Indian Institute of Technology, Guwahati

Pre-Final Year : 6^{th} Semester

Measure Theory

Animesh Renanse

Instructed by
Prof. Pratyoosh Kumar

January 21, 2021

Contents

1 Introduction

Measure Theory is a field of mathematics which deals with formalizing the notion of a *measure* of a given quantity. Traditional notion of a *measure* fails in lots of complicated scenario. Hence, we need a rigorous definition of it. The work in this direction was forwarded by Lebesgue and others in the dawn of 20^{th} century.

1.1 Few introductory definitions

These are few of the basic definitions that one might remember from real analysis.

- **Limit Points** : $x \in X$ is called a *limit point* of a subset $S \subseteq X$ if $\forall\, r > 0$, $\exists\, a \neq x$ such that $a \in S \cap B_r(x)$. That is, ball of any size r around x contains atleast one point of S.
- **Isolated Points** : $y \in S$ is called an *isolated point* of a subset $S \subseteq X$ if $\exists\, r > 0$ such that $(B_r(y) \setminus \{y\}) \cap S = \Phi$. That is, $B_r(y)$ contains no other point of S apart from y.
  - Also note that every point of closure $\overline{S}$ is either a limit point or an isolated point of S.
  - More specifically, *any subset of* $\mathbb{R}^d$ *is closed if and only if it contains all of it's limit points.*
- **Perfect Set** : A is called a perfect set if $A = A'$ where A' is the set of all *limit points* of A. More conveniently, if A does not contain any isolated points then it is a perfect set. $\mathbb{R}$ is a perfect set.
- **Symmetric Difference** : A and B are two sets then symmetric difference is $A\Delta B = (A \setminus B) \cup (B \setminus A)$.
- **Power Set** : Collection of all subsets of a set S, written as $P(S)$.
- **Lower Bound** : A *lower bound* of a subset S of a poset $(P, \leq)$ is an element $a \in P$ such that $a \leq x$ for all $x \in S$.
- **Infimum** : A lower bound $p \in P$ is called an *infimum* of S if for all lower bounds y of S in P, $y \leq p$.
- **Upper Bound** : An *upper bound* of a subset S of a poset P is an element $b \in P$ such that $b \geq x$ for all $x \in S$.
- **Supremum** : An upper bound $u \in P$ is called a *supremum* of S if for all upper bounds z of S in P, $z \geq u$.
- **Lower Sum** : $l(f, \mathcal{P})$ is the sum of the minimum functional values at the partition. That is,

  $$l(f,\mathcal{P}) = \sum_{i=0}^{n-1} m_i(a_{i+1} - a_i)$$

  where $m_i = \inf\{f(x) \mid x \in [a_{i-1}, a_i]\}$.
- **Upper Sum** : Similarly,

  $$u(f,\mathcal{P}) = \sum_{i=0}^{n-1} M_i(a_{i+1} - a_i)$$

  where $M_i = \sup\{f(x) \mid x \in [a_{i-1}, a_i]\}$.
  Remember that the function is *Riemann Integrable* if $l(f, \mathcal{P}) = u(f, \mathcal{P})$.
- **Countable Sets** : Note the following,
  1. *Cardinality* : Sets X and Y have the same cardinality if there exists a bijection from X to Y.
  2. *Finite Set* : A set is finite if it is empty or it has the same cardinality as $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
  3. *Countably Infinite* : If the set has the same cardinality as $\mathbb{N}$.
  4. *Enumeration* : An enumeration of a countably infinite set X is a bijection of $\mathbb{N}$ onto X. That is, an enumeration is an infinite sequence $\{x_n\}$ such that each of the x_i's are in X and each element of X is x_i for some i.
  5. *Countable* : A set is countable if it is finite or countably infinite. For example, $\mathbb{N}$ is countable, $\mathbb{Q}$ is also countable (!), $\mathbb{R} \setminus \mathbb{Q}$ (irrationals) is not countable, $\mathbb{R}$ is not countable.
- **Totally Bounded** : A subset $B \subseteq X$ is *totally bounded* when it can be covered by a finite number of r-balls for all $r > 0$. That is,

  $$\forall\, r > 0,\ \exists N \in \mathbb{N},\ \exists a_1, \ldots, a_N \in X \text{ such that } B \subseteq \bigcup_{n=1}^{N} B_r(a_n)$$

- **Compact Set** : A set K is said to be *compact* when given any cover of balls of possibly unequal radii, there is a finite sub-collection of them that still covers the set K. That is,

$$K \subseteq \bigcup_i B_{r_i}(a_i) \implies \exists\, i_1, \ldots, i_N,\ \ K \subseteq \bigcup_{n=1}^{N} B_{r_{i_n}}(a_{i_n})$$

  Note that *compact metric spaces are totally bounded (!)*. Also, *compact sets are closed.*

The problem begins with Riemann Integrable functions when we see that functions like Dirichlet function (1 on irrational and 0 on rational points) can become *measurable* even when the function is not continuous! This motivates the need of a formal notion of a *measure.*

1.2 Homework - I

1. Every open set in $\mathbb{R}$ can be written as disjoint union of open intervals.

   Proof. Let $G \subseteq \mathbb{R}$ be a open subset. Now by definition of an open subset, we have that for any $x \in G$, there exists atleast one open subset U such that $x \in U \subseteq G$. Now consider the following union of all such open subsets of x,

$$U_x = \bigcup_{x \in U \subseteq G} U$$

   It's now easy to see that U_x is the largest such subset of G, as any other $V \subseteq G$ such that $x \in V$ is by definition contained in U_x. Moreover, U_x is an interval as it is an arbitrary union of open intervals. Now, define the following relation on G:

$$y \sim x \iff y \in U_x$$

   Now we clearly have that $x \in U_x$ (reflexive); for $y \sim U_x$ we have $U \subseteq U_x$ such that $x, y \in U$, hence $x \in U_y$ (symmetric); for $x \in U_y$ and $y \in U_z$, we have that $x, y, z \in U_y$, since $z \in U_y \subseteq G$ so $U_y \subseteq U_z$, so $x \in U_z$ (transitive). Hence $\sim$ is an equivalence relation, hence $\sim$ partitions the set G. Denote the set of all equivalence classes as $\mathcal{I}$ so we get

$$G = \bigcup_{I \in \mathcal{I}} I$$

   such that $I_1 \cap I_2 = \Phi$ for any $I_1, I_2 \in \mathcal{I}$. Now note that for any $I \in \mathcal{I}$ is open because each I is generated by the relation $\sim$ such that $y \sim x$ iff $y \in U_x$. Hence for any $z \in I$, we have $z \in U_x \subseteq G$ where U_x is open. Therefore, we have $G = \cup_{I \in \mathcal{I}} I$ for disjoint open intervals in $\mathcal{I}$. ■

2. Prove that every non-empty perfect subset of $\mathbb{R}$ (or $\mathbb{R}^n$) is uncountable. That is, if $A = A'$ then A is uncountable.

