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Introduction

You've probably seen mathematical models of populations before. The old standbys, logistic and exponential equations, are useful because they capture how a population's rate of growth is dependent on its current conditions. If you want to know how many bacteria will be in a Petri dish ten hours from now, these models work fine. But consider the following questions:

- Is it more important to preserve a hawk population's nesting habitat or hunting ground?
- Which age groups contribute the most to Baltimore's birth rate?
- How does the size of fishing nets relate to the long-term stability of a haddock population?

Real-world populations of any kind — including humans — cannot be understood in a single number. To answer practical questions about changes in an organism's life cycle, scientists need to look at the internal structure of a population. Consider a population of butterflies. Some may be eggs, others caterpillars, some are pupae, and some are adults ready to breed. We can capture this information in a **population vector**, where the ith component is the number of organisms in the ith life stage. Our butterfly population vector might look like this:

$$\mathbf{p} = [1000, 500, 400, 200]$$

You could count the total population size as 2100 (all organisms), 1100 (excluding eggs), or 200 (only adults), depending on what question you are investigating.

It is a fact of life that not every organism survives to the next life stage. Some proportion of the eggs are not viable, some proportion of the caterpillars will be eaten by birds, and so on. **Stage-structured matrix population models** allow scientists to investigate how these factors interact with each other to contribute to population changes over time ("population dynamics"). The idea works by modeling population change as... you guessed it, a system of linear equations!

Constructing the Transition Matrix

We are interested in modeling the population change between time steps t and $t+1$. We'll just count the female butterflies and assume that the total population is just double that. This simplifies the math since it means we can assume every organism is capable of reproducing. Consider the following system of equations where $p_i(t)$ is the ith component of the population vector at time t:

$p_2(t+1) = 0.5p_1(t)$ — Half of the eggs become caterpillars.
$p_3(t+1) = 0.75p_2(t)$ — Three quarters of the caterpillars pupate.
$p_4(t+1) = 0.5p_3(t)$ — Half of the pupae become adults.
$p_1(t+1) = (0.2 \cdot 30)p_4(t)$ — One fifth of the adult butterflies survive to reproduce, and those lay thirty eggs each.

The coefficients in each equation can be determined by observation in the wild. Note that the coefficient representing reproduction is usually combined into a single number called "fecundity" which in this case would equal 6.

The transition matrix L records the coefficients of the system so that it can be represented as a matrix equation $L(\mathbf{p}(t)) = \mathbf{p}(t+1)$, where $\mathbf{p}(t)$ is the population vector at time t. You can think of "transitioning" from one population vector to the next. (The L stands for either "Leslie" or "Lefkovitch," the people who are credited with developing the first matrix population models.)

Kinds of Matrix Model

There are two main ways of setting up matrix population models. In **stage-structured matrix models,** like the butterfly model, the components of the population vector represent the population at consecutive life stages. For a mammal, this might look like baby, subadult, and adult. In **age-structured matrix models**, the components represent the population at various ages, for example [0-2 years, 2-4 years, 4-6 years, 8+ years]. The mathematical tools underlying the models are the same, but notice that an organism cannot get younger even though some (like plants) can "revert" to earlier life stages. The choice of model to use will determine what the transition matrix looks like and the kinds of conclusions you can make.

This kind of modeling can be useful outside of ecology and demographics. Any phenomenon where the transition from one time-step to the next can be modeled as a system of linear equations can be modeled in this manner. Such models are commonly used in physics, economics, and sociology, to name just a few fields.

Project Scenario: Fisheries Management

Your government regulates offshore fishing to ensure that fish populations and the marine environment remain healthy and stable. Overfishing could lead to economic and ecological catastrophe, so it is important to have an accurate understanding of how many fish are alive and how they reproduce. The government has asked you to help analyze the results of a recent census of the cod population. Cod is a kind of fish that lives in the Atlantic and Pacific oceans and is economically important as a food source. Scientists have developed an age-structured population model which is represented by the following transition matrix:

$$\begin{bmatrix} 0 & 1 & 1.9 & 1.7 & 0 \\ 0.61 & 0 & 0 & 0 & 0 \\ 0 & 0.33 & 0 & 0 & 0 \\ 0 & 0 & 0.21 & 0 & 0 \\ 0 & 0 & 0 & 0.13 & 0.08 \end{bmatrix}$$

The five population categories are 0-2 years old, 2-4 years old, 4-6 years old, 6-8 years old, and older than 8 years. The time step of the model is 2 years.

Scientists have also sampled the population to determine an initial population vector of $[289, 211, 120, 76, 51]$.

