

# Numerical Analysis homework # 2

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## I. Analysis of the Linear Interpolation for $\frac{1}{x}$

### i) Determining $\xi(x)$ Explicitly

The function values at the nodes are  $f(x_0) = f(1) = 1$  and  $f(x_1) = f(2) = \frac{1}{2}$ .

$$p_1(f; x) = f(x_0) \frac{x - x_1}{x_0 - x_1} + f(x_1) \frac{x - x_0}{x_1 - x_0}$$

$$p_1(f; x) = -(x - 2) + \frac{1}{2}(x - 1) = -\frac{1}{2}x + \frac{3}{2}$$

$$R_1(x) = \frac{1}{x} - \left(-\frac{1}{2}x + \frac{3}{2}\right) = \frac{(x - 1)(x - 2)}{2x}$$

$$f''(x) = 2x^{-3} = \frac{2}{x^3}$$

$$\frac{(x - 1)(x - 2)}{2x} = \frac{f''(\xi(x))}{2}(x - 1)(x - 2)$$

Since  $x \in (1, 2)$ , the term  $(x - 1)(x - 2)$  is non-zero, allowing us to simplify the equation:

$$\frac{1}{2x} = \frac{f''(\xi(x))}{2} \implies f''(\xi(x)) = \frac{1}{x}$$

Substituting the expression for the second derivative,  $f''(\xi(x)) = \frac{2}{(\xi(x))^3}$ :

$$\frac{2}{(\xi(x))^3} = \frac{1}{x} \implies (\xi(x))^3 = 2x$$

Thus:

$$\xi(x) = \sqrt[3]{2x}, \quad x \in (1, 2)$$

### ii) Extension and Calculation of Extrema

Since  $\xi(x)$  is a **monotonically** increasing function on  $[1, 2]$ , its minimum and maximum values occur at the endpoints, its maximum value occurs at the left endpoint  $x = 1$ :

$$\min_{x \in [1, 2]} \xi(x) = \xi(1) = \sqrt[3]{2 \cdot 1} = \sqrt[3]{2}$$

$$\max_{x \in [1, 2]} \xi(x) = \xi(2) = \sqrt[3]{2 \cdot 2} = \sqrt[3]{4}$$

$$\max_{x \in [1, 2]} f''(\xi(x)) = \max_{x \in [1, 2]} \left(\frac{1}{x}\right) = \frac{1}{1} = 1$$

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## II. Construction of a Non-negative Interpolating Polynomial

To find a polynomial  $p \in \mathbb{P}_{2n}^+$  such that it interpolates the given data points  $(x_i, f_i)$  for  $i = 0, 1, \dots, n$ , where  $f_i \geq 0$ .  
 Let  $l_k(x)$  for  $k = 0, 1, \dots, n$  be the **Lagrange basis polynomials** corresponding to the distinct nodes  $\{x_j\}_{j=0}^n$ . These polynomials satisfy the property  $l_k(x_j) = \delta_{kj}$ .

### Construction A

Consider the polynomial defined as a sum of squares:

$$p(x) = \sum_{k=0}^n f_k l_k^2(x)$$

The degree of each basis polynomial  $l_k(x)$  is  $n$ , which implies the degree of  $l_k^2(x)$  is  $2n$ . Consequently, the degree of  $p(x)$  is at most  $2n$ .

Given that  $f_k \geq 0$  and  $l_k^2(x) \geq 0$  for any real  $x$ , the sum  $p(x)$  is non-negative for all  $x \in \mathbb{R}$ . Thus,  $p(x) \in \mathbb{P}_{2n}^+$ .  
 Verifying the interpolation conditions at the nodes  $x_i$ :

$$p(x_i) = \sum_{k=0}^n f_k l_k^2(x_i) = f_i l_i^2(x_i) + \sum_{k \neq i} f_k l_k^2(x_i) = f_i (1)^2 + \sum_{k \neq i} f_k (0)^2 = f_i$$

This construction satisfies all requirements.

