Energy, Momentum, and Angular Momentum in Classical Electrodynamics

Randy S

Abstract Article 98002 introduced the stress-energy tensor for the electromagnetic field interacting with a charged particle in flat spacetime. This article highlights the associated conservation laws, shows how the conservation of momentum can be used to infer the Lorentz force equation that governs the particle's behavior, and clarifies the relationship of the stress-energy tensor to the "spin" of the electromagnetic field.

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1 Review of the equations of motion

Maxwell's equations are introduced in article 31738. This article uses the same notation and conventions.

The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are two parts of a single field called the electromagnetic (EM) field. This single field is represented by the **Faraday tensor**, whose components are denoted $F_{ab}(x)$. The Faraday tensor is **antisymmetric**, which means

$$F_{ab}(x) = -F_{ba}(x). \tag{1}$$

The relationship between F_{ab} and $\mathbf{E}, \mathbf{B}$ is described in article 31738.

Maxwell's equations can be written as two equations. The first equation

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \tag{2}$$

does not depend on a spacetime metric. The second equation

$$\partial_a F^{ab} = -J^b \tag{3}$$

does depend on a spacetime metric, which is assumed here to be the Minkowski metric. The J on the right-hand side accounts for charges and currents.

The Lorentz force equation[1]

$$m\ddot{X}_b = q\dot{X}^a(\tau) F_{ab}\Big(X(\tau)\Big) \tag{4}$$

is the **equation of motion** governing the behavior of a charged particle in this model. This form of the Lorentz force equation is introduced in article 98002.

[1]Here, the worldline parameter is the particle's proper time τ.

2 Conservation of stress-energy

Write η_{ab} for the components of the Minkowski metric. Whenever the EM field satisfies Maxwell's equations (2) and (3) with $J = 0$, the quantity[2,3]

$$\boxed{T^{ab} \equiv \frac{1}{4}\eta^{ab}F_{\bullet\times}F^{\bullet\times} - \eta_{\bullet\times}F^{a\bullet}F^{b\times}} \tag{5}$$

satisfies

$$\partial_a T^{ab} = 0 \qquad (\text{if } J = 0). \tag{6}$$

This is local conservation law for the free EM field. It holds for any allowed behavior of the EM field[4] in any region of spacetime where charges and currents are absent ($J = 0$). From article 98002, we recognize the quantity T^{ab} defined in (5) as the **stress-energy tensor** for the EM field by itself ($J = 0$). The goal here is to derive the conservation law (6).

To derive (6), start by applying ∂_a to the definition (5) *without* using the equations of motion (2) or (3). For the first term in (5), use the identity

$$F_{\bullet\times}F^{\bullet\times} = \eta^{ab}\eta^{cd}F_{ac}F_{bd} \tag{7}$$

to get

$$\partial_a\left(\frac{1}{4}\eta^{ab}F_{\bullet\times}F^{\bullet\times}\right) = \frac{1}{2}F_{\bullet\times}(\partial^b F^{\bullet\times}). \tag{8}$$

For the second term in (5), the result of applying ∂_a is

$$\partial_a\big(-\eta_{\bullet\times}F^{a\bullet}F^{b\times}\big) = -\eta_{\bullet\times}(\partial_a F^{a\bullet})F^{b\times} - \eta_{\bullet\times}F^{a\bullet}\partial_a F^{b\times}.$$

[2]To make the equation easier to parse, this equation uses the symbols $\bullet$ and $\times$ for summed indices.

[3]When the number of spacetime dimensions is $\neq 4$, the definition of T^{ab} in equation (5) would need an overall dimensionful coefficient to make the conventional units of F_{ab} (article 00669) consistent with the conventional units of energy (equation (14)). To avoid cluttering the equations, this article omits that coefficient.

[4]In this model, a behavior of the EM field is allowed if and only if it satisfies the equations of motion (2) and (3).

