Numerical Analysis homework # 2

Wang Hengning 3230104148 *

Computational mathematics 2301, Zhejiang University

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I. Analysis of the Linear Interpolation for $\frac{1}{x}$

i) Determining $\xi(x)$ Explicitly

The function values at the nodes are $f(x_0) = f(1) = 1$ and $f(x_1) = f(2) = \frac{1}{2}$.

The Lagrange form is:

$$p_1(f;x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0}$$

$$p_1(f;x) = 1 \cdot \frac{x-2}{1-2} + \frac{1}{2} \cdot \frac{x-1}{2-1} = -(x-2) + \frac{1}{2}(x-1) = -\frac{1}{2}x + \frac{3}{2}$$

The remainder term for the linear interpolation, $R_1(x) = f(x) - p_1(f;x)$, is given by:

$$R_1(x) = f(x) - p_1(f;x) = \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1), \quad \xi(x) \in (x_0, x_1) = (1,2)$$

First, we calculate the explicit expression for the remainder $R_1(x)$:

$$R_1(x) = \frac{1}{x} - \left(-\frac{1}{2}x + \frac{3}{2}\right) = \frac{1}{x} + \frac{x}{2} - \frac{3}{2} = \frac{2+x^2-3x}{2x} = \frac{(x-1)(x-2)}{2x}$$

$$f''(x) = 2x^{-3} = \frac{2}{x^3}$$

$$\frac{(x-1)(x-2)}{2x} = \frac{f''(\xi(x))}{2}(x-1)(x-2)$$

Since $x \in (1,2)$, the term $(x-1)(x-2)$ is non-zero, allowing us to simplify the equation:

$$\frac{1}{2x} = \frac{f''(\xi(x))}{2} \implies f''(\xi(x)) = \frac{1}{x}$$

Substituting the expression for the second derivative, $f''(\xi(x)) = \frac{2}{(\xi(x))^3}$:

$$\frac{2}{(\xi(x))^3} = \frac{1}{x} \implies (\xi(x))^3 = 2x$$

Thus, the explicit form for $\xi(x)$ is:

$$\xi(x) = \sqrt[3]{2x}, \quad x \in (1,2)$$

*Electronic address: whning@zju.edu.cn

ii) Extension and Calculation of Extrema

The function $\xi(x) = \sqrt[3]{2x}$ can be continuously extended to the closed interval $[x_0, x_1] = [1, 2]$.

Since $\xi(x)$ is a monotonically increasing function on $[1, 2]$, its minimum and maximum values occur at the endpoints:

$$\min_{x\in[1,2]} \xi(x) = \xi(1) = \sqrt[3]{2\cdot 1} = \sqrt[3]{2}$$

$$\max_{x\in[1,2]} \xi(x) = \xi(2) = \sqrt[3]{2\cdot 2} = \sqrt[3]{4}$$

Finally, we determine the maximum value of $f''(\xi(x))$ on $[1, 2]$. From the previous step, we established that $f''(\xi(x)) = \frac{1}{x}$.

The function $\frac{1}{x}$ is monotonically decreasing on $[1, 2]$, so its maximum value occurs at the left endpoint $x = 1$:

$$\max_{x\in[1,2]} f''(\xi(x)) = \max_{x\in[1,2]} \left(\frac{1}{x}\right) = \frac{1}{1} = 1$$

II. Construction of a Non-negative Interpolating Polynomial

We are tasked with finding a polynomial $p \in \mathbb{P}^+_{2n}$ such that it interpolates the given data points (x_i, f_i) for $i = 0, 1, \ldots, n$, where $f_i \geq 0$.

Let $l_k(x)$ for $k = 0, 1, \ldots, n$ be the Lagrange basis polynomials corresponding to the distinct nodes $\{x_j\}_{j=0}^n$. These polynomials satisfy the property $l_k(x_j) = \delta_{kj}$.

First Construction

Consider the polynomial defined as a sum of squares:

$$p(x) = \sum_{k=0}^{n} f_k l_k^2(x)$$

The degree of each basis polynomial $l_k(x)$ is n, which implies the degree of $l_k^2(x)$ is $2n$. Consequently, the degree of $p(x)$ is at most $2n$.

Given that $f_k \geq 0$ and $l_k^2(x) \geq 0$ for any real x, the sum $p(x)$ is non-negative for all $x \in \mathbb{R}$. Thus, $p(x) \in \mathbb{P}^+_{2n}$.

