The Electromagnetic Field and Maxwell's Equations

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Abstract Two long-range interactions exist in nature. One is gravity. The other is electromagnetism. This article introduces the field that mediates electromagnetism, with emphasis on the structure of the equations that govern its behavior (Maxwell's equations). Everything is formulated for an arbitrary number of spacetime dimensions.

Contents

1 Introduction

The electromagnetic (EM) field is typically represented by a pair of fields, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. The behavior of these fields is governed by Maxwell's equations. Maxwell's equations may be written in several different ways, the most iconic of which is probably this one:[1]

$$\boxed{\begin{aligned} \nabla\cdot\mathbf{E} &= \rho & \qquad \nabla\times\mathbf{B} &= \frac{\partial\mathbf{E}}{\partial t}+\mathbf{J} \\ \nabla\cdot\mathbf{B} &= 0 & \qquad \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}. \end{aligned}} \tag{1}$$

The quantities ρ and $\mathbf{J}$ are the charge density and current density, respectively. This might be the most familiar way of writing Maxwell's equations, but it is not necessarily the best way. This article re-introduces Maxwell's equations in a way that makes their hidden Lorentz symmetry more evident.[2]

[1] This article uses **natural units** (article 37431).

[2] Their Lorentz symmetry is demonstrated in article 00418, using the formulation introduced here.

2 Units

Different systems of units are convenient for different purposes. The purpose of this article is to elucidate the general mathematical structure of the EM field and of Maxwell's equations. For this purpose, we can streamline the equations by using **natural units**, which this article uses exclusively. Equations (1) are written in natural units. In this convention, the electric and magnetic fields are expressed in the same units:[3]

$$[\mathbf{E}] = [\mathbf{B}].$$

Other unit systems that have this property include the **Heaviside-Lorentz** system and the **Gaussian** system. In the **Standard International** system of units, the electric and magnetic fields have different units:

$$[\mathbf{E}] \neq [\mathbf{B}].$$

These various systems of units are described and compared in article 00669.

[3] The notation $[X]$ indicates the unit used to express the value of X.

3 Notation, part 1

Let D be the number of spatial dimensions. Equations (1) are specific to $D = 3$. For most of this article, D will be arbitrary. Use x as an abbreviation for the set of spacetime coordinates:[4]

$$x \equiv \{x^0,\, x^1,\, x^2,\, ...,\, x^D\}.$$

The extra coordinate x^0 is the "time" coordinate.[5] The notation $f(x)$ means that f is a function of all $1 + D$ spacetime coordinates. The abbreviation

$$\partial_a \equiv \frac{\partial}{\partial x^a} \qquad\qquad a \in \{0, 1, 2, ..., D\}$$

will be used for the partial derivative with respect to x^a. Square brackets around a set of indices will be used to denote the **completely antisymmetric** combination, like this:[6]

$$\begin{aligned} M_{[ab]} &\equiv M_{ab} - M_{ba} \\ M_{[abc]} &\equiv M_{abc} + M_{bca} + M_{cab} - M_{acb} - M_{cba} - M_{bac}. \end{aligned} \tag{2}$$

When $D = 3$, boldface will be used to denote a quantity with three components, as in equations (1).

[4] In this equation, each superscript is an index, not an exponent.

[5] Article 48968 explains the reason for the scare-quotes.

[6] This notation is standard, but a different normalization factor is sometimes used.

4 The electromagnetic (EM) field

The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are two parts of a single field called the electromagnetic (EM) field. This single field is represented by the **Faraday tensor**.[7] Its components are denoted $F_{ab}(x)$. It is **antisymmetric**, which means

$$\boxed{F_{ab}(x) = -F_{ba}(x).} \tag{3}$$

In particular, the diagonal components of F_{ab} are zero.[8]

The distinction between electric and magnetic parts is related to the distinction between time and space:

> The time-space components F_{k0} constitute the **electric** part, and the space-space components F_{jk} constitute the **magnetic** part.

Section 8 will explain this in more detail. Coordinate transformations that mix the time and space coordinates with each other also mix the electric and magnetic fields with each other. This is why they are regarded as parts of a single field.