   Proof. Take $A \subseteq \mathbb{R}$ to be a perfect subset. Since A it is perfect, therefore, it must contain all of it's limit points or, equivalently, contains no isolated points. Clearly, then, A cannot be finite, but can only be countably infinite or uncountable. If it is uncountable, then the proof is over. If A is countably infinite, then we can write A as the following :

$$A = \{a_1, a_2, \ldots\}.$$

   Construct a ball around a_{i_1} of any radius $r_1 > 0$. Since A is perfect, therefore $\exists\, a_{i_2} \in B_{r_1}(a_{i_1}) \cap A = C_1$. Similarly, for some $r_2 > 0$, we have $a_{i_3} \in B_{r_2}(a_{i_2}) \cap B_{r_1}(a_{i_1}) \cap A = C_2$ such that $a_{i_1} \notin C_2$ and so on. In general, we would have the following,

$$a_{i_{n+1}} \in \left(\bigcap_{j=1}^{n} B_{r_j}(a_{i_j}) \right) \cap A = C_n.$$

   Now, consider $C = \cap_n C_n$. Since $C_{n+1} \subseteq C_n$, therefore $C \neq \Phi$. But, $a_i \notin C$ for any $i \in \mathbb{N}$ as $a_i \notin C_{i+1}$. Therefore we have a contradiction. Hence A cannot by countably infinite, it must only be uncountable. ■

3. In the definition of Lebesgue Outer Measure on $\mathbb{R}$, one can instead take $\mathscr{C}_A$ to be collection of infinite sequences of the any form from $\{[a_n, b_n]\}, \{(a_n, b_n)\}$ or $\{(a_n, b_n]\}$.

   Proof. Refer Proof of Proposition 9. ■

4. Show the following:

$$\bigcup_{n=1}^{N} E_n = \bigcup_{n=1}^{N} \left(E_n \cap \left(\bigcup_{k<n} E_k \right)^{c} \right)$$

Proof. Take $x \in \bigcup_{n=1}^{N} E_n$. Then $\exists\ E_k$ for some a such that $x \in E_a$. Now, clearly, $x \in E_a \subseteq \left(\bigcup_{k<a} E_k\right)^c$, hence $x \in \left(E_a \cap \left(\bigcup_{k<a} E_k\right)^{\mathrm{c}}\right)$. Hence, we have $\bigcup_{n=1}^{N} E_n \subseteq \bigcup_{n=1}^{N} \left(E_n \cap \left(\bigcup_{k<n} E_k\right)^{\mathrm{c}}\right)$. The converse is easy to see too. ■

1.3 Measure of an Interval

In the definition of Riemann Integral, we see that a function is Riemann Integrable if $u(f, \mathcal{P}) = l(f, \mathcal{P})$. But it depends upon the definition of the *length of an interval*.
Note that in the traditional notion of *measure*, the number of points has no contribution in length. Hence the question becomes:

How to measure a set A in $\mathbb{R}$?

For this, we define *Outer Measure*.

1.3.1 Outer Lebesgue Measure

The concept of this topic began from the following question : If we consider the cauchy sequence of Riemann Integrable functions on $\mathbb{R}$, will it converge to a Riemann Integrable function?
We follow the following notion of measure (atleast for the first part of the course). We do approximation by approximation.

Suppose $A \subseteq \mathbb{R}$. We define $m^*(A)$ as the *Outer Lebesgue Measure* defined by

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} l(I_i) \mid A \subseteq \bigcup_i I_i \right\}$$

where $I_i = [a_i, b_i)$. Since we are taking infimum of the the collection, hence Note that it is approximating measure from outside. Also note that this is different from the definition of measurable sets.
Outer measure is a function $m^* : P(\mathbb{R}) \to [0, \infty]$.

2 Measures

2.1 Algebras & Sigma-Algebras

$\boxed{\star}$ **Definition 1.** (**Algebra/Field**) Let X be an arbitrary set. A collection $\mathscr{A} \subseteq P(X)$ of subsets of X is an algebra on X if:

- $X \in \mathscr{A}$.
- $A \in \mathscr{A} \implies A^c \in \mathscr{A}$.
- For each finite sequence $A_1, A_2, \ldots, A_n \in \mathscr{A}$ implies that

$$\bigcup_{i=1}^{n} A_i \in \mathscr{A}$$

- For each finite sequence $A_1, A_2, \ldots, A_n \in \mathscr{A}$ implies that

$$\bigcap_{i=1}^{n} A_i \in \mathscr{A}$$

$\boxed{\star}$ **Definition 2.** (**σ-Algebra/σ-Field**) Let X be an arbitrary set. A collection $\mathscr{A} \subseteq P(X)$ of subsets of X is a σ-algebra on X if:

- $X \in \mathscr{A}$.
- $A \in \mathscr{A} \implies A^c \in \mathscr{A}$.
- For each infinite sequence $\{A_i\}$ such that $A_i \in \mathscr{A}$, it implies that

$$\bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$$

- For each infinite sequence $\{A_i\}$ such that $A_i \in \mathscr{A}$, it implies that

$$\bigcap_{i=1}^{\infty} A_i \in \mathscr{A}$$

✓ **Proposition 1.** Let X be a set. Then the intersection of an arbitrary non-empty collection of σ-algebras on X is a σ-algebra on X.

Proof. Consider a collection $\mathscr{C}$ of σ-algebras on X. Denote $\mathscr{A} = \bigcap \mathscr{C}$ as intersection of all σ-algebras in $\mathscr{C}$. We can now easily see that any subset in $\mathscr{A}$ would be present in every σ-algebra present in collection $\mathscr{C}$, hence, it would obey all properties of a σ-algebras. Therefore, $\mathscr{A}$ is a σ-algebra. ∎

→ **Corollary 1.** Let X be a set and let $\mathscr{F} \subseteq P(X)$ be a family of subsets of X. Then there exists a smallest σ-algebra on X that includes $\mathscr{F}$.

Proof. Consider any given family $\mathscr{F} \subseteq P(X)$ and just take intersection of the family $\mathscr{C}$ of all σ-algebras which contains $\mathscr{F}$ to construct this smallest σ-algebra. ∎

$\boxed{\star}$ **Definition 3.** (**Generated σ-algebra**) The smallest σ-algebra on X containing a given family $\mathscr{F} \subseteq P(X)$ of subsets is called the σ-algebra *generated* by $\mathscr{F}$, denoted as $\sigma(\mathscr{F})$.

$\boxed{\star}$ **Definition 4.** (**Borel σ-algebra on** $\mathbb{R}^d$) It is the σ-algebra on $\mathbb{R}^d$ generated by the collection of all open subsets of $\mathbb{R}^d$, denoted as $\mathscr{B}(\mathbb{R}^d)$.

$\boxed{\star}$ **Definition 5.** (**Borel Subsets of** $\mathbb{R}^d$) Any $A \subseteq \mathbb{R}^d$ is called a Borel subset of $\mathbb{R}^d$ if $A \in \mathscr{B}(\mathbb{R}^d)$.

✓ **Proposition 2.** The Borel σ-algebra on $\mathbb{R}$, $\mathscr{B}(\mathbb{R})$, of Borel subsets of $\mathbb{R}$ is generated by each of the following collection of sets:

1. The collection of all closed subsets of $\mathbb{R}$.
2. The collection of all subintervals of $\mathbb{R}$ of the form $(-\infty, b]$.
3. The collection of all subintervals of $\mathbb{R}$ of the form $(a, b]$.

