Assignment 9 - MA212M

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1 Answer 1

Given:

- $X_1, X_2, \cdots, X_n \sim U(\theta_1, \theta_2)$
- $-\infty < \theta_1 < \theta_2 < \infty$

To find: Moment Estimator of (θ_1, θ_2)

The algorithm to find Moment Estimator is as follows:

1. Calculate:
$$\begin{aligned}\mu_1' &= g_1(\theta_1, \cdots, \theta_k) \\ \mu_2' &= g_2(\theta_1, \cdots, \theta_k) \\ \vdots &= \vdots \\ \mu_k' &= g_n(\theta_1, \cdots, \theta_k)\end{aligned}$$
where $\theta = (\theta_1, \cdots, \theta_k)$ are the parameters of the distribution from which X is drawn.

2. After calculating μ_j' $\forall i \in \{1, \cdots, k\}$, calculate the inverse functions, i.e.:
$$\begin{aligned}\theta_1 &= h_1(\mu_1', \cdots, \mu_k') \\ \theta_2 &= h_2(\mu_1', \cdots, \mu_k') \\ \vdots &= \vdots \\ \theta_k &= h_k(\mu_1', \cdots, \mu_k')\end{aligned}$$

3. Now just approximate μ_j' by the Monte Carlo Estimate as following:
$$\alpha_j = \frac{1}{n}\sum_{i=1}^{n} X_i^j$$
Therefore the Methods of Moments Estimator for θ_i becomes:
$$\hat{\theta}_i = h_i(\alpha_1, \cdots, \alpha_k)$$

Now to find the Moment Estimator for (θ_1, θ_2):

$$\begin{aligned}\mathbb{E}\left[X^1\right] = \mu_1' &= \frac{1}{2}(\theta_1 + \theta_2) \\ \mathbb{E}\left[X^2\right] = \mu_2' &= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \\ &= \int_{\theta_1}^{\theta_2} \frac{1}{\theta_2 - \theta_1} x^2 dx \\ &= \frac{1}{\theta_2 - \theta_1}\left[\frac{x^3}{3}\right]_{\theta_1}^{\theta_2} \\ &= \frac{1}{3}\left(\theta_1^2 + \theta_2^2 + \theta_1\theta_2\right)\end{aligned}$$

Now, to find the inverse functions h_1 and h_2:
Manipulating the equations of μ_1' and μ_2' to get:

$$\begin{aligned} 4\mu_1^2 - 3\mu_2 &= \theta_1\theta_2 \\ &= \theta_1.\left(2\mu_1 - \theta_1\right) \\ &\implies \theta_1^2 - 2\mu_1\theta_1 + 4\mu_1^2 - 3\mu_2 = 0 \\ &\implies \theta_1 = \frac{2\mu_1 \pm \sqrt{4\mu_1^2 - 4.\left(4\mu_1^2 - 3\mu_2\right)}}{2} \end{aligned}$$

Let $m = \sqrt{4\mu_1^2 - 4.\left(4\mu_1^2 - 3\mu_2\right)}$.

Therefore,

$$\begin{aligned} \theta_1 &= \mu_1 \pm \frac{m}{2} \\ \theta_2 &= \mu_1 \mp \frac{m}{2} \end{aligned}$$

Now, to calculate α_1 and α_2:

$$\begin{aligned} \alpha_1 &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ \alpha_2 &= \frac{1}{n}\sum_{i=1}^{n} x_i^2 \end{aligned}$$

Therefore, the Method of Moments Estimator for (θ_1, θ_2) is (Note that $\theta_2 > \theta_1$):

$$\begin{aligned} \hat{\theta_1} &= \frac{2\alpha_1 - \sqrt{4\alpha_1^2 - 4\left(4\alpha_1^2 - 3\alpha_2\right)}}{2} \\ \hat{\theta_2} &= \frac{2\alpha_1 + \sqrt{4\alpha_1^2 - 4\left(4\alpha_1^2 - 3\alpha_2\right)}}{2} \end{aligned}$$

2 Answer 2

Given:

- $X_1, \cdots, X_{10} \sim P_\theta$
- $f_P\left(x\right) = \frac{1}{2}\left(\frac{1}{\theta}e^{-\frac{x}{\theta}} + \frac{1}{10}e^{-\frac{x}{10}}\right)$
- $0 < x < \infty$
- $$\sum_{i=1}^{10} X_i = 150.$$

To Find: Method of Moments Estimator for θ.

