APPLIED SUPERCONDUCTIVITY LABORATORY

Introduction to Superconducting Quantum Circuits

- Superconducting Circuit Quantization Methods -

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Lecture Overview

Week 1. Introduction to Superconducting Quantum Circuits
Week 2. Review of Mathematics and Microwave Engineering
Week 3. Review of Classical and Quantum Mechanics
Week 4. Review of Superconductivity
Week 5. Quantum Harmonic/Anharmonic Oscillators and Light-Matter Interaction
Week 6. Circuit Quantization Methods
Week 7. Parametrically Pumped Josephson Devices
Week 8. Design and Analysis of Superconducting Resonators
Week 9. Design and Analysis of Superconducting Qubits
Week 10. Design and Analysis of Single-Qubit Device: 3D Cavity
Week 11. Design and Analysis of Single-Qubit Device : 2D Chip
Week 12. Design and Analysis of Two-Qubit Device
Week 13. Design and Analysis of Josephson Parametric Amplifier
Week 14. Term Project
Week 15. Term Project

overall backgrounds, terminologies of quantum computing

mathematical and engineering backgrounds
general superconductivity

Quantum circuit analysis

design and analysis of superconducting RF devices

Keywords in Superconducting Circuit Quantization Methods

Black-Box Quantization (BBQ) Method

Black-Box Circuit
Foster's Theorem
Weakly Anharmonic Qubit

Lumped Port
Impedance Responses
Perturbation Theory
Numerical Diagonalization

Dressed Mode Frequencies
Lamb Shift
Anharmonicity
Cross Kerr Shift

Energy Participation Ratio (EPR) Method

Participation ratio
Orthogonality
Conservation

Lumped RLC Element
Eigenmode Analysis
Perturbation Theory
Numerical Diagonalization

Dressed Mode Frequencies
Lamb Shift
Anharmonicity
Cross Kerr Shift

Two-Qubit System

Coupled System
Energy Level Shift

Dispersively Coupled Resonator

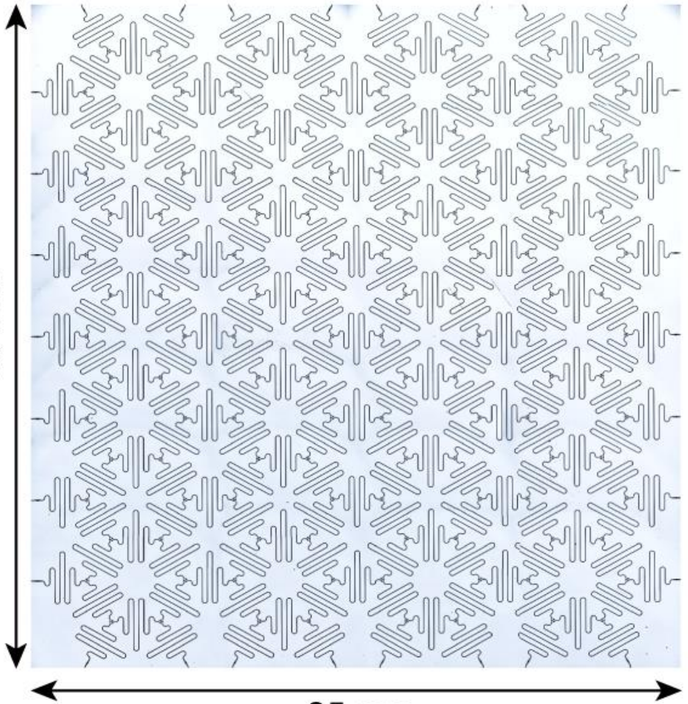
Effective Coupling Strength
ZZ Coupling

Introduction to Circuit Quantization Method

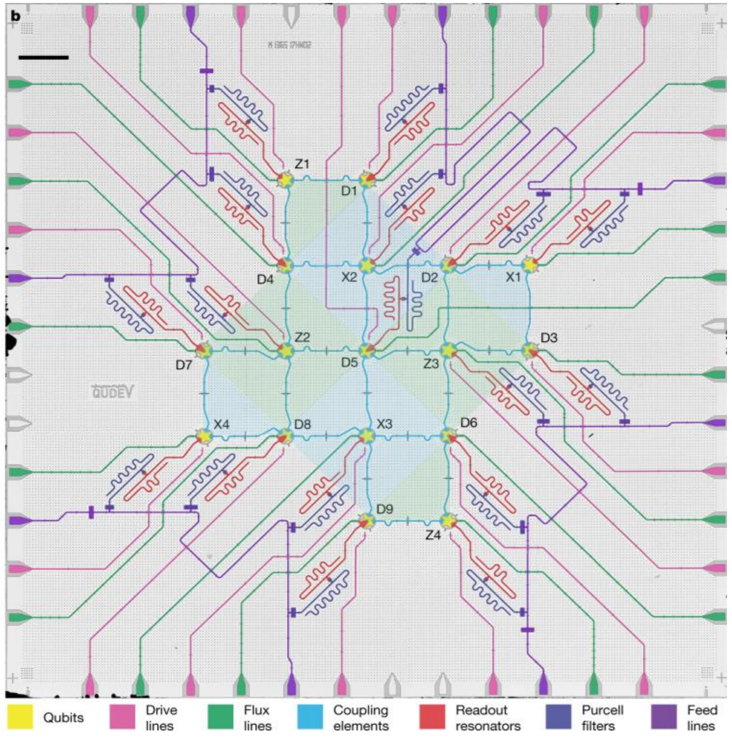
What is Circuit Quantization Method

- A method to quantum mechanically model superconducting circuits consisting of artificial atoms
- In previous lectures, we have studied how to canonically quantize selected simple circuits (e.g., a qubit and a resonator)
- Driving question: *can you REALLY quantize an arbitrary, complex superconducting circuit containing more than two qubits*?

(a)



(b)



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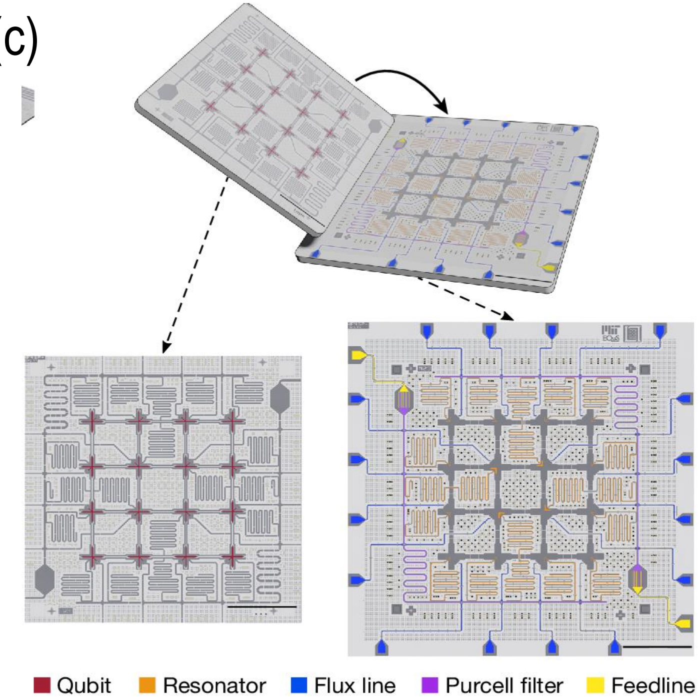


Fig. Example images of large-scale superconducting circuits. (a) ~200-resonator device from Princeton (A Houck group) (b) 16-qubit device from ETH Zurich (A Wallraff group), (c) 16-qubit device from MIT (WD Oliver group)

Images from AJ Kollár *et al.*, *Nature*, **571**, 45-50 (2019); S Krinner and N Lacroix *et al.*, *Nature*, **605**, 669-674 (2022); AH Karamlou *et al.*, *Nature*, **629**, 561-566 (2024).

