APPLIED SUPERCONDUCTIVITY LABORATORY

Introduction to Superconducting Quantum Circuits

- Review of Mathematical Methods for Quantum Computing -

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Keywords in Mathematical Methods for Quantum Computing

Mathematical Methods: Number Systems

Complex Number Polar Coordinates Euler Expressions

Mathematical Methods: Series Expansion

Taylor Expansion Jacobi-Anger Expansion Fourier Series Fourier Transform

Mathematical Methods: Linear Algebra

Hilbert Space Bra-Ket Notation Dirac Notation

Adjoint Operation Linear Operator Unitary Operator Hermitian Operator

Inner Product Tensor Product Taylor Expansion of Operators

Basis State Kronecker Delta Levi-Civita Eigenvalue Eigenstate

Mathematical Methods: Differential Equation

Ordinary Differential Equation Partial Differential Equation Green's Function

Mathematical Methods: Vector Calculus

Divergence Theorem Stokes' Theorem

Mathematical Methods: Dimensional Analysis

Physical Unit Conversion

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Introduction to Number Systems

- Major Number Systems
  - Natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$
  - Integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
  - Rational numbers: $\mathbb{Q} = \{\frac{a}{b}\}$ where a and b are integers
  - Irrational numbers: $\mathbb{I}$ where elements can NOT be expressed with $\frac{a}{b}$
  - Real numbers: $\mathbb{R} = \mathbb{Q} \cup \mathbb{I} = \{x : x \in \mathbb{Q} \text{ or } x \in \mathbb{I}\}$
  - Complex numbers: $\mathbb{C} = \{a + bj\}$ where a and b are real

NOTE: all quantum states are expressed with complex numbers

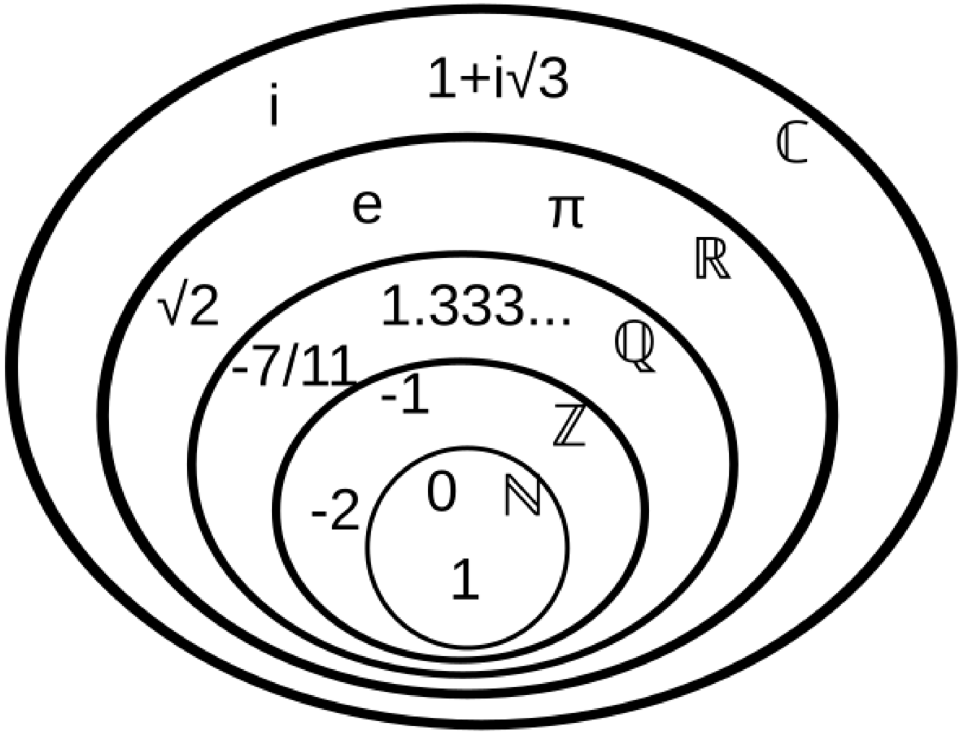

Figure. Number system set diagram

- Number System Set Inclusions
  - $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

- Imaginary Unit (j)
  - The imaginary unit is a solution of $x^2 = -1$
  - $1i = \sqrt{-1}$ (which is same as $1j = \sqrt{-1}$)

NOTE
most of electrical engineers prefer $1j$ notation rather than $1i$, since i is used to denote the electrical current

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Brief Review of Number Systems: Complex Number Expressions

- Complex Numbers: Real Part + Imaginary Part

$$z = a + bj \iff z^* = a - bj \text{ (conjugate of } z\text{)}$$

- Polar Coordinates for Complex Numbers

$$z = (r, \theta)$$

where
$r = \sqrt{a^2 + b^2}$ and $\theta = \arctan(\frac{b}{a})$

- Euler Expressions for Complex Numbers

$$z = re^{j\theta}$$

where Euler's formula is
$$e^{j\theta} = \cos\theta + j\sin\theta$$
Euler's identity is
$$e^{j\pi} + 1 = 0$$

