Likelihood Ratio Test

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1 Likelihood Ratio Test

This test is designed for the case when we want to test Null Hypothesis $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. As using UMP/MP level α tests does not exist in this problem.

1.1 Algorithm

1. We want to test $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$.
2. Consider:
$$\Lambda(\mathbf{x}) = \frac{sup_{\theta\in\Theta_0} L(\theta, \mathbf{x})}{sup_{\theta\in\Theta_0\cup\Theta_1} L(\theta, \mathbf{x})}.$$
where $\Lambda(\mathbf{x})$ **is called Likelihood Ratio Test Statistic.**
3. Likelihood level α test is given by,
$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } \Lambda(\mathbf{x}) < k \\ \gamma & \text{if } \Lambda(\mathbf{x}) = k \\ 0 & \text{if } \Lambda(\mathbf{x}) > k \end{cases}.$$
where γ and k are such that $\mathbb{E}_\theta(\varphi(\mathbf{X})) \leq \alpha$ for all $\theta \in \Theta_0$.

1.2 Discussion

- $sup_{\theta\in\Theta_0} L(\theta, \mathbf{x})$ can be considered as **the maximum value of the Likelihood Function over Θ_0 when $\mathbf{X} = \mathbf{x}$ is observed**
- Similarly, $sup_{\theta\in\Theta_0\cup\Theta_1} L(\theta, \mathbf{x})$ can be considered as **the maximum value of the Likelihood Function over $\Theta_0 \cup \Theta_1$ when $\mathbf{X} = \mathbf{x}$ is observed.**
- Clearly $\Lambda(\mathbf{x}) \in [0, 1]$.
- The main point of the whole algorithm/test is to *reject H_0 when $\Lambda(\mathbf{x})$ is small.* This is when likelihood under Θ_0 is lower than that of the likelihood under $\Theta_0 \cup \Theta_1$. This means that *observed values are more likely under Θ_1 rather than Θ_0. Hence we reject H_0.*

2 Examples

Question: Let $X_1, X_2, \cdots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where σ is known. Let μ_0 be a real number. We are interested to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$.

In this question, $\Theta_0 = \{\mu_0\}$ and $\Theta_1 = \mathbb{R}/\{\mu_0\}$. Hence $\Theta_0 \cup \Theta_1 = \mathbb{R}$.

Now, step 2 of the algorithm,

$$sup_{\mu\in\Theta_0} L(\mu) = L(\mu_0) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_0)^2\right].$$

and,

$$sup_{\mu\in\Theta_0\cup\Theta_1} L(\mu) = L(\overline{x}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\overline{x})^2\right].$$

Therefore, we can calculate $\Lambda(\mathbf{x})$

$$\Lambda(\mathbf{x}) = \frac{sup_{\theta\in\Theta_0} L(\theta,\mathbf{x})}{sup_{\theta\in\Theta_0\cup\Theta_1} L(\theta,\mathbf{x})} = \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2-(x_i-\overline{x})^2\right].$$

Expanding it,

$$\begin{aligned}
\Lambda(\mathbf{x}) &= \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(2x_i-\overline{x}-\mu_0)(\overline{x}-\mu_0)\right]\\
&= \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}2x_i(\overline{x}-\mu_0)-\sum_{i=1}^{n}(\overline{x}+\mu_0)(\overline{x}-\mu_0)\right)\right]\\
&= \exp\left[-\frac{1}{2\sigma^2}\left(2\overline{x}n(\overline{x}-\mu_0)-n\overline{x}^2+n\mu_0^2\right)\right]
\end{aligned}$$

Hence,

$$\Lambda(\mathbf{x}) = \exp\left[-\frac{n}{2\sigma^2}(\overline{x}-\mu_0)^2\right] < k \iff |\ \overline{x}-\mu_0\ | > k_1 \text{ for some } k_1 \in \mathbb{R}.$$

Therefore, now we can write the Likelihood Ratio Level α test as:

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } |\ \overline{x}-\mu_0\ | > k_1 \\ 0 & \text{otherwise} \end{cases}$$

remember that $X_i \sim N\left(\mu,\sigma^2\right)$
where k_1 is such that,

$$\begin{aligned}
\mathbb{E}_{\mu_0}(\varphi(\mathbf{x})) &= P_{\mu_0}\left[|\ \overline{X}-\mu_0\ | > k_1\right] = \alpha\\
&= P_{\mu_0}\left[\frac{\sqrt{n}}{\sigma}\ |\ \overline{X}-\mu_0\ | > \frac{\sqrt{n}}{\sigma}k_1\right] = \alpha\\
&= P_{\mu_0}\left[\frac{\sqrt{n}}{\sigma}\left(\overline{X}-\mu_0\right) > \frac{\sqrt{n}}{\sigma}k_1\right] = \frac{\alpha}{2}
\end{aligned}$$

which implies that

$$\frac{\sqrt{n}}{\sigma}k_1 = z_{\frac{\alpha}{2}}.$$

Note that we followed the same procedure as in finding UMP/MP test from previous topic.

Therefore, **the Likelihood Ratio Level α test is given by:**

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } \frac{\sqrt{n}}{\sigma}\left(\overline{X}-\mu_0\right) > z_{\frac{\alpha}{2}} \\ 0 & \text{otherwise} \end{cases}.$$