Homework 1

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Q1.

Code:

```
import random

def estimate_pi(num_samples=1000000):
    inside_circle = 0
    for _ in range(num_samples):
        x = random.random()
        y = random.random()
        if x**2 + y**2 <= 1:
            inside_circle += 1
    return 4 * inside_circle / num_samples

print("Estimated π:", estimate_pi(1000000))
```

Output:

```
Estimated π: 3.13734
```

Q2.

Homework 1

Q2.

a.

$$\mathrm{Cov}(x+y) = E\left[\left((x+y) - (\mu_x + \mu_y)\right)\left((x+y) - (\mu_x + \mu_y)\right)^T\right]$$

$$= E[(x-\mu_x)(x-\mu_x)^T] + E[(y-\mu_y)(y-\mu_y)^T]$$

$$+ E[(x-\mu_x)(y-\mu_y)^T] + E[(y-\mu_y)(x-\mu_x)^T]$$

$$= \Sigma_{xx} + \Sigma_{yy} + K_{xy} + K_{yx}$$

where $K_{yx} = K_{xy}^T$

b.

$$K_{xy} = E[xy^T] - \mu_x \mu_y^T$$

If x and y are independent,

then $E[xy^T] = E[x]E[y]^T = \mu_x \mu_y^T$

so $K_{xy} = 0_{m \times n}$

c.

Combining (a) and (b): for independent x and y,

$$\mathrm{Cov}(x+y) = \Sigma_{xx} + \Sigma_{yy}$$

Q3.

Q3.

a. Let X = number of sensors that detect the object

$X \sim \text{Binomial}(n=10, p=0.1)$

$\therefore P(X=k) = \binom{10}{k}(0.1)^k(0.9)^{10-k}, \quad k=0,1,2,\ldots,10$

b.

$P(x \geq 1) = 1 - P(x=0)$

$\because P(x=0) = (0.9)^{10}$

$\therefore P(x \geq 1) = 1 - (0.9)^{10}$

$(0.9)^{10} \approx 0.3487, \quad 1 - 0.3487 \approx 0.6513$

Q4.

Q4.

Let $x = (x_1, x_2)^T$, $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\Sigma = R \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix} R^T = \begin{bmatrix} \cos^2\theta + 0.25\sin^2\theta & 0.75\sin\theta\cos\theta \\ 0.75\sin\theta\cos\theta & \sin^2\theta + 0.25\cos^2\theta \end{bmatrix}$$

$\because \cos 30° = \frac{\sqrt{3}}{2}$, $\sin 30° = \frac{1}{2}$

$$\therefore \Sigma = \begin{bmatrix} 0.8125 & 0.3247595 \\ 0.3247595 & 0.4375 \end{bmatrix}, \ |\Sigma| = 0.25$$

pdf of $x \sim N(\mu, \Sigma)$ is

$$f(x) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right] = \frac{1}{\pi} \exp\left[-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right]$$

Let $z = R^T(x-\mu)$, then

$$f(x) = \frac{1}{\pi} \exp\left[-\frac{1}{2}(z_1^2 + 4z_2^2)\right], \quad \Sigma^{-1} = R\,\mathrm{diag}(1,4)\,R^T$$

Q5.

Q5.

Assume probability of boy $= p = \frac{1}{2}$, girl $= q = \frac{1}{2}$

X = number of family

$\therefore P(X=k) = (q)^{k-1} p \quad , k = 1, 2, 3 \ldots$

That is

$P(X=k) = (\frac{1}{2})^{k-1} (\frac{1}{2}) = (\frac{1}{2})^{k}$

$\because E[X] = \frac{1}{p}$

$\because p = \frac{1}{2} \quad \therefore E[X] = \frac{1}{\frac{1}{2}} = 2$