Assignment 10 - MA212M

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1 Answer 1

Given:

- $$X_1, X_2, \cdots, X_{25} \sim N(\mu, 81)$$
- $$\overline{X} = \frac{1}{25}\sum_{i=1}^{25} x_i = 81.2$$

To Find: 95% Confidence Interval for μ.

So to find $100(1-\alpha)\%$ Confidence Interval, we first need to find a function of θ which is $g(\theta)$ and two statistics $T_1(x)$ and $T_2(x)$ such that $P(T_1(x) \leq g(\theta) \leq T_2(x)) = 1-\alpha$.

Note that the sample mean $\overline{X}$ follows a Normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.

Therefore,

$$\overline{X} \sim N\left(\mu, \frac{81}{25}\right)$$
$$Z = \frac{5\left(\overline{X}-\mu\right)}{9} \sim N(0,1)$$

Now note that the distribution of Z is free from μ, the parameter whose confidence interval we want to find. Thus, the $\alpha\%$ confidence interval will be:

$$P\left(-z_{\frac{\alpha}{2}} \leq Z \leq z_{\frac{\alpha}{2}}\right) = 1-\alpha$$
$$P\left(-z_{\frac{\alpha}{2}} \leq \frac{5\left(\overline{X}-\mu\right)}{9} \leq z_{\frac{\alpha}{2}}\right) = 1-\alpha$$
$$P\left(\overline{X} - \frac{9}{5}z_{\frac{\alpha}{2}} \leq \mu \leq \overline{X} + \frac{9}{5}z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

Therefore, the 95% confidence interval for μ would be:

$$\left[\overline{X} - \frac{9}{5}z_{\frac{\alpha}{2}}, \overline{X} + \frac{9}{5}z_{\frac{\alpha}{2}}\right] \quad \text{for } 100(1-\alpha)\% \text{ confidence interval}$$
$$\left[81.2 - \frac{9}{5}\times 1.96, 81.2 + \frac{9}{5}\times 1.96\right] \text{Note that } z_{\frac{\alpha}{2}} \text{ is } P\left(Z > z_{\frac{\alpha}{2}}\right) = \frac{\alpha}{2}, \text{ hence:}$$
$$\implies [77.67, 84.73]$$

2 Answer 2

Given:

- $$X_1, X_2, \ldots, X_n \sim N\left(\mu, \sigma^2\right).$$
- $$\begin{aligned} \overline{X} &= \frac{1}{25}\sum_{i=1}^{25} X_i = 81.2 \\ S^2 &= \frac{1}{24}\sum_{i=1}^{25}\left(X_i - \overline{X}\right)^2 = 81 \end{aligned}$$

To Find: 95% Confidence Interval for the mean μ.

We know that:

$$\begin{aligned} \overline{X} &\sim N\left(\mu, \frac{\sigma^2}{25}\right) \\ \implies \frac{5\left(\overline{X}-\mu\right)}{\sigma} &\sim N(0,1) \\ \frac{24S^2}{\sigma^2} &\sim \chi^2_{24} \\ \implies \sqrt{\frac{S^2}{\sigma^2}} &\sim \sqrt{\frac{\chi^2_{24}}{24}} \end{aligned}$$

Therefore, if we divide those two distributions, we get:

$$Z = \frac{5\left(\overline{X}-\mu\right)}{S} \sim t_{24}$$

Hence, we now have a function of μ which is only a function of μ and no other parameters. Also, since Student t distribution is symmetric about 0, therefore we can get total $1-\alpha$ probability just like we did in standard normal distribution.

$$\begin{aligned} P\left(-t_{\frac{\alpha}{2},24} \le Z \le t_{\frac{\alpha}{2},24}\right) &= 1-\alpha \\ P\left(-t_{\frac{\alpha}{2},24} \le \frac{5\left(\overline{X}-\mu\right)}{S} \le t_{\frac{\alpha}{2},24}\right) &= 1-\alpha \\ P\left(\overline{X} - \frac{S}{5}t_{\frac{\alpha}{2},24} \le \mu \le \overline{X} + \frac{S}{5}t_{\frac{\alpha}{2},24}\right) &= 1-\alpha \end{aligned}$$

