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Sign Conventions in General Relativity

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Abstract General relativity involves several quantities whose overall signs are matters of convention. For some of these quantities, almost all of the sources surveyed use the same sign conventions. The variability in conventions for the metric tensor and the curvature tensor is greater. This article highlights some sign conventions that appear to be standard and describes some of the effects of changing the signs of the metric and curvature tensors. This can be used as a reference when comparing results from different sources.

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1 The sign of the metric tensor

The **metric tensor**[1] is one of the most basic quantities in general relativity. It mediates the gravitational interaction,[2,3] and it defines the geometry of spacetime.[4] It defines geometry by providing an **inner product** $g_{ab}V^aW^b$ between vector fields V, W, where g_{ab} are the components of the metric tensor, V^a and W^b are the components of the vector fields, and sums over the repeated indices a, b are implied. Spacetime has **lorentzian signature**, so the sign of the quantity

$$g_{ab}V^aV^b \tag{1}$$

depends on whether V is timelike or spacelike. Since the sign of (1) depends on V anyway, the overall sign of the metric tensor itself is a matter of convention.[5] In the **mostly plus** convention, (1) is positive for spacelike vectors. In the **mostly minus** convention, (1) is positive for timelike vectors. Both of these **signature conventions** are widely used. Mostly-plus seems to be preferred in classical general relativity,[6] and mostly-minus seems to be preferred in relativistic quantum field theory,[7] but both conventions are used in both contexts.[8]

[1] It may also be called the **metric tensor field** or **metric field** when we want to emphasize that it can vary throughout space and time (article 09894), or just the **metric** when we want to be more concise.

[2] Article 99922

[3] Models involving spinor fields need a more basic quantity called a **frame field** (Freedman and van Proeyen (2012), section 7.4.2; and Perry (2009), page 15), and then the metric tensor is expressed in terms of the frame field.

[4] Article 48968

[5] It's technically a matter of convention even when the signature is **euclidean**, in which case the sign of (1) is the same for all V, but in that case the convention that makes (1) positive for all $V \neq 0$ is the universal standard.

[6] Section 4

[7] Sources that use the mostly-minus convention include Schwartz (2013) (section 2.1.2), Banks (2008) (page 2), Peskin and Schroeder (1995) (page xix), Mandl and Shaw (1993) (page 28), Donoghue *et al* (1992) (page 505), Barger and Phillips (1987) (page 549), and Itzykson and Zuber (1980) (page 5). Sources that use the mostly-plus convention include Srednicki (2007) (chapter 1) and Weinberg (1995) (page 56).

[8] Both conventions are also used in classical electrodynamics. One that uses the mostly-plus convention is Zangwill (2012) (page 959), and one that uses the mostly-minus convention is Jackson (1975) (page 535).

2 Conventions that seem to be standard

This section highlights some sign conventions that seem to be standard, at least in general relativity.

The standard[9] convention for the sign of the connection coefficients Γ^c_{ab} is[10]

$$\nabla_a V^b = \partial_a V^b + \Gamma^b_{a\bullet} V^\bullet, \tag{2}$$

where ∇ is the covariant derivative (the Levi-Civita connection). With this convention, the geodesic equation is $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$. The standard[11] convention for the relationship between the Ricci tensor R_{ab} and the connection coefficients is[10,12]

$$R_{ab} = \partial_\bullet \Gamma^\bullet_{ab} - \partial_a \Gamma^\bullet_{\bullet b} + \Gamma^\times_{\times\bullet}\Gamma^\bullet_{ab} - \Gamma^\times_{a\bullet}\Gamma^\bullet_{\times b}. \tag{3}$$

The standard[9] convention for the relationship between the Ricci tensor and the curvature scalar R is[13]

$$R = g^{ab} R_{ab}. \tag{4}$$

The standard convention for the Einstein tensor is

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R.$$

In the gravitational field equation, the relative sign of the term with the stress-energy tensor T_{ab} is determined by the physical requirement that test objects should be attracted to positive masses.[14] The standard[9] convention for the sign of the cosmological constant Λ is such that Λ and T_{ab} occur only in the combination $\kappa T_{ab} u^a u^b + \Lambda$ whenever the gravitational field equation is contracted with $u^a u^b$ for a timelike unit vector u.

