APPLIED SUPERCONDUCTIVITY LABORATORY

# *Introduction to Superconducting Quantum Circuits*

## *- Review of Classical Mechanics for Quantum Computing -*

Department of Electrical and Computer Engineering
Seoul National University

Seong Hyeon Park and Seungyong Hahn

15th August, 2024

# Keywords in Classical Mechanics for Quantum Computing

**Classical Mechanics: Newtonian**

Conservative Force
Newton's Laws of Motion
Potential Energy
Kinetic Energy

**Classical Mechanics: Lagrangian**

Principle of Least Action
Euler-Lagrangian Equation

**Classical Mechanics: Hamiltonian**

Phase Space
Canonical Momentum
Legendre Transformation
Poisson Braket
Canonical Transformation

**Classical Mechanics: Simple Harmonic Oscillator**

Ladder Operator

## What is Classical Mechanics?

- Physical theory describing the motion of objects
- The development of classical mechanics involved substantial change in the methods and philosophy of physics

## Why is Classical Mechanics Important for Studying Quantum Computing?

- Jargons, analogy, formula,…etc. of superconducting quantum circuits are based on quantum and classical mechanics
- In this lecture, we will briefly review selected topics in classical mechanics

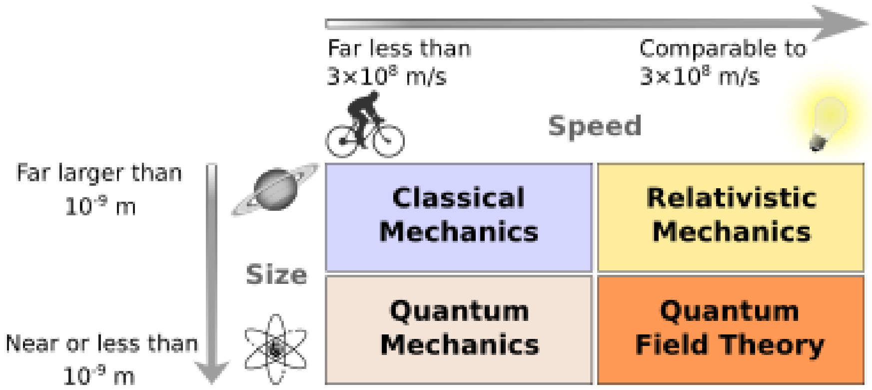
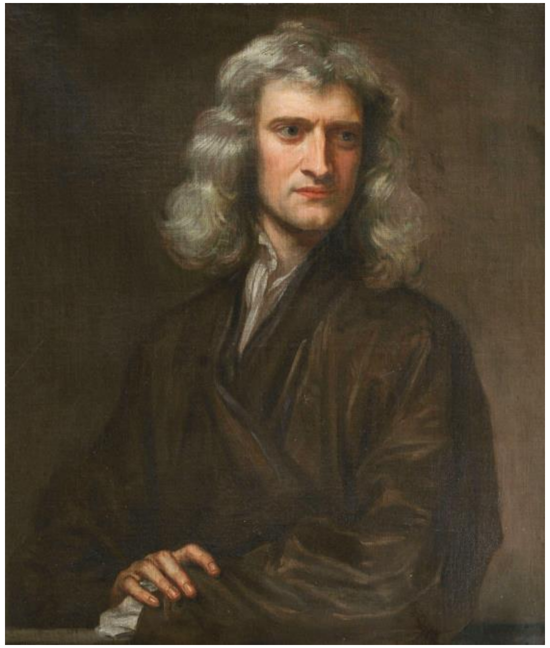
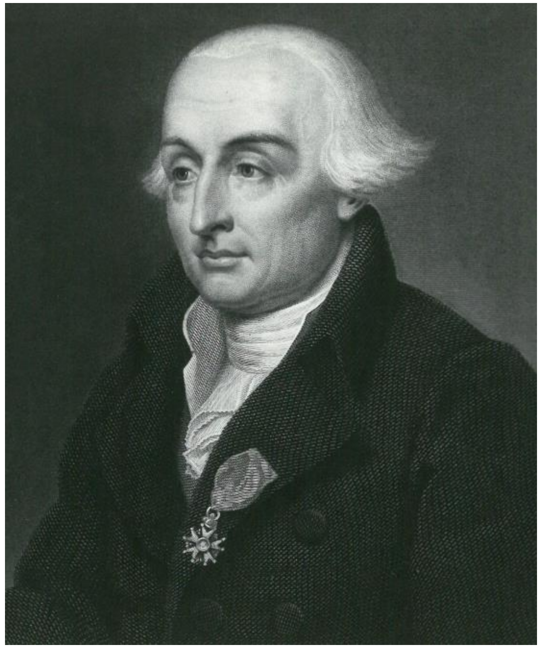



Fig. Domain of validity for classical mechanics



Isaac Newton (1642-1727)



Joseph-Louis Lagrange (1736-1810)

William Rowan Hamilton (1805-1865)

Image from: https://en.wikipedia.org/wiki/Classical_mechanics

# Brief Review of Newtonian Mechanics: (1) Newton's Laws of Motion

- Newton's Laws of Motion
  - Force $\boldsymbol{F}$: a vector quantity [N]
  - Mass $m$: a scalar quantity [kg]
  - Acceleration $\boldsymbol{a}$: a vector quantity [m/s$^2$]
  - Velocity $\boldsymbol{v}$: a vector quantity [m/s]
  - Momentum $\boldsymbol{p} = m\boldsymbol{v}$: a vector quantity
  - Equation of motion: $\sum \boldsymbol{F} = m\boldsymbol{a}$

NOTE:
Electrical circuit variables (ex: charge) can be also represented by the above variables

- Concept of Conservative Force
  - The work done by conservative force depends only on the initial and final positions (independent of path)
- Examples of Conservative Force
  - Gravity, elastic spring, electrostatic force