Overall, this gives the identity

$$\partial_a T^{ab} = \Omega^b - \eta_{\bullet\times}(\partial_a F^{a\bullet})F^{b\times} \tag{9}$$

with

$$\Omega^b \equiv \frac{1}{2}F_{\bullet\times}(\partial^b F^{\bullet\times}) - \eta_{\bullet\times}F^{a\bullet}\partial_a F^{b\times}. \tag{10}$$

To get a more enlightening expression for Ω^b, rewrite this second term like this:

$$\begin{aligned}
-\eta_{\bullet\times}F^{a\bullet}\partial_a F^{b\times} &= -F_{a\bullet}\partial^a F^{b\bullet} \\
&= \frac{1}{2}F_{\bullet a}(\partial^a F^{b\bullet} + \partial^\bullet F^{ab}) \\
&= \frac{1}{2}F_{\bullet\times}(\partial^\times F^{b\bullet} + \partial^\bullet F^{\times b}).
\end{aligned}$$

The antisymmetry property (1) was used to get the third expression, and one summed index was relabeled to get the last expression. Use this in equation (10) to get

$$\Omega^b = \frac{1}{2}F_{\bullet\times}(\partial^b F^{\bullet\times} + \partial^\times F^{b\bullet} + \partial^\bullet F^{\times b}).$$

Use this in (9) to get the identity

$$\partial_a T^{ab} = \frac{1}{2}F_{\bullet\times}(\partial^b F^{\bullet\times} + \partial^\times F^{b\bullet} + \partial^\bullet F^{\times b}) - \eta_{\bullet\times}(\partial_a F^{a\bullet})F^{b\times}. \tag{11}$$

This is an *identity*: it holds whether or not the field satisfies Maxwell's equations (2) and (3). If the field does satisfy Maxwell's equations (2) and (3), then it reduces to

$$\boxed{\partial_a T^{ab} = \eta_{\bullet\times}J^\bullet F^{b\times}.} \tag{12}$$

When $J = 0$, this gives the conservation law (6).

3 Energy and momentum of the EM field

From here thorugh section 10, we will consider the EM field by itself ($J = 0$). The timelike coordinate x^0 will also be denoted t, so the spacetime coordinates are $x = (t, \mathbf{x}) = (t, x^1, x^2, ..., x^D)$.

If F_{ab} goes to zero at spatial infinity, then integrating (6) over all space gives

$$\frac{d}{dt}\int d^Dx\; T^{0b} = 0 \qquad \text{(if } J = 0\text{)}. \tag{13}$$

This is a collection of conservation laws, one for each value of the index b. The quantity T^{00} is called the **energy density** of the EM field, and the quantities $T^{0k} = T^{k0}$ with $k \in \{1, ..., D\}$ are called the (components of the) **momentum density** of the EM field.

Equation (13) implies that the **(total) energy**

$$E \equiv \int d^Dx\; T^{00}(x) \tag{14}$$

and **(total) momentum**

$$P^k \equiv \int d^Dx\; T^{0k}(x) \qquad\qquad k \geq 1 \tag{15}$$

satisfy the conservation laws

$$\frac{d}{dt}E = 0 \qquad\qquad \frac{d}{dt}P^k = 0 \qquad\qquad \text{(if } J = 0\text{)}. \tag{16}$$

Remember that the symbol E here refers to energy, whereas the symbol E_k refers to the components of the electric field vector.

4 Energy and momentum in terms of E and B

We can use equation (5) in (14) and (15) to get explicit expressions for the energy and momentum in terms of the electric and magnetic fields. The components of the electric and magnetic fields are

$$E_k \equiv F_{k0} \qquad\qquad B_{jk} \equiv F_{jk}. \tag{17}$$

For the energy, this gives

$$\boxed{E = \int d^D x\; T^{00} \qquad\qquad T^{00} = \frac{1}{2}\left(\sum_k (E_k)^2 + \sum_{j<k}(B_{jk})^2\right).} \tag{18}$$

For the momentum, it gives

$$\boxed{P^j = \int d^D x\; T^{0j} \qquad\qquad T^{0j} = \sum_k B_{jk}(x)E_k(x).} \tag{19}$$

These expressions for T^{00} and T^{0j} are the energy density and momentum density of the EM field. This expression for the momentum density is also called the **Poynting vector**.

