Анмех Дз №5

№1

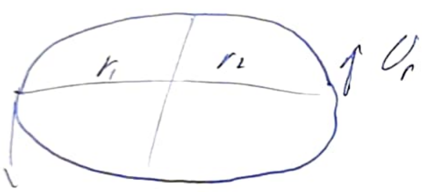
$U(r) = \frac{kr^2}{2}$ $\qquad r^2 = x^2+y^2+z^2$

(1) $\mathcal{L} = \frac{m}{2}(\dot x^2+\dot y^2+\dot z^2) - \frac{k}{2}(x^2+y^2+z^2)$

$$\begin{cases} -\frac{\partial \mathcal{L}}{\partial x} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot x}\right) = 0 \\ -\frac{\partial \mathcal{L}}{\partial y} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot y}\right) = 0 \\ -\frac{\partial \mathcal{L}}{\partial z} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot z}\right) = 0 \end{cases} \Rightarrow$$

(3) E; M

$U_{eff} = \frac{M^2}{2mr^2} + U(r)$



$E = \frac{mv_r^2}{2} + \frac{M^2}{2mr^2} + U(r)$

$v_r = 0$ в точках поворота

$E = \frac{M^2}{2mr^2} - \gamma\frac{m_1 m_2}{r} \quad \Big/ \cdot \frac{r^2}{E}$

$r^2 + \frac{\gamma m_1 m_2}{E} r - \frac{M^2}{2mE} = 0$

$r_1 + r_2 = -\frac{\gamma m_1 m_2}{E} = 2a$

$r^2 = \frac{-1 \pm \sqrt{1 + \frac{M^2 k}{mE^2}}}{k/E}$

$r = \sqrt{}$

$\frac{k}{2}2x + \frac{m}{2}2\ddot x = 0$

$\ddot x + \frac{k}{m}x = 0$

$\omega^2 = \frac{k}{m}$

$\lambda^2 = -\frac{k}{m}$

$\lambda = \pm\sqrt{-\frac{k}{m}}$

$x = C_1 e^{\sqrt{-\frac{k}{m}}t} + C_2 e^{-\sqrt{-\frac{k}{m}}t}$

(2) $x = A\cos(\omega t + \varphi_0)$

$y = B\sin(\omega t + \varphi_0)$

$z = C\cos(\omega t + \gamma)$

$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ - эллипс

$E = \frac{M^2}{2mr^2} - \frac{kr^2}{2} \quad \Big/ \cdot \frac{r^2}{E}$

$r^2 = \frac{M^2}{2mE} - \frac{kr^4}{2E}$

$r^2 + \frac{k}{2E}r^4 - \frac{M^2}{2mE} = 0$

$\frac{k}{2E}r^4 + r^2 - \frac{M^2}{2mE} = 0$

$D = b^2 - 4ac = 1 + 4\frac{M^2}{2mE}\frac{k}{2E} =$

$= 1 + \frac{M^2 k}{mE^2}$

$$E = \frac{mv_r^2}{2} + U_{eff}(r) = \frac{mv_r^2}{2} + \frac{M^2}{2mr^2} + \frac{kr^2}{2}$$

$$v_r = 0$$

$$E = \frac{M^2}{2mr^2} + \frac{kr^2}{2} \Big/ \cdot \frac{r^2}{E}$$

$$r^2 = \frac{M^2}{2mE} + \frac{k}{2E} r^4$$

$$D = b^2 - 4ac = 1 - \frac{4M^2}{2mE}\frac{k}{2E} =$$

$$\frac{k}{2E} r^4 - r^2 + \frac{M^2}{2mE} = 0$$

$$= 1 - \frac{M^2 k}{mE^2}$$

$$r_{1,2}^2 = \frac{1 \pm \sqrt{1 - \frac{M^2 k}{mE^2}}}{k/E}$$

$$2a = r_1 + r_2 = \sqrt{\frac{1 + \sqrt{1 - \frac{M^2k}{mE^2}}}{k/E}} + \sqrt{\frac{1 - \sqrt{1 - \frac{M^2k}{mE^2}}}{k/E}}$$

$$mv_1 r_1 = m v_2 r_2$$

$$b = a\sqrt{1 - \varepsilon^2}$$

$$E = \frac{mv^2}{2} + \frac{kr^2}{2} =$$

$$\varepsilon = \sqrt{1 - \frac{|E|}{|U_{min}|}}$$

№2

$U(r) = -\frac{\beta}{r^2} \quad \beta > 0$

$\varphi = \int\limits_{r_{min}}^{r} \frac{M}{\sqrt{2m}} \frac{dr}{r^2\left[E - \frac{M^2}{2mr^2} + \frac{\beta}{r^2}\right]^{1/2}} = \int\limits_{r_{min}}^{r} \frac{M}{\sqrt{2m}} \frac{dr}{r^2\left[E + \frac{1}{r^2}\left[\beta - \frac{M^2}{2m}\right]\right]^{1/2}} =$

