1. Rotations in 3D, so(3), and su(2).
* version 2.1 *

Matthew Foster

January 20, 2021

Contents

1.1 Rotation groups in 3D **1**
1.1.1 SO(2) $\simeq$ U(1) . . . 1
1.1.2 SO(3) . . . 3

1.2 Lie algebra: formal definition **4**

1.3 su(2) $\simeq$ so(3); irreducible representations **5**
1.3.1 Spin 1/2 . . . 5
1.3.2 Generic j . . . 6
1.3.3 Quadratic Casimir . . . 7

1.4 Tensor representations of so(3) **7**
1.4.1 Traceless symmetric tensors . . . 7
1.4.2 Spherical tensors . . . 8

1.5 Tensor representations of su(2) **9**

The discussion here largely follows chapter I of [1]. In addition to serving as a review, the last two sections collect a variety of useful facts and formulae for su(2) = so(3) representations.

1.1 Rotation groups in 3D

1.1.1 SO(2) $\simeq$ U(1)

The set of rotations about a given fixed axis in space define an **abelian**, **continuous** group. Consider a 3D vector $V = \{V_x, V_y, V_z\}$. A rotation in the xy plane (around the z axis) is represented by the matrix

$$\hat{R}(\theta) \equiv \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{1.1.1}$$

An active counterclockwise rotation by the angle θ is implemented by the matrix multiplication

$$V' \equiv \hat{R}(\theta)\, V = \begin{bmatrix} \cos(\theta)V^x - \sin(\theta)V^y \\ \sin(\theta)V^x + \cos(\theta)V^y \\ V^z \end{bmatrix} = \begin{bmatrix} V'^x \\ V'^y \\ V'^z \end{bmatrix} . \tag{1.1.2}$$

A **continuous group** G is a set of transformations satisfying group axioms,[1] in which each element $g(x) \in G$ can be specified by a continuously variable parameter or set of parameters $x = \{x_i\}$. Without loss of generality, we will consider real-valued parameters

[1] A **group** is a set $G = \{g_\alpha\} = \{g_1, g_2, \ldots\}$ along with a binary rule ("multiplication") $G \times G \to G$, satisfying the properties

Figure 1.1: Rotation of a 3D vector around the z-axis.

($x_i \in \mathbb{R}\ \forall\ i$). Unlike a finite group such as the set S_n of permutations of n objects, a continuous group clearly has an uncountably infinite number of elements. Instead, we can define the **dimension** d of a continuous group as the number of parameters needed to specify all group transformations, i.e. the $\{x_i\} = \{x_1, x_2, \ldots, x_d\}$. We can think of each group element $g(x)$ labeled by $x = \{x_i\}$ as a point on a d-dimensional **group manifold**, a notion we will explore a bit in module 2.

Rotations about a fixed axis have $d = 1$, specified by the angle θ. This group is also **compact**, meaning that we can label any element of the group by restricting θ to a finite interval of the real line: $\theta \in [-\pi, \pi)$. Equivalently, the group manifold is the circle. The inverse of a given rotation $\hat{R}(\theta)$ is a rotation of the same "strength" and opposite chirality:

$$\hat{R}(\theta)\,\hat{R}(-\theta) = \hat{1}_3,$$

where $\hat{1}_3$ is the 3×3 identity matrix.

Continuous symmetry transformations are usually represented by **unitary** operators in quantum mechanics. For a system with a finite n-dimensional Hilbert space, this simply means an $n \times n$ matrix $\hat{U}$ that satisfies

$$\hat{U}\,\hat{U}^\dagger = \hat{U}^\dagger\,\hat{U} = \hat{1}_n, \tag{1.1.3}$$

where the dagger † means the transpose conjugate. Eq. (1.1.2) is a unitary transformation. In addition, the elements of $\hat{R}(\theta)$ are purely real. This means that the inverse is the transpose:

$$\hat{R}^{-1}(\theta) = \hat{R}(-\theta) = \hat{R}^{\mathsf{T}}(\theta). \tag{1.1.4}$$

A unitary transformation satisfying Eq. (1.1.4) is said to be **orthogonal**. For a more in-depth review of orthogonal transformations in relation to matrix algebra, see e.g. Chapter 4 in [2].

We can write Eq. (1.1.1) in another way. Let us define three 3×3 matrices:

$$\hat{J}_x \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{J}_y \equiv \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \hat{J}_z \equiv \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.1.5}$$

The xy-plane rotation can then be expressed as

$$\hat{R}(\theta) = \exp\left(\theta\,\hat{J}_z\right). \tag{1.1.6}$$

Here the exponential function of a matrix is defined through the expansion

$$e^{\hat{A}} \equiv \hat{1}_3 + \hat{A} + \frac{1}{2!}\hat{A}^2 + \ldots$$

1. There exists an **identity element** $e \in G$ such that
$$g_\alpha \times e = e \times g_\alpha = g_\alpha,$$
2. For any element $g_\alpha \in G$, there exists an **inverse** element g_α^{-1} also in G, such that
$$g_\alpha \times g_\alpha^{-1} = g_\alpha^{-1} \times g_\alpha = e.$$

• **Exercise**: Check Eq. (1.1.6) by expanding in powers of θ and using

$$\hat{J}_z^2 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly we have

$$\left[\hat{R}(\theta), \hat{R}(\theta')\right] = 0 \tag{1.1.7}$$

for arbitrary θ and θ'. Here $[\hat{A}, \hat{B}] = \hat{A}\,\hat{B} - \hat{B}\,\hat{A}$ denotes the matrix commutator. Eq. (1.1.7) implies that the group of planar rotations is **abelian**: all possible group transformations commute. It means that the order in which a sequence of different rotations is applied to a vector in the plane does not matter:

$$\hat{R}(\theta_1)\,\hat{R}(\theta_2)\,\hat{R}(\theta_3)\,V = \hat{R}(\theta_2)\,\hat{R}(\theta_1)\,\hat{R}(\theta_3)\,V = \hat{R}(\theta_3)\,\hat{R}(\theta_1)\,\hat{R}(\theta_2)\,V = \hat{R}(\theta_1+\theta_2+\theta_3)\,V.$$

The last equality follows from Eq. (1.1.6). Higher dimensional continuous abelian groups are also possible, e.g. the set of all diagonal $n \times n$ matrices.[2]

The abelian group of rotations in a plane is denoted SO(2), meaning the special[3] orthogonal group acting on a vector (or its projection into the plane) in two dimensions. It is also denoted by U(1), the unitary group formed by the composition of complex phase factors; it also called the circle group.

1.1.2 SO(3)

If we consider rotations about *different* axes in 3 or more spatial dimensions, then story is different. To see this, we consider **infinitesimal rotations**, that is, continuous group elements that are arbitrarily close to the group identity. The latter is a unique point on the group manifold, conventionally labeled by the origin of the group parameter coordinate system $\{x^i\} = 0$. Eq. (1.1.6) implies that

$$\hat{R}_{xy}(\theta_3) \simeq \hat{1}_3 + \theta_3\,\hat{J}_z, \quad |\theta_3| \ll 1. \tag{1.1.8}$$

For a right-handed coordinate system in 3D [Fig. 1.1], we can represent infinitesimal yz and zx plane rotations using the other "generators" in Eq. (1.1.5),

$$\hat{R}_{yz}(\theta_1) \simeq \hat{1}_3 + \theta_1\,\hat{J}_x, \quad \hat{R}_{zx}(\theta_2) \simeq \hat{1}_3 + \theta_2\,\hat{J}_y, \quad |\theta_{1,2}| \ll 1. \tag{1.1.9}$$

The corresponding finite yz and zx plane rotations are exponentiations of these.

A rotation within *any* plane (or about any axis perpendicular to it) in 3D can be composed from a product of three finite rotations about at least two different axes; a concrete construction is the Euler angle scheme used in classical mechanics texts [2], but we will not need this here. We will instead focus on products of infinitesimal rotations. Consider a product of yz and zx plane rotations, applied in opposite orders. The difference between these is given by

$$\hat{R}_{yz}(\theta_1)\,\hat{R}_{zx}(\theta_2) - \hat{R}_{zx}(\theta_2)\,\hat{R}_{yz}(\theta_1) \simeq \theta_1\theta_2\left[\hat{J}_x, \hat{J}_y\right]. \tag{1.1.10}$$

Order therefore matters for the composition of rotations about orthogonal axes, with different results for different orders. For infinitesimal transformations, we need to know the **commutation relations** between the **generators** $\hat{J}_{x,y,z}$ of different rotations. For rotations in 3D, these are given by

$$\left[\hat{J}_x, \hat{J}_y\right] = \hat{J}_z, \quad \left[\hat{J}_y, \hat{J}_z\right] = \hat{J}_x, \quad \left[\hat{J}_z, \hat{J}_x\right] = \hat{J}_y. \tag{1.1.11}$$

• **Exercise**: Verify Eq. (1.1.11) using Eq. (1.1.5).

A more succinct statement employs the rank-3 **Levi-Civita** tensor. This is the unique completely antisymmetric third rank tensor with three indices $\{i, j, k\}$ which run over three values $i \in \{1, 2, 3\}$ or $i \in \{x, y, z\}$:

$$\begin{aligned} \epsilon_{ijk} &= -\,\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \\ &= \epsilon_{jki} = \epsilon_{kij}. \end{aligned} \tag{1.1.12}$$

[2] The representations of higher dimensional continuous abelian groups are however completely reducible.

[3] Special orthogonal transformations are defined to have determinant one. This includes rotations, but excludes discrete orthogonal transformations with determinant minus one, as well as products of these with rotations. Discrete orthogonal transformations with determinant minus one are reflections and (in odd spatial dimensionality) inversion.

In other words, it is antisymmetric under the exchange of any two indices, and symmetric under cyclic permutations. It is normalized such that

$$\epsilon_{123} = 1.$$

Obviously $\epsilon_{ijk} = 0$ for any two or more indices set equal to each other. Then Eq. (1.1.11) can be written as

$$\left[\hat{J}_i, \hat{J}_j\right] = \epsilon_{ijk}\, \hat{J}_k. \tag{1.1.13}$$

Eq. (1.1.13) specifies the **Lie algebra** associated to the group of rotations in three spatial dimensions. The group is denoted SO(3) (special [3] orthogonal in 3D), and the Lie algebra by so(3). A continuous group generated by a nontrivial Lie algebra (i.e., a Lie algebra with nontrivial commutation relations) is said to be **non-abelian**. The key data is encoded in the **structure constants** or non-vanishing commutation relations. For so(3), these are the components of the Levi-Civita tensor ϵ_{ijk}.

The Lie algebra encodes most aspects of what we want to know about a continuous group. A generic element of the group can be expressed as the exponentiation of a linear combination of generators, with coefficients specified by the group manifold coordinates. E.g., a generic SO(3) rotation can be written as

$$\hat{R}(\boldsymbol{\theta}) \equiv \exp\left(\theta_i\, \hat{J}_i\right), \tag{1.1.14}$$

where we assume Einstein index notation (repeated indices are summed over appropriate values—here $i \in \{1, 2, 3\}$). We can compose two arbitrary rotations using only the commutation relations, via the Baker-Campbell-Hausdorff matrix identity

$$\exp(\hat{A})\, \exp(\hat{B}) = \exp\left\{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{12}\left(\left[\hat{A}, [\hat{A}, \hat{B}]\right] + \left[\hat{B}, [\hat{B}, \hat{A}]\right]\right) + \ldots\right\} \equiv \exp(\hat{C}). \tag{1.1.15}$$

An explicit solution for $\hat{C}$ in terms of $\hat{A}$ and $\hat{B}$ is equivalent to a global group map that takes any two points on the group manifold, and returns a third. Because the result depends only on the commutator algebra, this map is a property of the group itself, and not the particular representation in which the generators $\{\hat{J}_i\}$ have been defined. In practice, the composition of generic group transformations is complicated, since it requires understanding both local (geometrical) and global (topological) aspects of the group manifold; explicit formulae exist only for simple cases.

In physics we typically don't need to understand group composition in detail. Instead, we usually want to classify and understand different representations of the Lie algebra. The representation in which we have defined the 3×3 matrix generators $\{\hat{J}_i\}$ in Eq. (1.1.5) is called the **vector** or **defining** representation. It is interesting to note that the numerical elements of the three generators are in one-to-one correspondence with those of the Levi-Civita symbol itself, i.e.

$$[J_j]_{ik} = \epsilon_{ijk}. \tag{1.1.16}$$

Thus the structure constants themselves also form a representation, known as the **adjoint representation**. It is the representation to which the elements of the Lie algebra themselves belong, and plays a crucial role in subsequent developments. The algebra so(3) is special because the defining and adjoint representations are the same. This is not true for su(2) (reviewed below) or most other Lie algebras.

In quantum theory, one often wants to classify different objects by symmetry. These can be matrix or differential operators acting on a Hilbert space in one-body quantum mechanics, or composite field operators in a many-particle quantum field theory. The key question then arises: if we have an "elementary" object ϕ_i that transforms in a certain way under a continuous group operation, how can we understand the transformation properties of a "composite" object such as a tensor of m indices, built from ϕ_i:

$$T_{i_1 i_2 \ldots i_m} \equiv \phi_{i_1}\, \phi_{i_2}\, \times \cdots \times \phi_{i_m}.$$

(In this case the tensor T is completely symmetric). Instead of determining the transformation properties of T directly from ϕ, it is better to take a detour to study the classification of so-called **irreducible representations** of the Lie algebra itself. The idea is that the commutation algebra Eq. (1.1.13) and the group composition hold abstractly, independent of representation. Any "composite" object like T should decompose into pieces that transform under different representations of the Lie algebra. These different representations are characterized by certain data such as the angular momentum quantum numbers (reviewed below), and these numbers determine the physical properties of the objects of interest in quantum theory.

1.2 Lie algebra: formal definition

A Lie algebra is technically a vector space L: a collection of objects with notions of addition, scalar multiplication, additive inverse and identity, and completeness. We do *not* impose an inner product structure (yet). In addition, a Lie algebra is equipped with a

bilinear operation $[\,,\,]$ ($L \times L \to L$), called a "Lie product," "Lie bracket," or "commutator," that satisfies the conditions

$$[x,y] = -[y,x] = z, \qquad x,y,z \in L \qquad \text{definition, antisymmetry.} \tag{1.2.1a}$$

$$[ax,y] = a[x,y], \qquad x,y \in L, \quad a \in \{\mathbb{R},\mathbb{C},\ldots\} \qquad \text{scalar multiplication.} \tag{1.2.1b}$$

$$[x_1+x_2,y] = [x_1,y]+[x_2,y], \qquad x_1,x_2,y \in L \qquad \text{linearity.} \tag{1.2.1c}$$

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0, \qquad x,y,z \in L \qquad \text{Jacobi identity.} \tag{1.2.1d}$$

The second and third conditions follow from the idea that if two elements x and y belong to L (a linear vector space), then so does $ax+by$, where a,b are scalar coefficients. The fourth (Jacobi) identity is very important. It holds identically for any representation of Lie algebra elements as $n \times n$ matrices, but we are insisting that this condition generalizes to the abstract definition of the algebra (wherein we are not required to view the Lie product as a matrix commutator).

1.3 su(2) $\simeq$ so(3); irreducible representations

As should be familiar from the theory of spin angular momentum in quantum mechanics, the commutation algebra for the generators of SU(2) transformations on 2-component complex vectors (or "spinors") satisfy the same Lie algebra as so(3). One says that su(2) = so(3), although the corresponding groups SO(3) and SU(2) are not globally equivalent.

1.3.1 Spin 1/2

For spin 1/2, one defines the Hermitian generators[4]

$$\hat{T}_i \equiv \frac{1}{2}\hat{\sigma}_i, \quad \left[\hat{T}_i,\hat{T}_j\right] = i\epsilon_{ijk}\,\hat{T}_k, \tag{1.3.1}$$

where the Pauli matrices are defined in the standard basis,

$$\hat{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.3.2}$$

The spin 1/2 representation is especially tractable because manipulations with Pauli matrices are facilitated by their *anticommutation* relations,

$$\{\hat{\sigma}_i,\hat{\sigma}_j\} = 2\hat{1}_2\,\delta_{i,j}, \tag{1.3.3}$$

where $\{\hat{A},\hat{B}\} \equiv \hat{A}\,\hat{B}+\hat{B}\,\hat{A}$ is the matrix anticommutator. Technically, we can view Eq. (1.3.3) as a **Clifford algebra**. Such an algebra of operators always arises in the **spinor representations** of the orthogonal group so(n) (although the latter is not equivalent to any other Lie algebra for $n > 6$).

For the Lie algebra su(2), spin 1/2 is the **defining** representation. By contrast, the su(2) adjoint representation is the defining or 3-vector representation of so(3). Since it is the representation in which the generators themselves transform, the dimensionality of the adjoint is always equal to d, the dimension of the group.

We summarize some properties of the spin 1/2 representation, most of which should already be familiar:

1. A generic spinor state $|\hat{n}\rangle \equiv \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle$ always "points somewhere" on the Bloch sphere:

$$\hat{n}\cdot\hat{\boldsymbol{\sigma}}\,|\hat{n}\rangle = |\hat{n}\rangle\,,$$

where $\hat{n} = \hat{n}(\alpha,\beta)$ is a real unit vector and $\hat{\boldsymbol{\sigma}} \equiv \hat{\sigma}_i\,\hat{i}$ ($i \in \{x,y,z\}$) is the vector of Pauli matrices. The ket $|\hat{n}\rangle$ is a combination of the spin up and down base kets with complex coefficients; in the standard basis [Eq. (1.3.2)]

$$|\uparrow\rangle \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\downarrow\rangle \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- **Exercise**: Determine the real unit vector $\hat{n}$ as a function of the complex coefficients $\{\alpha,\beta\}$.

[4] Note that the so(3) generators defined via Eq. (1.1.5) were taken to be real and thus anti-Hermitian, hence the extra factor of i in Eq. (1.3.1) relative to Eq. (1.1.13).

2. A finite SU(2) transformation can be written as

$$e^{-i\frac{1}{2}\boldsymbol{\theta}\cdot\hat{\boldsymbol{\sigma}}} = \hat{1}_2 \cos\left(\frac{\theta}{2}\right) - i\,\hat{\theta}\cdot\hat{\boldsymbol{\sigma}} \sin\left(\frac{\theta}{2}\right), \tag{1.3.4}$$

where $\theta = |\boldsymbol{\theta}|$ is the norm of $\boldsymbol{\theta} = \theta_1\,\hat{1} + \theta_2\,\hat{2} + \theta_3\,\hat{3}$.

3. Some useful Pauli matrix identities

$$\hat{\sigma}_i\,\hat{\sigma}_j = \delta_{i,j}\hat{1}_2 + i\,\epsilon_{ijk}\,\hat{\sigma}_k, \tag{1.3.5a}$$

$$[\hat{\sigma}_i]_{a,b}\,[\hat{\sigma}_i]_{c,d} + \delta_{a,b}\,\delta_{c,d} = 2\delta_{a,d}\,\delta_{c,b}, \qquad \text{SU(2) Fierz identity} \tag{1.3.5b}$$

$$[\hat{\sigma}_i]_{a,b}\,[\hat{\sigma}_i]_{c,d} = \delta_{a,d}\,\delta_{c,b} + [\hat{\sigma}_2]_{a,c}\,[\hat{\sigma}_2]_{b,d}. \qquad \text{Sp(2) Fierz identity} \tag{1.3.5c}$$

In the last two equations, one sums over the three generators on the left-hand side. We will prove Fierz identities for generic Lie algebras later. Here Sp(2n) denotes the **symplectic** Lie group, another "classical" family that we will eventually discuss. The three simplest Lie algebras are all equivalent, so(3) = su(2) = sp(2).

1.3.2 Generic j

We assume the existence of a generic representation in which the su(2) generators $\hat{T}_i$ are $n \times n$ Hermitian matrices. We rewrite Eq. (1.3.1) in terms of raising and lowering operators:

$$\hat{T}_\pm \equiv \hat{T}_x \pm i\hat{T}_y, \qquad \left[\hat{T}_z, \hat{T}_\pm\right] = \pm\hat{T}_\pm, \quad \left[\hat{T}_+, \hat{T}_-\right] = 2\hat{T}_z, \tag{1.3.6}$$

We assume the existence of a **highest weight state** $|j\rangle$ that satisfies

$$\hat{T}_z\,|j\rangle = j\,|j\rangle\,, \quad \hat{T}_+\,|j\rangle = 0, \quad \text{highest weight state.} \tag{1.3.7}$$

Then we consider

$$\hat{T}_z\,\hat{T}_-\,|j\rangle = \left(\hat{T}_-\,\hat{T}_z - \hat{T}_-\right)|j\rangle = (j-1)\,\hat{T}_-\,|j\rangle\,, \tag{1.3.8}$$

so that $\hat{T}_-\,|j\rangle$ is an eigenstate of $\hat{T}_z$ with eigenvalue $(j-1)$. We define

$$|k-1\rangle \equiv \hat{T}_-\,|k\rangle\,, \tag{1.3.9}$$

where $\langle k|k\rangle$ is not generally normalized to one. Then, in order to obtain a <u>finite</u> representation, we must have for some q

$$\hat{T}_-\,|q\rangle = 0, \quad \text{lowest weight state.} \tag{1.3.10}$$

What is q?

$$\hat{T}_z\,\hat{T}_+\,|k\rangle = \left(\hat{T}_+\,\hat{T}_z + \hat{T}_+\right)|k\rangle = (k+1)\hat{T}_+\,|k\rangle\,.$$

Define

$$\hat{T}_+\,|k\rangle \equiv r_k\,|k+1\rangle \tag{1.3.11}$$

so that

$$\hat{T}_+\,\hat{T}_-\,|k+1\rangle = \left(\hat{T}_-\,\hat{T}_+ + 2\hat{T}_z\right)|k+1\rangle = [r_{k+1} + 2(k+1)]\,|k+1\rangle \tag{1.3.12}$$

implying the recursion relation

$$r_{k-1} = r_k + 2k. \tag{1.3.13}$$

Eq. (1.3.7) implies that $r_j = 0$ (highest weight state assumption), so that

$$\begin{aligned}
r_{j-1} &= 2j \\
r_{j-2} &= 2j + 2(j-1) \\
r_{j-3} &= 2j + 2(j-1) + 2(j-2) \\
&\vdots \\
r_{j-m} &= 2jm - m(m-1).
\end{aligned}$$

or

$$\Rightarrow r_k = j(j+1) - k(k+1). \tag{1.3.14}$$

Returning to Eq. (1.3.10),

$$\begin{aligned} \hat{T}_+ \hat{T}_- |q\rangle &= \left(\hat{T}_- \hat{T}_+ + 2\hat{T}_z\right) |q\rangle \\ &= (r_q + 2q) |q\rangle = 0, \end{aligned} \tag{1.3.15}$$

so that

$$j(j+1) - q(q+1) + 2q = 0. \tag{1.3.16}$$

This has two possible solutions $q = \{j+1, -j\}$. By assumption $q = j - n$ with n some positive integer. Thus $2j \in \mathbb{Z}_+$ (the set of positive integers), and we recover the usual integer and half-integer representations of su(2): $j = 1/2$ (defining), $j = 1$ (adjoint), $j = 3/2$, etc.

1.3.3 Quadratic Casimir

Recall that the squared norm of the vector of angular momentum generators takes a constant value for an irreducible representation of su(2). This is the first example of a **quadratic Casimir operator** that can be used to distinguish representations. The Casimir is

$$\hat{C} \equiv \hat{T}_x^2 + \hat{T}_y^2 + \hat{T}_z^2 = \hat{T}_z^2 + \frac{1}{2}\left(\hat{T}_+ \hat{T}_- + \hat{T}_- \hat{T}_+\right). \tag{1.3.17}$$

Unlike the Lie algebra that exists for abstract generators using the Lie bracket, the Casimir operator can only be defined for a representation: we need to define the ordinary (matrix) product between two generators to write $\hat{C}$ down. Its action on a generic state in the representation defined above is [Using Eq. (1.3.14)]

$$\hat{C} |k\rangle = \left[k^2 + \frac{1}{2}(r_{k-1} + r_k)\right] |k\rangle = j(j+1) |k\rangle. \tag{1.3.18}$$

All states in the representation can be labeled by the eigenvalue $j(j+1)$, since the Casimir commutes with all generators:

$$\left[\hat{C}, \hat{T}_j\right] = \hat{T}_i \left[\hat{T}_i, \hat{T}_j\right] + \left[\hat{T}_i, \hat{T}_j\right] \hat{T}_i = i\epsilon_{ijk}\left(\hat{T}_i \hat{T}_k + \hat{T}_k \hat{T}_i\right) = 0. \tag{1.3.19}$$

1.4 Tensor representations of so(3)

In this section we briefly review integer j representations of su(2) = so(3) in terms of objects that frequently appear in physical calculations.

1.4.1 Traceless symmetric tensors

In the defining $j = 1$ representation of so(3), matrices act on 3-component vectors. We can obtain other representations by tensoring together multiple vector indices. Consider a product of two different vectors V_i and U_j,

$$T_{ij} \equiv U_i V_j.$$

This is a **rank-2** tensor with 9 independent components, and does not transform irreducibly under SO(3) rotations (defined to act identically on both i and j). We can easily decompose T_{ij} into irreducible representations using contractions and (anti)symmetrization:

1. **Trivial** or **scalar** representation, formed from the dot product of two vectors:

$$\delta^{ij} T_{ij} = T_{ii}, \qquad \text{scalar, } j = 0\text{, 1 component.} \tag{1.4.1a}$$

   (Einstein summation over i).

2. Vector representation, formed using the Levi-Civita tensor

$$\epsilon^{ijk} T_{jk} = (\mathbf{U} \times \mathbf{V})^i, \qquad \text{vector, } j = 1\text{, 3 components.} \tag{1.4.1b}$$

3. Finally there is a traceless symmetric tensor representation,

$$\tilde{T}_{(ij)} \equiv \frac{1}{2}\left(T_{ij}+T_{ji}\right)-\frac{1}{3}\delta_{ij}T_{kk}, \qquad \text{rank-2 tensor, } j=2,\ 5 \text{ components.} \tag{1.4.1c}$$

Thus the unsymmetrized tensor decomposes into a direct sum of scalar, vector, and symmetrized, traceless tensor components.[5]

In Eq. (1.4.1c), we have placed the indices in parentheses to indicate symmetrization. More generally, it is useful define complete symmetrization or antisymmetrization operations as follows. Given an arbitrary rank-n tensor $T_{i_1 i_2 \cdots i_n}$,

$$\begin{aligned} T_{(i_1 i_2 \ldots i_n)} &\equiv \frac{1}{n!}\left[T_{i_1 i_2 \cdots i_n}+(\text{permutations})\right], && \text{complete symmetrization,} \\ T_{[i_1 i_2 \ldots i_n]} &\equiv \frac{1}{n!}\left[T_{i_1 i_2 \cdots i_n}+(\text{alternating permutations})\right], && \text{complete antisymmetrization.} \end{aligned} \tag{1.4.2}$$

The representation in Eq. (1.4.1b) [(1.4.1c)] involves $T_{[jk]}$ [$T_{(jk)}$]. The antisymmetric $T_{[jk]}$ is converted into a vector using the Levi-Civita tensor, which is invariant under SO(3) transformations. This is obvious because ϵ^{ijk} has only one independent component, and must therefore transform in the scalar $j=0$ representation. In Eq. (1.4.1c), the irreducible representation differs from $T_{(jk)}$ by the removal of the scalar trace; this is indicated by the tilde.

We see in the above that the delta function δ^{ij} and the Levi-Civita tensor ϵ^{ijk} play special roles in constructing irreducible representations of so(3). Both are tensors are themselves group invariants, and both can be used to "tie up indices" and create lower rank representations from unsymmetrized higher rank tensors. In generic Lie algebras the Levi-Civita tensor generalizes to a higher rank completely antisymmetric tensor, and this is always a scalar under group transformations. By contrast, the Kronecker delta is special to the orthogonal algebra so(n), and acts as a "metric" for tracing over pairs of n-component vector indices.[6]

All other integer representations of su(2) can be realized in terms of completely symmetrized, traceless tensors. The j representation is associated to a rank-j tensor,

$$\tilde{T}_{(i_1 i_2 \cdots i_j)}; \qquad \tilde{T}_{(k k i_3 \cdots i_j)}=0. \tag{1.4.3}$$

A completely symmetric tensor with j indices that range over m values has as many independent components as there are distinguishable states that place j bosons into m different states. This is given by

$$\frac{(j+m-1)!}{j!(m-1)!}.$$

Here $m=3$; moreover, we must reduce the number of independent components by the number of constraints imposed by tracelessness. Therefore $\tilde{T}_{(i_1 i_2 \cdots i_j)}$ has

$$\frac{(j+2)!}{j!2!}-\frac{(j)!}{(j-2)!2!}=2j+1$$

components, as expected.

1.4.2 Spherical tensors

Finally, we recall that one can switch from the symmetrized, traceless Cartesian tensors in Eq. (1.4.3) to spherical tensors $\{\mathcal{O}_{jm}\}$. These are labeled by the representation j and the $\hat{T}_z$ eigenvalue $m \in \{-j,-j+1,\ldots,j-1,j\}$ ("magnetic quantum number"). The components of these are linear combinations of $\tilde{T}_{(i_1 i_2 \cdots i_j)}$ components. An example of spherical tensors are the spherical harmonics $Y_{l,m}$, which can be expressed using Legendre polynomials and the components of a unit vector. I.e., we can write

$$\hat{n} \equiv \{\Pi_x, \Pi_y, \sigma\}, \quad \Pi_x^2+\Pi_y^2+\sigma^2=1.$$

[5] Note that we are not making a mathematical distinction between "upper" and "lower" indices in the above equations. The integer representations of so(3) are "purely real," meaning there is no notion of a conjugate representation that appears e.g. in su($n \geq 3$). In the latter case, we will use lower (upper) indices to represent the defining (conjugate) representations. Here we simply try to mimic this for aesthetics.

[6] For su(n), there is in general no such rank-2 metric that can be used to trace together pairs of defining representation indices. Instead, one must contract one defining and one conjugate index. The latter turns out to be equivalent to contracting with the rank-n completely antisymmetric tensor $\epsilon^{i_1 i_2 \cdots i_n}$. This difference between su(n) and so(n) complicates large-n treatments of quantum magnets, since the notions of a "singlet" are different. See the discussion in the next Sec. 1.5.

Defining $\Pi_\pm \equiv \Pi_x \pm i\Pi_y$, one has

$$
\begin{aligned}
Y_{0,0} &= \frac{1}{\sqrt{4\pi}}, & Y_{2,2} &= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{15}{8}}(\Pi_+)^2, \\
& & Y_{2,1} &= -\frac{1}{\sqrt{4\pi}}\sqrt{\frac{15}{2}}\Pi_+\,\sigma, \\
Y_{1,1} &= -\frac{1}{\sqrt{4\pi}}\sqrt{\frac{3}{2}}\Pi_+, & Y_{2,0} &= \frac{1}{\sqrt{4\pi}}\sqrt{5}\left(\frac{3}{2}\sigma^2 - \frac{1}{2}\right), \\
Y_{1,0} &= \frac{1}{\sqrt{4\pi}}\sqrt{3}\sigma, & Y_{2,-1} &= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{15}{2}}\Pi_-\,\sigma, \\
Y_{1,-1} &= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{3}{2}}\Pi_-, & Y_{2,-2} &= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{15}{8}}(\Pi_-)^2,
\end{aligned}
\tag{1.4.4}
$$

etc. Spherical harmonic functions obtain by writing $\Pi_\pm = \sin(\theta)\exp(\pm i\phi)$ and $\sigma = \cos(\theta)$, where θ (ϕ) is the polar (azimuthal) angle, respectively.

1.5 Tensor representations of su(2)

In the previous section, we considered tensors formed by taking a direct product of vector ($j = 1$) representations. A generic rank-n tensor $T_{i_1 i_2 \cdots i_n}$ ($i_j \in \{x, y, z\}$) can be decomposed into irreducible components with $0 \le j \le n$; the $j = n$ case is non-degenerate and given by the completely symmetrized, traceless version $\tilde{T}_{(i_1 i_2 \cdots i_n)}$ [Eq. (1.4.3)] (assuming the latter doesn't vanish for the particular T we start with).

We cannot obtain half-integer representations by tensoring together vector indices. Instead, in this section we review tensoring together spin-1/2 (spinor) indices. As should be well-known from the theory of angular momentum addition, one can generate any representation of su(2), integer or half-integer, by tensoring together spinor induces. The fact that any representation can be built from spinor indices generalizes to so(n), as we will later demonstrate.

Imagine spinor wavefunctions ψ_a and ϕ_b, where $a, b \in \{\uparrow, \downarrow\}$ transform in the defining spin-1/2 representation of su(2). We can form a second-rank "spinor tensor" (direct product) $T_{ab} \equiv \psi_a \phi_b$. Similar to the second-rank so(3) tensor in Eq. (1.4.1), this can be decomposed into irreducible representations as follows:

1. Scalar representation, formed using the antisymmetric "metric" (rank-2 antisymmetric tensor) $\epsilon^{ab} = -\epsilon^{ba}$ ($\epsilon^{12} = +1$):

$$\epsilon^{ab}\, T_{ab} = \epsilon^{ab}\, T_{[ab]} = T_{\uparrow\downarrow} - T_{\downarrow\uparrow}, \qquad \text{scalar, } j = 0\text{, 1 component ("spin singlet").} \tag{1.5.1a}$$

   Note that $\epsilon^{ab} = i\,[\hat{\sigma}_2]_{a,b}$.

2. Vector representation

$$T_{(ab)} = \begin{cases} T_{\uparrow\uparrow}, & m = 1 \\ \frac{1}{2}\left(T_{\uparrow\downarrow} + T_{\downarrow\uparrow}\right), & m = 0 \\ T_{\downarrow\downarrow}, & m = -1 \end{cases}, \qquad \text{vector, } j = 1\text{, 3 components ("spin triplet").} \tag{1.5.1b}$$

   Here m is the $\hat{T}_z$ eigenvalue for the product representation. Of course the spin-1 components listed here (corresponding to kets $\{|\uparrow\uparrow\rangle, \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), |\downarrow\downarrow\rangle\}$ are spherical tensor components of the vector; the Cartesian x, y components are linear combinations of the top and bottom weights.

There is no analogue of the invariant Kronecker delta function for spinor indices [c.f. Eq. (1.4.1a)]. The only invariant tensor that we can use to "tie up" spinor indices is ϵ^{ab}.[7] Thus for a rank-$2n$ spinor tensor $T_{a_1 a_2 \cdots a_{2n-1} a_{2n}}$, we can form a scalar (e.g.) via

$$\epsilon^{a_1 a_2}\epsilon^{a_3 a_4} \times \cdots \times \epsilon^{a_{2n-1} a_{2n}} T_{a_1 a_2 \cdots a_{2n-1} a_{2n}}.$$

In physics language, we are pairing up spin-1/2 indices to form a collection of singlets; for $2n$ spin-(1/2)s, there are many (how many?) ways to do this.

[7] We are ignoring the conjugate representation $\phi^\dagger \to \phi^{*\,a}$, which can be used to form a singlet with ψ_b: $\phi^{*\,a}\psi_a$ (Einstein sum). The main point here is that for su(2) = sp(2), the conjugate and defining representations are in fact equivalent, since we can "raise the index" on ψ_b using ϵ^{ab}. We will define conjugate representations for su(n) in due time.

We can form an irreducible representation for arbitrary j by considering a symmetric rank-n tensor of spinor indices:

$$T_{(a_1 a_2 \cdots a_n)}, \quad j = n/2 \text{ representation.} \tag{1.5.2}$$

One way to see that this is irreducible is that the "contraction" of any two indices with ϵ^{ab} is automatically zero. We can also count the number of components, which is

$$\frac{(n+1)!}{n!1!} = n+1,$$

consistent with $n = 2j$.

We will see that the above generalizes to su(n) and sp($2n$), but in different ways. For su(n), the analogue of ϵ^{ab} is the rank-n completely antisymmetric tensor,

$$\epsilon^{a_1 a_2 \cdots a_n}, \quad a_i \in \{1, 2, \ldots, n\}. \tag{1.5.3}$$

This means that one needs n su(n) "spins" (defining representation indices) to form a singlet. Such a (rank-$2n$) invariant tensor also exists for sp($2n$). In addition, however, sp($2n$) possesses a generalized $2n \times 2n$ rank-2 antisymmetric tensor

$$\epsilon^{a_1 a_2} \quad \Rightarrow \quad \begin{bmatrix} \hat{0} & \hat{1}_n \\ -\hat{1}_n & \hat{0} \end{bmatrix}, \quad a_i \in \{1, 2, \ldots, 2n-1, 2n\}, \tag{1.5.4}$$

and this can be used to tie up two spinor indices to form a singlet (for any n).

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

[2] Herbert Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Massachusetts, 1980).

2. Lie groups as manifolds. SU(2) and the three-sphere.
* version 1.5 *

Matthew Foster

January 20, 2021

Contents

2.1 The Haar measure **1**

2.2 The group manifold for SU(2): S_3 **3**

2.3 Left- and right- group translations on SU(2): Isometries of S_3 **4**

This is the only module in which I will discuss Lie groups and their geometry; subsequent modules will not make use of the results derived here. We will employ some basic notions from differential geometry; an excellent reference is [1]. Introductions to the Haar measure can be found in Refs. [2] and [3].

2.1 The Haar measure

For a Lie group G, each element $g(x)$ is associated to a point on the group manifold, labeled by some coordinates $x = \{x^a\}$, $a \in \{1, 2, \ldots, d\}$. Here d denotes the group dimension. In this module, we follow differential geometry conventions for upper ("contravariant") and lower ("covariant" or dual) indices on tensors. The coordinate x^a is a scalar field defined over the manifold, but its differential variation dx^a is a contravariant vector (or equivalently, a dual basis vector) [1].

A **compact** Lie group has a group manifold is that is closed (has no boundary). The closure of the group under group multiplication implies the existence of a **group map** ϕ: $G \times G \to G$, which takes an arbitrary ordered pair of points (y, x) to a third point z. I.e., if

$$g(z) = g(y) \times g(x) \neq g(x) \times g(y), \tag{2.1.1}$$

then

$$z^a \equiv \phi^a(y; x) \neq \phi^a(x; y), \quad \text{group map.} \tag{2.1.2}$$

The group map $z^a = \phi^a(y; x)$ is an additional global structure defined over the entire manifold. We will see that it implies the existence of a unique integration measure and associated Riemannian metric for the group, tying the abstract structure of the group to a specific manifold geometry.

First, it is helpful to partially "gauge fix" (constrain) the choice of manifold coordinates.

- Let $g(x = 0) = 1$, where "1" denotes the group identity element.
- Let $g(\tilde{x}) \times g(x) = g(x) \times g(\tilde{x}) = 1 \Rightarrow g(\tilde{x}) = g^{-1}(x)$. Thus $\tilde{x}$ locates the inverse of $g(x)$.

Then

$$x^a = \phi^a(x^a; 0) = \phi^a(0; x^a), \qquad \text{multiplication by the group identity.} \tag{2.1.3a}$$

$$(\alpha + \beta)\, x^a = \phi^a(\alpha x; \beta x) = \phi^a(\beta x; \alpha x), \quad \alpha, \beta \in \mathbb{R}, \qquad \text{U(1) composition.} \tag{2.1.3b}$$

$$0 = \phi^a(x, \tilde{x}) = \phi^a(\tilde{x}, x), \qquad \text{group inverse.} \tag{2.1.3c}$$

$$\phi^a(\delta y; \delta x) \simeq \phi^a(\delta x; \delta y) \simeq \delta x^a + \delta y^a + \boldsymbol{O}\left(\delta^2\right), \qquad \text{composition of infinitesimal elements.} \tag{2.1.3d}$$

Figure 2.1: Lie group G as a manifold; group map.

In the last equation, we assume that δx and δy are "close" to the group identity 1.

In physical applications like random matrix theory or matrix field theories, one must define an appropriate measure for performing integration over matrix components. In the special case that the matrix represents a group transformation, one may wish to integrate over all possible group transformations. Taking the circle group as an example, we anticipate that a "natural" integration measure is invariant under group transformations. For non-abelian groups, this is called the Haar measure. Let us denote it via

$$d_{\mathsf{H}}[g(x)] \equiv \sqrt{\gamma(x)}\, d^d x, \qquad \gamma(x) = \det\left[\gamma_{ab}(x)\right], \tag{2.1.4}$$

where $\gamma_{ab}(x)$ is the associated **group metric**. We insist that the Haar measure is invariant under **left-** and **right-** group translations:

$$I(y) \equiv \int_G d_{\mathsf{H}}[g(x)]\, f\left[g(y) \times g(x)\right] = \int_G d_{\mathsf{H}}[g(x)]\, f\left[g(x) \times g(y)\right] = \int_G d_{\mathsf{H}}[g(x)]\, f\left[g(x)\right], \tag{2.1.5}$$

where $f(g)$ denotes some arbitrary function of the group element g. Therefore we require that

$$\begin{gathered} I(y) = \int_G d_{\mathsf{H}}[g(x)]\, f\left[g(y) \times g(x)\right] = \int_G d_{\mathsf{H}}[g(\tilde{y}) \times g(z)]\, f\left[g(z)\right] = \int_G d_{\mathsf{H}}[g(z)]\, f\left[g(z)\right], \\ g(z) = g(y) \times g(x), \quad z^a = \phi^a(y; x), \\ g(x) = g(\tilde{y}) \times g(z), \quad x^a = \phi^a(\tilde{y}, z). \end{gathered} \tag{2.1.6}$$

Now changing variables from x to z gives

$$\sqrt{\gamma(x)}\, d^d x = \sqrt{\gamma(x)} \left| \frac{\partial x^a}{\partial z^b} \right| d^d z, \tag{2.1.7}$$

where the $|\cdots|$ is the Jacobian. Comparing this to Eq. (2.1.6), we conclude that[1]

$$\sqrt{\gamma(z)} = \sqrt{\gamma(x)} \left| \frac{\partial \phi^a(\tilde{y}, z)}{\partial z^b} \right|. \tag{2.1.8}$$

We can compute the metric determinant $\gamma(z)$ by setting $x = 0$, so that $g(z) = g(y)$. Furthermore, we know that at the group identity $x = 0$,

$$d_{\mathsf{H}}[g(x \to 0)] = d^d x$$

(up to an arbitrary scale factor), because infinitesimal group transformations commute [c.f. Eq. (2.1.3d)]. We conclude that

$$\boxed{\sqrt{\gamma(z)} = \left| \frac{\partial \phi^a(\tilde{y}, z)}{\partial z^b} \right|_{y=z}.} \tag{2.1.9}$$

[1] Note that Eq. (2.1.8) is *not* merely the transformation law for tensor densities under coordinate diffeomorphisms. The invariant integration measure in that case is

$$\sqrt{\gamma(x)}\, d^d x = \sqrt{h(z)}\, d^d z,$$

where $h = |\det h_{ab}|$ and $h_{ab} = \gamma_{cd}(\partial x^c/\partial z^a)(\partial x^d/\partial z^b)$. The equivalent statement is

$$\sqrt{h(z)} = \sqrt{\gamma(x)} \left| \frac{\partial x^a}{\partial z^b} \right|.$$

For a generic coordinate diffeomorphism, the density $h(z)$ is a *different function* of its arguments from $\gamma(x)$. Invariance of the group integration measure instead constrains $h(z) = \gamma(z)$ on the left-hand side of Eq. (2.1.8).

2.2 The group manifold for SU(2): S_3

We can use the spin 1/2 representation to derive an explicit formula for the group map. Recall the explicit form an SU(2) transformation [Eq. (1.3.4)],

$$\hat{g}(\boldsymbol{\theta}) = \hat{1}_2 \cos\left(\frac{\theta}{2}\right) - i\hat{\theta}\cdot\hat{\boldsymbol{\sigma}} \sin\left(\frac{\theta}{2}\right). \tag{2.2.1}$$

We introduce $\mathbb{R}^4$ (embedding) coordinates

$$x^0 \equiv \cos\left(\frac{\theta}{2}\right), \quad x^i \equiv \sin\left(\frac{\theta}{2}\right)\hat{\theta}^i \quad \Rightarrow \quad \left(x^0\right)^2 = 1 - x^i x^i, \tag{2.2.2}$$

(Einstein summation implied on $i \in \{1,2,3\}$). If we assign a Euclidean metric to the 4D $\{x^0, x^i\}$ space, then the group of SU(2) transformations clearly resides on the unit 3-sphere, denoted S_3. We will see that the Haar measure confirms this.

The group element now reads

$$\hat{g}(x) = \hat{1}_2\, x^0 - i x^i \hat{\sigma}_i. \tag{2.2.3}$$

The group map obtains from

$$\hat{g}(y)\,\hat{g}(x) = \hat{1}_2\left(x^0 y^0 - x^i y^i\right) - i\hat{\sigma}_i\left(y^0 x^i + x^0 y^i + \epsilon^{ijk} y^j x^k\right) \equiv g(z), \tag{2.2.4}$$

so that

$$z^0 = \left(x^0 y^0 - x^i y^i\right), \quad z^i = \left(y^0 x^i + x^0 y^i + \epsilon^{ijk} y^j x^k\right). \tag{2.2.5}$$

• **Exercise**: Check that $(z^0)^2 = 1 - z^i z^i$.

Therefore

1. The group map is

$$\boxed{\phi^i(y;x) = y^0 x^i + x^0 y^i + \epsilon^{ijk} y^j x^k.} \tag{2.2.6}$$

2. In the chosen coordinate system, a group element $g(x)$ is related to its inverse $g(\tilde{x})$ via $\tilde{x}^i = -x^i$.

Therefore

$$\begin{aligned}
\phi^i(-y;z) &= \sqrt{1-y^k y^k}\, z^i - \sqrt{1-z^k z^k}\, y^i - \epsilon^{ijk} y^j z^k, \quad (\text{for } z^0 > 0),\\
\frac{\partial \phi^i(-y;z)}{\partial z^j} &= \sqrt{1-y^k y^k}\,\delta^i_j + \frac{y^i\, z_j}{\sqrt{1-z^k z^k}} - \epsilon^{ikj} y^k,\\
\Rightarrow \left.\frac{\partial \phi^i(-y;z)}{\partial z^j}\right|_{y=z} &= \sqrt{1-z^k z^k}\,\delta^i_j + \frac{z^i\, z_j}{\sqrt{1-z^k z^k}} + \epsilon^{ijk} z^k.
\end{aligned} \tag{2.2.7}$$

The determinant of this finally gives [via Eq. (2.1.9)]

$$\boxed{\sqrt{\gamma(z)} = \frac{1}{z^0} = \frac{1}{\sqrt{1-z^i z^i}}.} \tag{2.2.8}$$

This is the integration measure for a 3-sphere in terms of constrained Euclidean coordinates, as can be seen from the manifestly invariant integral

$$\int d^4z\; \delta(z^0 z^0 + z^i z^i - 1) = \int \frac{d^3 z}{\sqrt{1-z^i z^i}}.$$

The group metric can be obtained via the pullback,

$$\gamma_{ij}\, dz^i\, dz^j = \left[\frac{\partial z^0}{\partial z^i}\frac{\partial z^0}{\partial z^j} + \delta_{i,j}\right] dz^i\, dz^j = \left[\delta_{i,j} + \frac{z^i\, z^j}{1-z^k z^k}\right] dz^i\, dz^j. \tag{2.2.9}$$

• **Exercise**: Use Eq. (2.2.9) to verify Eq. (2.2.8). Evaluate the determinant using the formula

$$\det\hat{\gamma} = e^{\mathsf{Tr}\log\hat{\gamma}},$$

writing $\hat{\gamma} = \hat{1}_3 + \delta\hat{\gamma}$, and expanding the log in powers of $\delta\hat{\gamma}$.

2.3 Left- and right- group translations on SU(2): Isometries of S_3

We have shown that the group manifold for SU(2) is S_3. In this section, we demonstrate that continuously generated coordinate transformations that are symmetries of the group manifold ("isometries") encode invariance of the geometry under left- and right-group transformations. To see this, we consider the generators of symmetry transformations on S_3.

Just as S_2 is invariant under SO(3) transformations, S_3 is invariant under SO(4). The generators of SO(4) can be chosen so as to implement SO(2) rotations in each of the orthogonal planes formed by any two coordinate axes in 4D. There are $4 \times 3/2 = 6$ independent planes, and thus 6 generators.[2] We can work out the commutation relations between the generators by thinking about the Hilbert space operators that implement SO(4) rotations in 4D. Recall that in 3D, the z-component angular momentum operator L^z implements rotations in the xy plane and is defined as

$$L^z = -ix\,\partial_y + iy\,\partial_x. \tag{2.3.1}$$

Similarly, we can define the following generators for SO(2) rotations in the 6 orthogonal planes in 4D:

$$\begin{aligned} K^a &= -\,ix^0\partial_a + ix^a\partial_0, \qquad && (x^0\text{-}x^a \text{ plane}),\\ J^a &= -\,i\epsilon^{abc}x^b\partial_c, && (x^b\text{-}x^c \text{ plane}), \end{aligned} \tag{2.3.2}$$

$a, b, c \in \{1, 2, 3\}$. The commutation relations are

$$\begin{array}{c} \underline{\text{so(4) Lie algebra}} \\ [J^a, J^b] = i\epsilon^{abc}J^c, \\ [J^a, K^b] = i\epsilon^{abc}K^c, \\ [K^a, K^b] = i\epsilon^{abc}J^c. \end{array} \tag{2.3.3}$$

Like so(3) = su(2), the so(4) algebra is special, in that it is equivalent to two mutually commuting copies of su(2): so(4) = su(2) $\otimes$ su(2). We will show later that so($2n$+1) [so($2n$)] always has one 2^n dimensional [two 2^{n-1} dimensional] spinor representation(s), in addition to representations formed by tensoring together vector indices. The so(3) and so(4) algebras are sufficiently small that there is essentially nothing else: all representations are be formed by tensoring together spinor su(2) indices for these special algebras. In the case of so(4), the two independent su(2)s can be thought of as "left" and "right" representations, distinguished by chirality. The same decomposition holds for the non-compact version SO(3,1), which governs the Lorentz group of boosts and rotations in 4 spacetime dimensions. The left and right represenations realize left- and right-handed Weyl fermions in 3+1-D relativistic quantum field theory.

We define the linear combinations

$$S^a_\pm \equiv \frac{1}{2}\left(J^a \pm K^a\right). \tag{2.3.4}$$

Then so(4) = su(2) $\otimes$ su(2) follows from

$$[S^a_\sigma, S^b_{\sigma'}] = i\,\delta_{\sigma,\sigma'}\,\epsilon^{abc}S^c_\sigma, \tag{2.3.5}$$

where $\sigma, \sigma' \in \{+, -\}$.

• **Exercise**: Derive Eqs. (2.3.3) and (2.3.5).

We have obtained Eqs. (2.3.3) and (2.3.5) using the representation in Eq. (2.3.2) as differential operators acting in a (flat) four-dimensional Hilbert space. Just as the angular momentum operators in 3D depend only on angular coordinates, these so(4) generators can also be expressed in terms of differential operators acting on the unit 3-sphere, the su(2) group manifold. Technically the generators on S_3 are **Killing vector fields**, and can be derived by either (1) pulling back and raising the 4D dual vectors to the unit sphere, or (2) solving Killing's equation. In a homework problem, you will derive the explicit form of these in some coordinate system on S_3.

Claim: The isometries generated by these commuting su(2)s on the group manifold S_3 correspond to the diffeomorphisms induced by left- and right-multiplying SU(2) group transformations.

Proof: Consider the diffeomorphisms induced by left- and right-su(2) group transformations. Let $y^a = \sin(\theta)\,\hat{y}^a$, $y^0 = \cos(\theta)$.

[2] As we will show in modules 2A and 6, the generators of SO(n) in the defining n-vector representation are in one-to-one correspondence with the set of independent, Hermitian antisymmetric $n \times n$ matrices. There are $n(n-1)/2$ of these.

1. Left group translation: $g(x) \to g(y) \times g(x)$. Then

$$x^a \to \phi^a(y;x) = \cos(\theta)\, x^a + \sin(\theta)\, \hat{y}^a\, x^0 + \sin(\theta)\, \epsilon^{abc} \hat{y}^b x^c. \tag{2.3.6a}$$

2. Right group translation: $g(x) \to g(x) \times g(y)$. Then

$$x^a \to \phi^a(x;y) = \cos(\theta)\, x^a + \sin(\theta)\, \hat{y}^a\, x^0 - \sin(\theta)\, \epsilon^{abc} \hat{y}^b x^c. \tag{2.3.6b}$$

Thus for (e.g.) $\hat{y} = \hat{1}$,

$$\begin{aligned} x^1 &\to \cos(\theta)\, x^1 + \sin(\theta)\, x^0, \\ x^2 &\to \cos(\theta)\, x^2 \mp \sin(\theta)\, x^3, \\ x^3 &\to \cos(\theta)\, x^3 \pm \sin(\theta)\, x^2, \end{aligned} \tag{2.3.7}$$

where the "plus" ("minus") denotes left- (right-)translation.

We compare these to the diffeomorphism induced by the SO(4) rotation

$$x \to \exp\left(\mp i 2\theta S^1_\pm\right)\, x = \exp\left(\mp i\theta J^1\right)\, \exp\left(-i\theta K^1\right) x. \tag{2.3.8}$$

The first transformation $[\exp\left(-i\theta K^1\right) x]$ rotates x in the x^0-x^1 plane, so that

$$\begin{aligned} x^0 &\to \cos(\theta)\, x^0 - \sin(\theta)\, x^1 \\ x^1 &\to \sin(\theta)\, x^0 + \cos(\theta)\, x^1. \end{aligned}$$

The second transformation rotates in the x^2-x^3 plane,

$$\begin{aligned} x^2 &\to \cos(\theta)\, x^2 \mp \sin(\theta)\, x^3, \\ x^3 &\to \pm \sin(\theta)\, x^2 + \cos(\theta)\, x^3. \end{aligned}$$

so that the combination gives exactly Eq. (2.3.7). Q.E.D.

Therefore, we have shown that the so(4) generators that implement symmetry transformations on the three-sphere also implement left and right SU(2) $\otimes$ SU(2) group translations. The generators (Killing vector fields) $S^a_\pm$ thus form two mutually commuting sets of differential operators that can be used to implement group map transformations across the group manifold. **These conclusions generalize to any Lie group G: there must exist two sets of Killing vector fields that satisfy the associated Lie algebra, but mutually intercommute. Group map transformations are implemented by "exponentiating" linear combinations of these Killing fields.**

Finally, we note that the above manipulations have swept some technicalities under the rug. The coordinates that we are using on the group manifold are $x^{1,2,3}$; $x^0 = \pm\sqrt{1 - x^j x^j}$ is constrained. Therefore the transformations in Eq. (2.3.7) are actually *nonlinear* transformations on these coordinates. A more useful view is to think of $x^{1,2,3}$ as scalar fields defined over the manifold. The explicit form of the differential operators $\{K^a, J^b\}$ $(a, b \in \{1, 2, 3\})$ expressed solely in terms of $x^{1,2,3}$ will be derived in homework. Using these, one can check that Eq. (2.3.8) does indeed reproduce Eq. (2.3.7).

References

[1] Sean Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Addison-Wesley 2003); arXiv:gr-qc/9712019.

[2] M. Chaichian and R. Hagedorn, *Symmetries in Quantum Mechanics* (IoP Publishing, Bristol, UK, 1998).

[3] Konstantin Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, England, 1999).

2A. SU(n), SO(n), and Sp($2n$) Lie groups
* version 1.5 *

Matthew Foster

January 20, 2021

Contents

2A.1 Unitary U(n) and SU(n) **1**
2A.1.1 SU(n) tensors 2

2A.2 Orthogonal O(n) and SO(n) **2**
2A.2.1 SO(n) tensors 2

2A.3 Symplectic Sp($2n$) **3**
2A.3.1 Sp($2n$) tensors 3

Here we introduce unitary, orthogonal, and symplectic Lie groups via their defining representations. This is a brief "first pass" to acquaint the reader, not a systematic description of these classical Lie groups or their representations. We will return to this subject after developing roots and weights.

2A.1 Unitary U(n) and SU(n)

Consider an n-component complex vector $|\psi\rangle \to [\alpha_1\, \alpha_2\, \cdots \alpha_n]^{\mathsf{T}}$. Here $\{\alpha_i\} \in \mathbb{C}$ are the complex components expressed in some basis, and T denotes the matrix transpose. A generic unitary U(n) transformation is encoded in the $n \times n$ matrix $\hat{U}$ via

$$|\psi\rangle \to \hat{U}\, |\psi\rangle\,,$$

where

$$\hat{U}^\dagger \hat{U} = \hat{1}_n. \tag{2A.1.1}$$

$\hat{U}$ can be expressed in terms of Hermitian generators $\{\hat{T}_a\}$ via

$$\hat{U} = \exp\left(-i\theta^a\, \hat{T}_a\right), \quad \hat{T}_a^\dagger = \hat{T}_a, \tag{2A.1.2}$$

where $\{\theta^a\} \in \mathbb{R}^n$ are the group coordinates. A special unitary SU(n) transformation has determinant one,[1]

$$\det \hat{U} = 1 \quad \Leftrightarrow \quad \mathsf{Tr}\left[\hat{T}_a\right] = 0. \tag{2A.1.3}$$

In other words, SU(n) has group dimension $d = n^2 - 1$, since it excludes the $n \times n$ identity matrix as a generator. A U(n) transformation can always be written as the product of an abelian U(1) transformation (equivalent to an overall phase) and a non-abelian SU(n) transformation.

[1] Here we have used the formula

$$\det \hat{A} = \exp(\mathsf{Tr} \log \hat{A}).$$

2A.1.1 SU(n) tensors

We will show later that completely antisymmetric and completely symmetric tensors form different classes of irreducible representations:

$$\begin{aligned} T^{[i_1 i_2 \cdots i_k]}: &\quad \text{completely antisymmetric rank-}k\text{ tensor } (k \le n)\text{; irreducible rep. of SU}(n), \\ T^{(i_1 i_2 \cdots i_k)}: &\quad \text{completely symmetric rank-}k\text{ tensor; irreducible rep. of SU}(n). \end{aligned} \tag{2A.1.4}$$

Mixed symmetry tensors can be classified and decomposed via Young tableaux, as we will eventually see.

2A.2 Orthogonal O(n) and SO(n)

Orthogonal O(n) transformations can be defined as a subgroup of the U(n) transformations in Eq. (2A.1.1). In addition to being unitary, we restrict

$$\hat{U}^\dagger = \hat{U}^\mathsf{T}, \quad \text{(real elements)}. \tag{2A.2.1}$$

A special orthogonal transformation SO(n) has determinant one, and excludes reflections, inversion (for n = odd), and products of these with rotations ("improper rotations").

Eq. (2A.2.1) is obviously basis-dependent. We can define orthogonal transformations more generally via

$$\hat{U}^\dagger \hat{U} = \hat{1}_n \tag{2A.2.2a}$$

$$\hat{U}^\mathsf{T} \hat{M} \hat{U} = \hat{M}, \tag{2A.2.2b}$$

$$-\hat{M}\, \hat{T}_a^\mathsf{T}\, \hat{M} = \hat{T}_a, \quad \text{SO}(n)\text{ generators}, \tag{2A.2.2c}$$

where the $n \times n$ matrix $\hat{M}$ satisfies

$$\hat{M} = \hat{M}^\mathsf{T} = \hat{M}^\dagger; \quad \hat{M}^2 = \hat{1}_n. \tag{2A.2.3}$$

Thus, in addition to being unitary, an O(n) transformation preserves the form of a <u>symmetric</u> "matrix metric" $\hat{M}$. The SO(n) generators $\{\hat{T}_a\}$ satisfy the generalized antisymmetry requirement, Eq. (2A.2.2c). The group dimension $d = n(n-1)/2$ for SO(n), since this is the number of independent matrices that can be chosen to satisfy this constraint.

In Eq. (2A.2.1), $\hat{M} = \hat{1}_n$. The "metric contraction" with two n-component vectors A_i and B_j is in this case the conventional dot product,

$$A^\mathsf{T} \hat{M} B = A^i M_{ij} B^j = A^i \delta_{ij} B^j.$$

In analogy with differential geometry, we can think of M_{ij} (here the Kronecker delta) as a covariant metric used to "lower the index" on B^j, so that we can form the inner product $A^i B_i$. If we make an orthogonal group basis transformation using $\hat{U}$ that satisfies Eq. (2A.2.1), then obviously the form of $\hat{M} = \hat{1}_n$ is left invariant. This is not the case if we allow a more general unitary basis transformation, i.e. that generated by a non-orthogonal $\hat{U}$:

$$\hat{M} \to \hat{U}^\mathsf{T} \hat{M} \hat{U} \equiv \hat{M}' \neq \hat{M}.$$

• Regardless, the symmetry constraint $\hat{M}'^\mathsf{T} = \hat{M}'$ is basis independent.

2A.2.1 SO(n) tensors

Consider a tensor formed from a direct product of defining representation (vector) indices. A necessary but not sufficient condition for a rank-k tensor T to transform irreducibly under SO(n) is that it is traceless under contraction of any pair of indices with M_{ij}:

$$M_{i_p i_q}\, T^{i_1 i_2 \cdots i_k} = 0 \quad \forall\, p, q \in \{1, 2, \ldots, k\}\ (p \neq q). \tag{2A.2.4}$$

Completely antisymmetric tensors $T^{[i_1 i_2 \cdots i_k]}$ $(k \le n)$ satisfy this condition and indeed correspond to irreducible representations. *Traceless* completely symmetric tensors $\tilde{T}^{(i_1 i_2 \cdots i_k)}$ $[M_{i_1 i_2} \tilde{T}^{(i_1 i_2 \cdots i_k)} = 0]$ form a separate class of irreducible representations.

2A.3 Symplectic Sp($2n$)

The set of symplectic Sp($2n$) transformations form a subgroup of the unitary Lie group U($2n$); symplectic transformations and algebras are only defined for even-dimensional spaces. Similar to Eq. (2A.2.2), Sp($2n$) transformations can be defined via

$$\hat{U}^\dagger \hat{U} = \hat{1}_{2n} \tag{2A.3.1a}$$

$$\hat{U}^\mathsf{T}\, \hat{\epsilon}\, \hat{U} = \hat{\epsilon}, \tag{2A.3.1b}$$

$$\hat{\epsilon}\, \hat{T}_a^\mathsf{T}\, \hat{\epsilon} = \hat{T}_a, \quad \text{Sp}(2n) \text{ generators}, \tag{2A.3.1c}$$

where the $2n \times 2n$ matrix $\hat{\epsilon}$ satisfies

$$\hat{\epsilon} = -\hat{\epsilon}^\mathsf{T} = -\hat{\epsilon}^\dagger; \quad \hat{\epsilon}^2 = -\hat{1}_n. \tag{2A.3.2}$$

• The symmetry constraint $\hat{\epsilon}^\mathsf{T} = -\hat{\epsilon}$ is basis independent. Nevertheless, we can always find a basis in which we can view $\hat{\epsilon}$ as the block $2n \times 2n$ two-dimensional Levi-Civita tensor [c.f. Eq. (1.5.4)],

$$\hat{\epsilon} \to \begin{bmatrix} 0 & \hat{1}_n \\ -\hat{1}_n & 0 \end{bmatrix}. \tag{2A.3.3}$$

An alternative notation is to define $\hat{\sigma}_2 \equiv -i\,\hat{\epsilon}$, where we view $\hat{\sigma}_2$ as the block antisymmetric Pauli matrix

$$\hat{\sigma}_2 \to \begin{bmatrix} 0 & -i\,\hat{1}_n \\ i\,\hat{1}_n & 0 \end{bmatrix}, \qquad \begin{aligned} \hat{U}^\mathsf{T}\, \hat{\sigma}^2\, \hat{U} &= \hat{\sigma}^2, \\ -\hat{\sigma}^2\, \hat{T}_a^\mathsf{T}\, \hat{\sigma}^2 &= \hat{T}_a. \end{aligned} \tag{2A.3.4}$$

The group dimension $d = n(2n+1)$ for Sp($2n$), since this is the number of independent matrices that can be chosen to satisfy the constraint. This follows because the symplectic condition is a generalized *symmetry* constraint on the generators,

$$\hat{\sigma}^2\, \hat{T}_a = \left(\hat{\sigma}^2\, \hat{T}_a\right)^\mathsf{T}. \tag{2A.3.5}$$

The "antisymmetric metric" $\hat{\epsilon}$ can be used to "tie up" indices transforming in the defining representation of Sp($2n$). If ψ^i and ϕ^j are two such representations ($i, j \in \{1, 2, \ldots, 2n\}$), then

$$\psi^\mathsf{T}\, \hat{\epsilon}\, \phi = \psi^i\, \epsilon_{ij}\, \phi^j \tag{2A.3.6}$$

is invariant. The simplest example applies to spin-1/2 spinors in su(2) = sp(2), where this contraction gives the usual singlet state [Eq. (1.5.1a)]. Sp($2n$) is different from SU(n) or SU($2n$), in that one can always form a group invariant by tensoring together two defining rep. indices. Since the defining representation of Sp($2n$) is in some sense a generalization of spin-1/2, we sometimes refer to it as a "spinor" representation.

2A.3.1 Sp($2n$) tensors

Consider a tensor formed from a direct product of defining representation indices. A necessary but not sufficient condition for a rank-k tensor T to transform irreducibly under Sp($2n$) is that it is "traceless" under contraction of any pair of indices with ϵ_{ij}:

$$\epsilon_{i_p i_q}\, T^{i_1 i_2 \cdots i_k} = 0 \quad \forall\, p, q \in \{1, 2, \ldots, k\}\ (p \neq q). \tag{2A.3.7}$$

Completely symmetric tensors $T^{(i_1 i_2 \cdots i_k)}$ satisfy this condition and indeed correspond to irreducible representations. *Traceless* completely antisymmetric tensors $\tilde{T}^{[i_1 i_2 \cdots i_k]}$ ($k \leq 2n$) [$\epsilon_{i_1 i_2} \tilde{T}^{[i_1 i_2 \cdots i_k)]} = 0$] correspond to a different set of irreducible representations.

3. First example of a higher rank Lie algebra: su(3)
* version 1.7 *

Matthew Foster

January 20, 2021

Contents

3.1 Gell-Mann, Cartan-Weyl bases **1**

3.2 The adjoint representation **3**

3.2.1 Cartan subalgebra H and root H^* spaces 4

The discussion here largely follows chapter II of [1].

3.1 Gell-Mann, Cartan-Weyl bases

The Lie group SU(3) is defined as the group of determinant one unitary transformations that can be applied to a 3-component, complex vector V (or ket $|V\rangle$ if you prefer). The standard basis for the su(3) generators in the **defining** (or "3") representation in the old physics literature is due to Gell-Mann,

$$\begin{aligned}
\hat{\lambda}_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\hat{\lambda}_6 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{\lambda}_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \\
\hat{\lambda}_4 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \\
\hat{\lambda}_8 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.
\end{aligned} \tag{3.1.1}$$

We have displayed these in an unconventional way in order to highlight the following points:

- The first row forms an SU(2) subalgebra,

  $$\left[\hat{\lambda}_i, \hat{\lambda}_j\right] = 2i\epsilon_{ijk}\hat{\lambda}_k, \quad i,j,k \in \{1,2,3\}.$$

  In this basis, these generators block embed the 2×2 Pauli matrices $\hat{\sigma}_{1,2,3}$ so as to act only on the first 2 components of $|V\rangle$, which transforms as a spin 1/2 under this subalgebra.

- Similarly, $\hat{\lambda}_6$ and $\hat{\lambda}_7$ are embeddings of $\hat{\sigma}_1$ and $\hat{\sigma}_2$ acting on the bottom two components of $|V\rangle$.

- $\hat{\lambda}_4$ and $\hat{\lambda}_5$ are embeddings of $\hat{\sigma}_1$ and $\hat{\sigma}_2$ in the *outer* off-diagonal elements (3,1) and (1,3).

Now, a complete basis for 3×3 matrices has 9 elements, but in su(3) we exclude the identity matrix. This is because the latter merely determines the phase of the determinant for any group transformation. Thus although we have 3 different embeddings of $\hat{\sigma}_1$ and $\hat{\sigma}_2$ amongst $\{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_6, \hat{\lambda}_7, \hat{\lambda}_4, \hat{\lambda}_5\}$, we have only two diagonal generators $\hat{\lambda}_3$ and $\hat{\lambda}_8$. Note however that we can form the linear combinations

$$\frac{\sqrt{3}}{2}\hat{\lambda}_8 - \frac{1}{2}\hat{\lambda}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \frac{\sqrt{3}}{2}\hat{\lambda}_8 + \frac{1}{2}\hat{\lambda}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{3.1.2}$$

i.e. embeddings of $\hat{\sigma}_3$ that give su(2) subalgebras when combined with $\{\hat{\lambda}_6, \hat{\lambda}_7\}$ and $\{\hat{\lambda}_4, \hat{\lambda}_5\}$, respectively.

We can therefore view the su(3) algebra as a combination of three different su(2) subalgebras. The key point is that these are not independent, and not only because the needed "$\hat{\sigma}_3$ s" are shared between only two diagonal generators $\{\hat{\lambda}_3, \hat{\lambda}_8\}$. The non-trivial structure of su(3) arises because of non-trivial commutation relations between the off-diagonal elements. All representations of su(2) are one-dimensional: there is a highest weight state, and then you use the lowering operator to move down a chain of descendants until you reach the lowest weight state. In su(3) we will see that we can still always define a highest weight state in any irreducible representation, but that

- Acting with only one su(2) lowering operator accesses only a subset (1D chain) of the states (also called **weights**) in the representation, and that
- Acting with different su(2)s allows access to all weights. Representations divide into different classes (**conjugacy classes**), and each conjugacy class forms an infinite **regular lattice** of weights. Different representations in the same class correspond to different finite subsets of the lattice.

All irreducible representations of a given Lie algebra reside on lattices with a common spatial dimensionality r; r is the **rank** of the algebra. su(n) has rank $n-1$, so all representations of su(3) can be described by weights residing on lattices in the plane. A major goal of this course is to identify all possible representations of a given algebra, to develop a geometrical picture of any such representation as a section of a weight lattice, and to acquire tools to obtain the explicit weights and weight multiplicities (degeneracies).

As we will discuss in more detail below, the rank r is equal to the maximum number of independent commuting generators. The set of such generators is the **Cartan subalgebra**, and the weights of a given representation can be labeled by the *mutual* eigenvalues of these. In the Gell-Mann basis for su(3) it is spanned by $\hat{\lambda}_3$ and $\hat{\lambda}_8$,

$$\left[\hat{\lambda}_3, \hat{\lambda}_8\right] = 0. \tag{3.1.3}$$

It is useful to switch from the Gell-Mann to the **Cartan-Weyl** basis.[1] In the latter, we express all off-diagonal generators in terms of raising and lowering operators. We define

$$\begin{aligned} \hat{T}_\pm &\equiv \frac{1}{2}\left(\hat{\lambda}_1 \pm i\hat{\lambda}_2\right), \quad \hat{T}_z \equiv \frac{1}{2}\hat{\lambda}_3, \\ \hat{U}_\pm &\equiv \frac{1}{2}\left(\hat{\lambda}_6 \pm i\hat{\lambda}_7\right), \\ \hat{V}_\pm &\equiv \frac{1}{2}\left(\hat{\lambda}_4 \pm i\hat{\lambda}_5\right), \\ Y &\equiv \frac{1}{\sqrt{3}}\hat{\lambda}_8. \end{aligned} \tag{3.1.4}$$

We associate these 3×3 matrices to abstract Lie algebra generators $\{t_+, t_-, u_+, u_-, v_+, v-, t_z, y\}$. We can determine the structure constants (non-trivial Lie brackets) for the latter by computing all possible matrix commutators between the elements in Eq. (3.1.4). Via Eq. (3.1.2), the brackets between raising and lowering operators in the three su(2) subalgebras are given by

$$\begin{aligned} [t_+, t_-] &= 2t_z, \\ [u_+, u_-] &= 2\left(\tfrac{3}{4}y - \tfrac{1}{2}t_z\right), \\ [v_+, v_-] &= 2\left(\tfrac{3}{4}y + \tfrac{1}{2}t_z\right). \end{aligned} \tag{3.1.5}$$

The only non-trivial Lie bracket between <u>raising operators</u> is

$$[t_+, u_+] = v_+. \tag{3.1.6}$$

[1] We are following Cahn [1] in terms of our naming scheme for raising and lower operators. Later we will switch to the conventional root vector notation, e.g. $t_\pm \to e_{\pm\alpha_1}$, etc.

There are several non-trivial brackets between different types of raising and lowering operators,

$$\begin{aligned}[t_+, v_-] &= -\,u_-,\\ [u_+, v_-] &= t_-,\end{aligned} \tag{3.1.7}$$

as well as the "Hermitian conjugate" relations (defined with respect to the matrices of the defining representation)

$$\begin{aligned}[v_+, t_-] &= -\,u_+,\\ [v_+, u_-] &= t_+.\end{aligned}$$

Finally, the Lie brackets between the Cartan subalgebra elements $\{t_z, y\}$ and the raising and lowering operators $\{t_\pm, u_\pm, v_\pm\}$ are given by

$$\begin{aligned}[t_z, t_\pm] &= \pm\, t_\pm, & [y, t_\pm] &= 0,\\ [t_z, u_\pm] &= \mp\tfrac{1}{2} u_\pm, & [y, u_\pm] &= \pm\, u_\pm,\\ [t_z, v_\pm] &= \pm\tfrac{1}{2} v_\pm, & [y, v_\pm] &= \pm\, v_\pm.\end{aligned} \tag{3.1.8}$$

• **Exercise**: Verify Eqs. (3.1.6)–(3.1.8).