[9] All of the sources surveyed in section 4 use this convention.

[10] The symbols $\bullet$ and $\times$ is used here as indices, and sums over these indices are implied.

[11] Almost all of the sources surveyed in section 4 use this convention, with only one exception.

[12] ∂_a denotes the partial derivative with respect to the ath coordinate.

[13] Given the conventions (2)-(3), the sign of the curvature scalar (4) depends on the sign of the metric tensor.

[14] Article 99922

3 A property of the standard set of conventions

This section confirms that the conventions (2)-(4) gives $R > 0$ for the curvature scalar of the standard metric on a two-dimensional sphere when the mostly-plus convention is used for the metric tensor.[15]

Using notation that was introduced in article 21808, the standard metric on a two-dimensional sphere is defined by the line element

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2,$$

where θ, ϕ are the independent coordinates defined in a neighborhood near the point $\theta = \pi/2$. The connection coefficients may be derived using the lagrangian method described in article 33547. That leads to the geodesic equations

$$\ddot{\phi} + 2\frac{\cos\theta}{\sin\theta}\,\dot{\theta}\dot{\phi} = 0 \qquad\qquad \ddot{\theta} - (\cos\theta\,\sin\theta)\,\dot{\phi}^2 = 0,$$

and comparing this to the general geodesic equation $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$ gives[16]

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta} \qquad\qquad \Gamma^\theta_{\phi\phi} = -\cos\theta\,\sin\theta.$$

The convention (3) gives

$$\begin{aligned}
R_{\theta\theta} &= \partial_\bullet\Gamma^\bullet_{\theta\theta} - \partial_\theta\Gamma^\bullet_{\bullet\theta} + \Gamma^\bullet_{\bullet\times}\Gamma^\times_{\theta\theta} - \Gamma^\bullet_{\theta\times}\Gamma^\times_{\bullet\theta} = -\partial_\theta\Gamma^\phi_{\phi\theta} - \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\phi\theta} = 1 \\
R_{\phi\phi} &= \partial_\bullet\Gamma^\bullet_{\phi\phi} - \partial_\phi\Gamma^\bullet_{\bullet\phi} + \Gamma^\bullet_{\bullet\times}\Gamma^\times_{\phi\phi} - \Gamma^\bullet_{\phi\times}\Gamma^\times_{\bullet\phi} = \partial_\theta\Gamma^\theta_{\phi\phi} + \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi} - 2\Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi} = \sin^2\theta
\end{aligned}$$

with implied sums over $\bullet$ and $\times$, and then using (4) for the curvature scalar gives[17]

$$R = R_{\theta\theta} + \frac{1}{\sin^2\theta}R_{\phi\phi} = 2.$$

[15] This is also worked out in section 8.6 in Blau (2022). Article 96560 uses a different coordinate system to calculate the curvature scalar for a sphere S^D (unit sphere in $D+1$-dimensional euclidean space) for arbitrary D.

[16] These are independent of the sign convention for the metric tensor.

[17] This depends on the sign convention for the metric tensor.

4 A small survey

This table surveys some of the conventions used in a small sample of sources.[18] All of these sources use the standards listed in section 2, with one exception that uses the opposite sign in (3), as indicated in the table. One of these sources (Lee (1997)) is a book about Riemannian geometry, two are my own articles (included here to compare my own conventions), and the rest are books about general relativity.[19]