Verifying the interpolation conditions at the nodes x_i:

$$p(x_i) = \sum_{k=0}^{n} f_k l_k^2(x_i) = f_i l_i^2(x_i) + \sum_{k\neq i} f_k l_k^2(x_i) = f_i(1)^2 + \sum_{k\neq i} f_k(0)^2 = f_i$$

This construction satisfies all requirements.

Second Construction

Alternatively, consider the polynomial formed by squaring a sum:

$$p(x) = \left(\sum_{k=0}^{n} \sqrt{f_k} l_k(x)\right)^2$$

Let $S(x) = \sum_{k=0}^{n} \sqrt{f_k} l_k(x)$. Since each $l_k(x)$ is of degree n, the degree of $S(x)$ is at most n. Therefore, the degree of $p(x) = [S(x)]^2$ is at most $2n$.

The squared form inherently ensures that $p(x) \geq 0$ for all real x, so $p(x) \in \mathbb{P}^+_{2n}$.

Evaluating at the interpolation nodes x_i:

$$p(x_i) = \left(\sum_{k=0}^{n} \sqrt{f_k} l_k(x_i)\right)^2 = \left(\sqrt{f_i} l_i(x_i)\right)^2 = \left(\sqrt{f_i}\cdot 1\right)^2 = f_i$$

This alternative also fulfills all the problem's conditions.

III. Analysis of Divided Differences for $f(x) = e^x$

Proof by Induction

Let the statement be $P(n) : f[t, t+1, \ldots, t+n] = \frac{(e-1)^n}{n!} e^t$.

The base case $n = 0$ is trivial, as $f[t] = e^t = \frac{(e-1)^0}{0!} e^t$.

Assume $P(n)$ holds for some $n \geq 0$. For $n+1$, the recursive definition of divided differences is:

$$f[t, \ldots, t+n+1] = \frac{f[t+1, \ldots, t+n+1] - f[t, \ldots, t+n]}{n+1}$$

By the inductive hypothesis, the numerator is:

$$\frac{(e-1)^n}{n!} e^{t+1} - \frac{(e-1)^n}{n!} e^t = \frac{(e-1)^n}{n!} e^t (e-1) = \frac{(e-1)^{n+1}}{n!} e^t$$

Substituting this back confirms $P(n+1)$:

$$f[t, \ldots, t+n+1] = \frac{1}{n+1} \left(\frac{(e-1)^{n+1}}{n!} e^t \right) = \frac{(e-1)^{n+1}}{(n+1)!} e^t$$

Thus, the formula holds for all integers $n \geq 0$.

Determination of ξ

From the result above, setting $t = 0$ yields:

$$f[0, 1, \ldots, n] = \frac{(e-1)^n}{n!}$$

Corollary 2.22 provides an alternative expression for some $\xi \in (0, n)$:

$$f[0, 1, \ldots, n] = \frac{f^{(n)}(\xi)}{n!}$$

For $f(x) = e^x$, the n-th derivative is $f^{(n)}(x) = e^x$. Equating the two expressions gives:

$$\frac{(e-1)^n}{n!} = \frac{e^\xi}{n!}$$

Solving for ξ directly gives:

$$\xi = n \ln(e-1)$$

To compare ξ with the midpoint $n/2$, we analyze the term $\ln(e-1)$. Since $e \approx 2.718$ and $e^{1/2} = \sqrt{e} \approx 1.648$, we have $e - 1 > e^{1/2}$. Applying the natural logarithm to this inequality shows that $\ln(e-1) > 1/2$. Therefore, the position of ξ is determined by:

$$\xi = n \ln(e-1) > n \cdot \frac{1}{2} = \frac{n}{2}$$

The point ξ is located to the right of the midpoint of the interval $(0, n)$.