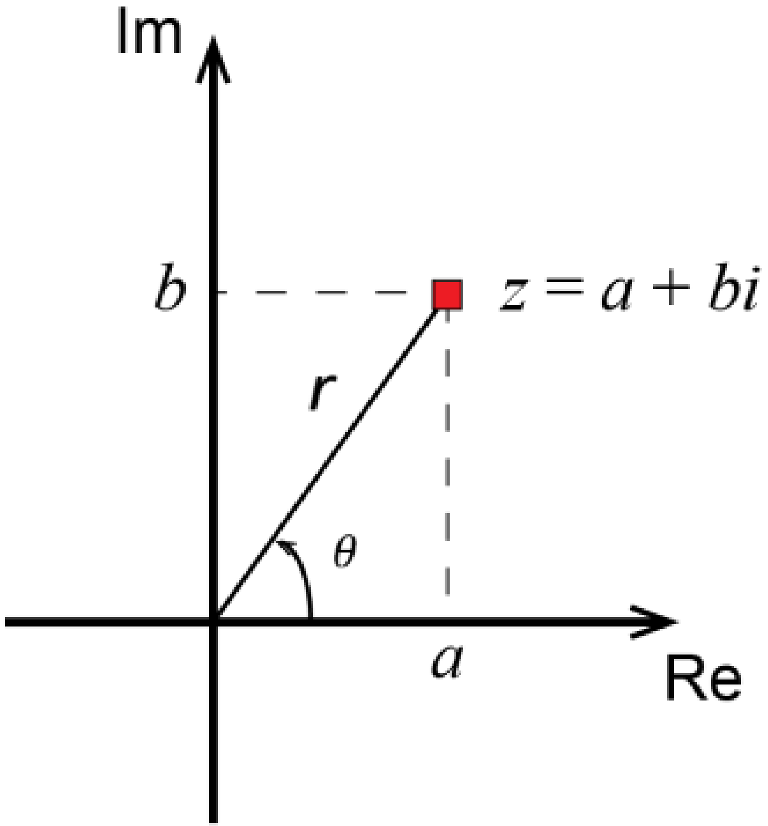

Figure. Polar form of a complex number $z = a + bj$

- Example: $z = 2 + 2\sqrt{3}j$

$$z = 2 + 2\sqrt{3}j \rightarrow r = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = 4 \text{ and } \theta = \arctan\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3} \rightarrow z = 4e^{j\frac{\pi}{3}}$$

Brief Review of Series Expansion: (1) Taylor Expansion

- Definition of Taylor Expansion
  - Approximation of a function $f(x)$ as a series of infinite sum of polynomial terms
- Formula of Taylor Expansion
  - For a function $f(x)$ that is infinitely differentiable at $x = a$, the Taylor series expansion around a is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

in simplified notation:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dx^n} f(x) \bigg|_{x=a} (x-a)^n$$

where $n!$ is factorial
$n! = 1 \times 2 \times \cdots \times n-1 \times n$

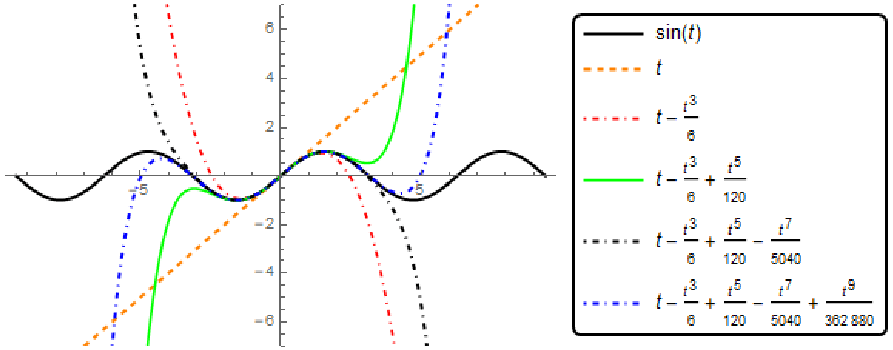

Figure. Truncated Taylor series of sine function up to 9th order

- Truncation of Taylor Expansion
  - In reality, infinite sum of Taylor series can not be computed…
  - For simplicity, only few low-order terms are only considered in practice
- Example $f(\theta) = \cos\theta$ around $\theta = 0$

$$f(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{d\theta^n} \cos\theta \bigg|_{\theta=0} \theta^n = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

Brief Review of Series Expansion: (2) Jacobi-Anger Expansion

- Definition of Jacobi-Anger Expansion
  - Approximation of an exponential function with trigonometric terms as a series of infinite sum of Bessel functions
- Formula of Jacobi-Anger Expansion
  - For a function $e^{jx\cos\theta}$,

$$e^{jx\cos\theta} = \sum_{n=-\infty}^{\infty} j^n J_n(x) e^{jn\theta}$$

  - For a function $e^{jx\sin\theta}$,

$$e^{jx\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{jn\theta}$$

where $J_n(x)$ is Bessel function of the first kind and $\Gamma(n)$ is Gamma function

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} \qquad \Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

- Some Useful Expressions of Jacobi-Anger Expansion

NOTE: these expressions will be utilized, when we analyze Josephson junction after

$$\cos(z\cos\theta) \equiv J_0(z) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(z)\cos(2n\theta), \quad \cos(z\sin\theta) \equiv J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos(2n\theta),$$

$$\sin(z\cos\theta) \equiv -2\sum_{n=1}^{\infty}(-1)^n J_{2n-1}(z)\cos[(2n-1)\theta], \quad \sin(z\sin\theta) \equiv 2\sum_{n=1}^{\infty} J_{2n-1}(z)\sin[(2n-1)\theta].$$

