Assignment 8 - MA212M

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1 Answer 1

Given:

- $(X, Y) \sim N_2(\mu, \Sigma)$
- $Var(X) = Var(Y)$

To show that $X+Y$ and $X-Y$ are independent.

If we can show that $Cov(X+Y, X-Y) = 0$, then it'll prove that $X+Y$ and $X-Y$ are independent.

First of all, using the fact that variance is same for both,

$$\begin{aligned} \mathbb{E}\left[(X-\mathbb{E}(X))^2\right] &= \mathbb{E}\left[(Y-\mathbb{E}(Y))^2\right] \\ \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 &= \mathbb{E}\left[Y^2\right] - \mathbb{E}[Y]^2 \end{aligned} \tag{1}$$

Now, expanding the $Cov(X+Y, X-Y)$

$$\begin{aligned} Cov(X+Y, X-Y) &= \mathbb{E}\left[(X+Y-\mathbb{E}(X+Y))(X-Y-\mathbb{E}(X-Y))\right] \\ &= \mathbb{E}\left[(X+Y)(X-Y)\right] - \mathbb{E}[X+Y]\,\mathbb{E}[X-Y] \\ &= \mathbb{E}\left(X^2\right) - \mathbb{E}\left[Y^2\right] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 \\ &= 0 \text{ Using (1)} \end{aligned}$$

Hence proved that $X+Y$ and $X-Y$ are independent.

2 Answer 2

Since we have a Bi variate Normal distribution therefore, after using the definition of correlation coefficient ρ, we get:

$$\mu = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$\Sigma = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$$

2.1 Answer 2.a

We know that if $\mathbf{u} = (a, b) \in \mathbb{R}^2/(0,0)$ and $\mathbf{X} \sim BND$, then

$$\mathbf{u}^T\mathbf{X} \sim N\left(\mathbf{u}^T\mu, \mathbf{u}^T\Sigma\mathbf{u}\right).$$

Hence, if we let $\mathbf{u}^T = \left[1, 1\right]$, then, we'll get:

$$\begin{aligned} X+Y &\sim N\left(0-1, \left[1,1\right]\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\ &\sim N(-1, 3) \end{aligned}$$

Therefore, $P(X+Y>0)$ becomes,

$$X+Y=Z\sim N(-1,3)$$
$$\frac{Z+1}{\sqrt{3}}\sim N(0,1)$$

therefore, if $X+Y>0$, then $Z>\frac{1}{\sqrt{3}}$, hence

$$P(X+Y>0)=P\left(Z>\frac{1}{\sqrt{3}}\right)=1-\Phi\left(\frac{1}{\sqrt{3}}\right).$$

2.2 Answer 2.b

If $aX+Y$ and $X+2Y$ are independent, then if we can show that $Cov(aX+Y,X+2Y)=0$, that will prove that they are independent.

$$\begin{aligned}
Cov(aX+Y,X+2Y) &= \mathbb{E}\left[(aX+Y)(X+2Y)\right]-\mathbb{E}\left[aX+Y\right]\mathbb{E}\left[X+2Y\right]\\
&= \mathbb{E}\left[aX^2+2aXY+XY+2Y^2\right]-a\mathbb{E}\left[X\right]^2-2a\mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]-\mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]-2\mathbb{E}\left[Y\right]^2\\
&= a\left(\mathbb{E}\left(X^2\right)-\mathbb{E}\left[X\right]^2\right)+(2a+1)\left(\mathbb{E}\left[XY\right]-\mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]\right)+2\left(\mathbb{E}\left[Y^2\right]-\mathbb{E}\left[Y\right]^2\right)\\
&= a\left(Var(X)\right)+(2a+1)Cov(X,Y)+2Var(Y)\\
&= a\times 1+(2a+1)\times -1+2\times 4
\end{aligned}$$

to be independent, this should be zero, hence:

$$\begin{aligned}
a\times 1+(2a+1)\times -1+2\times 4 &= 0\\
-a-1+8 &= 0\\
a &= 7
\end{aligned}$$