IV. Newton Interpolation and Minimum Approximation

Newton Form of the Interpolating Polynomial

The given data points are $(0, 5)$, $(1, 3)$, $(3, 5)$, and $(4, 12)$. The divided differences are computed and organized in the following table:

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, \ldots, x_{i+2}]$	$f[x_i, \ldots, x_{i+3}]$
0	5			
		-2		
1	3		1	
		1		1/4
3	5		2	
		7		
4	12			

The Newton form of the interpolating polynomial $p_3(x)$ is constructed using the top diagonal of the table.

$$p_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

Substituting the coefficients and nodes gives:

$$p_3(x) = 5 - 2x + 1 \cdot x(x - 1) + \frac{1}{4}x(x - 1)(x - 3)$$

Approximation of the Minimum

To find the minimum, we first find the derivative of the polynomial $p_3(x)$. Expanding the polynomial simplifies differentiation:

$$p_3(x) = 5 - 2x + (x^2 - x) + \frac{1}{4}(x^3 - 4x^2 + 3x) = \frac{1}{4}x^3 - \frac{9}{4}x + 5$$

The derivative is:

$$p_3'(x) = \frac{3}{4}x^2 - \frac{9}{4} = \frac{3}{4}(x^2 - 3)$$

Setting the derivative to zero, $p_3'(x) = 0$, yields the critical points.

$$x^2 - 3 = 0 \implies x = \pm\sqrt{3}$$

The problem suggests a minimum in the interval $(1, 3)$. Since $\sqrt{3} \approx 1.732$ is within this interval, we select it as our candidate. To confirm it is a minimum, we check the second derivative:

$$p_3''(x) = \frac{3}{2}x$$

At $x = \sqrt{3}$, we have $p_3''(\sqrt{3}) = \frac{3\sqrt{3}}{2} > 0$, which confirms a local minimum. Thus, the approximate location of the minimum is:

$$x_{\min} \approx \sqrt{3}$$

V. Divided Differences with Repeated Nodes for $f(x) = x^7$

Computation of the Divided Difference

To handle the repeated nodes, we require the derivatives of $f(x) = x^7$.

$$f'(x) = 7x^6, \quad f''(x) = 42x^5$$

The values needed for the divided difference table are:

$$f'(1) = 7, \quad f''(1) = 42, \quad f'(2) = 448$$

We construct the divided difference table for the nodes $z = \{0, 1, 1, 1, 2, 2\}$. For repeated nodes $z_i = z_{i+k}$, the higher-order differences are defined using derivatives, such as $f[z_i, \ldots, z_{i+k}] = f^{(k)}(z_i)/k!$.

z_i	$f[z_i]$	1st ord.	2nd ord.	3rd ord.	4th ord.	5th ord.
0	0					
		1				
1	1		6			
		$f'(1) = 7$		15		
1	1		$f''(1)/2 = 21$		42	
		$f'(1) = 7$		99		30
1	1		120		102	
		127		201		
2	128		321			
		$f'(2) = 448$				
2	128					

The final entry in the table gives the desired value.

$$f[0, 1, 1, 1, 2, 2] = 30$$

Determination of ξ

The Mean Value Theorem for divided differences relates the divided difference to a derivative.

$$f[z_0, \ldots, z_n] = \frac{f^{(n)}(\xi)}{n!} \quad \text{for some } \xi \in (\min(z_i), \max(z_i))$$

For this problem, we have $n = 5$ and the nodes are within the interval $(0, 2)$. The 5th derivative of $f(x)$ is calculated as:

$$f^{(5)}(x) = (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3)x^2 = 2520x^2$$

By substituting the computed divided difference and $n = 5$ into the theorem, we get an equation for ξ.

$$30 = \frac{f^{(5)}(\xi)}{5!} = \frac{2520\xi^2}{120}$$

We solve this equation for ξ:

$$\xi^2 = \frac{30 \cdot 120}{2520} = \frac{3600}{2520} = \frac{10}{7}$$

Since ξ must be in $(0, 2)$, we take the positive square root.

$$\xi = \sqrt{\frac{10}{7}} = \frac{\sqrt{70}}{7}$$

VI. Hermite Interpolation and Error Analysis

Estimate of $f(2)$

The Hermite interpolation problem is defined by the nodes $z = \{0, 1, 1, 3, 3\}$. We use the divided difference method to find the interpolating polynomial $H_4(x)$. The required derivative values are $f'(1) = -1$ and $f'(3) = 0$. The divided difference table is constructed as follows:

z_i	$f[z_i]$	1st	2nd	3rd	4th
0	1				
		1			
1	2		-2		
		-1		$2/3$	
1	2		0		$-5/36$
		-1		$1/4$	
3	0		$1/2$		
		0			
3	0				

The coefficients from the top diagonal are $1, 1, -2, 2/3, -5/36$. The Newton form of the polynomial is:

$$H_4(x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3)$$

We estimate $f(2)$ by evaluating $H_4(2)$.

