square$

The above result therefore show why almost all the time it is enough to work with stalks in geometry. Let us now define an injective and surjective map of sheaves.

Definition 22.2.0.3. (**Injective & surjective maps**) Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves on X. A map of sheaves $f : \mathcal{F} \to \mathcal{G}$ is said to be
  1. *injective* if for all opens $U \subseteq X$, the local homomorphism $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective,
  2. *surjective* if for all opens $U \subseteq X$ and all $s \in \mathcal{G}(U)$, there exists an open covering $\{U_i\}_{i \in I}$ such that $\rho_{U,U_i}(s) \in \mathrm{Im}\,(f_{U_i})$,
  3. *bijective* if f is injective and surjective.

Heuristically, one may understand the notion of f being surjection by saying that every local section of $\mathcal{G}$ is locally constructible by the image of $\mathcal{F}$ under the map f.

For each map of sheaves, we can also define two corresponding sheaves which are global algebraic analogues of the local algebraic constructions.

Definition 22.2.0.4. (**Quotient sheaf**) Let X be a topological space and $\mathcal{F}$ be a sheaf on X. For a subsheaf $\mathcal{S} \subseteq \mathcal{F}$, one defines the quotient sheaf $\mathcal{F}/\mathcal{S}$ as the sheafification of the presheaf F/S defined on open sets $U \subseteq X$ by

$$F/S(U) := \mathcal{F}(U)/\mathcal{S}(U).$$

Definition 22.2.0.5. (Image & kernel sheaves) Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves over X and $f : \mathcal{F} \to \mathcal{G}$ be a morphism. Then,

1. *image sheaf* is the sheafification of the presheaf $\operatorname{Im}(f)$ defined on open sets $U \subseteq X$ by

$$(\operatorname{Im}(f))(U) := \operatorname{Im}(f_U),$$

and we denote it by the same symbol, $\operatorname{Im}(f)$,

2. *kernel sheaf* is the sheafification of the presheaf $\operatorname{Ker}(f)$ defined on open sets $U \subseteq X$ by

$$(\operatorname{Ker}(f))(U) := \operatorname{Ker}(f_U)$$

and we denote it by the same symbol, $\operatorname{Ker}(f)$.

In both the above definitions, the important aspect is the sheafification of the canonical presheaves.

The main point is that one can check all the three notions introduced in Definition 22.2.0.3 for $f : \mathcal{F} \to \mathcal{G}$ by checking on stalks $f_x : \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$.

Theorem 22.2.0.6. [1] *Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves over X. Then, a map $f : \mathcal{F} \to \mathcal{G}$ is*

1. *injective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$,*
2. *surjective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$,*
3. *bijective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective for all $x \in X$,*
4. *an isomorphism if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective for all $x \in X$[2].*
5. *an isomorphism if and only if $f : \mathcal{F} \to \mathcal{G}$ is bijective.*

Proof. The proof is more of an exercise to get a familiarity with the flexibility of sheaf language. The main idea almost everywhere is to do some local calculations and use sheaf axioms to construct a unique section out of local sections.

1. (L $\Rightarrow$ R) We wish to show that f_x is injective. Suppose for two $(U, s)_x, (V, t)_y \in \mathcal{F}_x$ we have $f_x((U, s)_x) = f_x((V, t)_x) \in \mathcal{G}_x$, which translates to $(U, f_U(s))_x = (V, f_U(t))_x$. We wish to show that $(U, s)_x = (V, t)_y$. By definition of equality on stalks, we obtain open $W \subseteq U \cap V$ containing x such that

$$\rho_{U,W}(f_U(s)) = \rho_{V,W}(f_V(t)).$$

By the fact that f is a natural transformation, we further translate the above equality to

$$f_W(\rho_{U,W}(s)) = f_W(\rho_{V,W}(t)).$$

---

[1] Exercise II.1.2, II.1.3 and II.1.5 of Hartshorne.

[2] In general, we should write "*... if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism*", but since we are in the setting of abelian sheaves and bijective homomorphism of abelian groups is an isomorphism, so we can get away with this.

By injectivity of homomorphism f_W, we obtain

$$\rho_{U,W}(s) = \rho_{V,W}(t)$$

in $\mathcal{F}(W)$. Hence by the definition of equality on stalks, we obtain $(U,s)_x = (V,t)_x$.

(R $\Rightarrow$ L) Pick any open $U \subseteq X$. We wish to show that $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. Let $s \in \mathcal{F}(U)$ be such that $f_U(s) = 0$. Thus for all $x \in U$, we have $(U, f_U(s))_x = 0$. Further, by definition of the map f_x, we obtain $f_x((U,s))_x = (U, f_U(s))_x = 0$. By injectivity of f_x, we obtain $(U,s)_x = 0$ for all $x \in U^3$. By definition of equality on stalks, we obtain an open cover $\{W_x\}_{x \in U}$ such that $x \in W_x$ and $s|_{W_x} := \rho_{U,W_x}(s) = 0$. Since f is a natural transformation, we therefore obtain that $\{s|_{W_x}\}_{x \in U}$ is a matching family, i.e. on intersections of W_x, W_y, the corresponding sections agree. Hence, there is a unique glue of $\{s|_{W_x}\}_{x \in U}$ denote $t \in \mathcal{F}(U)$. Since each $s|_{W_x} = 0$, therefore we have two glues of the family over U, one is 0 and the other is s. By uniqueness of the glue, it follows that $s = 0$.

2. (L $\Rightarrow$ R) Pick any $x \in X$. We wish to show that $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective. Pick any $(V,t)_x \in \mathcal{G}_x$. We wish to show that for some open $U \ni x$, we have $(U,s)_x \in \mathcal{F}_x$ such that

$$(V,t)_x = (U, f_U(s))_x.$$

Since $t \in \mathcal{G}(V)$, therefore by surjectivity of f that there exists an open cover $\{V_i\}_{i \in I}$ of V such that

$$\rho_{V,V_i}(t) \in \mathrm{Im}\,(f_{V_i}).$$

Therefore we may pick $s_i \in \mathcal{F}(V_i)$ such that

$$
\begin{aligned}
\rho_{V,V_i}(t) &= f_{V_i}(s_i) \\
&= f_{V_i}(\rho_{V_i,V_i}(s_i)) \\
&= \rho_{V_i,V_i}(f_{V_i}(s_i)).
\end{aligned}
$$

Thus, $(V,t)_x$ and $(V_i, f_{V_i}(s_i))_x$ are same.

