Introduction to Superconducting Quantum Circuits

- Review of Microwave Engineering for Quantum Computing -

Department of Electrical and Computer Engineering
Seoul National University

Seong Hyeon Park and Seungyong Hahn

23rd October, 2024

Keywords in Microwave Engineering for Quantum Computing

Selected Microwave Key Modules

Amplifier	Filter
Local Oscillator	Frequency Multiplexer
Circulator	Isolator
Attenuator	Bias-Tee
Directional Coupler	Mixer
Arbitrary Wave Generator	IQ Mixer

Microwave Resonators

2D Microstrip	2D Coplanar Waveguide
2D Inverted Microstrip	3D Cavity
Resonant Frequency	Quality Factor
Characteristic Impedance	

Transmission Line Theory

Wave Propagation	Characteristic Impedance
Insertion Loss	Return Loss
Input Impedance	Impedance Matching
Transverse Magnetic Mode	Transverse Electric Mode
Transverse Electromagnetic Mode	

Microwave network Analysis

Foster's Theorem	Brune's Theorem
One/Two/Multi-port Network	Scattering Matrix
Transmission Matrix	Impedance/Admittance Matrix

Introduction to Microwave Engineering

What Is Microwave Engineering?

- The study of microwave circuits, components, and systems (in general, electromagnetic wave frequency > 100 MHz)
- High frequency (= short wavelength) offers distinct advantages: (1) small footprint, (2) high-speed data transmission rate
- Breakthroughs in wireless communication were demonstrated by Guglielmo Giovanni Maria Marconi

Why Is Microwave Engineering Important to Study Superconducting Quantum Circuits?

- Control and readout of superconducting qubit, resonator, amplifier are in range of 1 GHz to 20 GHz

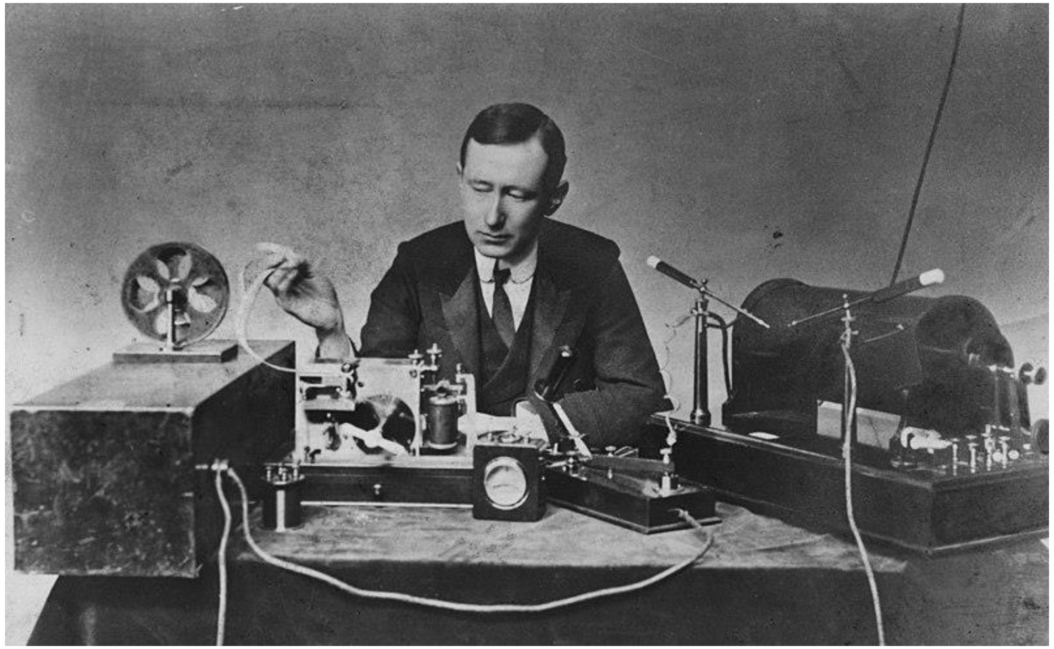


Fig. Guglielmo Giovanni Maria Marconi (1874 – 1937)

Image from: https://en.wikipedia.org/wiki/Guglielmo_Marconi

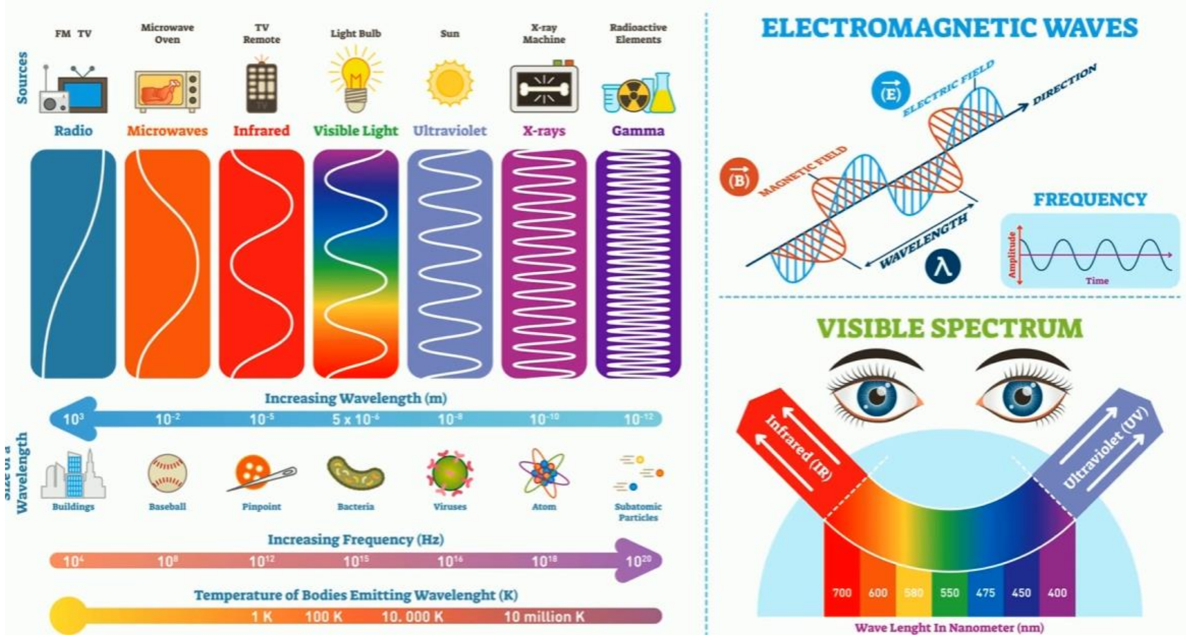


Fig. Electromagnetic Spectrum

Image from: https://www.youtube.com/watch?app=desktop&v=7F6fT5p5oFk&ab_channel=MooMooMathandScience

SH Park
<pajoheji0909@snu.ac.kr>

Microwave Components in Superconducting Qubit Experiments (1/3)

- What's Inside the Dilution Refrigerator?
  - Dilution refrigerator: extreme cryocooling system to cooldown the sample around T = 10 mK (0.01 K = -293.14°C !!!)

