№1

(1) $U(x) = \frac{m\omega^2 x^2}{2} + \frac{m\beta x^4}{2}$

$U'(x) = m\omega^2 x + m\beta x^3 = 0$ $\qquad U_m(x_0) = \frac{m\omega^2}{2}\left(-\frac{\omega^2}{\beta}\right) + \frac{m\beta}{4}\frac{\omega^4}{\beta^2} =$

$x(\omega^2 + \beta x^2) = 0$ $\qquad = \frac{m\omega^4}{4\beta}$

$x^2 = -\frac{\omega^2}{\beta}$ $\qquad$ Условие применимости поправки: $E \ll \frac{m\omega^4}{4\beta}$

$x_0 = \pm\sqrt{\frac{\omega^2}{\beta}}$

$$\delta T = -\sqrt{2m}\frac{\partial}{\partial E}\int_{-\sqrt{\frac{2E}{m\omega^2}}}^{\sqrt{\frac{2E}{m\omega^2}}} \frac{m\beta x^4}{4}\frac{dx}{\sqrt{E - \frac{m\omega^2 x^2}{2}}} = \frac{-\sqrt{2m}\, m\beta}{4}\frac{\partial}{\partial E}\int_{-\#}^{\#}\frac{x^4\, dx}{\sqrt{E}\sqrt{1-\frac{m\omega^2}{2E}x^2}} =$$

$$= \underbrace{\frac{-2\sqrt{2m}\, m\beta}{4}}_{\psi}\frac{\partial}{\partial E}\int_0^{\#}\frac{x^4\, dx}{\sqrt{E}\sqrt{1-\frac{m\omega^2}{2E}x^2}} = \psi\frac{\partial}{\partial E}\int_0^{\#}\frac{1}{\sqrt{E}}\frac{1}{2}\left(\sqrt{\frac{2E}{m\omega^2}}\right)^5\frac{u^2\, du}{\sqrt{u}\sqrt{1-u}} =$$

$$= \psi\frac{\partial}{\partial E}\int_0^1 \frac{1}{2}\frac{4\sqrt{2}}{m^2\sqrt{m}\omega^5}E^2\, \underline{u^{3/2}(1-u)^{-1/2}\, du} =$$

$$B\left(\tfrac{5}{2};\tfrac{1}{2}\right) = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} = \frac{\frac{3}{4}\sqrt{\pi}\cdot\sqrt{\pi}}{2} = \frac{3}{8}\pi$$

$$= \frac{\psi\, 2E\, 4\sqrt{2}}{2\, m^2\sqrt{m}\omega^5}\cdot\frac{3}{8}\pi = \frac{-2\sqrt{2m}\, m\beta E\, 4\sqrt{2}}{4\, m^2\sqrt{m}\,\omega^5}\cdot\frac{3}{8}\pi = -\frac{3\beta E\pi}{2m\omega^5}$$

$$\delta T = -\frac{3\pi\beta E}{2m\omega^5}$$

(2) $U(x) = U_0\left(\frac{1}{e^{2ax}} - \frac{2}{e^{ax}}\right) - Ve^{ax} = U_0\left(\frac{1-2e^{ax}}{e^{2ax}}\right) - Ve^{ax}$

$$U'(x) = U_0\left(\frac{-2ae^{ax}e^{2ax} - (1-2e^{ax})2ae^{2ax}}{e^{4ax}}\right) - Vae^{ax} = 0$$

$$U_0\left(\frac{-2ae^{3ax} - 2ae^{2ax} + 4ae^{3ax}}{e^{4ax}}\right) - Vae^{ax} = 0$$