In the special case $D = 3$, the energy density and momentum density (the integrands of (18) and (19)) may be written

$$T^{00} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} \qquad\qquad T^{0k} = (\mathbf{E}\times\mathbf{B})_k,$$

using the notation from article 31738.

5 Static fields can have momentum!

According to the expressions for T^{0j} shown in the previous section, even static (time-independent) fields can have momentum.[5] This might seem counterintuitive, because our experience with models of tangible objects suggests that momentum involves *motion*. In fact, the word *momentum* comes from the Latin word for "to move." Old words are sometimes given new meanings to reflect new understanding, though, and this is one of those cases. In modern physics, *momentum* is defined to be the conserved quantity associated with translation symmetry.[6] With this definition, nonzero momentum doesn't necessarily imply motion, as illustrated by the equations shown in the previous section: a time-independent configuration of the EM field can indeed have nonzero momentum.[7] If this is still troubling, notice that a Lorentz boost can change the magnitude of the momentum even if the fields are independent of the time coordinate.

An electric field by itself does not have nonzero momentum (or angular momentum), nor does a magnetic field by itself. To have nonzero momentum (or angular momentum), the combination $\sum_k B_{jk}E_k$ must be nonzero. It is nonzero in the "paradox" described in section 17-4 of Feynman *et al* (1989): the solenoid supplies the magnetic field, and the charges attached to the disk supply the electric field.

[5] Equation (24) shows that static fields can have angular momentum, too.

[6] The relationship between translation symmetry and the stress-energy tensor (5) is explained in article 32191 by explaining the relationship between the **Hilbert** stress-energy tensor (which is used here) and the **canonical** stress-energy tensor associated with translation symmetry via Noether's theorem.

[7] Readers who have some preliminary knowledge of quantum field theory might wonder whether this is explained by some kind of "moving photon" picture of static fields in quantum electrodynamics (QED). No, it isn't. A completely time-independent state of the EM field can have nonzero momentum in QED, too, just like it can in classical electrodynamics, and trying to relate this to the phenomena we call *photons* does not provide any deeper explanation than the one given here.

6 Angular momentum of the EM field

Suppose $J = 0$, as in the preceding sections. The stress-energy tensor (5) for the EM field is symmetric in $a \leftrightarrow b$:

$$T^{ab}(x) = T^{ba}(x).$$

We can use this together with the conservation law (6) to deduce the related conservation law

$$\partial_a M^{abc} = 0 \tag{20}$$

with

$$M^{abc} = x^b T^{ac} - x^c T^{ab}. \tag{21}$$

As explained in article 49705, the quantities M^{0jk} with $j, k \in \{1, 2, ..., D\}$ are the components of the **angular momentum density** – for the EM field in this case. The integral[8]

$$\begin{aligned} J^{jk} &\equiv \int d^D x \; M^{0jk} \\ &\equiv \int d^D \; (x^j T^{0k} - x^k T^{0j}) \end{aligned} \tag{22}$$

over the spatial coordinates is the **total angular momentum**, and (20) implies that it is conserved:

$$\frac{d}{dt} J^{jk} = 0.$$

Language alert: the name *total angular momentum* is used for two different things. In this article, *total* refers to the fact that each component of J^{jk} includes the contributions from all parts of the system – the *total system*. In the context of quantum physics, *total* often refers instead to a special rotation-invariant combination of all of the components of J^{jk}.

[8]This J should not be confused with the current density in equation (3), which has only one index.