Proof. To show all of these, consider the three σ-algebras $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ corresponding to conditions 1,2 & 3 respectively and try to prove $\mathscr{A}_3 \subseteq \mathscr{A}_2 \subseteq \mathscr{A}_1 \subseteq \mathscr{B}(\mathbb{R})$ together with $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{A}_3$. The first three inclusions are trivial to see. For the case that $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{A}_3$, simply note that any open subset can be made by unions of the sets of form $(a, b]$ and by Homework-I,1, each open set is union of open subsets. ■

✓ **Proposition 3.** The σ-algebra $\mathscr{B}(\mathbb{R}^d)$ of Borel subsets of $\mathbb{R}^d$ is generated by each of the following collections:

1. The collection of all closed subsets of $\mathbb{R}^d$.
2. The collection of all closed half-spaces in $\mathbb{R}^d$ that have the form $\{(x_1, \dots, x_d) \mid x_i \leq b\}$ for some index i and some $b \in \mathbb{R}$.
3. The collection of all rectangles in $\mathbb{R}^d$ that have the form

$$\{(x_1, \dots, x_d) \mid a_i < x_i \leq b_i \text{ for } i = 1, \dots, d\}$$

Proof. Almost the same as in Proposition 2. $\mathscr{A}_1 \subseteq \mathscr{B}(\mathbb{R}^d)$ trivially by definition. $\mathscr{A}_2 \subseteq \mathscr{A}_1$ as $\{(x_1, \dots, x_d) \mid x_i \leq b\}$ is closed itself. $\mathscr{A}_3 \subseteq \mathscr{A}_2$ by the observation that $\{(x_1, \dots, x_d) \mid a_i < x_i \leq b_i\}$ is made by the difference of two subsets of the form $\{(x_1, \dots, x_d) \mid x_i \leq b_i\}$ and $\{(x_1, \dots, x_d) \mid x_i > a_i\}$, the latter is the complement of a certain subset in $\mathscr{A}_2$, moreover, $\{(x_1, \dots, x_d) \mid a_i < x_i \leq b_i \text{ for } i = 1, \dots, d\}$ is then constructed by intersection of d such subsets. Finally, $\mathscr{B}(\mathbb{R}^d) \subseteq \mathscr{A}_3$ can be seen via the fact that open subsets in $\mathbb{R}^d$ are made by union of rectangles of type 3 and as such, they are called open subsets. ■

Conditions for an algebra to become a σ-algebra.

✓ **Proposition 4.** Let X be a set and let $\mathscr{A}$ be an algebra on X. Then, $\mathscr{A}$ is a σ-algebra on X if **either**

- $\mathscr{A}$ is closed under the formation of **unions** of *increasing* sequence of sets, or,
- $\mathscr{A}$ is closed under the formation of **intersections** of *decreasing* sequence of sets.

Proof. Take any countably infinite collection of subsets $A_1, A_2, \dots \in \mathscr{A}$ where $\mathscr{A}$ is an algebra. Due to the definition of an algebra, we have that $C_n = \bigcup_{i=1}^n A_i \in \mathscr{A}$ for any $n \geq 1 \in \mathbb{Z}_+$. Now note that $C_1 \subseteq C_2 \subseteq \dots$, that is, the sequence $\{C_n\}$ forms an increasing sequence of sets. Hence, by the requirement of the question, we have that $\bigcup_{i=1}^\infty C_i \in \mathscr{A}$. But then we also have that $\bigcup_{i=1}^\infty A_i \subseteq \bigcup_{i=1}^\infty C_i \in \mathscr{A}$. Hence we have the required condition for part 1. For part 2, we can see that $C_1^c \supseteq C_2^c \supseteq \dots$ is a decreasing sequence of sets. Then we must have, by the requirement of the question, that $\bigcap_{i=1}^\infty C_i^c = \left(\bigcup_{i=1}^\infty C_i\right)^c \in \mathscr{A}$. But then by definition of algebra, we must have $\bigcup_{i=1}^\infty C_i \in \mathscr{A}$, which already contains the countably infinite union $\bigcup_{i=1}^\infty A_i$. ■

2.2 Measures

$\boxed{\star}$ **Definition 6.** (**Countably Additive Function**) Let X be a set and $\mathscr{A}$ be a σ-algebra on X. Function $\mu : \mathscr{A} \longrightarrow [0, +\infty]$ is said to be *countably additive* **if** it satisfies:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

for **each** infinite sequence $\{A_i\}$ of **disjoint** sets in $\mathscr{A}$.

$\boxed{\star}$ **Definition 7.** (**Measure**) A *measure* on $\mathscr{A}$ is a function $\mu : \mathscr{A} \to [0, +\infty]$ that is **countably additive** and satisfies:

$$\mu(\Phi) = 0.$$

Remark. This is sometimes also referred as *countably additive measure* on $\mathscr{A}$.

$\boxed{\star}$ **Definition 8.** We have following definitions to compactly represent above definitions:

1. (***Measure Space***) If X is a set, $\mathscr{A}$ is a σ-algebra on X and if μ is a measure on $\mathscr{A}$, then the triple $(X, \mathscr{A}, \mu)$ is called a *measure space*.
2. (***Measurable Space***) If X is a set and $\mathscr{A}$ is a σ-algebra on X, then the pair $(X, \mathscr{A})$ is called a *measurable space*.

✓ **Proposition 5.** Let $(X, \mathscr{A}, \mu)$ be a measure space and let $A, B \in \mathscr{A}$ such that $A \subseteq B$. Then,

- We have $\mu(A) \leq \mu(B)$.
- Additionally, if A satisfies that $\mu(A) < +\infty$, then:

$$\mu(B - A) = \mu(B) - \mu(A).$$

Proof. Note that A and $B \cap A^c$ are disjoint sets in the sigma algebra $\mathscr{A}$. Hence we can write, by countably additive property of μ, that:

$$\begin{aligned} \mu(A \cup (B \cap A^c)) &= \mu(B) \\ &= \mu(A) + \mu(B \cap A^c) \end{aligned}$$

Since $\mu(B \cap A^c) \geq 0$, hence $\mu(A) \leq \mu(B)$. Moreover, if $\mu < \infty$, then we can additionally write $\mu(B \cap A^c) = \mu(B) - \mu(A)$. ■

$\boxed{\star}$ **Definition 9.** Let μ be a measure on a measurable space $(X, \mathscr{A})$. Then,

- (***Finite Measure***) If $\mu(X) < +\infty$.
- (***σ-Finite Measure***) If $X = \bigcup_i A_i$ where $A_i \in \mathscr{A}$ such that $\mu(A_i) < +\infty$ for all $i \in \mathbb{N}$.

Remark. In other words, a subset $A \in \mathscr{A}$ is *σ-finite* if it is a union of a countable sequence of sets that are in $\mathscr{A}$ and are of finite measure under μ.

2.2.1 Elementary Properties of Measures

✓ **Proposition 6.** Let $(X, \mathscr{A}, \mu)$ be a measure space. If $\{A_k\}$ is an **arbitrary** sequence of sets that belong to $\mathscr{A}$, then,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

Proof. Denote $B_1 = A_1$ and $B_i = A_i \cap \left(\bigcup_{k=1}^{i-1} A_k\right)^c$. Note that B_i and B_j are disjoint for distinct i and j. Since $\{A_k\} \in \mathscr{A}$, therefore $\{B_i\} \in \mathscr{A}$. Moreover, $\bigcup_{i=1}^{\infty} B_i = \bigcup_{k=1}^{\infty} A_k$ by construction. We then get,

$$\begin{aligned} \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) &= \sum_{i=1}^{\infty} \mu(B_i) \\ &\leq \sum_{i=1}^{\infty} \mu(A_i) \quad (\because\ B_i \subseteq A_i \text{ by construction.}) \end{aligned}$$

Hence proved. ■

✓ **Proposition 7.** Let $(X, \mathscr{A}, \mu)$ be a measure space.

