Numerical Analysis homework # 1

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I. Analysis of the Bisection Method on the Interval $[1.5, 3.5]$

Width of the interval

Given the initial interval $[a_0, b_0] = [1.5, 3.5]$, the initial width is $W_0 = b_0 - a_0 = 2$. The interval width is halved at each step, so the width at step n, W_n, is:

$$W_n = \frac{W_0}{2^n} = \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

Supremum of the error

At step n, let the interval be $[a_n, b_n]$ and the midpoint $c_n = (a_n + b_n)/2$. The root r satisfies $r \in [a_n, b_n]$. The distance $|r - c_n|$ is maximized when r is located at an endpoint of the interval. This gives the supremum of the error as 1/2 the interval width.

$$\sup |r - c_n| = c_n - a_n = b_n - c_n = \frac{b_n - a_n}{2} = \frac{W_n}{2^1} = \frac{W_{n-1}}{2^2} = \frac{1}{2^n}$$

II. Proof for the Number of Steps for a Given Relative Error

Denote the root by r, we need to find the number of steps n such that $|r - c_n|/|r|$ is no greater than ϵ.

$$\frac{|r - c_n|}{|r|} \leq \epsilon$$

First, the absolute error is bounded by half the interval width at step n.

$$|r - c_n| \leq \frac{b_n - a_n}{2} = \frac{b_0 - a_0}{2^{n+1}}$$

Given that $r \in [a_0, b_0]$ and $a_0 > 0$, the magnitude of the root is bounded below by a_0.

$$|r| \geq a_0$$

Therefore, we can conclude that:

$$\frac{|r - c_n|}{|r|} \leq \frac{(b_0 - a_0)/2^{n+1}}{a_0} = \frac{b_0 - a_0}{a_0 \cdot 2^{n+1}}$$

To guarantee the desired accuracy, we enforce this upper bound to be less than or equal to ϵ.

$$\frac{b_0 - a_0}{a_0 \cdot 2^{n+1}} \leq \epsilon$$

$$\frac{b_0 - a_0}{\epsilon \cdot a_0} \leq 2^{n+1}$$

Taking the logarithm of both sides yields:

$$\log\left(\frac{b_0 - a_0}{\epsilon \cdot a_0}\right) \leq \log(2^{n+1})$$

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$$\log(b_0 - a_0) - \log\epsilon - \log a_0 \leq (n+1)\log 2$$

Rearranging terms :

$$n \geq \frac{\log(b_0 - a_0) - \log\epsilon - \log a_0}{\log 2} - 1$$

This completes the proof.

III. Application of Newton's Method

The problem is to find a root for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. We have the derivative of $p(x)$:

$$p'(x) = 12x^2 - 4x$$

The iteration formula for this specific problem is:

$$x_{k+1} = x_k - \frac{4x_k^3 - 2x_k^2 + 3}{12x_k^2 - 4x_k}$$

Starting with $x_0 = -1$, we perform 4 iterations. The results are organized in the following table.

Iteration, k	x_k	$p(x_k)$	$p'(x_k)$
0	-1.000000	-3.000000	16.000000
1	-0.812500	-0.465820	11.171875
2	-0.770804	-0.020138	10.212886
3	-0.768832	-0.000044	10.168568
4	-0.768828	-0.000000	10.168472

After four iterations, the approximation of the root is x_4.

$$x_4 \approx -0.768828$$

IV. Convergence Analysis of a Modified Newton's Method

Let r be the root, with $f(r) = 0$, and let the error be $e_n = x_n - r$. The error recurrence relation derived from the iteration formula is:

$$e_{n+1} = x_{n+1} - r = e_n - \frac{f(x_n)}{f'(x_0)}$$

Expanding $f(x_n) = f(r + e_n)$ as a Taylor series around r gives:

$$f(x_n) = f'(r)e_n + \frac{f''(r)}{2}e_n^2 + O(e_n^3)$$

$$e_{n+1} = e_n - \frac{f'(r)e_n + \frac{f''(r)}{2}e_n^2 + O(e_n^3)}{f'(x_0)}$$

$$e_{n+1} = \left(1 - \frac{f'(r)}{f'(x_0)}\right)e_n - \left(\frac{f''(r)}{2f'(x_0)}\right)e_n^2 + O(e_n^3)$$

By comparing with the form $e_{n+1} = Ce_n^s$, we identify the exponent of the dominant term as s and its coefficient as C.Therefore, this iteration formula is **linearly convergent**.

