Neyman-Pearson Lemma

Animesh Renanse

May 14, 2020

1 Most Powerful Test

Def: Consider the **collection C of all level α tests** for $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$. A test belonging to C with power function $\beta(.)$ is called **uniformly most powerful (UMP) level α test** if :

$$\beta(\theta) \geq \beta^*(\theta) \text{ for all } \theta \in \Theta_1.$$

where $\beta^*(.)$ is the power function of any other test in C. *Note the $\theta \in \Theta_1$ in above equation*

If the alternative hypothesis is simple (that means that Θ_1 is singleton), the test is called **most powerful (MP) level α test.**

Remark: Note what we are doing here is that, *across all $\theta \in \Theta_1$, we are finding that test amongst all tests in C with level α which returns maximum value of it's own test function $\beta(.)$.* In this way, we maximize the $\beta(.)$ for all parameters from Θ_1, which is the parameter space of Alternative Hypothesis. Therefore, in reality, we are maximizing the value of $1 - P\,(\text{Type-II Error})$, in-turn, **minimizing P (Type-II Error)**.
At the same time, we are **keeping a bound on probability of Type-I Error** because level being α, i.e. $P\,(\text{Type-I Error}) \leq \alpha$

Remark: When $H_1 : \theta = \theta_1$ for some fixed θ_1, that is, H_1 is simple, it boils down to checking whether $\beta(\theta_1) \geq \beta^*(\theta_1)$ amongst all tests in C. That is the reason why word *uniformly* is removed from the definition when H_1 is simple.

2 Neyman-Pearson Lemma

Theorem: Let $\theta_0 \neq \theta_1$ be two fixed numbers in Θ. The **MP level α test for $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$** is given by,

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } L(\theta_1) > kL(\theta_0) \\ \gamma & \text{if } L(\theta_1) = kL(\theta_0) \\ 0 & \text{if } L(\theta_1) < kL(\theta_0) \end{cases},$$

where $k \geq 0$ and $\gamma \in [0, 1]$ such that $\beta(\theta_0) = \mathbb{E}_{\theta_0}(\varphi(\mathbf{X})) = \alpha$. **Here $L(.)$ is the Likelihood Function.**

Note: This theorem says that if Θ_0 and Θ_1 are singleton sets, then the above $\varphi(\mathbf{x})$ is the most powerful test *automatically.* Since being *most powerful* means that the value of $\beta(.)$ is greater than or equal to the value of all other tests, therefore it has been set to the maximum possible value of $\beta(.)$ under the constraints of the definition, which is α.

Remark: In the theorem, both Null and Alternative Hypotheses are simple.

Remark: $L(\theta_1) > kL(\theta_0)$ can be expresed as $\frac{L(\theta_1)}{L(\theta_0)} > k$ **if** $L(\theta_0) > 0$**.** Hence the **MP test rejects the null hypothesis for large values of** $\frac{L(\theta_1)}{L(\theta_0)}$.

3 Examples

Question: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where σ is known. let $\mu_0 < \mu_1$ be two real numbers. We are interested to test $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1$.

First of all,

$$\frac{L(\mu_1)}{L(\mu_0)} = \exp\left[\frac{1}{2\sigma^2}\left\{2n\overline{x}(\mu_1 - \mu_0) + n\left(\mu_0^2 - \mu_1^2\right)\right\}\right].$$

Now, if $\frac{L(\mu_1)}{L(\mu_0)} > k \iff \overline{x} > k_1$ for some constant k_1, as $\mu_1 > \mu_0$.