First, calculating μ_1' :

$$\begin{aligned}
\mathbb{E}\left[X\right] = \mu_1' &= \int_{-\infty}^{\infty} x f_P\left(x\right) dx \\
&= \int_0^{\infty} \frac{1}{2}\left(\frac{1}{\theta}e^{-\frac{x}{\theta}} + \frac{1}{10}e^{-\frac{x}{10}}\right).x.dx \\
&= \frac{1}{2\theta}\int_0^{\infty} xe^{-\frac{x}{\theta}} dx + \frac{1}{20}\int_0^{\infty} xe^{-\frac{x}{10}} dx \\
&= \frac{\theta}{2}\Gamma\left(2\right) + 5\Gamma\left(2\right) \\
&= \Gamma\left(2\right)\left(\frac{\theta}{2} + 5\right)
\end{aligned}$$

Now, finding the inverse function h_1:

$$\theta = 2\left(\frac{\mu_1'}{\Gamma\left(2\right)} - 5\right)$$

Approximating μ_1' by α_1:

$$\alpha_1 = \frac{1}{10}\sum_{i=1}^{10} x_i = 15$$

Hence, the M.M.E. of θ will be:

$$\begin{aligned}
\hat{\theta} &= 2\left(\frac{\alpha_1}{\Gamma\left(2\right)} - 5\right) \\
&= 2\left(\frac{15}{2} - 5\right) \\
&= 5
\end{aligned}$$

3 Answer 3

Given:

- $$X_1, X_2, \cdots, X_n \sim P_\theta.$$
- $$f_P(x;\theta) = \frac{1}{2}e^{-|x-\theta|} \text{ for } x \in \mathbb{R}.$$
- $$\theta \in \mathbb{R}.$$

To Find: Maximum Likelihood Estimator of θ.

Finding the Likelihood Function $L(\theta)$:

$$L(\theta) = \frac{1}{2^n} \exp\left[-\sum_{i=1}^{n} \mid x_i - \theta \mid\right]$$

To maximize $L(\theta)$ means to minimize $\sum \mid x_i - \theta \mid$.

Let's assume that $x_1, \cdots, x_n$ are arranged in ascending order. Thus we can formally write the target to minimize as:

$$L_1(\theta) = -\sum_{i=1}^{n} \mid x_i - \theta \mid$$

Now, two cases arise:

- **CASE I:** $\theta < x_1$

  If that's the case, then:

  $$L_1(\theta) = -\sum_{i=1}^{n} (x_i - \theta)$$

- **CASE II:** $\theta > x_n$

  If that's the case then:

  $$L_1(\theta) = \sum_{i=1}^{n} (x_i - \theta)$$

Remember that we want to minimize the value of $L_1(\theta)$. Therefore, we see now that in CASE I that the function $L_1(\theta)$ is decreasing whereas in CASE II, $L_1(\theta)$ is increasing.

Naturally, there will be a point of minima for θ between $\theta < x_1$ and $\theta > x_n$.

Thus our task now is to find i such that $x_i < \theta < \theta + d < x_{i+1}$ for some $d > 0$ and $i \in \{1, 2, \ldots, n-1\}$. Hence, we can write $L_1(\theta + d)$ as:

$$\begin{aligned}
L_1(\theta + d) &= -\sum_{k=1}^{n} \mid x_k - \theta - d \mid \\
&= -\sum_{k=1}^{i} (x_k - \theta - d) + \sum_{k=i+1}^{n} (x_k - \theta - d) \\
&= i\theta + id - \sum_{k=1}^{i} x_k - (n-i)\theta - (n-i)d + \sum_{k=1}^{n} x_k \\
&= (2i - n)d - \sum_{k=1}^{i} (x_k - \theta) + \sum_{k=i+1}^{n} (x_k - \theta) \\
&= (2i - n)d + L_1(\theta) \\
\implies L_1(\theta + d) - L_1(\theta) &= (2i - n)d
\end{aligned}$$

Therefore,

$$L_1(\theta + d) - L_1(\theta) = \begin{cases} < 0 & \text{if } i < \frac{n}{2} \\ = 0 & \text{if } i = \frac{n}{2} \\ > 0 & \text{if } i > \frac{n}{2} \end{cases}$$

Hence, $L_1(\theta)$ is decreasing for $i < \frac{n}{2}$, constant if $i = \frac{n}{2}$ and increasing for $i > \frac{n}{2}$.

Therefore, if n is even, then the M.L.E. of θ would be any $\theta \in \left[x_{\frac{n}{2}}, x_{\frac{n}{2}+1}\right]$.

Whereas, if n is odd, then the M.L.E of θ would be $x_{\frac{n+1}{2}}$.