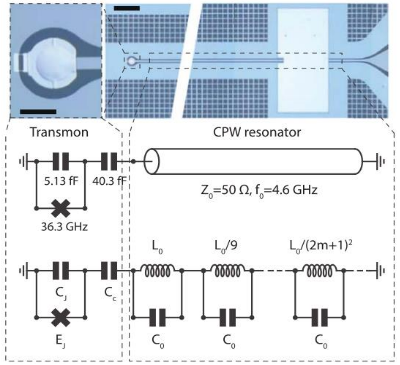
Introduction to Circuit Quantization Method

- Selected Methods for Superconducting Circuit Quantization
  - Lumped cQED and quasi-lumped cQED: based on ideal lumped-element circuit models
  - Black-box quantization (BBQ): based on impedance responses from driven-mode simulations
  - Energy participation ratio (EPR): based on electromagnetic field distributions from eigenmode simulations

Limited but computationally efficient analysis

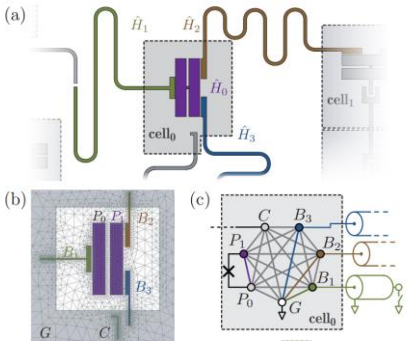
general but computationally expensive analysis

Lumped cQED



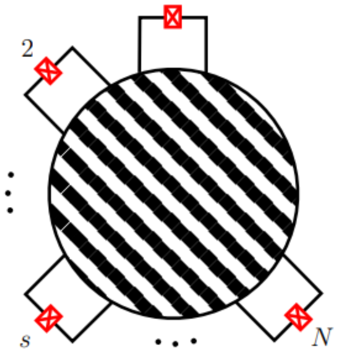
M.F. Gely, *New J. Phys.*, **22**, 013025 (2020).

Quasi-Lumped cQED



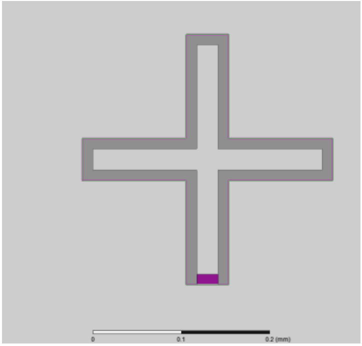
Z.K Minev et al., arXiv:2103.10344

Black-Box Quantization



SE Niggs *et al.*, *Phys. Rev. Lett.*, **108**, 240502 (2012).

Energy Participation Ratio



ZK Minev *et al.*, *npj Quantum Inf.*, **7**, 131 (2021)

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Introduction to Circuit Quantization Method

- Normal-Mode Hamiltonian by Perturbation Theory and Numerical Diagonalization
  - The ultimate goal of circuit quantization method is to derive the Hamiltonian of a superconducting circuit as

$$\mathcal{H}/\hbar = \sum_{n=1}^{N}\left(\omega_n \hat{a}_n^\dagger \hat{a}_n + \frac{\alpha_n}{2}\hat{a}_n^\dagger \hat{a}_n\left(\hat{a}_n^\dagger \hat{a}_n - 1\right)\right) + \sum_{n\neq m}\chi_{mn}\,\hat{a}_m^\dagger \hat{a}_m \hat{a}_n^\dagger \hat{a}_n$$

ω_n : mode frequency
α_n: self-Kerr (anharmonicity for qubit-mode) of mode n
χ_{mn}: cross-Kerr between mode m and mode n

  - Perturbation theory: approximated nonlinear part of Josephson junctions as $E_J[1-\cos\hat{\varphi}] - \frac{E_J}{2!}\hat{\varphi}^2 \approx -\frac{E_J}{4!}\hat{\varphi}^4$
  - Pros: simple and fast
  - Cons: inaccurate

  - Numerical diagonalization: nonlinear part truncated to include higher order terms $E_J[1-\cos\hat{\varphi}] - \frac{E_J}{2!}\hat{\varphi}^2 \approx -\frac{E_J}{4!}\hat{\varphi}^4 + \cdots$
  - Pros: accurate
  - Cons: slow and large memory for diagonalization

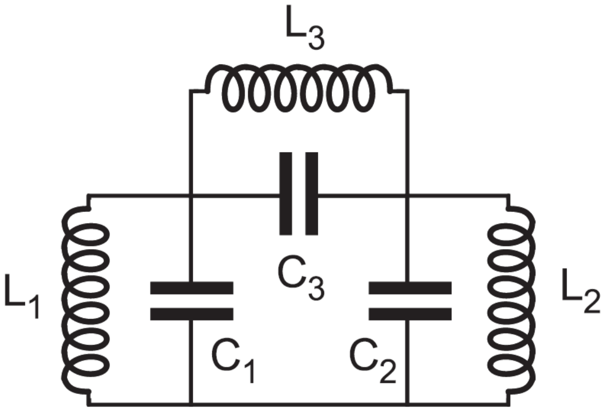
Canonical Quantization Method: (1) Lumped cQED

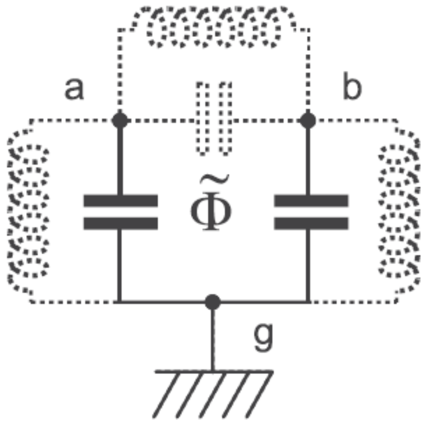
- Review: Example of Lossless Linear Circuit
  - Kirchhoff's law for branch:

$$\sum_{\text{all } b \text{ around } l} \Phi_b = \tilde{\Phi}_l$$

  - Kirchhoff's law for node:

$$\sum_{\substack{\text{all } b \text{ arriving}\\ \text{at } n}} Q_b = \tilde{Q}_n$$





NOTE: From the Kirchhoff's laws, boundary conditions for a circuit are established

  - Potential energy ε_{pot} and kinetic energy ε_{kin} are

$$\mathcal{E}_{\text{pot}} = \frac{1}{2}\vec{\phi}^t[\boldsymbol{L}^{-1}]\vec{\phi} + \sum_b \frac{1}{L_b}(\phi_n - \phi_{n'})\tilde{\Phi}_b \qquad \mathcal{E}_{\text{kin}} = \frac{1}{2}\vec{\dot{\phi}}^t[\boldsymbol{C}]\vec{\dot{\phi}}$$

  - Lagrangian $\mathcal{L} = \varepsilon_{\text{kin}} - \varepsilon_{\text{pot}}$ is

$$\mathcal{L}\left(\phi_a, \dot{\phi}_a, \phi_b, \dot{\phi}_b\right) = \frac{C_1\dot{\phi}_a^2}{2} + \frac{C_2\dot{\phi}_b^2}{2} + \frac{C_3\left(\dot{\phi}_a - \dot{\phi}_b\right)^2}{2} - \left[\frac{\phi_a^2}{2L_1} + \frac{\phi_b^2}{2L_2} + \frac{\left(\phi_a - \phi_b + \tilde{\Phi}\right)^2}{2L_3}\right]$$

  - Canonical conjugate momentum q_n for the node flux n is $q_n = \dfrac{\partial\mathcal{L}}{\partial\,\dot{\phi}_n}$
  - The Hamiltonian $\mathcal{H}$ of a circuit is

$$\mathcal{H}\left(\phi_a, q_a, \phi_b, q_b\right) = \frac{1}{C_1C_2 + C_1C_3 + C_2C_3}\left[\frac{\left(C_2 + C_3\right)q_a^2}{2} + \frac{\left(C_1 + C_3\right)q_b^2}{2} + C_3q_aq_b\right] + \left[\frac{\phi_a^2}{2L_1} + \frac{\phi_b^2}{2L_2} + \frac{\left(\phi_a - \phi_b + \tilde{\Phi}\right)^2}{2L_3}\right]$$

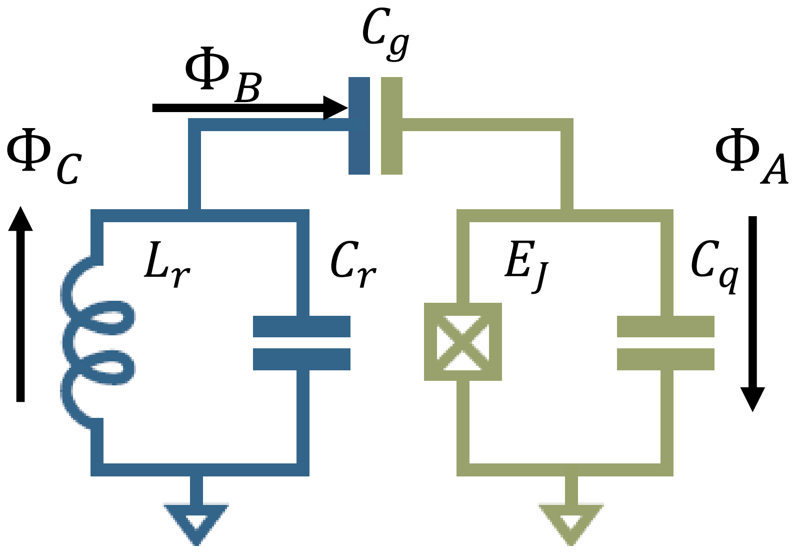
$\vec{\phi}$: node flux column vector
$\vec{\phi}^{\,t}$: transpose of a node flux column vector
Φ_b: flux of branch b
$\boldsymbol{L}$: inductance matrix of a circuit
$\boldsymbol{C}$: capacitance matrix of a circuit

