Submanifolds and Boundaries

Randy S

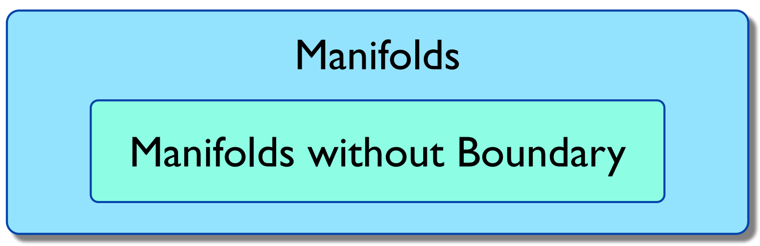
Abstract Article 93875 reviews the definitions of *topological manifold* and *smooth manifold* for manifolds that don't have boundaries. This article explains how those definitions may be extended to allow boundaries. This article also introduces the concept of a *submanifold*, a manifold S that is a subset of another manifold M with a special relationship between the (topological or smooth) structures of S and M. If M is an n-dimensional manifold and S is an $(n-2)$-dimensional submanifold without boundary, then S may or may not be the boundary of an $(n-1)$-dimensional submanifold Σ of M. When such a Σ exists, it is called a **Seifert hypersurface** for S. This article uses the concept of a Seifert hypersurface to define the **linking number** of S with a given a closed loop in M. This generalizes the more familiar concept of *linking number* between two closed loops (knots) when $n = 3$.

Contents

© 2018-2024 Randy S
For noncommercial use only

1 Conventions

In this article, the unqualified word *map* means *continuous map*, and the unqualified word *manifold* means a finite-dimensional topological manifold with boundary. The boundary may be empty,[1] in which case it's a manifold without boundary. This language convention can be summarized in a Venn diagram:



Many math texts – including many of the sources cited in this article – use a different convention in which the word *manifold* by itself implies *without boundary*. Beware of this when consulting those sources for more details.

In this article, the statement $A \subset B$ is synonymous with $A \subseteq B$. (The case $A = B$ is not automatically excluded.)

Some references to Lee (2011) are paired with references to the earlier edition Lee (2000), because the earlier edition is freely accessible online.

[1]Badzioch (2018), example 13.17

2 Topological spaces and subspaces

A **topological space** M is a set together with a **topological structure**, also called a **topology**. The topological structure consists of a collection of subsets of M designated as **open sets**, satisfying the conditions reviewed in article 93875.

If M is a topological space, then any subset $S \subset M$ may be promoted to a topological space by giving it the **subspace topology**, defined by declaring a subset of S to be an open set if and only if it has the form $S \cap U$ for some open set $U \subset M$ in M's topology.[2,3,4] With that topology, S is called a **subspace** of M.

If U is an open set in M's topology, then a set of the form $S \cap U$ is often called **relatively open** to remind us that it is not necessarily an open set in M's topology even though it is an open set (by definition) in the topology of the subspace $S \subset M$.[5] The next paragraph describes an example.

A *topological manifold* is a special kind of topological space, and the extra conditions associated with those special spaces allow the concept of a *boundary* to be defined. This will be done in section 3. To prepare, here's an important example of a subspace. Choose a positive integer n, and start with the topological space $M = \mathbb{R}^n$. Define the **half-space** $\mathbb{H}^n$ to be the subset

$$\mathbb{H}^n \equiv \big\{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1 \geq 0\big\}$$

equipped with the subspace topology.[6] For any $r > 0$, the subset $U \subset \mathbb{R}^n$ defined by $U \equiv \big\{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < r\big\}$ is an open set in the topology of $\mathbb{R}^n$, so the intersection $\mathbb{H}^n \cap U$ is a open set in the subspace topology for $\mathbb{H}^n$ (in other words, it's a *relatively* open set), even though it's not an open set in the topology of $\mathbb{R}^n$.

---

[2] Lee (2011), chapter 3, page 49 (also Lee (2000), chapter 3, pages 39-40)

[3] It's also called the **relative topology**.

[4] This collection of open sets of S automatically satisfies the required conditions (Lee (2011), exercise 3.1).

[5] Lee (2013), appendix A, page 601

[6] Lee (2013), page 25; Cohen (2023), definition 3.9

3 Topological manifolds with boundaries

A **topological manifold** is a topological space whose topological structure satisfies some additional conditions. Article 93875 reviews the additional conditions for topological manifolds without boundaries. To allow boundaries, one of those conditions must be modified. This section explains how it must be modified.

For a topological space M to be an n-dimensional topological manifold without boundary, every point of M must have a neighborhood that is homeomorphic[7] to $\mathbb{R}^n$. The condition for a **topological manifold with boundary** is similar except that now every point has a neighborhood that is homeomorphic to a (relatively) open set in the subspace $\mathbb{H}^n \subset \mathbb{R}^n$ that was defined in section 2.[8] This still allows a point to have a neighborhood homeomorphic to an open set of $\mathbb{R}^n$, but it doesn't require all points to have such neighborhoods. Points that do are called **interior points**, and points that don't are called **boundary points**.[9] The set of all interior points is the manifold's **interior**, and the set of all boundary points is the manifold's **boundary** and is denoted ∂M.[10]

If M is an n-dimensional manifold, then its interior is an n-dimensional manifold without boundary,[11] and M's boundary ∂M is an $(n-1)$-dimensional manifold that does not have a boundary:[12]

$$\dim(\partial M) = \dim(M) - 1 \tag{1}$$
$$\partial(\partial M) = \varnothing. \tag{2}$$

In words, equation (2) says that the boundary of a boundary is empty.

[7]Article 93875 defines *homeomorphism* (equivalence of topological manifolds) and *diffeomorphism* (equivalence of smooth manifolds).

[8]Tu (2011), definition 22.6; Badzioch (2018), definition 13.11

[9]Lee (2013), theorem 1.37

[10]Badzioch (2018), definition 13.14

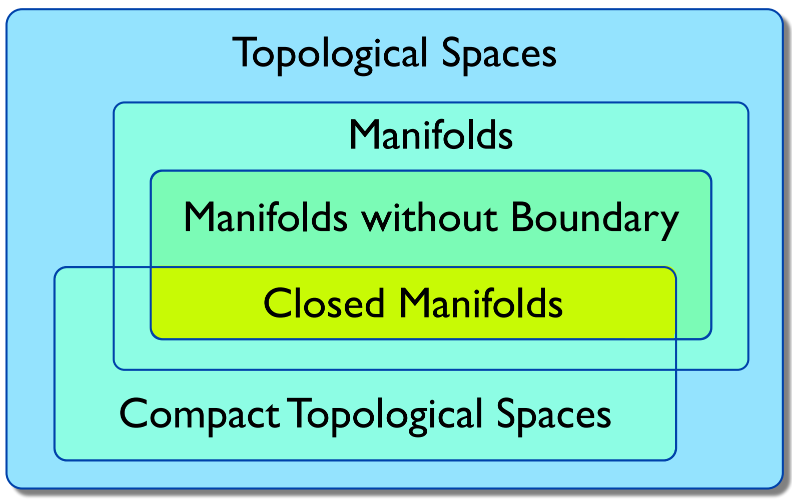
[11]Badzioch (2018), proposition 13.19

[12]Lee (2013), proposition 1.38; Tu (2011), section 22.3

4 Compact spaces and closed manifolds

A topological space X is called **compact** if any collection of open sets that covers X includes a finite number of open sets that already cover X.[13] Examples: an n-dimensional sphere S^n is compact, but n-dimensional infinite space $\mathbb{R}^n$ is not. Compactness is a topological invariant: if two spaces are homeomorphic to each other, then either they are both compact or they are both non-compact.[14]

A compact manifold without boundary is often called a **closed manifold**:[15,16]



The n-dimensional sphere S^n an example of a closed manifold. Deleting a single point from a closed manifold gives a non-compact manifold that still doesn't have a boundary. Example: deleting a point from S^n gives $\mathbb{R}^n$.