3.2 The adjoint representation

The generators of a Lie algebra transform in the **adjoint** representation. This should already be familiar from the notion of a basis or symmetry ("similarity") transformation in quantum mechanics, where an $n \times n$ matrix $\hat{M}$ transforms according to

$$\hat{M} \to \hat{M}' \equiv \hat{U}^\dagger\, \hat{M}\, \hat{U}, \qquad \hat{U} \equiv \exp(i\alpha^a\, \hat{T}_a). \tag{3.2.1}$$

Here $\hat{T}_a$ is an $n \times n$ Hermitian matrix [i.e., a generator of su(n)], and the coordinates $\{\alpha^a \in \mathbb{R}\}$ specify the transformation $\hat{U}$. Since the matrices $\{\hat{T}_a\}$ plus the identity form a basis for $n \times n$ matrices, Eq. (3.2.1) can equivalently be interpreted as a transformation law for the generators themselves,

$$\begin{aligned}\hat{T}_a \to \hat{T}'_a &\equiv \exp(-i\alpha^b\, \hat{T}_b)\, \hat{T}_a\, \exp(i\alpha^c\, \hat{T}_c)\\ &= \hat{T}_a + i\alpha^b \left[\hat{T}_a, \hat{T}_b\right] + \frac{1}{2!}(i\alpha^b)(i\alpha^c) \left[\left[\hat{T}_a, \hat{T}_b\right], \hat{T}_c\right] + \frac{1}{3!}(i\alpha^b)(i\alpha^c)(i\alpha^d) \left[\left[\left[\hat{T}_a, \hat{T}_b\right], \hat{T}_c\right], \hat{T}_d\right] + \ldots,\end{aligned} \tag{3.2.2}$$

where we have used the Baker-Campbell-Hausdorff formula on the second line. The latter implies that the similarity transformation can be assumed for the abstract Lie algebra L, since the corresponding t'_a is expressed soley in terms of linear combinations of nested Lie brackets.

Consider the transformation on a basis element $x_j \in L$:

$$a_{d_i}(x_j) \equiv [x_i, x_j] = i f_{ijk}\, x_k, \tag{3.2.3}$$

where the f_{ijk} are the structure constants (and the big i is the square root of minus one). For a Lie algebra with dimension d, the operator a_{d_i} can be represented by a $d \times d$ matrix $\widehat{a}_{d_i}$. The argument (x_j) on the left-hand side of Eq. (3.2.3) specifies the jth column of $\widehat{a}_{d_i}$, while the row entries are the structure constants $i f_{ijk}$ for each x_k:

$$\left[\widehat{a}_{d_i}\right]_{k,j} = i f_{ijk}. \tag{3.2.4}$$

We can show that a_{d_x} satisfies the Lie algebra L. Assume that $[x, y] = z$ for some $x, y, z, \in L$. Then we want to see that $[a_{d_x}, a_{d_y}] = a_{d_z}$. Examine the action of these operators on some other element $w \in L$:

$$\begin{aligned}[a_{d_x}, a_{d_y}]w &= a_{d_x}\left[a_{d_y}(w)\right] - a_{d_y}\left[a_{d_x}(w)\right] = [x, [y, w]] - [y, [x, w]] = [x, [y, w]] + [y, [w, x]]\\ &= -\,[w, [x, y]] = [z, w] = a_{d_z}(w).\end{aligned} \tag{3.2.5}$$

On the second line, we have used the Jacobi identity [Eq. (1.2.1d)].

In the case of su(3), the adjoint representation is 8-dimensional. We associate each of $\{t_+, t_-, u_+, u_-, v_+, v-, t_z, y\}$ to the basis vectors $\{|1\rangle, |2\rangle, \ldots, |8\rangle\}$,

$$a\, t_+ + b\, t_- + c\, u_+ + d\, u_- + e\, v_+ + f\, v_- + g\, t_z + h\, y \to \begin{bmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\end{bmatrix}.$$

Then (e.g.)

$$\widehat{a}d_{t_+} \to \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & t_+ \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_- \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_+ \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & u_- \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & v_+ \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_- \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & t_z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \\ t_+ & t_- & u_+ & u_- & v_+ & v_- & t_z & y & \end{bmatrix}. \tag{3.2.6}$$

• **Exercise**: Find the matrix representation for $\widehat{a}d_{v_-}$.

3.2.1 Cartan subalgebra H and root H^* spaces

In su(3), the abelian subalgebra t_z and y form a basis for the **Cartan subalgebra $\boldsymbol{H}$**. Any linear combination of these is an element of H (a linear vector space). Consider the adjoint representation of a generic Cartan subalgebra element $a\,t_z + b\,y$. Eq. (3.1.8) implies that the adjoint representation is purely diagonal,

$$\widehat{a}d_{a\,t_z+b\,y} \to \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_+ \\ 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & t_- \\ 0 & 0 & -\frac{a}{2}+b & 0 & 0 & 0 & 0 & 0 & u_+ \\ 0 & 0 & 0 & \frac{a}{2}-b & 0 & 0 & 0 & 0 & u_- \\ 0 & 0 & 0 & 0 & \frac{a}{2}+b & 0 & 0 & 0 & v_+ \\ 0 & 0 & 0 & 0 & 0 & -\frac{a}{2}-b & 0 & 0 & v_- \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \\ t_+ & t_- & u_+ & u_- & v_+ & v_- & t_z & y & \end{bmatrix}. \tag{3.2.7}$$

This is a defining attribute of the Cartan-Weyl basis for L: raising and lowering operators are eigenvectors of a generic Cartan subalgebra element $h \in H$ in the adjoint representation. In su(3), this forces us to choose $\{t_\pm, u_\pm, v_\pm\}$ for the basis elements in L outside of H. By contrast, there is no unique basis for H, since while there are three pairs of raising and lowering operators, at most two "$\hat{\sigma}_3$ s" can be chosen independently [see the discussion following Eq. (3.1.2), above].

For any Cartan-Weyl basis element $x_b \in L$ and a generic element of the Cartan subalgebra $h \in H$, we have

$$a_{d_h}(x_b) = \alpha_b(h)\, x_b. \tag{3.2.8}$$

If $x_b \in H$, then $\alpha_b(h) = 0$. Otherwise x_b is a raising or lowering operator. In this case,

1. The eigenvalue α_b is called a **root**, and it is a linear functional on the elements of the Cartan subalgbra.
2. The associated element $x_b \in L$ is called a **root vector**.

It is helpful to think of the space of roots as being dual to the Cartan subalgebra. More precisely, we define $\boldsymbol{H}^*$: The space of linear functionals on H. This is a dual vector space, elements of which are linear combinations of the roots $\{\alpha_b\}$. Eq. (3.2.7) implies that the roots of su(3) can be represented via

$$\begin{aligned} a_{d_{a\,t_z+b\,y}}(t_\pm) &= \alpha_{t_\pm}\,(a\,t_z + b\,y)\,t_\pm &&= \pm a\,t_\pm, \\ a_{d_{a\,t_z+b\,y}}(u_\pm) &= \alpha_{u_\pm}\,(a\,t_z + b\,y)\,u_\pm &&= \mp\left(\frac{a}{2} - b\right) u_\pm, \\ a_{d_{a\,t_z+b\,y}}(v_\pm) &= \alpha_{v_\pm}\,(a\,t_z + b\,y)\,v_\pm &&= \pm\left(\frac{a}{2} + b\right) v_\pm. \end{aligned} \tag{3.2.9}$$

The number of basis elements in L is d, the dimension of the group. The Cartan subalgebra and dual root spaces have dimension r, where $r < d$ is the rank. Although $d = 8$ for su(3), the rank $r = 2$ is much smaller. All irreducible representations can be represented by intersecting weight chains drawn in a 2D plane. The presence of three roots suggests that weights arrange into strings oriented along three different directions in the plane, and this naturally suggests weight lattices of triangular symmetry, as we will soon confirm. It is remarkable that the geometry of the root system translates into precise rules that map out all states in a given irreducible representation. In order to see this connection, we must introduce an inner product on the root space H^*.

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

4. Killing form, root space inner product, and commutation relations
* version 1.6 *

Matthew Foster

January 20, 2021

Contents

4.1 Weights in the Cartan-Weyl basis; rank-r bases for H and H^* **1**

4.2 The Killing form **2**
4.2.1 su(3) . . . 2

4.3 Root representation in the Cartan subalgebra **3**
4.3.1 Weight inner products and root geometry . . . 3
4.3.2 Some definitions . . . 4

4.4 The structure of simple Lie algebras **4**
4.4.1 Root facts . . . 4
4.4.2 Construction of the Cartan subalgebra . . . 5
4.4.3 Commutation relations . . . 5

The discussion here largely follows chapters III and IV of [1].

4.1 Weights in the Cartan-Weyl basis; rank-r bases for H and H^*

As introduced in module 3 with the example of su(3), a Lie algebra L of dimension d expressed in the Cartan-Weyl basis consists of raising and lowering operators (**root vectors**), coupled with a rank $r < d$ abelian Cartan subalgebra $H \in L$. Recall that a **root** is the eigenvalue associated to the action of a generic element $h \in H$ on a root vector in the adjoint representation:

$$a_{d_h}(x_b) = [h, x_b] = \alpha_b(h)\, x_b. \tag{4.1.1}$$

Here x_b is the root vector and α_b is the root, a linear functional on H. The roots belong to the dual vector space H^*. Both H and H^* are linear vector spaces with dimension r.

A generic element ρ of H^* is a linear combination of roots with complex coefficients, referred to as a **weight**; H^* is also referred to as the **space of weights**. The action of ρ on a generic element $k \in H$ is determined by its expansion in terms of roots:

$$\rho \equiv \sum_b c_b\, \alpha_b, \quad \{c_b \in \mathbb{C}\}, \tag{4.1.2}$$

$$\rho(h) = \sum_b c_b\, \alpha_b(h). \tag{4.1.3}$$

Roots refer to specific elements of H^* that correspond to the root vectors in the Cartan-Weyl basis. The total number of root vectors is $d - r$, and the number of these that correspond to "raising operators" (called **positive root vectors**) is half this value. It will turn out the number of positive root vectors is always greater than the rank r, except for the case of $\mathrm{su}(2) = \mathrm{so}(3) = \mathrm{sp}(2)$ (for which they are equal).

For example, in su(3) we have three positive root vectors $\{t_+, u_+, v_+\}$; $r = 2$ and so $d = 3 + 3 + 2 = 8$. A generic Lie group transformation requires the specification of d parameters; in this sense all elements of L (including all positive root vectors)

are independent. However, since the number of positive roots is greater than r, it is clear that the roots form an *overcomplete* basis in H^* (thinking of the latter as a linear vector space). If we view the positive roots as "vectors with arrows" pointing somewhere in the r-dimensional H^*, then overcompleteness means that we cannot choose all vectors to be "orthogonal."

Our goal is to bring out the structure of H^* in terms of root geometry. A key tool is the definition of a weight inner product. To define an inner product on H^*, we must first link it with H.

4.2 The Killing form

The first step is to define the **Killing form**, which is the trace normalization for generators in the adjoint representation. Let L be a Lie algebra and $a, b \in L$. Then

$$(a, b) \equiv \mathsf{Tr}\left[\widehat{a}_{d_a}\, \widehat{a}_{d_b}\right], \quad \text{Killing form } L \times L \to \mathbb{C}. \tag{4.2.1}$$

The Killing form itself is *not* a conventional inner product, since there is no requirement that (a, a) be positive definite.[1] If we assume that all $\{\widehat{a}_{d_a}\}$ are Hermitian, then $(a, a) \geq 0$, but we are not required to make this choice.

How do we evaluate the Killing form?

1. We could construct the explicit matrix representations for all adjoint operators in some chosen basis for the Lie algebra, $\{x_i\} \in L \Rightarrow \{\widehat{a}_{d_i}\}$, and then compute the traces of all bilinear products.

2. An easier way is to note that $a_{d_a} a_{d_b}(x) = [a, [b, x]]$. Applied to basis elements $\{x_i\}$,

$$a_{d_i} a_{d_j}(x_k) = [x_i, [x_j, x_k]] = \sum_l c_{ijk}^{(l)}\, x_l, \tag{4.2.2}$$

where $c_{ijk}^{(k)}$ is the contribution to the trace from the kth basis element. (Recall that in the adjoint representation, e.g.

$$x_1 \to |x_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x_1 \to \langle x_1| = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

acting in the trace on the right and left, respectively.)

4.2.1 su(3)

Using the commutation algebra in Eqs. (3.1.5)–(3.1.8) and Eq. (4.2.2), we can determine all non-vanishing Killing forms in the Cartan-Weyl basis for su(3). For example,

$$\begin{aligned}
[t_z, [t_z, t_\pm]] &= t_\pm && \Rightarrow c_{t_z t_z t_\pm}^{(t_\pm)} = 1, \\
[t_z, [t_z, u_\pm]] &= \frac{1}{4} u_\pm && \Rightarrow c_{t_z t_z u_\pm}^{(u_\pm)} = \frac{1}{4}, \\
[t_z, [t_z, v_\pm]] &= \frac{1}{4} v_\pm && \Rightarrow c_{t_z t_z v_\pm}^{(v_\pm)} = \frac{1}{4}, \\
[t_z, [t_z, t_z]] &= 0 && \Rightarrow c_{t_z t_z t_z}^{(t_z)} = 0, \\
[t_z, [t_z, y]] &= 0 && \Rightarrow c_{t_z t_z y}^{(y)} = 0,
\end{aligned} \tag{4.2.3}$$

so that $(t_z, t_z) = 3$. The non-vanishing Killing forms in su(3) are

$$\begin{aligned}
(t_z, t_z) &= 3, \\
(y, y) &= 4, \\
(t_+, t_-) &= 6, \\
(u_+, u_-) &= 6, \\
(v_+, v_-) &= 6.
\end{aligned} \tag{4.2.4}$$

• **Exercise**: Verify Eq. (4.2.4).

The Killing forms in Eq. (4.2.4) do not give any hint of the relationship between the different su(2)s, since products involving different "species" of root vectors vanish, e.g. $(t_{\sigma_1}, u_{\sigma_2}) = 0$ for $\sigma_{1,2} \in \{+, -\}$. This result generalizes to all Lie algebras (see below).

[1] Note that Eq. (4.2.1) involves a product of $\widehat{a}_{d_a}$ and $\widehat{a}_{d_b}$, not $(\widehat{a}_{d_a})^\dagger$ and $\widehat{a}_{d_b}$.

4.3 Root representation in the Cartan subalgebra

Now we establish a link between the Cartan subalgebra H and the dual root space H^*:
Lemma: For any weight ρ in H^*, there exists a unique element h_ρ in H, such that for every $k \in H$,

$$\rho(k) = (h_\rho, k). \tag{4.3.1}$$

In words, we can identify an element of the Cartan subalgebra $h_\rho \in H$, such that the action of the weight $\rho \in H^*$ on any element $k \in H$ is given by the Killing form of h_ρ and k. Thus it relates the action of a dual vector ρ on a vector k to the Killing form between two vectors h_ρ and k. This Lemma is only guaranteed to work for semi-simple Lie algebras, defined precisely in Sec. 4.3.2, below. The requirement is that the "overlap matrix" (h_i, h_j) for any basis in H is invertible, and this is true for semi-simple algebras.

• su(3) example: From Eq. (3.2.9), the positive roots can be taken to be

$$\begin{aligned} \alpha_1\,(a\,t_z + b\,y) &= a, \\ \alpha_2\,(a\,t_z + b\,y) &= -\frac{a}{2} + b, \\ \alpha_3\,(a\,t_z + b\,y) &= \frac{a}{2} + b, \end{aligned} \tag{4.3.2}$$

where we have relabeled $\{t_+, u_+, v_+\} \to \{1, 2, 3\}$. Define

$$h_{\alpha_i} \equiv c_i\,t_z + d_i\,y. \tag{4.3.3}$$

Eq. (4.2.4) implies that

$$(h_{\alpha_i}, t_z) = 3c_i, \quad (h_{\alpha_i}, y) = 4d_i. \tag{4.3.4}$$

Asserting that Eq. (4.3.1) reproduces Eq. (4.3.2) for $\rho = \alpha_{1,2,3}$, we arrive at

$$h_{\alpha_1} = \frac{1}{3}t_z, \quad h_{\alpha_2} = -\frac{1}{6}t_z + \frac{1}{4}y, \quad h_{\alpha_3} = \frac{1}{6}t_z + \frac{1}{4}y. \tag{4.3.5}$$

4.3.1 Weight inner products and root geometry

Define H_0^* to be the space of weights formed from real linear combinations of the roots. This is a subspace of H^* (root combinations with complex coefficients). Assume that α and β are weights in H_0^*. Let $h_{\alpha,\beta}$ be the associated elements in H. We define the **weight inner product** between α and β via

$$\boxed{\langle \alpha, \beta \rangle = (h_\alpha, h_\beta), \quad \text{weight inner product } H_0^* \times H_0^* \to \mathbb{R}.} \tag{4.3.6}$$

Why do we introduce an inner product structure only for elements of H_0^*?

- We want something that gives us geometrical information about the overcomplete basis of roots; as demonstrated in Eq. (4.2.4), the Killing form of the Cartan-Weyl basis elements does not tell us this.
- There are no "naturally distinguished" elements of the Cartan subalgebra H.

The roots are eigenvalue functionals that are independent of the basis chosen for H. Eq. (4.3.1) translates these into special elements of H, and the inner product Eq. (4.3.6) extracts the geometrical root data. Eq. (4.3.6) will enable us to map out irreducible representations by calculating the depth of weight chains generated by lowering with a given root vector (module 5).

In su(3), Eqs. (4.2.4) and (4.3.5) give

$$\begin{aligned} \langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = \langle \alpha_3, \alpha_3 \rangle &= 1/3, \\ \langle \alpha_1, \alpha_2 \rangle &= -1/6, \\ \langle \alpha_1, \alpha_3 \rangle = \langle \alpha_2, \alpha_3 \rangle &= 1/6. \end{aligned} \tag{4.3.7}$$

These results imply that we can visualize $\alpha_{1,2,3}$ as three equal-length vectors in the plane, with $\pi/3$ angles between $\alpha_{1,2}$ and α_3, and a $2\pi/3$ angle between α_1 and α_2. Incorporating the negative roots, we obtain the picture in Fig. 4.1.

Figure 4.1: Root geometry in su(3), as implied by the weight inner products in Eq. (4.3.7).

4.3.2 Some definitions

1. A **subalgebra** is a subspace M of L that is closed under the Lie bracket.

2. An **ideal** is a special subalgebra J of L, such that if $x \in J$ and $y \in L$, then $[x, y] \in J$.
   Example: The generators of spatial translations in a Hilbert space (momentum operators) $[\hat{P}_i, \hat{P}_j] = 0$ form an abelian, ideal subalgebra of the 3D Euclidean "Poincaré" algebra. The latter consists of translation and rotation generators,

$$\begin{aligned} [\hat{P}_i, \hat{P}_j] &= 0, \\ [\hat{L}_i, \hat{P}_j] &= i\epsilon_{ijk}\hat{P}_k, \\ [\hat{L}_i, \hat{L}_j] &= i\epsilon_{ijk}\hat{L}_k. \end{aligned} \tag{4.3.8}$$

   Example: consider $u(n) = u(1) \otimes \mathrm{su}(n)$, the generators of $n \times n$ unitary matrices. The u(1) generator $\hat{1}_n$ is an abelian ideal. The set of traceless Hermitian su(n) generators is a non-abelian ideal.

3. A **simple** Lie algebra has no ideals except the full algebra and 0 (null).

4. A **semi-simple** Lie algebra has no abelian ideals. Thus neither the Poincaré nor u(n) algebras are semi-simple, while su(3) is simple. In general, a semi-simple algebra is a sum of simple non-abelian ideals.

4.4 The structure of simple Lie algebras

4.4.1 Root facts

Our exploration of su(3) in module 3 suggested that a Lie algebra L of dimension d can be partitioned in the Cartan-Weyl basis via the Lie bracket relations with a generic Cartan subalgebra element:

- **Cartan subalgebra H basis.** This consists of r independent elements $\{h_i\}$, $i \in \{1, \ldots, r\}$; r is the rank. The $\{h_i\}$ mutually commute,

$$a_{d_{h_i}}(h_j) = [h_i, h_j] = 0. \tag{4.4.1}$$

- **Root vectors.** A root vector e_α satisfies [Eq. (4.1.1)]

$$a_{d_h}(e_\alpha) = [h, e_\alpha] = \alpha(h)\, e_\alpha, \quad h \in H, \tag{4.4.2}$$

  where $\alpha \in H^*$ is a root (a functional on the Cartan subalgebra H). Eq. (4.4.2) is the analog of the su(2) equation $[s^z, s^\pm] = (\pm 1)s^\pm$, where s^z is the sole Cartan subalgebra element, and $s^\pm$ denote positive and negative root vectors, i.e. raising and lowering operators when acting on some representation.

In the case of su(2), raising and lowering is by one unit. For a generic Lie algebra, Eq. (4.4.2) implies that one moves up and down weight chains via increments determined by the roots.

We assert the following facts about roots, some of which we will prove later:

1. If α (e_α) is a root (root vector), then $-\alpha$ ($e_{-\alpha}$) is also a root (root vector). Because α is a functional and not a number, there is no unique way to define "raising" and "lowering," but we can always make some choice to regard (e.g.) e_α ($e_{-\alpha}$) as the raising (lowering) operator in a string of weights appearing in some irreducible representation. In su(3), we found three different root vector pairs $\{t_\pm\}, \{u_\pm\}, \{v_\pm\}$, c.f. Eq. (3.1.8); the associated roots were respectively relabeled $\pm\alpha_{1,2,3}$ in Eqs. (4.3.2) and (4.3.7).

2. Each root α_i uniquely corresponds to a particular root vector e_{α_i}. In other words, the roots are non-degenerate.

3. If α is a root, then 2α is not a root. It means that α and $-\alpha$ are highest and lowest weight states in L itself, if we act on either with $e_{\pm\alpha}$. Along with a middle element with weight 0 (corresponding to some particular $h_\alpha \in H$, defined below), the weight chain $\{-\alpha, 0, \alpha\}$ is the adjoint representation for the su(2) subalgebra associated to the roots $\pm\alpha$.

   In pictures, this means that each pair of roots $\pm\alpha$ are "spokes" sticking out in opposite directions from a central hub that corresponds to the Cartan subalgebra. The 2D (rank $r = 2$) example of su(3) is shown in Fig. 4.1. Each su(2) subalgebra $\{e_\alpha, e_{-\alpha}, h_\alpha\}$, allows motion through the hub along the spokes $\pm\alpha$. What is missing from Fig. 4.1 are the allowed moves *between spokes corresponding to different roots.*

4.4.2 Construction of the Cartan subalgebra

The Cartan subalgebra is constructed by finding the **regular** element of a Lie algebra L. $h \in L$ is regular if $\widehat{a}_{d_h}$ has as <u>few</u> zero eigenvalues as possible. In su(3), Eq. (3.2.7) implies that t_z is the regular element, since $\widehat{a}_{d_{t_z}}$ has only two zero eigenvalues, while y has four.

The Cartan subalgebra H is then defined as the maximal abelian (commuting) subalgebra of L containing the regular element. In su(3), we add y [Eq. (3.2.7)] to get a rank-2 subalgebra.

4.4.3 Commutation relations

The above establishes Lie bracket relations between elements of H with themselves and with root vectors. What remains to be determined are brackets between two root vectors e_α and e_β. Assume $h \in H$ and consider

$$\begin{aligned} a_{d_h}([e_\alpha, e_\beta]) &= [h, [e_\alpha, e_\beta]] \\ &= -[e_\alpha, [e_\beta, h]] - [e_\beta, [h, e_\alpha]] = [\alpha(h) + \beta(h)]\,[e_\alpha, e_\beta], \end{aligned} \tag{4.4.3}$$

where we have used the Jacobi identity [Eq. (1.2.1d)]. Eq. (4.4.3) applies for *any* h in the Cartan subalgebra. Then one of the following must be true:

$$\alpha + \beta = 0, \quad [e_\alpha, e_\beta] \neq 0, \qquad \Rightarrow \qquad [e_\alpha, e_{-\alpha}] \in H, \qquad \text{or} \tag{4.4.4a}$$

$$\alpha + \beta \neq 0, \quad [e_\alpha, e_\beta] = 0, \qquad \text{or} \tag{4.4.4b}$$

$$\alpha + \beta \neq 0, \quad [e_\alpha, e_\beta] \propto e_{\alpha+\beta}. \tag{4.4.4c}$$

The first case corresponds to $[s^+, s^-] = 2s^z$; $[e_\alpha, e_{-\alpha}] \in H$ can be regarded as the analog of s^z for this su(2) subalgebra. The second and third cases prove that a Lie bracket between two root vectors e_α and e_β either vanishes, or gives another root vector with root $\alpha + \beta$. If all brackets between root vectors with $\alpha + \beta \neq 0$ were to vanish, we would simply have a collection of independent su(2)s. In a simple Lie algebra of rank $r > 1$, there are nonvanishing brackets.

We can refine Eq. (4.4.4a) and make a connection to Eqs. (4.3.1) and (4.3.6), which involve root representation in H. First we prove two lemmas.

Lemma:

$$(e_\alpha, e_\beta) = 0 \ \text{ unless } \ \alpha + \beta = 0. \tag{4.4.5}$$

Proof: Evaluate the Killing form in the Cartan-Weyl basis, so that the trace is over root vectors and r Cartan subalgebra basis elements. For $h_i \in H$,

$$\begin{aligned} a_{d_{e_\alpha}} a_{d_{e_\beta}}(h_i) = [e_\alpha, [e_\beta, h_i]] &= -\beta(h_i)\,[e_\alpha, e_\beta] \\ &= \begin{cases} 0, & \text{if } [e_\alpha, e_\beta] = 0, \\ -\beta(h_i)\,\mathcal{C}_{\alpha,\beta}\, e_{\alpha+\beta}, & \text{if } [e_\alpha, e_\beta] \neq 0, \end{cases} \end{aligned} \tag{4.4.6}$$

where $\mathcal{C}_{\alpha,\beta}$ is a numerical constant. The only possible contribution to the trace (Killing form) must have $e_{\alpha+\beta} \in H$, i.e. $\alpha + \beta = 0$ [c.f. Eq. (4.2.2)]. For a root vector e_γ,

$$a_{d_{e_\alpha}} a_{d_{e_\beta}}(e_\gamma) = [e_\alpha, [e_\beta, e_\gamma]] = \mathcal{C}_{\beta,\gamma}[e_\alpha, e_{\beta+\gamma}] = \mathcal{C}_{\beta,\gamma}\,\mathcal{C}_{\alpha,\beta+\gamma}\; e_{\alpha+\beta+\gamma}. \tag{4.4.7}$$

Again, the only contribution to the trace must have $\alpha + \beta = 0$. Q.E.D.

Lemma: Invariance of the Killing form

$$(a, [b, c]) = ([a, b], c)\,. \tag{4.4.8}$$

Proof:

$$(a, [b, c]) = \mathsf{Tr}\left[\widehat{a}_{d_a}\left[\widehat{a}_{d_b}, \widehat{a}_{d_c}\right]\right] = \mathsf{Tr}\left[\left[\widehat{a}_{d_a}, \widehat{a}_{d_b}\right]\widehat{a}_{d_c}\right] = ([a, b], c)\,.$$

Now consider

$$\begin{aligned} ([e_\alpha, e_{-\alpha}], h) &= (e_\alpha, [e_{-\alpha}, h]) = \alpha(h)\,(e_\alpha, e_{-\alpha})\,, \\ \Rightarrow\ \alpha(h) &= \left(\frac{[e_\alpha, e_{-\alpha}]}{(e_\alpha, e_{-\alpha})}, h\right). \end{aligned} \tag{4.4.9}$$

Eq. (4.3.1) then implies that the root α is represented in the Cartan subalgebra by the element

$$\boxed{h_\alpha \equiv \frac{[e_\alpha, e_{-\alpha}]}{(e_\alpha, e_{-\alpha})}, \qquad \text{root } \alpha \text{ representation in the Cartan subalgebra } H.} \tag{4.4.10}$$

The number of positive roots is always larger than the rank r for $r > 1$, so that the set $\{h_\alpha\}$ forms an overcomplete basis for the Cartan subalgebra.

We apply Eq. (4.4.10) to su(3), relabeling $\{t_\pm, u_\pm, v_\pm\} \to \{e_{\pm\alpha_1}, e_{\pm\alpha_2}, e_{\pm\alpha_3}\}$. Eqs. (3.1.5) and (4.2.4) give

$$h_{\alpha_1} = \frac{t_z}{3}, \quad h_{\alpha_2} = \frac{3y - 2t_z}{12}, \quad h_{\alpha_3} = \frac{3y + 2t_z}{12},$$

identical to Eq. (4.3.5).

We can summarize the Lie brackets for all elements of a Lie algebra L in the Cartan-Weyl basis as follows. Let $\Sigma \equiv L - H$ denote the root vector space. Then we can write

$$[h_i, h_j] = 0, \qquad h_{i,j} \in H, \tag{4.4.11a}$$

$$[h_i, e_\alpha] = \alpha(h_i)\, e_\alpha, \qquad h_i \in H, \quad e_\alpha \in \Sigma, \tag{4.4.11b}$$

$$[e_\alpha, e_\beta] = \begin{cases} (e_\alpha, e_{-\alpha})\, h_\alpha, & \alpha + \beta = 0, \\ N_{\alpha,\beta}\, e_{\alpha+\beta}, & \alpha + \beta \neq 0,\ e_{\alpha+\beta} \in \Sigma, \\ 0, & \alpha + \beta \neq 0,\ e_{\alpha+\beta} \notin \Sigma. \end{cases} \tag{4.4.11c}$$

The structure constants $\{N_{\alpha,\beta} \in \mathbb{C}\}$ are the nontrivial data that connect different su(2)s in L. In module 5 we will prove that $N_{\alpha,\beta} \neq 0$ if $\alpha + \beta$ is a root.

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

5. Roots and weights
* version 2.0 *

Matthew Foster

January 20, 2021

Contents

5.1 Representations I **1**
5.1.1 Matrix representations and weight vectors . . . 1
5.1.2 Raising and lowering; chains of weights . . . 3
5.1.3 The Master Weight Depth Formula (MWDF) . . . 4

5.2 Root geometries **6**
5.2.1 Weyl reflections . . . 6
5.2.2 Root chains and inner products . . . 7

5.3 Constructing the adjoint representation using the MWDF **10**
5.3.1 Simple roots and the Cartan matrix . . . 10
5.3.2 Higher level roots . . . 12
5.3.3 G_2 example . . . 14
5.3.4 Dynkin diagrams . . . 16

The discussion here largely follows chapters V–VII of [1].

5.1 Representations I

5.1.1 Matrix representations and weight vectors

In module 4, we discussed how a general Lie algebra L expressed in the Cartan-Weyl basis is a complex of interwoven su(2)s, formed from raising and lowering operators (root vectors $\{e_\alpha, e_{-\alpha}\}$) and a basis $\{h_i\} \in H$ of mutually commuting Cartan subalgebra elements, $[h_i, h_j] = 0$. The number of independent basis elements for H is the rank r, and the group dimension $d = r + 2 \times$ (# of positive root vectors).

For each pair of raising and lowering operators, we can construct a unique element $h_\alpha \in H$ that serves as the "S_z" for that particular su(2): [c.f. Eq. (4.4.11)]

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)\, e_\alpha = \langle \alpha, \alpha \rangle\, e_\alpha, \tag{5.1.1a}$$

$$[h_\alpha, e_{-\alpha}] = -\,\alpha(h_\alpha)\, e_{-\alpha} = -\langle \alpha, \alpha \rangle\, e_{-\alpha}, \tag{5.1.1b}$$

$$[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})\, h_\alpha. \tag{5.1.1c}$$

Here $(e_\alpha, e_{-\alpha})$ is the Killing form for this pair of raising and lowering operators; the last equation defines the H element h_α. The root $\alpha = \alpha(h)$ belongs to the **space of weights** H^*, and is a functional on a generic Cartan subalgebra element $h \in H$. When we choose $h = h_\alpha$, the resulting eigenvalue is the weight space inner product [Eq. (4.3.6)]

$$\alpha(h_\alpha) = (h_\alpha, h_\alpha) = \langle \alpha, \alpha \rangle, \qquad H_0^* \times H_0^* \to \mathbb{R}.$$

Recall that while we can construct an h_α for every root vector pair $\{e_\alpha, e_{-\alpha}\}$ in L, this set is overcomplete [unless $r = 1$, su(2)]. In the Cartan subalgebra, it is important not to confuse the overcomplete $\{h_\alpha\}$ with the basis $\{h_i\}$; a prescription for constructing the latter will be described later.

Now, consider an N-dimensional irreducible unitary matrix representation of L, so that

$$a \in L \to \hat{A}, \quad \text{an } N \times N \text{ matrix.}$$

The Cartan subalgebra elements map to

$$[h, h'] \to [\hat{H}, \hat{H}'] = 0. \tag{5.1.2}$$

In a unitary representation, we can without loss of generality (WLoG) assume that each basis element $h_i \in H$ $(1 \leq i \leq r)$ corresponds to a Hermitian matrix $\hat{H}_i = \hat{H}_i^\dagger$. Then there is a basis for vectors in the representation that diagonalizes the mutually commuting $\{\hat{H}_i\}$. An element of this basis is a **weight vector** ϕ^a, where $0 \leq a \leq N-1$ (N is the dimension of the representation). In the eigenbasis of $\{H_i\}$, we can take (e.g.)

$$\phi^0 \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \phi^1 \to \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad \phi^{N-1} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{5.1.3}$$

A weight vector satisfies

$$\hat{H}_i \, \phi^a = \lambda_i^a \, \phi^a, \quad 1 \leq i \leq r, \tag{5.1.4}$$

where the eigenvalue $\lambda_i^a \in \mathbb{R}$. Thus each weight vector is specified by the set of eigenvalues obtained by acting with any of the $\{\hat{H}_i\}$. This is similar to quantum mechanics, where one seeks to identify a physical state by labeling it via a complete set of commuting observables [2], e.g. mutually diagonalizable Hermitian matrices. Even in a given irreduible representation, however, a weight vector labeled by $\{\lambda_i^a\}$ is not necessarily unique. Later we will show that there can be degeneracies; this is different from the representation theory of su(2).

Instead of employing the explicit $\{\lambda_i^a\}$ obtained in some particular basis, we recast Eq. (5.1.4) for a generic Cartan subalgebra element $h \to \hat{H}$:

$$\hat{H} \, \phi^a = M^a(h) \, \phi^a, \tag{5.1.5}$$

where the functional $M^a(h)$ is called the **weight**, and resides in the same space H^* as the roots.

• <u>su(3) example</u>: A basis for H in the defining representation is given by [Eqs. (3.1.4) and (3.1.1)]

$$\hat{T}_z = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}. \tag{5.1.6}$$

In this basis, we denote the weight vectors as

$$\phi^A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \phi^B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \phi^C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.1.7}$$

Then

$$\begin{aligned} M^A(at_z + by) &= \frac{1}{2}a + \frac{1}{3}b &&= \frac{2}{3}\alpha_{t_+} + \frac{1}{3}\alpha_{u_+} \\ M^B(at_z + by) &= -\frac{1}{2}a + \frac{1}{3}b &&= -\frac{1}{3}\alpha_{t_+} + \frac{1}{3}\alpha_{u_+} \\ M^C(at_z + by) &= -\frac{2}{3}b &&= -\frac{1}{3}\alpha_{t_+} - \frac{2}{3}\alpha_{u_+}. \end{aligned} \tag{5.1.8}$$

On the right-hand-sides, we have re-expressed the weights in terms of two of the positive roots using Eq. (3.2.9).

Figure 5.1: Action of root vector operators on a weight vector in the defining representation of su(3).

5.1.2 Raising and lowering; chains of weights

Assume that ϕ^a is a weight vector with weight M^a, Eq. (5.1.5). Then

$$\hat{H}\,\hat{E}_{\pm\alpha}\,\phi^a = \left[\hat{E}_{\pm\alpha}\,\hat{H} \pm \alpha(h)\,\hat{E}_{\pm\alpha}\right]\phi^a = \left[M^a(h) \pm \alpha(h)\right]\hat{E}_{\pm\alpha}\,\phi^a, \tag{5.1.9}$$

so that $\hat{E}_{\pm\alpha}\,\phi^a$, if non-zero, is a weight vector with weight $(M^a \pm \alpha)$.
This suggests the following picture: all weights belong to **weight strings** or chains, consisting of 1 or more weights separated by root intervals. These chains can be pictured as parallel 1D sublattices embedded in an r-dimensional Euclidean space. Acting with different pairs of root vectors, we translate through an r-dimensional **weight lattice**. Since the number of positive root vectors is greater than r for any $r > 1$, different sets of parallel sublattices generated by different roots cannot all be mutually orthogonal.

We consider two examples from su(3).

• Defining ("3") representation: The weights are given by Eq. (5.1.8), associated to the weight vectors in Eq. (5.1.7). The root vector operators have the matrix representations [Eqs. (3.1.4) and (3.1.1)]

$$\{\hat{T}_+, \hat{T}_-\} = \left\{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\right\}, \quad \{\hat{U}_+, \hat{U}_-\} = \left\{\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\right\}, \quad \{\hat{V}_+, \hat{V}_-\} = \left\{\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}\right\}. \tag{5.1.10}$$

The action of these on a vector expressed in the weight vector basis is shown in Fig. 5.1. Consistent with this picture, Eq. (5.1.8) implies that

$$M^A = M^B + \alpha_{t_+}, \quad M^B = M^C + \alpha_{u_+}. \tag{5.1.11}$$

Fig. 5.1 also suggests that

$$M^A = M^C + \alpha_{v_+}. \tag{5.1.12}$$

Eqs. (5.1.11) and (5.1.12) then imply that

$$\alpha_{v^+} = \alpha_{t^+} + \alpha_{u_+}. \tag{5.1.13}$$

This is indeed correct, see Eq. (3.2.9). The defining representation can therefore be depicted as the triangle shown in Fig. 5.2.

Figure 5.2: The defining representation of su(3).

Figure 5.3: The adjoint representation of su(3).

• Adjoint representation: This is the representation in which the roots $\{t_\pm, u_\pm, v_\pm\}$ themselves are weights, along with a 2D basis for the Cartan subalgebra. Recall the commutation relations from module 3 between root vectors of different su(2)s, Eqs. (3.1.6) and (3.1.7):

$$\begin{aligned} ad_{t_+}(u_+) &= [t_+, u_+] &&= v_+, \\ ad_{t_+}(v_-) &= [t_+, v_-] &&= -u_-, \\ ad_{u_+}(v_-) &= [u_+, v_-] &&= t_-, \\ ad_{v_+}(t_-) &= [v_+, t_-] &&= -u_+, \\ ad_{v_+}(u_-) &= [v_+, u_-] &&= t_+. \end{aligned} \tag{5.1.14}$$

Previously we found that the root scalar product applied to su(3) implied the root geometry shown in Fig. 4.1 (where $\alpha_{\pm 1,2,3} \equiv \{\alpha_{t_\pm}, \alpha_{u_\pm}, \alpha_{v_\pm}\}$. Using the same color conventions as we used for $t_\pm$ (blue), $u_\pm$ (green), and $v_\pm$ (orange) "moves" in the defining representation [Fig. 5.2], Fig. 4.1 and Eq. (5.1.14) imply that the adjoint representation can be depicted as in Fig. 5.3. Note the following two points:

1. The coefficients (± 1) in Eq. (5.1.14) do not enter this geometric description of the representation. Only the "fusion rule" is important, i.e. whether the Lie bracket vanishes or not, and if not which root vector appears on the right-hand-side.

2. Given the root geometry from Fig. 4.1, Eq. (5.1.14) and Fig. 5.3 suggest that if $\{\alpha, \beta, \alpha+\beta\}$ are all roots, then $[e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}$ *and* $N_{\alpha,\beta} \neq 0$. In other words, if $\alpha+\beta$ is a root, then we can access it by acting with ad_{e_α} on β or with ad_{e_β} on α. We might say that every possible move is allowed in the root system. You will prove this in general in a homework problem.

5.1.3 The Master Weight Depth Formula (MWDF)

Consider a highest weight vector ϕ^0 with weight M^*, in a weight string generated by acting with some particular root vectors $e_{\pm\alpha}$. Assume that the weight string terminates after lowering q times with $\hat{E}_{-\alpha}$, i.e. the weights are $\{M^*, M^*-\alpha, M^*-2\alpha, \ldots, M^*-q\alpha\}$. We will show that the depth q can be related to the inner product of the weight M^* with the root α.

Then

$$\hat{H}\,\phi^0 = M^*(h)\,\phi^0, \quad h \in H, \tag{5.1.15a}$$

$$\hat{E}_\alpha\,\phi^0 = 0, \tag{5.1.15b}$$

$$\hat{H}\,\left(\hat{E}_{-\alpha}\right)^j\,\phi^0 = [M^*(h) - j\,\alpha(h)]\left(\hat{E}_{-\alpha}\right)^j\,\phi^0, \qquad 0 \le j \le q, \tag{5.1.15c}$$

$$\left(\hat{E}_{-\alpha}\right)^{q+1}\,\phi^0 = 0. \tag{5.1.15d}$$

Assume for simplicity that a normalization for the root vector operators is chosen such that

$$(e_\alpha, e_{-\alpha}) = 1 \quad \Rightarrow \quad [e_\alpha, e_{-\alpha}] = h_\alpha.$$

This choice is always possible, and is obviously independent of the root normalization. Denote

$$\phi^j \equiv \left(\hat{E}_{-\alpha}\right)^j \phi^0, \qquad \hat{E}_\alpha \phi^j \equiv r_j \phi^{j-1}. \tag{5.1.16}$$

Then

$$\begin{aligned}\hat{E}_\alpha \phi^k &= r_k \phi^{k-1} = \hat{E}_\alpha \hat{E}_{-\alpha} \phi^{k-1} = \left(\hat{E}_{-\alpha} \hat{E}_\alpha + \hat{H}_\alpha\right) \phi^{k-1} = \{r_{k-1} + [M^*(h_\alpha) - \alpha(h_\alpha)(k-1)]\} \phi^{k-1} \\ &= [r_{k-1} + \langle M^*, \alpha\rangle - (k-1)\langle\alpha,\alpha\rangle] \phi^{k-1}, \end{aligned} \tag{5.1.17}$$

$$\begin{aligned}\hat{E}_{-\alpha} \phi^k &= \phi^{k+1} = \frac{1}{r_{k+1}} \hat{E}_{-\alpha} \hat{E}_\alpha \phi^{k+1} = \frac{1}{r_{k+1}} \left(\hat{E}_\alpha \hat{E}_{-\alpha} - \hat{H}_\alpha\right) \phi^{k+1} = \frac{1}{r_{k+1}} \{r_{k+2} - [M^*(h_\alpha) - \alpha(h_\alpha)(k+1)]\} \phi^{k+1} \\ &= \frac{1}{r_{k+1}} [r_{k+2} - \langle M^*, \alpha\rangle + (k+1)\langle\alpha,\alpha\rangle] \phi^{k+1}, \end{aligned} \tag{5.1.18}$$

so that

$$\begin{aligned} r_k &= r_{k-1} + \langle M^*, \alpha\rangle - (k-1)\langle\alpha,\alpha\rangle, \\ &= r_{k+1} - \langle M^*, \alpha\rangle + k\langle\alpha,\alpha\rangle. \end{aligned} \tag{5.1.19}$$

Since $r_0 = 0$, we obtain

$$r_k = k\,\langle M^*, \alpha\rangle - \frac{k(k-1)}{2}\langle\alpha,\alpha\rangle. \tag{5.1.20}$$

On the other hand,

$$\hat{E}_\alpha \phi^{q+1} = r_{q+1} \phi^q = 0$$

implies that $r_{q+1} = 0$, so that

$$r_q = -\langle M^*, \alpha\rangle + q\langle\alpha,\alpha\rangle. \tag{5.1.21}$$

Together Eqs. (5.1.20) and (5.1.21) lead to

$$q = \frac{2\langle M^*, \alpha\rangle}{\langle\alpha,\alpha\rangle}. \tag{5.1.22}$$

A more useful formula does not require that we start with the highest weight. Assume that weight M is somewhere in a string generated by $\hat{E}_\alpha$ (raising) and $\hat{E}_{-\alpha}$ (lowering) of length $m+p+1$, such that there are p weights (m weights) above (below) M. Then $q = m + p$, so that

- For weight M in the string of length $p + 1 + m$ $(m, p \geq 0)$,
  $\{M + p\alpha, M + (p-1)\alpha, \ldots, M + \alpha, M, M - \alpha, \ldots, M - (m-1)\alpha, M - m\alpha\}$,

$$m - p = \langle M, \alpha^\vee\rangle, \qquad \underline{\text{Master Weight Depth Formula}}\ \text{(MWDF)},$$

$$\alpha^\vee \equiv \frac{2\alpha}{\langle\alpha,\alpha\rangle}, \qquad \textbf{coroot}\ \text{associated to root}\ \alpha. \tag{5.1.23}$$

Eq. (5.1.23) is one of the most important formulae we will derive in this course. We have christened it the "Master Weight Depth Formula" (MWDF); apparently it has no proper appellation.[1] Since the product of any weight M and any **coroot** $\alpha^\vee$ is always an integer, one says that the coroots are dual to the weights.

• **Exercise**: Verify that Eq. (5.1.23) holds for $M = M^A$ in the defining representation of su(3), for two cases:

1. $\alpha = \alpha_{t_+}$.
2. $\alpha = \alpha_{v_-}$.

From Fig. 5.2, what are m and p in each case? The relevant data is in Eqs. (5.1.8) and (4.3.7). In the latter, $\alpha_{1,2,3} \to \{\alpha_{t_+}, \alpha_{u_+}, \alpha_{v_+}\}$.

• **Exercise**: Verify that Eq. (5.1.23) holds for $M = \alpha_{u_-}$ in the adjoint representation of su(3), for three cases:

[1] Georgi [3] refers to it as the "master formula," while others call it the depth equation.

1. $\alpha = \alpha_{v_+}$.

2. $\alpha = \alpha_{t_-}$.

3. $\alpha = \alpha_{u_+}$.

From Fig. 5.3, what are m and p in each case? The relevant data is in Eq. (4.3.7). In the latter, $\alpha_{1,2,3} \to \{\alpha_{t_+}, \alpha_{u_+}, \alpha_{v_+}\}$.

5.2 Root geometries

In this section, we will state and/or prove various important facts about root systems (= adjoint representations). The goal is to generalize beyond su(3) to generic semi-simple Lie algebras.

5.2.1 Weyl reflections

Consider a weight chain with $q+1$ weights such that q is even. Since the number of weights is odd, there is a particular weight at the center of the chain; denote it as M_0. Obviously $m = p = q/2$ for M_0, where m (p) denotes the number of weights below (above) it. We can denote a generic weight in the chain as

$$M = M_0 + n\alpha, \qquad 0 \le |n| \le q/2, \tag{5.2.1}$$

where $n = (m-p)/2$ and there are m weights below (p weights above) M. Consider a **Weyl reflection** about the chain center,

$$M \to M_0 - n\alpha = M - 2n\alpha = M - (m-p)\alpha \tag{5.2.2}$$

so that

$$M \to M - \langle M, \alpha^\vee \rangle\, \alpha, \qquad \text{Weyl reflection } \equiv S_\alpha. \tag{5.2.3}$$

• **Exercise**: Verify that Eq. (5.2.3) also holds for a reflection about the center of an α-weight chain with $q+1$ weights, with q odd.

For each positive root α we can define an associated Weyl reflection operation S_α. With respect to the su(2) generated by $\{e_{-\alpha}, h_\alpha, e_\alpha\}$ [Eq. (5.1.1)], the adjoint representation decomposes into "parallel" strings of weights of varying lengths. Applying the Weyl reflection interchanges the weights along these strings. **We assert that a valid root system is isometric under all Weyl reflections, i.e. the "picture" of the root system is invariant.** This is illustrated for su(3) in Fig. 5.4. Note that a weight that is "orthogonal" to α is left invariant according to Eq. (5.2.3).

The product of all Weyl reflections forms the **Weyl group**, which is a subgroup of the isometry group for the root system.[2] We will examine the Weyl group in more detail when we derive Weyl's character formula in module 8.

[2] Note however that the Weyl group itself contains more than Weyl reflections, since the product of two reflections is equivalent to a rotation. The Weyl group for any root system can be analyzed with finite group theory.

Figure 5.4: Weyl reflections $S_{1,2,3}$ in the su(3) root system (adjoint representation). Reflection S_i is associated with positive root α_i [Eq. (5.2.3)], and $\alpha_{1,2,3} \leftrightarrow \alpha_{t_+,u_+,v_+}$ (c.f. Figs. 4.1 and 5.3).

5.2.2 Root chains and inner products

We assert the following facts about roots, some of which will be proven below.

I. If α is a root, then

$$\langle \alpha, \alpha \rangle \equiv |\alpha|^2 = (h_\alpha, h_\alpha) > 0. \tag{5.2.4}$$

II. If α is a root, so is $-\alpha$. These are associated to the "raising" and "lowering" root vector operators $e_{\pm\alpha}$.

III. The *only* allowed multiples of α in the root system are $\{-\alpha, 0, \alpha\}$, with the center weight belonging to the Cartan subalgebra H.

IV. Root systems come in two different varieties. Either (a) all roots have the same norm $|\alpha|^2$, or (b) each root has one of two lengths, "short" or "long." A Lie algebra in which all roots have the same length is said to be **simply-laced**.

V. If $\{\alpha, \beta, \alpha+\beta\}$ are all roots, then

$$[e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}, \qquad N_{\alpha,\beta} \neq 0. \tag{5.2.5}$$

In other words, "all possible moves are allowed in the root system."

VI. There is only one linearly independent root vector associated to each root. The **weight space** is defined as the degeneracy of a particular weight in an irreducible representation. The weight space of all weights in the adjoint is one-dimensional, except for the r-dimensional Cartan subalgebra with weight 0.

VII. In the adjoint representation, the maximum length of any weight chain is 4.

VIII. If α and β are roots, then $\langle \alpha, \beta \rangle \in \mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers. The set of roots forms an overcomplete basis for H^*, but we can choose a subset $\{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_r\}$ to use as a complete basis. Then any weight ρ can be expanded as

$$\rho = \sum_{j=1}^{r} \kappa_j^{(\rho)}\, \overline{\alpha}_j, \quad \kappa_j^{(\rho)} \in \mathbb{Q}. \tag{5.2.6}$$

We comment on or prove some of these statements (I.)–(VIII.) as follows.

(II.) **Proof**: Recall that

$$(e_\alpha, e_\beta) = 0$$

unless $\alpha + \beta = 0$ [Eq. (4.4.5)]. If $-\alpha$ is not a root, then

$$(e_\alpha, x) = 0 \;\; \forall\, x \in L.$$

However, a semi-simple Lie algebra has a non-degenerate Killing form, so that the determinant of the "overlap matrix" of all Cartan-Weyl basis elements (x_i, x_j) cannot vanish. Therefore $-\alpha$ must be a root if α is.

(III.) This establishes the generalization of the picture we had for su(3): a generic root system is a set of su(2)s $\{-\alpha_i, 0, \alpha_i\}$ that can be drawn as "spokes" passing through a central hub (the Cartan subalgebra H). The root geometry can be embedded in r Euclidean dimensions. The point group symmetry is implied by the root inner products $\{\langle \alpha_i, \alpha_j \rangle\}$. Roots can have one or two different norms $|\alpha|^2$, allowing us to distinguish "short" and "long" roots (IV.). In **simply-laced** root systems like su(3), all roots have the same length. An example with short and long roots is the G_2 root system, shown in Fig. 5.5.

(V.) This will be proven in homework, using the MWDF [Eq. (5.1.23)].

(VII.) **Proof:** Assume that a string with 5 weights exists in the adjoint representation. WLoG we can write this as

$$\{\beta - 2\alpha, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha\}.$$

Now, by assertion [(III.)]

$$2\alpha = (\beta + 2\alpha) - \beta \text{ is } \underline{\text{not}} \text{ a root, nor is}$$
$$2(\beta + \alpha) = (\beta + 2\alpha) + \beta.$$

Figure 5.5: The root system for the rank-two exceptional Lie algebra G_2.

These conditions imply that $\beta + 2\alpha$ is a string of length 1 with respect to the actions of $\hat{E}_{\pm\beta}$, i.e. $ad_{e_{\pm\beta}}(e_{\beta+2\alpha}) = 0$. Similarly

$$\begin{aligned} -2\alpha &= (\beta - 2\alpha) - \beta \text{ is } \underline{\text{not}} \text{ a root, nor is} \\ 2(\beta - \alpha) &= (\beta - 2\alpha) + \beta, \end{aligned}$$

so that $\beta - 2\alpha$ is a string of length 1 with respect to the actions of $\hat{E}_{\pm\beta}$. Eq. (5.1.23) then implies that

$$(m - p)_{\beta+2\alpha;\beta} = 0 = \langle \beta + 2\alpha, \beta^\vee \rangle, \qquad (m' - p')_{\beta-2\alpha;\beta} = 0 = \langle \beta - 2\alpha, \beta^\vee \rangle, \tag{5.2.7}$$

where the subscripts on the left-hand-sides mean that m and p are determined for the weights $\beta \pm 2\alpha$ in β-strings. Combining these equations gives

$$\frac{\langle \beta, \beta \rangle}{\langle \beta, \beta \rangle} = 0,$$

which is inconsistent [(I.)]. Q.E.D.

(VIII.) This will be proven in homework, using the MWDF [Eq. (5.1.23)].

Further information about root geometry can be extracted from the MWDF Eq. (5.1.23). Consider

$$(m - p)_{\alpha;\beta} = \langle \alpha, \beta^\vee \rangle \equiv M, \qquad (m' - p')_{\beta;\alpha} = \langle \beta, \alpha^\vee \rangle \equiv N, \tag{5.2.8}$$

where again the subscript $\{\alpha; \beta\}$ means that m and p are determined for weight α in a string generated by $e_{\pm\beta}$. In Eq. (5.2.8), M and N are some integers. Then we have

$$\frac{MN}{4} = \frac{\langle \alpha, \beta \rangle^2}{|\alpha|^2 |\beta|^2}, \qquad |\alpha|^2 \equiv \langle \alpha, \alpha \rangle, \tag{5.2.9}$$

so that

$$\frac{MN}{4} \equiv \cos^2(\theta) \leq 1, \tag{5.2.10}$$

Figure 5.6: The four rank-two root systems. The Cartan matrix $\hat{A}$ is given for each; it is defined in Sec. 5.3.1, below.

and equal to one only for $\beta = \pm\alpha$. Given that a weight string can have at most 4 weights in the adjoint representation, it implies that

$$|M|, |N| \in \{0, 1, 2, 3\} \quad \Rightarrow \quad |M||N| \in \{0, 1, 2, 3, 4, 6, 9\},$$

where the possibilities in red are disallowed because of Eq. (5.2.10). We therefore conclude that the only allowed values for $\cos\theta$ in Eq. (5.2.10) are

$$\cos\theta \in \left\{0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}, \pm 1\right\} \quad \Leftrightarrow \quad \theta \in \left\{0, \pi; \frac{\pi}{6}, \frac{5\pi}{6}; \frac{\pi}{4}, \frac{3\pi}{4}; \frac{\pi}{3}, \frac{2\pi}{3}; \frac{\pi}{2}\right\}. \tag{5.2.11}$$

It turns out that this severely restricts the possibilities for root geometries. In the next subsection, we will obtain further constraints by considering a basis for H^*. For now, we claim that the full list of root systems is exhausted by four "classical" families of algebras, and 5 particular "exceptional" algebras. The classical algebra families are

$$\begin{aligned} A_n &= \mathrm{su}(n+1), & n \geq 1, \\ B_n &= \mathrm{so}(2n+1), & n \geq 1, \\ C_n &= \mathrm{sp}(2n), & n \geq 1, \\ D_n &= \mathrm{so}(2n), & n \geq 2. \end{aligned} \tag{5.2.12}$$

Here the subscript n indicates the rank. The exceptional algebras are

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8. \tag{5.2.13}$$

The classical families $\{A_n, B_n, C_n, D_n\}$ routinely crop up as symmetries in quantum mechanics and quantum field theory. The exceptional algebras are more exotic, but have found applications e.g. in string theory and in the theory of massive integrable 1+1-D quantum field theories.

We can also organize algebras by rank r. For small r, there are numerous "accidental" equivalences between members of the classical families. For example, there is only one unique algebra at $r = 1$, $\mathrm{su}(2) = \mathrm{so}(3) = \mathrm{sp}(2)$ ($A_1 = B_1 = C_1$). The four rank-two root systems are shown in Fig. 5.6. As discussed in module 2, $\mathrm{so}(4) = \mathrm{su}(2) \times \mathrm{su}(2)$, so that $D_2 = A_1 \times A_1$. $A_2 = \mathrm{su}(3)$ is the second example. The third is $B_2 = C_2$, which corresponds to the accidental equivalence $\mathrm{so}(5) = \mathrm{sp}(4)$. As we will show later, in this case the defining representation of $\mathrm{sp}(4)$ is the **spinor representation** of $\mathrm{so}(5)$, similar to the half-integer representations of $\mathrm{su}(2) = \mathrm{so}(3)$. The final rank-2 root system is that of the exceptional algebra G_2.

Figure 5.7: Positive and simple roots for su(3). Relative to Fig. 5.3 we relabel $\{t_\pm, u_\pm, v_\pm\} \Rightarrow \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$. The simple roots are $\overline{\alpha}_{1,2}$. The positive roots are $\{\overline{\alpha}_1, \overline{\alpha}_2, \alpha_3\}$, where $\alpha_3 = \overline{\alpha}_1 + \overline{\alpha}_2 \equiv \theta$, the highest root.

5.3 Constructing the adjoint representation using the MWDF

5.3.1 Simple roots and the Cartan matrix

We can construct the adjoint representation for any Lie algebra given some minimal data about the root system. The needed information concerns the **simple roots**, which are defined as follows.

For some Lie algebra L, choose a basis in H_0^* (the space of real combinations of the roots) as a subset of the roots $\{\tilde{\alpha}_i\}$, $i \in \{1, 2, \ldots, r\}$; r is the rank. Any Cartan-Weyl basis weight (root or H-basis element) β can be expanded as [Proposition (VIII.), Sec. 5.2.2]

$$\beta = \sum_{i=1}^{r} q_i \, \tilde{\alpha}_i, \quad q_i \in \mathbb{Q}. \tag{5.3.1}$$

Let us denote the complete set of roots as Δ and the total number of roots by $|\Delta|$; $|\Delta| + r = d$, the dimension of the Lie group. We introduce "lexicographic ordering" on Δ:

- β is a **positive root** if $q_1 > 0$, *or* $q_1 = 0$ and $q_2 > 0$, *or* $q_1 = q_2 = 0$, and $q_3 > 0$, $\ldots$, *or* $q_1 = q_2 = \cdots = q_{r-1} = 0$ and $q_r > 0$. Denote the set of positive roots as Δ_+; for $\beta \in \Delta_+$, we write $\beta > 0$. By definition, all basis roots $\{\tilde{\alpha}_i\}$ are positive.
- If $\Delta_+ \equiv \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ denotes the set of positive roots, then the set of negative roots Δ_- is defined as $\{-\alpha_1, -\alpha_2, \ldots, -\alpha_p\}$. Let $|\Delta_\pm|$ denote the number of positive or negative roots, so that $|\Delta_+| = |\Delta_-| = |\Delta|/2$.
- If α and β are roots, then $\alpha > \beta$ if $\alpha - \beta$ is positive.

Note that a Cartan subalgebra basis element has $q_i = 0$ for all $i \in \{1, \ldots, r\}$.

This naturally (if arbitrarily) orders the set of all roots. In particular, there will be a **highest root** θ that satisfies $\theta > \alpha$ for all roots $\alpha \neq \theta$. The highest root cannot be "raised" by acting with any positive root vector,

$$ad_{e_\alpha}(e_\theta) = [e_\alpha, e_\theta] = 0, \quad \alpha \in \Delta_+. \tag{5.3.2}$$

A **simple root** is denoted with a bar $\overline{\rho}$, and satisfies two conditions:

1. It is positive $\overline{\rho} \in \Delta_+$,
2. It cannot be written as a sum of positive roots, $\overline{\rho} \neq \alpha + \beta$ for $\alpha, \beta \in \Delta_+$.

• su(3) example: The root system is shown in Fig. 5.7, where relative to Fig. 5.3 we have relabeled $\{t_\pm, u_\pm, v_\pm\} \Rightarrow \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$. The black line divides the algebra into positive and negative roots. Positive roots $\overline{\alpha}_1$ and $\overline{\alpha}_2$ are also simple, but $\alpha_3 = \overline{\alpha}_1 + \overline{\alpha}_2$ is not. In fact, $\alpha_3 = \theta$, the highest root.

Let Π denote the set of simple roots. Let $\overline{\alpha}, \overline{\beta} \in \Pi$ ($\overline{\alpha} \neq \overline{\beta}$). We assert the following simple root facts:

I. $\overline{\alpha} - \overline{\beta}$ and $\overline{\beta} - \overline{\alpha}$ are not roots, which implies that [Eq. (4.4.11c)]

$$a_{d_{e_{-\overline{\beta}}}}(e_{\overline{\alpha}}) = [e_{-\overline{\beta}}, e_{\overline{\alpha}}] = 0, \quad a_{d_{e_{-\overline{\alpha}}}}(e_{\overline{\beta}}) = [e_{-\overline{\alpha}}, e_{\overline{\beta}}] = 0. \tag{5.3.3}$$

In other words, simple roots are lowest weight states in mutually-generated strings.

Proof: Either $\overline{\alpha} - \overline{\beta}$ or $\overline{\beta} - \overline{\alpha}$ is positive. Then either $\overline{\alpha} = (\overline{\alpha} - \overline{\beta}) + \overline{\beta}$ or $\overline{\beta} = (\overline{\beta} - \overline{\alpha}) + \overline{\alpha}$ is not simple (contradiction).

II. $\langle \overline{\alpha}, \overline{\beta} \rangle \leq 0$, i.e. the angle between simple roots is obtuse (or 90°).

Proof: The MWDF Eq. (5.1.23) implies that

$$(m-p)_{\overline{\alpha};\overline{\beta}} = -p_{\overline{\alpha};\overline{\beta}} = \frac{2\langle \overline{\alpha}, \overline{\beta} \rangle}{\langle \overline{\beta}, \overline{\beta} \rangle} \quad \Rightarrow \quad \langle \overline{\alpha}, \overline{\beta} \rangle \leq 0, \tag{5.3.4}$$

since $\langle \overline{\beta}, \overline{\beta} \rangle > 0$ [Proposition (I.), Sec. 5.2.2].

III. The set of simple roots Π is linearly independent, and can be employed as a basis for the positive roots.

Proof: Assume that this is not true, and write $\Pi = \Pi_1 \oplus \Pi_2$. Linear dependence means

$$\sum_{\overline{\alpha}_i \in \Pi_1} a_i \, \overline{\alpha}_i - \sum_{\overline{\beta}_j \in \Pi_2} b_j \, \overline{\beta}_j = 0, \quad a_i, b_j \geq 0 \ \ \forall \ \ \{i, j\}. \tag{5.3.5}$$

Take the inner product with respect to the first term to get

$$\sum_{\overline{\alpha}_i, \overline{\alpha}_j \in \Pi_1} a_i a_j \, \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = \sum_{\overline{\alpha}_i \in \Pi_1, \overline{\beta}_j \in \Pi_2} a_i b_j \, \langle \overline{\alpha}_i, \overline{\beta}_j \rangle. \tag{5.3.6}$$

The left-hand side is a positive squared-norm, but the right-hand side is a sum of negative terms (contradiction).

IV. Every positive root $\rho > 0$ can be written as a sum of simple roots with non-negative, integer coefficients $\{\kappa_i^{(\rho)}\}$:

$$\begin{aligned} &\rho = \sum_{i=1}^{r} \kappa_i^{(\rho)} \, \overline{\alpha}_i, \qquad \rho \in \Delta_+, \ \ \kappa_i^{(\rho)} \in \mathbb{N}_0, && \text{Decomposition of a positive root in terms of simple roots with non-negative integer coefficients.} \\ &k \equiv \sum_{i=1}^{r} \kappa_i^{(\rho)} \in \{1, 2, \ldots\}, && \textbf{Level } k \text{ of the positive root } \rho. \end{aligned} \tag{5.3.7}$$

Here $\mathbb{N}_0$ denotes the set of natural numbers including zero.

Proof: Eq. (5.3.7) trivially holds if ρ itself is simple; the level $k = 1$ in this case. If ρ is not simple ($k \geq 2$), it can be expressed as a sum of two positive roots, $\rho = \lambda + \gamma$, $\{\lambda, \gamma\} \in \Delta_+$. Either of these are simple, or can be further decomposed. Continue this bipartitioning process until all components are simple roots, and this gives the explicit decomposition of ρ.

In order to construct higher level roots from simple roots, it is extremely useful to introduce the **Cartan matrix** A_{ij},

$$\boxed{A_{ij} \equiv \langle \overline{\alpha}_i, \overline{\alpha}_j^{\vee} \rangle = \frac{2\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle}{\langle \overline{\alpha}_j, \overline{\alpha}_j \rangle}, \qquad \text{Cartan matrix.}} \tag{5.3.8}$$

We emphasize that the Cartan matrix is generally *not* symmetric. The Cartan matrix encodes two different types of information: (1) the simple root geometry (angles between and relative lengths of simple roots), and (2) via the MWDF Eq. (5.1.23), the "height" of positive weight strings, starting from a simple root. In particular

Attributes of the Cartan matrix

1. The matrix elements of A_{ij} are integers. For $i \neq j$ the elements are non-positive integers

$$A_{ij} = -p_{\overline{\alpha}_i;\overline{\alpha}_j}, \tag{5.3.9}$$

where $p_{\overline{\alpha}_i;\overline{\alpha}_j} + 1$ is the length of the weight string generated by "raising" $\overline{\alpha}_i$ with the root vector $e_{\overline{\alpha}_j}$: $\{\overline{\alpha}_i, \overline{\alpha}_i + \overline{\alpha}_j, \ldots \overline{\alpha}_i + (p_{\overline{\alpha}_i;\overline{\alpha}_j})\,\overline{\alpha}_j\}$.

From Proposition (VII.) in Sec. 5.2.2, we conclude that

$$-3 \leq A_{ij} \leq 0, \quad i \neq j. \tag{5.3.10}$$

In particular, $A_{ij} = 0$ for $i \neq j$ implies that $\overline{\alpha}_i + \overline{\alpha}_j$ is not a root, so that

$$A_{ij} = 0 \quad \Leftrightarrow \quad \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = 0 \quad \Leftrightarrow \quad [e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] = 0, \quad i \neq j. \tag{5.3.11}$$

• Orthogonal simple roots correspond to a vanishing root vector commutator.

Each diagonal element $A_{ii} = 2$ (no summation on i), since this is the depth of the su(2) subalgebra generated by lowering with $e_{-\overline{\alpha}_i}$,

$$A_{ii} = m_{\overline{\alpha}_i;\overline{\alpha}_i} = 2.$$

2. The angle between simple roots $\overline{\alpha}_i$ and $\overline{\alpha}_j$ is encoded in the product

$$\frac{A_{ij}A_{ji}}{4} = \frac{\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle^2}{|\overline{\alpha}_i|^2 |\overline{\alpha}_j|^2} = \cos^2\theta \leq 1, \tag{5.3.12}$$

[c.f. Eq. (5.2.8)–(5.2.10)]. Combining Eq. (5.2.11) and Proposition (II.) above, we know that

$$\theta \in \{90°, 120°, 135°, 150°\} \text{ for } i \neq j \quad \Rightarrow \quad A_{ij} \in \{0, -1, -2, -3\}, \text{ for } i \neq j, \tag{5.3.13}$$

consistent with Eq. (5.3.10).

3. The relative squared lengths of simple roots $\overline{\alpha}_i$ and $\overline{\alpha}_j$ are encoded in the ratio

$$\frac{A_{ij}}{A_{ji}} = \frac{|\overline{\alpha}_i|^2}{|\overline{\alpha}_j|^2}. \tag{5.3.14}$$

4. $\det \hat{A} \neq 0$, since the simple roots form a basis for H_0^*.

• su(3) example: The simple roots are $\overline{\alpha}_1$ and $\overline{\alpha}_2$, see Fig. 5.7. From Eq. (4.3.7),

$$\langle \overline{\alpha}_1, \overline{\alpha}_1 \rangle = \langle \overline{\alpha}_2, \overline{\alpha}_2 \rangle = 1/3, \quad \langle \overline{\alpha}_1, \overline{\alpha}_2 \rangle = -1/6,$$

leading to the Cartan matrix

$$\hat{A}_{A_2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \tag{5.3.15}$$

Cartan's notation A_n for su(n+1) was introduced in Eq. (5.2.12).

5.3.2 Higher level roots

We can iteratively construct the complete set of positive roots from the simple ones. A generic positive root $\rho \in \Delta_+$ can be expanded as in Eq. (5.3.7). Using the MWDF Eq. (5.1.23), we rewrite this equation as

$$p_{\rho;\overline{\alpha}_i} = m_{\rho;\overline{\alpha}_i} - \sum_{j=1}^{r} \kappa_j^{(\rho)} A_{ji}, \tag{5.3.16}$$

where A_{ji} is the Cartan matrix, and $p_{\rho;\overline{\alpha}_i}$ ($m_{\rho;\overline{\alpha}_i}$) denotes the number of weights above (below) ρ in the chain generated by $e_{\pm\overline{\alpha}_i}$.

The higher root construction algorithm now proceeds as follows.

Root building algorithm

1. At level $k=1$ we have the simple roots $\{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_r\}$. Let $\rho = \overline{\alpha}_{j_0}$ (a particular simple root), so that

$$p_{\rho;\overline{\alpha}_i} = -A_{j_0\, i}, \quad (i \neq j_0). \tag{5.3.17}$$

$\Rightarrow$ The number of weights above ρ in the chain generated by raising with $e_{\overline{\alpha}_i}$ is given by $-A_{j_0\, i}$. We label each simple root $\rho = \overline{\alpha}_{j_0}$ by a <u>row</u> of the Cartan matrix

$$[A_{j_0\,1}\, A_{j_0\,2} \cdots A_{j_0\,r}] \equiv [\Lambda_1^{(1)}\, \Lambda_2^{(1)} \cdots \Lambda_r^{(1)}]. \tag{5.3.18}$$

The integers $\{\Lambda_i^{(1)}\}$ are the **Dynkin labels** for the level-1 weight $\rho = \overline{\alpha}_{j_0}$.

- <u>su(3)</u>: Using Eq. (5.3.15), the Dynkin labels of the simple roots are

$$\underset{(\overline{\alpha}_1)}{\begin{bmatrix} 2 & -1 \end{bmatrix}} \quad \underset{(\overline{\alpha}_2)}{\begin{bmatrix} -1 & 2 \end{bmatrix}} \tag{5.3.19}$$

2. To find all level $k=2$ roots, examine the Dynkin labels $[\Lambda_1 \cdots \Lambda_r]$ for each simple root $\overline{\alpha}_{j_0}$. Whenever $A_{j_0\, j_1} < 0$, $\rho = \overline{\alpha}_{j_0} + \overline{\alpha}_{j_1}$ is a valid positive root. For each valid positive root ρ at $k=2$, record the sum of the corresponding rows of the Cartan matrix,

$$\sum_{j=1}^{r} \kappa_j^{(\rho)} A_{ji} = A_{j_0\, i} + A_{j_1\, i} \quad \Leftrightarrow \quad [(A_{j_0\,1} + A_{j_1\,1}) \cdots (A_{j_0\,r} + A_{j_1\,r})] \equiv [\Lambda_1^{(2)}\, \Lambda_2^{(2)} \cdots \Lambda_r^{(2)}]. \tag{5.3.20}$$

- <u>su(3)</u>: There is one root α_3 at level 2 with Dynkin coefficients

$$\underset{(\overline{\alpha}_1 + \overline{\alpha}_2)}{\begin{bmatrix} 1 & 1 \end{bmatrix}} \tag{5.3.21}$$

Each $k=2$ root can arise in multiple ways, e.g. by raising $\overline{\alpha}_{j_0}$ with $e_{\overline{\alpha}_{j_1}}$ or by raising $\overline{\alpha}_{j_1}$ with $e_{\overline{\alpha}_{j_0}}$. For each $k=2$ root ρ, record the set of non-zero $\{m_{\rho;\overline{\alpha}_i}\}$ ($m_{\rho;\overline{\alpha}_i} \in \{0,1\}$, $i \in \{1,\ldots,r\}$), which specify whether a level-1 simple root can be accessed by acting with each $e_{-\overline{\alpha}_i}$ on ρ.

- <u>su(3)</u>:

$$\rho = \alpha_3: \quad m_{\alpha_3;\overline{\alpha}_1} = m_{\alpha_3;\overline{\alpha}_2} = 1. \tag{5.3.22}$$

3. To find all level $k=3$ roots, proceed as follows. Consider each $k=2$ root ρ. To determine if $\rho + \overline{\alpha}_i$ is a valid root, compute

$$p_{\rho;\overline{\alpha}_i} = m_{\rho;\overline{\alpha}_i} - \Lambda_i^{(2)}, \tag{5.3.23}$$

where $m_{\rho;\overline{\alpha}_i} \in \{0,1\}$ and the Dynkin coefficient $\Lambda_i^{(2)}$ were computed in step (2.), above. If $p_{\rho;\overline{\alpha}_i} > 0$, then $\rho + \overline{\alpha}_i$ is a valid root.

- <u>su(3)</u>: The only $k=2$ root is α_3. We have

$$\begin{aligned} p_{\alpha_3;\overline{\alpha}_1} &= m_{\alpha_3;\overline{\alpha}_1} - \Lambda_1^{(2)} = 1 - 1 = 0, \\ p_{\alpha_3;\overline{\alpha}_2} &= m_{\alpha_3;\overline{\alpha}_2} - \Lambda_2^{(2)} = 1 - 1 = 0. \end{aligned} \tag{5.3.24}$$

We conclude that $\alpha_3 = \theta$, the highest root in su(3). This algebra terminates at level $k=2$.

