Topology

Solution to selected problems

Animesh Renanse

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Answer 2.7.4: The question defined any $O \subseteq M$ as *open* if $O \in \mathfrak{U}(p)$ for all $p \in O$. We need to show that this defines a topology $\mathcal{M}$ on M. $\mathfrak{U}(p)$ follows properties 2-5 in Proposition 3.
To check if $M \in \mathcal{M}$: For any $U \in \mathfrak{U}(p)$ and since $U \subseteq M$ hence $M \in \mathfrak{U}(p)$ and hence M is open.
To check $\Phi \in \mathcal{M}$: For any $U \in \mathfrak{U}(p)$ then $\exists\ V \in \mathfrak{U}(p)$ with $V \subseteq U$. Now since $U \setminus V \subseteq U \in \mathfrak{U}(p)$. Therefore $U \setminus V \in \mathfrak{U}(p)$ and since $V \in \mathfrak{U}(p)$ which implies $U \setminus V \cap V = \Phi \in \mathfrak{U}(p)$.
To check if $O_1, \ldots, O_n \in \mathfrak{U}(p)$ then $O_1 \cap \ldots \cap O_n \in \mathfrak{U}(p)$: This is just the 3^{rd} property in Proposition 3.
To check if $O_1, \cdots \in \mathfrak{U}(p)$ then $O_1 \cup \cdots \in \mathfrak{U}(p)$: Denote $O = \bigcup_{i\in I} O_i$. From the 5^{th} property, we have for each O_i a $V_i \in \mathfrak{U}(p)$ with $V_i \subseteq O_i$ and $V_i \in \mathfrak{U}(p)\ \forall\ q \in V_i$. Therefore, $V = \bigcup_{i\in I} V_i \subseteq O$. Note that $V \in \mathfrak{U}(q)\ \forall\ q \in V_i\ \forall\ i \in I$. Since $p \in V_i$ for any i therefore $\bigcup_{i\in I} V_i \in \mathfrak{U}(p)$. Since $V \subseteq O$, hence $O = \bigcup_{i\in I} O_i \in \mathfrak{U}(p)$, proving the first part.

Now, to determine the neighborhoods of such a topology. Note the definition of neighborhood : $U \subseteq M$ is called a neighborhood of $p \in M$ if $\exists$ *open* $O \subseteq M$ such that $p \in O \subseteq U$.
Consider any $p \in M$ and an *open* subset containing p, that is $p \in O \in \mathfrak{U}(p)$. Now since for any $O \in \mathfrak{U}(p)$, we have $p \in O$ therefore any *open* $O \in \mathfrak{U}(p)$ is by definition the neighborhood. Therefore

$$\mathfrak{U}(p) \subseteq \overline{\mathfrak{U}(p)}$$

Now consider any $U \in \overline{\mathfrak{U}(p)}$ which is a neighborhood of p. By definition of neighborhood, U must contain an *open* subset containing p. Therefore $\exists\ O' \subseteq U$ such that $p \in O'$ where O' is *open*, hence $O' \in \mathfrak{U}(p)$. Since $O' \subseteq U$ which implies that $U \in \mathfrak{U}(p)$ for any $U \in \overline{\mathfrak{U}(p)}$. Therefore,

$$\overline{\mathfrak{U}(p)} \subseteq \mathfrak{U}(p)$$

so that $\overline{\mathfrak{U}(p)} = \mathfrak{U}(p)$. ■
