Numerical Analysis homework # 3

Wang Hengning 3230104148 *

Computational mathematics 2301, Zhejiang University

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Problem I: Determining $p(x)$ for a C^2 cubic spline $s(x)$

Let $s_0(x) = p(x) = ax^3 + bx^2 + cx + d$ on $[0, 1]$ and $s_1(x) = (2 - x)^3$ on $[1, 2]$. The condition $s(0) = p(0) = 0$ implies $d = 0$. The spline $s \in \mathbb{S}_3^2$ requires C^2 continuity at $x = 1$.

For $p(x) = ax^3 + bx^2 + cx$:

$$p'(x) = 3ax^2 + 2bx + c, \quad p''(x) = 6ax + 2b$$

For $s_1(x) = (2 - x)^3$:

$$s_1'(x) = -3(2 - x)^2, \quad s_1''(x) = 6(2 - x)$$

At $x = 1$, the values for s_1 are $s_1(1) = 1$, $s_1'(1) = -3$, and $s_1''(1) = 6$. The C^2 continuity conditions $p^{(k)}(1) = s_1^{(k)}(1)$ for $k = 0, 1, 2$ yield the system:

$$\begin{cases} p(1) = a + b + c = 1 \\ p'(1) = 3a + 2b + c = -3 \\ p''(1) = 6a + 2b = 6 \end{cases}$$

Solving this system gives $a = 7$, $b = -18$, and $c = 12$.

$$p(x) = 7x^3 - 18x^2 + 12x$$

— A natural cubic spline requires $s''(x) = 0$ at the endpoints $x = 0$ and $x = 2$.

$$s''(2) = s_1''(2) = 0$$

$$s''(0) = p''(0) = 6a(0) + 2b = 2(-18) = -36$$

Since $s''(0) = -36 \neq 0$, $s(x)$ is **not** a natural cubic spline.

Problem II: Interpolating with a quadratic spline $s \in \mathbb{S}_2^1$

(a) A quadratic spline $s \in \mathbb{S}_2^1$ on n knots $x_1, \ldots, x_n$ consists of $n - 1$ quadratic pieces $s_i \in \mathbb{P}_2$. Each piece has 3 coefficients, giving $3(n - 1)$ total degrees of freedom (DoF). The constraints are:

1. Interpolation: $s_i(x_i) = f_i$ and $s_i(x_{i+1}) = f_{i+1}$ for $i = 1, \ldots, n - 1$. ($2(n - 1)$ conditions)
2. C^1 continuity: $s_{i-1}'(x_i) = s_i'(x_i)$ at the $n - 2$ interior knots $x_2, \ldots, x_{n-1}$. ($n - 2$ conditions)

Total constraints: $2(n - 1) + (n - 2) = 3n - 4$. Free parameters: DoF − Constraints = $(3n - 3) - (3n - 4) = 1$. Therefore, one additional condition is needed to determine s uniquely.

— (b) Represent $p_i(x) = s|_{[x_i, x_{i+1}]} \in \mathbb{P}_2$ centered at x_i:

$$p_i(x) = c_0 + c_1(x - x_i) + c_2(x - x_i)^2$$

$$p_i'(x) = c_1 + 2c_2(x - x_i)$$

*Electronic address: whning@zju.edu.cn

The conditions $p_i(x_i) = f_i$ and $p_i'(x_i) = m_i$ yield $c_0 = f_i$ and $c_1 = m_i$. Let $h_i = x_{i+1} - x_i$. The condition $p_i(x_{i+1}) = f_{i+1}$ determines c_2:

$$f_i + m_i h_i + c_2 h_i^2 = f_{i+1} \implies c_2 = \frac{f_{i+1} - f_i - m_i h_i}{h_i^2}$$

$$p_i(x) = f_i + m_i(x - x_i) + \left(\frac{f_{i+1} - f_i - m_i h_i}{h_i^2}\right)(x - x_i)^2$$

— (c) The C^1 continuity $s_{i-1}'(x_i) = s_i'(x_i) = m_i$ is required for $i = 2, \ldots, n$. Let $h_{i-1} = x_i - x_{i-1}$. Using the result from (b) for $p_{i-1}(x)$:

$$p_{i-1}'(x) = m_{i-1} + 2\left(\frac{f_i - f_{i-1} - m_{i-1}h_{i-1}}{h_{i-1}^2}\right)(x - x_{i-1})$$

