Month of Topology & Abstract Algebra - IITG

Topology - Definitions, Propositions, Theorems & Proofs

Animesh Renanse

January 2, 2021

Target : *Reach Topological Manifolds*

1 Topological Spaces & Continuity

Moving forward from the basic notion of Open & Closed Sets, Continuity & Completeness in Metric Spaces. Most of the prerequisites are from point-set topology background in Functional Analysis.

1.1 Basic Topological Constructions

$\boxed{\star}$ **Definition 1.** ***(Topological Space)*** Let M be a set. Then a subset $\mathcal{M} \subseteq 2^M$ of the power set of M is called a topology if it satisfies the following properties:

1. The empty set Φ and M are in $\mathcal{M}$.
2. If $\{O_i\}_{i\in I}$ with $O_i \in \mathcal{M}$ is an **arbitrary** collection of elements in $\mathcal{M}$, then union of all such sets should also be in $\mathcal{M}$. That is
$$\bigcup_{i\in I} O_i \in \mathcal{M}$$
3. If $O_1, O_2, \ldots, O_n \in \mathcal{M}$ are **finitely** many elements in $\mathcal{M}$, then there intersection should also be in $\mathcal{M}$. That is
$$O_1 \cap O_2 \cap \cdots \cap O_n \in \mathcal{M}$$

Remark.

- A set M together with a topology $\mathcal{M} \subseteq 2^M$ is called a topological space $(M, \mathcal{M})$. The sets O are called the **open** subsets of $(M, \mathcal{M})$.
- The power set $\mathcal{M} = 2^M$ is called the **Finest Topology of** M.
- If one takes topology to be $\mathcal{M} = \{\Phi, M\}$, it is called the **Coarsest** or **Trivial Topology of** M.
- The collection $\mathcal{M}_{\text{cofinite}} \subseteq 2^M$ of all those subsets which have finite complements and Φ is called the **Cofinite Topology on** M.

$\boxed{\star}$ **Definition 2.** ***(Closed Subset)*** A subset $A \subseteq M$ of a topological space $(M, \mathcal{M})$ is called closed if $M \;/ A$ is open.

✓ **Proposition 1.** If $\{\mathcal{M}_i\}_{i\in I}$ are topologies on M, then
$$\mathcal{M} = \bigcap_{i\in I} \mathcal{M}_i$$
is also a topology on M.

---The following distinction of topologies helps in comparing them.

$\boxed{\star}$ **Definition 3.** ***(Finer & Coarser)*** Let $\mathcal{M}_1, \mathcal{M}_2 \subseteq 2^M$ be topologies on M. Then $\mathcal{M}_1$ is called *finer* than $\mathcal{M}_2$ if $\mathcal{M}_2 \subseteq \mathcal{M}_1$. In this case, $\mathcal{M}_2$ is called *coarser* than $\mathcal{M}_1$.

---The coarsest topology containing open subsets.

✓ **Proposition 2.** Let $S \subseteq 2^M$ be a subset of the power set of M containing Φ and M. Then,

1. There exists a unique topology $\mathcal{M}(S)$ which is **coarser than every other topology** containing S.
2. This topology $\mathcal{M}(S)$ can be obtained by the following two step procedure:
   (a) Take all finite intersections of subsets in S.
   (b) Then take unions of all the resulting subsets.

$\boxed{\star}$ **Definition 4.** ***(Basis & Subbasis)*** Let $(M, \mathcal{M})$ be a topological space and let $\mathcal{S}, \mathcal{B} \subseteq \mathcal{M}$ be subsets containing Φ and M. Then,

1. The set $\mathcal{B}$ is called a *basis* of $\mathcal{M}$ if every open subset in $\mathcal{M}$ is a union of subsets from $\mathcal{B}$.
2. The set $\mathcal{S}$ is called a *subbasis* of $\mathcal{M}$ if the collection of all finite intersections of sets from $\mathcal{S}$ forms a basis of $\mathcal{M}$.

$\boxed{\star}$ **Definition 5.** ***(Subspace Topology)*** Let $(M, \mathcal{M})$ be a topological space and $N \subseteq M$ a subset. Then $\mathcal{M}|_N$ defined by,

$$\mathcal{M}|_N = \{O \cap N | O \in \mathcal{M}\} \subseteq 2^N$$

is a topology for N.

1.2 Neighborhoods, Interiors and Closures

To define the neighborhood system of a point for a topological space in the same way as we know for metric space.

$\boxed{\star}$ **Definition 6.** ***(Neighborhood)*** Let $(M, \mathcal{M})$ be a topological space and $p \in M$. Then,

1. A subset $U \subseteq M$ is called neighborhood of p if there exists an open subset $O \subseteq M$ with $p \in O \subseteq U$.
2. The collection of *all* neighborhoods of p is called the **neighborhood system** (or neighborhood filter) of p, denoted by $\mathfrak{U}(p)$.

Basic properties of Neighborhoods in Topology.