The following sections show how to write Maxwell's equations in terms of the Faraday tensor.

[7] It's also called the **field strength** tensor and the **EM field** tensor.

[8] The argument x will not always be written, but it's implied.

5 Maxwell's equations, part 1

The functions $F_{ab}(x)$ depend on the "time" coordinate x^0, so they describe the *behavior* of the EM field. Maxwell's equations are the **equations of motion** for the EM field, which means they specify which behaviors are physically allowed. They can be written in a natural way as just two equations: one that depends on the metric structure of spacetime (section 6), and one that does not. This section introduces the one that does not.

The metric-independent half of Maxwell's equations says that a given behavior $F_{ab}(x)$ is physically allowed only if

$$\boxed{\partial_{[a}F_{bc]} = 0.} \tag{4}$$

The square brackets around the subscripts represent the completely antisymmetric combination (equation (2)). Using the fact that F_{ab} is already antisymmetric, we can write equation (4) more explicitly as

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0.$$

Notice the simple pattern: the subscripts are cyclically permuted from one term to the next. Section 10 will show that equation (4) repreduces both equations on the last row of (1) when $D = 3$.

6 Notation, part 2

This article assumes that spacetime is flat (article 48968), so we can use a coordinate system in which the metric has components

$$\eta^{ab} \equiv \begin{cases} 1 & \text{if } a = b = 0, \\ -1 & \text{if } a = b \neq 0, \\ 0 & \text{if } a \neq b. \end{cases}$$

This is the **Minkowski metric**, which is also implicitly assumed in equations (1). The standard summation convention will be used, so that a sum is implied over any index that occurs both as a superscript and as a subscript within the same term. Example: in the expression $\eta^{ab}\partial_b$, a sum over b is implied. The standard abbrevation

$$\partial^a \equiv \eta^{ab}\partial_b$$

will be used. Explicitly:

$$\partial^0 = \partial_0 \qquad \qquad \partial^k = -\partial_k \;\; \text{for } k \in \{1, 2, ..., D\}.$$

Using the metric η^{ab}, we can define a raised-index version of the Faraday tensor by[9]

$$F^{ab} \equiv \eta^{ac}\eta^{bd}F_{cd}. \tag{5}$$

Explicitly:

$$F^{0k} = -F_{0k} \qquad \qquad F^{jk} = F_{jk} \qquad \qquad \text{for } j, k \in \{1, 2, ..., D\}.$$

[9] Remember that sums are implied over the repeated indices c and d.

7 Maxwell's equations, part 2

The metric-dependent half of Maxwell's equations says that a given behavior is allowed only if[10]

$$\boxed{\partial_a F^{ab} = -J^b.} \tag{6}$$

Section 11 will show that this reproduces both equations on the first row of (1) when $D = 3$.

The quantities J^b can all be mixed with each other by coordinate transformations, so they are naturally regarded as different components of a single entity. They are the components of the **current density**, often abbreviated **current**. The timelike component J^0 is the **charge density**. In this article, the components J^b are prescribed functions of space and time. In a more complete model, J^b would instead be expressed in terms of other dynamic entities that are governed by other equations of motion.[11] Either way, J^b is not entirely arbitrary: equation (6) implies that it must at least obey

$$\boxed{\partial_b J^b = 0.} \tag{7}$$

This is a local conservation law, also called a **continuity equation**. To derive it, apply ∂_b to equation (6) and use the fact that F^{ab} is antisymmetric (equations (3) and (5)).[12]

Equations (4) and (6) are collectively called **Maxwell's equations**. They are the equations of motion that govern the behavior of the EM field.

[10] This sign convention is common but not universal. It matches the sign convention used in d'Inverno (1995), Griffiths (1989), Nakahara (1990), and Weinberg (1995), even though the explicit minus-sign is sometimes hidden by writing $\partial_a F^{ba}$ instead of $\partial_a F^{ab}$ on the left-hand side of the equation. The opposite sign convention is used in Jackson (1975). The sign convention in equation (6) is correlated with the conventions described in the next section.

[11] In article 98002, those other dynamic entities are point particles, and their behavior is governed by the **Lorentz force equation**.