Evaluating at $x = x_i$:

$$\begin{aligned} p_{i-1}'(x_i) &= m_{i-1} + 2\left(\frac{f_i - f_{i-1} - m_{i-1}h_{i-1}}{h_{i-1}^2}\right)h_{i-1} \\ &= m_{i-1} + \frac{2(f_i - f_{i-1})}{h_{i-1}} - 2m_{i-1} = -m_{i-1} + \frac{2(f_i - f_{i-1})}{h_{i-1}} \end{aligned}$$

The continuity condition $m_i = p_{i-1}'(x_i)$ gives the recurrence relation for $i = 2, \ldots, n$:

$$m_i = -m_{i-1} + \frac{2(f_i - f_{i-1})}{h_{i-1}}$$

Given $m_1 = f'(a)$, m_2 can be computed, then m_3, and so on, sequentially determining $m_2, \ldots, m_n$.

Problem III: Determining a natural cubic spline $s(x)$

Let $s_1(x) = 1 + c(x+1)^3$ on $[-1, 0]$ and $s_2(x) = ax^3 + bx^2 + dx + e$ on $[0, 1]$. The spline must be C^2 continuous at $x = 0$ and satisfy $s''(-1) = 0$ and $s''(1) = 0$.

Derivatives for $s_1(x)$:

$$s_1'(x) = 3c(x+1)^2, \quad s_1''(x) = 6c(x+1)$$

The condition $s''(-1) = s_1''(-1) = 0$ is satisfied for any $c \in \mathbb{R}$.

Derivatives for $s_2(x)$:

$$s_2'(x) = 3ax^2 + 2bx + d, \quad s_2''(x) = 6ax + 2b$$

Continuity at $x = 0$, $s_1^{(k)}(0) = s_2^{(k)}(0)$, determines e, d, b in terms of c:

$$\begin{aligned} C^0 &: \quad s_1(0) = 1 + c \implies s_2(0) = e = 1 + c \\ C^1 &: \quad s_1'(0) = 3c \implies s_2'(0) = d = 3c \\ C^2 &: \quad s_1''(0) = 6c \implies s_2''(0) = 2b = 6c \implies b = 3c \end{aligned}$$

So, $s_2(x) = ax^3 + 3cx^2 + 3cx + (1 + c)$. The remaining natural condition $s''(1) = s_2''(1) = 0$ gives:

$$s_2''(1) = 6a(1) + 2b = 6a + 2(3c) = 6a + 6c = 0 \implies a = -c$$

This determines $s_2(x)$:

$$s_2(x) = -cx^3 + 3cx^2 + 3cx + (1 + c) = c(-x^3 + 3x^2 + 3x + 1) + 1$$

— To find c, use the condition $s(1) = -1$:

$$s(1) = s_2(1) = c(-1 + 3 + 3 + 1) + 1 = 6c + 1$$

$$6c + 1 = -1 \implies 6c = -2 \implies c = -\frac{1}{3}$$

Problem IV: Natural cubic spline interpolation of $f(x) = \cos(\frac{\pi x}{2})$

(a) The function is $f(x) = \cos(\frac{\pi x}{2})$ on $[-1, 1]$. The knots are $x_1 = -1$, $x_2 = 0$, $x_3 = 1$. The data is:

$$f_1 = f(-1) = 0, \quad f_2 = f(0) = 1, \quad f_3 = f(1) = 0$$

Let $s(x)$ be the spline (s_1 on $[-1, 0]$, s_2 on $[0, 1]$) and $M_i = s''(x_i)$. Natural boundary conditions: $M_1 = 0$, $M_3 = 0$. Knot spacing: $h_1 = x_2 - x_1 = 1$, $h_2 = x_3 - x_2 = 1$. The system equation for M_2 is:

$$\frac{h_1}{6}M_1 + \frac{h_1 + h_2}{3}M_2 + \frac{h_2}{6}M_3 = \frac{f_3 - f_2}{h_2} - \frac{f_2 - f_1}{h_1}$$