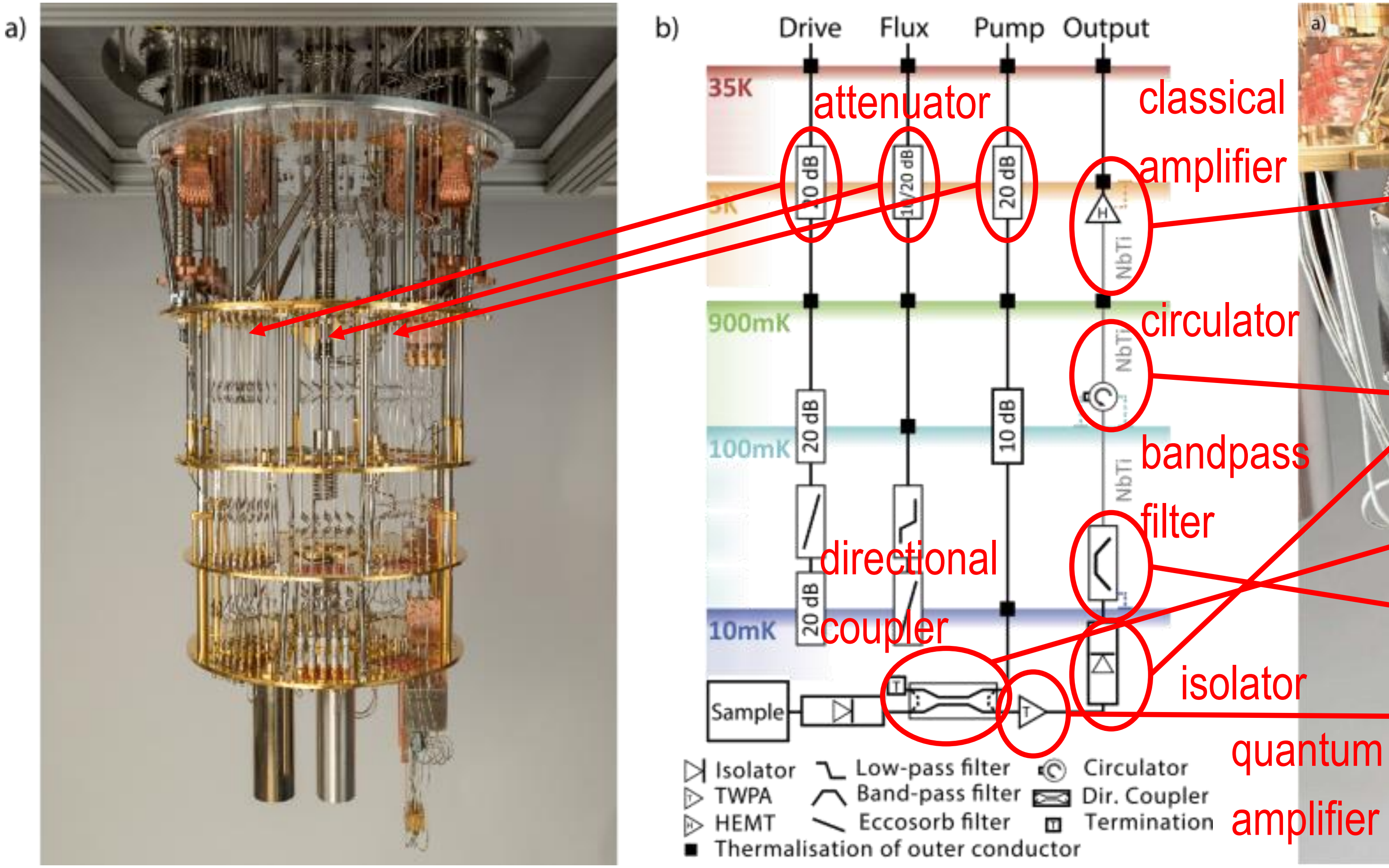


Fig. (a) Cabled dilution refrigerator for qubit experiments. (b) Schematic illustration.

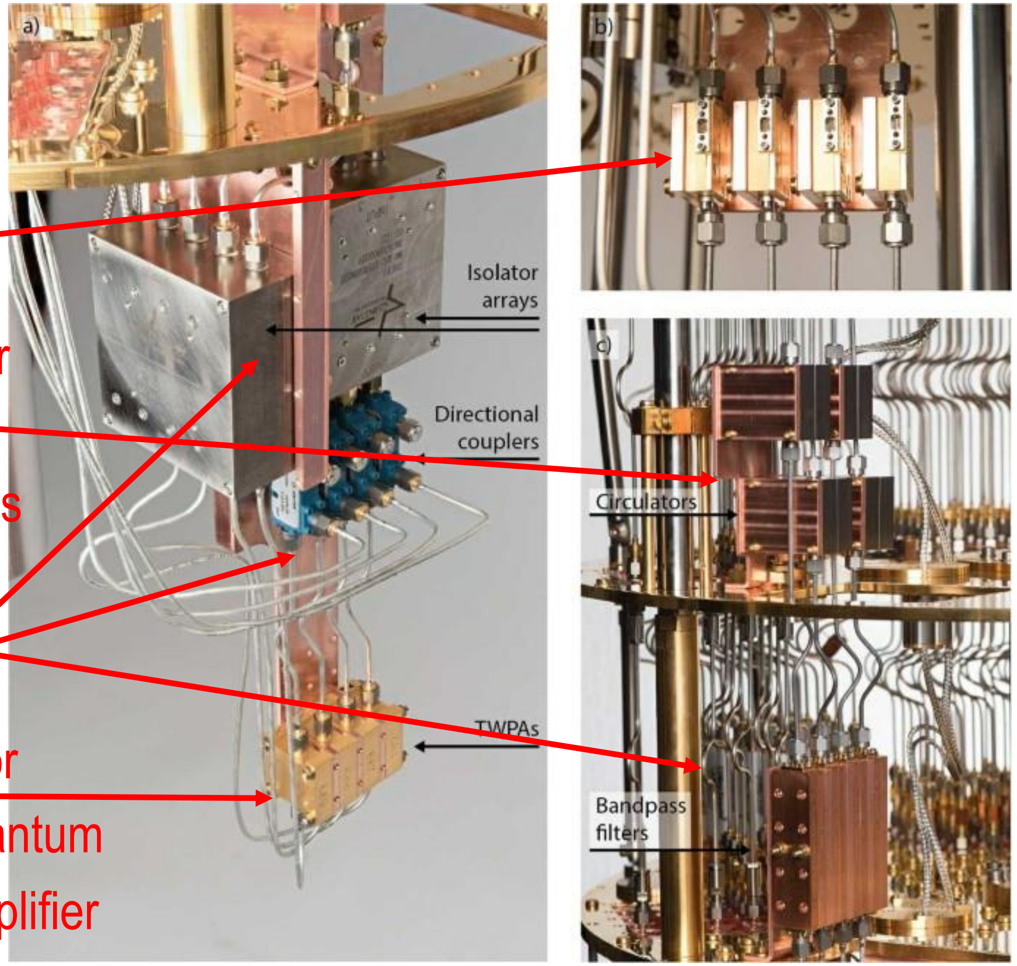


Fig. Components at (a) 10 mK stage, (b) 4 K stage, and (c) 100 mK stage

Image from: S Krinner *et al*., "Engineering cryogenic setups for 100-qubit scale superconducting circuit systems," *EPJ Quantum Technol*., **6**, 2 (2019).

Microwave Components in Superconducting Qubit Experiments (2/3)

- **Attenuator**
  - To suppress heat load (intrusion) from room temperature environments
  - Typical installation: every stage of the dilution refrigerator at input lines
- **Classical Amplifier (High Electron Mobility Transistor, HEMT)**
  - To amplify the output signal with low noise
  - Typical installation: 4 K stage and room temperature environment
- **Circulator / Isolator**
  - Circulator: to route signals between port #1~#3 with minimal loss
  - Isolator: to allow signals in one direction
  - Typical installation: 10 mK stage or 100 mK stage
- **Directional Coupler**
  - To circulate signals in one direction with minimal loss
  - Typical installation: 10 mK stage or 100 mK stage
- **Band-pass / high-pass / low-pass Filter**
  - To selectively pass or reject signals from the unwanted frequency regime
  - Typical installation: every stage is possible