The **Heine–Borel theorem** says that a subset of $\mathbb{R}^n$ is compact if and only if it satisfies both of these conditions:[17] it's a closed set, and it's bounded.[18]

[13] Tu (2011), section A.8; Eschrig (2011), section 2.4

[14] Lee (2011), corollary 4.33 (also Lee (2000), theorem 4.18); Tanaka (2020)

[15] Lee (2013), text above proposition 1.38; Badzioch (2018), note 20.10

[16] This usage of the word "closed" should not be confused with the more basic concept of a *closed set*: a topological structure is defined in terms of *open sets*, and a *closed set* is the complement of an open set (article 93875).

[17] Lee (2011), theorem 4.40 (also Lee (2000), proposition A.6 and theorem A.8); Tanaka (2020)

[18] A subset of $\mathbb{R}^n$ is called **bounded** if it is contained in some open ball (Tu (2011), section A.9).

5 Topological submanifolds

If M is a manifold, then any subset $S \subset M$ may be endowed with the subspace topology. That makes S a subspace of M, but it doesn't necessarily make S a manifold.[19] If it does – if the subspace topology for S satisfies the additional conditions required for a manifold – then the subspace S is called a **topological submanifold** of M, or just a **submanifold** if the qualifier *topological* is already clear from the context.

The manifold M is called the **ambient manifold** for S.[20] If the submanifold S is k-dimensional and the ambient manifold is n-dimensional, then $k \leq n$. The difference $n - k$ is called the **codimension** of S. If a manifold M has a nonempty boundary ∂M, then ∂M is a submanifold of M with codimension 1.[21]

Let X and M be topological spaces. An injective map $f : X \to M$ is called a **topological embedding** if its image $f(X)$ is homeomorphic to X when the subspace topology is used for $f(X) \subset M$.[22,23] A subspace $S \subset M$ of a manifold M is a submanifold if and only if it is the image of a topological embedding.[24]

[19] A subspace S of a manifold M automatically satisfies two of the conditions required for manifolds (proposition 3.11 in Lee (2011) says that it's automatically Hausdorff and second countable, properties that it inherits from the manifold M through the subspace topology), but it doesn't necessarily satisfy the third required condition (it might not be necessarily locally euclidean, even though M is). Article 93875 mentions an example.

[20] Lee (2013) introduces the name *ambient* in the context of smooth manifolds (chapter 5, page 99), but it can also be applied more generally in the context of topological manifolds.

[21] Lee (2000), problem 2-18

[22] Lee (2011), chapter 3, page 54 (and Lee (2000), chapter 3, page 40); Daverman and Venema (2009), page xiv

[23] The remark after definition 11.11 in Tu (2011) explains why using the subspace topology is important. When the topological structure of $f(X)$ is not specified, the subspace topology is usually intended.

[24] Proof: Suppose that S is the image of a topological embedding $f : X \to M$. Then S is homeomorphic to X, which is a manifold, so S is also a manifold. Conversely, suppose that S is a submanifold. Then the inclusion map $i : S \to M$ is a topological embedding whose image is S.

6 Smooth manifolds with boundaries

An n-dimensional **smooth manifold** M is an n-dimensional topological manifold equipped with a **smooth structure**, which is enough extra structure for defining derivatives.[25] The data that defines M's smooth structure is a collection of **charts** satisfying the conditions reviewed in article 93875 for manifolds without boundary. When M doesn't have a boundary, each chart is a pair (U, σ), where $U \subset M$ is an open set in M's topological structure, and σ is a homeomorphism from U to an open subset of $\mathbb{R}^n$. When M has a boundary, σ is a homeomorphism from U to an open subset of the half-space $\mathbb{H}^n$ instead.[26] This is consistent with how the boundary is accommodated in the definition of *topological manifold*.[27]

If M is a smooth manifold with boundary, then its boundary ∂M is also a smooth manifold.[28] Equations (1) and (2) still hold.

Every smooth manifold M admits a **boundary defining function**. This is a smooth function

$$f : M \to [0, \infty) \subset \mathbb{R}$$

for which $\partial M = f^{-1}(0)$ and for which $df \neq 0$ at all points in the interior of M.[29]

If M is an n-dimensional smooth manifold with boundary, then two copies of M can be glued together along ∂M to obtain an n-dimensional smooth manifold without boundary called the **double** of M.[30] More generally, if A and B are smooth manifolds whose boundaries are diffeomorphic to each other, then A and B can be glued together along their boundaries to obtain a smooth manifold without boundary.[31]

[25] Article 93875
[26] Tu (2011), section 22.2, page 251; Michor (2008), section 10.8; Lee (2013), pages 27-28
[27] Section 3
[28] Tu (2011), section 22.3; Michor (2008), section 10.8
[29] Lee (2013), theorem 5.43
[30] Lee (2013), example 9.32 (for smooth manifolds); Badzioch (2018), definition 20.11 (for topological manifolds)
[31] Lee (2013), theorem 9.29 (for smooth manifolds); Badzioch (2018), proposition 20.12 (for topological manifolds)

7 Manifolds with corners

If A and B are topological manifolds, then their cartesian product $A \times B$ is also a topological manifold, and its boundary is[32]

$$\partial(A \times B) = (\partial A \times B) \cup (A \times \partial B).$$

For smooth manifolds, boundaries and the cartesian product don't always play quite so nicely together.

They do play nicely together if no more than one of the two manifolds in the product has a boundary. If A and B are smooth manifolds, at least one of which doesn't have a boundary, then their cartesian product $A \times B$ is also a smooth manifold.[33] As an example, consider a two-dimensional disk D and a circle S^1. The boundary ∂D of D is another circle, and S^1 does not have a boundary. The product $D \times S^1$ is a solid torus, which is a smooth manifold with boundary. Its boundary $(\partial D) \times S^1 = S^1 \times S^1$ is a two-dimensional torus, which is also a smooth manifold.

In contrast, if A and B are smooth manifolds that both have non-empty boundaries, then $A \times B$ is *not* a smooth manifold with boundary. Instead, it is something called a **smooth manifold with corners**.[34] As an example, consider a two-dimensional disk D and a line segment I. The boundary of I is a pair of points.[35] Their cartesian product, $D \times I$, is a cylinder. The subset $(\partial D) \times (\partial I)$ is a pair of circles on which the boundary of $D \times I$ is not smooth: on those circles, the boundary of the cylinder has a corner in one of its two dimensions.