source	metric tensor	curvature tensor	Ricci tensor ($R_{ab}=\downarrow$)	grav'l field equation ($G_{ab}=\downarrow$)
Lee (1997)	N/A	$R_{[ab]c}{}^{d}=\partial_a\Gamma^d_{bc}\cdots$	$\partial_\bullet\Gamma^\bullet_{ab}\cdots$	N/A
articles 03519, 99922	mostly −	$R_{[ab]c}{}^{d}=\partial_a\Gamma^d_{bc}\cdots$	$\partial_\bullet\Gamma^\bullet_{ab}\cdots$	$\kappa T_{ab}+g_{ab}\Lambda$
d'Inverno (1995)	mostly −	$R^a{}_{b[cd]}=\partial_c\Gamma^a_{bd}\cdots$	$\partial_\bullet\Gamma^\bullet_{ab}\cdots$	$\kappa T_{ab}+g_{ab}\Lambda$
Martin (1988)	mostly −	$R^a{}_{b[cd]}=\partial_c\Gamma^a_{bd}\cdots$	$\partial_\bullet\Gamma^\bullet_{ab}\cdots$	$\kappa T_{ab}+g_{ab}\Lambda$
Penrose & Rindler (1986)	mostly −	$R_{[ab]c}{}^{d}=\partial_a\Gamma^d_{bc}\cdots$	$\partial_a\Gamma^\bullet_{b\bullet}\cdots$	$-\kappa T_{ab}-g_{ab}\Lambda$
Blau (2022)	mostly +	$R^a_{b[cd]}=\partial_c\Gamma^a_{bd}\cdots$	$\partial_\bullet\Gamma^\bullet_{ab}\cdots$	$\kappa T_{ab}-g_{ab}\Lambda$
Schutz (1985)	mostly +	$R^a{}_{b[cd]}=\partial_c\Gamma^a_{bd}\cdots$	$\partial_\bullet\Gamma^\bullet_{ab}\cdots$	$\kappa T_{ab}-g_{ab}\Lambda$
Stephani *et al* (2003)	mostly +	$R^a{}_{b[cd]}=\partial_c\Gamma^a_{bd}\cdots$	$\partial_\bullet\Gamma^\bullet_{ab}\cdots$	$\kappa T_{ab}-g_{ab}\Lambda$
Wald (1984)	mostly +	$R_{[ab]c}{}^{d}=\partial_b\Gamma^d_{ac}\cdots$	$\partial_\bullet\Gamma^\bullet_{ab}\cdots$	$\kappa T_{ab}-g_{ab}\Lambda$

In the curvature-tensor column, square brackets indicate the pair of subscripts that correspond to the subscripts in the expression $[\nabla_a,\nabla_b]$ in section 5.

[18] In Lee (1997), see pages 52, 118, 124. In d'Inverno (1995), see pages 86-87, 108, 143, 322. In Martin (1988), see pages 80, 98-99, 140, 153. In Penrose & Rindler (1986), see pages 3, 200, 210, 234-235. In Blau (2022), see (5.3), (8.5), (8.39), (8.42), (19.46). In Schutz (1985), see pages 155, 169, 173-174, 199. In Stephani *et al* (2003), see (1.1), (2.71a), (2.79), (2.83), (3.1), (3.48). In Wald (1984), see pages xi, 40, 48, 51, 72, 99.

[19] This is only a small subset of the many books that introduce general relativity. This subset might be biased by factors like readability (partly subjective) and availability on library shelves at the times I happened to be walking by. I don't select books based on what sign convention(s) they use, but I do select books based on what subjects they cover, and that might be correlated with sign conventions. Example: Wald (1984) uses the mostly-plus convention everywhere except in the chapter about spinor fields, where he uses the mostly-minus convention instead. A survey of sign conventions used in introductions to supergravity might be revealing, but that would require accounting for additional sign conventions, like the one used in the definition of the Clifford algebra (article 03910).

5 The sign of the curvature tensor

The quantities g_{ab}, Γ^c_{ab}, and R_{ab} are all invariant under permutations $a \leftrightarrow b$ of their subscripts, but the curvature tensor is not invariant under all permutations of its three subscripts. This adds another degree of variability among conventions for the curvature tensor beyond its overall sign. Section 4 listed some examples.