$$U_0\,\frac{2a(e^{3ax} - e^{2ax})}{e^{4ax}} = Vae^{ax}$$

$$2U_0\,\frac{e^{ax}-1}{e^{2ax}} = Ve^{ax}$$

$$2U_0\frac{e^{ax}-1}{e^{2ax}}=Ve^{ax}$$

$$2U_0(e^{ax}-1)=Ve^{3ax}$$

$$2(e^{ax}-1)=\frac{V}{U_0}e^{3ax}=0$$

т.к $\frac{V}{U_0}<<1$

$$e^{ax}=1$$

$$\left[\begin{array}{l}x=\frac{1}{a}\\ x=0\end{array}\right.$$

$$U_m\left(\frac{1}{a}\right)=U_0\left(\frac{1-2}{e^2}\right)-V=$$

$$=\frac{-U_0}{e^2}-V$$

$E>>-\frac{U_0}{e^2}-V$ - Граница применимости

$$\delta T=-\sqrt{2m}\frac{\partial}{\partial E}\int Ve^{ax}\frac{dx}{(E-U_m)^{1/2}}=-\sqrt{2m}V\frac{\partial}{\partial E}\int\frac{e^{ax}dx}{\left(E-U_0\left(\frac{1-2e^{ax}}{e^{2ax}}\right)\right)^{1/2}}$$

$$=-\sqrt{2m}V\frac{\partial}{\partial E}\int\frac{e^{2ax}dx}{(Ee^{2ax}+2U_0e^{ax}-U_0)^{1/2}}=-\sqrt{2m}V\cdot$$

$$\cdot\frac{\partial}{\partial E}\int\frac{1}{\sqrt{E}}\frac{e^{2ax}dx}{\left(e^{2ax}+\frac{2U_0}{E}e^{ax}-\frac{U_0}{E}\right)^{1/2}}=-\sqrt{2m}V\frac{\partial}{\partial E}\int\frac{1}{\sqrt{E}}\frac{e^{2ax}dx}{\left(\left(e^{ax}+\frac{U_0}{E}\right)^2-\frac{U_0}{E}-\frac{U_0^2}{E^2}\right)^{1/2}}=-\sqrt{2m}V\frac{\partial}{\partial E}\int\frac{1}{\sqrt{E}}\frac{e^{2ax}dx}{\left(\left(e^{ax}+\frac{U_0}{E}\right)^2-\left(\frac{U_0}{E}+\frac{U_0^2}{E^2}\right)\right)^{1/2}}$$

№2

$U(x) = U_0 - a|x|^n$ $\quad n > 2$, $a > 0$ $\qquad U(x) \approx E - a|x|^n + \varepsilon$

$\varepsilon = U_0 - E \qquad \varepsilon \ll U_0$

$$T \approx \sqrt{2m} \int \frac{dx}{\sqrt{E - U(x)}} = \sqrt{2m} \int \frac{dx}{\sqrt{a|x|^n - \varepsilon}} \ominus$$

$u = \frac{a|x|^n}{\varepsilon} \qquad x = \sqrt[n]{\frac{\varepsilon}{a}} \sqrt[n]{u}$

$$du = \frac{a}{\varepsilon} n x^{n-1} dx = \frac{a}{\varepsilon} n \frac{\frac{\varepsilon}{a} u}{\left(\frac{\varepsilon}{a}\right)^{1/n} u^{1/n}} dx = \cancel{n\left(\frac{\varepsilon}{a}\right)^{1/n} u^{1-\frac{1}{n}} dx} = \cancel{n\left(\frac{\varepsilon}{a}\right)^{1/n} u^{\frac{n-1}{n}} dx}$$

$$= n\left(\frac{a}{\varepsilon}\right)^{1/n} u^{1-\frac{1}{n}} dx$$

$$dx = \frac{du}{n\left(\frac{a}{\varepsilon}\right)^{1/n} u^{\frac{n-1}{n}}}$$

$$\ominus \frac{\sqrt{2m}}{\sqrt{\varepsilon}} \int \frac{du}{n\left(\frac{a}{\varepsilon}\right)^{1/n} u^{\frac{n-1}{n}} \sqrt{u-1}} = \frac{\sqrt{2m}}{n} \varepsilon^{-\frac{1}{2}+\frac{1}{n}} a^{-\frac{1}{n}} \underbrace{\int \frac{du}{u^{1-\frac{1}{n}} \sqrt{u-1}}}_{\text{Бета ф-ция}}$$