$$\frac{1}{6}(0) + \frac{1+1}{3}M_2 + \frac{1}{6}(0) = \frac{0-1}{1} - \frac{1-0}{1}$$

$$\frac{2}{3}M_2 = -2 \implies M_2 = -3$$

On $[-1, 0]$, $s_1''(x)$ passes through $M_1 = 0$ and $M_2 = -3$:

$$s_1''(x) = M_1\frac{0-x}{h_1} + M_2\frac{x-(-1)}{h_1} = 0 - 3(x+1) = -3(x+1)$$

Integrating twice: $s_1(x) = -\frac{1}{2}(x+1)^3 + A(x+1) + B$. $s_1(-1) = 0 \implies B = 0$. $s_1(0) = 1 \implies -\frac{1}{2} + A = 1 \implies A = \frac{3}{2}$.

$$s_1(x) = -\frac{1}{2}(x+1)^3 + \frac{3}{2}(x+1)$$

On $[0, 1]$, $s_2''(x)$ passes through $M_2 = -3$ and $M_3 = 0$:

$$s_2''(x) = M_2\frac{1-x}{h_2} + M_3\frac{x-0}{h_2} = -3(1-x) + 0 = -3(1-x)$$

Integrating twice: $s_2(x) = -\frac{1}{2}(1-x)^3 + C(1-x) + D$. $s_2(1) = 0 \implies D = 0$. $s_2(0) = 1 \implies -\frac{1}{2} + C = 1 \implies C = \frac{3}{2}$.

$$s_2(x) = -\frac{1}{2}(1-x)^3 + \frac{3}{2}(1-x)$$

— (b) The total bending energy is $E(g) = \int_{-1}^{1}[g''(x)]^2dx$. For the spline $s(x)$:

$$E(s) = \int_{-1}^{0}[s_1''(x)]^2dx + \int_{0}^{1}[s_2''(x)]^2dx$$

$$E(s) = \int_{-1}^{0}[-3(x+1)]^2dx + \int_{0}^{1}[-3(1-x)]^2dx$$

$$E(s) = 9\int_{-1}^{0}(x+1)^2dx + 9\int_{0}^{1}(1-x)^2dx$$

$$E(s) = 9\left[\frac{(x+1)^3}{3}\right]_{-1}^{0} + 9\left[\frac{(1-x)^3}{-3}\right]_{0}^{1} = 3(1) - 3(-1) = 6$$

(i) For the quadratic interpolant $g_1(x) = ax^2 + bx + c$: The conditions $g_1(-1) = 0$, $g_1(0) = 1$, $g_1(1) = 0$ yield:

$$\begin{cases} a - b + c = 0 \\ c = 1 \\ a + b + c = 0 \end{cases} \implies a = -1, b = 0, c = 1$$

$g_1(x) = 1 - x^2$, so $g_1''(x) = -2$.

$$E(g_1) = \int_{-1}^{1}(-2)^2dx = 4\int_{-1}^{1}dx = 4(1-(-1)) = 8$$

(ii) For the function $f(x)$ itself, $g_2(x) = f(x) = \cos(\frac{\pi x}{2})$:

$$g_2''(x) = f''(x) = -\frac{\pi^2}{4}\cos(\frac{\pi x}{2})$$

$$E(f) = \int_{-1}^{1} \left[-\frac{\pi^2}{4} \cos(\frac{\pi x}{2}) \right]^2 dx = \frac{\pi^4}{16} \int_{-1}^{1} \cos^2 \left(\frac{\pi x}{2} \right) dx$$

Using $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$:

$$E(f) = \frac{\pi^4}{16} \int_{-1}^{1} \frac{1+\cos(\pi x)}{2} dx = \frac{\pi^4}{32} \left[x + \frac{\sin(\pi x)}{\pi} \right]_{-1}^{1}$$

$$E(f) = \frac{\pi^4}{32}[(1+0) - (-1+0)] = \frac{2\pi^4}{32} = \frac{\pi^4}{16} \approx 6.0875$$

We verify the minimal energy property: $E(s) = 6 < E(f) \approx 6.0875$ and $E(s) = 6 < E(g_1) = 8$.

