书面作业6.1

第1部分 基础

第2部分 理论

T1 实数集R上的二元运算*：a*b=a+b-a∘ b，+、-、∘ 为一般的加法、减法、乘法运算，请问代数结构<R; *>是否有单位元、零元与幂等元，如果有单位元，哪些元素有逆元?

T2 证明：有限半群存在幂等元.

提示：注意元素"有限"、"结合律", 则可以构造出元素相等关系，从而可以进一步构造出幂等元.

T3 设h是代数结构V_1=<S；o>到V_2=<S′；o′>的同态映射，h的同态像为h(S)⊆ S′, 证明:

(1) <h(S)；o′>为V_2的子代数;

(2) h是V_1到<h(S)；o′>的满同态映射;

(3) 如果V_1关于运算o有单位元e或零元z，则同态像h(S)中有关于o′的单位元h(e)或零元h(z).

T4 设f，g都是< S；*>到< S′；*′>的同态，并且*′ 运算均满足交换律和结合律，证明:如下定义的函数h：S→S′：h(x)=f(x)*′g(x)是<S；*>到<S′；*′>的同态.

T5 给定代数结构 A=<X；∘ >、B=<Y；*> 和 C=<Z；×>.设 f：X→Y 是从A到B的同态，且 g：Y→Z 是从B到C的同态，试证明gof：X→Z必定是从A到C的同态, gof为函数f,g的复合.

T6 复数的加、乘运算可以转换为矩阵的加、乘运算，请从代数结构同构的角度进行证明.

提示：设复数的集合 C={a+bi|a+bi 为复数, a,b ∈ R}, 相应地可以定义 2×2 矩阵集合：

$$M=\left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| \ a, b\in R \right\}.$$

T7 代数结构间的同构关系是等价关系.

T8 已知代数结构<Z; +>以及<C; $+_3$>, 其中， Z为整数集合，C={0, 1, 2}. +，$+_3$为Z、C上的一般加法、加模3运算. 请定义<Z; +>到<C; $+_3$>的同态映射φ，并按照同态基本定理，构造相应同态三角形，并给出解释.

T9 If <A;+>is a algebraic structure, where the binary operation + is associative, and <A;+>has an identity, and its element has an inverse, then <A;+> is called a group(群).

A ring(环) is an algebra with the structure<A; +, *>, where <A;+> is a commutative group(交换群, i.e. , <A;+> is a group and the operation + is commutative) , <A; *> is a monoid (独异点/单位半群), and the operation * distributes over + from the left and the right（即*对+满足左/右分配律）.

If <A; +, *> is a ring with the additional property that <A – {0}; *>is a commutative group, then it's called a field(域). Finite field, also known as Galois Field(named after Evariste Galois), refers to a field in which there exists finitely many elements. The most popular and widely used application of Galois Field is in Cryptography(密码学). Since each byte of data are represented as a vector in a finite field, encryption and decryption（加密与解密） using mathematical arithmetic is very straightforward and is easily manipulable.

Now, let N_5 = {0, 1, 2, 3, 4}, and let $+_5$ and $*_5$ be the two operations of addition mod 5（加模 5 求余）and multiplication mod 5（乘模 5 求余）, respectively. Please show that <N_5; $+_5$, $*_5$>is a field.

T10 (定义满足某些性质的二元运算) Let A = {a, b}. For each of the following problems, find an operation table satisfying the given condition for a binary operation ∘ on A.

a. <A; ∘> is a group（群的定义请参考 T8）.

b. <A; ○> is a monoid but not a group.

c. <A; ○> is a semigroup(半群) but not a monoid.

T11 Show that there is an epimorphism(满同态) between the set B of binary numerals(二进制数) with the usual binary addition(一般二进制加法) defined on B and the set N of natural numbers with the usual addition on N.（提示：注意到二进制与十进制之间的对应关系）

T12 Find the three homomorphisms(定义 3 个同态映射) that exist from the algebra $<N_3; +_3>$ to the algebra $<N_6; +_6>$ where , where $+_3$,$+_6$ is the operation of addition mod 3 or 6. （提示：$+_3$,$+_6$是加模 3，加模 6 运算，注意定义需要满足同态方程）

T13 Suppose we need a function $f : N_8 \to N_8$ with the property that f (1) = 3; and also, f must be a homomorphism(同态) from the algebra $<N_8; +_8>$ to itself, where $+_8$ is the operation of addition mod 8. Please finish the definition of f. (提示：利用需要满足的同态方程来定义）

第3部分 综合应用