$T(\varepsilon) \sim \varepsilon^{-\frac{1}{2}+\frac{1}{n}}$

при $n = 2$ $\quad T(\varepsilon) \sim \varepsilon^{0}$

С лекционным случаем не сходится

$T \sim \log\left(\frac{1}{\varepsilon}\right) \leftarrow$ Из лекции

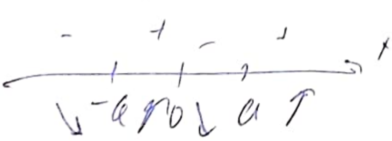
Результат не сходится т.к при $n=2$ интеграл расходится и посчитать его можно приближенно

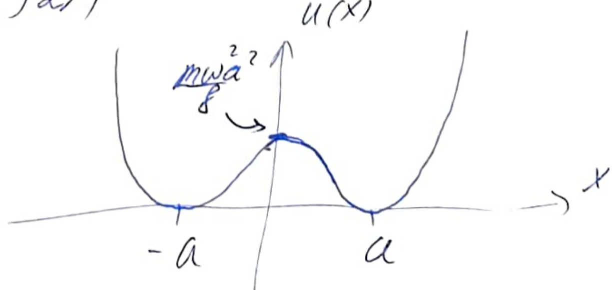
№3

$$U(x) = \frac{m\omega^2}{8a^2}(x^2 - a^2)^2$$

(1) $$U'(x) = \frac{m\omega^2}{8a^2}\left(2(x^2 - a^2)2x\right)$$

$$\left[\begin{array}{l} x = \pm a \\ x = 0 \end{array}\right.$$





$x = 0$ - максимум

$x = \pm a$ - минимумы

Если $E > 0$, то движение финитно

1) $E > \frac{m\omega^2 a^2}{8}$

2) $0 < E < \frac{m\omega^2 a^2}{8}$

Разные финитные движения

(2)

$$U(a+x) = U(a) + U'(a)x + \frac{U''(a)}{2}x^2$$

$$U'(a) = 0$$

$$U''(a) = \frac{m\omega^2}{2a^2}\left(x^3 - a^2x\right)'' = \frac{m\omega^2}{2a^2}\left(3x^2 - a^2\right) = \frac{m\omega^2}{2a^2}\cdot 2a^2 = m\omega^2$$

$$\Downarrow$$

$$U(a+x) \approx \frac{m\omega^2}{2}x^2 \leftarrow \text{Гармонический осциллятор}$$

$$\Downarrow$$

$$T = \frac{2\pi}{\omega}$$

(3)

$$U(x) = U(0) + U'(0)x + \frac{U''(0)}{2}x^2 = \frac{m\omega^2 a^2}{8} - \frac{m\omega^2}{4}x^2$$

$$U'(0) = 0$$

$$U''(0) = \frac{m\omega^2}{2a^2}(-a^2) = -\frac{m\omega^2}{2}$$

$$T = \sqrt{2m}\int\frac{dx}{\left(E + \frac{m\omega^2}{4}x^2 - \frac{m\omega^2 a^2}{8}\right)^{1/2}} = \sqrt{2m}\int\frac{dx}{\left(\frac{m\omega^2}{4}x^2 - \mathcal{E}\right)^{1/2}} \circleddash$$

$$\mathcal{E} = \frac{m\omega^2 a^2}{8} - E$$

$$\circleddash\ \frac{\sqrt{2m}}{\sqrt{\mathcal{E}}}\int\frac{dx}{\left(\frac{m\omega^2}{4\mathcal{E}}x^2 - 1\right)^{1/2}} = \sqrt{\frac{2m}{\mathcal{E}}}\int\frac{du}{2\sqrt{u}\left(\frac{m\omega^2}{4\mathcal{E}}\right)^{1/2}(u-1)^{1/2}} \circleddash$$

