Numerical Analysis homework # 1

Wang Hengning 3230104148 *

Computational mathematics 2301, Zhejiang University

Due time: September 19, 2025

I. Analysis of the Bisection Method on the Interval $[1.5, 3.5]$

Width of the interval

Given the initial interval $[a_0, b_0] = [1.5, 3.5]$, the initial width is $W_0 = b_0 - a_0 = 2$. The interval width is halved at each step, so the width at step n, W_n, is:

$$W_n = \frac{W_0}{2^n} = \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

Supremum of the error

At step n, let the interval be $[a_n, b_n]$ and the midpoint $c_n = (a_n + b_n)/2$. The root r satisfies $r \in [a_n, b_n]$. The distance $|r - c_n|$ is maximized when r is located at an endpoint of the interval. This gives the supremum of the error as 1/2 the interval width.

$$\sup |r - c_n| = c_n - a_n = b_n - c_n = \frac{b_n - a_n}{2} = \frac{W_n}{2^1} = \frac{W_{n-1}}{2^2} = \frac{1}{2^n}$$

II. Proof for the Number of Steps for a Given Relative Error

Denote the root by r, we need to find the number of steps n such that $|r - c_n|/|r|$ is no greater than ϵ.

$$\frac{|r - c_n|}{|r|} \leq \epsilon$$

First, the absolute error is bounded by half the interval width at step n.

$$|r - c_n| \leq \frac{b_n - a_n}{2} = \frac{b_0 - a_0}{2^{n+1}}$$

Given that $r \in [a_0, b_0]$ and $a_0 > 0$, the magnitude of the root is bounded below by a_0.

$$|r| \geq a_0$$

Therefore, we can conclude that:

$$\frac{|r - c_n|}{|r|} \leq \frac{(b_0 - a_0)/2^{n+1}}{a_0} = \frac{b_0 - a_0}{a_0 \cdot 2^{n+1}}$$

To guarantee the desired accuracy, we enforce this upper bound to be less than or equal to ϵ.

$$\frac{b_0 - a_0}{a_0 \cdot 2^{n+1}} \leq \epsilon$$

$$\frac{b_0 - a_0}{\epsilon \cdot a_0} \leq 2^{n+1}$$

Taking the logarithm of both sides yields:

$$\log\left(\frac{b_0 - a_0}{\epsilon \cdot a_0}\right) \leq \log(2^{n+1})$$

*Electronic address: `whning@zju.edu.cn`

$$\log(b_0 - a_0) - \log\epsilon - \log a_0 \leq (n+1)\log 2$$

Rearranging terms :

$$n \geq \frac{\log(b_0 - a_0) - \log\epsilon - \log a_0}{\log 2} - 1$$

This completes the proof.

III. Application of Newton's Method

The problem is to find a root for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. We have the derivative of $p(x)$:

$$p'(x) = 12x^2 - 4x$$

The iteration formula for this specific problem is:

$$x_{k+1} = x_k - \frac{4x_k^3 - 2x_k^2 + 3}{12x_k^2 - 4x_k}$$

Starting with $x_0 = -1$, we perform 4 iterations. The results are organized in the following table.

Iteration, k	x_k	$p(x_k)$	$p'(x_k)$
0	-1.000000	-3.000000	16.000000
1	-0.812500	-0.465820	11.171875
2	-0.770804	-0.020138	10.212886
3	-0.768832	-0.000044	10.168568
4	-0.768828	-0.000000	10.168472

After four iterations, the approximation of the root is x_4.

$$x_4 \approx -0.768828$$

IV. Convergence Analysis of a Modified Newton's Method

Let r be the root, with $f(r) = 0$, and let the error be $e_n = x_n - r$. The error recurrence relation derived from the iteration formula is:

$$e_{n+1} = x_{n+1} - r = e_n - \frac{f(x_n)}{f'(x_0)}$$

Expanding $f(x_n) = f(r + e_n)$ as a Taylor series around r gives:

$$f(x_n) = f'(r)e_n + \frac{f''(r)}{2}e_n^2 + O(e_n^3)$$

$$e_{n+1} = e_n - \frac{f'(r)e_n + \frac{f''(r)}{2}e_n^2 + O(e_n^3)}{f'(x_0)}$$

$$e_{n+1} = \left(1 - \frac{f'(r)}{f'(x_0)}\right)e_n - \left(\frac{f''(r)}{2f'(x_0)}\right)e_n^2 + O(e_n^3)$$

By comparing with the form $e_{n+1} = Ce_n^s$, we identify the exponent of the dominant term as s and its coefficient as C.Therefore, this iteration formula is **linearly convergent**.

$$s = 1$$

$$C = 1 - \frac{f'(r)}{f'(x_0)}$$

V. Convergence of the Iteration $x_{n+1} = \tan^{-1} x_n$

The iteration $x_{n+1} = \tan^{-1} x_n$ has a unique fixed point α that solves $\tan^{-1}\alpha = \alpha$, which is $\alpha = 0$. If $x_0 = 0$, the sequence converges trivially. Thus, we only consider the case where $x_0 \neq 0$. To analyze convergence, we consider the absolute value of the terms. For any $x \neq 0$, it is a known property that:

$$|\tan^{-1} x| < |x|$$

Applying this property to the iteration for any $x_n \neq 0$ gives a strictly **contractive** relationship.

$$|x_{n+1}| = |\tan^{-1} x_n| < |x_n|$$

This shows that the sequence of absolute values, $\{|x_n|\}$, is **strictly decreasing** and **bounded below by 0**. Therefore, the sequence of absolute values must converge to 0.

$$\lim_{n\to\infty} |x_n| = 0$$

This implies that the sequence $\{x_n\}$ itself converges to 0 for all initial values $x_0 \in (-\pi/2, \pi/2)$.