✓ **Proposition 3.** Let $(M, \mathcal{M})$ be a topological space and $p \in M$. Then,

1. A subset $O \subseteq M$ is a neighborhood of all of it's points if and only if O is open.
2. For $U \in \mathfrak{U}(p)$ and $U \subseteq U'$ then we trivially have $U' \in \mathfrak{U}(p)$.
3. For $U_1, \ldots, U_n \in \mathfrak{U}(p)$, we have $U_1 \cap \cdots \cap U_n \in \mathfrak{U}(p)$.
4. For $U \in \mathfrak{U}(p)$ we have $p \in U$.
5. For $U \in \mathfrak{U}(p)$ there exists a $V \in \mathfrak{U}(p)$ with $V \subseteq U$ and $V \in \mathfrak{U}(q)$ for all $q \in V$.

$\boxed{\star}$ **Definition 7.** ***(Neighborhood Basis)*** Let $(M, \mathcal{M})$ be a topological space and $p \in M$. Then, a subset $\mathfrak{B}(p) \subseteq \mathfrak{U}(p)$ of neighborhoods of $p \in M$ is called a neighborhood basis of p if for every $U \in \mathfrak{U}(p)$, there is a $B \in \mathfrak{B}(p)$ with $B \subseteq U$.

To capture the phenomenon of having different size of neighborhood bases, we have the following construction.

$\boxed{\star}$ **Definition 8.** ***(First & Second Countability)*** Let $(M, \mathcal{M})$ be a topological space.

1. The space M is called first countable (at $p \in M$) if every point (the point p) has a countable neighborhood basis.
2. The space M is called second countable if $\mathcal{M}$ has a countable basis.

Constructing more subsets of M given an arbitrary $A \subseteq M$.

$\boxed{\star}$ **Definition 9.** ***(Interior, Closure & Boundary)*** Let $(M, \mathcal{M})$ be a topological space and $A \subseteq M$.

1. A point $p \in M$ is called **inner point** of A if $A \in \mathfrak{U}(p)$.
2. The **interior** A° of A is the set of all inner points of A.
3. A point $p \in M$ is called **boundary point** of A if for every neighborhood $U \in \mathfrak{U}(p)$, we have
$$A \cap U \neq \Phi \neq (M \ / A) \cap U$$
4. The **boundary** ∂A of A is the set of all boundary points of A.
5. The **closure** A^{cl} of A is the set of all points $p \in M$ such that all $U \in \mathfrak{U}(p)$ satisfy
$$U \cap A \neq \Phi$$

Alternative characterizations of Interior, Closure & Boundary.

✓ **Proposition 4.** Let $(M, \mathcal{M})$ be a topological space and $A \subseteq M$. Then,

1. The interior A° of A is the largest open subset inside A.
2. The closure A^{cl} of A is the smallest closed subset containing A.
3. The boundary ∂A of A is closed and $\partial A = A^{\mathrm{cl}} \ / A^\circ$.

The *maximum* and *minimum* sizes of subsets.

$\boxed{\star}$ **Definition 10.** ***(Dense & Nowhere Dense)*** Let $(M, \mathcal{M})$ be a topological space. Then,

1. A subset $A \subseteq M$ is called dense if $A^{\mathrm{cl}} = M$.
2. A subset $A \subseteq M$ is called nowhere dense if $(A^{\mathrm{cl}})^\circ = \Phi$.

Properties of Closures, Open Interiors & Boundaries with respect to unions, intersections and complements.

✓ **Proposition 5.** Let $(M, \mathcal{M})$ be a topological space and let $A, B \subseteq M$ be subsets. Then, one has,

1. $\Phi^\circ = \Phi = \Phi^{\mathrm{cl}}$ and $M^\circ = M = M^{\mathrm{cl}}$ as well as $\partial\Phi = \Phi = \partial M$.

   Proof. Since $\Phi \in \mathfrak{U}(\Phi)$, thus, $\Phi^\circ = \Phi$ and similarly for the closure. For boundary of Φ, note that for $p \in \partial\Phi$, $U \in \mathfrak{U}(p)$ should be such that $U \cap \Phi \neq \Phi \neq (M \ / \Phi) \cap U$. Since $U \cap \Phi \neq \Phi \implies U = \Phi$. Also, for $q \in \partial M$, $U \in \mathfrak{U}(q)$ should be such that $U \cap M \neq \Phi \neq U \cap (M \ / M) = U \cap \Phi \implies U = \Phi$. ■

2. $A^\circ \subseteq A \subseteq A^{\mathrm{cl}}$ and
$$(A^{\mathrm{cl}})^{\mathrm{cl}} = A^{\mathrm{cl}} \ , \ \ (A^\circ)^\circ = A^\circ \ , \ \ \partial(\partial A) \subseteq \partial A$$