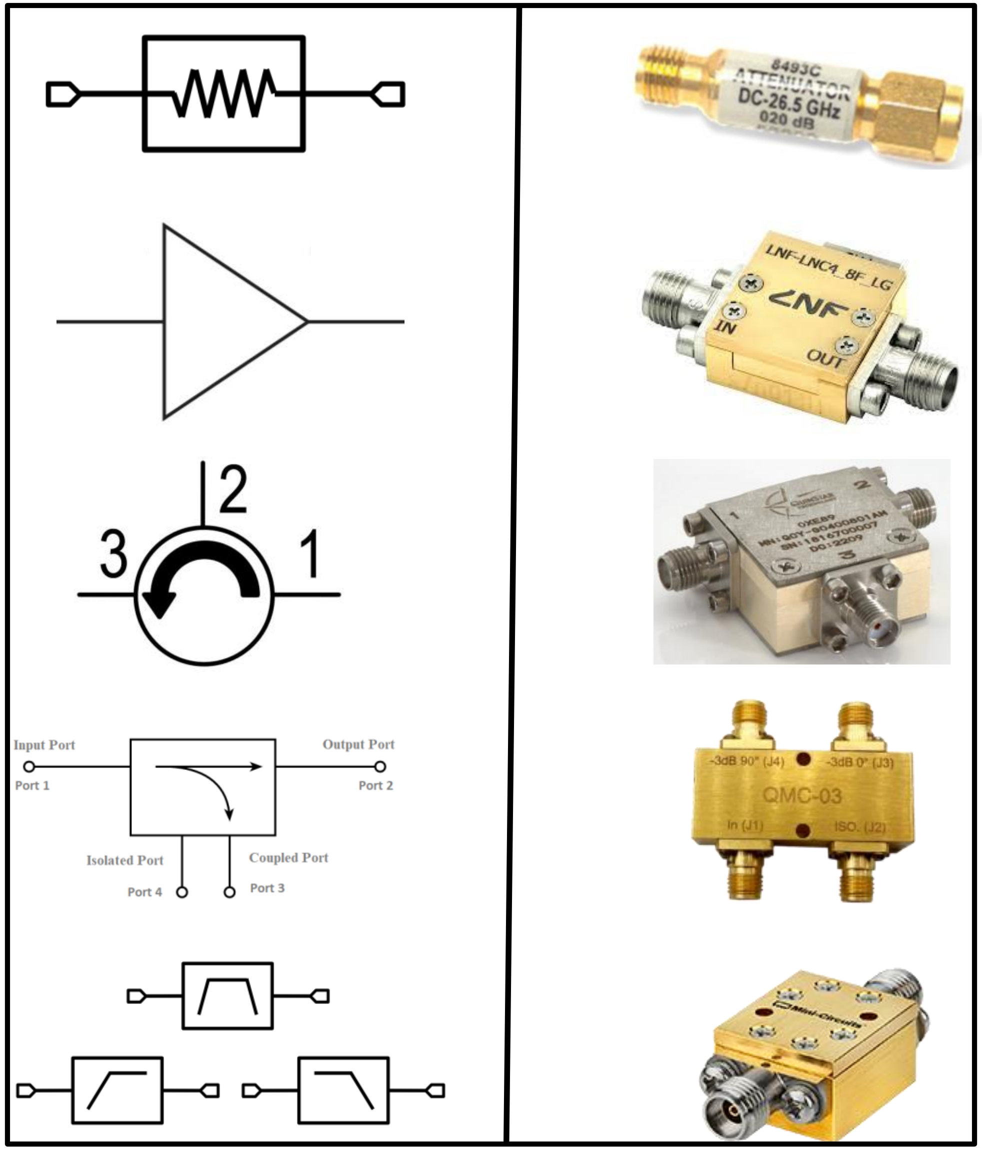


Fig. (left) circuit symbols and (right) commercially available examples

SH Park
<pajoheji0909@snu.ac.kr>

Microwave Components in Superconducting Qubit Experiments (3/3)

- Bias-Tee
  - To merge RF signal and DC signal from two input lines
  - Typical installation: 10 mK stage
- Quantum-Limited Noise Parametric Amplifier
  - To amplify the output signal with quantum-limited level noise!
  - Typical installation: 10 mK stage
- Infrared Filter
  - To protect the sample from high-energy photon (high-frequency radiation)
  - Typical installation: 10 mK stage or 100 mK stage
- Local Oscillator / Arbitrary Wave Generator / Multiplexer / IQ Mixer
  - To control and readout superconducting qubit
  - Typical installation: room temperature environment
  - Recently, there are various commercially available products that combines all the features in one device

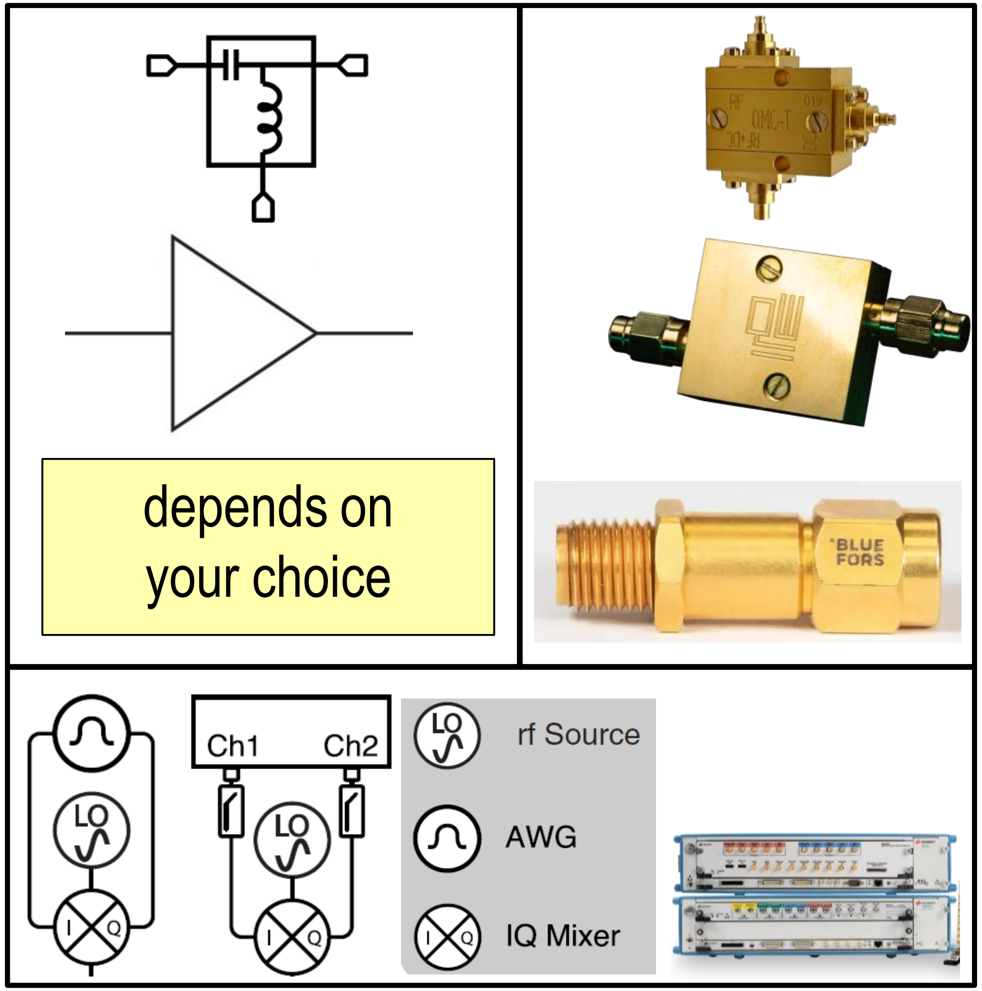


Fig. (left) circuit symbols and (right) commercially available examples

NOTE:
- design, fabrication, and operation of quantum-limited noise amplifier are very active research field!