For each $k=3$ root ρ, record the set of non-zero $\{m_{\rho;\overline{\alpha}_i}\}$ ($m_{\rho;\overline{\alpha}_i} \in \{0,1,2\}$, $i \in \{1,\ldots,r\}$), which specify the number of lower-level roots below ρ accessed by acting repeatedly with each $e_{-\overline{\alpha}_i}$ on ρ.

The algorithm iterates until the highest root θ is reached.

5.3.3 G_2 example

Let us illustrate the algorithm with a less trivial rank-2 example: the exceptional Lie algebra G_2. The Cartan matrix is (as quoted in Fig. 5.6)

$$\hat{A}_{G_2} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}. \tag{5.3.25}$$

1. At $k = 1$, we have the simple roots

$$\underset{(\overline{\alpha}_1)}{\begin{bmatrix} 2 & -3 \end{bmatrix}} \quad \underset{(\overline{\alpha}_2)}{\begin{bmatrix} -1 & 2 \end{bmatrix}} \tag{5.3.26a}$$

2. At $k = 2$, the sum $\rho^{(2)} \equiv \overline{\alpha}_1 + \overline{\alpha}_2$ is a valid root since [Eq. (5.3.17)]

$$\begin{aligned} p_{\overline{\alpha}_2;\overline{\alpha}_1} &= 1, \\ p_{\overline{\alpha}_1;\overline{\alpha}_2} &= 3. \end{aligned} \tag{5.3.26b}$$

The Dynkin labels obtain from summing those of $\overline{\alpha}_{1,2}$,

$$\underset{(\overline{\alpha}_1+\overline{\alpha}_2)}{\begin{bmatrix} 1 & -1 \end{bmatrix}} \tag{5.3.26c}$$

The depths are

$$m_{\rho^{(2)};\overline{\alpha}_1} = m_{\rho^{(2)};\overline{\alpha}_2} = 1. \tag{5.3.26d}$$

3. At $k = 3$, we compute

$$\begin{aligned} p_{\rho^{(2)};\overline{\alpha}_1} &= m_{\rho^{(2)};\overline{\alpha}_1} - 1 = 0, \\ p_{\rho^{(2)};\overline{\alpha}_2} &= m_{\rho^{(2)};\overline{\alpha}_2} + 1 = 2. \end{aligned} \tag{5.3.26e}$$

$\rho^{(3)} = \rho^{(2)} + \overline{\alpha}_2$ is the only $k = 3$ root,

$$\underset{(\overline{\alpha}_1+2\overline{\alpha}_2)}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \tag{5.3.26f}$$

The depths are

$$m_{\rho^{(3)};\overline{\alpha}_1} = 0, \qquad m_{\rho^{(3)};\overline{\alpha}_2} = 2. \tag{5.3.26g}$$

4. At $k = 4$,

$$\begin{aligned} p_{\rho^{(3)};\overline{\alpha}_1} &= m_{\rho^{(3)};\overline{\alpha}_1} - 0 = 0, \\ p_{\rho^{(3)};\overline{\alpha}_2} &= m_{\rho^{(3)};\overline{\alpha}_2} - 1 = 1. \end{aligned} \tag{5.3.26h}$$

$\rho^{(4)} = \rho^{(3)} + \overline{\alpha}_2$ is the only $k = 4$ root,

$$\underset{(\overline{\alpha}_1+3\overline{\alpha}_2)}{\begin{bmatrix} -1 & 3 \end{bmatrix}} \tag{5.3.26i}$$

The depths are

$$m_{\rho^{(4)};\overline{\alpha}_1} = 0, \qquad m_{\rho^{(4)};\overline{\alpha}_2} = 3. \tag{5.3.26j}$$

5. At $k = 5$,

$$\begin{aligned} p_{\rho^{(4)};\overline{\alpha}_1} &= m_{\rho^{(4)};\overline{\alpha}_1} + 1 = 1, \\ p_{\rho^{(4)};\overline{\alpha}_2} &= m_{\rho^{(4)};\overline{\alpha}_2} - 3 = 0. \end{aligned} \tag{5.3.26k}$$

$\rho^{(5)} = \rho^{(4)} + \overline{\alpha}_1$ is the only $k = 5$ root,

$$\underset{(2\overline{\alpha}_1+3\overline{\alpha}_2)}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \tag{5.3.26l}$$

The depths are

$$m_{\rho^{(5)};\overline{\alpha}_1} = 1, \qquad m_{\rho^{(5)};\overline{\alpha}_2} = 0. \tag{5.3.26m}$$

Figure 5.8: Positive roots at successive levels k in G_2. For a rank-2 algebra such as this, we can graphically distinguish "left-moves" (raising by $e_{\overline{\alpha}_1}$) from "right-moves" (raising by $e_{\overline{\alpha}_2}$). Branching diagrams can be constructed for the adjoint representation of any Lie algebra, although planar figures become cumbersome for higher rank algebras.

6. At $k = 6$,

$$\begin{aligned} p_{\rho^{(5)};\overline{\alpha}_1} &= m_{\rho^{(5)};\overline{\alpha}_1} - 1 = 0, \\ p_{\rho^{(5)};\overline{\alpha}_2} &= m_{\rho^{(5)};\overline{\alpha}_2} - 0 = 0. \end{aligned} \tag{5.3.26n}$$

We conclude that there are no $k = 6$ roots. $\rho^{(5)} = 2\overline{\alpha}_1 + 3\overline{\alpha}_2 \equiv \theta$ is the highest root.

The results of the positive-root-finding algorithm can be indicated with a branching diagram, shown in Fig. 5.8. Since the roots are non-degenerate [Proposition (VI.), Sec. 5.2.2] and each positive root ρ has a corresponding negative root $-\rho$, we have a complete inventory of the roots for G_2. There are $|\Delta| = 12$ roots, so the group dimension $d = 12 + 2 = 14$.

To get a geometrical picture of the algebra, we examine the Cartan matrix in Eq. (5.3.25). The angle between simple roots $\overline{\alpha}_1$ and $\overline{\alpha}_2$ is encoded in the product [Eq. (5.3.12)]

$$\frac{A_{12}A_{21}}{4} = \cos^2\theta = \frac{3}{4} \quad \Rightarrow \quad \theta = 150^\circ. \tag{5.3.27}$$

Figure 5.9: The adjoint representation of the exceptional Lie algebra G_2.

$$\hat{A}_{A_1\times A_1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\hat{A}_{A_2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\hat{A}_{C_2} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$\hat{A}_{G_2} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Figure 5.10: The Dynkin diagrams for rank-2 algebras.

The relative squared lengths of simple roots $\overline{\alpha}_1$ and $\overline{\alpha}_2$ are encoded in the ratio [Eq. (5.3.14)]

$$\frac{A_{12}}{A_{21}} = \frac{|\overline{\alpha}_1|^2}{|\overline{\alpha}_2|^2} = 3, \tag{5.3.28}$$

so that $|\overline{\alpha}_1| = \sqrt{3}|\overline{\alpha}_2|$. Combining this information with the root inventory, we immediately get the picture shown in Fig. 5.9. We note that the roots come in two varieties, short and long, and that the highest root θ is a long root. For non-simply-laced algebras with short and long roots, it turns out that the highest root is always long.

5.3.4 Dynkin diagrams

The Cartan matrix A_{ij} [Eq. (5.3.8)] succinctly encodes the entire structure of a Lie algebra. One can use Dynkin coefficients $\begin{bmatrix}\Lambda_1 & \Lambda_2 & \cdots & \Lambda_r\end{bmatrix}$ to define all possible highest weight states in finite-dimensional irreducible representations. Then one employs a variant of the root-building algorithm presented above to determine all weights in the representation (module 7). Weight space multiplicities obtain via Freudenthal's formula (module 8). All commutation relations can in principle be reconstructed from A_{ij}; we will sketch how in module 7.

We close this module with an alternative way of encoding the Cartan subalgebra data for a given Lie algebra. The **Dynkin diagram** for a Lie algebra L with Cartan matrix $\hat{A}$ is constructed as follows.

- For each simple root, place an open dot $\circ$.
- Fill in (darken $\bullet$) dots corresponding to short roots.
- Connect the ith and j^{th} dots with
$$A_{ij}A_{ji} = 4\cos^2\theta \in \{0,1,2,3\}$$
straight lines. Thus orthogonal simple roots ($\langle\overline{\alpha}_i,\overline{\alpha}_j\rangle = 0$ corresponding to $[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] = 0$) are disconnected, while simple root pairs separated by 120°, 135°, or 150° are respectively connected by one, two, or three lines.

The Dynkin diagram for a simple Lie algebra is fully connected, but can be divided into two pieces by cutting one single, double, or triple "bond." The Dynkin diagram for a semi-simple Lie algebra consists of disjoint pieces, each of which corresponds to a simple Lie algebra.

The rank-2 examples are shown in Fig. 5.10.

Figure 5.11: The Dynkin diagram for a rank-3 algebra [$B_3 = \mathrm{so}(5)$].

We can also reconstruct the Cartan matrix from a Dynkin diagram. A rank-3 example is shown in Fig. 5.11. $\overline{\alpha}_3$ is apparently a short root. $A_{12}A_{21} = 1$, so that $A_{12} = A_{21} = -1$. $\overline{\alpha}_1$ and $\overline{\alpha}_3$ are not connected, so that $A_{13} \propto A_{31} \propto \langle \overline{\alpha}_1, \overline{\alpha}_3 \rangle = 0$. Finally $A_{23}A_{32} = 2$, which implies that $\{A_{23}, A_{3,2}\} \in \{-1, -2\}$. Since $\overline{\alpha}_3$ is short, we conclude that

$$\frac{A_{23}}{A_{32}} = \frac{|\overline{\alpha}_2|^2}{|\overline{\alpha}_3|^2} = 2.$$

Therefore the Cartan matrix corresponding to Fig. 5.11 is

$$\hat{A}_{B_3} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}. \tag{5.3.29}$$

This is the Cartan matrix for so(7) = B_3.

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

[2] R. Shankar, *Principles of Quantum Mechanics*, 2nd ed. (Plenum Press, New York, 1994).

[3] Howard Georgi, *Lie Algebras in Particle Physics*, 2nd ed. (Westview Press, Boulder, Colorado, 1999).

6. Cartan classification of Hamiltonians: the 10-fold way
* version 1.2 *

Matthew Foster

January 20, 2021

Contents

6.1 Random matrix theory and Cartan's classification: overview and modern applications **1**

6.2 Spinless fermions **2**
6.2.1 Hamiltonian . . . 2
6.2.2 Symmetries . . . 3
6.2.3 Examples . . . 4
6.2.3.1 Spinless superconductor without time-reversal invariance: Class D . . . 4
6.2.3.2 Spinless time-reversal invariant superconductor: Class BDI . . . 5
6.2.3.3 Metal with broken time-reversal symmetry: "Unitary" class A . . . 6
6.2.3.4 Spinless metal with time-reversal symmetry: "Orthogonal" class AI . . . 6

6.3 Spin-1/2 fermions **6**
6.3.1 Hamiltonian . . . 6
6.3.2 Examples . . . 7
6.3.2.1 Spin-1/2 superconductor with no physical symmetries: Class D . . . 7
6.3.2.2 Spin-1/2 superconductor with time-reversal symmetry and spin-orbit coupling: Class $DIII$ . . . 7
6.3.2.3 Spin-1/2 metal with time-reversal invariance and spin-orbit coupling: "Symplectic" class AII . . . 8
6.3.2.4 Spin-1/2 metal with time-reversal invariance and spin SU(2) symmetry: "Orthogonal" class AI . . . 8

6.4 The Cartan classification **10**

This material is reviewed in (e.g.) [1, 2], which detail the classification of (strong, "symmetry-protected") topological insulators and superconductors in arbitrary spatial dimensions. These references clarified and expanded the connection between equivalence classes of bulk topological band structures on one hand, and robust surface state theories with anomalous local symmetries on the other. For a different approach that gives the same results, see [3]. Like bulk topological band structures, surface state theories also form equivalence classes, since these are immune to the most detrimental effects of quenched disorder (topological protection from Anderson localization) [1, 2, 4].

6.1 Random matrix theory and Cartan's classification: overview and modern applications

There are various situations in quantum physics where it makes sense to study an *ensemble of Hamiltonians*, instead of one particular Hamiltonian.[1] The simplest version applies to a system with a finite N-dimensional Hilbert space, such that the Hamiltonian $\hat{h}$ is an $N \times N$ Hermitian matrix. In **random matrix theory**, one studies ensemble-averaged properties of such matrices, such as the density of states and correlation functions between eigenvalues.

Random matrix theory [5, 6] was originally invented to study highly excited levels in complex nuclei. The idea was that many particles interact with each other in a confined volume, leading to an effectively random Hamiltonian constrained only by key

[1] Note this is different from the usual ergodic hypothesis for thermodynamics, where one averages over *states* for a fixed Hamiltonian.

symmetries. These should be symmetries unrelated to the Poincaré group of spacetime translations, boosts, and rotations, since all of these are broken if we consider the "effective single particle" Hamiltonian seen by a typical nucleon interacting with many others. Much later the same ideas were applied to irregularly shaped quantum dots formed in solids. A quantum dot is a meso- or nanoscopic "artificial atom" with an irregular boundary that can trap a single electron, such that the corresponding classical problem would exhibit conservative chaotic dynamics (point particle moving in a chaotic billiard).

Cartan showed that $N \times N$ random Hermitian matrices can be divided into 10 symmetry classes. Each class views a particular Hamiltonian $\hat{h}$ as either (1) an element of a Lie algebra (typically in the defining representation), or (2) an element of a Lie algebra coset representation. A coset is just a Lie algebra with some subalgebra removed. Different classes are realized by imposing different discrete symmetries that hold for every member of the class (such as time-reversal symmetry). These are imposed on the otherwise random form of $\hat{h}$. The 10 classes correspond to a special set of highly symmetric manifolds, the Riemannian symmetric spaces.

In fact, this "10-fold way" classification system has recently found much wider application in condensed matter physics. It can be used to classify "dirty" quantum solids, wherein electrons scatter randomly off of different types of *quenched disorder* such as impurities, vacancies, grain boundaries, etc. It turns out that many properties of such systems are independent of the details of disorder, and depend only on the spatial dimensionality d and the symmetry class [7].[2]

The 10-fold way is also useful for classifying topological band structures [1, 2, 4]. Here one considers a d-dimensional clean (= perfectly crystalline) non-interacting or mean-field model of electrons in a solid. If the system is fully gapped (i.e. $M \in \{1, 2, 3, \ldots\}$ bands are completely filled at zero temperature, with all others completely empty), then it turns out that the system can exhibit nontrivial topology with interesting physical implications. Such systems are called (strong) topological insulators and superconductors, and are predicted to host $(d-1)$-dimensional surface states with remarkable properties [4, 8]. The latter circulate along the $(d-1)$-dimensional physical boundary of the bulk d-dimensional material sample. Why is the classification useful here? The answer is precisely because these systems are topological. It means that key properties such as the existence of surface states hold for an equivalence class of band structures, wherein you can move around the details of the energy bands in any fashion so long as you do not (1) close the energy gap or (2) break a defining symmetry.

In what follows, we will discuss only the classification scheme itself, seeing how it arises in concrete examples. We will not pursue calculations in random matrix theory, disordered and/or topological quantum phases. One of the most interesting directions is the connection between bulk topology and the physics of topological surface states in the presence of surface perturbations, such as quenched disorder and interactions. But that's another story...

6.2 Spinless fermions

6.2.1 Hamiltonian

Consider a tight-binding model of *spinless*, non-interacting fermions hopping on a lattice in d dimensions. This can be written generically through the second-quantized Hamiltonian

$$\hat{H} = \sum_{ij} \left[t_{ij} c_i^\dagger c_j + \frac{1}{2} \Delta_{ij} c_i^\dagger c_j^\dagger + \frac{1}{2} \Delta_{ij}^\dagger c_i c_j \right]. \tag{6.2.1}$$

Here c_i and $c_i^\dagger$ are fermion annihilation and creation operators satisfying canonical anticommutation relations

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}. \tag{6.2.2}$$

Let us assume that the lattice has a finite number of N sites. Then $i, j \in \{1, 2, \ldots, N\}$. The $N \times N$ "hopping" matrix t_{ij} is Hermitian ($\hat{t}^\dagger = \hat{t}$), and encodes how electrons "tunnel" from site to site on the lattice. It can be translationally invariant in a clean system, or random in a dirty one. Typically we would let t_{ij} connect only sites that are within some fixed spatial radius, e.g. up to nearest-neighbor or next-nearest-neighbor hopping, but it is not necessary to specify this here. The anomalous terms proportional to $\hat{\Delta}$ and $\hat{\Delta}^\dagger$ arise in the mean-field description of superconductors, where Δ_{ij} is the superconducting gap parameter. It can also be translationally invariant or not. A normal metal or insulator has $\hat{\Delta} = \hat{\Delta}^\dagger = 0$.

It will prove useful to rewrite Eq. (6.2.1) in a more compact way. Let us introduce the "Majorana" spinor

$$\chi \equiv \begin{bmatrix} c \\ (c^\dagger)^\mathsf{T} \end{bmatrix} \quad \Rightarrow \quad \chi_{i,\sigma} = \begin{cases} c_i, & \sigma = 1 \\ c_i^\dagger, & \sigma = 2 \end{cases} \tag{6.2.3}$$

[2] We emphasize however that for an extended system in d spatial dimensions, most physical properties do not obtain from random matrix theory. The latter is useful for calculating ensemble-averaged observables in systems that are effectively "zero dimensional," such as a nucleus. Only the 10-fold classification scheme is useful for $d > 0$; physical properties need to be computed using other tools. For systems that are Anderson *delocalized*, these tools nevertheless always have a strongly Lie algebraic character, e.g. matrix non-linear sigma models [7], affine Lie current algebras, effective replica or graded supersymmetric Hamiltonians via transfer matrix, etc.

We collect all of the annihilation and creation operators and stack them in a single $2N$-component column vector; the result is χ. It carries two different types of indices: a site index $i \in \{1, 2, \ldots, N\}$, and a "particle-hole" index $\sigma \in \{1, 2\}$, such that $\sigma = 1$ ($\sigma = 2$) corresponds to an annihilation (creation) operator. In the first equality of Eq. (6.2.3), we have suppressed the site index. In this case, we view c ($c^\dagger$) as a column (row) vector in the ordered list of lattice sites. The symbol $(c^\dagger)^\mathsf{T}$ means to take the row vector $\begin{bmatrix} c_1^\dagger & c_2^\dagger & \cdots & c_N^\dagger \end{bmatrix}$ and transpose it to get a column, so that χ is a stack of two column vectors.

We introduce a set of block $2N \times 2N$ Pauli matrices acting on particle-hole space,

$$\hat{\sigma}_1 = \begin{bmatrix} 0 & \hat{1}_N \\ \hat{1}_N & 0 \end{bmatrix}, \quad \hat{\sigma}_2 = \begin{bmatrix} 0 & -i\hat{1}_N \\ i\hat{1}_N & 0 \end{bmatrix}, \quad \hat{\sigma}_3 = \begin{bmatrix} \hat{1}_N & 0 \\ 0 & -\hat{1}_N \end{bmatrix}. \tag{6.2.4}$$

Then, we note that

$$\chi^\dagger = \begin{bmatrix} c^\dagger & c^\mathsf{T} \end{bmatrix} = \chi^\mathsf{T} \hat{\sigma}_1. \tag{6.2.5}$$

Thus $\chi^\dagger$ is not independent of χ; for this reason we refer to χ as a "Majorana" fermion field.[3] Eq. (6.2.1) can then be rewritten as

$$H = \frac{1}{2} \chi^\mathsf{T} \hat{M}_P \, \hat{h} \, \chi, \qquad \hat{M}_P = \hat{\sigma}_1, \qquad \hat{h} = \begin{bmatrix} \hat{t} & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\hat{t}^\mathsf{T} \end{bmatrix}. \tag{6.2.6}$$

The matrix $\hat{h} = \hat{h}^\dagger$ is the effective single-particle Hamiltonian; for a superconductor ($\hat{\Delta} \neq 0$), this is the **Bogoliubov-de Gennes** Hamiltonian.

6.2.2 Symmetries

The next step is to define symmetry conditions for $\hat{h}$. Since we wish to apply the classification scheme of random matrix theory, we exclude spatial symmetries like translations and rotations. Alternatively, we can imagine that these are explicitly broken by the presence of impurities or other forms of quenched disorder. What other symmetries can $\hat{h}$ possess? In fact, there is one symmetry that is "automatic:" **particle-hole** symmetry. Consider the following manipulation, wherein we anticommute the χ fields on the left and right-hand sides of H in Eq. (6.2.6):

$$\begin{aligned} H &= \frac{1}{2} \chi_i \left[\hat{M}_P\right]_{ij} \left[\hat{h}\right]_{jk} \chi_k \\ &= -\frac{1}{2} \chi_k \left[\hat{h}^\mathsf{T}\right]_{kj} \left[\hat{M}_P^\mathsf{T}\right]_{ji} \chi_i + (\text{const.}) \\ &= -\frac{1}{2} \chi^\mathsf{T} \hat{M}_P \hat{M}_P \, \hat{h}^\mathsf{T} \hat{M}_P^\mathsf{T} \chi \\ &= \frac{1}{2} \chi^\mathsf{T} \hat{M}_P \left(-\hat{M}_P \, \hat{h}^\mathsf{T} \, \hat{M}_P \right) \chi. \end{aligned} \tag{6.2.7}$$

Going from the first to second lines in this equation, we pick up an additive constant (const.) due to the non-trivial anticommutator Eq. (6.2.2); we drop this term in the following because it plays no role in the argument. To get the final equality, we have used the facts that $\hat{M}_P^2 = \hat{1}_{2N}$ and $\hat{M}_P^\mathsf{T} = \hat{M}_P$. We see that without loss of generality, we can take $\hat{h}$ to satisfy

$$-\hat{M}_P \, \hat{h}^\mathsf{T} \, \hat{M}_P = \hat{h}, \quad \hat{M}_P = \begin{cases} +\hat{M}_P^\mathsf{T}, & P^2 = +1 \\ -\hat{M}_P^\mathsf{T}, & P^2 = -1 \end{cases} \qquad \text{particle-hole symmetry } P;\ P^2 = \pm 1. \tag{6.2.8}$$

The notation $P^2 = +1$ ($P^2 = -1$) is used for the case of symmetric $\hat{M}_P$ (antisymmetric $\hat{M}_P$); these are unitarily inequivalent (cannot be interchanged by a basis transformation).[4] In our case, $\hat{M}_P = \hat{\sigma}_1 = \hat{M}_P^\mathsf{T}$ (symmetric). Physically, this "particle-hole" symmetry[5] appears because the anomalous terms in Eq. (6.2.1) satisfy (e.g.)

$$\frac{1}{2} \sum_{ij} \Delta_{ij} c_i^\dagger c_j^\dagger = -\frac{1}{2} \sum_{ij} \Delta_{ij} c_j^\dagger c_i^\dagger = \frac{1}{2} \sum_{ij} \left(-\Delta_{ji}\right) c_i^\dagger c_j^\dagger, \tag{6.2.9}$$

[3] The term "Majorana fermion" strictly speaking should apply only to a spinor field χ in a Lorentz covariant quantum field theory, wherein χ satisfies a condition as in Eq. (6.2.5). Here we are abusing high energy field theorist nomenclature to apply the terminology to a lattice model. In either case, these are just ordinary fermions satisfying a reality condition. There is something else that frequently gets called a "Majorana fermion," but is really a Majorana zero mode bound to a topological defect. These objects aren't fermions at all, since they satisfy non-abelian braiding statistics, and might be useful for topological quantum computation [9].

[4] The notational motivation for $P^2 = \pm 1$ and $T^2 = \pm 1$ obtains by considering the action of two successive particle-hole or time-reversal transformations on a fermion field operator in second quantization. The field must return to itself after two applications of the symmetry transformation, up to an overall sign. This sign determines the "square" of the symmetry operation.

[5] In high-energy physics, particle-hole symmetry is referred to as "charge conjugation" symmetry and denoted by C, in order to distinguish it from parity.

wherein we first anticommute the two creation operators, and then relabel the summation indices. As a result,

$$\hat{\Delta} = -\hat{\Delta}^{\mathsf{T}}, \tag{6.2.10}$$

i.e. the pairing gap must be an antisymmetric function of space. This is the well-known constraint that spinless fermions can only pair in odd angular momentum channels (p-wave, f-wave, etc.).

• **Exercise**: Verify that Eq. (6.2.8) implies Eq. (6.2.10). What happens if we take a more general Δ_{ij} that does not satisfy Eq. (6.2.10)?

Another important symmetry condition is **time-reversal invariance.** For our spinless lattice fermions in Eq. (6.2.1), time-reversal can be encoded as the transformation

$$c_i \to c_i, \quad c_j^\dagger \to c_j^\dagger, \quad i \to -i. \tag{6.2.11}$$

What this means is that the creation and annihilation operators are left unchanged,[6] but all complex-valued parameters in the model Eq. (6.2.1) are complex conjugated. The transformation law for χ [Eq. (6.2.3)] is also trivial,

$$\chi \to \chi, \qquad \chi^\dagger \to \chi^\dagger, \qquad i \to -i, \tag{6.2.12}$$

leading via Eq. (6.2.6) to

$$\hat{M}_T\, \hat{h}^*\, \hat{M}_T = \hat{h}, \quad \hat{M}_T = \begin{cases} +\hat{M}_T^{\mathsf{T}}, & T^2 = +1 \\ -\hat{M}_T^{\mathsf{T}}, & T^2 = -1 \end{cases} \qquad \text{time-reversal symmetry } T;\ T^2 = \pm 1. \tag{6.2.13}$$

In our case,

$$\hat{M}_T = \hat{1}_{2N} = \hat{M}_T^{\mathsf{T}}. \tag{6.2.14}$$

The antisymmetric case arises for spin-1/2 fermions, as we shall see below.

Finally, we can define a third possible symmetry, which is the product of time-reversal and particle-hole. Combining Eqs. (6.2.8) and (6.2.13) gives **chiral** (sometimes called "sublattice" [1, 2]) symmetry,

$$-\hat{M}_S\, \hat{h}\, \hat{M}_S = \hat{h}, \quad \hat{M}_S = \hat{M}_S^\dagger = \hat{M}_S^{-1} \qquad \text{chiral symmetry } S. \tag{6.2.15}$$

In our case

$$\hat{M}_S = \hat{\sigma}_1. \tag{6.2.16}$$

As suggested by Eqs. (6.2.8) and (6.2.13), for each of time-reversal and particle-hole symmetries, there are two inequivalent types. These are distinguished by the symmetry or antisymmetry of the conjugating matrices $\hat{M}_{P,T} = \pm\hat{M}_{P,T}$. The chiral symmetry S is simply present or not, and is always present if both P and T are.

The complete Cartan classification is formed by taking all possible combinations of T, P, and S: each of $\mathcal{O} \in \{T, P\}$ is absent, present with $\mathcal{O}^2 = +1$, or present with $\mathcal{O}^2 = -1$. That gives 9 possibilities. The final possibility has neither T nor P, but does possess chiral symmetry S. The classes are listed in Table 1, along with example theoretical and or experimental realizations in two- and three-dimensional condensed matter systems.

• A word of caution: Although the set of $\{T, P, S\}$ exhausts the possibilities, it is not always true that T and P correspond to the *physical* symmetries for a given model. In particular, we will see that if there is an additional continuous symmetry present, it can change the character of the *effective form* of T or P used in the classification Table 1. In fact, if there is a continuous symmetry, one should first block diagonalize $\hat{h}$ using that symmetry, and then reconsider the classification *for the irreducible blocks*. Nevertheless, we can realize all possible classes by considering spinless and spin-1/2 Hamiltonians with different physical symmetries.

6.2.3 Examples

6.2.3.1 Spinless superconductor without time-reversal invariance: Class D

Consider a system with Hamiltonian H [Eq. (6.2.1)] and no physical symmetries. You might expect this to correspond to a metal with broken time-reversal symmetry ("unitary class", see below), but this is not the case. Recall that any H cast in the form of

[6] Note that Eq. (6.2.11) encodes the transformation in second quantization on the operators and parameters. It is *not* a transformation on wavefunctions. A wavefunction ψ has to be complex-conjugated under a time-reversal transformation, but an annihilation operator c_i does not become a creation operator. There is a symmetry operation that exchanges creation and annihilation operations; this is in fact the second-quantized version of particle-hole symmetry [Eq. (6.2.8)].

the Bogoliubov-de Gennes equation in Eq. (6.2.6) is automatically invariant under P, Eq. (6.2.8) with $\hat{M}_P = \hat{\sigma}_1$. Physically, a superconductor has *less symmetry* than a metal, because it lacks U(1) electric charge symmetry; the latter is broken spontaneously. If there are no other symmetries, then we can only impose Eq. (6.2.8); explicitly,

$$-\hat{\sigma}_1\,\hat{h}^{\mathsf{T}}\,\hat{\sigma}_1 = \hat{h}, \qquad P^2 = 1 \text{ particle-hole symmetry.} \tag{6.2.17}$$

Now, $\hat{h}$ is a $2N \times 2N$ Hermitian matrix. It can therefore be expressed as a linear combination of Hermitian generators for the group U($2N$), acting on vectors in the defining $2N$ representation. We write this as

$$\hat{h} \in \mathrm{u}(2N),$$

since $\hat{h}$ is an element of the Lie algebra. Using Eqs. (2A.2.2) and (2A.2.3) from module 2A, Eq. (6.2.17) implies that

$$\hat{h} \in \mathrm{so}(2N), \qquad \text{spinless superconductor with } P^2 = +1\text{, broken time-reversal: class } D. \tag{6.2.18}$$

In other words, $\hat{h}$ is formed from a subset of u($2N$) generators, and these are precisely the generators of the subgroup so($2N$). In the Cartan classification, this is an example of class D, since D_n corresponds to the Lie algebra so($2N$) [Eq. (5.2.12)].

6.2.3.2 Spinless time-reversal invariant superconductor: Class *BDI*

We can also consider a time-reversal invariant spinless superconductor. Then we must also enforce Eq. (6.2.13),

$$\hat{h}^{\mathsf{T}} = \hat{h}, \qquad T^2 = 1 \text{ time-reversal symmetry.} \tag{6.2.19}$$

Consider the *complementary* condition

$$-\hat{h}^{\mathsf{T}} = \hat{h}. \tag{6.2.20}$$

If we were to enforce particle-hole [Eq. (6.2.17)] and the "anti-time-reversal symmetry" in Eq. (6.2.20), then we would have

$$-\hat{h}^{\mathsf{T}} = \hat{h}, \tag{6.2.21a}$$

$$\hat{\sigma}_1\,\hat{h}\,\hat{\sigma}_1 = \hat{h}. \tag{6.2.21b}$$

We can make a change of basis using $\hat{h}' \equiv \hat{U}^\dagger\,\hat{h}\,\hat{U}$, where $\hat{U} = (1/\sqrt{2})(\hat{1}_{2N} - i\hat{\sigma}_2)$, leading to

$$-\hat{h}'^{\mathsf{T}} = \hat{h}', \tag{6.2.22a}$$

$$\hat{\sigma}_3\,\hat{h}'\,\hat{\sigma}_3 = \hat{h}'. \tag{6.2.22b}$$

• **Exercise**: Verify Eq. (6.2.22).
Eq. (6.2.22) implies that we can make the σ-space decomposition

$$\hat{h}' = \begin{bmatrix} \hat{h}_N^{(1)} & 0 \\ 0 & \hat{h}_N^{(2)} \end{bmatrix}, \quad \left[\hat{h}_N^{(i)}\right]^{\mathsf{T}} = -\hat{h}_N^{(i)} \;\; (i \in \{1,2\}). \tag{6.2.23}$$

Since the $N \times N$ matrices $\hat{h}_N^{(1)}$ and $\hat{h}_N^{(2)}$ appear in separate blocks of $\hat{h}'$, they represent independent, mutually commuting algebras. Moreover, each satisfies an orthogonal condition. Enforcing particle-hole [Eq. (6.2.17)] and "anti-time-reversal symmetry" [Eq. (6.2.20)] would therefore give $\hat{h} \in \mathrm{so}(N) \times \mathrm{so}(N)$. But this isn't what we want: we want to enforce time-reveral symmetry Eq. (6.2.19). We therefore conclude that

$$\hat{h} \in \mathrm{so}(2N)\,/\,\mathrm{so}(N) \times \mathrm{so}(N), \qquad \text{spinless superconductor with } T^2 = +1,\ P^2 = +1\text{: class } BDI. \tag{6.2.24}$$

Here the quotient notation g/h for Lie algebras g and $h \in g$ means the subset (coset) of g generators that are <u>not</u> in the subalgebra h. The Cartan label for this class is *BDI*. The corresponding manifold SO($2N$)/SO(N) × SO(N) is also known as the orthogonal Grassmannian.

6.2.3.3 Metal with broken time-reversal symmetry: "Unitary" class A

Next we consider a metal, Eqs. (6.2.1) and (6.2.6) with $\hat{\Delta} = 0$. In this case $\hat{h}$ is block diagonal in σ-space. Equivalently, we can say that $\hat{h}$ commutes with the generator of U(1) electric charge rotation. An electric charge rotation is

$$c_i \to \exp(i\theta)\, c_i, \quad c_i^\dagger \to \exp(-i\theta)\, c_i^\dagger, \qquad \Rightarrow \qquad \chi \to \exp\left(i\theta\hat{\sigma}_3\right)\, \chi, \tag{6.2.25}$$

so that invariance of Eq. (6.2.6) requires that

$$\hat{\sigma}_3\, \hat{h}\, \hat{\sigma}_3 = \hat{h}, \qquad \text{electric U(1) invariance.} \tag{6.2.26}$$

Formally, we can determine the structure of $\hat{h}$ by combining the built-in particle-hole condition Eq. (6.2.17) with electric U(1) invariance Eq. (6.2.26). The latter implies the block decomposition

$$\hat{h} = \begin{bmatrix} \hat{h}_N^{(1)} & 0 \\ 0 & \hat{h}_N^{(2)} \end{bmatrix}. \tag{6.2.27}$$

By itself, this would imply $\hat{h} \in \mathrm{u}(N) \times \mathrm{u}(N)$. Imposing Eq. (6.2.17) constrains

$$-\left[\hat{h}_N^{(2)}\right]^{\mathsf{T}} = \hat{h}_N^{(1)}, \tag{6.2.28}$$

which implies that the blocks are not independent. Since there is no further constraint on $\hat{h}_N^{(1)}$, we conclude that

$$\boxed{\hat{h} \in \mathrm{u}(N), \qquad \text{metal with broken } T\text{: class } A.} \tag{6.2.29}$$

Again the Cartan notation comes from the classification of Lie algebras, since $A_n = \mathrm{su}(n+1)$.

Here we have an example of the warning given at the end of Sec. 6.2.2: the combination of the discrete P symmetry with continuous U(1) symmetry effectively gives a subblock Hamiltonian with *no* T, P, or S symmetry. Class A is also called the "unitary" class, and is one of the original three random matrix classes investigated a half-century ago by Wigner and Dyson. (The others are the "orthogonal" AI and "symplectic" AII classes, described below; see also Table 1). The additional 7 classes in Table 1 were only systematically introduced into condensed matter physics in the 90s by Altland and Zirnbauer [7].

6.2.3.4 Spinless metal with time-reversal symmetry: "Orthogonal" class AI

Finally, we consider the combination of P, T, and electric U(1) symmetry. As shown above, P and electric U(1) give $\hat{h} \in u(N)$. The "anti-time-reversal" invariance in Eq. (6.2.20) would give $\hat{h} \in \mathrm{so}(N)$. Therefore

$$\boxed{\hat{h} \in \mathrm{u}(N)\,/\,\mathrm{so}(N), \qquad \text{spinless metal with } T^2 = +1\text{: class } AI.} \tag{6.2.30}$$

This is the "orthogonal" Wigner-Dyson class, c.f. Table 1.

6.3 Spin-1/2 fermions

6.3.1 Hamiltonian

We can also consider spin-1/2 fermions. The Bogoliubov-de Gennes Hamiltonian is

$$\hat{H} = \sum_{ij} \left[t_{is\,js'} c_{i,s}^\dagger c_{j,s'} + \frac{1}{2}\Delta_{is\,js'} c_{i,s}^\dagger c_{j,s'}^\dagger + \frac{1}{2}\Delta_{is\,js'}^\dagger c_{i,s} c_{j,s'} \right]. \tag{6.3.1}$$

Here $s, s' \in \{\uparrow, \downarrow\}$ denote the spin-1/2 state along the z-axis in spin space. The hopping $\hat{t}$ and pairing $\hat{\Delta}$ amplitudes are now matrices in both position and spin space. Spin-dependent hopping arises with spin-orbit coupling, while spin-triplet pairing can break spin SU(2) symmetry in $\Delta_{is\,js'}$. Again we re-express this using a "Majorana" field that collects all degrees of freedom,

$$\chi \equiv \begin{bmatrix} c \\ i\hat{s}_2 \left(c^\dagger\right)^{\mathsf{T}} \end{bmatrix} \quad \Rightarrow \quad \chi_{i,\sigma,s} = \begin{cases} c_{i,s}, & \sigma = 1 \\ i\left[\hat{s}_2\right]_{s,s'} c_{i,s'}^\dagger, & \sigma = 2 \end{cases} \tag{6.3.2}$$

Here $\hat{s}_2$ is the antisymmetric Pauli matrix acting in spin space. Recall that $i\hat{s}_2$ serves as a "metric" for the $s = 1/2$ representation of SU(2), see Sec. 1.5 in module 1. It means that under a spin SU(2) transformation, χ defined in Eq. (6.3.2) transforms like c:

$$c \to \exp\left(-i\frac{\hat{\boldsymbol{s}}}{2}\cdot\boldsymbol{\theta}\right)c \quad \Rightarrow \quad \chi \to \exp\left(-i\frac{\hat{\boldsymbol{s}}}{2}\cdot\boldsymbol{\theta}\right)\chi. \tag{6.3.3}$$

• **Exercise**: Verify the χ-field spin SU(2) transformation in Eq. (6.3.3).
We note that

$$\chi^\dagger = \begin{bmatrix} c^\dagger & -ic^\mathsf{T}\hat{s}_2 \end{bmatrix} = \begin{bmatrix} c^\mathsf{T} & -ic^\dagger\hat{s}_2 \end{bmatrix}\hat{s}_2\hat{\sigma}_2 = \chi^\mathsf{T}\hat{s}_2\hat{\sigma}_2. \tag{6.3.4}$$

The Hamiltonian can be written as

$$H = \frac{1}{2}\chi^\mathsf{T}\hat{M}_P\,\hat{h}\,\chi, \qquad \hat{M}_P = \hat{\sigma}_2\hat{s}_2, \qquad \hat{h} = \begin{bmatrix} \hat{t} & -i\,\hat{\Delta}\,\hat{s}_2 \\ i\,\hat{s}_2\,\hat{\Delta}^\dagger & -\hat{s}_2\,\hat{t}^\mathsf{T}\,\hat{s}_2 \end{bmatrix}. \tag{6.3.5}$$

The key symmetry conditions are

1. P in Eq. (6.2.8) with

$$\hat{M}_P = \hat{\sigma}_2\hat{s}_2 = \hat{M}_P^\mathsf{T}, \qquad P^2 = +1 \text{ particle-hole "symmetry."} \tag{6.3.6}$$

2. T in Eq. (6.2.13) with

$$\hat{M}_T = \hat{s}_2, \qquad T^2 = -1 \text{ time-reversal symmetry.} \tag{6.3.7}$$

This arises due to the Kramers degeneracy of states related by time-reversal in a spin-1/2 system. Time-reversal is encoded in the transformation

$$c \to i\hat{s}_2 c, \qquad c^\dagger \to c^\dagger(-i)\hat{s}_2, \qquad i \to -i. \tag{6.3.8}$$

We can also define a chiral condition from the product of P and T, but we don't need this in the examples that follow.

6.3.2 Examples

6.3.2.1 Spin-1/2 superconductor with no physical symmetries: Class *D*

We again consider a superconductor with no physical symmetries. Apriori,

$$\hat{h} \in \mathrm{u}(4N),$$

where $4N$ arises from the direct product of N lattice, 2 particle-hole, and 2 spin degrees of freedom. Imposing the automatic particle-hole condition in Eqs. (6.2.8) and (6.3.6) gives

$$\hat{h} \in \mathrm{so}(4N), \qquad \text{spin-1/2 superconductor with } P^2 = +1\text{, broken time-reversal, and spin-orbit coupling: class } D. \tag{6.3.9}$$

This is the same class as the spinless case without physical symmetries, Eq. (6.2.18).

6.3.2.2 Spin-1/2 superconductor with time-reversal symmetry and spin-orbit coupling: Class *DIII*

If we add time-reversal symmetry to the class D superconductor, we get the additional constraint implied by Eqs. (6.2.13) and (6.3.7). The "anti-time-reversal" condition would be

$$-\hat{s}_2\,\hat{h}^\mathsf{T}\,\hat{s}_2 = \hat{h}. \tag{6.3.10}$$

Combining this with particle-hole [Eqs. (6.2.8) and (6.3.6)] would give the additional constraint

$$\hat{\sigma}_2\,\hat{h}\,\hat{\sigma}_2 = \hat{h}. \tag{6.3.11}$$

Using a basis transformation $\hat{h}' \equiv \hat{U}^\dagger\,\hat{h}\,\hat{U}$, $\hat{U} = (1/\sqrt{2})(\hat{1}_{4N} + i\hat{\sigma}_1)$, Eqs. (6.3.10) and (6.3.11) become

$$-\hat{s}_2\hat{\sigma}_1\,\hat{h}'^\mathsf{T}\,\hat{s}_2\hat{\sigma}_1 = \hat{h}', \tag{6.3.12a}$$

$$\hat{\sigma}_3\,\hat{h}'\,\hat{\sigma}_3 = \hat{h}'. \tag{6.3.12b}$$

These imply the σ-space decomposition

$$\hat{h}' = \begin{bmatrix} \hat{h}^{(1)}_{2N} & 0 \\ 0 & \hat{h}^{(2)}_{2N} \end{bmatrix}, \qquad -\hat{s}_2 \left[\hat{h}^{(2)}_{2N}\right]^{\mathsf{T}} \hat{s}_2 = \hat{h}^{(1)}_{2N}. \tag{6.3.13}$$

The latter condition implies $\hat{h}^{(2)}_{2N}$ is not independent. Since there are no constraints on $\hat{h}^{(1)}_{2N}$, we would conclude $\hat{h} \in \mathrm{u}(2N)$. For the opposite case of time-reversal and particle-hole, we get

$$\hat{h} \in \mathrm{so}(4N) \,/\, \mathrm{u}(2N), \qquad \text{spin-1/2 superconductor with } T^2 = -1,\ P^2 = +1, \text{ and spin-orbit coupling: class } \mathit{DIII}. \tag{6.3.14}$$

Thus fermion quasiparticles in a spin-1/2 superconductor with time-reversal invariance and spin-orbit coupling [no spin SU(2) symmetry] belong to class *DIII*. Class *DIII* is realized by the 3D topological superfluid ^{3}He B, see Table 1 and Refs. [1,2,4,10]. Note that the matrix structure of this class is different from that of the spinless, time-reversal invariant superconductor (Sec. 6.2.3.2), owing to the $T^2 = -1$ version of time-reversal in the case of *DIII*.

6.3.2.3 Spin-1/2 metal with time-reversal invariance and spin-orbit coupling: "Symplectic" class *AII*

Next, we consider a metal with time-reversal symmetry and no spin SU(2) symmetry. We can view this as a restriction of class *DIII*, where we impose charge U(1) invariance:

$$\hat{\sigma}_3\, \hat{h}\, \hat{\sigma}_3 = \hat{h}. \tag{6.3.15}$$

Adding P and "anti-time-reversal" conditions gives Eqs. (6.3.10) and (6.3.11). The combination of Eqs. (6.3.15) and (6.3.11) imply that $\hat{h}$ is an identity matrix in σ-space, i.e. $\hat{h} \in \mathrm{u}(2N)$. Adding Eq. (6.3.10) would restrict

$$\hat{h} \in \mathrm{sp}(2N), \tag{6.3.16}$$

where we have used Eq. (2A.3.4) in module 2A. The combination of electric U(1) [Eq. (6.3.15)] and P [Eqs. (6.2.8) and (6.3.6)] alone implies

$$\hat{h} = \begin{bmatrix} \hat{h}^{(1)}_{2N} & 0 \\ 0 & \hat{h}^{(2)}_{2N} \end{bmatrix}, \qquad -\hat{s}_2 \left[\hat{h}^{(2)}_{2N}\right]^{\mathsf{T}} \hat{s}_2 = \hat{h}^{(1)}_{2N}, \tag{6.3.17}$$

so that $\hat{h} \in \mathrm{u}(2N)$. Quotienting out Eq. (6.3.16) finally gives

$$\hat{h} \in \mathrm{u}(2N) \,/\, \mathrm{sp}(2N), \qquad \text{spin-1/2 metal with } T^2 = -1 \text{ and spin-orbit coupling: class } \mathit{AII}. \tag{6.3.18}$$

This is the "symplectic" Wigner-Dyson class, c.f. Table 1. The recently-discovered $\mathbb{Z}_2$ topological insulators [8] in 2D and 3D reside in class *AII*.

6.3.2.4 Spin-1/2 metal with time-reversal invariance and spin SU(2) symmetry: "Orthogonal" class *AI*

Finally, we consider a metal with electric U(1) invariance, time-reversal symmetry, and spin SU(2) symmetry. The physical symmetries are

$$\hat{s}_2\, \hat{h}^{\mathsf{T}}\, \hat{s}_2 = \hat{h}, \qquad T^2 = -1 \text{ time-reversal invariance}, \tag{6.3.19a}$$

$$\hat{\sigma}_3\, \hat{h}\, \hat{\sigma}_3 = \hat{h}, \qquad \text{electric U(1) invariance}, \tag{6.3.19b}$$

$$-\hat{s}_2\hat{\sigma}_2\, \hat{h}^{\mathsf{T}}\, \hat{s}_2\hat{\sigma}_2 = \hat{h}, \qquad P^2 = +1 \text{ particle-hole (automatic)}, \tag{6.3.19c}$$

$$\hat{s}_i\, \hat{h}\, \hat{s}_i = \hat{h}, \quad i \in \{1,2,3\}, \qquad \text{spin SU(2) invariance}. \tag{6.3.19d}$$

Apriori $\hat{h} \in \mathrm{u}(4N)$, but the last spin SU(2) condition implies it must be proportional to the identity in spin space. We can then replace the above conditions with

$$\hat{h}^{\mathsf{T}} = \hat{h}, \qquad \underline{\textit{effective}}\ T^2 = +1 \text{ time-reversal invariance}, \tag{6.3.20a}$$

$$\hat{\sigma}_3\, \hat{h}\, \hat{\sigma}_3 = \hat{h}, \qquad \text{electric U(1) invariance}, \tag{6.3.20b}$$

$$-\hat{\sigma}_2\, \hat{h}^{\mathsf{T}}\, \hat{\sigma}_2 = \hat{h}, \qquad \underline{\textit{effective}}\ P^2 = -1 \text{ particle-hole (automatic)}. \tag{6.3.20c}$$

Here we see an example wherein the physical T and P are replaced with different *effective* T and P, after imposing spin SU(2) symmetry. In fact the P symmetry can be eliminated by further exploiting the electric U(1) invariance; these conditions imply that

$$\hat{h} = \begin{bmatrix} \hat{h}_N^{(1)} & 0 \\ 0 & \hat{h}_N^{(2)} \end{bmatrix}, \qquad -\left[\hat{h}_N^{(2)}\right]^{\mathsf{T}} = \hat{h}_N^{(1)}, \tag{6.3.21}$$

leading to $\hat{h} \in \mathrm{u}(N)$. Finally, we remove the subalgebra associated to the "anti-time-reversal" condition $-\hat{h}^{\mathsf{T}} = \hat{h}$, leading to

$$\hat{h} \in \mathrm{u}(N) \,/\, \mathrm{so}(N), \qquad \text{spin-1/2 metal with } T^2 = -1 \text{ and spin SU(2) symmetry: class } AI. \tag{6.3.22}$$

In this case, the spinless (Sec. 6.2.3.4) and spinful classes for time-reversal invariant metals are identical, so long as spin SU(2) symmetry is imposed in the latter case. This is *not* true for superconductors, however. The generic class for spinless time-reversal invariant superconductors is class *BDI* [Eq. (6.2.24)]. A spin-1/2 time-reversal invariant superconductor with spin SU(2) symmetry instead resides in class *CI*, see Table 1.

• **Exercise**: Demonstrate that a spin SU(2) invariant superconductor without time-reversal symmetry (neglecting Zeeman effects) has

$$\hat{h} \in \mathrm{sp}(2N), \tag{6.3.23}$$

while a spin SU(2) invariant superconductor with physical time-reversal symmetry [Eqs. (6.2.13) and (6.3.7)] has

$$\hat{h} \in \mathrm{sp}(2N) \,/\, \mathrm{u}(N). \tag{6.3.24}$$

Verify that these correspond to classes C and CI in Table 1, with the prescribed *effective* versions of T and/or P.

6.4 The Cartan classification

Table 1: The "10-fold way:" the random matrix classification of Hamiltonians. The Cartan designation for the Riemannian symmetric space AI, DIII, etc. is given in the first column. Example realizations from condensed matter physics are listed in the second. These consist of two- or three-dimensional phases in non-interacting or mean field fermion models, or real (interacting) experimental systems in the same class. Here "SC" abbreviates superconductor. Real experimental systems are highlighted in blue. The defining discrete symmetries are listed next: T (time-reversal), P (particle-hole), and S (chiral or "sublattice"). As emphasized in the text, these are not necessarily the microscopic symmetries, if the system has an additional continuous symmetry. The matrix structure is listed in the penultimate column. The final column asks whether a gapped, strong topological phase (insulator or superconductor) can be realized in this class in three spatial dimensions.

Class	Example	T	P	S	Structure	Strong top. phase in 3D?
A "Unitary"	Metal with broken T due to (e.g.) magnetic impurities, or magnetic field: 2D integer quantum Hall effect	0	0	0	u(N)	no
D	SC with broken T; 2D topological p+ip SC: 5/2 state in fractional quantum Hall effect (?); superfluid ^{3}He A; 2D thermal quantum Hall effect	0	1	0	so($2N$)	no
C	SC with broken T and spin su(2) symmetry; 2D spin quantum Hall effect	0	-1	0	sp($2N$)	no
AI "Orthogonal"	Metal with T and spin su(2) symmetry: 3D P-doped Si, 2D Si MOSFET MIT	1	0	0	u(N) / so(N)	no
BDI	Orthogonal metal with chiral symmetry; spinless p-wave SC with T	1	1	1	so($2N$) / so(N)×so(N)	no
CI	SC with T and spin su(2) symmetry: 2D high-T_c cuprate SCs, 3D topological SC	1	-1	1	sp($2N$) / u(N)	yes
AII "Symplectic"	Metal with T and spin-orbit coupling; $\mathbb{Z}_2$ topological insulators in 2D, 3D	-1	0	0	u($2N$) / sp($2N$)	yes
DIII	SC with T and spin-orbit coupling: 3D topological SC, superfluid ^{3}He B	-1	1	1	so($2N$) / u(N)	yes
CII	Symplectic metal with chiral symmetry	-1	-1	1	sp($4N$) / sp($2N$)×sp($2N$)	yes
AIII	Unitary metal with chiral symmetry; SC with T and spin U(1) symmetry (e.g. p-wave); 3D topological SC	0	0	1	u($2N$) / u(N)×u(N)	yes

References

[1] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).

[2] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New J. Phys. **12**, 065010 (2010).

[3] A. Kitaev, AIP Conf. Proc. No. 1134 (AIP, New York, 2009), p. 22.

[4] For a review, see e.g. C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. **88**, 035005 (2016).

[5] M. Mehta, *Random Matrices*, 3rd ed. (Elsevier Academic Press, Amsterdam, 2004).

[6] K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, England, 1997).

[7] F. Evers and A. D. Mirlin, Rev. Mod. Phys. **80**, 1355 (2008).

[8] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).

[9] For a recent review, see e.g. J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).

[10] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, Oxford, 2003).

7. The classical and exceptional Lie algebras
* version 1.5 *

Matthew Foster

January 20, 2021

Contents

7.1 su(n): A_{n-1} **1**
7.1.1 Example: su(4) = A_3. 4

7.2 sp($2n$): C_n **5**

7.3 so($2n$): D_n **9**

7.4 so($2n+1$): B_n **13**

7.5 Classical Lie algebras to rank 4; equivalent algebras **16**

7.6 The exceptional algebras **16**

In Sec. 5.3, we demonstrated how the root system (adjoint representation) for a generic Lie algebra can be constructed from the Cartan matrix or Dynkin diagram. The latter encodes the inner products and norm ratios of the simple roots. In this module, we assemble the Cartan matrix from the simple roots for the four classical Lie algebra families A_n, B_n, C_n, and D_n. The needed data is extracted from the defining representation for each algebra. We will also state (but not derive) the results for the five exceptional algebras G_2, F_4, and $E_{6,7,8}$.

In the next module 8, we will show how the Cartan matrix can be used to obtain the full set of weights in any (finite dimensional) irreducible representation. In addition, the weights of the defining representation for the orthogonal algebras B_n [so($2n+1$)] and D_n [so($2n$)] turn out to give a useful alternative basis for H^*, since these prove to be orthonormal. We will exploit this fact later in module 10 to construct spinor representations.

The discussion here largely follows chapter VIII of [1].

7.1 su(n): A_{n-1}

Consider the space of all $n \times n$ matrices with real or complex elements. A basis for this space can be defined as follows:

$$\left(\hat{E}_{ab}\right)_{ij} = \delta_{ai}\delta_{bj} \quad \Rightarrow \quad \hat{E}_{ab} \to \begin{array}{c} \\ \\ \vdots \\ a-1 \\ a \\ a+1 \\ \vdots \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \cdots & b-1 & b & b+1 & \cdots \end{array} \\ \left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & 1 & & \\ & & & & \\ & & & & \\ & & & & \end{array}\right] \end{array} . \tag{7.1.1}$$

All unmarked entries are zero. These elements satisfy the multiplication rule

$$\hat{E}_{ab} \cdot \hat{E}_{cd} \to \left(\hat{E}_{ab}\right)_{ij} \left(\hat{E}_{cd}\right)_{jk} = \delta_{ai}\, \delta_{bj}\, \delta_{cj}\, \delta_{dk} \to \delta_{bc}\, \hat{E}_{ad}, \tag{7.1.2}$$

so that

$$[\hat{E}_{ab}, \hat{E}_{cd}] = \delta_{bc}\, \hat{E}_{ad} - \delta_{da}\, \hat{E}_{cb}. \tag{7.1.3}$$

Note that

$$\sum_c [\hat{E}_{ab}, \hat{E}_{cc}] = 0, \tag{7.1.4}$$

which implies that the identity

$$\hat{1}_n = \sum_c \hat{E}_{cc} \tag{7.1.5}$$

generates a one-dimensional abelian ideal. We want to construct root systems for generic semi-simple Lie algebras (no abelian ideals); we should therefore restrict ourselves to traceless $n \times n$ matrices, i.e. the su(n)= A_{n-1} Lie algebra. In this case, an arbitrary element of the Cartan subalgebra can be written as

$$\hat{H} = \sum_{i=1}^{n} \lambda_i\, \hat{E}_{ii}, \qquad \sum_{i=1}^{n} \lambda_i = 0. \tag{7.1.6}$$

Now

$$\widehat{a}_{dh}\,(e_{ab}) \to [\hat{H}, \hat{E}_{ab}] = \lambda_a\, \hat{E}_{ab} - \lambda_b\, \hat{E}_{ab}, \tag{7.1.7}$$

which implies that e_{ab} is a root vector:

$$[h, e_{ab}] = \alpha_{ab}(h)\, e_{ab}, \qquad \alpha_{ab}\left(\sum_i \lambda_i e_{ii}\right) = \lambda_a - \lambda_b. \tag{7.1.8}$$

We identify the positive root vectors with set of upper triangular $\{\hat{E}_{ab}\}$. For a generic Cartan subalgebra element $h \in H$ defined via Eq. (7.1.6), we claim that the set of simple roots Π can be associated to the first diagonal root vectors:

root vector	root $\alpha(h)$
$e_{12} \equiv e_{\overline{\alpha}_1}$	$\overline{\alpha}_1(h) = \lambda_1 - \lambda_2$
$e_{23} \equiv e_{\overline{\alpha}_2}$	$\overline{\alpha}_2(h) = \lambda_2 - \lambda_3$
$\vdots$	$\vdots$
$e_{n-1,n} \equiv e_{\overline{\alpha}_{n-1}}$	$\overline{\alpha}_{n-1}(h) = \lambda_{n-1} - \lambda_n$

(7.1.9)

Proof: A generic positive root is

$$\alpha_{ij}(h) = \lambda_i - \lambda_j, \quad i < j. \tag{7.1.10}$$

This can be written as

$$\alpha_{ij}(h) = (\lambda_i - \lambda_{i+1}) + (\lambda_{i+1} - \lambda_{i+2}) + (\lambda_{i+2} - \lambda_{i+3}) + \ldots + (\lambda_{j-1} - \lambda_j) = \sum_{l=i}^{j-1} \overline{\alpha}_l. \tag{7.1.11}$$

Therefore every positive root can be expressed as a linear combination of the $\{\overline{\alpha}_i\}$ with non-negative integral coefficients. Moreover, $\overline{\alpha}_i$ cannot be expressed as a linear combination of the other proposed simple roots. To see this, assume the converse, i.e. that $\overline{\alpha}_i$ is positive, but not simple. Then

$$\overline{\alpha}_i(h) = \lambda_i - \lambda_{i+1} = \sum_{j \neq i} \kappa_j^{(i)} \overline{\alpha}_j(h) = \ldots + \kappa_{i-1}^{(i)}(\lambda_{i-1} - \lambda_i) + \kappa_{i+1}^{(i)}(\lambda_{i+1} - \lambda_{i+2}) + \ldots \tag{7.1.12}$$

Since all the $\kappa_j^{(i)}$ are positive, there is no way to get λ_{i+1} with a negative coefficient (contradiction).

For the simple roots, we therefore have the associated Lie brackets

$$[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] = \delta_{j,i+1}\, e_{i,i+2} - \delta_{j,i-1}\, e_{i-1,i+1}, \tag{7.1.13a}$$

$$[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_j}] = \delta_{ij}\left(e_{ii} - e_{i+1,i+1}\right), \tag{7.1.13b}$$

$$[h, e_{\overline{\alpha}_i}] = (\lambda_i - \lambda_{i+1})\, e_{\overline{\alpha}_i}, \quad h \text{ defined as in Eq. (7.1.6).} \tag{7.1.13c}$$

Here the root vector associated to $-\overline{\alpha}_j$ is $e_{-\overline{\alpha}_j} \to \hat{E}_{j+1,j}$. Eq. (7.1.13b) is consistent with the fact that simple roots are lowest weight states in mutually generated strings of the adjoint representation [Eq. (5.3.3)].

It is useful to construct the $\{h_{\overline{\alpha}_i}\}$ ($i \in \{1, 2, \ldots, n-1\}$), i.e. the elements of the Cartan subalgebra H that represent the simple roots. Consider the Killing form between generic elements of H. We use the algorithm described in module 4 [Eq. (4.2.2)]. For

$$h \equiv \sum_i \lambda_i e_{ii}, \quad h' \equiv \sum_j \lambda'_j e_{jj}, \tag{7.1.14}$$

we have the Killing form

$$(h, h') = \sum_{\substack{a,b=1\\ a\neq b}}^{n} \Big([h, [h', e_{ab}]]\Big)_{\text{coeff. of } e_{ab}} = \sum_{a,b=1}^{n} (\lambda_a - \lambda_b)(\lambda'_a - \lambda'_b) = 2n \sum_{a=1}^{n} \lambda_a \lambda'_a, \tag{7.1.15}$$

since $\sum_b \lambda_b = 0$. Therefore, since

$$(h_{\overline{\alpha}_i}, h') = \overline{\alpha}_i(h') = \lambda'_i - \lambda'_{i+1}, \tag{7.1.16}$$

we conclude that

$$h_{\overline{\alpha}_i} = \frac{e_{ii} - e_{i+1,i+1}}{2n} \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_i)} e_{jj}, \quad \lambda_j^{(\overline{\alpha}_i)} = \frac{1}{2n}\left(\delta_{i,j} - \delta_{i+1,j}\right). \tag{7.1.17}$$

Recall that [Eq. (4.4.10)]

$$h_{\overline{\alpha}_i} = \frac{[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}]}{(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i})}. \tag{7.1.18}$$

Eqs. (7.1.13b), (7.1.17) and (7.1.18) therefore give the Killing form of the simple root vectors,

$$(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}) = 2n. \tag{7.1.19}$$

We can now compute the scalar products between simple roots, using

$$\begin{aligned}[h_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle e_{\overline{\alpha}_j} \\ &= \left(\lambda_j^{(\overline{\alpha}_i)} - \lambda_{j+1}^{(\overline{\alpha}_i)}\right) e_{\overline{\alpha}_j} = \frac{1}{2n}\left(\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1} + \delta_{i+1,j+1}\right) e_{\overline{\alpha}_j}.\end{aligned} \tag{7.1.20}$$

Therefore

$$\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = \frac{1}{2n}\left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right), \qquad \Big| \qquad \text{simple root scalar products in } A_{n-1}. \tag{7.1.21}$$

The Cartan matrix is defined as [Eqs. (5.3.8) and (5.3.9)]

$$A_{ij} = \frac{2\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle}{\langle \overline{\alpha}_j, \overline{\alpha}_j \rangle} = -p_{\overline{\alpha}_i;\overline{\alpha}_j} (i \neq j), \tag{7.1.22}$$

where $p_{\overline{\alpha}_i;\overline{\alpha}_j} \in \{0, 1, 2, 3\}$ determines the number of states above $\overline{\alpha}_i$ in the root string generated by $e_{\overline{\alpha}_j}$. For A_{n-1}, Eq. (7.1.21) implies that

$$\boxed{A_{ij} = 2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1} \quad \Rightarrow \quad \hat{A} \to \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{bmatrix}, \qquad \text{Cartan matrix for } A_{n-1}.} \tag{7.1.23}$$

The corresponding Dynkin diagram is shown in Fig. 7.1. Because all roots have the same length, su$(n) = A_{n-1}$ is simply laced.

Figure 7.1: su$(n) = A_{n-1}$.

7.1.1 Example: su(4) = A_3.

Using the root-building algorithm from Sec. 5.3.2, we construct the roots for A_3.

1. At $k = 1$, we have the simple roots

$$\underset{(\overline{\alpha}_1)}{\begin{bmatrix} 2 & -1 & 0 \end{bmatrix}} \quad \underset{(\overline{\alpha}_2)}{\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}} \quad \underset{(\overline{\alpha}_3)}{\begin{bmatrix} 0 & -1 & 2 \end{bmatrix}} \tag{7.1.24a}$$

2. At $k = 2$, the sum $\rho_1^{(2)} \equiv \overline{\alpha}_1 + \overline{\alpha}_2$ is a valid root since [Eq. (7.1.22)]

$$\begin{aligned} p_{\overline{\alpha}_1;\overline{\alpha}_2} &= 1, \\ p_{\overline{\alpha}_2;\overline{\alpha}_1} &= 1. \end{aligned} \tag{7.1.24b}$$

The sum $\rho_2^{(2)} \equiv \overline{\alpha}_2 + \overline{\alpha}_3$ is a valid root since

$$\begin{aligned} p_{\overline{\alpha}_2;\overline{\alpha}_3} &= 1, \\ p_{\overline{\alpha}_3;\overline{\alpha}_2} &= 1. \end{aligned} \tag{7.1.24c}$$

The Dynkin labels for these level 2 roots are

$$\underset{\rho_1^{(2)}}{\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}} \quad \underset{\rho_2^{(2)}}{\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}} \tag{7.1.24d}$$

The non-zero depths are

$$m_{\rho_1^{(2)};\overline{\alpha}_1} = m_{\rho_1^{(2)};\overline{\alpha}_2} = 1, \quad m_{\rho_2^{(2)};\overline{\alpha}_2} = m_{\rho_1^{(2)};\overline{\alpha}_3} = 1. \tag{7.1.24e}$$

3. At $k = 3$, we compute

$$\begin{aligned} p_{\rho_1^{(2)};\overline{\alpha}_1} &= m_{\rho_1^{(2)};\overline{\alpha}_1} - 1 = 0, \\ p_{\rho_1^{(2)};\overline{\alpha}_2} &= m_{\rho_1^{(2)};\overline{\alpha}_2} - 1 = 0, \\ p_{\rho_1^{(2)};\overline{\alpha}_3} &= m_{\rho_1^{(2)};\overline{\alpha}_3} + 1 = 1, \\ p_{\rho_2^{(2)};\overline{\alpha}_1} &= m_{\rho_1^{(2)};\overline{\alpha}_1} + 1 = 1, \\ p_{\rho_2^{(2)};\overline{\alpha}_2} &= m_{\rho_1^{(2)};\overline{\alpha}_2} - 1 = 0, \\ p_{\rho_2^{(2)};\overline{\alpha}_3} &= m_{\rho_1^{(2)};\overline{\alpha}_3} - 1 = 0. \end{aligned} \tag{7.1.24f}$$

Thus $\rho^{(3)} \equiv \rho_1^{(2)} + \overline{\alpha}_3 = \rho_2^{(2)} + \overline{\alpha}_1 = \overline{\alpha}_1 + \overline{\alpha}_2 + \overline{\alpha}_3$ is the only $k = 3$ root,

$$\underset{\rho^{(3)}}{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}} \tag{7.1.24g}$$

The depths are

$$m_{\rho^{(3)};\overline{\alpha}_1} = 1, \qquad m_{\rho^{(3)};\overline{\alpha}_2} = 0, \qquad m_{\rho^{(3)};\overline{\alpha}_3} = 1. \tag{7.1.24h}$$

Figure 7.2: Root tree for su(4) = A_3. Roots are specified by their Dynkin labels. The level $k = 1$ simple roots appear at the top; linear combinations of these give the other positive roots. The labels on diagonal lines indicate the simple root that gives the level $(k+1)^{\text{th}}$ root from the kth one.

4. At $k = 4$,

$$\begin{aligned} p_{\rho^{(3)};\overline{\alpha}_1} &= m_{\rho^{(3)};\overline{\alpha}_1} - 1 = 0, \\ p_{\rho^{(3)};\overline{\alpha}_2} &= m_{\rho^{(3)};\overline{\alpha}_2} - 0 = 0, p_{\rho^{(3)};\overline{\alpha}_3} = \quad m_{\rho^{(3)};\overline{\alpha}_3} - 1 = 0. \end{aligned} \tag{7.1.24i}$$

We conclude that there are no level $k = 4$ roots. $\rho^{(3)} = \overline{\alpha}_1 + \overline{\alpha}_2 + \overline{\alpha}_3 \equiv \theta$ is the highest root.

The results of the algorithm can be indicated with the branching diagram in Fig. 7.2. Since the roots are non-degenerate [Proposition (VI.), Sec. 5.2.2] and each positive root ρ has a corresponding negative root $-\rho$, we have a complete inventory for the set of roots Δ for A_3. There are $|\Delta| = 12$ roots, so the group dimension $d = 12 + 3 = 15$, as expected for su(4).

To get a geometrical picture of the algebra, we examine the Cartan matrix in Eq. (7.1.23). The angle between simple roots $\{\overline{\alpha}_1, \overline{\alpha}_2\}$ and between $\{\overline{\alpha}_2, \overline{\alpha}_3\}$ is 120°, while $\overline{\alpha}_1$ and $\overline{\alpha}_3$ are orthogonal. Combined with the fact that the algebra is simply laced and the inventory of roots (Fig. 7.2), one finds that the roots correspond to the twelve vertices of a certain quasiregular polyhedron (a cuboctahedron). See Fig. 7.3.

7.2 sp($2n$): C_n

Next we consider C_n, the Lie algebra associated to symplectic Sp($2n$) transformations. From module 2A, Eqs. (2A.3.1) and (2A.3.3), the defining conditions for a symplectic transformation $\hat{U}$ on a $2n$-component complex vector are

$$\hat{U}^\dagger \hat{U} = \hat{1}_{2n}, \tag{7.2.1a}$$

$$\hat{U}^\mathsf{T} \hat{\epsilon} \hat{U} = \hat{\epsilon}, \tag{7.2.1b}$$

Figure 7.3: Root geometry for su(4) = A_3. The twelve roots correspond to the vertices of the green polyhedron (a cuboctahedron). The vertices of the embedded red (blue) tetrahedron correspond to the four weights of the defining "4" (conjugate "$\overline{4}$") representation, as demonstrated module 8. The box arrays on the right are the corresponding Young tableaux, also discussed in module 8.

where the $2n \times 2n$ matrix $\hat{\epsilon}$ is a block rank-2 Levi-Civita symbol,

$$\hat{\epsilon} \to \begin{bmatrix} 0 & \hat{1}_n \\ -\hat{1}_n & 0 \end{bmatrix}. \tag{7.2.2}$$

If we express $\hat{U}$ as the exponentiation of an (antihermitian) $2n \times 2n$ matrix $\hat{Y}$

$$\hat{U} = \exp(\hat{Y}),$$

then the symplectic condition translates to

$$\hat{Y}^{\mathsf{T}} = \hat{\epsilon}\,\hat{Y}\,\hat{\epsilon}. \tag{7.2.3}$$

As with $\hat{\epsilon}$, we decompose this into $n \times n$ blocks,

$$\hat{Y} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_3 & \hat{Y}_4 \end{bmatrix}. \tag{7.2.4}$$

Eq. (7.2.3) then implies that

$$\begin{aligned} \hat{Y}_4 &= -\hat{Y}_1^{\mathsf{T}}, \\ \hat{Y}_2 &= \hat{Y}_2^{\mathsf{T}}, \\ \hat{Y}_3 &= \hat{Y}_3^{\mathsf{T}}. \end{aligned} \tag{7.2.5}$$

Let $1 \le j, k \le n$. We can define the following basis elements for $\hat{Y}$,

$$\hat{E}^1_{jk} \equiv \hat{E}_{jk} - \hat{E}_{k+n,j+n}, \tag{7.2.6a}$$

$$\hat{E}^2_{jk} \equiv \hat{E}_{j,k+n} + \hat{E}_{k,j+n} = \hat{E}^2_{(jk)}, \tag{7.2.6b}$$

$$\hat{E}^3_{jk} \equiv \hat{E}_{j+n,k} + \hat{E}_{k+n,j} = \hat{E}^3_{(jk)}. \tag{7.2.6c}$$

The parentheses here denote symmetrization [Eq. (1.4.2)]. The total number of such elements is

$$n^2 + n(n+1) = n(2n+1),$$

which is the correct group dimension [Eq. (2A.3.5)].

Explicitly for $n = 5$ [sp(10)], we have (e.g.)

$$\hat{E}^1_{34} \to \left[\begin{array}{ccccc|ccccc} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right], \tag{7.2.7a}$$

$$\hat{E}^2_{34} \to \left[\begin{array}{ccccc|ccccc} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right], \tag{7.2.7b}$$

$$\hat{E}^3_{25} \to \left[\begin{array}{ccccc|ccccc} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0 \end{array}\right]. \tag{7.2.7c}$$

Using Eq. (7.1.3), one can verify the following Lie brackets implied by the corresponding commutation relations:

$$[e^1_{jk}, e^1_{lp}] = \delta_{kl}\, e^1_{jp} - \delta_{jp}\, e^1_{lk}, \tag{7.2.8a}$$
$$[e^1_{jk}, e^2_{lp}] = \delta_{kl}\, e^2_{jp} + \delta_{pk}\, e^2_{jl}, \tag{7.2.8b}$$
$$[e^1_{jk}, e^3_{lp}] = -\,\delta_{jp}\, e^3_{kl} - \delta_{jl}\, e^3_{kp}, \tag{7.2.8c}$$
$$[e^2_{jk}, e^2_{lp}] = 0, \tag{7.2.8d}$$
$$[e^3_{jk}, e^3_{lp}] = 0, \tag{7.2.8e}$$
$$[e^2_{jk}, e^3_{lp}] = \delta_{kl}\, e^1_{jp} + \delta_{jp}\, e^1_{kl} + \delta_{pk}\, e^1_{jl} + \delta_{jl}\, e^1_{kp}. \tag{7.2.8f}$$

• **Exercise**: Verify all of the Lie brackets in Eq. (7.2.8).

A basis for the n-dimensional Cartan subalgebra H is given by the diagonal traceless elements $\{e^1_{jj}\}$, $1 \le j \le n$; a generic element $h \in H$ is

$$h = \sum_{j=1}^{n} \lambda_j\, e^1_{jj}. \tag{7.2.9}$$

Then

$$[h, e^1_{jk}] = (\lambda_j - \lambda_k)\, e^1_{jk}, \tag{7.2.10a}$$
$$[h, e^2_{jk}] = (\lambda_j + \lambda_k)\, e^2_{jk}, \tag{7.2.10b}$$
$$[h, e^3_{jk}] = -\,(\lambda_j + \lambda_k)\, e^3_{jk}, \tag{7.2.10c}$$

so that the collection of $\{e^{1,2,3}_{jk}\}$ excluding diagonal $\{e^1_{jj}\}$ is the set of root vectors.