Therefore, the $100\left(1-\alpha\right)\%$ Confidence Interval for μ is:

$$\left[\overline{X} - \frac{S}{5}t_{\frac{\alpha}{2},24}, \overline{X} + \frac{S}{5}t_{\frac{\alpha}{2},24}\right]$$

Therefore, when $\alpha = 0.05$:

$$\begin{aligned} &\left[81.2 - \frac{9}{5}\times 2.064, 81.2 + \frac{9}{5}\times 2.064\right] \\ &\implies [77.4848, 84.9152] \end{aligned}$$

3 Answer 3

Given:

- $$X_1, X_2, \ldots, X_n \sim N(\mu, 16)$$
- $\overline{X}$ is the sample mean
- $\left(\overline{X} - 1, \overline{X} + 1\right)$ is the 90% Confidence Interval for μ.

To Find: The smallest sample size n for the above conditions.

The Confidence Interval for μ is:

$$\left[\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}, \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}\right]$$

Therefore, $\frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} = 1$, hence:

$$\begin{aligned}
\frac{4}{\sqrt{n}} z_{\frac{\alpha}{2}} &= 1 \\
\sqrt{n} &= 4 z_{\frac{\alpha}{2}} \\
n &= 16 z_{\frac{\alpha}{2}}^2
\end{aligned}$$

Since $\alpha = 0.1$, therefore:

$$\begin{aligned}
n &= 16 z_{0.05}^2 \\
n &= 16 \times (1.65)^2 \\
n &= 43.56
\end{aligned}$$

Therefore, $n = 44$.

4 Answer 4

Given:

- $$\begin{aligned} X_1, X_2, \ldots, X_m &\sim N\left(\mu_1, \sigma^2\right) \\ Y_1, Y_2, \ldots, Y_n &\sim N\left(\mu_2, \sigma^2\right) \end{aligned}$$

To Find: The $100\left(1-\alpha\right)\%$ Confidence interval when σ is known and when σ is unknown.

Remember that:

$$\begin{aligned} \overline{X} &\sim N\left(\mu_1, \frac{\sigma^2}{m}\right) \\ \frac{(m-1)\,S_X^2}{\sigma^2} &\sim \chi^2_{m-1} \\ \overline{Y} &\sim N\left(\mu_2, \frac{\sigma^2}{n}\right) \\ \frac{(n-1)\,S_Y^2}{\sigma^2} &\sim \chi^2_{n-1} \end{aligned}$$

4.1 When σ is known

Note that:

$$\overline{X}-\overline{Y} \sim N\left(\mu_1-\mu_2, \frac{\sigma^2}{m}+\frac{\sigma^2}{n}\right)$$

$$Z = \frac{\overline{X}-\overline{Y}-\left(\mu_1-\mu_2\right)}{\sqrt{\frac{\sigma^2}{m}+\frac{\sigma^2}{n}}} \sim N\left(0,1\right)$$

Therefore, $100\left(1-\alpha\right)\%$ confidence interval will be:

$$\begin{aligned} P\left(-z_{\frac{\alpha}{2}} \leq Z \leq z_{\frac{\alpha}{2}}\right) &= 1-\alpha \\ P\left(-z_{\frac{\alpha}{2}} \leq \frac{\overline{X}-\overline{Y}-\left(\mu_1-\mu_2\right)}{\sqrt{\frac{\sigma^2}{m}+\frac{\sigma^2}{n}}} \leq z_{\frac{\alpha}{2}}\right) &= 1-\alpha \\ P\left(\overline{X}-\overline{Y}-\sqrt{\frac{\sigma^2}{m}+\frac{\sigma^2}{n}}z_{\frac{\alpha}{2}} \leq \mu_1-\mu_2 \leq \overline{X}-\overline{Y}+\sqrt{\frac{\sigma^2}{m}+\frac{\sigma^2}{n}}z_{\frac{\alpha}{2}}\right) &= 1-\alpha \end{aligned}$$