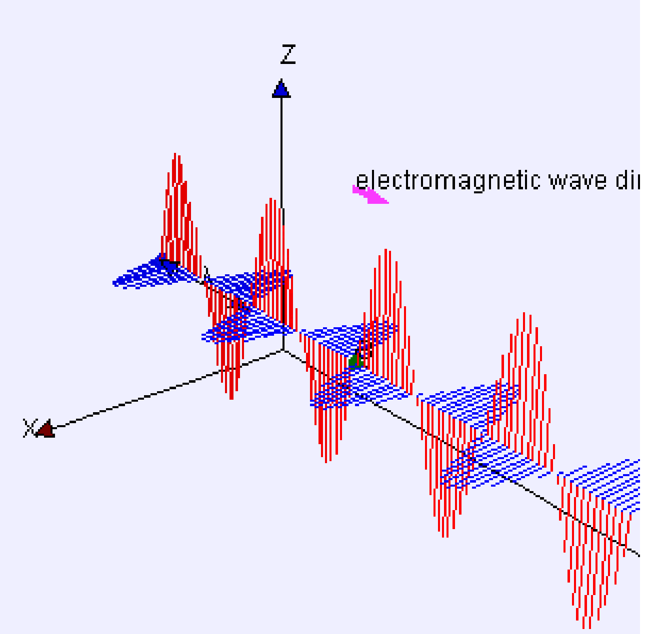
SH Park
<pajoheji0909@snu.ac.kr>

Transmission Line Theory: Electromagnetic Field and Wave Propagation

- Typical Types of Electromagnetic Field Profiles in Transmission Line
  - Transverse Electric mode (TE): no electric field along the propagating direction, $E_z = 0\ (H_z \neq 0)$
  - Transverse Magnetic mode (TM): no magnetic field along the propagating direction, $H_z = 0\ (E_z \neq 0)$
  - Transverse Electromagnetic mode (TEM): no electric and magnetic fields along the propagating direction, $E_z = H_z = 0$
  - Quasi Transverse Electromagnetic mode (quasi-TEM): small electric and magnetic fields along the propagating direction
- Characteristic Parameters of Wave Propagation
  - Frequency f: frequency of a wave
  - Angular frequency $\omega = 2\pi \times f$
  - Wavelength $\lambda = \frac{v_p}{f}$: length of a wave
  - Wave number $\beta = \frac{2\pi}{\lambda}$: spatial frequency of a wave
  - Phase velocity $v_p = \frac{\omega}{\beta}$: velocity with which the wave propagates
  - Group velocity $v_g = \frac{\partial\omega}{\partial\beta}$: velocity with which the overall envelope propagates

In vacuum, $v_p = c$
where c: speed of light

See D Cheng, *Field and Wave Electromagnetics*, Ch. 7



$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

μ_0: vacuum permeability
ε_0: vacuum permittivity

Fig. Electromagnetic wave propagating in the y-axis. Maxwell's equation.

Image from: https://en.wikipedia.org/wiki/Electromagnetic_radiation

Transmission Line Theory: Circuit Representation

- Definition of Transmission Line
  - A structure designed to conduct high-frequency electromagnetic waves from one point to another with minimal loss
  - Types of transmission lines: coaxial cable, coplanar waveguide, microstrip,…
- Governing Equations of Transmission Line
  - To describe the voltage $V(x,t)$ and current $I(x,t)$ with distance x and time t, the distributed element model of a transmission line was developed (right Fig.(a))
  - Distributed resistance R, inductance L, capacitance C, and conductance G are expressed in per unit length
  - The paired equations of $V(x,t)$ and $I(x,t)$ are:

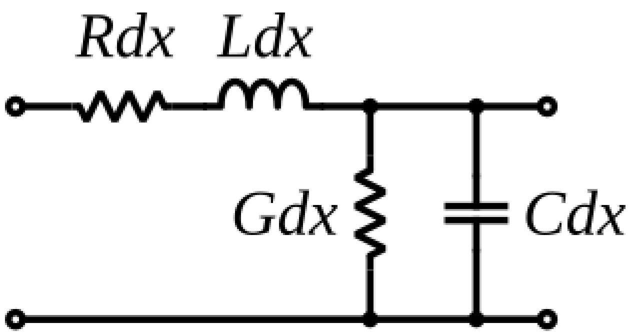
$$\frac{\partial V(x,t)}{\partial x} = -(R + j\omega L) I(x,t) \qquad \frac{\partial I(x,t)}{\partial x} = -(G + j\omega C) V(x,t)$$

  - Simplified equations are:

$$\frac{\partial^2 V(x,t)}{\partial x^2} = \gamma^2 V(x,t) \qquad \frac{\partial^2 I(x,t)}{\partial x^2} = \gamma^2 I(x,t)$$

where $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$ is propagation constant

(a)



(b)

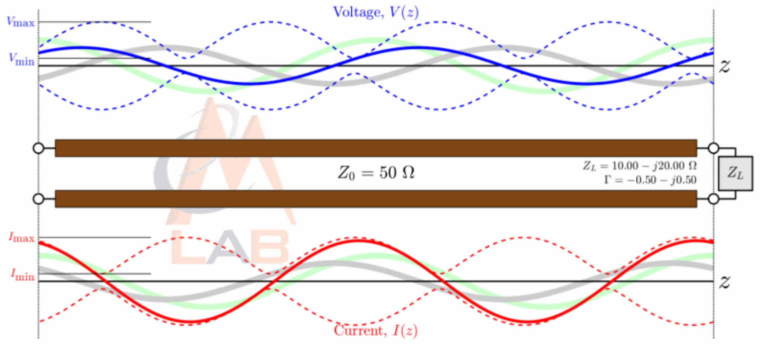


Fig. (a) Distributed element model and (b) animation of electromagnetic wave properties in a transmission line.

NOTE: superconducting transmission lines can be treated as lossless line, where R and G are neglected

Image from: https://empossible.net/academics/emp4301_5302/ and https://en.wikipedia.org/wiki/Transmission_line

Transmission Line Theory: Characteristic Parameters

Characteristic Parameters of Transmission Line

- Characteristic impedance Z_0: the ratio of voltage to current for a wave in a transmission line

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \xrightarrow{\text{Lossless line}} Z_0 \approx \sqrt{\frac{L}{C}}$$

NOTE: most of transmission lines feature $Z_0 = 50\ \Omega$
See the history of Z_0 being 50 Ω at:
https://resources.altium.com/p/mysterious-50-ohm-impedance-where-it-came-and-why-we-use-it

- Input impedance Z_{in}: the impedance measured at a distance l from the load Z_L

$$Z_{\text{in}} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

- Reflection coefficient Γ: the ratio of the reflected wave to the incident wave

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Impedance Matching

- To maximize signal transfer (minimize signal reflection), the input impedance needs to be matched to 50 Ω

NOTE: there are so many techniques to match the impedance (Smith chart, stub load,…)
See D Pozar, *Microwave and RF design of wireless systems* textbook for further details

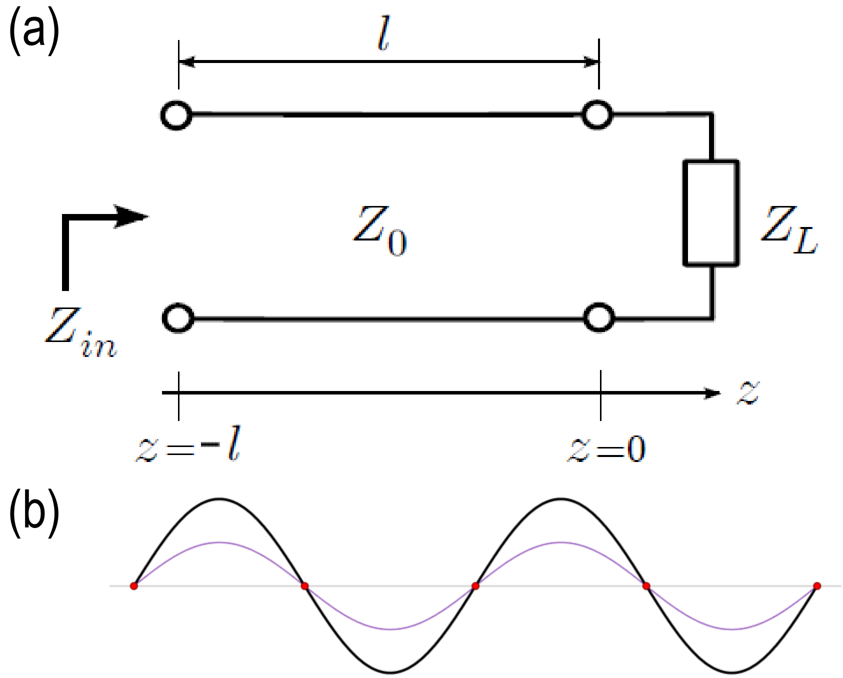


Fig. (a) Schematic of a transmission line terminated by a load. (b) Incident (blue), reflected (red), and net (black) waves are shown.