   Proof. For any $p \in A^\circ$, we have $p \in A \in \mathfrak{U}(p)$, thus, $A^\circ \subseteq A$. The A is necessarily in A^{cl} as for any point $p \in A$ and any $U \in \mathfrak{U}(p)$ have $\{p\}$ in common. Thus $A^\circ \subseteq A \subseteq A^{\mathrm{cl}}$. Since A° is the largest open set in A, then $(A^\circ)^\circ$ is also the largest open set in A°. Thus $(A^\circ)^\circ = A^\circ$. Similarly for closure. For boundary, using Proposition 4, we see that ∂A is closed with $\partial A = A^{\mathrm{cl}} \ / A^\circ$. Therefore, $\partial(\partial A) = (\partial A)^{\mathrm{cl}} \ / (\partial A)^\circ = \partial A \ / (\partial A)^\circ \subseteq \partial A$. ■

3. For $A \subseteq B$,
$$A^\circ \subseteq B^\circ \ , \ \ A^{\mathrm{cl}} \subseteq B^{\mathrm{cl}}$$
4. $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ \ , \ \ \partial(A \cup B) \subseteq \partial A \cup \partial B \ , \ \ A^{\mathrm{cl}} \cup B^{\mathrm{cl}} = (A \cup B)^{\mathrm{cl}}$.
5. $A^\circ \cap B^\circ = (A \cap B)^\circ \ , \ \ (A \cap B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cap B^{\mathrm{cl}}$.

   Proof. Note that $A^\circ \subseteq A$, $B^\circ \subseteq B$ and $A \cap B \subseteq A, B$, thus $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$ and $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$ hence proving the first part. For second part, note that $A \cap B \subseteq A, B$, thus $(A \cap B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cap B^{\mathrm{cl}}$. ■

6. $(M \ / A)^\circ = M \ / A^{\mathrm{cl}} \ , \ \ \partial(M \ / A) = \partial A \ , \ \ (M \ / A)^{\mathrm{cl}} = M \ / A^\circ$.

Proof. Note that the boundary points have a symmetric definition (Definition 9) w.r.t. A or $M \;/ A$, hence $\partial(M \;/ A) = \partial A$. Now, let $p \in M \;/ A^\circ$. Since $p \notin A^\circ$, thus, $A \notin \mathfrak{U}(p)$, that is for all $U \in \mathfrak{U}(p)$, $U \cap (M \;/ A) \neq \Phi$. This means that $p \in (M \;/ A)^{\mathrm{cl}}$, hence showing $(M \;/ A)^{\mathrm{cl}} = M \;/ A^\circ$. We can use this to show the last part in the following way,

$$
\begin{aligned}
& M \;/ (M \;/ A)^\circ = (M \;/ (M \;/ A)^\circ)^{\mathrm{cl}} \\
& \qquad\qquad\quad\;\; = A^{\mathrm{cl}} \\
\Longrightarrow \quad & M \;/ A^{\mathrm{cl}} = M \;/ (M \;/ (M \;/ A)^\circ) \\
& \qquad\qquad\quad\;\; = (M \;/ A)^\circ
\end{aligned}
$$

Hence Proved. ■

1.3 Continuous Maps

Definition of continuous maps between topological spaces is derived in the same way as we learned for Metric Spaces.

$\boxed{\star}$ **Definition 11.** ***(Continuity)*** Let $f : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a map between topological spaces.

1. The map f is called continuous at $p \in M$ if for every neighborhood $U \in \mathfrak{U}(f(p))$ of $f(p)$ the $f^{-1}(U)$ is a neighborhood of p.
2. The map f is called continuous if the pre-image of every open subset of N is open in M.
3. The set of continuous maps is denoted by

$$\mathscr{C}(M, N) = \{f : M \longrightarrow N \mid f \text{ is continuous}\}.$$

Basic results of continuity.

✓ **Proposition 6.** Let $f : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a map between topological spaces. Then the following statements are equivalent:

1. The map f is continuous at every point.
2. The map f is continuous.
3. The subset $f^{-1}(A) \subseteq M$ is closed for every closed $A \subseteq N$.
4. The subset $f^{-1}(O)$ is open for every $O \in \mathcal{S}$ in a subbasis $\mathcal{S}$ of N.

Proof. For 1. $\to$ 2. : Let $O \subseteq N$ be open. Let $p \in f^{-1}(O)$, therefore $f(p) \in O$.Thus, $O \in \mathfrak{U}(f(p))$ and $f^{-1}(O) \in \mathfrak{U}(p)$. Since this is true for all $p \in f^{-1}(O)$, we thus have $f^{-1}(O)$ that is open.
For 2. $\to$ 1. : Let $U \in \mathfrak{U}(f(p))$, thus there is an open set $O \subseteq U$ with $f(p) \in O$. Since f is continuous, so $f^{-1}(O)$ is open. Hence, $f^{-1}(O) \subseteq f^{-1}(U)$, that is, $f^{-1}(U) \in \mathfrak{U}(p)$. Hence proved.
For 2. $\to$ 3. : Let A be an open set in N. Since f is continuous at every point in M, therefore $f^{-1}(A)$ is open. This implies that if $B = N \ / A$ is closed then $f^{-1}(B)$ is closed. ■

Composition of maps.