2.3 Answer 2.c

$$\begin{pmatrix} X+Y\\ 2X-Y\end{pmatrix}\sim N_2\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right)$$

where,

$\mu_1=\left[1,1\right]\begin{bmatrix}0\\-1\end{bmatrix}$

$\mu_2=\left[2,-1\right]\begin{bmatrix}0\\-1\end{bmatrix}$

$\sigma_1=Var(X+Y)=Var(X)+Var(Y)+2Cov(X,Y)=1+4-2=3$

$\sigma_2=Var(2X,-Y)=4+4+4=12$

$$\rho=\frac{Cov(X+Y,2X-Y)}{\sqrt{Var(X+Y)\,Var(2X-Y)}}=\frac{2Var(X)-Cov(X,Y)+2Cov(X,y)-Var(Y)}{\sqrt{3\times 12}}=-\frac{1}{2}$$

Then, we can represent $f_{X_1|X_2}(x_1|x_2)$ as:

$$f_{X_1|X_2}(x_1\mid x_2)=\frac{1}{\sigma_{1|2}\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x_1-\mu_{1|2}}{\sigma_{1|2}}\right)^2\right]\forall x_1\in\mathbb{R}$$

where,

$$\mu_{1|2} = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(x_2 - \mu_2\right)$$
$$\sigma_{1|2}^2 = \sigma_1^2\left(1 - \rho^2\right)$$

3 Answer 3

Given

$$\begin{aligned}\mu_x &= 0\\ \sigma_x^2 &= 1\\ \mu_y &= 0\\ \sigma_y^2 &= 1\\ \text{Correlation Coefficient} &= \rho\end{aligned}$$

To find: $\mathbb{E}\left[X^2Y^2\right]$

We can use the fact that $\mathbb{E}\mathbb{E}\left[X \mid Y\right] = \mathbb{E}\left[X\right]$

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}\left(X^2Y^2 \mid Y\right)\right] = \mathbb{E}\left[X^2Y^2\right] &= \mathbb{E}\left[Y^2\mathbb{E}\left[X^2 \mid Y\right]\right]\\ &= \mathbb{E}\left[Y^2\left\{Var\left(X|Y\right) + \mathbb{E}\left[X|Y\right]^2\right\}\right]\\ &= \mathbb{E}\left[Y^2\left\{1\left(1-\rho^2\right) + 0 + \left(\rho Y\right)^2\right\}\right]\\ &= \left(1-\rho^2\right)\mathbb{E}\left[Y^2\right] + \rho^2\mathbb{E}\left[Y^4\right]\end{aligned}$$

We can easily find $\mathbb{E}\left[Y^2\right]$ as it is equal to $Var\left(Y\right) + \mathbb{E}\left[Y\right]^2 = 0 + 1^2 = 1$

To find $\mathbb{E}\left(Y^4\right)$, knowing that $Y \sim N\left(0,1\right)$, we get:

$$\begin{aligned}\mathbb{E}\left[Y^4\right] &= \int_{-\infty}^{\infty} y^4\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy\\ &= \frac{2}{\sqrt{2\pi}}\int_0^{\infty} y^4e^{-\frac{y^2}{2}}dy\end{aligned}$$

let $\frac{y^2}{2} = t$, therefore, $dt = ydy$, hence,

$$\begin{aligned}&= \frac{2}{\sqrt{2\pi}}\int_0^{\infty} y^4e^{-t}\frac{dt}{y}\\ &= \frac{2}{\sqrt{2\pi}}\int_0^{\infty} y^3e^{-t}dt\\ &= \frac{2}{\sqrt{2\pi}}\int_0^{\infty} \left(2t\right)^{\frac{3}{2}}e^{-t}dt\\ &= \frac{4}{\sqrt{\pi}}\int_0^{\infty} t^{\frac{3}{2}}e^{-t}dt\\ &= \frac{4}{\sqrt{\pi}}\Gamma\left(\frac{5}{2}\right)\end{aligned}$$

Therefore, $\mathbb{E}\left[X^2Y^2\right]$ is:

$$\begin{aligned}\mathbb{E}\left[X^2Y^2\right] &= 1-\rho^2+\frac{4\rho^2}{\sqrt{\pi}}\Gamma\left(\frac{5}{2}\right)\\ &= 1-\rho^2+\frac{4\rho^2}{\sqrt{\pi}}\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\\ &= 1+2\rho^2\end{aligned}$$

4 Answer 4

Given:

$$\mu=\begin{bmatrix}5\\10\end{bmatrix},\ \Sigma=\begin{bmatrix}1 & 5\rho\\ 5\rho & 25\end{bmatrix}$$

where $\rho>0$ is the correlation coefficient.