Therefore, if σ is known, then the $100\left(1-\alpha\right)$ Confidence Interval for $\mu_1-\mu_2$ is:

$$\left[\overline{X}-\overline{Y}-\sqrt{\frac{\sigma^2}{m}+\frac{\sigma^2}{n}}, \overline{X}-\overline{Y}+\sqrt{\frac{\sigma^2}{m}+\frac{\sigma^2}{n}}\right]$$

4.2 When σ is unknown

When σ is unknown, then first note that:

$$\frac{\sqrt{m}\left(\overline{X}-\mu_1\right)}{\sigma} \sim N\left(0,1\right)$$
$$\frac{\left(m-1\right)S_X^2}{\sigma^2} \sim \chi^2_{m-1}$$
$$\frac{\left(n-1\right)S_Y^2}{\sigma^2} \sim \chi^2_{n-1}$$

Since χ^2 distribution is additive, therefore:

$$\frac{\left(m-1\right)S_X^2}{\sigma^2}+\frac{\left(n-1\right)S_Y^2}{\sigma^2} \sim \chi^2_{m+n-2}$$

Therefore, if we divide the distribution of difference in sample mean with the sum of sample variance, we get:

$$\frac{\overline{X}-\overline{Y}-\left(\mu_1-\mu_2\right)}{\sigma\sqrt{\frac{1}{m}+\frac{1}{n}}} \times \sigma\sqrt{\frac{\left(m+n-2\right)}{\left(m-1\right)S_X^2+\left(n-1\right)S_Y^2}} \sim t_{m+n-2}$$
$$\frac{\overline{X}-\overline{Y}-\left(\mu_1-\mu_2\right)}{S_P}\sqrt{\frac{mn}{m+n}} \sim t_{m+n-2}$$

Hence, the $100\left(1-\alpha\right)\%$ Confidence Interval for the $\mu_1-\mu_2$ is:

$$\left[\overline{X}-\overline{Y}-S_P\sqrt{\frac{m+n}{mn}}t_{\frac{\alpha}{2}}, \overline{X}-\overline{Y}+S_P\sqrt{\frac{m+n}{mn}}t_{\frac{\alpha}{2}}\right]$$

5 Answer 5

Given:

- $$\begin{aligned} X_1, X_2, \ldots, X_7 &\sim N\left(\mu_1, \sigma^2\right) \\ Y_1, Y_2, \ldots, Y_7 &\sim N\left(\mu_2, \sigma^2\right) \end{aligned}$$

- $$\begin{aligned} \overline{X} &= 4.8 \\ S_X^2 &= 8.38 \\ \overline{Y} &= 5.4 \\ S_Y^2 &= 7.62 \end{aligned}$$

- σ is unknown.

To Find: 95% Confidence Interval for $\mu_1 - \mu_2$.

We found the $100\,(1-\alpha)\,\%$ confidence interval for this case in question 4. Therefore, to use that:

$$\begin{aligned} S_P^2 &= \frac{(m-1)\,S_X^2 + (n-1)\,S_Y^2}{m+n-2} \\ &= \frac{1}{2}\,(8.38 + 7.62) \\ &= 8 \end{aligned}$$

Now,

$$t_{\frac{\alpha}{2}} = t_{0.025} = 2.179$$

Therefore, the C.I. is:

$$\left[4.8 - 5.4 - \frac{\sqrt{14 \times 8}}{7} \times 2.179, 4.8 - 5.4 + \frac{\sqrt{14 \times 8}}{7} \times 2.179\right]$$
$$[-3.89, 2.69]$$