Black Holes: How Quickly Do Infalling Objects Disappear?

Randy S

Abstract Consider this common question: if a hovering observer releases a beacon, letting it fall toward the event horizon, does the hovering observer see the object freeze at the horizon? Or does the hovering observer see the object disappear? This article uses the metric of an ideal non-rotating black hole to answer the question quantitatively.

Contents

1 Introduction

Consider a radially falling object in the Schwarzschild metric. Use the standard coordinate system in which the line element for radial motion is given by equation (6) in section 4, with coordinates t, r. Suppose the object is dropped from a radius $r_0 > R$, where R is the Schwarzschild radius. How quickly does the object disappear, for all practical purposes, according to an observer hovering at $r_H \gg R$?

(In this article, H denotes the distant hovering observer and F denotes the falling object. A subscript 0 denotes an initial value for the falling object.)

To define "disappear," suppose that the object is emitting light with a frequency f_0 and with a constant power P_0, both according to its own clock. For simplicity, pretend that all of the light travels directly toward the distant observer, so that none of the power is wasted. The distant hovering observer will receive a lower frequency f_H and a lower power P_H, both of which continue to decrease as the object falls, due to the increasing redshift as the object approaches the event horizon. The object "disappears" when the frequency or power (or both) are too low to be observed by any practical devices.

Let t be the coordinate-time,[1] which coincides with the distant observer's proper time in the standard coordinate system. This article derives the result

$$f_H(t) = Q(t) f_0 \qquad\qquad P_H(t) = Q^2(t) P_0 \tag{1}$$

with

$$Q(t) \leq \frac{r_0}{R} \exp\left(\left(\frac{r_0}{R}\right)^2 - \frac{ct}{R}\right) \tag{2}$$

where c is the speed of light. The exponentially-decreasing dependence on t implies that the falling object will disappear very quickly (according to the distant observer's clock) as it approaches the event horizon.

[1] As emphasized in article 48968, time is defined by the metric (6), not by how we name the coordinates, but referring to t as "coordinate time" will be convenient.

2 Example

For a numeric example, consider a solar-mass black hole, for which $R \approx 3$ km, and suppose that the object is dropped from $r_0 = 1.8R$.

Before the object is dropped, suppose that it is temporarily held at r_0. If the light emitted has wavelength 470 nm (a blue wavelength) according to that object's clock, then the light received by the distant observer will be[2]

$$(470 \text{ nm}) \times \left(1 - \frac{R}{r_0}\right)^{-1/2} = 705 \text{ nm}$$

(a red wavelength) according to the distant observer's clock.

Now suppose that the object is dropped from r_0. After 1 millisecond has elapsed on the distant observer's clock (so that $ct/R \approx 100$), we have

$$Q(t) < 10^{-40},$$

so the wavelength received by the distant observer will be more than 10^{40} times the emitted wavelength, and the power received will be less than 10^{-80} times the emitted power. For all practical purposes, according to the distant observer, the object has disappeared less than a millisecond after being dropped.

[2] I chose $r_0 = 1.8R$ so that $1 - R/r_0 = 4/9$, to make it easy.

3 Approach

Consider radial motion only, and use units in which the speed of light is $c = 1$. Suppose we have equations

$$t = t_F(\tau) \qquad\qquad r = r_F(\tau)$$

for the coordinates of the falling (F) object as a function of its own proper time τ. Suppose also that we have an equation

$$t = t_L(r)$$

for a null geodesic[3] (L = light) expressed by giving the t-coordinate as a function of the r-coordinate. As the object falls, it emits light that has a constant frequency and constant power according to its own clock. Consider an observer hovering at r_H, and let t_H at which a given parcel of light reaches the observer, expressed as a function of the falling object's proper time τ at which that parcel was emitted. Then

$$t_H(\tau) = t_F(\tau) + \Delta_L\big(r_F(\tau)\big). \tag{3}$$

The first term on the right-hand side is the coordinate-time at which the parcel of light was emitted, and the second term is the elapsed coordinate-time during the light's journey from r_F to r_H. The term Δ_L depends only on r_F because the metric (6) is invariant under translations in t. The factor $Q(t)$ that relates f_H to f_0 in equation (1) is

$$Q\big(t(\tau)\big) = (\dot{t}_H)^{-1} \tag{4}$$

where the overhead dot means a derivative with respect to τ.