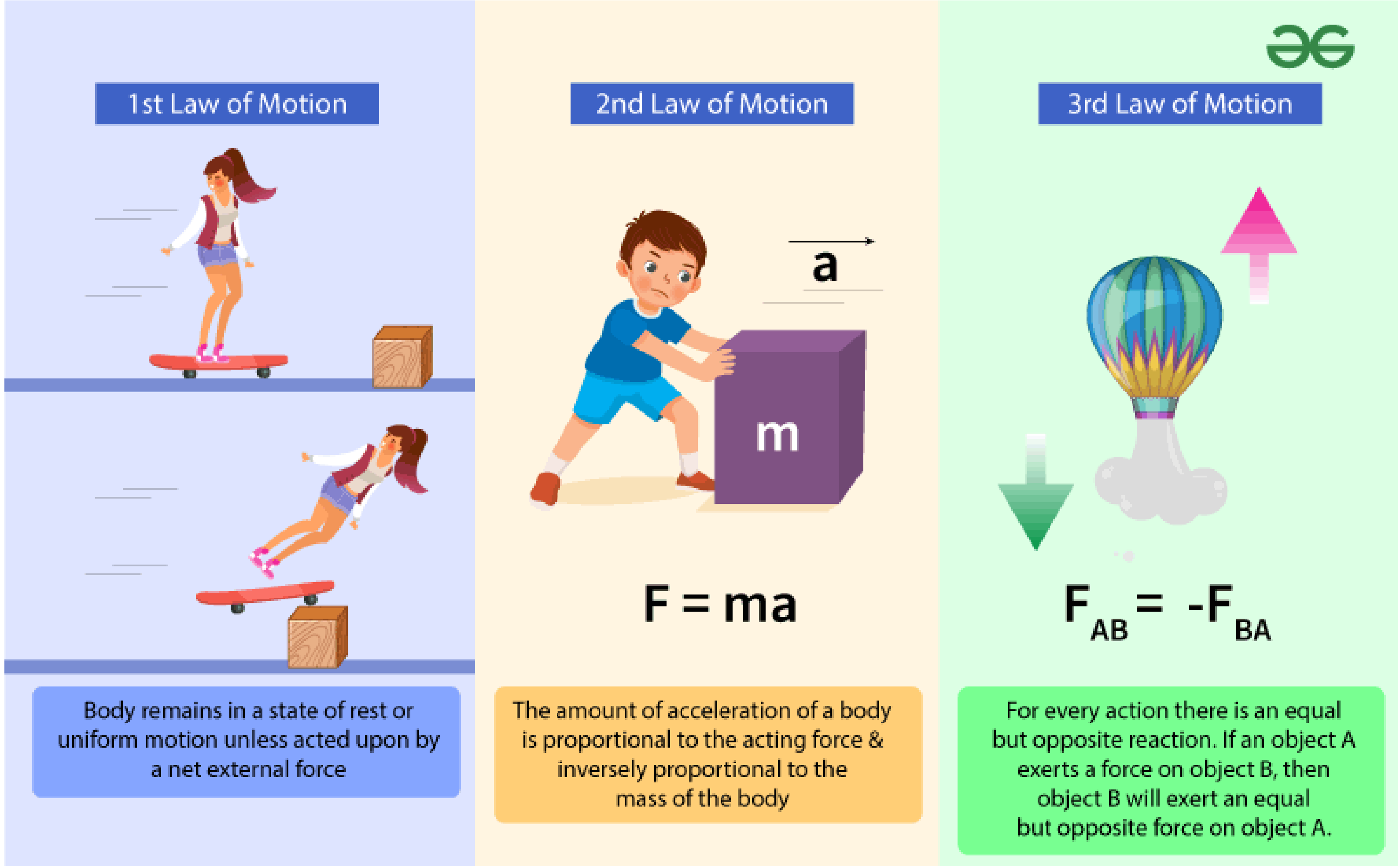
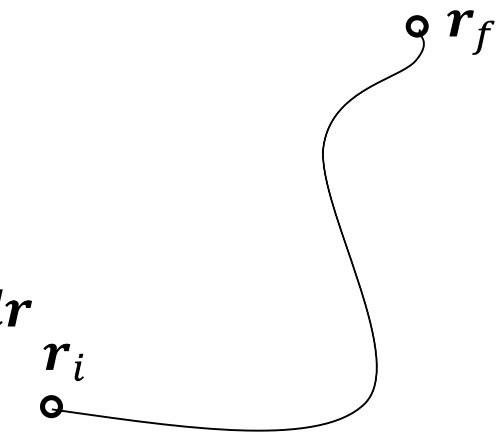



Fig. Definition, formula, and example of Newton's laws of motion.

Image from: https://www.geeksforgeeks.org/newtons-laws-of-motion/

# Brief Review of Newtonian Mechanics: (2) Potential and Kinetic Energy




- **Definition of Potential Energy**
  - Total work done by conservative forces, defined by virtue of an object's position relative to others
- **Formula of Potential Energy**
  - For conservative force $\boldsymbol{F}$ and path between the initial point $\boldsymbol{r}_i$ and the final point $\boldsymbol{r}_f$,
  - The difference of the potential energy $U$ is: $U(\boldsymbol{r}_f) - U(\boldsymbol{r}_i) = -W$ (negative work) $= -\int_{\boldsymbol{r}_i}^{\boldsymbol{r}_f} \boldsymbol{F} \cdot d\boldsymbol{r}$

negative notation (typical, but not necessary)

- **Definition of Kinetic Energy**
  - Energy due to the motion
- **Formula of Kinetic Energy**
  - For a point object (assuming that its mass exist at one point) with mass $m$ and velocity $\boldsymbol{v}$,
  - The kinetic energy $T$ is: $T = \frac{1}{2} m\boldsymbol{v}^2 = \frac{\boldsymbol{p}^2}{2m}$
- **Law of Conservation of Energy**
  - The total energy (sum of $U$ and $T$) of an isolated system remains constant
  - The kinetic energy can be converted to the potential energy, vice versa