✓ **Proposition 7.** Let $f : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ and $g : (N, \mathcal{N}) \longrightarrow (K, \mathcal{K})$ be maps between topological spaces.

1. If f is continuous at $p \in M$ and g is continuous at $f(p) \in N$ then $g \circ f$ is continuous at p.
2. If f and g are continuous then $g \circ f$ is continuous.

Proof. First note that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Therefore, for any point $p \in K$ and it's neighborhood $U \in \mathfrak{U}(p)$, $(g \circ f)^{-1}(U) \in \mathfrak{U}(f^{-1}(g^{-1})(U))$ as f and g are continuous. ■

Additional features of maps.

$\boxed{\star}$ **Definition 12.** ***(Open & Closed Maps)*** Let $f : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a map between topological spaces.

1. The map f is called open if $f(O) \subseteq N$ is open for all open $O \subseteq M$.
2. The map f is called closed if $f(A) \subseteq N$ is closed for all closed $A \subseteq M$.

Note that these three notions (continuous, open and closed) of maps are not mutually exclusive, that is, a map which is open is not generally not closed and/or continuous.

Isomorphism between topological spaces.

$\boxed{\star}$ **Definition 13.** ***(Homeomorphism)*** Let $f : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a map between topological spaces.

1. The map f is called homeomorphism if f is bijective, continuous and if f^{-1} is also continuous.
2. If there is a homeomorphism $f : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ then the spaces $(M, \mathcal{M})$ and $(N, \mathcal{N})$ are called **homeomorphic**.
3. The map f is called an **embedding** if f is injective and if

$$f : (M, \mathcal{M}) \longrightarrow \Big(f(M), \mathcal{N}|_{f(M)}\Big)$$

is a homeomorphism.

Equivalent characterizations of homeomorphisms

✓ **Proposition 8.** Let $f : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a map between topological spaces. Then the following statements are equivalent:

1. The map f is a homeomorphism.
2. The map f is continuous, bijective and open.
3. The map f is continuous, bijective and closed.
4. The map f is continuous and there exists a continuous map $g : N \longrightarrow M$ with $g \circ f = \mathrm{id}_M$ and $f \circ g = \mathrm{id}_N$ where $\mathrm{id}_M : M \to M$ is the identity map.

Proof. From 1. $\to$ 2. : Map f is trivially continuous and bijective. Consider an open subset $O \subseteq M$. Now since f^{-1} is continuous, therefore $(f^{-1})^{-1}(O) = f(O) \subseteq N$ is open, so f is open.
From 1. $\to$ 3. : Using Proposition 6, we can proceed as above.
From 2. $\to$ 1. : We just need to show that f^{-1} is also continuous. We also have that for any open $O \subseteq M$, $f(O) \subseteq N$ is open and for any open $O \subseteq N$, $f^{-1}(O) \subseteq M$ is open. We need that for any open subset $O \subseteq M$, $(f^{-1})^{-1}(O) \subseteq N$ is open, which is trivial to see now as $(f^{-1})^{-1}$ is f.
From 3. $\to$ 1. : Using Proposition 6, we can proceed as above.
From 1. $\to$ 4. : Simply let $g = f^{-1}$. ∎

1.4 Connectedness

Let's further discuss more properties of Topological spaces, especially of *connectedness* and *path-connectedness.*

Motivation for definition of Connectedness.

✓ **Proposition 9.** Consider the closed interval $M = [0, 1]$ with it's usual topology. Suppose we have two open subsets $O_1, O_2 \subseteq [0, 1]$ with $O_1 \cup O_2 = [0, 1]$ and $O_1 \cap O_2 = \Phi$. Then necessarily O_1 and O_2 are just $[0, 1]$ and Φ.

$\boxed{\star}$ **Definition 14.** ***(Connectedness)*** Let $(M, \mathcal{M})$ be a topological space. Then M is called connected if there are no two open, disjoint subsets $O_1, O_2 \subseteq M$ with $O_1 \cup O_2 = M$ beside M and Φ.
A subset $A \subseteq M$ is called connected if $(A, \mathcal{M}|_A)$ is connected.

Remark. It's easy to see now that the unit interval $[0, 1]$ is connected.

✓ **Proposition 10.** Let $A \subseteq \mathbb{R}$ and $a, b \in A$. If A is connected, then $[a, b] \subseteq A$.

Proof. Assume $z \in [a, b]$ such that $z \notin A$. Clearly, $(-\infty, z)$ and (z, ∞) are open subsets of $\mathbb{R}$. Using the definition of subspace topology, it implies that $(-\infty, z) \cap A$ and $(z, \infty) \cap A$ are open in A. Note that since A is connected and $((-\infty, z) \cap A) \cup ((z, \infty) \cap A) = A$ and both are disjoint, therefore, $(-\infty, z) \cap A$ and $(z, \infty) \cap A$ are A and Φ. But if $(-\infty, z) \cap A$ is A and $(z, \infty) \cap A$ is Φ, then $a \in A$ but $b \notin A$ and vice-versa, which leads to the contradiction opposing $a, b \in A$. ∎

Continuity obeys Connectedness.