Image from: https://en.wikipedia.org/wiki/Transmission_line

Introduction to Microwave Resonators

- **Definition of Microwave Resonator**
  - A device or system that exhibits electromagnetic resonance or resonant behavior
- **Characteristics of Microwave Resonator**
  - Resonant frequency (ω_r): The specific frequency where the resonator naturally oscillates with maximum amplitude
  - Quality-factor (Q_r): A dimensionless parameter representing how underdamped a resonator is, indicating energy loss rate
    - Low Q resonator = high loss, fast-decaying, and broad bandwidth resonator
    - High Q resonator = low loss, long-live, and narrow bandwidth resonator
  - Decay rate $(\kappa_r = \omega_r/Q_r)$: Rate of energy loss in the resonator
  - Characteristic impedance $(Z_r = \sqrt{L/C})$: the ratio of voltage to current

NOTE:
high Q resonator for quantum memory
low Q resonator for rapid measurement

- **Microwave Resonator in Superconducting Quantum Circuits**
  - Microwave resonator can be treated as
    quantum harmonic oscillator with the equidistant energy difference of $\hbar\omega_r$
  - Microwave resonator can be realized in 2D or 3D
    2D: microstrip, coplanar waveguide, …
    3D: rectangular cavity, cylindrical cavity, …

$\hbar$: reduced Planck's constant
ω_r: angular resonant frequency

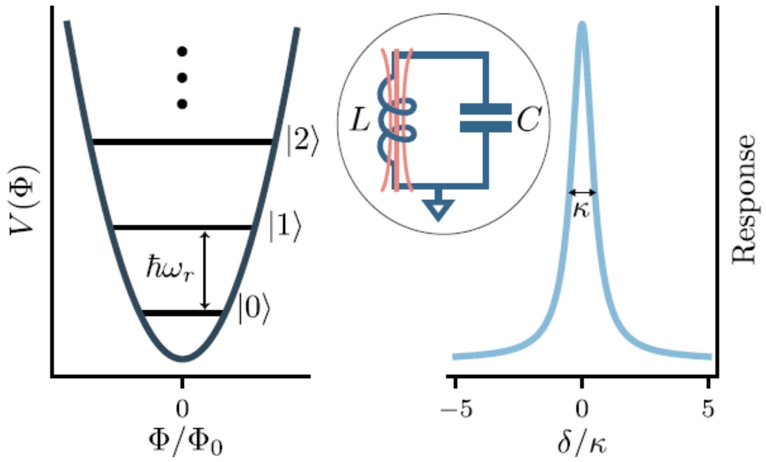
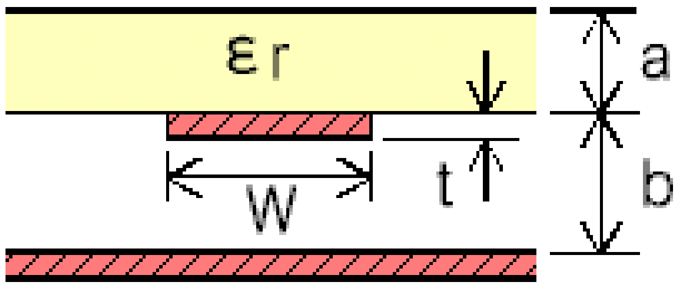


Fig. Harmonic potential of electrical LC resonator.

Image from: A Blais *et al.*, "Circuit Quantum Electrodynamics," *Rev. Mod. Phys.*, **93** 025005 (2021).

Microwave Resonators: (1) 2D Microstrip and Inverted Microstrip

- **Geometric Features of Microstrip**
  - Conductor is separated from ground layers by substrate
  - Microstrip supports quasi-TEM mode
  - Substrate thickness h, permittivity ε_r, and conductor width w are important to design the microstrip's Z_r
- **Geometric Features of Inverted Microstrip**
  - The microstrip is suspended and separated from ground layers by air (or another dielectric substrate)



NOTE:
microstrip is not preferred for qubit circuit design…
But, quantum-limited noise amplifier often employs it

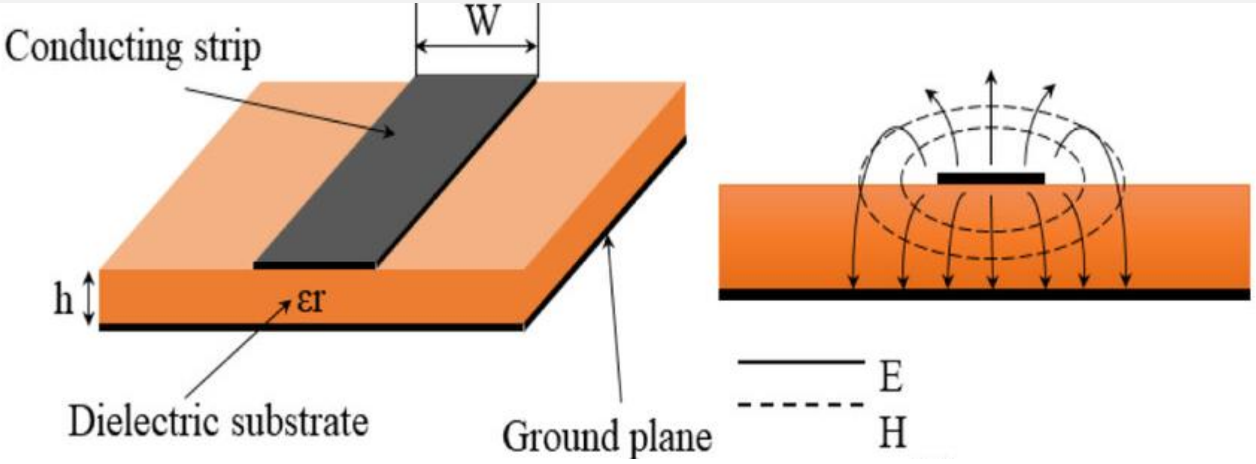


Fig. Microstrip structure. (a) Geometry and (b) electric and magnetic field distribution

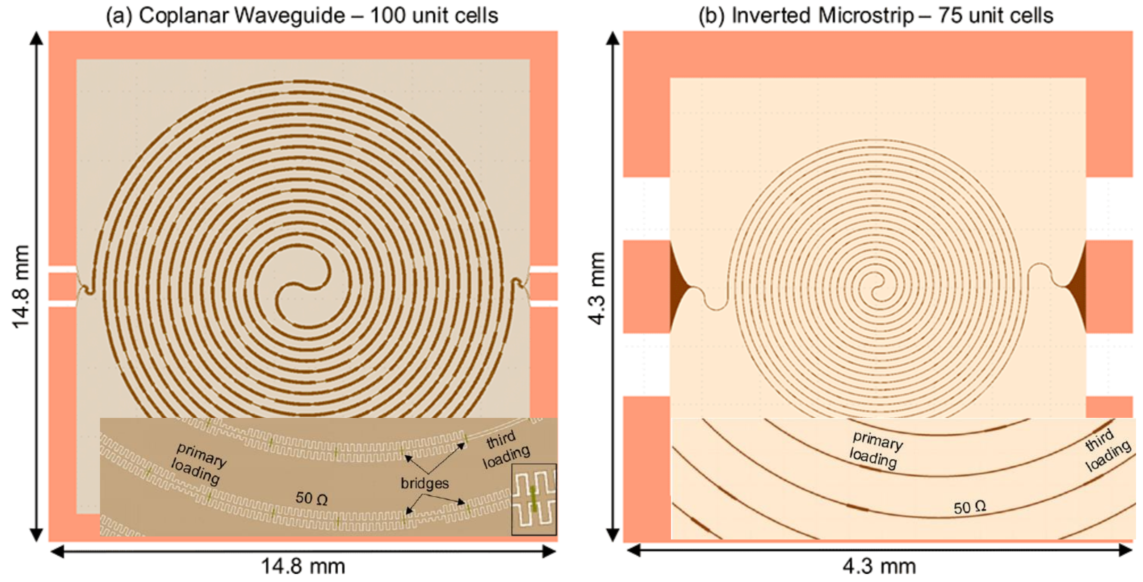


Fig. Layouts of (a) CPW based and (b) microstrip based quantum-limited noise amplifier