Also, we know that,

$$p\left(4<Y<16 \mid X=5\right)=0.954.$$

Now, to find $P\left(Y \mid X\right)$,

$$\begin{aligned}\mu_{Y|X} &= 10+5\rho\left(x-5\right)\\ \sigma^2_{Y|X} &= 25\left(1-\rho^2\right)\end{aligned}$$

$$f_{Y|X}\left(Y|X=x\right)=\frac{1}{25\left(1-\rho^2\right)}\exp\left[-\frac{1}{2}\left(\frac{\left(y-10-5\rho\left(x-5\right)\right)^2}{25\left(1-\rho^2\right)}\right)\right]$$

Therefore,

$$\begin{aligned}Y \mid X=5=Z &\sim N\left(10,25\left(1-\rho^2\right)\right)\\ \frac{Z-10}{5\sqrt{1-\rho^2}} &\sim N\left(0,1\right)\end{aligned}$$

Therefore, if $4<Z<16$, then, $\frac{-6}{5\sqrt{1-\rho^2}}<\frac{Z-10}{5\sqrt{1-\rho^2}}<\frac{6}{5\sqrt{1-\rho^2}}$.

Let $c=\frac{6}{5\sqrt{1-\rho^2}}$, then,

$$\begin{aligned}P\left(4<Z<16\right)=2\Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right)-1 &= 0.954\\ \implies \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) &= 0.977\end{aligned}$$

Thus,

$$\begin{aligned}\frac{6}{5\sqrt{1-\rho^2}} &= \Phi^{-1}\left(0.977\right)\approx 2\\ \implies \rho^2 &= 1-\frac{9}{25}\\ =\rho^2 &= \frac{16}{25}\\ \implies \rho &= \frac{4}{5}\end{aligned}$$

5 Answer 5

Since $\rho = 0$, therefore, X and Y are independent.

Since both X and Y are standard normal, therefore we can write:

$$\begin{aligned} P\left(-c < X < c, -c << Y < c\right) &= P\left(-c < X < c\right) P\left(-c < Y < c\right) \\ &= \left(2\Phi\left(c\right) - 1\right)^2 = 0.95 \\ &\implies \Phi\left(c\right) = 0.98734 \\ &\implies c \approx 2.24 \end{aligned}$$

6 Answer 6

Each velocity component is a random variable

$$\begin{aligned} V_x, V_y, V_z &\sim N\left(0, \frac{kT}{m}\right) \\ \sqrt{\frac{m}{kT}}V_x, \sqrt{\frac{m}{kT}}V_y, \sqrt{\frac{m}{kT}}V_z &\sim N\left(0, 1\right) \end{aligned}$$

To find: PDF of the velocity $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$

We know that sum of squares of n Random variables from Standard Normal forms χ_n^2 distribution, i.e.

$$\begin{aligned} \frac{m}{kT}V_x^2 + \frac{m}{kT}V_y^2 + \frac{m}{kT}V_z^2 &\sim \chi_n^2 \\ V_x^2 + V_y^2 + V_z^2 &\sim \frac{kT}{m}\chi_n^2 \end{aligned}$$

Since we need distribution of $\sqrt{V_x^2 + V_y^2 + V_z^2}$, therefore, effectively, we need the PDF of $\sqrt{\frac{kT}{m}\chi_n^2}$.

Remember, if $V \sim \chi_n$ distribution, then:

$$f_V\left(v\right) = \frac{1}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} v^{n-1} e^{-\frac{v^2}{2}}$$

where $v \in [0, \infty)$.

Therefore, $\sqrt{\frac{m}{kT}}\sqrt{V_x^2 + V_y^2 + V_z^2}$ will follow,

$$f_Z\left(z\right) = \frac{1}{\sqrt{2}\Gamma\left(\frac{3}{2}\right)} \left(\sqrt{\frac{m}{kT}}z\right)^2 e^{-\frac{m}{2kT}z^2}$$

only if $z \in [0, \infty]$, else 0.

7 Answer 7

Given (let W denote wife's Random Variable and H denote husband's Random Variable)

$$\begin{aligned} \mu_W &= 66.8,\ \sigma_W = 2 \\ \mu_H &= 70,\ \sigma_H = 2 \\ \rho &= 0.68 \end{aligned}$$

To find: $P(W > H)$

Therefore, we need to find the distribution of $W - H$, to do that, we can use the fact that linear combination of the R.V.s in a BND also follows a Normal Distribution.

