N1

$$L = \frac{m\dot{x}^2}{2} e^{\gamma t}$$

$x(0) = 0$
$\dot{x}(0) = v$

$$\underbrace{\frac{\partial L}{\partial x}}_{0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \Rightarrow \frac{d}{dt}\left(m\dot{x}e^{\gamma t}\right)$$

$$m\gamma e^{\gamma t}\dot{x} + m e^{\gamma t}\ddot{x} = 0$$

$$\ddot{x} + \gamma\dot{x} = 0$$

$$\gamma\dot{x} = -\ddot{x}$$

$$-\int_0^t \gamma\, dt = \int_v^V \frac{d\dot{x}}{\dot{x}}$$

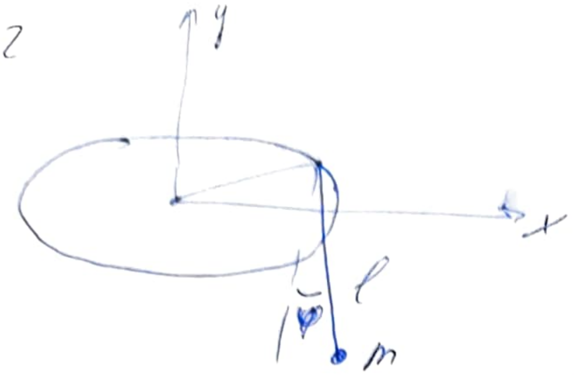
$$-\gamma t = \ln\left(\frac{V}{v}\right)$$

$$V = v e^{-\gamma t}$$

$$x = -\frac{v}{\gamma}\left(e^{-\gamma t} - e^{-\gamma \cdot 0}\right) = \frac{v}{\gamma}\left(1 - e^{-\gamma t}\right)$$

Лагранжиан соответствует движению с вязким трением $F_{тр} = kv$

N 2



$$x_0(t) = a_x \cos(\omega t)$$
$$y_0(t) = a_y \sin(\omega t)$$
$$\Downarrow$$
$$x = a_x \cos(\omega t) + l \sin\varphi$$
$$y = a_y \sin(\omega t) - l\cos\varphi$$

1)

$$\dot{x} = -a_x \omega \sin(\omega t) + l\dot{\varphi}\cos\varphi$$
$$\dot{y} = a_y \omega \cos(\omega t) + l\dot{\varphi}\sin\varphi$$
$$U = mgy = mg(a_y \sin(\omega t) - l\cos\varphi)$$
$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}\Big(a_x^2\omega^2\sin^2(\omega t) + l^2\dot{\varphi}^2\cos^2\varphi - 2a_x\omega l\dot{\varphi}\cdot$$
$$\cdot \sin(\omega t)\cos\varphi + a_y^2\omega^2\cos^2(\omega t) + l^2\dot{\varphi}^2\sin^2\varphi + 2a_y\omega l\dot{\varphi}\cos(\omega t)\sin\varphi\Big) =$$
$$= \frac{m}{2}\Big(l^2\dot{\varphi}^2 + 2\omega l\dot{\varphi}(a_y\cos(\omega t)\sin\varphi - a_x\sin(\omega t)\cos\varphi) +$$
$$+ \omega^2(a_x^2\sin^2(\omega t) + a_y^2\cos^2(\omega t))\Big)$$
$$L = T - U = \frac{m}{2}\Big(l^2\dot{\varphi}^2 + 2\omega l\dot{\varphi}(a_y\cos(\omega t)\sin\varphi -$$
$$- a_x\sin(\omega t)\cos\varphi) + \omega^2(a_x^2\sin^2(\omega t) + a_y^2\cos^2(\omega t))\Big)$$
$$+ mg(l\cos\varphi - a_y\sin\omega t)$$

