

Ito's process

$$\begin{aligned}
\int_0^t A_u dW_u &\iff A_t dW_t \\
\int_0^t B_u du &\iff B_t dt \\
X_t &= X_0 + \underbrace{\int_0^t A_u du + \int_0^t B_u dW_u}_{\text{Full form}} \\
dX_t &= \underbrace{A_t dt + B_t dW_t}_{\text{Short form}} \\
\int_0^t dW_u &= W_t \quad \int_0^t W_u dW_u = \frac{W_t^2 - t}{2}
\end{aligned} \tag{1}$$

Ito's Lemma

$$\begin{aligned}
dX_t &= A_t dt + B_t dW_t \\
Y_t &= f(X_t, t, W_t) \\
&\implies \\
dY_t &= \frac{\delta f}{\delta t} dt + \frac{\delta f}{\delta W} dW_t + \frac{\delta f}{\delta X} dX_t + \frac{1}{2} \left[\frac{\delta^2 f}{\delta X^2} (dX_t)^2 + \frac{\delta^2 f}{\delta W^2} (dW_t)^2 + 2 \frac{\delta^2 f}{\delta X \delta W} dX_t dW_t \right] \\
&\begin{cases} dW_t \cdot dW_t = dt \\ dt \cdot dt = 0 \\ dW_t \cdot dt = 0 \end{cases}
\end{aligned} \tag{3}$$

Black-Scholes model

$$\begin{cases} dS_t = \mu S_t dt + \delta S_t dW_t \\ dB_t = r B_t dt \end{cases} \implies \begin{cases} S_t = S_0 \cdot e^{\mu t} \cdot e^{\delta W_t - \frac{\delta^2}{2} t} \\ B_t = B_0 \cdot e^{rt} \end{cases} \tag{4}$$

Discounting

$$X_0 = e^{-Tr} \mathbb{E}(X_T) \tag{5}$$

Properties

I_t - martingale, $\Delta \geq 0 \implies \mathbb{E}(I_{t+\Delta} | \mathbb{F}_t) = I_t$

$$I_t = \int_0^t A_u dW_u \quad J_t = \int_0^t B_u dW_u$$

$$\mathbb{Cov}(I_t, J_t) = \int_0^t \mathbb{E}(A_u, B_u) du$$

$$\mathbb{Var}(I_t) = \int_0^t \mathbb{E}(A_u^2) du \tag{2}$$

$$\mathbb{E}\left(\int_0^t A_u du\right) = \int_0^t \mathbb{E}(A_u) du$$

$$\mathbb{E}\left(\int_0^t A_u dW_u\right) = 0$$

Limits in L2

$$\begin{cases} \mathbb{E}(R_n) \rightarrow \mu \\ \mathbb{Var}(R_n) \rightarrow 0 \end{cases} \implies \begin{cases} R_n \xrightarrow{I^2} \mu \\ R_n \xrightarrow{L'} \mu \\ R_n \xrightarrow{P} \mu \end{cases} \tag{6}$$