[12] Article 19253 explains why the same conservation law can be derived either by using the equations of motion for the EM field or by using the equations of motion for the entities that carry the current.

8 The electric and magnetic fields

The relationship between the Faraday tensor and the fields **E** and **B** in equations (1) depends on sign conventions. This article uses the convention

$$\boxed{E_k \equiv F_{k0}} \tag{8}$$

for the components of the electric field **E**. This makes sense for any D.

For the magnetic field, the most natural representation uses two subscripts:

$$\boxed{B_{jk} \equiv F_{jk}.} \tag{9}$$

In D-dimensional space for most values of D, the magnetic field does not have D components: the magnetic field is *not a vector*.[13] When $D = 3$, the magnetic field has 3 components, and we can use the convention

$$\boxed{\mathbf{B} = (B_1, B_2, B_3) \equiv (F_{23}, F_{31}, F_{12})} \tag{10}$$

to relate the two-index formulation (9) to the traditional formulation that treats it as a vector. This will be used in sections 10 and 11.

Thanks to the antisymmetry condition (3), the components F_{ab} with $a < b$ determine all of the others. For this reason, when counting components, we might as well only count those with $a < b$. In D-dimensional space, the electric part has D components, and the magnetic part has[14] $\binom{D}{2} = (D^2 - D)/2$ components. In the special case $D = 3$, the electric and magnetic parts have the same number of components. For general D, they do not.[15]

[13] This is acknowledged in the footnote on page 198 in section 5.1.2 of Griffiths (1989).

[14] This is the number of ways of choosing 2 distinct index-values from a set of D distinct index-values.

[15] When $D = 1$, the Faraday tensor doesn't have a magnetic part. The magnetic field is defined only if $D \geq 2$.

9 Splitting spacetime into space and time

Sometimes using a notation that distinguishes the spatial components is useful. Let

$$\mathbf{x} \equiv (x^1, x^2, ..., x^D)$$

denote the set of space coordinates, excluding the time coordinate.[16] The time coordinate x^0 will also be denoted t. Indices[17] from the beginning of the alphabet $(a, b, ...)$ will be used for both time and space components, and indices from the middle of the alphabet $(j, k, ...)$ will be used only for space components. The abbreviation

$$\nabla_k \equiv \frac{\partial}{\partial x^k} \qquad\qquad k \in \{1, 2, ..., D\} \tag{11}$$

will be used for the components of the spatial gradient, as in equations (1):

$$\nabla = (\nabla_1,\ \nabla_2,\ ...,\ \nabla_D).$$

The **divergence**

$$\nabla \cdot \mathbf{U} \equiv \sum_k \nabla_k U_k$$

and the antisymmetric combinations

$$\nabla_j U_k - \nabla_k U_j \tag{12}$$

are defined for any D-component vector $\mathbf{U} = (U_1, ..., U_D)$. In the realistic case $D = 3$, these antisymmetric combinations are traditionally arranged as though they were the components of a 3-component vector $\nabla \times \mathbf{U}$ called the **curl** of $\mathbf{U}$, but this is not necessary and often causes more trouble than it saves. For any D, including $D = 3$, we can leave the antisymmetric combinations (12) as they are.[18]

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[16] In this article, boldface symbols denote quantities with D components.

[17] "Indices" is the plural form of "index": one index, two indices. "Indicee" is not a word (article 18505).

[18] Article 81674 addresses this in more detail – for the cross-product instead of the curl, but the idea is the same.

10 Recovering the iconic equations, part 1

The iconic equations (1) involve expressions that are defined only when $D = 3$, but equations (4) and (6) make sense for arbitrary D. This section shows that when $D = 3$, equation (4) implies the two lower equations in (1).

The antisymmetry condition (3) implies that the combination $\partial_{[a}F_{bc]}$ is *identically* zero whenever two or more of the indices are equal to each other, so equation (4) is empty in that case. That leaves only two cases to consider: no time-index, and one time-index.