Images from U Vool and M Devoret, *Int. J. Circ. Theor. Appl.*, **45**, 897-934 (2017)

Canonical Quantization Method: (1) Lumped cQED

- Review: Single-Qubit Capacitively Coupled to a Resonator
  - By associating the flux of each node, generalized coordinate for the circuit can be established with Kirchhoff's voltage and current laws
  - The Lagrangian of the system can be expressed as

$$\mathcal{L} = \underbrace{\frac{C_q}{2}\dot{\Phi}_A^2 + \frac{C_g}{2}\left(\dot{\Phi}_A + \dot{\Phi}_B\right)^2 + E_J \cos\left(\left(\frac{2\pi}{\Phi_0}\right)\Phi_A\right)}_{\mathcal{L}\text{ of a qubit}} + \underbrace{\frac{C_r}{2}\dot{\Phi}_C^2 - \frac{1}{2L_r}\Phi_C}_{\mathcal{L}\text{ of a resonator}}$$



  - The Hamiltonian of the system can be derived as

$$\mathcal{H} = \frac{C_r + C_g}{2\left(C_q C_g + C_r C_g + C_q C_r\right)^2} Q_A^2 + \frac{C_g}{\left(C_q C_g + C_r C_g + C_q C_r\right)^2} Q_A Q_c - E_J \cos\left(\left(\frac{2\pi}{\Phi_0}\right)\Phi_A\right) + \frac{C_q + C_g}{2\left(C_q C_g + C_r C_g + C_q C_r\right)^2} Q_c^2 + \frac{\Phi_C}{2L}$$

  - The above $\mathcal{H}$ can be further approximated with assumptions (see previous lecture note: light-matter interaction)
  - There are open-source projects about lumped cQED analysis (such as QuCAT, scQubits, CircuitQ,…etc.)

You may accurately calculate every capacitance or inductance in a simple circuit…
As circuit becomes larger, there are parasitic capacitance and inductance → an alternative approach needs to be performed

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Canonical Quantization Method: (2) Black-Box Quantization

Characteristics of BBQ Method

- An arbitrary black-box circuit is assumed as a Foster's network
- Foster's network is a series-connected, parallel RLC circuit of modes M

$$Z(\omega) = \sum_{p=1}^{M} \left(j\omega C_p + \frac{1}{j\omega L_p} + \frac{1}{R_p} \right)^{-1} \text{ and } Y(\omega) = \frac{1}{Z(\omega)} = \sum_{p=1}^{M} \left(j\omega C_p + \frac{1}{j\omega L_p} + \frac{1}{R_p} \right)$$

- A Josephson junction is replaced by linear and nonlinear parts
- Linear part: inductance L_J and capacitance C_J
- Nonlinear part: $E_J[1 - \cos\hat{\varphi}] - \frac{E_J}{2!}\hat{\varphi}^2 \approx -\frac{E_J}{4!}\hat{\varphi}^4 + \frac{E_J}{6!}\hat{\varphi}^6 - \frac{E_J}{8!}\hat{\varphi}^8 + \cdots$

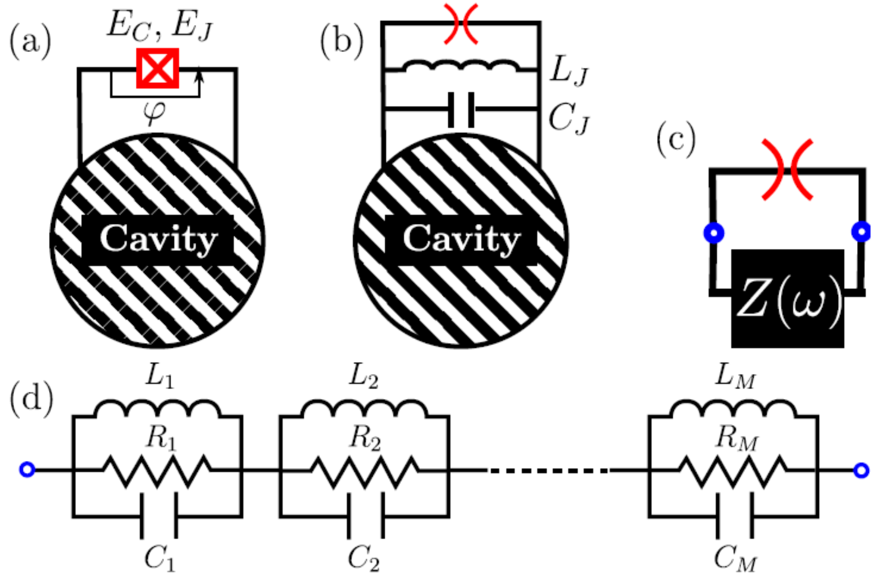

Fig. (a) Schematic of a Josephson junction coupled to an arbitrary linear circuit. (b) The Josephson element is replaced by a parallel combination of a linear and nonlinear elements. (c) An arbitrary linear circuit is represented by the impedance responses seen by the Josephson junction. (d) Foster's network.

Key Assumptions of BBQ Method

- Linear low-loss black-box → condition for lossy Foster's network theorem
- Weakly anharmonic qubit → neglecting charge dispersion effect in qubit (transmon qubit is assumed)
- Impedance matrix to model the black-box network → necessitating full-wave driven-mode simulations
- Linear and nonlinear parts of a Josephson junction → Linear Hamiltonian and nonlinear Hamiltonian of a Josephson junction

Images from SE Niggs *et al.*, "Black-box superconducting circuit quantization," *Phys. Rev. Lett.*, **108**, 240502 (2012).

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Canonical Quantization Method: (2) Black-Box Quantization

Post-Processing in BBQ Method

- From L_J, C_J and $Z(\omega)$ data, the Hamiltonian of the device can be derived as

$$\mathcal{H}/\hbar = \sum_{n=1}^{N}\left(\omega_n \hat{a}_n^\dagger \hat{a}_n + \frac{\alpha_n}{2}\hat{a}_n^\dagger \hat{a}_n\left(\hat{a}_n^\dagger \hat{a}_n - 1\right)\right) + \sum_{n\neq m}\chi_{mn}\,\hat{a}_m^\dagger \hat{a}_m \hat{a}_n^\dagger \hat{a}_n$$

Perturbation Theory

- Mode frequency ω_n can be identified by zero-crossing frequency of $Y(\omega)$ of a black-box circuit
- Mode capacitance $C_n = \frac{1}{2}\mathrm{Im}\left[\frac{\partial Y(\omega)}{\partial \omega}\right]$
- Mode inductance $L_n = \frac{1}{C_n \omega_n^2}$
- Effective mode impedance $Z_n^{\mathrm{eff}} = \frac{1}{\omega_n C_n}$
- Lamb shift $\Delta_n = -\frac{e^2}{2L_J}\left[Z_n^{\mathrm{eff}}\sum_p Z_p^{\mathrm{eff}} - \frac{\left(Z_p^{\mathrm{eff}}\right)^2}{2}\right]$
- Self kerr $\alpha_n = -\frac{L_n}{L_J}\frac{C_J}{C_n}E_c$
- Cross Kerr $\chi_{mn} = -2\sqrt{\alpha_m \alpha_n}$

Numerical Diagonalization

- From the full Hamiltonian, linear and nonlinear part can be separated as

$$\hat{H}_{\mathrm{full}} = \sum_{n=1}^{N}\omega_n \hat{a}_n^\dagger \hat{a}_n - \sum_{j=1}^{M}\sum_{k=2} E_j \frac{(-1)^k}{(2k)!}\hat{\varphi}_j^{2k}$$

$$\hat{\varphi}_j = \sum_{n=1}^{N}\varphi_{j,n}^{\mathrm{ZPF}}\left(\hat{a}_n^\dagger + \hat{a}_n\right),\ \varphi_{j,n}^{\mathrm{ZPF}} = \sqrt{\frac{\hbar}{2}Z_n^{\mathrm{eff}}}$$