Image from: AA Shahri *et al*., "A high sensitivity microwave glucose sensor," *Meas. Sci. Technol.*, **32,** 075104 (2021).
Image from Bk Tan et al., "Engineering the thin film characteristics for optimal performance of superconducting kinetic inductance amplifiers using a rigorous modelling technique," *Open Research Europe*, 2022

Microwave Resonators: (2) 2D Coplanar Waveguide (CPW)

- **Geometric Features of CPW**
  - Center conductor separated by gaps from two ground planes
  - CPW supports quasi-TEM mode (similar to microstrip)
  - Substrate permittivity, gap, and conductor width are important to design the CPW's Z_r

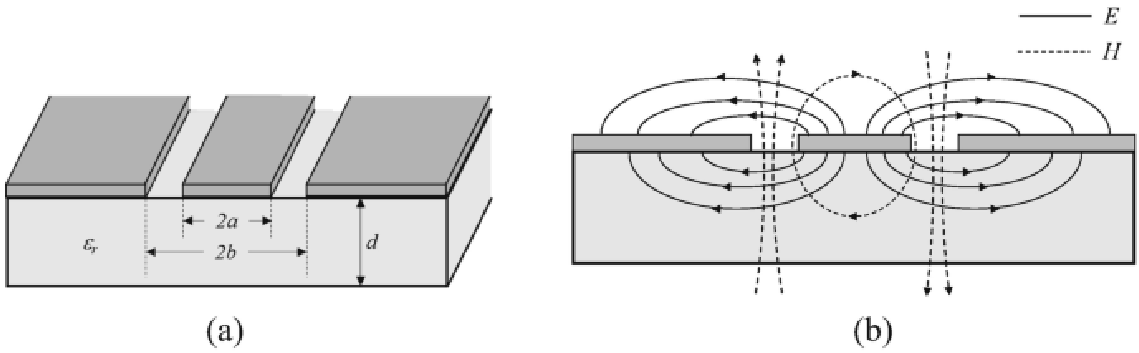


Fig. CPW structure. (a) Geometry and (b) electric and magnetic field distribution

- **Characteristics of CPW**
  - Single planar surface: all conductors are on the same plane
  - Ease of fabrication: compatible with standard photolithography
  - Integration: easily integrated with other components and circuits
  - Half-wavelength resonance: Open to the both ends
  - Quarter-wavelength resonance: Open and short to the ends

NOTE:
CPW is popular design choice for superconducting quantum circuits but CPW usually exhibits lower Q_r than cavity resonator
(*The mechanism of coherent loss will be discussed later!)

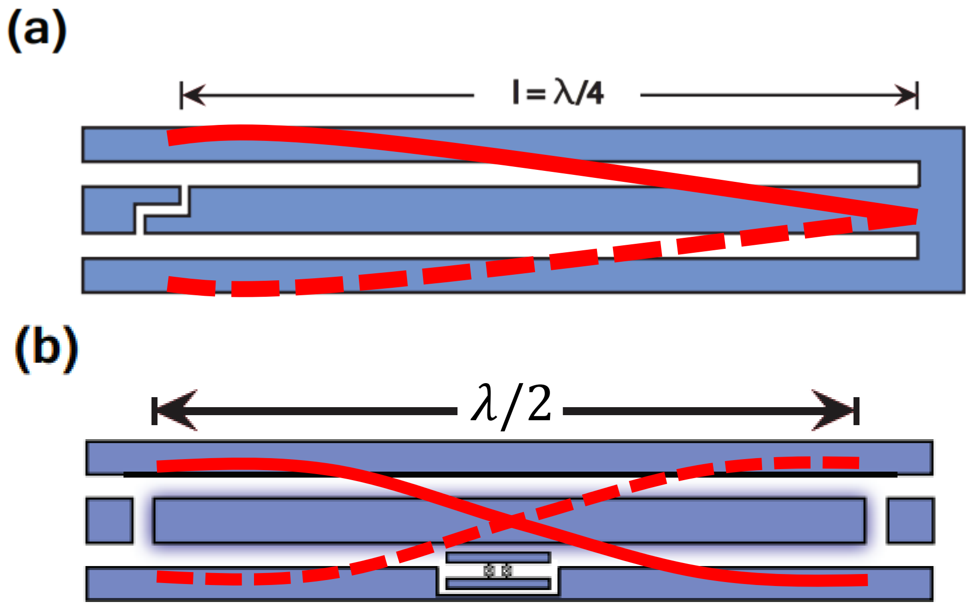


Fig. Schematic layouts of (a) quarter- and (b) half-wavelength CPW resonator. The amplitude of the electric field is shown by red lines.

Image from JS Rieh, Introduction to Terahertz Electronics, Springer, 2021.
Image from P Krantz, Parametrically pumped superconducting circuits, MS dissertation, Chalmers University of Technology, 2013

Microwave Resonators: (2) 2D Coplanar Waveguide (CPW)

- Effective Permittivity of CPW
  - For substrate permittivity ε_r, thickness h, gap s, and center conductor width w
  - The effective permittivity $\varepsilon_{\mathrm{eff}}$ is:

$$\varepsilon_{\mathrm{eff}} \approx \frac{\varepsilon_r + 1}{2}$$

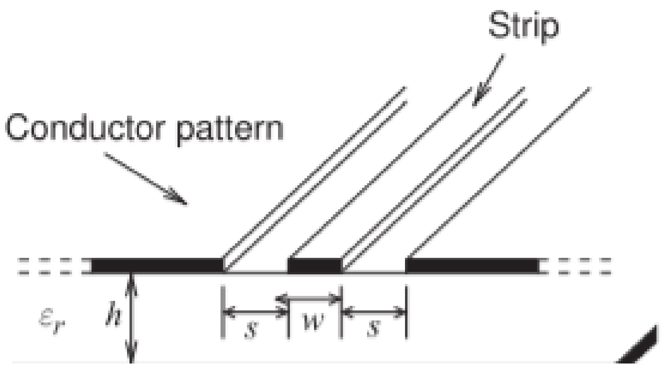


Fig. Geometric features of CPW structure.

- Characteristic Impedance of CPW
  - For substrate permittivity ε_r, thickness h, gap s, and center conductor width w
  - The characteristic impedance Z_r is:

$$Z_r \approx \frac{30\pi}{\sqrt{\varepsilon_{\mathrm{eff}}}} \frac{K(k')}{K(k)}$$

$K(k)$: elliptic integral, $k = \frac{w}{w+2s}, k' = \sqrt{1-k^2}$

For simplicity, $\frac{K(k')}{K(k)} \approx \frac{1}{\pi} \ln\left(2\frac{1+\sqrt{k'}}{1-\sqrt{k'}}\right)$ can be assumed for $0.707 \leq k \leq 1$

ONLINE CALCULATOR:
https://www.microwaves101.com/calculators/864-coplanar-waveguide-calculator

NOTE:
As a guideline, for silicon substrate, gap $s = 6\ \mu$m and width $w = 10\ \mu$m are good choices to match $Z_r = 50\ \Omega$
More rigorous calculations to incorporate the conductor thickness effect can be found in J Gao, PhD dissertation, CALTECH, 2008

Image from https://eng.libretexts.org/Bookshelves/Electrical_Engineering/Electronics/Microwave_and_RF_Design_II_-_Transmission_Lines_(Steer)/03%3A_Planar_Transmission_Lines/3.08%3A_Co-Planar_Waveguide

Microwave Resonators: (3) 3D Cavity

- **Geometric Features of Cavity Resonator**
  - A closed metallic enclosure that confines electromagnetic waves
  - The cavity acts as a resonant circuit, storing electromagnetic energy
  - Popular choices: cylindrical, rectangular, elliptical,…
- **Characteristics of Cavity Resonator**
  - TE or TM modes are determined by the geometric features
  - For cylindrical cavity,
    - TE011 mode is preferred for quantum memory-like applications
    - TM010 mode is preferred for particle accelerator applications
  - Cavity resonators usually exhibit high Q due to bulk and clean superconductor properties

NOTE:
Cavity resonator with high Q is also important to build particle accelerator (recall that high Q means low energy loss)