With no time-index, equation (4) reduces to

$$\nabla_j B_{k\ell} + \nabla_k B_{\ell j} + \nabla_\ell B_{jk} = 0 \tag{13}$$

using notation that was defined in equations (9) and (11). When $D = 3$, equation (13) reproduces the lower-left equation in (1). To see why, recall that the left-hand side of (13) is identically zero unless all of the indices are distinct, so the indices in (13) must be $1, 2, 3$. Combine this with (10) to see that equation (13) reproduces the lower-left equation in (1) when $D = 3$.

For the case with one time-index, use $(a, b, c) = (j, k, 0)$ in equation (4) to get

$$\nabla_j E_k - \nabla_k E_j + \frac{\partial}{\partial t} B_{jk} = 0. \tag{14}$$

using notation defined in equations (8) and (9). When $D = 3$, this reproduces the lower-right equation in (1).

11 Recovering the iconic equations, part 2

The $b = 0$ component of equation (6) is

$$\nabla \cdot \mathbf{E} = J^0, \tag{15}$$

and the $b = k > 0$ components of equation (6) are[19]

$$\frac{\partial}{\partial t} E_k + \sum_j \nabla_j B_{jk} = -J^k. \tag{16}$$

When $D = 3$, these become the two equations on the top row of (1), with $\rho \equiv J^0$.

Together with section 10, this shows that equations (4) and (6) reproduce the iconic equations (1) when $D = 3$.

[19] A sum is implied when an index occurs both as a superscript and subscript in the same term. In (16), the sum is written explicitly because both occurrences of j are subscripts.

12 Conservation of charge

The distinction between "time" and "space" coordinates corresponds to a distinction between the electric and magnetic parts of the EM field (section 8). Similarly, the continuity equation (7) may be written

$$\frac{\partial}{\partial t} J^0 + \nabla \cdot \mathbf{J} = 0 \tag{17}$$

where the components of $\mathbf{J}$ are J^k with $k \geq 1$. The component J^0 is the **charge density**, and the quantity $\mathbf{J}$ is called the **current density**[20] Equation (17) says that the rate at which the charge density is changing at any given point in space must be balanced by the influx or outflux of current to/from that point.

Equation (17) is a *local* conservation law: it holds at each point in space. Integrating (17) over all of the space coordinates $(x^1, ..., x^D)$ gives

$$\frac{d}{dt} Q + \int d^D x \; \nabla \cdot \mathbf{J} = 0 \tag{18}$$

with

$$Q \equiv \int d^D x \; J^0. \tag{19}$$

The quantity Q is the **total charge**. If the current density $\mathbf{J}$ goes to zero at spatial infinity, then the fundamental theorem of calculus says that the last term on the right-hand side is zero, leaving

$$\frac{d}{dt} Q = 0. \tag{20}$$

This says that the total charge must conserved in order to be consistent with Maxwell's equations.

[20] The name *current density* is also used for the combination $(J^0, \mathbf{J})$, as in section 7.

13 Electromagnetic waves

In any region of spacetime where J^b is zero, equation (6) reduces to $\partial_a F^{ab} = 0$, which is equivalent to

$$\partial^a F_{ab} = 0, \tag{21}$$

so the behavior of the EM field in such a region is governed by equations (4) and (21). The easiest solutions of these equations are constants – any configuration in which the components of $F_{ab}(x)$ are all independent of all of the coordinates $x = (x^0, x^1, ..., x^D)$. Another easy type of solution is a **plane wave**, such as the solution

$$\begin{aligned} F_{20}(x) &= \alpha \cos(\omega x^0 - \omega x^1) \\ F_{12}(x) &= \alpha \cos(\omega x^0 - \omega x^1) \end{aligned}$$

for some constants α, ω, with all other components equal to zero (except those that are related to these by the antisymmetry condition (3)). Notice that the electric and magnetic components are **in phase** with each other: they both attain their maximum magnitudes at the same values of $x^0 - x^1$.

For realistic solutions, the field F_{ab} is nonzero only within a bounded region of space (which may grow with time), but simple solutions like constant fields and plane waves can be good approximations within sufficiently small regions.

Apply ∂^a to equation (4) and use (21) to get

$$\partial^a \partial_a F_{bc} = 0.$$

This says that in any region of spacetime where the charge and current densities are zero, each component of the EM field obeys the **wave equation**.

14 References

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