Assignment 11 - MA212M

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1 Answer 1

Given:

- $X_1, X_2, \cdots, X_n \sim N\left(\mu, \sigma^2\right)$
- σ is known.

1.1 Answer 1.a

To Find: Most Powerful level α test for $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1$, where $\mu_1 < \mu_0$.

Since both $\Theta_0 = \mu_0$ and $\Theta_1 = \mu_1$ are singleton, therefore, we can use Neyman Pearson Lemma.

Hence, we first need to find $\frac{L(\mu_1)}{L(\mu_0)}$

$$\frac{L(\mu_1)}{L(\mu_0)} = \exp\left(\frac{1}{2\sigma^2}\left\{2n\overline{x}\left(\mu_1 - \mu_0\right) + n\left(\mu_0^2 - \mu_1^2\right)\right\}\right)$$

Therefore, if $\frac{L(\mu_1)}{L(\mu_0)} > k$, this implies that

$$\overline{x} < k_1 \quad \text{where } k_1 > 0, \text{ as } \mu_1 < \mu_0$$

Thus the M.P. level α test is

$$\varphi(x) = \begin{cases} 1 & \text{if } \overline{x} < k_1 \\ \gamma & \text{if } \overline{x} = k_1 \\ 0 & \text{if } \overline{x} > k_1 \end{cases}$$

this should be such that

$$\mathbb{E}_{\mu_0}\left[\varphi(x)\right] = \alpha$$

$$\begin{aligned} \mathbb{E}_{\mu_0}\left[\varphi(x)\right] &= P\left(\overline{X} < k_1\right) + \gamma P\left(\overline{X} = k_1\right) \\ &= P\left(\overline{X} < k_1\right) + 0 \quad \text{as } \overline{X} \text{ follows a Normal distribution} \\ &= P\left(\overline{X} < k_1\right) \quad \text{where } \overline{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right) \\ &\implies \frac{\sqrt{n}\left(\overline{X} - \mu_0\right)}{\sigma} \sim N(0,1) \\ &\implies P\left(\overline{X} < k_1\right) = \Phi\left(\frac{\sqrt{n}\left(k_1 - \mu_0\right)}{\sigma}\right) = \alpha \\ &\implies k_1 = \frac{z_\alpha \sigma}{\sqrt{n}} + \mu_0 \end{aligned}$$

Also, since $P\left(\overline{X} = k_1\right) = 0$ therefore, $\gamma = 0$.

Hence,

$$\varphi(x) = \begin{cases} 1 & \text{if } \overline{x} - \mu_0 < \frac{z_\alpha}{\sqrt{n}}\sigma \\ 0 & \text{otherwise} \end{cases}$$

1.2 Answer 1.b

To Find: Uniformly Most Probable level α test for $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$.

Note that the statement of this question is exactly same as that for that of part 1.a, the U.M.P. level α test is same as in 1.a

2 Answer 2

Given:

- $\phi(.)$ be a M.P. level α test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$

To Show: $\beta(\theta_0) \leq \beta(\theta_1)$ where $\beta(.)$ is the power function of the Most Powerful test.

Consider

$$\phi(x) = \alpha \quad \text{for all } x$$

as a test.

Clearly, $\mathbb{E}_{\theta_0}[\phi(x)] = \alpha$

Hence, $\phi(x)$ is the most powerful level α test. Since $\phi(x)$ is the M.P. level α test, therefore,

$$\begin{aligned} &\beta(\theta) \geq \beta^*(\theta) \quad \text{where } \theta \in \Theta_1 \\ \implies &\beta(\theta_1) \geq \beta^*(\theta_1) \\ \implies &\beta(\theta_1) \geq \alpha \end{aligned}$$

where $\beta^*(.)$ represents power function for all other level α tests.

Note that $\beta(\theta_0)$ represents the probability of rejecting H_0 when θ_0 is the true parameter, i.e. H_0 is correct. Which is nothing but Type-I Error.

Since

$$\begin{aligned} &\beta(\theta_0) = \mathbb{E}_{\theta_0}[\phi(x)] \leq \alpha \\ &\beta(\theta_0) \leq \alpha \end{aligned}$$

Therefore:

$$\beta(\theta_0) \leq \beta(\theta_1)$$

3 Answer 3

Given:

- X_1 and X_2 are two random samples drawn from a PDF $f(x)$, $x \in \mathbb{R}$.
- Consider

$$f_0(x) = \frac{3}{64}x^2 I_{(0,4)}(x)$$
$$f_1(x) = \frac{3}{16}\sqrt{x} I_{(0,4)}(x)$$

To Find: The most powerful level α test for testing $H_0 : f(x) = f_0(x)$ against $H_1 : f(x) = f_1(x)$