- If $\{A_k\}$ is an **increasing** sequence of sets in $\mathscr{A}$, then

$$\mu\left(\bigcup_k A_k\right) = \lim_k \mu(A_k).$$

- If $\{A_k\}$ is a **decreasing** sequence of sets in $\mathscr{A}$ **and** if $\mu(A_n) < +\infty$ holds for some n, then

$$\mu\left(\bigcap_k A_k\right) = \lim_k \mu(A_k).$$

Proof. Consider $\{A_i\} \in \mathscr{A}$ be an increasing sequence of subsets of X. Define $B_1 = A_1$ and $B_i = A_i - A_{i-1} = A_i \cap A_{i-1}^c$. Clearly, B_i are disjoint for distinct i. Hence, we can now write,

$$\mu\left(\bigcup_i B_i\right) = \sum_{i=1}^{\infty} \mu(B_i).$$

Moreover, $\bigcup_i B_i = \bigcup_k A_k$ and $B_i = \bigcup_{j=1}^{i} A_j$, therefore we can modify the above as

$$\mu\left(\bigcup_k A_k\right) = \lim_{k\to\infty} \sum_{i=1}^{k} \mu\left(\bigcup_{j=1}^{i} A_j\right) = \lim_{k\to\infty} \mu\left(\bigcup_{i=1}^{k}\bigcup_{j=1}^{i} A_j\right) = \lim_{k\to\infty} \mu(A_k)$$

Now, consider $\{A_k\}$ to be a decreasing sequence of subsets in $\mathscr{A}$. Take $n \in \mathbb{N}$ such that $\mu(A_n) < +\infty$. Denote $B_i = A_n - A_{n+i}$. We have each B_i to be disjoint by construction for any two i's and $\{B_i\}$ is increasing. Moreover, $\bigcup_k B_k = A_n - \bigcap_i A_{n+i}$. Hence by first part, we have:

$$\begin{aligned}
\mu\left(\bigcup_k B_k\right) &= \mu\left(A_n - \bigcap_i A_{n+i}\right) \\
&= \mu(A_n) - \mu\left(\bigcap_i A_{n+i}\right) \quad \text{(Since } \{A_k\} \text{ is decreasing)} \\
&= \lim_{k\to\infty} \mu(B_k) = \lim_{k\to\infty} \mu(A_n - A_{n+k}) \quad \text{(Part 1.)} \\
&= \mu(A_n) - \lim_{k\to\infty} \mu(A_{n+k})
\end{aligned}$$

Hence, we get the second result (from 2^{nd} and 4^{th} equality). ∎

The following partial converse of Proposition 7 helps in checking whether finitely additive measure is countably additive.

✓ **Proposition 8.** Let $(X, \mathscr{A})$ be a measurable space and let μ be a **finitely additive** measure on $(X, \mathscr{A})$. Then μ is a countably additive measure if **either**

- The following holds for each **increasing sequence** $\{A_k\}$ of sets that belong to $\mathscr{A}$:

$$\lim_k \mu(A_k) = \mu\left(\bigcup_k A_k\right)$$

or,

- The following holds for each **decreasing sequence** $\{A_k\}$ that belongs to $\mathscr{A}$ which **satisfies** $\bigcap_k A_k = \Phi$:

$$\lim_k \mu(A_k) = 0.$$

Proof. Consider any disjoint sequence of sets $\{A_i\}$ in $\mathscr{A}$. Let the first part be true. We can construct an increasing sequence from $\{A_i\}$ by considering a new sequence $\{B_i\}$ where $B_k = \bigcup_{j=1}^{k} A_j$. Now, finite additivity of μ means that $\mu(B_k) = \mu\left(\bigcup_{j=1}^{k} A_j\right) = \sum_{j=1}^{k} \mu(A_j)$. Also, by the condition of part 1, we have $\lim_{k\to\infty} \mu(B_k) = \mu(\bigcup_k B_k) = \mu(\bigcup_k A_k)$. Hence proved part 1.

For part 2, we simply have for any disjoint sequence $\{A_i\}$ in $\mathscr{A}$, a decreasing sequence $\{B_k\}$ given as $B_k = \bigcup_{i=k}^{\infty} A_i$. Again, we can write,

$$
\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mu\left(\left(\bigcup_{i=1}^{k} A_i\right) \bigcup \left(\bigcup_{j=k+1}^{\infty} A_j\right)\right) \\
&= \mu\left(\left(\bigcup_{i=1}^{k} A_i\right) \bigcup B_{k+1}\right) \\
&= \mu\left(\bigcup_{i=1}^{k} A_i\right) + \mu\left(B_{k+1}\right) \quad \text{(Since these two are disjoint.)} \\
&= \sum_{i=1}^{k} \mu\left(A_i\right) + \mu\left(B_{k+1}\right)
\end{aligned}
$$

Taking $\lim\limits_{k\to\infty}$ both sides then yields us

$$
\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right) + 0
$$

as $\lim\limits_{k\to\infty} \mu\left(B_k\right) = 0$ is given in the assumption of part 2. ■

2.3 Outer Measures

$\boxed{\star}$ **Definition 10.** (**Outer Measure**) Let X be a set and let $\mathscr{P}(X)$ be the collection of all subsets of X. An *outer measure* on X is a function $\mu^* : \mathscr{P}(X) \longrightarrow [0, +\infty]$ such that:

- For the empty set Φ,
$$\mu^*(\Phi) = 0$$
- If $A \subseteq B \subseteq X$, then
$$\mu^*(A) \leq \mu^*(B).$$
- If $\{A_n\}$ is an infinite sequence of subsets of X, then
$$\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n)$$

$\boxed{\star}$ **Definition 11.** (**Lebesgue Outer Measure on** $\mathbb{R}$) For each subset $A \subseteq \mathbb{R}$, let $\mathscr{C}_A$ be the *set of all infinite sequences* $\{(a_i, b_i)\}$ of *bounded open intervals* such that $A \subseteq \bigcup_i (a_i, b_i)$. That is,

$$\mathscr{C}_A = \{\{(a_i, b_i)\} \mid A \subseteq \cup_i (a_i, b_i) \text{ and } a_i, b_i \in \mathbb{R}\}$$

Then, $\lambda^* : \mathscr{P}(\mathbb{R}) \longrightarrow [0, +\infty]$ is the Lebesgue outer measure, defined by:

$$\lambda^*(A) = \inf\left\{\sum_i (b_i - a_i) \;\middle|\; \{(a_i, b_i)\} \in \mathscr{C}_A\right\} \tag{1}$$

To verify that λ^* is indeed an outer measure.

✓ **Proposition 9.** Lebesgue outer measure on $\mathbb{R}$ is an outer measure and it assigns to each subinterval of $\mathbb{R}$ it's length.

Proof. Denote $\mathscr{C}_A = \{\{(a_i, b_i)\} \mid A \subseteq \cup_i (a_i, b_i)\}$. To show that λ^* is an outer measure, we first need to show that $\lambda^*(\Phi) = 0$. For that, consider the set of all infinite sequences $\{(a_i, b_i)\} \in \mathscr{C}_\Phi$, that is (trivially) $\Phi \subseteq \cup_i (a_i, b_i)$, such that $\sum_i (b_i - a_i) < \epsilon$ for all $\epsilon > 0$. Then, if we denote $\mathscr{L}_A = \{\sum_i (b_i - a_i) \mid \{(a_i, b_i)\} \in \mathscr{C}_A\}$, then $\inf \mathscr{L}_\Phi = 0$ as for any lower bound l of $\mathscr{L}_A$, if $l > 0$ then $\exists\, \{(a_i, b_i)\} \in \mathscr{C}_\Phi$ such that $\sum_i (b_i - a_i) < l$, hence $l \leq 0$, or $\inf \mathscr{L}_\Phi = 0$.
Second, we need to show that if $A \subseteq B \subseteq X$, then $\lambda^*(A) \leq \lambda^*(B)$. For this, consider $A \subseteq B$. Clearly, we have that $\mathscr{C}_B \subseteq \mathscr{C}_A$, therefore $\mathscr{L}_B \subseteq \mathscr{L}_A$ and hence $\inf \mathscr{L}_B \geq \inf \mathscr{L}_A$.
Third, we need to show that for any infinite sequence $\{A_n\}$ of subsets of X,