$$\frac{m\omega^2}{4\mathcal{E}}x^2 = u \qquad x = \sqrt{u}\left(\frac{4\mathcal{E}}{m\omega^2}\right)^{1/2}$$

$$\frac{m\omega^2}{4\mathcal{E}}2x\,dx = du$$

$$\cancel{du = \frac{m\omega^2}{4\mathcal{E}}\cdot} dx = \frac{du}{2x\frac{m\omega^2}{4\mathcal{E}}} = \frac{du}{2\sqrt{u}\left(\frac{4\mathcal{E}}{m\omega^2}\right)^{1/2}\frac{m\omega^2}{4\mathcal{E}}} =$$

$$= \frac{du}{2\sqrt{u}\left(\frac{m\omega^2}{4\mathcal{E}}\right)^{1/2}}$$

$$\circleddash\ \sqrt{\frac{2m}{\mathcal{E}}}\,\frac{1}{2}\left(\frac{4\mathcal{E}}{m\omega^2}\right)^{1/2}\int_{u_{turn}}^{u^*}\frac{du}{\sqrt{u}\,(u-1)^{1/2}} = \frac{1}{\omega}\int_{u_{turn}}^{u^*}\frac{du}{\underbrace{(u^2-u)^{1/2}}_{\text{полный квадрат}}} =$$

$$= \frac{1}{\omega}\int_{u_{turn}}^{u^*}\frac{du}{\left((u-\frac{1}{2})^2 - \frac{1}{4}\right)^{1/2}} = \frac{1}{\omega}\underbrace{\ln\left((u-\tfrac{1}{2}) + \sqrt{(u-\tfrac{1}{2})^2 - \tfrac{1}{4}}\right)}_{\operatorname{arccosh}\left(\frac{u}{2}\right)}\Bigg|_{u_{turn}}^{u^*} =$$

$$= \frac{1}{\omega}\operatorname{arccosh}\left(\frac{u}{2}\right)\Big|^{u}_{u_{turn}} = \frac{1}{\omega}\operatorname{arccosh}\left(x\sqrt{\frac{m\omega^2}{4E}}\right)\Big|^{x}_{x_{turn}}$$

Ответ также пропорционален $\frac{1}{\omega}$, но коэффициент будет задаваться точками поворота и зависеть от U_m, поэтому коэф не будет равным 2

(4)

$$U(x+a) = U(a) + U'(a)x + \underbrace{\frac{U''(a)}{2}x^2}_{\frac{m\omega^2}{2}x^2} + \underbrace{\frac{U'''(a)}{6}x^3}_{\frac{m\omega^2}{2a}\cdot 6a} = \frac{m\omega^2}{2}x^2 + \frac{m\omega^2}{2a}x^3$$

Ф-ция нечётная, поэтому нужно рассматривать 2-ую поправку по периоду

$$\delta T = \frac{\sqrt{2m}}{2}\frac{\partial^2}{\partial E^2}\boxed{\int\left(\frac{m\omega^2}{2a}x^3\right)^2\frac{dx}{\left(E-\frac{m\omega^2x^2}{2}\right)^{1/2}}}$$

$$\int\left(\frac{m\omega^2x^3}{2a}\right)^2\frac{dx}{\left(E-\frac{m\omega^2x^2}{2}\right)^{1/2}} = 2\frac{m^2\omega^4}{4a^2\sqrt{E}}\int\frac{x^6dx}{\left(1-\frac{m\omega^2}{2E}x^2\right)^{1/2}} =$$

$$= \frac{2m^2\omega^4}{4a^2\sqrt{E}}\left(\frac{2E}{m\omega^2}\right)^{7/2}\int_0^1\frac{x^6dx}{(1-x^2)^{1/2}} = \frac{2m^2\omega^4}{4a^2\sqrt{E}}\left(\frac{2E}{m\omega^2}\right)^{7/2}\frac{1}{2}\int_0^1 dy\, y^{5/2}(1-y)^{-1/2} =$$

