Testing of Hypothesis

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1 Testing of Hypothesis

So far we have discussed how we can gather a point or interval estimate of unknown parameters.

Now we will look how we can figure out whether a given statement is true or not.

2 Examples

2.1 Clinical Trial

- Pharmaceutical companies use Hypothesis test to see whether a particular drug is efficient or not.
- To do so, they administer two sets of patients, one with the drug (test group) and other with a placebo (control group).
- Assume that the drug is a cough syrup
- Let μ_1 be the expected number of expectorations per hour after a patient has used a placebo.
- Let μ_2 be the expected number of expectorations after the patient has used the cough syrup.
- We want to know if $\mu_2 < \mu_1$.
- Note that here two expectations are compared with NO reference number
- Let $X_1, X_2, ..X_{n_1}$ be the n_1 i.i.d. Random Variables with distribution $P(\mu_1)$.
- Let $Y_1, Y_2, \cdots Y_{n_2}$ be the n_2 i.i.d. Random Variables with distribution $P(\mu_2)$.
- We want to test whether $\mu_2 < \mu_1$ or $\mu_2 = \mu_1$.
- Intuitive idea might be to compare sample means $\overline{X}$ and $\overline{Y}$.

2.2 Coin Toss

- A coin is tossed 80 times and Heads is observed 55 times. Can we conclude that the coin is significantly fair?
- Here $n = 80$ and $X_1, X2, \cdots X_n \sim$ *Bernoulli(p)* .
- We want to test if $p \neq 0.5$ or $p = 0.5$
- Note that the sample mean here is $\overline{X} = \frac{55}{80} = 0.6875$
- If X_i follows Bernoulli distribution then we can say that

$$T_n = \frac{\sqrt{n}\left(\overline{X_n} - p\right)}{\sqrt{p\,(1-p)}} \approx N(0,1).$$

- After putting in the values, we get T_n to be equal to 3.3541
- It then seems quite reasonable to conclude that hypothesis that $p = 0.5$ isn't true as a value of 3.3.541 seems quite unlikely for a $N(0, 1)$ distribution.
- Note that we are saying that 3.3541 is *quite unlikely* for a $N(0, 1)$ distribution. But how can we formalize that?
- We can write that *we are rejecting* $p = 0.5$ *if the observation belong to the set*

  $$\{x :| T_n |> C\} ..$$

  But what value of C should we choose?

3 Definitions

Def: A hypothesis is a statement about the unknown parameters.

Def: Suppose one wants to choose between two reasonable hypotheses $H_0 : \theta \in \Theta_0$ and $H_1 : \theta \in \Theta_1$, where $\Theta_0 \subset \Theta$, $\Theta_1 \subset \Theta$ and $\Theta_0 \cup \Theta_1 = \Phi$. We call H_1 the alternative hypothesis and H_0 the null hypothesis.

Remark: Note that the role of these two hypothesis H_1 and H_0 are asymmetric, therefore one needs to be careful about them.

Approach: We will consider a reasonable statistic and will make the choice based on it.

Def: Let R be a subset of $\mathbb{R}^n$ such that we reject H_0 if $x \in R$. Then R is called **Rejection Region** or **Critical Region** whereas R^C is called **Acceptance Region**.
Note that by size of Critical Region R, one means the probability of R under Null Hypothesis H_0*, i.e.*

$$\textit{Size of Critical Region} = P(R) \textit{ under } H_0.$$

Def: The error committed by *Rejecting* H_0 *when it is actually True* is called **Type-I Error** and the error committed by *Accepting* H_0 *when it is actually False* is called **Type-II Error**.

One can say that Type-I Error is False Negative and Type-II Error is False Positive.

Ultimate AIM: To choose R such that the probabilities of errors of the above two types is **as small as possible**.

3.1 Example Use

Question: Let $X_1, X_2, \cdots X_9 \sim^{i.i.d.} N(\theta, 1)$. Suppose that we want to test $H_0 : \theta = 5.5$ against $H_1 : \theta = 7.5$. Let us consider two critical regions

$R_1 = \left\{x \in \mathbb{R}^9 : \overline{x} > 6\right\}$ and $R_2 = \left\{x \in \mathbb{R}^9 : \overline{x} > 7\right\}$.

Let us compute the probability of errors for region R_1.

$$P\left(Type - IError\right) = P_{\theta=5.5}(\overline{X} > 6) = 1 - P(\overline{X} < 6).$$

Since $\overline{X}$ follows a $N(\theta, \frac{1}{9})$, therefore $P(\overline{X} < 6) = P(\overline{X} - \theta < 6 - \theta) = P(\frac{\overline{X}-6}{\frac{1}{\sqrt{9}}} < \frac{6-\theta}{\frac{1}{\sqrt{9}}}) = P(3\left(\overline{X} - 5.5\right) < 3\left(6 - 5.5\right)) = \Phi(3\left(6 - 5.5\right)) = 0.93319$

Therefore $P(Type - IError) = 1 - 0.93319 = 0.06681$

Similarly, the probability of Type-II error will be:

$$P(Type - IIError) = P_{\theta=7.5}(\overline{X} \leq 6) = \Phi(3\left(6 - 7.5\right)) \approx 0.$$

Similarly, we can compute for R_2.

3.2 Remarks

- If we take $R = \phi$, then $P(Type - I) = 0$ and $P(Type - II) = 1$
- Similarly, if we take $R = \mathbb{R}^n$, then $P(Type - I) = 1$ and $P(Type - II) = 0$
- Notice that if we try to reduce the probability of one type of error, then the probability of other type of error increases
- In this type of optimization problem, we can use some combination of two functions and then try to minimize the combination.
- However, for *Hypothesis Testing* the **APPROACH** is as follows:

  Put a bound on the probability of Type-I Error and try to minimize the probability of Type-II Error.