2) Ур-ние Лагранжа:

$$\frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 0$$
$$\frac{m}{2}\Big(2\omega l\dot{\varphi}(a_y\cos(\omega t)\cos\varphi + a_x\sin(\omega t)\sin\varphi)\Big) -$$
$$- mgl\sin\varphi - \frac{d}{dt}\Big[\frac{m}{2}\Big(2l^2\dot{\varphi} + 2\omega l(a_y\cos\omega t\cdot$$
$$\cdot\sin\varphi - a_x\sin(\omega t)\cos\varphi)\Big)\Big] = 0$$

$$\frac{d}{dt}\left(\frac{m}{2}\left(2l^2\dot\varphi + 2\omega l\left(a_y\cos\omega t\sin\varphi - a_x\sin(\omega t)\cos\varphi\right)\right)\right) =$$

$$= ml^2\ddot\varphi + \frac{2m\omega}{2}l\left(-a_y\omega\sin\omega t\sin\varphi + a_y\cos\omega t\cos\varphi\,\dot\varphi -\right.$$

$$\left.- a_x\omega\cos\omega t\cos\varphi + a_x\sin(\omega t)\sin\varphi\,\dot\varphi\right)$$

$$\cancel{m\omega l\dot\varphi\left(a_y\cos(\omega t)\cos\varphi + a_x\sin(\omega t)\sin\varphi\right)} =$$

$$= ml^2\ddot\varphi - \cancel{m\omega l\dot\varphi\left(a_y\cos\omega t\cos\varphi + a_x\sin\omega t\sin\varphi\right)} +$$

$$+ m\omega^2 l a_y\sin(\omega t)\sin\varphi + m\omega^2 l a_x\cos(\omega t)\cos\varphi = 0$$

$$\cancel{m}\omega^2 l a_y\sin(\omega t)\sin\varphi + \cancel{m}\omega^2 l a_x\cos(\omega t)\cos\varphi = \cancel{m}l^2\ddot\varphi$$

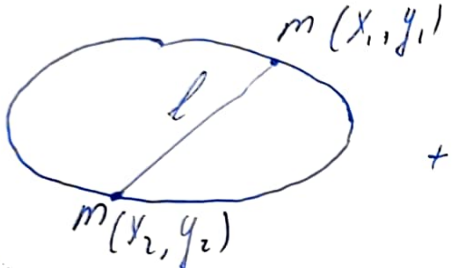
$$\omega^2 a_y\sin(\omega t)\sin\varphi + \omega^2 a_x\cos(\omega t)\cos\varphi = l\ddot\varphi$$

№3

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \begin{matrix} x=a\cos\varphi \\ y=b\sin\varphi \end{matrix} \qquad \begin{matrix} \dot{x}=-a\dot{\varphi}\sin\varphi \\ \dot{y}=b\dot{\varphi}\cos\varphi \end{matrix}$$

$$L=\frac{m}{2}(\dot{x}^2+\dot{y}^2)=\underline{\frac{m}{2}\left(a^2\dot{\varphi}^2\sin^2\varphi+b^2\dot{\varphi}^2\cos^2\varphi\right)}$$

№4



$$L=\frac{m}{2}(\dot{x}_1^2+\dot{y}_1^2)+\frac{m}{2}(\dot{x}_2^2+\dot{y}_2^2)+\lambda_1\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)+$$
$$+\lambda_2\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1\right)+\lambda_3\left((x_1-x_2)^2+(y_1-y_2)^2-l^2\right)$$

Ур-ния движения

$$\begin{cases} \frac{\partial L}{\partial x_1}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)=0 \\ \frac{\partial L}{\partial y_1}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right)=0 \\ \frac{\partial L}{\partial x_2}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)=0 \\ \frac{\partial L}{\partial y_2}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_2}\right)=0 \end{cases} \quad + \text{ условия } \quad \begin{cases} \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \\ \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1 \\ (x_1-x_2)^2+(y_1-y_2)^2=l^2 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 2\lambda_1\frac{1}{a^2}x_1+2\lambda_3(x_1-x_2)=m\ddot{x}_1 \\ 2\lambda_1\frac{1}{b^2}y_1+2\lambda_3(y_1-y_2)=m\ddot{y}_1 \\ 2\lambda_2\frac{1}{a^2}x_2-2\lambda_3(x_1-x_2)=m\ddot{x}_2 \\ 2\lambda_2\frac{1}{b^2}y_2-2\lambda_3(y_1-y_2)=m\ddot{y}_2 \end{cases} + \begin{cases} \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \\ \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1 \\ (x_1-x_2)^2+(y_1-y_2)^2=l^2 \end{cases}$$