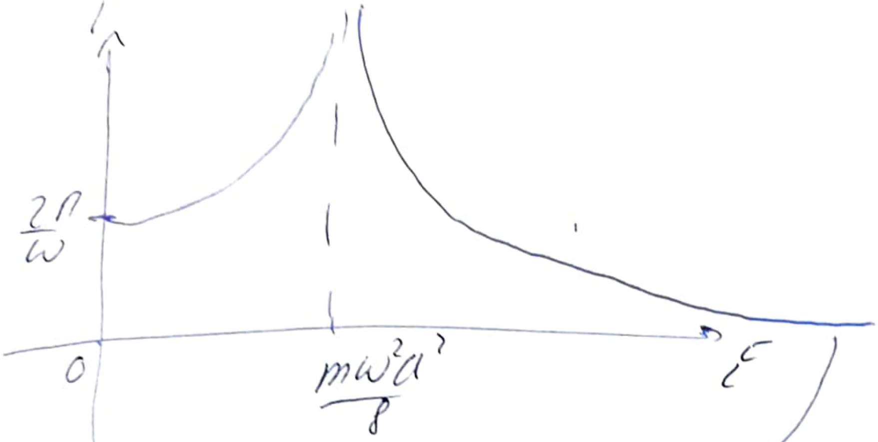
$$= \frac{m^2\omega^4}{4a^2\sqrt{E}}\left(\frac{2E}{m\omega^2}\right)^{7/2}\underbrace{B\left(\frac{7}{2},\frac{1}{2}\right)} = \frac{m^2\omega^4\, 2^{7/2}E^3\sqrt{E}}{4a^2\sqrt{E}\, m^3\sqrt{m}\,\omega^7}\frac{5\pi}{16} =$$

$$= \frac{8\sqrt{2}E^3}{4a^2m\sqrt{m}\,\omega^3}\cdot\frac{5\pi}{16} = \frac{5\sqrt{2}\pi E^3}{8a^2m\sqrt{m}\,\omega^3}$$

$$\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} = \frac{\frac{15}{8}\sqrt{\pi}\sqrt{\pi}}{6} = \frac{5\pi}{16}$$

$$\delta T = \frac{\sqrt{2m}}{2}\frac{5\sqrt{2}\,3\pi E\cdot 2}{8a^2m\sqrt{m}\,\omega^3} =$$

$$= \frac{15\pi E}{4a^2m\omega^3}$$



(5) $E \gg U_{max}$

$$T = \sqrt{2m}\int_{x_1}^{x_2} \frac{dx}{\left(E - \frac{m\omega^2}{8a^2}(x^2-a^2)^2\right)} \approx \sqrt{2m}\int \frac{dx}{\left(E-\frac{m\omega^2}{8a^2}x^4\right)^{1/2}} =$$

$$= \sqrt{\frac{2m}{E}}\left(\frac{8a^2E}{m\omega^2}\right)^{1/4}\int_{-1}^{1}\frac{du}{(1-u^4)^{1/2}} = \sqrt{\frac{2m}{E}}\left(\frac{8a^2E}{m\omega^2}\right)^{1/4}\frac{2}{4}\underbrace{\int_0^1 t^{-3/4}(1-t)^{-1/2}\,dt}_{B\ \text{ф-ция}}$$

$t = u^4 \qquad \sqrt[4]{t} = u$

$dt = 4u^3\,du$

$du = \frac{1}{4}u^{-3}$

$$\frac{(E)^{1/4}}{E^{1/2}} \Rightarrow E^{-1/4}$$

$$T \sim E^{-1/4}$$