Beware that the boundary of a smooth manifold with corners is typically not a smooth manifold with corners.[36]

[32] Badzioch (2018), text below example 13.18, and exercise E13.5

[33] Lee (2013), proposition 1.45

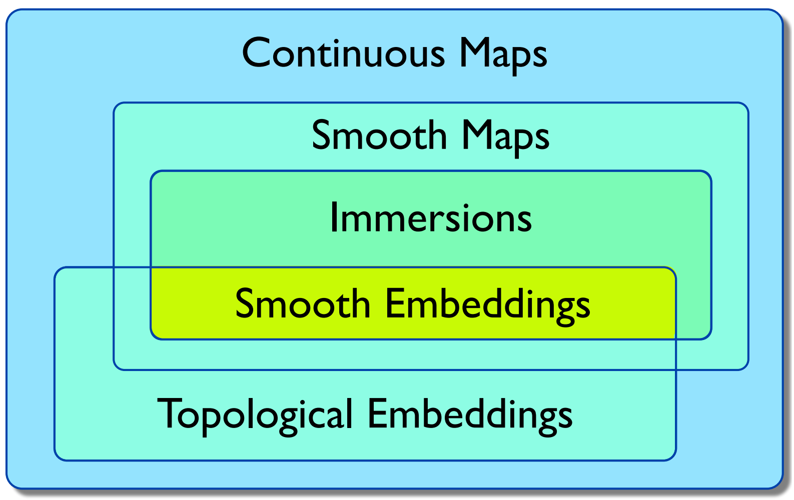
[34] This is defined in chapter 16 in Lee (2013) and in `https://ncatlab.org/nlab/show/manifold+with+boundary`.

[35] A point is a zero-dimensional smooth manifold. Any finite number of points is a zero-dimensional smooth manifold with that many disconnected components.

[36] Lee (2013), text above equation (16.7)

8 Immersions and embeddings: preview

Section 10 will define a concept of *submanifold* appropriate for the category of smooth manifolds. As prerequisites for that definition, section 9 will define a special kind of smooth map called a *(smooth) immersion*, and section 10 will define a further specialization called a *smooth embedding*. Roughly, a smooth map $X \to M$ that inserts a copy of X into M is a smooth immersion if it allows that copy to intersect itself (but not to be tangent to itself), and it's a smooth embedding if it doesn't. This Venn diagram summarizes the relationships:



As indicated by the diagram, some topological embeddings are not smooth,[37] and some topological embeddings that are smooth maps are not smooth embeddings.[38]

Every smooth immersion is locally a smooth embedding. More precisely, if $f : X \to M$ is a smooth immersion, then every point $p \in X$ has a neighborhood $U \subset X$ for which $f : U \to M$ is a smooth embedding.[39]

[37] One example is a map $S^1 \to \mathbb{R}^2$ whose image is a square.

[38] Lee (2013), example 4.18

[39] Lee (2013), theorem 4.25 (also proposition 5.22); Cohen (2023), proposition 3.4

9 Immersions and submersions

For a smooth manifold M without boundary, article 09894 defines a **scalar field** to be a smooth map from M to $\mathbb{R}$ and defines a **(tangent) vector field** to be a special kind of map v (called a **derivation**) from the set of scalar fields to itself. Those definitions also work for a smooth manifold with boundary. A **tangent vector** at a point $p \in M$ can be defined as the map from scalar fields to $\mathbb{R}$ given by applying the map v and then evaluating the resulting scalar field at p.[40] At each point p of an m-dimensional smooth manifold M, the set of tangent vectors forms an m-dimensional vector space, even if p is on the boundary of M.[41]

Let M and N smooth manifolds with m and n dimensions, respectively. Two special types of smooth map are defined by what they do to tangent vectors:[42]

- A **(smooth) immersion** is a smooth map $f : M \to N$ that maps the tangent space at each point $p \in M$ to an m-dimensional space of tangent vectors at the point $f(p) \in N$. This requires $m \leq n$.
- A **(smooth) submersion** is a smooth map $f : M \to N$ that maps the tangent space at each point $p \in M$ to an n-dimensional space of tangent vectors at the point $f(p) \in N$. This requires $m \geq n$.

One example of an immersion is a smooth map $S^1 \to \mathbb{R}^2$ whose image is a figure-eight (intersects itself).[43] One example of a submersion is the smooth map $\mathbb{R} \to S^1$ defined by identifying all points of $\mathbb{R}$ that differ from each other by an integer. More generally, the bundle projection $\pi : E \to B$ of a fiber bundle[44] is a submersion from the total space E to the base space B.[45]

[40] Lee (2013), text surrounding equation (3.4)
[41] Lee (2013), proposition 3.12
[42] Lee (2013), text above proposition 4.1; Tu (2011), section 8.8; Gallot *et al* (2004), paragraph 1.18
[43] Lee (2013), example 4.19
[44] Article 70621 reviews the concept of a *fiber bundle*.
[45] Gallot *et al* (2004), paragraph 1.92

10 Smooth embedded submanfolds

Consider two smooth manifolds X and M and a map $f : X \to M$. Even if f is an injective immersion, the subspace $f(X) \subset M$ might not be a manifold,[46] and even if it is a manifold, it might not be homeomorphic to X, because points that are separated from each other in X might not be separated from each other in $f(X) \subset M$.[47] This section defines a more restricted type of smooth map for which X and $f(X)$ are homeomorphic to each other and that allows $f(X)$ to inherit a smooth structure from X, making them diffeomorphic to each other.

Let X and M be smooth manifolds. A map $f : X \to M$ is called a **smooth embedding** if it is both a topological embedding and a smooth immersion.[48,49]

Given a smooth embedding $f : X \to M$, we can define a smooth structure for its image $S \equiv f(X)$ like this:[50] for each of X's charts (U, σ), we can define a chart for S by

$$\Big(f(U),\ \sigma\big(f^{-1}(\cdot)\big)\Big).$$

Section 11 will show that these charts for S are all smoothly compatible with each other, so they define a smooth structure for S. When equipped with this smooth structure, S is called a **smooth embedded submanifold** of the ambient manifold M.[51]

The smooth structure defined above makes S diffeomorphic to X,[52] but calling S a *submanifold* of M suggests that the smooth structures of S and M should also be consistent with each other. Section 11 will show that they are.

---

[46]One example is an immersion $\mathbb{R} \to \mathbb{R}^2$ whose image is a figure-eight with $\lim_{r\to\infty} f(r) = \lim_{r\to-\infty} f(r) = f(0)$. This is described more explicitly in Lee (2013), example 4.19. Example 11.9 in Tu (2011) is similar.

[47]One example is an immersion $\mathbb{R} \to \mathbb{R}^2$ whose image is a circle with $\lim_{r\to\infty} f(r) = \lim_{r\to-\infty} f(r)$.

[48]Lee (2013), chapter 4, page 85; Tu (2011), definition 11.11

[49]Some authors write *imbedding/imbedded* instead of *embedding/embedded* (Kirby and Siebenmann (1977), essay I, section 2, page 6). They are synonymous.

[50]Section 6 introduced this notation. Section 11 will describe this smooth structure for S in more detail.

[51]Section 12 will show that this definition is equivalent to the one in Lee (2013). That's important because this article cites Lee (2013) for several results.