The Ricci tensor R_{ab} is normally defined in terms of the curvature tensor. For the purpose of comparing conventions across different sources, sections 2 and 4 expressed R_{ab} in terms of the connection coefficients instead, so that the reader isn't forced to trace through a more obtuse set of conventions associated with the curvature tensor. To illustrate just how subtle that can be, this section compares two different definitions of the curvature tensor that implicitly use opposite sign conventions even though they both look equally natural.

The covariant derivative along a vector field X will be denoted ∇_X, or ∇_a when the vector field is ∂_a (the partial derivative with respect to the ath coordinate).[20] To make the equations easier to parse, I'll use the abbreviation

$$[\nabla_X, \nabla_Y]Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$$

for any tensor field Z. Lee (1997) uses the convention[21]

$$[\nabla_a, \nabla_b]\partial_c = R_{abc}{}^{\bullet}\partial_{\bullet} \qquad \text{(Lee).} \tag{5}$$

In contrast, Wald (1984) uses the convention[22]

$$[\nabla_a, \nabla_b]\omega_c = R_{abc}{}^{\bullet}\omega_{\bullet} \qquad \text{(Wald),} \tag{6}$$

where ω_c is a covector field (one-form field) written using abstract index notation.[23] Equations (5) and (6) both look natural, but they define opposite sign conventions

[20] Article 09894

[21] Page 118 in Lee (1997) writes this as $R(\partial_a, \partial_b)\partial_c = R_{abc}{}^d\partial_d$ with $R(X,Y) \equiv [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$, where $[X,Y]$ denotes the Lie bracket of two vector fields X, Y. Using the identity $[\partial_a, \partial_b] = 0$ gives (5).

[22] Wald (1984), equation 3.2.3

[23] Wald (1984), section 2.4

for the curvature tensor. This follows from the fact that ∂_a is a vector field and ω_a is a covector field. Both are written using subscripts, but for different reasons: the subscript on ∂_a specifies the vector field's direction, whereas the subscript on ω_a is an "abstract index" that emulates the standard notation for the field's components.[24]

This hidden difference in the overall sign of the curvature tensor can be exposed by rewriting equations (5) and (6) in terms of components. For equation (6), that's just a matter of inserting parentheses, like this:

$$\big([\nabla_a,\nabla_b]\omega\big)_c = R_{abc}{}^{\bullet}\omega_{\bullet} \qquad\qquad \text{(Wald).} \tag{7}$$

The left-hand side is the cth component of the covector field $[\nabla_a,\nabla_b]\omega$, and the $\omega_\bullet$ on the right-hand side is the $\bullet$th component of the covector field ω. To write equation (5) in terms of components, consider a general vector field $V = V^\bullet\partial_\bullet$. Using equation (5) gives[25]

$$[\nabla_a,\nabla_b]V^c\partial_c = V^\bullet[\nabla_a,\nabla_b]\partial_\bullet = V^\bullet R_{ab\bullet}{}^c\partial_c = R_{abd}{}^cV^d\partial_c.$$

Compare the first and last expressions to get

$$\big([\nabla_a,\nabla_b]V\big)^c = R_{ab\bullet}{}^cV^\bullet \qquad\qquad \text{(Lee).} \tag{8}$$

To expose the fact that equations (7) and (8) define opposite sign conventions for the curvature tensor, use the fact that $V^\bullet\omega_\bullet$ is a scalar field, which implies

$$\begin{aligned} 0 &= [\nabla_a,\nabla_b](V^\bullet\omega_\bullet) \\ &= \big([\nabla_a,\nabla_b]V\big)^\bullet\omega_\bullet + V^\bullet\big([\nabla_a,\nabla_b]\omega\big)_\bullet. \\ &\overset{\text{Lee}}{=} R_{ab\bullet}{}^{\times}\, V^\bullet\omega_\times + \overset{\text{Wald}}{R_{ab\bullet}{}^{\times}}\, V^\bullet\omega_\times. \end{aligned}$$

This shows that Lee (1997) and Wald (1984) use opposite sign conventions for the curvature tensor, even though their definitions (5) and (6) both look natural.

[24] Wald (1984), section 2.4

[25] The first step uses the identity shown in exercise 7.1 in Lee (1997).

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