Indian Institute of Technology Guwahati
Mathematical Statistics (MA212M)
Problem Set 08

1. Let (X, Y) be bivariate normal such that $Var(X) = Var(Y)$. Show that the two random variables $X + Y$ and $X - Y$ are independent.

2. Let (X, Y) be bivariate normal with parameters $\mu_x = 0$, $\sigma_x^2 = 1, \mu_y = -1$, $\sigma_y^2 = 4$, and $\rho = -1/2$.
   (a) Find $P(X + Y > 0)$.
   (b) Find the constant a for which $aX + Y$ and $X + 2Y$ are independent.
   (c) Find $P(X + Y > 0|2X - Y = 0)$.

3. Let $(X,\, Y)$ be bivariate normal with parameters $\mu_x = 0$, $\sigma_x^2 = 1, \mu_y = 0$, $\sigma_y^2 = 1$ and correlation coefficient ρ. Using conditional expectation, find $E(X^2Y^2)$.

4. Let $(X,\, Y)$ be bivariate normal with parameters $\mu_x = 5$, $\sigma_x^2 = 1, \mu_y = 10$, $\sigma_y^2 = 25$ and correlation coefficient ρ, where $\rho > 0$. If it is known that the conditional probability of $Y \in (4,\, 16)$ given $X = 5$ is 0.954, determine the value of ρ. (Ans: 0.8)

5. Let$(X,\, Y)$ be bivariate normal with parameters $\mu_x = 0$, $\sigma_x^2 = 1$, $\mu_y = 0, \sigma_y^2 = 1$, $\rho = 0$. Find the real constant c such that
$$P(-c < X < c, -c < Y < c) = 0.95\,.$$
   You can use that $\Phi(2.24) = 0.987$.

6. Assume that the velocity components V_x, V_y, V_z of any molecule of a gas are mutually independent random variables, each being $N(0, \frac{kT}{m})$ where k is Boltzmann's constant, T is the absolute temperature of the gas and m the mass of a molecule. Find the PDF of the velocity $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$.

7. Suppose that the heights of married couples can be explained by a bivariate normal distribution. If the wives have a mean height of 66.8 inches and a standard deviation of 2 inches while the heights of the husbands have a mean of 70 inches and a standard deviation of 2 inches. The correlation between the heights is 0.68. What is the probability that for a randomly selected couple the wife is taller than her husband? Use the fact that $\Phi(2) = 0.977$. (Ans: 0.023)

8. Let X and Y have the bivariate normal distribution. The following facts are known: $\mu_x = 1$, $\sigma_x = 2$ and the best estimate of Y based on X is given by $3X + 7$. The minimum mean square error is 28. Find μ_y, σ_y and the correlation coefficient ρ between X and Y. (Ans: $\mu_y = 10$, $\sigma_y = 8$, $\rho = 3/4$).

9. Let $X_1,\, X_2,\, \ldots,\, X_n$ be n i.i.d. $N(0,\, 1)$ random variables. Find $E(Y)$ and $Var(Y)$, where $Y = \sum_{i=1}^n X_i^2$.

10. Let $X \sim \chi_n^2$ and $Y \sim N(0,\, 1)$. Also assume that X and Y are independent random variables. Find the distribution of $T = \frac{Y}{\sqrt{X/n}}$. [Note: The distribution of T is called the t-distribution with degree of freedom n and is denoted by $T \sim t_n$.]

11. Let $X \sim \chi_n^2$ and $Y \sim \chi_m^2$. Also assume that X and Y are independent random variables. Find the distribution of $F = \frac{X/n}{Y/m}$. [Note: The distribution of F is called the F-distribution with degrees of freedom n and m, respectively. It is denoted by $F \sim F_{n,m}$.]