$$\lambda^*\left(\bigcup_n A_n\right) \leq \sum_n \lambda^*(A_n)$$

For this, consider the Lebesgue outer measure of A_n, that is, $\lambda^*(A_n)$. We must have, that for any infinite sequence $\{(a_{n,i}, b_{n,i})\} \in \mathscr{C}_{A_n}$, that

$$\sum_{i=1}^{\infty} (b_{n,i} - a_{n,i}) \geq \lambda^*(A_n).$$

Hence, consider that the difference is upper bounded according to n, that is the sequence $\{(a_{n,i}, b_{n,i})\} \in \mathscr{C}_{A_n}$ is such that,

$$\sum_{i=1}^{\infty} (b_{n,i} - a_{n,i}) - \lambda^*(A_n) \leq \epsilon/2^n.$$

Now, we can cover the entire $\bigcup_i A_i$ by the union of the above intervals, that is,

$$\bigcup_i A_i \subseteq \bigcup_n \bigcup_i (a_{n,i}, b_{n,i}).$$

Now, we know that

$$\lambda^*\left(\bigcup_i A_i\right) = \inf \mathscr{L}_{\cup_i A_i}.$$

But since

$$\sum_n \sum_i (b_{n,i} - a_{n,i}) \in \mathscr{L}_{\cup_i A_i},$$

and

$$\sum_n \left(\sum_i (b_{n,i} - a_{n,i}) - \lambda^* (A_n) \right) \leq \sum_n \epsilon/2^n$$

which is equal to

$$\sum_n \sum_i (b_{n,i} - a_{n,i}) - \sum_n \lambda^* (A_n) \leq \epsilon \times 1$$

or,

$$\sum_n \sum_i (b_{n,i} - a_{n,i}) \leq \sum_n \lambda^* (A_n) + \epsilon$$

and since $\lambda^* (\bigcup_i A_i) = \inf \mathscr{L}_{\bigcup_i A_i}$, therefore,

$$\lambda^* \left(\bigcup_i A_i \right) \leq \sum_n \sum_i (b_{n,i} - a_{n,i}) \leq \sum_n \lambda^* (A_n)$$

Hence proved.

Now, we need to show that λ^* assigns each subinterval it's length.
For this first show that $\lambda^* ([a, b]) \leq b - a$. This is easy to show if we take,

$$[a, b] = \bigcup_i (a_i, b_i)$$

where $(a_1, b_1) = (a, b)$, $(a_i, b_i) = (a - \epsilon/2^i, a)$ for all even i and $(a_j, b_j) = (b, b + \epsilon/2^j)$ for all odd j. Now,

$$\begin{aligned}
\sum_i (b_i - a_i) &= (b - a) + \sum_{i=2,4,\ldots} \epsilon/2^i + \sum_{i=3,5,\ldots} \epsilon/2^i \\
&= b - a + \sum_{i=1,2,\ldots} \epsilon/2^i \\
&= b - a + \epsilon
\end{aligned}$$

therefore $\lambda^* ([a, b]) = \inf \mathscr{L}_{[a,b]} \leq b - a + \epsilon$ for all $\epsilon > 0$, hence $\lambda^* ([a, b]) \leq b - a$.
Now, to show the converse that $b - a \leq \lambda^* ([a, b])$, we first note that $[a, b]$ is compact, so for any infinite cover $\{(a_i, b_i)\} \in \mathscr{C}_{[a,b]}$, there exists a finite subcover $\{(a_i, b_i)\}_{i=1}^n$ of $[a, b]$. Now, since λ^* is an outer measure, therefore,

$$b - a \leq \sum_{i=1}^n \lambda^* ((a_i, b_i)) \leq \sum_{i=1}^\infty \lambda^* ((a_i, b_i)) \in \mathscr{L}_{[a,b]}$$

Therefore, $b - a$ is a lower bound of $\mathscr{L}_{[a,b]}$ and hence $b - a \leq \inf \mathscr{L}_{[a,b]} = \lambda^* ([a, b])$.
Hence $\lambda^* ([a, b]) = b - a$.
Now since, one can construct subintervals of the form $(a, b]$ or $[a, b)$ from the following manner:

$$(a, b] \subseteq (a, b) \bigcup \left(\bigcup_n [b, b + \epsilon/2^n] \right)$$

from which we get that $\lambda^* ((a, b]) \leq b - a$ and also,

$$[a, b] \subseteq (a, b] \bigcup \left(\bigcup_n [a - \epsilon/2^n, a] \right)$$

which yields $b - a \leq \lambda^* ((a, b])$. Similarly for $(-\infty, b]$ to show that $\lambda^* ((-\infty, b]) = +\infty$. ■

2.4 Exercises - II

1. *Proof.* Note that $\mathbb{Q}$ is dense, that is $\overline{\mathbb{Q}} = \mathbb{R}$. Therefore, $\overline{\mathbb{Q} \cap [0,1]} = [0,1]$. Now, since *closure of union is union of closures*[1], then we get,

$$\mathbb{Q} \cap [0,1] \subseteq \overline{\mathbb{Q} \cap [0,1]} = [0,1] \subseteq \bigcup_{n=1}^{m} \overline{(a_n, b_n]} = \bigcup_{n=1}^{m} [a_n, b_n]$$

Now since λ^* is an outer measure, therefore,

$$\lambda^* ([0,1]) = 1 \leq \lambda^* \left(\bigcup_{n=1}^{m} [a_n, b_n] \right) \leq \sum_{n=1}^{m} l(I_n)$$

because $[0,1]$ is a subinterval, closure of unions is union of closures and infinite subadditivity implies finite subadditivity. Hence proved. ■

2. *Proof.* Since $\lambda^*(A) = \inf \mathscr{L}_A$ where

$$\mathscr{L}_A = \left\{ \sum_i (b_i - a_i) \mid \{(a_i, b_i)\} \in \mathscr{C}_A \right\}$$

and

$$\mathscr{C}_A = \{\{(a_i, b_i)\} \mid A \subseteq \cup_n (a_n, b_n)\}.$$

Therefore, for any $\epsilon > 0$, and if we write $\lambda^*(A) + \epsilon$, then $\exists \ \{(a_i, b_i)\} \in \mathscr{C}_A$ such that $\lambda^*(A) + \epsilon \geq \sum_i (b_i - a_i) \geq \lambda^* \left(\bigcup_i (a_i, b_i)\right)$. We can clearly see that $\bigcup_i (a_i, b_i)$ is union of open intervals and since arbitrary union of open intervals is open, therefore, we have the union of the sequence $\{(a_i, b_i)\}$ as the needed open interval. ■

3. *Proof.* We have $A \subseteq \mathbb{R}$. For this, we have

$$\lambda^*(A) = \inf \mathscr{L}_A$$

Therefore, for any $\{(a_i, b_i)\} \in \mathscr{C}_A$,

$$\lambda^*(A) \leq \sum_i (b_i - a_i).$$

Now, consider,

$$B^n = \bigcup_i (a_i^n, b_i^n) \text{ for any } \{(a_i^n, b_i^n)\} \in \mathscr{C}_A$$

Now, since B^n covers A, that is $A \subseteq B^n$ for all n so B^n is open as arbitrary union of open sets is open, therefore,

$$A \subseteq \bigcap_n B^n = C$$

where $\bigcap_n B^n = C$ is clearly G_δ. Now we need to show both have the same outer measure. For this, we first trivially have that $\mathscr{C}_C \subseteq \mathscr{C}_A$ because $A \subseteq C$ and any infinite cover of C would then cover A. But moreover, we also see that any infinite sequence $\{(a_i^n, b_i^n)\} \in \mathscr{C}_A$ for some n, we have

$$C = \bigcap_n B^n \subseteq B^n = \bigcup_i (a_i^n, b_i^n).$$

That is, any infinite sequence in $\mathscr{C}_A$ also covers C. Therefore $\mathscr{C}_A \subseteq \mathscr{C}_C$. Hence $\mathscr{C}_A = \mathscr{C}_C$. So that $\mathscr{L}_A = \mathscr{L}_C$ and hence $\lambda^*(A) = \lambda^*(C)$. ■