$$f(2) \approx H_4(2) = 1 + 2 - 2(2)(1) + \frac{2}{3}(2)(1)^2 - \frac{5}{36}(2)(1)^2(2-3)$$

$$H_4(2) = 3 - 4 + \frac{4}{3} - \frac{5}{36}(-2) = -1 + \frac{4}{3} + \frac{10}{36} = -1 + \frac{24}{18} + \frac{5}{18} = \frac{-18 + 24 + 5}{18} = \frac{11}{18}$$

Maximum Possible Error

The error in Hermite interpolation for a function $f \in C^5[0, 3]$ is given by the formula:

$$E_4(x) = f(x) - H_4(x) = \frac{f^{(5)}(\xi)}{5!} \prod_{i=0}^{4}(x - z_i)$$

where ξ is in the interval spanned by the nodes, $(0, 3)$. We want to bound the error at $x = 2$. The node product at this point is:

$$\Psi(2) = (2-0)(2-1)(2-1)(2-3)(2-3) = (2)(1)(1)(-1)(-1) = 2$$

The error at $x = 2$ is therefore:

$$|E_4(2)| = \left| \frac{f^{(5)}(\xi)}{5!} \cdot \Psi(2) \right| = \left| \frac{f^{(5)}(\xi)}{120} \cdot 2 \right| = \frac{|f^{(5)}(\xi)|}{60}$$

Given that $|f^{(5)}(x)| \leq M$ on $[0, 3]$, we can bound the error.

$$|E_4(2)| \leq \frac{M}{60}$$

The maximum possible error for the estimate $f(2) \approx 11/18$ is $M/60$.

VII. Relation Between Finite and Divided Differences

Forward Difference

We prove the identity $\Delta^k f(x) = k!h^k f[x_0, x_1, \dots, x_k]$ by induction on k, where $x_j = x + jh$. For the base case $k = 1$, we use the definitions of the operators.

$$\Delta f(x) = f(x + h) - f(x) = f(x_1) - f(x_0)$$

The right-hand side is:

$$1!h^1 f[x_0, x_1] = h\frac{f(x_1) - f(x_0)}{x_1 - x_0} = h\frac{f(x_1) - f(x_0)}{h} = f(x_1) - f(x_0)$$

The identity holds for $k = 1$. Now, assume the proposition is true for an integer $k \geq 1$. We examine the case for $k + 1$.

$$\Delta^{k+1} f(x) = \Delta^k f(x + h) - \Delta^k f(x)$$

Applying the inductive hypothesis to both terms on the right-hand side yields:

$$\Delta^{k+1} f(x) = k!h^k f[x_1, \dots, x_{k+1}] - k!h^k f[x_0, \dots, x_k] = k!h^k \left(f[x_1, \dots, x_{k+1}] - f[x_0, \dots, x_k] \right)$$

Using the recursive property of divided differences, we have:

$$f[x_0, \dots, x_{k+1}] = \frac{f[x_1, \dots, x_{k+1}] - f[x_0, \dots, x_k]}{x_{k+1} - x_0}$$

Since $x_{k+1} - x_0 = (x + (k + 1)h) - x = (k + 1)h$, we can write:

$$\Delta^{k+1} f(x) = k!h^k \left((x_{k+1} - x_0) f[x_0, \dots, x_{k+1}] \right) = k!h^k (k + 1)h f[x_0, \dots, x_{k+1}]$$

This simplifies to the desired result for $k + 1$.

$$\Delta^{k+1} f(x) = (k + 1)!h^{k+1} f[x_0, \dots, x_{k+1}]$$

By the principle of induction, the formula is proven for all $k \geq 1$.

Backward Difference

A similar proof by induction can be constructed. Alternatively, we can use the established relationship between the backward and forward difference operators, $\nabla f(x) = \Delta f(x - h)$. By repeated application, this leads to the identity $\nabla^k f(x) = \Delta^k f(x - kh)$. We apply the proven forward difference formula to the function evaluated at the point $z = x - kh$:

$$\Delta^k f(z) = k!h^k f[z, z + h, \dots, z + kh]$$

Substituting $z = x - kh$ into the nodes gives the set:

$$\{x - kh, (x - kh) + h, \dots, (x - kh) + kh\} = \{x - kh, x - (k - 1)h, \dots, x\}$$

This set of nodes is precisely $\{x_{-k}, x_{-k+1}, \dots, x_0\}$. Therefore, we have:

$$\nabla^k f(x) = \Delta^k f(x - kh) = k!h^k f[x_{-k}, x_{-k+1}, \dots, x_0]$$

Since the divided difference is a symmetric function of its arguments, the order of the nodes is irrelevant.

$$\nabla^k f(x) = k!h^k f[x_0, x_{-1}, \dots, x_{-k}]$$

This completes the proof.