7 Orbital angular momentum and spin: introduction

Recall that the magnetic field B can be written in terms of a gauge field A like this (article 98002):

$$B_{jk}(x) = \nabla_j A_k(x) - \nabla_k A_j(x). \tag{23}$$

Also recall that rotational symmetry is associated with the conservation of angular momentum.[9] When the coordinate system is rotated, the components of the gauge field are affected in two ways: the argument x of each component is affected, and the different components are mixed with each other. These two effects of a rotation can be associated with two different contributions to the total angular momentum, called **orbital angular momentum** and **spin**, respectively. This section explains how to define these contributions in a gauge-invariant way[10] and highlights the fact that they are both *nonlocal*, even though their sum is local.

This article is about classical electrodynamics, not quantum electrodynamics (QED), but one aspect of QED is worth mentioning here: the nonlocality of the spin-observable is consistent with the phenomena we call **photons** in QED. Photons are often called *particles*, partly because they can be approximately localized – but the key word here is *approximately*. The **Reeh-Schlieder theorem**[11] implies, among other things, that the concept of a strictly localized particle is not consistent with the general principles of relativistic quantum field theory (which includes QED). To borrow a slogan from Peres (2002), this is "well known to those who know things well," even though misleading/misguided stories about pointlike particles in QED are still abundant. The main result of this section – namely that the observable representing spin is nonlocal – doesn't contradict the fact that a single photon in QED has nonzero spin. Photons are nonlocal, too, despite what many popular stories say.[12]

[9] Articles 12342, 49705, and 32191

[10] Observables must be invariant under gauge transformations (article 98002).

[11] Witten (2018) reviews the proof of the Reeh-Schlieder theorem.

[12] Occasionally, physicists use the term "point particles" as a (careless) way of alluding to the fact that in relativistic quantum field theory, interactions are local in spacetime – in contrast to string theory, where interactions are nonlocal. Locality in this sense is an input to the Reeh-Schlieder theorem, which says that particles can't be strictly localized!

8 Orbital angular momentum and spin: nonlocality

Start with the Hilbert stress-energy tensor (5), which is gauge-invariant. We will be interested in the components[13]

$$T^{0k} = -F^{0\ell}F^{k}{}_{\ell} = \sum_{\ell} B_{k\ell}E_{\ell}$$

with $j, k \in \{1, 2, ..., D\}$, because these are the components that occur in the angular momentum density M^{0jk}. Use this expression for T^{0k} in (22) to get[14]

$$\boxed{J^{jk} = \int d^D x \ \left(\sum_{\ell}(x^j B_{k\ell} - x^k B_{j\ell})E_\ell\right).} \tag{24}$$

Use (23) in (24) to get

$$J^{jk} = J_0^{jk} + J_1^{jk} \tag{25}$$

with

$$J_0^{jk} \equiv \int d^D x \ \mathbf{E}\cdot(x^j\nabla_k\mathbf{A} - x^k\nabla_j\mathbf{A}) \tag{26}$$

$$J_1^{jk} \equiv \int d^D x \ \big((\mathbf{E}\cdot\nabla A_j)x^k - (\mathbf{E}\cdot\nabla A_k)x^j\big). \tag{27}$$

Recall that we're assuming $J = 0$ (section 3). If the field satisfies its equations of motion, including $\nabla\cdot\mathbf{E} = 0$, then we can use

$$\begin{aligned}\int d^D x \ (\mathbf{E}\cdot\nabla A_j)x^k &= \int d^D x \ \nabla\cdot(\mathbf{E}A_j x^k) - \int d^D x \ (\mathbf{E}A_j)\cdot\nabla x^k \\ &= 0 - \int d^D x \ E_k A_j\end{aligned}$$

[13] A sum over j is implied when the index j occurs both as a superscript and subscript in the same term. In the last expression, the sum is written explicitly because j no longer occurs as a superscript.

[14] For $D = 3$, this can be written $\mathbf{J} = \int d^3x \ \mathbf{x}\times(\mathbf{E}\times\mathbf{B})$ with $\mathbf{J} = (J^{23}, J^{31}, J^{12})$.

to get this simpler expression for J_1^{jk}

$$J_1^{jk} = \int d^D x \; (E_j A_k - E_k A_j). \tag{28}$$

The quantities J_0^{jk} and J_1^{jk} are associated with the orbital angular momentum and spin, respectively, but they don't qualify as observables because they are not gauge-invariant. The angular momentum (24) is gauge-invariant, but the individual contributions J_0^{jk} and J_1^{jk} are not.