The simple roots can be associated to the root vectors

root vector	root $\alpha(h)$
$e^1_{12} \equiv e_{\overline{\alpha}_1}$	$\overline{\alpha}_1(h) = \lambda_1 - \lambda_2$
$e^1_{23} \equiv e_{\overline{\alpha}_2}$	$\overline{\alpha}_2(h) = \lambda_2 - \lambda_3$
$\vdots$	$\vdots$
$e^1_{n-1,n} \equiv e_{\overline{\alpha}_{n-1}}$	$\overline{\alpha}_{n-1}(h) = \lambda_{n-1} - \lambda_n$
$e^2_{n,n} \equiv e_{\overline{\alpha}_n}$	$\overline{\alpha}_n(h) = 2\lambda_n.$

(7.2.11)

By definition, we take $\overline{\alpha}_1 > \overline{\alpha}_2 > \ldots > \overline{\alpha}_n$ (lexicographic ordering). Then positive roots are associated to root vectors $\{e^1_{jk}\}$ with $j < k$, and to $\{e^2_{jk}\}$. Negative root vectors are $\{e^1_{jk}\}$ $(j > k)$ and $\{e^3_{jk}\}$. The positive root associated to e^1_{jk} $(j < k)$ is expressed in terms of simple roots identically as in Eq. (7.1.11). For e^2_{ij}, the decomposition is (taking $j \ge i$ WLoG)

$$\lambda_i + \lambda_j = (\lambda_i - \lambda_j) + 2(\lambda_j - \lambda_n) + 2\lambda_n = \sum_{l=i}^{j-1} \overline{\alpha}_l + 2\sum_{l=j}^{n-1} \overline{\alpha}_l + \overline{\alpha}_n. \tag{7.2.12}$$

The commutation relations between the simple root vectors are given by

$$
\begin{aligned}
[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \delta_{j,i+1}\, e^1_{i,i+2} - \delta_{j,i-1}\, e^1_{i-1,i+1}, && 1 \le i,j \le n-1, && (7.2.13a)\\
[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_n}] &= 2\delta_{i,n-1}\, e^2_{n-1,n}, && 1 \le i \le n-1, && (7.2.13b)\\
[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_j}] &= \delta_{ij}\, \left(e^1_{i,i} - e^1_{i+1,i+1}\right), && 1 \le i,j \le n-1, && (7.2.13c)\\
[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_n}] &= 0, && 1 \le i \le n-1, && (7.2.13d)\\
[e_{\overline{\alpha}_n}, e_{-\overline{\alpha}_n}] &= 4\, e^1_{nn}. && && (7.2.13e)
\end{aligned}
$$

The Killing form between Cartan subalgebra elements

$$
h \equiv \sum_i \lambda_i e^1_{ii}, \quad h' \equiv \sum_j \lambda'_j e^1_{jj}, \tag{7.2.14}
$$

is given by

$$
\begin{aligned}
(h,h') &= \sum_{\substack{p,q=1\\ p\neq q}}^{n} \Big([h,[h',e^1_{pq}]] \Big)_{\text{coeff. of } e^1_{pq}} + \sum_{p\ge q=1}^{n} \Big([h,[h',e^2_{pq}]] \Big)_{\text{coeff. of } e^2_{pq}} + \sum_{p\ge q=1}^{n} \Big([h,[h',e^3_{pq}]] \Big)_{\text{coeff. of } e^3_{pq}}\\
&= \sum_{p,q=1}^{n} (\lambda_p - \lambda_q)(\lambda'_p - \lambda'_q) + 2 \sum_{p\ge q=1}^{n} (\lambda_p + \lambda_q)(\lambda'_p + \lambda'_q)\\
&= \sum_{p,q=1}^{n} (\lambda_p - \lambda_q)(\lambda'_p - \lambda'_q) + \sum_{p,q=1}^{n} (\lambda_p + \lambda_q)(\lambda'_p + \lambda'_q) + 4\sum_{p=1}^{n} \lambda_p \lambda'_p\\
&= 4(n+1) \sum_{p=1}^{n} \lambda_p \lambda'_p.
\end{aligned} \tag{7.2.15}
$$

Now since

$$
(h_{\overline{\alpha}_i}, h') = \overline{\alpha}_i(h') = \begin{cases} \lambda'_i - \lambda'_{i+1}, & 1 \le i \le n-1,\\ 2\lambda'_n, & i = n, \end{cases} \tag{7.2.16}
$$

we conclude that

$$
\begin{aligned}
h_{\overline{\alpha}_i} &= \frac{1}{4(n+1)} \left(e^1_{ii} - e^1_{i+1,i+1}\right) \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_i)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_i)} = \frac{1}{4(n+1)} \left(\delta_{i,j} - \delta_{i+1,j}\right), \qquad 1 \le i \le n-1,\\
h_{\overline{\alpha}_n} &= \frac{1}{2(n+1)} e^1_{nn} \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_n)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_n)} = \frac{1}{2(n+1)} \delta_{j,n}.
\end{aligned} \tag{7.2.17}
$$

Eqs. (7.2.13c), (7.2.13e), (7.2.17) and (7.1.18) therefore give the Killing forms of the simple root vectors,

$$
\begin{aligned}
(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}) &= 4(n+1), \qquad 1 \le i \le n-1,\\
(e_{\overline{\alpha}_n}, e_{-\overline{\alpha}_n}) &= 8(n+1).
\end{aligned} \tag{7.2.18}
$$

We can now compute the scalar products between simple roots, using

$$
\begin{aligned}
[h_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle e_{\overline{\alpha}_j}\\
&= \left(\lambda_j^{(\overline{\alpha}_i)} - \lambda_{j+1}^{(\overline{\alpha}_i)}\right) e_{\overline{\alpha}_j} = \frac{1}{4(n+1)} \left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right) e_{\overline{\alpha}_j}, \qquad 1 \le i,j \le n-1,\\
[h_{\overline{\alpha}_n}, e_{\overline{\alpha}_j}] &= \langle \overline{\alpha}_n, \overline{\alpha}_j \rangle e_{\overline{\alpha}_j}\\
&= \left(\lambda_j^{(\overline{\alpha}_n)} - \lambda_{j+1}^{(\overline{\alpha}_n)}\right) e_{\overline{\alpha}_j} = -\frac{1}{2(n+1)} \delta_{j+1,n}\, e_{\overline{\alpha}_j}, \qquad 1 \le j \le n-1,\\
[h_{\overline{\alpha}_n}, e_{\overline{\alpha}_n}] &= \langle \overline{\alpha}_n, \overline{\alpha}_n \rangle e_{\overline{\alpha}_n}\\
&= \frac{1}{(n+1)} e_{\overline{\alpha}_n}.
\end{aligned} \tag{7.2.19}
$$

Figure 7.4: $\mathrm{sp}(2n) = C_n$.

Therefore

$$\begin{aligned}
\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle &= \frac{1}{4(n+1)} \left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right), \quad 1 \le i,j \le n-1, \\
\langle \overline{\alpha}_i, \overline{\alpha}_n \rangle &= -\frac{1}{2(n+1)} \delta_{i,n-1}, \qquad\qquad 1 \le i \le n-1, \\
\langle \overline{\alpha}_n, \overline{\alpha}_n \rangle &= \frac{1}{(n+1)},
\end{aligned} \qquad \text{simple root scalar products in } C_n. \tag{7.2.20}$$

The last equation implies that $\overline{\alpha}_n$ is a long root, with $|\overline{\alpha}_n|^2 = 2|\overline{\alpha}_j|^2$, $1 \le j \le n-1$. C_n is not simply laced.

The corresponding Cartan matrix is given by [via Eq. (7.1.22)]

$$\hat{A} \to \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -2 & 2 \end{bmatrix}, \qquad \text{Cartan matrix for } C_n. \tag{7.2.21}$$

The Dynkin diagram is shown in Fig. 7.4.

7.3 so$(2n)$: D_n

Next we consider D_n, the Lie algebra associated to special orthogonal SO$(2n)$ transformations on even-dimensional vectors. From Eq. (2A.2.2) in module 2A, the defining condition for an orthogonal transformation $\hat{U}$ on a $2n$-component vector is

$$\hat{U}^{\mathsf{T}} \hat{U} = \hat{1}_{2n}. \tag{7.3.1}$$

If we express $\hat{U}$ as the exponentiation of an (antihermitian) $2n \times 2n$ matrix $\hat{Y}$

$$\hat{U} = \exp(\hat{Y}),$$

then we have the antisymmetry condition

$$-\hat{Y}^{\mathsf{T}} = \hat{Y}. \tag{7.3.2}$$

Since we would like to identify Cartan subalgebra elements with diagonal matrices, Eq. (7.3.2) is inconvenient. Instead, we make a unitary (but *not* orthogonal) basis transformation,

$$\hat{U}' \equiv \hat{V}^\dagger \, \hat{U} \, \hat{V}, \qquad \hat{V}^\dagger \hat{V} = \hat{1}_{2n}. \tag{7.3.3}$$

Then

$$\begin{aligned}
&\hat{U}'^{\mathsf{T}} \hat{M} \hat{U}' = \hat{M}, \\
&\hat{M} \equiv \hat{V}^{\mathsf{T}} \hat{V} = \hat{M}^{\mathsf{T}}, \quad \hat{M}^\dagger \hat{M} = \hat{1}_{2n}.
\end{aligned} \tag{7.3.4}$$

For $\hat{U}' = \exp(\hat{Y}')$,

$$-\hat{M}^\dagger \, \hat{Y}'^{\mathsf{T}} \, \hat{M} = \hat{Y}'. \tag{7.3.5}$$

We choose $\hat{V}$ to have the block form

$$\hat{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} i\hat{1}_n & -i\hat{1}_n \\ -\hat{1}_n & -\hat{1}_n \end{bmatrix}, \quad \Rightarrow \quad \hat{M} = \hat{M}^\dagger = \begin{bmatrix} 0 & \hat{1}_n \\ \hat{1}_n & 0 \end{bmatrix}. \tag{7.3.6}$$

As with $\hat{M}$, we decompose $\hat{Y}'$ into $n \times n$ blocks,

$$\hat{Y}' = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_3 & \hat{Y}_4 \end{bmatrix}. \tag{7.3.7}$$

Eq. (7.3.5) then implies that

$$\begin{aligned} \hat{Y}_4 &= -\hat{Y}_1^{\mathsf{T}}, \\ \hat{Y}_2 &= -\hat{Y}_2^{\mathsf{T}}, \\ \hat{Y}_3 &= -\hat{Y}_3^{\mathsf{T}}. \end{aligned} \tag{7.3.8}$$

This is almost the same as the symplectic case [Eq. (7.2.5)], except the off-diagonal blocks are now antisymmetric.

Let $1 \leq j, k \leq n$. We can define the following basis elements for $\hat{Y}$,

$$\hat{E}^1_{jk} \equiv \hat{E}_{jk} - \hat{E}_{k+n,j+n}, \tag{7.3.9a}$$

$$\hat{E}^2_{jk} \equiv \hat{E}_{j,k+n} - \hat{E}_{k,j+n} = \hat{E}^2_{[jk]}, \tag{7.3.9b}$$

$$\hat{E}^3_{jk} \equiv \hat{E}_{j+n,k} - \hat{E}_{k+n,j} = \hat{E}^3_{[jk]}. \tag{7.3.9c}$$

The parentheses here denote antisymmetrization [Eq. (1.4.2)]. The total number of such elements is

$$n^2 + n(n-1) = n(2n-1),$$

which is the correct group dimension [Eq. (7.3.2)].

Relative to the $n = 5$ symplectic example in Eq. (7.2.7),

$$\hat{E}^2_{34} \to \left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{7.3.10a}$$

$$\hat{E}^3_{25} \to \left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{7.3.10b}$$

Using Eq. (7.1.3), the Lie brackets are given by [c.f. Eq. (7.2.8)]

$$[e^1_{jk}, e^1_{lp}] = \delta_{kl}\, e^1_{jp} - \delta_{jp}\, e^1_{lk}, \tag{7.3.11a}$$
$$[e^1_{jk}, e^2_{lp}] = \delta_{kl}\, e^2_{jp} - \delta_{pk}\, e^2_{jl}, \tag{7.3.11b}$$
$$[e^1_{jk}, e^3_{lp}] = \delta_{jp}\, e^3_{kl} - \delta_{jl}\, e^3_{kp}, \tag{7.3.11c}$$
$$[e^2_{jk}, e^2_{lp}] = 0, \tag{7.3.11d}$$
$$[e^3_{jk}, e^3_{lp}] = 0, \tag{7.3.11e}$$
$$[e^2_{jk}, e^3_{lp}] = \delta_{kl}\, e^1_{jp} + \delta_{jp}\, e^1_{kl} - \delta_{pk}\, e^1_{jl} - \delta_{jl}\, e^1_{kp}. \tag{7.3.11f}$$

• **Exercise**: Verify the Lie brackets in Eq. (7.3.11).

A generic Cartan subalgebra H element is given by

$$h = \sum_{j=1}^{n} \lambda_j\, e^1_{jj}. \tag{7.3.12}$$

Then

$$[h, e^1_{jk}] = (\lambda_j - \lambda_k)\, e^1_{jk}, \tag{7.3.13a}$$
$$[h, e^2_{jk}] = (\lambda_j + \lambda_k)\, e^2_{jk}, \tag{7.3.13b}$$
$$[h, e^3_{jk}] = -\,(\lambda_j + \lambda_k)\, e^3_{jk}. \tag{7.3.13c}$$

The simple roots can be associated to the root vectors

root vector	root $\alpha(h)$
$e^1_{12} \equiv e_{\overline{\alpha}_1}$	$\overline{\alpha}_1(h) = \lambda_1 - \lambda_2$
$e^1_{23} \equiv e_{\overline{\alpha}_2}$	$\overline{\alpha}_2(h) = \lambda_2 - \lambda_3$
$\vdots$	$\vdots$
$e^1_{n-1,n} \equiv e_{\overline{\alpha}_{n-1}}$	$\overline{\alpha}_{n-1}(h) = \lambda_{n-1} - \lambda_n$
$e^2_{n-1,n} \equiv e_{\overline{\alpha}_n}$	$\overline{\alpha}_n(h) = \lambda_{n-1} + \lambda_n$.

(7.3.14)

By definition, we take $\overline{\alpha}_1 > \overline{\alpha}_2 > \ldots > \overline{\alpha}_n$ (lexicographic ordering). Then positive roots are associated to root vectors $\{e^1_{jk}\}$ with $j < k$, and to $\{e^2_{jk}\}$. Negative root vectors are $\{e^1_{jk}\}$ $(j > k)$ and $\{e^3_{jk}\}$. The positive root associated to e^1_{jk} $(j < k)$ is expressed in terms of simple roots identically as in Eq. (7.1.11). For e^2_{ij}, the decomposition is (taking $j > i$ WLoG)

$$\lambda_i + \lambda_j = (\lambda_i - \lambda_j) + 2(\lambda_j - \lambda_{n-1}) + (\lambda_{n-1} - \lambda_n) + (\lambda_{n-1} + \lambda_n) = \sum_{l=i}^{j-1} \overline{\alpha}_l + 2\sum_{l=j}^{n-2} \overline{\alpha}_l + \overline{\alpha}_{n-1} + \overline{\alpha}_n. \tag{7.3.15}$$

The commutation relations between the simple root vectors are given by

$$[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] = \delta_{j,i+1}\, e^1_{i,i+2} - \delta_{j,i-1}\, e^1_{i-1,i+1}, \qquad 1 \le i, j \le n-1, \tag{7.3.16a}$$
$$[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_n}] = \delta_{i,n-2}\, e^2_{n-2,n}, \qquad 1 \le i \le n-1, \tag{7.3.16b}$$
$$[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_j}] = \delta_{ij}\, \left(e^1_{i,i} - e^1_{i+1,i+1}\right), \qquad 1 \le i, j \le n-1, \tag{7.3.16c}$$
$$[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_n}] = 0, \qquad 1 \le i \le n-1, \tag{7.3.16d}$$
$$[e_{\overline{\alpha}_n}, e_{-\overline{\alpha}_n}] = e^1_{n-1,n-1} + e^1_{n,n}. \tag{7.3.16e}$$

The Killing form between Cartan subalgebra elements

$$h \equiv \sum_i \lambda_i e^1_{ii}, \quad h' \equiv \sum_j \lambda'_j e^1_{jj}, \tag{7.3.17}$$

is given by

$$(h, h') = 4(n-1) \sum_{p=1}^{n} \lambda_p \lambda'_p. \tag{7.3.18}$$

• **Exercise**: Verify Eq. (7.3.18).

Now since

$$(h_{\overline{\alpha}_i}, h') = \overline{\alpha}_i(h') = \begin{cases} \lambda'_i - \lambda'_{i+1}, & 1 \le i \le n-1, \\ \lambda'_{n-1} + \lambda'_n, & i = n, \end{cases} \tag{7.3.19}$$

we conclude that

$$\begin{aligned} h_{\overline{\alpha}_i} &= \frac{1}{4(n-1)}\left(e^1_{ii} - e^1_{i+1,i+1}\right) \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_i)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_i)} = \frac{1}{4(n-1)}\left(\delta_{i,j} - \delta_{i+1,j}\right), \qquad 1 \le i \le n-1, \\ h_{\overline{\alpha}_n} &= \frac{1}{4(n-1)}\left(e^1_{n-1,n-1} + e^1_{n,n}\right) \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_n)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_n)} = \frac{1}{4(n-1)}\left(\delta_{n-1,j} + \delta_{n,j}\right). \end{aligned} \tag{7.3.20}$$

Eqs. (7.3.16c), (7.3.16e), (7.3.20) and (7.1.18) therefore give the Killing forms of the simple root vectors,

$$(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}) = 4(n-1), \qquad 1 \le i \le n. \tag{7.3.21}$$

We can now compute the scalar products between simple roots, using

$$\begin{aligned} [h_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \frac{1}{4(n-1)}\left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right) e_{\overline{\alpha}_j}, \qquad 1 \le i,j \le n-1, \\ [h_{\overline{\alpha}_n}, e_{\overline{\alpha}_j}] &= -\frac{1}{4(n-1)}\,\delta_{j,n-2}\, e_{\overline{\alpha}_j}, \qquad 1 \le j \le n-1, \\ [h_{\overline{\alpha}_n}, e_{\overline{\alpha}_n}] &= \frac{2}{4(n-1)} e_{\overline{\alpha}_n}. \end{aligned} \tag{7.3.22}$$

Therefore

$$\begin{aligned} \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle &= \frac{1}{4(n-1)}\left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right), \quad 1 \le i,j \le n-1, \\ \langle \overline{\alpha}_i, \overline{\alpha}_n \rangle &= -\frac{1}{4(n-1)}\delta_{i,n-2}, \qquad 1 \le i \le n-1, \\ \langle \overline{\alpha}_n, \overline{\alpha}_n \rangle &= \frac{2}{4(n-1)}, \end{aligned} \qquad \text{simple root scalar products in } D_n. \tag{7.3.23}$$

Since all roots have the same length, D_n is simply laced.

The corresponding Cartan matrix is given by [via Eq. (7.1.22)]

$$\boxed{\hat{A} \to \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 2 \end{bmatrix}, \qquad \text{Cartan matrix for } D_n.} \tag{7.3.24}$$

The Dynkin diagram is shown in Fig. 7.5.

Figure 7.5: $\mathrm{so}(2n) = D_n$.

7.4 so$(2n+1)$: B_n

Finally we consider $B_n \Leftrightarrow \mathrm{so}(2n+1)$. We can view the B_n generators in the defining representation as equivalent to those for D_n, but with an added "zeroth" row and "zeroth" column preceding the $2n \times 2n$ generators of so$(2n)$; i.e. a vector index i runs from $0 \le i \le 2n$.

As in Sec. 7.3, we begin by rotating the orthogonal condition $\hat{U}^{\mathsf{T}}\hat{U} = \hat{1}_{2n+1}$ to a different basis. Applying the transformation as in Eq. (7.3.3), we choose

$$\hat{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & \hat{0}_{1,n} & \hat{0}_{1,n} \\ \hat{0}_{n,1} & i\hat{1}_n & -i\hat{1}_n \\ \hat{0}_{n,1} & -\hat{1}_n & -\hat{1}_n \end{bmatrix}, \quad \Rightarrow \quad \hat{M} = \hat{V}^{\mathsf{T}}\hat{V} = \begin{bmatrix} 1 & \hat{0}_{1,n} & \hat{0}_{1,n} \\ \hat{0}_{n,1} & 0 & \hat{1}_n \\ \hat{0}_{n,1} & \hat{1}_n & 0 \end{bmatrix}. \tag{7.4.1}$$

Here $\hat{0}_{1,n}$ ($\hat{0}_{n,1}$) denotes an n-fold row (column) of zeroes. The $(2n+1)\times(2n+1)$ generator $\hat{Y}'$ satisfies the condition as in Eq. (7.3.5), which allows the block decomposition [c.f. Eqs. (7.3.7) and (7.3.8)]

$$\hat{Y}' = \begin{bmatrix} b^1 & \hat{c}^1_{1,n} & \hat{c}^2_{1,n} \\ \hat{d}^1_{n,1} & \hat{Y}_1 & \hat{Y}_2 \\ \hat{d}^2_{n,1} & \hat{Y}_3 & \hat{Y}_4 \end{bmatrix}. \tag{7.4.2}$$

Eq. (7.3.5) then implies that

$$\begin{aligned} \hat{Y}_4 &= -\hat{Y}_1^{\mathsf{T}}, \\ \hat{Y}_2 &= -\hat{Y}_2^{\mathsf{T}}, \\ \hat{Y}_3 &= -\hat{Y}_3^{\mathsf{T}}, \\ b^1 &= 0, \\ \hat{c}^1_{1,n} &= -\left[\hat{d}^2_{n,1}\right]^{\mathsf{T}}, \\ \hat{c}^2_{1,n} &= -\left[\hat{d}^1_{n,1}\right]^{\mathsf{T}}. \end{aligned} \tag{7.4.3}$$

The basis elements for $\hat{Y}$ include $\{\hat{E}^1_{jk}, \hat{E}^2_{jk}, \hat{E}^3_{jk}\}$ $(1 \le j,k \le n)$ from Eq. (7.3.9), and

$$\hat{E}^4_j \equiv \hat{E}_{0,j} - \hat{E}_{j+n,0}, \tag{7.4.4a}$$

$$\hat{E}^5_j \equiv \hat{E}_{j,0} - \hat{E}_{0,j+n}, \tag{7.4.4b}$$

both with $1 \le j \le n$. The total number of such elements is

$$n(2n-1) + 2n = n(2n+1).$$

Relative to the example in Eq. (7.3.10), $\{\hat{E}^1_{jk}, \hat{E}^2_{jk}, \hat{E}^3_{jk}\}$ are now block embedded in 11×11 matrices (after one initial row and one initial column of zeroes), while (e.g.)

$$\hat{E}^4_4 \to \left[\begin{array}{c|ccccc|ccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{7.4.5a}$$

$$\hat{E}_3^5 \to \left[\begin{array}{c|ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{7.4.5b}$$

In addition to the Lie brackets in Eq. (7.3.11), we have

$$[e^1_{jk}, e^4_l] = -\,\delta_{jl}\, e^4_k, \tag{7.4.6a}$$
$$[e^1_{jk}, e^5_l] = \delta_{kl}\, e^5_j, \tag{7.4.6b}$$
$$[e^2_{jk}, e^4_l] = \delta_{jl}\, e^5_k - \delta_{kl}\, e^5_j, \tag{7.4.6c}$$
$$[e^2_{jk}, e^5_l] = 0, \tag{7.4.6d}$$
$$[e^3_{jk}, e^4_l] = 0, \tag{7.4.6e}$$
$$[e^3_{jk}, e^5_l] = \delta_{jl}\, e^4_k - \delta_{kl}\, e^4_j, \tag{7.4.6f}$$
$$[e^4_j, e^4_k] = -\,e^3_{jk}, \tag{7.4.6g}$$
$$[e^5_j, e^5_k] = -\,e^2_{jk}, \tag{7.4.6h}$$
$$[e^4_j, e^5_k] = -\,e^1_{kj}. \tag{7.4.6i}$$

• **Exercise**: Verify the Lie brackets in Eq. (7.4.6).

A generic Cartan subalgebra H element is given by

$$h = \sum_{j=1}^{n} \lambda_j\, e^1_{jj}. \tag{7.4.7}$$

Then

$$[h, e^1_{jk}] = (\lambda_j - \lambda_k)\, e^1_{jk}, \tag{7.4.8a}$$
$$[h, e^2_{jk}] = (\lambda_j + \lambda_k)\, e^2_{jk}, \tag{7.4.8b}$$
$$[h, e^3_{jk}] = -\,(\lambda_j + \lambda_k)\, e^3_{jk}, \tag{7.4.8c}$$
$$[h, e^4_j] = -\,\lambda_j\, e^4_j, \tag{7.4.8d}$$
$$[h, e^5_j] = \lambda_j e^5_j. \tag{7.4.8e}$$

The simple roots can be associated to the root vectors

root vector	root $\alpha(h)$
$e^1_{12} \equiv e_{\overline{\alpha}_1}$	$\overline{\alpha}_1(h) = \lambda_1 - \lambda_2$
$e^1_{23} \equiv e_{\overline{\alpha}_2}$	$\overline{\alpha}_2(h) = \lambda_2 - \lambda_3$
$\vdots$	$\vdots$
$e^1_{n-1,n} \equiv e_{\overline{\alpha}_{n-1}}$	$\overline{\alpha}_{n-1}(h) = \lambda_{n-1} - \lambda_n$
$e^5_n \equiv e_{\overline{\alpha}_n}$	$\overline{\alpha}_n(h) = \lambda_n.$

(7.4.9)

By definition, we take $\overline{\alpha}_1 > \overline{\alpha}_2 > \ldots > \overline{\alpha}_n$ (lexicographic ordering). The positive roots are associated to root vectors $\{e^1_{jk}\}$ with $j < k$, $\{e^2_{jk}\}$, and $\{e^5_j\}$. Negative root vectors are $\{e^1_{jk}\}$ $(j > k)$, $\{e^3_{jk}\}$, and $\{e^4_j\}$. The positive root associated to e^1_{jk} $(j < k)$ is expressed in terms of simple roots identically as in Eq. (7.1.11). For e^2_{ij}, the decomposition is (taking $j > i$ WLoG)

$$\lambda_i + \lambda_j = (\lambda_i - \lambda_j) + 2(\lambda_j - \lambda_n) + 2\lambda_n = \sum_{l=i}^{j-1} \overline{\alpha}_l + 2\sum_{l=j}^{n-1} \overline{\alpha}_l + 2\overline{\alpha}_n. \tag{7.4.10}$$

Eq. (7.4.10) has *almost* the same form as the symplectic C_n case, Eq. (7.2.12); the difference is the prefactor of two for $\overline{\alpha}_n$ in Eq. (7.4.10). In fact we will see that B_n and C_n are closely related (although not identical, except for $B_2 = C_2$). For e_i^5, the decomposition is

$$\lambda_i = (\lambda_i - \lambda_n) + \lambda_n = \sum_{l=i}^{n-1} \overline{\alpha}_l + \overline{\alpha}_n. \tag{7.4.11}$$

The commutation relations between the simple root vectors are given by

$$\begin{aligned}
[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \delta_{j,i+1}\, e^1_{i,i+2} - \delta_{j,i-1}\, e^1_{i-1,i+1}, && 1 \le i,j \le n-1, && (7.4.12a)\\
[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_n}] &= \delta_{i,n-1}\, e^5_{n-1}, && 1 \le i \le n-1, && (7.4.12b)\\
[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_j}] &= \delta_{ij}\, \left(e^1_{i,i} - e^1_{i+1,i+1}\right), && 1 \le i,j \le n-1, && (7.4.12c)\\
[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_n}] &= 0, && 1 \le i \le n-1, && (7.4.12d)\\
[e_{\overline{\alpha}_n}, e_{-\overline{\alpha}_n}] &= e^1_{nn}. && && (7.4.12e)
\end{aligned}$$

The Killing form between Cartan subalgebra elements

$$h \equiv \sum_i \lambda_i e^1_{ii}, \quad h' \equiv \sum_j \lambda'_j e^1_{jj}, \tag{7.4.13}$$

is given by

$$(h, h') = 4\left(n - \frac{1}{2}\right) \sum_{p=1}^{n} \lambda_p \lambda'_p. \tag{7.4.14}$$

• **Exercise**: Verify Eq. (7.4.14).

Now since

$$(h_{\overline{\alpha}_i}, h') = \overline{\alpha}_i(h') = \begin{cases} \lambda'_i - \lambda'_{i+1}, & 1 \le i \le n-1, \\ \lambda'_n, & i = n, \end{cases} \tag{7.4.15}$$

we conclude that

$$\begin{aligned}
h_{\overline{\alpha}_i} &= \frac{1}{4\left(n-\frac{1}{2}\right)} \left(e^1_{ii} - e^1_{i+1,i+1}\right) \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_i)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_i)} = \frac{1}{4\left(n-\frac{1}{2}\right)} \left(\delta_{i,j} - \delta_{i+1,j}\right), \qquad 1 \le i \le n-1,\\
h_{\overline{\alpha}_n} &= \frac{1}{4\left(n-\frac{1}{2}\right)} e^1_{n,n} \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_n)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_n)} = \frac{1}{4\left(n-\frac{1}{2}\right)} \delta_{n,j}.
\end{aligned} \tag{7.4.16}$$

Eqs. (7.4.12c), (7.4.12e), (7.4.16) and (7.1.18) therefore give the Killing forms of the simple root vectors,

$$(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}) = 4\left(n - \frac{1}{2}\right), \qquad 1 \le i \le n. \tag{7.4.17}$$

We can now compute the scalar products between simple roots, using

$$\begin{aligned}
[h_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \frac{1}{4\left(n-\frac{1}{2}\right)} \left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right) e_{\overline{\alpha}_j}, \qquad 1 \le i,j \le n-1,\\
[h_{\overline{\alpha}_n}, e_{\overline{\alpha}_j}] &= -\frac{1}{4\left(n-\frac{1}{2}\right)} \delta_{j,n-1}\, e_{\overline{\alpha}_j}, \qquad 1 \le j \le n-1,\\
[h_{\overline{\alpha}_n}, e_{\overline{\alpha}_n}] &= \frac{1}{4\left(n-\frac{1}{2}\right)} e_{\overline{\alpha}_n}.
\end{aligned} \tag{7.4.18}$$

Figure 7.6: $\mathrm{so}(2n+1) = B_n$.

Therefore

$$\begin{aligned}
\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle &= \frac{1}{4\left(n-\frac{1}{2}\right)} \left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right), \quad 1 \le i,j \le n-1, \\
\langle \overline{\alpha}_i, \overline{\alpha}_n \rangle &= -\frac{1}{4\left(n-\frac{1}{2}\right)} \delta_{i,n-1}, \qquad 1 \le i \le n-1, \\
\langle \overline{\alpha}_n, \overline{\alpha}_n \rangle &= \frac{1}{4\left(n-\frac{1}{2}\right)}.
\end{aligned} \qquad \text{simple root scalar products in } B_n. \tag{7.4.19}$$

The last equation implies that $\overline{\alpha}_n$ is a short root, with $|\overline{\alpha}_n|^2 = |\overline{\alpha}_j|^2/2$, $1 \le j \le n-1$. Like C_n, B_n is not simply laced.

The corresponding Cartan matrix is given by [via Eq. (7.1.22)]

$$\hat{A} \to \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -2 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}, \qquad \text{Cartan matrix for } B_n. \tag{7.4.20}$$

The Dynkin diagram is shown in Fig. 7.6.

The Cartan matrices/Dynkin diagrams for C_n [Eq. (7.2.21), Fig. 7.4] and B_n [Eq. (7.4.20), Fig. 7.6] are almost identical. In both cases, the only non-vanishing simple root inner products involve "nearest-neighbors" $\overline{\alpha}_i$ and $\overline{\alpha}_{i+1}$, $1 \le i \le n-1$. The first $n-1$ simple roots share the same length, while $\overline{\alpha}_n$ is a long (short) root for C_n (B_n). In fact, the root systems for B_n and C_n have the same geometry, except that long and short roots are interchanged.

On the other hand, we will see that D_n and B_n have more in common in terms of the n simplest (so called fundamental[1]) representations, described in module 8. In particular, both D_n and B_n possess fundamental spinor representations, which are qualitatively distinct and cannot be built by tensoring together defining vector indices. By contrast, every representation of C_n or A_n can be built by tensoring together indices transforming in the the defining $2n$-dimensional [$(n+1)$-dimensional] representation.

• **Exercise**: Find the inventory of roots and plot the 3D root geometry for B_3 and C_3, following the same approach used for A_3 in Sec. 7.1.1.

7.5 Classical Lie algebras to rank 4; equivalent algebras

In Table 1, we show the classical Lie algebras through rank four. Various "accidental" equivalences between different algebras of the same rank are indicated, and easily recognized by the identical Dynkin diagrams. All four families are distinct for each rank above three. Of course, equivalence of the Lie algebras does not imply equivalence of the corresponding symmetry groups.

7.6 The exceptional algebras

The four classical families studied above do not exhaust the possibilities for (semi-)simple Lie algebras. It is perhaps surprising that there are only five additional possibilities corresponding to particular algebras of fixed rank. This can be shown by a careful determination of **allowable** Dynkin diagrams. The argument is explicated in chapter IX of Cahn [1] and chapter 20 of Georgi [2];

[1] Note that in the physics literature, the terms "defining" and "fundamental" are often used interchangably. However, for a given Lie algebra L, there exists a *set* of "simplest" representations (which always includes the defining one), and mathematicians and mathematical physicists refer to each member of this set as a fundamental representation. We will define fundamental representations precisely in module 8, using the **fundamental weights**.

Table 1: Classical Lie algebras to rank 4.

Rank	Dynkin	Cartan	Compact	# of positive roots
1		A_1	$\mathrm{su}(2) = \mathrm{sp}(2) = \mathrm{so}(3)$	1
2		A_2	$\mathrm{su}(3)$	3
		$B_2 = C_2$	$\mathrm{so}(5) = \mathrm{sp}(4)$	4
		$D_2 = A_1 \times A_1$	$\mathrm{so}(4) = \mathrm{su}(2) \times \mathrm{su}(2)$	2
3		$A_3 = D_3$	$\mathrm{su}(4) = \mathrm{so}(6)$	6
		B_3	$\mathrm{so}(7)$	9
		C_3	$\mathrm{sp}(6)$	9
4		A_4	$\mathrm{su}(5)$	10
		B_4	$\mathrm{so}(9)$	16
		C_4	$\mathrm{sp}(8)$	16
		D_4	$\mathrm{so}(8)$	12

we won't belabor it here. Instead, we simply state the Cartan matrices and Dynkin diagrams for the five exceptional algebras G_2, F_4, and $E_{6,7,8}$. Our labeling conventions are the same as chapter 13 of [3].

(7.6.1)

Cartan matrices for the exceptional Lie algebras G_2, F_4, E_6, E_7, and E_8

$$\hat{A}_{G_2} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}, \qquad \hat{A}_{E_6} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix},$$

$$\hat{A}_{F_4} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \qquad \hat{A}_{E_7} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}, \tag{7.6.2}$$

$$\hat{A}_{E_8} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}.$$

Figure 7.7: The five exceptional algebras.

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

[2] Howard Georgi, *Lie Algebras in Particle Physics*, 2nd ed. (Westview Press, Boulder, Colorado, 1999).

[3] Phillipe Di Francesco, Pierre Mathieu, David Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1996).

8. Highest-weight representations
* version 2.5 *

Matthew Foster

April 30, 2021

Contents

8.1 Weights in 3 bases **1**
8.1.1 Roots, coroots, and fundamental weights . . . 2
8.1.2 Marks and comarks . . . 4

8.2 Mostly su($n+1$) **5**
8.2.1 Elementary tensors representations: complete (anti)symmetrization . . . 5
8.2.1.1 Complete antisymmetrization: the fundamental weight representations . . . 6
8.2.1.2 Complete symmetrization . . . 7
8.2.2 su(3) again . . . 8
8.2.3 The Levi-Civita symbol, $\overline{n+1}$ representation of su($n+1$), and conjugate representations . . . 8
8.2.4 The highest root for A_n, B_n, C_n, and D_n . . . 11
8.2.5 su(4) . . . 12

8.3 so($2n+1$), so($2n$), and sp($2n$) **12**
8.3.1 so($2n+1$) . . . 12
8.3.2 sp($2n$) . . . 14
8.3.3 so($2n$) . . . 16

8.4 Lattices and conjugacy classes **18**
8.4.1 $A_n = \mathrm{su}(n+1)$ . . . 18
8.4.2 $B_n = \mathrm{so}(2n+1)$ . . . 19
8.4.3 $C_n = \mathrm{sp}(2n)$ . . . 19
8.4.4 $D_{2m} = \mathrm{so}(4m)$, $m \in \mathbb{Z}_+$. . . . 20
8.4.5 $D_{2m+1} = \mathrm{so}(4m+2)$, $m \in \mathbb{Z}_+$. . . . 20

8.5 Tensors and tableaux for su(n) **20**
8.5.1 Young tableaux . . . 21
8.5.2 Young tableau for a generic highest weight representation; tensors . . . 21
8.5.3 Dimension rule: "Factors over hooks" . . . 23
8.5.4 Semistandard Young tableaux: enumerating the states . . . 25

8.6 Dual Coxeter numbers and quadratic form matrices **26**

The discussion here follows chapter X of [1], chapter 13 of [2], and elements of Chapters 10–13 in [3].

8.1 Weights in 3 bases

In this module we focus on generic irreducible **highest-weight representations**, wherein we assume the existence of some state that cannot be raised using simple root vector operators, and we use the Master Weight Depth Formula (MWDF) Eq. (5.1.23) to obtain all other weights below it in the representation.

The emphasis so far has been on expressing weights in terms of the n simple roots $\{\overline{\alpha}_j\}$ for a rank-n Lie algebra. In the adjoint representation, the set of positive roots consists of simple root combinations with non-negative integral coefficients. For a general representation, however, the coefficients are only required to be rational numbers. In order to specify a generic highest-weight representation, it is advantageous to develop a basis for the space of weights H^* in which expansion coefficients are integral.

8.1.1 Roots, coroots, and fundamental weights

In fact, let us specify three different bases, each of which gives certain advantages. Let Λ be a weight in some irreducible representation. Then

Bases for a generic weight $\Lambda \in H^*$

$$\begin{aligned} \Lambda &= \sum_{j=1}^{n} \kappa_j^{(\Lambda)}\, \overline{\alpha}_j, && \text{Expansion in terms of simple roots,} \\ &= \sum_{j=1}^{n} \kappa_j^{\vee(\Lambda)}\, \overline{\alpha}_j^{\vee}, && \text{Expansion in terms of simple coroots,} \\ &= \sum_{j=1}^{n} \Lambda_j\, \omega_j, && \text{Expansion in terms of fundamental weights.} \end{aligned} \tag{8.1.1}$$

As defined in Eq. (5.1.23), the coroot $\overline{\alpha}_i^{\vee}$ associated to a root $\overline{\alpha}_i$ is defined via

$$\overline{\alpha}_i^{\vee} \equiv \frac{2\overline{\alpha}_i}{\langle \overline{\alpha}_i, \overline{\alpha}_i \rangle}. \tag{8.1.2}$$

The Cartan matrix is expressed as the scalar product between roots and coroots, $A_{ij} = \langle \overline{\alpha}_i, \overline{\alpha}_j^{\vee} \rangle$ [Eq. (5.3.8)]. The expansion coefficients $\{\kappa_i^{(\Lambda)}, \kappa_j^{\vee(\Lambda)}\} \in \mathbb{Q}$ (the field of rational numbers) [Prop. VIII., Sec. 5.2.2].

The **fundamental weights** $\{\omega_i\}$ $(1 \leq i \leq n)$ are defined to be dual to the set of simple coroots, i.e. these are mutually orthonormal:

$$\langle \omega_i, \overline{\alpha}_j^{\vee} \rangle = \delta_{ij}. \tag{8.1.3}$$

Since the simple coroots span H^*, the fundamental weights do as well. The coefficients $\{\Lambda_j\}$ for the expansion of a weight Λ in terms of fundamental weights are the **Dynkin labels**, which are guaranteed to be integers. This follows from taking the inner product of the expansion with a simple coroot,

$$\langle \Lambda, \overline{\alpha}_i^{\vee} \rangle = \Lambda_i = (m - p)_{\Lambda;\overline{\alpha}_i}, \tag{8.1.4}$$

where m and p respectively denote the number of weights below and above Λ in the chain generated by acting with $\hat{E}_{\pm\overline{\alpha}_i}$ [Eq. (5.1.23)].

We then immediately see that a highest weight Λ is defined by a set of non-negative Dynkin coefficients $\{\Lambda_i \geq 0\}$, since by definition $p_{\Lambda;\overline{\alpha}_i} = 0\ \forall\ \overline{\alpha}_i$. Thus we can encode any representation by the n non-negative Dynkin coefficients of its highest weight. On the other hand, the expansion of weights in terms of simple roots is also useful, since it encodes the geometric picture for the representation connected to root scalar products. To relate these, we write the overlap matrices

$$\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = A_{ij} \frac{|\overline{\alpha}_j|^2}{2}, \tag{8.1.5a}$$

$$\langle \overline{\alpha}_i^{\vee}, \overline{\alpha}_j^{\vee} \rangle = \frac{2}{|\overline{\alpha}_i|^2} A_{ij}, \tag{8.1.5b}$$

$$\langle \omega_i, \omega_j \rangle = F_{ij}. \tag{8.1.5c}$$

In these equations A_{ij} denotes the Cartan matrix. The symmetric overlap matrix between fundamental weights F_{ij} is called the **quadratic form matrix**. Its relation to the Cartan matrix follows from

$$\begin{aligned} \Lambda_j &= \langle \Lambda, \overline{\alpha}_j^{\vee} \rangle = \sum_{i=1}^{n} \kappa_i^{(\Lambda)}\, A_{ij}, \\ \kappa_j^{\vee(\Lambda)} &= \langle \Lambda, \omega_j \rangle = \sum_{i=1}^{n} \Lambda_i\, F_{ij}, \\ \Rightarrow\ \Lambda_j &= \sum_{l=1}^{n} \kappa_l^{\vee(\Lambda)} \frac{2}{|\overline{\alpha}_l|^2} A_{lj} = \sum_{i=1}^{n} \Lambda_i \sum_{l=1}^{n} F_{il} \frac{2}{|\overline{\alpha}_l|^2} A_{lj} \ \Rightarrow\ \sum_{l=1}^{n} F_{il} \frac{2}{|\overline{\alpha}_l|^2} A_{lj} = \delta_{ij}, \end{aligned} \tag{8.1.6}$$

so that

$$(\hat{F}^{-1})_{ij} = \frac{2}{|\overline{\alpha}_i|^2} A_{ij} = \langle \overline{\alpha}_i^\vee, \overline{\alpha}_j^\vee \rangle. \tag{8.1.7}$$

The quadratic form matrices for all 9 classes of semi-simple Lie algebras are known explicitly. These are tabulated for the four classical Lie algebra families in Sec. 8.6; the exceptional cases can be found in section 13.A of [2].

For the moment, we note that Eq. (8.1.6) implies that

$$\Lambda = \kappa^{(\Lambda)} \hat{A}, \quad \kappa^{(\Lambda)} = \Lambda \hat{A}^{-1}, \tag{8.1.8}$$

where we view Λ and $\kappa^{(\Lambda)}$ as row vectors of coefficients. Consider the **defining** or **3** representation of su(3), previously analyzed in Sec. 5.1. In Eq. (5.1.8), we found that the weights in this representation can be expressed as

$$\begin{aligned} M^A &= \frac{2}{3}\overline{\alpha}_1 + \frac{1}{3}\overline{\alpha}_2, \\ M^B &= -\frac{1}{3}\overline{\alpha}_1 + \frac{1}{3}\overline{\alpha}_2, \\ M^C &= -\frac{1}{3}\overline{\alpha}_1 - \frac{2}{3}\overline{\alpha}_2, \end{aligned} \tag{8.1.9}$$

where $\alpha_{t_+} \to \overline{\alpha}_1$ and $\alpha_{u_+} \to \overline{\alpha}_2$. Using the su(3) Cartan matrix from Eq. (5.3.15), the associated Dynkin coefficients are [via Eq. (8.1.8)]

$$\begin{aligned} M^A &\to (1,0), \\ M^B &\to (-1,1), \\ M^C &\to (0,-1). \end{aligned} \tag{8.1.10}$$

We can obtain these another way, using the MWDF Eq. (5.1.23). This is the complement to the positive-root-building algorithm we used in Secs. 5.3.2 and 7.1.1. Now, we subtract simple roots to generate all of the weights from the highest one:

1. We start with the highest weight state

$$\begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{8.1.11a}$$

The depths are given by the Dynkin coefficients,

$$m_{\Lambda;\overline{\alpha}_1} = 1, \quad m_{\Lambda;\overline{\alpha}_2} = 0. \tag{8.1.11b}$$

2. At the second depth level, we know that $\Lambda^{(2)} \equiv \Lambda - \overline{\alpha}_1$ is a valid weight. Its Dynkin coefficients are

$$\begin{bmatrix} -1 & 1 \end{bmatrix}. \tag{8.1.11c}$$

Figure 8.1: Weight trees for the defining "3" (1,0), conjugate "$\bar{3}$" (0,1), and adjoint (1,1) representations of su(3). The ordered pair labels (Λ_1, Λ_2) ($\Lambda_i \in \mathbb{Z}$) are Dynkin coefficients. The simple root labels $\{\overline{\alpha}_i\}$ indicate the difference between the weights above and below the connecting line.

Figure 8.2: Weight geometries for the defining "3" (1,0), conjugate "$\bar{3}$" (0,1), and adjoint (1,1) representations of su(3). The 3 and $\bar{3}$ representations are also faintly superimposed over the adjoint. These three representations belong to the three different conjugacy classes (shifted triangular lattices) of su(3) weights.

The heights are

$$p_{\Lambda^{(2)};\overline{\alpha}_1} = 1, \quad p_{\Lambda^{(2)};\overline{\alpha}_2} = 0, \tag{8.1.11d}$$

so that the depths below this level are

$$\begin{aligned} m_{\Lambda^{(2)};\overline{\alpha}_1} &= p_{\Lambda^{(2)};\overline{\alpha}_1} + \Lambda_1^{(2)} = 1 - 1 = 0, \\ m_{\Lambda^{(2)};\overline{\alpha}_2} &= p_{\Lambda^{(2)};\overline{\alpha}_2} + \Lambda_2^{(2)} = 0 + 1 = 1. \end{aligned} \tag{8.1.11e}$$

3. At the third depth level, we know that $\Lambda^{(3)} \equiv \Lambda^{(2)} - \overline{\alpha}_2$ is a valid weight. Its Dynkin coefficients are

$$\begin{bmatrix} 0 & -1 \end{bmatrix}. \tag{8.1.11f}$$

The heights are

$$p_{\Lambda^{(3)};\overline{\alpha}_1} = 0, \quad p_{\Lambda^{(3)};\overline{\alpha}_2} = 1, \tag{8.1.11g}$$

so that the depths below this level are

$$m_{\Lambda^{(3)};\overline{\alpha}_1} = p_{\Lambda^{(3)};\overline{\alpha}_1} + 0 = 0, \quad m_{\Lambda^{(3)};\overline{\alpha}_2} = p_{\Lambda^{(3)};\overline{\alpha}_2} - 1 = 0. \tag{8.1.11h}$$

We conclude that $\Lambda^{(3)}$ is the lowest weight state.

In addition to the defining representation (1,0) $\Leftrightarrow \omega_1$, the $A_2 = \text{su}(3)$ algebra has a second fundamental representation (0,1) $\Leftrightarrow \omega_2$. This turns out to be the **conjugate** or "$\bar{\mathbf{3}}$" representation. The weight trees for the 3, $\bar{3}$, and adjoint representations of su(3) are depicted in Fig. 8.1. We can use Eq. (8.1.8) to obtain the expansion coefficients of all of these weights in terms of simple roots; the simple root geometry then implies the geometry of the representations shown in Fig. 8.2. Note that these three representations (1,0), (0,1), and (1,1) correspond to regular fragments of three different triangular lattices. The lattices are generated by the same lattice vectors $\overline{\alpha}_1$ and $\overline{\alpha}_2$, but from different (shifted) origins. The different lattices are the three **conjugacy classes** of su(3), exemplified by these representations. All other representations reside on one of these lattices.

Conjugate representations will be defined precisely below, but the key features appear in Figs. 8.1 and 8.2. Namely, the top weight of the conjugate $\bar{3}$ representation is minus the bottom weight of the defining 3 representation (and vice-versa). All weights are in one-to-one correspondence between a representation and its conjugate, being related by inversion ($\Lambda \to -\Lambda$). Finally, the adjoint representation is self-conjugate, since its lowest weight (-1,-1) is minus its highest one.

8.1.2 Marks and comarks

The highest weight state of the adjoint representation for any rank-n Lie algebra is the highest root θ, introduced in Sec. 5.3.1. It can be expanded as in Eq. (8.1.1),

$$\theta = \sum_{j=1}^{n} a_j\, \overline{\alpha}_j = \sum_{j=1}^{n} a_j^\vee\, \overline{\alpha}_j^\vee = \sum_{j=1}^{n} \theta_j\, \omega_j. \tag{8.1.12}$$

Figure 8.3: Dynkin diagrams for the four classical families $A_n = \mathrm{su}(n+1)$, $B_n = \mathrm{so}(2n+1)$, $C_n = \mathrm{sp}(2n)$, and $D_n = \mathrm{so}(2n)$. The three numbers $(m, a_m, a_m^\vee)$ that label each node specify the fundamental weight ω_m, the mark for the associated root a_m, and the comark for the associated coroot $a_m^\vee$. Here we set $|\alpha|^2 = 2$, where α is a long root, so that marks and comarks are identical for simply-laced algebras.

In this case, the root and coroot coefficients are called **marks** $\{a_j\}$ and **comarks** $\{a_j^\vee\}$, respectively. The comarks play a key role in the construction of dominant highest-weight representations in **affine Lie algebras**, which are equivalent to certain exactly solvable 2D conformal field theories (Kac-Moody current algebras). The marks and comarks can be computed from the Dynkin coefficients of θ via Eq. (8.1.8). For su(3), $\theta = (1, 1)$. We will derive the $\{\theta_j\}$ for the four classical families in Sec. 8.2.4, below.

Another key piece of data for any Lie algebra is the **dual Coxeter number** g, defined via

$$g = 1 + \sum_{j=1}^{n} a_j^\vee. \tag{8.1.13}$$

In module 9, we will show that g is half the value of the quadratic Casimir operator, evaluated in the adjoint representation. The dual Coxeter numbers for the four classical families are given in Sec. 8.6.

We can use Dynkin diagrams to enumerate the fundamental weights for each rank-n algebra. These are shown in Fig. 8.3, along with the corresponding marks and comarks. Here and in Sec. 8.6, we use a normalization scheme such that long roots have $|\alpha|^2 = 2$.

8.2 Mostly su$(n+1)$

8.2.1 Elementary tensors representations: complete (anti)symmetrization

We will build up more complex representations from simpler ones by tensoring together the latter. If we have two representations ϕ_i and η_i such that for $x \in L$,

$$\begin{aligned} \left(\hat{X}^{(1)}\phi\right)_i &= X^{(1)}_{ij}\,\phi_j, \\ \left(\hat{X}^{(2)}\eta\right)_i &= X^{(2)}_{ij}\,\eta_j, \end{aligned} \tag{8.2.1}$$

then the action on the tensor (direct) product is

$$\left(\hat{X}\,\phi\otimes\eta\right)_{ij} = X^{(1)}_{ik}\,\phi_k\,\eta_j + X^{(2)}_{jk}\,\phi_i\,\eta_k. \tag{8.2.2}$$

Thus if ϕ and η are weight vectors with weights $M^{(1)}$ and $M^{(2)}$, respectively, then $\phi_i\,\eta_j$ is also a weight vector with weight $M^{(1)}+M^{(2)}$. In particular, if we tensor together two highest weight representations with highest weights $\Lambda^{(1)}$ and $\Lambda^{(2)}$, respectively, then the highest weight of the tensor product is given by $\Lambda^{(1)} + \Lambda^{(2)}$.

In what follows, we will exploit the following fact, which we do not prove here:

• **Claim:** The weight space[1] (degeneracy) of the highest weight is always one-dimensional for an irreducible representation.

If we tensor together m different copies of the defining representation in some Lie algebra (i.e. form an m-fold direct product), the resulting rank-m tensor $T_{i_1 i_2 \cdots i_m}$ will *not* belong to an irreducible representation. As discussed in the previous section, any highest-weight representation in a rank-n Lie algebra can be specified by the n non-negative Dynkin labels of its highest weight $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$. We can in fact generate most or all[2] representations by appropriately **symmetrizing** and **antisymmetrizing** groups of indices of the unsymmetrized $T_{i_1 i_2 \cdots i_m}$. The simplest cases correspond to complete antisymmetrization or complete symmetrization. We now show how to obtain the Dynkin coefficients for su$(n+1)$ tensors in these cases.

8.2.1.1 Complete antisymmetrization: the fundamental weight representations

Consider the defining "$n+1$" representation of $A_n = \text{su}(n+1)$. As in su(3) (Figs 8.1 and 8.2), the Dynkin labels for the $n+1$ representation are $(\Lambda_1, \Lambda_2, \Lambda_3, \ldots, \Lambda_n) = (1, 0, 0, \ldots, 0)$. Equivalently, we say that the defining representation is equal to the first fundamental weight,

$$\Lambda = \omega_1.$$

Using the $n \times n$ Cartan matrix for A_n [Eq. (7.1.23)]

$$\hat{A}_{A_n} = \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{bmatrix}, \tag{8.2.3}$$

we can generate the n descendants of the highest weight using the MWDF algorithm in Eq. (8.1.11). This gives the defining representation weight tree shown in Fig. 8.4. Note that the mth step down the chain is generated by lowering with the simple root $\overline{\alpha}_m$ $(1 \leq m \leq n)$. This is also easily seen from the explicit matrix representation of the simple root vectors $\{\hat{E}_{\overline{\alpha}_i}\}$ for A_n, Eqs. (7.1.1) and (7.1.9). Since there are $n+1$ different weights in Fig. 8.4, all weight spaces in the defining representation are one-dimensional (non-degenerate).

We can view the $n+1$ representation as a rank-1 tensor T_i, $i \in \{1, 2, \ldots, n+1\}$. Now, consider the completely antisymmetrized rank-m tensor

$$T_{[i_1 i_2 \cdots i_m]}, \quad 1 \leq m \leq n+1. \tag{8.2.4}$$

This has

$$\frac{(n+1)!}{m!(n+1-m)!} \tag{8.2.5}$$

independent components. We assert that this m-fold tensor is automatically an irreducible representation. The argument is twofold:

1. We cannot simplify the index structure further, i.e. symmetrize any subgroup of indices,
2. We cannot contract any two indices to form a simpler tensor.

The second fact is equivalent to the statement that there is no natural "metric" for su$(n+1)$ $(n \geq 2)$ that can be used to contract pairs of defining representation indices to produce an invariant scalar, unlike in sp$(2n)$ or so(n) (module 2A). [We will discuss contractions using the rank-$(n+1)$ Levi-Civita symbol when we introduce conjugate representations, below.]

What are the Dynkin coefficients for $T_{[i_1 i_2 \cdots i_m]}$?

- $m = 2$: second-rank, antisymmetric tensor $T_{[ij]}$ with $(n+1)n/2$ components. The highest weight possible for each index i and j is $\omega_1 = (1, 0, 0, \ldots, 0)$, associated to $i = 1$ or $j = 1$. However, due to the antisymmetrization, we cannot have $i = j$. Thus, the component with the highest possible weight is $T_{12} = -T_{21}$. From Fig. 8.4, this has weight

$$(1, 0, 0, \ldots, 0) + (-1, 1, 0, \ldots, 0) = (0, 1, 0, \ldots, 0) \to \omega_2. \tag{8.2.6a}$$

Thus the case $m = 2$ gives a realization of the representation with highest weight $\Lambda = \omega_2$.

[1] Recall from module 5 that the **weight space** is the degeneracy of a particular weight Λ in some irreducible representation. If we consider all possible representations in the same conjugacy class, then the weight space must be specified for all "sites" of the corresponding infinite weight lattice. Each weight space is therefore characterized by a non-negative integer. This is not to be confused with H^*, which is the space of weights, i.e. the space in which different weights are expressed as linear combinations of simple roots, simple coroots, or fundamental weights. The terminology here is a bit unfortunate.

[2] All representations in $A_n = \text{su}(n+1)$ and $C_n = \text{sp}(2n)$ can be obtained by tensoring together copies of the defining representation, and (anti-)symmetrizing. In $B_n = \text{so}(2n+1)$ and $D_n = \text{so}(2n)$, however, there are spinor representations that cannot be obtained by tensoring together vector indices. We will demonstrate this in Sec. 8.3.

Figure 8.4: Weight tree for the defining representation of su$(n+1)$ (A_n): $(1,0,0,\cdots,0) \Leftrightarrow \omega_1$. The simple root labels $\{\overline{\alpha}_i\}$ indicate the difference between the weights above and below the connecting line.

- $m=3$: $T_{[ijk]}$ with $(n+1)n(n-1)/3!$ components. The component with the highest possible weight is $T_{123} = -T_{213} = T_{231}$ (etc.). The highest weight is

$$(1,0,0,0,\ldots,0) + (-1,1,0,0,\ldots,0) + (0,-1,1,0,\ldots,0) = (0,0,1,\ldots,0) \to \omega_3. \tag{8.2.6b}$$

$\vdots$

- $m=n$: $T_{[i_1 i_2 \cdots i_n]}$ with $n+1$ components. The highest weight is

$$(1,0,0,\ldots,0,0,0) + (-1,1,0,\ldots,0,0,0) + \ldots + (0,0,0,\ldots,0,-1,1) = (0,0,0,\ldots,0,0,1) \to \omega_n. \tag{8.2.6c}$$

- $m=n+1$: $T_{[i_1 i_2 \cdots i_n i_{n+1}]}$. This is proportional to the rank-$(n+1)$ Levi-Civita symbol $\epsilon_{i_1 \cdots i_{n+1}}$ and has only one unique component (which may be zero); see also Sec. 8.2.3, below. The weight is

$$(1,0,0,\ldots,0,0,0) + (-1,1,0,\ldots,0,0,0) + \ldots + (0,0,0,\ldots,0,0,-1) = (0,0,0,\ldots,0,0,0) = 0. \tag{8.2.6d}$$

- Thus for su$(n+1)$, the fundamental weight ω_m corresponds precisely to a rank-m completely antisymmetric tensor:

$$T_{[i_1 i_2 \cdots i_m]}, \quad 1 \le m \le n: \text{ irreducible representation of su}(n+1) \text{ with highest weight } \omega_m. \tag{8.2.7}$$

The same conclusion will apply to all but one (two) fundamental weights in $B_n = \text{so}(2n+1)$ $[D_n = \text{so}(2n)]$; the outliers are the fundamental spinor representations. In $C_n = \text{sp}(2n)$, the fundamental weight ω_m also corresponds to a rank-m completely antisymmetric tensor $\tilde{T}_{[i_1 \cdots i_m]}$ $(1 \le m \le n)$; the tilde indicates that, in addition, all traces with the antisymmetric "symplectic metric" ϵ^{ij} have been removed [c.f. Eqs. (7.2.1) and (7.2.2)].

8.2.1.2 Complete symmetrization

Another class of representations are the completely symmetric rank-m tensors,

$$T_{(i_1 i_2 \cdots i_m)}, \quad m \ge 1. \tag{8.2.8}$$

This has

$$\frac{(m+n)!}{m!n!} \tag{8.2.9}$$

independent components. For su$(n+1)$, completely symmetric tensors are also irreducible representations; the argument is the same as the antisymmetric case. The highest weight state of the rank-m tensor is $T_{11\cdots 1}$, with weight $\Lambda = (m,0,0,\cdots,0)$:

$$T_{(i_1 i_2 \cdots i_m)}, \quad m \ge 1: \text{ irreducible representation of su}(n+1) \text{ with highest weight } m\,\omega_1. \tag{8.2.10}$$

A completely symmetric rank-m tensor forms an irreducible representation with weight $m\,\omega_1$ for C_n as well. For the orthogonal algebras one has to in addition remove all possible traces to get irreducible representations.

Thus we have identified tensor representations with highest weight Λ given by the fundamental weights $\Lambda \in \{\omega_i\}$ (completely antisymmetric) or by $\Lambda = m\,\omega_1$ (completely symmetric). Other irreducible representations of su$(n+1)$ can be built from tensors with mixed indices; one technique (Young tableaux) will be demonstrated below in Sec. 8.5.

Figure 8.5: Symmetric second rank tensor (2,0) $\Leftrightarrow$ $T_{(ij)}$ representation of su(3). This is in the conjugacy class of the conjugate $\bar{3}$ $\Leftrightarrow$ (0,1) representation. There are $3(4)/2 = 6$ states in this representation, so all weight spaces are one dimensional.

8.2.2 su(3) again

We next construct the weight trees and geometries for a couple of more complex su(3) representations. The fundamental representations ω_1 and ω_2 were depicted in Figs. 8.1 and 8.2, along with the adjoint representation $\omega_1+\omega_2 \to (1,1)$. The second-rank symmetric tensor representation $T_{(ij)} \Leftrightarrow (2,0)$ is shown in Fig. 8.5. A mixed tensor representation with Dynkin labels $(2,1)$ is shown in Fig. 8.6. This representation turns out to have 15 total states, but only 12 distinct weights. In this case, each of the "inner triangle" weights is doubly degenerate, as we will prove later in module 9.

Figure 8.6: (2,1) representation of su(3). This is in the conjugacy class of the defining 3 $\Leftrightarrow$ (1,0) representation. There are 15 states in this representation, but only 12 distinct weights. Each of the weights in the inner triangle are doubly degenerate, as we will show using Freudenthal's formula in module 9.

8.2.3 The Levi-Civita symbol, $\overline{n+1}$ representation of su$(n+1)$, and conjugate representations

We argued above that completely (anti)symmetric tensors are automatic irreducible representations of su$(n+1)$. Part of the argument was that there is no natural "metric" with which we can "tie up" pairs of defining representation indices to form a "singlet" (group invariant) in A_n. In fact, there is one very important tensor that is a group invariant: the rank-$(n+1)$ completely antisymmetric tensor (Levi-Civita symbol[3]):

$$\epsilon^{i_1 i_2 \cdots i_{n+1}} = \epsilon^{[i_1 i_2 \cdots i_{n+1}]}; \quad \epsilon^{1,2,3,\cdots n,n+1} = +1. \tag{8.2.11}$$

Here we write this object with all indices "upstairs," as opposed to the tensor in Eq. (8.2.4) built out of defining, "downstairs" ω_1-indices. Following conventions from general relativity and differential geometry, the idea is that we can only "contract" (sum

[3]Here we use the terms "Levi-Civita symbol" and "Levi-Civita tensor" interchangeably. There is a distinction for coordinate tensors on curved manifolds, but our indices here are constructed from the defining representation in a linear vector space. This is independent of the arbitrary coordinates used to parameterize the corresponding Lie group manifold.

pairwise) over one upper and one lower index, but never pairs of lower or upper indices. If we have a rank $m \geq n+1$ tensor, then we can contract

$$\begin{aligned} \epsilon^{i_1 i_2 \cdots i_{n+1}}\, T_{i_1 \cdots i_{n+1} i_{n+2} \cdots i_m} &= \epsilon^{i_1 i_2 \cdots i_{n+1}}\, T_{[i_1 \cdots i_{n+1}] i_{n+2} \cdots i_m} \\ &\equiv \tilde{T}_{i_{n+2} i_{n+3} \cdots i_m}, \qquad\qquad m \geq n+1. \end{aligned} \tag{8.2.12}$$

Here repeated indices are summed (Einstein). The square brackets on the right-hand side of the first line mean complete antisymmetrization: the Levi-Civita symbol projects out this combination of components. In particular, if $m = n+1$, then $\tilde{T}$ (if non-zero) is guaranteed to be a scalar, i.e. to not transform under SU$(n+1)$ transformations. The reason is simply because the Levi-Civita symbol has only one unique component. Thus $(n+1)$ different ω_1-indices are needed to form a "singlet" in su$(n+1)$.

We can also use the Levi-Civita symbol to "raise" indices on a completely antisymmetric rank-$m \leq n$ tensor:

$$\epsilon^{i_1 i_2 \cdots i_m i_{m+1} \cdots i_{n+1}}\, T_{[i_1 \cdots i_m]} \equiv T^{[i_{m+1} \cdots i_{n+1}]}, \quad m \leq n. \tag{8.2.13}$$

This gives an alternative tensor representation for the ω_m representation as an antisymmetric tensor with $(n+1-m)$ indices. In particular, the **conjugate $\overline{n+1}$ representation** is defined via

$$T^i \equiv \epsilon^{j_1 j_2 \cdots j_n i} T_{[j_1 j_2 \cdots j_n]}, \quad \overline{n+1} = \omega_n \text{ conjugate representation of su}(n+1). \tag{8.2.14}$$

The $T^{[i_1 \cdots i_m]}$ can be viewed as the completely antisymmetrized tensor of m $(\overline{n+1})$-representation indices; this is itself an irreducible representation with highest weight $\Lambda = \omega_{n+1-m}$ $(1 \leq m \leq n)$. The defining $n+1$ representation is therefore equivalent to the nth-rank tensor $T^{[i_1 \cdots i_n]}$. The precise relation obtains by introducing the Levi-Civita symbol with all lower indices,

$$\epsilon_{i_1 i_2 \cdots i_{n+1}} = \epsilon_{[i_1 i_2 \cdots i_{n+1}]}; \quad \epsilon_{1,2,3,\cdots n,n+1} = +1, \tag{8.2.15}$$

so that

$$T_i = \frac{1}{n!}\, \epsilon_{i j_1 j_2 \cdots j_n} T^{[j_1 j_2 \cdots j_n]} = \frac{1}{n!}\, \epsilon_{i j_1 \cdots j_n} \epsilon^{k j_1 \cdots j_n}\, T_k = T_i. \tag{8.2.16}$$

We can also consider tensors with mixed upper and lower indices. The simplest is the Kronecker delta δ^i_j, which is automatically an invariant tensor. A general second-rank tensor $T^i{}_j$ instead decomposes into

$$\text{scalar } (0): \qquad T_0 \equiv T^i{}_i, \tag{8.2.17a}$$

$$\text{adjoint } (1,1): \quad \tilde{T}^i{}_j \equiv T^i{}_j - \frac{\delta^i_j}{(n+1)} T^k{}_k. \tag{8.2.17b}$$

Thus the pairwise contraction of defining $n+1$ and conjugate $\overline{n+1}$ indices gives a scalar, but this can be alternatively viewed as the contraction with the $n+1$-fold Levi-Civita tensor:

$$U^i V_i = \epsilon^{i_1 i_2 \cdots i_n i_{n+1}}\, U_{[i_1 \cdots i_n]}\, V_{i_{n+1}}. \tag{8.2.18}$$

These are familiar from the action of symmetry transformations on a complex wavefunction or field operator in quantum mechanics and quantum field theory. If

$$\psi \to \hat{U}\psi, \quad \psi^\dagger \to \psi^\dagger \hat{U}^\dagger, \qquad \hat{U}^\dagger \hat{U} = \hat{1}, \tag{8.2.19}$$

then we have the following transformation properties for the *bilinears*:

$$\begin{aligned} \psi^\dagger \psi &\to \psi^\dagger \psi, && \text{invariant scalar,} \\ \psi^\dagger \hat{M} \psi &\to \psi^\dagger (\hat{U}^\dagger \hat{M} \hat{U}) \psi, && \text{adjoint representation for } \mathsf{Tr}(\hat{M}) = 0. \end{aligned}$$

If ψ_i $[(\psi^\dagger)^j]$ transforms in the $(n+1)$ $[(\overline{n+1})]$ representation of SU$(n+1)$, then $M_j{}^i$ can be decomposed into a linear combination of su$(n+1)$ generators. The similarity transformation $\hat{M} \to \hat{U}^\dagger \hat{M} \hat{U}$ acts in the adjoint representation. In Sec. 8.5, we will connect tensors with different combinations of (anti)symmetrized upper and lower indices with more complex irreducible representations of su$(n+1)$.

Eq. (8.2.19) gives an alternative way to define the $\overline{n+1}$ representation. If we express the SU$(n+1)$ transformation as

$$\psi_i \to U_i{}^j \psi_j, \quad \hat{U} = \exp(i\hat{T}^a \theta^a), \tag{8.2.20}$$

where $\{\hat{T}^a\}$ are the generators acting on the $n+1$ representation, then

$$(\psi^\dagger)^i \to (\psi^\dagger)^j (U^*)^i{}_j, \quad \hat{U}^* = \exp\left[i(-\hat{T}^a)^*(\theta^a)^*\right]. \tag{8.2.21}$$

We restrict the generators to the Cartan subalgebra, so that we can take $\hat{T}^a \in \{\hat{H}^{(n+1)}_{\overline{\alpha}_i}\}$. Then $\hat{T}^a = (\hat{T}^a)^\dagger$ and $\theta^a \in \mathbb{R}$. Moreover, the $\hat{H}^{(n+1)}_{\overline{\alpha}_i}$ can be taken to be diagonal and hence symmetric (module 7). We conclude that the Cartan subalgebra generators of $\overline{n+1}$ are (c.f. Fig. 8.4)

$$\hat{H}^{(\overline{n+1})}_{\overline{\alpha}_i} = -\hat{H}^{(n+1)}_{\overline{\alpha}_i}. \tag{8.2.22}$$

Explicitly, if the weight vectors and weights of the defining $n+1$ representation are

$$\begin{aligned}
|1\rangle &= \begin{bmatrix}1\\0\\\vdots\\0\\0\end{bmatrix} \quad \Leftrightarrow \quad \Lambda^{(1)} = (1,0,0,\cdots,0,0), \qquad \text{highest weight state,}\\
|2\rangle &= \begin{bmatrix}0\\1\\\vdots\\0\\0\end{bmatrix} \quad \Leftrightarrow \quad \Lambda^{(2)} = (-1,1,0,\cdots,0,0),\\
&\vdots \qquad\qquad \vdots\\
|n+1\rangle &= \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} \quad \Leftrightarrow \quad \Lambda^{(n+1)} = (0,0,0,\cdots,0,-1), \quad \text{lowest weight state,}
\end{aligned} \tag{8.2.23}$$

then the weight vectors and weights of the conjugate $\overline{n+1}$ representation are

$$\begin{aligned}
|1\rangle &= \begin{bmatrix}1\\0\\\vdots\\0\\0\end{bmatrix} \quad \Leftrightarrow \quad -\Lambda^{(1)} = (-1,0,0,\cdots,0,0), \quad \textbf{lowest} \text{ weight state,}\\
|2\rangle &= \begin{bmatrix}0\\1\\\vdots\\0\\0\end{bmatrix} \quad \Leftrightarrow \quad -\Lambda^{(2)} = (1,-1,0,\cdots,0,0),\\
&\vdots \qquad\qquad \vdots\\
|n+1\rangle &= \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} \quad \Leftrightarrow \quad -\Lambda^{(n+1)} = (0,0,0,\cdots,0,1), \quad \textbf{highest} \text{ weight state.}
\end{aligned} \tag{8.2.24}$$

Therefore the weights of $\overline{n+1}$ are minus those of $n+1$; the highest-weight state of $n+1$ corresponds to the lowest weight state of $\overline{n+1}$. This ensures that the weight of each term $\{U_1 V^1, U_2 V^2, \ldots, U_{n+1} V^{n+1}\}$ is zero, a necessary condition for $U_i V^i$ to be a scalar.

Eq. (8.2.22) can be generalized to all irreducible representations. A representation Λ (defined by the Dynkin coefficients of its highest weight) is **self-conjugate** if its lowest weight is equal to minus its highest weight. Otherwise, the representation is said to be **complex**, and there exists a **conjugate** representation Λ^* which has weights equal to minus those of Λ.

Figure 8.7: Weight trees for the three fundamental representations of su(4): (1,0,0) $\Leftrightarrow \omega_1$ (defining "4") (0,1,0) $\Leftrightarrow \omega_2$, and (0,0,1) $\Leftrightarrow \omega_1$ (conjugate "$\overline{4}$"). The simple root labels $\overline{\alpha}_{1,2,3}$ indicate the difference between the weights above and below the connecting line. The (0,1,0) representation is self-conjugate.

• **Claim:** *All* representations of $B_n = \mathrm{so}(2n+1)$, $C_n = \mathrm{sp}(2n)$, G_2, F_4, E_7, and E_8 are self-conjugate. This is related to the absence of reflection symmetry for the Dynkin diagrams of these algebras [2]. It turns out that all $D_{2n} = \mathrm{so}(4n)$ representations are also self-conjugate. By contrast, $A_n = \mathrm{su}(n+1)$, $D_{2m+1} = \mathrm{so}(4m+2)$, and E_6 have both self-conjugate and pairwise conjugate representations.

8.2.4 The highest root for A_n, B_n, C_n, and D_n

With the basics of tensor-building and conjugate representations in place, we can now define the adjoint representation for each of the classical families in terms of the Dynkin coefficients for the highest root θ. For $A_n = \mathrm{su}(n+1)$, the adjoint representation corresponds to a traceless product of $n+1$ and $\overline{n+1}$ representations, $\tilde{T}^i{}_j$ in Eq. (8.2.17b). Tracelessness does not force the highest-weight state $\tilde{T}^{n+1}{}_1$ to vanish [see Eqs. (8.2.23) and (8.2.24)]. Therefore

$$\theta = \omega_1 + \omega_n \Leftrightarrow (1,0,0,\ldots,0,1), \quad \text{Highest root of } A_n. \tag{8.2.25}$$

For the orthogonal algebras $B_n = \mathrm{so}(2n+1)$ and $D_n = \mathrm{so}(2n)$, the generators acting in the defining vector representation are completely antisymmetric matrices [Eq. (7.3.2)]. The completely antisymmetric tensor $T_{[ij]}$ has weight ω_2:

$$\theta = \omega_2 \Leftrightarrow (0,1,0,\ldots,0,0), \quad \text{Highest root of } B_n \text{ and } D_n. \tag{8.2.26}$$

Finally, in $C_n = \mathrm{sp}(2n)$ the symmetric second rank tensor $T_{(ij)}$ is an irreducible representation, because it cannot be contracted with the $2n \times 2n$ antisymmetric "metric" ϵ^{ij} [Eq. (7.2.2)]. [Note that ϵ^{ij} is a block rank-2 Levi-Civita symbol; this should not be confused

Figure 8.8: Weight geometries for the first two fundamental representations of su(4): (1,0,0) $\Leftrightarrow \omega_1$ (defining) and (0,1,0) $\Leftrightarrow \omega_2$. Weights correspond to the vertices of the polyhedra. The (0,1,0) representation corresponds to the second rank antisymmetric tensor $T_{[ij]}$. The roots are indicated with arrows.

with the rank-$2n$ tensor $\epsilon^{i_1 i_2 \cdots i_{2n}}$. Both are group invariants for Sp($2n$).] In the defining $2n$ representation, the symplectic generators in fact satisfy a generalized symmetry constraint, see Eq. (2A.3.5). We therefore claim that $T_{(ij)}$ is in fact the adjoint representation,

$$\theta = 2\omega_1 \Leftrightarrow (2, 0, 0, \ldots, 0, 0), \quad \text{Highest root of } C_n. \tag{8.2.27}$$

The Dynkin coefficients for the highest root of the exceptional algebras are given in Sec. 13.A of [2].