> E_j : Junction energy of a Josephson junction j
> $\varphi_{j,n}^{\mathrm{ZPF}}$: zero-point phase fluctuation of a junction j at mode n
> M: maximum level for truncation of cosine potential of a junction j

- By numerically diagonalizing $\hat{\mathcal{H}}_{\mathrm{full}}$, the normal mode Hamiltonian are calculated

Canonical Quantization Method: (2) BBQ for Single-Qubit Device

■ Steps to Perform BBQ Method

(1) Design (or import) superconducting device layouts in FEM

Note: air needs to be defined around the chip for full-wave simulations

(2) Set boundary conditions for superconducting layers

Note: Perfect electric conductor is a typical choice

(3) Set lumped ports for Josephson junctions

Note: L_J and C_J can be modeled as a lumped LC boundary condition

(4) Set initial mesh seeds and setup

Note: fine mesh configurations are required for lumped-ports and compact geometries

(5) Set FEM simulation setups in driven-mode (frequency domain)

Note: frequency step needs to be fine enough

(6) Simulate the device

(7) Calculate $Z(\omega)$ of the black-box circuit

(8) Post-process $Z(\omega)$ to find out the normal-mode Hamiltonian!

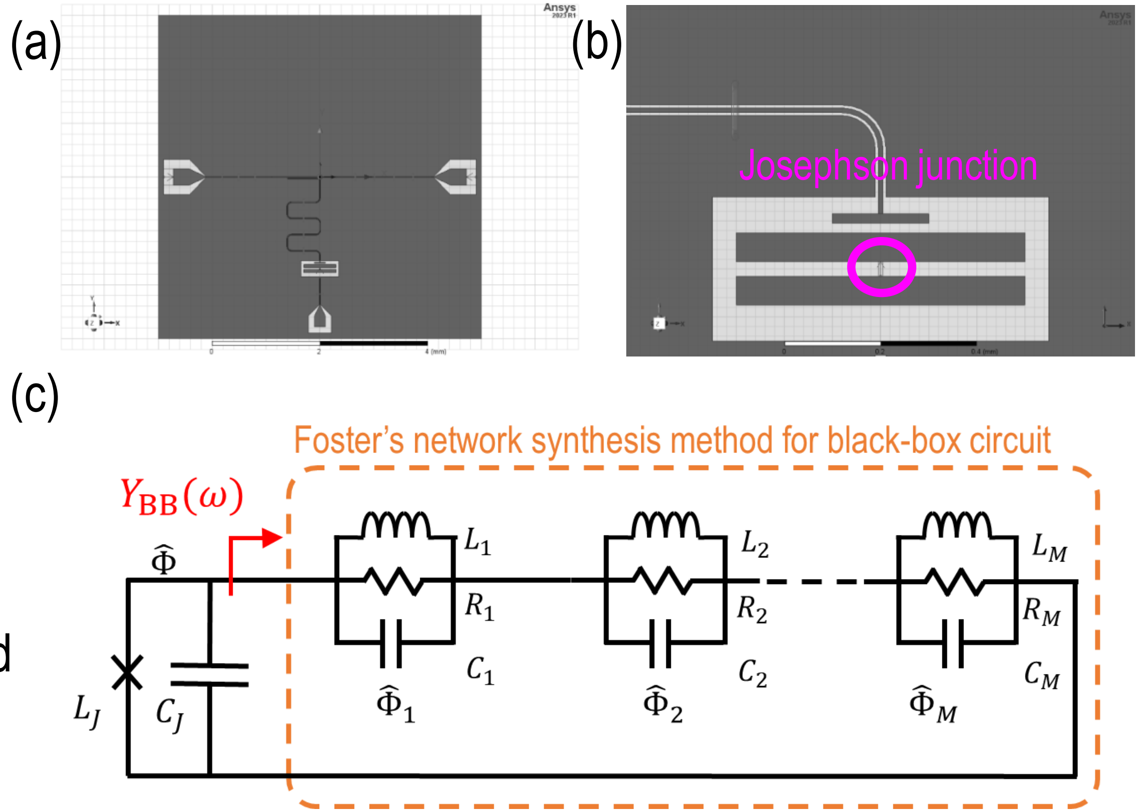

Fig. (a) Overall geometric features of an example device consisting of a transmon qubit and a readout resonator, coupled to a transmission line. (b) Enlarged view of the qubit. (c) An arbitrary linear circuit is represented by an equivalent Foster's network, seen by the Josephson junction of the qubit.

Images from SE Niggs *et al.*, *Phys. Rev. Lett.*, **108**, 240502 (2012) and SH Park, PhD Dissertation, Seoul National University, (2024)

Canonical Quantization Method: (2) BBQ for Single-Qubit Device

- Example Admittance Plots for BBQ Method

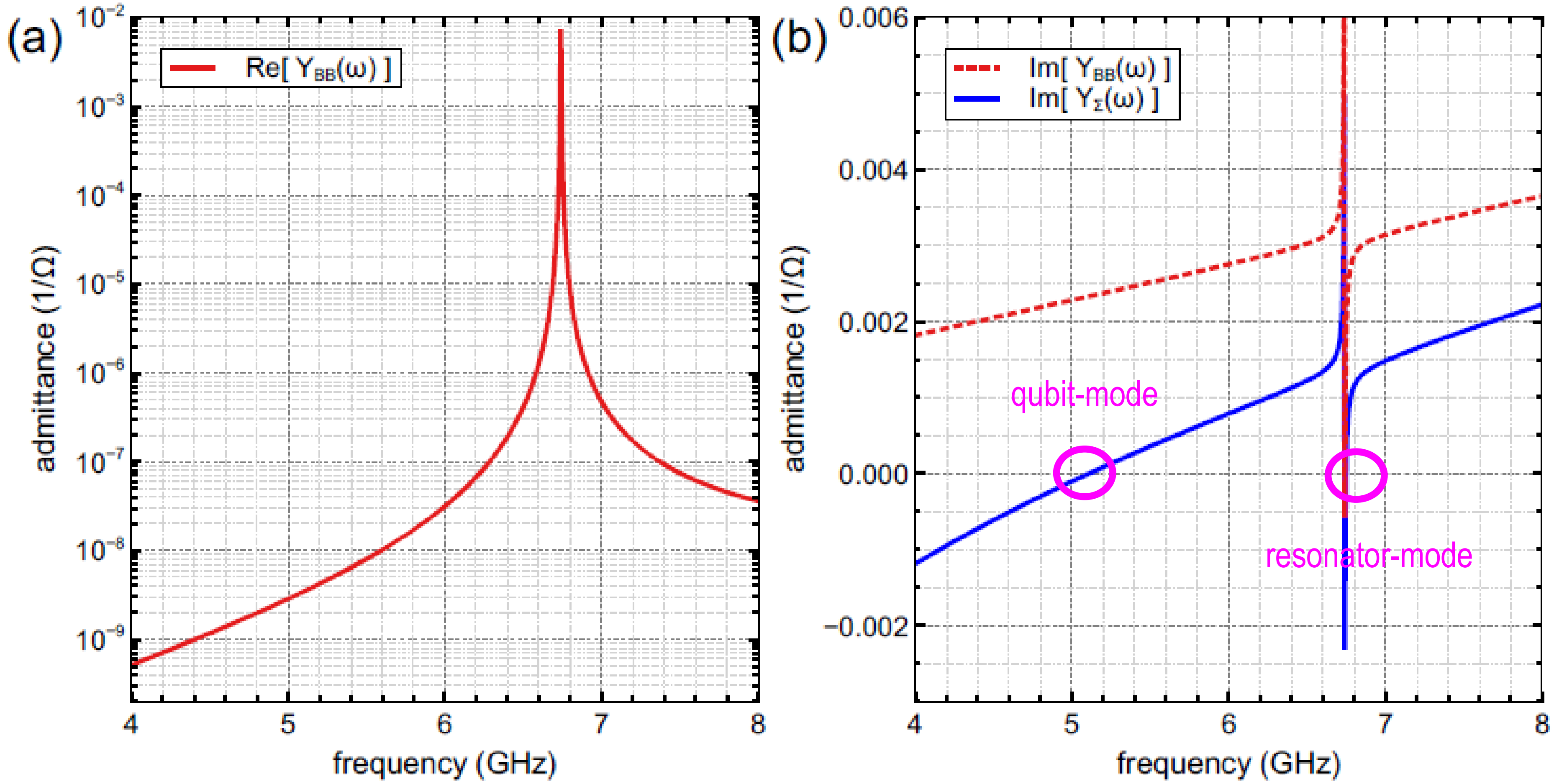

Fig. (a) Real and (b) imaginary parts of $Y_{BB}(\omega)$. The total imaginary part is calculated as $Y_{\Sigma}(\omega) = Y_{BB}(\omega) + Y_{JJ}(\omega)$ while the Josephson junction's admittance is $Y_{JJ}(\omega) = j\omega C_J + \frac{1}{j\omega L_J}$. The zero crossing frequencies are 5.110 GHz and 6.751 GHz, indicating qubit mode and readout resonator mode, respectively.