Here, $\mathbf{u}^T = \begin{bmatrix}1, -1\end{bmatrix}$, $\mu = \begin{bmatrix}66.8 \\ 70\end{bmatrix}$ and $\Sigma = \begin{bmatrix}4 & 2.72 \\ 2.72 & 4\end{bmatrix}$.

Since $\begin{pmatrix}W \\ H\end{pmatrix} \sim N_2(\mu, \Sigma)$, therefore:

$$\begin{aligned} \mathbf{u}^T\mathbf{X} &\sim N_2\left(\mathbf{u}^T\mu, \mathbf{u}^T\Sigma\mathbf{u}\right) \\ W - H &\sim N_2\left(\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}66.8 \\ 70\end{bmatrix}, \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}4 & 2.72 \\ 2.72 & 4\end{bmatrix}\begin{bmatrix}1 \\ -1\end{bmatrix}\right) \\ W - H &\sim N_2(-3.2, 2.56) \\ \frac{W - H + 3.2}{\sqrt{2.56}} &\sim N_2(0, 1) \\ \frac{Z + 3.2}{1.6} &\sim N_2(0, 1) \end{aligned}$$

where $Z = W - H$.

Therefore, if $W - H > 0$, then $Z > 2$.

Hence,

$$\begin{aligned} P(W - H > 0) = P(Z > 2) &= 1 - \Phi(2) \\ &= 1 - 0.977 = 0.023 \end{aligned}$$

8 Answer 8

Given:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2(\mu, \Sigma)$$
$$\mu_x = 1 \text{ and } \sigma_x = 2$$
$$\mathbb{E}[Y \mid X] = 3X + 7$$
$$\mathbb{E}\left[(Y - \mathbb{E}[Y \mid X])^2\right] = 28$$

First of all, to get μ_y:

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y \mid X]] \\
&= \mathbb{E}[3X + 7] \\
&= 3\mathbb{E}[X] + 7 \\
&= 3 \times 1 + 7 = 10
\end{aligned}$$

Now, to find $Var(Y)$:

$$\begin{aligned}
\mathbb{E}\left[(Y - \mathbb{E}[Y \mid X])^2\right] &= \mathbb{E}\left[\mathbb{E}\left[(Y - \mathbb{E}[Y|X])^2 |X\right]\right] \\
&= \mathbb{E}[Var(Y \mid X)] \\
&= \mathbb{E}\left[\sigma_y^2\left(1 - \rho^2\right)\right] \text{ (because } X \text{ and } Y \text{ are in BND)} \\
&= \sigma_y^2\left(1 - \rho^2\right) = 28
\end{aligned}$$

Now, $\mathbb{E}[Y \mid X = x] = 3x + 7$, also, $\mathbb{E}[Y \mid X = x] = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x)$.

Hence,

$$\begin{aligned}
10 + \rho\frac{\sigma_y}{2}(x - 1) &= 3x + 7 \\
&\implies \rho\frac{\sigma_y}{2} = 3 \\
&\implies \rho\sigma_y = 6
\end{aligned}$$

Using above equation to get σ_y as follows:

$$\begin{aligned}
\sigma_y^2 - (\rho\sigma_y)^2 &= 28 \\
\sigma_y^2 &= 64 \\
\sigma_y &= 8
\end{aligned}$$

For ρ,

$$\begin{aligned}
\rho\sigma_y &= 6 \\
&\implies \rho = \frac{6}{8} = 0.75
\end{aligned}$$

9 Answer 9

Given:

$$X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(0,1)$$
$$Y = \sum_{i=1}^{n} X_i^2$$

To find: $\mathbb{E}[Y]$ and $Var(Y)$.