### Construction B

Alternatively, consider the polynomial formed by squaring a sum:

$$p(x) = \left( \sum_{k=0}^n \sqrt{f_k} \times l_k(x) \right)^2$$

Let  $S(x) = \sum_{k=0}^n \sqrt{f_k} l_k(x)$ . Since each  $l_k(x)$  is of degree  $n$ , the degree of  $S(x)$  is at most  $n$ . Therefore, the degree of  $p(x) = [S(x)]^2$  is at most  $2n$ .

The squared form inherently ensures that  $p(x) \geq 0$  for all real  $x$ , so  $p(x) \in \mathbb{P}_{2n}^+$ .  
 Evaluating at the interpolation nodes  $x_i$ :

$$p(x_i) = \left( \sum_{k=0}^n \sqrt{f_k} l_k(x_i) \right)^2 = \left( \sqrt{f_i} l_i(x_i) \right)^2 = \left( \sqrt{f_i} \cdot 1 \right)^2 = f_i$$

This alternative also fulfills all the problem's conditions.

## III. Analysis of Divided Differences for $f(x) = e^x$

### Proof by Induction

Let the statement be  $P(n) : f[t, t+1, \dots, t+n] = \frac{(e-1)^n}{n!} e^t$ . The base case  $n = 0$  is trivial, as  $f[t] = e^t = \frac{(e-1)^0}{0!} e^t$ . Assume  $P(n)$  holds for some  $n \geq 0$ . For  $n+1$ , the recursive definition of divided differences is:

$$f[t, \dots, t+n+1] = \frac{f[t+1, \dots, t+n+1] - f[t, \dots, t+n]}{n+1}$$

By the inductive hypothesis, the numerator is:

$$\frac{(e-1)^n}{n!} e^{t+1} - \frac{(e-1)^n}{n!} e^t = \frac{(e-1)^n}{n!} e^t (e-1) = \frac{(e-1)^{n+1}}{n!} e^t$$

Substituting this back confirms  $P(n+1)$ :

$$f[t, \dots, t+n+1] = \frac{1}{n+1} \left( \frac{(e-1)^{n+1}}{n!} e^t \right) = \frac{(e-1)^{n+1}}{(n+1)!} e^t$$

Thus, the formula holds for all integers  $n \geq 0$ .

## Determination of $\xi$

From the result above, setting  $t = 0$  yields:

$$f[0, 1, \dots, n] = \frac{(e-1)^n}{n!}$$

Corollary 2.22 provides an alternative expression for some  $\xi \in (0, n)$ :

$$f[0, 1, \dots, n] = \frac{f^{(n)}(\xi)}{n!}$$

For  $f(x) = e^x$ , the  $n$ -th derivative is  $f^{(n)}(x) = e^x$ . Equating the two expressions gives:

$$\frac{(e-1)^n}{n!} = \frac{e^\xi}{n!}$$

Solving for  $\xi$  directly gives:

$$\xi = n \ln(e-1) > n \ln(1.7) = \frac{n}{2} \ln(2.89) > \frac{n}{2} \ln(e) = \frac{n}{2}$$

The point  $\xi$  is located to the right of the midpoint of the interval  $(0, n)$ .

## IV. Newton Interpolation and Minimum Approximation

### Newton Form of the Interpolating Polynomial

The given data points are  $(0, 5)$ ,  $(1, 3)$ ,  $(3, 5)$ , and  $(4, 12)$ . The divided differences are computed and organized in the following table:

$i$	$x_i$	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, \dots, x_i]$	$f[x_{i-3}, \dots, x_i]$
0	0	5			
1	1	3	-2		
2	3	5	1	1	
3	4	12	7	2	1/4

The Newton form of the interpolating polynomial  $p_3(x)$  is constructed using the top diagonal of the table.

$$p_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

Substituting the coefficients and nodes gives:

$$p_3(x) = 5 - 2x + 1 \cdot x(x - 1) + \frac{1}{4}x(x - 1)(x - 3)$$

### Approximation of the Minimum

To find the minimum, we first find the derivative of the polynomial  $p_3(x)$ . Expanding the polynomial simplifies differentiation:

$$p_3(x) = 5 - 2x + (x^2 - x) + \frac{1}{4}(x^3 - 4x^2 + 3x) = \frac{1}{4}x^3 - \frac{9}{4}x + 5$$