Canonical Quantization Method: (3) Energy Participation Ratio (EPR) Method

■ Characteristics of EPR Method

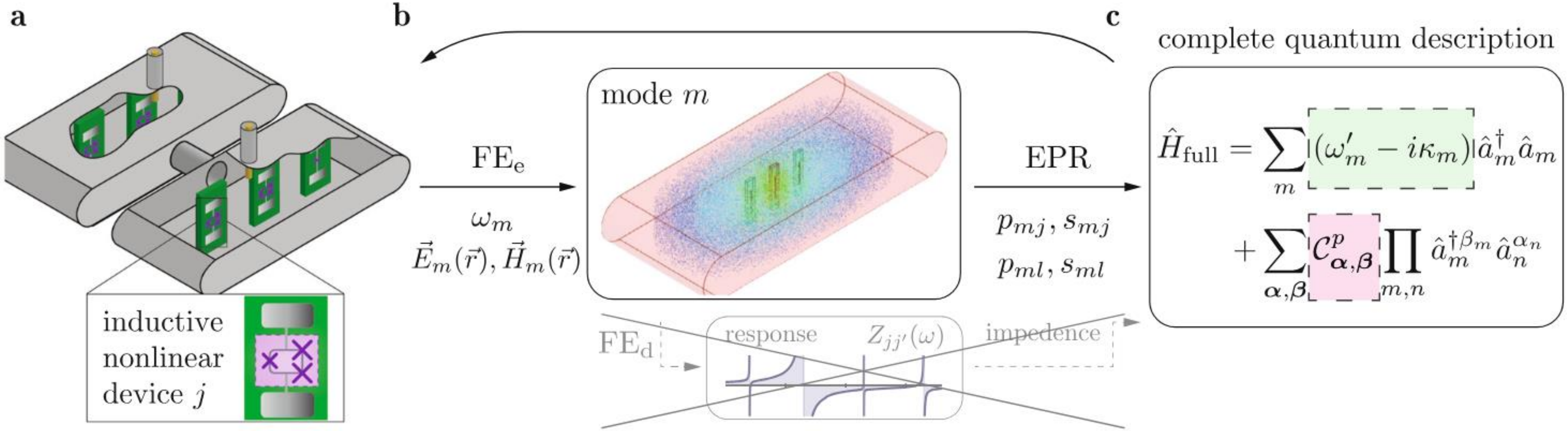

Fig. (a) Illustration of an example quantum device. (b) Electric and magnetic fields of the eigenmode m without the impedance matrix calculation. (c) The Hamiltonian is calculated from the eigenmode simulations via the energy participation ratio p_{mj} and its sign S_{mj} of each Junction j at mode m.

- □ Instead of impedance responses in BBQ, the EPR method employs participation ratio p_{mj} to derive the Hamiltonian
- □ Instead of driven-mode simulations in BBQ, the EPR method employs eigenmode simulations to calculate p_{mj}
- □ A Josephson junction is replaced by linear and nonlinear parts, similar to the BBQ method

■ Key Assumptions of EPR Method

- □ Same as the BBQ method

Images from ZK Minev *et al*., "Energy-participation quantization of Josephson circuits," *npj Quantum Inf.*, **7**, 131 (2021).

Canonical Quantization Method: (3) Energy Participation Ratio (EPR) Method

- Definition of the Energy Participation Ratio in EPR Method
  - The Energy Participation Ratio (EPR) p_{mj} quantifies the fraction of energy of mode m stored in junction element j as

$$p_{mj} = \frac{\text{Inductive energy stored in junction } j}{\text{Inductive energy stored in mode } m} = \frac{\langle \psi_m | \frac{1}{2} E_j \hat{\varphi}_j^2 | \psi_m \rangle}{\langle \psi_m | \frac{1}{2} \mathcal{H}_{\text{lin}} | \psi_m \rangle}$$

  - p_{mj} is a dimensionless positive-definite number between 0 and 1
- Characteristics of the Energy Participation Ratio in EPR Method
  - Canonical quantization: participation ratio can be converted into the zero-point phase fluctuation with a sign value s_{mj}

$$\varphi_{mj} = s_{mj} \sqrt{p_{mj} \frac{\hbar \omega_m}{2E_j}} \text{ and } s_{mj} \in \{-1, +1\}$$

  - Normalization: for each mode m, the total participation ratio of a junction j is bounded by unity

$$\sum_{m=1}^{M} p_{mj} = 1$$

Orthogonality: participation ratios satisfy an orthogonality condition due to the orthogonal nature of eigenmodes in the system

$$\sum_{m=1}^{M} s_{mj} s_{mj'} \sqrt{p_{mj} p_{mj'}} = 0$$

ZK Minev *et al.*, "Energy-participation quantization of Josephson circuits," *npj Quantum Inf.*, **7**, 131 (2021).

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Canonical Quantization Method: (3) Energy Participation Ratio (EPR) Method

Post-Processing in EPR Method

- From L_J, C_J and p_{mj} data, the Hamiltonian of the device can be derived as

$$\mathcal{H}/\hbar = \sum_{n=1}^{N}\left(\omega_n \hat{a}_n^\dagger \hat{a}_n + \frac{\alpha_n}{2}\hat{a}_n^\dagger \hat{a}_n\left(\hat{a}_n^\dagger \hat{a}_n - 1\right)\right) + \sum_{n\neq m}\chi_{mn}\,\hat{a}_m^\dagger \hat{a}_m \hat{a}_n^\dagger \hat{a}_n$$

Perturbation Theory

- Diagonal matrix of eigenfrequency $\mathbf{\Omega} = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_N \end{bmatrix}$
- Diagonal matrix of junction energy $\mathbf{E_J} = \begin{bmatrix} E_{J,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & E_{J,N} \end{bmatrix}$
- Matrix of participation ratios $\mathbf{P} = \begin{bmatrix} p_{11} & \cdots & p_{1J} \\ \vdots & \ddots & \vdots \\ p_{N1} & \cdots & p_{NJ} \end{bmatrix}$
- Kerr matrix $\mathbf{X} = \frac{\hbar}{4}\mathbf{\Omega}\mathbf{P}\left(\mathbf{E_J}\right)^{-1}\mathbf{P}\mathbf{\Omega}$
- Self Kerr is diagonal element of $\mathbf{X}$ as $\alpha_n = \frac{1}{2}[\mathbf{X}]_{nn}$
- Cross Kerr is off-diagonal element of $\mathbf{X}$ as $\chi_{mn} = [\mathbf{X}]_{mn}$
- Lamb shift is $\Delta_m = \frac{1}{2}\sum_{n=1}^{N}[\mathbf{X}]_{mn}$

Numerical Diagonalization

- From the full Hamiltonian, linear and nonlinear part can be separated as

$$\hat{H}_{\mathrm{full}} = \sum_{n=1}^{N}\omega_n \hat{a}_n^\dagger \hat{a}_n - \sum_{j=1}^{M}\sum_{k=2} E_j \frac{(-1)^k}{(2k)!}\hat{\varphi}_j^{2k}$$

$$\hat{\varphi}_j = \sum_{n=1}^{N}\varphi_{j,n}^{\mathrm{ZPF}}\left(\hat{a}_n^\dagger + \hat{a}_n\right),$$

E_j : Junction energy of a Josephson junction j
$\varphi_{j,n}^{\mathrm{ZPF}}$: zero-point phase fluctuation of a junction j at mode n
M: maximum level for truncation of cosine potential of a junction j

- By numerically diagonalizing $\widehat{\mathcal{H}}_{\mathrm{full}}$, the normal mode Hamiltonian are calculated

Canonical Quantization Method: (3) EPR for Single-Qubit Device

■ Steps to Perform EPR Method

(1) Design (or import) superconducting device layouts in FEM

Note: air needs to be defined around the chip

(2) Set boundary conditions for superconducting layers

Note: Perfect electric conductor (PEC) is a typical choice

(3) Replace Josephson junctions with lumped LC boundary conditions

(4) Set initial mesh seeds and setup

Note: fine mesh configurations are required for lumped elements and compact geometries

(5) Set FEM simulation setups in eigenmode

Note: desired number of modes are need to be defined

(6) Simulate the device

(7) Calculate p_{mj} of the device

(8) Post-process p_{mj} to find out the normal-mode Hamiltonian!