[52]Lee (2013) shows this in the proof of proposition 5.2 for $\partial S = \varnothing$, and the same proof works for $\partial S \neq \varnothing$.

11 Consistency of S's and M's smooth structures

Let S be a smooth embedded submanifold of M as defined in section 10. This section describes the smooth structure of S more carefully and shows that it is consistent with the smooth structure of the ambient manifold M, as the name *submanifold* suggests.

The smooth structure of the n-dimensional manifold M is a maximal smooth atlas α_M[53] consisting of charts (U_M, σ_M), where:

- U_M is an open set in the topological structure of M,
- σ_M is a homeomorphism from U_M to a relatively open subset of $\mathbb{H}^n \subset \mathbb{R}^n$.

Define X and f as in section 10. The smooth structure of the k-dimensional manifold X is a maximal smooth atlas α_X consisting of charts (U_X, σ_X), where:

- U_X is an open set in the topological structure of X,
- σ_X is a homeomorphism from U_X to a relatively open subset of $\mathbb{H}^k \subset \mathbb{R}^k$.

The topology of $S = f(X) \subset M$ is the subspace topology. This means that if U_M is an open subset of M, then $U_M \cap S$ is an open subset of S whenever it's not empty. The fact that f is continuous then implies that $f^{-1}(U_M \cap S)$ is an open subset of X. For each chart in α_M with domain U_M, let U_X be the open subset of X given by $U_X = f^{-1}(U_M \cap S)$, and define a chart (U_S, σ_S) by[54]

$$U_S = f(U_X) \qquad\qquad \sigma_S(\cdot) = \sigma_X(f^{-1}(\cdot)),$$

as in section 10. To show that this defines a smooth structure for S, we need to show that these charts are all smoothly compatible with each other.[55] If (U_S, σ_S) and (U'_S, σ'_S) are any two of these charts, then

$$\sigma'_S\left(\sigma_S^{-1}(\cdot)\right) = \sigma'_X\left(f^{-1}\left(f\left(\sigma_X^{-1}(\cdot)\right)\right)\right) = \sigma'_X\left(\sigma_X^{-1}(\cdot)\right),$$

[53] Mnemonic: α stands for "atlas."

[54] The definition of σ_S implies $\sigma_S(U_S) = \sigma_X(U_X)$.

[55] Pages 12 and 28 in Lee (2013) define **smoothly compatible** for manifolds with boundary.

so the fact that X's charts are smoothly compatible with each other implies that these charts for S are also smoothly compatible with each other. This shows that they define a smooth atlas for S. Denote this smooth atlas by $\alpha_{S,f}$. To show that this smooth structure for S is consistent with the smooth structure of the ambient space M, use the premise that the map $f : X \to M$ is smooth. The premise that f is smooth means[56] that $\sigma_M(f(\sigma_X^{-1}(\cdot)))$ is a smooth map from $\sigma_X(U_X) \subset \mathbb{H}^k$ to $\sigma_M(U_M) \subset \mathbb{H}^n$. The definition of σ_S implies

$$\sigma_M(f(\sigma_X^{-1}(\cdot))) = \sigma_M(\sigma_S^{-1}(\cdot)),$$

so $\sigma_M(\sigma_S^{-1}(\cdot))$ is also a smooth map from $\sigma_S(U_S) = \sigma_X(U_X)$ to $\sigma_M(U_M)$. The fact that $\sigma_M(\sigma_S^{-1}(\cdot))$ is smooth shows that the smooth structure $\alpha_{S,f}$ for S is consistent with the smooth structure α_M for M.[57]

To reinforce this conclusion, remember that the purpose of giving a manifold a smooth structure is to allow defining the concept of a smooth function from that manifold to $\mathbb{R}$. Saying that a function $g : M \to \mathbb{R}$ is **smooth** means[58] that the composite function $g(\sigma_M^{-1}(\cdot))$ from $\sigma_M(U_M) \subset \mathbb{H}^n$ to $\mathbb{R}$ is smooth, for each chart (U_M, σ_M) in M's smooth structure. We already deduced that the function

$$h(\cdot) \equiv \sigma_M(\sigma_S^{-1}(\cdot))$$

is smooth, so the composite function

$$g(\sigma_S^{-1}(\cdot)) = g(\sigma_M^{-1}(\sigma_M(\sigma_S^{-1}(\cdot)))) = g(\sigma_M^{-1}(h(\cdot)))$$

is a smooth function from $\sigma_S(U_S)$ to $\mathbb{R}$. This shows that if $g : M \to \mathbb{R}$ is smooth with respect to M's smooth structure, then g restricted to $S \subset M$ is smooth with respect to S's smooth structure, too.[59] In other words, the smooth structure $\alpha_{S,f}$ for S is consistent with the smooth structure α_M for M.

[56] Lee (2013), page 34

[57] Theorem 5.8 in Lee (2013) expresses this consistency another way when $\partial S = \partial M = \varnothing$.

[58] Lee (2013), pages 32-33

[59] This conclusion is a special case of theorem 5.53(a) in Lee (2013).

12 Equivalence to the definition in Lee (2013)

In Lee (2013),[60,61] a *smooth embedded submanifold* of M is defined to be a subset $S \subset M$ together with a topology and smooth structure for which the inclusion map is a smooth embedding. The **inclusion map** $i : S \to M$ is defined by $i(s) = s \in M$ for all $s \in S$.

That definition of *smooth embedded submanifold* is equivalent to the one in section 10. Proof:

- Proposition 5.49(b) in Lee (2013) says that if S satisfies the definition in section 10, then it satisfies the one in Lee (2013).[62]

- Conversely, suppose that S satisfies the definition in Lee (2013). Take the map f in section 10 to be the inclusion map i, and give S a topology and smooth structure that makes i a smooth embedding. (This is logically sound, because the existence of such a topology and smooth structure for S is a premise of the definition in Lee (2013).) Then S manifestly satisfies the definition in section 10.

For later use, here's a related result that I'll call the **inclusion-embedding lemma**: if S is any subset of M and has any smooth structure α_S (not necessarily related to the smooth structure of M), then α_S is the same as the smooth structure $\alpha_{S,f}$ defined in section 10 when $X = S$ and when f is the inclusion map $i : S \to M$.[63]

[60] Lee (2013), chapter 5, page 120

[61] Chapter 5 in Lee (2013) starts with a definition that assumes $\partial S = \varnothing$ (pages 98-99), but the definition on page 120 allows both $\partial S \neq \varnothing$ and $\partial M \neq \varnothing$.

[62] Proposition 5.49(b) in Lee (2013) allows both $\partial S \neq \varnothing$ and $\partial M \neq \varnothing$.

[63] To prove this, let (U_S, σ_S) be a chart in α_S. When $X = S$ and $f = i$, U_S is also the domain of a chart $(U_S, \tilde{\sigma}_S)$ in $\alpha_{S,f}$ with $\tilde{\sigma}_S(\cdot) \equiv \sigma_X(f^{-1}(\cdot))$, which satisfies $\tilde{\sigma}_S(p) = \sigma_X(f^{-1}(p)) = \sigma_S(i^{-1}(p)) = \sigma_S(p)$ for all $p \in U_S$, so the two charts are equal. These charts generate the smooth structure $\alpha_{S,f}$, so $\alpha_{S,f}$ and α_S are equal.