To get *observables* corresponding to the orbital angular momentum and spin, let $G(x)$ be a **Green's function** for **Poisson's equation**, which means[15]

$$\nabla^2 \int d^D y \; G(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) = f(\mathbf{x}) \tag{29}$$

for all sufficiently well-behaved functions $f(\mathbf{x})$, and $G(\mathbf{x}) \to 0$ zero when $|\mathbf{x}| \to \infty$. Given such a "function," define

$$\boxed{\tilde{A}_j(t,\mathbf{x}) \equiv \sum_k \int d^D y \; G(\mathbf{x}-\mathbf{y}) \nabla_k B_{kj}(t,\mathbf{y}).} \tag{30}$$

This is manifestly gauge-invariant, because it's expressed entirely in terms of the magnetic field B. Using equation (23), we can also write it in terms of the gauge field A like this:

$$\tilde{A}_j(t,\mathbf{x}) = A_j(t,\mathbf{x}) - \nabla_j \int d^D y \; G(\mathbf{x}-\mathbf{y}) \nabla\cdot\mathbf{A}(t,\mathbf{y}). \tag{31}$$

The field $\tilde{\mathbf{A}}$ with these components is called the **transverse part** of $\mathbf{A}$, because it (clearly) satisfies $\nabla\cdot\tilde{\mathbf{A}} = 0$. Equation (31) also shows that it satisfies

$$\nabla_j \tilde{A}_k(x) - \nabla_k \tilde{A}_j(x) = B_{jk}(x), \tag{32}$$

[15] The arguments are written in boldface to indicate that only the spatial coordinates are involved.

so we can repeat the steps that led to equations (26) and (28) but with $\tilde{\mathbf{A}}$ in place of $\mathbf{A}$. The result is

$$J^{jk} = \tilde{J}_0^{jk} + \tilde{J}_1^{jk}$$

with

$$\tilde{J}_0^{jk} \equiv \int d^D x \; \mathbf{E} \cdot (x^j \nabla_k \tilde{\mathbf{A}} - x^k \nabla_j \tilde{\mathbf{A}}) \tag{33}$$

$$\tilde{J}_1^{jk} \equiv \int d^D x \; (E_j \tilde{A}_k - E_k \tilde{A}_j). \tag{34}$$

These are the observables corresponding to the field's **orbital angular momentum** and **spin**, respectively. They look superficially like the previous expressions (26) and (28), but with two key differences: the definition (30) implies that $\tilde{J}_0^{jk}$ and $\tilde{J}_1^{jk}$ are gauge-invariant (they qualify as observables), and it also implies that they are nonlocal – because $\tilde{\mathbf{A}}$ at *each* point in space depends on the magnetic field at *all* points in space.

Even though the observables (33) and (34) are individually nonlocal, their sum J^{jk} is local, because the term involving G in (31) cancels in the sum. By construction, their sum J^{jk} is the total angular momentum (22), which is (the integral of) a local function of the electric and magnetic fields.

9 Orbital angular momentum and spin: conservation

Rotational symmetry is associated with the conservation of a system's *total* angular momentum. The angular momenta of different parts of the system are generally not separately conserved, because the different parts can exchange angular momentum with each other. One exception occurs for the **free** EM field (with no charges or currents), which has been the focus of this article so far.[16] In this case, the orbital and spin parts defined above are separately conserved. This section outlines the proof that the spin part $\tilde{J}_1^{jk}$ is separately conserved. We already know from section 6 that the total angular momentum J^{jk} is conserved, so this implies that the orbital part $\tilde{J}_0^{jk}$ is also separately conserved.