(R $\Rightarrow$ L) We wish to show that $f : \mathcal{F} \to \mathcal{G}$ is surjective. Pick any open set $V \subseteq X$ and $t \in \mathcal{G}(V)$. We wish to find an open cover $\{W_i\}$ of V such that $s_i \in \mathcal{F}(V_i)$ and $f_{V_i}(s_i) = \rho_{V,V_i}(t)$. Since we have $(V,t)_x \in \mathcal{G}_x$ for all $x \in V$, therefore by surjectivity of each $f_x : \mathcal{F}_x \to \mathcal{G}_x$, we obtain germs $(W_x, s_x)_x \in \mathcal{F}_x$ such that $(W_x, f_{W_x}(s_x))_x = (V,t)_x$ for all $x \in V$. By shrinking W_x and restricting s_x, we may assume $\{W_x\}$ covers V. Thus we have an open cover of V such that for all $s_x \in \mathcal{F}(W_x)$, we have $f_{W_x}(s_x) = \rho_{V,W_x}(t)$.

3. Trivially follows from 1. and 2.

4. (L $\Rightarrow$ R) Use the fact that taking stalks is a functor (Lemma 22.2.0.1).

   (R $\Rightarrow$ L) Let $g_x : \mathcal{G}_x \to \mathcal{F}_x$ be the inverse homomorphism of f_x for each $x \in X$. Using

---

^{3}We could be done right here by Lemma 22.2.0.2.

g_x, we can easily construct a sheaf homorphism $g : \mathcal{G} \to \mathcal{F}$ which will be the inverse of f. Indeed, consider the following map for any open $U \subseteq X$

$$g_U : \mathcal{G}(U) \longrightarrow \mathcal{F}(U)$$
$$t \longmapsto s$$

where $s \in \mathcal{F}(U)$ is formed as the unique glue of the matching family

$$\{s_x \in \mathcal{F}(U_x)\}_{x \in U}$$

where $(U, t)_x = (U_x, f_{U_x}(s_x))_x$ for each $x \in U$ and $U_x \subseteq U$. In particular, $s_x = g_x((U_x, \rho_{U,U_x}(t))_x)$. This is obtained via the bijectivity of f_x. Consequently, g is a sheaf homomorphism, which is naturally the inverse of f.

5. Follows from 3. and 4.

$\square$

The following theorem further tells us that our intuition about algebra can be globalized, and equality of sheaf morphisms can be checked on each stalk.

Theorem 22.2.0.7. *Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves over X. Then, a map $f : \mathcal{F} \to \mathcal{G}$*

1. is injective if and only if the kernel sheaf $\mathrm{Ker}\,(f)$ is the zero sheaf,
2. is surjective if and only if the image sheaf $\mathrm{Im}\,(f)$ is $\mathcal{G}$,
3. is equal to another map $g : \mathcal{F} \to \mathcal{G}$ if and only if $f_x = g_x$ for all $x \in X$.

Proof. The main idea in most of the proofs below is to either use the definition or the universal property of sheafification.

1. (L $\Rightarrow$ R) Let $f : \mathcal{F} \to \mathcal{G}$ be injective. We wish to show that $\mathrm{Ker}\,(f) = 0$. Since the kernel presheaf $\ker f = 0$, therefore its sheafification $\mathrm{Ker}\,(f) = 0$.
   (R $\Rightarrow$ L) Let $\mathrm{Ker}\,(f) = 0$. We wish to show that f is injective. Suppose to the contrary that f is not injective. We have that $(\mathrm{Ker}\,(f))_x = 0$ for all $x \in X$. Thus there exists an open set $U \subseteq X$ such that $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is not injective. Hence, there exists, $0 \neq s \in \mathcal{F}(U)$ such that $f_U(s) = 0$. Thus, we have an element $(U, s)_x \in (\ker f)_x = (\mathrm{Ker}\,(f))_x = 0$ for all $x \in U$. Hence $s = 0$ by Lemma 22.2.0.2, which is a contradiction.

2. (L $\Rightarrow$ R) Let $f : \mathcal{F} \to \mathcal{G}$ be a surjective map. In order to show that $\mathrm{Im}\,(f) = \mathcal{G}$, we will show that $\mathcal{G}$ satisfies the universal property of sheafification (Theorem 22.1.0.1). For this, consider a sheaf $\mathcal{H}$ and a presheaf map $h : \mathrm{im}\,(f) \to \mathcal{H}$. Consider the inclusion map $\iota : \mathrm{im}\,(f) \hookrightarrow \mathcal{G}$. We will construct a unique sheaf map $\tilde{h} : \mathcal{G} \to \mathcal{H}$ which will be natural such that $\tilde{h} \circ \iota = h$. Pick any open set $U \subseteq X$. We wish to define the map

$$\tilde{h}_U : \mathcal{G}(U) \longrightarrow \mathcal{H}(U).$$

Take $t \in \mathcal{G}(U)$. By surjectivity of f, there exists a covering $\{U_i\}$ of U and matching family $s_i \in \mathcal{F}(U_i)$ for all i such that

$$f_{U_i}(s_i) = \rho_{U,U_i}(t) =: t_i.$$

We shall construct an element $\tilde{h}_U(t) \in \mathcal{H}(U)$. Indeed, we first claim that

$$\{h_{U_i}(t_i) \in \mathcal{H}(U_i)\}_i$$

is a matching family. This can be shown by keeping the following diagram in mind and the fact that $\{s_i\}$ is a matching family:

$$
\begin{array}{ccccc}
\mathcal{H}(U_i) & \xleftarrow{\ h_{U_i}\ } & \mathrm{im}\,(f_{U_i}) & \xleftarrow{\ f_{U_i}\ } & \mathcal{F}(U_i) \\
{\scriptstyle \rho_{U_i,U_i\cap U_j}}\big\downarrow & & {\scriptstyle \rho_{U_i,U_i\cap U_j}}\big\downarrow & & {\scriptstyle \rho_{U_i,U_i\cap U_j}}\big\downarrow \;. \\
\mathcal{H}(U_i\cap U_j) & \xleftarrow[\ \tilde{h}_{U_i\cap U_j}\ ]{} & \mathrm{im}\,(f_{U_i\cap U_j}) & \xleftarrow[\ f_{U_i\cap U_j}\ ]{} & \mathcal{F}(U_i\cap U_j)
\end{array}
$$