Brief Review of Fourier Analysis: (1) Fourier Series

- **Definition of Fourier Series**
  - Approximation of a periodic function as a sum of trigonometric terms
- **Formula of Fourier Series**
  - For a function $f(t)$ with period $2l$,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{jn\pi t}{l}}$$

where c_n coefficient is

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(t) e^{-\frac{jn\pi t}{l}} dx$$

- **Properties of Fourier Series**
  - For functions $f_1(t) = \sum a_n e^{\frac{jn\pi t}{l}}$ and $f_2(t) = \sum b_n e^{\frac{jn\pi t}{l}}$,
  - Linearity: $af_1(t) + bf_2(t) = a\left(\sum a_n e^{\frac{jn\pi t}{l}}\right) + b\left(\sum b_n e^{\frac{jn\pi t}{l}}\right)$
  - Shift in time: $f_1(t - t_0) = \left(\sum a_n e^{\frac{jn\pi (t-t_0)}{l}}\right)$
- **Example** $f(t)\begin{cases} 0, & 0 < x < 0.5 \\ 1, & 0.5 < x < 1 \end{cases}$ and its period 1

$$f(t) = \frac{1}{2} + \frac{1}{j\pi}\left(e^{2j\pi t} - e^{-2j\pi t} + \frac{1}{3}\left(e^{6j\pi t} - e^{-6j\pi t}\right) + \frac{1}{5}\left(e^{10j\pi t} - e^{-10j\pi t}\right) + \cdots\right)$$

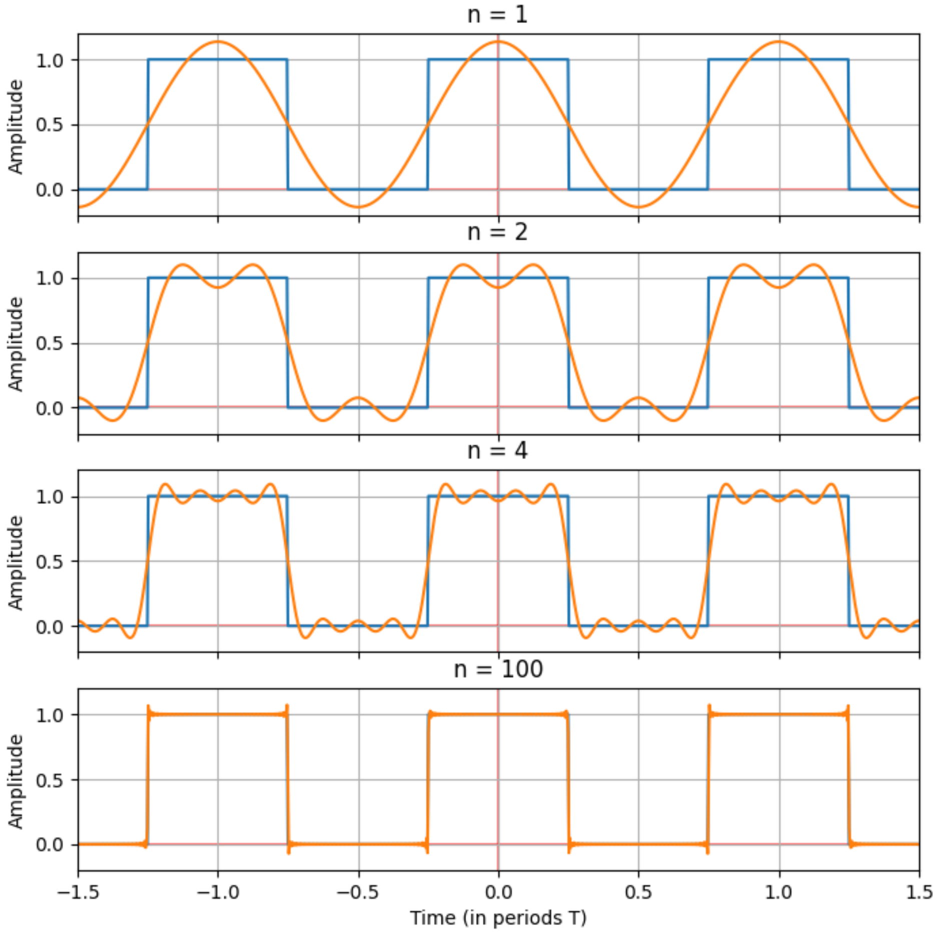

Figure. Fourier series approximation of squared pulse up to $N = 100$

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Brief Review of Fourier Analysis: (2) Fourier Transform

Definition of Fourier Transform

- Conversion of an arbitrary time-domain signal $f(t)$ into its frequency-domain $F(\omega)$