4. *Proof.* Take any $B \subseteq \mathbb{R}$. We are given a subset $A \subseteq \mathbb{R}$ such that $\lambda^*(A) = 0$. Note that

$$\lambda^*(B) \leq \lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B) = \lambda^*(B)$$

Hence we already have that $\lambda^*(B) = \lambda^*(A \cup B)$. Now to show that $\lambda^*(A \cup B) = \lambda^*(B - A)$, first note that $B - A = B \cap A^c$. Clearly, $B \cap A^c \subseteq B$, therefore,

$$\lambda^*(B \cap A^c) \leq \lambda^*(B) = \lambda^*(A \cup B).$$

Now, to show the converse to complete the proof, we note that $A \cup B = A \cup (B \cap A^c)$, hence, by countable sub-additivity, we finally have,

$$\lambda^*(A \cup (B \cap A^c)) = \lambda^*(A \cup B) = \lambda^*(B) \leq \lambda^*(A) + \lambda^*(B \cap A^c) = \lambda^*(B \cap A^c)$$

Hence the proof is complete. ■

[1] Proof done in diary : 3^{rd} *March, 2018* - The date of reference.

5. *Proof.* Take $A \subseteq \mathbb{R}$ such that it is countable. Now, A can either be finite or countably infinite. Let's first take A to be finite, so that $\exists n \in \mathbb{N}$ such that
$$A = \{a_1, a_2 \dots, a_n\}.$$
Now, consider $\{(a_i, b_i)\} \in \mathscr{C}_A$ such that it covers each point with an exponentially decreasing interval, as follows:
$$\{I_j\} = \left\{(a_1 - \frac{\epsilon}{2}, a_1 + \frac{\epsilon}{2}), \dots, (a_n - \frac{\epsilon}{2}, a_n + \frac{\epsilon}{2}), (a_1 - \frac{\epsilon}{2^2}, a_1 + \frac{\epsilon}{2^2}), \dots\right\}$$
In other words,
$$\{(a_i, b_i)\} = \left\{\left(a_{i\%n+1} - \frac{\epsilon}{2^{\lceil i\%n \rceil}}, a_{i\%n+1} + \frac{\epsilon}{2^{\lceil i\%n \rceil}}\right)\right\}$$
We clearly have that this sequence $\{(a_i, b_i)\} \in \mathscr{C}_A$ and since $\lambda^*(A) = \inf \mathscr{L}_A$, therefore we must have that
$$\lambda^*(A) \le \sum_i (b_i - a_i) = \sum_i \frac{2\epsilon}{2^{\lceil i\%n \rceil}} = 4\epsilon = \epsilon'$$
Now since $\lambda^* : \mathscr{P}(\mathbb{R}) \to [0, +\infty]$, therefore, $\lambda^*(A) = 0$ as $\epsilon > 0$ is arbitrary. So far, we have proved that finite set has outer measure zero. For countably infinite set, we would have $A = \{a_1, a_2, \dots\}$. On this, consider the cover $\{(a_i, b_i)\} = \left\{\left(a_i - \frac{\epsilon}{2^{i+1}}, a_i + \frac{\epsilon}{2^{i+1}}\right)\right\}$. Clearly, this sequence $\{(a_i, b_i)\}$ is in $\mathscr{C}_A$. Moreover,
$$\lambda^*(A) \le \sum_i (b_i - a_i) = \sum_i \frac{\epsilon}{2^i} = \epsilon$$
Now since $\epsilon > 0$ is arbitrary, therefore, $\lambda^*(A) = 0$ even for countably infinite set A.
Now since $\lambda^*(\mathbb{R}) \neq 0$, therefore $\mathbb{R}$ is not countable. ■

6. *Proof.* Isn't it very trivial? That is just the definition of Lebesgue Outer Measure. ■

7. *Proof.* Consider $\mathscr{C}_A = \{\{(a_i, b_i)\} \mid A \subseteq \bigcup_i (a_i, b_i)\}$ and $\mathscr{C}_{\alpha A} = \{\{(c_i, d_i)\} \mid \alpha A \subseteq \bigcup_i (c_i, d_i)\}$. We will show that
$$\text{To Show} : \mathscr{C}_{\alpha A} = \alpha \mathscr{C}_A = \left\{\{\alpha(a_i, b_i)\} \mid A \subseteq \bigcup_i (a_i, b_i)\right\}.$$
First, take any infinite sequence $\{(a_i, b_i)\} \in \mathscr{C}_A$. Then $\{\alpha(a_i, b_i)\} \in \mathscr{C}_{\alpha A}$ because if $A \subseteq \bigcup_i (a_i, b_i)$, then $\alpha A \subseteq \bigcup_i \alpha(a_i, b_i)$ where $\alpha(a_i, b_i) = (\alpha a_i, \alpha b_i)$ if $\alpha > 0$ or $= (\alpha b_i, \alpha a_i)$ if $\alpha < 0$. Hence,
$$\alpha \mathscr{C}_A \subseteq \mathscr{C}_{\alpha A}.$$
To see the converse, first note that the case of $\alpha = 0$ is trivial, so we would take $\alpha \neq 0$. Now, take $\{(c_i, d_i)\} \in \mathscr{C}_{\alpha A}$. Hence $\alpha A \subseteq \bigcup_i (c_i, d_i)$. Now, $\frac{1}{\alpha} \cdot \alpha A = A \subseteq \bigcup_i \frac{1}{\alpha}(c_i, d_i)$. Therefore, $\{\frac{1}{\alpha}(c_i, d_i)\} \in \mathscr{C}_A \implies \{(c_i, d_i)\} \in \alpha \mathscr{C}_A$. Hence $\mathscr{C}_{\alpha A} \subseteq \alpha \mathscr{C}_A$. Hence $\mathscr{C}_{\alpha A} = \alpha \mathscr{C}_A$. Hence, we get that $\mathscr{L}_{\alpha A} = |\alpha| \mathscr{L}_A$ because $\sum_i l(\alpha I_n) = \sum_i \alpha(b_i - a_i)$ if $\alpha > 0$ or $= \sum_i \alpha(a_i - b_i)$ if $\alpha < 0$. Hence we can write $\alpha \mathscr{L}_A = \{\sum_i |\alpha|(b_i - a_i) \mid \{(a_i, b_i)\} \in \mathscr{C}_A\}$. Now, $\inf \mathscr{L}_{\alpha A} = \inf |\alpha| \mathscr{L}_A \implies \lambda^*(\alpha A) = |\alpha| \lambda^*(A)$. ■

2.5 Lebesgue Measurability.

$\boxed{\star}$ **Definition 12.** (**μ^*-measurable subset**) Let X be a set and let μ^* be an *outer measure* on X. A subset $B \subseteq X$ is μ^*-measurable if:

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

holds **for all** subsets $A \subseteq X$.

$\boxed{\star}$ **Definition 13.** (**Lebesgue Measurable subset of** $\mathbb{R}$) A subset $B \subseteq \mathbb{R}$ is called a Lebesgue measurable subset of $\mathbb{R}$ if B **is λ^*-measurable.** That is, for **any** $A \subseteq \mathbb{R}$, we must have:

$$\lambda^*(A) = \lambda^*(A \cap B) + \lambda^*(A \cap B^c)$$

Remark. Important to note are the following:

- Due to sub-additivity of μ^* and $A \subseteq (A \cap B) \cup (A \cap B^c)$, we already have that

$$\mu^*(A) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

  for *any* subsets $A, B \subseteq X$.