SH Park
<pajoheji0909@snu.ac.kr>

# Brief Review of Newtonian Mechanics: (3) Harmonic Oscillator Example

- Definition of Harmonic Oscillator
  - Special type of periodic trigonometric oscillation towards the equilibrium point
- Properties of Harmonic Oscillator
  - Conservation of the total energy for ideal harmonic oscillators
  - In real oscillators, damping (frictional or dragging force) exists
- Example of Harmonic Oscillator: Mass on an Ideal Spring
  - In Newtonian mechanics, the restoring force $\boldsymbol{F}$ at the distance $\boldsymbol{x}$ is

$$\boldsymbol{F} = m\frac{d^2\boldsymbol{x}}{dt^2} = -k\boldsymbol{x}$$

where $k$: spring constant [N/m]

  - The above equation is the second-order differential equation
  - The solution for $\boldsymbol{x}$ is

$$\boldsymbol{x}(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

where $\omega$: angular frequency, defined as $\omega = \sqrt{\frac{k}{m}}$

  - The coefficients $c_1$ and $c_2$ can be determined by the initial conditions

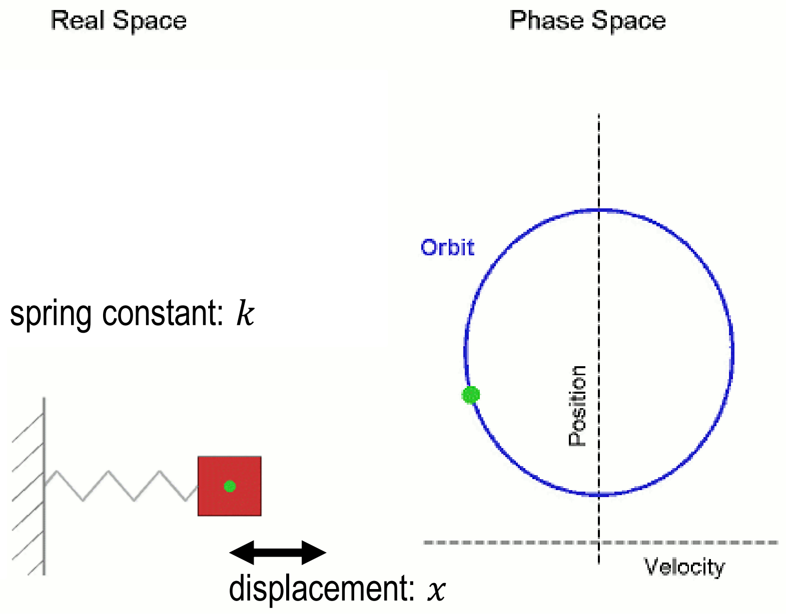



Fig. Motion of harmonic oscillator in real space and phase space.

Image from: https://en.wikipedia.org/wiki/Simple_harmonic_motion

NOTE:
Harmonic oscillator is very important concept
for quantum mechanics AND superconducting circuits

# Brief Review of Lagrangian Mechanics: (1) Principles of Least Action

- **Philosophy of Lagrangian Mechanics**
  - The fundamental idea of the Lagrangian mechanics is to reformulate the equations of motion in terms of the dynamical variables that describe the degrees of freedom (no explicit force terms)
  - For complex physical systems, obtaining exact force is difficult

- **Characteristics of Lagrangian Mechanics**
  - Generalized coordinates $\{q_k\}$: fully specify the motion of the system → potential energy $U(q,t)$
  - Generalized velocity coordinates $\{\dot{q}_k\}$: total time derivatives of the generalized coordinates → kinetic energy $T(\dot{q},t)$
  - Lagrangian $\mathcal{L}(q,\dot{q},t) = T - U$: defined as the difference between the kinetic energy $T$ and the potential energy $U$
  - Lagrangian is scalar quantity! While complex vector calculation is required with Newtonian mechanics

- **Principles of Least Action**
  - Path taken between two states is the one for which the action is minimized
  - The action $S$ is defined as

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt = \int_{t_1}^{t_2} T - U\, dt$$

The path $x(t)$ that makes $S$ stationary satisfies the Euler-Lagrange equation!

example: harmonic oscillator (in previous slide)
Describe the system with $q$ and $\dot{q}$ ONLY

- potential energy $U = \frac{1}{2}kx^2$ (here, $q$ is $x$)
- kinetic energy $T = \frac{1}{2}m\dot{x}^2$
- Lagrangian $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

SH Park
<pajoheji0909@snu.ac.kr>

# Brief Review of Lagrangian Mechanics: (2) Euler-Lagrangian Equation

## Definition of Euler-Lagrangian Equation

- Second-order ODEs, whose solutions are stationary points of the given action

## Formula of Euler-Lagrangian Equation

- For $n$-dimensional generalized coordinate vector $\boldsymbol{q} = \{q_1(t), \dots, q_n(t)\}$ and speed vector $\dot{\boldsymbol{q}} = \{\dot{q}_1(t), \dots, \dot{q}_n(t)\}$
- The Lagrangian $\mathcal{L}$ is dependent on $\boldsymbol{q}, \dot{\boldsymbol{q}}, t$ that $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ can be expressed
- The Euler-Lagrangian equations are defined as

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$

where $i = 1, \dots, n$

Derivation of the Euler-Lagrangian equation is beyond the scope of this lecture (also, not necessary)

See Fowles, *Analytical Mechanics,* (2004) for the proof

example: harmonic oscillator

- $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
- The Euler-Lagrangian equation is
- $\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = -kx - \frac{d}{dt}(m\dot{x})$
  $= -kx - m\ddot{x} = 0$
- $\therefore kx + m\ddot{x} = 0$ (equation of motion)
- same result from Newtonian mechanics