To relate this to power, we can use a simplistic "photon" model for the light. Then (4) is proportional to the number of photons received per second as a function of τ, given that photons are emitted at a constant rate on the falling object's clock. The power received is proportional to hf (the energy in one photon) times the

[3] **Null geodesic** is a shorter name for **lightlike geodesic** (article 48968).

number of photons received per second, where h is Planck's constant, so the power is proportional to Q^2.

Equation (3) implies

$$\dot{t}_H(\tau) = \dot{t}_F(\tau) + \Delta'_L\big(r_F(\tau)\big)\dot{r}_F(\tau) \tag{5}$$

where Δ'_L is the derivative of $\Delta_L(r)$ with respect to its argument r_F. Now if we express τ as a function of the coordinate time t, then we have Q as a function of the coordinate time t, which coincides with the distant observer's proper time if $r_H \gg R$.

4 Geodesic equations

For radial motion with $r > R$, the proper-time equation for an ideal non-rotating black hole (article 24902) can be written

$$d\tau^2 = a\,dt^2 - a^{-1}\,dr^2 \tag{6}$$

with

$$a(r) \equiv 1 - \frac{R}{r}$$

where R is the Schwarzschild radius. The equations of motion for an object with non-zero mass are (article 33547)[4]

$$\ddot{r} = -a'/2 \qquad\qquad a\dot{t} = \gamma \tag{7}$$

where γ is a constant and where each overhead dot is a derivative with respect to τ. Substitute this expression for $\dot{t}^2$ into (6) to deduce

$$1 = a^{-1}\gamma^2 - a^{-1}\dot{r}^2. \tag{8}$$

At the point where the object is dropped, we have $\dot{r} = 0$, so this gives

$$\gamma^2 = a(r_0) \tag{9}$$

if the object is dropped from r_0. Use this in the second of equations (7)

$$\boxed{\dot{t}^2 = \frac{a(r_0)}{a^2(r)}.} \tag{10}$$

Use (9) in (8) to get

$$\boxed{\dot{r}^2 = a(r_0) - a(r),} \tag{11}$$

[4] We won't need the equation for $\ddot{r}$.

and multiply both sides of this by $1/\dot{t}^2$ to get

$$\boxed{(dr/dt)^2 = \frac{a(r_0) - a(r)}{a(r_0)} a^2(r).} \tag{12}$$

The equation for a radial null geodesic is easier to derive, because for this special metric, any radial null worldline is a geodesic. Set $d\tau = 0$ in (6) to get

$$\boxed{(dr/dt)^2 = a^2(r) \qquad\qquad \text{(null geodesic).}} \tag{13}$$

Solving equation (11) gives us $r = r_F(\tau)$, and then substituting this into (10) gives us $t = t_F(\tau)$. Solving equation (13) gives us $t = t_L(r)$. Altogether, this is everything we need for equation (3).

5 World-line of the light

The quantity

$$\Delta'_L(r_F)$$

in equation (5) can be evaluated exactly. For an outgoing parcel of light, equation (13) reduces to

$$dr/dt = a(r),$$

which is the same as

$$dt/dr = 1/a(r) = \frac{r}{r-R}. \tag{14}$$

The solution is

$$t = \text{constant} + r + R\log(r-R) \equiv t_L(r),$$

which can be verified by taking the derivative with respect to r and comparing to (14). The elapsed coordinate-time for the light to travel from r_F to r_H is therefore

$$\Delta_L(r_F) = r_H - r_F + R\big(\log(r_H - R) - \log(r_F - R)\big),$$

and the derivative of this with respect to r_F is

$$\Delta'_L(r_F) = -\left(1 + \frac{R}{r_F - R}\right) = \frac{-1}{a(r_F)}. \tag{15}$$

This could have been inferred directly from (14) with no calculation, because only the r_F-end of the journey changes as a function of r_F. The result is negative because the elapsed coordinate-time decreases with increasing r_F (because the distance traveled decreases witih increasing r_F). In equation (5), the negative sign is canceled by the sign of $\dot{r}_F < 0$.