★ Due to the above fact, *all that remains to be shown to ascertain that $B \subseteq \mathbb{R}$ is μ^*-measurable* is to show the following converse:

$$\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c).$$

  for all $A \subseteq X$.

✓ **Proposition 10.** Let X be a set and let μ^* be an outer measure on X. Then each subset $B \subseteq X$ that satisfies $\mu^*(B) = 0$ **or** that satisfies $\mu^*(B^c) = 0$ is μ^*-measurable.

Proof. This result actually proves that for subset $B \subseteq X$ which has zero outer measure under μ^*, any other subset $A \subseteq X$ would be such that $\mu^*(A \cap B) = 0$(!) After proving this, and from the remark above, we would just be left to show that if $\mu^*(B) = 0$, then $\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c)$. We show the former here, from which the latter follows naturally.

Consider $B \subseteq X$ such that $\mu^*(B) = 0$. It's true that $A \cap B \subseteq B$. Now since μ^* is an outer measure on X, therefore, we must have $\mu^*(A \cap B) \leq \mu^*(B) = 0$. This implies that $\mu^*(A \cap B) = 0$. Now, we would see that the required condition follows naturally from the previous. First, note the following:

$$A \cap B \subseteq A \text{ and } A \cap B^c \subseteq A.$$

Hence, we can write:

$$\mu^*(A \cap B) \leq \mu^*(A) \text{ and } \mu^*(A \cap B^c) \leq \mu^*(A).$$

Now if $\mu^*(B) = 0$, then $\mu^*(A \cap B) = 0$ and then in the second inequality, we would have:

$$\mu^*(A \cap B^c) + \mu^*(A \cap B) \leq \mu^*(A) + 0$$

Or, if $\mu^*(B^c) = 0$, then $\mu^*(A \cap B^c) = 0$ and then in the first inequality, we would have:

$$\mu^*(A \cap B) + \mu^*(A \cap B^c) \leq \mu^*(A) + 0.$$

Hence, B is μ^*-measurable for any $B \subseteq X$ which satisfies that either $\mu^*(B) = 0$ or $\mu^*(B^c) = 0$. ■

The following theorem is a fundamental fact about outer measures.

$\boxed{\circledast}$ **Theorem 1.** Let X be a set, let μ^* be an outer measure on X and let $\mathscr{M}_{\mu^*}$ be the collection of all μ^*-measurable subsets of X. Then,

- $\mathscr{M}_{\mu^*}$ is a σ-algebra.
- The restriction of μ^* to $\mathscr{M}_{\mu^*}$ is a measure on $\mathscr{M}_{\mu^*}$.

Proof. **Act 1.** *$\mathscr{M}_{\mu^*}$ is an algebra.*
First, it is clear that $X, \Phi \in \mathscr{M}_{\mu^*}$ from Proposition 10, because $\mu^*(\Phi) = \mu^*(X^c) = 0$. Now, if $B \in \mathscr{M}_{\mu^*}$, then $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)\ \ \forall\ A \subseteq X$. But if we replace B by B^c in the above, we would get the same equation, hence $B^c \in \mathscr{M}_{\mu^*}$. So $\mathscr{M}_{\mu^*}$ is closed under complements. Now, to show closed nature under finite unions, we take any two subsets $B_1, B_2 \in \mathscr{M}_{\mu^*}$ and show that $A \cup B \in \mathscr{M}_{\mu^*}$. First we have

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c)\\ &= \mu^*(A \cap B_2) + \mu^*(A \cap B_2^c)\end{aligned}$$

for any $A \subseteq X$. Now, we see that from the fact that $B_1 \in \mathscr{M}_{\mu^*}$,

$$\begin{aligned}\mu^*(A \cap (B_1 \cup B_2)) &= \mu^*(A \cap (B_1 \cup B_2) \cap B_1) + \mu^*(A \cap (B_1 \cup B_2) \cap B_1^c)\\ &= \mu^*(A \cap B_1) + \mu^*(A \cap B_2 \cap B_1^c)\end{aligned}$$

Similarly, we have from the fact $B_2 \in \mathscr{M}_{\mu^*}$,

$$\begin{aligned}\mu^*(A \cap (B_1 \cup B_2)^c) &= \mu^*(A \cap (B_1 \cup B_2)^c \cap B_2) + \mu^*(A \cap (B_1 \cup B_2)^c \cap B_2^c)\\ &= \mu^*(A \cap B_1^c \cap B_2^c \cap B_2) + \mu^*(A \cap B_1^c \cap B_2^c \cap B_2^c)\\ &= \mu^*(\Phi) + \mu^*(A \cap B_1^c \cap B_2^c)\\ &= \mu^*(A \cap (B_1 \cup B_2)^c)\end{aligned}$$

Now, adding the above results yield,

$$\begin{aligned}\mu^*(A \cap (B_1 \cup B_2)^c) + \mu^*(A \cap (B_1 \cup B_2)) &= \mu^*(A \cap (B_1 \cup B_2)^c) + \mu^*(A \cap B_1) + \mu^*(A \cap B_2 \cap B_1^c)\\ &= \mu^*(A \cap B_1^c \cap B_2^c) + \mu^*(A \cap B_1^c \cap B_2) + \mu^*(A \cap B_1)\\ &= \mu^*(A \cap B_1^c) + \mu^*(A \cap B_1)\\ &= \mu^*(A).\end{aligned}$$

Hence, $B_1 \cup B_2$ is μ^*-measurable, so $B_1 \cup B_2 \in \mathscr{M}_{\mu^*}$. Now, we can, for a finite collection of subsets in $\mathscr{M}_{\mu^*}$, we can proceed like above, to show that $\mathscr{M}_{\mu^*}$ is closed under finite union, hence showing that $\mathscr{M}_{\mu^*}$ is an algebra.

Act 2. *$\mathscr{M}_{\mu^*}$ is a σ-algebra.*
All that is left to show that $\mathscr{M}_{\mu^*}$ is a σ-algebra is to show that it is closed under countable union. We have already proved closed nature under finite union. We extend it via induction principle. Suppose $\{B_i\}$ is a sequence of disjoint subsets in $\mathscr{M}_{\mu^*}$. For this, we first prove[2] using induction that, for all $A \subseteq X$ and $n \in \mathbb{N}$,

$$\text{To Prove}: \mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \left(\bigcap_{i=1}^{n} B_i^c\right)\right) \tag{2}$$

For the case when $n = 1$, we see that it Eq. 2 reduces to $\mu^*(A) = \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c)$. But since $B_i \in \mathscr{M}_{\mu^*}\ \forall\ i \in \mathbb{N}$, therefore this is trivially true. Now, by the induction principle, we assume that Eq. 2 is true uptill n and then we try to prove it for $n+1$ step. For this, since $B_{n+1} \in \mathscr{M}_{\mu^*}$ is disjoint to all other B_i's, we have,

$$\begin{aligned}\mu^*\left(A \cap \bigcap_{i=1}^{n} B_i^c\right) &= \mu^*\left(\left(A \cap \bigcap_{i=1}^{n} B_i^c\right) \cap B_{n+1}\right) + \mu^*\left(\left(A \cap \bigcap_{i=1}^{n} B_i^c\right) \cap B_{n+1}^c\right)\\ &= \mu^*(A \cap B_{n+1}) + \mu^*\left(A \cap \bigcap_{i=1}^{n+1} B_i^c\right)\end{aligned}$$

where the last line follows from the fact that each B_i is disjoint to other B_j's, hence each B_j^c would contain B_i and therefore $B_{n+1} \subseteq \bigcap_{i=1}^{n} B_i^c$. Now, substituting the above equation in Eq. 2 gives,