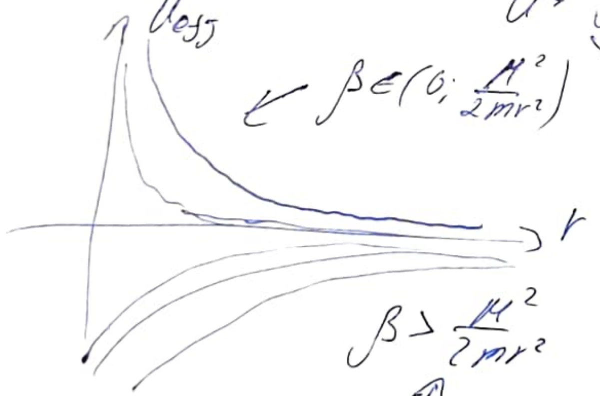
$\theta = \frac{1}{r}$

$= -\int\limits_{\theta_{min}}^{\theta} \frac{M}{\sqrt{2m}} \frac{d\theta}{\left[E + \left[\beta - \frac{M^2}{2m}\right]\theta^2\right]^{1/2}} = -\int\limits_{\theta_{min}}^{\theta} \frac{M}{\sqrt{2mE}} \frac{d\theta}{\left[1 + \frac{\beta - \frac{M^2}{2m}}{E}\theta^2\right]^{1/2}} =$

$= -\int\limits_{u_{min}}^{u} \frac{M}{\sqrt{2mE}} \left(\frac{E}{\beta - \frac{M^2}{2m}}\right)^{1/2} \frac{du}{\left[1 + u^2\right]^{1/2}} = -\frac{M}{\sqrt{2m}} \cdot \frac{1}{\sqrt{\beta - \frac{M^2}{2m}}} \ln\left(u + \sqrt{u^2 + 1}\right)\Big|_{u_{min}}^{u} =$

$= \frac{M}{\sqrt{2m}} \frac{1}{\sqrt{\beta - \frac{M^2}{2m}}} \ln\left(\frac{u_{min} + \sqrt{u_{min}^2 + 1}}{u + \sqrt{u^2 + 1}}\right) = \frac{M}{\sqrt{2m}} \frac{1}{\sqrt{\beta - \frac{M^2}{2m}}} \ln\left(\frac{\frac{\psi}{r_{min}} + \sqrt{\frac{\psi^2}{r_{min}^2} + 1}}{\frac{\psi}{r} + \sqrt{\frac{\psi^2}{r^2} + 1}}\right)$

$u = \underbrace{\sqrt{\frac{\beta - \frac{M^2}{2m}}{E}}}_{\psi} \cdot \frac{1}{r}$ при $r_{min} \to 0$



$\Downarrow$

$\ln\left(\frac{\psi}{r_{min}} + \sqrt{\frac{\psi^2}{r_{min}^2} + 1}\right) \to 0$

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$\ln \to \infty$

$\Downarrow$

Число оборотов бесконечно большое

Подходит, т.к
при $\beta \in (0; \frac{M^2}{2mr^2})$ частица
движется инфинитно

№3

$U = \frac{-\gamma}{r^n}$ $\qquad$ $U_{eff} = \frac{M^2}{2mr^2} - \frac{\gamma}{r^n}$

при $n=2$ условие

но γ такое не как в прошлой задаче

$$\varphi = \int_{r_1}^{r_2} \frac{M}{\sqrt{2m}} \frac{dr}{r^2\left[E - \frac{M^2}{2mr^2} + \frac{\gamma}{r^n}\right]^{1/2}} = -\frac{M}{\sqrt{2m}} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\left[E - \underline{\frac{M^2}{2m}\theta^2} + \gamma\theta^n\right]^{1/2}} =$$

можно пренебречь при $n>2$

$\theta = \frac{1}{r}$ $\qquad$ $r_1 \to \infty$

$r = \frac{1}{\theta}$

$$= -\frac{M}{\sqrt{2m}} \int_{r_1}^{r_2} \frac{d\theta}{\left[E + \gamma\theta^n\right]^{1/2}} = -\frac{M}{\sqrt{2mE}} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\left[1 + \frac{\gamma}{E}\theta^n\right]^{1/2}} =$$

$$= \frac{-M}{\sqrt{2mE}} \int_{\theta_1}^{\theta_2} \frac{\left(\frac{E}{\gamma}\right)^{1/n} d\tilde{\theta}}{\left(1 + \tilde{\theta}^n\right)^{1/2}} = \frac{-M\left(\frac{E}{\gamma}\right)^{1/n}}{\sqrt{2mE}} \frac{1}{n} \int_{\theta_1}^{\theta_2} \underline{\psi^{1/n-1} (1+\psi)^{-1/2} d\psi} =$$

$\psi = \theta^n$ $\qquad$ $\psi^{1/n} = \theta$

$d\psi = n\theta^{n-1} d\theta$

$d\theta = \frac{d\psi}{n\theta^{n-1}} = \frac{d\psi\,\theta}{n\psi} = \frac{1}{n} d\psi\, \psi^{1/n-1}$

Интеграл выражается в терминах бета функции и является конечным (не возникает асимптотик)

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Число оборотов при $n>2$ конечно

при $n>2$ γ может быть любым $\gamma > 0$

при $n<2$ движение инфинитно

при $n=2$ так же как и в прошлой дз возникает асимптотика