Elements of Econometrics. 2022-2023.
Class 3. Properties of Regression Estimators

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Let $Y_i = \beta_1 + \beta_2 X_i + u_i$ as we know $b_2 = \hat{\beta}_2 = \dfrac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$

Decomposition $b_2 = \beta_2 + \dfrac{\sum(X_i - \bar{X})(u_i - \bar{u})}{\sum(X_i - \bar{X})^2} = \beta_2 + \sum a_i u_i$

Where $a_i = \dfrac{X_i - \bar{X}}{\sum_{j=1}^{n}(X_j - \bar{X})^2}$ are ***'magic coefficients'***

the properties of 'magic coefficients'

1) $\sum_{i=1}^{n} a_i = 0$

2) $\sum_{i=1}^{n} a_i^2 = \dfrac{1}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \dfrac{1}{\sum_{i=1}^{n} x_i^2}$ where $x_i = X_i - \bar{X}$

3) $\sum_{i=1}^{n} a_i X_i = 1$

Problem 2. Let the regression be $Y_t = \beta_1 + \beta_2 X_t + u_t;\ t = 1, 2, ..., T$ where $E(u_t) = 0;\ E(u_t^2) = \sigma^2$ and $E(u_s u_t) = 0$ if $s \neq t$. Prove that $b_2 = \hat{\beta}_2 = \dfrac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$ is unbiased estimator of β_2

Now the same in concise form $b_2 = \dfrac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$

Problem 3. Let the regression be $Y_t = \beta_1 + \beta_2 X_t + u_t;\ t = 1, 2, ..., T$ where $E(u_t) = 0;\ E(u_t^2) = \sigma^2$ and $E(u_s u_t) = 0$ if $s \neq t$. Derive formula for decomposition of regression estimator $b_2 = \dfrac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$ into fixed and random components: $b_2 = \beta_2 + \dfrac{\mathrm{Cov}(X,u)}{\mathrm{Var}(X)}$

Problem 4. Under the same assumptions prove that OLS estimator of the slope coefficient is unbiased.

Properties of estimators of the constant term

Problem 5. Demonstrate that $b_1 = \beta_1 + \sum_{i=1}^{n} c_i u_i$, where $c_1 = \dfrac{1}{n} - a_i \bar{X}$ and a_i is 'magic coefficients' (see above)

Problem 6 Prove that OLS estimator of the intercept b_1 is unbiased estimator of β_1.

Variance of regression estimators

Problem 7. Prove that the variance of OLS estimator of the slope coefficient is $\sigma_{b_2}^2 = \dfrac{\sigma_u^2}{\sum(X_i - \bar{X})^2}$

Problem 8. Prove that the variance of OLS estimator of the intercept is $\sigma_{b_1}^2 = \sigma_u^2 \left\{ \dfrac{1}{n} + \dfrac{\bar{X}^2}{\sum(X_i - \bar{X})^2} \right\}$