Formula of Fourier Transform

- For a function $f(t)$, the transformed function $F(\omega)$ is

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

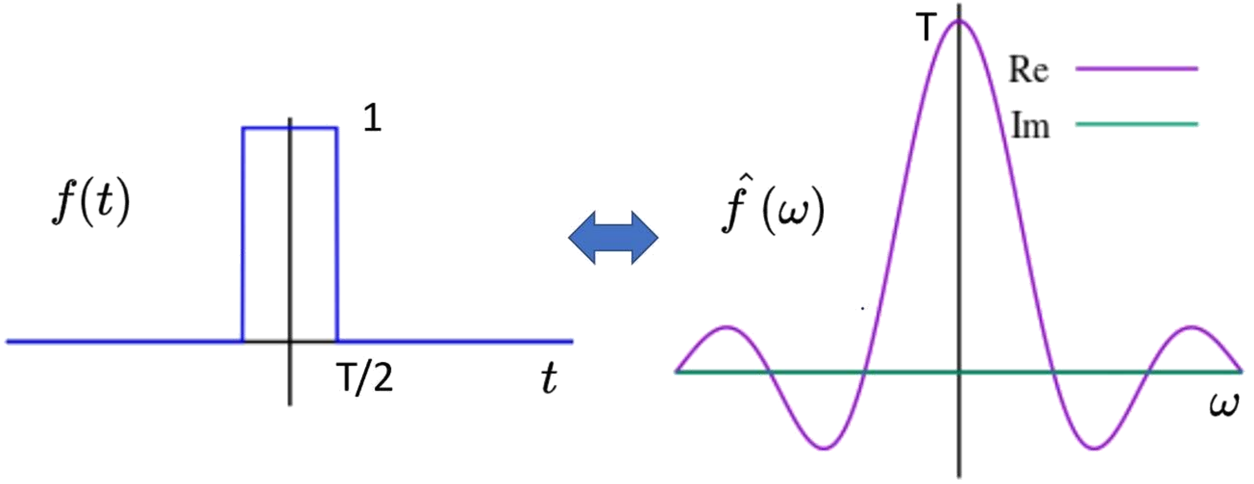

Figure. Square pulse in time domain and its Fourier transformed function in frequency domain

Properties of Fourier Transform

- Linearity: $af_1(t) + bf_{2(t)} \Leftrightarrow aF_1(\omega) + bF_2(\omega)$
- Time shifting: $f(t - t_0) \Leftrightarrow e^{-jt_0\omega} F(\omega)$
- Time scaling: $f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$

NOTE: these expressions are useful,
when we control a superconducting qubit with RF pulse

Some Useful Examples of Fourier Transform

- square pulse in time domain ⇔ sinc in frequency domain
- Gaussian pulse in time domain ⇔ Gaussian (but different amplitude) in frequency domain
- Gaussian modulated cosine (frequency: ω_0) pulse in time domain ⇔ two Gaussian (with ω_0 and -ω_0) in frequency domain

Brief Review of Linear Algebra: (1) Bra-Ket Notation (Dirac Notation)

- Definition of Bra-Ket Notation (also Known as Dirac Notation)
  - A standard notation for describing quantum states in the mathematical framework
- Key Components of Bra-Ket Notation
  - Ket vector $|\psi\rangle$: a column vector
  - Bra vector $\langle\psi|$: a row vector, which is conjugate transpose (symbol: †, dagger) of a Ket vector

Ket

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Bra

$$\begin{aligned} \langle\psi| &= |\psi\rangle^\dagger \\ &= (a_1^*, a_2^*, \ldots, a_n^*) \end{aligned}$$

- Inner Product Using Bra-Ket Notation

$$\int \psi_a^* \psi_b = \langle\psi_a|\psi_b\rangle = (a_1^*, a_2^*, \ldots, a_n^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1} a_i^* b_i$$

NOTE: inner product can be also expressed as $\langle\psi_a, \psi_b\rangle$ instead of $\langle\psi_a|\psi_b\rangle$

- Why Do We Use Bra-Ket Notation?
  - Simple notation to express the quantum state

Brief Review of Linear Algebra: (2) Inner Product

Definition of Inner Product

- The inner product (or dot product) in a Hilbert space returns scalar by multiplying two vectors

$$\langle \psi_a | \psi_b \rangle = (a_1^*, a_2^*, \dots, a_n^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1} a_i^* b_i$$

Properties of Inner Product

- Linearity: $\langle c_1\psi_1 + c_2\psi_2 | \psi_3 \rangle = c_1\langle \psi_1 | \psi_3 \rangle + c_2\langle \psi_2 | \psi_3 \rangle$

where c_1 and c_2 are scalar constants

- Conjugate symmetric: $\langle \psi_a | \psi_b \rangle = \langle \psi_b | \psi_a \rangle^*$
- Positive definite: $\langle \psi_a | \psi_a \rangle \geq 0$, with equality if and only if $\psi_a = 0$

Characteristics of Inner Product for Quantum Computing

- Inner product represents projection of a quantum state to the other state
- Square of inner product $|\langle \psi | \phi \rangle|^2$ indicates the probability of a system to be in state $|\psi\rangle$, given that the system is in state $|\phi\rangle$

Example of Inner Product

- For vectors $|\psi\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\langle \psi | \phi \rangle = (1 \quad -i) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 - i$

Brief Review of Linear Algebra: (3) Tensor Product

- Definition of Tensor Product
  - The tensor product of two vector spaces V and W results in a new vector space, denoted $V \otimes W$
  - For vectors $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ in V and $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$ in W, the tensor product $|\psi\rangle \otimes |\varphi\rangle$ is

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

- Properties of Tensor Product
  - Linearity: $(a|\psi_1\rangle + b|\psi_2\rangle) \otimes |\phi\rangle = a(|\psi_1\rangle \otimes |\phi\rangle) + b(|\psi_2\rangle \otimes |\phi\rangle)$
  - Associativity: $(|\psi_1\rangle \otimes |\psi_2\rangle) \otimes |\psi_3\rangle = |\psi_1\rangle \otimes (|\psi_2\rangle \otimes |\psi_3\rangle)$
- Characteristics of Tensor Product for Quantum Computing
  - Tensor product represents the combined state of two quantum systems
- Example of Tensor Product
  - For qubit #1 in state $|0\rangle$ and qubit #2 in state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, the tensor product of qubit #1 and qubit #2 is

$$|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Brief Review of Linear Algebra: (4) Hilbert Space

Definition of Hilbert Space

- Hilbert space is a complete, infinite-dimensional vector space equipped with an inner product

Properties of Hilbert Space

- Vector space: A collection of vectors where vector addition and scalar multiplication are defined
- Inner product: Dot product (integration over the entire vector space) of two vectors
- Orthogonality: two vectors $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, if the inner product is zero $\langle\psi|\phi\rangle=0$
- Norm: length of a vector $|\psi\rangle$ is calculated as $\| \psi \| = \sqrt{\langle\psi|\psi\rangle}$
- Basis: Any vector in the Hilbert space can be expressed as a combination of the basis vectors

NOTE: Here, we consider a qubit as two-level system.
A qubit can be considered as m-level system for some cases.