✓ **Proposition 11.** Let $f : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a continuous map between topological spaces. If M is connected then $f(M)$ is connected too.

Proof. Suppose $f(M)$ is not connected even after M being connected and let $O_1, O_2 \subseteq f(M)$ be open and disjoint such that $O_1 \cup O_2 = f(M)$. Clearly, $U_1 = f^{-1}(O_1) \subseteq M$ and $U_2 = f^{-1}(O_2) \subseteq M$ are both open as well due to continuity of f. Since $M = f^{-1}(f(M)) = f^{-1}(O_1 \cup O_2) = f^{-1}(O_1) \cup f^{-1}(O_2) = U_1 \cup U_2$ and $\Phi = f^{-1}(\Phi) = f^{-1}(O_1 \cap O_2) = f^{-1}(O_1) \cap f^{-1}(O_2) = U_1 \cap U_2$, hence we have a contradiction to connectedness of M. ∎

Motivated from Proposition 10, we have the Topological Intermediate Value Theorem as the following corollary.

$\to$ **Corollary 1.** ***(Intermediate Value Theorem)*** Let $f : (M, \mathcal{M}) \longrightarrow \mathbb{R}$ be a continuous function on a connected topological space M. If $a, b \in f(M)$ then also $[a, b] \subseteq f(M)$.

Connectedness from continuous joining of two points.

$\boxed{\star}$ **Definition 15.** ***(Path - Connectedness)*** Let $(M, \mathcal{M})$ be a topological space.

1. A **path** in M is a continuous map γ from interval $[0,1]$ to M,
$$\gamma : [0,1] \longrightarrow M$$
2. The space M is called **path-connected** if for any $p, q \in M$ one finds a path γ with
$$\gamma(0) = p \text{ and } \gamma(1) = q$$

Path-connectedness $\Longrightarrow$ Connectedness.

✓ **Proposition 12.** Let $(M, \mathcal{M})$ be a path-connected topological space. Then M is connected too.

Proof. Suppose M is not connected. Let $O_1, O_2 \subseteq M$ be open, disjoint and such that $O_1 \cup O_2 = M$. Let $p \in O_1$ and $q \in O_2$ and them being connected by the continuous path $\gamma : [0,1] \longrightarrow M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Since γ is continuous, so $\gamma^{-1}(O_1)$ and $\gamma^{-1}(O_2)$ are open, disjoint and non-empty. Also $[0,1] = \gamma^{-1}(M) = \gamma^{-1}(O_1 \cup O_2) = \gamma^{-1}(O_1) \cup \gamma^{-1}(O_2)$. Hence, we have a contradiction to the connectedness of $[0,1]$. Therefore, our assumption that when M is not connected then $(M, \mathcal{M})$ would be path-connected is wrong. ■

Remark. Path-connectedness is a stronger property than connectedness.

Even if $(M, \mathcal{M})$ is not path connected, we can talk about the largest subset containing a given point $p \in M$ which is path-connected.

✓ **Proposition 13.** Let $(M, \mathcal{M})$ be a topological space.

1. If $\{C_i\}_{i \in I}$ is a family of path-connected subsets of M such that $\bigcap_{i \in I} C_i \neq \Phi$ then
$$\bigcup_{i \in I} C_i$$
is again path-connected.
2. If $A \subseteq B \subseteq A^{\text{cl}} \subseteq M$ and A is a connected subset then B is connected as well. In particular, if A is connected, then A^{cl} is connected too.
3. The union of all connected subsets of M which contain p is connected and closed and is denoted by $\mathfrak{C}(p)$.
4. The union of all path-connected subsets of M which contain p is denoted by $\Pi(p)$ which is path-connected and
$$\Pi(p) \subseteq \mathfrak{C}(p).$$

Proof. First write $C = \bigcup_{i \in I} C_i$ and let $O_1, O_2 \subseteq M$ such that $(O_1 \cap C) \cap (O_2 \cap C) = \Phi$ and $C \subseteq O_1 \cup O_2$, that is, we are referring to the subspace topology of C. Now, since $C \subseteq O_1 \cup O_2$, therefore $C_i \subseteq O_1 \cup O_2$ for all $i \in I$. Since C_i is connected for each i, therefore we have that either $C_i \cap O_1$ or $C_i \cap O_2$ as Φ. WLOG, let us assume $C_i \cap O_1 = \Phi$ and since $C_i \subseteq O_1 \cup O_2$, therefore $C_i \subseteq O_2 \cap C_i$ for all $i \in I$. Hence $C \subseteq O_2 \cap C$, hence for any open set in the subspace topology $\mathcal{M}|_C$, the only one which covers whole of C is C and Φ itself. Hence proved the connectedness of C. The proof for path-connectedness is much simpler

The general logical technique used here (part 2) is the following : Prove the contrapositive by contradiction. That is, for statements P and Q such that $P \to Q$, it is equivalent to $\neg Q \to \neg P$ and prove this by contradiction, that is assume negation and derive contradiction, more simply since for any statements A and B we have that $\neg(B \to A) \Leftrightarrow B \wedge \neg A$, in our case, we hence assume that $\neg(\neg Q \to \neg P) \Leftrightarrow \neg Q \wedge P$ is true and derive a contradiction.