8.2.5 su(4)

The su(3) algebra exhibits most, but not all general characteristics of su($n+1$) representations. The root system for su(4) was constructed in Sec. 7.1.1. Unlike su(3), the defining and conjugate representations do not exhaust the fundamental weights. In su(4), the second fundamental weight $\omega_2 \Leftrightarrow (0, 1, 0)$ is associated to rank-2 antisymmetric tensor $T_{[ij]}$. This representation is self-conjugate. The weight trees for all three fundamental representations are shown in Fig. 8.7. Using Eq. (8.1.8) to reconstruct the simple root expansion coefficients, one obtains the weight geometries shown in Fig. 8.8. The geometry of the defining ω_1, conjugate ω_3, and adjoint $\theta = \omega_1 + \omega_3$ representations were depicted in Fig. 7.3. The fundamental representations $\{\omega_m\}$ $(1 \le m \le n)$ plus the adjoint representation $\theta = \omega_1 + \omega_n$ exhaust the $n+1$ different conjugacy classes of su($n+1$), as we discuss in Sec. 8.4, below.

- **Exercise**: Verify the weight geometries in Fig. (8.8).
- **Exercise**: Construct the weight tree for the (0,2,0) representation of su(4).

8.3 so($2n+1$), so($2n$), and sp($2n$)

We now explore fundamental highest-weight representations for the other classical families. At rank two, we have the special cases $D_2 = A_1 \times A_1$ and $B_2 = C_2$ (Table 1 in module 7). The representations of the former are trivial, while those of the latter can be inferred from C_n at higher ranks. We therefore begin with rank three.

8.3.1 so($2n+1$)

Consider so(7) $= B_3$. The defining vector representation is $(1, 0, 0) \Leftrightarrow \omega_1$. The adjoint representation (second-rank antisymmetric tensor $T_{[ij]}$) is $(0, 1, 0) \Leftrightarrow \omega_2$. Can we associate $(0, 0, 1) \Leftrightarrow \omega_3$ with a third-rank antisymmetric tensor? To determine this, we compute the weight tree for $(1, 0, 0)$ using the Cartan matrix [Eq. (7.4.20)]

$$\hat{A}_{B_3} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}. \tag{8.3.1}$$

The results are shown in Fig. 8.9.

As in A_n, all completely antisymmetric tensors are irreducible representations of B_n or D_n, since we cannot trace out any pair of indices with the symmetric "metric" δ^{ij} (i.e. the usual vector dot product). We will compute the Dynkin coefficients for the antisymmetric tensor representations of so(7) from the $(1, 0, 0)$ weight tree, as we did in Eq. (8.2.6).

- $m = 2$: second-rank, antisymmetric tensor $T_{[ij]}$ with 21 components. From Fig. 8.9, this has highest weight

$$(1, 0, 0) + (-1, 1, 0) = (0, 1, 0) \to \omega_2, \tag{8.3.2a}$$

  consistent with the identification of (0,1,0) with the adjoint representation.

- $m = 3$: $T_{[ijk]}$ with 35 components. The highest weight is

$$(1, 0, 0) + (-1, 1, 0) + (0, -1, 2) = (0, 0, 2) \to 2\omega_3. \tag{8.3.2b}$$

Thus the third-rank antisymmetric tensor is associated to $(0, 0, 2) \to 2\omega_3$, and is *not* a fundamental weight. Note that there is no need to consider higher-rank antisymmetric tensors, since these are equivalent to the first three via the rank-7 Levi-Civita symbol:

$$\begin{aligned} T_{[i_1 i_2 i_3]} &\equiv \epsilon_{i_1 i_2 i_3 i_4 i_5 i_6 i_7}\, T_{[i_1 i_2 i_3 i_4]} \Leftrightarrow (0, 0, 2) \to 2\omega_3, \\ T_{[i_1 i_2]} &\equiv \epsilon_{i_1 i_2 i_3 i_4 i_5 i_6 i_7}\, T_{[i_1 i_2 i_3 i_4 i_5]} \Leftrightarrow (0, 1, 0) \to \omega_2, \\ T_{i_1} &\equiv \epsilon_{i_1 i_2 i_3 i_4 i_5 i_6 i_7}\, T_{[i_1 i_2 i_3 i_4 i_5 i_6]} \Leftrightarrow (1, 0, 0) \to \omega_1, \\ T &\equiv \epsilon_{i_1 i_2 i_3 i_4 i_5 i_6 i_7}\, T_{[i_1 i_2 i_3 i_4 i_5 i_6 i_7]} \Leftrightarrow 0 \text{ (scalar)}. \end{aligned} \tag{8.3.3}$$

Figure 8.9: Weight trees for the first and last fundamental representations of so(7): (1,0,0) $\Leftrightarrow \omega_1$ (defining) and (0,0,1) $\Leftrightarrow \omega_3$. The simple root labels $\overline{\alpha}_{1,2,3}$ indicate the difference between the weights above and below the connecting line. The (0,0,1) representation has $2^3 = 8$ total states, and corresponds to the *fundamental spinor* representation of so(7). We will discuss spinor representations of B_n and D_n in detail in module 10. Like all representations of B_n, these representations are self-conjugate.

In B_n, C_n, and D_n, the defining representation is self-conjugate. There is thus no need to distinguish between upper and lower indices. (In fact all representations of B_n, C_n, and D_{2m} are self-conjugate, but this is not true for D_{2m+1}).

The question remains: what is the $(0,0,1) \to \omega_3$ representation of so(7) $= B_3$? Clearly it cannot be obtained by any combination of symmetrizing or antisymmetrizing vector indices. The weight tree is shown in Fig. 8.9. There are $2^3 = 8$ different weights in $(0,0,1)$; all turn out to be non-degenerate. The highest weight representation ω_3 is the **fundamental spinor** representation of so(7). We will discuss spinor representations of B_n and D_n in detail in module 10.

Using the Eq. (8.1.8) and the root geometry for B_3, one can determine the weight geometries for the fundamental vector $(1,0,0)$ and spinor $(0,0,1)$ representations. These are depicted in Fig. 8.10. An interesting and important fact is that the positive

Figure 8.10: Weight geometries for the first and last fundamental representations of so(7): (1,0,0) $\Leftrightarrow \omega_1$ (defining) and (0,0,1) $\Leftrightarrow \omega_3$ (fundamental spinor). Weights correspond to the vertices of the polyhedra. The defining vector representation (1,0,0) also includes the central weight (0,0,0); in both subfigures, this is indicated by the black dot. The other (1,0,0) weights are also indicated by arrows in both subfigures. A special feature of the defining representation for both $B_n = \text{so}(2n+1)$ and $D_n = \text{so}(2n)$ is that the positive weights are orthonormal [c.f. Eqs. (8.3.4) and (8.3.5)]. Note that roots are not depicted in either subfigure.

weights of the vector representation (Fig. 8.9)

$$\begin{aligned}\Lambda^{(1)} &\equiv (1,0,0) = \omega_1,\\ \Lambda^{(2)} &\equiv (-1,1,0),\\ \Lambda^{(3)} &\equiv (0,-1,2),\end{aligned} \tag{8.3.4}$$

are **orthonormal**:

$$\langle \Lambda^{(a)}, \Lambda^{(b)} \rangle = \delta^{a,b}, \tag{8.3.5}$$

(adopting a standard normalization of the roots such that $|\alpha|^2 = 2$ for long roots—see module 9). In Eqs. (8.3.4) and (8.3.5), we define positive weights in the vector representation as the highest weight and its descendants above the central zero weight.

• **Exercise**: Prove Eq. (8.3.5) using the Dynkin coefficients for each weight and the quadratic form matrix F_{ij} for so(7), given by Eq. (8.6.2b).

By contrast, the weights of the fundamental spinor representation ($\equiv \{\Xi^{(a)}\}$) correspond to the vertices of the cube,

$$\Xi^{(a)} \in \frac{1}{2}\left(s_1\Lambda^{(1)} + s_2\Lambda^{(2)} + s_3\Lambda^{(3)}\right), \tag{8.3.6}$$

where $s_{1,2,3} \in \pm 1$. See Fig. 8.10. These facts generalize to $B_n = \mathrm{so}(2n+1)$ for $n \geq 3$:

<u>Highest fundamental weight representations of $B_n = \mathrm{so}(2n+1)$</u>

- The n positive weights $\{\Lambda^{(a)} > 0\}$ of the defining vector representation ω_1 are orthonormal [Eq. (8.3.5)].
- The fundamental weight ω_m representation ($1 \leq m \leq n-1$) corresponds to a rank-m antisymmetric tensor $T_{[i_1\cdots i_m]}$.
- The fundamental weight ω_n is the highest weight state of the **fundamental spinor** representation.
  This has 2^n non-degenerate weights that correspond to the vertices of an n-dimensional hypercube.
- The spinor representation cannot be obtained by tensoring together vector indices.

(8.3.7)

We will prove these statements in modules 9 and 10. Finally, we note that symmetric traceless tensors of vector indices are also irreducible representations:

$$\tilde{T}_{(i_1\cdots i_m)}, \quad \tilde{T}_{(iii_3\cdots i_m)} = 0, \quad m \geq 1: \quad \text{irreducible representation of so}(2n+1) \text{ with highest weight } m\,\omega_1. \tag{8.3.8}$$

The weights that correspond to irreducible representations built entirely from vector indices reside in a single conjugacy class of B_n. This includes any combination of the fundamental weights $\{\omega_m\}$ with $1 \leq m \leq n-1$. The fundamental spinor representation resides in a separate conjugacy class. Higher-rank spinor- or mixed spinor- and vector-index tensors belong to one of these two classes.

8.3.2 sp($2n$)

The defining $2n$ representation of sp($2n$) has highest weight ω_1. Using the Cartan matrix [Eq. (7.2.21)]

$$\hat{A}_{C_n} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -2 & 2 \end{bmatrix}, \tag{8.3.9}$$

we can obtain the weight tree shown in Fig. 8.11.

• **Exercise**: Derive the weight tree shown in Fig. 8.11.

Unlike A_n, B_n, and D_n, an antisymmetrized tensor $T_{[i_1 i_2\cdots i_m]}$ does not automatically correspond to an irreducible representation of C_n. This is because any pair of indices can be contracted with the rank-2 antisymmetric "metric" ϵ_{ij} [Eq. (7.2.2)], which is a group invariant. Here we disregard the distinction between upper and lower indices, because the defining and all other irreducible representations of sp($2n$) are self-conjugate. We can define *traceless* antisymmetric tensors

$$\tilde{T}_{[i_1\cdots i_m]}, \quad \epsilon_{pq}\,\tilde{T}_{[pqi_3\cdots i_m]} = 0, \qquad 1 \leq m \leq n, \tag{8.3.10}$$

Figure 8.11: Weight tree for the defining representation of sp($2n$) (C_n): $(1, 0, 0, \cdots, 0) \Leftrightarrow \omega_1$. The simple root labels $\{\overline{\alpha}_i\}$ indicate the difference between the weights above and below the connecting line. Like all representations of C_n, the defining representation is self-conjugate.

and these do correspond to irreducible representations. There is no need to consider a rank $m > n$ antisymmetric tensor, since if it is non-zero it is equivalent to a rank $2n - m < n$ tensor, obtained by contracting indices with the rank-$2n$ Levi-Civita symbol. The rank-m traceless tensor $\tilde{T}_{[i_1 \cdots i_m]}$ has

$$\frac{(2n)!}{m!(2n-m)!} - \frac{(2n)!}{(m-2)!(2n-m+2)!} = 2(n+1-m)\frac{(2n+1)!}{m!(2n-m+2)!} \tag{8.3.11}$$

independent components. On the left-hand-side of this equation, the second term is the number of constraints imposed by the trace condition $\epsilon_{pq}\,\tilde{T}_{[pqi_3 \cdots i_m]} = 0$.

We determine the Dynkin coefficients for traceless antisymmetric tensors as in Eq. (8.2.6).

- $m = 2$: $\tilde{T}_{[ij]}$. Tracelessness does not require that $\tilde{T}_{12} = -\tilde{T}_{21}$ is equal to zero. Fig. 8.11 therefore implies that the highest weight is

$$(1, 0, 0, \ldots, 0) + (-1, 1, 0, \ldots, 0) = (0, 1, 0, \ldots, 0) \to \omega_2. \tag{8.3.12a}$$

- $m = 3$: $\tilde{T}_{[ijk]}$. The highest weight is

$$(1, 0, 0, 0, \ldots, 0) + (-1, 1, 0, 0, \ldots, 0) + (0, -1, 1, 0, \ldots, 0) = (0, 0, 1, \ldots, 0) \to \omega_3. \tag{8.3.12b}$$

$\vdots$

- $m = n$: $\tilde{T}_{[i_1 i_2 \cdots i_n]}$. The highest weight is

$$(1, 0, 0, \ldots, 0, 0, 0) + (-1, 1, 0, \ldots, 0, 0, 0) + \ldots + (0, 0, 0, \ldots, 0, -1, 1) = (0, 0, 0, \ldots, 0, 0, 1) \to \omega_n. \tag{8.3.12c}$$

- $m = n + 1$: $\tilde{T}_{[i_1 i_2 \cdots i_n i_{n+1}]}$. The weight is

$$(0, 0, 0, \ldots, 0, 0, 1) + (0, 0, 0, \ldots, 0, 1, -1) = (0, 0, 0, \ldots, 0, 1, 0) \to \omega_{n-1}. \tag{8.3.12d}$$

This is just the $m = n - 1$ tensor in disguise. More generally, the rank-$2n$ Levi-Civita symbol converts

$$\epsilon_{[i_1 i_2 \cdots i_{n+m} j_1 \cdots j_{n-m}]}\,\tilde{T}_{[i_1 \cdots i_{n+m}]} \equiv \tilde{T}_{[j_1 \cdots j_{n-m}]}, \quad 1 \le m \le n. \tag{8.3.12e}$$

As argued above in Sec. 8.2.4, completely symmetric tensors are automatically irreducible representations. We conclude that

$$\begin{array}{l}\underline{\text{Elementary tensor representations of } C_n = \mathrm{sp}(2n)\text{:}} \\ \bullet\ \tilde{T}_{[i_1 i_2 \cdots i_m]}, \quad \epsilon_{pq}\,\tilde{T}_{[pqi_3\cdots i_m]} = 0, \quad 1 \le m \le n: \text{ irreducible representation with highest weight } \omega_m. \\ \bullet\ T_{(i_1 i_2 \cdots i_m)}, \quad m \ge 1: \text{ irreducible representation with highest weight } m\,\omega_1.\end{array} \tag{8.3.13}$$

Thus, similar to su$(n+1)$, all fundamental representations of C_n can be identified with (traceless) antisymmetric tensors. All irreducible representations of C_n can be formed by tensoring together copies of the defining ω_1 representation, symmetrizing and/or antisymmetrizing, and removing all traces.

• **Exercise**: Show that the top n weights $\{\Lambda^{(a)}\}$ of the defining $2n$ representation satisfy

$$\langle \Lambda^{(a)}, \Lambda^{(b)} \rangle = \frac{1}{2}\delta^{a,b}. \tag{8.3.14}$$

To do this, use the Dynkin coefficients for the weights shown in Fig. 8.11 and the explicit form of the quadratic form matrix F_{ij} given by Eq. (8.6.2c).

8.3.3 so$(2n)$

Finally we consider $D_n = \mathrm{so}(2n)$. The defining $2n$-component vector representation has highest weight ω_1. The descendant weights obtain via the MWDF, using the Cartan matrix [Eq. (7.3.24)]

$$\hat{A}_{D_n} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & 2 \end{bmatrix}. \tag{8.3.15}$$

The result is the weight tree shown in Fig. 8.12.

As for B_n, we expect there will be spinor representations of D_n corresponding to fundamental weights. To find them, we first search for completely antisymmetric tensor representations, following the same logic in Eqs. (8.2.6), (8.3.2), and (8.3.12). One finds that the first $n-2$ ranks correspond to fundamental weights; a traceless symmetric tensor of rank m corresponds as usual to weight $m\,\omega_1$:

$$\begin{array}{l}\underline{\text{Elementary tensor representations of } D_n = \mathrm{so}(2n)\text{:}} \\ \bullet\ T_{[i_1 i_2 \cdots i_m]}, \quad 1 \le m \le n-2: \text{ irreducible representation with highest weight } \omega_m. \\ \bullet\ \tilde{T}_{(i_1 i_2 \cdots i_m)}, \quad \delta_{pq}\,\tilde{T}_{(pqi_3\cdots i_m)} = 0, \quad m \ge 1: \text{ irreducible representation with highest weight } m\,\omega_1.\end{array} \tag{8.3.16}$$

As for B_n and C_n, we do not distinguish upper and lower indices because the defining representation is self-conjugate.

What about rank-$(n-1)$ and rank-n tensors? The Dynkin labels for the highest weight state of $T_{[i_1\cdots i_{n-1}]}$ are given by [Eq. (8.3.16) and Fig. 8.12]

$$\bullet\ T_{[i_1 i_2 \cdots i_{n-1}]}: \quad (0,0,0,\dots,0,1,0,0) + (0,0,0,\dots,0,-1,1,1) = (0,0,0,\dots,0,1,1) \to \omega_{n-1} + \omega_n. \tag{8.3.17}$$

For the rank-n tensor $T_{[i_1\cdots i_n]}$, there are *two* possibilities according to Fig. 8.12:

$$\bullet\ T_{[i_1 i_2 \cdots i_n]}: \quad \begin{array}{l} (0,0,0,\dots,0,1,1) + (0,0,0,\dots,0,-1,1) = (0,0,0,\dots,0,0,2) \to 2\omega_n, \\ (0,0,0,\dots,0,1,1) + (0,0,0,\dots,0,1,-1) = (0,0,0,\dots,0,2,0) \to 2\omega_{n-1}. \end{array} \tag{8.3.18}$$

We see that $T_{[i_1 i_2 \cdots i_n]}$ splits into two representations. The interpretation is that there are really two independent rank-n antisymmetric tensors. Define

$$\begin{aligned} T^{(I)}_{[i_1 i_2 \cdots i_n]} &\equiv T_{[i_1 i_2 \cdots i_n]}, \\ T^{(II)}_{[i_1 i_2 \cdots i_n]} &\equiv \frac{1}{n!}\epsilon_{i_1\cdots i_n j_1 \cdots j_n}\, T_{[j_1 j_2 \cdots j_n]}. \end{aligned} \tag{8.3.19}$$

Note that

$$\frac{1}{n!}\epsilon_{i_1\cdots i_n j_1\cdots j_n} T^{(II)}_{[j_1 j_2\cdots j_n]} = \left(\frac{1}{n!}\right)^2 \epsilon_{i_1\cdots i_n j_1\cdots j_n}\, \epsilon_{j_1\cdots j_n k_1\cdots k_n}\, T^{(I)}_{[k_1 k_2\cdots k_n]} \tag{8.3.20}$$

$$= T^{(I)}_{[i_1 i_2\cdots i_n]}, \tag{8.3.21}$$

where we have used the fact (not proven here) that

$$\epsilon_{i_1\cdots i_n j_1\cdots j_n}\, \epsilon_{j_1\cdots j_n k_1\cdots k_n} = n! \sum_P (-1)^P \delta_{i_1,P(k_1)}\delta_{i_2,P(k_2)} \times \cdots \times \delta_{i_n,P(k_n)},$$

and P denotes a permutation of the n symbols $\{k_1, k_2, \ldots, k_n\}$. Thus $T^{(I)}$ and $T^{(II)}$ are "dual" to each other. One can form the combinations [4]

$$T^{\pm}_{[i_1 i_2\cdots i_n]} \equiv T^{(I)}_{[i_1 i_2\cdots i_n]} \pm T^{(II)}_{[i_1 i_2\cdots i_n]}. \tag{8.3.22}$$

Then T^+ (T^-) is self-dual (anti-self-dual), and these are independent representations.

Therefore the rank $n-1$ and n antisymmetric tensors do not correspond to fundamental weights. In D_n, there are *two* fundamental spinor representations. We will prove the following statements in module 10:

<u>Fundamental spinor representations of $D_n = \mathrm{so}(2n)$:</u>

- The fundamental weights ω_{n-1} and ω_n (associated to the "ears" of the D_n Dynkin diagram in Fig. 8.3) are the highest weight states of the two **fundamental spinor** representations.
- Each spinor representation has 2^{n-1} non-degenerate weights. These can be viewed as the bipartitioning of the 2^n-dimension $\mathrm{so}(2n+1)$ fundamental spinor representation.
- The spinor representations cannot be obtained by tensoring together vector indices.

(8.3.23)

Although the fundamental spinor representations cannot be constructed by tensoring together vector indices, the converse is not true: *all* representations can be constructed by tensoring together fundamental spinor indices. The key to the construction is the **Clifford algebra**, introduced in module 10. The semi-simple algebra $D_2 = A_1 \times A_1 = \mathrm{su}(2)\times\mathrm{su}(2)$ gives an extreme example

Figure 8.12: Weight tree for the defining representation of $\mathrm{so}(2n)$ (D_n): $(1, 0, 0, \cdots, 0) \Leftrightarrow \omega_1$. The simple root labels $\{\overline{\alpha}_i\}$ indicate the difference between the weights above and below the connecting line.

of this, since the two fundamental weights $\omega_{1,2}$ are spin-1/2 spinor representations. In high-energy (and recently, condensed matter) physics, these two inequivalent representations of so(4) correspond to left- and right-handed Weyl fermions.

For $D_{2m} = \text{so}(4m)$ ($m \in \{1, 2, \ldots\}$), the two fundamental spinor representations are each self-conjugate. For $D_{2m+1} = \text{so}(4m+2)$ ($m \in \{1, 2, \ldots\}$), the two fundamental spinor representations are pairwise conjugates. The spinor representations belong to different conjugacy classes than tensors built from vector indices, but the overall class structure of D_n is slightly more involved than it is for the other algebras. It will be discussed in the next section.

• **Exercise**: Show that the two spinor representations are self-conjugate (pairwise conjugate) for so(8) [so(10)] by computing the weight trees from the Dynkin coefficients

a. so(8): $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$,

b. so(10): $(0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 1)$.

How many distinct weights do you find in each case?

8.4 Lattices and conjugacy classes

In a rank-n Lie algebra, all roots can be expressed as linear combinations of n simple roots with integral coefficients. The MWDF Eq. (5.1.23) then implies that all weights in a given irreducible representation consist of a section of an infinite n-dimensional lattice generated by the simple root vectors. The weight geometries computed thus far are displayed in Figs. 8.2, 8.5, and 8.6 [$A_2 = \text{su}(3)$], Figs. 7.3 and 8.8 [$A_3 = \text{su}(4)$], and Fig. 8.10 [$B_3 = \text{so}(7)$]. These figures make it clear that different representations of the same algebra can reside on different lattices. These differ only by a fractional lattice vector displacement relative to the origin, which we assign to the zero weight (i.e., the Cartan subalgebra of the adjoint representation).

In other words, the set of all representations divides into different **conjugacy classes** (or congruence classes), and these classes label the different lattices. We can crystallize this idea by defining two lattices:

$$\text{The weight lattice:} \quad P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \ldots + \mathbb{Z}\omega_n, \tag{8.4.1a}$$

$$\text{The root lattice:} \quad Q = \mathbb{Z}\overline{\alpha}_1 + \mathbb{Z}\overline{\alpha}_2 + \ldots + \mathbb{Z}\overline{\alpha}_n. \tag{8.4.1b}$$

All weights of all highest weight representations reside in the weight lattice P. Only those representations that include the zero weight reside in the root lattice Q. We label the conjugacy class of Q by zero; this automatically includes the adjoint representation of any semi-simple Lie algebra. The lattices P and Q can be viewed as abelian groups, and the number of conjugacy classes is equal to the number of distinct elements of the coset P/Q, which is a finite (factor) group. The order $|P/Q|$ is equal to the determinant of the Cartan matrix. For the classical families, one has

algebra	$\lvert P/Q \rvert$
A_n	$n+1$
B_n	2
C_n	2
D_n	4

(8.4.2)

We can assign conjugacy classes using the **congruence vector** v, an n-component vector with integer elements that is fixed for a given algebra. For a weight Λ, the conjugacy classes are defined via

$$\Lambda \cdot v \equiv \sum_{i=1}^{n} \Lambda_i \, v_i \qquad \text{mod } |P/Q| \,. \tag{8.4.3}$$

In this equation, the $\{\Lambda_i\}$ are the Dynkin coefficients of weight Λ.

8.4.1 $A_n = \mathbf{su}(n+1)$

The congruence vector is

$$v = \{1, 2, 3, \ldots, n\}, \tag{8.4.4}$$

so that

$$\Lambda \cdot v = \Lambda_1 + 2\Lambda_2 + 3\Lambda_3 + \ldots + n\Lambda_n \qquad \text{mod } (n+1). \tag{8.4.5}$$

Figure 8.13: su(3) representations in each of the algebra's three conjugacy classes. Classes 0, 1, and 2 include the adjoint, defining 3, and conjugate $\bar{3}$ representations, respectively (see Fig. 8.2). Here we display the weights of three higher-dimensional representations: (3,0), (2,1), and (2,0). All reside on triangular lattices generated by the simple root vectors $\overline{\alpha}_1$ and $\overline{\alpha}_2$, but the origin for the class 1 or 2 lattice is shifted relative to class 0. The adjoint origin (the zero weight associated to the Cartan subalgebra) is indicated by the axes drawn in each subfigure.

Thus the fundamental weight representation with highest weight ω_m $(1 \leq m \leq n)$ resides in class m, while the adjoint representation resides in class 0. The defining $n+1$ (conjugate $\overline{n+1}$) representation is in class 1 (n).

The three classes of su(3) were previously exhibited in Fig. 8.2. Fig. 8.13 shows three other representatives of the su(3) conjugacy classes.

8.4.2 $B_n = \mathbf{so}(2n+1)$

We first note that all irreducible representations of B_n are self-conjugate, which means the lowest weight is equal to minus the highest weight in every representation. This in part explains the simpler conjugacy class structure relative to A_n.

The congruence vector is

$$v = \{0, 0, 0, \ldots, 0, 1\}, \tag{8.4.6}$$

so that

$$\Lambda \cdot v = \Lambda_n \qquad \text{mod } (2). \tag{8.4.7}$$

Recall that ω_n corresponds to the fundamental spinor representation. This resides in class 1. All other fundamental weight representations ω_m $(1 \leq m \leq n-1)$ reside in class 0. Moreover, all representations built from the first $n-1$ fundamental weights also reside in class 0, as does any representation in which Λ_n is even. These are the irreducible representations that obtain by tensoring together vector indices. Irreducible representations built from an even (odd) number of spinor indices and any number of vector indices reside in class 0 (1).

8.4.3 $C_n = \mathbf{sp}(2n)$

All irreducible representations of C_n are self-conjugate.

The congruence vector is

$$v = \{1, 2, 3, \ldots, n\}, \tag{8.4.8}$$

so that

$$\Lambda \cdot v = \Lambda_1 + 2\Lambda_2 + 3\Lambda_3 + \ldots + n\Lambda_n \qquad \text{mod } (2). \tag{8.4.9}$$

The odd (even) fundamental weight representation ω_m with $m \in \{1, 3, 5, \ldots\}$ $(m \in \{2, 4, 6, \ldots\})$ resides in class 1 (0). The adjoint representation $\theta = 2\omega_1$ resides in class 0 (as expected).

8.4.4 $D_{2m} = \mathbf{so}(4m)$, $m \in \mathbb{Z}_+$.

All irreducible representations of D_n with $n = 2m$ (even) are self-conjugate. This leads to a simpler conjugacy class structure than D_{2m+1}.

The congruence vector is

$$v = \{0, 0, 0, \ldots, 0, 1, 1\}, \tag{8.4.10}$$

so that

$$\Lambda \cdot v = \Lambda_{n-1} + \Lambda_n \mod (4), \qquad n = 2m. \tag{8.4.11}$$

The fundamental representations $\{\omega_m\}$ with $\{1 \leq m \leq n-2\}$ reside in class 0. This includes the adjoint representation ω_2. The fundamental spinor representations ω_{n-1} and ω_n reside in class 1. Completely antisymmetric tensors of rank $n-1$ and n reside in class 2 [Eqs. (8.3.17) and (8.3.18)].

8.4.5 $D_{2m+1} = \mathbf{so}(4m+2)$, $m \in \mathbb{Z}_+$.

Representations of D_n with $n = 2m+1$ (odd) can be self- or pairwise-conjugate.

The congruence vector is

$$v = \{2, 4, 6, \ldots, 2n-4, n-2, n\}, \qquad n = 2m+1. \tag{8.4.12}$$

so that

$$\Lambda \cdot v = 2\Lambda_1 + 4\Lambda_2 + 6\Lambda_3 + \ldots + 2(n-2)\Lambda_{n-2} + (n-2)\Lambda_{n-1} + n\Lambda_n \mod (4). \tag{8.4.13}$$

Thus the fundamental highest weight representations divide up as follows:

class	fundamental weight representations
0	$\omega_2,\ \omega_4,\ \omega_6,\ \ldots, \omega_{n-3}$
2	$\omega_1,\ \omega_3,\ \omega_5,\ \ldots, \omega_{n-2}$
$(2m-1) \mod (4)$	ω_{n-1}
$(2m+1) \mod (4)$	ω_n

(8.4.14)

Completely antisymmetric tensors of even (odd) rank less than $n-2$ reside in class 0 (2). Completely symmetric tensors of any even (odd) rank also reside in class 0 (2). The rank-$(n-1)$ completely antisymmetric tensor [Eq. (8.3.17)] resides in class 0, while the rank-n antisymmetric tensor [Eq. (8.3.17)] divides into two representations, both in class 2.

• **Exercise**: Prove two facts about the (0,2,0) and (0,0,2) partitions of the third-rank antisymmetric tensor of D_3:

1. The representations (0,2,0) and (0,0,2) reside in the same conjugacy class (for example, share a weight),
2. The (0,2,0) and (0,0,2) representations are pairwise conjugate.

To do this, compute the weight trees for these representations explicitly. Are there degenerate weights?

The fundamental spinor representations $\{\omega_{n-1}, \omega_n\}$ respectively reside in conjugacy classes $\{1, 3\}$ or $\{3, 1\}$, depending on m.

8.5 Tensors and tableaux for su(n)

For the four classical Lie algebra families, we have seen how to associate completely antisymmetric and completely symmetric tensors (with suitable traces removed in B_n, C_n, and D_n) with the fundamental weights $\{\omega_m\}$, or simple combinations thereof. The orthogonal algebra B_n (D_n) has in addition one (two) fundamental spinor representation(s). We will consider spinor representations and their tensor products in module 10.

Completely antisymmetric, symmetric, or spinor tensors are the most frequently encountered representations in many-body quantum physics, be it high energy or condensed matter. These can arise e.g. as composite operators built from products of elementary boson or fermion fields with indices transforming in some physical symmetry space (spin, particle-hole, color, flavor, replica, etc.). They can also arise as order parameters for effective low-energy field theories with a spontaneously broken continuous symmetry. Nevertheless, occasionally one must confront more complicated representations corresponding to mixed symmetry tensors.

Figure 8.14: Complete set of Young tableaux with 3 boxes (top) and 5 boxes (bottom). Each n-box tableau corresponds to particular partition of n.

The standard method for dealing with generic tensor representations in $A_{n-1} = \mathrm{su}(n)$ is **Young tableaux**. These also appear in the classification of representations of the symmetric permutation group S_n (of order $n!$). An advantage of Young tableaux is that they can be used to find all states in a representation, including degeneracies; the MWDF algorithm that we have been using only determines the weights, but not the dimensionality of the associated weight spaces. In the next module, we will however develop a recursive algorithm (Freudenthal) for determining weight degeneracies for any representation of any semi-simple Lie algebra, as well as the total dimension of such a representation (Weyl).

A disadvantage of Young tableaux is that they are mainly useful for su(n), whereas we have tried to emphasize tools that work the same way for all Lie algebras.

8.5.1 Young tableaux

A **Young tableau**[4] is a top- and left-justified array of boxes. The boxes are arranged into rows. If the mth row has p_m boxes, then the $(m+1)^{\text{th}}$ row can have p_{m+1} boxes with $0 \leq p_{m+1} \leq p_m$. In other words, the number of boxes in each successive row must either decrease or stay the same.

Young tableaux enumerate the **partitions of the positive integers (natural numbers)** $\mathbb{N}$. A partition is a decomposition of a number $n \in \mathbb{N}$ into a sum over components $\{n_i \in \mathbb{N}\}$, such that $1 \leq n_i \leq n$. For example, there are 5 partitions of the number 4:

$$\begin{aligned} 4 &= 1+1+1+1 \\ &= 1+1+2 \\ &= 2+2 \\ &= 1+3 \\ &= 4. \end{aligned}$$

The set of all Young tableux with n boxes is in one-to-one correspondence with the partitions of n, see Fig. 8.14.

8.5.2 Young tableau for a generic highest weight representation; tensors

There is a canonical construction for assembling a Young tableau from a highest weight representation of A_{n-1}, using the expansion of the weight in terms of an orthonormal basis (instead of the simple roots, coroots, or fundamental weights). This can be found e.g. in Sec. 13.3.2 of [2]. The derivation leads to the conclusion that a highest weight can be viewed in terms the *integral partitions* of its Dynkin labels. Because Young tableaux precisely enumerate partitions of the positive integers, they are an efficient tool for su(n). Here we will only state the result and provide examples of particular classes of tensor representations.

The Young tableau for a generic highest-weight representation of su(n) is shown in Fig. 8.15. If the highest weight has Dynkin labels $\Lambda \to \{\Lambda_1, \Lambda_2, \ldots, \Lambda_{n-1}\}$, then the tableau has $\Lambda_{n-1} \geq 0$ columns of length $(n-1)$, $\Lambda_{n-2} \geq 0$ columns of length $(n-2)$, etc. We can interpret each box in the tableau as a particular index i_m in a rank-R tensor $(1 \leq m \leq R)$. All indices belong to the defining ("n") representation $(1 \leq i_m \leq n)$. From Fig. 8.15, the rank R is

$$\begin{aligned} R &= \Lambda_1 + 2\Lambda_2 + 3\Lambda_3 + \ldots + (n-1)\Lambda_{n-1} \\ &= \sum_{i=1}^{n-1} v_i \, \Lambda_i, \end{aligned} \tag{8.5.1}$$

[4] Unlabled tableaux are sometimes called Young diagrams.

Figure 8.15: Young tableau for a generic finite-dimensional, irreducible representation of su(n) $= A_{n-1}$. The highest weight is assumed to possess the Dynkin labels $\Lambda \to \{\Lambda_1, \Lambda_2, \ldots, \Lambda_{n-1}\}$.

where $v = \{1, 2, \ldots, n-1\}$ is the congruence vector for su(n) [Eq. (8.4.4)]. Eq. (8.5.1) implies that **all Young tableaux with the same number of boxes (mod n) belong to the same conjugacy class.**

The rank-R tensor $T_{i_1 i_2 \cdots i_R}$ corresponding to the tableau in Fig. 8.15 is made an irreducible representation by the following prescription:

1. Take a generic rank-R tensor, and symmetrize the indices within each row of the tableau.
2. Now antisymmetrize indices within each column.

A generic tensor of this form is obviously cumbersome to work with; in the remainder we will specialize to simpler cases.

The mth fundamental representation (rank-m fully antisymmetric tensor) with highest weight $\Lambda = \omega_m$ corresponds to a single column of m boxes, (a) in Fig. 8.16. The rank-p fully symmetric tensor with highest weight $\Lambda = p\,\omega_1$ corresponds to a single row of p boxes. This displays the general scheme: indices in rows are (pre)symmetrized, while indices in columns are antisymmetrized. Moreover, we cannot have a column of length greater than n. A length-n column corresponds to a set of n indices that can be contracted with the rank-n Levi-Civita symbol; for this reason, we always omit columns of length n.

We can also exhibit tableaux for tensors built from conjugate $\bar{n}$ ("upstairs") indices (see Sec. 8.2.3, above). The rank-m fully antisymmetric tensor $T^{[i_1 \cdots i_m]}$ has weight ω_{n-m}, and thus corresponds to a single column of $n-m$ boxes, (c) in Fig. 8.16. A fully symmetrized rank-p conjugate-index tensor $T^{(i_1 \cdots i_p)}$ has highest weight $\Lambda = (0, 0, \ldots, 0, p) = p\,\omega_{n-1}$. This corresponds to the $(n-1) \times p$ block tableau in Fig. 8.16(d). It is always important to remember that the boxes of correspond to defining n-representation indices. In particular, the rank-1 conjugate representation T^i corresponds to a single <u>maximal column</u> of $(n-1)$ boxes.

More complicated tensors can be built by mixing defining-n and conjugate-$\bar{n}$ indices. Two simple classes that correspond to irreducible representations are mixed, traceless, mutually (anti)symmetric tensors:

- $\tilde{T}^{(j_1 \cdots j_q)}_{(i_1 \cdots i_p)}$; $\tilde{T}^{(k j_2 \cdots j_q)}_{(k i_2 \cdots i_p)} = 0$: bisymmetrized rank-$p$ $\times$ rank-q traceless tensor. The highest weight is

$$(p, 0, 0, \ldots, 0, q) \Leftrightarrow p\,\omega_1 + q\,\omega_{n-1}. \tag{8.5.2}$$

Figure 8.16: Young tableau for simple tensors. (a) Fundamental representation $T_{[i_1 \cdots i_m]} \Leftrightarrow \omega_m$. (b) Fully symmetrized representation $T_{(i_1 \cdots i_p)} \Leftrightarrow p\,\omega_1$. (c) Fully antisymmetrized rank-m tensor built from conjugate indices, $T^{[i_1 \cdots i_m]} \Leftrightarrow T_{[j_1 \cdots j_{n-m}]} \Leftrightarrow \omega_{n-m}$. (d) Fully symmetrized rank-p tensor built from conjugate indices, $T^{(i_1 \cdots i_p)} \Leftrightarrow p\,\omega_{n-1}$.

Figure 8.17: Subfigures (a) and (b) can be associated to tensors with mixed lower and upper indices. Subfigure (a) corresponds to a symmetrized traceless tensor $\tilde{T}^{(j_1\cdots j_q)}_{(i_1\cdots i_p)} \Leftrightarrow p\,\omega_1 + q\,\omega_{n-1}$ [Eq. (8.5.2)]. Subfigure (b) corresponds to an antisymmetrized traceless tensor $\tilde{T}^{[j_1\cdots j_q]}_{[i_1\cdots i_p]} \Leftrightarrow \omega_p + \omega_{n-q}$ $(p+q \neq n)$ [Eq. (8.5.3)]. Subfigure (c) shows a generic block tableau corresponding to $p\,\omega_q$ [Eq. (8.5.4)].

- $\tilde{T}^{[j_1\cdots j_q]}_{[i_1\cdots i_p]}$; $\tilde{T}^{[kj_2\cdots j_q]}_{[ki_2\cdots i_p]} = 0$: biantisymmetrized rank-p $\times$ rank-q traceless tensor. The highest weight is

$$(0,0,\ldots,0,\overset{\substack{p^{\text{th}}\\ \text{place}}}{1},0,\ldots,0,\overset{\substack{(n-q)^{\text{th}}\\ \text{place}}}{1},0,\ldots,0,0) \Leftrightarrow \omega_p + \omega_{n-q}, \qquad (p+q \neq n). \tag{8.5.3}$$

The corresponding tableaux correspond to the simple "gluing" of the defining rank-p and conjugate rank-q tableaux, Figs. 8.17(a,b).

As a final example, we consider the tableau corresponding to the highest weight

$$(0,0,\ldots,0,\overset{\substack{q^{\text{th}}\\ \text{place}}}{p},0,\ldots,0,0) \Leftrightarrow p\,\omega_q. \tag{8.5.4}$$

The tableau is the "block" shown in Fig. 8.17(c). The general prescription says that we can view this irreducible representation as a tensor of $p \times q$ defining indices, first symmetrized in the rows of the tableau, and finally antisymmetrized in the columns. I.e.,

$$T_{[i_{1,1}i_{2,1}\cdots i_{q,1}][i_{1,2}i_{2,2}\cdots i_{q,2}]\cdots[i_{1,p}i_{2,p}\cdots i_{q,p}]}, \tag{8.5.5}$$

where $i_{m,n}$ is an index appearing in the mth row and n^{th} column of the tableau. Here the indices are grouped according to columns, each completely antisymmetric. The "presymmetrization" of indices sharing rows [i.e., $(i_{1,1}i_{1,2}\cdots i_{1,p})$] is necessary to obtain an irreducible representation.

8.5.3 Dimension rule: "Factors over hooks"

We can encode any irreducible representation of su(n) by a Young tableau. Yet this doesn't appear to give a particularly more transparent picture for a representation than its explicit tensor form (at least for simple cases). Nor do tableaux clearly convey geometrical information (e.g. the weight geometry of a representation). Why bother introducing them at all?

The main advantage is a number of simple tableau-based algorithms that allow efficient calculation of data pertaining to su(n) representations and their products. We will present a couple of these schemes here, and more later in the course. At this stage these are presented as "cookbook" recipes, rather than derived.

The dimension of an su(n) irreducible representation can be computed from its tableau using the "factors over hooks" rule. A hook is a line passing up through a particular column of a tableau up to some particular row, turning right, and passing out through that row. See Figs. 8.18(a)–(c).

To compute the dimension, there is the following prescription. Given a young tableau with R boxes,

1. Write "n" in the upper-left corner box.
2. Fill in rows with monotonically increasing sequences. I.e., the first row of length p should read $\{n, n+1, n+2, \ldots, n+p\}$.
3. Fill in columns with monotonically decreasing sequences. I.e., the first column of length q should read $\{n, n-1, n-2, \ldots, n-q+1\}$.

Figure 8.18: Ingredients in the "factors over hooks" rule for computing the dimension d_Λ of an irreducible representation (labeled by the Dynkin coefficients $\{\Lambda_i\}$ if its highest weight state, or the associated tableau). (a)–(c) show examples of "hooks." The hook in (a), (b), or (c) passes through 5, 4, or 3 boxes, respectively. Subfigures (d)–(f) [(g)] show su(3) [su(n)] tableaux enumerated according to the factors over hooks scheme. The dimensions are computed in the text.

4. The dimension d_Λ of the representation is give by

$$d_\Lambda = \frac{\prod_{\text{boxes}} (\text{factor in each box})}{\prod_{\text{hooks}} (\text{number of boxes intersected by each hook})} \equiv \frac{F_\Lambda}{H_\Lambda}. \tag{8.5.6}$$

This seemingly ad-hoc scheme is in fact a special case of the Weyl character formula, which we will derive for any Lie algebra in module 9.

Let us verify the factors over hooks rule for a number of cases. First we consider tableau for su(3). Fig. 8.18(d) corresponds to the representation $\Lambda = 4\omega_1$, i.e. a rank-4 fully symmetric tensor. This has

$$d_{(4,0)} = \frac{(4+2)!}{4!2!} = 15$$

independent components. On the other hand, the numerator and denominator of Eq. (8.5.6) are given by

$$\begin{aligned} F_{(4,0)} = 3\times 4\times 5\times 6, \qquad H_{(4,0)} = 4\times 3\times 2\times 1, \\ \Rightarrow d_{(4,0)} = 15, \end{aligned} \tag{8.5.7}$$

in agreement with the above. Fig. 8.18(e) is the su(3) adjoint representation with dimension $d_{(1,1)} = 8$. Eq. (8.5.6) gives

$$\begin{aligned} F_{(1,1)} = 3\times 4\times 2, \qquad H_{(1,1)} = 3\times 1\times 1, \\ \Rightarrow d_{(1,1)} = 8. \end{aligned} \tag{8.5.8}$$

Fig. 8.18(f) is the su(3) representation (2,1) with weight geometry shown in Fig. 8.6. We will show in module 9 that this has $d_{(2,1)} = 15$. Eq. (8.5.6) gives

$$\begin{aligned} F_{(2,1)} = 3\times 4\times 5\times 2, \qquad H_{(2,1)} = 4\times 2\times 1\times 1, \\ \Rightarrow d_{(2,1)} = 15. \end{aligned} \tag{8.5.9}$$

Finally, consider the su(n) tableau shown in Fig. 8.18(g). This corresponds to the traceless biantisymmetric tensor $\tilde{T}^{[j_1\cdots j_q]}_{[i_1\cdots i_p]} \Leftrightarrow \omega_p + \omega_{n-q}$, Eq. (8.5.3). The number of components of such a tensor is

$$\begin{aligned} d_{\omega_p+\omega_{n-q}} &= \frac{n!}{p!(n-p)!}\frac{n!}{q!(n-q)!} - \frac{n!}{(p-1)!(n-p+1)!}\frac{n!}{(q-1)!(n-q+1)!} \\ &= \frac{n!}{p!(n-p+1)!}\frac{n!}{q!(n-q+1)!}\left[(n-p+1)(n-q+1) - pq\right] \\ &= \frac{n!(n+1)!(n+1-p-q)}{p!(n-p+1)!q!(n-q+1)!}. \end{aligned} \tag{8.5.10}$$

The second term on the right-hand-side of the first line of Eq. (8.5.10) accounts for the traceless constraint. On the other hand, Eq. (8.5.6) gives

$$\begin{aligned} F_{\omega_p+\omega_{n-q}} &= \frac{n!}{q!}\frac{(n+1)!}{(n-p+1)!}, \\ H_{\omega_p+\omega_{n-q}} &= \frac{(n-q+1)!}{(n-p-q+1)!}(n-p-q)!p! \\ \Rightarrow d_{\omega_p+\omega_{n-q}} &= \frac{n!(n+1)!(n-p-q+1)}{p!(n-p+1)!q!(n-q+1)!}. \end{aligned} \tag{8.5.11}$$

8.5.4 Semistandard Young tableaux: enumerating the states

We can also generate all states of an irreducible su(n) representation explicitly using **semistandard** Young tableaux. From a given tableau corresponding to a particular highest weight representation, states are enumerated by filling in boxes with different sequences of positive integers. The prescription is to generate all possible labelings adhering to the following rules. Let $c_{i,j}$ be the integer appearing in the box in the ith row and j^{th} column. Then

$$1 \le c_{i,j} \le n, \qquad c_{i,j} \le c_{i,j+1}, \qquad c_{i,j} < c_{i+1,j}. \tag{8.5.12}$$

Figure 8.19: Semistandard tableaux for su(3) representations (a) (1,0) (3), (b) (0,1) ($\bar{3}$), and (c) (1,1) (adjoint).

I.e., each successive element of a row is greater than or equal to the previous entry, while successive elements of a column are strictly greater than previous entries.

We apply this algorithm to the defining 3, conjugate $\bar{3}$, and adjoint representations of su(3) in Fig. 8.19.

Each semistandard tableau corresponds to a particular state in the representation, irrespective of the dimension of the associated weight space. The weights themselves can be reconstructed as follows. For an su(n) semistandard tableau, a box with number p ($1 \leq p \leq n$) is associated to the weight

$$\omega_p - \omega_{p-1}, \qquad \omega_0 = \omega_n \equiv 0. \tag{8.5.13}$$

The weight of the semistandard tableau obtains by summing the contributions of all boxes, c.f. Fig. 8.19 for su(3) examples.

8.6 Dual Coxeter numbers and quadratic form matrices

Here we list the quadratic form matrices and dual Coxeter numbers for the four classical Lie algebra families. In this section we use a standard normalization scheme for the root scalar products, such that long roots are normalized to 2: $\langle \alpha, \alpha \rangle = 2$. This means that roots and coroots are identical for simply laced algebras. This normalization scheme is different from the one employed to define the classical Lie algebras in module 7, and will be explained in module 9.

The dual Coxeter numbers are

algebra	group dimension d	g
A_n	$n(n+2)$	$n+1$
B_n	$n(2n+1)$	$2n-1$
C_n	$n(2n+1)$	$n+1$
D_n	$n(2n-1)$	$2n-2$

(8.6.1)

The quadratic form matrix [Eq. (8.1.5c)] for each of these families is given by

$$\left(\hat{F}_{A_n}\right)_{ij} = \frac{1}{(n+1)} \min(i,j)\,[n+1-\max(i,j)]\,,$$

$$\hat{F}_{A_n} = \frac{1}{n+1}\begin{bmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \cdots & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & \cdots & 6 & 3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}, \tag{8.6.2a}$$

$$\left(\hat{F}_{B_n}\right)_{ij} = \begin{cases} \min(i,j), & 1 \le i,j \le n-1, \\ \min(i,j)/2, & 1 \le i \le n-1 \text{ and } j=n, \text{ or } 1 \le j \le n-1 \text{ and } i=n, \\ n/4, & i=j=n, \end{cases}$$

$$\hat{F}_{B_n} = \frac{1}{2}\begin{bmatrix} 2 & 2 & 2 & \cdots & 2 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n/2 \end{bmatrix}, \tag{8.6.2b}$$

$$\left(\hat{F}_{C_n}\right)_{ij} = \frac{1}{2}\min(i,j),$$

$$\hat{F}_{C_n} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}, \tag{8.6.2c}$$

$$\left(\hat{F}_{D_n}\right)_{ij} = \begin{cases} \min(i,j), & 1 \le i,j \le n-2, \\ \min(i,j)/2, & 1 \le i \le n-2 \text{ and } j \in \{n-1,n\}, \text{ or } 1 \le j \le n-2 \text{ and } i \in \{n-1,n\}, \\ (n-2)/4, & i=n-1 \text{ and } j=n, \text{ or } j=n-1 \text{ and } i=n, \\ n/4, & i=j=n-1, \text{ or } i=j=n, \end{cases}$$

$$\hat{F}_{D_n} = \frac{1}{2}\begin{bmatrix} 2 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 & 3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-2) & n-2 & n-2 \\ 1 & 2 & 3 & \cdots & n-2 & n/2 & (n-2)/2 \\ 1 & 2 & 3 & \cdots & n-2 & (n-2)/2 & n/2 \end{bmatrix}. \tag{8.6.2d}$$

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

[2] Phillipe Di Francesco, Pierre Mathieu, David Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1996).

[3] Howard Georgi, *Lie Algebras in Particle Physics*, 2nd ed. (Westview Press, Boulder, Colorado, 1999).

[4] Anthony Zee, *Group Theory in a Nutshell for Physicists* (Princeton University Press, Princeton, New Jersey, 2016).

8B. Informal lecture: Some physics applications of highest weight representations
* version 1.0 *

Matthew Foster

October 27, 2016

8B.1 Non-abelian gauge theory

- Gauge theory: couple a matter field (usually fermions) to a (4-vector) potential $A_\mu = \{A_0, \vec{A}\}$. In the Standard model, gauge bosons mediate the strong, weak, and electromagnetic forces.
- What is a gauge theory mathematically? Make a global symmetry local.
- Consider a theory with a global, continuous symmetry. Can be abelian [U(1) or O(2)] or non-abelian Lie group G.
- Under symmetry transformation, group acts on some "indices" carried by the matter field. E.g., the phase a complex fermion field, or the color of a quark field.
- Given the space in which the group acts, why should we need to make a global choice of coordinates? I.e., if I am measuring quark colors on earth and you measure them on the moon, why should we have to "synchronize" our coordinate frames. My green could be your blue.
- Formally, one can "gauge" a symmetry, introducing a gauge field $A_\mu(t, \mathbf{r})$ such that one can choose bases differently at all points in spacetime $(t, \mathbf{r})$. By itself, this doesn't introduce any physics, only a tremendous amount of mathematical redundancy.
- If all we are doing is choosing different bases, then all configurations of A_μ are "gauge equivalent" to zero. But there is another possibility. There can be gauge field configurations with some kind of gauge-invariant "curvature" present (e.g., magnetic or electric flux). In a quantum gauge theory, the physical gauge flux is carried by transverse-polarized gauge quanta (photons in QED, gluons in QCD, etc).

Simplest non-abelian example: Quantum chromodynamics (QCD), theory of quarks interacting via the strong force, carried by the gluons.

- Lagrangian density (fermion sector)

$$L = \bar{\psi}\left[\hat{\gamma}^\mu\left(i\frac{\partial}{\partial x^\mu} + g\hat{A}_\mu\right) - m\right]\psi = \bar{\psi}^{\sigma,i}\left\{(\hat{\gamma}^\mu)_{\sigma,\sigma'}\left[i\frac{\partial}{\partial x^\mu}\delta_i{}^j + gA^a_\mu\left(\hat{T}^a\right)_i{}^j\right] - m\,\delta_\sigma{}^{\sigma'}\delta_i{}^j\right\}\psi_{\sigma',j}. \tag{8B.1.1}$$

Quark field $\psi \to \psi_{\sigma',j}$ carries two indices:

1. Spinor index σ

   $\sigma \in \{1, 2, 3, 4\}$ SO(3,1) $\sim$ SO(4) Lorentz group; σ is a direct sum of left- and right- handed Weyl components

   By the way,

   $$\hat{\gamma}^\mu\,\hat{\gamma}^\nu + \hat{\gamma}^\nu\,\hat{\gamma}^\mu = -2\eta^{\mu\nu}\hat{1}_\sigma$$

   is a (non-compact) **Clifford algebra**. The Euclidean spacetime versions offer a natural basis of operators acting on the spinor representation(s) of SO(5) [or SO(4)]. Module 10.

2. Color index i

   $i \in \{1, 2, \ldots, n\}$ defining representation of (e.g.) su(n).

- With $\hat{A}_\mu = 0$, theory is invariant under global (indept. of spacetime) SU(n) transformations on color. Also invariant under Lorentz transforms (which however act on both spinor indices and spacetime arguments $\Rightarrow$ relativistic spin-orbit coupling).

– The fermion field ψ (Dirac adjoint field $\bar{\psi}$) transforms in the defining "n" (conjugate "$\bar{n}$") representation of the color symmetry group, e.g. su(n).

– The gauge field $\hat{A}_\mu = A^a_\mu \hat{T}^a$ transforms in the adjoint representation under **global** color symmetry transformations. (It also transforms like a vector under Lorentz transformations).

– Local spacetime color symmetry:

$$\begin{aligned} \psi(x) &\to \hat{U}(x)\,\psi(x), \\ \bar{\psi}(x) &\to \bar{\psi}(x)\,\hat{U}^\dagger(x), \\ \hat{A}_\mu(x) &\to \hat{U}(x)\,\hat{A}_\mu(x)\,\hat{U}^\dagger(x) + \frac{i}{g}\hat{U}(x)\frac{\partial}{\partial x^\mu}\hat{U}^\dagger(x) \end{aligned} \tag{8B.1.2}$$

For a local symmmetry transformation, $\hat{A}_\mu(x)$ does **not** transform in the adjoint rep. There is a correction.

QCD of quarks and gluons: color group is SU(3). 3 independent types of each quark. (That means 3 different colors of "up" quarks. There are also 5 other flavors of quarks; the 6 flavors divide into 3 generations. Flavors have nothing to do with color, however, and can be thought of as yet another index.) There are 8 independent types of gluons, corresponding to states of the adjoint representation of the color group.

Thus we get physical implementations of the defining (1,0), conjugate (0,1) and adjoint (1,1) representations of su(3). Dynamics for the gauge field: additional term in Lagrangian involving the (non-abelian) field strength tensor

$$\hat{F} = d\hat{A} + g\hat{A} \wedge \hat{A}.$$

N.B., can also get classical general relativity from very similar construction. **Gauge the Lorentz group SO(3,1)!**.

- Gauge fields: Christoffel symbols or spin connections.
- Field strength tensor: Riemann tensor.
- Problem: quantum fluctuations become too strong on short scales; "straightforward" quantization of GR gives a nonrenormalizable QFT that fails to make predictions in the ultraviolet (highest energy, shortest distance) limit. Not completely unexpected, since one might need to deal with quantum fluctuations of spacetime itself, and that isn't captured by a perturbative quantization of GR. Way out? Strings (but...no SUSY evidence; landscape problem)? Loop quantum gravity? Something else?

8B.2 Generalized quantum magnets

- In condensed matter, frequently want to understand the physics of quantum Heisenberg antiferromagnets with spin SU(2) symmetry. E.g.,

$$H = J_1 \sum_{\langle \mathbf{r},\mathbf{r'}\rangle} \hat{S}^a_{\mathbf{r}} \hat{S}^a_{\mathbf{r'}} + J_2 \sum_{\langle\langle \mathbf{r},\mathbf{r'}\rangle\rangle} \hat{S}^a_{\mathbf{r}} \hat{S}^a_{\mathbf{r'}} + \ldots, \tag{8B.2.1}$$

where (e.g.) $2\hat{S}^a_{\mathbf{r}}$ is a 2×2 Pauli matrix acting on a spin-1/2 degree of freedom sitting on site $\mathbf{r}$ in some lattice. Coupling ("exchange") constants J_1, J_2, etc. encode interactions: nearest-neighbor exchange, next-nearest-neighbor, etc.

- Quantum magnets with antiferromagnetic exchange ($J_1 > 0$) can have complicated ground states, e.g. Néel state on a bipartite lattice, root-3 state on triangular lattice, columnar states, valence bond solids, spin liquids, etc. Frustration and quantum fluctuations mean that the ground state is never a simple product state (unlike in a simple quantum ferromagnet).
- In low dimensions, quantum fluctuations are strong and Hamiltonian such as Eq. (8B.2.1) can be strongly coupled. Perturbative calculations fail. **What to do?**
- No small parameter: **invent one.** For example, can replace SU(2) $\Rightarrow$ SU(n); hope to use $1/n$ as small parameter.
- <u>Problem</u>: for spin-1/2 SU(2) magnet, can always form a singlet from two nearest-neighbor sites

$$|0\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right].$$

But if we put a defining "n" representation "spin" on each site, we know that we need n sites to form a singlet in su(n). Either give up notion of local singlets, or modify construction. Local singlets are good if we want to find valence bond solids or spin liquids...

- One way: put a defining representation ψ_i on half of the sites ("A" sites on bipartite lattice) and a conjugate representation ϕ^i on the other half ("B" sites on a bipartite lattice). Then we can form a scalar by pairing one A and one B site: $\phi^i \psi_i$.

- How to make this concrete? What is the Hilbert space on each site, and how does the Hamiltonian act on it? Answer: **(E.g.) slave fermions**. The generalization of Eq. (8B.2.1) to SU(n) is

$$H = J_1 \sum_{\langle \mathbf{r}, \mathbf{r'} \rangle} \hat{T}^a_{\mathbf{r}} \hat{T}^a_{\mathbf{r'}} + \ldots, \tag{8B.2.2}$$

where

$$\hat{T}^a = c^{\dagger\, i} \left[\hat{t}^a\right]_i{}^j c_j, \tag{8B.2.3}$$

and c_j is a fermion operator that transforms in the defining n (ω_1) representation. $\left[\hat{t}^a\right]_i{}^j$ is the explicit matrix generator in this representation. Can easily verify that

$$[\hat{T}^a, \hat{T}^b] = i f^{abc} \hat{T}^c \tag{8B.2.4}$$

if $[\hat{t}^a, \hat{t}^b] = i f^{abc} \hat{t}^c$ is the su(n) Lie algebra commutation relations.

- That gives us the operators in the Hamiltonian. What about the states?

$$\text{Defining:} \quad \psi_i \Leftrightarrow c_i^\dagger |0\rangle \quad \text{set of all states with one and only one particle per site} \tag{8B.2.5}$$

$$\text{Conjugate:} \quad \phi^i \Leftrightarrow c_{i_1}^\dagger c_{i_2}^\dagger \times \cdots \times c_{i_{n-1}}^\dagger |0\rangle \quad \text{set of all states with } n-1 \text{ particles per site} \tag{8B.2.6}$$

We can then form a singlet

$$\epsilon^{i_1 \cdots i_n} \psi_{i_1} \phi_{i_2 \cdots i_n}.$$

- The above treats different sites asymmetrically. What if we want all sites to have the same degrees of freedom? Take $n = 2m$, and assign $\psi_{[i_1 \cdots i_m]} \Leftrightarrow \omega_m$ representation to each site. The singlet is

$$\epsilon^{i_1 \cdots i_m i_{m+1} \cdots i_n} \psi_{[i_1 \cdots i_m]}(\mathbf{r})\, \psi_{[i_{m+1} \cdots i_{2m}]}(\mathbf{r'})$$

- Using slave fermions in the defining representation, Eq. (8B.2.3) still holds for the Hamiltonian. The states on each site have m fermions,

$$\psi_{[i_1 \cdots i_m]} \Leftrightarrow c_{i_1}^\dagger c_{i_2}^\dagger \times \cdots \times c_{i_m}^\dagger |0\rangle\,. \tag{8B.2.7}$$

Other schemes:

- Slave bosons. Natural Fock space states are completely symmetric tensors (representations with weights $m\omega_1$).

- More complicated slave fermions–add an additional "flavor" index (or more). Can realize generic tensor representation of SU(n) on each site.

- Use a different group. E.g. Sp($2n$). Then we can form a singlet from two defining representation indices, using the antisymmetric "metric" ϵ^{ij}.

8B.3 Non-linear sigma models

8B.4 Kac-Moody 2D conformal field theories

9. Casimir operators, characters, dimension and strange formulae
* version 1.8 *

Matthew Foster

April 30, 2021

Contents

9.1 Quadratic Casimir operator **1**
9.1.1 Standard normalization for Killing form and Casimir . . . 4
9.1.2 Index of a representation; some examples . . . 6
9.1.2.1 su(2) . . . 7
9.1.2.2 su(3) . . . 7

9.2 Freudenthal's recursion formula for weight space dimensions **8**

9.3 Weyl group **11**

9.4 Weyl's character formula: derivation **13**
9.4.1 Generating functions . . . 13
9.4.2 Freudenthal redux . . . 16
9.4.3 Resumming Freudenthal . . . 16
9.4.4 *finis* . . . 18

9.5 Dimension and strange formulae **20**
9.5.1 Dimensions for generic representations of the classical algebras . . . 21

The discussion here follows chapters XI–XIII of [1] and Secs. 13.2 and 13.4 of [2].

9.1 Quadratic Casimir operator

An essential datum for each irreducible su(2) representation is the eigenvalue $C^{(2j)} = j(j+1)$ of the **quadratic Casimir operator** (see Sec. 1.3.3),

$$\hat{C} \equiv \hat{T}_z \hat{T}_z + \frac{1}{2}\left(\hat{T}_+ \hat{T}_- + \hat{T}_- \hat{T}_+\right), \tag{9.1.1}$$

where $2j \in \{0, 1, 2, \ldots\}$ is the Dynkin label for the spin-j representation (i.e., the Dynkin coefficient of the highest weight). The key properties of the Casimir operator are

1. It gives the same eigenvalue for all states in an irreducible representation,

$$\hat{C}\,|j,m\rangle = j(j+1)\,|j,m\rangle\,, \quad \hat{T}_z\,|j,m\rangle = m\,|j,m\rangle\,, \quad m \in \{-j, -j+1, \ldots, j-1, j\}. \tag{9.1.2a}$$

2. It commutes with all generators,

$$[\hat{C}, \hat{T}_+] = [\hat{C}, \hat{T}_-] = [\hat{C}, \hat{T}_z] = 0. \tag{9.1.2b}$$

Eigenvalues of an su(2)-invariant Hamiltonian can depend only on $C^{(2j)} = j(j+1)$, not on the "magnetic quantum number" m that distinguishes different weight vectors. Although each representation is uniquely labeled by its Casimir eigenvalue, it is important to remember that $\hat{C}$ is not defined abstractly for the Lie algebra $A_1 = \text{su}(2)$, but only for a particular representation in terms of the explicit bilinear sum of matrix generators in Eq. (9.1.1).[1]

For a generic rank-n Lie algebra, it turns out that one can construct exactly n Casimir operators satisfying the axioms in Eq. (9.1.2) [3]. We will discuss only the generalization of the quadratic Casimir in Eq. (9.1.1), which is unique. The other higher order Casimir operators are built from sums of m-fold products, where $m \geq 3$ is the order of each factor in the operator.[2]

We define the quadratic Casimir operator for a rank-n Lie algebra L as an operator acting on states of a generic finite-dimensional, irreducible representation as follows:

$$\hat{C} \equiv \sum_{i,j=1}^{n} \mathcal{A}_{ij}^{-1} \hat{H}_{\overline{\alpha}_i} \hat{H}_{\overline{\alpha}_j} + \sum_{\alpha \neq 0} \frac{\hat{E}_\alpha \hat{E}_{-\alpha}}{(e_\alpha, e_{-\alpha})}, \qquad \text{Quadratic Casimir operator,} \tag{9.1.3a}$$

$$\mathcal{A}_{ij} \equiv \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = A_{ij} \frac{|\overline{\alpha}_j|^2}{2}. \tag{9.1.3b}$$

Just as in su(2) [Eq. (9.1.1)], the operator $\hat{C}$ is a sum of bilinears of Cartan subalgebra elements $\{\hat{H}_{\overline{\alpha}_i}\}$ or root vector raising and lowering operators $\{\hat{E}_\alpha, \hat{E}_{-\alpha}\}$. The overlap matrix $\mathcal{A}_{ij}$ is a symmetrized version of the Cartan matrix $A_{ij} = \langle \overline{\alpha}_i, \overline{\alpha}_j^\vee \rangle$ [Eq. (5.3.8)]. Note that the sum over root vector operators in $\hat{C}$ includes all nonzero roots, not just positive or simple ones. The number of terms in the sum Eq. (9.1.3a) is equal to N_θ, the dimension of the adjoint representation: [recall that θ labels the highest weight (root) of the adjoint representation, Secs. 5.3.1 and 8.2.4]

$$N_\theta = n + |\Delta| = n + |\Delta_+| + |\Delta_-| = d. \tag{9.1.4}$$

In this equation, n denotes the rank, $|\Delta|$ ($|\Delta_\pm| = |\Delta|/2$) is the number of roots (number of positive or negative roots), and d is the dimension of the corresponding continuous group G generated by the Lie algebra L.

Claim I: The quadratic $\hat{C}$ is a pure number in any irreducible representation. I.e., it commutes with all $\{\hat{H}_{\overline{\alpha}_i}, \hat{E}_\alpha\}$.

Before we prove this statement, we examine some of its consequences.

1. Trace in the adjoint representation. Consider the trace

$$(C) \equiv \mathsf{Tr}\left[\hat{C}\right]_\theta, \tag{9.1.5}$$

where $\mathsf{Tr}[\cdots]_\Lambda$ denotes the trace in representation Λ; here θ indicates the adjoint representation. Since the trace in the adjoint representation is the Killing form $\mathsf{Tr}[\widehat{a}_{d_x}\, \widehat{a}_{d_y}]_\theta = (x, y)$, we have

$$(C) = \sum_{i,j=1}^{n} \mathcal{A}_{ij}^{-1} \left(h_{\overline{\alpha}_i}, h_{\overline{\alpha}_j}\right) + \sum_{\alpha \neq 0} = \sum_{i,j=1}^{n} \mathcal{A}_{ij}^{-1} \mathcal{A}_{ji} + \sum_{\alpha \neq 0} = N_\theta. \tag{9.1.6}$$

2. Action on the highest-weight state ϕ_Λ of an irreducible representation. Let Λ denote the associated highest weight. Then

$$\begin{aligned} \hat{C}\phi_\Lambda &= \sum_{i,j=1}^{n} \langle \Lambda, \overline{\alpha}_i \rangle \mathcal{A}_{ij}^{-1} \langle \overline{\alpha}_j, \Lambda \rangle \phi_\Lambda + \sum_{\alpha > 0} \left[\frac{\hat{E}_{-\alpha} \hat{E}_\alpha}{(e_\alpha, e_{-\alpha})} + \hat{H}_\alpha \right] \phi_\Lambda \\ &= \left[\langle \Lambda, \Lambda \rangle + \sum_{\alpha > 0} \langle \Lambda, \alpha \rangle \right] \phi_\Lambda. \end{aligned} \tag{9.1.7}$$

On the first line of Eq. (9.1.7), we have restricted the sum over root vectors to positive roots (because ϕ_Λ is a highest weight state annihilated by all $\hat{E}_{-\alpha}$ with $\alpha < 0$), and we have employed the definition for h_α in Eq. (4.4.10). On the second line, we have removed the resolution of the identity in the rank-n space of weights H^*. For a linearly independent, but not necessarily orthonormal basis of states $\{|i\rangle\}$, the identity can be expressed as

$$\hat{1}_n = \sum_{i,j=1}^{n} \mathcal{O}_{ij}^{-1} |i\rangle \langle j|, \qquad \mathcal{O}_{ij} = \langle i | j \rangle. \tag{9.1.8}$$

[1] To write Eq. (9.1.1), we need to define the (matrix) product between generators. The only product operation defined for the Lie algebra itself is the Lie bracket, which becomes a matrix commutator in a finite-dimensional representation.

[2] The order $m+1$ of the $(m+1)$-fold products summed in a particular Casimir is called the *exponent* m. For each algebra A_n, B_n, C_n, and D_n, the n **exponents of the algebra** are tabulated (e.g.) in Sec. 13A of [2].

It will prove useful to rewrite Eq. (9.1.7) in a more compact way. We define the **Weyl vector** for a Lie algebra as the weight

$$\rho_{\mathsf{W}} \equiv \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{j=1}^{n}\omega_j, \qquad \text{Weyl vector for a rank-}n\text{ Lie algebra.} \tag{9.1.9}$$

The second equality implies that the Dynkin coefficients of the Weyl vector are $(\Lambda_1,\ldots,\Lambda_n) = (1,1,1,\ldots,1)$. We will prove this in Sec. 9.3, below. Eq. (9.1.7) then becomes

$$\hat{C}\phi_\Lambda \equiv C^\Lambda\,\phi_\Lambda = \langle\Lambda,\Lambda+2\rho_{\mathsf{W}}\rangle\,\phi_\Lambda. \tag{9.1.10}$$

<u>Proof:</u> The quadratic $\hat{C}$ commutes with all generators $\{\hat{H}_{\overline{\alpha}_i},\hat{E}_\alpha\}$ in an irreducible representation. Exploiting the Lie bracket structure of a generic Lie algebra [Eq. (4.4.11)], we have

$$\begin{aligned}\left[\hat{C},\hat{H}_{\overline{\alpha}_i}\right] &= \sum_{\alpha\neq 0}\frac{\hat{E}_\alpha\left[\hat{E}_{-\alpha},\hat{H}_{\overline{\alpha}_i}\right]+\left[\hat{E}_\alpha,\hat{H}_{\overline{\alpha}_i}\right]\hat{E}_{-\alpha}}{(e_\alpha,e_{-\alpha})}\\ &= \sum_{\alpha\neq 0}\frac{\hat{E}_\alpha\,\hat{E}_{-\alpha}\langle\alpha,\overline{\alpha}_i\rangle-\langle\alpha,\overline{\alpha}_i\rangle\hat{E}_\alpha\,\hat{E}_{-\alpha}}{(e_\alpha,e_{-\alpha})} = 0,\end{aligned} \tag{9.1.11a}$$

$$\begin{aligned}\left[\hat{C},\hat{E}_\beta\right] &= \sum_{i,j=1}^{n}\mathcal{A}^{-1}_{ij}\langle\beta,\overline{\alpha}_j\rangle\left(\hat{H}_{\overline{\alpha}_i}\,\hat{E}_\beta+\hat{E}_\beta\,\hat{H}_{\overline{\alpha}_i}\right)+\sum_{\alpha\neq 0}\frac{\hat{E}_\alpha\left[\hat{E}_{-\alpha},\hat{E}_\beta\right]+\left[\hat{E}_\alpha,\hat{E}_\beta\right]\hat{E}_{-\alpha}}{(e_\alpha,e_{-\alpha})}\\ &= \sum_{i,j=1}^{n}\mathcal{A}^{-1}_{ij}\langle\beta,\overline{\alpha}_j\rangle\left(\hat{H}_{\overline{\alpha}_i}\,\hat{E}_\beta+\hat{E}_\beta\,\hat{H}_{\overline{\alpha}_i}\right)+\sum_{\substack{\alpha\neq 0\\ \alpha\neq\beta}}\frac{N_{-\alpha,\beta}\,\hat{E}_\alpha\,\hat{E}_{\beta-\alpha}}{(e_\alpha,e_{-\alpha})}+\sum_{\substack{\alpha\neq 0\\ \alpha\neq-\beta}}\frac{N_{\alpha,\beta}\,\hat{E}_{\alpha+\beta}\,\hat{E}_{-\alpha}}{(e_\alpha,e_{-\alpha})}-\hat{E}_\beta\,\hat{H}_\beta-\hat{H}_\beta\,\hat{E}_\beta.\end{aligned} \tag{9.1.11b}$$

Next we invoke invariance of the Killing form [Eq. (4.4.8)], as well as Eq. (4.4.5) to obtain

$$\begin{aligned}(e_\alpha,[e_\beta,e_\gamma]) &= N_{\beta,\gamma}\,(e_\alpha,e_{-\alpha})\,\delta_{\beta+\gamma+\alpha,0} = -N_{-\beta-\alpha,\beta}\,(e_\alpha,e_{-\alpha})\,\delta_{\beta+\gamma+\alpha,0}\\ &= ([e_\alpha,e_\beta],e_\gamma) = N_{\alpha,\beta}\,(e_{\alpha+\beta},e_{-\alpha-\beta})\,\delta_{\alpha+\beta+\gamma,0},\\ \Rightarrow\frac{N_{\alpha-\beta,\beta}}{(e_{\beta-\alpha},e_{\alpha-\beta})} &= -\frac{N_{-\alpha,\beta}}{(e_\alpha,e_{-\alpha})}.\end{aligned} \tag{9.1.12}$$

In this equation, we have used $N_{\alpha,\beta} = -N_{\beta,\alpha}$, which follows from the antisymmetric property of the Lie bracket [Eq. (1.2.1a)]; on the last line we have sent $\alpha\to-\alpha$ on both sides. We can therefore rewrite the red term in Eq. (9.1.11b) as follows:

$$\sum_{\substack{\alpha\neq 0\\ \alpha\neq-\beta}}\frac{N_{\alpha,\beta}\,\hat{E}_{\alpha+\beta}\,\hat{E}_{-\alpha}}{(e_\alpha,e_{-\alpha})} = \sum_{\substack{\alpha'\neq 0\\ \alpha'\neq\beta}}\frac{N_{\alpha'-\beta,\beta}\,\hat{E}_{\alpha'}\,\hat{E}_{\beta-\alpha'}}{(e_{\alpha'-\beta},e_{-\alpha'+\beta})} = -\sum_{\substack{\alpha'\neq 0\\ \alpha'\neq\beta}}\frac{N_{-\alpha',\beta}\,\hat{E}_{\alpha'}\,\hat{E}_{\beta-\alpha'}}{(e_{\alpha'},e_{-\alpha'})}, \tag{9.1.13}$$

where we have defined $\alpha'\equiv\alpha+\beta$. The two terms in Eq. (9.1.11b) explicitly involving the structure constants therefore exactly cancel, leaving

$$\left[\hat{C},\hat{E}_\beta\right] = \hat{E}_\beta\left[\sum_{i,j=1}^{n}\mathcal{A}^{-1}_{ij}\langle\beta,\overline{\alpha}_j\rangle\hat{H}_{\overline{\alpha}_i}-\hat{H}_\beta\right]+\left[\sum_{i,j=1}^{n}\mathcal{A}^{-1}_{ij}\langle\beta,\overline{\alpha}_j\rangle\hat{H}_{\overline{\alpha}_i}-\hat{H}_\beta\right]\hat{E}_\beta = 0. \tag{9.1.14}$$

The final equality follows from the representation of $\hat{H}_\beta$ in the linearly independent basis $\{H_{\overline{\alpha}_j}\}$; e.g., consider a state $|f\rangle$ multiplied by the resolution of the identity in Eq. (9.1.8):

$$|f\rangle = \sum_{i,j=1}^{n}\mathcal{O}^{-1}_{ij}\,|i\rangle\,\langle j\,|f\rangle\,.$$

Q.E.D.