4 Answer 4

Given:

- Samples drawn from a distribution P_θ

$$\{3, 3, 3, 3, 3, 7, 7, 7\}.$$

- $$P(3) = \theta \text{ and } P(7) = 1 - \theta.$$

To Find: MME and MLE of θ

For Methods of Moments Estimator for θ:

$$\begin{aligned}
\mathbb{E}[X] = \mu_1' &= 3\theta + 7(1 - \theta) \\
&= 7 - 4\theta \\
\implies \theta &= \frac{7 - \mu_1'}{4}
\end{aligned}$$

Also,

$$\alpha_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{8} \times 36 = 4.5$$

Hence, the M.M.E. for θ would be:

$$\begin{aligned}\hat{\theta} &= \frac{7-\alpha_1}{4} \\ &= \frac{7-4.5}{4} \\ &= \frac{25}{40} = \frac{5}{8} = 0.625\end{aligned}$$

Now, to find the Maximum Likelihood Estimator for θ:

First, the Likelihood Function is:

$$L(\theta) = \theta^5 (1-\theta)^3$$

We need to maximize this Likelihood Function $L(\theta)$, therefore:

$$\begin{aligned}\frac{d}{d\theta} L(\theta) &= 5\theta^4 (1-\theta)^3 - 3\theta^5 (1-\theta)^2 = 0 \\ &= \theta^4 (1-\theta)^2 [5-8\theta] = 0 \\ &\implies \theta = 0, 1, \frac{5}{8}\end{aligned}$$

To find the maximum, we check second order derivative, at the end of which, we see that if $\theta = \frac{5}{8}$, then $\frac{d^2}{d\theta^2} L\left(\theta = \frac{5}{8}\right) > 0$.

Hence, M.L.E. of θ is:

$$\hat{\theta} = \frac{5}{8} = 0.625.$$

5 Answer 5

Given:

- $$X_1, X_2, \ldots, X_n \sim P_\theta.$$
- The PMF for P_θ is:

$$f(x;\theta) = \begin{cases} \frac{1-\theta}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 2 \\ \frac{\theta}{2} & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}.$$

To Find: Maximum Likelihood Estimator of $\theta \in (0, 1)$.

The Likelihood Function is:

$$L(\theta) = \prod_{i=1}^{n} \left[\left(\frac{1-\theta}{2}\right)^{\frac{(x_i-2)(x_i-3)}{2}} \left(\frac{1}{2}\right)^{-(x_i-1)(x_i-3)} \left(\frac{\theta}{2}\right)^{\frac{(x_i-1)(x_i-2)}{2}} \right]$$

Yikes.........

6 Answer 6

Given:

- $$X_1, X_2, \ldots, X_n \sim P_N.$$

- PMF of P_N is:

$$P(X = k) = \begin{cases} \frac{1}{N} & \text{if } k = 1, 2, \ldots, N \\ 0 & \text{otherwise} \end{cases}.$$

- N is a positive integer.

To Find: Maximum Likelihood Estimator of N.

Firstly, the Likelihood Function $L(N)$ is:

$$L(N) = \frac{1}{N^n} \quad \text{if } x_1, x_2, \ldots, x_n \text{ are positive integers} \leq N$$

We need to maximize the Likelihood Function $L(N)$.

Maximizing $\frac{1}{N^n}$ is equivalent to minimizing N^n if all $x_1, x_2, \ldots, x_n$ are positive integers **and** are $\leq N$.

Now, if $x_1, x_2, \ldots, x_n$ are positive integers $\leq N$, or, $N \geq x_1, x_2, \ldots, x_n$, therefore it's easy to see that N is minimized at the closest value of x_i to N, which is $\max(x_1, x_2, \ldots, x_n)$.

Hence, the Maximum Likelihood Estimator $\hat{N}$ is:

$$\hat{N} = \max(x_1, x_2, \ldots, x_n)$$

7 Answer 7

Given:

- First M fishes are caught from the pond, tagged and returned to the pond. Next n fishes are caught at random out of which x fishes are found to be tagged.

To Find: Maximum Likelihood Estimator for N, the total number of fishes in the pond.

Since our task was to maximize the number of fishes that we tag, hence we first find the probability of tagging x fishes.

Firstly, the probability of tagging x fishes, where X is the Random Variable denoting the number of fishes tagged so far:

$$P(X = x) = \frac{{}^{M}C_{x} \, {}^{N-M}C_{n-x}}{{}^{N}C_{n}}$$

The Likelihood Function for N thus becomes:

$$L(N) = \frac{{}^{M}C_{x}\,{}^{N-M}C_{n-x}}{{}^{N}C_{n}} \quad \text{where } N \geq M, \text{ i.e. } N \in \{M, M+1, \dots\}$$

Now, to maximize $L(N)$, calculating $\frac{L(N+1)}{L(N)}$:

$$\begin{aligned} \frac{L(N+1)}{L(N)} &= \frac{{}^{M}C_{x}\,{}^{N+1-M}C_{n-x}}{{}^{N+1}C_{n}} \Big/ \frac{{}^{M}C_{x}\,{}^{N-M}C_{n-x}}{{}^{N}C_{n}} \\ &= \frac{{}^{M}C_{x}\,{}^{N-M+1}C_{n-x}\,{}^{N}C_{n}}{{}^{M}C_{x}\,{}^{N-M}C_{n-x}\,{}^{N+1}C_{n}} \\ &= \frac{(N-M+1)(N+1-n)}{(N-M+1-n+x)(N+1)} \end{aligned}$$

