Derivation of Bicubic Interpolation Polynomial

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The BIMOND algorithms, given function values $f(x_i, y_j)$ for the input values x_i, $i = 1, \ldots, n_x$, and y_j, $j = 1, \ldots, n_y$, provides the values of the derivatives f_x, f_y, and f_{xy} at these input points. It is clear that we can write [1] the interpolated surface and its derivatives as

$$
\begin{aligned}
f(x, y) &= \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j \\
f_x(x, y) &= \sum_{i=1}^{3} \sum_{j=0}^{3} a_{ij} i x^{i-1} y^j \\
f_y(x, y) &= \sum_{i=0}^{3} \sum_{j=1}^{3} a_{ij} x^i j y^{j-1} \\
f_{xy}(x, y) &= \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} i x^{i-1} j y^{y-1}
\end{aligned}
$$

Let the corners of the interpolated surface be at $f(0, 0)$, $f(h, 0)$, $f(0, k)$, and $f(h, k)$; let the coefficients and function values be written as

$$
\begin{aligned}
\alpha =& (a_{00} \quad a_{10} \quad a_{20} \quad a_{30} \quad a_{01} \quad a_{11} \quad a_{21} \quad a_{31} \quad a_{02} \quad a_{12} \quad a_{22} \quad a_{32} \quad a_{03} \quad a_{13} \quad a_{23} \quad a_{33})^T \\
x =& (f(0,0) \quad f(h,0) \quad f(0,k) \quad f(h,k) \quad f_x(0,0) \quad f_x(h,0) \quad f_x(0,k) \quad f_x(h,k) \\
& f_y(0,0) \quad f_y(h,0) \quad f_y(0,k) \quad f_y(h,k) \quad f_{xy}(0,0) \quad f_{xy}(h,0) \quad f_{xy}(0,k) \quad f_{xy}(h,k))^T
\end{aligned}
$$

such that we can write the derivatives and function values as a system of equations $A\alpha = x$, where

$$
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & h & h^2 & h^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & k & 0 & 0 & 0 & k^2 & 0 & 0 & 0 & k^3 & 0 & 0 & 0 \\
1 & h & h^2 & h^3 & k & hk & h^2k & h^3k & k^2 & hk^2 & h^2k^2 & h^3k^2 & k^3 & hk^3 & h^2k^3 & h^3k^3 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2h & 3h^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & k & 0 & 0 & 0 & k^2 & 0 & 0 & 0 & k^3 & 0 & 0 \\
0 & 1 & 2h & 3h^2 & 0 & k & 2hk & 3h^2k & 0 & k^2 & 2hk^2 & 3h^2k^2 & 0 & k^3 & 2hk^3 & 3h^2k^3 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & h & h^2 & h^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2k & 0 & 0 & 0 & 3k^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & h & h^2 & h^3 & 2k & 2hk & 2h^2k & 2h^3k & 3k^2 & 3hk^2 & 3h^2k^2 & 3h^3k^3 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2h & 3h^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2k & 0 & 0 & 0 & 3k^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2h & 3h^2 & 0 & 2k & 4hk & 6h^2k & 0 & 3k^2 & 6hk^2 & 9h^2k^2
\end{bmatrix}
$$

References

[1] *Bicubic interpolation.* (2015, August 4). Retrieved from `https://en.wikipedia.org/wiki/Bicubic_interpolation`.