The derivative is:

$$p'_3(x) = \frac{3}{4}x^2 - \frac{9}{4} = \frac{3}{4}(x^2 - 3)$$

Setting the derivative to zero,  $p'_3(x) = 0$ , yields the critical points.

$$x^2 - 3 = 0 \implies x = \pm\sqrt{3}$$

To confirm it is a minimum, we check the second derivative:

$$p''_3(x) = \frac{3}{2}x$$

At  $x = \sqrt{3}$ , we have  $p''_3(\sqrt{3}) = \frac{3\sqrt{3}}{2} > 0$ , which confirms a local minimum. Thus, the approximate location of the minimum is:

$$x_{\min} \approx \sqrt{3}$$

## V. Divided Differences with Repeated Nodes for $f(x) = x^7$

### Computation of the Divided Difference

To handle the repeated nodes, we require the derivatives of  $f(x) = x^7$ .

$$f'(x) = 7x^6, \quad f''(x) = 42x^5$$

The values needed for the divided difference table are:

$$f'(1) = 7, \quad f''(1) = 42, \quad f'(2) = 448$$

We construct the divided difference table for the nodes  $z = \{0, 1, 1, 1, 2, 2\}$ . For repeated nodes  $z_i = z_{i+k}$ , the higher-order differences are defined using derivatives, such as  $f[z_i, \dots, z_{i+k}] = f^{(k)}(z_i)/k!$ .

$i$	$z_i$	$f[z_i]$	1st	2nd	3rd	4th	5th
0	0	0					
1	1	1	1				
2	1	1	7	6			
3	1	1	7	21	15		
4	2	128	127	120	99	42	
5	2	128	448	321	201	102	30

The final entry in the table gives the desired value.

$$f[0, 1, 1, 1, 2, 2] = 30$$

### Determination of $\xi$

The Mean Value Theorem for divided differences relates the divided difference to a derivative.

$$f[z_0, \dots, z_n] = \frac{f^{(n)}(\xi)}{n!} \quad \text{for some } \xi \in (\min(z_i), \max(z_i))$$

For this problem, we have  $n = 5$  and the nodes are within the interval  $(0, 2)$ . The 5th derivative of  $f(x)$  is calculated as:

$$f^{(5)}(x) = 2520x^2$$

By substituting the computed divided difference and  $n = 5$  into the theorem, we get an equation for  $\xi$ .

$$30 = \frac{f^{(5)}(\xi)}{5!} = \frac{2520\xi^2}{120}$$

We solve this equation for  $\xi$ :

$$\xi^2 = \frac{30 \cdot 120}{2520} = \frac{3600}{2520} = \frac{10}{7}$$

Since  $\xi$  must be in  $(0, 2)$ , we take the positive square root.

$$\xi = \sqrt{\frac{10}{7}} = \frac{\sqrt{70}}{7}$$

## VI. Hermite Interpolation and Error Analysis

### Estimate of $f(2)$

The Hermite interpolation problem is defined by the nodes  $z = \{0, 1, 1, 3, 3\}$ . We use the divided difference method to find the interpolating polynomial  $H_4(x)$ . The required derivative values are  $f'(1) = -1$  and  $f'(3) = 0$ . The divided difference table is constructed as follows:

$i$	$z_i$	$f[z_i]$	1st	2nd	3rd	4th
0	0	1				
1	1	2	1			
2	1	2	-1	-2		
3	3	0	-1	0	2/3	
4	3	0	0	1/2	1/4	-5/36

The coefficients from the top diagonal are  $1, 1, -2, 2/3, -5/36$ . The Newton form of the polynomial is:

$$H_4(x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3)$$

$$H_4(2) = -1 + \frac{4}{3} + \frac{10}{36} = -1 + \frac{24}{18} + \frac{5}{18} = \frac{-18 + 24 + 5}{18} = \frac{11}{18}$$