9.1.1 Standard normalization for Killing form and Casimir

The Dynkin coefficients for any weight in an irreducible representation are guaranteed to be integers by the MWDF, Eq. (5.1.23). Dynkin coefficients are independent of the normalization of simple roots, simple coroots, or fundamental weights; the same is true of the Cartan matrix $A_{ij} = \langle \overline{\alpha}_i, \overline{\alpha}_j^\vee \rangle = 2\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle / \langle \overline{\alpha}_j, \overline{\alpha}_j \rangle$, the rows of which give the Dynkin coefficients for the simple roots.

By contrast, the quadratic form matrix $F_{ij} = \langle \omega_i, \omega_j \rangle$ [Eq. (8.1.5c)] and the quadratic Casimir eigenvalue in Eq. (9.1.10) do depend upon the normalization scheme. In certain applications (notably affine Lie algebras in conformal field theory (CFT) [2]), it is crucial to adopt a consistent normalization. We define the "standard" normalization scheme used in CFT and elsewhere in this subsection, and will employ it throughout the remainder of these lectures (unless otherwise noted).

Recall that all roots (including simple roots) come in one or two varieties for semi-simple Lie algebras: simply-laced algebras have all roots with the same length; otherwise there are "short" and "long" roots (Prop. IV., Sec. 5.2.2). The inner products of simple roots were computed for the four classical families in module 7, using a particular normalization scheme for the generators of the defining representation in each case. The results for $A_{n-1} = \mathrm{su}(n)$ [Eq. (7.1.21)], $B_n = \mathrm{so}(2n+1)$ [Eq. (7.4.19)], $C_n = \mathrm{sp}(2n)$ [Eq. (7.2.20)], and $D_n = \mathrm{so}(2n)$ [Eq. (7.3.23)] show that each <u>long simple root</u> $\overline{\alpha}_i^{(\mathsf{long})}$ for algebras B_n and C_n (all simple roots for simply-laced algebras, A_n and D_n) is normalized to

$$\left\langle \overline{\alpha}_i^{(\mathsf{long})}, \overline{\alpha}_i^{(\mathsf{long})} \right\rangle = |\overline{\alpha}_i^{(\mathsf{long})}|^2 = \frac{1}{g}, \tag{9.1.15}$$

where g is the **dual Coxeter number** defined in module 8, Eq. (8.1.13). The tabulated values for g in each case appear in Eq. (8.6.1), and match Eqs. (7.1.21), (7.4.19), (7.2.20), and (7.3.23).

The standard normalization scheme resets $|\overline{\alpha}_i^{(\mathsf{long})}|^2 = 2$ for all algebras, i.e. all long roots normalized to two. The highest root θ (Sec. 8.2.4) is always a long root, so that this scheme always has $|\theta|^2 = 2$. To achieve this, we renormalize the following objects:

- <u>Renormalized Killing form</u>:

$$(x, y)_2 \equiv \frac{1}{2g}(x, y) = \frac{1}{2g}\mathsf{Tr}\left[\widehat{a}_{d_x}\, \widehat{a}_{d_y}\right]_\theta . \tag{9.1.16a}$$

  The trace in the adjoint representation for a product of generators is $2g$ times the renormalized Killing form.

- <u>Renormalized root representation in H, the Cartan subalgebra</u>: Let $h_\alpha \in H$ represent the root $\alpha \in \Delta$; Δ denotes the set of roots. h_α is explicitly defined in the old normalization scheme via Eq. (4.4.10). Then we define the renormalized version

$$h_\alpha^{(2)} \equiv 2g\, h_\alpha \qquad \Rightarrow \qquad (h_\alpha^{(2)}, k)_2 = (h_\alpha, k) = \alpha(k). \tag{9.1.16b}$$

  Thus the eigenvalue of a root α acting on a generic element $k \in H$ is unchanged.

- <u>Root scalar product:</u> Let α and β belong to the set of roots Δ. Then we have the renormalized root scalar product

$$\langle \alpha, \beta \rangle_2 \equiv \left(h_\alpha^{(2)}, h_\beta^{(2)}\right)_2 = 2g\, \langle \alpha, \beta \rangle. \tag{9.1.16c}$$

Comparing Eqs. (9.1.15) and (9.1.16c) guarantees that

$$\left\langle \overline{\alpha}_i^{(\mathsf{long})}, \overline{\alpha}_i^{(\mathsf{long})} \right\rangle_2 = 2, \tag{9.1.17}$$

as desired. We also define a renormalized Casimir operator [c.f. Eq. (9.1.3a)],

$$\hat{C}_2 \equiv \sum_{i,j=1}^{n} \mathcal{A}_{2\,ij}^{-1} \hat{H}_{\overline{\alpha}_i}^{(2)}\, \hat{H}_{\overline{\alpha}_j}^{(2)} + \sum_{\alpha \neq 0} \frac{\hat{E}_\alpha\, \hat{E}_{-\alpha}}{(e_\alpha, e_{-\alpha})_2}, \qquad \text{Quadratic Casimir operator (standard normalization),} \tag{9.1.18a}$$

$$\mathcal{A}_{2\,ij} \equiv \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle_2 = A_{ij}\, \frac{\langle \overline{\alpha}_j, \overline{\alpha}_j \rangle_2}{2}. \tag{9.1.18b}$$

It is evident that

$$\hat{C}_2 = 2g\, \hat{C}. \tag{9.1.19}$$

Now, the analog of the trace in Eqs. (9.1.5) and (9.1.6) is

$$(C_2)_2 \equiv \frac{1}{2g}\mathsf{Tr}\left[\hat{C}_2\right]_\theta = (C) = N_\theta. \tag{9.1.20}$$

Eqs. (9.1.10) and (9.1.20) then imply that

$$\boxed{C_2^\theta = \langle \theta, \theta + 2\rho_{\mathsf{w}} \rangle_2 = 2g. \qquad \begin{array}{l}\text{Standard Casimir in the adjoint representation} \\ = \text{twice the dual Coxeter number.}\end{array}} \tag{9.1.21}$$

If we express θ in terms of its comarks [Eq. (8.1.12)],

$$\theta = \sum_{j=1}^{n} a_j^\vee \, \overline{\alpha}_j^\vee, \tag{9.1.22}$$

then using $\langle \omega_i, \overline{\alpha}_j^\vee \rangle = \langle \omega_i, \overline{\alpha}_j^\vee \rangle_2 = \delta_{ij}$, we get

$$\langle \theta, \rho_{\mathsf{w}} \rangle_2 = \sum_{j=1}^{n} a_j^\vee. \tag{9.1.23}$$

Here we have used the fact that the Weyl vector ρ_{w} always has Dynkin coefficients $(1, 1, 1, \ldots, 1)$ [Eq. (9.1.9)]. Combined with the fact that $\langle \theta, \theta \rangle_2 = 2$, Eqs. (9.1.21) and (9.1.23) give

$$g = 1 + \sum_{j=1}^{n} a_j^\vee, \tag{9.1.24}$$

which is the formula for the dual Coxeter number given in terms of comarks by Eq. (8.1.13).

We can compute Eq. (9.1.21) explicitly, using (a) the Dynkin coefficients of the highest root $\theta = (\Lambda_1^{(\theta)}, \Lambda_2^{(\theta)}, \ldots, \Lambda_n^{(\theta)})$ (Sec. 8.2.4), (b) the Dynkin coefficients of the Weyl vector $\rho_{\mathsf{w}} = (1, 1, \ldots, 1)$ [Eq. (9.1.9)], and (c) the quadratic form matrix F_{ij}, already tabulated in the standard normalization for each algebra in Eq. (8.6.2). The Casimir for the adjoint representation is then

$$C_2^\theta = 2 + 2 \sum_{i,j=1}^{n} \Lambda_i^{(\theta)} F_{ij}. \tag{9.1.25a}$$

For su$(n+1)$, $\Lambda_1^{(\theta)} = \Lambda_n^{(\theta)} = 1$ and all others vanish [Eq. (8.2.25)], while F_{ij} has the form in Eq. (8.6.2a). This leads to

$$C_2^{\theta\,(A_n)} = 2 + 2 \sum_{j=1}^{n} (F_{1j} + F_{nj}) = 2 + \frac{2}{n+1}\,[n(n+1)] = 2(n+1). \tag{9.1.25b}$$

For sp$(2n)$, $\Lambda_1^{(\theta)} = 2$ and all others vanish [Eq. (8.2.27)], while F_{ij} has the form in Eq. (8.6.2c):

$$C_2^{\theta\,(C_n)} = 2 + 2^2 \sum_{j=1}^{n} F_{1j} = 2(n+1). \tag{9.1.25c}$$

For so$(2n)$, $\Lambda_2^{(\theta)} = 1$ and all others vanish [Eq. (8.2.26)], while F_{ij} has the form in Eq. (8.6.2d):

$$C_2^{\theta\,(D_n)} = 2 + 2 \sum_{j=1}^{n} F_{2j} = 2 + [3 \times 2 + (n-3)4] = 2(2n-2). \tag{9.1.25d}$$

Eqs. (9.1.25b)–(9.1.25d) agree with the previously quoted results for $2g$ in Eq. (8.6.1).

For a generic representation with highest weight $\Lambda = (\Lambda_1, \cdots, \Lambda_n)$, the standard Casimir is

$$C_2^\Lambda = \langle \Lambda, \Lambda + 2\rho_{\mathsf{w}} \rangle_2 = \langle \Lambda, \Lambda \rangle_2 + 2 \sum_{i,j=1}^{n} \Lambda_i \, F_{i,j}, \qquad \text{Standard Casimir for highest weight representation } \Lambda. \tag{9.1.26}$$

For the defining representation $\Lambda = \omega_1$, one finds the following

Algebra	Casimir $\langle \omega_1, \omega_1 + 2\rho_{\mathsf{w}} \rangle_2$
$A_n = \mathrm{su}(n+1)$	$\frac{n(n+2)}{n+1}$
$B_n = \mathrm{so}(2n+1)$	$2n$
$C_n = \mathrm{sp}(2n)$	$n + \frac{1}{2}$
$D_n = \mathrm{so}(2n)$	$2n - 1$

(9.1.27)

• **Exercise**: Show that the standard Casimir for a rank-q fully antisymmetric tensor representation of A_n is given by

$$\langle \Lambda, \Lambda + 2\rho_{\mathsf{w}} \rangle_2 = \frac{q(n+1-q)(n+2)}{n+1}, \quad 1 \leq q \leq n. \tag{9.1.28}$$

• **Exercise**: Show that the standard Casimir for a rank-q fully antisymmetric tensor representation of C_n is given by

$$\langle \Lambda, \Lambda + 2\rho_{\mathsf{w}} \rangle_2 = \frac{1}{2} q \left[2(n+1) - q \right], \quad 1 \leq q \leq n. \tag{9.1.29}$$

9.1.2 Index of a representation; some examples

Let x and y be elements of the Lie algebra, $x, y \in L$. For a highest weight representation Λ, define

$$((x,y))_\Lambda \equiv \mathsf{Tr} \left[\hat{X}\, \hat{Y} \right]_\Lambda, \tag{9.1.30}$$

where the right-hand side is the trace normalization of the product of generators in the irreducible representation. The **index** l_Λ for representation Λ is then defined via

$$((x,y))_\Lambda = l_\Lambda\, (x,y)_2. \qquad \text{Index of a representation.} \tag{9.1.31}$$

The index relates the trace normalization in some arbitrary irreducible representation Λ to that in the adjoint representation (i.e., the renormalized Killing form). Using the renormalized Casimir operator constructed in the representation Λ, Eqs. (9.1.18a), (9.1.26), and (9.1.31) imply that

$$\begin{aligned} ((C_2)) &= l_\Lambda (C_2)_2 = l_\Lambda\, N_\theta \\ &= \langle \Lambda, \Lambda + 2\rho_{\mathsf{w}} \rangle_2\, N_\Lambda, \end{aligned} \tag{9.1.32}$$

which leads to

$$l_\Lambda = \frac{\langle \Lambda, \Lambda + 2\rho_{\mathsf{w}} \rangle_2\, N_\Lambda}{N_\theta}. \tag{9.1.33}$$

Here N_Λ is the dimension of the representation Λ (i.e., the total number of states, including degeneracies); N_θ is the dimension of the adjoint representation. Note that

$$l_\theta = \langle \theta, \theta + 2\rho_{\mathsf{w}} \rangle = 2g, \tag{9.1.34}$$

consistent with Eq. (9.1.16a). The index of the defining representation l_{ω_1}, the adjoint representation $l_\theta = 2g$ [Eq. (8.6.1)], as well as the dimension N_{ω_1} and the dimension of the adjoint N_θ for the classical families are [via Eq. (9.1.27)]

Algebra	N_{ω_1}	N_θ	l_{ω_1}	$l_\theta = 2g$
$A_n = \mathrm{su}(n+1)$	$n+1$	$n^2 + 2n$	1	$2(n+1)$
$B_n = \mathrm{so}(2n+1)$	$2n+1$	$n(2n+1)$	2	$2(2n-1)$
$C_n = \mathrm{sp}(2n)$	$2n$	$(2n)(n+1/2)$	1	$2(n+1)$
$D_n = \mathrm{so}(2n)$	$2n$	$n(2n-1)$	2	$4(n-1)$

(9.1.35)

For the orthogonal algebras, it is interesting to compute the Casimirs and indices for the spinor representations.

• **Exercise**: Show that the standard Casimir for the ω_n fundamental spinor representation of $B_n = \mathrm{so}(2n+1)$ is given by

$$\langle \omega_n, \omega_n + 2\rho_{\mathsf{w}} \rangle_2 = \frac{n}{4}(2n+1). \tag{9.1.36}$$

• **Exercise**: Show that the standard Casimir for the ω_{n-1} and ω_n fundamental spinor representations of $D_n = \mathrm{so}(2n)$ is given by

$$\langle \omega_{n-1}, \omega_{n-1} + 2\rho_{\mathsf{W}} \rangle_2 = \langle \omega_n, \omega_n + 2\rho_{\mathsf{W}} \rangle_2 = \frac{n}{4}(2n-1). \tag{9.1.37}$$

Using the facts (proven later) that $N_{\omega_n} = 2^n$ in B_n and $N_{\omega_{n-1}} = N_{\omega_n} = 2^{n-1}$ for D_n, the indices are given by

Algebra	Fundamental spinor rep. index(ices)
$B_n = \mathrm{so}(2n+1)$	$l_{\omega_n} = 2^{n-2}$
$D_n = \mathrm{so}(2n)$	$l_{\omega_{n-1}} = l_{\omega_n} = 2^{n-3}$

(9.1.38)

9.1.2.1 su(2)

In su(2), the Cartan matrix is a single number: $A = \langle \theta, \theta \rangle = 2$. For a simply-laced algebra, $\alpha^\vee = 2\alpha/|\alpha|^2 = \alpha$ in the standard normalization scheme. Then the quadratic form matrix is $F = \langle \omega_1, \omega_1 \rangle = A^{-1} = 1/2$. The Weyl vector is $\rho_{\mathsf{W}} = \omega_1$. The spin j representation has highest weight $\Lambda = 2j$. The Casimir for this representation is

$$C_2^{(2j)} = 2j(2j+2)\langle \omega_1, \omega_1 \rangle_2 = 2j(j+1). \tag{9.1.39}$$

This is twice the usual normalization in quantum mechanics. The index is

$$l_{2j} = 2j(j+1)\frac{2j+1}{3} = \frac{2j(j+1)(2j+1)}{3}. \tag{9.1.40}$$

The index of the first few j values is

j	l_{2j}
0	0
1/2	1
1	4
3/2	10
2	20
5/2	35

(9.1.41)

Eq. (9.1.41) implies that the su(2) generators in the defining $j = 1/2$ representation are

$$\hat{T}_{x,y,z} = \frac{1}{\sqrt{2}}\hat{\sigma}^{x,y,z}, \qquad \left[\hat{T}_i, \hat{T}_j\right] = i f^{ijk}\hat{T}_k. \tag{9.1.42}$$

where $\hat{\sigma}^{x,y,z}$ are the standard Pauli matrices. The structure constants are $f^{ijk} = \sqrt{2}\epsilon^{ijk}$, which explains the "extra" factor of two in Eq. (9.1.39).

9.1.2.2 su(3)

A special feature of su(3) is that any highest weight representation can be associated to a traceless bisymmetrized tensor $\tilde{T}^{(j_1\cdots j_q)}_{(i_1\cdots i_p)}$, $\tilde{T}^{(kj_2\cdots j_q)}_{(ki_2\cdots i_p)} = 0$. The Dynkin coefficients of the associated highest weight are (p,q) [c.f. Eq. (8.5.2)]. The dimension is the number of independent elements of this tensor,

$$N_{p,q} = \frac{(p+2)!}{p!2!}\frac{(q+2)!}{q!2!} - \frac{(p+1)!}{(p-1)!2!}\frac{(q+1)!}{(q-1)!2!} = \frac{(p+1)(q+1)(p+q+2)}{2}. \tag{9.1.43}$$

This result can also be obtained from the "factors over hooks" rule. The Casimir for this representation is

$$\begin{aligned} C_2^{(p,q)} &= \langle p\,\omega_1 + q\,\omega_2, (p+2)\omega_1 + (q+2)\omega_2 \rangle \\ &= \frac{2}{3}\left[p(p+2) + q(q+2)\right] + \frac{1}{3}\left[p(q+2) + q(p+2)\right] = \frac{2}{3}\left[p^2 + q^2 + pq + 3(p+q)\right]. \end{aligned} \tag{9.1.44}$$

Thus the index is

$$l_{p,q} = \frac{C_2^{(p,q)} N_{p,q}}{N_\theta} = \frac{1}{24}(p+1)(q+1)(p+q+2)\left[p^2 + q^2 + pq + 3(p+q)\right]. \tag{9.1.45}$$

The Casimir and index for a few low-dimensional representations are

HWS	conjugacy class	$N_{p,q}$	$C_2^{(p,q)}$	$l_{p,q}$
$(0,0)$	0	1	0	0
$(1,0)$	1	3	$8/3$	1
$(1,1)$	0	8	6	6
$(2,0)$	2	6	$20/3$	5
$(2,1)$	1	15	$32/3$	20
$(2,2)$	0	27	16	54

(9.1.46)

Eqs. (9.1.35), (9.1.38), (9.1.41), and (9.1.46) suggest that the index is always a non-negative integer; this is indeed the case.

9.2 Freudenthal's recursion formula for weight space dimensions

The master weight depth formula Eq. (5.1.23) allows the determination of all weights in a finite-dimensional, highest weight irreducible representation of a Lie algebra. It does *not* determine the total number of states in a given representation, however, because some weights can be degenerate. For example, the (2,1) representation of su(3) shown in Fig. 9.1 has only 12 distinct weights. On the other hand, Eq. (9.1.43) implies that $N_{2,1} = 15$. Therefore between one to three of the weights in (2,1) must be degenerate. Recall that the **weight space** refers to the set of states sharing the same weight, i.e. the same Dynkin coefficients.

In this section we will determine a recursive algorithm (Freudenthal's formula) to determine the dimension of all weight spaces in a given representation. Let M be a weight in the representation with highest weight Λ. Consider the trace of the Casimir [Eq. (9.1.26)] within the possibly degenerate weight space associated to M,

$$\begin{aligned}\mathsf{Tr}\left[\hat{C}_2^{\Lambda}\right]_M &= n_M\,\langle\Lambda,\Lambda+2\rho_{\mathsf{w}}\rangle_2\\ &= n_M\langle M,M\rangle_2+\sum_{\alpha>0}\frac{\mathsf{Tr}_M\left[\hat{E}_\alpha\,\hat{E}_{-\alpha}+\hat{E}_{-\alpha}\,\hat{E}_\alpha\right]}{(e_\alpha,e_{-\alpha})_2},\end{aligned}\tag{9.2.1}$$

where n_M is the dimension of the weight space labeled by M. In order to evaluate the second term, we consider the su(2) generated by

$$\begin{aligned}[\hat{H}_\alpha^{(2)},\hat{E}_{\pm\alpha}] &= \pm\,\langle\alpha,\alpha\rangle_2\,\hat{E}_{\pm\alpha},\\ [\hat{E}_\alpha,\hat{E}_{-\alpha}] &= (e_\alpha,e_{-\alpha})_2\,\hat{H}_\alpha^{(2)}.\end{aligned}\tag{9.2.2}$$

It is useful to rewrite this in terms of standard su(2) generators,

$$[\hat{T}_z,\hat{T}_\pm]=\pm\hat{T}_\pm,\qquad [\hat{T}_+,\hat{T}_-]=2\hat{T}_z.\tag{9.2.3}$$

Let

$$\hat{T}_\pm\equiv A\,\hat{E}_{\pm\alpha},\quad \hat{T}_z\equiv B\,\hat{H}_\alpha,\quad A=\sqrt{\frac{2}{\langle\alpha,\alpha\rangle_2(e_\alpha,e_{-\alpha})_2}},\quad B=\frac{1}{\langle\alpha,\alpha\rangle_2}.\tag{9.2.4}$$

For the particular root α, we have the su(2) Casimir in Eq. (9.1.1),

$$\frac{1}{2}\left(\hat{T}_+\hat{T}_-+\hat{T}_-\hat{T}_+\right)+\hat{T}_z\hat{T}_z=\frac{A^2}{2}\left[\hat{E}_\alpha\,\hat{E}_{-\alpha}+\hat{E}_{-\alpha}\,\hat{E}_\alpha\right]+B^2\,\hat{H}_\alpha^2.\tag{9.2.5}$$

Now, each state in the n_M-fold degenerate weight space of M belongs to <u>some</u> irreducible representation of this su(2) with highest weight $t\in\mathbb{Z}/2$, Casimir eigenvalue $t(t+1)$. We choose a basis within the weight space such that each weight vector belongs to a distinct α-generated su(2) representation. The weight space may contain multiple distinguishable copies of the same irreducible representation and associated highest weight t. It may also possess one or more isolated states, i.e. states with weight M that cannot be raised or lowered by $\hat{E}_{\pm\alpha}=\hat{T}_\pm/A$.

Consider the state in the weight space of M that sits in the α-generated su(2) with highest weight t. Denote this state by the weight vector $\phi_t^{(M)}$. Suppose that $\phi_t^{(M+K\alpha)}$ is the associated highest weight:

$$\hat{T}_z\,\phi_t^{(M+K\alpha)}=t\,\phi_t^{(M+K\alpha)}=B\langle\alpha,M+K\alpha\rangle_2\,\phi_t^{(M+K\alpha)}\quad\Rightarrow\quad t=B\langle\alpha,M+K\alpha\rangle_2.\tag{9.2.6}$$

Then

$$\left[\hat{E}_\alpha\,\hat{E}_{-\alpha}+\hat{E}_{-\alpha}\,\hat{E}_\alpha\right]\phi_t^{(M)}=\left\{-\frac{2B^2}{A^2}\,\langle M,\alpha\rangle_2^2+\frac{2B}{A^2}\langle\alpha,M+K\alpha\rangle_2+\frac{2B^2}{A^2}\left[\langle\alpha,M\rangle_2^2+K^2\langle\alpha,\alpha\rangle_2^2+2K\langle M,\alpha\rangle_2\langle\alpha,\alpha\rangle_2\right]\right\}\phi_t^{(M)}$$
$$=\frac{2B}{A^2}\left[(2K+1)\,\langle M,\alpha\rangle_2+K(K+1)\,\langle\alpha,\alpha\rangle_2\right]\phi_t^{(M)}. \tag{9.2.7}$$

Next we note that the number of α-generated su(2) representations with label K marking the highest weight is

$$n_{M+\alpha K}-n_{M+\alpha(K+1)}. \tag{9.2.8}$$

The first term is the degeneracy of the weight space for the presumed highest weight. The second term is the degeneracy of the weight space one step above the highest weight; taking the difference eliminates the contributions of degenerate states with weight M that originate by lowering from a *higher* highest weight state than that assumed. To get the total contribution from the weight space trace on the right-hand side of Eq. (9.2.1), we sum over all possible values of K. I.e., we sum over all possible highest weight states that can give M as a descendant:

$$\sum_{\alpha>0}\frac{\mathsf{Tr}_M\left[\hat{E}_\alpha\,\hat{E}_{-\alpha}+\hat{E}_{-\alpha}\,\hat{E}_\alpha\right]}{(e_\alpha,e_{-\alpha})_2}=\sum_{\alpha>0}\sum_{K\geq 0}\left[n_{M+\alpha K}-n_{M+\alpha(K+1)}\right]\frac{2B}{A^2(e_\alpha,e_{-\alpha})_2}\left[(2K+1)\,\langle M,\alpha\rangle_2+K(K+1)\,\langle\alpha,\alpha\rangle_2\right]$$
$$=\sum_{\alpha>0}\sum_{K\geq 0}\left[n_{M+\alpha K}-n_{M+\alpha(K+1)}\right]\left[(2K+1)\,\langle M,\alpha\rangle_2+K(K+1)\,\langle\alpha,\alpha\rangle_2\right]$$
$$=\sum_{\alpha>0}\left\{n_M\langle M,\alpha\rangle_2+\sum_{K\geq 1}n_{M+\alpha K}\left[2\,\langle M,\alpha\rangle_2+2K\,\langle\alpha,\alpha\rangle_2\right]\right\}$$
$$=\left\{n_M\langle M,2\rho_{\mathsf{w}}\rangle_2+\sum_{\alpha>0}\sum_{K\geq 1}2n_{M+\alpha K}\left[\langle M,\alpha\rangle_2+K\,\langle\alpha,\alpha\rangle_2\right]\right\}$$
$$=n_M\left[\langle\Lambda,\Lambda+2\rho_{\mathsf{w}}\rangle_2-\langle M,M\rangle_2\right]. \tag{9.2.9}$$

Solving for n_M leads to

$$\boxed{n_M=\frac{\displaystyle\sum_{\alpha>0}\sum_{K\geq 1}2\,n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle_2}{\langle\Lambda+M+2\rho_{\mathsf{w}},\Lambda-M\rangle_2},\qquad\text{Freudenthal's recursion formula for the dimension of weight space } M.} \tag{9.2.10}$$

An algorithm to evaluate Eq. (9.2.10) is the following:

- Start with the highest weight of the representation Λ. This is guaranteed to be non-degenerate, so $n_\Lambda=1$.
- Proceed to each weight that differs from Λ by one root. To compute the inner products in the numerator and denominator of Eq. (9.2.10), one can express all weights in terms of Dynkin coefficients and use the quadratic form matrix [Eq. (8.6.2)]. Alternatively, to compute $\langle\Lambda,\Lambda'\rangle$ one can express

$$\Lambda=\sum_{i=1}^n\Lambda_i\omega_i,\quad\Lambda'=\sum_{j=1}^n\kappa_j^{(\Lambda')}\,\overline{\alpha}_j, \tag{9.2.11}$$

so that

$$\langle\Lambda,\Lambda'\rangle=\sum_{i=1}^n\Lambda_i\,\kappa_i^{(\Lambda')}\,\frac{|\overline{\alpha}_i|_2^2}{2}. \tag{9.2.12}$$

Here we have exploited the orthonormality of the fundamental weights and simple coroots; the factor of $|\overline{\alpha}_i|^2/2$ appears because we have expanded Λ' in terms of simple roots. Note that for a simply-laced algebra, $|\overline{\alpha}_i|_2^2=2$.

Figure 9.1: (2,1) representation of su(3). There are 15 states in this representation, but only 12 distinct weights. Each of the weights in the inner triangle are doubly degenerate.

We will illustrate Eq. (9.2.10) for the (2,1) representation of the simply-laced algebra su(3), Fig. 9.1.

1. The highest weight state has $n_{2,1} = 1$.
2. Consider $M = (0, 2)$. It has

$$\begin{aligned} \Lambda + M + 2\rho_{\mathsf{w}} = (2,1) + (0,2) + (2,2) = (4,5), \qquad \Lambda - M = \overline{\alpha}_1, \\ \Rightarrow \quad \langle \Lambda + M + 2\rho_{\mathsf{w}}, \Lambda - M \rangle_2 = 4, \\ K = 1: \quad 2\, n_{M+\overline{\alpha}_1} \langle M + \overline{\alpha}_1, \overline{\alpha}_1 \rangle_2 = 2\,(0+2) = 4, \end{aligned} \tag{9.2.13}$$

so that $n_{0,2} = 1$.

3. $M = (-2, 3)$.

$$\begin{aligned} \Lambda + M + 2\rho_{\mathsf{w}} = (2,1) + (-2,3) + (2,2) = (2,6), \qquad \Lambda - M = 2\overline{\alpha}_1, \\ \Rightarrow \quad \langle \Lambda + M + 2\rho_{\mathsf{w}}, \Lambda - M \rangle_2 = 4, \\ K = 1: \quad 2\, n_{M+\overline{\alpha}_1} \langle M + \overline{\alpha}_1, \overline{\alpha}_1 \rangle_2 = 0, \\ K = 2: \quad 2\, n_{M+2\overline{\alpha}_1} \langle M + 2\overline{\alpha}_1, \overline{\alpha}_1 \rangle_2 = 2\,(-2+4) = 4, \end{aligned} \tag{9.2.14}$$

so that $n_{-2,3} = 1$.

4. $M = (3, -1)$.

$$\begin{aligned} \Lambda + M + 2\rho_{\mathsf{w}} = (2,1) + (3,-1) + (2,2) = (7,2), \qquad \Lambda - M = \overline{\alpha}_2, \\ \Rightarrow \quad \langle \Lambda + M + 2\rho_{\mathsf{w}}, \Lambda - M \rangle_2 = 2, \\ K = 1: \quad 2\, n_{M+\overline{\alpha}_2} \langle M + \overline{\alpha}_2, \overline{\alpha}_2 \rangle_2 = 2\,(-1+2) = 2, \end{aligned} \tag{9.2.15}$$

so that $n_{3,-1} = 1$.

5. $M = (1, 0)$.

$$\begin{aligned} \Lambda + M + 2\rho_{\mathsf{w}} = (2,1) + (1,0) + (2,2) = (5,3), \qquad \Lambda - M = \overline{\alpha}_1 + \overline{\alpha}_2, \\ \Rightarrow \quad \langle \Lambda + M + 2\rho_{\mathsf{w}}, \Lambda - M \rangle_2 = 8, \\ \overline{\alpha}_1, \quad K = 1: \quad 2\, n_{M+\overline{\alpha}_1} \langle M + \overline{\alpha}_1, \overline{\alpha}_1 \rangle_2 = 2(2+1) = 6, \\ \overline{\alpha}_2, \quad K = 1: \quad 2\, n_{M+\overline{\alpha}_2} \langle M + \overline{\alpha}_2, \overline{\alpha}_2 \rangle_2 = 2(2) = 4, \\ \overline{\alpha}_1 + \overline{\alpha}_2, \quad K = 1: \quad 2\, n_{M+\overline{\alpha}_1+\overline{\alpha}_2} \langle M + \overline{\alpha}_1 + \overline{\alpha}_2, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle_2 = 2(3) = 6, \end{aligned} \tag{9.2.16}$$

so that $n_{1,0} = (6+4+6)/8 = 2$.

Figure 9.2: The same (2,1) representation of su(3) as shown in Fig. 9.1. Here we depict the decomposition into irreducible su(2) representations generated by $\hat{E}_{\pm\overline{\alpha}_1}$, i.e. the decomposition into parallel strings consisting of weights separated by simple root $\overline{\alpha}_1$ moves. The doubly-degenerate weights arise due to overlapping representations.

We could proceed to evaluate the remaining weight space dimensions, but it is much easier to simply infer them using Weyl reflections, see Fig. 9.3. We conclude that the weights (1,0), (-1,1), and (0,-1) are each doubly-degenerate, while each weight in the "outer layer" (see Fig. 9.1) has a one-dimensional weight space. This gives a total of 12 + 3 = 15 weights, in agreement with Eq. (9.1.43).

The partitioning of the (2,1) representation into different and sometimes overlapping su(2) representations generated by $\hat{E}_{\pm\overline{\alpha}_1}$ is depicted in Fig. 9.2.

• **Exercise**: Determine the weight tree and the dimension of all weight spaces for the su(3) representation (2,2), which has 27 total states [Eq. (9.1.46)]. Is there a pattern to the degeneracies in terms of the weight geometry?

• **Exercise**: Use Freudenthal's formula to show that all states of the fundamental spinor representation (0,0,1) of so(7) correspond to distinct weights, i.e. that all weight spaces are one-dimensional. The weight tree (geometry) is shown in Fig. 8.9 (Fig. 8.10).

9.3 Weyl group

• • • Throughout the remainder of the course, unless stated otherwise we will employ the "standard normalization" defined by Eq. (9.1.17). We will drop explicit "2" subscripts to lighten the notation.

The **Weyl reflection** S_α along a root α was introduced in Sec. 5.2.1, in the context of root system geometries. The action of S_α upon a generic weight M in an irreducible representation is given by [Eq. (5.2.3)]

$$S_\alpha\, M = M - \langle M, \alpha^\vee\rangle\, \alpha, \quad S_\alpha^2 = 1, \qquad \text{Weyl reflection } \equiv S_\alpha. \tag{9.3.1}$$

With respect to the su(2) generated by $\{\hat{E}_{\pm\alpha}, \hat{H}_\alpha\}$, an irreducible representation decomposes into parallel strings of weights of varying lengths. Applying the Weyl reflection interchanges the weights along these strings. Weyl reflections for su(3) are depicted in Fig. 9.3. A **simple Weyl reflection** $S_{\overline{\alpha}_i}$ gives

$$S_{\overline{\alpha}_i}\, M = M - \Lambda_i^{(M)}\, \overline{\alpha}_i, \qquad \text{Simple Weyl reflection}, \tag{9.3.2}$$

where $\Lambda_i^{(M)}$ is the corresponding Dynkin coefficient. Acting on another simple root gives

$$S_{\overline{\alpha}_i}\, \overline{\alpha}_j = \overline{\alpha}_j - A_{ji}\, \overline{\alpha}_i, \tag{9.3.3}$$

where $A_{ji} \equiv \langle \overline{\alpha}_j, \overline{\alpha}_i^\vee\rangle$ is the Cartan Matrix [Eq. (5.3.8)].

We prove the following facts about Weyl reflections.

1. Invariance of the scalar product.

$$\langle S_\alpha\, M, M'\rangle = \langle M - \langle M, \alpha^\vee\rangle\alpha, M'\rangle = \langle M, M'\rangle - \langle M, \alpha^\vee\rangle\langle M', \alpha\rangle = \langle M, S_\alpha\, M'\rangle. \tag{9.3.4}$$

Figure 9.3: Weyl reflections $S_{1,2,3}$ for su(3).

2. The simple Weyl reflection $S_{\overline{\alpha}_i}$ permutes positive roots, except for $\overline{\alpha}_i \to -\overline{\alpha}_i$. Consider a positive root $\beta \in \Delta_+$ [Eq. (5.3.7)]

$$\beta \equiv \sum_{i=1}^{n} \kappa_i^{(\beta)} \overline{\alpha}_i, \quad \kappa_i^{(\beta)} \in \mathbb{N}_0. \tag{9.3.5}$$

The action of a simple root Weyl reflection is

$$S_{\overline{\alpha}_i}\, \beta = \sum_{j \neq i}^{n} \kappa_j^{(\beta)} \overline{\alpha}_j + \overline{\alpha}_i \left[\sum_{j \neq i}^{n} \kappa_j^{(\beta)} \, |A_{ji}| - \kappa_i^{(\beta)} \right]. \tag{9.3.6}$$

Now, assuming that the positive root $\beta \neq \overline{\alpha}_i$, then at least one $\kappa_{j \neq i}^{(\beta)} > 0$. But recall that in the "root-building" application of the MWDF [Eqs. (5.3.26) and (7.1.24)], each positive root obtains by adding some combination of simple roots to a particular simple root. It means that if one expansion coefficient in Eq. (9.3.6) is positive, then so are all others. Therefore $S_{\overline{\alpha}_i}\, \beta$ is a positive root.

Using Weyl reflections, we can prove the equivalence of the two expressions for the Weyl vector ρ_{W} in Eq. (9.1.9). Let

$$\sigma \equiv \frac{1}{2} \sum_{\alpha > 0} \alpha, \qquad \rho \equiv \sum_{j=1}^{n} \omega_j. \tag{9.3.7}$$

Since a simple Weyl reflection $S_{\overline{\alpha}_i}$ permutes the positive roots except $\overline{\alpha}_i$,

$$S_{\overline{\alpha}_i}\, \sigma = \frac{1}{2} \sum_{\substack{\alpha > 0 \\ \alpha \neq \overline{\alpha}_i}} \alpha - \frac{1}{2} \overline{\alpha}_i = \sigma - \overline{\alpha}_i. \tag{9.3.8}$$

Then

$$\langle S_{\overline{\alpha}_i}\, \sigma, \overline{\alpha}_i^{\vee} \rangle = \langle \sigma, \overline{\alpha}_i^{\vee} \rangle - 2 = \langle \sigma, S_{\overline{\alpha}_i}\, \overline{\alpha}_i^{\vee} \rangle = -\langle \sigma, \overline{\alpha}_i^{\vee} \rangle, \tag{9.3.9}$$

or

$$\langle \sigma, \overline{\alpha}_i^{\vee} \rangle = \Lambda_i = 1, \qquad i \in \{1, 2, \ldots, n\}. \tag{9.3.10}$$

In this equation $\{\Lambda_i\}$ denote the Dynkin coefficients of σ. We conclude that $\sigma = \rho = \rho_{\mathsf{W}}$ in Eq. (9.3.7).

The product of all Weyl reflections forms the **Weyl group** W, which is a subgroup of the isometry group for the root system. Any weight geometry associated to an irreducible representation must be left invariant under a Weyl group transformation. Note that the Weyl group contains more than Weyl reflections, since the product of two reflections is equivalent to a rotation. The product of an even (odd) even number of Weyl reflections is a "proper rotation" (an "improper rotation") with determinant equal to $+1$ (-1) (thinking of S_α as an orthogonal transformation on n-dimensional vectors in H_0^*, the space of distinct weights).

We can choose a "basis" for the Weyl group by considering compositions of simple root Weyl reflections $\{S_{\overline{\alpha}_j}\}$, $1 \le j \le n$ (and n denotes the rank of the algebra). Every element $w \in W$ can be written as a product of simple Weyl reflections,

$$w = S_{\overline{\alpha}_{i_1}} S_{\overline{\alpha}_{i_2}} \times \cdots \times S_{\overline{\alpha}_{i_p}}. \tag{9.3.11}$$

The Weyl group is important because it relates weights based on *point group symmetries* of the representation geometry. In module 8, we mainly stressed tools that do not make this geometry manifest. We focused on expanding weights in terms of the fundamental weights $\{\omega_i\}$ with Dynkin expansion coefficients, the MWDF algorithm for constructing the weight tree, Young tableaux, etc. The geometry of a representation is instead encoded in the expansion of a generic weight in terms of simple roots and the geometry of the root system. Weyl group transformations algebraically encode this geometrical information.

An obvious but important statement is the following. Given the set of simple roots, the entire root geometry can be constructed by acting on simple roots with elements of the Weyl group. In other words,

$$\Delta = \left\{ w\,\overline{\alpha}_1, w\,\overline{\alpha}_2, \ldots, w\,\overline{\alpha}_n \middle| w \in W \right\}. \tag{9.3.12}$$

Here Δ denotes the set of all roots. Evidently any set $\{w\,\overline{\alpha}_i\}$ for a particular, fixed Weyl element w could serve as a basis of simple roots. Thus we can use the Weyl group to change the basis for the simple roots.

Another obvious implication is that

$$n_M = n_{(w\,M)}, \quad w \in W. \tag{9.3.13}$$

Here n_M is the dimension of the weight space for weight M in some irreducible representation [as appears in Freudenthal's formula Eq. (9.2.10)] and w is a generic Weyl group element.

The set of weights that can be obtained by acting with all possible Weyl group operations on some particular weight M is called the **Weyl orbit** of M. We can choose one particular weight in this orbit with non-negative Dynkin coefficients to represent the entire orbit. **Proof:** Suppose that M' is the highest weight in the orbit, and suppose that it has a negative Dynkin coefficient: $\Lambda_i = \langle M', \overline{\alpha}_i^\vee \rangle < 0$. But then

$$S_{\overline{\alpha}_i}\, M' = M' - \langle M', \overline{\alpha}_i^\vee \rangle \overline{\alpha}_i = M' + |\Lambda_i|\, \overline{\alpha}_i \tag{9.3.14}$$

is by definition (lexicographic ordering, Sec. 5.3.1) greater than M' (contradiction). A weight with non-negative Dynkin coefficients is said to be **dominant**.

9.4 Weyl's character formula: derivation

9.4.1 Generating functions

In the following, we will need to view H_0^* (the space of roots with real coefficients) in two different ways:

1. As a linear vector space of dimension n (where n denotes the rank), with the linearly independent simple root basis. A generic element is a "vector" (weight)

$$\beta = \sum_{j=1}^{n} \kappa_j^{(\beta)} \overline{\alpha}_j, \qquad \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = \mathcal{A}_{ij} = A_{ij} \frac{|\overline{\alpha}_j|^2}{2}. \tag{9.4.1}$$

2. As an infinite dimensional Hilbert space, where $|\beta\rangle$ corresponding to a generic weight is itself a <u>basis element</u>. The inner product between two such states is

$$\langle \beta | \lambda \rangle = \delta^{(n)}(\beta - \lambda).$$

A generic state is a <u>linear function</u>,

$$|f\rangle = \int d^n\beta\, f(\beta)\, |\beta\rangle\,. \tag{9.4.2}$$

We can view the action of a Weyl group transformation $w \in W$ in these two different ways:

1. As an orthogonal transformation on an n-component vector,

$$w\,\beta = \sum_j \kappa_j^{(\beta)} (w\,\overline{\alpha}_j), \tag{9.4.3}$$

where $w\,\overline{\alpha}_j$ is the transformed basis vector.

2. As a unitary Hilbert space operator $\hat{U}_w$ such that

$$\hat{U}_w\,|\beta\rangle = |w\,\beta\rangle\,. \tag{9.4.4}$$

The action on a generic function is

$$\hat{U}_w\,|f\rangle = \int d^n\beta\, f(\beta)\,|w\,\beta\rangle = \int d^n\beta\, f(w^{-1}\beta)\,|\beta\rangle\,, \tag{9.4.5}$$

so that

$$(w\,f)\,(\beta) \equiv \langle\beta|\,\hat{U}_w\,|f\rangle = f(w^{-1}\beta). \tag{9.4.6}$$

Now we define two different functions on H_0^*:

$$\chi(\beta) = \langle\beta|\chi\rangle \equiv \sum_M n_M\, e^{\langle M,\beta\rangle}, \qquad \text{Character of a representation.} \tag{9.4.7a}$$

$$Q(\beta) = \langle\beta|Q\rangle \equiv \prod_{\alpha>0}\left[e^{\frac{1}{2}\langle\alpha,\beta\rangle} - e^{-\frac{1}{2}\langle\alpha,\beta\rangle}\right], \qquad \text{(Auxiliary) } Q\text{-function.} \tag{9.4.7b}$$

The character function $\chi(\beta)$ involves a sum over all weights $\{M\}$ of an irreducible representation; n_M is the weight space dimension [Eq. (9.2.10)]. The Q-function involves a product over the positive roots.

The character formula is invariant under a Weyl transformation:

$$(w\,\chi)\,(\beta) = \chi(w^{-1}\beta) = \sum_M n_M\, e^{\langle w\,M,\beta\rangle} = \sum_M n_{w^{-1}\,M}\, e^{\langle M,\beta\rangle} = \chi(\beta), \tag{9.4.8}$$

where we have used Eq. (9.3.13).

Let $\hat{U}_{\overline{\alpha}_i}$ implement a simple Weyl reflection [Eq. (9.3.2)],

$$\hat{U}_{\overline{\alpha}_i}\,|\beta\rangle = |S_{\overline{\alpha}_i}\beta\rangle\,. \tag{9.4.9}$$

Then

$$\begin{aligned}(S_{\overline{\alpha}_i}Q)\,(\beta) &= \langle\beta|\,\hat{U}_{\overline{\alpha}_i}\,|Q\rangle = \prod_{\alpha>0}\left[e^{\frac{1}{2}\langle S_{\overline{\alpha}_i}\alpha,\beta\rangle} - e^{-\frac{1}{2}\langle S_{\overline{\alpha}_i}\alpha,\beta\rangle}\right]\\ &= \prod_{\substack{\alpha>0\\ \alpha\neq\overline{\alpha}_i}}\left[e^{\frac{1}{2}\langle\alpha,\beta\rangle} - e^{-\frac{1}{2}\langle\alpha,\beta\rangle}\right]\left[e^{-\frac{1}{2}\langle\overline{\alpha}_i,\beta\rangle} - e^{\frac{1}{2}\langle\overline{\alpha}_i,\beta\rangle}\right] = -Q(\beta).\end{aligned} \tag{9.4.10}$$

Here we have used the fact that $S_{\overline{\alpha}_i}$ permutes the positive roots *except* for $\overline{\alpha}_i \to -\overline{\alpha}_i$. More generally, can show that

$$(w\,Q)\,(\beta) = Q(w^{-1}\beta) = (\det w)\;Q(\beta), \qquad w\in W. \tag{9.4.11}$$

Here $\det w$ is the determinant of the Weyl group element w, represented as an $n\times n$ matrix acting on the linear vector space H_0^* spanned by $\{\overline{\alpha}_i\}$. This determinant is guaranteed to be equal to plus or minus one for a proper or improper rotation, respectively. Eq. (9.4.11) implies that $Q(\beta)$ is an "alternating" function on weight space. Equivalently, $|Q\rangle$ is an eigenvector of a generic Weyl transformation $\hat{U}_w$ with eigenvalue $\det w$,

$$\hat{U}_w\,|Q\rangle = (\det w)\,|Q\rangle\,. \tag{9.4.12}$$

Define

$$\hat{U}_\sigma \equiv \sum_{w\in W}(\det w)\,\hat{U}_w, \qquad \text{"Alternating" (and un-normalized) projector.} \tag{9.4.13}$$

This has the properties

$$\hat{U}_w\,\hat{U}_\sigma = \hat{U}_\sigma\,\hat{U}_w = (\det w)\,\hat{U}_\sigma, \tag{9.4.14a}$$

$$(\sigma\,f)\,(\beta) \equiv \langle\beta|\,\hat{U}_\sigma\,|f\rangle = \sum_{w\in W}(\det w)\,f(w^{-1}\beta), \tag{9.4.14b}$$

$$\hat{U}_\sigma^2 = N_W\,\hat{U}_\sigma, \tag{9.4.14c}$$

where N_W is the order (number of elements) of the Weyl group.

Let us attempt a decomposition for $Q(\beta)$ using $\hat{U}_\sigma$:

$$Q(\beta) \equiv (\sigma\, R)\,(\beta), \tag{9.4.15}$$

where $R(\beta)$ is some other function. Eq. (9.4.14a) guarantees that $Q(\beta)$ constructed in this way is alternating. If we expand the product in Eq. (9.4.7b), we will get terms of the form $\exp\left[\left\langle \sum_{\alpha>0} (\pm 1)^{m_\alpha}\, \frac{\alpha}{2}, \beta\right\rangle\right]$, where $m_\alpha \in \{0,1\}$. The term with all $m_\alpha = 0$ or all $m_\alpha = 1$ is $\exp\left(\langle \pm\rho_{\mathsf{w}}, \beta\rangle\right)$ [Eq. (9.1.9)]. We therefore assume an expansion of the form

$$R(\beta) \equiv c_0\, f_{\rho_{\mathsf{w}}}(\beta) + \sum_{\alpha>0} c_\alpha\, f_{\rho_{\mathsf{w}}-\alpha}(\beta), \qquad f_M(\beta) \equiv \exp\left[\langle M, \beta\rangle\right]. \tag{9.4.16}$$

Note that

$$(\sigma\, f_M)\,(\beta) = \sum_{w\in W} (\det w)\, \exp\left[\langle wM, \beta\rangle\right] \tag{9.4.17a}$$

$$(\sigma\, f_{w\,M})\,(\beta) = (\det w)\, (\sigma\, f_M)\,(\beta). \tag{9.4.17b}$$

Eq. (9.4.17b) implies that there is no point including different weights related by Weyl transformations in the expansion Eq. (9.4.16), since their contributions to $Q(\beta)$ are equivalent up to a sign.

We can therefore restrict the sum in Eq. (9.4.16) to weights $M = \rho_{\mathsf{w}} - \alpha$ with non-negative Dynkin coefficients [c.f. Eq. (9.3.14) and the surrounding discussion]. In fact, $(\sigma\, f_M)\,(\beta) = 0$ for any weight M with at least one vanishing Dynkin label.

Proof:

- Suppose the Dynkin coefficient $\Lambda_i^{(M)} = 0$. Then [Eq. (9.3.2)]

$$S_{\overline{\alpha}_i} M = M.$$

- The function $f_M(\beta)$ is invariant,

$$(S_{\overline{\alpha}_i}\, f_M)\,(\beta) = f_M\,(S_{\overline{\alpha}_i}\, \beta) = f_{S_{\overline{\alpha}_i} M}(\beta) = f_M(\beta) \quad \Rightarrow \quad \left(\sigma\, f_{S_{\overline{\alpha}_i} M}\right)(\beta) = (\sigma\, f_M)\,(\beta).$$

- But then Eq. (9.4.17b) implies that

$$(\sigma\, S_{\overline{\alpha}_i}\, f_M)\,(\beta) = \left(\sigma\, f_{S_{\overline{\alpha}_i} M}\right)(\beta) = -\,(\sigma\, f_M)\,(\beta).$$

  Therefore $(\sigma\, f_M)\,(\beta) = 0$.

Now, the Dynkin coefficients of the Weyl vector ρ_{w} are $(1, 1, \ldots, 1)$; therefore, any weight of the form $M = \rho_{\mathsf{w}} - \alpha$ is guaranteed to have a negative or zero Dynkin coefficient. We conclude that $R(\beta)$ in Eq. (9.4.16) reduces to

$$R(\beta) = c_0\, \exp\left[\langle \rho_{\mathsf{w}}, \beta\rangle\right]. \tag{9.4.18}$$

Then

$$Q(\beta) = c_0 \sum_{w\in W} (\det w)\, e^{\langle w\,\rho_{\mathsf{w}}, \beta\rangle} = c_0\, e^{\langle \rho_{\mathsf{w}}, \beta\rangle} + (\text{terms involving Weyl transformations of } \rho_{\mathsf{w}}). \tag{9.4.19}$$

If we compare this to Eq. (9.4.7b), we conclude that $c_0 = 1$; therefore

$$\boxed{Q(\beta) = (\sigma\, f_{\rho_{\mathsf{w}}})\,(\beta) = \sum_{w\in W} (\det w)\, e^{\langle w\,\rho_{\mathsf{w}}, \beta\rangle} = \sum_{w\in W} (\det w)\, e^{\langle \rho_{\mathsf{w}}, w\,\beta\rangle}.} \tag{9.4.20}$$

The last equality holds because $\det w^{-1} = \det w$ for an orthogonal Weyl group transformation w.

9.4.2 Freudenthal redux

Freudenthal's formula [Eq. (9.2.10)] can be rewritten as

$$[\langle\Lambda,\Lambda\rangle - \langle M,M\rangle + \langle 2\rho_{\mathsf{w}},\Lambda - M\rangle]\, n_M = 2\sum_{\alpha>0}\sum_{K\geq 1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle. \tag{9.4.21}$$

It is useful to recast this in slightly different form.
Claim:

$$\sum_{K=-\infty}^{\infty} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle = 0. \tag{9.4.22}$$

In this equation, it is understood that $n_{M+\alpha K} = 0$ for weights in the chain generated by $\hat{E}_{\pm\alpha}$ that do not belong to the highest weight representation Λ.
Proof: The MWDF Eq. (5.1.23) implies that

$$\begin{aligned}
\sum_{K=-\infty}^{\infty} n_{M+\alpha K}\,\langle M+K\alpha,\alpha^\vee\rangle &= \sum_{K=-\infty}^{\infty} n_{M+\alpha K}\,[(m-p)_{M;\alpha} + 2K] \\
&= n_{M+p\alpha}\,[(m-p)+2p] + n_{M-m\alpha}\,[(m-p)-2m] \\
&\quad + \\
&\quad n_{M+(p-1)\alpha}\,[(m-p)+2(p-1)] + n_{M-(m-1)\alpha}\,[(m-p)-2(m-1)] \\
&\quad + \dots \\
&= n_{M+p\alpha}\,[2(m-p)+2p-2m] \\
&\quad + \\
&\quad n_{M+(p-1)\alpha}\,[2(m-p)+2(p-1)-2(m-1)] + \dots \\
&= 0.
\end{aligned} \tag{9.4.23}$$

Here we have used invariance of the weight space dimensions under the S_α Weyl reflection. Thus Eq. (9.4.21) can be written as

$$\begin{aligned}
[\langle\Lambda,\Lambda\rangle - \langle M,M\rangle + \langle 2\rho_{\mathsf{w}},\Lambda - M\rangle]\, n_M &= \sum_{\alpha>0}\sum_{K\geq 1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle - \sum_{\alpha>0} n_M\,\langle M,\alpha\rangle - \sum_{\alpha>0}\sum_{K\leq -1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle \\
&= \sum_{\alpha>0}\sum_{K\geq 1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle + \sum_{\alpha<0}\sum_{K\geq 1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle - 2n_M\,\langle M,\rho_{\mathsf{w}}\rangle, \\
&= \sum_{\alpha\neq 0}\sum_{K\geq 1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle - 2n_M\,\langle M,\rho_{\mathsf{w}}\rangle.
\end{aligned} \tag{9.4.24}$$

We can add the $K=0$ term to the right-hand side, since

$$\sum_{\alpha\neq 0} n_M\,\langle M,\alpha\rangle = n_M\,\left[\langle M,2\rho_{\mathsf{w}}\rangle - \langle M,2\rho_{\mathsf{w}}\rangle\right] = 0.$$

Therefore

$$[\langle\Lambda+\rho_{\mathsf{w}},\Lambda+\rho_{\mathsf{w}}\rangle - \langle\rho_{\mathsf{w}},\rho_{\mathsf{w}}\rangle - \langle M,M\rangle]\, n_M = \sum_{\alpha\neq 0}\sum_{K\geq 0} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle. \tag{9.4.25}$$

9.4.3 Resumming Freudenthal

Multiplying both sides of Eq. (9.4.25) by $\exp\left[\langle M,\beta\rangle\right]$ and summing over M, we get

$$[\langle\Lambda+\rho_{\mathsf{w}},\Lambda+\rho_{\mathsf{w}}\rangle - \langle\rho_{\mathsf{w}},\rho_{\mathsf{w}}\rangle]\,\chi(\beta) - \sum_M \langle M,M\rangle n_M e^{\langle M,\beta\rangle} = \sum_M\sum_{\alpha\neq 0}\sum_{K\geq 0} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle\, e^{\langle M,\beta\rangle}. \tag{9.4.26}$$

If we express the weight β in terms of <u>the overcomplete sum over all roots</u> $\alpha\in\Delta$

$$\beta \equiv \sum_{\alpha\in\Delta} \beta_\alpha\,\alpha, \tag{9.4.27}$$

then

$$\frac{\partial^p}{\partial(\beta_\alpha)^p} e^{\langle M,\beta\rangle} = \langle M,\alpha\rangle^p e^{\langle M,\beta\rangle} \quad \Rightarrow \quad \sum_{\alpha\in\Delta} \frac{\partial^2}{\partial(\beta_\alpha)^2} e^{\langle M,\beta\rangle} = \sum_{\alpha\in\Delta} \langle M,\alpha\rangle\langle\alpha,M\rangle e^{\langle M,\beta\rangle}. \tag{9.4.28}$$

Now, recall that the scalar product between two weights is defined via the Killing form:

$$\langle M,N\rangle = (h_M, h_N).$$

The Killing form can be evaluated by summing over all elements of the Lie algebra [Eq. (4.2.2)],

$$(x,y) = \sum_{\overline{\alpha}_i} [x,[y,h_{\overline{\alpha}_i}]]|_{h_{\overline{\alpha}_i}} + \sum_{\alpha\neq 0} [x,[y,e_\alpha]]|_{e_\alpha}\,, \tag{9.4.29}$$

where $\mathcal{O}|_x$ means take the coefficient of the Lie algebra x. Thus

$$\langle M,N\rangle = (h_M,h_N) = \sum_{\alpha\neq 0} \alpha(h_M)\,\alpha(h_N) = \sum_{\alpha\neq 0} \langle M,\alpha\rangle\langle\alpha,N\rangle. \tag{9.4.30}$$

Therefore Eq. (9.4.28) implies that

$$\sum_{\alpha\in\Delta} \frac{\partial^2}{\partial(\beta_\alpha)^2} e^{\langle M,\beta\rangle} = \langle M,M\rangle\, e^{\langle M,\beta\rangle}, \tag{9.4.31}$$

and Eq. (9.4.26) becomes

$$\left[\langle\Lambda+\rho_{\mathsf{w}},\Lambda+\rho_{\mathsf{w}}\rangle - \langle\rho_{\mathsf{w}},\rho_{\mathsf{w}}\rangle - \sum_{\alpha\in\Delta}\frac{\partial^2}{\partial(\beta_\alpha)^2}\right]\chi(\beta) = \sum_{\alpha\neq 0}\sum_M I_{M,\alpha}(\beta), \tag{9.4.32a}$$

$$I_{M,\alpha}(\beta) \equiv \sum_{K\geq 0} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle\, e^{\langle M,\beta\rangle}. \tag{9.4.32b}$$

The left-hand side of Eq. (9.4.32a) is a differential equation for the character function $\chi(\beta)$, viewed as a function of the "coordinates" $\{\beta_\alpha\}$ in Eq. (9.4.27). We want to express the right-hand side of Eq. (9.2.10) in terms of the generating functions $\chi(\beta)$ and $Q(\beta)$, Eq. (9.4.7).

Consider $I_{M,\alpha}(\beta)$, Eq. (9.4.32b). For a fixed weight M in the weight string generated by $\hat{E}_{\pm\alpha}$, we must determine the contribution of each spin-j representation of this su(2) [Eq. (9.2.2)] to $I_{M,\alpha}(\beta)$. The spin-j representation has an odd ($j\in\mathbb{Z}$) or even [$j\in(2\mathbb{Z}+1)/2$] number of weights. Let M_0 locate the center of this representation. For integer j ($2j+1$ odd), M_0 is the center weight; for half-integer j ($2j+1$ even), M_0 is not itself a weight. Then a generic weight M lying somewhere in this representation can be written as

$$M = M_0 + q\alpha, \qquad q\in\{-j,-j+1,\ldots,j-1,j\}. \tag{9.4.33}$$

Suppose one copy of the spin-j representation with some (undetermined) M_0 contributes to $I_{M,\alpha}(\beta)$ in Eq. (9.4.32b). The contribution is

$$I^{(j)}_{M,\alpha}(\beta) \equiv \sum_{K\geq 0}^{j-q} (1)\,\langle M_0+(K+q)\alpha,\alpha\rangle\, e^{\langle M_0+q\alpha,\beta\rangle} = \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle} \sum_{K\geq 0}^{j-q} (K+q)\, e^{q\langle\alpha,\beta\rangle}, \tag{9.4.34}$$

where we've used the fact that $\langle M_0,\alpha\rangle = 0$; the latter is obvious for the case where M_0 is itself the center weight [Eq. (5.1.23)]. We can sum over the nondegenerate weights [Eq. (9.4.33)] in this spin-j representation to get

$$I^{(j)}_\alpha(\beta) \equiv \sum_M I^{(j)}_{M,\alpha}(\beta) = \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle} \sum_{q=-j}^{j}\sum_{K\geq 0}^{j-q} (K+q)\, e^{q\langle\alpha,\beta\rangle}. \tag{9.4.35}$$

Consider

$$\left[e^{\langle\alpha,\beta\rangle}-1\right] I_\alpha^{(j)}(\beta) = \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle}\left[\sum_{q=-j}^{j}\sum_{K\geq 0}^{j-q}(K+q)\, e^{(q+1)\langle\alpha,\beta\rangle} - \sum_{q=-j-1}^{j-1}\sum_{K\geq 0}^{j-q-1}(K+q+1)\, e^{(q+1)\langle\alpha,\beta\rangle}\right]$$

$$= \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle}\left[\begin{array}{l} j\, e^{(j+1)\langle\alpha,\beta\rangle} + \displaystyle\sum_{q=-j}^{j-1}\sum_{K\geq 0}^{j-q-1}(K+q)\, e^{(q+1)\langle\alpha,\beta\rangle} + \sum_{q=-j}^{j-1} j\, e^{(q+1)\langle\alpha,\beta\rangle} \\ - \displaystyle\sum_{q=-j}^{j-1}\sum_{K\geq 0}^{j-q-1}(K+q+1)\, e^{(q+1)\langle\alpha,\beta\rangle} - \sum_{K\geq 0}^{2j}(K-j)\, e^{-j\langle\alpha,\beta\rangle} \end{array}\right] \tag{9.4.36}$$

$$= \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle}\left[j\, e^{(j+1)\langle\alpha,\beta\rangle} + \sum_{q=-j}^{j-1} j\, e^{(q+1)\langle\alpha,\beta\rangle} - \sum_{q=-j}^{j-1}(j-q)\, e^{(q+1)\langle\alpha,\beta\rangle}\right]$$

$$= \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle}\sum_{q=-j}^{j} q\, e^{(q+1)\langle\alpha,\beta\rangle}.$$

Thus

$$I_\alpha^{(j)}(\beta) = \frac{1}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}\sum_{q=-j}^{j}\langle q\alpha,\alpha\rangle\, e^{\langle M_0+q\alpha+\alpha,\beta\rangle} = \frac{1}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}\sum_{q=-j}^{j}\langle M,\alpha\rangle\, e^{\langle M+\alpha,\beta\rangle}, \tag{9.4.37}$$

where we have used Eq. (9.4.33) and the fact that $\langle M_0,\alpha\rangle = 0$. Eq. (9.4.37) is the contribution to the right-hand side of Eq. (9.4.32a) of *all weights* in the spin-j su(2) generated by $\hat{E}_{\pm\alpha}$, with central weight M_0. **The key result is that Eq. (9.4.37) does not depend explicitly on M_0, but only implicitly through the (summed) weight M [Eq. (9.4.33)].**

We can therefore rewrite the right-hand side of Eq. (9.4.32a) as

$$\text{RHS} \equiv \sum_{\alpha\neq 0}\sum_{M} I_{M,\alpha}(\beta) = \sum_{\alpha\neq 0}\sum_{M_0}\sum_{j} n_{M_0,j}^{(\alpha)}\, I_\alpha^{(j)}(\beta). \tag{9.4.38}$$

For each root α, we must sum over all central weights M_0 and all representations j; there is an (undetermined) degeneracy factor $n_{M_0,j}^{(\alpha)}$ that can be zero, one, or more if there are multiple spin-j su(2) representations with the same central weight that contribute. At this point it looks like we have just introduced many more unknowns, but now a minor miracle ensues. Combining Eqs. (9.4.37) and (9.4.38), we get

$$\text{RHS} = \sum_{\alpha\neq 0}\frac{1}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}\sum_{M_0}\sum_{j}\sum_{q=-j}^{j} n_{M_0,j}^{(\alpha)}\,\langle M,\alpha\rangle\, e^{\langle M+\alpha,\beta\rangle} = \sum_{\alpha\neq 0}\frac{1}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}\sum_{M} n_M\,\langle M,\alpha\rangle\, e^{\langle M+\alpha,\beta\rangle}, \tag{9.4.39}$$

since the sum over representation centers M_0, representations j, and weights $q\in\{-j,\ldots,j\}$ weighted by the degeneracy $n_{M_0,j}^{(\alpha)}$ is just a particular decomposition of the sum over all weights M (again weighted by the appropriate degeneracy).

As an example, consider the (2,1) representation of su(3) shown in Fig. 9.2. For simple root $\overline{\alpha}_1$, the center weights $\{M_0\}$ enumerate the four parallel horizontal weight strings. Each horizontal string consists of one or two different spin-j representations. Summing over all M_0, all j, and states within each j is equivalent to summing over all weights times the appropriate degeneracies. Note that in this case, $n_{M_0,j}^{(\overline{\alpha}_1)} = 1$ for all M_0 and j.

9.4.4 *finis*

Replacing the right-hand side of Eq. (9.4.32a) with Eq. (9.4.39), we have

$$\left[\langle\Lambda+\rho_{\mathsf{w}},\Lambda+\rho_{\mathsf{w}}\rangle - \langle\rho_{\mathsf{w}},\rho_{\mathsf{w}}\rangle - \sum_{\alpha\in\Delta}\frac{\partial^2}{\partial(\beta_\alpha)^2}\right]\chi(\beta) = \sum_{\alpha\neq 0}\sum_{M}\frac{n_M\,\langle M,\alpha\rangle\, e^{\langle M+\alpha,\beta\rangle}}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}. \tag{9.4.40}$$

Eq. (9.4.7b) implies that

$$Q^2(\beta) = \prod_{\alpha>0} \left[e^{\langle\alpha,\beta\rangle} - 1\right]\left[1 - e^{-\langle\alpha,\beta\rangle}\right] = \prod_{\alpha\neq 0} \left[e^{\langle\alpha,\beta\rangle} - 1\right](-1)^{|\Delta_+|}, \tag{9.4.41}$$

where $|\Delta_+|$ denotes the number of positive roots. Then

$$\frac{\partial}{\partial\beta_\alpha}\log[Q^2(\beta)] = \sum_{\alpha'\neq 0} \frac{\langle\alpha',\alpha\rangle\, e^{\langle\alpha',\beta\rangle}}{\left[e^{\langle\alpha',\beta\rangle} - 1\right]}, \tag{9.4.42}$$

where β_α is an expansion coefficient for weight β in the overcomplete set of all roots, Eq. (9.4.27). Combining the above with Eq. (9.4.28), we get

$$\begin{aligned}
\sum_{\alpha\neq 0}\left\{\frac{\partial}{\partial\beta_\alpha}\log[Q^2(\beta)]\right\}\left\{\frac{\partial}{\partial\beta_\alpha}\chi(\beta)\right\} &= \sum_M n_M \sum_{\alpha'\neq 0}\sum_{\alpha\neq 0} \frac{\langle\alpha',\alpha\rangle\langle\alpha,M\rangle e^{\langle M+\alpha',\beta\rangle}}{\left[e^{\langle\alpha',\beta\rangle} - 1\right]} \\
&= \sum_M n_M \sum_{\alpha'\neq 0} \frac{\langle\alpha',M\rangle e^{\langle M+\alpha',\beta\rangle}}{\left[e^{\langle\alpha',\beta\rangle} - 1\right]} \\
&= \frac{1}{Q}\sum_{\alpha\neq 0}\left[\frac{\partial^2}{\partial(\beta_\alpha)^2}\left(Q\chi\right)(\beta) - \chi\frac{\partial^2}{\partial(\beta_\alpha)^2}Q(\beta) - Q\frac{\partial^2}{\partial(\beta_\alpha)^2}\chi(\beta)\right].
\end{aligned} \tag{9.4.43}$$

On the second line, we have used Eq. (9.4.30). Eq. (9.4.40) becomes

$$\left[\langle\Lambda+\rho_{\mathsf{w}},\Lambda+\rho_{\mathsf{w}}\rangle - \langle\rho_{\mathsf{w}},\rho_{\mathsf{w}}\rangle\right]\left(Q\chi\right)(\beta) = \sum_{\alpha\neq 0}\left[\frac{\partial^2}{\partial(\beta_\alpha)^2}\left(Q\chi\right)(\beta) - \chi\frac{\partial^2}{\partial(\beta_\alpha)^2}Q(\beta)\right]. \tag{9.4.44}$$

Eq. (9.4.20) implies that

$$\begin{aligned}
\sum_{\alpha\neq 0}\frac{\partial^2}{\partial(\beta_\alpha)^2}Q(\beta) &= \sum_{\alpha\neq 0}\sum_{w\in W}(\det w)\,\langle w\,\rho_{\mathsf{w}},\alpha\rangle\langle\alpha,w\,\rho_{\mathsf{w}}\rangle\, e^{\langle w\,\rho_{\mathsf{w}},\beta\rangle} \\
&= \sum_{w\in W}(\det w)\,\langle w\,\rho_{\mathsf{w}},w\,\rho_{\mathsf{w}}\rangle\, e^{\langle w\,\rho_{\mathsf{w}},\beta\rangle} \\
&= \langle\rho_{\mathsf{w}},\rho_{\mathsf{w}}\rangle\, Q(\beta).
\end{aligned} \tag{9.4.45}$$

Therefore Eq. (9.4.44) becomes

$$\boxed{\sum_{\alpha\neq 0}\frac{\partial^2}{\partial(\beta_\alpha)^2}\left(Q\chi\right)(\beta) = \langle\Lambda+\rho_{\mathsf{w}},\Lambda+\rho_{\mathsf{w}}\rangle\,\left(Q\chi\right)(\beta).} \tag{9.4.46}$$

Note that $(Q\chi)(\beta)$ is an alternating function [Eqs. (9.4.8) and (9.4.11)]. Eqs. (9.4.46), (9.4.45), and (9.4.20) suggest the following simple ansatz (guess):

$$\left(Q\chi\right)(\beta) = \sum_{w\in W}(\det w)\, e^{\langle\Lambda+\rho_{\mathsf{w}},w\,\beta\rangle}. \tag{9.4.47}$$

This obviously satisfies Eq. (9.4.46), but it is not the unique solution. The precise form of Eq. (9.4.47) can be proven following a similar line of argument used to obtain Eq. (9.4.20) from Eqs. (9.4.15) and (9.4.16).

Combining Eqs. (9.4.47) and (9.4.20), we finally obtain

$$\boxed{\chi(\beta) = \sum_M n_M e^{\langle M,\beta\rangle} = \frac{\displaystyle\sum_{w\in W}(\det w)\, e^{\langle\Lambda+\rho_{\mathsf{w}},w\,\beta\rangle}}{\displaystyle\sum_{w\in W}(\det w)\, e^{\langle\rho_{\mathsf{w}},w\,\beta\rangle}}, \qquad \text{Weyl's character formula.}} \tag{9.4.48}$$

Eq. (9.4.48) is essentially a resummation of Freudenthal's recursion formula Eq. (9.2.10), assisted by the Weyl group.