Example of Hilbert Space

- Single qubit system: 2-dimensional space with basis $|0\rangle$ and $|1\rangle$
- Two qubit system: 4-dimensional space with basis $|00\rangle, |01\rangle, |10\rangle$, and $|11\rangle$
- n qubit system: 2^n-dimensional space with basis $|00\cdots00\rangle, |00\cdots01\rangle, \cdots, |11\cdots11\rangle$

Brief Review of Linear Algebra: (5) Linear Operators

- Definition of Linear Operator
  - A mapping function $\hat{L}$ between two vector spaces
  - In typical quantum computing problems, a linear operator $\hat{L}$ maps two vectors within the same Hilbert space

- Properties of Linear Operator
  - Linearity: $\hat{L}(a|\psi\rangle + b|\phi\rangle) = a\,\hat{L}|\psi\rangle + b\hat{L}|\psi\rangle$
  - Associativity: $\langle\psi|\hat{L}|\phi\rangle = \langle(\psi\hat{L})|\phi\rangle = \langle\psi|(\hat{L}\phi)\rangle$

where a and b are scalar constants

- Characteristics of Linear Operator for Quantum Computing
  - In a finite-dimensional Hilbert space, linear operators can be represented as matrices
  - For a Ket vector in a n-dimensional space, a linear operator $\hat{L}$ can be represented by an $n \times n$ matrix

- Example of Linear Operator
  - Qubit NOT gate (also known as Pauli X gate): flips qubit state between $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
  - Matrix representation of the NOT gate: $\hat{X} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ that $\hat{X}|0\rangle = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$ and vice versa

Brief Review of Linear Algebra: (6) Basis States

- Definition of Basis States
  - Basis is a set of linearly independent vectors that span a vector space

- Properties of Basis States
  - Orthogonality: Basis states are orthogonal, meaning $\langle 0|1\rangle=0$ and $\langle 1|0\rangle=0$
  - Normalization: Basis states are normalized, meaning $\langle 0|0\rangle=1$ and $\langle 1|1\rangle=1$
  - Superposition: Any state $|\psi\rangle$ can be expressed as a linear combination of basis states

- Characteristics of Basis States for Quantum Computing
  - A qubit's state can be represented by a superposition of basis states $|0\rangle$ and $|1\rangle$ as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
  - Measuring a qubit in superposition collapses it to one of the basis states, $|0\rangle$ or $|1\rangle$, with probabilities of $|\alpha|^2$ or $|\beta|^2$

  - Otherwise, one can define qubit's basis states with $|+\rangle$ and $|-\rangle$ states
  - A qubit's state can be represented by a superposition of basis states $|+\rangle$ and $|-\rangle$ as $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$
  - Measuring a qubit in superposition collapses it to one of the basis states, $|+\rangle$ or $|-\rangle$, with probabilities of $|\alpha|^2$ or $|\beta|^2$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Brief Review of Linear Algebra: (7) Kronecker Delta and Levi-Civita

- Definition of Kronecker Delta
  - The Kronecker delta, denoted as δ_{ij}, is a function of two variables (usually integers) that is 0 or 1, defined as

$$\delta_{ij} = \begin{cases} 0, & \text{if i} \neq \text{j} \\ 1, & \text{if i} = \text{j} \end{cases}$$

- Definition of Levi-Civita
  - The Levi-Civita, denoted as ε_{ijk}, is a function of two variables (usually integers) that is -1, 0 or 1, defined as

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ \;\;0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases}$$

  - The generalized Levi-Civita in n-dimensions is defined as

$$\varepsilon_{a_1 a_2 a_3 \ldots a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an even permutation of } (1,2,3,\ldots,n) \\ -1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an odd permutation of } (1,2,3,\ldots,n) \\ \;\;0 & \text{otherwise} \end{cases}$$

- Why Do We Use Kronecker Delta and Levi-Civita?
  - Simplified expressions of matrix operations

Brief Review of Linear Algebra: (8) Eigenvalues and Eigenstates

- Definition of Eigenvalue
  - The constant factor λ by which the eigenstate is scaled when the operator $\hat{O}$ is applied
- Definition of Eigenstate
  - The quantum state $|\psi\rangle$ that, when an operator $\hat{O}$ is applied to it, results in the state being scaled by its eigenvalue.

$$\hat{O}|\psi\rangle = \underbrace{\lambda}_{\text{eigenvalue}}\underbrace{|\psi\rangle}_{\text{eigenstate}}$$