VIII. Derivative of Divided Differences

Partial Derivative with respect to x_0

By the definition of the partial derivative, we have:

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] = \lim_{h \to 0} \frac{f[x_0 + h, x_1, \ldots, x_n] - f[x_0, x_1, \ldots, x_n]}{h}$$

Due to the symmetry property of divided differences, we can reorder the arguments of the first term in the numerator.

$$f[x_0 + h, x_1, \ldots, x_n] = f[x_1, \ldots, x_n, x_0 + h]$$

The expression inside the limit can then be rewritten as:

$$\frac{f[x_1, \ldots, x_n, x_0 + h] - f[x_0, x_1, \ldots, x_n]}{(x_0 + h) - x_0}$$

This is precisely the recursive definition for a higher-order divided difference over the nodes $\{x_0, x_1, \ldots, x_n, x_0 + h\}$.

$$\frac{\partial}{\partial x_0} f[x_0, \ldots, x_n] = \lim_{h \to 0} f[x_0, x_1, \ldots, x_n, x_0 + h]$$

Since f is differentiable, the divided difference is a continuous function of its arguments. We can thus evaluate the limit by substitution.

$$\lim_{h \to 0} f[x_0, x_1, \ldots, x_n, x_0 + h] = f[x_0, x_1, \ldots, x_n, x_0]$$

Using the symmetry property one last time to group the repeated nodes gives the final result.

$$\frac{\partial}{\partial x_0} f[x_0, \ldots, x_n] = f[x_0, x_0, x_1, \ldots, x_n]$$

Generalization for other variables

The divided difference $f[x_0, \ldots, x_n]$ is a symmetric function of its arguments. This allows the result to be generalized to the partial derivative with respect to any other variable x_i. We can permute the arguments to place x_i in the first position without changing the function's value.

$$f[x_0, \ldots, x_i, \ldots, x_n] = f[x_i, x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$$

Let Z be the set of all other variables $\{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. Differentiating $f[x_i, Z]$ with respect to x_i is directly analogous to the case for x_0.

$$\frac{\partial}{\partial x_i} f[x_i, Z] = f[x_i, x_i, Z]$$

Substituting the set Z back gives the general formula for any $i \in \{0, 1, \ldots, n\}$.

$$\frac{\partial}{\partial x_i} f[x_0, \ldots, x_n] = f[x_i, x_i, x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$$

This result represents the divided difference over the original set of nodes with the node x_i repeated.

IX. A Min-Max Polynomial Problem

Let $P_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$. The problem is to determine the value of $\min_{a_1, \ldots, a_n} \max_{x \in [a,b]} |P_n(x)|$. We can factor out the fixed leading coefficient $|a_0|$:

$$\max_{x \in [a,b]} |P_n(x)| = |a_0| \max_{x \in [a,b]} |x^n + \frac{a_1}{a_0} x^{n-1} + \cdots + \frac{a_n}{a_0}|$$

Minimizing over $a_1, \ldots, a_n$ is equivalent to minimizing over the coefficients of the resulting monic polynomial. Let $\mathcal{P}_n^M$ be the set of all monic polynomials of degree n. The problem is transformed into:

$$|a_0| \min_{p \in \mathcal{P}_n^M} \max_{x \in [a,b]} |p(x)|$$

We use the linear transformation $x = \frac{b-a}{2}t + \frac{a+b}{2}$ to map the interval $t \in [-1,1]$ onto $x \in [a,b]$. A monic polynomial $p(x) \in \mathcal{P}_n^M$ can be expressed in terms of t:

$$p(x) = \left(\frac{b-a}{2}t + \frac{a+b}{2}\right)^n + \cdots = \left(\frac{b-a}{2}\right)^n t^n + \text{lower degree terms in } t$$

This implies that $p(x)$ can be written as $(\frac{b-a}{2})^n q(t)$, where $q(t)$ is a monic polynomial in t of degree n. The min-max problem over $[a,b]$ becomes a corresponding problem over $[-1,1]$.

