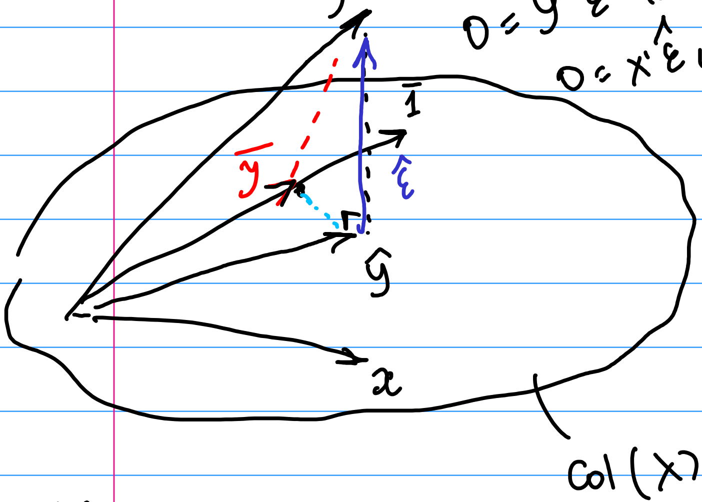
$$\begin{cases} y_1 = 1\cdot\beta_0 + \beta_1 x_{11} + \dots + \beta X_{k1} + \varepsilon_1 \\ \vdots \\ y_n = \dots \end{cases} \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{k1} \\ \vdots & & \ddots & \\ 1 & x_{1n} & & x_{kn} \end{bmatrix} \qquad \beta = \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_k \end{bmatrix}$$

$0 = 1'\hat{\varepsilon} = \Sigma\hat{\varepsilon}$ (a)

$0 = \hat{y}'\hat{\varepsilon}$ (d)

$0 = X'\hat{\varepsilon}$ (c)

$$y = X\beta + \varepsilon \qquad \varepsilon \sim Norm(0, \sigma_\varepsilon^2 \cdot \mathbb{1})$$



$$y_i = \beta_0 + \beta_1 x_{1i} + \dots + \beta_k x_{ki} + \varepsilon_i$$

$$\hat{\varepsilon} = y - \hat{y}$$

$$\hat{y} = \underset{\hat{\beta}}{\arg\min} \|y - \hat{y}\|^2 = \underset{\hat{\beta}}{\arg\min} \|\hat{\varepsilon}\|^2$$

$$a'b = 0 \iff \langle a, b\rangle = 0 \iff a \perp b$$

$$X'\hat{\varepsilon} = 0$$

$$X'(y - \hat{y}) = 0$$

$$X'(y - X\hat{\beta}) = 0$$

$$X'y = X'X\hat{\beta} \qquad \hat{\beta} = (X'X)^{-1}X'y$$

$$Var(\hat{\beta}) = \sigma^2 (X'X)^{-1}$$

- $y_i = \beta x_i + \varepsilon_i \qquad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\hat{\beta} = \left((x_1 \dots x_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right)^{-1} (x_1 \dots x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \frac{\Sigma x_i y_i}{\Sigma x_i^2}$$

- $y_i = \alpha + \varepsilon_i \qquad X = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$$\hat{\alpha} = \frac{\Sigma y_i}{n} = \bar{y}$$

$$\hat{\beta} = \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\Sigma (x_i - \bar{x})^2}$$

Problem 1 $y_i = \alpha + \beta x_i + \varepsilon_i$

a) $\frac{1}{n} \sum \hat{\varepsilon}_i = 0$

$$\sum \hat{\varepsilon}_i = \sum (y_i - (\hat{\alpha} + \hat{\beta} x)) =$$

$$= \sum (y_i - (\bar{y} - \hat{\beta}\bar{x} + \hat{\beta} x_i)) =$$

$$= \underbrace{\sum (y_i - \bar{y})}_{0} - \hat{\beta} \underbrace{\sum (x_i - \bar{x})}_{0} = 0$$

$$\sum y_i = n \cdot \bar{y}$$

b) $1/n \sum y_i = 1/n \sum \hat{y}_i$

$$\sum y_i = \sum (\hat{y}_i + \hat{\varepsilon}_i) = \sum \hat{y}_i + \underbrace{\sum \hat{\varepsilon}_i}_{0} = \sum \hat{y}_i$$

c) $\frac{1}{n} \sum X_i \cdot \hat{\varepsilon}_i = 0$ $\qquad \hat{E}(X \hat{\varepsilon}) = \hat{Cov}(x, \hat{\varepsilon})$

$$\sum \hat{\varepsilon}_i X_i = \sum \hat{\varepsilon}_i (x_i - \bar{x}) =$$

$\{ \bar{x} \sum \hat{\varepsilon}_i = 0 \}$

$$= \sum \Big((y_i - \bar{y}) - \hat{\beta} (x_i - \bar{x}) \Big) (x_i - \bar{x}) =$$