## Maximum Possible Error

The error in Hermite interpolation for a function  $f \in C^5[0, 3]$  is given by the formula:

$$E_4(x) = f(x) - H_4(x) = \frac{f^{(5)}(\xi)}{5!} \prod_{i=0}^4 (x - z_i)$$

where  $\xi$  is in the interval spanned by the nodes,  $(0, 3)$ . We want to bound the error at  $x = 2$ . The node product at this point is:

$$\Psi(2) = (2-0)(2-1)(2-1)(2-3)(2-3) = (2)(1)(1)(-1)(-1) = 2$$

The error at  $x = 2$  is therefore:

$$|E_4(2)| = \left| \frac{f^{(5)}(\xi)}{5!} \cdot \Psi(2) \right| = \left| \frac{f^{(5)}(\xi)}{120} \cdot 2 \right| = \frac{|f^{(5)}(\xi)|}{60}$$

Given that  $|f^{(5)}(x)| \leq M$  on  $[0, 3]$ , we can bound the error.

$$|E_4(2)| \leq \frac{M}{60}$$

The maximum possible error for the estimate  $f(2) \approx 11/18$  is  $M/60$ .

## VII. Relation Between Finite and Divided Differences

### Forward Difference

We prove the identity  $\Delta^k f(x) = k!h^k f[x_0, x_1, \dots, x_k]$  by induction on  $k$ , where  $x_j = x + jh$ . For the base case  $k = 1$ , we use the definitions of the operators.

$$\Delta f(x) = f(x+h) - f(x) = f(x_1) - f(x_0)$$

The right-hand side is:

$$1!h^1 f[x_0, x_1] = h \frac{f(x_1) - f(x_0)}{x_1 - x_0} = h \frac{f(x_1) - f(x_0)}{h} = f(x_1) - f(x_0)$$

The identity holds for  $k = 1$ . Now, assume the proposition is true for an integer  $k \geq 1$ . We examine the case for  $k + 1$ .

$$\Delta^{k+1} f(x) = \Delta^k f(x+h) - \Delta^k f(x)$$

Applying the inductive hypothesis to both terms on the right-hand side yields:

$$\Delta^{k+1} f(x) = k!h^k f[x_1, \dots, x_{k+1}] - k!h^k f[x_0, \dots, x_k] = k!h^k (f[x_1, \dots, x_{k+1}] - f[x_0, \dots, x_k])$$

$$f[x_0, \dots, x_{k+1}] = \frac{f[x_1, \dots, x_{k+1}] - f[x_0, \dots, x_k]}{x_{k+1} - x_0}$$

$$\Delta^{k+1} f(x) = k!h^k ((x_{k+1} - x_0)f[x_0, \dots, x_{k+1}]) = k!h^k (k+1)hf[x_0, \dots, x_{k+1}]$$

This simplifies to the desired result for  $k + 1$ .

$$\Delta^{k+1} f(x) = (k+1)!h^{k+1} f[x_0, \dots, x_{k+1}]$$

By the principle of induction, the formula is proven for all  $k \geq 1$ .

## Backward Difference

A similar proof by induction can be constructed. Alternatively, we can use the established relationship between the backward and forward difference operators,  $\nabla f(x) = \Delta f(x - h)$ . By repeated application, this leads to the identity  $\nabla^k f(x) = \Delta^k f(x - kh)$ . We apply the proven forward difference formula to the function evaluated at the point  $z = x - kh$ :

$$\Delta^k f(z) = k!h^k f[z, z + h, \dots, z + kh]$$

$$\{x - kh, (x - kh) + h, \dots, (x - kh) + kh\} = \{x - kh, x - (k - 1)h, \dots, x\}$$

This set of nodes is precisely  $\{x_{-k}, x_{-k+1}, \dots, x_0\}$ . Therefore, we have:

$$\nabla^k f(x) = \Delta^k f(x - kh) = k!h^k f[x_{-k}, x_{-k+1}, \dots, x_0]$$

$$\nabla^k f(x) = k!h^k f[x_0, x_{-1}, \dots, x_{-k}]$$

This completes the proof.