Combine equations (30) and (34) to get this expression for the spin part:

$$\tilde{J}_1^{jk} = \int d^D x\, d^D y\ E_j(\mathbf{x}) G(\mathbf{x}-\mathbf{y}) \sum_\ell \nabla_\ell B_{\ell k}(\mathbf{y}) - (j \leftrightarrow k).$$

To calculate the time-derivative of this, use the $J = 0$ version of Maxwell's equation (3) to rewrite the time-derivative of the E-factor, and use Maxwell's other equation (2) to rewrite the time-derivative of the B-factor.[17] This gives

$$\begin{aligned} \frac{d}{dt}\tilde{J}_1^{jk} &= \int d^D x\, d^D y\ \left(-\sum_\ell \nabla_\ell B_{\ell j}(\mathbf{x})\right) G(\mathbf{x}-\mathbf{y}) \sum_\ell \nabla_\ell B_{\ell k}(\mathbf{y}) - (j \leftrightarrow k) \\ &\quad + \int d^D x\, d^D y\ E_j(\mathbf{x}) G(\mathbf{x}-\mathbf{y}) \sum_\ell \nabla_\ell \big(\nabla_k E_\ell(\mathbf{y}) - \nabla_\ell E_k(\mathbf{y})\big) - (j \leftrightarrow k). \end{aligned}$$

The first line on the right-hand side is zero because of the antisymmetry in j, k. In the second line, the first term in large parentheses is zero because the $J = 0$ version of Maxwell's equation (3) implies $\nabla \cdot \mathbf{E} = 0$, and the remainder is zero because of (29) and the antisymmetry in j, k. Altogether, this shows that the spin part is separately conserved when charges and currents are absent.

[16] Charges and currents will be incorporated starting in section 11.

[17] Article 31738 shows how those equations are expressed in terms of the electric and magnetic fields.

10 Orbital angular momentum and spin: a warning

The stress-energy tensor that occurs in general relativity is called the **Hilbert** stress-energy tensor. The stress-energy tensor shown in equation (5) is the Hilbert stress-energy tensor for the EM field. As explained in article 32191, this differs from the **canonical** stress-energy tensor that is associated with Noether's theorem, namely

$$T_C^{ab} = \frac{1}{4}\eta^{ab}F^{cd}F_{cd} - F^{ac}\partial^b A_c. \tag{35}$$

The Hilbert tensor is symmetric and gauge-invariant, but the canonical stress-energy tensor is not. Article 32191 explains that the canonical and Hilbert stress-energy tensors are the same modulo an identically-conserved term when the field satisfies its equations of motion. That doesn't make them interchangeable, though, especially not with respect to angular momentum:

- If the canonical stress-energy tensor (35) were used in the definition (21) of the angular momentum density M, then the conservation law (20) would not hold, because (20) relies on T^{ab} being symmetric.

- If the canonical stress-energy tensor (35) were used in the definition (22) of the total angular momentum J^{jk}, then the spin term J_1^{jk} in (25) would be absent.

These issues shouldn't cause any concern, because the *canonical* stress-energy tensor is not an observable: observables must be invariant under gauge transformations (article 98002), and the canonical stress-energy tensor is not. That's why this article uses the *Hilbert* stress-energy tensor instead: it is gauge-invariant, it arises automatically in general relativity, and it accounts for all of the field's angular momentum – including the spin term – when used in the definition (22).

11 How to infer the Lorentz force equation

Maxwell's equations govern the behavior of the EM field under the influence of any given current J. The Lorentz force equation describes how J is influenced by the EM field, so that the influence goes both ways. The behavior of J is not uniquely determined by Maxwell's equations, but Maxwell's equations do constrain the behavior of J to some degree. One such constraint (the conservation law $\partial_a J^a = 0$) was derived in article 31738. This section shows that with the help of an extra assumption, we can also deduce the Lorentz force equation.