$$\min_{p\in\mathcal{P}_n^M} \max_{x\in[a,b]} |p(x)| = \min_{q\in\mathcal{P}_n^M} \max_{t\in[-1,1]} \left|\left(\frac{b-a}{2}\right)^n q(t)\right| = \left(\frac{b-a}{2}\right)^n \min_{q\in\mathcal{P}_n^M} \max_{t\in[-1,1]} |q(t)|$$

A fundamental theorem of approximation theory states that the minimum value of the maximum absolute value for a monic polynomial of degree n on $[-1,1]$ is achieved by the monic Chebyshev polynomial $\tilde{T}_n(t) = T_n(t)/2^{n-1}$.

$$\min_{q\in\mathcal{P}_n^M} \max_{t\in[-1,1]} |q(t)| = \max_{t\in[-1,1]} |\tilde{T}_n(t)| = \frac{1}{2^{n-1}}$$

Substituting this result back, we obtain the final solution.

$$\min_{a_1,\dots,a_n} \max_{x\in[a,b]} |P_n(x)| = |a_0| \left(\frac{b-a}{2}\right)^n \frac{1}{2^{n-1}} = |a_0| \frac{(b-a)^n}{2^{2n-1}}$$

X. A Minimization Property of Scaled Chebyshev Polynomials

We proceed by contradiction. Assume there exists a polynomial $p(x) \in \mathbb{P}_n^a$ such that $\|p\|_\infty < \|\hat{p}_n\|_\infty$. Define a new polynomial $Q(x) = \hat{p}_n(x) - p(x)$. Since $\hat{p}_n$ and p are both polynomials of degree n, $Q(x)$ is a polynomial of degree at most n.

The Chebyshev polynomial $T_n(x)$ attains its extrema on $[-1,1]$ at the $n+1$ points $x_k = \cos(k\pi/n)$ for $k = 0, 1, \dots, n$, where $T_n(x_k) = (-1)^k$. The polynomial $\hat{p}_n(x) = T_n(x)/T_n(a)$ therefore satisfies:

$$\hat{p}_n(x_k) = \frac{(-1)^k}{T_n(a)}$$

Since $a > 1$, $T_n(a) > 0$. The maximum absolute value of $\hat{p}_n(x)$ on $[-1,1]$ is thus $\|\hat{p}_n\|_\infty = 1/T_n(a)$.

At these points x_k, we evaluate $Q(x_k)$:

$$Q(x_k) = \hat{p}_n(x_k) - p(x_k)$$

From our initial assumption, $|p(x_k)| \le \|p\|_\infty < \|\hat{p}_n\|_\infty = |\hat{p}_n(x_k)|$. This implies that $p(x_k)$ cannot change the sign of $\hat{p}_n(x_k)$. Thus, for $k = 0, \dots, n$:

$$\text{sign}(Q(x_k)) = \text{sign}(\hat{p}_n(x_k)) = \text{sign}((-1)^k)$$

Since $Q(x)$ alternates in sign across the $n+1$ points x_k, by the Intermediate Value Theorem, $Q(x)$ must have at least n distinct roots in the interval $(-1,1)$.

Furthermore, both $\hat{p}_n(x)$ and $p(x)$ are in $\mathbb{P}_n^a$, which means $\hat{p}_n(a) = 1$ and $p(a) = 1$. Therefore, $Q(x)$ has an additional root at $x = a$:

$$Q(a) = \hat{p}_n(a) - p(a) = 1 - 1 = 0$$

Since $a > 1$, this root is distinct from the n roots found within $(-1,1)$. This gives $Q(x)$ at least $n+1$ distinct roots.

We have established that $Q(x)$ is a polynomial of degree at most n with at least $n+1$ roots. This is only possible if $Q(x)$ is identically zero. If $Q(x) \equiv 0$, then $p(x) \equiv \hat{p}_n(x)$, which implies $\|p\|_\infty = \|\hat{p}_n\|_\infty$. This contradicts our initial assumption that $\|p\|_\infty < \|\hat{p}_n\|_\infty$. The assumption must be false, and therefore, for any $p \in \mathbb{P}_n^a$, we must have $\|\hat{p}_n\|_\infty \le \|p\|_\infty$.