## VIII. Derivative of Divided Differences

### Partial Derivative with respect to $x_0$

By the definition of the partial derivative, we have:

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \dots, x_n] = \lim_{h \rightarrow 0} \frac{f[x_0 + h, x_1, \dots, x_n] - f[x_0, x_1, \dots, x_n]}{h}$$

$$f[x_0 + h, x_1, \dots, x_n] = f[x_1, \dots, x_n, x_0 + h]$$

$$\frac{f[x_1, \dots, x_n, x_0 + h] - f[x_0, x_1, \dots, x_n]}{(x_0 + h) - x_0}$$

This is precisely the recursive definition for a higher-order divided difference over the nodes  $\{x_0, x_1, \dots, x_n, x_0 + h\}$ .

$$\frac{\partial}{\partial x_0} f[x_0, \dots, x_n] = \lim_{h \rightarrow 0} f[x_0, x_1, \dots, x_n, x_0 + h]$$

$$\lim_{h \rightarrow 0} f[x_0, x_1, \dots, x_n, x_0 + h] = f[x_0, x_1, \dots, x_n, x_0]$$

$$\frac{\partial}{\partial x_0} f[x_0, \dots, x_n] = f[x_0, x_0, x_1, \dots, x_n]$$

### For other variables

The divided difference  $f[x_0, \dots, x_n]$  is a symmetric function of its arguments. This allows the result to be generalized to the partial derivative with respect to any other variable  $x_i$ .

$$f[x_0, \dots, x_i, \dots, x_n] = f[x_i, x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$$

Let  $Z$  be the set of all other variables  $\{x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ .

$$\frac{\partial}{\partial x_i} f[x_i, Z] = f[x_i, x_i, Z]$$

Substituting the set  $Z$  back gives the general formula for any  $i \in \{0, 1, \dots, n\}$ .

$$\frac{\partial}{\partial x_i} f[x_0, \dots, x_n] = f[x_i, x_i, x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$$

This result represents the divided difference over the original set of nodes with the node  $x_i$  repeated.

## IX. A Min-Max Polynomial Problem

Let  $P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ . The problem is to determine the value of  $\min_{a_1, \dots, a_n} \max_{x \in [a, b]} |P_n(x)|$ . We can factor out the fixed leading coefficient  $|a_0|$ :

$$\max_{x \in [a, b]} |P_n(x)| = |a_0| \max_{x \in [a, b]} \left| x^n + \frac{a_1}{a_0}x^{n-1} + \dots + \frac{a_n}{a_0} \right|$$

Minimizing over  $a_1, \dots, a_n$  is equivalent to minimizing over the coefficients of the resulting monic polynomial. Let  $\mathcal{P}_n^M$  be the set of all monic polynomials of degree  $n$ . The problem is transformed into:

$$|a_0| \min_{p \in \mathcal{P}_n^M} \max_{x \in [a, b]} |p(x)|$$

We use the linear transformation  $x = \frac{b-a}{2}t + \frac{a+b}{2}$  to map the interval  $t \in [-1, 1]$  onto  $x \in [a, b]$ . A monic polynomial  $p(x) \in \mathcal{P}_n^M$  can be expressed in terms of  $t$ :

$$p(x) = \left( \frac{b-a}{2}t + \frac{a+b}{2} \right)^n + \dots = \left( \frac{b-a}{2} \right)^n t^n + \text{lower degree terms in } t$$

This implies that  $p(x)$  can be written as  $\left( \frac{b-a}{2} \right)^n q(t)$ , where  $q(t)$  is a monic polynomial in  $t$  of degree  $n$ . The min-max problem over  $[a, b]$  becomes a corresponding problem over  $[-1, 1]$ .

$$\min_{p \in \mathcal{P}_n^M} \max_{x \in [a, b]} |p(x)| = \min_{q \in \mathcal{P}_n^M} \max_{t \in [-1, 1]} \left| \left( \frac{b-a}{2} \right)^n q(t) \right| = \left( \frac{b-a}{2} \right)^n \min_{q \in \mathcal{P}_n^M} \max_{t \in [-1, 1]} |q(t)|$$

A fundamental theorem of approximation theory states that the minimum value of the maximum absolute value for a monic polynomial of degree  $n$  on  $[-1, 1]$  is achieved by the monic Chebyshev polynomial  $\tilde{T}_n(t) = T_n(t)/2^{n-1}$ .