Hence the MP level α test is given by:

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } \overline{x} > k_1 \\ \gamma & \text{if } \overline{x} = k_1 \\ 0 & \text{if } \overline{x} < k_1 \end{cases},$$

where k_1 and γ are such that,

$$\mathbb{E}_{\mu_0}(\varphi(\mathbf{x})) = \alpha.$$

i.e.,

$$\mathbb{E}_{\mu_0}(\varphi(\mathbf{X})) = P_{\mu_0}\left[\overline{X} > k_1\right] = \alpha$$
$$\implies k_1 = \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$$

Now note that $P_{\mu_0}\left(\overline{X} = k_1\right) = 0$ as $\overline{X}$ is a (continuous) Normal Distribution. It implies that $\gamma = 0$, **which makes it a non-randomized test.**

Hence the MP level α test is given by:

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } \overline{x} > \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha \\ 0 & \text{otherwise} \end{cases}$$

Remark: Note the way of solving the problem:

- We try to simplify $\frac{L(\mu_1)}{L(\mu_0)} > k$ so that we can *condition on a* ***statistic whose distribution under*** H_0 ***is known or can be found.***
- If this statistic is a continuous random variable , **then we will have a non-randomized test**. Otherwise, we may need to consider $\gamma \in (0, 1)$, making the test a randomized one.

Question: Let $X_1, X_2, .., X_n \overset{i.i.d.}{\sim} Bernoulli\,(\theta)$. Let $0 < \theta_1 < \theta_0 < 1$ be two real numbers. We are interested to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$.

First of all,

$$\frac{L(\theta_1)}{L(\theta_0)} = \left(\frac{\theta_1}{\theta_0} \times \frac{1-\theta_0}{1-\theta_1}\right)^t > k \iff t < k_1$$

where $t \sim T = \sum_{i=1}^{n} X_i \sim Bin\,(n, \theta_0)$ and for some constant k_1 as $\theta_0 > \theta_1$. Hence the MP level α test is:

$$\varphi(\mathbf{x}) = \begin{cases} 1 \text{ if } t < k_1 \\ \gamma \text{ if } t = k_1 \\ 0 \text{ if } t > k_1 \end{cases},$$

where $\mathbb{E}_{\theta_0}(\varphi(\mathbf{x})) = \alpha = P_{\theta_0}(T < k_1) + \gamma P_{\theta_0}(T = k_1)$

Now, take $\hat{K} \in \{1, 2, \ldots, n\}$ such that

$$P_{\theta_0}\left(T < \hat{K}\right) \le \alpha < P_{\theta_0}\left(T \le \hat{K}\right).$$

so that if we take $k_1 = \hat{K}$, we get::

$$\gamma = \frac{\alpha - P_{\theta_0}\left(T < \hat{K}\right)}{P_{\theta_0}\left(T = \hat{K}\right)}.$$

Therefore, the MP level α test is given by:

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } t < \hat{K} \\ \frac{\alpha - P_{\theta_0}(T<\hat{K})}{P_{\theta_0}(T=\hat{K})} & \text{if } t = \hat{K} \\ 0 & \text{if } t > \hat{K} \end{cases}.$$

Question: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma_2)$, where σ is known. Let μ_0 be a real number. We are interested to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$.

We know that the MP level α test for testing $H_0 : \mu = \mu_0$ against $H_1 : \mu = m_1\,(> \mu_0)$ is given by:

$$\varphi_1(\mathbf{x}) = \begin{cases} 1 & \text{if } \frac{\sqrt{n}}{\sigma}(\overline{x} - \mu_0) > z_\alpha \\ 0 & \text{otherwise} \end{cases}.$$

In the similar way, the MP level α test for testing $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1\,(< \mu_0)$ is given by:

$$\varphi_2(\mathbf{x}) = \begin{cases} 1 & \text{if } \frac{\sqrt{n}}{\sigma}(\overline{x} - \sigma_0) < -z_\alpha \\ 0 & \text{otherwise} \end{cases}.$$

Now, notice that:

- For $\mu_1 > \mu_0$, the $\varphi_1(.)$ has the maximum power among level α tests.
- For $\mu_1 < \mu_0$, the $\varphi_2(.)$ has the maximum power among level α tests.
- As these two tests are different, therefore UMP level α test for $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ does not exist for all $\alpha \in (0, 1)$

Remark: There are cases when UMP test does not exist.

Remark: However, the problem of hypothesis testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ is practically quite meaningful. Hence we need some alternatives. One such alternative called **Likelihood Ratio Test** which depends on the concept of Maximum Likelihood Estimator (MLE) is discussed next.