## Lagrangian in Quantum Computing

- The Lagrangian will be utilized as the fundamental analysis tool for superconducting quantum circuits
- To analyze superconducting qubits and resonators, deriving the Lagrangian is a key procedure

# Brief Review of Lagrangian Mechanics: (3) Simple Pendulum Example

- Simple Pendulum Problem using Newtonian Mechanics
  - The total force $\boldsymbol{F}$ on the object of mass $m$ is

$$\boldsymbol{F} = m\boldsymbol{a} = -mg\sin\theta(t)$$

where $g$: gravity [m/s$^2$]

NOTE:
acceleration along the tangential axis (red line in the figure) ONLY

  - Relation between the tangential axis (red) and the angle $\theta$ is

$$\boldsymbol{s} = l\boldsymbol{\theta},$$

$$\boldsymbol{a} = \frac{d^2\boldsymbol{s}}{dt^2} = l\frac{d^2\boldsymbol{\theta}}{dt^2},$$

where $l$: length of pendulum [m]

  - Thus, the equation of motion is $\therefore\ l\dfrac{d^2\theta}{dt^2} + g\sin\theta = 0$

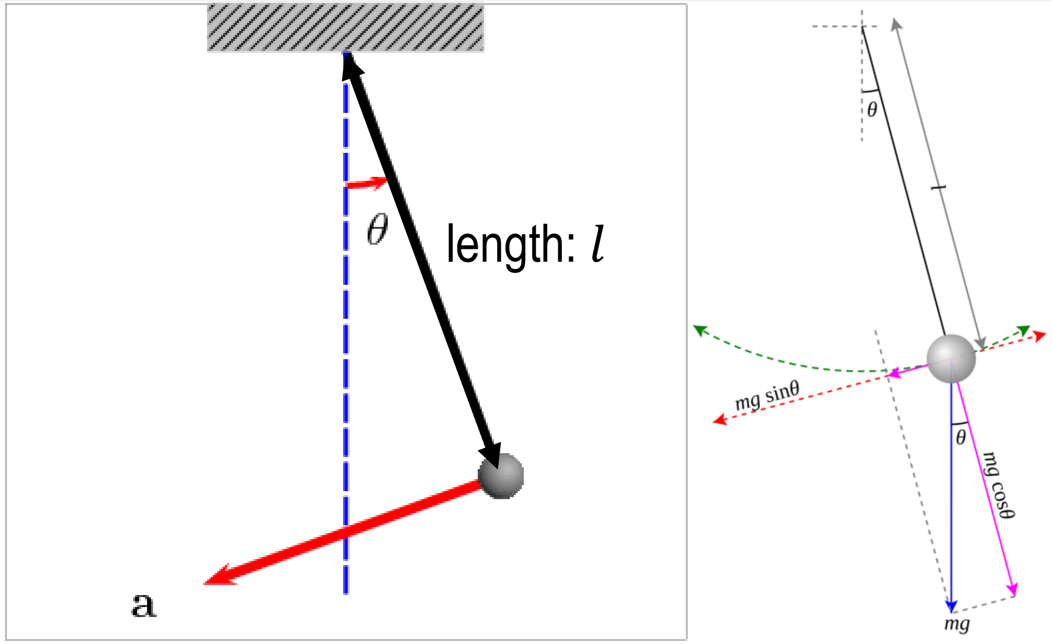


Fig. Motion of pendulum and force diagram under the gravity.

Image from: https://en.wikipedia.org/wiki/Pendulum_(mechanics)

- Simple Pendulum Problem using Lagrangian Mechanics
  - Assuming the coordinates with respect to $\theta$ only, the kinetic energy, potential energy, and Lagrangian are

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \text{ and } U = -mgl\cos\theta \Rightarrow \text{Lagrangian is } \mathcal{L}(\theta,\dot{\theta},t) = T - U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

  - From the Euler-Lagrangian equation,

$$\frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = -mgl\sin\theta - ml^2\ddot{\theta} = 0 \Rightarrow \therefore\ l\frac{d^2\theta}{dt^2} + g\sin\theta = 0$$

# Brief Review of Hamiltonian Mechanics: Introduction

- Philosophy of Hamiltonian Mechanics
  - Recall that Lagrangian mechanics allows us to find the equations of motion in terms of generalized coordinates and velocities
  - Extend the method using the canonical momenta (we will learn about this in the next slide) instead of generalized velocities

- Characteristics of Hamiltonian Mechanics
  - The Hamiltonian, denoted by $\mathcal{H}$, is the sum of kinetic energy $T$ and potential energy $U$

$$\mathcal{H} = T + U$$

Recall that Lagrangian is defined as $\mathcal{L} = T - U$

  - If the generalized coordinates for the system are time-independent, the Hamiltonian is also time-independent (conserved)
  - The Hamiltonian is also scalar

- Summary of Newtonian, Lagrangian, and Hamiltonian Mechanics
  - Newtonian: directly based on forces and accelerations → intuitive approach but cumbersome for complex systems
  - Lagrangian: difference between kinetic and potential energy → simplified problems with generalized coordinates
  - Hamiltonian: total energy of the system → deeper insight into the conservation laws and symmetries of the system

# Brief Review of Hamiltonian Mechanics: (1) Canonical Conjugate Momentum

- **Definition of Canonical Conjugate Momentum**
  - Canonical coordinates: can describe a physical system with generalized position quantities $\{q_i\}$ and velocity quantities $\{\dot{q}_i\}$
  - Conjugate momentum: partial derivative of the Lagrangian, with respect to the generalized velocity
- **Formula of Canonical Conjugate Momentum**

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

for $n$-dimensional generalized coordinates, there will be $n$ canonical conjugate momentum

- **Example of Canonical Conjugate Momentum for Harmonic Oscillator**
  - Coordinate $q$: $x$ (and recall that the Lagrangian is $\mathcal{L}(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$)
  - Momentum $p$: $\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$