Thus we get a unique glue which we define to be the image of $\tilde{h}_U$ for the section $t \in \mathcal{G}(U)$, denoted $\tilde{h}_U(t) \in \mathcal{H}(U)$. Uniqueness and naturality follows from construction.
(R $\Rightarrow$ L) We have that $(\mathrm{im}\,(f))^{++} = \mathcal{G}$. Pick any open set $U \subseteq X$ and a section $t \in \mathcal{G}$. We wish to find an open cover $\{U_i\}_{i\in I}$ of U and $s_i \in \mathcal{F}(U_i)$ such that $f_{U_i}(s_i) = \rho_{U,U_i}(t)$ for all $i \in I$. Indeed, by Corollary 22.1.0.2, we obtain that $\mathcal{G}_x = \mathrm{im}\,(f)_x$ for all $x \in X$. Hence for the chosen (U,t), we obtain for each $x \in U$, by appropriately shrinking and restricting, an open set $W_x \subseteq U$ containing x and a section $s_x \in \mathcal{F}(W_x)$ satisfying $\rho_{U,W_x}(t) = f_{W_x}(s_x)$.

3. (L $\Rightarrow$ R) Trivial.
(R $\Rightarrow$ L) Suppose for all $x \in X$ we have $f_x = g_x : \mathcal{F}_x \to \mathcal{G}_x$. We wish to show that $f = g$. Pick an open set $U \subseteq X$ and consider $s \in \mathcal{F}(U)$. We wish to show that $f_U(s) = g_U(s)$. For each $x \in U$, we have $(U,s)_x \in \mathcal{F}_x$ and by the fact that $f_x = g_x$, we further have

$$(U, f_U(s))_x = (U, g_U(s))_x.$$

Hence for all $x \in U$, there exists open $x \in W_x \subseteq U$ such that

$$\rho_{U,W_x}(f_U(s)) = \rho_{U,W_x}(g_U(s)).$$

It is then an easy observation that both $\{\rho_{U,W_x}(g_U(s))\}_{x\in U}$ and $\{\rho_{U,W_x}(f_U(s))\}_{x\in U}$ forms the same matching family. Hence we have a unique glue by sheaf axiom of $\mathcal{G}$ to obtain $f_U(s) = g_U(s)$ in $\mathcal{G}(U)$.

$\square$

Lemma 22.2.0.8. *Let X be a topological space. Then, the following are equivalent:*
1. *The following is an exact sequence of sheaves over X*

$$\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}'',$$

that is, $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$.
2. *The following is an exact sequence of stalks for each $x \in X$*

$$\mathcal{F}'_x \xrightarrow{f_x} \mathcal{F}_x \xrightarrow{g_x} \mathcal{F}''_x.$$

Proof. (1. ⇒ 2.) Pick any $(U, s)_x \in \mathcal{F}_x$ which is in $\ker g_x$. Thus, there exists $V \subseteq U$ open such that $\rho_{U,V}(g_U(s)) = g_V(\rho_{U,V}(s)) = 0$. Thus $\rho_{U,V}(s) \in \mathcal{F}(V)$ is in $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$ and thus $(V, \rho_{U,V}(s))_x = (U, s)_x \in \mathcal{F}_x$ is in $\mathrm{im}\,(f_x)$. Conversely, for $(U, f_x(t))_x \in \mathrm{im}\,(f_x)$, we see that since $g \circ f = 0$, then $(U, g_x(f_x(t)))_x = 0$.

(2. ⇒ 1.) This is immediate, by looking at a section of $\mathcal{F}$ at any open set (use Remark 22.1.0.4). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

22.3 Sheaves are étale spaces

Another important and in some sense dual viewpoint of sheaves over X is that they can be equivalently defined as a certain type of bundle over X and all such bundles arises only from a sheaf. This is important because this viewpoint naturally extends the usual concepts of covering spaces, bundles and vector bundles to that of sheaves. In particular, a lot of classical constructs in algebraic topology can be equivalently be seen as specific instantiates of the notion of étale space of the sheaf.

Definition 22.3.0.1. (**Étale space**) Let X be a topological space and let $\pi : E \to X$ be a bundle over X. Then (E, π, X) is said to be étale over X or just étale if for all $e \in E$, there exists an open set $V \ni e$ of E such that $p(V)$ is open and $p|_V : V \to p(V)$ is a homeomorphism, that is, if p is a local homeomorphism. A morphism of étale spaces $(E_1, \pi_1, X), (E_2, \pi_2, X)$ over X is given by a continuous map $f : E_1 \to E_2$ such that $\pi_2 \circ f = \pi_1$. Denote the category of étale spaces over X by $\mathbf{\acute{E}t}/X$.