Problem V: The quadratic B-spline $B_i^2(x)$

(a) We derive $B_i^2(x)$ from the Cox-de Boor recursion (Definition 3.23) for $n = 1$:

$$B_i^2(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} B_i^1(x) + \frac{t_{i+2} - x}{t_{i+2} - t_i} B_{i+1}^1(x)$$

The linear B-splines (hat functions) $B_i^1(x)$ and $B_{i+1}^1(x)$ are given by Definition 3.21 and Example 3.24.

$$B_i^1(x) = \begin{cases} \frac{x-t_{i-1}}{t_i-t_{i-1}} & x \in [t_{i-1}, t_i] \\ \frac{t_{i+1}-x}{t_{i+1}-t_i} & x \in [t_i, t_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

$$B_{i+1}^1(x) = \begin{cases} \frac{x-t_i}{t_{i+1}-t_i} & x \in [t_i, t_{i+1}] \\ \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} & x \in [t_{i+1}, t_{i+2}] \\ 0 & \text{otherwise} \end{cases}$$

The support of $B_i^2(x)$ is $[t_{i-1}, t_{i+2}]$ (Lemma 3.27). On $[t_{i-1}, t_i]$: $B_{i+1}^1(x) = 0$.

$$B_i^2(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} \left(\frac{x - t_{i-1}}{t_i - t_{i-1}} \right) = \frac{(x - t_{i-1})^2}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})}$$

On $[t_i, t_{i+1}]$:

$$B_i^2(x) = \left(\frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} \right) \left(\frac{t_{i+1} - x}{t_{i+1} - t_i} \right) + \left(\frac{t_{i+2} - x}{t_{i+2} - t_i} \right) \left(\frac{x - t_i}{t_{i+1} - t_i} \right)$$

On $[t_{i+1}, t_{i+2}]$: $B_i^1(x) = 0$.

$$B_i^2(x) = \frac{t_{i+2} - x}{t_{i+2} - t_i} \left(\frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} \right) = \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})}$$

This matches the expressions in Example 3.25.

— (b) The derivative formula for B-splines (Theorem 3.34) for $n = 2$ is:

$$\frac{d}{dx} B_i^2(x) = \frac{2B_i^1(x)}{t_{i+1} - t_{i-1}} - \frac{2B_{i+1}^1(x)}{t_{i+2} - t_i}$$

The hat functions $B_i^1(x)$ and $B_{i+1}^1(x)$ are continuous (Definition 3.21). Since $\frac{d}{dx} B_i^2(x)$ is a linear combination of continuous functions, it is continuous for all x, including the interior knots t_i and t_{i+1}. This is consistent with $B_i^2 \in \mathbb{S}_2^1$ (Corollary 3.35).

— (c) We seek x^* such that $\frac{d}{dx} B_i^2(x^*) = 0$. From (b), this requires:

$$\frac{B_i^1(x^*)}{t_{i+1} - t_{i-1}} = \frac{B_{i+1}^1(x^*)}{t_{i+2} - t_i}$$

The support is $[t_{i-1}, t_{i+2}]$. The derivative is zero at the endpoint t_{i-1} (since $B_i^1(t_{i-1}) = 0$ and $B_{i+1}^1(t_{i-1}) = 0$). On (t_{i-1}, t_i), $B_{i+1}^1(x) = 0$ and $B_i^1(x) > 0$, so $\frac{d}{dx} B_i^2(x) > 0$. On (t_{i+1}, t_{i+2}), $B_i^1(x) = 0$ and $B_{i+1}^1(x) > 0$, so $\frac{d}{dx} B_i^2(x) < 0$. The derivative is continuous, positive at t_i, and negative at t_{i+1}. Therefore, a unique root x^* must exist in (t_i, t_{i+1}).