9.5 Dimension and strange formulae

The Weyl character χ is a functional defined for any irreducible representation; in Eq. (9.4.48) it is evaluated for an arbitrary element $\beta \in H_0^*$. If we set

$$\beta = t\,\rho_{\mathsf{w}} \tag{9.5.1}$$

with $t \in \mathbb{R}$, then using Eq. (9.4.7b)

$$\begin{aligned}\chi\left(t\,\rho_{\mathsf{w}}\right) &= \frac{\sum\limits_{w\in W}\left(\det w\right) e^{\langle w\,\rho_{\mathsf{w}},\, t(\Lambda+\rho_{\mathsf{w}})\rangle}}{\sum\limits_{w\in W}\left(\det w\right) e^{\langle w\,\rho_{\mathsf{w}},\, t\,\rho_{\mathsf{w}}\rangle}} = \frac{Q\left[t\left(\Lambda+\rho_{\mathsf{w}}\right)\right]}{Q\left(t\,\rho_{\mathsf{w}}\right)} \\ &= \prod_{\alpha>0}\frac{\sinh\left[\frac{1}{2}\langle\alpha,\, t\left(\Lambda+\rho_{\mathsf{w}}\right)\rangle\right]}{\sinh\left[\frac{1}{2}\langle\alpha,\, t\,\rho_{\mathsf{w}}\rangle\right]}.\end{aligned} \tag{9.5.2}$$

Therefore

$$\lim_{t\to 0}\chi\left(t\,\rho_{\mathsf{w}}\right) = \sum_M n_M = N_\Lambda \simeq \lim_{t\to 0}\frac{\prod\limits_{\alpha>0}\langle\alpha,\, t\left(\Lambda+\rho_{\mathsf{w}}\right)\rangle}{\prod\limits_{\alpha>0}\langle\alpha,\, t\,\rho_{\mathsf{w}}\rangle}, \tag{9.5.3}$$

where N_Λ is the dimension of the representation. We thus have the **dimension formula**

$$\boxed{N_\Lambda = \prod_{\alpha>0}\frac{\langle\alpha,\Lambda+\rho_{\mathsf{w}}\rangle}{\langle\alpha,\rho_{\mathsf{w}}\rangle}, \qquad \text{Dimension } N_\Lambda \text{ of highest weight representation } \Lambda.} \tag{9.5.4}$$

We can obtain another formula from Eq. (9.5.2). Expanding again for small t,

$$\begin{aligned}\chi\left(t\,\rho_{\mathsf{w}}\right) &\simeq \prod_{\alpha>0}\frac{\langle\alpha,\left(\Lambda+\rho_{\mathsf{w}}\right)\rangle\left[1+\frac{t^2}{24}\langle\alpha,\left(\Lambda+\rho_{\mathsf{w}}\right)\rangle^2+\ldots\right]}{\langle\alpha,\rho_{\mathsf{w}}\rangle\left[1+\frac{t^2}{24}\langle\alpha,\rho_{\mathsf{w}}\rangle^2+\ldots\right]} \\ &= N_\Lambda\left\{1+\frac{t^2}{48}\sum_{\alpha\neq 0}\left[\langle\left(\Lambda+\rho_{\mathsf{w}}\right),\alpha\rangle\langle\alpha,\left(\Lambda+\rho_{\mathsf{w}}\right)\rangle-\langle\rho_{\mathsf{w}},\alpha\rangle\langle\alpha,\rho_{\mathsf{w}}\rangle\right]+\ldots\right\} \\ &= N_\Lambda\left\{1+\frac{t^2}{48}\left[\langle\left(\Lambda+\rho_{\mathsf{w}}\right),\left(\Lambda+\rho_{\mathsf{w}}\right)\rangle-\langle\rho_{\mathsf{w}},\rho_{\mathsf{w}}\rangle\right]+\ldots\right\},\end{aligned} \tag{9.5.5}$$

where we have used Eq. (9.4.30). On the other hand,

$$\chi(t\,\rho_{\mathsf{w}}) = \sum_M n_M e^{\langle M,\, t\,\rho_{\mathsf{w}}\rangle} = \sum_M n_M\left[1+t\langle M,\,\rho_{\mathsf{w}}\rangle+\frac{t^2}{2!}\langle M,\,\rho_{\mathsf{w}}\rangle^2+\ldots\right] \tag{9.5.6}$$

The terms linear in t sum to zero due to Weyl invariance. If we take $\Lambda = \theta$ (the highest root, i.e. we are studying the adjoint representation), then the sum over M is replaced by a sum over non-zero roots (since the inner product with the zero weight specifying the Cartan subalgebra gives zero). Therefore

$$\chi(t\,\rho_{\mathsf{w}}) = N_\theta + \frac{t^2}{2!}\sum_{\alpha\neq 0}\langle\alpha,\,\rho_{\mathsf{w}}\rangle^2+\ldots = N_\theta+\frac{t^2}{2}\langle\rho_{\mathsf{w}},\rho_{\mathsf{w}}\rangle+\ldots, \tag{9.5.7}$$

where we have used Eq. (9.4.30) and the non-degeneracy of the roots $n_\alpha = 1$ (Proposition (VI.), Sec. 5.2.2]. Eqs. (9.5.5) and (9.5.7) imply that

$$\langle\rho_{\mathsf{w}},\rho_{\mathsf{w}}\rangle = \frac{N_\theta}{24}\langle\theta,\theta+2\rho_{\mathsf{w}}\rangle. \tag{9.5.8}$$

Here $N_\theta = d$ is the dimension of the adjoint representation (or the Lie group). In the standard normalization we obtain

$$\boxed{\langle \rho_{\mathsf{W}}, \rho_{\mathsf{W}} \rangle_2 = \frac{N_\theta\, g}{12}, \qquad \text{Freudenthal-de Vries strange formula.}} \tag{9.5.9}$$

Here g denotes the dual Coxeter number [Eq. (9.1.21)].

9.5.1 Dimensions for generic representations of the classical algebras

There are different methods to evaluate Eq. (9.5.4) for a generic highest weight representation Λ in a particular Lie algebra. We will use a strategy that works for all algebras and can be easily evaluated numerically.

The simplest case (unsurprisingly) is $A_{n-1} = \mathrm{su}(n)$. In Sec. 7.1, we showed that the positive roots can be enumerated $\{\alpha_{ij}\}$ with $1 \le i < j \le n$. These are expressed in terms of the simple roots via Eq. (7.1.11),

$$\alpha_{ij} = \sum_{k=i}^{j-1} \overline{\alpha}_k = \sum_{k=i}^{j-1} \overline{\alpha}_k^\vee. \tag{9.5.10}$$

Then

$$\langle \Lambda + \rho_{\mathsf{W}}, \alpha_{ij} \rangle = \sum_{k=i}^{j-1} (\Lambda_k + 1) = (j-i) + \sum_{k=i}^{j-1} \Lambda_k \equiv 2\,\mathcal{F}(\Lambda; i, j). \tag{9.5.11a}$$

Therefore

$$\boxed{N_\Lambda = \prod_{i=1}^{n-1} \prod_{j>i}^{n} \left[\frac{\mathcal{F}(\Lambda; i, j)}{\mathcal{F}(0; i, j)} \right], \qquad \text{Dimension formula for } \mathrm{su}(n) = A_{n-1}.} \tag{9.5.12}$$

- **Exercise**: Use Eq. (9.5.12) to show that the $\Lambda = (p, q)$ representation of su(3) has $N_{p,q}$ given by Eq. (9.1.43).
- **Exercise**: Use Eq. (9.5.12) to show that the $\Lambda = \omega_p$ representation of su(n) has dimension

$$N_p = \frac{n!}{p!(n-p)!}, \tag{9.5.13}$$

consistent with a rank-p antisymmetric tensor [c.f. Eq. (8.2.7)].

The dimension formula for sp($2n$) can be similarly obtained using Eqs. (7.1.11) and (7.2.12). The result is

$$\boxed{N_\Lambda = \prod_{i=1}^{n} \left[\frac{\mathcal{F}(\Lambda; i, n+1)}{\mathcal{F}(0; i, n+1)} \right] \prod_{j>i}^{n} \left\{ \frac{\mathcal{F}(\Lambda; i, j)\,[\mathcal{F}(\Lambda; i, j) + 2\mathcal{F}(\Lambda; j, n+1)]}{\mathcal{F}(0; i, j)\,[\mathcal{F}(0; i, j) + 2\mathcal{F}(0; j, n+1)]} \right\}, \qquad \text{Dimension formula for } \mathrm{sp}(2n) = C_n.} \tag{9.5.14}$$

- **Exercise**: Derive Eq. (9.5.14).
- **Exercise**: Use Eq. (9.5.14) to verify that the fundamental representation $\Lambda = \omega_p$ $(1 \le p \le n)$ has dimension

$$N_{\omega_p} = 2(n+1-p)\frac{(2n+1)!}{p!(2n-p+2)!}, \tag{9.5.15}$$

consistent with a "traceless" fully antisymmetric rank-p tensor [Eq. (8.3.11)].

We can also obtain generic dimension formulae for so($2n$) $= D_n$ and so($2n+1$) $= B_n$, but we won't transcribe the results here. The most important are the dimensions of the fundamental spinor representations, which are $N_{\omega_n} = 2^n$ for the ω_n representation of B_n, and $N_{\omega_{n-1}} = N_{\omega_n} = 2^{n-1}$ for the independent ω_{n-1} and ω_n representations of D_n.

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

[2] Phillipe Di Francesco, Pierre Mathieu, David Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1996).

[3] Jürgen Fuchs, *Affine Lie Algebras and Quantum Groups* (Cambridge University Press, Cambridge, England, 1995).

Lie Algebra Rep. Theory III: Spinor Irreps of SO(N) and Clifford Algebras

BRIEF REVIEW: <u>ELEMENTS OF LIE ALGEBRA THEORY</u>

Chevalley Basis:

$$e_i \equiv e_{\bar\alpha_i} \qquad [h_i, e_j] = A_{ji}\, e_j$$
$$f_i \equiv e_{-\bar\alpha_i} \qquad [h_i, f_j] = -A_{ji}\, f_j$$
$$h_i \equiv h_{\bar\alpha_i^\vee} \qquad [e_i, f_j] = \delta_{ij}\, \frac{|\bar\alpha_i|^2}{2} (e_{\bar\alpha_i}, e_{-\bar\alpha_i})\, h_i$$

IRREP: RANK n L.A. $\Lambda_{HWS} = \sum_{j=1}^{n} K_j^{(\Lambda)} \bar\alpha_j = \sum_{j=1}^{n} \Lambda_j \omega_j$

WEIGHT VECTOR ϕ_Λ: $\hat H_i \phi_\Lambda = \langle \Lambda, \bar\alpha_i^\vee \rangle \phi_\Lambda = \Lambda_i \phi_\Lambda$

↑ DYNKIN COEFF. ARE CSA E'VALS IN THE CHEVALLEY BASIS

CASIMIR

$$\hat C = \sum_{i,j=1}^{n} \langle \bar\alpha_i, \bar\alpha_j \rangle_2^{-1} \hat H^{(2)}_{\bar\alpha_i} \hat H^{(2)}_{\bar\alpha_j} + \sum_{\alpha \neq 0} (e^{(2)}_\alpha, e^{(2)}_{-\alpha})_2\, \hat E^{(2)}_\alpha \hat E^{(2)}_{-\alpha} = \sum_a \hat T^a \hat T^a$$

$$\mathrm{Tr}_\Lambda[\hat C] = N_\Lambda \langle \Lambda, \Lambda + 2\rho \rangle_2 = N_{Adj}\, \ell_\Lambda \quad \Rightarrow \quad \ell_\Lambda = \frac{N_\Lambda}{N_{Adj}} \langle \Lambda, \Lambda + 2\rho \rangle_2$$

↑ INDEX (TR. NORM) : $\mathrm{Tr}_\Lambda[\hat T^a \hat T^b] = \ell_\Lambda \delta^{ab}$

<u>ADJOINT:</u>

$$\ell_\Theta = \langle \Theta, \Theta + 2\rho \rangle_2 = 2g$$

GEO 4, p. 71-75 **EX:** SO(2n)

FUNDAMENTAL IRREP:

$$h_{\bar\alpha_i^\vee} \Rightarrow \hat E^1_{ii} - \hat E^1_{i+1,i+1}, \quad 1 \le i \le n-1$$

$$h_{\bar\alpha_n^\vee} \Rightarrow \hat E^1_{n-1,n-1} + \hat E^1_{n,n}$$

$$\hat E^1_{jk} \equiv \hat E_{jk} - \hat E_{k+n,j+n} \rightarrow$$

$$\hat E^2_{jk} \equiv \hat E_{j,n+k} - \hat E_{k,n+j} \rightarrow$$

ex: SO(8)

$$\hat H_{\bar\alpha_1^\vee} = \mathrm{diag}(1\ -1\ 0\ 0\ -1\ 1\ 0\ 0)$$
$$\hat H_{\bar\alpha_2^\vee} = \mathrm{diag}(0\ 1\ -1\ 0\ 0\ -1\ 1\ 0)$$
$$\hat H_{\bar\alpha_3^\vee} = \mathrm{diag}(0\ 0\ 1\ -1\ 0\ 0\ -1\ 1)$$
$$\hat H_{\bar\alpha_4^\vee} = \mathrm{diag}(0\ 0\ 1\ 1\ 0\ 0\ -1\ -1)$$

$\hat E_{-\bar\alpha_1} =$

$\hat E_{-\bar\alpha_2} =$

$\hat E_{-\bar\alpha_3} =$

$\hat E_{-\bar\alpha_4} =$

SIMPLE ROOT LOWERING OPERATORS

$\Rightarrow$ GEN CSA ELEM : $\hat{H} = \left[\begin{array}{cccc|cccc} \Lambda^{(1)} & & & & & & & \\ & \Lambda^{(2)} & & & & & & \\ & & \Lambda^{(3)} & & & & & \\ & & & \Lambda^{(4)} & & & & \\ \hline & & & & -\Lambda^{(1)} & & & \\ & & & & & -\Lambda^{(2)} & & \\ & & & & & & -\Lambda^{(3)} & \\ & & & & & & & -\Lambda^{(4)} \end{array}\right]$

GEO 4, p. 90E

<u>SO(8)</u>

$1\ 0\ 0\ 0 \quad \phi_1$

$\downarrow \bar{\alpha}_1$

$-1\ 1\ 0\ 0 \quad \phi_2$

$\downarrow \bar{\alpha}_2$

$0\ -1\ 1\ 1 \quad \phi_3$

$\bar{\alpha}_3 \swarrow \quad \searrow \bar{\alpha}_4$

$\phi_4 \quad 0\ 0\ -1\ 1 \qquad 0\ 0\ 1\ -1 \quad \phi_5 = \bar{\phi}_8$

$\searrow \bar{\alpha}_4 \quad \swarrow \bar{\alpha}_3$

$0\ 1\ -1\ -1 \quad \phi_6 = \bar{\phi}_7$

$\downarrow \bar{\alpha}_2$

$1\ -1\ 0\ 0 \quad \phi_7 = \bar{\phi}_6$

$\downarrow \bar{\alpha}_1$

$-1\ 0\ 0\ 0 \quad \phi_8 = \bar{\phi}_5$

MORE CONV. ORDERING

$\phi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; $\Lambda^{(1)} \equiv \{1,0,0,0\}_D$ ← DYNKIN

$\Leftarrow$ A SIMULTANEOUS EIGENVECTOR OF $\{\hat{H}_{\bar{\alpha}_1}, \hat{H}_{\bar{\alpha}_2}, \hat{H}_{\bar{\alpha}_3}, \hat{H}_{\bar{\alpha}_4}\}$

$\phi_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; $\Lambda^{(2)} = \{-1,1,0,0\}_D$

$\phi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; $\Lambda^{(3)} = \{0,-1,1,1\}_D$

$\phi_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; $\Lambda^{(4)} = \{0,0,-1,1\}_D$

$\phi_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; $\Lambda^{(5)} = \{0,0,1,-1\}_D = -\Lambda^{(4)}$

$\phi_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$; $\Lambda^{(6)} = \{0,1,-1,-1\} = -\Lambda^{(3)}$, $\phi_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$; $\Lambda^{(7)} = \{1,-1,0,0\} = -\Lambda^{(2)}$, $\phi_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; $\Lambda^{(8)} = \{-1,0,0,0\} = -\Lambda^{(1)}$

THERE ARE <u>THREE</u> NATURAL BASES FOR THE WEIGHT SPACE H_0^*

① SIMPLE ROOTS $\Lambda = \sum_{j=1}^{n} K_j^{(\Lambda)} \bar{\alpha}_j$; $\langle \bar{\alpha}_i, \bar{\alpha}_j^\vee \rangle = A_{ij}$

TYPICALLY NON-INTEGER (UNLESS IN THE SAME CONJ. CLASS AS THE ADJOINT)

② FUND. WEIGHTS $\Lambda = \sum_{j=1}^{n} \Lambda_j \omega_j$; $\langle \omega_i, \omega_j \rangle = F_{ij}$

DYNKIN

NEW * ③ (LIN. INDEP. OF SUBSET) WEIGHTS OF THE FUND. IRREP. $\Lambda = \sum_{j=1}^{n} \Gamma_j \Lambda^{(j)}$; $\langle \Lambda^{(j)}, \Lambda^{(K)} \rangle = M^{jK}$

WHICH WEIGHT $\Rightarrow$ THIS IS <u>NOT</u> A DYNKIN LABEL.

LET'S COMPUTE $\langle \Lambda^{(j)}, \Lambda^{(K)} \rangle$ FOR $SO(8)$:

SO(2n)

$\Lambda^{(1)} = \{1\,0\,0\,0\}_D$
$\Lambda^{(2)} = \{-1\,1\,0\,0\}_D$
$\Lambda^{(3)} = \{0\,-1\,1\,0\}_D$
$\Lambda^{(4)} = \{0\,-1\,1\,1\}_D$

$$\langle \Lambda^{(j)}, \Lambda^{(K)} \rangle = \sum_{m,n=1}^{4} \Lambda_m^{(j)} \Lambda_n^{(K)} F_{mn}$$

CFT BOOK, CH 13:

$$F = \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

(RECALL: QUAD FORM MTX ENTIRELY DET. BY DYNKIN DIAGRAM $(\hat{F}^{-1})_{ij} = A_{ij} \frac{2}{|\bar{\alpha}_i|^2}$)
GEO 4A, p. 1

MATHEMATICA: $\langle \Lambda^{(j)}, \Lambda^{(K)} \rangle = \delta^{ij}$ ORTHONORMAL!

CLAIM: GENERALIZES TO $SO(2n) \Rightarrow \{\Lambda^{(j)}\}$ ARE ORTHONORMAL BASIS IN ROOT/WEIGHT SPACE H_0^*

GEO 4, p. 72:
WE CAN EXPRESS THE ROOTS IN TERMS OF $\{\Lambda^{(j)}\}$
EVALS OF GEN CSA ELEM IN FUND. REP.! $\hat{H} = \begin{bmatrix} \Lambda^{(1)} & & \\ & \Lambda^{(n)} & \\ & & \Lambda^{(0)} \end{bmatrix}$

I. POSITIVE ROOTS

① $\alpha^1_{jK} = \Lambda^{(j)} - \Lambda^{(K)} \iff C^1_{jK}, \quad j<K$ — $\frac{n(n-1)}{2}$ OF THESE

② $\alpha^2_{jK} = \Lambda^{(j)} + \Lambda^{(K)} \iff C^2_{[jK]}, \quad j<K$ — $\frac{n(n-1)}{2}$ OF THESE

$\Rightarrow 2[n(n-1)] + n = 2n^2 - n = \frac{1}{2}2n(2n-1)$ ✓

II. SIMPLE ROOTS

① $\bar{\alpha}_1 = \Lambda^{(1)} - \Lambda^{(2)}$
$\bar{\alpha}_2 = \Lambda^{(2)} - \Lambda^{(3)}$
$\vdots$
$\bar{\alpha}_{n-1} = \Lambda^{(n-1)} - \Lambda^{(n)}$

$\Rightarrow$ CLEARLY, $|\bar{\alpha}_i|^2 = 2 \;\forall i \in \{1,\dots,n\}$ ✓ SO(2n) IS SIMPLY-LACED.

$\bar{\alpha}_i = \bar{\alpha}_i^\vee \;\forall i$

② $\bar{\alpha}_n = \Lambda^{(n-1)} + \Lambda^{(n)} = \alpha^2_{n-1,n}$

III. FUND. WEIGHTS

① $\omega_i = \sum_{j=1}^{i} \Lambda^{(j)}, \quad 1 \le i \le n-2$ — C.F. GEO 4A, p. 16

② $\omega_{n-1} = \frac{1}{2}\sum_{j=1}^{n-1} \Lambda^{(j)} - \frac{1}{2}\Lambda^{(n)}$; ③ $\omega_n = \frac{1}{2}\sum_{j=1}^{n} \Lambda^{(j)}$

SO(2n)

APPLICATION: DERIVATION OF QUAD. FORM MATRIX

① $1 \le i,j \le n-2$

$$F_{ij} = \sum_{p=1}^{i} \sum_{q=1}^{j} \delta^{p,q} = \min(i,j) \checkmark$$

② $1 \le i \le n-2$

$$F_{i,n-1} = \frac{1}{2} \sum_{p=1}^{i} \sum_{q=1}^{n-1} \delta^{p,q} = i/2 \checkmark$$

CFT BOOK CH 13 ✓

③ $1 \le i \le n-2$

$$F_{i,n} = i/2 \checkmark$$

④ $$F_{n-1,n} = \frac{1}{4} \sum_{p=1}^{n-1} \sum_{q=1}^{n-1} \delta^{p,q} - \frac{1}{4} = \frac{n-2}{4} \checkmark$$

⑤ $$F_{n,n} = \frac{n}{4} = F_{n-1,n-1} \checkmark$$

REVIEW: CONJUGACY CLASSES OF SO(2n)

A $n = \text{odd} \Rightarrow n-1$ IS EVEN

CLASS	WEIGHTS
0	• $\omega_1, \omega_3, \dots, \omega_{n-3}$
2	• (Adj) $\omega_2, \omega_4, \dots, \omega_{n-2}$
$(n-2) \bmod 4$	• ω_{n-1}
$[(n-2)+2] \bmod 4$	• ω_n

B $n = \text{EVEN} \Rightarrow n-1$ IS ODD

0	• ω_1, (Adj) $\omega_2, \omega_3, \dots, \omega_{n-2}$
1	• ω_{n-1}, ω_n

GEORGI

21 SO(2n+1) AND SPINORS

FUND. IRREP.: GEO 4, p. 90

SO(2n+1)

CAHN'S TREATMENT OF B_n (GEO 4) – p. 75–81

GEN CSA ELEMENT $\hat{H} = \left[\begin{array}{c|c} 0 & 0\,0\cdots 0 \\ \hline \begin{matrix}0\\0\\\vdots\\0\end{matrix} & \begin{matrix}\Lambda^{(1)} & & & \\ & \Lambda^{(2)} & & \\ & & \ddots & \\ & & & \Lambda^{(n)}\end{matrix} \\ \hline \begin{matrix}0\\0\\\vdots\\0\end{matrix} & \begin{matrix}-\Lambda^{(1)} & & & \\ & -\Lambda^{(2)} & & \\ & & \ddots & \\ & & & -\Lambda^{(n)}\end{matrix}\end{array}\right]$

a) ϕ_1, ϕ_{n+1}: $1\,0\,0\cdots 0$ $\downarrow \bar{\alpha}_1$ $\quad -1\,0\,0\,0\cdots 0$ $\uparrow \bar{\alpha}_1$

b) ϕ_2, ϕ_{n+2}: $-1\,1\,0\cdots 0$ $\downarrow \bar{\alpha}_2$ $\quad 1\,-1\,0\,0\cdots 0$

c) ϕ_3, ϕ_{n+3}: $0\,-1\,1\,0\cdots 0$

$\vdots$ $\downarrow \bar{\alpha}_{n-2}$ $\quad \vdots$ $\bar{\alpha}_{n-2} \uparrow$

$0\,0\,0\cdots 0\,-1\,1\,0$ $\downarrow \bar{\alpha}_{n-1}$ $\quad 0\cdots 0\,1\,-1\,0$ $\bar{\alpha}_{n-1} \uparrow$

ϕ_n, ϕ_{2n}: $0\,0\,0\cdots 0\,0\,-1\,2$ $\downarrow \bar{\alpha}_n$ $\quad 0\cdots 0\,0\,1\,-2$ $\nearrow \bar{\alpha}_n$

ϕ_0: $0\,0\,0\cdots 0\,0\,0\,0$

Positive Roots:

$\frac{n(n-1)}{2}$ LONG ① $\alpha^1_{jk} = \Lambda^{(j)} - \Lambda^{(k)}$, $j<k$

$\frac{n(n-1)}{2}$ LONG ② $\alpha^2_{jk} = \Lambda^{(j)} + \Lambda^{(k)}$, $j<k$

n SHORT ③ $\alpha^s_j = \Lambda^{(j)}$

$\Rightarrow$ FUND. IRREP. WEIGHTS $\{\Lambda^{(j)}\}$ ARE ALSO [NON-SIMPLE] POSITIVE ROOTS!

• SPECIAL TO SO(2n+1)

$\Rightarrow 2[n^2] + n = \dim(G)$ ✓

SIMPLE Roots:

LONG ① $\bar{\alpha}_i = \alpha^1_{i,i+1} = \Lambda^{(i)} - \Lambda^{(i+1)}$, $i \in \{1,\dots,n-1\}$

SHORT ② $\bar{\alpha}_n = \alpha^s_n = \Lambda^{(n)}$

CLAIM: $\langle \Lambda^{(a)}, \Lambda^{(b)} \rangle = \delta^{ab}$ ORTHONORMAL!

PROOF: FOR CSA, CAN TAKE BLOCK DIAGONAL STRUCTURE

$$\hat{H}_a = \left[\begin{array}{c|c|c|c|c} 0 & & & & \\ \hline & \hat{H}^{(1)}_a & & & \\ \hline & & \hat{H}^{(2)}_a & & \\ \hline & & & \ddots & \\ \hline & & & & \hat{H}^{(n)}_a \end{array}\right]$$

, WHERE $\hat{H}^{(k)}_a = \delta^k_a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

THUS, IN THE FUND. REP, WE CAN LABEL WEIGHTS BY THE EIGENVALUES $\{-1, 0, 1\}$

WEIGHT VECTOR	DYNKIN	ORTHONORMAL
SO(2n+1) ϕ_0 :	$\{\hat{H}_{\bar{\alpha}_i^\vee}\} = \{0,0,\dots 0\}$	$\{\hat{H}_a\} = \{0,0,\dots,0\}$
ϕ_1 :	$= \{1,0,0,\dots,0\}$	$= \{1,0,0,\dots,0\}$
ϕ_{n+1} :	$= \{-1,0,0,\dots,0\}$	$= \{-1,0,0,\dots,0\}$
ϕ_2 :	$= \{-1,1,0,0,\dots,0\}$	$= \{0,1,0,\dots,0\}$
ϕ_{n+2} :	$= \{1,-1,0,0,\dots,0\}$	$= \{0,-1,0,\dots,0\}$
$\vdots$		
ϕ_n :	$= \{0,0,\dots,0,-1,2\}$	$= \{0,0,0,\dots,1\}$
ϕ_{2n} :	$= \{0,0,\dots,0,1,-2\}$	$= \{0,0,0,\dots,-1\}$

$\Rightarrow$ CAN TAKE $\{\Lambda^{(a)}\}$ (n LIN. INDEPT. FUND. IRREP. WEIGHTS) TO BE ORTHONORMAL

$$\langle \Lambda^{(a)}, \Lambda^{(b)} \rangle = \delta^{ab}$$

POS. ROOTS

① $\alpha^1_{jk} = \Lambda^{(j)} - \Lambda^{(k)}$, $j<k$

② $\alpha^2_{jk} = \Lambda^{(j)} + \Lambda^{(k)}$, $j<k$

③ $\alpha^5_j = \Lambda^{(j)}$

SIMPLE ROOTS

① $\bar{\alpha}_i = \alpha^1_{i,i+1} = \Lambda^{(i)} - \Lambda^{(i+1)}$, $i \in \{1,\dots,n-1\}$

② $\bar{\alpha}_n = \alpha^5_n = \Lambda^{(n)}$

$\bar{\alpha}_i = \bar{\alpha}_i^\vee$, $i \in \{1,\dots,n\}$

$\bar{\alpha}_n^\vee = 2\bar{\alpha}_n$

FUND. WEIGHTS:

① $\omega_i = \sum_{j=1}^{i} \Lambda^{(j)}$, $i \in \{1,\dots,n-1\}$

② $\omega_n = \frac{1}{2} \sum_{j=1}^{n} \Lambda^{(j)}$ FUND. SPINOR IRREP.

APPLICATION: QUAD. FORM MATRIX

① $\langle \omega_i, \omega_j \rangle = \min(i,j)$, $i,j \in \{1,\dots,n-1\}$ ✓

② $\langle \omega_i, \omega_n \rangle = i/2$, $i \in \{1,\dots,n-1\}$ ✓

③ $\langle \omega_n, \omega_n \rangle = \frac{n}{4}$ ✓

FUND. SPINOR IRREP:

so(2n+1)

$$\omega_n = \frac{1}{2}\sum_{j=1}^{n} \Lambda^{(j)}$$

NOW, $\{\Lambda^{(a)}\}$ ARE ORTHONORMAL ROOTS (NON-SIMPLE) [p.6] → NOT TRUE FOR SO(2N).

• ALSO •

∴ BY WEYL REFL. ALONG THESE AXES, GENERATE $2^n - 1$ ADDL. WEIGHTS WITH DEG. 1 EACH (SAME AS THE HWS ω_n).

SPINOR IRREP. WEIGHTS:

- $\Lambda_{\sigma_1 \cdots \sigma_n} \in \{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \cdots, \pm\frac{1}{2}\}$ IN ORTHONORMAL BASIS.

∴ $\Lambda = \omega_n$ IS AT LEAST 2^n-DIM.

CLEARLY SELF-CONJUGATE, B/C $\{\Lambda_{\sigma_1 \cdots \sigma_n}\} \to \{-\Lambda_{\sigma_1 \cdots \sigma_n}\}$ (SET OF WEIGHTS IN IRREP) $\parallel$ $\{\Lambda_{\sigma_1' \cdots \sigma_n'}\}$ UNDER COMPLEX CONJUGATION.

∴ $\Lambda = \omega_n$ IS REAL OR PSEUDOREAL

CLAIM: $\Lambda = \omega_n$ HAS 2^n WEIGHTS.

PROOF #1: WEYL CHARACTER FORMULA $\quad \dim \Lambda = \prod_{\alpha > 0} \frac{\langle \alpha, \Lambda + \delta \rangle}{\langle \alpha, \delta \rangle}$

$$\delta = \sum_{j=1}^{n} \omega_j = \frac{1}{2}\sum_{\alpha>0} \alpha \quad \text{WEYL VECTOR}$$

$$\delta = \sum_{j=1}^{n-1}\sum_{k=1}^{j} \Lambda^{(k)} + \frac{1}{2}\sum_{j=1}^{n} \Lambda^{(j)}$$

$$\Lambda = \omega_n = \frac{1}{2}\sum_{j=1}^{n} \Lambda^{(j)}$$

① $\langle \alpha^1_{jk}, \Lambda + \delta \rangle = \langle \Lambda^{(j)} - \Lambda^{(k)}, \sum_{p=1}^{n}\sum_{q=1}^{p} \Lambda^{(q)} \rangle = \sum_{p=1}^{n}\sum_{q=1}^{p} (\delta^{j,q} - \delta^{k,q})$

$= \langle \alpha^1_{jk}, \delta \rangle \qquad = \sum_{p=j}^{n} - \sum_{p=k}^{n} = k - j$

② $\langle \alpha^2_{jk}, \Lambda + \delta \rangle = n - j + 1 + n - k + 1 = 2(n+1) - j - k$

$\langle \alpha^2_{jk}, \delta \rangle = 2n + 1 - j - k$

③ $\langle \alpha^5_j, \Lambda + \delta \rangle = n - j + 1$

$\langle \alpha^5_j, \delta \rangle = n - j + \frac{1}{2}$

MATHEMATICA [Pochammer FCN.S]

$$\therefore \dim \omega_n = \prod_{j=1}^{n} \prod_{k=j+1}^{n} \left(\frac{2(n+1)-j-k}{2n+1-j-k} \right) \prod_{j=1}^{n} \left(\frac{n-j+1}{n-j+\frac{1}{2}} \right) = 2^n \checkmark$$

SO(2n+1)

ex: $n=3$ $(SO(7))$

SO(7) ROOT SYSTEM

ROOT SYSTEM — SEE ALSO WZW 28 PART 1, p.1 (FCC)

$\therefore$ ALL CONJ. CLASS #1 IRREPS OF SO(7) [INCL. $\{\omega_1, \dots, \omega_{n-1}\}$] RESIDE ON AN FCC LATTICE.

T_i $\quad T_{[i_1 i_2 \cdots i_{n-1}]}$

(NOTE: $T_{[i_1 \cdots i_{n-1} i_n]} \Leftrightarrow (0,0,0,\dots,0,2)_{\text{DYNKIN}}$)

2^3 WEIGHTS OF THE ω_3 SPINOR IRREP. OF SO(7).

$\therefore$ ALL CONJ. CLASS #2 (SPINOR) IRREPS OF SO(7) [INCL. ω_n] RESIDE ON A SIMPLE CUBIC LATTICE

$\Updownarrow$

FOCK SPACE FOR n-SPIN $\frac{1}{2}$ MODEL.

EXPLICIT CONSTRUCTION: FUNDAMENTAL SPINOR IRREP. OF SO(2n+1)

$$\Lambda_{HWS} = \omega_n = \frac{1}{2} \sum_{j=1}^{n} \Lambda^{(j)}$$

$$\Lambda_{\sigma_1 \cdots \sigma_n} \equiv \frac{1}{2} \sum_{j=1}^{n} \sigma_j \Lambda^{(j)}$$ GENERIC WEIGHT ; $\sigma_j \in \{-1, 1\}$

WEIGHT VECTORS: $|\sigma_1 \sigma_2 \cdots \sigma_n\rangle$ n-SPIN DIRECT PRODUCT

CARTAN SUBALGEBRA: $\hat{H}_a = \frac{1}{2} \hat{\sigma}_a^3$

PAULI SPIN OP. ON a^{TH} SPIN.

IN ADDITION, WE HAVE A NATURAL COMPLETE OPERATOR BASIS : $\{\hat{\sigma}_j^{1,2,3}\}$ SPIN OPS

CONSTRUCTING ROOT (LADDER) OPERATORS

GEO 4, p. 76-77:

IDENTIFY

- $C^1_{jk} \equiv C_{\Lambda^{(j)} - \Lambda^{(k)}}$, $C^2_{jk} \equiv C_{\Lambda^{(j)} + \Lambda^{(k)}}$ $(j<k)$, $C^3_{jk} \equiv C_{-\Lambda^{(j)} - \Lambda^{(k)}}$ $(j<k)$, $C^4_j \equiv C_{-\Lambda^{(j)}}$, $C^5_j \equiv C_{\Lambda^{(j)}}$

GENERIC CSA ELEMENT: $h = \sum_j \Lambda^{(j)} h_j = \sum_j \Lambda^{(j)} e^1_{jj}$

SO(2n+1)

① $[e_{\Lambda^{(j)}}, e_{-\Lambda^{(k)}}] = [e^5_j, e^4_k] = e^1_{jk} = \begin{cases} e_{\Lambda^{(j)}-\Lambda^{(k)}}, & j \neq k \\ h_j, & j = k \end{cases}$

② $[e_{\Lambda^{(j)}}, e_{\Lambda^{(k)}}] = -e^2_{jk} = -e_{\Lambda^{(j)}+\Lambda^{(k)}}, \quad j<k$

③ $[e_{-\Lambda^{(j)}}, e_{-\Lambda^{(k)}}] = -e^3_{jk} = -e_{-\Lambda^{(j)}-\Lambda^{(k)}}, \quad j<k$

$\Rightarrow$ $e_{\pm\Lambda^{(j)} \pm \Lambda^{(k)}}$ CAN BE CONSTRUCTED USING THE 2n ROOTS $\{e_{\pm\Lambda^{(j)}}\}$

$\therefore$ CLEARLY, WE CANNOT IDENTIFY $e_{\Lambda^{(j)}} \leftrightarrow \sigma^+_j$, $e_{-\Lambda^{(j)}} \leftrightarrow \sigma^-_j$, B/C $[\sigma^a_i, \sigma^b_j] \propto \delta_{ij}$ INDEPT. SU(2)s!

INSTEAD, WE REQUIRE NON-LOCAL COMMUTATORS.

SOL'N: JORDAN-WIGNER STRINGS!

$\sigma^+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$\sigma^- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

- $\sigma^+\sigma^- = 2(1+\sigma^3)$
- $\sigma^-\sigma^+ = 2(1-\sigma^3)$
- $\sigma^3\sigma^\pm = \pm\sigma^\pm$
- $[\sigma^+, \sigma^-] = 4\sigma^3$

LET

$$e_{\Lambda^{(j)}} \equiv \frac{1}{2\sqrt{2}} \prod_{i=1}^{j-1} \sigma^3_i \sigma^+_j \; ; \; e_{-\Lambda^{(k)}} \equiv \frac{1}{2\sqrt{2}} \prod_{i=1}^{j-1} \sigma^3_i \sigma^-_j$$

① $[e_{\Lambda^{(j)}}, e_{-\Lambda^{(j)}}] = \frac{1}{2}\sigma^3_j = h_j$ ✓

② $\{e_{\Lambda^{(j)}}, e_{\Lambda^{(k)}}\} = \{e_{-\Lambda^{(j)}}, e_{-\Lambda^{(k)}}\} = 0 \; ; \; \{e_{\Lambda^{(j)}}, e_{-\Lambda^{(k)}}\} = \delta^{jk}\,\frac{1}{2}$ FERMIONS!

① AND ② ARE FAMILIAR FROM SPIN-$\frac{1}{2}$ AND FERMION MODELS.

THE NON-LOCAL COMMUTATORS ARE NOT:

$$[e_{\Lambda^{(j)}}, e_{-\Lambda^{(k)}}]^{(j<k)} = \frac{1}{8}\left[\prod_{i=1}^{j-1}\sigma^3_i\sigma^+_j, \prod_{l=1}^{k-1}\sigma^3_l\sigma^-_k\right] = \frac{1}{4}\sigma^+_j \prod_{l=j}^{k-1}\sigma^3_l\,\sigma^-_k$$

$$= -\frac{1}{4}\sigma^+_j \prod_{l=j+1}^{k-1}\sigma^3_l\,\sigma^-_k \equiv e_{\Lambda^{(j)}-\Lambda^{(k)}} \quad (j<k)$$

CHECK:

i) $$[h_j, e_{\Lambda^{(m)}-\Lambda^{(n)}}] = -\frac{1}{8}\left[\sigma^3_j, \sigma^+_m \prod_{p=m+1}^{n-1}\sigma^3_p\,\sigma^-_n\right]$$

$$= -\frac{1}{4}(\delta^{jm}+\delta^{jn})\,\sigma^3_j\,\sigma^+_m \prod_{p=m+1}^{n-1}\sigma^3_p\,\sigma^-_n$$

$$= -\frac{1}{4}(\delta^{jm}-\delta^{jn})\,\sigma^+_m \prod_p \sigma^3_p\,\sigma^-_n$$

$$= (\delta^{jm}-\delta^{jn})\,e_{\Lambda^{(m)}-\Lambda^{(n)}}$$

$$= \langle \Lambda^{(j)}, \Lambda^{(m)}-\Lambda^{(n)}\rangle\, e_{\Lambda^{(m)}-\Lambda^{(n)}} \checkmark$$

$(j<k)$ $\quad [\mathcal{C}_{\Lambda^{(j)}-\Lambda^{(k)}}, \mathcal{C}_{\Lambda^{(\ell)}}] = -\frac{1}{8}\sigma_j^+ \left(\prod_{m=j+1}^{k-1}\sigma_m^3\right)\sigma_k^- \left(\prod_{n=1}^{\ell-1}\sigma_n^3\right)\sigma_\ell^+ \frac{1}{\sqrt{2}}$

$SO(2n+1)$

$$+\frac{1}{8}\left(\prod_{n=1}^{\ell-1}\sigma_n^3\right)\sigma_\ell^+ \sigma_j^+ \left(\prod_{m=j+1}^{k-1}\sigma_m^3\right)\sigma_k^- \frac{1}{\sqrt{2}}$$

(-1) IF $j\le \ell-1$ $\qquad$ (-1) IF $j+1\le \ell\le k-1$

$$\underset{\text{IF } j\le\ell-1}{(-1)}\cdot\underset{\text{IF } j+1\le\ell\le k-1}{(-1)}\cdot\frac{1}{8}\sigma_j^+\left(\prod_{m=j+1}^{k-1}\sigma_m^3\right)\left(\prod_{n=1}^{\ell-1}\sigma_n^3\right)\left[\sigma_k^-\sigma_\ell^+ + 4\delta_{k,\ell}\sigma_\ell^3\right]\frac{1}{\sqrt{2}}$$

$$\overset{A}{\underset{\text{IF } j\le\ell-1}{(-1)}}\cdot\overset{B}{\underset{\text{IF } j+1\le\ell\le k-1}{(-1)}}\cdot\overset{C}{\underset{\text{IF } k\le\ell-1}{(-1)}}\cdot\frac{1}{8}\sigma_j^+\left(\prod_m\sigma_m^3\right)\sigma_k^-\left(\prod_n\sigma_n^3\right)\sigma_\ell^+\frac{1}{\sqrt{2}}$$

$$-\frac{1}{8}\sigma_j^+\left(\prod_m\sigma_m^3\right)\left(\prod_n\sigma_n^3\right)4\delta_{k,\ell}\sigma_k^3\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{4}\sigma_j^+\left(\prod_{m=j+1}^{k-1}\sigma_m^3\right)\cancel{\sigma_k^-}\left(\prod_{n=1}^{k-1}\sigma_n^3\right)\cancel{\sigma_k^+}\,\delta_{k,\ell}\,2\left[1-\cancel{\sigma_k^3}\right]\frac{1}{\sqrt{2}}$$

$$\cancel{-\frac{1}{2}\sigma_j^+\left(\prod_{m=1}^{j}\sigma_m^3\right)\delta_{k,\ell}\sigma_k^3\frac{1}{\sqrt{2}}}$$

$$= -\frac{1}{2}\sigma_j^+\left(\prod_{m=1}^{j}\sigma_m^3\right)\delta_{k,\ell}\frac{1}{\sqrt{2}} = \delta_{k,\ell}\frac{1}{2\sqrt{2}}\left(\prod_{m=1}^{j-1}\sigma_m^3\right)\sigma_j^+$$

$$= \delta_{k,\ell}\,\mathcal{C}_{\Lambda^{(j)}} \checkmark$$

GEO 4, p. 76:

$$[\mathcal{C}^1_{jk}, \mathcal{C}^5_\ell] = \delta_{k,\ell}\,\mathcal{C}^5_j \checkmark$$

SUMMARY:

CARTAN SUBALGEBRA ① $h_j = \frac{1}{2}\sigma_j^3$; $\qquad [h_j, \mathcal{C}_{\pm\Lambda^{(k)}}] = \delta_{jk}\,\mathcal{C}_{\pm\Lambda^{(k)}}(\pm 1)$

ORTHONORMAL ROOTS ② $\mathcal{C}_{\pm\Lambda^{(j)}} = \frac{1}{2\sqrt{2}}\left(\prod_{i=1}^{j-1}\sigma_i^3\right)\sigma_j^\pm$; JORDAN-WIGNER FERMIONS! $\qquad$ $j<k$: $[\mathcal{C}_{\Lambda^{(j)}}, \mathcal{C}_{-\Lambda^{(k)}}] = \mathcal{C}_{\Lambda^{(j)}-\Lambda^{(k)}}$

COMPOSITE ROOTS ③ $\mathcal{C}_{\Lambda^{(j)}-\Lambda^{(k)}} = -\frac{1}{4}\sigma_j^+\left(\prod_{m=j+1}^{k-1}\sigma_m^3\right)\sigma_k^-$; $\qquad [h_i, \mathcal{C}_{\Lambda^{(j)}-\Lambda^{(k)}}] = (\delta_{ij}-\delta_{ik})\mathcal{C}_{\Lambda^{(j)}-\Lambda^{(k)}}$

$= 2\,\mathcal{C}_{\Lambda^{(j)}}\mathcal{C}_{-\Lambda^{(k)}}$ $(j<k)$

SINCE $\{\mathcal{C}_{\Lambda^{(j)}}, \mathcal{C}_{-\Lambda^{(k)}}\} = \frac{1}{2}\delta^{jk}$ $\qquad [\mathcal{C}_{\Lambda^{(j)}-\Lambda^{(k)}}, \mathcal{C}_{\Lambda^{(\ell)}}] = \delta_{k,\ell}\,\mathcal{C}_{\Lambda^{(j)}}$

$\Rightarrow$ ALL OTHER COM. RELATIONS OBTAIN VIA

$\{h_j, \mathcal{C}_{\pm\Lambda^{(j)}}\}$ (SET OF — NOT ANTICOMM.)

FERMION REPRESENTATION: GENERATORS OF $SO(2n+1)$ IN THE W_n SPINOR IRREP.

$SO(2n+1)$

$$C_j^\dagger \equiv \sqrt{2}\, \mathcal{C}_{\Lambda^{(j)}} \quad ; \quad C_j \equiv \sqrt{2}\, \mathcal{C}_{-\Lambda^{(j)}} \quad ; \quad \{C_i, C_j^\dagger\} = \delta_{ij}$$

- $[h_j, \mathcal{C}_{\pm\Lambda^{(k)}}] = \pm\langle W_j^{(j)}, \Lambda^{(k)}\rangle \mathcal{C}_{\pm\Lambda^{(k)}}$
- $[\mathcal{C}_{\Lambda^{(j)}}, \mathcal{C}_{-\Lambda^{(j)}}] = h_j$
- $[\mathcal{C}_{\Lambda^{(j)}}, \mathcal{C}_{-\Lambda^{(k)}}] = \mathcal{C}_{\Lambda^{(j)}-\Lambda^{(k)}}$ $(j \neq k)$
- $[\mathcal{C}_{\Lambda^{(j)}-\Lambda^{(k)}}, \mathcal{C}_{\Lambda^{(l)}}] = \delta_{kl}\, \mathcal{C}_{\Lambda^{(j)}}$

① $h_j + \frac{1}{2} = C_j^\dagger C_j$, $1 \le j \le n$ (rank) $[so(2n+1)]$

② $\sqrt{2}\, \mathcal{C}_{\Lambda^{(j)}} = C_j^\dagger$, $\sqrt{2}\, \mathcal{C}_{-\Lambda^{(j)}} = C_j$

③ $\mathcal{C}_{\Lambda^{(j)}-\Lambda^{(k)}} = C_j^\dagger C_k$, $j \neq k$

④ $\mathcal{C}_{\Lambda^{(j)}+\Lambda^{(k)}} = -C_j^\dagger C_k^\dagger$, $j<k$; $\mathcal{C}_{-\Lambda^{(j)}-\Lambda^{(k)}} = -C_k C_j$

$\therefore$ FOR <u>ANY</u> 1D (SPINLESS) FERMION CHAIN, THERE EXISTS A CONSERVED NON-LOCAL QUANTITY THAT TAKES THE SAME FIXED VALUE FOR <u>ALL</u> STATES IN THE FOCK SPACE: THE QUADRATIC CASIMIR.

THE ACTION OF ANY OPERATORS IN ①–④ CAN BE VISUALIZED AS RAISING AND LOWERING ALONG WEIGHT STRINGS IN THE 2^n-NODE n-HYPERCUBE.

$n=3$ CUBE

$SO(7)$

$|\sigma_1\sigma_2\sigma_3\rangle$

$n=4$ TESSERACT

$SO(9)$

$|\sigma_1\sigma_2\sigma_3\sigma_4\rangle$

APPLICATION:

(RANK $n-1$) $SU(n) \subset SO(2n+1)$ (RANK n) : A NON-MAXIMAL SUBALGEBRA

- SLAVE COMPLEX FERMIONS $\rightarrow SU(n)$
- SLAVE REAL FERMIONS $\rightarrow SO(N)$

LET $\hat{T}_a \rightarrow [T_a]^\sigma{}_\rho$ $n \times n$ GENERATOR OF $SU(n)$ (FUND. IRREP.)

$$\Rightarrow M_a \equiv \sum_{i,j \in \{1-n\}} C_i^\dagger [T_a]^i{}_j C^j$$

$2n \times 2n$ GENERATOR OF $SU(n)$, $a \in \{1, \dots, n^2-1\}$

SO(2n+1)

ALL IRREPS OF $SO(2n+1)$ ARE SELF-CONJUGATE $\Leftrightarrow$

LACKS REFL. SYMMETRY

CONJ. CLASSES: GEO 4A, p. 6c

CONGRUENCE VECTOR

$$\Lambda \cdot V = \Lambda_n \mod 2$$

n^{TH} DYNKIN LABEL

CLASS	BASIC FUND. WEIGHTS
0	$\omega_1, \omega_2, \omega_3, \dots, \omega_{n-1}$ (adj)
1	ω_n

- $\Lambda = \omega_1$: REAL REP. $\hat{T}^a = -\hat{R}\,\hat{T}^{a*}\hat{R}^\dagger$; $\hat{R} = \hat{R}^T$ AND $\hat{R}\hat{R}^\dagger = \hat{1}$
- $\Lambda = \omega_n$: MAY BE REAL $(\hat{R} = \hat{R}^T)$ OR PSEUDOREAL $(\hat{R} = -\hat{R}^T)$

EXPLICIT CONST: HERMITIAN GENERATORS IN $\Lambda = \omega_n$ (2^n-DIM. IRREP.)

A. $-h_j^* = -h_j = -\frac{1}{2}\sigma_j^3$ CS

B. $e_j^{1,2} \equiv \frac{1}{2\sqrt{2}}\left(\prod_{i=1}^{j-1}\sigma_i^3\right)\sigma_j^{1,2}$ HERM. ROOT VECTOR COMBOS

- INVT. TENSOR FOR $\Lambda = \omega_n$ SPINOR IRREP. $(\omega_n \times \omega_n)$

LET $$R \equiv \prod_{\substack{j \in \{1,3,\dots\} \\ \text{ODD}}} \sigma_j^2 \prod_{\substack{k \in \{2,4,\dots\} \\ \text{EVEN}}} \sigma_k^1 = \prod_{k \text{ EVEN}} \sigma_k^1 \prod_{j \text{ ODD}} \sigma_j^2$$

$$\Rightarrow -R\,h_j^* R^\dagger = h_j \checkmark$$

$$\Rightarrow -R\,e_j^{1*} R^\dagger = -R\,e_j^1 R^\dagger = e_j^1 \checkmark$$

$$\Rightarrow -R\,e_j^{2*} R^\dagger = R\,e_j^2 R^\dagger = e_j^2 \checkmark$$

NOW,

$$\hat{R}^T = \hat{R} \times \begin{cases} \text{(a) } n \text{ IS ODD: } \prod_{j \in \{1,3,\dots,n\}} (-1) = (-1)^{\frac{n+1}{2}} = (-1)^{\frac{n(n+1)}{2}} & (n \text{ IS ODD}) \\ \text{(b) } n \text{ IS EVEN: } \prod_{j \in \{1,3,\dots,n-1\}} (-1) = (-1)^{\frac{n}{2}} = (-1)^{\frac{n}{2}(n+1)} & [(n+1) \text{ IS ODD}] \end{cases}$$

$1, 2, \dots, \frac{n+1}{2}$; $1, 2, \dots, n/2$

$$\therefore \hat{R}^T = \hat{R}\,(-1)^{\frac{n(n+1)}{2}}$$

n	1	2	3	4	5	6 …
$\frac{n(n+1)}{2} \mod 2$	1	1	0	0	1	1 …

$SO(2n+1)$

$\therefore$ SPINOR IRREP. IS PSEUDOREAL : $SO(3), SO(5), SO(11), SO(13), \ldots$

REAL : $SO(7), SO(9), SO(15), SO(17), \ldots$

1.	$SO(3) \sim SU(2)$	$R = \sigma^2$	PR	(DEFINING REP. $Sp(2)$!)
2.	$SO(5) \sim Sp(4)$	$R = \sigma_1^2 \sigma_2^1$	PR	(DEFINING REP. $Sp(4)$!)
3.	$SO(7)$	$R = \sigma_1^2 \sigma_2^1 \sigma_3^2$	R	
4.	$SO(9)$	$R = \sigma_1^2 \sigma_2^1 \sigma_3^2 \sigma_4^1$	R	
5.	$SO(11)$	$R = \sigma_1^2 \sigma_2^1 \sigma_3^2 \sigma_4^1 \sigma_5^2$	PR	

AFFINE LIE ALGEBRAS (DiF/M/S CH. 14)

14.1.1 CONSTRUCTION

SOME FINITE L.A. (e.g. A_n, B_n, C_n, D_n)

"LOOP ALGEBRA": $\tilde{\mathfrak{g}} \equiv \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$

SET OF LAURENT POLYNOMIALS IN t (SOME VAR.)

GENERATORS $J_n^a \equiv J^a \otimes t^n$

$$[J_n^a, J_m^b] = i f^{abc} J_{m+n}^c$$

CENTRAL EXTENSION: INTRODUCE $\hat{k}$ S.T. $[J_n^a, \hat{k}] = 0 \quad \forall a, n$

$$[J_n^a, J_m^b] = i f^{abc} J_{m+n}^c + \hat{k} n \delta^{ab} \delta_{m+n,0}$$

CARTAN-WEYL BASIS:

CONV. FOR FINITE L.A.: • KILLING FORM $K(X,Y) \equiv \frac{1}{2g} \mathrm{Tr}[\hat{a}_{d_X} \cdot \hat{a}_{d_Y}]$; $K(J^a, J^b) = \delta^{ab}$ ORTHONORMAL BASIS.

$g = \sum_{j=1}^{r} a_j^\vee + 1$ DUAL COXETER NUMBER

COMARKS: COEFF. OF θ (HW ROOT) IN COROOT EXPANSION

GEO 4:

HW ROOT: $\theta \equiv \sum_{i=1}^{r} a_i \bar{\alpha}_i = \sum_{i=1}^{r} a_i^\vee \bar{\alpha}_i^\vee = \begin{cases} (1,0,\dots,0,1)_D & A_r \text{ p.90}_A \\ (0,1,0,\dots,0)_D & B_r \text{ p.90} \\ (2,0,0,\dots,0)_D & C_r \text{ p.82} \\ (0,1,0,\dots,0)_D & D_r \text{ p.90}_E \end{cases}$

(a_i: MARKS; $a_i^\vee$: COMARKS)

SEE CFT BOOK p. 540–545 FOR TABULATED $\{a_j\}$ AND $\{a_j^\vee\}$.

• COXETER # g, CASIMIR, INDEX OF IRREP.:

- θ IS NEC. A LONG ROOT
- CONV.: LONG ROOTS NORM. TO 2

DEF'N. OF ρ – p. 2 OF GEO 4A

$$(\theta, \theta + 2\rho) = 2 + (\theta, \rho) \cdot 2 = 2 + 2 \sum_{ij} a_i^\vee \langle \bar{\alpha}_i^\vee, \omega_j \rangle$$
$$= 2 + 2(g-1) = 2g$$

INDEX: $\chi_\lambda \equiv \frac{\dim|\lambda| \, (\lambda, \lambda + 2\rho)}{2 \dim \mathfrak{g}}$ DIFFERS FROM DEF'N. IN GEO 4 ON p. 92 BY CONV. FACTOR OF 2.

ADJ. REP.: $\chi_\theta = g$

• CARTAN SUBALGEBRA $\mathfrak{h}$: ORTHONORMAL BASIS H^i S.T. $K(H^i, H^j) = \delta^{ij}$

DUAL ROOT SPACE h^*; REP IN h: $H^\alpha \equiv \alpha \cdot H$; $\alpha = \{\alpha(H^1), \alpha(H^2), \dots, \alpha(H^r)\}$

$$H^\alpha = \frac{[E^\alpha, E^{-\alpha}]}{K(E^\alpha, E^{-\alpha})} \; ; \quad K(E^\alpha, E^{-\alpha}) = \frac{2}{|\alpha|^2}$$ (= 1 FOR <u>LONG</u> ROOTS)

- SCALAR PRODUCT ON ROOT SPACE: $(\gamma, \beta) = K(H^\gamma, H^\beta) = \gamma^i \beta^i$ (EXP. IN ON h-BASIS)

$\hat{g}$: C-W. FOR AFFINE L.A.: $[\hat{k}, \text{EVERYTHING}] = 0$

L_0 EIGENVALUE: <u>GRADE</u>

$$[H_n^i, H_m^j] = \hat{k}\, n\, \delta^{ij} \delta_{n+m,0}$$

- WE WILL ADD $[L_0, J_n^a] = -n J_n^a$ BELOW.

$$[H_n^i, E_m^\alpha] = \alpha^i E_{m+n}^\alpha$$

$$[E_n^\alpha, E_m^\beta] = K(E^\alpha, E^{-\alpha})\left[H_{n+m}^\alpha + \hat{k}\, n\, \delta_{n+m,0}\right] = K(E^\alpha, E^{-\alpha})\left[H_{n+m} \cdot \alpha + \hat{k}\, n\, \delta_{n+m,0}\right], \beta = -\alpha$$

$$= N_{\alpha,\beta} E_{n+m}^{\alpha+\beta} \qquad , \alpha+\beta \in \Delta$$

SET OF ROOTS

$$= 0, \quad \text{ELSE}$$

- ABELIAN SUBALGEBRA ("CARTAN"): $\{H_0^1, \dots, H_0^r, \hat{k}\}$

$\hat{Ad}_{H_0^i} \Rightarrow$

$$\begin{array}{c} \left[\begin{array}{cccccccccccccccccccc}
\alpha_1^i & & & & & & & & & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & & & & & & & & & \\
 & & \alpha_1^i & & & & & & & & & & & & & & & & & \\
 & & & -\alpha_1^i & & & & & & & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & & & & & & & \\
 & & & & & -\alpha_1^i & & & & & & & & & & & & & & \\
 & & & & & & \ddots & & & & & & & & & & & & & \\
 & & & & & & & +\alpha_{|\Delta_+|}^i & & & & & & & & & & & & \\
 & & & & & & & & \ddots & & & & & & & & & & & \\
 & & & & & & & & & \alpha_{|\Delta_+|}^i & & & & & & & & & & \\
 & & & & & & & & & & -\alpha_{|\Delta_+|}^i & & & & & & & & & \\
 & & & & & & & & & & & \ddots & & & & & & & & \\
 & & & & & & & & & & & & -\alpha_{|\Delta_+|}^i & & & & & & & \\
 & & & & & & & & & & & & & 0 & & & & & & \\
 & & & & & & & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & & & & & & & 0 & & & & \\
 & & & & & & & & & & & & & & & & 0 & & & \\
 & & & & & & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & & & & & & 0 & \\
 & & & & & & & & & & & & & & & & & & & 0
\end{array}\right] \end{array}$$

$\{E_n^{\alpha_1}\}$
$\{E_n^{-\alpha_1}\}$
$\vdots$
$\{E_n^{\alpha_{|\Delta_+|}}\}$
$\{E_n^{-\alpha_{|\Delta_+|}}\}$
$\{H_n^1\}$
$\{H_n^2\}$
$\vdots$
$\{H_n^r\}$
$\leftarrow \hat{k}$
$\leftarrow L_n$ (SEE BELOW)

Grade Zero Finite $\mathfrak{g}$ Subalgebra:

$$[H_0^i, H_0^j] = 0$$

$$[H_0^i, E_0^\alpha] = \alpha^i E_0^\alpha$$

$$[E_0^\alpha, E_0^\beta] = K(E^\alpha, E^{-\alpha})\, H_0^\alpha \qquad , \ \beta = -\alpha$$

$$= N_{\alpha,\beta} E_0^{\alpha+\beta} \qquad , \ \alpha+\beta \in \Delta$$

$$= 0 \qquad , \text{ ELSE}$$

MWDF for Affine Weight Strings

SU(2) - Generated String: Choose $E_m^\alpha, E_n^{-\alpha}$ s.t. $K(E^\alpha, E^{-\alpha}) = 1$

$$[E_n^\alpha, E_{-n}^{-\alpha}] = H_0^\alpha + Kn$$

$$[H_0^\alpha, E_{\pm n}^{\pm\alpha}] = \pm\langle\alpha,\alpha\rangle E_{\pm n}^{\pm\alpha} \qquad ; \qquad [-L_0, E_{\pm n}^{\pm\alpha}] = \pm n\, E_{\pm n}^{\pm\alpha}$$

- Grade

$$[H_0^\alpha, H_0^\alpha] = 0$$

Following Geo 4, p. 31-32:

Assume Irrep. w/ $\hat{H}_0^\alpha \phi^a = M^a(H^\alpha)\, \phi^a = \langle M^a, \alpha\rangle \phi^a$

Highest Weight State: $\hat{H}_0^\alpha \phi_0 = \langle M^*, \alpha\rangle \phi_0 \quad ; \quad \hat{E}_n^\alpha \phi_0 = 0$

$$\text{Let} \quad \phi_k \equiv (\hat{E}_{-n}^{-\alpha})^k \phi_0 \quad ; \quad \hat{E}_n^\alpha \phi_k \equiv r_k \phi_{k-1}$$

$$\Rightarrow \hat{E}_n^\alpha \phi_{k'} = r_{k'} \phi_{k'-1} = (\hat{E}_{-n}^{-\alpha} \hat{E}_n^\alpha + \hat{H}_0^\alpha + Kn)\, \phi_{k'-1} = (r_{k'-1} + \langle\alpha, M^* - (k'-1)\alpha\rangle + Kn)\, \phi_{k'-1}$$

$$\therefore \ r_m = r_{m-1} + \langle M^*, \alpha\rangle - (m-1)\langle\alpha,\alpha\rangle + Kn$$

- Affine scalar product: $\langle\hat{\lambda},\hat{\mu}\rangle = \langle\lambda,\mu\rangle + K_\lambda n_\mu + K_\mu n_\lambda \quad ; \quad \hat{\lambda} = (\lambda; K_\lambda; n_\lambda) \ , \ \hat{\mu} = (\mu; K_\mu; n_\mu)$

Here, $\hat{M}^* = (M^*; K; n_{M^*}) \quad ; \quad \hat{\alpha} = (\alpha; 0; n)$ (p. 5)

$$\therefore \; r_m = r_{m-1} + \langle \hat{M}^*, \hat{\alpha} \rangle - (m-1) \langle \hat{\alpha}, \hat{\alpha} \rangle$$

$$r_0 = 0$$
$$r_1 = \langle \hat{M}^*, \hat{\alpha} \rangle + 0$$
$$r_2 = 2\langle \hat{M}^*, \hat{\alpha} \rangle - \langle \hat{\alpha}, \hat{\alpha} \rangle$$
$$r_3 = 3\langle \hat{M}^*, \hat{\alpha} \rangle - (1+2)\langle \hat{\alpha}, \hat{\alpha} \rangle$$
$$\vdots$$
$$r_m = m \langle \hat{M}^*, \hat{\alpha} \rangle - \frac{m(m-1)}{2} \langle \hat{\alpha}, \hat{\alpha} \rangle$$

<u>TERMINATION</u>: STRING OF LENGTH $q+1$ ELEMENTS

$$\therefore \; r_{q+1} = 0 = (q+1)\langle \hat{M}^*, \hat{\alpha} \rangle - \tfrac{1}{2}(q+1)q \langle \hat{\alpha}, \hat{\alpha} \rangle \quad \Rightarrow \quad q = \frac{2\langle \hat{M}^*, \hat{\alpha} \rangle}{\langle \hat{\alpha}, \hat{\alpha} \rangle}$$

FOR $\hat{M}$ A WEIGHT IN A STRING GENERATED BY $\hat{E}^{\alpha}_{n}$ (RAISING) AND $\hat{E}^{-\alpha}_{-n}$ (LOWERING) OF LENGTH $m+p+1$, $\hat{M}+p\hat{\alpha}, \hat{M}+(p-1)\hat{\alpha}, \ldots, \hat{M}, \ldots, \hat{M}-m\hat{\alpha}$, WE OBTAIN THE AFFINE MWAF

LEVEL OF WEIGHT $\hat{M}$
GRADE OF ROOT $\hat{\alpha}$

$$(m-p)_{\hat{M};\hat{\alpha}} = \frac{2\langle \hat{M}, \hat{\alpha} \rangle}{\langle \hat{\alpha}, \hat{\alpha} \rangle} = \langle \hat{M}, \hat{\alpha}^{\vee} \rangle = \frac{2\left[\langle M, \alpha \rangle + k n\right]}{\langle \alpha, \alpha \rangle}$$

$\hookrightarrow (-1)$: "GRADE" IS EVAL OF $\underline{\underline{L_0}}$, NOT $-L_0$.

$\hat{M} \equiv \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + \ell\delta \Rightarrow (m-p)_{\hat{M};\hat{\alpha}_i} = \langle \hat{M}, \hat{\alpha}_i^{\vee} \rangle = \lambda_i$, $i \in \{0,1,\ldots,r\}$ AFFINE DYNKIN LABELS

BY CONTRAST, H^i_n AND H^i_{-n} GENERATE A HEISENBERG ALGEBRA:

$$[H^i_n, H^i_{-n}] = \hat{k} n \quad \text{(NO SUM ON } i\text{)} \quad ; \quad [-L_0, H^i_{\pm n}] = \pm n H^i_{\pm n}$$

IRREP. w/ $\hat{H}^{\alpha}_0 \phi^a = \langle M, \alpha \rangle \phi^a$; $\hat{k}\phi^a = k\phi^a$; $-\hat{L}_0 \phi^a = r_M \phi^a$

HWS: $\hat{H}^i_n \phi_0 = 0$.

LET $\phi_\ell \equiv (\hat{H}^i_{-n})^{\ell} \phi_0$; $\phi_{\ell-1} \times r_\ell = \hat{H}^i_n \phi_\ell$

$$\Rightarrow \hat{H}^i_n \phi_\ell = r_\ell \phi_{\ell-1} = (\hat{H}^i_{-n}\hat{H}^i_n + kn)\phi_{\ell-1} = (r_{\ell-1} + kn)\phi_{\ell-1}$$

$$\therefore \; r_m = m \langle \hat{M}^*, \widehat{n\delta} \rangle$$

$\Rightarrow$ NEVER TERMINATES!
$(\langle \hat{\delta}, \hat{\delta} \rangle = 0)$

WE ALSO SEEM TO HAVE $\hat{\mathrm{Ad}}_{\hat{k}} \to [0]$? $\hat{k}$ DOES NOT PLAY A ROLE IN THE ADJ. REP. ?

TO "RESOLVE" GENERATORS $\{E_n^\alpha\}$ AND $\{H_n^i\}$ IN THE <u>GRADE</u> n, WE INTRODUCE

$$L_0 \equiv -t\frac{d}{dt}$$

WHICH AUGMENTS THE COMM. REL. ON p. 2 WITH

$$[L_0, J_n^a] = -n J_n^a \quad ; \quad [L_0, \hat{k}] = 0 \qquad (*)$$

$\Rightarrow$ "MAXIMAL" CARTAN SUBALGEBRA: $\{H_0^1, \ldots, H_0^r, \hat{k}, L_0\}$

$\hat{\mathrm{Ad}}_{L_0} \Rightarrow$

<u>AFFINE LIE ALGEBRA:</u> $\equiv \hat{g}$

COMMUTATION REL. ON p. 2, TOGETHER WITH $(*)$ ABOVE.

$$\hat{g} = \tilde{g} \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}L_0$$

LOOP ALG. (p. 1) — CENTRAL EXT. (SCHWINGER) — HAMILTONIAN – IN WZW, WE GENERALIZE TO THE ENTIRE VIRASORO ALGEBRA.

ex): AFFINE U(1) $\quad [H_n, H_m] = \hat{k}\, n\, \delta_{m+n,0}$ $\quad$ HEISENBERG (FREE BOSON) ALGEBRA.

$[L_0, H_n] = -n H_n$

14.1.2 KILLING FORM

INVC. OF THE KILLING FORM FOR FINITE LA g: $K([Z,X],Y) + K(X,[Z,Y]) = 0$

WE ASSUME THIS COND. ALSO APPLIES IN THE AFFINE CASE.

$\Rightarrow$ IMPLIES

a.) $X, Y \in \{J_n^a\}$; $Z = L_0$:

$$K([L_0, J_n^a], J_m^b) + K(J_n^a, [L_0, J_m^b]) = (-n-m)\, K(J_n^a, J_m^b) = 0$$

$$\therefore \quad K(J_n^a, J_m^b) \propto \delta_{n+m,0}$$

LOOP ALG. REP, p. 1

b.) $K(J_n^a, J_m^b) = \delta_{n+m,0}\, K(J^a \cancel{t^n}, J^b \cancel{t^m}) = \delta_{m+n,0}\, \delta^{a,b}$

c.) $K([J_n^a, J_m^b], \hat{k}) + \cancel{K(J_m^b, 0)} = 0$

$-n \neq m$: $\quad K(i f^{abc} J_{n+m}^c, \hat{k}) = 0 \quad \Rightarrow \quad K(J_n^a, \hat{k}) = 0 \quad (n \neq 0)$

$m = -n$: $\quad K(i f^{abc} J_0^c + \hat{k}\, n\, \delta^{ab}, \hat{k}) = 0 \quad \Rightarrow \quad K(J_0^a, \hat{k}) = K(\hat{k}, \hat{k}) = 0$

d.)

$$K([J_n^a, J_m^b], L_0) + K(J_m^b, [J_n^a, L_0]) = 0$$

$$\Rightarrow K(i f^{abc} J_{n+m}^c + \hat{k}\, n\, \delta^{ab} \delta_{n+m,0}, L_0) + n\, K(J_m^b, J_n^a) = 0$$

$$i f^{abc} K(J_{n+m}^c, L_0) + n\, \delta^{ab} \delta_{n+m,0} K(\hat{k}, L_0) + n\, \delta_{n+m,0}\, \delta^{ab} = 0$$

$\therefore \quad K(J_n^a, L_0) = 0 \;, \quad K(\hat{k}, L_0) = -1 \quad \Rightarrow$ CANNOT IGNORE $\hat{k}$ COMPLETELY IN ADJ. REP.!

SUMMARY:

① $K(J_n^a, J_m^b) = \delta^{ab} \delta_{n+m,0}$

② $K(J_n^a, \hat{k}) = 0$

③ $K(\hat{k}, \hat{k}) = 0$

④ $K(J_n^a, L_0) = 0$

⑤ $K(\hat{k}, L_0) = -1$!!

⑥ $K(L_0, L_0) \equiv 0$ (CONVENTION)

AFFINE KILLING FORM

NOTE: $K(L_0, L_0)$ MAY BE SHIFTED TO ANY DESIRED VALUE VIA $L_0' \equiv L_0 - \frac{E_0^2}{2}\hat{k}$

$$\Rightarrow K(L_0', L_0') = -E_0^2 K(L_0, \hat{k}) = E_0^2$$

<u>AFFINE WEIGHT:</u> $\hat{\lambda} \equiv (\lambda; k_\lambda; n_\lambda) = (\lambda(H_0^1), \lambda(H_0^2), \cdots, \lambda(H_0^r); \underset{\hat{k}\text{ E.V.}}{k_\lambda}; \underset{-L_0\text{ E.V.}}{n_\lambda})$

- GRADE

FINITE COMPONENTS OF THE WEIGHT SUCH THAT $\langle \lambda, w\rangle = (H_\lambda, H_w)$

$= \lambda(H_0^1)w(H_0^1) + \ldots + \lambda(H_0^r)w(H_0^r)$

(TREATS WEIGHTS AS VECTORS w/ COMP. IN CART. COORD. $\mathbb{R}^r$ SPACE) → TYPICALLY NOT USEFUL IN PRACTICE, WHERE WE INSTEAD RELY UPON THE EXP. IN TERMS OF FUND. WEIGHTS, COROOTS, w/ ELEM. $\langle \bar{\alpha}_i, \bar{\alpha}_j^\vee\rangle = A_{ij}$.

- WE ASSUME THE SAME ISOMORPHISM B/T THE CART. SUB. AND THE DUAL ROOT/WEIGHT SPACE.

USING THE RESULTS ON THE BOTTOM OF p. 4:

$$\langle \hat{\lambda}, \hat{\mu}\rangle = \langle \lambda, \mu\rangle + k_\lambda n_\mu + k_\mu n_\lambda \qquad \text{AFFINE WEIGHT SCALAR PRODUCT}$$

- GRADE OF $\hat{\mu}$ — - GRADE OF $\hat{\lambda}$

WEIGHTS IN THE ADJOINT REP: SINCE $\hat{Ad}_{\hat{k}} \to$ "[0]" (DESPITE CRUCIAL "ANOMALY" $K(\hat{k}, L_0) = -1$), <u>AFFINE ROOTS</u> TAKE THE FORM

$$\hat{\beta} = (\beta; 0; n) \Rightarrow (\hat{\beta}, \hat{\alpha}) = (\beta, \alpha) \quad ; \quad \hat{\beta}, \hat{\alpha} \in \hat{\Delta} \text{ (SET OF AFFINE ROOTS)}$$

DIF/M/S NOTATION FOR $\langle\, ,\rangle$

IN PARTICULAR,

$$E_n^\alpha \Leftrightarrow \hat{\alpha} = (\alpha; 0; n) \quad , \quad \alpha \in \Delta \text{ AND } n \in \mathbb{Z}.$$
$$H_n^i \Leftrightarrow \hat{n\delta} = (0; 0; n) \quad , \quad n \in \mathbb{Z} \text{ AND } n \neq 0.$$

DEFINE: $\alpha \equiv (\alpha; 0; 0)$ PURE FINITE ROOT ASSOC. TO E_0^α

$\delta \equiv (0; 0; 1)$ "IMAGINARY," PURE AFFINE ROOT ASSOC. TO $\{H_1^i\}$, $i \in \{1, \ldots, r\}$

THEN ① $E_n^\alpha \Leftrightarrow \hat{\alpha} = \alpha + n\delta$ NON-DEGEN.
② $H_n^i \Leftrightarrow \hat{n\delta} = n\delta, n \neq 0$ r-DEGEN.
$\Big\} \equiv \hat{\Delta}$ FULL SET OF AFFINE ROOTS

<u>NOTE:</u> $\langle \delta, \delta\rangle = 0$; $\delta \equiv$ AN IMAGINARY ROOT

$\langle m\delta, n\delta\rangle = 0 \Rightarrow m\delta$ IS IMAGINARY w/ DEGEN. r.

$\langle \hat{\alpha}, \hat{\alpha}'\rangle = \langle \alpha, \alpha'\rangle \Rightarrow \langle \hat{\alpha}, \hat{\alpha}\rangle = \langle \alpha, \alpha\rangle \neq 0$; $\hat{\alpha} \equiv$ A REAL ROOT w/ DEGEN. 1.