For our case, let $A \subseteq B \subseteq A^{\text{cl}} \subseteq M$ and A is connected but B is not connected. Hence we can find two open subsets $O_1, O_2 \subseteq M$ such that $B \subseteq O_1 \cup O_2$ and $(B \cap O_1) \cap (B \cap O_2) = \Phi$ and each intersection being non-empty themselves. Since $A \subseteq B$, then $A \subseteq O_1 \cup O_2$ and $(A \cap O_1) \cap (A \cap O_2) = \Phi$. We still need to show that $A \cap O_k$ is not empty to establish the contradiction to completeness of A. For that, since $B \subseteq A^{\text{cl}}$, thus for any element $p \in B$, $p \in A^{\text{cl}}$ and hence there exists open $U \in \mathfrak{U}(p)$ such that $U \cap A \neq \Phi$. Hence, choosing $b_1 \in B \cap O_1$ and $b_2 \in B \cap O_2$, we first note that $O_1 \in \mathfrak{U}(b_1)$ and $O_2 \in \mathfrak{U}(b_2)$, hence $O_1 \cap A \neq \Phi \neq O_2 \cap A$. Hence the contradiction.

Part 3 and 4 follows from 1 and 2. ■

Properties of $\mathfrak{C}(p)$ and $\Pi(p)$.

$\boxed{\star}$ **Definition 16.** ***(Connected Components)*** Let $(M, \mathcal{M})$ be a topological space and $p \in M$.

1. The subset $\mathfrak{C}(p)$ is called the **connected component** of p.
2. The subset $\Pi(p)$ is called the **path-connected component** of p.

Equivalence of *being in same connected component of* M.

✓ **Proposition 14.** Let $(M, \mathcal{M})$ be a topological space. Then,

1. $q \in \mathfrak{C}(p)$ holds if and only if $p \in \mathfrak{C}(q)$.
2. If $q \in \mathfrak{C}(p)$ and $x \in \mathfrak{C}(q)$, then $x \in \mathfrak{C}(p)$.
3. Hence, belonging in same connected component is an **equivalence relation**. Similarly for path-connected component too.

Proof. It's trivial to see Part 1 (symmetric) as since $q \in \mathfrak{C}(p)$, therefore $\mathfrak{C}(p) \subseteq \mathfrak{C}(q)$. Hence $p \in \mathfrak{C}(q)$. For part 2 (transitive), since $q \in \mathfrak{C}(x)$ (Part 1) therefore $\mathfrak{C}(x) \subseteq \mathfrak{C}(q)$. Similarly, since $p \in \mathfrak{C}(q)$, hence $\mathfrak{C}(q) \subseteq \mathfrak{C}(p)$. Hence we get that $\mathfrak{C}(x) \subseteq \mathfrak{C}(q) \subseteq \mathfrak{C}(p) \implies x \in \mathfrak{C}(p)$. ■

Easing the constraint of Global Connectedness.

$\boxed{\star}$ **Definition 17.** ***(Local Connectedness & Total Disconnectedness)*** Let $(M, \mathcal{M})$ be a topological space.

1. If for every point $p \in M$, every neighborhood $U \in \mathfrak{U}(p)$ contains a (path-)connected neighborhood of p then $(M, \mathcal{M})$ is called **Locally (Path-)Connected**.
2. If $\mathfrak{C}(p) = \{p\}$ for all $p \in M$ then $(M, \mathcal{M})$ is called **Totally Disconnected**.

✓ **Proposition 15.** Let $(M, \mathcal{M})$ be a topological space.

1. If M is locally connected then the connected component $\mathfrak{C}(p)$ of $p \in M$ is open.
2. The space M is locally connected if and only if the connected open subsets form a basis of the topology.
3. If M is locally path-connected then for all $p \in M$ we have
$$\mathfrak{C}(p) = \Pi(p).$$
4. Suppose M is locally path-connected. Then M is connected if and only if M is path-connected.

