Ito's process

$$\begin{aligned}
&\int_0^t A_u dW_u \iff A_t dW_t \\
&\int_0^t B_u du \iff B_t dt \\
&X_t = \underbrace{X_0 + \int_0^t A_u du + \int_0^t B_u dW_u}_{\text{Full form}} \\
&dX_t = \underbrace{A_t dt + B_t dW_t}_{\text{Short form}}
\end{aligned} \tag{1}$$

$\int_0^t dW_u = W_t \quad \int_0^t W_u dW_u = \frac{W_t^2 - t}{2}$

Properties

I_t - martingale, $\Delta \geq 0 \implies \mathbb{E}\left(I_{t+\Delta} | \mathbb{F}_t\right) = I_t$

$I_t = \int_0^t A_u dW_u \quad J_t = \int_0^t B_u dW_u$

$$\begin{aligned}
&\mathbb{C}\text{ov}\left(I_t, J_t\right) = \int_0^t \mathbb{E}\left(A_u, B_u\right) du \\
&\mathbb{V}\text{ar}\left(I_t\right) = \int_0^t \mathbb{E}\left(A_u^2\right) du \\
&\mathbb{E}\left(\int_0^t A_u du\right) = \int_0^t \mathbb{E}\left(A_u\right) du \\
&\mathbb{E}\left(\int_0^t A_u dW_u\right) = 0
\end{aligned} \tag{2}$$

Ito's Lemma

$$\begin{aligned}
&dX_t = A_t dt + B_t dW_t \\
&Y_t = f(X_t, t, W_t) \\
&\implies \\
&dY_t = \frac{\delta f}{\delta t} dt + \frac{\delta f}{\delta W} dW_t + \frac{\delta f}{\delta X} dX_t + \frac{1}{2}\left[\frac{\delta^2 f}{\delta X^2}(dX_t)^2 + \frac{\delta^2 f}{\delta W^2}(dW_t)^2 + 2\frac{\delta^2 f}{\delta X \delta W} dX_t dW_t\right] \\
&\begin{cases} dW_t \cdot dW_t = dt \\ dt \cdot dt = 0 \\ dW_t \cdot dt = 0 \end{cases}
\end{aligned} \tag{3}$$

Black-Scholes model

$$\begin{cases} dS_t = \mu S_t dt + \delta S_t dW_t \\ dB_t = r B_t dt \end{cases} \quad \begin{cases} S_t = S_0 \cdot e^{\mu t} \cdot e^{\delta W_t - \frac{\delta^2}{2} t} \\ B_t = B_0 \cdot e^{rt} \end{cases} \tag{4}$$

Discounting

$$X_0 = e^{-Tr} \mathbb{E}\left(X_T\right) \tag{5}$$

Limits in L2

Series $R_n \xrightarrow{L^2} R$ if

$$\begin{aligned}
&\begin{cases} \forall n : \mathbb{E}\left(R_n^2\right) < +\infty \\ \lim\limits_{n \to +\infty} \mathbb{E}\left((R_n - R)^2\right) = 0 \end{cases} \\
&\begin{cases} \mathbb{E}\left(R_n\right) \to \mu \\ \mathbb{V}\text{ar}\left(R_n\right) \to 0 \end{cases} \implies \begin{cases} R_n \xrightarrow{L^2} \mu \\ R_n \xrightarrow{L^1} \mu \\ R_n \xrightarrow{P} \mu \end{cases}
\end{aligned} \tag{6}$$