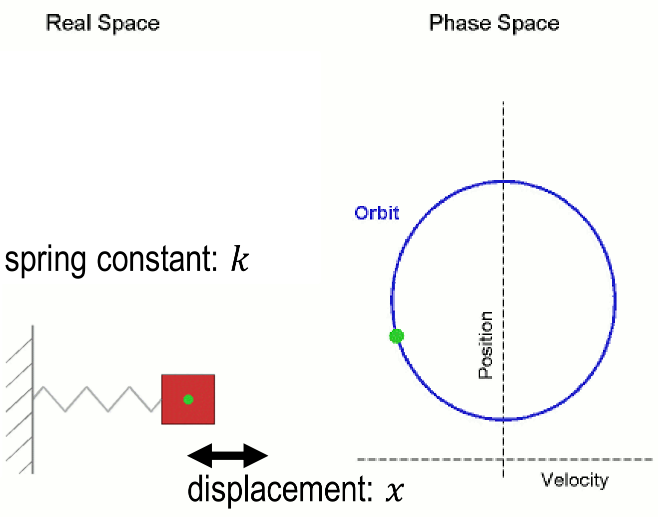



Fig. Motion of harmonic oscillator

- **Canonical Conjugate Momentum in Quantum Computing**
  - Assume a superconducting $LC$ parallel circuit
  - From the electrical circuit theory, flux $\Phi(t) = \int_{t_0}^{t} V(t')dt'$ and $Q(t) = CV(t)$

where $t_0$: reference time

  - Flux $\Phi$ = coordinate and charge $Q$ = canonical conjugate momentum

NOTE: Quantum mechanical analysis of electrical circuits will be introduced later

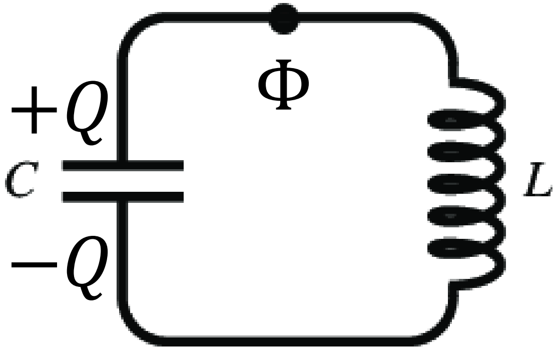



Fig. Parallel LC circuit

SH Park
<pajoheji0909@snu.ac.kr>

# Brief Review of Hamiltonian Mechanics: (2) Phase Space

## Definition of Phase Space

- A multidimensional space in which all possible states of a physical system can be represented
- The phase space can be defined by the generalized coordinates $q_i$ and their canonical conjugate momenta $p_i$

## Characteristics of Phase Space

- For $n$-dimensional generalized coordinates, the phase space has $n$-canonical coordinates and $n$-canonical momenta
- If the Lagrangian does not depend on the $k^{\text{th}}$ coordinate variable $q_k$, $q_k$ is cyclic canonical variable
- Without $q_k$ dependence ($q_k$ is cyclic), the following relations are satisfied

$$\frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \text{and} \quad p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \text{constant}$$

## Example of Phase Space: Harmonic Oscillator

- The Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
- Generalized coordinate = $\{x\}$ and canonical conjugate momentum = $\{m\dot{x}\}$
- The corresponding phase space has $\{x, m\dot{x}\}$ canonical variables

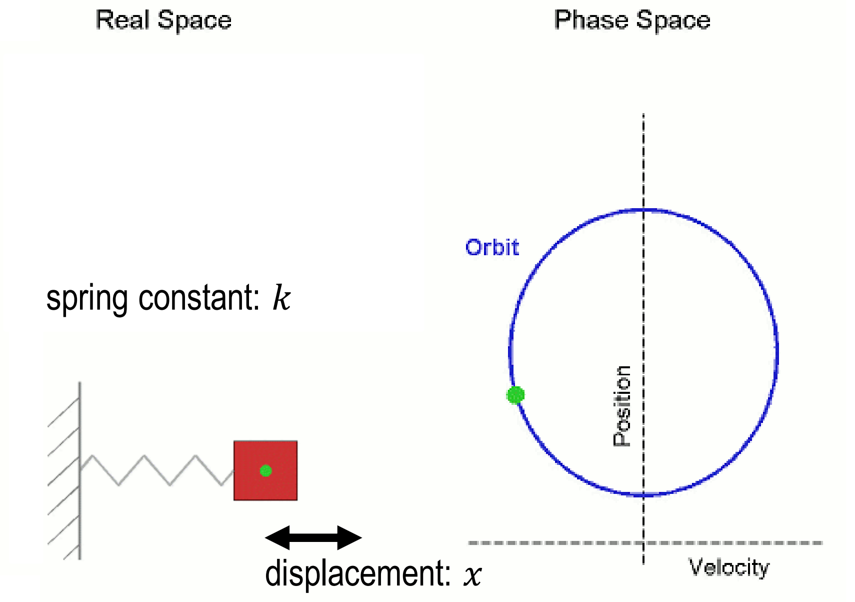


Fig. Motion of harmonic oscillator in real space and phase space.