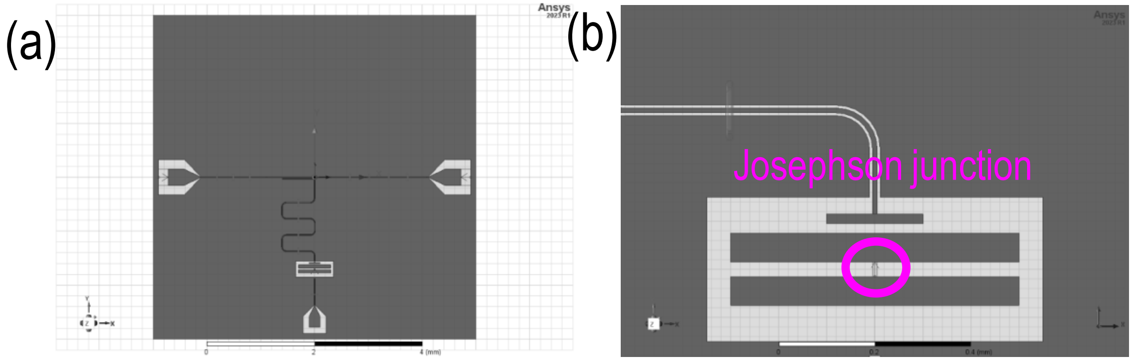

Fig. (a) Overall geometric features of an example device consisting of a transmon qubit and a readout resonator, coupled to a transmission line. (b) Enlarged view of the qubit.

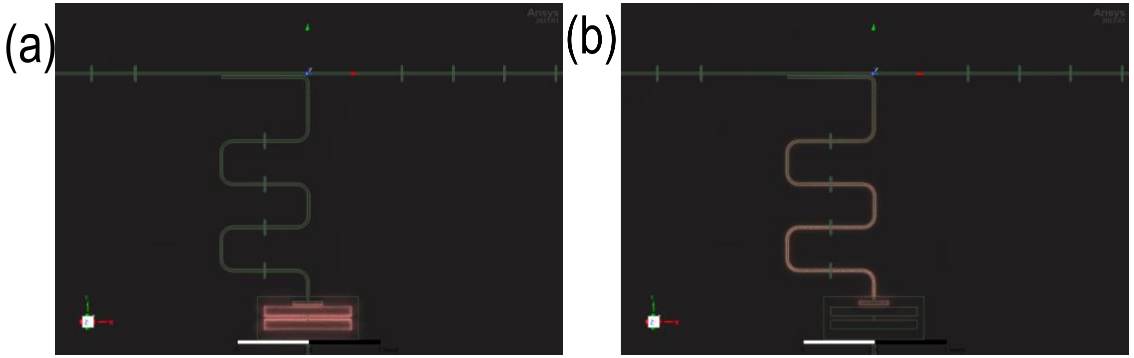

Fig. Normalized electric field distributions of (a) qubit mode and (b) readout resonator mode from eigenmode FEM simulations.

NOTE: You can perform the EPR method with ease by using the pyEPR package ! (https://github.com/zlatko-minev/pyEPR)

Images from SH Park, PhD Dissertation, Seoul National University, (2024)

Canonical Quantization Method: (4) BBQ vs EPR for Single-Qubit Device

■ Perturbation Theory vs Numerical Diagonalization

Table. Comparison between the perturbation theory (PT) and the numerical diagonalization (ND). The parentheses of ND indicate the truncated (cosine approximation order, Fock state excitation order)

Circuit Quantization Methodology	$\omega_q/2\pi$ [GHz]	$\omega_r/2\pi$ [GHz]	$\alpha_q/2\pi$ [MHz]	$\chi_{qr}/2\pi$ [MHz]
BBQ PT	4.856	6.750	-254.4	-3.141
EPR PT	4.851	6.751	-254.7	-2.845
BBQ ND (6, 6)	4.840	6.749	-270.5	-2.269
EPR ND (6, 6)	4.838	6.752	-270.8	-2.057
BBQ ND (6, 8)	4.839	6.749	-290.8	-2.292
EPR ND (6, 8)	4.837	6.752	-291.2	-2.079
BBQ ND (6, 10)	4.839	6.749	-293.2	-2.296
EPR ND (6, 10)	4.837	6.752	-293.6	-2.082
BBQ ND (8, 8)	4.839	6.749	-290.8	-2.292
EPR ND (8, 8)	4.837	6.752	-291.2	-2.079
BBQ ND (8, 10)	4.838	6.749	-293.2	-2.296
EPR ND (8, 10)	4.837	6.752	-293.6	-2.082
BBQ ND (12, 10)	4.838	6.749	-293.2	-2.296
EPR ND (12, 10)	4.837	6.752	-293.6	-2.082
BBQ ND (12, 14)	4.838	6.749	-293.5	-2.297
EPR ND (12, 14)	4.837	6.752	-293.9	-2.083
BBQ ND (20, 20)	4.838	6.749	-293.5	-2.297
EPR ND (20, 20)	4.837	6.752	-293.9	-2.083

PT shows overestimated χ and underestimated α

ND converges when higher order levels are considered!