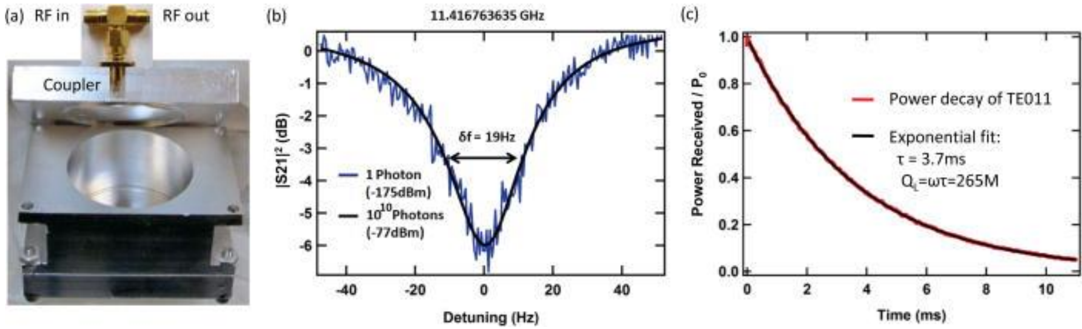


Fig. (a) Superconducting aluminum cylindrical cavity TE011 mode. (b) Transmission at its resonance. (c) Power decay measurement with Q > 2×10^8

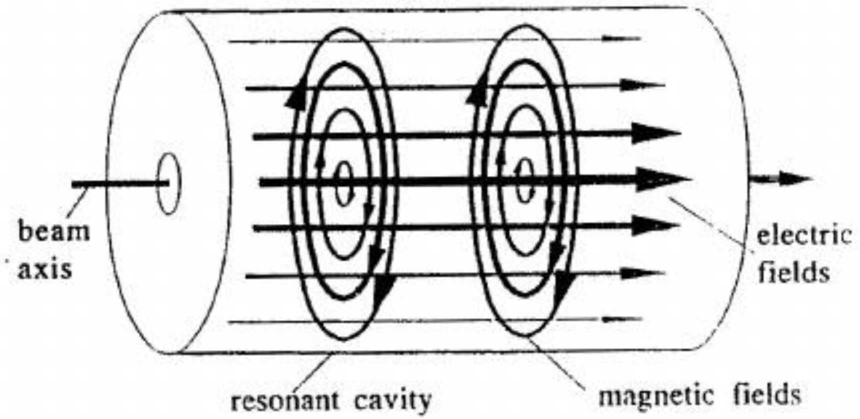
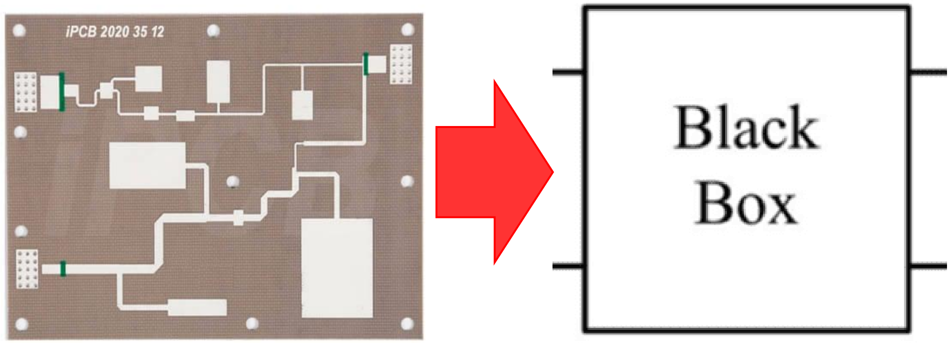


Fig. Superconducting cylindrical cavity TM 010 mode. The charged beam can be accelerated along the beam line, along the resonant electric fields.

Image from M Reagor et al., "Reaching 10 ms single photon lifetimes for superconducting aluminum cavities," *Appl. Phys. Lett.*, **102**, 192604 (2013).

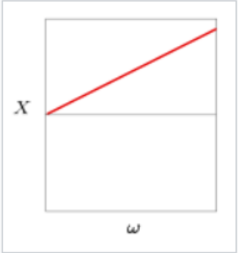
Microwave Network Analysis: Black-Box Circuit and Foster's Theorem

- **Definition of Microwave Network Analysis**
  - A complex, multi-mode microwave circuit is difficult to extract its electrical circuit from the analytic modeling
  - Instead, by modeling the circuit as a 'black-box' and analyzing its impedance response, an accurate circuit representation can be obtained
  - To synthesize such black-box, Foster's theorem and Brune's theorem are the popular methods

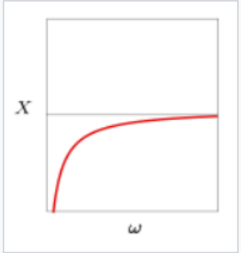


NOTE: Brune's theorem is known to be more accurate but requires rigorous calculation. See: https://doi.org/10.1016/j.aeue.2016.09.007

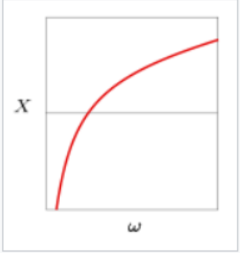
- **Definition of Foster's Theorem**
  - Passive elements (L, C) feature monotonically increasing reactance $Z_L = j\omega L$ and $Z_c = -\frac{1}{j\omega C}$ with increasing frequency
  - Reactance of the black-box features zeros and poles as frequency ➔ series connected LC parallel circuits



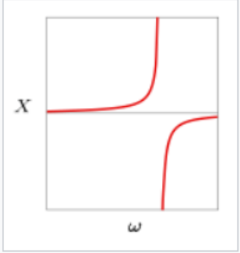
Plot of the reactance of an inductor against frequency



Plot of the reactance of a capacitor against frequency



Plot of the reactance of a series LC circuit against frequency



Plot of the reactance of a parallel LC circuit against frequency

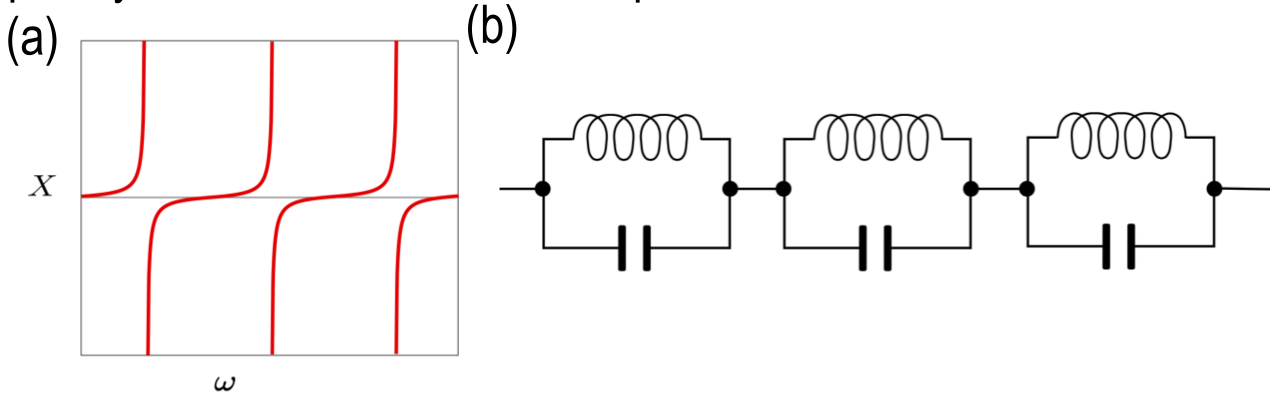


Fig. (a) Plot of the reactance of Foster's theorem and (b) equivalent circuit representation.