$$\min_{q \in \mathcal{P}_n^M} \max_{t \in [-1, 1]} |q(t)| = \max_{t \in [-1, 1]} |\tilde{T}_n(t)| = \frac{1}{2^{n-1}}$$

$$\min_{a_1, \dots, a_n} \max_{x \in [a, b]} |P_n(x)| = |a_0| \left( \frac{b-a}{2} \right)^n \frac{1}{2^{n-1}} = |a_0| \frac{(b-a)^n}{2^{2n-1}}$$

## X. A Minimization Property of Scaled Chebyshev Polynomials

We proceed by contradiction. Assume there exists a polynomial  $p(x) \in \mathbb{P}_n^a$  such that  $\|p\|_\infty < \|\hat{p}_n\|_\infty$ . Define a new polynomial  $Q(x) = \hat{p}_n(x) - p(x)$ . Since  $\hat{p}_n$  and  $p$  are both polynomials of degree  $n$ ,  $Q(x)$  is a polynomial of degree at most  $n$ .

The Chebyshev polynomial  $T_n(x)$  attains its extrema on  $[-1, 1]$  at the  $n+1$  points  $x_k = \cos(k\pi/n)$  for  $k = 0, 1, \dots, n$ , where  $T_n(x_k) = (-1)^k$ . The polynomial  $\hat{p}_n(x) = T_n(x)/T_n(a)$  therefore satisfies:

$$\hat{p}_n(x_k) = \frac{(-1)^k}{T_n(a)}, \quad \forall k = 0, 1, \dots, n, x_k = \cos\left(\frac{k}{n}\pi\right).$$

Since  $a > 1$ ,  $T_n(a) > 0$ . The maximum absolute value of  $\hat{p}_n(x)$  on  $[-1, 1]$  is thus  $\|\hat{p}_n\|_\infty = 1/T_n(a)$ .

At these points  $x_k$ , we evaluate  $Q(x_k)$ :

$$Q(x_k) = \hat{p}_n(x_k) - p(x_k)$$

From our initial assumption,  $|p(x_k)| \leq \|p\|_\infty < \|\hat{p}_n\|_\infty = |\hat{p}_n(x_k)|$ . This implies that  $p(x_k)$  cannot change the sign of  $\hat{p}_n(x_k)$ . Thus, for  $k = 0, \dots, n$ :

$$\text{sign}(Q(x_k)) = \text{sign}(\hat{p}_n(x_k)) = \text{sign}((-1)^k)$$

Since  $Q(x)$  alternates in sign across the  $n+1$  points  $x_k$ , by the Intermediate Value Theorem,  $Q(x)$  must have at least  $n$  distinct roots in the interval  $(-1, 1)$ .

Furthermore, both  $\hat{p}_n(x)$  and  $p(x)$  are in  $\mathbb{P}_n^a$ , which means  $\hat{p}_n(a) = 1$  and  $p(a) = 1$ . Therefore,  $Q(x)$  has an additional root at  $x = a$ :

$$Q(a) = \hat{p}_n(a) - p(a) = 1 - 1 = 0$$

Since  $a > 1$ , this root is distinct from the  $n$  roots found within  $(-1, 1)$ . This gives  $Q(x)$  at least  $n+1$  distinct roots.

We have established that  $Q(x)$  is a polynomial of degree at most  $n$  with at least  $n+1$  roots. This is only possible if  $Q(x)$  is identically zero. If  $Q(x) \equiv 0$ , then  $p(x) \equiv \hat{p}_n(x)$ , which implies  $\|p\|_\infty = \|\hat{p}_n\|_\infty$ . This contradicts our initial assumption that  $\|p\|_\infty < \|\hat{p}_n\|_\infty$ . The assumption must be false, and therefore, for any  $p \in \mathbb{P}_n^a$ , we must have  $\|\hat{p}_n\|_\infty \leq \|p\|_\infty$ .