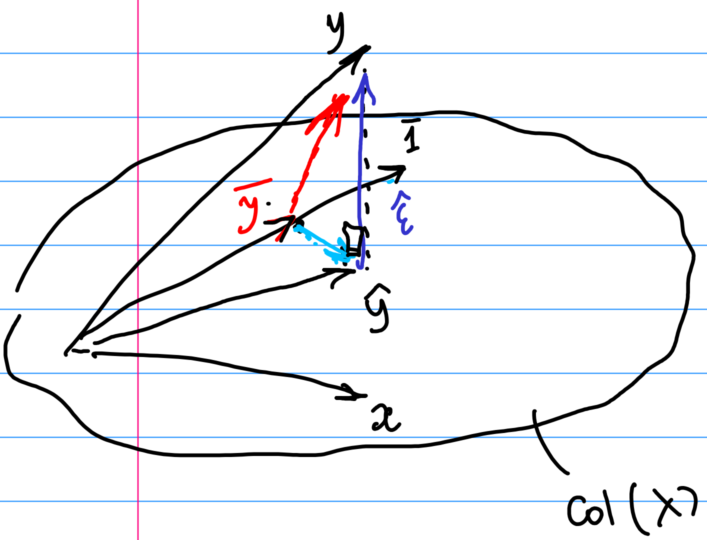
$$= \sum (y_i - \bar{y})(x_i - \bar{x}) - \hat{\beta} \cdot \sum (x_i - x)^2 = 0$$

$$\hat{\beta} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

d) $\frac{1}{n} \sum \hat{y}_i \hat{\varepsilon}_i = 0$

$$\sum (\hat{\alpha} + \hat{\beta} x_i) \hat{\varepsilon}_i = \alpha \underbrace{\sum \hat{\varepsilon}_i}_{0} + \hat{\beta} \underbrace{\sum x_i \hat{\varepsilon}_i}_{0} = 0$$

e)



$TSS = ESS + RSS$

$\|\hat{\varepsilon}\|^2 = RSS = \sum \hat{\varepsilon}_i^2$

$\|\hat{y} - \bar{y}\|^2 = ESS = \sum (\hat{y}_i - \bar{y})^2$

$\|y - \bar{y}\|^2 = TSS = \sum (y_i - \bar{y})^2$

by Pythagorean Thm

$$TSS = \sum (y_i - \bar{y})^2 = \sum \left((y_i - \hat{y}_i) + (\hat{y}_i - \bar{y})\right)^2 =$$

$$\underbrace{\sum (y_i - \hat{y}_i)^2}_{RSS} + \underbrace{\sum (\hat{y}_i - \bar{y})^2}_{ESS} + 2 \cdot \sum (y_i - \hat{y}_i)(\hat{y}_i - \bar{y}) =$$

$$RSS + ESS + 2 \underbrace{\sum \hat{\varepsilon}_i \cdot \hat{y}_i}_{0} = RSS + ESS$$

$$R^2 = \frac{ESS/n}{TSS/n} \quad \begin{matrix} - \ Var(\hat{y}) \\ - \ Var(y) \end{matrix}$$

explained variance

a) $R^2 = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS}$ — $Var(\hat{\varepsilon})$ – unexplained variance

b) $R^2 = \frac{ESS}{TSS} = \hat{\rho}^2_{x,y}$

$$R^2 = \frac{\Sigma(\hat{y}_i - \bar{y})^2}{\Sigma(y_i - \bar{y})^2} = \frac{\Sigma(\hat{\alpha} + \hat{\beta} x_i - \bar{y})^2}{\Sigma(y_i - \bar{y})^2} =$$

$$= \frac{\Sigma(\bar{y} - \hat{\beta}\bar{x} + \hat{\beta} x_i - \bar{y})^2}{\Sigma(y_i - \bar{y})^2} = \frac{\hat{\beta}^2 \cdot \Sigma(x_i - \bar{x})^2}{\Sigma(y_i - \bar{y})^2} =$$

$$= \frac{\left(s_{x,y}/s_x^2\right)^2 \cdot s_x^2}{s_y^2} = \frac{s_{x,y}^2}{s_x^2 \, s_y^2} = \hat{\rho}^2_{x,y}$$

$R^2 \in [0, 1]$

1) $\hat{\beta} = 0$ $\quad$ $ESS = 0$ $\quad$ $R^2 = 0$

2) X_i fit perfectly y_i: $\quad$ $RSS = 0$ $\quad$ $R^2 = 1$

Problem 3

$$y_i = \alpha + \beta x_i + \varepsilon_i \qquad \hat{\alpha}, \hat{\beta} - \text{OLS}$$

$$\hat{\beta} = \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\Sigma (x_i - \bar{x})^2} \qquad \tilde{\beta} - \text{OLS}$$

(Assume $\alpha = 0$)

$$RSS = \Sigma \hat{\varepsilon}_i^2 = \Sigma (y_i - \tilde{\beta} \cdot x_i)^2 \rightarrow \min_{\tilde{\beta}}$$

$$\frac{\partial RSS}{\partial \tilde{\beta}} = 2 \Sigma (y_i - \tilde{\beta} x_i) \cdot (-x_i) = 0$$

$$\Sigma (y_i - \tilde{\beta} x_i) x_i = 0$$

$$\Sigma y_i x_i = \tilde{\beta} \Sigma x_i^2 \qquad \tilde{\beta} = \frac{\Sigma x_i y_i}{\Sigma x_i^2}$$

$$E(\tilde{\beta}) = E\left(\frac{\Sigma x_i y_i}{\Sigma x_i^2}\right) = \frac{\Sigma E(x_i y_i)}{\Sigma x_i^2} = \frac{\Sigma x_i E(y_i)}{\Sigma x_i^2} =$$

$$\frac{\Sigma x_i E(\alpha + \beta x_i + \varepsilon_i)}{\Sigma x_i^2} = \underbrace{\alpha \frac{\Sigma x_i}{\Sigma x_i^2}} + \beta + \frac{\Sigma x_i \cancel{E(\varepsilon_i)}^{0}}{\Sigma x_i^2}$$

$\tilde{\beta} \nearrow$ biased (else)

$\tilde{\beta} \searrow$ unbiased (if $\alpha = 0$ or $\Sigma x_i = 0$) $\qquad E(\tilde{\beta}) = \beta$

$$Var(\tilde{\beta}) < Var(\hat{\beta})$$