Случай окружности

l - хорда

$$\frac{\frac{l}{2}}{a} = \sin\left(\frac{\varphi_2-\varphi_1}{2}\right) \text{ - из прямоугольного } \Delta$$

$$\varphi_2-\varphi_1 = 2\arcsin\left(\frac{l}{2a}\right)$$

Можно упростить лагранжиан уменьшив количество обобщённых координат, описав движение в полярных координатах углами φ_1 и φ_2

$$L = \frac{ma^2}{2}\left(\dot{\varphi}_1^2+\dot{\varphi}_2^2\right) + \lambda\left(\varphi_2-\varphi_1-2\arcsin\left(\frac{l}{2a}\right)\right)$$

$$\begin{cases} \frac{\partial L}{\partial \varphi_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) = 0 \\ \frac{\partial L}{\partial \varphi_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) = 0 \\ \varphi_2-\varphi_1-2\arcsin\left(\frac{l}{2a}\right) = 0 \end{cases} \Rightarrow \begin{cases} -\lambda - ma^2\ddot{\varphi}_1 = 0 \\ \lambda - ma^2\ddot{\varphi}_2 = 0 \\ \varphi_2-\varphi_1-2\arcsin\left(\frac{l}{2a}\right) = 0 \end{cases}$$

$$-ma^2\ddot{\varphi}_1 - ma^2\ddot{\varphi}_2 = 0$$

$$\Updownarrow$$

$$\ddot{\varphi}_1 = \ddot{\varphi}_2$$

$$\left.\begin{matrix} \lambda = ma^2\ddot{\varphi}_2 \\ \lambda = -ma^2\ddot{\varphi}_1 \end{matrix}\right\} \Rightarrow \begin{matrix} \ddot{\varphi}_1 = \ddot{\varphi}_2 \\ \lambda = 0 \end{matrix}$$

$$\cancel{ma^2}\,\ddot{\varphi}_1 = 0$$

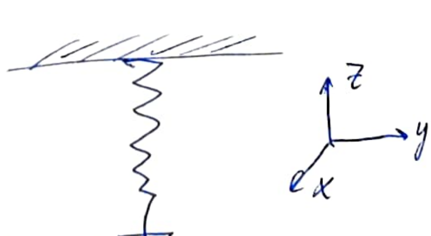
$$\cancel{ma^2}\,\omega_0 = C$$

$$\varphi_1 = \omega_0 t + \varphi_0$$

$$\varphi_2 = \omega_0 t + \varphi_0 + 2\arcsin\left(\frac{l}{2a}\right)$$

N5

$$U = \frac{k}{2}(r-r_0)^2$$



$$T = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

$$U = mgz + \frac{k}{2}\left(\sqrt{x^2+y^2+z^2}-r_0\right)^2$$

$$L = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - mgz - \frac{k}{2}\left(\sqrt{x^2+y^2+z^2}-r_0\right)^2$$

$$
\begin{cases}
\dfrac{\partial L}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = 0 \\
\dfrac{\partial L}{\partial y} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{y}}\right) = 0 \\
\dfrac{\partial L}{\partial z} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{z}}\right) = 0
\end{cases}
$$

$$
\begin{cases}
m\ddot{x} = -k\left(\sqrt{x^2+y^2+z^2} - r_0\right)\dfrac{2x}{2\sqrt{x^2+y^2+z^2}} = -kx\left(1 - \dfrac{r_0}{\sqrt{x^2+y^2+z^2}}\right) \\
m\ddot{y} = -ky\left(1 - \dfrac{r_0}{\sqrt{x^2+y^2+z^2}}\right) \\
m\ddot{z} = -kz\left(1 - \dfrac{r_0}{\sqrt{x^2+y^2+z^2}}\right) - mg
\end{cases}
$$