The extra assumption is that the complete system, including both EM field and matter,[18] has a conserved total momentum of the form $p^k + \int T^{0k}$ where p^k is the matter's contribution to the momentum and where $\int T^{0k}$ is the EM field's contribution to the momentum, as in section 3.[19] The assumption that this total momentum is conserved implies

$$\frac{dp^k}{dt} = -\frac{d}{dt}\int d^D x\; T^{0k}. \tag{36}$$

The right-hand side of this equation can be evaluated using only Maxwell's equations, so this gives us an equation for the force (dp^k/dt) that the EM field exerts on the matter.

To evaluate the right-hand side of (36), start with the result (12), which can also be written

$$\partial_a T^{ab} = g^{ab} F_{ac} J^c. \tag{37}$$

Setting $b = k \in \{1, ..., D\}$ and integrating (37) over all of space gives

$$\frac{d}{dt}\int d^D x\; T^{0k} = -\int d^D x\; F_{kc} J^c = -\int d^D x\; \Big(E_k J^0 + \sum_j B_{kj} J^j\Big). \tag{38}$$

[18] In the context of electrodynamics in flat spacetime, *matter* means everything other than the electromagnetic field. In the context of general relativity, *matter* typically means everything (including the electromagnetic field) other than the metric field.

[19] This assumption is consistent with equations (40)-(41) in the next section.

Substitute this into the right-hand side of (36) to get

$$\frac{dp^k}{dt} = \int d^D x \; \Big(E_k J^0 + \sum_j B_{kj} J^j\Big). \tag{39}$$

If we further assume that the matter consists of a single pointlike charged particle, so that the current J is non-zero only at one point in space at any given time, then the integral on the right-hand side of (39) is just the integrand with E and B evaluated at that one time-dependent point. In that case, equation (39) is the Lorentz force equation (article 54711), which describes how the EM field influences the behavior of the charged particle.

12 The full stress-energy tensor

The previous section showed how the Lorentz force equation can be inferred from the conservation of the stress-energy tensor of the full system (particle and EM field). The next section goes the other direction, using the equations of motion for the full system to show that the full stress-energy tensor is conserved.[20]

When a single charged particle is present in addition to the EM field, the full stress-energy tensor is

$$\boxed{T^{ab}(x) \equiv T_F^{ab}(x) + T_X^{ab}(x).} \tag{40}$$

The first term T_F^{ab} is the field-only part that was previously denoted T^{ab} (equation (5)). The new particle-only term T_X^{ab} was derived in article 41182, with the result

$$\boxed{T_X^{ab}(x) \equiv m \int d\tau\; \dot{X}^a(\tau)\dot{X}^b(\tau)\,\delta^{D+1}\big(x - X(\tau)\big)} \tag{41}$$

when the metric is the Minkowski metric. The delta-function $\delta^{D+1}\big(x - X(\tau)\big)$ enforces the constraint that $T_X^{ab}(x)$ can only be non-zero at points x that are on the world-line $X(\tau)$ of the particle.

[20] This provides the context for the analysis in article 41182.

13 Conservation of the full stress-energy tensor

If $F_{ab}(x)$ and $X^a(\tau)$ satisfy Maxwell's equations and the Lorentz force equation (equations (2)-(4)), then the local conservation law

$$\boxed{\partial_a T^{ab}(x) = 0} \tag{42}$$

holds at every point x in space-time, where T^{ab} is the full stress-energy tensor defined in (40).[21]

To derive (42), start with the stress-energy tensor T^{ab} defined by equation (40). This is the sum of two parts, a field-only part T_F^{ab} and a particle-only part T_X^{ab}. We will calculate the result of applying ∂_a to each of these two parts separately, and then we will see that the results cancel each other, leaving zero. This gives the conservation law (42).