■ BBQ vs EPR with Varying Junction Inductance

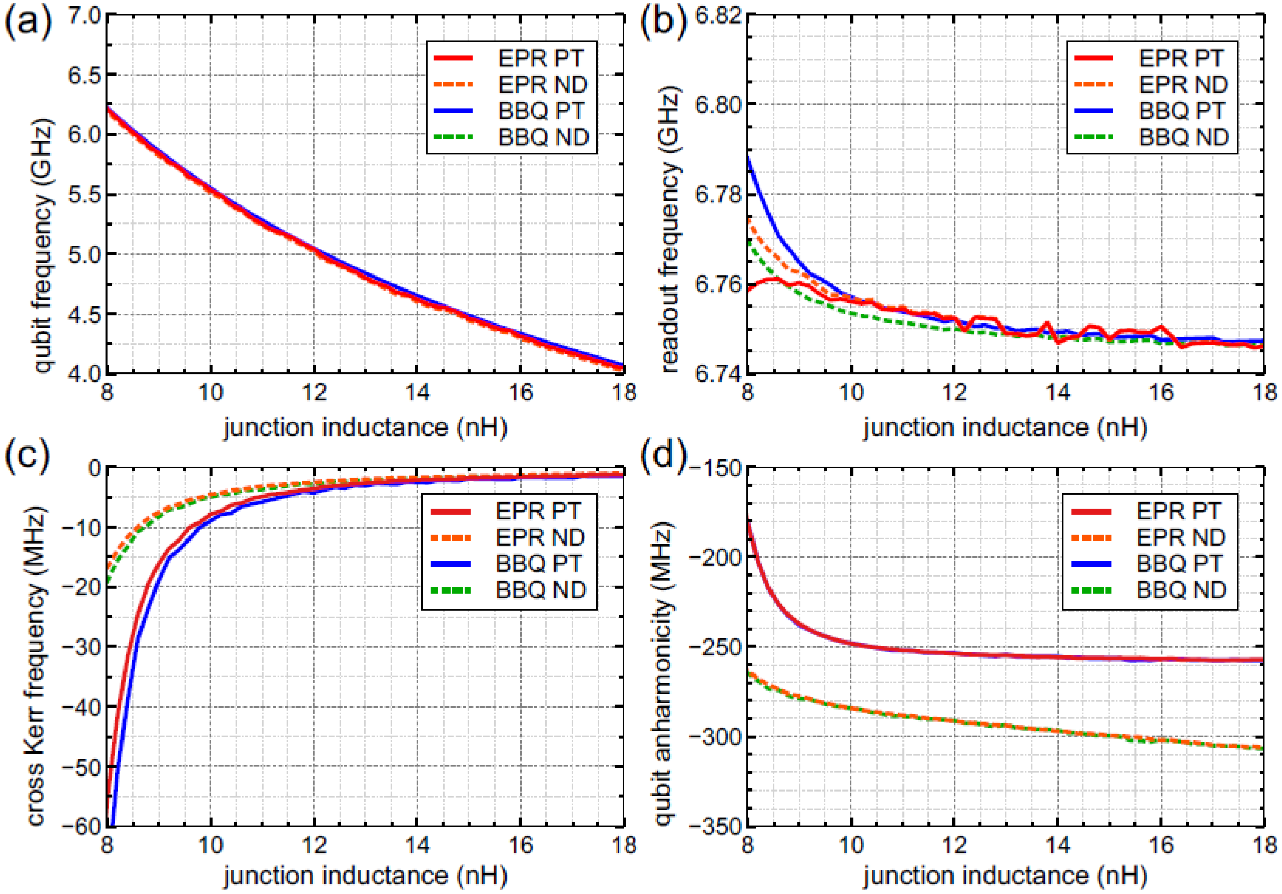

Fig. circuit parameters by BBQ and EPR methods as a function of the junction inductance. (a) Qubit frequency, (b) readout resonator frequency, (c) qubit-readout resonator cross Kerr shift frequency, and (d) qubit anharmonicity.

Images from SH Park, PhD Dissertation, Seoul National University, (2024)

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Canonical Quantization Method: (5) Two-Qubit Device

- Definition of ZZ Coupling between Two Qubits
  - ZZ coupling ν_{ZZ} is a static coupling between coupled qubits that results in an energy shift for the qubit pair

$$\nu_{ZZ} = (\widetilde{\omega}_{11} - \widetilde{\omega}_{10} - \widetilde{\omega}_{01} + \widetilde{\omega}_{00}) = \chi_{12} \text{ between qubits}$$

$\widetilde{\omega}_{ij}$: dressed state frequency when qubits are in i and j states

- Characteristics of ZZ Coupling between Two Qubits
  - ZZ coupling impacts on two-qubit gate fidelity
  - suppressing ν_{ZZ} is desired for fixed-frequency qubit devices
- Simple Equations for ZZ Coupling between Two Qubits
  - For two transmon qubits coupled by a bus resonator coupler, the effective coupling J between qubits is

$$J = \frac{g_1 g_2(\omega_1 + \omega_2 - \omega_r)}{2(\omega_1 - \omega_r)(\omega_2 - \omega_r)}$$

  - Then, ν_{ZZ} can be calculated as

$$\nu_{zz} = 2J^2\left(\frac{1}{\omega_1 - \omega_2 + \alpha_1} - \frac{1}{\omega_1 - \omega_2 - \alpha_1}\right)$$

when $|\omega_{Q1} - \omega_{Q2}| < |\alpha|$,

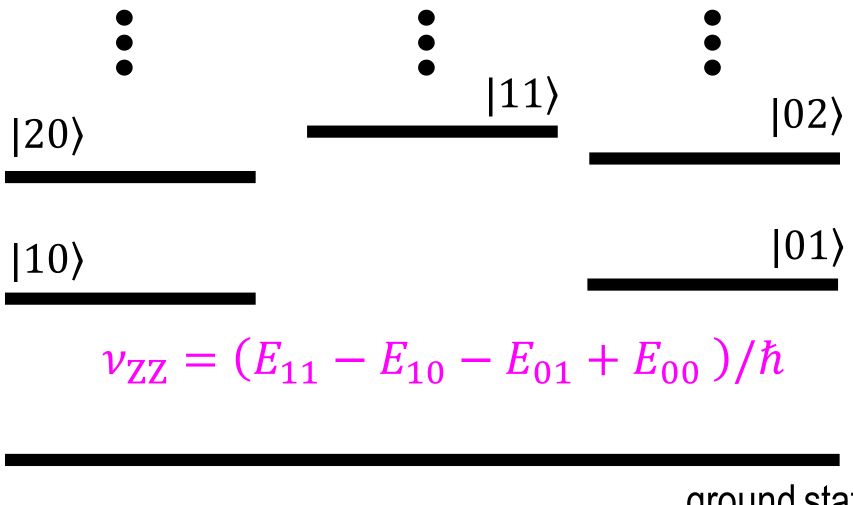

Fig. Energy level diagram of two qubit system.

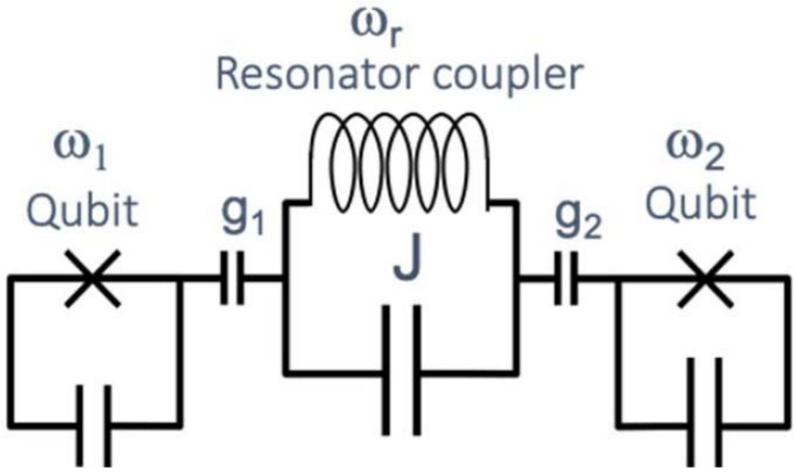

Fig. Schematic of two transmon qubits coupling by a bus resonator coupler

Images from H Paik *et al.*, "Coupler characterization of superconducting transmons qubits for cross-resonance gate," *IEEE IEDM*, 2020.