# Brief Review of Hamiltonian Mechanics: (3) Legendre Transformation

- Definition of Legendre Transformation
  - Transformation of the Lagrangian $\mathcal{L}$ into the Hamiltonian $\mathcal{H}$

- Formula of Legendre Transformation
  - For $n$-generalized coordinates $q_i$ and the corresponding momenta $p_i$, the Hamiltonian $\mathcal{H}$ can be obtained as follows

Recall that $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n, t) = \sum_{i=1} p_i \dot{q}_i - \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$

- Characteristics of Legendre Transformation

$$\frac{\partial \mathcal{H}}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} \text{ and } \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

$$-\dot{p}_i = \frac{\partial \mathcal{H}}{\partial q_i} \text{ and } \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

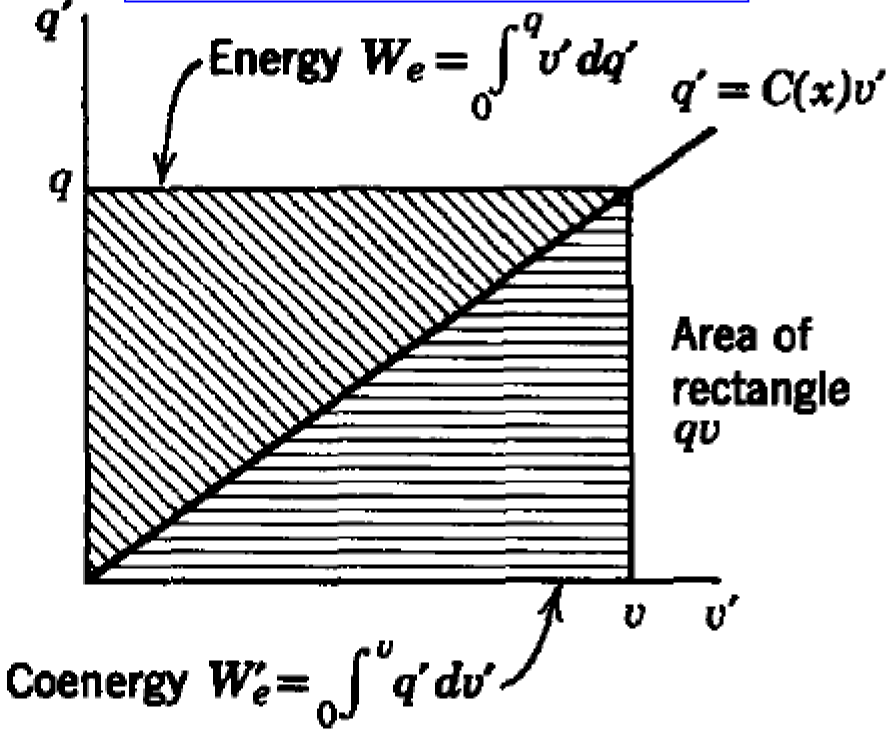
You can derive these relations from the Legendre transformation

- Legendre Transformation in Quantum Computing
  - Legendre transformation from the Lagrangian of superconducting quantum circuit to derive the circuit's Hamiltonian

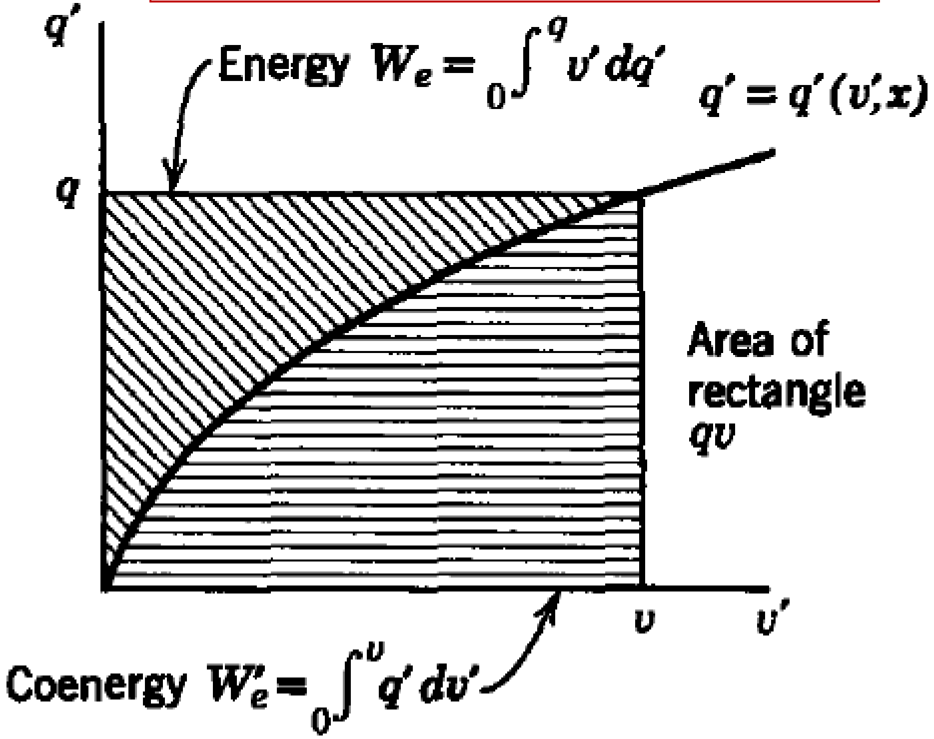
# Similar Technique in Electrical Engineering: Coenergy Method

- Comparison of Energy $W_e$ and Coenergy $W_e'$ between Linear and Nonlinear Systems
  - By definition, $W_e + W_e'$ is equal to $qv$ in electric system and $\lambda i$ in magnetic system
  - Only in the *linear* system, $W_e = W_e'$; if a system is *non-linear*, $W_e \neq W_e'$
  - Even if a system is linear, the mathematical expression of $W_e$ is *completely different* from that of $W_e'$

*For Electrically Linear System*




*For Electrically Non-linear System*




SH Park
<pajoheji0909@snu.ac.kr>

# Brief Review of Hamiltonian Mechanics: (4) Poisson Bracket

## Definition of Poisson Bracket

- An operator that measures the infinitesimal change of one observable quantity with respect to another in phase space

## Formula of Poisson Bracket

- For two functions $f(q,p)$ and $g(q,p)$, the Poisson bracket is defined as

$$\{f,g\} = \sum_i \left( \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

where for $i = 1, \dots, n$,
$q_i$ and $p_i$ are the generalized coordinates and momenta