The Likelihood function is:

$$L(x_1, x_2) = f(x_1) f(x_2)$$

Since both $\Theta_0 = f_0$ and $\Theta_1 = f_1$ are singleton, thus, using Neyman-Pearson Lemma, we get the M.P. level α test function as

$$\varphi(x) = \begin{cases} 1 & \text{if } \frac{L(\theta_1)}{L(\theta_0)} > k \\ \gamma & \text{if } \frac{L(\theta_1)}{L(\theta_0)} = k \\ 0 & \text{if } \frac{L(\theta_1)}{L(\theta_0)} < k \end{cases}$$

Here, $\frac{L(\theta_1)}{L(\theta_0)}$ is:

$$\frac{L(f_1)}{L(f_0)} = 16 (x_1 x_2)^{\frac{3}{2}}$$

Therefore, if $\frac{L(f_1)}{L(f_0)} > k \implies x_1 x_2 > k_1$ for some $k_1 > 0$.

Consider now a new random variable under H_0:

$$Y = -6 \ln \frac{X_1}{4}$$

Note that $\frac{dx_1}{dy} = -\frac{2}{3}e^{-\frac{y}{6}}$

This gives us the PDF of Y as

$$f_Y(y) = \frac{3}{64}\left(4e^{-\frac{y}{6}}\right)\frac{2}{3}e^{-\frac{y}{6}} \quad \text{if } y > 0$$
$$= \frac{1}{2}e^{-y} \quad \text{if } y > 0$$

Now, note that

$$Y \sim \chi_2^2$$

Also, if $Y_1, Y_2 \sim \chi^2_2$, then $Y_1 + Y_2 \sim \chi^2_4$.

Hence, we can now write

$$\begin{aligned} x_1, x_2 > k_1 &\implies \frac{x_1}{4}\frac{x_2}{4} > k_2 \\ &\implies \ln\frac{x_1}{4} + \ln\frac{x_2}{4} > k_3 \\ &\implies -6\ln\frac{x_1}{4} - 6\ln\frac{x_2}{4} < k_4 \end{aligned}$$

Now, since $\mathbb{E}_{f_0}\left[\varphi\left(x\right)\right] = \alpha$ Therefore (note that $\gamma = 0$ as in the previous question)

$$\begin{aligned} \mathbb{E}_{f_0}\left[\varphi\left(x\right)\right] = P\left(Z < k_4\right) \quad \text{where } Z \sim \chi^2_4 \\ \implies k_4 = \chi^2_{4;1-\alpha} \end{aligned}$$

Hence,

$$\varphi\left(x\right) = \begin{cases} 1 & \text{if } -6\ln\frac{x_1}{4} - 6\ln\frac{x_2}{4} < \chi^2_{4;1-\alpha} \\ 0 & \text{otherwise} \end{cases}$$

4 Answer 4

Given:

- $X_1, X_2, \ldots, X_n \sim P\left(\lambda\right)$ where $\lambda > 0$.

TO Find: Most Powerful level α test for $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda = \lambda_1\ (> \lambda_0)$

To use Neyman Pearson Lemma, we first need the Likelihood:

$$L\left(\lambda\right) = \left(\frac{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}{\prod_{i=1}^n x_i!}\right)$$

Hence,

$$\frac{L\left(\lambda_1\right)}{L\left(\lambda_0\right)} = \left(\frac{\lambda_1}{\lambda_0}\right)^{\sum_{i=1}^n x_i} \times e^{-n\left(\lambda_1 - \lambda_0\right)}$$

This implies that,

$$\begin{aligned} \frac{L\left(\lambda_1\right)}{L\left(\lambda_0\right)} = \left(\frac{\lambda_1}{\lambda_0}\right)^{\sum_{i=1}^n x_i} \times e^{-n\left(\lambda_1 - \lambda_0\right)} > k \\ \implies \sum_{i=1}^n x_i > k_1 \quad \text{as } \lambda_1 > \lambda_0 \end{aligned}$$

Therefore, the M.P. level α test would be:

$$\varphi\left(x\right) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i > k_1 \\ \gamma & \text{if } \sum_{i=1}^n x_i = k_1 \\ 0 & \text{if } \sum_{i=1}^n x_i < k_1 \end{cases}$$

such that $\mathbb{E}_{\lambda_0}\left[\varphi\left(x\right)\right] = \alpha$.