14.1.3 Simple Roots, Cartan Matrix, Dynkin Diagrams

Simple Roots: $r+1$ basis elements with which to construct any root w/ strictly positive (or negative) coeffs.

Choose:

$$\bar{\alpha}_0 \equiv (-\theta; 0; 1) = -\theta + \delta \quad \text{(highest root; } \langle\theta,\theta\rangle = 2\text{)} \qquad \Rightarrow E_1^{-\theta},\ E_{-1}^{\theta}$$

$$\bar{\alpha}_i \equiv (\bar{\alpha}_i; 0; 0) \quad , i \in \{1,\dots,r\} \quad \text{ordinary finite roots} \qquad \Rightarrow E_0^{\bar{\alpha}_i},\ E_0^{-\bar{\alpha}_i}$$

Assoc. Root Vectors:

Set of positive roots:

$$\hat{\Delta}_+ = \{\alpha \mid \alpha \in \Delta_+\} \cup \{\alpha + n\delta \mid n>0, \alpha\in\Delta\} \cup \{n\delta \mid n>0\}$$

$\Updownarrow$ E_0^{α} w/ $\alpha\in\Delta_+$; • $\alpha = \sum_{i=1}^{r} k_i^{(\alpha)} \bar{\alpha}_i$ as usual

$\Updownarrow$ E_n^{α} w/ $n>0$; • $\alpha + n\delta = n\bar{\alpha}_0 + \sum_{i=1}^{r} k_i^{(\alpha+n\delta)} \bar{\alpha}_i$

$\Updownarrow$ H_n^i w/ $n>0$ ($i\in\{1,\dots,r\}$ degen.!); • $n\delta = n\bar{\alpha}_0 + n\sum_{i=1}^{r} a_i \bar{\alpha}_i$ ← marks (p. 1)

$\therefore \{E_{n\geq 2}^{\alpha}, H_{n\geq 1}^{i}\}$ do not correspond to simple roots.

⇒ There is no Highest Root!

⇒ The Adjoint Rep. is not a Highest Weight Irrep.!

Extended Cartan Matrix:

Def: Affine Coroot $\hat{\alpha}^{\vee} \equiv \frac{2}{|\hat{\alpha}|^2}(\alpha; 0; n) = \frac{2}{|\alpha|^2}(\alpha; 0; n) = (\alpha^{\vee}; 0; \frac{2}{|\alpha|^2} n)$

$$\boxed{\hat{A}_{ij} \equiv \langle \bar{\alpha}_i, \bar{\alpha}_j^{\vee}\rangle \quad ; \quad 0\leq i,j\leq r}$$

with $\bar{\alpha}_0^{\vee} = \bar{\alpha}_0$; $\bar{\alpha}_i^{\vee} = (\bar{\alpha}_i^{\vee}; 0; 0)$ $i\in\{1,\dots,r\}$

$$\hat{A} = \begin{bmatrix} \hat{A}_{00} & \hat{A}_{01} & \cdots & \hat{A}_{0r} \\ \hat{A}_{10} & & & \\ \vdots & & A_{\text{finite}} & \\ \hat{A}_{r0} & & & \end{bmatrix}$$

$$\hat{A}_{00} = \langle\theta,\theta\rangle = 2$$

$$\hat{A}_{0j} = \langle\bar{\alpha}_0, \bar{\alpha}_j^{\vee}\rangle = -\langle\theta, \bar{\alpha}_j^{\vee}\rangle = -\sum_{i=1}^{r} a_i A_{ij}$$

$$\hat{A}_{i0} = \langle\bar{\alpha}_i, \bar{\alpha}_0\rangle = -\langle\bar{\alpha}_i, \theta\rangle = -\sum_{j=1}^{r} a_j^{\vee} A_{ij}$$

• We define $a_0 = a_0^{\vee} \equiv 1$ (θ is a long root and $\langle\theta,\theta\rangle = 2$)

$$\therefore \ ① \sum_{i=0}^{r} a_i \hat{A}_{ij} = -\sum_{l=1}^{r} a_l A_{lj} + \sum_{i=1}^{r} a_i A_{ij} = 0 \ ; \ ② \sum_{j=0}^{r} \hat{A}_{ij} a_j^{\vee} = -\sum_{l=1}^{r} A_{il} a_l^{\vee} + \sum_{j=1}^{r} A_{ij} a_j^{\vee} = 0$$

$\Rightarrow$ LIN. DEPC. OF ROWS OR COLUMNS IN $\hat{A}$: $\hat{A}$ HAS ONE ZERO EIGENVALUE (SEE e.g. GEO 4A, p.6)

(i.e. $\hat{A}$ IS <u>NOT</u> INVERTIBLE, ALTHOUGH IT MAY BE DIAGONALIZABLE)

THE ZERO REFLECTS THE $\bar{\alpha}_0 = \bar{\alpha}_0^\vee$ ROOT, WHICH HAS A FINITE PART $-\theta$ LIN. DEPT. ON THE $\{\bar{\alpha}_i\}$ $i \in \{1,\dots,r\}$ AND AN AFFINE PART δ WITH ZERO NORM.

IN FACT, $(a_0 = a_0^\vee \equiv 1)$

- $$\delta = \sum_{i=0}^{r} a_i \bar{\alpha}_i = \delta - \cancel{\theta} + \cancel{\sum_{i=1}^{r} a_i \bar{\alpha}_i} \checkmark$$

$$= \sum_{i=0}^{r} a_i^\vee \bar{\alpha}_i^\vee = \delta - \cancel{\theta} + \cancel{\sum_{i=1}^{r} a_i^\vee \bar{\alpha}_i^\vee} \checkmark$$

$\therefore$ FORMALLY, THE PURE IMAG. ROOT δ WITH $\langle \delta, \delta \rangle = 0$ PLAYS THE ROLE OF THE "AFFINE HWS."

$\Rightarrow$ ADJOINT REP. IS <u>NOT</u> A HIGHEST WEIGHT IRREP.!

- DUAL COXETER #: $g = \sum_{i=0}^{r} a_i^\vee$ (p.1)

<u>ex:</u> $(SU(N))_{\mathfrak{h}}$: $\hat{A}_{0j} = \hat{A}_{j0} = -\langle \bar{\alpha}_j, \theta \rangle = -(0 \dots 0 \overset{j^{TH}\text{ PLACE}}{1} 0 \dots 0) \cdot (1 0 \cdots 0 1)_{DYN}$

OR $\widehat{SU(N)}$ (WE HAVE NOT FIXED $\hat{\mathfrak{h}} \Leftrightarrow \mathfrak{h}$) $= -\delta_{j,1} - \delta_{j,N-1}$

$$\hat{A}_{ij} \Rightarrow \left[\begin{array}{c|cccccc} 2 & -1 & 0 & \cdots & & 0 & -1 \\ \hline -1 & 2 & -1 & & & & 0 \\ 0 & -1 & 2 & -1 & & & \\ 0 & 0 & -1 & \ddots & & & \vdots \\ \vdots & \vdots & & & \ddots & -1 & 0 \\ 0 & & & & -1 & 2 & -1 \\ -1 & 0 & & \cdots & 0 & -1 & 2 \end{array}\right]$$

= HOPPING HAMILTONIAN w/ PBC!

ZERO (TRANSL) MODE : $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ \\ \vdots \\ 1 \end{bmatrix}$

DYNKIN DIAGRAM:

* EXCEPTIONAL CASE: $\widehat{SU(2)}$: $A = [2]$; $\theta = \bar{\alpha} = \bar{\alpha}^\vee \rightarrow \hat{A}_{01} = -1 \cdot (2) \Rightarrow \hat{A} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

DYNKIN: $\hat{A}_{r=1}$

SEE DiF/M/S p. 563 FOR FULL SET OF AFFINE DYNKIN DIAGRAMS.

14.1.5 FUNDAMENTAL WEIGHTS

WE HAVE ALREADY EST. THE FORM OF AFFINE WEIGHTS $\hat{\lambda} = (\lambda\,;\, k_\lambda\,;\, n_\lambda)$

WITH $\langle \hat{\lambda}, \hat{\mu} \rangle = \langle \lambda, \mu \rangle + k_\lambda n_\mu + k_\mu n_\lambda$

FUND. WEIGHTS: ASSUMED TO BE EIGENSTATES OF L_0 w/ ZERO EIGENVALUE:

① $\hat{\omega}_0 \equiv (0\,;\,1\,;\,0)$ "BASIC FUND. WEIGHT"

② $\hat{\omega}_i \equiv (\omega_i\,;\,a_i^\vee\,;\,0)\,,\quad i \in \{1,\dots,r\}$

FROM p. 6, THE DUAL SIMPLE COROOTS ARE

① $\bar{\alpha}_0^\vee \equiv (-\theta\,;\,0\,;\,1) = -\theta + \delta$

② $\bar{\alpha}_i^\vee \equiv (\bar{\alpha}_i^\vee\,;\,0\,;\,0)\,,\quad i \in \{1,\dots,r\}$

CHECK:

$\langle \hat{\omega}_0, \bar{\alpha}_0^\vee \rangle = 1$ ✓ ; $\langle \hat{\omega}_i, \bar{\alpha}_0^\vee \rangle = -a_i^\vee + a_i^\vee = 0$ ✓ $\quad i \in \{1 — r\}$

$i \in \{1 — r\}$: $\langle \hat{\omega}_0, \bar{\alpha}_i^\vee \rangle = 0$ ✓ ; $\langle \hat{\omega}_i, \bar{\alpha}_j^\vee \rangle = \delta_{ij}$ ✓

LET $\omega_i \equiv (\omega_i\,;\,0\,;\,0)$

THEN $\hat{\omega}_i = a_i^\vee \hat{\omega}_0 + \omega_i\,,\quad i \in \{1,\dots,r\}$

SCALAR PRODUCTS: FUND. WEIGHTS

① $\langle \hat{\omega}_0, \hat{\omega}_0 \rangle = \langle \hat{\omega}_0, \hat{\omega}_i \rangle = 0\,,\quad i \in \{1,\dots,r\}$ ($\hat{\omega}_0$ IS IMAGINARY)

② $\langle \hat{\omega}_i, \hat{\omega}_j \rangle = \langle \omega_i, \omega_j \rangle = F_{ij}\,,\quad i,j \in \{1,\dots,r\}$

EXPANSION OF AFFINE WEIGHTS; LEVEL k

PREV., WE WROTE $\hat{\lambda} = (\lambda(H_0^1),\dots,\lambda(H_0^r)\,;\,k_\lambda\,;\,n_\lambda)$ (p. 5)

NOW, WE EXPRESS $\hat{\lambda}$ IN TERMS OF AFFINE DYNKIN LABELS $\hat{\lambda} \equiv [\lambda_0, \lambda_1, \dots, \lambda_r]$

(THIS NOTATION DOES NOT TRACK THE L_0 EIGENVALUE)

$$\hat{\lambda} \equiv \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + \underbrace{\ell}_{-\text{GRADE}}\delta \quad ;\ \ell \in \mathbb{R}$$

(IN PRINCIPLE, L_0 E'VALUE MIGHT NOT BE QUANTIZED $\to \infty$-D L.A.)

$$= \lambda_0 \hat{\omega}_0 + \sum_{i=1}^{r} \lambda_i \omega_i + \sum_{i=1}^{r} \underbrace{a_i^\vee}_{\text{COMARKS}} \underbrace{\lambda_i}_{\text{DYNKIN}} \hat{\omega}_0 + \ell\delta$$

$$= (\lambda_0 + \langle \lambda, \theta \rangle)\,\hat{\omega}_0 + \underbrace{\lambda}_{\text{FINITE WEIGHT COMPONENT}} + \ell\delta$$

WE DEFINE THE LEVEL k VIA $\quad k \equiv \hat{\lambda}(\hat{k}) = \lambda_0 + \langle \lambda, \theta \rangle = \sum_{i=0}^{r} a_i^\vee \lambda_i$

EIGENVALUE OF $\hat{k}$ — SINCE $a_0 = a_0^\vee = 1$ (p. 6-7)

$\therefore \quad \lambda_0 = k - \langle \lambda, \theta \rangle$ ZEROTH DYNKIN LABEL IN TERMS OF $\{\lambda_1, \dots, \lambda_r\}$ AND k

$\therefore \quad \hat{\lambda} = \lambda + k\hat{\omega}_0 + \ell\delta$ GENERIC AFFINE WEIGHT ; $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \neq \sum_{i=1}^{r} \lambda_i \hat{\omega}_i$!

NOTE: ANOTHER WAY TO OBTAIN THE $\hat{k}$ EIGENVALUE OF $\hat{\lambda}$:

$$k = \hat{\lambda}(\hat{k}) = \langle \hat{\lambda}, \delta \rangle = \left\langle \sum_{i=0}^{r} \lambda_i \hat{\omega}_i , \sum_{i=0}^{r} a_i^\vee \bar{\alpha}_i^\vee \right\rangle + \ell \cancel{\langle \delta, \delta \rangle}^{\,0} = \sum_{i=0}^{r} a_i^\vee \lambda_i$$

"HIGHEST WEIGHT" IN ADJ. REP. (IN FORMAL SENSE ONLY) — p. 7 ; DEF'N OF δ, p. 7 ; COMARKS ; DYNKIN (AS USUAL)

DYNKIN LABELS: $[\lambda_0 \lambda_1 \cdots \lambda_r] \Leftrightarrow \hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + \ell\delta \overset{(?)}{=} \sum_{i=0}^{r} k_i^\vee \bar{\alpha}_i^\vee + k\hat{\omega}_0$

(A.) FUND. WEIGHTS (ALL HAVE $\ell = 0$) — ℓ - GRADE

$\hat{\omega}_0 = [1\,0\,0\,0 \cdots 0]$ $\quad k = 1, \ell = 0$

$\hat{\omega}_1 = [0\,1\,0\,0 \cdots 0]$ $\quad k = a_1^\vee, \ell = 0$

$\hat{\omega}_2 = [0\,0\,1\,0 \cdots 0]$ $\quad k = a_2^\vee, \ell = 0$

$\vdots \qquad \vdots$

$\hat{\omega}_r = [0\,0\,0\,0 \cdots 1]$ $\quad k = a_r^\vee, \ell = 0$

PROOF: i) $\hat{\omega}_0 = \cancel{\lambda}^{\,0} + k\hat{\omega}_0 \; ; \rightarrow k = 1 = \lambda_0 + \cancel{\langle \lambda, \theta \rangle}^{\,0}$ ✓

ii) $\hat{\omega}_i = \lambda + k\hat{\omega}_0 = \omega_i + a_i^\vee \hat{\omega}_0$

$\rightarrow k = a_i^\vee = \lambda_0 + \langle \omega_i, \theta \rangle = \lambda_0 + a_i^\vee$

$\therefore \lambda_0 = 0$ ✓

(B.) SIMPLE ROOTS : $\hat{\bar{\alpha}}_0 = -\theta + \delta \quad ; \; \hat{\bar{\alpha}}_0^\vee = \bar{\alpha}_0$

$\hat{\bar{\alpha}}_i = \bar{\alpha}_i \;, \; i \in \{1, \dots, r\} \; ; \; \hat{\bar{\alpha}}_i^\vee = \bar{\alpha}_i^\vee$

$\lambda_j = \langle \hat{\lambda}, \bar{\alpha}_j^\vee \rangle$

$\bar{\alpha}_0 = [\hat{A}_{00}\; \hat{A}_{01} \cdots \hat{A}_{0r}]$ $\quad k = 0, \lambda = 1$

$\bar{\alpha}_i = [\hat{A}_{i0}\; \hat{A}_{i1} \cdots \hat{A}_{ir}]$ $\quad k = 0, \ell = 0$

$i \in \{1, \dots, r\}$

AFFINE CARTAN MATRIX ROWS!

PROOF: i) $\bar{\alpha}_0 = (-\theta; 0; 1) \Rightarrow \langle \bar{\alpha}_0, \bar{\alpha}_0 \rangle = 2 = \hat{A}_{00}$

ii) $\langle \bar{\alpha}_0, \bar{\alpha}_j^\vee \rangle = -\sum_{i=1}^{r} a_i A_{ij} = \hat{A}_{0j} \;, \; j \in \{1, \dots, r\}$

iii) $\langle \bar{\alpha}_i, \bar{\alpha}_0^\vee \rangle = -\sum_{j=1}^{r} A_{ij} a_j^\vee = \hat{A}_{i0} \;, \; i \in \{1, \dots, r\}$

iv) $\langle \bar{\alpha}_i, \bar{\alpha}_j^\vee \rangle = A_{ij} = \hat{A}_{ij} \;, \; i, j \in \{1, \dots, r\}$

(p. 6)

(c.) Affine Weyl Vector

$$\hat{\rho} \equiv \sum_{i=0}^{r} \hat{\omega}_i = [1\ 1\ 1 \cdots 1]_{k=g,\ \ell=0}$$

PROOF: $k = 1 + \sum_{i=1}^{r} a_i^\vee = \sum_{i=0}^{r} a_i^\vee = g$ (p.7)

- Cannot be written as a sum over the (Infinite) Set of Positive Roots, Divided by Two. (c.f. GEO 4A, p.2)

"Dominant" Weight $\equiv$ All Dynkin Labels $\lambda_i \geq 0$ (Non-negative integers $\mathbb{Z}$)

But: $\lambda_0 = k - \langle \lambda, \theta \rangle$ $\Rightarrow$ Possibilities for Dominant Weights Constrained by the Level k!

Set of All Dominant Weights at Level k: $\equiv P_+^k$

Note that if $\hat{\lambda} \in P_+^k$, then finite part $\lambda \in P_+^k$ (but not vice-versa)

14.1.6 Affine Weyl Group

Def'n: Affine Weyl Reflection [Real Root $\hat{\alpha}$]

$$\hat{\alpha} \equiv (\alpha; 0; m)$$
$$\hat{\alpha}^\vee = (\alpha^\vee; 0; m\tfrac{2}{|\alpha|^2})$$
$$\hat{\lambda} \equiv (\lambda; k; n)$$

$$\boxed{S_{\hat{\alpha}}\hat{\lambda} \equiv \hat{\lambda} - \langle \hat{\lambda}, \hat{\alpha}^\vee \rangle \hat{\alpha}}$$

Explicitly,

$$S_{\hat{\alpha}}\hat{\lambda} = (\lambda; k; n) - \left[\langle \lambda, \alpha^\vee \rangle + km\tfrac{2}{|\alpha|^2}\right](\alpha; 0; m)$$

$$= \left(\lambda - \langle \lambda, \alpha^\vee \rangle \alpha - km\alpha^\vee; k; n - \left[\langle \lambda, \alpha^\vee \rangle + km\tfrac{\langle \alpha^\vee, \alpha^\vee \rangle}{2}\right]m\right)$$

$$= \left(S_\alpha[\lambda + km\alpha^\vee]; k; n - m\langle \lambda, \alpha^\vee \rangle - \tfrac{k}{2}\langle m\alpha^\vee, m\alpha^\vee \rangle\right)$$

$$= \left(S_\alpha[\lambda + km\alpha^\vee]; k; n + \tfrac{1}{2k}\left[|\lambda|^2 - |\lambda + km\alpha^\vee|^2\right]\right)$$

Properties:

① $S_{\hat{\alpha}}\hat{\alpha} = (-\alpha; 0; m - 2m) = -\hat{\alpha}$ ✓

② $S_{\hat{\alpha}}\hat{\delta} = (0; 0; 1) = \hat{\delta}$ INVT. ✓

③ $S^2_{\hat{\alpha}}\hat{\lambda} = \Big(S_\alpha\big[S_\alpha(\lambda + km\alpha^\vee) + km\alpha^\vee\big]; k; n - m\langle \lambda, \alpha^\vee \rangle - \tfrac{k}{2}\langle m\alpha^\vee, m\alpha^\vee \rangle$
$\qquad + m\langle \lambda + km\alpha^\vee, \alpha^\vee \rangle - \tfrac{k}{2}\langle m\alpha^\vee, m\alpha^\vee \rangle\Big)$

$$= \left(\lambda + \cancel{km\alpha^\vee} - \cancel{km\alpha^\vee}; k; n - \cancel{m\langle \lambda, \alpha^\vee \rangle} - k\langle \cancel{m\alpha^\vee}, m\alpha^\vee \rangle + \cancel{m\langle \lambda}\cancel{+ km\alpha^\vee}, \alpha^\vee \rangle\right)$$

$$= \hat{\lambda}$$

$\therefore S^2_{\hat{\alpha}} = 1$ ✓

Decomposition: Finite Weyl Refl. × Coroot Translation

$$t_{\alpha^\vee} \equiv S_\alpha S_{\widehat{\alpha+\delta}} = S_{\widehat{-\alpha+\delta}} S_\alpha$$

$$S_\alpha S_{\widehat{\alpha+\delta}} \bullet \hat{\lambda} = (\lambda + K\alpha^\vee; K; n - \langle \lambda, \alpha^\vee \rangle - \tfrac{K}{2}\langle \alpha^\vee, \alpha^\vee \rangle)$$

$$S_{\widehat{-\alpha+\delta}} S_\alpha \bullet \hat{\lambda} = (S_{-\alpha}[S_\alpha \lambda - K\alpha^\vee]; K; n + \langle S_\alpha \lambda, \alpha^\vee \rangle - \tfrac{K}{2}\langle \alpha^\vee, \alpha^\vee \rangle)$$

$$= (\lambda + K\alpha^\vee; K; n - \langle \lambda, \alpha^\vee \rangle - \tfrac{K}{2}\langle \alpha^\vee, \alpha^\vee \rangle)$$

$$\therefore \; t_{\alpha^\vee} = S_\alpha S_{\widehat{\alpha+\delta}} = S_{\widehat{-\alpha+\delta}} S_\alpha \; ; \; t_{\alpha^\vee}\hat{\lambda} = (\lambda + K\alpha^\vee; K; n - \langle \lambda, \alpha^\vee \rangle - \tfrac{K}{2}\langle \alpha^\vee, \alpha^\vee \rangle)$$

$$S_{\hat{\alpha}} = S_\alpha (t_{\alpha^\vee})^m \qquad \text{FOR } \hat{\alpha} = (\alpha; 0; m)$$

COROOT TRANS. (INVT. SUBGROUP OF $\hat{W}_{EYL}$)

COROOT TRANS $t_{\alpha^\vee}$:

① $t_{\alpha^\vee} t_{\beta^\vee} = t_{\alpha^\vee + \beta^\vee}$ (ABELIAN) COMPOSITION OF TRANSLATIONS

PROOF:

$$t_{\alpha^\vee} t_{\beta^\vee} \hat{\lambda} = t_{\alpha^\vee}(\lambda + K\beta^\vee; K; n - \langle \lambda, \beta^\vee \rangle - \tfrac{K}{2}\langle \beta^\vee, \beta^\vee \rangle)$$

$$= (\lambda + K[\alpha^\vee + \beta^\vee]; K; n - \langle \lambda, \beta^\vee \rangle - \tfrac{K}{2}\langle \beta^\vee, \beta^\vee \rangle - \langle \lambda + \beta^\vee K, \alpha^\vee \rangle - \tfrac{K}{2}\langle \alpha^\vee, \alpha^\vee \rangle)$$

$$= (\lambda + K[\alpha^\vee + \beta^\vee]; K; n - \langle \lambda, [\alpha^\vee + \beta^\vee] \rangle - \tfrac{K}{2}\langle \alpha^\vee + \beta^\vee, \alpha^\vee + \beta^\vee \rangle) = t_{\alpha^\vee + \beta^\vee} \checkmark$$

$$\Rightarrow (t_{\alpha^\vee})^m = t_{m\alpha^\vee}$$

② $S_\alpha t_{\beta^\vee} S_\alpha = t_{S_\alpha \beta^\vee}$

PROOF:

$$S_\alpha t_{\beta^\vee} S_\alpha \hat{\lambda} = S_\alpha (S_\alpha \lambda + K\beta^\vee; K; n - \langle S_\alpha \lambda, \beta^\vee \rangle - \tfrac{K}{2}\langle \beta^\vee, \beta^\vee \rangle)$$

$$= (\lambda + K S_\alpha \beta^\vee; K; n - \langle \lambda, S_\alpha \beta^\vee \rangle - \tfrac{K}{2}\langle S_\alpha \beta^\vee, S_\alpha \beta^\vee \rangle) = t_{S_\alpha \beta^\vee} \checkmark$$

③ FOR $w \in \hat{W}$ AFFINE WEYL GROUP, $w t_{\alpha^\vee} w^{-1} = t_{w\alpha^\vee}$

$w \in \hat{W}$ $\alpha^\vee$: FINITE COROOT.

$Q^\vee$: GROUP OF COROOT LAT. TRANS. IS INVT. UNDER INNER AUTOMORPHISM $w \in \hat{W}$ (GEO 4B, p. 8)

$$\Rightarrow w' t_{\alpha^\vee} w t_{\beta^\vee} = w' w \, t_{w^{-1}\alpha^\vee + \beta^\vee}$$

BASIS

$S_i \equiv S_{\bar{\alpha}_i}$, $i \in \{1, \dots, r\}$ SAME AS FINITE GEN.

$S_0 \equiv S_{\bar{\alpha}_0} = S_{-\theta} t_{-\theta} = S_\theta t_{-\theta}$

$\Big\}$ $r+1$ SIMPLE WEYL REFL.S

SIMPLE WEYL REFL.S IN TERMS OF DYNKIN LABELS

① $S_0\hat{\lambda} = \hat{\lambda} - \lambda_0 \bar{\alpha}_0 = \left[-\lambda_0, \lambda_1 - \lambda_0 \hat{A}_{01}, \dots, \lambda_r - \lambda_0 \hat{A}_{0r}\right]_{n' = \underset{\substack{\Uparrow \\ \text{GRADE}}}{n} - \lambda_0}^{k' = k}$

② $S_i\hat{\lambda} = \hat{\lambda} - \lambda_i \bar{\alpha}_i = \left[\lambda_0 - \lambda_i \hat{A}_{i0}, \lambda_1 - \lambda_i \hat{A}_{i1}, \dots, \lambda_r - \lambda_i \hat{A}_{ir}\right]_{n' = n}^{k' = k}, \quad i \in \{1, \dots, r\}$

ex): $\widehat{SU(2)}$ $\quad \hat{A} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$S_0\hat{\lambda} = \left[-\lambda_0, \lambda_1 + 2\lambda_0\right]_{n' = n - \lambda_0}^{k' = k} \quad ; \quad S_1\hat{\lambda} = \left[\lambda_0 + 2\lambda_1, -\lambda_1\right]_{n' = n}^{k' = k}$

ex): $\widehat{SU(3)}$ $\quad \hat{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$S_0\hat{\lambda} = \left[-\lambda_0, \lambda_1 + \lambda_0, \lambda_2 + \lambda_0\right]_{n' = n - \lambda_0}^{k' = k}$

$S_1\hat{\lambda} = \left[\lambda_0 + \lambda_1, -\lambda_1, \lambda_2 + \lambda_1\right]_{n' = n}^{k' = k}$

$S_2\hat{\lambda} = \left[\lambda_0 + \lambda_2, \lambda_1 + \lambda_2, -\lambda_2\right]_{n' = n}^{k' = k}$

NOTE ALSO THAT, SINCE $\hat{A}_{0j} = \langle \bar{\alpha}_0, \bar{\alpha}_j^\vee \rangle = -\lambda_j^{(\theta)}$ FINITE DYNKIN COEFF. FOR HW ROOT θ $\quad j \in \{1 - r\}$

$\hat{A}_{i0} = \langle \bar{\alpha}_i, \bar{\alpha}_0 \rangle = -\lambda_i^{(\theta)} \cdot \frac{|\bar{\alpha}_i|^2}{2} \quad i \in \{1 - r\}$

$\therefore$ ① $S_0\hat{\lambda} = \left[-\lambda_0, \lambda_1 + \lambda_0 \lambda_1^{(\theta)}, \dots, \lambda_r + \lambda_0 \lambda_r^{(\theta)}\right]_{n' = n - \lambda_0}^{k' = k}$

② $S_i\hat{\lambda} = \left[\lambda_0 + \lambda_i \lambda_i^{(\theta)} \frac{|\bar{\alpha}_i|^2}{2}, \dots \right]_{n' = n}^{k' = k}$

Simple Affine Weyl Refl.

① $S_0 \hat{\lambda} = S_\theta(\lambda - k\theta; k; n - \langle\lambda,\theta\rangle - \frac{k}{2}\langle\theta,\theta\rangle) = (S_\theta\lambda + k\theta; k; n + \langle\lambda,\theta\rangle - k)$

$= (\lambda - \langle\lambda,\theta\rangle\theta + k\theta; k; n + \langle\lambda,\theta\rangle - k)$

$\hat{\lambda} \equiv (\lambda; k; n)$ ← Finite Dynkin labels.

$\lambda_0 \equiv k - \langle\lambda,\theta\rangle$ ∴ $S_0\hat{\lambda} = (\lambda + \lambda_0\theta; k; n - \lambda_0)$ (p. 9) ⇒ $S_0\hat{\omega}_0 = \hat{\omega}_0 - \bar{\alpha}_0$; $S_0\hat{\omega}_i = \hat{\omega}_i$, $i \in \{1,\dots,r\}$

$= \hat{\lambda} - \lambda_0\bar{\alpha}_0$

More general: $S_{\bar{\alpha}_i}\hat{\omega}_j = \hat{\omega}_j - \delta_{ji}\bar{\alpha}_i$, $i,j \in \{0,1,\dots,r\}$

② $S_i\hat{\lambda} = (S_{\bar{\alpha}_i}\lambda; k; n)$

$= \hat{\lambda} - \lambda_i\bar{\alpha}_i$, $i \in \{1,\dots,r\}$

Simple Weyl Refl. on simple roots:

a.) $S_0\bar{\alpha}_0 = (-\theta + \langle+\theta,\theta\rangle\theta; 0; 1 - \langle\theta,\theta\rangle) = (\theta; 0; -1) = -\bar{\alpha}_0$ ✓

b.) $S_0\bar{\alpha}_i = (\bar{\alpha}_i - \langle\bar{\alpha}_i,\theta\rangle\theta; 0; \langle\bar{\alpha}_i,\theta\rangle) = (\bar{\alpha}_i + \hat{A}_{i0}\theta; 0; -\hat{A}_{i0})$, $i \in \{1,\dots,r\}$

c.) $S_i\bar{\alpha}_0 = (-\theta + \langle\theta,\bar{\alpha}_i^\vee\rangle\bar{\alpha}_i; 0; 1) = (-\theta - \hat{A}_{0i}\bar{\alpha}_i; 0; 1)$, $i \in \{1,\dots,r\}$

d.) $S_i\bar{\alpha}_j = (\bar{\alpha}_j - \langle\bar{\alpha}_j,\bar{\alpha}_i^\vee\rangle\bar{\alpha}_i; 0; 0) = (\bar{\alpha}_j - \hat{A}_{ji}\bar{\alpha}_i; 0; 0)$, $i,j \in \{1,\dots,r\}$

Consistent w/ $S_{\bar{\alpha}_i}\bar{\alpha}_j = \bar{\alpha}_j - \hat{A}_{ji}\bar{\alpha}_i$; $i,j \in \{0,1,2,\dots,r\}$. ✓

Affine Weyl Chambers and Orbits

Affine Weyl Chamber: $\hat{C}_w \equiv \{\hat{\lambda} \mid \langle w\hat{\lambda},\bar{\alpha}_i\rangle \geq 0,\ i \in \{0,1,\dots,r\}$ and $w \in \hat{W}\}$

↑ i.e. all Dynkin labels in Fund. Chamber $\in \mathbb{Z}_+$

$w = 1$: Fund. Chamber; $\hat{\lambda}_{\text{Fund. Ch.}} = \sum_{i=0}^{r} \lambda_i\omega_i + \ell\delta$, $\lambda_i \geq 0$ and $\ell \in \mathbb{R}$.

since $S_{\hat{\alpha}}^2 = 1$ (p. 11),

$\langle S_{\hat{\alpha}}\hat{\lambda}, S_{\hat{\alpha}}\hat{\lambda}\rangle = \langle\hat{\lambda},\hat{\lambda}\rangle$ ⇒ All states in a Weyl Orbit have same norm.

$\hat{W}$-orbit: <u>infinite</u> # of weights with a unique representative in the Fund. Chamber.

14.1.7 EXAMPLES

I. $\widehat{SU(2)}$

p.7: $\hat{A} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$\bar{\alpha}_0 = (-\bar{\alpha}_1; 0; 1)_{k=0\ \ell=1} = [2 \quad -2] = \bar{\alpha}_0^\vee$

$\bar{\alpha}_1 = (\bar{\alpha}_1; 0; 0)_{k=0\ \ell=0} = [-2 \quad 2] = \bar{\alpha}_1^\vee$

$|\bar{\alpha}_0|^2 = |\bar{\alpha}_1|^2 = 2$; $\langle \bar{\alpha}_0, \bar{\alpha}_1 \rangle = \hat{A}_{01} = -2$

FINITE SU(2) ROOT-WEIGHT LATTICE:

$\hat{\omega}_0 = [1 \quad 0]_{k=1, \ell=0}$

$\hat{\omega}_1 = [0 \quad 1]_{k=1, \ell=0}$

$\langle \hat{\omega}_0, \hat{\omega}_0 \rangle = \langle \hat{\omega}_0, \hat{\omega}_1 \rangle = 0$; $\langle \hat{\omega}_1, \hat{\omega}_1 \rangle = \langle \omega_1, \omega_1 \rangle = \frac{1}{2}$

$\langle \omega_1, \bar{\alpha}_1 \rangle = 1$

$\Rightarrow \omega_1 = \frac{1}{2}\bar{\alpha}_1$

ROOTS: $\hat{\Delta} = \{ \pm\bar{\alpha}_1 , \pm\bar{\alpha}_1 + n\delta, n\delta \mid n \in \mathbb{Z}, n \neq 0 \}$

REAL ROOT STRINGS TERMINATE! AFFINE MWDF (p. 28)

$\delta = \bar{\alpha}_0 + \bar{\alpha}_1$ (p.7)

$\Rightarrow \hat{\Delta} = \{ n\bar{\alpha}_0 + m\bar{\alpha}_1 \mid |n-m| \leq 1 , n, m \in \mathbb{Z} \text{ AND } (n,m) \neq (0,0) \}$

AFFINE! IN A SIMPLE LA, IF β IS A ROOT, THEN 2β IS NOT A ROOT; HERE $n\delta = n(\bar{\alpha}_0 + \bar{\alpha}_1)$ IS A ROOT FOR ANY $n \in \mathbb{Z}, n \neq 0$.

AFFINE WEYL GROUP:

$s_0 \hat{\lambda} = (\lambda_1\omega_1 + \lambda_0 2\omega_1 ; k; n - \lambda_0) = (\lambda_1'\omega_1 ; k'; n')$; $\lambda_1' = 2\lambda_0 + \lambda_1$, $k' = k$, $n' = n - \lambda_0$

$k = \lambda_0 + \langle \lambda_1\omega_1, \bar{\alpha}_1 \rangle = \lambda_0 + \lambda_1$ $\qquad \Rightarrow \lambda_0' = \lambda_0 + \lambda_1 - (2\lambda_0 + \lambda_1) = -\lambda_0$

$s_1 \hat{\lambda} = (-\lambda_1\omega_1 ; k; n) = (\lambda_1'\omega_1 ; k'; n')$; $\lambda_1' = -\lambda_1$, $k' = k$, $n' = n$

$\Rightarrow \lambda_0' = \lambda_0 + \lambda_1 - (-\lambda_1) = \lambda_0 + 2\lambda_1$

$$\therefore\ S_0\,[\lambda_0,\lambda_1]^{k=\lambda_0+\lambda_1}_{\ell=n} = [-\lambda_0,\ 2\lambda_0+\lambda_1]^{k=\lambda_0+\lambda_1}_{\ell=n-\lambda_0} = (\lambda_1\omega_1+\lambda_0\bar{\alpha}_1;\,k;\,n-\lambda_0)$$

$$\therefore\ S_1\,[\lambda_0,\lambda_1]^{k=\lambda_0+\lambda_1}_{\ell=n} = [\lambda_0+2\lambda_1,\ -\lambda_1]^{k=\lambda_0+\lambda_1}_{\ell=n} = (-\lambda_1\omega_1;\,k;\,n)$$

$$\therefore\ S_0S_1\,[\lambda_0,\lambda_1]^{k=\lambda_0+\lambda_1}_{\ell=n} = [-\lambda_0-2\lambda_1,\ 3\lambda_1+2\lambda_0]^{k=\lambda_0+\lambda_1}_{\ell=n-\lambda_0-2\lambda_1} = ([3\lambda_1+2\lambda_0]\omega_1;\,k;\,n-[\lambda_0+2\lambda_1])$$

SINCE $\lambda_0 = k-\lambda_1$

$$S_0\,[\lambda_0,\lambda_1]_{\ell=n} = [\lambda_1-k,\ 2k-\lambda_1]_{\ell=n-\lambda_0}$$

$$S_1\,[\lambda_0,\lambda_1]_{\ell=n} = [\lambda_1+k,\ -\lambda_1]_{\ell=n}$$

$$S_0S_1\,[\lambda_0,\lambda_1]_{\ell=n} = [-\lambda_1-k,\ \lambda_1+2k]_{\ell=n-\lambda_1-k} \Rightarrow t_{\bar{\alpha}_1^\vee} \equiv S_0S_1$$

SINCE THE FINITE PART
$\lambda_1\omega_1 \to \lambda_1\omega_1 + 2k\omega_1$
$= \lambda_1\omega_1 + k\bar{\alpha}_1$ ✓

$$\Rightarrow\ t_{\bar{\alpha}_1^\vee}\,\hat{\lambda} = (\lambda_1\omega_1 + k\bar{\alpha}_1^\vee;\,k;\,n - \langle\lambda_1\omega_1,\bar{\alpha}_1^\vee\rangle - \tfrac{k}{2}\langle\bar{\alpha}_1^\vee,\bar{\alpha}_1^\vee\rangle)$$ ✓

<u>GENERIC WEYL REFL.</u>: $S_{(\bar{\alpha}_1;0;m)} = S_{\bar{\alpha}_1}\,(t_{\bar{\alpha}_1^\vee})^m$

$$= S_1\,(S_0S_1)^m$$

GEO 4a, p. 2: "COXETER GROUP" PROPERTY:

$$(S_0S_1)^{m_{01}} = 1\ ,\quad m_{01} = \frac{\pi}{\pi-\theta_{01}} \Rightarrow \infty \quad \text{FOR } \theta_{01}\to\pi$$

AFFINE WEYL CHAMBERS OF $\widehat{SU(2)}$: CFT BOOK p. 570.

ACTION ON ROOTS: $\hat{\alpha} = [-\lambda_1,\lambda_1]^{k=0}_{\ell=n}$

$$S_0\,\hat{\alpha} = [\lambda_1,-\lambda_1]^{k=0}_{\ell=n+\lambda_1}$$

$$S_1\,\hat{\alpha} = [\lambda_1,-\lambda_1]^{k=0}_{\ell=n}$$

$$S_0S_1\,\hat{\alpha} = [-\lambda_1,\lambda_1]^{k=0}_{\ell=n-\lambda_1}$$

AFFINE WEYL GROUP
ACTION: ROOT SYSTEM

II. $\widehat{SU(3)}$

$$\hat{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} \bar{\alpha}_0 &= (-\bar{\alpha}_1 - \bar{\alpha}_2 \,;\, 0 \,;\, 1) &&= [2 \;\; -1 \;\; -1] \\ \bar{\alpha}_1 &= (\bar{\alpha}_1 \,;\, 0 \,;\, 0) &&= [-1 \;\; 2 \;\; -1] \\ \bar{\alpha}_2 &= (\bar{\alpha}_2 \,;\, 0 \,;\, 0) &&= [-1 \;\; -1 \;\; 2] \end{aligned}$$

AFFINE SIMPLE WEYL REFL. : (p. 12A)

$$S_0 \hat{\lambda} = [-\lambda_0, \lambda_0 + \lambda_1, \lambda_0 + \lambda_2]^{k'=k}_{n'=n-\lambda_0}$$

$$S_1 \hat{\lambda} = [\lambda_0 + \lambda_1, -\lambda_1, \lambda_1 + \lambda_2]^{k'=k}_{n'=n}$$

$$S_2 \hat{\lambda} = [\lambda_0 + \lambda_2, \lambda_1 + \lambda_2, -\lambda_2]^{k'=k}_{n'=n}$$

e.g., LET $\hat{\lambda} = (\alpha \,;\, 0 \,;\, n)$, A ROOT

$$\Rightarrow \lambda_0 = \langle \hat{\lambda}, \bar{\alpha}_0^\vee \rangle = -\langle \lambda, \theta \rangle = -\sum_{i=1}^{r} \lambda_i a_i^\vee \,; \text{ FOR } SU(N),\ a_i^\vee = 1 \ \forall\, i \in \{1, \dots, N-1\}$$

$$\therefore\ \hat{\lambda} = [-\lambda_1 - \lambda_2, \lambda_1, \lambda_2]^0_n$$

$$\Rightarrow S_0 \hat{\lambda} = [\lambda_1 + \lambda_2, -\lambda_2, -\lambda_1]^0_{n+\lambda_1+\lambda_2}$$

$$\therefore\ S_0 \bar{\alpha}_1 = [1, 1, -2]^0_1 \quad , \quad S_0 \bar{\alpha}_2 = [1, -2, 1]^0_1$$

$$= (-\bar{\alpha}_2 \,;\, 0 \,;\, 1) \qquad\qquad = (-\bar{\alpha}_1 \,;\, 0 \,;\, 1)$$

$\underline{\underline{S_0}}$:

14.3 Highest Weight Representations

HWS: $|\hat{\lambda}\rangle = |(\lambda;\kappa;n)\rangle$ such that

$$E_0^{\alpha}|\hat{\lambda}\rangle = 0 \quad , \alpha \in \Delta_+$$
$$E_n^{\pm\alpha}|\hat{\lambda}\rangle = 0 \quad , n>0$$
$$H_n^i\,|\hat{\lambda}\rangle = 0 \quad , n>0$$

UPPER INDEX: NOT A DYNKIN LABEL – SEE p. 2

1. $H_0^i|\hat{\lambda}\rangle = \lambda^i|\hat{\lambda}\rangle, \; i\in\{1,\dots,r\}$
2. $\hat{\kappa}\,|\hat{\lambda}\rangle = \kappa\,|\hat{\lambda}\rangle$

$\Big\}\; \lambda_0 = \kappa - \langle\lambda,\theta\rangle$

3. $L_0|\hat{\lambda}\rangle = 0$ (CONVENTION; CAN BE RE-DEFINED TO ANY VALUE VIA $L_0' \equiv L_0 - \{n^{(\hat{\lambda})};(\hat{\kappa}/\kappa)\tilde{n}^{(\hat{\lambda})}\}$)

$$\Rightarrow |\hat{\lambda}\rangle = |(\lambda;\kappa;0)\rangle = |[\lambda_0,\{\lambda_i\}]\rangle_{n=0}$$

CHEVALLEY BASIS:

$$[H_0^0 = (\hat{\kappa} - \theta^i H_0^i)]|\hat{\lambda}\rangle = (\kappa - \langle\theta,\lambda\rangle)|\hat{\lambda}\rangle = \lambda_0|\hat{\lambda}\rangle$$
$$H_0^{\bar{\alpha}_i^\vee}|\hat{\lambda}\rangle = \lambda(\bar{\alpha}_i^\vee)|\hat{\lambda}\rangle = \langle\lambda,\bar{\alpha}_i^\vee\rangle|\hat{\lambda}\rangle = \lambda_i|\hat{\lambda}\rangle$$

ALL STATES IN VERMA MODULE ASSOC. TO $|\hat{\lambda}\rangle$ HAVE $\hat{\kappa}\to\kappa$, FIXED.

p. 28: MWDF FOR WEIGHT STRINGS GENERATED BY REAL ROOTS $\pm\hat{\alpha} = (\pm\alpha;0;\pm n)$: $\Rightarrow E_n^{\alpha}, E_{-n}^{-\alpha}$

$$(m-p)_{\hat{\lambda};\hat{\alpha}} = \langle\hat{\lambda},\hat{\alpha}^\vee\rangle = \frac{2\left[\langle\lambda,\alpha\rangle + \underbrace{\kappa}_{\text{LEVEL OF }\hat{\lambda}}\underbrace{n}_{(-1)\cdot\text{GRADE OF }\hat{\alpha}}\right]}{\langle\alpha,\alpha\rangle}$$

FOR A HWS:

① $(m)_{\hat{\lambda};\bar{\alpha}_0} = \langle\hat{\lambda},\bar{\alpha}_0\rangle = [-\langle\lambda,\theta\rangle + \kappa] = \lambda_0$ # DESC. IN $\bar{\alpha}_0$ CHAIN

② $(m)_{\hat{\lambda};\bar{\alpha}_i} = \langle\hat{\lambda},\bar{\alpha}_i^\vee\rangle = \langle\lambda,\bar{\alpha}_i^\vee\rangle = \lambda_i$ # DESC. IN $\bar{\alpha}_i$ CHAIN
$i\in\{1,\dots,r\}$

$\therefore\; \{\lambda_i\}\in\mathbb{Z}_+ \;,\; i\in\{0,1,\dots,r\}$ FOR $\hat{\lambda}\equiv$ HWS

NOTE: $\langle\lambda,\theta\rangle = \langle\lambda,\sum_{i=1}^{r} a_i^\vee\bar{\alpha}_i^\vee\rangle = \sum_{i=1}^{r} \underbrace{a_i^\vee}_{\text{COMARKS: POS. INTEGERS (GEO 4, p. 48)}}\underbrace{\lambda_i}_{\text{POS. INT. FOR HWS (OR ZERO)}} = (m-p)_{\lambda;\theta}$ = POSITIVE INTEGER

$$\therefore \quad k \in \mathbb{Z}_+ \quad ; \quad k \geq \langle \lambda, \theta \rangle = \sum_{i=1}^{r} a_i^\vee \lambda_i$$

MWDF FOR REAL ROOTS: $\hat{\lambda} = (\lambda; k; 0) = [\lambda_0 \, \lambda_1 \cdots \lambda_r]$ IS A HWS; $\lambda_0 = k - \langle \lambda, \theta \rangle$

① $(m)_{\hat{\lambda};\bar{\alpha}_0} = \langle \hat{\lambda}, \bar{\alpha}_0 \rangle = \lambda_0$ $\qquad \bar{\alpha}_0 \Leftrightarrow E_1^{-\theta}, E_{-1}^{\theta}$

② $(m)_{\hat{\lambda};\bar{\alpha}_i} = \langle \hat{\lambda}, \bar{\alpha}_i^\vee \rangle = \lambda_i \;, \; i \in \{1, \ldots, r\}$ $\qquad \bar{\alpha}_i \Leftrightarrow E_0^{\bar{\alpha}_i}, E_0^{-\bar{\alpha}_i}$

IMPLIED NULL VECTORS:

$$(E_{-1}^{\theta})^{\lambda_0 + 1} \, |\hat{\lambda}\rangle = 0$$

$$(E_0^{-\bar{\alpha}_i})^{\lambda_i + 1} \, |\hat{\lambda}\rangle = 0$$

THESE CONDITIONS DO <u>NOT</u> IMPLY A FINITE HW IRREP., IN CONTRAST TO THE FINITE CASE

<u>MWDF, NULL VECTORS, FINITE IRREPS FOR SIMPLE LIE ALGEBRAS: SU(3) EXAMPLE:</u>

CONSIDER THE (2, 1) IRREP. ON p. 89B OF GEO 4

$(m)_{(2,1);\bar{\alpha}_1} = 2 \quad \Rightarrow \quad (E^{-\bar{\alpha}_1})^3 \, |(2,1)\rangle = 0$

$(m)_{(2,1);\bar{\alpha}_2} = 1 \quad \Rightarrow \quad (E^{-\bar{\alpha}_2})^2 \, |(2,1)\rangle = 0$

$\therefore$ MWDF AND WEYL GROUP IMPLY NULL VECTORS TRUNCATE WEIGHT LATTICE TO FINITE REGION FOR A HWS IRREP. OF A SIMPLE L.A.

IN THE AFFINE CASE, PRESENCE OF THE PURELY IMAGINARY ROOT $\delta = \sum_{i=0}^{r} a_i \bar{\alpha}_i = \sum_{i=0}^{r} a_i^\vee \bar{\alpha}_i^\vee$ IMPLIES THAT A DESC. OF ANY WEIGHT OBTAINED BY SUBTRACTING ANY MULTIPLE OF δ IS ALSO A WEIGHT; δ-STRINGS DO <u>NOT</u> TERMINATE.

(SEE ALSO p. 2B)

Dominant Highest Weight Irreps at Fixed Level k

For fixed k, $k \in \mathbb{Z}_+$ and $k \geq \langle \lambda, \theta \rangle = \sum_{i=1}^{r} \lambda_i a_i^\vee$,
there exist only a finite # of dominant, HW irreps.

<u>ex 1</u>: $SU(N)_{k=1}$

All comarks $a_i^\vee = 1$, $i \in \{0, 1, \ldots, r = N-1\}$

$\Rightarrow$ at $k=1$, dominant irreps given by the fund. weights (p. 9)

- grade

0. $\hat{\omega}_0 = [1\,0 \cdots 0]$ $k=1; \ell=0$ $\equiv$ "The Basic Representation" (Vacuum $\Leftrightarrow$ $\mathbb{I}$ for thy. w/ $k=1$)
1. $\hat{\omega}_1 = [0\,1\,0 \cdots 0]$ $k=1; \ell=0$

$\vdots$

r. $\hat{\omega}_{N-1} = [0\,0\,0 \cdots 0\,1]$ $k=1; \ell=0$

Also true for $Sp(2N)$, where $r=N$: dom. weights = fund. weights $\{\hat{\omega}_0, \hat{\omega}_1, \ldots, \hat{\omega}_N\}$,
since $a_i^\vee = 1$ $\forall i$ in this case as well (CFT book, p. 541-542).

<u>ex 2</u>: $SO(2r+1)_{k=1}$; see also Geo 4 p. 90

CFT book, p. 541:

$\hat{\omega}_0 = [1\,0\,0 \cdots 0]$ $k=1; \ell=0$
$\hat{\omega}_1 = [0\,1\,0 \cdots 0]$ $k=1; \ell=0$
$\hat{\omega}_2 = [0\,0\,1 \cdots 0]$ $k=2; \ell=0$

$\vdots$

$\hat{\omega}_{r-1} = [0\,0\,0 \cdots 0\,1\,0]$ $k=2; \ell=0$
$\hat{\omega}_r = [0\,0\,0 \cdots 0\,0\,1]$ $k=1; \ell=0$

$\Rightarrow$ at $k=1$, only $\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_r$ correspond to HW states in dominant irreps!

<u>Check</u>: comarks (simple LA)

HWS: $\theta = (0\,1\,0 \cdots 0)$ [asym 2nd rank tensor]

$\Rightarrow \theta = \omega_2 = \sum_{j=1}^{r} a_j^\vee \bar{\alpha}_j^\vee$

$\Rightarrow \langle \theta, \omega_i \rangle = a_i^\vee = \langle \omega_2, \omega_i \rangle = F_{2i} = (1\,2\,2 \cdots 2\,1)$ CFT book p. 541 ✓

<u>ex 3</u>: $SU(3)_{k=2}$

We have

$\hat{\omega}_0 = [1\,0\,0]$ $k=1, \ell=0$; $\hat{\omega}_2 = [0\,0\,1]$ $k=1, \ell=0$
$\hat{\omega}_1 = [0\,1\,0]$ $k=1, \ell=0$

$\Rightarrow$ AT $k=2$,

$[2\,0\,0]$, $[1\,1\,0]$
$[0\,2\,0]$, $[1\,0\,1]$
$[0\,0\,2]$, $[0\,1\,1]$

DEF'N: <u>INTEGRABLE REPRESENTATION</u> $\equiv$ REP. DECOMPOSES INTO FINITE IRREPS OF SU(2), FURTHER WRITTEN AS A SUM OF FINITE-DIM. WEIGHT SPACES.

<u>EX. OF INTEGRABLE IRREPS:</u>
- DOMINANT HIGHEST WEIGHT IRREPS
- ADJOINT REP. (NOT A HIGHEST WEIGHT REP.)

<u>CLAIM:</u> DOMINANT HIGHEST-WEIGHT REPS ARE <u>UNITARY</u>.

e.g.,
$$|E^{\alpha}_{-n}|\hat{\lambda}\rangle|^2 = \langle\hat{\lambda}|E^{-\alpha}_{n}E^{\alpha}_{-n}|\hat{\lambda}\rangle = K(E^{\alpha},E^{-\alpha})\langle\hat{\lambda}|[H^{-\alpha}_{0}+\hat{K}n]|\hat{\lambda}\rangle$$
$$= K(E^{\alpha},E^{-\alpha})\langle\hat{\lambda}|[-\langle\alpha,\lambda\rangle+Kn]|\hat{\lambda}\rangle$$
$$= \underset{>0}{K(E^{\alpha},E^{-\alpha})}\,\underset{>0}{\langle\hat{\lambda}|\hat{\lambda}\rangle}\,[\underbrace{(K-\langle\theta,\alpha\rangle)}_{\lambda_0>0} + \underset{\geq 0}{K(n-1)} + \langle\theta-\alpha,\lambda\rangle]$$

$\langle\theta-\alpha,\lambda\rangle$: $\quad \theta-\alpha \equiv \tilde{\alpha} = \sum_i K_i^{(\tilde{\alpha})}\bar{\alpha}_i$
$K_i^{(\tilde{\alpha})} \in \mathbb{Z}^+$ (GEO 1, p. 48: $\tilde{\alpha}$ IS A POSITIVE ROOT)

$$\lambda = \sum_j \lambda_j \omega_j$$

$$\langle\tilde{\alpha},\lambda\rangle = \sum_i K_i^{(\tilde{\alpha})}\frac{|\bar{\alpha}_i|^2}{2}\sum_j \lambda_j\langle\bar{\alpha}_i^{\vee},\omega_j\rangle = \sum_i K_i^{(\tilde{\alpha})}\frac{|\bar{\alpha}_i|^2}{2}\lambda_i \geq 0 \quad (\{\lambda_i\}\in\mathbb{Z}^+)$$

$\therefore\ |E^{\alpha}_{-n}|\hat{\lambda}\rangle|^2 \geq 0$ ✓

<u>HOW TO DERIVE THE STRUCTURE OF A DOMINANT HW IRREP:</u>

APPLY AFFINE MWDF TO OBTAIN DESC. OF THE HIGHEST WEIGHT $\hat{\lambda} = \sum_{j=0}^{n}\lambda_j\hat{\omega}_j + \cancel{k}\hat{\delta}$ $\quad (m-p)_{\hat{\lambda};\bar{\alpha}_i} = \langle\hat{\lambda},\bar{\alpha}_i^{\vee}\rangle = \lambda_i = \sum_j K_j^{(\hat{\lambda})}\hat{A}_{ji}$

$\Uparrow$ AFFINE CARTAN MATRIX.

MULTIPLICITY: GENERALIZED FREUDENTHAL FORMULA
$$\underset{\text{HWS}}{\Pi_{\hat{\lambda}}}(\underset{\text{WEIGHT IN IRREP}}{\hat{\lambda}'}) = \frac{2\sum_{\hat{\alpha}>0}\text{mult.}(\hat{\alpha})\sum_{k=1}^{\infty}\langle\hat{\lambda}'+k\hat{\alpha},\hat{\alpha}\rangle\,\Pi_{\hat{\lambda}}(\hat{\lambda}'+k\hat{\alpha})}{|\hat{\lambda}+\hat{\rho}|^2 - |\hat{\lambda}'+\hat{\rho}|^2}$$

$$\text{FREUDENTHAL:}\quad \Pi_{\hat\lambda}(\hat\lambda') = \frac{2\sum_{\hat\alpha>0}\text{mult.}(\hat\alpha)\sum_{k=1}^{\infty}\langle\hat\lambda'+k\hat\alpha,\hat\alpha\rangle\,\Pi_{\hat\lambda}(\hat\lambda'+k\hat\alpha)}{|\hat\lambda+\hat\rho|^2-|\hat\lambda'+\hat\rho|^2}$$

DIFF. K! NOT THE LEVEL!!

NATURAL GEN. OF FORMULA ON p. 96 OF GEO 4, WITH ACCOUNT TAKEN OF THE POSITIVE ROOT MULTIPLICITY; RECALL THAT $\text{mult.}((\alpha;0;n)) = 1$ FOR $\alpha\neq 0$,
$\text{mult.}((0;0;n)) = r$ FOR $n\neq 0$.

ex): BASIC REP. OF $SU(2)_1$: $\hat A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ (DESC. OF THE IDENTITY PRIMARY II)

GRADE ℓ	
0	$[1\ 0]_1$ (MULTIPLICITY)
-1	$[-1\ 2]_1 \leftarrow [1\ 0]_{i_1} \leftarrow [3\ -2]_1$
-2	$[-1\ 2]_{i_1} \leftarrow [1\ 0]_{i_2} \leftarrow [3\ -2]_{i_1}$
-3	$[-1\ 2]_{i_2} \leftarrow [1\ 0]_{i_3} \leftarrow [3\ -2]_{i_2}$
-4	$[-3\ 4]_1 \leftarrow [-1\ 2]_{i_3} \leftarrow [1\ 0]_{i_4} \leftarrow [3\ -2]_{i_3} \leftarrow [5\ -4]_1$
	$\vdots$

ONLY 4 UNKNOWN MULTIPLICITIES TO GRADE -4, THANKS TO WEYL REFL.

① $S_0[\lambda_0\ \lambda_1]^K_{\ell=n} = [-\lambda_0+2\lambda_0+\lambda_1]^K_{\ell=n-2}$

② $S_1[\lambda_0\ \lambda_1]^K_{\ell=n} = [\lambda_0+2\lambda_1, -\lambda_1]^K_{\ell=n}$

→ p. 15

(p. 14-15)

$\widehat{SU(2)}$: $\bar\alpha_0 = -2(\hat\omega_1-\hat\omega_0)+\delta$ $\qquad \langle\delta,\delta\rangle=0$, $\langle\hat\omega_0,\hat\omega_0\rangle=0$

$\bar\alpha_1 = 2(\hat\omega_1-\hat\omega_0)$ $\qquad \langle\hat\omega_0,\delta\rangle=1$, $\langle\hat\omega_0,\hat\omega_1\rangle=0$

$\langle\hat\omega_1,\delta\rangle=1$, $\langle\hat\omega_1,\hat\omega_1\rangle=\frac12$

$\hat\lambda=\hat\omega_0$; $\hat\lambda+\hat\rho = 2\hat\omega_0+\hat\omega_1$

$\Rightarrow |\hat\lambda+\hat\rho|^2 = \frac12$

① i_1 : $\hat\lambda' = [1\ 0]_{\ell=-1} = \hat\omega_0-\delta$

$\Rightarrow \hat\lambda'+\hat\rho = 2\hat\omega_0+\hat\omega_1-\delta$

$\Rightarrow |\hat\lambda'+\hat\rho|^2 = -2\cdot 2+\frac12-2 = -6+\frac12 \quad\Rightarrow \text{DEN} = +6$

$\text{NUM} = 2\cdot 1\cdot[\underset{\substack{\Uparrow\\ [-1\ 2]_{\ell=-1}\\ \hat\alpha=\bar\alpha_1\\ p=1}}{2} + \underset{\substack{\Uparrow\\ [1\ 0]_{\ell=0}\\ \hat\alpha=\bar\alpha_0+\bar\alpha_1\\ p=1}}{1}] = 6$, p = FREUDENTHAL SUM INDEX

$\therefore\ i_1 = 1$ ✓

A Trivial (Irr. Rep.) KM Descendant of the Vacuum: The Holomorphic Energy-Momentum $T(z)$

GEO 4C

CFT 2, p. 10: $$L_{-2} = \frac{1}{2(k+g)}\left[\sum_{m\le -1} J^a_m J^a_{-2+|m|} + \sum_{m\ge 0} J^a_{-2-m} J^a_m\right]$$

CFT 1, p. 59: $$T(z)\big|_{z\to 0} = (L_{-2}\,\mathbb{I})$$

$$\therefore\ T_{(z\to 0)} = \frac{1}{2(k+g)}\left(\left(J^a_{-1}J^a_{-1} + \cancel{J^a_{-2}J^a_0}^{\,0} + \cancel{J^a_{-3}J^a_1}^{\,0} + \dots \right)\mathbb{I}\right)$$

VAC. IS INVT.

$$\therefore\ T_{(z\to 0)} = (L_{-2}\,\mathbb{I}) = \frac{1}{2(k+g)}\left(J^a_{-1}J^a_{-1}\,\mathbb{I}\right)$$

p. 21-23: $SU(2)_1$, KM DESC. OF $\mathbb{I}$:

| -GRADE λ | | |

② i_2: $\hat{\lambda}' = [1\ 0]_{\ell=-2} = \hat{\omega}_0 - 2\delta$

$\Rightarrow \hat{\lambda}'+\rho = 2\hat{\omega}_0 + \hat{\omega}_1 - 2\delta$

$\Rightarrow |\hat{\lambda}'+\rho|^2 = -8 + \frac{1}{2} - 4$ $\qquad \Rightarrow$ DEN $= 12$

$$\text{NUM} = 2\cdot 1\left[\begin{array}{cccccccc}
\underset{\substack{[-1\ 2]_{\ell=-2}\\ \hat{\alpha}=\bar{\alpha}_1\\ \rho=1}}{2} & + & \underset{\substack{[3\ -2]_{\ell=-1}\\ \hat{\alpha}=\bar{\alpha}_0\\ \rho=1}}{3} & + & \underset{\substack{[1\ 0]_{\ell=-1}\\ \hat{\alpha}=\bar{\alpha}_0+\bar{\alpha}_1\\ \rho=1}}{1} & + & \underset{\substack{[-1\ 2]_{\ell=-1}\\ \hat{\alpha}=\bar{\alpha}_0+2\bar{\alpha}_1\\ \rho=1}}{(-1+4)} \\
 & + & \underset{\substack{[1\ 0]_{\ell=0}\\ \hat{\alpha}=2\bar{\alpha}_0+2\bar{\alpha}_1\\ \rho=1}}{2} & + & \underset{\substack{[1\ 0]_{\ell=0}\\ \hat{\alpha}=\bar{\alpha}_0+\bar{\alpha}_1\\ \rho=2}}{1} & & &
\end{array}\right]$$

AFFINE!
2β IS NOT A ROOT
IF β IS IN A SIMPLE LA;
HERE
$\hat{\alpha} = n\bar{\alpha}_0 + m\bar{\alpha}_1$; $|n-m| \le 1$; $(n,m) \neq (0,0)$ (p.14)
$\Rightarrow n\delta = n(\bar{\alpha}_0+\bar{\alpha}_1)$ IS A ROOT FOR ANY $n \in \mathbb{Z}$; $n \neq 0$.

$= 24$

$\therefore i_2 = 2$ ✓

③ i_3: $\hat{\lambda}' = [1\ 0]_{\ell=-3}$

$\hat{\lambda}'+\hat{\rho} = 2\hat{\omega}_0 + \hat{\omega}_1 - 3\delta$

$\Rightarrow |\hat{\lambda}'+\hat{\rho}|^2 = -12 - 6 + \frac{1}{2}$ $\qquad \Rightarrow$ DEN $= 2\cdot 9$

NUM $= 2\cdot 1\cdot[2A + B + 2C + D + E + F + G + H]$

$$\begin{array}{ccccc}
 & & [1\ 0]_1 & & \\
 & \nearrow & H & & \\
[-1\ 2]_1 & \leftarrow & [1\ 0]_1 & \leftarrow & [3\ -2]_1 \\
G & \nearrow & F & \nearrow & E \\
[-1\ 2]_1 & \leftarrow & [1\ 0]_2 & \leftarrow & [3\ -2]_1 \\
D & \nearrow & C & \nearrow & B \\
[-1\ 2]_2 & \leftarrow & [1\ 0]_{i_3} & & \\
A & & & &
\end{array}$$

Ⓐ $\hat{\lambda}'+\bar{\alpha}_1$	Ⓑ $\hat{\lambda}'+\bar{\alpha}_0$	Ⓒ $\hat{\lambda}'+\bar{\alpha}_0+\bar{\alpha}_1$	Ⓓ $\hat{\lambda}'+\bar{\alpha}_0+2\bar{\alpha}_1$
$\hat{\alpha}=\bar{\alpha}_1$	$\hat{\alpha}=\bar{\alpha}_0$	$\hat{\alpha}=\bar{\alpha}_0+\bar{\alpha}_1$	$\hat{\alpha}=\bar{\alpha}_0+2\bar{\alpha}_1$
$\rho=1$	$\rho=1$	$\rho=1$	$\rho=1$
2	3	1	3

Ⓔ $\hat{\lambda}'+\bar{\alpha}_0\cdot 2+\bar{\alpha}_1$	Ⓕ $\hat{\lambda}'+2(\bar{\alpha}_0+\bar{\alpha}_1)$	Ⓕ $\hat{\lambda}'+2(\bar{\alpha}_0+\bar{\alpha}_1)$	Ⓖ $\hat{\lambda}'+2\bar{\alpha}_0+3\bar{\alpha}_1$	Ⓗ $\hat{\lambda}'+3(\bar{\alpha}_0+\bar{\alpha}_1)$
$\hat{\alpha}=2\bar{\alpha}_0+\bar{\alpha}_1$	$\hat{\alpha}=2(\bar{\alpha}_0+\bar{\alpha}_1)$	$\hat{\alpha}=\bar{\alpha}_0+\bar{\alpha}_1$	$\hat{\alpha}=2\bar{\alpha}_0+3\bar{\alpha}_1$	$\hat{\alpha}=3(\bar{\alpha}_0+\bar{\alpha}_1)$
$\rho=1$	$\rho=1$	$\rho=2$	$\rho=1$	$\rho=1$
4	2	1	4	3

(H) $\hat{\lambda}' + 3(\bar{\alpha}_0 + \bar{\alpha}_1)$ $\Rightarrow$ NUM $= 2\,[2\cdot 2 + 3 + 2 + 3 + 4 + 2 + 1 + 4 + 3 + 1]$

$\hat{\alpha} = \bar{\alpha}_0 + \bar{\alpha}_1$ $\quad = 2\,[12 + 9 + 4 + 2]$

$p = 3$ $\quad = 2 \cdot 9^3$

1

$\therefore\ i_3 = 3$ ✓

i.e., $\hat{\lambda} = (0; 1; -m)$
m in

IN THIS EXAMPLE, ONLY NEED THE MULTIPLICITIES OF $[1\ 0]$ AT GRADES $\{0, -1, -2, -3, -4, \ldots\}$ TO DETERMINE THE DIMENSIONS OF ALL (EXPLICITLY LISTED) WEIGHT SPACES, THANKS TO THE AFFINE WEYL GROUP.

FOR THE CASE OF THE DOMINANT HIGHEST WEIGHT $[1\,0]_{\ell=0}$ IN $SU(2)_1$, THE FIRST FEW MULTIPLICITIES ARE $1, 1, 2, 3, 5, 7, 11, 15, \ldots$
WE ENCODE THESE NUMBERS IN A <u>STRING FUNCTION</u>

DEF'N: $\Omega_{\hat{\lambda}} \equiv$ WEIGHT SYSTEM (∞ IRREP./VERMA MODULE) ASSOC. TO THE DOMINANT HWS $\hat{\lambda}$.

DEF'N: $\Omega_{\hat{\lambda}}^{MAX} \equiv$ SUBSET OF WEIGHTS $\hat{\mu} \in \Omega_{\hat{\lambda}}$ SUCH THAT $\hat{\mu} + \delta \neq \Omega_{\hat{\lambda}}$,
i.e. EACH $\hat{\mu}$ RESIDES AT THE TIP OF AN INFINITE VERTICAL WEIGHT STRING WITH FIXED (AFFINE) DYNKIN COEFF., BUT EVER DECREASING GRADES $\ell \to -\infty$.
IN THE $SU(2)_1$ EXAMPLE ON p. 21, $[-1\ 2]_{\ell=-1}$, $[3\ -2]_{\ell=-1}$, $[3\ 4]_{\ell=-4}$, AND $[5\ -4]_{\ell=-4}$ BELONG TO $\Omega_{\hat{\lambda}}^{max}$, WHILE $[-1\ 2]_{\ell \leq -2}$, $[1\ 0]_{\ell \leq -1}$, etc. DO NOT.

- IN THE WZW CFT, EACH UNIQUE STATE IN $\Omega_{\hat{\lambda}}^{max}$ CORRESPONDS TO AT LEAST ONE VIRASORO PRIMARY FIELD; THERE ARE INFINITELY MANY VIRASORO VERMA MODULES FUSED TOGETHER IN <u>EACH</u> KM VERMA MODULE $\Omega_{\hat{\lambda}}$; CFT CONSISTS OF ALL $\hat{\lambda} \in P_+^K$ AND THEIR KM DESCENDANTS.

DEF'N: <u>STRING FUNCTION</u> $\equiv$ GEN. FCN. ENCODES MULTIPLICITIES OF ALL IMAG. ROOT DESCENDANTS OF A WEIGHT $\hat{\mu} \in \Omega_{\hat{\lambda}}^{max}$, i.e. OF THE STATES $\hat{\mu}, \hat{\mu} - \delta, \hat{\mu} - 2\delta, \ldots$

$$\sigma_{\hat{\mu}}^{(\hat{\lambda})}(q) \equiv \sum_{n=0}^{\infty} \mathrm{mult}_{\hat{\lambda}}(\hat{\mu} - n\delta)\, q^n$$

ex): STRING FCN. FOR DOM. HW $\hat{\lambda} = [1\ 0]_{k=0}$ IN $SU(2)_1$ EXAMPLE ON p. 21.

<u>CLAIM</u>: $$\sigma^{([10])}_{[10]}(q) = \varphi^{-1}(q) = \prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n)\, q^n$$

$\varphi(q)$: EULER FCN.

OF PARTITIONS OF n INTO POS. INTEGERS UNDER SUMMATION

SEE ALSO CFT 1, p 48; SAME SEQUENCE ARISES IN COUNTING DESC. STATES IN A GENERIC VIRASORO VERMA MODULE {BEFORE QUOTIENTING-OUT NULL DESC.}

⇒ CFT BOOK p. 158, 203-204

IN PRINCIPLE, NEED TO KNOW $\sigma^{(\hat{\lambda})}_{\hat{\mu}}(q)$ FOR EVERY $\hat{\mu} \in \Omega^{max}_{\hat{\lambda}}$, i.e. THERE ARE AN INF. NUMBER OF STRING FCN.S NEC. TO DET. MULT. OF ALL WEIGHT SPACES IN $\Omega_{\hat{\lambda}}$.

<u>BUT</u>: WEIGHT MULTIPLICITIES CONST. ALONG WEYL ORBITS:

$$\sigma^{\hat{\lambda}}_{w\hat{\mu}}(q) = \sigma^{\hat{\lambda}}_{\hat{\mu}}(q)$$

∴ SUFFICIENT TO DETERMINE $\sigma^{\hat{\lambda}}_{\hat{\mu}}(q)$ FOR $\hat{\mu} \in P^K_+$, i.e. THOSE WEIGHTS IN THE AFFINE FUND. WEYL CHAMBER AT LEVEL K (SEE p. 570 CFT BOOK FOR $\widehat{SU(2)}$ EXAMPLE)

FURTHER CONSTRAINTS:

- <u>CLAIM</u>: ALL STATES $\hat{\beta} \in \Omega_{\hat{\lambda}}$ IN SAME CONGRUENCE CLASS AS $\hat{\lambda}$

⇒ # STRING FCN.S NEEDED TO RESOLVE ALL MULTIPLICITIES IN $\Omega_{\hat{\lambda}}$
= # $\hat{\mu} \in P^K_+$ IN THE SAME CONGRUENCE CLASS AS $\hat{\lambda}$.

Vacuum and Current Descendants

At level k, the Vacuum (Identity Field $\mathbb{I}$) is assoc. to the Dominant Weight

$$\text{"}\hat{\omega}_0^k\text{"}: \quad [k\,0\,0\cdots 0]_{\ell=0}^{k=k} \equiv \hat{\mathbb{I}}_k = k\hat{\omega}_0$$

At -grade $\ell=-1$ in $\Omega_{\hat{\mathbb{I}}_k}$, the "1st Descendant" obtains by subtracting

$$\bar{\alpha}_0 = [\hat{A}_{00}\ \hat{A}_{01}\cdots\hat{A}_{0r}]_{\ell=1}^{k=0} = [2\ -\lambda_1^{(\theta)}\ -\lambda_2^{(\theta)}\cdots -\lambda_r^{(\theta)}]_{\ell=1}^{k=0}$$

where $\{\lambda_i^{(\theta)}\}$ denotes the set of finite Dynkin coeff. for the root θ, the HWS of the Adjoint Rep.

Thus we have the Graded Module $\Omega_{\hat{\mathbb{I}}_k}$:

-grade ℓ				
0		$[k\,0\,0\cdots 0]_1$		Vacuum (scalar)
-1	$[k-2\ \lambda_1^{(\theta)}\cdots\lambda_r^{(\theta)}]_1$	$[k\,0\,0\cdots 0]_r$ (Cartan subalgebra)	$[k+2\ -\lambda_1^{(\theta)}\cdots -\lambda_r^{(\theta)}]$	Adjoint irrep. only
-2	$[k-2\ \lambda_1^{(\theta)}\cdots\lambda_r^{(\theta)}]$	$[k\,0\,0\cdots 0]$	$[k+2\ -\lambda_1^{(\theta)}\cdots -\lambda_r^{(\theta)}]_1$	$\oplus$ {other finite $\mathfrak{g}$ irreps at level -2}
$\vdots$		$\vdots$	$\vdots$	$\vdots$ $\oplus\ldots$

E_{-1}^{θ}

The -grade $\ell=-1$ desc. transform in the Adjoint Rep., and may be assoc. to the Kac-Moody Current Components in the Assoc. WZW Model:

$$J^a(z) = (J_{-1}^a\,\mathbb{I})(z) \qquad (\text{CFT } \underline{2}, \text{ p. } 7)$$