## Characteristics of Poisson Bracket

- Anti-symmetry: $\{f,g\} = -\{g,f\}$
- Linearity: $\{af + bg,\ h\} = a\{f,h\} + b\{g,h\}$
- Leibniz Rule: $\{q_i, p_j\} = \delta_{ij}$

where $a$ and $b$: constants
where $\delta_{ij}$: Kronecker-delta function

## Poisson Bracket and Hamiltonian Dynamics

- For any function $f(q,p)$ in phase space, its time evolution is

$$\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$$

NOTE: The Poisson bracket is the classical analogy of the commutator in quantum mechanics

# Brief Review of Hamiltonian Mechanics: (5) Canonical Transformation

## Definition of Canonical Transformation

- A generalized coordinate transformation that changes the generalized coordinates $q_i$ and conjugate momenta $p_i$ in $\mathcal{H}$ to a new set of coordinates $Q_i$ and momenta $P_i$ in $\mathcal{H}'$, which satisfy canonical relations
- For generating function $F$ (mapping function), the transformed Hamiltonian is

$$\mathcal{H}'(Q_1, \dots, Q_n, P_1, \dots, P_n) = \mathcal{H}(q_1, \dots, q_n, p_1, \dots, p_n) + \frac{\partial F}{\partial t}$$

## Formula of Canonical Transformation

- Type 1: $F_1(q, Q, t)$

$$p_i = -\frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$$

Trivial example of $F_1 = \sum q_i Q_i$ where $Q_i = p_i$ and $P_i = -q_i$

- Type 2: $F_2(q, P, t) - \sum Q_i P_i$

$$p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_2}{\partial P_i}$$

Trivial example of $F_2 = \sum q_i P_i$ where $Q_i = q_i$ and $P_i = p_i$

- Type 3: $F_3(p, Q, t) + \sum q_i p_i$

$$q_i = -\frac{\partial F_3}{\partial p_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$$

Trivial example of $F_3 = \sum p_i Q_i$ where $Q_i = -q_i$ and $P_i = -p_i$

- Type 4: $F_4(p, P, t) + \sum q_i p_i - \sum Q_i P_i$

$$q_i = -\frac{\partial F_4}{\partial p_i}, Q_i = \frac{\partial F_4}{\partial P_i}$$

Trivial example of $F_4 = \sum p_i P_i$ where $Q_i = p_i$ and $P_i = -q_i$

# Brief Review of Hamiltonian Mechanics: (6) Simple Harmonic Oscillator

## Example of Canonical Transformation

- Kinetic energy $T$, potential energy $U$, and Lagrangian $\mathcal{L}$ of the system are

$$T = \frac{1}{2}m\dot{x}^2, U = \frac{1}{2}kx^2, \mathcal{L}(x,\dot{x}) = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

- Canonical momentum $p$ and Hamiltonian $\mathcal{H}$ of the system are

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad \mathcal{H} = p\dot{x} - \mathcal{L},$$

$$\therefore \mathcal{H}(x,p) = \frac{kx^2}{2} + \frac{p^2}{2m} = \frac{m\omega^2 x^2}{2} + \frac{p^2}{2m}$$

where $\omega \equiv \sqrt{\frac{k}{m}}$ oscillator's eigenfrequency

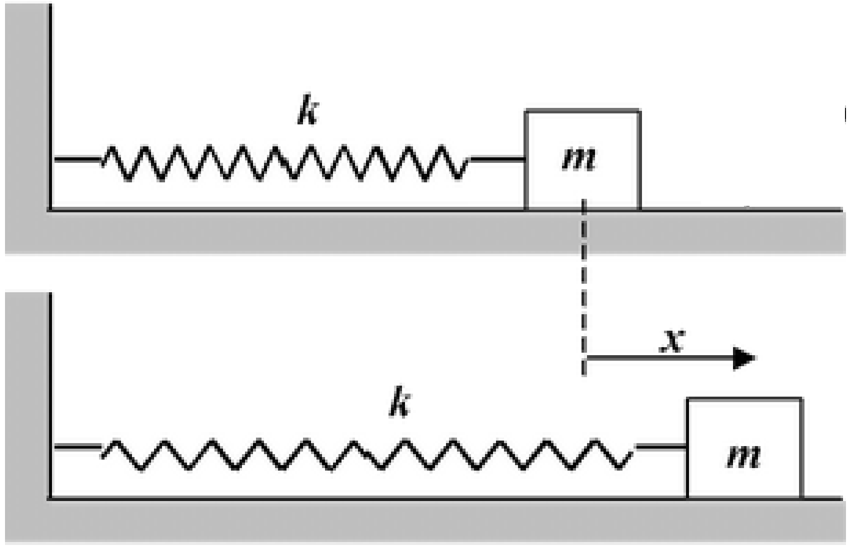



Fig. Motion of mass on a spring (harmonic oscillator)

- Let the generating function $F_1(q,Q) = \frac{m\omega q^2}{2}\cot Q$,

$$p = \frac{\partial F_1(q,Q)}{\partial q_i} = m\omega q \cot Q,$$

$$P = -\frac{\partial F_1(q,Q)}{\partial Q} = \frac{m}{2}\frac{\omega q^2}{\sin^2 Q},$$

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q,$$

$$p = \sqrt{2m\omega P}\cos Q,$$

$$\mathcal{H}'(P,Q) = \omega P(\cos^2 Q + \sin^2 Q)$$

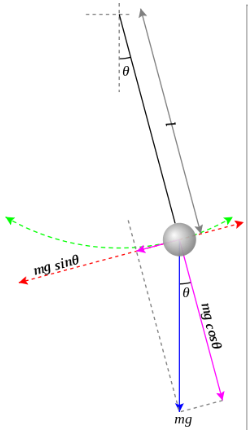
$$\boxed{\therefore \mathcal{H}' = \omega P}$$