- Formula of Eigenvalues and Eigenstates
  - For n-dimensional Hilbert space, $\hat{O}$ is $n \times n$ matrix
  - By solving the linear equation of $\det(\hat{O} - \lambda\,\hat{I}) = 0$, eigenvalues and eigenstates of the operator $\hat{O}$ can be obtained
- Example of Eigenvalues and Eigenstates
  - For the Pauli-Z gate $\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, eigenstate $|0\rangle$ and its corresponding $\lambda = 1$, while eigenstate $|1\rangle$ and its $\lambda = -1$

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Brief Review of Linear Algebra: (9) Adjoint Operation

- Definition of Adjoint Operation
  - Adjoint of a linear operator $\hat{A}$, denoted as $\hat{A}^\dagger$: complex conjugate transpose of $\hat{A}$ ➔ transpose($\hat{A}^*$)

- Formula of Adjoint Operation
  - For $n \times n$ matrix $\hat{A}$, let the (i, j) element representation as $\hat{A}_{ij}$
  - The (j, i) element of the $\hat{A}^\dagger$ is $\left(\hat{A}^\dagger\right)_{ji} = \left(\hat{A}_{ij}\right)^*$

- Characteristics of Adjoint Operation
  - The adjoint of a linear operator $\hat{A}$ satisfies the following relation for all vectors in a Hilbert space:

$$\langle\psi|\hat{A}|\phi\rangle = \langle\psi|\hat{A}\phi\rangle = \langle\psi\,\hat{A}^\dagger|\phi\rangle$$

  - An operator $\hat{A}$ is unitary operator, if $\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A} = \hat{I}$
  - An operator $\hat{A}$ is Hermitian operator, if $\hat{A} = \hat{A}^\dagger$

Details of unitary and Hermitian operators will be discussed next slides

- Example of Adjoint Operation
  - For an operator $\hat{A} = \begin{pmatrix} 2 & 1+j \\ -1 & 2-j \end{pmatrix}$, the adjoint $\hat{A}^\dagger = \begin{pmatrix} 2 & -1 \\ 1-j & 2+j \end{pmatrix}$

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Brief Review of Linear Algebra: (10) Unitary Operators

- Definition of Unitary Operator
  - A linear operator $\hat{U}$ is unitary, if it satisfies: $\hat{U}^\dagger \hat{U} = \hat{I}$, where $\hat{I}$ is the identity operator
- Properties of Unitary Operator
  - Norm preservation: An unitary operator $\hat{U}$ preserves the norm (length) of vectors
  - Reversibility: An unitary operator $\hat{U}$ is reversible that its inverse $\hat{U}^{-1}$ exists and is defined as $\hat{U}^{-1} = \hat{U}^\dagger$
  - Inner product preservation: An unitary operator $\hat{U}$ preserves the inner product as $\langle \psi | \phi \rangle = \langle \hat{U}\psi | \hat{U}\phi \rangle$
- Characteristics of Unitary Operator for Quantum Computing
  - The probability remains unchanged under unitary operators
  - Any quantum states preserve their initial states after two identical unitary operators
- Example of Unitary Operator
  - Hadamard gate: turns $|0\rangle$ or $|1\rangle$ into the superposition of $0\rangle$ and $|1\rangle$
  - Matrix representation of the Hadamard gate: $\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
  - Proof: $\hat{H}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\hat{H}^\dagger \hat{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}$

Brief Review of Linear Algebra: (11) Hermitian Operators

- Definition of Hermitian Operator
  - A linear operator $\hat{H}$ is Hermitian (also known as self-adjoint), if it is equal to its own adjoint: $\hat{H} = \hat{H}^\dagger$
- Properties of Hermitian Operator
  - Real eigenvalues: Hermitian operator $\hat{H}$ has real (not complex) eigenvalues as

    $\hat{H}|\psi\rangle = \lambda|\psi\rangle$, where λ is the eigenvalue and real value
  - Orthogonal eigenvectors: Eigenvectors corresponding to different eigenvalues of an Hermitian operator are orthogonal

    $\hat{H}|\psi_1\rangle = \lambda_1|\psi_1\rangle$ and $\hat{H}|\psi_2\rangle = \lambda_2|\psi_2\rangle \rightarrow \langle\psi_1|\psi_2\rangle = 0$, if $\lambda_1 \neq \lambda_2$
  - Diagonalization: Hermitian operator $\hat{H}$ can be diagonalized by an unitary operator $\hat{U}$ as

$$\hat{U}\hat{H}\hat{U} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ & & 0 \\ 0 & \ddots & \vdots \\ \vdots & & \\ 0 & \cdots & \lambda_n \end{pmatrix}$$, where λ_i are the eigenvalues of $\hat{H}$

- Characteristics of Hermitian Operator for Quantum Computing
  - Hermitian operators represent observable physical quantities, such as position, momentum, and energy
  - Hamiltonian is one of the Hermitian operators, since Hamiltonian represents the total energy of a quantum system

Brief Review of Linear Algebra: (12) Taylor Expansion of an Operator

- **Definition of Taylor Expansion of Operator**
  - Similar to the Taylor expansion of a function, operators can be also approximated by the Taylor expansion
- **Formula of Taylor Expansion**
  - For an operator $\hat{A}$, the exponential of the operator is

$$e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{\left(\hat{A}\right)^n}{n!} = \hat{A} + \frac{1}{2!}\hat{A}\hat{A} + \frac{1}{3!}\hat{A}\hat{A}\hat{A} + \cdots$$