13 The boundary as an embedded submanifold

If M is a smooth manfold, then its boundary ∂M admits a smooth structure that makes it a smooth embedded submanifold of M.[64] This section describes that smooth structure.

To construct an appropriate smooth structure for ∂M, let (U_M, σ_M) be a chart in M's smooth atlas. If U_M intersects ∂M, then define

$$U_{\partial M} \equiv U_M \cap \partial M \qquad\qquad \sigma_{\partial M} = \sigma_M\big|_{U_{\partial M}},$$

where $\sigma\big|_U$ denotes the restriction of the map σ to the domain U. If $(U_{\partial M}, \sigma_{\partial M})$ and $(U'_{\partial M}, \sigma'_{\partial M})$ are any two of these charts, then

$$\sigma'_{\partial M}\left(\sigma_{\partial M}^{-1}(\cdot)\right) = \sigma'_M\left(\sigma_M^{-1}(\cdot)\right),$$

so the fact that M's charts are smoothly compatible with each other implies that these charts for ∂M are also smoothly compatible with each other. This shows that they define a smooth structure for ∂M.

To relate this to the definition of *smooth embedded submanifold* in section 10, we need to show that ∂M is the image of a smooth embedding $f : X \to M$. We can do this by setting $X = \partial M$ and taking f to be the inclusion map $i : \partial M \to M$. Then the fact that ∂M has the subspace topology implies that f is a topological embedding.[65] To show that it's also a smooth immersion (and therefore a smooth embedding), use the fact that σ_M and $\sigma_{\partial M}$ are homeomorphisms from U_M and $U_{\partial M}$ to $\mathbb{H}^n$ and $\partial\mathbb{H}^n$, respectively. Now the fact that the inclusion map $\partial\mathbb{H}^n \to \mathbb{H}^n$ is a smooth immersion implies that f is, too. This shows that f is a smooth embedding. Finally, when $X = \partial M$ has the smooth structure $\alpha_{\partial M}$ that was constructed above, the inclusion-embedding lemma[66] says that $\alpha_{\partial M}$ is the same as the smooth structure $\alpha_{\partial M,f}$ defined in section 10. This shows that the boundary is a smooth embedded submanifold.

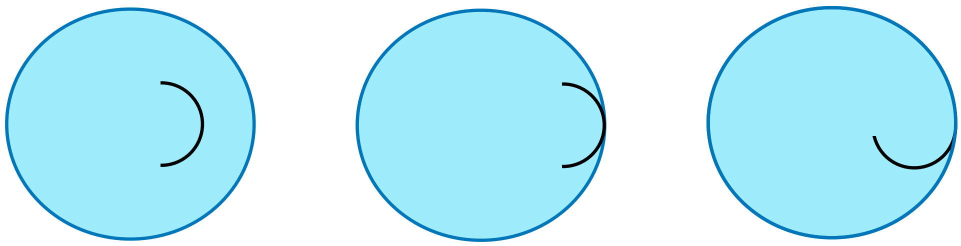
[64] Lee (2013), theorem 5.11 (and the sentence after this says it's unique); Hirsch (1976), chapter 1, section 4

[65] Lee (2013), proposition A.17(d)

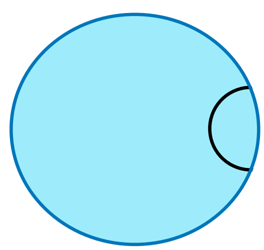
[66] Section 12

14 Neat submanifolds: preview

When the ambient manifold M has a non-empty boundary ∂M, a submanifold S may be situated relative to ∂M in a variety of ways. A few of them are illustrated here,[67] using a blue 2-dimensional disk for M (so that ∂M is a circle) and a black 1-dimensional arc for S (so that ∂S is a pair of points):



In the left picture, S does not intersect ∂M. In the middle picture, the interior of S is tangent to ∂M. In the right picture, the interior of S approaches ∂M tangentially. Another possibility is $S \subset \partial M$ (not illustrated here), which includes the important case $S = \partial M$ that was treated in section 13. Section 15 will explore a different important case called a **neat submanifold**, illustrated here:



A neat submanifold is one for which $\partial S = S \cap \partial M$, and the interior of S approaches ∂M only transversely, not tangentially.[68]

[67] Figure 5.3 in Kupers (2019) and figure 1-6 in Hirsch (1976) use similar illustrations.

[68] A **neat embedding** is a smooth embedding whose image is a neat submanifold (Kupers (2019), definition 5.2.5).

15 Neat submanifolds

As in section 2, define $\mathbb{H}^n$ to consist of all points $(x_1, ..., x_n)$ in $\mathbb{R}^n$ with $x_1 \geq 0$. For any k in the range $1 \leq k \leq n$, define $\mathbb{H}^{k,n}$ to consist of all points $(x_1, ..., x_n)$ in $\mathbb{H}^n$ with $x_{k+1} = x_{k+2} = \cdots = x_n = 0$. Calling S a *neat submanifold* of M means roughly that the relationship $S \subset M$ looks locally like the relationship $\mathbb{H}^{k,n} \subset \mathbb{H}^n$.

One way to make this precise is to start with a subset $S \subset M$ and then construct a smooth structure $\alpha_{S,M}$ for S directly from M's smooth structure, assuming that the subset S satisfies a special condition to ensure that the construction will work. This section uses that approach to define *neat submanifold*.[69,70] This section also shows that a neat submanifold defined this way is the image of a smooth embedding f and that the smooth structure $\alpha_{S,M}$ is the same as the smooth structure $\alpha_{S,f}$ that was defined in section 11.

Start with an n-dimensional smooth manifold M and a subset $S \subset M$. Given any chart (U_M, σ_M) in M's smooth structure, define

$$U_S \equiv U_M \cap S \qquad\qquad \sigma_S = \sigma_M\big|_{U_S}.$$

For most choices of the subset $S \subset M$, the "charts" (U_S, σ_S) do not give a smooth structure for the subset S, but they do if S satisfies this special condition:[71,72] each point $p \in S$ has a neighborhood of the form $U_S = U_M \cap S$ for which[73]

$$\sigma_S(U_S) \equiv \sigma_M(U_S) \subset \mathbb{H}^{k,n}. \tag{3}$$

[69]This approach is used in Hirsch (1976), chapter 1, section 4; Cohen (2023), definition 3.11; and Freed (2013), definition 3.1. For manifolds without boundaries, this approach is used in Crainic (2017), definition 3.20; Gorodski (2012), page 113; and Adachi (1993), chapter 1, page 10.

[70]Kirby and Siebenmann (1977) use a similar approach to define **clean submanifold**, a generalization of *neat submanifold* that works for manifolds with corners (essay I, section 2, pages 12-13).

[71]Lee (2013) calls this the ***k*-slice property** (text above theorem 5.51, for $\partial M = \varnothing$), and Tu (2011) uses the name **regular submanifold** for a subset $S \subset M$ with this property (definition 9.1, for $\partial S = \partial M = \varnothing$).

[72]When $\partial M = \varnothing$, every smooth embedded submanifold satisfies this condition (Lee (2013), theorem 5.51).