[2] But why to prove Eq. 2? The motivation for Eq. 2 comes from Part 1. More specifically, notice in the equation where we added $\mu^*(A \cap (B_1 \cup B_2)^c)$ and $\mu^*(A \cap (B_1 \cup B_2))$. Note it's 2[nd] line, this is the case when $n = 2$ in Eq. 2 combined with the fact that B_i's are disjoint. Now why to take B_i's to be disjoint? The reason for this comes from the fact that for any infinite sequence of subsets $\{A_i\}$, one can construct infinite sequence of disjoint subsets, that is : $A_1, A_2 \cap A_1^c, A_3 \cap (A_1 \cup A_2)^c, \ldots$ and it's union is again $\bigcup_n A_n$. Hence if we prove that a disjoint infinite sequence is closed under union, then we could prove that any infinite sequence of subsets is closed under union too!

$$\begin{aligned}\mu^*(A) &= \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*(A \cap B_{n+1}) + \mu^*\left(A \cap \bigcap_{i=1}^{n+1} B_i^{\mathrm{c}}\right) \\ &= \sum_{i=1}^{n+1} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{n+1} B_i^{\mathrm{c}}\right)\end{aligned}$$

Hence, by induction principle, Eq. 2 is true for all $n \in \mathbb{N}$. Hence, now we can write,

$$\begin{aligned}\mu^*(A) &\geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{\infty} B_i^{\mathrm{c}}\right) \\ &= \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^{\mathrm{c}}\right)\end{aligned}$$

Now, to prove that $\bigcup_i B_i \in \mathscr{M}_{\mu^*}$, we need to show

$$\text{To Show}: \; \mu^*(A) \geq \mu^*\left(A \cap \bigcup_i B_i\right) + \mu^*\left(A \cap \left(\bigcup_i B_i\right)^{\mathrm{c}}\right)$$

This comes from previous result as follows:

$$\begin{aligned}\mu^*(A) &\geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^{c}\right) \\ &\geq \mu^*\left(\bigcup_{i=1}^{\infty}(A \cap B_i)\right) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^{c}\right) \\ &= \mu^*\left(A \cap \bigcup_{i=1}^{\infty} B_i\right) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^{c}\right)\end{aligned} \tag{3}$$

Therefore, $\bigcup_i B_i \in \mathscr{M}_{\mu^*}$. Now, as the previous footnote mentions, for every infinite sequence $\{C_i\}$ in $\mathscr{M}_{\mu^*}$, we have a disjoint sequence of subsets as $C_1, C_2 \cap C_1^{\mathrm{c}}, C_3 \cap C_2^{\mathrm{c}} \cap C_1, \ldots$. Now, this disjoint sequence is closed under union as we just showed and since union of this disjoint sequence is equal to the union of $\{C_i\}$, hence $\bigcup_i C_i \in \mathscr{M}_{\mu^*}$ for any sequence $\{C_i\}$ in $\mathscr{M}_{\mu^*}$. Thus, $\mathscr{M}_{\mu^*}$ is a σ-algebra.

Act 3. *μ^* restricted to $\mathscr{M}_{\mu^*}$ is a measure.*
Consider $\{B_n\}$ be an infinite sequence of subsets in $\mathscr{M}_{\mu^*}$. Now, by finite subadditivity, we trivially have

$$\mu^*\left(\bigcup_i B_i\right) \leq \sum_i \mu^*(B_i)$$

Moreover, from Part 2 and setting $A = \cup_i B_i$, we get:

$$\begin{aligned}\mu^*\left(\bigcup_i B_i\right) &\geq \sum_j \mu^*\left(\bigcup_i B_i \cap B_j\right) + \mu^*\left(\bigcup_i B_i \cap \left(\bigcup_i B_i\right)^{c}\right) \\ &= \sum_j \mu^*(B_j) + \mu^*(\Phi) \\ &= \sum_j \mu^*(B_j).\end{aligned}$$

We hence have the complete proof. ∎

✓ **Proposition 11.** Every interval of form $(-\infty, b]$ is Lebesgue measurable.

Proof. All we need to show that for $B = (-\infty, b]$ and for all $A \subseteq \mathbb{R}$,

$$\text{To Show : } \lambda^*(A) \geq \lambda^*(A \cap B) + \lambda^*(A \cap B^c).$$

Consider $\{(a_i, b_i)\} \in \mathscr{C}_A$. Therefore $A \subseteq \bigcup_i (a_i, b_i)$. Clearly, this implies that

$$\lambda^*(A) \leq \sum_i (b_i - a_i).$$

Equivalently, $\exists\, \epsilon > 0$ such that,

$$\lambda^*(A) + \epsilon \geq \sum_i (b_i - a_i).$$

Now, it's also clear that $(a_k, b_k) \cap B$ and $(a_k, b_k) \cap B^c$ are disjoint for any $k \in \mathbb{N}$. But we know that **for disjoint λ^*-measurable sets C and D, it's true that $\lambda^*(C \cup D) = \lambda^*(C) + \lambda^*(D)$, which can be extended for countable union**[3]. Hence, we can write:

$$(b_k - a_k) = \lambda^*((a_k, b_k)) = \lambda^*\left((a_k, b_k) \cap B \bigcup (a_k, b_k) \cap B^c\right) = \lambda^*((a_k, b_k) \cap B) + \lambda^*((a_k, b_k) \cap B^c).$$

Now since $A \subseteq \bigcup_i (a_i, b_i)$, therefore $A \cap B \subseteq \bigcup_i (a_i, b_i) \cap B$. Similarly, $A \cap B^c \subseteq \bigcup_i (a_i, b_i) \cap B^c$.
This implies that,

$$\lambda^*(A \cap B) \leq \lambda^*\left(\bigcup_i (a_i, b_i) \cap B\right)$$
$$\lambda^*(A \cap B^c) \leq \lambda^*\left(\bigcup_i (a_i, b_i) \cap B^c\right)$$

Adding these equations, we get the desired result:

$$\begin{aligned}
\lambda^*(A \cap B) + \lambda^*(A \cap B^c) &\leq \lambda^*\left(\bigcup_i (a_i, b_i) \cap B\right) + \lambda^*\left(\bigcup_i (a_i, b_i) \cap B^c\right) \\
&\leq \sum_i \lambda^*((a_i, b_i) \cap B) + \sum_i \lambda^*((a_i, b_i) \cap B^c) \\
&= \sum_i \lambda^*((a_i, b_i)) \quad \text{(Shown above.)} \\
&= \sum_i (b_i - a_i) \\
&\leq \lambda^*(A) + \epsilon
\end{aligned}$$

which is true for all $\epsilon > 0$, therefore

$$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \lambda^*(A).$$

■

✓ **Proposition 12.** Every Borel subset of $\mathbb{R}$ is Lebesgue Measurable.

Proof. From Proposition 11, we have that the subsets of the form $(-\infty, b]$ are Lebesgue measurable. Remember that a subset $B \subseteq \mathbb{R}$ is called borel if $B \in \mathscr{B}(\mathbb{R})$; that is, it is present in the smallest σ-algebra generated by open subsets of $\mathbb{R}$.
Now, since $(-\infty, b]$ is Lebesgue measurable, therefore $(-\infty, b] \in \mathscr{M}_{\lambda^*}$. But since σ-algebra $\mathscr{B}(\mathbb{R})$ is closed under complements, complement of open subset is closed and $(-\infty, b]$ is closed, therefore all intervals of form $(-\infty, b]$ are also in $\mathscr{B}(\mathbb{R})$. Since $\mathscr{B}(\mathbb{R})$ is the smallest σ-algebra containing intervals of form $(-\infty, b]$, therefore $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{M}_{\lambda^*}$. ■

---

[3] We need it's proof too.