Now, remember that if $X_i \sim P\left(\lambda_i\right)$, then $\sum_{i=1}^{n} X_i = P\left(\sum_{i=1}^{n} \lambda_i\right)$

$$\mathbb{E}_{\lambda_0}\left[\varphi\left(x\right)\right] = P\left(\sum_{i=1}^{n} X_i > k_1\right) + \gamma P\left(\sum_{i=1}^{n} X_i = k_1\right) = \alpha$$

Note that $\sum_{i=1}^{n} X_i \sim P\left(n\lambda_0\right) = Y$ under H_0, hence:

$$\begin{aligned}\mathbb{E}_{\lambda_0}\left[\varphi\left(x\right)\right] = P\left(Y > k_1\right) + \gamma P\left(Y = k_1\right) = \alpha \\ \text{Take} \quad P\left(Y > k_1\right) \leq \alpha \leq P\left(Y \geq k_1\right) \\ \gamma = \frac{\alpha - P\left(Y > k_1\right)}{P\left(Y = k_1\right)}\end{aligned}$$

5 Answer 5

Given:

- $X_1, X_2, \ldots, X_n \sim f\left(x;\theta\right)$.
- $$f\left(x;\theta\right) = \theta^{-1} e^{-\frac{x}{\theta}} I_{(0,\infty)}\left(x\right).$$
- $\theta > 0$ and $\alpha \in (0,1)$.

To Find: A Likelihood Ratio Test for $H_0 : \theta = \theta_0\left(> 0\right)$ against $H_1 : \theta \neq \theta_0$.

Firstly, to find $sup_{\theta\in\Theta_0} L\left(\theta, x\right)$:

$$sup_{\theta\in\Theta_0} L\left(\theta\right) = L\left(x;\theta_0\right) = \theta_0^{-n} e^{-\frac{1}{\theta_0}\sum_{i=1}^{n} x_i} \quad \text{where } x_i > 0 \ \forall \ i \in \{1,2,\ldots,n\}$$

Second, to find $sup_{\theta\in\Theta_0\cup\Theta_1} L\left(\theta, x\right)$.

$$sup_{\theta\in\Theta_0\cup\Theta_1} L\left(\theta, x\right) = L\left(x;\overline{x}\right) = \overline{x}^{-n} e^{-\frac{1}{\overline{x}}\sum_{i=1}^{n} x_i} \quad \text{where } x_i > 0 \ \forall \ i \in \{1,2,\ldots,n\}$$

Therefore

$$\begin{aligned}\Lambda\left(x\right) &= \left(\frac{\theta_0}{\overline{x}}\right)^{-n} e^{-\frac{1}{\theta_0} n\overline{x} + n} \\ &= \frac{\overline{x}^n}{\theta_0^n} e^{-\frac{1}{\theta_0}\overline{x}n + n}\end{aligned}$$

Now, we need to analyze this function closely to draw a statistic from $\Lambda\left(x\right) < k$.

$$\begin{aligned}\ln\Lambda\left(x\right) = n\ln\overline{x} - n\ln\theta_0 - \frac{1}{\theta_0}\left(\overline{x}n\right) + n \\ \frac{d}{d\overline{x}}\left(\ln\Lambda\left(x\right)\right) = \frac{n}{\overline{x}} - \frac{n}{\theta_0} \\ \frac{d^2}{d\overline{x}^2}\left(\ln\Lambda\left(x\right)\right) = -\frac{n}{\overline{x}^2} < 0\end{aligned}$$

Hence, $\Lambda(x)$ has a maxima at $\overline{x} = \theta_0$.

Therefore, if $\Lambda(x) < k \implies$ either $\overline{x} < k_1$ or $\overline{x} > k_2$

Now, LRT level α can be found as:

$$\varphi(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i < k_1 or \sum_{i=1}^n x_i > k_2 \\ 0 & \text{otherwise} \end{cases}$$

where $E_{\Theta_0}[\varphi(x)] = \alpha$, thus:

$$\begin{aligned} E_{\theta_0}[\varphi(x)] &= P\left(\sum_{i=1}^n X_i < k_1 or \sum_{i=1}^n X_i > k_2\right) \\ &= P\left(\sum_{i=1}^n X_i < k_1\right) + P\left(\sum_{i=1}^n X_i > k_2\right) \end{aligned}$$

Now, note that $X_i \sim Exp\left(\frac{1}{\theta}\right)$, thus $\sum_{i=1}^n X_i = Y = Gamma\left(n, \frac{1}{\theta}\right)$

Which will translate to

$$P(Y < k_1) + P(Y > k_2) = \alpha$$

We can choose these two probabilities as simply:

$$\begin{aligned} P(Y < k_1) &= \frac{\alpha}{2} \\ P(Y > k_2) &= \frac{\alpha}{2} \end{aligned}$$

as there are no restrictions on each of them, the only constraint is their sum is $= \alpha$.

Hence,

$$\begin{aligned} k_1 &= G_{k_1; \frac{\alpha}{2}} \\ k_2 &= G_{k_2; 1-\frac{\alpha}{2}} \end{aligned}$$