[73]Here, a smooth structure is understood to be defined by a maximal smooth altas, one that includes every chart that is smoothly compatible with it (article 93875). We could define the same smooth structure of M using a smooth atlas with fewer charts, but that might exclude charts that satisfy (3).

For the rest of this section, suppose that S satisfies this special condition.

Promote S to a topological space by giving it the subspace topology derived from M's topology. When combined with the condition (3), the fact that σ_M is a homeomorphism onto its image in $\mathbb{H}^n$ implies that σ_S is a homeomorphism onto its image in $\mathbb{H}^{k,n}$. To show that the charts (U_S, σ_S) define a smooth structure for S, we need to show that they are all smoothly compatible with each other. If (U_S, σ_S) and (U'_S, σ'_S) are any two of these charts, then

$$\sigma'_S\left(\sigma_S^{-1}(\cdot)\right) = \sigma'_M\left(\sigma_M^{-1}(\cdot)\right)$$

when both sides are regarded as a map from $\sigma_S(U_S)$ to $\sigma'_S(U'_S)$, so the fact that M's charts are smoothly compatible with each other implies that these charts for S are also smoothly compatible with each other. This shows that they define a smooth structure for S. A subset $S \subset M$ equipped with this smooth structure is called a **neat submanifold** of S.

The smooth structure constructed above will be denoted $\alpha_{S,M}$. It is manifestly consistent with M's smooth structure.

To relate this to the definition of *smooth embedded submanifold* in section 10, we need to show that S is the image of a smooth embedding $f : X \to M$. We can do this by setting $X = S$ and taking f to be the inclusion map $i : S \to M$. Then the fact that S has the subspace topology (as a subset of M) implies that f is a topological embedding,[74] and (3) implies that f is a smooth immersion because the inclusion map $\mathbb{H}^{k,n} \to \mathbb{H}^n$ is a smooth immersion. This shows that f is a smooth embedding. Finally, when $X = S$ has the smooth structure $\alpha_{S,M}$, the inclusion-embedding lemma[75] says that the smooth structures $\alpha_{S,f}$ and $\alpha_{S,M}$ are equal. This shows that a neat submanifold as defined above is a special case of a smooth embedded submanifold as defined in section 10.[76]

[74] Lee (2013), proposition A.17(d)

[75] Section 12

[76] In the special case $\partial S = \partial M = \varnothing$, this is theorems 11.13 and 11.14 in Tu (2011).

16 Realizing manifolds as submanifolds of $\mathbb{R}^k$

Every (topological or smooth) manifold is equivalent to a submanifold of $\mathbb{R}^k$ for some k. This section reviews some results that bound the required value of k.

- Every n-dimensional topological manifold is homeomorphic to a topological submanifold of $\mathbb{R}^{2n+1}$.[77,78]
- For $n \geq 2$, every n-dimensional smooth manifold can be smoothly immersed in $\mathbb{R}^{2n-1}$.[79,80,81] This is the **(strong) Whitney immersion theorem**.
- For $n \geq 1$, every n-dimensional smooth manifold is diffeomorphic to a smooth embedded submanifold of $\mathbb{R}^{2n}$.[82,83,84,85] This is the **(strong) Whitney embedding theorem**.

Related results: any smooth map from an n-dimensional smooth manifold without boundary into $\mathbb{R}^{2n}$ can be approximated arbitrarily well by an immersion,[86] and any smooth map of a compact n-dimensional smooth manifold into $\mathbb{R}^{2n+1}$ can be approximated arbitrarily well by a smooth embedding.[87]

[77] Davis and Petrosyan (2012), page 2; and `https://mathoverflow.net/questions/34658/`

[78] This is true even though some topological manifolds are not smoothable (article 93875), so this implies that some topological submanifolds of $\mathbb{R}^k$ are not smoothable (at least for some k).

[79] Lee (2013), theorem 6.20. Example: a Klein bottle can be smoothly immersed in $\mathbb{R}^3$. This is the usual picture of a Klein bottle in three-dimensional euclidean space (Lee (2011), figure 6.5), which necessarily intersects itself.

[80] For most n, the ambient manifold can have even fewer dimensions (theorem 6.11 in Cohen (2023), also mentioned in the text below theorem 6.20 in Lee (2013)). Example: every 3-dimensional smooth manifold can be smoothly immersed in $\mathbb{R}^4$.

[81] Harrison (2020) reports an analogous theorem for *totally non-parallel immersions*.

[82] Lee (2013), theorem 6.19. Example: a Klein bottle can be smoothly embedded in $\mathbb{R}^4$.

[83] Theorem 4.3 in Hirsch (1976) says that every n-dimensional smooth manifold with $n \geq 1$ is the image of a **neat embedding** into the half-space $\mathbb{H}^{2n+1}$, and theorem 6.3 in Cohen (2023) tightens this to $\mathbb{H}^{2n}$.

[84] The text below theorem 6.20 in Lee (2013) mentions that for some n, the ambient manifold can have even fewer dimensions: Example: every 3-dimensional smooth manifold can be smoothly embedded in $\mathbb{R}^5$.

[85] Results about isometric embeddings of riemannian manifolds into flat euclidean space (and generalizations to other signatures) are also known, like the **Nash embedding theorem** mentioned on page 66 in Lee (1997).

[86] Adachi (1993), theorem 2.5

[87] Lee (2013), corollary 6.17

17 Seifert surfaces

Let M be a smooth manifold, and define a **closed curve** to be the image of a smooth embedding $c : S^1 \to M$. When $M = \mathbb{R}^n$ with $n \geq 2$, every closed curve in M is the boundary of a two-dimensional submanifold[88] of M. This is intuitively clear when $n = 2$, and it's also intuitively clear when $n \geq 4$ because a closed curve in a four-dimensional euclidean space cannot be knotted. It might be more surprising when $n = 3$, because then a closed curve can be knotted, but it's still true: every closed curve in $\mathbb{R}^3$ is the boundary of a two-dimensional submanifold of $\mathbb{R}^3$. When $n = 3$, such a submanifold is called a **Seifert surface** for the given closed curve.[89,90]

This remains true when the n-dimensional ambient manifold M is generalized from $\mathbb{R}^n$ to any other simply-connected manifold, liks S^n. This is intuitively clear, because saying that M is *simply connected* means that any closed curve can be continuously morphed so that it's contained in an arbitrarily small neighborhood of a point,[91] so we can take that neighborhood to be homeomorphic to $\mathbb{R}^n$.

[88] References about Seifert surfaces tend to use the word *submanifold* by itself, without specifying which type of submanifold they mean (footnote 92 in section 18). They presumably mean *embedded submanifold* (as opposed to *immersed submanifold*, which would allow self-intersections), but to be safe, this article avoids using the explicit qualifier *embedded* when the cited sources don't use it.

[89] More generally, every collection of knots (which may be linked with each other) has a Seifert surface. A concise review of the proof is shown in Collins (2016), theorem 2.3. Several examples are depicted in van Wijk and Cohen (2006).

[90] The topology of a Seifert surface is not unique, because if one Seifert surface is given, then many other Seifert surfaces for the same knot may be constructed by adding more "handles" to it (`https://en.wikipedia.org/wiki/Handle_decomposition`). Hayden *et al* (2022) describes an example of a knot with two different Seifert surfaces that have the same intrinsic topology but that are not isotopic to each other, not even after adding an extra dimension the ambient space.