Clearly, covering spaces over X are étale spaces over X, but not all étale spaces over X are covering, of-course. We now wish to show that the sheafification functor factors through a functor mapping a presheaf to an étale space. In particular, we want to show the existence of functor F, G so that the following commutes

$$\begin{array}{ccc}
 & \mathbf{\acute{E}t}/X & \\
{\scriptstyle G}\nearrow & & \searrow{\scriptstyle F} \\
\mathbf{PSh}(X) & \underset{\xleftarrow{\hspace{2cm}}}{\overset{(-)^{++}}{\xrightarrow{\hspace{2cm}}}} & \mathbf{Sh}(X)
\end{array} \quad .$$

Construction 22.3.0.2. (*Étale space of a sheaf*) Let us now show the construction of the above functors:

1. (*The functor G*) Let P be a presheaf over X. The étale space $E := G(P)$ is given by the disjoint union of all stalks:

$$E := \coprod_{x \in X} P_x.$$

The topology on E is given by the initial topology of the map

$$\pi : E \longrightarrow X$$
$$s_x \longmapsto x.$$

In particular, E has a basis given by sets of the form $B_{U,s} \subseteq E$ where $B_{U,s} = \{s_x \in E \mid x \in U\}$ and $s \in P(U)$. Next, we wish to establish that π is a local homeomorphism. So take any $s_x \in E$ and consider the basic open set $B_{U,s} \ni s_x$. The map $\pi|_{B_{U,s}} : B_{U,s} \to \pi(B_{U,s})$ takes $s_x \mapsto x$. This is a homeomorphism because we can construct an inverse given by $x \mapsto s_x$. A simple calculation checks that this is continuous. Hence indeed, (E, π, X) is an étale space over X.

Next consider a map of presheaves $\varphi : F \to G$. We can construct a map of corresponding étale spaces as

$$\hat{\varphi} : (E_F, \pi_F, X) \longrightarrow (E_G, \pi_G, X)$$
$$s_x \longmapsto \varphi_x(s_x).$$

This map is continuous and a valid bundle map over X. This defines the functor G.

2. (*The functor F*) Let $\pi : E \to X$ be an étale space over X. Then, we can construct a sheaf $\mathcal{E}$ over X out of it. This is done in a very natural way by considering the set of sections over U of $\mathcal{E}$ to be quite literally the set of *cross-sections*[4] of map π on U. That is, define:

$$\mathcal{E}(U) := \{s : U \to E \mid \pi \circ s = \mathrm{id}_U\}.$$

The fact that this is indeed a sheaf can be seen by a general phenomenon that for any continuous map $f : X \to Y$, the set of all cross-sections of f over open subsets of Y assembles itself into a sheaf. Hence, we have constructed a sheaf $\mathcal{E}$ out of an étale space E over X.

Next consider a map of étale spaces $\xi : (E_1, \pi_1, X) \to (E_2, \pi_2, X)$. we can construct a map of corresponding sheaves $\tilde{\xi} : \mathcal{E}_1 \to \mathcal{E}_2$ by defining the following for open $U \subseteq X$:

$$\tilde{\xi}_U : \mathcal{E}_1(U) \longrightarrow \mathcal{E}_2(U)$$
$$s \longmapsto \xi \circ s.$$

One can check that this is indeed a valid sheaf morphism. This defines the functor F.

We then see that the categories $\mathbf{\acute{E}t}/X$ and $\mathbf{Sh}(X)$ are equivalent.

Theorem 22.3.0.3. [5] (*The étale viewpoint of sheaves*) *Let X be a topological space. The functors F and G as defined in Construction 22.3.0.2 defines an equivalence of categories*

$$\mathbf{Sh}(X) \equiv \mathbf{\acute{E}t}/X.$$

We will prove this result in many small lemmas below. We would first like to observe that for any étalé bundle E over X yields a sheaf by $F(E)$ whose stalks are bijective to fibres of E.

---

[4]In-fact, historically the notion of sheaf was really that of this étale space, and that is why to this day, we still use the terminology of "sections" of a sheaf over an open subset.
[5]Exercise II.1.13 of Hartshorne.

Lemma 22.3.0.4. *Let* (E, π, X) *be an étalé bundle over* X *and let* $\mathcal{E}$ *be the sheaf obtained by* $F((E, \pi, X))$. *Then, for any* $x \in X$, *the following is a bijection*

$$\tau_x : \mathcal{E}_x \longrightarrow E_x := \pi^{-1}(x)$$
$$(U, s)_x \longmapsto s(x).$$

Proof. We first show that τ_x is injective. Let $(U, s)_x, (V, t)_x$ be two germs such that $p = s(x) = t(x)$. We wish to show that s and t are equal on an open subset in $U \cap V$. As E is étaleé, therefore we have an open $A \subseteq E$ with $p \in A$ such that $\pi|_A : A \to \pi(A)$ is a homeomorphism. Consequently, we see that the open set $W = \pi(A) \cap U \cap V$ would do just fine.

We now show surjectivity. Pick $e \in E_x$. As E is étalé, we thus get an open set $A \ni e$ in E such that $\pi|_A : A \to \pi(A)$ is a homeomorphism. Denote the inverse of this homeomorphism by $g : \pi(A) \to A$. This is therefore a section of E over $\pi(A)$ where $x \in \pi(A)$. Consequently, $(\pi(A), g)_x \in \mathcal{E}_x$ is such that τ_x maps it to e. $\qquad\square$

Proof of Theorem 22.3.0.3. We first show that $F \circ G$ is naturally isomorphic to sheafification functor. Let $\mathcal{E}$ be a presheaf, $(E, \pi) = G(\mathcal{E})$ and $F(E, \pi) = \mathcal{E}'$. We wish to show that there is a natural isomorphism $\mathcal{E}^{++} \to \mathcal{E}'$. By Theorem 22.1.0.1 and 22.2.0.6, 3, it suffices to show that there is a map of presheaves $\mathcal{E} \to \mathcal{E}'$ which is isomorphism on stalks.

Consider the map $\varphi : \mathcal{E} \to \mathcal{E}'$ which on an open set $U \subseteq X$ gives the following map

$$\varphi_U : \mathcal{E}(U) \longrightarrow \mathcal{E}'(U)$$
$$s \longmapsto U \overset{f_{U,s}}{\to} E$$

where $f_{U,s} : U \to E$ maps as $x \mapsto (U, s)_x$. Since $f_{U,s}$ is just the stalk map, then as in Construction 22.3.0.2, $f_{U,s}$ is continuous. Now on the stalks, we get the following commutative diagram by Lemma 22.3.0.4:

$$\begin{array}{ccc} \mathcal{E}_x & \overset{\varphi_x}{\longrightarrow} & \mathcal{E}'_x \\ \downarrow & \diagdown {\scriptstyle \cong} & \\ E_x & & \end{array} \quad,$$

where the vertical map takes a germ $(U, s)_x$ and maps it to the element represented in $E_x = \mathcal{E}_x$, as $E_x = \pi^{-1}(x) = \{x \in E \mid \pi(e) = x\} = \{(U, s)_y \in E \mid \pi((U, s)_y) = y = x\}$. Consequently, the vertical map is a bijection and thus φ_x is a bijection. The naturality of this isomorphism can be checked trivially.