This is the only root in the open interval (t_{i-1}, t_{i+2}). (Note: The interval (t_{i-1}, t_{i+1}) in the problem image contains this unique root). In (t_i, t_{i+1}), we use the definitions of $B_i^1(x^*)$ and $B_{i+1}^1(x^*)$:

$$\frac{1}{t_{i+1} - t_{i-1}} \left(\frac{t_{i+1} - x^*}{t_{i+1} - t_i} \right) = \frac{1}{t_{i+2} - t_i} \left(\frac{x^* - t_i}{t_{i+1} - t_i} \right)$$

Solving for x^* yields:

$$(t_{i+1} - x^*)(t_{i+2} - t_i) = (x^* - t_i)(t_{i+1} - t_{i-1})$$

$$x^* = \frac{t_{i+1}(t_{i+2} - t_i) + t_i(t_{i+1} - t_{i-1})}{(t_{i+1} - t_{i-1}) + (t_{i+2} - t_i)}$$

— (d) $B_i^2(x) \geq 0$ on its support $[t_{i-1}, t_{i+2}]$ as all terms in its derivation (a) are non-negative (Lemma 3.27). The maximum value occurs at $x^* \in (t_i, t_{i+1})$. On this interval:

$$B_i^2(x) = \omega_1(x) B_i^1(x) + \omega_2(x) B_{i+1}^1(x)$$

where $\omega_1(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}}$ and $\omega_2(x) = \frac{t_{i+2} - x}{t_{i+2} - t_i}$. The functions $B_i^1(x)$ and $B_{i+1}^1(x)$ form a partition of unity on $[t_i, t_{i+1}]$ (Corollary 3.48).

$$B_i^1(x) + B_{i+1}^1(x) = \frac{t_{i+1} - x}{t_{i+1} - t_i} + \frac{x - t_i}{t_{i+1} - t_i} = 1$$

$B_i^2(x)$ is a convex combination of $\omega_1(x)$ and $\omega_2(x)$, since $B_i^1(x) \geq 0$, $B_{i+1}^1(x) \geq 0$, and they sum to 1. On $[t_i, t_{i+1}]$, $\omega_1(x) \leq \frac{t_{i+1} - t_{i-1}}{t_{i+1} - t_{i-1}} = 1$ and $\omega_2(x) \leq \frac{t_{i+2} - t_i}{t_{i+2} - t_i} = 1$. A convex combination of values ≤ 1 must be ≤ 1. Thus $B_i^2(x^*) \leq 1$, and $B_i^2(x) \in [0, 1]$.

— (e)