$$s = 1$$

$$C = 1 - \frac{f'(r)}{f'(x_0)}$$

V. Convergence of the Iteration $x_{n+1} = \tan^{-1} x_n$

The iteration $x_{n+1} = \tan^{-1} x_n$ has a unique fixed point α that solves $\tan^{-1} \alpha = \alpha$, which is $\alpha = 0$. If $x_0 = 0$, the sequence converges trivially. Thus, we only consider the case where $x_0 \neq 0$. To analyze convergence, we consider the absolute value of the terms. For any $x \neq 0$, it is a known property that:

$$|\tan^{-1} x| < |x|$$

Applying this property to the iteration for any $x_n \neq 0$ gives a strictly **contractive** relationship.

$$|x_{n+1}| = |\tan^{-1} x_n| < |x_n|$$

This shows that the sequence of absolute values, $\{|x_n|\}$, is **strictly decreasing** and **bounded below by 0**. Therefore, the sequence of absolute values must converge to 0.

$$\lim_{n\to\infty} |x_n| = 0$$

This implies that the sequence $\{x_n\}$ itself converges to 0 for all initial values $x_0 \in (-\pi/2, \pi/2)$.

VI. Analysis of a Continued Fraction

Assuming the sequence converges to a value x, this value must be a fixed point satisfying the relation:

$$x_n = \frac{1}{p + \frac{1}{p+\dots}} \implies x = \frac{1}{p+x}$$

The solutions are trivial.

$$x = \frac{-p \pm \sqrt{p^2+4}}{2}$$

Since the sequence of convergents x_n is always positive for $p > 1$, its limit x must be positive. We thus select the positive root.

$$x = \frac{-p + \sqrt{p^2+4}}{2}$$

To prove convergence, we analyze the iteration $x_{n+1} = g(x_n)$ with the function $g(x) = 1/(p+x)$. We will show that $g(x)$ is a contraction mapping for $x \geq 0$. First, we find the derivative.

$$g'(x) = -\frac{1}{(p+x)^2}$$

The magnitude of the derivative is **bounded**.For $p > 1$ and any $x \geq 0$, the denominator $(p+x)^2 > p^2$. This provides a uniform bound on the magnitude of the derivative.

$$|g'(x)| = \frac{1}{(p+x)^2} < \frac{1}{p^2}$$

Since $p > 1$, we have $p^2 > 1$, which implies that $1/p^2 < 1$ and $k = \frac{1}{p^2} < 1$. Because $|g'(x)| \leq k < 1$ for all $x \geq 0$, the function $g(x)$ is a contraction mapping on $[0, \infty)$. Therefore, by the **Contraction Mapping Theorem**, the sequence converges for any initial $x_0 \geq 0$.

VII. Bisection Method Analysis When Initial Interval Contains Zero

Analysis of the Relative Error Measure

Using the symbols from Problem II, the relative error ϵ_n and the absolute error bound $|\Delta\alpha|$ are defined by:

$$\epsilon_n = \left| \frac{x_n - \alpha}{\alpha} \right| = \frac{|\Delta\alpha|}{|\alpha|}$$

We know that the absolute error is bounded by:

$$|\Delta\alpha| \leq \frac{|b_0 - a_0|}{2^{n+1}}, \quad e_n \leq \epsilon$$

To guarantee the relative error ϵ_n is no greater than ϵ ($\epsilon_n \leq \epsilon$):

$$\frac{|\Delta\alpha|}{|\alpha|} \leq \frac{|b_0 - a_0|}{2^{n+1}|\alpha|} \leq \epsilon$$

This inequality can be rearranged to find the required number of steps n, implying that:

$$n \geq \frac{\log(|b_0 - a_0|) - \log \epsilon - \log |\alpha|}{\log 2} - 1$$

This result implies that the number of steps n explicitly depends on the magnitude of the root $|\alpha|$, so the relative error is not a good measure in this case. Especially when α is around 0, the relative error is ∞ in the limit case $\alpha \to 0$. Instead, the absolute error $e_n^{abs} = |x_n - \alpha|$ is a good stand-in.