We now wish to show that $G \circ F$ is naturally isomorphic to the identity functor on $\mathbf{\acute{E}t}/X$. Pick an étalé bundle (E, π) over X, denote $F(E, \pi) = \mathcal{E}$ and $G(\mathcal{E}) = (E', \pi')$. We wish to find a homeomorphism φ so that the following commutes:

$$\begin{array}{ccc} E' & \overset{\varphi}{\longrightarrow} & E \\ {\scriptstyle \pi'} \downarrow & \diagdown {\scriptstyle \pi} & \\ X & & \end{array} \quad.$$

Consider the following map

$$\varphi : E' \longrightarrow E$$
$$(U, s)_x \longmapsto s(x).$$

By Lemma 22.3.0.4, φ is a bijective map. We thus reduce to showing that φ is a continuous open map.

To show continuity, consider an open set $A \subseteq E$ and then observe that

$$\varphi^{-1}(A) = \{(U, s)_x \in E' \mid s(x) \in A\}$$
$$= \{(U, s)_x \in E' \mid x \in s^{-1}(A)\}$$
$$= \bigcup_{U \ni x, s:U \to E} B_{U,s}$$

and since $B_{U,s} \subseteq E'$ is a basic open, therefore φ is continuous.

Finally, to show that φ is open, one reduces to showing that if $s : U \to E$ is a continuous section of bundle (E, π) and $U \subseteq X$ is an open set, then $s(U)$ is an open set in E (by working with a basic open $B_{U,s} \subseteq E'$). This follows from the fact that since π is a local homeomorphism, therefore for each $e \in s(U)$, there exists an open set $A \ni e$ in E such that $s(U) \cap A \ni e$ and since $\pi : s(U) \cap A \to \pi(s(U) \cap A) = U \cap \pi(A)$ is a homeomorphism, we further get that $s(U) \cap A$ is open (as $U \cap \pi(A)$ is open). Consequently, $s(U)$ is open. $\quad\square$

Remark 22.3.0.5. (*The sheaf associated to a covering space*) By the above equivalence, each covering space space over X, which is an étale map, determines a unique sheaf (upto isomorphism). We analyze this sheaf. **TODO**.

22.4   Direct and inverse image

Let $f : X \to Y$ be a continuous map of topological spaces. Then one can derive two functors $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and $f^{-1} : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ which are adjoint of each other, called direct and inverse image functors respectively. While f_* is easy to define, it is usually the inverse image of a sheaf that causes trouble for its obscurity if one works with the definition that inverse image functor is left-adjoint to direct image functor. This is resolved by working with the corresponding étale spaces (Theorem 22.3.0.3). In this section we will show how to construct them.

Let us first define the direct image functor.

Definition 22.4.0.1. (**Direct image**) Let $f : X \to Y$ be a continuous map. Then, for any sheaf $\mathcal{F}$ on X, we can define its direct image under f as $f_*\mathcal{F}$ whose sections on open $V \subseteq Y$ are given by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V)).$$

This can easily be seen to be a sheaf. For any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on X, we can define the map of direct image sheaves as

$$(f_*\varphi)_V : f_*\mathcal{F}(V) \longrightarrow f_*\mathcal{G}(V)$$
$$s \longmapsto \varphi_{f^{-1}(V)}(s).$$

This defines a functor

$$f_* : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Y).$$

One defines the inverse image of a sheaf as follows:

Definition 22.4.0.2. (Inverse image) Let $f : X \to Y$ be a continuous map and let $\mathcal{G}$ be a sheaf over Y. Consider a presheaf F over X constructed by the data of $\mathcal{G}$ as follows. Let $U \subseteq X$ be open, then define

$$f^+\mathcal{G}(U) := \varinjlim_{\text{open } V \supseteq f(U)} \mathcal{G}(V),$$

where restriction maps of $f^+\mathcal{G}$ is given by the unique map obtained by universality of colimits. Then, $f^+\mathcal{G}$ is a presheaf over X and this construction is functorial again by universal property of colimits:

$$f^+ : \mathbf{PSh}(Y) \longrightarrow \mathbf{PSh}(X).$$

Let $f^{-1}\mathcal{G} = (f^+\mathcal{G})^{++}$ denote the sheafification of $f^+\mathcal{G}$. This sheaf is called the inverse sheaf of $\mathcal{G}$ under f. Now for any map of sheaves $\varphi : \mathcal{G} \to \mathcal{H}$ over Y, we get a corresponding map of inverse image sheaves $f^{-1}\varphi : f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{H}$ by composition of two functors. This yields a functor

$$f^{-1} : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(X).$$

As is visible, this definition is quite obscure if one likes elemental definitions. We thus give some general properties enjoyed by inverse sheaf.

Lemma 22.4.0.3. *Let $f : X \to Y$ be a continuous map and $\mathcal{G}$ be a sheaf over Y.*
1. *If f is open, then $f^{-1}\mathcal{G} = \mathcal{G}(f(-))$.*
2. *If f is constant to $y \in Y$, then $f^{-1}\mathcal{G}$ is the constant sheaf on X with sections $\mathcal{G}_y$.*
3. *If $X = \{x\}$ is a singleton space, then $f^{-1}\mathcal{G}$ is the constant sheaf on X with sections $\mathcal{G}_{f(x)}$.*
4. *If $x \in X$, then*

$$(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}.$$

Proof. 1. One notes that $f^+\mathcal{G}(U) := \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \mathcal{G}(f(U))$. The mapping $\mathcal{G}(f(-))$ is a sheaf, hence sheafifying it will yield the same sheaf.
2. We see that $f^+\mathcal{G}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \varinjlim_{V \ni y} \mathcal{G}(V) = \mathcal{G}_y$ and presheaves with constant values are sheaves, as restrictions are identity.