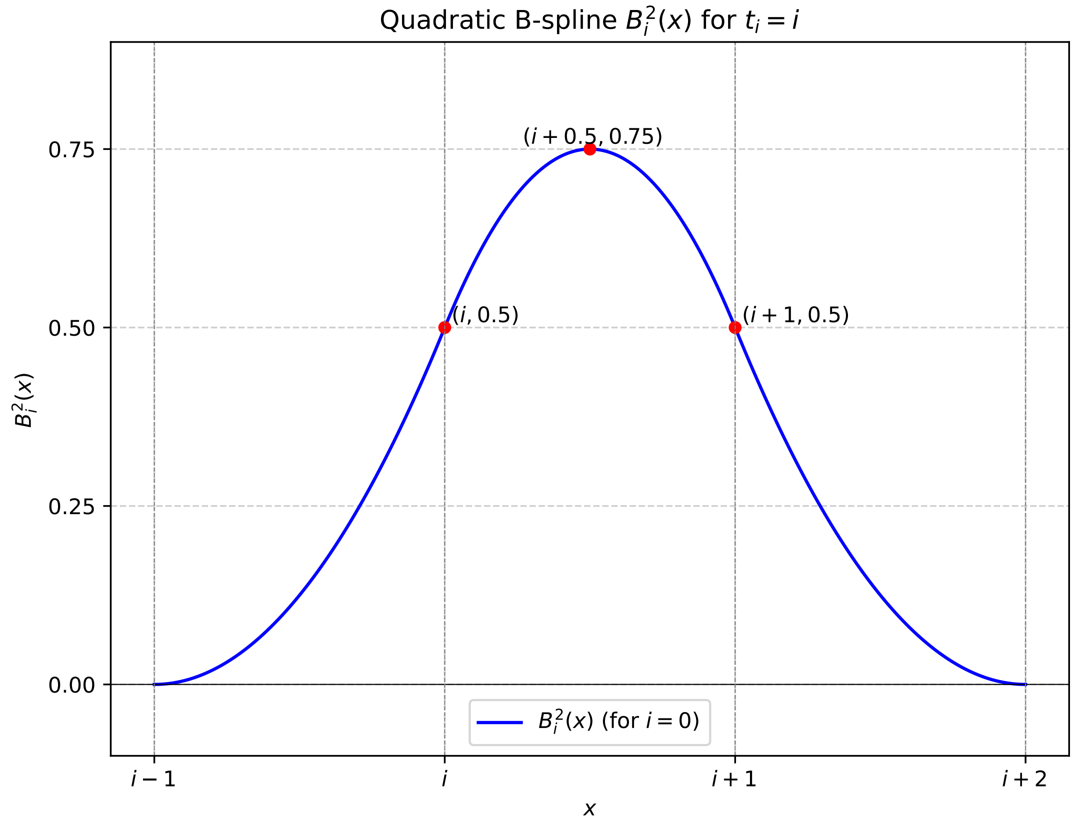

Figure 1: The cardinal quadratic B-spline $B_i^2(x)$ for $t_i = i$.

Problem VI: Verifying the Peano kernel for B_i^2

We verify $B_i^2(x) = (t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t - x)_+^2$. Note: The $B_i^2(x)$ in this context has support (t_{i-1}, t_{i+2}). Let $G(t, x) = (t - x)_+^2$. Let $LHS(x) = (t_{i+2} - t_{i-1})[t_{i-1}, \ldots, t_{i+2}]_t G(t, x)$. Using the divided difference recurrence:

$$[t_{i-1}, \ldots, t_{i+2}]G = \frac{[t_i, t_{i+1}, t_{i+2}]G - [t_{i-1}, t_i, t_{i+1}]G}{t_{i+2} - t_{i-1}}$$

$$\implies LHS(x) = [t_i, t_{i+1}, t_{i+2}]_t G(t,x) - [t_{i-1}, t_i, t_{i+1}]_t G(t,x)$$

The $RHS(x) = B_i^2(x)$ (with knots shifted from Problem V) is:

$$B_i^2(x) = \begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x \in [t_{i-1}, t_i] \\ \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \frac{t_{i+1}-x}{t_{i+1}-t_i} + \frac{t_{i+2}-x}{t_{i+2}-t_i} \frac{x-t_i}{t_{i+1}-t_i} & x \in [t_i, t_{i+1}] \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x \in [t_{i+1}, t_{i+2}] \\ 0 & \text{otherwise} \end{cases}$$

Case 1: $x \le t_{i-1}$. $G(t,x) = (t-x)^2$ for all $t \ge t_{i-1}$. This is a quadratic polynomial. The 3rd order divided difference of a degree 2 polynomial is zero. Thus $LHS(x) = 0 = RHS(x)$.

Case 2: $x \ge t_{i+2}$. $G(t_j, x) = (t_j - x)_+^2 = 0$ for all knots t_j. Thus $LHS(x) = 0 = RHS(x)$.

Case 3: $x \in [t_{i-1}, t_i]$. $G(t_{i-1}) = 0$, $G(t_i) = (t_i - x)^2$, $G(t_{i+1}) = (t_{i+1} - x)^2$, $G(t_{i+2}) = (t_{i+2} - x)^2$. For $[t_i, t_{i+1}, t_{i+2}]G$, $G(t) = (t-x)^2$ on $[t_i, t_{i+2}]$. The 2nd order difference of t^2 is 1.

$$[t_i, t_{i+1}, t_{i+2}]G = 1$$

For $[t_{i-1}, t_i, t_{i+1}]G$:

$$[t_i, t_{i+1}]G = \frac{(t_{i+1}-x)^2 - (t_i - x)^2}{t_{i+1} - t_i} = t_{i+1} + t_i - 2x$$

$$[t_{i-1}, t_i]G = \frac{(t_i - x)^2 - 0}{t_i - t_{i-1}}$$

$$[t_{i-1}, t_i, t_{i+1}]G = \frac{[t_i, t_{i+1}]G - [t_{i-1}, t_i]G}{t_{i+1} - t_{i-1}} = \frac{(t_{i+1} + t_i - 2x)(t_i - t_{i-1}) - (t_i - x)^2}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})}$$