Image from https://www.ipcb.com/microwave-circuit.html and https://en.wikipedia.org/wiki/Foster%27s_reactance_theorem

Microwave Network Analysis: Scattering (S) Matrix

Definition of S-Matrix

- S-parameter (S_{mn}) refers to the transmitted wave at port m induced by the incident wave at port n
- For N-port network,

$$S_{mn} = \frac{V_m^-}{V_n^+}$$

- For 2-port network,

$$\begin{pmatrix} V_1^- \\ V_2^- \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} V_1^+ \\ V_2^+ \end{pmatrix}$$

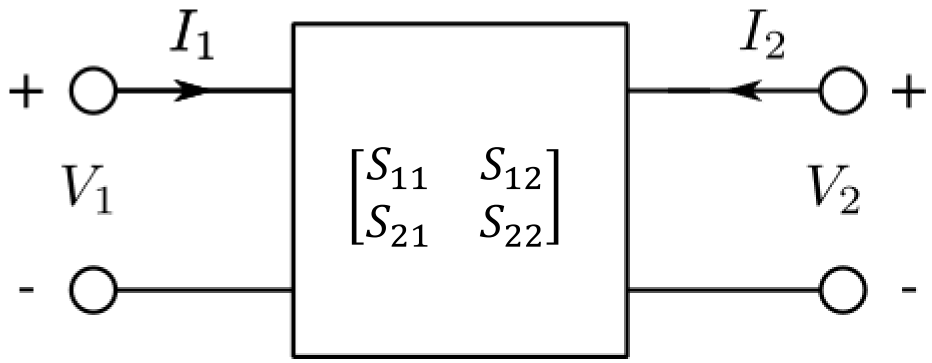


Fig. Voltage and currents in a 2-port microwave network.

S-Matrix in Quantum Computing

- S-matrix can be used to analyze the system's reflection and transmission characteristics

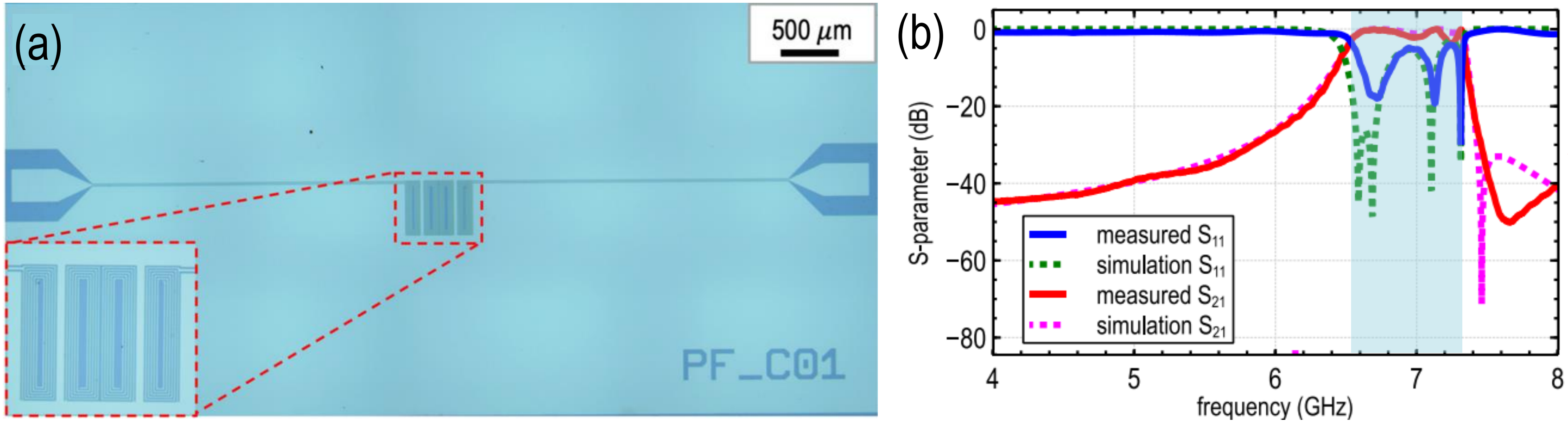


Fig. (a) Optical image of superconducting bandpass filter. (b) Measured and simulated S-matrix of the filter.

low S11 value in passband = no reflection
large S11 value out of the band = total reflection

large S21 value in passband = total transmission
low S21 value out of the band = no transmission

good bandpass filter features are measured

NOTE:
dB unit can be converted as $20\log_{10}(|S|)$

Image from https://resources.altium.com/p/advantages-abcd-parameters-analyzing-your-pcb
Image from SH Park *et al.*, "Characterization of broadband Purcell filters with compact footprint for fast multiplexed superconducting qubit readout," *Appl. Phys. Lett.*, **124**, 044003 (2024)

Microwave Network Analysis: ABCD / Z / Y - Matrix

Definition of Transmission (ABCD) Matrix

- Relations between the voltage and current at the input of an N-port network to the voltage and current measured at the output of the network
- ABCD matrix is useful to find the features of the cascaded network
- For 2-port network,

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$

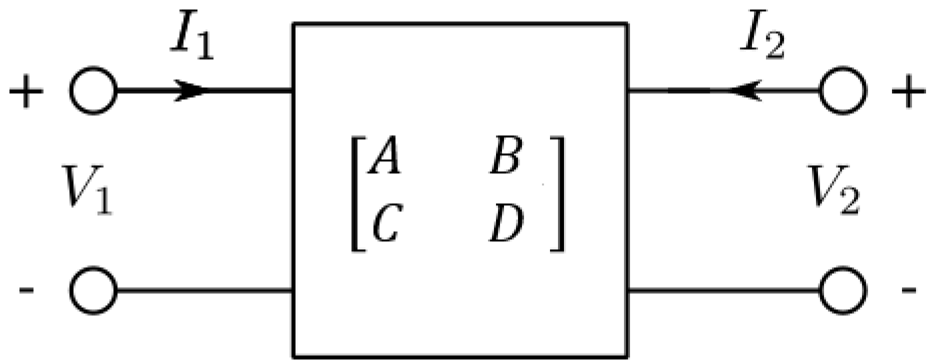


Fig. Voltage and currents in a 2-port microwave network.

Characteristics of Impedance (Z) Matrix

- Relations between the voltages and the currents of ports
- Z matrix can be determined as $Z = Y^{-1}$
- For 2-port network,

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

- For N-port network,

$$Z_{mn} = \left.\frac{V_n}{I_m}\right|_{I_k=0 \text{ for } k\neq m}$$

Characteristics of Admittance (Y) Matrix

- Relations between the currents and the voltages of ports
- Y matrix can be determined as $Y = Z^{-1}$
- For 2-port network,

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

- For N-port network,

$$Y_{mn} = \left.\frac{I_n}{V_m}\right|_{V_k=0 \text{ for } k\neq m}$$

NOTE:
Z matrix and Y matrix are usually employed to synthesize the black-box circuit using Foster's or Brune's theorem

Image from https://resources.altium.com/p/advantages-abcd-parameters-analyzing-your-pcb

Microwave Network Analysis: Conversion Between Matrices

Conversion Table Between Various Matrices for Two-Port Network

	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\|Y\|}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\|Y\|}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\|Y\|}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\|Y\|}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\|Z\|}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\|Z\|}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\|Z\|}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\|Z\|}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\|Z\|}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\|Y\|}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

$|Z| = Z_{11}Z_{22} - Z_{12}Z_{21}$; $|Y| = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$.

Table from D. Pozar, *Microwave Engineering*, John Wiley & Sons, 4th Edition, 2011.

See Also…

Textbooks:

[1] D Pozar, *Microwave and RF design of wireless systems*, John Wiley & Sons, 2000. * recommended

[2] CA Balanis, *Advanced Engineering Electromagnetics*, John Wiley & Sons, 2012.

Review Papers:

[1] JC Bardin, D Sank, O Naaman, E Jeffrey, "Quantum computing: An introduction for microwave engineers," *IEEE Microw. Mag.*, **21**, 24-44 (2020).

[2] JC Bardin, DH Slichter, DJ Reilly, "Microwaves in Quantum Computing," *IEEE J. Microw.*, **1**, 403-427 (2021).

[3] P Krantz et al., "A quantum engineer's guide to superconducting qubits," *Appl. Phys. Rev.*, **6**, 021318 (2019).

Open Courses:

[1] D Staelin, *Electromagnetics and Applications*, MIT OCW, 2005. [Online Available]

https://ocw.mit.edu/courses/6-013-electromagnetics-and-applications-fall-2005/