3. We see that $f^+\mathcal{G}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \varinjlim_{V \ni f(x)} \mathcal{G}(V) = \mathcal{G}_{f(x)}$ and presheaves with constant values are sheaves, as restrictions are identity.

4. By passing to the right adjoint, one observes that for $f : X \to Y$ and $g : Y \to Z$ continuous maps, one can obtain the following natural isomorphism of functors

$$(g \circ f)^{-1} \cong f^{-1} \circ g^{-1}.$$

Consider the composite $f \circ \iota$ where $\iota : \{x\} \hookrightarrow X$ is the inclusion map. Consequently, by 3. above, we obtain the following

$$\begin{aligned}
\mathcal{G}_{f(x)} &\cong (f \circ \iota)^{-1}(\mathcal{G})(\{x\}) \\
&\cong (\iota^{-1} \circ f^{-1})(\mathcal{G})(\{x\}) \\
&\cong \iota^{-1}(f^{-1}\mathcal{G})(\{x\}) \\
&\cong (f^{-1}\mathcal{G})_{f(x)}.
\end{aligned}$$

$\square$

The following is a fundamental duality between inverse and direct image functors.

Theorem 22.4.0.4. [6] *(Direct and inverse image adjunction) Let $f : X \to Y$ be a continuous map. Then the the inverse image functor is the left adjoint of direct image functor* [7]

$$\mathbf{Sh}(Y) \xrightarrow[\underset{f_*}{\longleftarrow}]{f^{-1}} \mathbf{Sh}(X) \ .$$

In particular, we have a natural bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}\left(f^{-1}\mathcal{F}, \mathcal{G}\right) \cong \mathrm{Hom}_{\mathbf{Sh}(Y)}\left(\mathcal{F}, f_*\mathcal{G}\right).$$

Proof. We will show the construction of unit and counit and the duality between $\mathcal{F} \to f_*\mathcal{G}$ and $f^{-1}\mathcal{F} \to \mathcal{G}$. $\square$

lete the proof
eorem 22.4.0.4,
er 22.

One situation that we will find ourselves a lot in algebraic geometry is when $f : X \to Y$ will be a closed immersion of topological spaces ($f : X \to f(X)$ is homeomorphism and $f(X) \subseteq Y$ is closed) and for a sheaf $\mathcal{F}$ over X, we would like to find $(f_*\mathcal{F})_{f(x)}$ for each point $x \in X$. This is a situation where the stalk of direct image can be calculated quite easily.

Lemma 22.4.0.5. *Let $f : X \to Y$ will be a closed immersion of topological spaces and $\mathcal{F}$ a sheaf over X. Then, there is a natural isomorphism*

$$(f_*\mathcal{F})_{f(x)} \cong \mathcal{F}_x.$$

Proof. From a straightforward unravelling of definitions of the two stalks, the result follows from the observation that each open set $U \ni x$ in X is in one-to-one correspondence with open set $f(U) \ni f(x)$ in Y. $\square$

---

[6] Exercise II.1.18 of Hartshorne.

[7] admirers of topoi may see this as a quintessential example of geometric map of topoi.

Remark 22.4.0.6. We wish to know how the inverse image of sheaves changes the stalk. Let $f : X \to Y$ be a continuous map and let $\mathcal{F}$ be a sheaf on Y. Consider the inverse sheaf $f^{-1}\mathcal{F}$ on X. Let $x \in X$. Then we have that (Lemma 22.4.0.3, 4)

$$(f^{-1}\mathcal{F})_x \cong \mathcal{F}_{f(x)}.$$

The importance of this is that, suppose $f : X \to Y$ is given together with $\mathcal{F}$ and $\mathcal{G}$ are sheaves over X and Y respectively and a map $\varphi^\flat : \mathcal{G} \to f_*\mathcal{F}$ over Y, which is equivalent to $\varphi^\sharp : f^{-1}\mathcal{G} \to \mathcal{F}$ over X. Now, most of the time, our interest in a sheaf is only limited to stalks (functions defined in *some* open subset around a point), therefore we are mostly interested in considering only the map induced at the level of stalks at a point $f(x) \in Y$:

$$\varphi^\flat_{f(x)} : \mathcal{G}_{f(x)} \longrightarrow (f_*\mathcal{F})_{f(x)}.$$

But the description of the stalk $(f_*\mathcal{F})_{f(x)}$ is usually not simple to derive. But dually, we may ask the map of stalks of the other map at $x \in X$, and we directly land into the stalks

$$\varphi^\sharp_x : \mathcal{G}_{f(x)} \cong (f^{-1}\mathcal{G})_x \longrightarrow \mathcal{F}_x.$$

However, this is a strange map as the stalks are of sheaves which are not on same space. In particular, this map is given as follows. For any open $V \ni f(x)$ in Y, we have the following maps:

$$\mathcal{G}(V) \xrightarrow{\varphi^\flat_V} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}_x \ .$$

Passing to colimits ($\varphi^\flat_V$ commutes with restrictions), one can see that we get the map $\varphi^\sharp_x : \mathcal{G}_{f(x)} \to \mathcal{F}_x$ back.

It is a good principle to keep in mind that if we wish to work with explicit local sections, then we should look for the "flat" map and if it is enough to work with germs, then we should look for the "sharp" map, even though the above remark telling us how to construct the map of stalks from the "flat" maps on each open set.

This map $\varphi^\sharp_x$ can be heuristically be defined as the map which on sections which makes sure that a non-invertible section remains non-invertible after going through the map. Hence we mostly work only with maps $f^{-1}\mathcal{G} \to \mathcal{F}$ if we are interested only at the stalk level (which is more than enough for us).

22.5 Category of abelian sheaves

We will discuss some basic properties of the category of sheaves of abelian groups over X, denoted $\mathbf{Sh}(X)$. This is important as we wish to calculate cohomology of its objects, hence we would require the notion of injective and projective resolutions of sheaves. We covered the homological methods necessary for this section in the Homological Methods, Chapter 21.

The main theorem is that the category of abelian sheaves on a topological space is a Grothendieck-abelian category (Theorem **??**) and thus it has enough injectives.