[91] Article 61813

18 Seifert hypersurfaces

This section reviews a nice generalization of the more familiar result that was reviewed in section 17. In this section, all manifolds are smooth.[92]

If an n-dimensional manifold M is closed, oriented, and 2-connected,[93] then any closed, oriented, codimension-2 submanifold $M_{n-2} \subset M$ is the boundary of an oriented and connected codimension-1 submanifold $M_{n-1} \subset M$.[94] Such an M_{n-1} is called a **Seifert hypersurface** for the given M_{n-2}.[95]

The n-sphere S^n is $(n-1)$-connected,[96] so when $n \geq 3$, the n-sphere $M = S^n$ is one example of an ambient manifold that satisfies the theorem's premise. In particular, when $n \geq 3$, every codimension-2 sphere S^{n-2} embedded in S^n has a Seifert hypersurface.[97] This is true even though the embedded $(n-2)$-dimensional sphere may be knotted.[98,99]

Seifert hypersurfaces answer the question: is the given submanifold a boundary of another submanifold inside the given ambient manifold? We could also ask whether a given manifold is the boundary of another compact manifold without confining it to any given ambient manifold. That's one of the questions addressed by subject called **cobordism**.[100]

---

[92]This convention is used in Michel and Weber (2014) (section 1.5), which is the source of the main result cited here. In that source, *submanifold* presumably means (smooth) embedded submanifold, but they don't specify this, and that convention is not universal. Page 10 in Adachi (1993) uses the shorter name *submanifold* for an embedded submanifold, but page 109 in Lee (2013) uses the shorter name *submanifold* for an immersed submanifold.

[93]A topological space is called **k-connected** if its first k homotopy groups are trivial (article 61813). The case $k = 1$ has a special name: *simply connected* means 1-connected.

[94]Michel and Weber (2014), theorem 11.0.1

[95]Michel and Weber (2014), definition 11.0.1

[96]Wright (2007), first corollary (page 5)

[97]Ranicki (2014), top of page XX (in the Introduction)

[98]For any $n \geq 3$, the image of an embedding $S^{n-2} \to S^n$ may be knotted (Kervaire and Weber (1978)).

[99]Page XXI in Ranicki (2014) says, "Seifert [hyper]surfaces are in fact the main geometric tool of high-dimensional knot theory..."

[100]Examples: a single point cannot be the boundary of any compact one-dimensional manifold, and the real projective space $\mathbb{RP}^2$ is not the boundary of any compact three-dimensional manifold (Freed (2013), proposition 1.32). More generally, a smooth closed manifold is a boundary if and only if all of its Steifel-Whitney numbers are zero (Milnor (1974), corollary 4.11; Freed (2013), theorem 2.24).

19 Intersection number

Fix an n-dimensional ambient smooth manifold $M \equiv \mathbb{R}^n$ with $n \geq 2$. Let C be an oriented one-dimensional closed loop in M, and let Σ be an oriented submanifold of codimension 1 in M with boundary $\partial\Sigma$. Think of C's orientation as a choice of a direction in which to travel around the curve C, and think of Σ's orientation as a choice of which side is the *front*. If C is not tangent to Σ anywhere, then we can define the **intersection number** $\eta(C, \Sigma) \equiv n_+ - n_-$, where n_+ (resp. n_-) is the number of times the oriented curve C passes through the oriented manifold Σ from back-to-front (resp. front-to-back).[101,102] The sign of $\eta(C, \Sigma)$ depends on the orientations of C and Σ.

Examples:

- If $n = 3$, then Σ is an oriented two-dimensional surface with one-dimensional boundary $\partial\Sigma$. Suppose that Σ is a disk, with a circle as its boundary, and suppose that the loop C wraps around $\partial\Sigma$ once. Then $\eta(C, \Sigma) = \pm 1$, where the sign depends on whether C pierces Σ from back-to-front or from front-to-back. If C wraps k times around $\partial\Sigma$ in the same direction, then $\eta(C, \Sigma) = \pm k$.

- If $n = 2$, then Σ is a curve with endpoints, and $\partial\Sigma$ is the pair of endpoints. If the loop C encircles one of these endpoints exactly once and doesn't encircle the other one, then the general definition implies $\eta(C, \Sigma) = \pm 1$, where the sign depends on which of the two endpoints is encircled and on the direction (clockwise or counterclockwise) in which it is encircled. If C circles k times around one of the endpoints, then $\eta(C, \Sigma) = \pm k$. If C circles around both endpoints without ever passing between them, then $\eta(C, \Sigma) = n_+ - n_- = 0$.

[101] Singer (2022)

[102] Definition 8.2 in Cohen (2023) defines the intersection number of two closed submanifolds P and Q of another closed manifold M, with $\dim P + \dim Q = \dim M$, assuming that P and Q intersect each other in only a finite number of points (which can always be arranged by adjusting P or Q slightly). The definition described in this section is essentially a special case of that one, with $\dim P = 1$ and $M = S^n$ (because topologically, $\mathbb{R}^n$ may be obtained from S^n by deleting a single point).

20 Linking number

The intersection number $\eta(C,\Sigma)$ that was defined in section 19 doesn't depend on Σ except through its oriented boundary, if the orientation of $\partial\Sigma$ is consistent with the orientation of Σ.[103] The number $\eta(C,\Sigma)$ is also invariant under smooth deformations of the C and $\partial\Sigma$, if they don't intersect each other during the deformation process.[104]

Since it depends only on the boundary $\partial\Sigma$, we could call it the **linking number** of C and $\partial\Sigma$. When the ambient manifold $\mathbb{R}^n$ is three-dimensional ($n=3$), this is the same[105] as the usual linking number of two closed curves in **knot theory**.[106,107] When $n\geq 4$, two closed curves cannot be linked with each other,[108] but the one-dimensional loop C can be linked with the $(n-2)$-dimensional boundary $\partial\Sigma$ of a $(n-1)$-dimensional manifold Σ.

[103] Robbin *et al* (2018), theorem 4.2.8; Seifert and Threlfall (1980), section 77

[104] Intuitively, this can be inferred from the preceding property.

[105] Meilhan (2018), theorem 2.2

[106] Livingston (1993) introduces knot theory. The ambient three-dimensional space is usually taken to be the three-sphere S^3.

[107] Pages 132-136 in Rolfsen (1976) list several ways to define the linking number of two closed curves in $\mathbb{R}^3$. Section 2 in Meilhan (2018) reviews some properties of the linking number of two closed curves in $\mathbb{R}^3$ that are not obvious from (some of) the definitions.

[108] Intuitively, this should be obvious: given two closed curved that are linked in 3d euclidean space, add a fourth dimension and "lift" one of the curves into the fourth dimension to unlink them.

21 Linking number: generalizations

Section 19 defined the intersection number $\eta(C,\Sigma)$ of a closed curve C with a codimension-1 submanifold Σ of $\mathbb{R}^n$ with $n \geq 2$, and section 20 used that to define the linking number of C with $\partial\Sigma$. That works because $\eta(C,\Sigma)$ depends on Σ only through its boundary $\partial\Sigma$.