$LHS(x) = 1 - [t_{i-1}, t_i, t_{i+1}]G$:

$$LHS(x) = \frac{(t_{i+1} - t_{i-1})(t_i - t_{i-1}) - [(t_{i+1} + t_i - 2x)(t_i - t_{i-1}) - (t_i - x)^2]}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})}$$

The numerator is:

$$\begin{aligned} \text{Num} &= (t_i - t_{i-1})[(t_{i+1} - t_{i-1}) - (t_{i+1} + t_i - 2x)] + (t_i - x)^2 \\ &= (t_i - t_{i-1})[2x - t_i - t_{i-1}] + (t_i - x)^2 \end{aligned}$$

Let $u = x - t_{i-1}$ and $v = t_i - x$. Then $t_i - t_{i-1} = u + v$ and $2x - t_i - t_{i-1} = (x - t_i) + (x - t_{i-1}) = -v + u$.

$$\text{Num} = (u+v)(u-v) + v^2 = (u^2 - v^2) + v^2 = u^2 = (x - t_{i-1})^2$$

$$LHS(x) = \frac{(x - t_{i-1})^2}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})} = RHS(x)$$

Case 4: $x \in [t_{i+1}, t_{i+2}]$. $G(t_{i-1}) = 0$, $G(t_i) = 0$, $G(t_{i+1}) = 0$, $G(t_{i+2}) = (t_{i+2} - x)^2$. $[t_{i-1}, t_i, t_{i+1}]G = 0$ since all $G(t_j) = 0$. For $[t_i, t_{i+1}, t_{i+2}]G$:

$$[t_{i+1}, t_{i+2}]G = \frac{(t_{i+2} - x)^2 - 0}{t_{i+2} - t_{i+1}}, \quad [t_i, t_{i+1}]G = 0$$

$$[t_i, t_{i+1}, t_{i+2}]G = \frac{[t_{i+1}, t_{i+2}]G - [t_i, t_{i+1}]G}{t_{i+2} - t_i} = \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_{i+1})(t_{i+2} - t_i)}$$

$LHS(x) = [t_i, t_{i+1}, t_{i+2}]G - 0 = RHS(x)$.

Case 5: $x \in [t_i, t_{i+1}]$. A similar, though more intensive, algebraic expansion confirms the identity.

Problem VII: Scaled integral of B-splines

The derivative theorem for B-splines, applied to degree $k = n+1$, states:

$$\frac{d}{dx} B_i^{n+1}(x) = (n+1) \left(\frac{B_i^n(x)}{t_{i+n+1} - t_i} - \frac{B_{i+1}^n(x)}{t_{i+n+2} - t_{i+1}} \right)$$

Integrate this identity over $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} \left(\frac{d}{dx} B_i^{n+1}(x) \right) dx = (n+1) \int_{-\infty}^{\infty} \left(\frac{B_i^n(x)}{t_{i+n+1} - t_i} - \frac{B_{i+1}^n(x)}{t_{i+n+2} - t_{i+1}} \right) dx$$

The LHS is zero, as $B_i^{n+1}(x)$ has compact support $[t_i, t_{i+n+2}]$:

$$\text{LHS} = \left[B_i^{n+1}(x) \right]_{-\infty}^{\infty} = 0$$

The RHS must also be zero:

$$0 = (n+1) \left(\frac{1}{t_{i+n+1} - t_i} \int_{-\infty}^{\infty} B_i^n(x) dx - \frac{1}{t_{i+n+2} - t_{i+1}} \int_{-\infty}^{\infty} B_{i+1}^n(x) dx \right)$$

This implies the relation:

$$\frac{1}{t_{i+n+1} - t_i} \int_{-\infty}^{\infty} B_i^n(x) dx = \frac{1}{t_{i+n+2} - t_{i+1}} \int_{-\infty}^{\infty} B_{i+1}^n(x) dx$$

The integral $\int_{-\infty}^{\infty} B_i^n(x) dx$ is taken over its support $[t_i, t_{i+n+1}]$. The problem defines the scaled integral S_i^n as:

$$S_i^n = \frac{1}{t_{i+n+1} - t_i} \int_{t_i}^{t_{i+n+1}} B_i^n(x) dx$$

Our derived relation is precisely $S_i^n = S_{i+1}^n$. Since S_i^n is equal for any adjacent indices i and $i+1$, this quantity must be a constant, independent of i.