Proof. Let $p, q \in M$ such that $q \in \mathfrak{C}(p)$. Since M is locally connected therefore the connected neighborhood $U \in \mathfrak{U}(q)$ is such that $U \cap \mathfrak{C}(p) \neq \Phi$. Now note that by Proposition 13, we have that $U \cup \mathfrak{C}(p)$ is again connected and contains p. Hence $U \subseteq U \cup \mathfrak{C}(p) \subseteq \mathfrak{C}(p)$, that is $\mathfrak{C}(p) \in \mathfrak{U}(q)$. Since q is arbitrary in $\mathfrak{C}(p)$, therefore by Proposition 3, $\mathfrak{C}(p)$ is open.
For 2, if the space M is locally connected then for each point, every nieghborhood has a connected neighborhood, which forms a neighborhood basis, hence it forms a basis for the whole topology. Similarly for the converse.
For 3, let M be locally path-connected, hence locally connected. From Proposition 13, we know that $\Pi(p) \subseteq \mathfrak{C}(p)$. Now suppose that $q \in \mathfrak{C}(p) \setminus \Pi(p)$. Clearly, $\Pi(q) \subseteq \mathfrak{C}(q) \subseteq \mathfrak{C}(p) \setminus \Pi(p)$. This means that

$$\mathfrak{C}(p) \setminus \Pi(p) = \bigcup_{q \in \mathfrak{C}(p) \setminus \Pi(p)} \Pi(q)$$

hence $q \in \mathfrak{C}(p) \setminus \Pi(p)$ is open as $\Pi(q)$ is open by extension of part 1. Hence we have a decomposition of $\mathfrak{C}(p)$ into disjoint subsets $\mathfrak{C}(p) \setminus \Pi(p)$ and $\Pi(p)$. Since $\mathfrak{C}(p)$ is connected, therefore one of these has to be empty but since $p \in \Pi(p)$ hence $\mathfrak{C}(p) \setminus \Pi(p) = \Phi$. The 4^{th} now follows as 3rd is true for all $p \in M$. ■

1.5 Separation Properties

We now discuss the axioms which collect features of how points in a topological space can be separated from each other.

$\boxed{\star}$ **Definition 18.** (**Separation Properties**) Let $(M, \mathcal{M})$ be a topological space.

1. The space M is called a T_0**-space** if for each pair of two different points $p \neq q$ in M, we find an open subset which contains only one of them.
2. The space M is called a T_1**-space** if for each pair of two different point $p \neq q$ in M, we find open subsets O_1 and O_2 with $p \in O_1$ and $q \in O_2$ but $p \notin O_2$ and $q \notin O_1$.
3. The space M is called a T_2**-space** or a **Hausdorff space** if for each pair of two different points $p \neq q$ in M, we find disjoint open subsets O_1 and O_2 with $p \in O_1$ and $q \in O_2$.
4. The space M is called a T_3**-space** if for every closed subset $A \subseteq M$ and every $p \in M \setminus A$, there are disjoint open subsets O_1 and O_2 with $A \subseteq O_1$ and $p \in O_2$.
5. The space M is called a T_4**-space** if for two disjoint closed subsets $A_1, A_2 \subseteq M$, there are disjoint open subsets O_1 and O_2 with $A_1 \subseteq O_1$ and $A_2 \subseteq O_2$.

Remark. It should be noted that $T_0 \;\not\Longrightarrow\; T_1$ but $T_1 \implies T_0$, $T_1 \;\not\Longrightarrow\; T_2$ but $T_2 \implies T_1$.

Basic Implications between axioms.

✓ **Proposition 16.** A topological space $(M, \mathcal{M})$ is a T_1-space if and only if every point $p \in M$ gives a closed subset $\{p\} \subseteq M$.

Proof. It's trivial to see that since $M \setminus \{p\}$ is neighborhood of all of it's points as M is T_1, hence $M \setminus \{p\}$ is open and hence $\{p\}$ is closed. Converse can be seen from this argument too. ∎

✓ **Proposition 17.** A topological space $(M, \mathcal{M})$ is a T_3-space if and only if for every $p \in M$ and every open neighborhood $O \in \mathfrak{U}(p)$ one finds an open $U \in \mathfrak{U}(p)$ with

$$p \in U \subseteq U^{\mathrm{cl}} \subseteq O.$$

This means that there is a neighborhood basis of closed subsets for each point $p \in M$.

Proof. Let $(M, \mathcal{M})$ be a T_3-space and $O \in \mathfrak{U}(p)$ be open. Clearly, $M \setminus O$ is closed, hence $p \in O$. Now since $(M, \mathcal{M})$ is T_3, hence there exists open O_1 and O_2 such that $M \setminus O \subseteq O_1$ and $p \in O_2$ and $O_1 \cap O_2 = \Phi$. Hence it is clear to see that now $M \setminus O_1 \subseteq O$ where trivially $M \setminus O_1$ is closed and since O_1 and O_2 are disjoint hence we get $O_2 \subseteq M \setminus O_1 \subseteq O$. Since O_2^{cl} is the smallest closed subset containing O_2, hence we get the desired result. ∎

✓ **Proposition 18.** A topological space $(M, \mathcal{M})$ is a T_4-space if and only if for every closed subset $A \subseteq M$ and every open $O \subseteq M$ with $A \subseteq O$, we have an open $U \subseteq M$ with

$$A \subseteq U \subseteq U^{\mathrm{cl}} \subseteq O$$

Proof. Let $(M, \mathcal{M})$ be a T_4-space and $A \subseteq M$ be a closed subset with $A \subseteq O$ where O is an open subset in M. Now we have that $M \setminus O$ and A are closed. Using the T_4 criterion, we have two open, disjoint subsets U and V such that $M \setminus O \subseteq U$ and $A \subseteq V$. Clearly, $M \setminus U \subseteq O$ and since U is open so $M \setminus U$ is closed. Since U and V are disjoint so $V \subseteq M \setminus U \subseteq O$. ∎

Motivated from unrelated nature of T_1, T_2 and T_3, T_4 axioms, we get the following definitions.