Now, if the function $\frac{L(N+1)}{L(N)}$ is increasing, then:

$$\frac{L(N+1)}{L(N)} > 1$$

Therefore,

$$\begin{aligned} N^2 - NM + N + N - M + 1 - nN + nM - n &> N^2 - NM + N - nN + xN + N - M + 1 - n + x \\ nM &> xN + x \\ N &< \frac{nM - x}{x} \\ N &< \frac{nM}{x} - 1 \end{aligned}$$

Similarity, if function $\frac{L(N+1)}{L(N)}$ is decreasing, then it would be < 1, which would transcribe to :

$$N > \frac{nM}{x} - 1$$

Hence, we see that N on left side of $\frac{nM}{x} - 1$, the ratio $\frac{L(N+1)}{L(N)}$ is increasing while decreasing on the right.

Therefore the maximum likelihood should occur at $N = \frac{nM}{x} - 1$

Now, few cases arises on $\frac{nM}{x}$

- **CASE I:** $\frac{nM}{x}$ is an Integer.

  If that's the case, then M.L.E. is not unique, as the Likelihood remains maximum for for both $\frac{Mn}{x} - 1$ and $\frac{Mn}{x}$ as the likelihood Function's value between them is same, but since N has to be an integer therefore, only $\frac{Mn}{x} - 1$ and $\frac{Mn}{x}$ are taken.

- **CASE II:** $\frac{Mn}{x}$ is not an Integer.

  Note that $\left[\frac{Mn}{x}\right] > \left[\frac{Mn}{x}\right] - 1 > \left[\frac{Mn}{x}\right] - 2$, therefore:

  $$L\left(\left[\frac{Mn}{x}\right]\right) < L\left(\left[\frac{Mn}{x}\right] - 1\right) > L\left(\left[\frac{Mn}{x}\right] - 2\right)$$

  Hence, M.L.E. for N in the case when $\frac{Mn}{x}$ is not an integer is $\left[\frac{Mn}{x}\right] - 1$

8 Answer 8

Given:

- $$X_1, X_2, \ldots, X_n \sim P_\theta$$

- PMF of P_θ, the lifetime of the integrated circuit is:

$$f(x, \theta) = \begin{cases} 2\lambda x e^{-\lambda x^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- X be the R.V. denoting the number of integrated circuits that fail before τ, where $\tau > 0$ is a known time.

To Find: Maximum Likelihood Estimator of the variance of X.

To find the probability of number of ICs which fail before τ, we should first find the probability of a I.C. failing before τ units if time:

$$\begin{aligned} P(X_i < \tau) &= \int_0^\tau f(x, \theta)\, dx \\ &= 2\lambda \int_0^\tau x e^{-\lambda x^2} dx \\ &= 1 - e^{-\lambda \tau^2} \end{aligned}$$

Now, the number of ICs that fail before τ, X is:

$$X \sim \text{Bin}\left(n, 1 - e^{-\lambda\tau^2}\right)$$

Therefore, the variance of X will become:

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= n e^{-\lambda\tau^2}\left(1 - e^{-\lambda\tau^2}\right) \end{aligned} \tag{1}$$

Remember that we want to find M.L.E. for (1), thus if we find the M.L.E. for λ, then we can use that to find M.L.E. for (1). Hence, we first try to find M.L.E for λ. The Likelihood function for λ then becomes:

$$L(\lambda) = 2^n \lambda^n \left(\prod_{i=1}^n x_i\right) e^{-\lambda \sum_{i=1}^n x_i^2}$$

To find critical points:

$$\begin{aligned} \frac{d}{d\lambda} L(\lambda) = n 2^n \lambda^{n-1} \left(\prod_{i=1}^n x_i\right) e^{-\lambda \sum_{i=1}^n x_i^2} - 2^n \lambda^n \left(\prod_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i^2\right) e^{-\lambda \sum_{i=1}^n x_i^2} = 0 \\ \implies \lambda = \frac{n}{\sum_{i=1}^n x_i^2} \end{aligned}$$

Hence, the Maximum Likelihood Estimator for λ will be:

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i^2}$$

Therefore, we can now use this M.L.E. of λ to find M.L.E. of another function of λ, which in this context is $\mathrm{Var}\,(X) = ne^{-\lambda\tau^2}\left(1 - \lambda e^{-\lambda\tau^2}\right)$, which then becomes:

$$\hat{\mathrm{Var}}(X) = ne^{-\hat{\lambda}\tau^2}\left(1 - \hat{\lambda} e^{-\hat{\lambda}\tau^2}\right)$$

which is the Maximum Likelihood Estimator for the Variance in the number of I.C. failing before time τ.