First of all, to find $\mathbb{E}[Y]$, note that,:

$$\mathbb{E}[Y] = \sum_{i=1}^{n} \mathbb{E}\left[X_i^2\right]$$

therefore, to find $\mathbb{E}\left[X_i^2\right]$:

$$\begin{aligned}\mathbb{E}\left[X_i^2\right] &= Var(X_i) + \mathbb{E}[X_i]^2 \\ &= 1 + 0^2 = 1\end{aligned}$$

Hence,

$$\mathbb{E}[Y] = n$$

To find $Var(Y)$, first note that since X_i and X_j are independent for $i \neq j$, thus $Cov(X_i, X_j) = 0$, which implies that:

$$Var(Y) = \sum_{i=1}^{n} Var\left(X_i^2\right)$$

Now we can easily find the variance of Y:

$$\begin{aligned}Var\left(X_i^2\right) &= \mathbb{E}\left[X_i^4\right] - \mathbb{E}\left[X_i^2\right]^2 \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^4 e^{-\frac{x^2}{2}} dx - 1 \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^4 e^{-\frac{x^2}{2}} dx - 1\end{aligned}$$

Let $\frac{x^2}{2} = t$, therefore $xdx = dt$

$$\begin{aligned}&= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} (2t)^{\frac{3}{2}} e^{-t} dt - 1 \\ &= \frac{4}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{3}{2}} e^{-t} dt - 1 \\ &= \frac{4}{\sqrt{\pi}} \Gamma\left(\frac{5}{2}\right) - 1 \\ &= \frac{4}{\sqrt{\pi}} \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} - 1 \\ &= 2\end{aligned}$$

Hence,

$$Var(Y) = 2n$$

10 Answer 10

Given:

$$\begin{aligned} X &\sim \chi^2_n \\ Y &\sim N(0,1) \\ Cov(X,Y) &= 0 \end{aligned}$$

To find: Distribution of $T = \frac{Y}{\sqrt{\frac{X}{n}}}$

Since X and Y are independent, therefore, $f_{X,Y}(x,y)$ can be found easily as follows:

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\ &= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \text{ for all } x > 0 \text{ and } y \in \mathbb{R} \end{aligned}$$

Now, $X = V$ and thus let $Y = T\sqrt{\frac{V}{n}}$. Therefore, we would like to find $f_{T,V}(t,v)$

$$f_{T,V}(t,v) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} v^{\frac{n}{2}-1} e^{-\frac{v}{2}} \times \frac{1}{\sqrt{2\pi}} e^{-t^2\frac{v}{n}}$$

Now, to get $f_T(t)$, note that J here is the Jacobian and $J = \sqrt{\frac{v}{n}}$:

$$\begin{aligned} f_t(t) &= \int_{-\infty}^{\infty} f_{T,V}(t,v)\sqrt{\frac{v}{n}}dv \\ &= \frac{1}{2^{\frac{n+1}{2}}\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}} \int_0^{\infty} v^{\frac{n}{2}-1} e^{-\frac{v}{2}} e^{-t^2\frac{v}{2n}} \left(\frac{v}{n}\right)^{\frac{1}{2}} dv \\ &= \frac{1}{2^{\frac{n+1}{2}}\Gamma\left(\frac{n}{2}\right)\sqrt{\pi n}} \int_0^{\infty} v^{\frac{n-1}{2}} e^{-\frac{v}{2}\left(1+\frac{t^2}{n}\right)} dv \end{aligned}$$

Let $u = \frac{v}{2}\left(1+\frac{t^2}{n}\right)$, therefore $\frac{1}{2}\left(1+\frac{t^2}{n}\right)dv = du$,

$$\begin{aligned} &= \frac{1}{2^{\frac{n+1}{2}}\Gamma\left(\frac{n}{2}\right)\sqrt{\pi n}} \int_0^{\infty} \frac{2}{1+\frac{t^2}{n}} \left(\frac{2u}{1+\frac{t^2}{n}}\right)^{\frac{n-1}{2}} e^{-u}du \\ &= \frac{1}{2^{\frac{n+1}{2}}\Gamma\left(\frac{n}{2}\right)\sqrt{\pi n}} \left(\frac{2}{1+\frac{t^2}{n}}\right)^{\frac{n+1}{2}} \int_0^{\infty} u^{\frac{n-1}{2}} e^{-u}du \\ &= \frac{1}{2^{\frac{n+1}{2}}\Gamma\left(\frac{n}{2}\right)\sqrt{\pi n}} \left(\frac{2}{1+\frac{t^2}{n}}\right)^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{\pi n}} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} \end{aligned}$$

11 Answer 11

Given:

$$\begin{aligned}X &\sim \chi_n^2\\ Y &\sim \chi_m^2\\ Cov\left(X, Y\right) &= 0\end{aligned}$$

