MiniQuiz 3.3: Divided Differences and Chebyshev Interpolation

Task 1: Completing the Divided Differences Table

The following data are given for a function $f(x)$ at four distinct nodes:

$$\begin{aligned} x_0 &= -0.3 \qquad & f(x_0) &= 1.2 \\ x_1 &= 0.0 \qquad & f(x_1) &= 1.6 \\ x_2 &= 0.4 \qquad & f(x_2) &= 2.1 \\ x_3 &= 0.7 \qquad & f(x_3) &= 2.8 \end{aligned}$$

- Construct the full divided-difference table using the recursive formulas:

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}, \qquad f[x_i, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

- Show each computation step and tabulate all divided differences up to $f[x_0, x_1, x_2, x_3]$.
- Write the resulting Newton interpolating polynomial $P_3(x)$.

Task 2: Chebyshev Interpolation of $\mathbf{f(x) = e^{-x^2}}$ — five nodes

Consider the function $f(x) = e^{-x^2}$ on the interval $[-1.2, 0.8]$.

1. Compute the $n = 5$ Chebyshev nodes on $[-1, 1]$:

$$t_j = \cos\left(\frac{(2j-1)\pi}{2n}\right), \qquad j = 1, 2, 3, 4, 5.$$

   Then rescale them to $[-1.2, 0.8]$ using the linear transformation:

$$x_j = \frac{(b-a)t_j + (a+b)}{2}, \qquad a = -1.2,\ b = 0.8.$$

   Order them as $x_0 < x_1 < x_2 < x_3 < x_4$ and report their numerical values to at least three decimal places.

2. Evaluate $f(x_i) = e^{-x_i^2}$ for each Chebyshev node and report values to three decimal places.
3. Construct the full divided-difference table for these five nodes (i.e., compute all first, second, third, and fourth divided differences). Show every algebraic step.
4. Write the Newton interpolating polynomial $P_4(x)$ (degree ≤ 4) in Newton form:

$$P_4(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, \ldots, x_4] \prod_{k=0}^{3} (x - x_k).$$

5. Evaluate $P_4(-0.5)$ and compare it with the exact value $f(-0.5) = e^{-(-0.5)^2}$. Compute the absolute error $|f(-0.5) - P_4(-0.5)|$.
6. Create a plot on $[-1.2, 0.8]$ that includes:
   - The exact function $f(x) = e^{-x^2}$ (smooth curve),
   - The five Chebyshev nodes $(x_j, f(x_j))$ marked distinctly,
   - The Newton interpolating polynomial $P_4(x)$ (separate curve; indicate in legend).

   Provide the plot with sufficiently fine sampling so differences between f and P_4 are visible.