$\boxed{\star}$ **Definition 19.** (***Regular & Normal Spaces***) Let $(M, \mathcal{M})$ be a topological space.

1. The space $(M, \mathcal{M})$ is called **regular** if it is T_1 and T_3.
2. The space $(M, \mathcal{M})$ is called **normal** if it is T_1 and T_4.

✓ **Proposition 19.** A **regular space is Hausdorff** (T_2) and a **normal space is regular**.

Proof. Let $(M, \mathcal{M})$ be a regular space. So it is T_1 and T_3 by definition. Also let $p \neq q$ in M, hence $\{p\}, \{q\} \subseteq M$ are closed subsets by Proposition 16. Hence T_3 implies T_4 trivially.

For $(M, \mathcal{M})$ being a normal space, we have that it is T_1 and T_4. By T_1, a point is closed, therefore T_4 implies T_3 by definition. ■

Remark. Note the following:

- $T_1 + T_3 \implies T_2$.
- $T_1 + T_4 \implies T_1 + T_3$.
- Hence, $T_1 + T_4 \implies T_1 + T_2 + T_3 + T_4$.
- This proposition shows the importance of T_2-space, hence the special name (Hausdorff) given to it.

Metric Spaces are Normal!

✓ **Proposition 20.** A metric space is normal and hence it is $T_1, T_2, T_3 \& T_4$.

Proof. Let M be a metric space with distance function $d(\cdot, \cdot)$. Consider two points $p, q \in M$. Let $r = d(p, q)$. Clearly, $q \notin B_{r/2}(p)$ and $p \notin B_{r/2}(q)$, hence M is T_1.

For proving T_4, consider two closed disjoint subsets $A, B \subseteq M$. For any $p \in A$, we have a r_p such that $B_{r_p}(p) \cap B = \Phi$. Note that $p \in M \setminus B$ and $M \setminus B$ is open. Similarly for $q \in B$ we have r_q such that $B_{r_q}(q) \cap A = \Phi$. Then we consider the following open sets

$$U = \bigcup_{p \in A} B_{r_p/2}(p) \text{ and } V = \bigcup_{q \in B} B_{r_q/2}(q)$$

Then, we have that $A \subseteq U$ and $B \subseteq V$. To show U and V are disjoint, we just need to show that every intersection $B_{r_p/2}(p) \cap B_{r_q/2}(q) = \Phi$, which trivially follows by triangle inequality. ■

✓ **Proposition 21.** Let $f, g : (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a continuous maps between topological spaces and assume that $(N, \mathcal{N})$ is Hausdorff. Then,

1. The coincidence set $\{p \in M \mid f(p) = g(p)\} \subseteq M$ is **closed**.
2. If $U \subseteq M$ is dense then $f|_U = g|_U$ implies $f = g$.

Proof. Consider $A = \{p \in M \mid f(p) = g(p)\}$, then $M \setminus A = \{q \in M \mid f(q) \neq g(q)\}$. Now take any point $p \in M \setminus A$ so that $f(p) \neq g(p)$. Since f and g are continuous, therefore $\forall\ U \in \mathfrak{U}(f(p))\ \&\ \forall\ V \in \mathfrak{U}(g(p))$, $f^{-1}(U)$ and $g^{-1}(V)$ are open neighborhoods of p. Moreover, since $(N, \mathcal{N})$ is Hausdorff, therefore there exists disjoint open sets O_1 and O_2 such that $f(p) \in O_1$ and $g(p) \in O_2$. Hence $f^{-1}(O_1), g^{-1}(O_2) \in \mathfrak{U}(p)$. Now since $p \in U := f^{-1}(O_1) \cup g^{-1}(O_2) \subseteq M \setminus A$ is open, hence $U \in \mathfrak{U}(p)$. Therefore $M \setminus A \in \mathfrak{U}(p)$ for all $p \in M \setminus A$ so $M \setminus A$ is open and A is closed.
In the second part, $f|_U = \{f(p) \mid p \in U \subseteq M\}$, similarly for $g|_U$. If $f|_U = g|_U$, then the incidence set $\{p \in U \mid f(p) = g(p)\}$ is the whole of U. Since incidence set is closed, so is the U^{cl}. But $U^{\text{cl}} = M$ since U is dense, hence $f = g$ for all points in $(M, \mathcal{M})$. ■

---

2 Construction of Topological Spaces