- **Characteristics of Taylor Expansion of Operator for Quantum Computing**
  - The time evolution of a quantum state is governed by the exponential of a time-independent Hamiltonian operator $\widehat{H}$ as

$$e^{-j\widehat{H}t/\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{j\widehat{H}t}{\hbar}\right)^n$$

where $\hbar$ is the reduced Planck's constant and t is time

  - Taylor expansion is used in perturbation theory to approximate the effects of small perturbations

Brief Review of Differential Equation: Ordinary Differential Equation (ODE)

Definition of ODE

- An equation involving a function of one independent variable and its derivatives
- The generalized expression of an ODE for the unknown function y is

$$\frac{d^n}{dx^n}y + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}y + \cdots + a_1(x)\frac{d}{dx}y + a_0(x)y = g(x)$$

where $a_i(x)$ and $g(x)$ are given functions

Properties of ODE

- Order: The highest derivative of the unknown function in the ODE
- Homogeneous ODE: If $g(x) = 0$ (otherwise, $g(x) \neq 0$, inhomogeneous ODE)

Mathematical Methods to Solve ODE

- Analytical methods: only some ODEs can be solved by exact analytical methods
- Numerical methods: most of ODEs are solved by numerical methods with approximations

Brief Review of Differential Equation: Partial Differential Equation (PDE)

- Definition of partial Derivative
  - Derivative of a function with respect to one of the independent variables, with the others being constant
  - The partial derivative of a function y at the point $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$ with respect to the i^{th} variable x_i is defined as

$$\frac{\partial}{\partial x_i} f(\boldsymbol{a}) = \lim_{\mathrm{h} \to 0} \frac{f(a_1, \cdots, a_i + h, \cdots, a_n) - f(a_1, \cdots, a_i, \cdots, a_n)}{h}$$

- Definition of PDE
  - An equation involving a function of multiple independent variables and its partial derivatives
  - The generalized expression of an ODE for the unknown function y with the independent variables x and t is

$$\frac{\partial^n}{\partial t^n} y + a_{n-1}(x,t) \frac{\partial^{n-1}}{\partial t^{n-1}} y + \cdots + a_1(x,t) \frac{\partial}{\partial t} y + a_0(x,t) y = g(x,t)$$

where $a_i(x,t)$ and $g(x,t)$ are given functions

- Example PDE in Quantum Computing
  - Schrödinger equation: time evolution of a quantum system can be represented by a linear PDE in time variable

$$j\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle$$

NOTE: You don’t have to solve Schrödinger equation by yourself.
There are various numerical tools to solve it.

Brief Review of Differential Equation: Green's Function

- Definition of Green's Function
  - Impulse response of an inhomogeneous linear differential operator $\hat{L}$ with specified boundary conditions
  - Green's function is often utilized to solve an inhomogeneous differential equation of $\hat{L}y(x) = f(x)$

- Solving Inhomogeneous Differential Equation Using Green's Function
  - For a given forcing term $f(x)$, the forcing term can be expressed with Dirac-delta function as

$$f(x) = \int_{\mathbb{R}^n} f(r)\delta(x-r)\,dr$$

Where Dirac-delta function is $\delta(x-r)\begin{cases}\infty, & \text{if } x = r \\ 0, & \text{if } x \neq r\end{cases}$ and satisfies $\int_{\mathbb{R}^n} \delta(r)\,dr = 1$

  - For a given linear differential operator $\hat{L}(x)$, the Green's function $G(x)$ can be obtained from the following relation

$$\hat{L}G(x,r) = \delta(x-r)$$

NOTE: Most of Green's functions corresponding to the specific $\hat{L}$ can be found in https://en.wikipedia.org/wiki/Green%27s_function

  - Using the above relation, the inhomogeneous differential equation satisfies the following relations

$$f(r)\hat{L}G(x,r) = f(r)\delta(x-r) \longrightarrow \hat{L}f(r)G(x,r) = f(r)\delta(x-r)$$

  - Thus, by integrating over the region $\mathbb{R}^n$, the unknown function $y(x)$ can be expressed as

$$\therefore\ y(x) = \int_{\mathbb{R}^n} f(r)G(x,r)\,dr$$

Brief Review of Vector Calculus: (1) Divergence Theorem

- Definition of Divergence
  - The divergence is the flux through the surface of a vector field $\mathbf{F}$, defined at $x = x_0$ as

$$\nabla \cdot \mathbf{F}\Big|_{x=x_0} = \lim_{V\to 0} \frac{1}{|V|} \oint_{\partial V} \mathbf{F} \cdot d\boldsymbol{a}$$

  - For cartesian coordinate, the divergence of a vector field $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

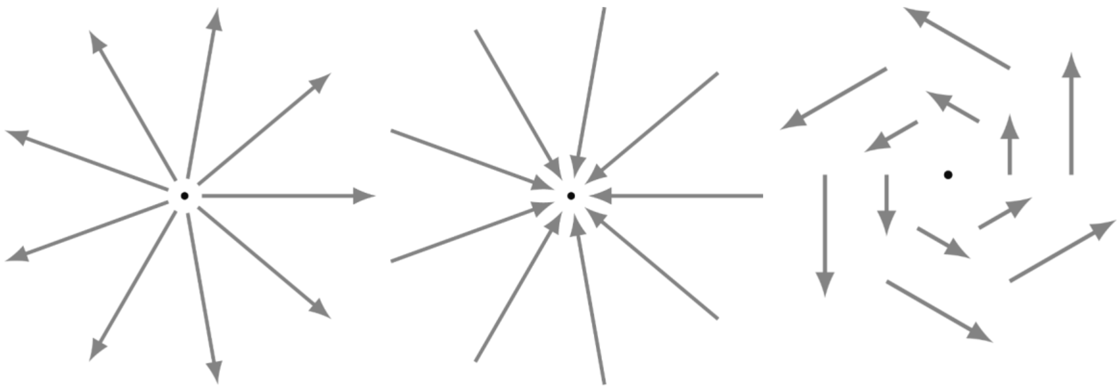

Figure. Example visualizations of divergence of $\mathbf{F}$

- Definition of Divergence Theorem (also Known as Gauss's Theorem)
  - The surface integral of a vector field over a closed surface is equal to the volume integral of the divergence over the region enclosed by the surface