6 Reduction to an integral

We can easily derive a version of equation (5) in which the parameter is r_F (abbreviated r from now on) instead of τ. Equations (10), (11), and (15) give[5]

$$\dot{t}_F + \Delta_L' \dot{r}_F = \frac{a_0^{1/2}}{a} + \frac{1}{a}(a_0 - a)^{1/2} = \frac{a_0^{1/2} + (a_0 - a)^{1/2}}{a} \tag{16}$$

with $a \equiv a(r)$ and $a_0 \equiv a(r_0)$. This is the quantity of interest expressed as a function of the falling object's coordinate r. As expected, it diverges as $r \to R$, so the frequency goes to zero as $r \to R$.

To finish, we need to express r as a function of the coordinate time t, so we need to solve equation (12). Define

$$h(r) \equiv a(r)b(r) \qquad\qquad b(r) \equiv \sqrt{\frac{a(r_0) - a(r)}{a(r_0)}}$$

so that equation (12) may be written

$$dr/dt = -h(r).$$

The minus sign ensures that r is a decreasing function of t (so that the object is falling toward the black hole). We can also think of this as

$$dt/dr = -1/h(r),$$

which is solved by

$$t = \int_r^{r_0} \frac{ds}{h(s)} = \int_r^{r_0} ds\ \frac{s}{(s-R)b(s)}. \tag{17}$$

To check this, just take the derivative of both sides with respect to r. The upper limit of the integration range is set to r_0 so that $t = 0$ when $r = r_0$, which represents the instant the object is dropped.

[5] The minus sign in (15) cancels the minus sign in $\dot{r}_F = -(a_0 - a)^{1/2}$.

The integral (17) diverges as $r \to R$. By adding and subtracting $R/(s-R)$ from the integrand, we can write the integral as

$$t = T(r) + S(r) \tag{18}$$

with

$$T(r) \equiv \int_r^{r_0} ds\ \frac{R}{s-R} \tag{19}$$

$$S(r) \equiv \int_r^{r_0} ds\ \left(\frac{s}{(s-R)b(s)} - \frac{R}{s-R}\right). \tag{20}$$

Here's why this is useful:

- The function $T(r)$ can be calculated exactly, and it diverges as $r \to R$.
- The function $S(r)$ remains finite as $r \to R$ and is therefore negligible compared to $T(r)$ when the falling object is very close to the horizon (when r is close to R), so we don't need to calculate it exactly.

7 The divergent part

The part that diverges as $r \to R$ is

$$T(r) = R\big(\log(r_0 - R) - \log(r - R)\big) > 0.$$

Use this in (18) to get

$$t \leq S_{\text{max}} + R\big(\log(r_0 - R) - \log(r - R)\big).$$

Re-arrange this to get

$$r \leq R + (r_0 - R)\exp\left(\frac{S_{\text{max}} - t}{R}\right) \equiv R + \rho(t).$$

This implies

$$a(r) \leq 1 - \frac{R}{R + \rho(t)} = \frac{\rho(t)}{R + \rho(t)}.$$

Equations (3), (4), and (16) say the redshift factor is

$$Q(t) = \frac{a(r)}{\sqrt{a_0} + \sqrt{a_0 - a}} \leq \frac{a(r)}{\sqrt{a_0}},$$

and combining this with the preceding inequality for $a(r)$ gives

$$\begin{aligned}
Q(t) &\leq \frac{1}{\sqrt{a_0}} \frac{\rho(t)}{R + \rho(t)} \\
&\leq \frac{1}{\sqrt{a_0}} \frac{\rho(t)}{R} \\
&= \left(\left(\frac{r_0}{R}\right)^2 - \frac{r_0}{R}\right)^{1/2} \exp\left(\frac{S_{\text{max}} - t}{R}\right) \\
&\leq \frac{r_0}{R} \exp\left(\frac{S_{\text{max}} - t}{R}\right)
\end{aligned}$$