22.5.1   Direct and inverse limits in $\mathbf{Sh}(X)$

Since Grothendieck-abelian categories have all colimits, therefore it also has direct limits. We now show that the direct limits in $\mathbf{Sh}(X)$ are obtained by sheafifying the corresponding direct limit in $\mathbf{PSh}(X)$.

Lemma 22.5.1.1. [8] *Let X be a topological space and $\{\mathcal{F}_i\}$ be a direct system of sheaves over X. Then, the direct limit $\varinjlim_i \mathcal{F}_i$ in $\mathbf{Sh}(X)$ is formed by sheafification of the presheaf $U \mapsto \varinjlim_i \mathcal{F}_i(U)$.*

Proof. Let F denote the presheaf obtained by $U \mapsto \varinjlim_i \mathcal{F}_i(U)$ and further denote $\mathcal{F} = F^{++}$, the sheafification of F. Note that we have $\mathcal{F}_i \xrightarrow{j_i} F \to \mathcal{F}$. We wish to show that $\mathcal{F}$ satisfies the universal property of direct limits in $\mathbf{Sh}(X)$. Indeed, take any other sheaf $\mathcal{G}$ for which there are maps $f_i : \mathcal{F}_i \to \mathcal{G}$ which further satisfies that for any $j \geq i$ in the direct set indexing the system, we have that the following triangle commutes:

$$
\begin{array}{ccc}
 & \mathcal{G} & \\
{}^{f_i}\nearrow & & \nwarrow{}^{f_j} \\
\mathcal{F}_i \xrightarrow[\varphi_{ij}]{} & & \mathcal{F}_j
\end{array} \quad .
$$

We wish to show that there exists a unique map $\tilde{f} : \mathcal{F} \to \mathcal{G}$ such that for all i, the following commutes:

$$
\begin{array}{ccc}
\mathcal{G} & \xleftarrow{\ \tilde{f}\ } & \mathcal{F} \\
{}^{f_i}\uparrow & & \uparrow \\
\mathcal{F}_i & \xrightarrow{\ j_i\ } & F
\end{array} \quad .
$$

But this is straightforward, as by the universal property of direct limits in $\mathbf{PSh}(X)$, we first have a map $f : F \to \mathcal{G}$ which makes the bottom left triangle in the above commute. Then, by the universal property of sheafification (Theorem 22.1.0.1), we get a corresponding $\tilde{f} : \mathcal{F} \to \mathcal{G}$ which makes the top right triangle in the above commute. Consequently, we have obtained $\tilde{f}$ which makes the square commute. $\qquad\square$

---

[8]Exercise II.1.10 of Hartshorne.

22.6 Classical Čech cohomology

Sheaf cohomology becomes an important path to any attempt at answering a sophisticated geometric question in topology and geometry. It is a tool which measures the obstructions faced in extending a local construction (which are usually not too difficult to make) to a global one (which are most of the time very difficult to make). To get a feel of why such questions and tools developed to them are important, one may look no further than basic analysis; say in case of $\mathbb{R}^n$, we wish to extend a local isometry from an open set of $\mathbb{R}^n$ to $\mathbb{R}^m$, into a global one between $\mathbb{R}^n$ and $\mathbb{R}^m$. Clearly the former is much, much easier than the latter. In the same vein, we wish to understand obstructions faced in making local-to-global leaps in the context of schemes, which covers almost all range of algebro-geometric situations.

Let X be a topological space and F be a presheaf over X. We will construct and discuss the Čech cohomology groups $\check{H}^q(X; F)$. After giving the basic constructions, we will specialize to the case of schemes in Foundational Algebraic Geometry, Chapter 1, to prove the Serre's theorem on invariance of affine refinements of cohomology of coherent sheaves.

We first construct the Čech cochain complex. Let X be a topological space and F be a presheaf. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a fixed open cover of X. We can then define for each $i = 0, 1, 2, \ldots$, a group called the group of i-cochains:

$$C^i(\mathcal{U}, F) := \prod_{\alpha_0, \ldots, \alpha_i \in I} F(U_{\alpha_0} \cap U_{\alpha_1} \ldots U_{\alpha_i}).$$

where the product runs over all $i + 1$ indices in I. A typical element $s \in C^i(\mathcal{U}, F)$ is called an i-cochain, whose part corresponding to $\beta_0, \ldots, \beta_i \in I$ is denoted by $s(\beta_0, \ldots, \beta_i)$. For example, the set of all 0-cochains is $\prod_{\alpha_0 \in I} F(U_{\alpha_0})$, which is equivalent to choosing a section for each element of the cover. Similarly, choosing an element from $C^1(\mathcal{U}, F)$ can be thought of as choosing a section which is common in exactly two elements of the cover $\mathcal{U}$.

Next, we give the sequence of groups $\{C^i(\mathcal{U}, F)\}_{i \in \mathbb{N} \cup 0}$ the structure of a cochain complex. Indeed, one defines the differential in quite an obvious manner, if one knows the construction of singular homology. Define a map

$$d : C^i(\mathcal{U}, F) \longrightarrow C^{i+1}(\mathcal{U}, F)$$
$$s = (s(\alpha_0, \ldots, \alpha_i)) \longmapsto ds$$

where the components of ds are given as follows for $\beta_0, \ldots, \beta_{i+1} \in I$:

$$(ds)(\beta_0, \ldots, \beta_{i+1}) := \sum_{j=0}^{i+1} (-1)^j \rho_j(s(\beta_0, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_{i+1}))$$

where ρ_j is the following restriction map of the presheaf F:

$$\rho_j : F(U_{\beta_0}, \ldots, U_{\beta_{j-1}}, U_{\beta_{j+1}}, U_{\beta_{i+1}}) \longrightarrow F(U_{\beta_0}, \ldots, U_{\beta_{j-1}}, U_{\beta_j}, U_{\beta_{j+1}}, U_{\beta_{i+1}}),$$

that is, the one where the open set U_{β_j} is dropped from intersection.