Canonical Quantization Method: (5) Two-Qubit Device

Geometric Features of a Two-Qubit Device

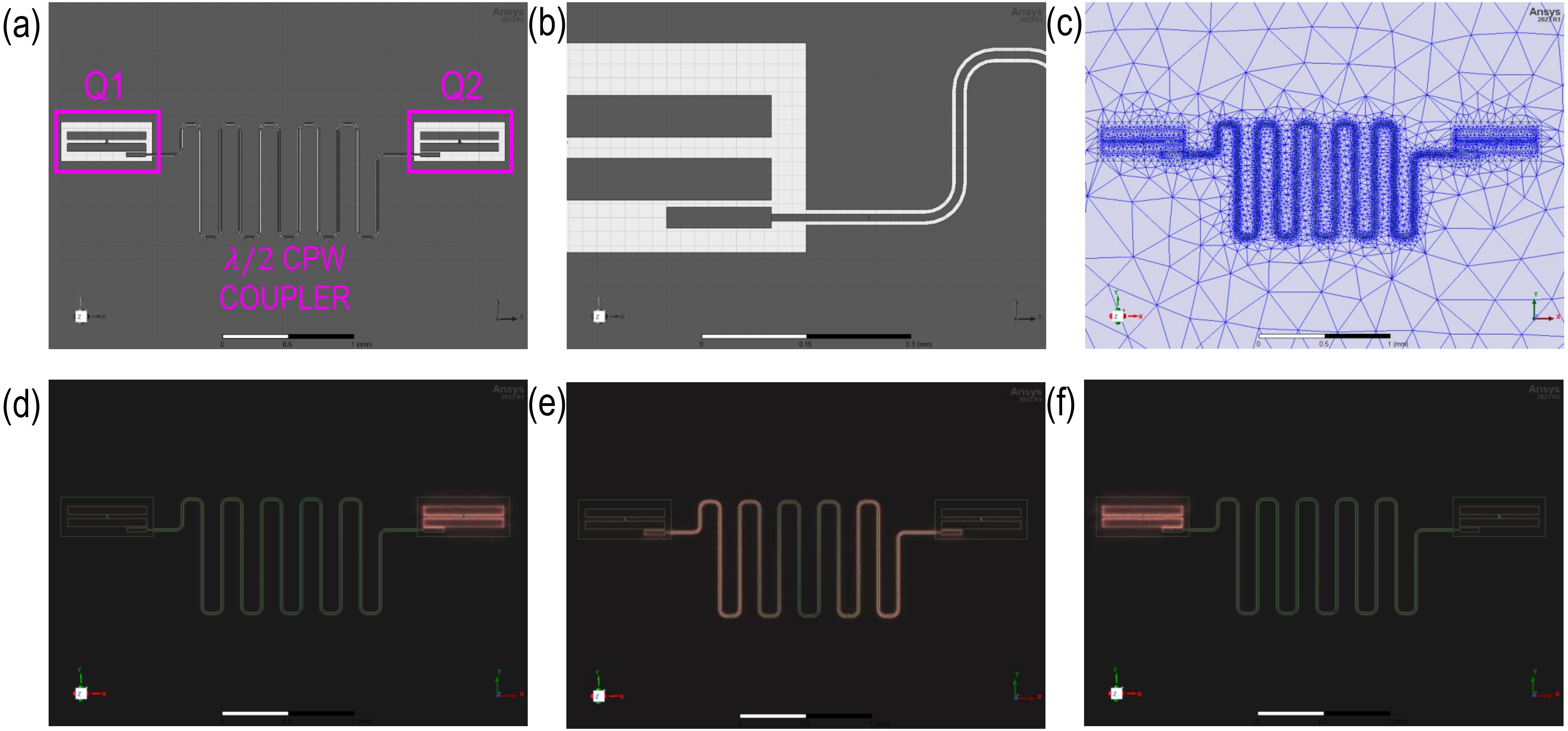

Fig. Planar two-qubit chip layout. (a) Overall geometric features of two qubits, coupled through a bus resonator. (b) Enlarged view of Q1. Here, Q1 and Q2 have the identical geometric features but different LJ values of 13.5 nH and 13 nH, respectively. (c) Mesh view of the chip. (d), (e), and (f) show the normalized electric field distributions of Q1 mode, Q2 mode, and resonator mode, obtained by eigenmode simulations.

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Canonical Quantization Method: (5) BBQ vs EPR for Two-Qubit Device

■ Perturbation Theory vs Numerical Diagonalization

Table. Comparison between the perturbation theory (PT) and the numerical diagonalization (ND). The parentheses of ND indicate the truncated (cosine approximation order, Fock state excitation order)

Circuit Quantization Methodology	$\omega_1/2\pi$ [GHz]	$\omega_2/2\pi$ [GHz]	$\omega_b/2\pi$ [GHz]	$\alpha_1/2\pi$ [MHz]	$\alpha_2/2\pi$ [MHz]	$\chi_{1b}/2\pi$ [MHz]	$\chi_{2b}/2\pi$ [MHz]	$\chi_{12}/2\pi$ [kHz]
BBQ PT	4.738	4.841	5.729	-250.6	-249.5	-4.462	-5.828	-3413
EPR PT	4.704	4.806	5.726	-245.5	-245.2	-4.203	-5.580	-2835
BBQ ND (6, 8)	4.731	4.823	5.728	-290.1	-288.2	-2.206	-2.608	171.1
EPR ND (6, 8)	4.689	4.791	5.727	-284.2	-284.3	-2.122	-2.548	145.2
BBQ ND (8, 8)	4.731	4.823	5.728	-290.1	-288.2	-2.206	-2.608	171.1
EPR ND (8, 8)	4.689	4.791	5.727	-284.2	-284.3	-2.122	-2.548	145.2
BBQ ND (8, 11)	4.730	4.823	5.728	-292.7	-290.6	-2.208	-2.610	165.2
EPR ND (8, 11)	4.689	4.791	5.727	-286.7	-286.7	-2.124	-2.550	132.2
BBQ ND (12, 11)	4.730	4.823	5.728	-292.7	-290.6	-2.208	-2.610	165.2
EPR ND (12, 11)	4.689	4.791	5.727	-286.7	-286.7	-2.124	-2.550	132.2
BBQ ND (12, 14)	4.730	4.823	5.728	-292.8	-290.8	-2.208	-2.610	165.0
EPR ND (12, 14)	4.689	4.791	5.727	-286.8	-286.8	-2.124	-2.550	131.8
BBQ ND (20, 20)	4.730	4.823	5.728	-292.9	-290.8	-2.208	-2.610	165.0
EPR ND (20, 20)	4.689	4.791	5.727	-286.8	-286.8	-2.124	-2.550	131.8

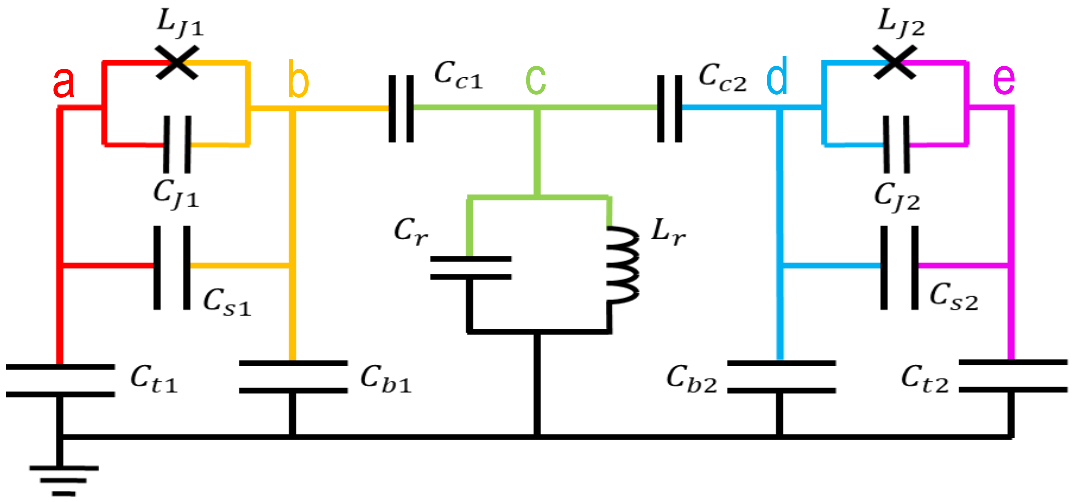

Fig. An equivalent lumped-element circuit diagram of the two-qubit device.

- The normal-mode Hamiltonian of the device is

$$\mathcal{H}/\hbar = \sum_{n=1}^{N}\left(\omega_n \hat{a}_n^\dagger \hat{a}_n + \frac{\alpha_n}{2}\hat{a}_n^\dagger \hat{a}_n(\hat{a}_n^\dagger \hat{a}_n - 1)\right) + \sum_{n\neq m} \chi_{mn}\, \hat{a}_m^\dagger \hat{a}_m \hat{a}_n^\dagger \hat{a}_n$$

$N = \{1,2,b\}$: for qubit 1, qubit 2, bus coupler, respectively

- The ZZ coupling χ_{12} between qubits is one of the most important parameters to analyze gate fidelity in future !
- χ_{12} by lumped-element model is 169 kHz
- PT shows overestimated χ_{12} with opposite sign !
- ND with higher order truncation shows good agreement

See Also…

Lumped-Element Papers:

[1] A Parra-Rodriguez *et al.*, "Canonical circuit quantization with linear nonreciprocal devices," *Phys. Rev.* B, **99**, 014514 (2019).

[2] ZK Minev *et al.*, "Circuit quantum electrodynamics (cQED) with modular quasi-lumped models," *arXiv:2103.10344* (2021).

[3] IL Egusquiza, A Parra-Rodriguez, "Algebraic canonical quantization of lumped superconducting networks," *Phys. Rev.* B, **106**, 024510 (2022).

[4] A Osborne *et al.*, "Symplectic Geometry and Circuit Quantization," *PRX Quantum*, **5**, 020309 (2024).

BBQ Papers:

[1] SE Niggs *et al.*, "Black-box superconducting circuit quantization," *Phys. Rev. Lett.*, **108**, 240502 (2012).

[2] F Solgun *et al.*, "Multiport impedance quantization," *Ann. Phys.*, **361**, 605-669 (2015).

[3] F Solgun *et al.*, "Direct Calculation of Interaction Rates in Multimode Circuit Quantum Electrodynamics," *Phys. Rev. Appl.*, **18**, 044025 (2022).

EPR Papers:

[1] ZK Minev *et al.*, "Energy-participation quantization of Josephson circuits," *npj Quantum Inf.*, **7**, 131 (2021).

[2] KH Yu *et al.*, "Using the inductive-energy participation ratio to characterize a superconducting quantum chip," *Phys. Rev. Appl.*, **21**, 034027 (2024).

Open Courses:

[1] Qiskit Global Summer School (2020): *for circuit quantization, see videos #16~ #21 (but every video is very useful)

https://youtube.com/playlist?list=PLOFEBzvs-VvrXTMy5Y2IqmSaUjfnhvBHR&si=xE-4OjWXYSxf5gY6