VI. Analysis of a Continued Fraction

Assuming the sequence converges to a value x, this value must be a fixed point satisfying the relation:

$$x_n = \frac{1}{p + \frac{1}{p+\dots}} \implies x = \frac{1}{p+x}$$

The solutions are trivial.

$$x = \frac{-p \pm \sqrt{p^2+4}}{2}$$

Since the sequence of convergents x_n is always positive for $p > 1$, its limit x must be positive. We thus select the positive root.

$$x = \frac{-p + \sqrt{p^2+4}}{2}$$

To prove convergence, we analyze the iteration $x_{n+1} = g(x_n)$ with the function $g(x) = 1/(p+x)$. We will show that $g(x)$ is a contraction mapping for $x \geq 0$. First, we find the derivative.

$$g'(x) = -\frac{1}{(p+x)^2}$$

The magnitude of the derivative is **bounded**.For $p > 1$ and any $x \geq 0$, the denominator $(p+x)^2 > p^2$. This provides a uniform bound on the magnitude of the derivative.

$$|g'(x)| = \frac{1}{(p+x)^2} < \frac{1}{p^2}$$

Since $p > 1$, we have $p^2 > 1$, which implies that $1/p^2 < 1$ and $k = \frac{1}{p^2} < 1$. Because $|g'(x)| \leq k < 1$ for all $x \geq 0$, the function $g(x)$ is a contraction mapping on $[0, \infty)$. Therefore, by the **Contraction Mapping Theorem**, the sequence converges for any initial $x_0 \geq 0$.

VII. Bisection Method Analysis when the Interval Contains Zero

Derivation of the Inequality for the Number of Steps

The upper bound for the relative error is:

$$\frac{|r - c_n|}{|r|} \leq \frac{b_0 - a_0}{|r| \cdot 2^{n+1}}$$

Under the condition $a_0 < 0 < b_0$, the root r can be arbitrarily close to 0. The supremum of the error bound is therefore not finite.

$$\sup_{r\in[a_0,b_0]} \left(\frac{b_0 - a_0}{|r| \cdot 2^{n+1}} \right) \to \infty \quad \text{as } r \to 0$$

Thus, no general inequality for n exists that can guarantee the relative error is bounded by a given ϵ.

Appropriateness of Relative Error

Relative error is an inappropriate measure when the root r is near zero. For a small but fixed absolute error $|r - c_n| = \delta > 0$, the relative error diverges as $r \to 0$.

$$\lim_{r \to 0} \frac{|r - c_n|}{|r|} = \lim_{r \to 0} \frac{\delta}{|r|} = \infty$$

Absolute error, $|r - c_n|$, is the proper measure in this case.

VIII. Newton's Method for Roots of Multiplicity k

Detection of a Multiple Zero

A multiple zero can be detected by two primary observations during the iteration process. First, the convergence of the sequence $\{x_n\}$ to the root r is slow. It degrades from quadratic to linear. Second, as the iterates x_n approach the root r, both the function values $f(x_n)$ and **the derivative values** $f'(x_n)$ **converge to 0**. For a simple root, $f'(x_n)$ would converge to a non-zero constant, $f'(r)$.

Proof of Restored Quadratic Convergence

We are given the modified Newton's iteration for a root of multiplicity k.

$$x_{n+1} = g(x_n) \quad \text{where} \quad g(x) = x - k\frac{f(x)}{f'(x)}$$

To prove quadratic convergence, we must show that $g'(r) = 0$. The derivative of $g(x)$ is:

$$g'(x) = 1 - k\left[\frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}\right] = 1 - k\left[1 - \frac{f(x)f''(x)}{[f'(x)]^2}\right]$$

$$g'(x) = 1 - k + k\frac{f(x)f''(x)}{[f'(x)]^2}$$

Since r is a root of multiplicity k, we can write $f(x) = (x - r)^k h(x)$ for some function $h(x)$ where $h(r) \neq 0$. The leading terms of the derivatives near $x = r$ are:

$$f(x) \approx (x - r)^k h(r)$$

$$f'(x) \approx k(x - r)^{k-1} h(r)$$

$$f''(x) \approx k(k - 1)(x - r)^{k-2} h(r)$$

$$\begin{aligned} \lim_{x \to r} \frac{f(x)f''(x)}{[f'(x)]^2} &= \frac{(x - r)^k h(r) \cdot k(k - 1)(x - r)^{k-2} h(r)}{[k(x - r)^{k-1} h(r)]^2} \\ &= \frac{k(k - 1)(x - r)^{2k-2} h(r)^2}{k^2 (x - r)^{2k-2} h(r)^2} = \frac{k - 1}{k} \end{aligned}$$

Substituting this back for $g'(r)$:

$$g'(r) = 1 - k + k\left(\frac{k - 1}{k}\right) = 1 - k + (k - 1) = 0$$

Since $g'(r) = 0$ (and assuming $g''(r) \neq 0$), the fixed-point iteration for $g(x)$ converges quadratically.

Acknowledgement

Honestly, the **translation** was done using LLM tools, but the answers and solutions were written by myself.