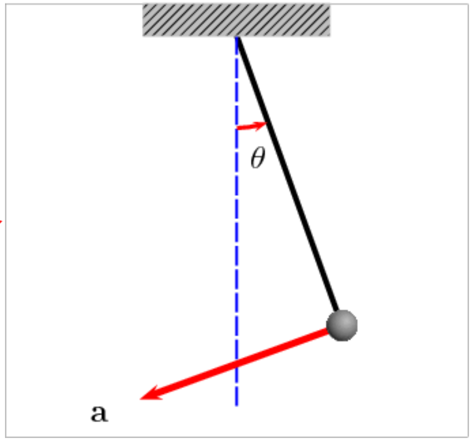
Implies that $Q$ is a cyclic coordinate for $\mathcal{H}'$ and the Hamiltonian means the total energy

NOTE: The above generating function will be utilized to solve quantum mechanical harmonic oscillator

# Brief Review of Classical Mechanics: Simple Pendulum Example

## Newtonian Mechanics




Newton's Force equation

- Equation of motion: $F = ma$

$$\therefore F = -mg\sin\theta$$

- From force vector analysis,

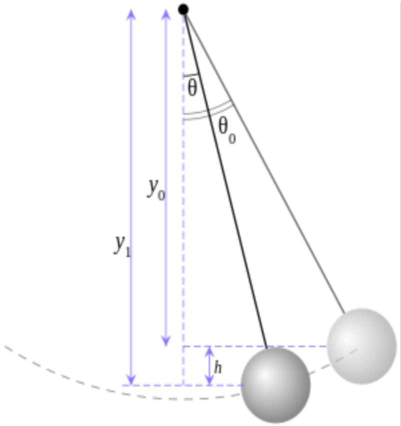
$$s = l\theta,$$

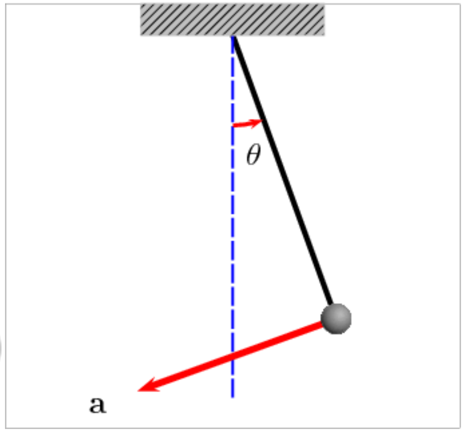
$$v = \frac{ds}{dt},$$

$$a = \frac{d^2s}{dt^2} = l\ddot{\theta}$$

$$\therefore l\ddot{\theta} = -g\sin\theta$$

## Lagrangian Mechanics




Generalized coordinate $\theta$ and velocity $\dot{\theta}$

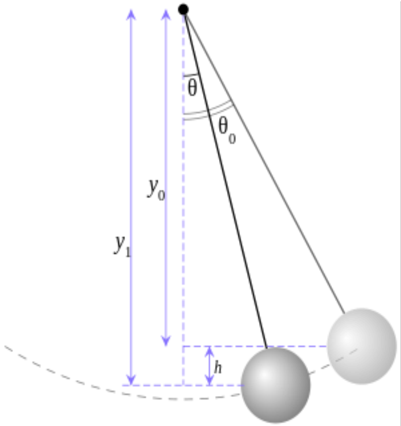
- Potential energy: $U = -mgl\cos\theta$
- Kinetic energy: $T = \frac{1}{2}m\left(l\dot{\theta}\right)^2$
- Lagrangian: $\mathcal{L} = T - U$

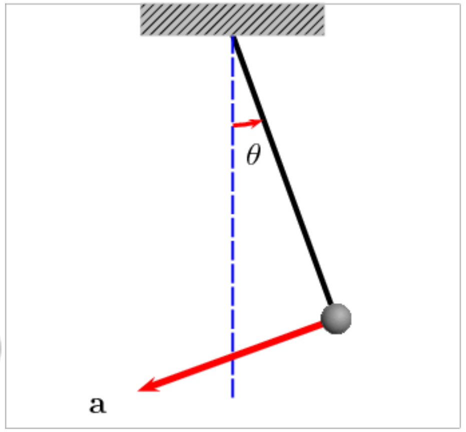
$$\mathcal{L} = \frac{1}{2}m\left(l\dot{\theta}\right)^2 + mgl\cos\theta$$

- The Euler-Lagrangian equation: $\frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = 0$

$$\therefore g\sin\theta + l\ddot{\theta} = 0$$

## Hamiltonian Mechanics




From the Lagrangian,

- Canonical conjugate momentum: $p = \frac{\partial\mathcal{L}}{\partial\dot{q}} = ml^2\dot{\theta}$
- Legendre transformation: $\mathcal{H} = \sum_{i=1} p_i\dot{q}_i - \mathcal{L}$

$$\mathcal{H} = ml^2\dot{\theta}^2 - \mathcal{L}$$

$$= \frac{1}{2}m\left(l\dot{\theta}\right)^2 - mgl\cos\theta$$

- From the Hamiltonian's characteristic: $-\dot{p} = \frac{\partial\mathcal{H}}{\partial q}$

$$-ml^2\ddot{\theta} = \frac{\partial\mathcal{H}}{\partial\theta}$$

$$\therefore g\sin\theta + l\ddot{\theta} = 0$$

# See Also…

## Textbooks:

[1] Grant R. Fowles and George L. Cassiday, *Analytical Mechanics*, Cengage Learning, 2004. * recommended

## Open Courses:

[1] Iain W. Stewart, *Advanced Classical Mechanics*, MIT OCW, 2014. [Online Available]
https://ocw.mit.edu/courses/8-09-classical-mechanics-iii-fall-2014/

[2] Sunil Golwala, *Lecture Notes on Classical Mechanics for Physics 106ab*, 2007 [Online Available]
https://sites.astro.caltech.edu/~golwala/ph106ab/ph106ab_notes.pdf