Questions and Predictions

Your job is to prepare a report or presentation to help government officials understand and interpret the results of the cod census. The officials have questions that they need answered so that they can make informed decisions about proposed fishing technologies. These include:

- How should the entries in matrix L be interpreted?
- What is the predicted population 20 years from now?
- How well does the model work for estimating past populations? What was the approximate population 4 years ago?

A new fishing technology is proposed that would change the cod's survival rates and fecundity. The new transition matrix can be written as $L' = L – F$, where L is the old transition matrix and F is a matrix representing the effects of fishing:

$$\begin{bmatrix} 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.04 & 0 & 0 & 0 \\ 0 & 0 & 0.02 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- A lobbyist presents some calculations to you showing that the matrix power $F^3 = \mathbf{0}$. They argue that, because of this, the new fishing technology will not have an impact after six years (three time-steps in the model). Use matrix algebra to prove or disprove this line of reasoning.
- Propose a sustainable fishing regime by altering F so that the cod population remains stable for many years. About how many fish will be harvested every year under this plan?
- Make a line chart showing population projections for all three models over the next 30 years, and discuss the long-term behavior of the models.

Your presentation may also include other hypotheses, predictions, or questions that you think are relevant to your analysis.

Checkpoints

This project consists of material that you will learn throughout Units 1 and 2. You will have three checkpoints for this project; you must complete each checkpoint before you can move on to the next.

Project 1 Checkpoint 1

It is recommended that you complete Checkpoint 1 after completing the lesson and practice problems for Section 2.1. For this checkpoint, you are going to provide information about how you intend to present your project.

Project 1 Checkpoint 2

For Checkpoint 2, you will provide your instructor will information about your progress on the project by answering questions. It is recommended that you complete this checkpoint at the beginning of Unit 2. Please wait for feedback from your instructor before moving on to Checkpoint 3.

Project 1 Submission (Checkpoint 3)

For Checkpoint 3, you will submit your completed project and complete a self-reflection survey. This checkpoint should be completed at the end of Unit 2.

Instructions

Your project should consist of five parts: Introduction, Mathematical Methods, Analysis, Conclusion, and References. Refer to the **Project Resources** page for helpful information on data analysis, mathematical writing, and evaluating internet sources.

Introduction

Describe the scenario, how it is modeled, and what questions or hypotheses guided your investigation. Briefly summarize any major results from your analysis, such as long-term predictions for the fish population.

Mathematical Methods

In this section, provide detailed descriptions explaining each of the following topics to an audience that is familiar with linear algebra but not with matrix population models. You may find it useful to use specific examples of population models from outside your study

1. Matrix Population Models: What are population vectors and transition matrices? How do you make predictions with the model? How well can you use the models to look backward in time?
2. Comparisons with Other Models: What are the benefits and drawbacks of matrix population models as compared to other population models you've learned about in the past (such as exponential models)?
3. Matrix Operations: How should scalar multiples and matrix sums of transition matrices be interpreted? How can taking multiples and sums affect (or not affect) long-term behavior? Use specific examples. You may find it useful to reference the lobbyist's proposed fishing technology here.
4. Linear Transformations and Rank: If we understand the transition matrix as representing a linear transformation, what space does it transform? What is the significance of the rank of the transition matrix in terms of understanding population structure and long-term population dynamics?

Analysis

Using the mathematical techniques presented in the previous section, analyze the fisheries management scenario described in the Project Scenario: Fisheries Management above. Present evidence for all the claims you make. Your analysis should include responses to all of the prompts written in the scenario and may address additional questions you have posed yourself.

1. Present and interpret the transition matrix and the new fishing scheme.
2. Make projections forward and backward in time and discuss them.
3. Use algebra to analyze the lobbyist's reasoning.
4. Propose a new fishing regime.
5. Plot and compare the three models.

Conclusion

In this section, summarize your findings and recommendations. Then reflect on the model, addressing the following:

1. What phenomena relevant to the cod population does the matrix model leave out? If these factors were considered, how might they affect your conclusions?
2. What are the benefits and limitations of modeling populations in this way — put differently, what does this model allow you to "see" that you couldn't otherwise? What does the model "hide"?
3. Is the usefulness of this model dependent on computing power? How might the model be made more effective with the use of computers?

4. How might this model be used as part of a solution to problems of global importance? How might the use of this model exacerbate those problems?

References (Bibliography)

If writing a report in Microsoft Word, you can include this directly at the end of the report. If you are presenting your project as a PowerPoint, video, or some other format, you will need to upload your bibliography as a separate Microsoft Word document. Microsoft Word can manage your sources and create a bibliography for you or you can choose to create one yourself using any standard citation structure such as MLA, APA, or Chicago-style.

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