Derivation of the Absolute Error Inequality

The absolute error is bounded by half the interval length after n steps:

$$e_n^{abs} = |\alpha - c_n| \leq \frac{b_n - a_n}{2} = \frac{b_0 - a_0}{2^{n+1}}$$

To ensure the absolute error $e_n^{abs} \leq \epsilon_{abs}$, we require:

$$\frac{b_0 - a_0}{2^{n+1}} \leq \epsilon_{abs}$$

Solving for n by taking the logarithm yields:

$$n \geq \frac{\log(b_0 - a_0) - \log(\epsilon_{abs})}{\log 2} - 1$$

VIII. Modified Newton's Method for Multiple Roots

A root α is considered a root of multiplicity k for a function $f(x) \in C^{k+1}$ if the function can be expressed in the form:

$$f(x) = (x - \alpha)^k g(x), \quad \text{with } g(x) \in C^1, g(\alpha) \neq 0.+ \tag{14}$$

Detection of a Multiple Zero

Considering the iterative process, we have

$$\begin{aligned} x_{n+1} - \alpha &= x_n - \alpha - \frac{f(x_n)}{f'(x_n)} \\ e_{n+1} &\approx e_n \frac{(k-1)g(x_n)}{kg(x_n) + (x_n - \alpha)g'(x_n)} \\ \implies \lim_{n\to\infty} \frac{e_{n+1}}{e_n} &= \frac{k-1}{k} < 1, \end{aligned} \tag{15}$$

The behavior indicated by (15) can be analyzed more rigorously. By the definition of a root of multiplicity k, Taylor's theorem for $f(x)$ and $f'(x)$ around α gives:

$$f(x_n) = \frac{f^{(k)}(\alpha)}{k!}(x_n - \alpha)^k + O\left((x_n - \alpha)^{k+1}\right)$$

$$f'(x_n) = \frac{f^{(k)}(\alpha)}{(k-1)!}(x_n - \alpha)^{k-1} + O\left((x_n - \alpha)^{k}\right)$$

From these expansions, a primary detection criterion emerges. For a multiple root ($k \geq 2$), it is evident that as $x_n \to \alpha$, both $f(x_n) \to 0$ and $f'(x_n) \to 0$. This is in direct contrast to a simple root ($k = 1$), for which $f'(\alpha) \neq 0$ and thus $\lim_{n\to\infty} f'(x_n) \neq 0$.

Furthermore, the ratio in the Newton's step can be approximated using the leading terms of the expansions:

$$\frac{f(x_n)}{f'(x_n)} \approx \frac{\frac{f^{(k)}(\alpha)}{k!}(x_n - \alpha)^k}{\frac{f^{(k)}(\alpha)}{(k-1)!}(x_n - \alpha)^{k-1}} = \frac{(k-1)!}{k!}(x_n - \alpha) = \frac{1}{k}(x_n - \alpha) = \frac{1}{k}e_n.$$

Substituting this directly into the error evolution formula provides a concise derivation for the linear convergence:

$$\begin{aligned}
e_{n+1} &= e_n - \frac{f(x_n)}{f'(x_n)} \\
&\approx e_n - \frac{1}{k}e_n = \left(1 - \frac{1}{k}\right)e_n = \left(\frac{k-1}{k}\right)e_n.
\end{aligned}$$

This rigorously confirms the result in (15) and shows that the observed linear convergence is a direct consequence of the fact that $f'(x_n)$ approaches zero at a rate precisely one order lower than $f(x_n)$.

Restoring Quadratic Convergence

To restore the quadratic rate of convergence, we can apply a modified Newton's method. The analysis of this modified process is as follows:

$$\begin{aligned}
x_{n+1} - \alpha &= x_n - k\frac{f(x_n)}{f'(x_n)} - \alpha \\
&= x_n - k\frac{(x_n-\alpha)g(x_n)}{(x_n-\alpha)g'(x_n) + kg(x_n)} - \alpha \\
&= (x_n - \alpha)\left[1 - \frac{kg(x_n)}{(x_n-\alpha)g'(x_n) + kg(x_n)}\right] \\
&= (x_n-\alpha)^2 g'(x_n)\frac{1}{(x_n-\alpha)g'(x_n) + kg(x_n)},
\end{aligned}$$

This leads to the following limit for the error ratio, confirming that quadratic convergence is recovered:

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{e_n^2} = \lim_{n\to\infty} \left|\frac{g'(x_n)}{(x_n-\alpha)g'(x_n) + kg(x_n)}\right| = \left|\frac{g'(\alpha)}{kg(\alpha)}\right|.$$

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Honestly, the **translation** was done using LLM tools, but the answers and solutions were written by myself. This version is edited out on 9th Oct.