XI. Proof of the Degree Elevation Property for Bernstein Polynomials

The proof begins by expanding the right-hand side (RHS) of the identity using the definition of a Bernstein basis polynomial, $b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$.

$$\text{RHS} = \frac{n-k}{n} b_{n,k}(t) + \frac{k+1}{n} b_{n,k+1}(t)$$

Substituting the definition yields:

$$\text{RHS} = \frac{n-k}{n}\binom{n}{k}t^k(1-t)^{n-k} + \frac{k+1}{n}\binom{n}{k+1}t^{k+1}(1-t)^{n-k-1}$$

Next, we expand the binomial coefficients and simplify the constant factors. For the first term's coefficient:

$$\frac{n-k}{n}\binom{n}{k} = \frac{n-k}{n}\frac{n!}{k!(n-k)!} = \frac{n-k}{n}\frac{n(n-1)!}{k!(n-k)(n-k-1)!} = \frac{(n-1)!}{k!(n-k-1)!} = \binom{n-1}{k}$$

For the second term's coefficient:

$$\frac{k+1}{n}\binom{n}{k+1} = \frac{k+1}{n}\frac{n!}{(k+1)!(n-k-1)!} = \frac{k+1}{n}\frac{n(n-1)!}{(k+1)k!(n-k-1)!} = \frac{(n-1)!}{k!(n-k-1)!} = \binom{n-1}{k}$$

Substituting these simplified coefficients back into the expression for the RHS:

$$\text{RHS} = \binom{n-1}{k}t^k(1-t)^{n-k} + \binom{n-1}{k}t^{k+1}(1-t)^{n-k-1}$$

We can factor out the common terms $\binom{n-1}{k}t^k(1-t)^{n-k-1}$.

$$\text{RHS} = \binom{n-1}{k}t^k(1-t)^{n-k-1}\left[(1-t)+t\right]$$

This simplifies to:

$$\text{RHS} = \binom{n-1}{k}t^k(1-t)^{(n-1)-k} = b_{n-1,k}(t)$$

The final expression is the definition of $b_{n-1,k}(t)$, which is the left-hand side (LHS) of the identity. This completes the proof.

XII. Proof of the Integral Property of Bernstein Basis Polynomials

Let $I_{n,k}$ denote the integral of the Bernstein basis polynomial $b_{n,k}(t)$ over the interval $[0,1]$.

$$I_{n,k} = \int_0^1 b_{n,k}(t)dt = \binom{n}{k}\int_0^1 t^k(1-t)^{n-k}dt$$

We apply integration by parts to the integral, with $u = (1-t)^{n-k}$ and $dv = t^k dt$. This gives $du = -(n-k)(1-t)^{n-k-1}dt$ and $v = t^{k+1}/(k+1)$.

$$I_{n,k} = \binom{n}{k}\left[\left.\frac{t^{k+1}(1-t)^{n-k}}{k+1}\right|_0^1 + \frac{n-k}{k+1}\int_0^1 t^{k+1}(1-t)^{n-k-1}dt\right]$$

The boundary term evaluates to zero for $k < n$. Thus, the expression simplifies to:

$$I_{n,k} = \frac{n-k}{k+1}\binom{n}{k}\int_0^1 t^{k+1}(1-t)^{n-(k+1)}dt$$

We analyze the coefficient:

$$\frac{n-k}{k+1}\binom{n}{k} = \frac{n-k}{k+1}\frac{n!}{k!(n-k)!} = \frac{n!}{(k+1)!(n-k-1)!} = \binom{n}{k+1}$$

Substituting this back, we establish a recurrence relation for the integral.

$$I_{n,k} = \binom{n}{k+1}\int_0^1 t^{k+1}(1-t)^{n-(k+1)}dt = I_{n,k+1}$$

This relation holds for $k = 0, 1, \ldots, n-1$, implying that the value of the integral $I_{n,k}$ is constant for all k. To find this constant value, we use the partition of unity property of Bernstein basis polynomials:

$$\sum_{k=0}^{n} b_{n,k}(t) = 1$$

Integrating this identity over $[0, 1]$ yields:

$$\int_0^1 \sum_{k=0}^{n} b_{n,k}(t)dt = \sum_{k=0}^{n} \int_0^1 b_{n,k}(t)dt = \int_0^1 1dt = 1$$

Since all $n + 1$ integrals in the sum are equal, we have:

$$(n + 1)I_{n,k} = 1$$

This leads to the final result, which is independent of k.

$$\int_0^1 b_{n,k}(t)dt = \frac{1}{n + 1}$$

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Honestly, the **translation** was done using LLM tools, but the answers and solutions were written by myself. This version is edited out on 9th Oct.