$$\underbrace{\int_V}_{\text{Volume Integral}} (\nabla \cdot \mathbf{F})\, dV = \underbrace{\int_{\partial V}}_{\text{Surface Integral}} \mathbf{F} \cdot d\,\mathbf{a}$$

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Brief Review of Vector Calculus: (2) Stokes' Theorem

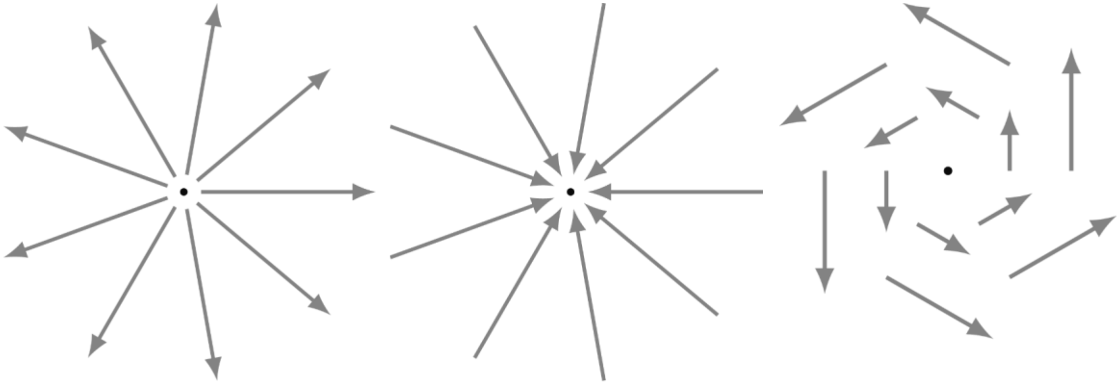
Definition of Curl

- The curl is the circulation of a vector field $\mathbf{F}$, defined at $x = x_0$ as

$$\nabla \times \mathbf{F}\Big|_{x=x_0} = \lim_{A\to 0} \frac{1}{|A|} \oint_{\partial A} \mathbf{F} \cdot d\boldsymbol{l}$$

- For cartesian coordinate, the curl of a vector field $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$



$\nabla \times \mathbf{F} = \mathbf{0}$ $\nabla \times \mathbf{F} = \mathbf{0}$ $\nabla \times \mathbf{F} \neq \mathbf{0}$

Figure. Example visualizations of curl of $\mathbf{F}$

Definition of Stokes' Theorem

- The line integral of a vector field over a loop is equal to the surface integral of its curl over the enclosed surface

$$\underbrace{\int_A (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{a}}_{\text{Surface Integral}} = \underbrace{\oint_{\partial A} \mathbf{F} \cdot d\boldsymbol{l}}_{\text{Line Integral}}$$

Brief Review of Dimensional Analysis: Physical Unit Conversion

- Definition of Physical Unit Conversion
  - Analysis of the relationships between physical quantities by identifying their base quantities and units of measurement
  - Using the international standard unit systems, intuitive analysis and comparison between variables are possible

- Physical Unit Conversion in Quantum Computing
  - In physics, particularly in superconducting quantum circuits, many variables have unfamiliar physical units
  - Example: qubit energy in [Hz] or in [rad/s] by reducing the physical constants as 1

$$\begin{aligned}\text{qubit energy [J]} &= \cancel{\hbar} \times \text{qubit angular frequency [rad/s]} \\ &= \cancel{h} \times \text{qubit frequency [Hz]}\end{aligned}$$

Why?
In practice, we measure the qubit energy levels by RF spectroscopy in frequency units

- Some Notable Physical Constants
  - Planck's constant $h \approx 6.63 \times 10^{-34}$ [J·s]
  - Charge of an electron $e \approx 1.60 \times 10^{-19}$ [C]
  - Speed of light in vacuum $c \approx 3.00 \times 10^{8}$ [m/s]
  - Impedance of vacuum $Z_\eta \approx 377$ [Ω]
  - Boltzmann constant $k_B \approx 1.38 \times 10^{-23}$ [J/K]
  - Vacuum permeability $\mu_0 \approx 1.26 \times 10^{-6}$ [H/m]
  - Vacuum permittivity $\varepsilon_0 \approx 8.85 \times 10^{-12}$ [F/m]
  - Magnetic flux quantum $\Phi_0 = \frac{h}{2e} \approx 2.07 \times 10^{-15}$ [Wb]

See Also…

- Textbooks:

[1] Mary L. Boas, *Mathematical Methods in the Physical Sciences*, John Wiley & Sons, 2006. * recommended

[2] Erwin O. Kreyszig, *Advanced Engineering Mathematics*, John Wiley & Sons, 2011.

[3] George B. Arfken, *Mathematical Methods for Physicists*, Elsevier, 2012.