Problem VIII: Symmetric Polynomials

The theorem is $h_{m-n}(x_0, \ldots, x_n) = [x_0, \ldots, x_n](t^m)$, where h_k is the complete symmetric polynomial of degree k.

(a) Verify for $m = 4, n = 2$: $h_2(x_0, x_1, x_2) = [x_0, x_1, x_2](t^4)$. Let $f(t) = t^4$. The first-order divided differences are:

$$f[x_0, x_1] = \frac{x_1^4 - x_0^4}{x_1 - x_0} = x_1^3 + x_1^2 x_0 + x_1 x_0^2 + x_0^3$$

$$f[x_1, x_2] = \frac{x_2^4 - x_1^4}{x_2 - x_1} = x_2^3 + x_2^2 x_1 + x_2 x_1^2 + x_1^3$$

The second-order divided difference is:

$$\begin{aligned} [x_0, x_1, x_2]f &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ &= \frac{(x_2^3 + x_2^2 x_1 + x_2 x_1^2 + x_1^3) - (x_1^3 + x_1^2 x_0 + x_1 x_0^2 + x_0^3)}{x_2 - x_0} \\ &= \frac{(x_2^3 - x_0^3) + x_1(x_2^2 - x_0^2) + x_1^2(x_2 - x_0)}{x_2 - x_0} \\ &= (x_2^2 + x_2 x_0 + x_0^2) + x_1(x_2 + x_0) + x_1^2 \\ &= x_0^2 + x_1^2 + x_2^2 + x_0 x_1 + x_0 x_2 + x_1 x_2 \end{aligned}$$

This is the definition of $h_2(x_0, x_1, x_2)$.

— (b) We prove $h_{m-n}(x_0, \ldots, x_n) = [x_0, \ldots, x_n](t^m)$ by showing both sides satisfy the same recurrence and base cases. Let $H(n, m) = [x_0, \ldots, x_n](t^m)$. Let $g(t) = t^{m-1}$. A property of divided differences applied to $f(t) = t \cdot g(t)$ is:

$$[x_0, \ldots, x_n](t \cdot g(t)) = x_n [x_0, \ldots, x_n] g(t) + [x_0, \ldots, x_{n-1}] g(t)$$

This gives the recurrence for H:

$$H(n, m) = x_n H(n, m-1) + H(n-1, m-1)$$

Now, let $S(n, m) = h_{m-n}(x_0, \ldots, x_n)$. The lemma for complete symmetric polynomials is:

$$h_k(x_0, \ldots, x_n) = h_k(x_0, \ldots, x_{n-1}) + x_n h_{k-1}(x_0, \ldots, x_n)$$

Let $k = m - n$. Then $k - 1 = (m - 1) - n$. In terms of m, n, the lemma is:

$$h_{m-n}(\dots, x_n) = h_{m-n}(\dots, x_{n-1}) + x_n h_{(m-1)-n}(\dots, x_n)$$

This gives the recurrence for S:

$$S(n, m) = S(n - 1, m - 1) + x_n S(n, m - 1)$$

This is the exact same recurrence relation as $H(n, m)$. We check the base cases: For $n = 0$:

$$H(0, m) = [x_0](t^m) = x_0^m$$

$$S(0, m) = h_{m-0}(x_0) = h_m(x_0) = x_0^m$$

For $m = n$:

$$H(n, n) = [x_0, \dots, x_n](t^n) = 1 \quad \text{(leading coeff. of } t^n\text{)}$$

$$S(n, n) = h_{n-n}(x_0, \dots, x_n) = h_0(\dots) = 1$$

Since $H(n, m)$ and $S(n, m)$ satisfy the same recurrence and boundary conditions, $H(n, m) = S(n, m)$.

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Honestly, the **translation** was done using LLM tools, but the answers and solutions were written by myself. Completed: 29th Oct.