The next section derives the result

$$S_{\max} \leq \frac{r_0^2}{R}, \tag{21}$$

which finally gives the result quoted in the introduction, expressed here in units where $c = 1$:

$$\boxed{Q(t) \leq \frac{r_0}{R} \exp\left(\left(\frac{r_0}{R}\right)^2 - \frac{t}{R}\right).}$$

8 The finite remainder

This section derives the inequality (21). Use the identity $b(R) = 1$ to see that the integrand of $S(r)$ (equation (20)) is finite when $s = R$. The integrand diverges when $s = r_0$, because $b(r_0) = 0$, but the integral is still finite. To prove this, start by writing $S(r)$ as

$$\begin{aligned} S(r) &= \int_r^{r_0} ds\ \frac{s - b(s)R}{(s - R)b(s)}. \\ &= \int_r^{r_0} ds\ \frac{s\sqrt{a_0} - R\sqrt{a_0 - a(s)}}{(s - R)\sqrt{a_0 - a(s)}}. \end{aligned}$$

Temporarily work in units $R = 1$ and multiply the numerator and denominator of the integrand by $\sqrt{r_0 s}$ to get

$$S(r) = \int_r^{r_0} ds\ \frac{s^{3/2}\sqrt{r_0 - 1} - \sqrt{r_0 - s}}{(s - 1)\sqrt{r_0 - s}}.$$

Define

$$u \equiv \sqrt{r_0 - s} \qquad\qquad \sigma \equiv \sqrt{r_0 - 1}$$

and use $u\,du = -ds$ to get

$$\begin{aligned} S(r) &= \int_0^{\sqrt{r_0 - r}} du\ \frac{(r_0 - u^2)^{3/2}\sigma - u}{\sigma^2 - u^2} \\ &= \int_0^{\sqrt{r_0 - r}} du\ \frac{(1 + \sigma^2 - u^2)^{3/2}\sigma - u}{\sigma^2 - u^2}. \end{aligned}$$

The integrand is finite for all $0 \leq u \leq \sqrt{r_0 - s}$, as long as $s \geq 1$. This proves that $S(r)$ is finite for all $1 \leq r \leq r_0$, which is $R \leq r \leq r_0$ after restoring factors of R.

To get an upper bound on $S(r)$, use

$$\begin{aligned}
S(r) &\leq \int_0^{\sqrt{r_0-r}} du\ \frac{(1+\sigma^2-u^2)^2\sigma - u}{\sigma^2-u^2} \\
&= \int_0^{\sqrt{r_0-r}} du\ \left(\frac{1}{\sigma+u} + 2\sigma + (\sigma^2-u^2)\sigma\right) \\
&\leq \int_0^{\sqrt{r_0-r}} du\ \left(\frac{1}{\sigma} + 2\sigma + \sigma^3\right) \\
&= \left(\frac{1}{\sigma} + 2\sigma + \sigma^3\right)\sqrt{r_0-r}.
\end{aligned}$$

The maximum value occurs for $r = 1$, so

$$S_{\max} \leq 1 + 2\sigma^2 + \sigma^4 = (1+\sigma^2)^2 = r_0^2.$$

After restoring factors of R, this becomes

$$\boxed{S_{\max} \leq \frac{r_0^2}{R}.}$$

9 References in this series

Article **24902** (https://cphysics.org/article/24902):
"The Ideal Non-Rotating Black Hole" (version 2022-02-05)

Article **33547** (https://cphysics.org/article/33547):
"Free-Fall, Weightlessness, and Geodesics" (version 2022-02-05)

Article **48968** (https://cphysics.org/article/48968):
"The Geometry of Spacetime" (version 2022-01-16)