This differential can be understood in the simple case of $i = 0$ as follows. Take $s = (s(\alpha_0)) \in C^0(\mathcal{U}, F)$. Then $ds \in C^1(\mathcal{U}, F)$ and it corresponds to a choice of a section in the intersection on each pair of open sets in $\mathcal{U}$. For $\beta_0, \beta_1 \in I$, this choice is given by

$$(ds)(\beta_0, \beta_1) = \rho_0(s(\beta_1)) - \rho_1(s(\beta_0)).$$

This is interpreted as "how much far away $s(\beta_1) \in U_{\beta_1}$ and $s(\beta_0) \in U_{\beta_0}$ are atleast in the intersection $U_{\beta_1} \cap U_{\beta_0}$".

Similarly, for a $s \in C^1(\mathcal{U}, F)$, we can think of it as a choice of a section on each intersecting pair of open sets of $\mathcal{U}$. Then, the differential $ds \in C^2(\mathcal{U}, F)$ for some $\beta_0, \beta_1, \beta_2 \in I$ gives

$$(ds)(\beta_0, \beta_1, \beta_2) = \rho_0(s(\beta_1, \beta_2)) - \rho_1(s(\beta_0, \beta_2)) + \rho_2(s(\beta_0, \beta_1)).$$

If this quantity is non zero, then it measures "how much the three elements $s(\beta_1, \beta_2) \in U_{\beta_1} \cap U_{\beta_2}$, $s(\beta_0, \beta_2) \in U_{\beta_0} \cap U_{\beta_2}$ and $s(\beta_0, \beta_1) \in U_{\beta_0} \cap U_{\beta_1}$ differs in the combined intersection $U_{\beta_0} \cap U_{\beta_1} \cap U_{\beta_2}$".

Now it is quite obvious that in order to measure the failure of an element of $C^i(\mathcal{U}, F)$ to "match up in one level above" will be measured by the homology of the cochain complex. Indeed that is what we do now.

For any $s \in C^i(\mathcal{U}, F)$, it is observed by doing the summation and using the fact that the restriction maps ρ are group homomorphisms that

$$d^2 = 0.$$

Hence, we have a cochain complex, called the **Čech cochain complex**(?? no, probably):

$$\cdots \xleftarrow{\ d\ } C^2(\mathcal{U}, F) \xleftarrow{\ d\ } C^1(\mathcal{U}, F) \xleftarrow{\ d\ } C^0(\mathcal{U}, F) \ .$$

The cohomology of this complex is denoted by

$$H^q(\mathcal{U}; F) := \frac{\operatorname{Im}(d)}{\operatorname{Ker}(d)}$$

for $C^{q+1}(\mathcal{U}, F) \leftarrow C^q(\mathcal{U}, F) \leftarrow C^{q-1}(\mathcal{U}, F)$.

22.7   Derived functor cohomology

Do this from Chapter 10 of Wedhorn and the main ideas are in Hartshorne.

We will here define the cohomology of abelian sheaves over a topological space as right derived functors of the left exact global-sections functor (see Section 21.2 for preliminaries

on derived functors).

Let X be a topological space. In Section 22.5, we showed that the category of abelian sheaves $\mathbf{Sh}(X)$ has enough injectives. We now use it to define cohomology of $\mathbf{Sh}(X)$.

Definition 22.7.0.1. (**Sheaf cohomology functors**) Let X be a topological space and $\mathbf{Sh}(X)$ be the category of abelian sheaves over X. The ith-cohomology functor $H^i(X, -) :$ $\mathbf{Sh}(X) \to \mathbf{AbGrp}$ is defined to be the ith-right derived functor of the global sections functor $\Gamma(-, X) : \mathbf{Sh}(X) \to \mathbf{AbGrp}$. In other words, $H^i(X, \mathcal{F})$ for $\mathcal{F} \in \mathbf{Sh}(X)$ is defined by choosing an injective resolution $0 \to \mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^\bullet$ in $\mathbf{Sh}(X)$ and then

$$H^i(X, \mathcal{F}) := h^i(\Gamma(X, \mathcal{I}^\bullet)).$$

Let us now give some properties of cohomology functors. **TODO : Do properties from Godement and the vanishing theorem of Grothendieck from Hartshorne.**

22.7.1 Restricting to $\mathbf{Mod}(\mathcal{O}_X)$

We would like to see the following theorem.

Theorem 22.7.1.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the right derived functors of $\Gamma(-, X) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{AbGrp}$ is equal to the restriction of the cohomology functors $H^i(X, -) : \mathbf{Sh}(X) \to \mathbf{AbGrp}$.*

Remember that $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives (Theorem 6.5.2.2). Above result is important because its relevance in rectifying the cohomology of $\mathcal{O}_X$-modules (which are of the only utmost interest in algebraic geometry) to that of the usual cohomology functors. Consequently, we may completely work inside the module category $\mathbf{Mod}(\mathcal{O}_X)$. In the process of proving it, we will show that a certain class of modules in $\mathbf{Mod}(\mathcal{O}_X)$ will take the role of injective resolutions in it (namely, *flasque* sheaves).

Do it from Section 2, Chapter Hartshorne, Chapter 22.

22.7.2 Čech-to-derived functor spectral sequence

We wish to now observe how the Čech cohomology and derived functor cohomology are related. This is done by a spectral sequence.

Chapter 23

Quadratic Forms

We review here the language and results of the theory of quadratic forms.

Chapter 24

Curiosities

In this chapter we discuss some line of thoughts that we tend to get while discussing problems with others. Some of these might be useless, but nonetheless, fun.

24.1 Basically compact spaces

A notion intermediate between compactness and local compactness.

Definition 24.1.0.1. (**Basically compact spaces**) A space X is basically compact if there exists a basis $\mathcal{B}$ of X such that for each $B \in \mathcal{B}$, $\overline{B} \subseteq X$ is compact.

Statistics:-

- # Parts = 5
- # Chapters = 24
- # Definitions = ??
- # Remarks = ??
- # Lemmas = ??
- # Propositions = ??
- # Theorems = ??

Milestones:-

- Started : Somewhere in middle of December 2022.
- First 100 pages of Chapter 1 : 24th September, 2023.
- Total 400 pages : 8th October, 2023.