That property of the intersection number $\eta(C,\Sigma)$ doesn't necessarily hold if we replace the ambient space $\mathbb{R}^n$ with another manifold M, so the concept of the *linking number* between C and $\partial\Sigma$ isn't always well-defined in that more general setting. As an example, suppose $M = S^1 \times S^1$, and consider two circles in M: one circle C that wraps once around one of the S^1 factors, and one circle C' that wraps once around the other S^1 factor. The circles C and C' intersect each other at a single point p. Take Σ to be a short segment of C' containing p, so that C and Σ intersect each other. The boundary $\partial\Sigma$ is a pair of points on C'. Take $\tilde{\Sigma}$ to be the part of C' that has those same endpoints but excludes the rest of Σ. Then Σ intersects C, but $\tilde{\Sigma}$ does not, so $\eta(C,\tilde{\Sigma}) \neq \eta(C,\Sigma)$ even though $\partial\tilde{\Sigma} = \partial\Sigma$.[109]

That problem does not occur if the ambient space M is a compact oriented manifold without boundary and the closed curve C is the boundary of a two-dimensional surface. More generally, if M is a compact oriented manifold without boundary and if S and Σ are submanifolds of M with $\dim S + \dim \Sigma = \dim M + 1$ such that the boundary $C \equiv \partial S$ intersects Σ in a finite number of points, then we can use the intersection number $\eta(C,\Sigma)$ to define the linking number of C with $\partial\Sigma$ without any ambiguity.[110] That definition assumes that the linked submanifolds are both boundaries.[111] The example in the previous paragraph violates this condition, because the loop C in that example was not the boundary of any surface in M.

Cohen (2023) reviews another approach to defining the *linking number* of two non-intersecting closed submanifolds of $\mathbb{R}^n$ whose dimensions add up to $n-1$.[112]

[109] The text below equation (1.1) in Horowitz and Srednicki (1990) describes a higher-dimensional example.

[110] Horowitz and Srednicki (1990), paragraph leading to equation (1.1)

[111] A submanifold which is a boundary of another submanifold is called **homologically trivial** in Horowitz and Srednicki (1990).

[112] Cohen (2023), definitions 9.4 and 9.5

22 References

Adachi, 1993. *Embeddings and immersions.* American Mathematical Society

Badzioch, 2018. "MTH 427/527 Introduction to Topology I" https://www.nsm.buffalo.edu/~badzioch/MTH427/lecture_notes.html

Cohen, 2023. "Bundles, Homotopy, and Manifolds" http://virtualmath1.stanford.edu/~ralph/book.pdf

Collins, 2016. "An algorithm for computing the Seifert matrix of a link from a braid representation" *Ensaios Matemáticos* **30**: 246-262, https://ensaios.sbm.org.br/wp-content/uploads/sites/14/2021/09/EM_30_Collins-1.pdf

Crainic, 2017. "Manifolds 2017" https://webspace.science.uu.nl/~crain101/manifolds-2017/lecture-notes-23-Oct.pdf

Daverman and Venema, 2009. *Embeddings in Manifolds.* American Mathematical Society

Davis and Petrosyan, 2012. "Manifolds and Poincaré complexes" https://jfdmath.sitehost.iu.edu/teaching/m623/dp.pdf

Freed, 2013. "Bordism: Old and New" https://web.ma.utexas.edu/users/dafr/bordism.pdf

Eschrig, 2011. *Topology and Geometry for Physics.* Springer

Gallot *et al*, 2004. *Riemannian Geometry.* Springer

Gorodski, 2012. "Appendix A: Smooth manifolds" https://www.ime.usp.br/~gorodski/teaching/mat5771/smoothmflds.pdf

Harrison, 2020. "Introducing totally nonparallel immersions" *Advances in Mathematics* **374**: 107337, https://arxiv.org/abs/1907.11312

Hayden *et al*, 2022. "Seifert surfaces in the 4-ball" https://arxiv.org/abs/2205.15283

Hirsch, 1976. *Differential Topology.* Springer-Verlag

Horowitz and Srednicki, 1990. "A Quantum Field Theoretic Description of Linking Numbers and Their Generalization" *Commun. Math. Phys.* **130**: 83-94, https://projecteuclid.org/euclid.cmp/1104187929

Kervaire and Weber, 1978. "A survey of multidimensional knots." Pages 61-134 in *Knot Theory: Proceedings, Plans-sur Bex, Switzerland 1977*, edited by Jean-Claude Hausmann (Springer)

Kirby and Siebenmann, 1977. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations.* Princeton University Press

Kupers, 2019. "Lectures on diffeomorphism groups of manifolds" https://people.math.harvard.edu/~kupers/teaching/272x/book.pdf

Lee, 1997. *Riemannian Manifolds.* Springer

Lee, 2000. *Introduction to Topological Manifolds.* Springer, https://archive.org/details/springer_10.1007-978-0-387-22727-6/

Lee, 2011. *Introduction to Topological Manifolds (Second Edition).* Springer

Lee, 2013. *Introduction to Smooth Manifolds (Second Edition).* Springer

Livingston, 1993. *Knot Theory.* The Mathematical Association of America

Meilhan, 2018. "Linking number and Milnor invariants" https://arxiv.org/abs/1812.03319

Michel and Weber, 2014. "Michel Kervaire work on knots in higher dimensions" https://arxiv.org/abs/1409.0704v1

Michor, 2008. *Topics in Differential Geometry.* Americal Mathematical Society, https://www.mat.univie.ac.at/~michor/listpubl.html#books

Milnor, 1974. *Characteristic Classes.* Princeton University Press

Ranicki, 2014. *High-dimensional knot theory.* Springer-Verlag

Robbin *et al*, 2018. "Introduction to Differential Topology" https://people.math.ethz.ch/~salamon/PREPRINTS/difftop.pdf

Rolfsen, 1976. *Knots and Links.* AMS Chelsea Publishing

Seifert and Threlfall, 1980. *A Textbook of Topology.* Academic Press

Singer, 2022. "Poincaré duality notes" https://www.homepages.ucl.ac.uk/~ucaheps/topics/pd_notes22.pdf

Tanaka, 2020. "Lecture 10: Compactness, II. Heine-Borel theorem" https://www.hiroleetanaka.com/pdfs/2020-fall-4330-topology-notes-10-heine-borel.pdf

Tu, 2011. *An Introduction to Manifolds (Second Edition).* Springer

van Wijk and Cohen, 2006. "Visualization of Seifert Surfaces" *IEEE Transactions on Visualization and Computer Graphics* **12**: 485-496

Wright, 2007. "Homotopy groups of spheres: a very basic introduction" https://public.websites.umich.edu/~alexmw/HomotopyGroupsOfSoheres.pdf

23 References in this series

Article **09894** (https://cphysics.org/article/09894):
"Tensor Fields on Smooth Manifolds" (version 2023-11-12)

Article **61813** (https://cphysics.org/article/61813):
"Homotopy, Homotopy Groups, and Covering Spaces" (version 2024-02-27)

Article **70621** (https://cphysics.org/article/70621):
"Principal Bundles and Associated Vector Bundles" (version 2024-02-25)

Article **93875** (https://cphysics.org/article/93875):
"From Topological Spaces to Smooth Manifolds" (version 2024-02-25)