The field-only part T_F^{ab} is given by (12), which can also be written

$$\partial_a T_F^{ab}(x) = -J_a(x) F^{ab}(x). \tag{43}$$

From article 98002, we have

$$J^a(x) = q \int d\tau\ \dot{X}^a(\tau)\, \delta^{D+1}\big(x - X(\tau)\big). \tag{44}$$

Now consider the matter-only part, $T_X^{ab}(x)$. Apply ∂_a to the expression (41) for $T_X^{ab}(x)$ to get

$$\partial_a T_X^{ab}(x) = m \int d\tau\ \dot{X}^a(\tau)\dot{X}^b(\tau)\, \partial_a \delta^{D+1}\big(x - X(\tau)\big). \tag{45}$$

To evaluate the right-hand side, use the identities

$$\frac{\partial}{\partial x^a}\delta^{D+1}\big(x - X\big) = -\frac{\partial}{\partial X^a}\delta^{D+1}\big(x - X\big)$$

[21] The quantity that was denoted T^{ab} in sections 2 through 11 is denoted T_F^{ab} from now on.

and

$$\frac{d}{d\tau}f\big(X(\tau)\big) = \dot{X}^a\frac{\partial}{\partial X^a}f\big(X\big)\bigg|_{X=X(\tau)}.$$

This gives

$$\partial_a T_X^{ab}(x) = -m\int d\tau\ \dot{X}^b(\tau)\,\frac{d}{d\tau}\delta^{D+1}\big(x - X(\tau)\big). \tag{46}$$

Now use integration-by-parts to get

$$\partial_a T_X^{ab}(x) = m\int d\tau\ \ddot{X}^b(\tau)\,\delta^{D+1}\big(x - X(\tau)\big). \tag{47}$$

Equation (47) holds whether or not the particle satisfies the Lorentz force equation. If the particle does satisfy the Lorentz force equation (4), then we can use this to rewrite the factor $\ddot{X}$, which gives

$$\partial_a T_X^{ab}(x) = q\int d\tau\ \dot{X}_a(\tau)F^{ab}(x)\,\delta^{D+1}\big(x - X(\tau)\big) = J_a(x)F^{ab}(x). \tag{48}$$

Compare this to equation (43) to conclude that

$$\partial_a T_F^{ab}(x) + \partial_a T_X^{ab}(x) = 0$$

whenever the dynamic variables (field and particle) all satisfy their equations of motion. This completes the derivation of the conservation law (42).

Thanks to the action principle (article 98002), the signs in Maxwell's equation (3) and the Lorentz force equation (4) are tied to each other. If we change the sign of F_{ab} in *both* of those equations, then we have not really changed anything: we have only redefined F_{ab}. That changes the *convention*, but it doesn't change the system's *behavior*. However, if we changed the sign of F_{ab} in only *one* of these two equations (either (3) or (4) but not both), then we would change the system's behavior – and the system would no longer satisfy the action principle. The action principle ties the signs together in a particular way. This is essential for the conservation law (42), for the fact that opposite charges attract each other, and for the fact that parallel currents attract each other.

14 Energy and momentum

As in section 3, equation (42) implies that the **energy** (14) and **momentum** (15) satisfy the conservation laws (16), where now the quantity T^{ab} in those equations is the stress-energy tensor (40) of the full system, including the particle and the EM field. The resulting expressions for the total energy and momentum are

$$E = E_F + E_X \qquad\qquad P^k = P_F^k + P_X^k, \tag{49}$$

where the subscript F indicates the field-only parts that were already displayed in equations (18) and (19), and the subscript X indicates the particle-only parts

$$E_X = \int d^D x\; T_X^{00} \qquad\qquad P_X^k = \int d^D x\; T_X^{0k}. \tag{50}$$

Evaluating these integrals leads to simple expressions for the particle's energy and momentum. These calculations are done in aritlce 41182, and the result is

$$E_X = m\frac{dX^0}{d\tau} \qquad\qquad P_X^k = m\frac{dX^k}{d\tau}. \tag{51}$$

This agrees with the equations that were introduced in article 77597, but now we have a more satisfying foundation for those equations, because now we see how they relate to conservation laws in a model that explicitly includes an interaction between particles: the interaction is mediated by the EM field.[22]

[22] We only included one particle in this analysis, but extending the analysis to multiple particles is straightforward, and then the EM field does mediate an interaction between the particles.

15 References

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