To find: Distribution of $F = \frac{\frac{X}{n}}{\frac{Y}{m}}$

We can transform the variables as follows:

$$\begin{aligned}X &= V\\ Y &= \frac{V}{F}\frac{m}{n}\end{aligned}$$

Since X and Y are independent, therefore $f_{X,Y}\left(x, y\right)$ can be found easily as follows:

$$\begin{aligned}f_{X,Y}\left(x, y\right) &= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}x^{\frac{n}{2}-1}e^{-\frac{x}{2}} \times \frac{1}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)}y^{\frac{m}{2}-1}e^{-\frac{y}{2}}\\ &= \frac{1}{2^{\frac{m+n}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}x^{\frac{n}{2}-1}y^{\frac{m}{2}-1}e^{-\frac{1}{2}(x+y)}\end{aligned}$$

Now, to find the Jacobian:

$$J = det\left(\begin{bmatrix}-\frac{V}{F^2}\frac{m}{n} & 0\\ \frac{1}{F}\frac{m}{n} & 1\end{bmatrix}\right) = -\frac{V}{F^2}\frac{m}{n}$$

Now to find $f_{F,V}\left(f, v\right)$:

$$\begin{aligned}f_{F,V}\left(f, v\right) &= \frac{1}{2^{\frac{m+n}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}v^{\frac{n}{2}-1}\left(\frac{v}{f}\frac{m}{n}\right)^{\frac{m}{2}-1}e^{-\frac{1}{2}\left(v+\frac{v}{f}\frac{m}{n}\right)} \mid J \mid\\ &= \frac{1}{2^{\frac{m+n}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}}\frac{v^{\frac{n}{2}+\frac{m}{2}-1}}{f^{\frac{m}{2}+1}}e^{-\frac{v}{2}\left(1+\frac{m}{nf}\right)}\end{aligned}$$

Therefore, now we can find find marginal distribution of $f_F\left(f\right)$ by integrating the other part out as follows:

$$\begin{aligned}f_F\left(f\right) &= \int_{-\infty}^{\infty} f_{F,V}\left(f, v\right)dv\\ &= \int_0^{\infty}\frac{1}{2^{\frac{m+n}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}}\frac{v^{\frac{n}{2}+\frac{m}{2}-1}}{f^{\frac{m}{2}+1}}e^{-\frac{v}{2}\left(1+\frac{m}{nf}\right)}dv\\ &= \frac{1}{2^{\frac{m+n}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}}\frac{1}{f^{\frac{m}{2}+1}}\int_0^{\infty} v^{\frac{n}{2}+\frac{m}{2}-1}e^{-\frac{v}{2}\left(1+\frac{m}{nf}\right)}dv\end{aligned}$$

Now let $\frac{v}{2}\left(1+\frac{m}{nf}\right)=u$. Hence we get that $dv=\frac{2}{1+\frac{m}{nf}}du$, therefore:

$$
\begin{aligned}
f_F\left(f\right) &= \frac{1}{2^{\frac{m+n}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}}\frac{1}{f^{\frac{m}{2}+1}}\int_0^\infty\left(\frac{2u}{1+\frac{m}{nf}}\right)^{\frac{n}{2}+\frac{m}{2}-1}e^{-u}\frac{2du}{1+\frac{m}{nf}} \\
&= \frac{1}{2^{\frac{m+n}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{m}{n}\right)^{\frac{m}{2}}\frac{1}{f^{\frac{m}{2}+1}}\left(\frac{2}{1+\frac{m}{nf}}\right)^{\frac{n}{2}+\frac{m}{2}}\int_0^\infty u^{\frac{n}{2}+\frac{m}{2}-1}e^{-u}du \\
&= \left(\frac{m}{n}\right)^{\frac{m}{2}}\frac{\Gamma\left(\frac{m}{2}+\frac{n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}f^{-1-\frac{m}{2}}\left(1+\frac{m}{nf}\right)^{-\frac{n}{2}-\frac{m}{2}} \\
&= \frac{1}{B\left(\frac{m}{2},\frac{n}{2}\right)}\left(\frac{n}{m}\right)^{\frac{n}{2}}f^{\frac{n}{2}-1}\left(1+\frac{nf}{m}\right)^{-\frac{m+n}{2}}
\end{aligned}
$$

if $f>0$, otherwise $f_F\left(f\right)=0$.