## XI. Proof of the Degree Elevation Property for Bernstein Polynomials

The proof begins by expanding the right-hand side (RHS) of the identity using the definition of a Bernstein basis polynomial,  $b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$ .

$$\begin{aligned} \text{RHS} &= \frac{n-k}{n} b_{n,k}(t) + \frac{k+1}{n} b_{n,k+1}(t) \\ \text{RHS} &= \frac{n-k}{n} \binom{n}{k} t^k (1-t)^{n-k} + \frac{k+1}{n} \binom{n}{k+1} t^{k+1} (1-t)^{n-k-1} \end{aligned}$$

For the first term's coefficient:

$$\frac{n-k}{n} \binom{n}{k} = \frac{n-k}{n} \frac{n!}{k!(n-k)!} = \frac{n-k}{n} \frac{n(n-1)!}{k!(n-k)(n-k-1)!} = \frac{(n-1)!}{k!(n-k-1)!} = \binom{n-1}{k}$$

For the second term's coefficient:

$$\frac{k+1}{n} \binom{n}{k+1} = \frac{k+1}{n} \frac{n!}{(k+1)!(n-k-1)!} = \frac{k+1}{n} \frac{n(n-1)!}{(k+1)k!(n-k-1)!} = \frac{(n-1)!}{k!(n-k-1)!} = \binom{n-1}{k}$$

$$\begin{aligned} \text{RHS} &= \binom{n-1}{k} t^k (1-t)^{n-k} + \binom{n-1}{k} t^{k+1} (1-t)^{n-k-1} \\ \text{RHS} &= \binom{n-1}{k} t^k (1-t)^{n-k-1} [(1-t) + t] \\ \text{RHS} &= \binom{n-1}{k} t^k (1-t)^{(n-1)-k} = b_{n-1,k}(t) = \text{LHS} \end{aligned}$$

The final expression is the definition of  $b_{n-1,k}(t)$ , which is the left-hand side (LHS) of the identity. This completes the proof.

## XII. Proof of the Integral Property of Bernstein Basis Polynomials

Let  $I_{n,k}$  denote the integral of the Bernstein basis polynomial  $b_{n,k}(t)$  over the interval  $[0, 1]$ .

$$I_{n,k} = \int_0^1 b_{n,k}(t) dt = \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} dt$$

We apply integration by parts to the integral, with  $u = (1-t)^{n-k}$  and  $dv = t^k dt$ . This gives  $du = -(n-k)(1-t)^{n-k-1} dt$  and  $v = t^{k+1}/(k+1)$ .

$$I_{n,k} = \binom{n}{k} \left[ \frac{t^{k+1}(1-t)^{n-k}}{k+1} \Big|_0^1 + \frac{n-k}{k+1} \int_0^1 t^{k+1}(1-t)^{n-k-1} dt \right]$$

The boundary term evaluates to zero for  $k < n$ . Thus, the expression simplifies to:

$$\begin{aligned} I_{n,k} &= \frac{n-k}{k+1} \binom{n}{k} \int_0^1 t^{k+1}(1-t)^{n-(k+1)} dt \\ \frac{n-k}{k+1} \binom{n}{k} &= \frac{n-k}{k+1} \frac{n!}{k!(n-k)!} = \frac{n!}{(k+1)!(n-k-1)!} = \binom{n}{k+1} \\ I_{n,k} &= \binom{n}{k+1} \int_0^1 t^{k+1}(1-t)^{n-(k+1)} dt = I_{n,k+1} \end{aligned}$$

This relation holds for  $k = 0, 1, \dots, n-1$ , implying that the value of the integral  $I_{n,k}$  is constant for all  $k$ . To find this constant value, we use the partition of unity property of Bernstein basis polynomials:

$$\sum_{k=0}^n b_{n,k}(t) = 1$$



Integrating this identity over  $[0, 1]$  yields:

$$\int_0^1 \sum_{k=0}^n b_{n,k}(t) dt = \sum_{k=0}^n \int_0^1 b_{n,k}(t) dt = \int_0^1 1 dt = 1$$

Since all  $n + 1$  integrals in the sum are equal, we have:

$$(n + 1)I_{n,k} = 1$$

This leads to the final result, which is independent of  $k$ .

$$\int_0^1 b_{n,k}(t) dt = \frac{1}{n + 1}$$

### Acknowledgement

Honestly, the **translation** was done using LLM tools, but the answers and solutions were written by myself. This version is edited out on 19th Oct.