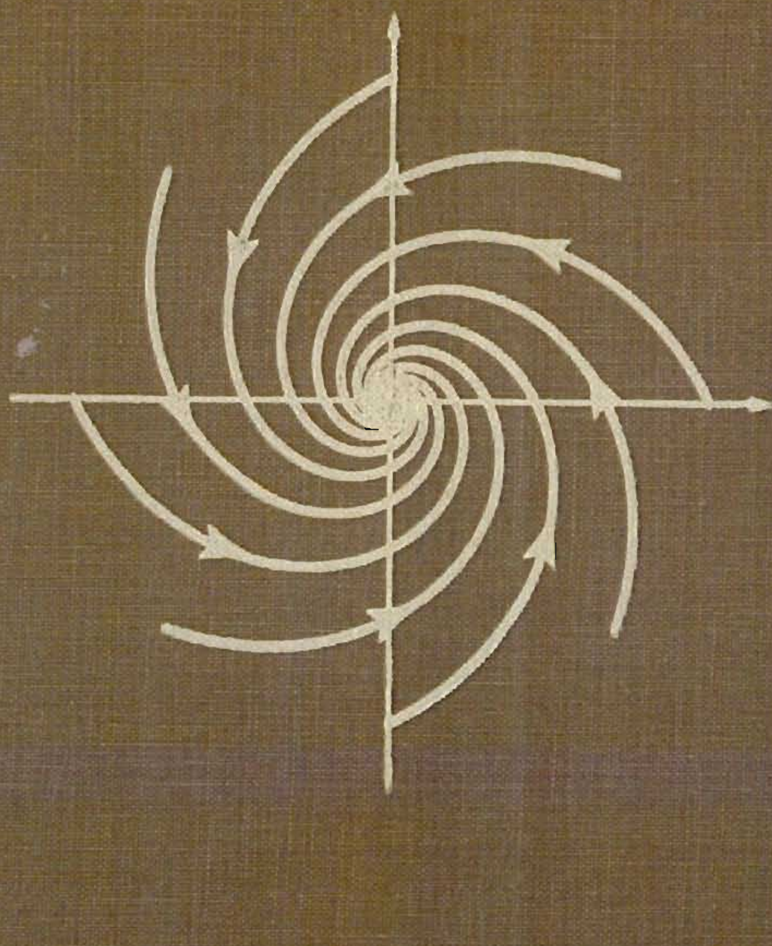
MATHEMATICAL METHODS for Scientists and Engineers

Donald A. McQuarrie



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for Scientists and Engineers

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Biographies and Historical Notes

Preface

I have written this book for students who have had one and a half or two years of calculus and little else. The most important prerequisite is that students realize the need to use mathematics in their studies or work. I should say at the outset that this is not a mathematics book in the sense that I do not prove many theorems and may have occasional lapses of the degree of rigor that would satisfy a pure mathematician. However, I have endeavored to present theorems that scientists and engineers might use in their work in a manner that is both accurate and intelligible. (I don't know who first said it, but there is a saying that, "Pure mathematicians don't trust applied mathematicians, and applied mathematicians don't understand pure mathematicians.") There are entire books that cover each of the topics that we discuss in a single chapter or even less, so we can give only an introduction to each. I have tried to make my treatment of each topic self-contained, but I would consider it a great success if you became interested enough in any topic that you sought out further study by going to a more detailed treatment—not because my treatment is opaque, but because you want to know more. The references at the end of each chapter should get you started in this direction.

The first chapter is a review of calculus, which some readers may find easy or superfluous, while others may find it to be helpful. Its purpose is to bring everyone up to speed and to provide practice for those whose math is rusty. Although the treatment is elementary, I introduce the ϵ-δ notation for the definition of limits and continuity and then discuss the idea of uniform continuity and uniform convergence of integrals. Even though your interest may not lie in mathematical rigor, you should be aware of when interchanging limiting operations is permissible. For example, when can we write

$$\frac{dF}{dx} = \int_0^\infty \frac{\partial f(x, t)}{\partial x} dt$$

if $F(x) = \int_0^\infty f(x, t)\, dt$? Chapter 2, in which we discuss series, is also a review of material that is treated in all calculus courses. The use and manipulation of series play such an important role throughout applied mathematics that it is important to appreciate the concept of convergence and when certain operations such as term-by-term differentiation and term-by-term integration are valid. In Chapter 3, we introduce a number of non-elementary functions, such as the gamma function, the error function, and the Dirac delta function, that are defined by certain integral expressions. Then, in Chapter 4, we discuss complex numbers, the complex plane, and, very briefly, the properties of functions of complex variables. We introduce vectors in Chapter 5 and illustrate the power of vector notation by applying it to a number of problems in analytic geometry that are fairly easy using vector notation but would be tedious without it. Functions of more than one variable are discussed in Chapter 6. This material leads into Chapter 7, where we discuss vector calculus, which is indispensable throughout the sciences and engineering. After discussing various coordinate systems in Chapter 8, we go on to linear algebra and vector spaces in Chapter 9 and then matrices and eigenvalue problems in Chapter 10. The next four chapters constitute a segment on differential equations, including nonlinear differential equations and phase space in Chapter 13 and special functions and Sturm-Liouville theory in Chapter 14. The next two chapters treat Fourier series and their application to solving partial differential equations by the method of separation of variables. We continue our study of partial differential equations in Chapter 17, where we discuss integral transforms, particularly Laplace transforms and Fourier transforms. The need to invert Laplace transforms leads naturally to functions of complex variables and integration in the complex plane, which we discuss in Chapter 18. Complex variable theory is one of the most profound and beautiful subjects in applied mathematics, and all science and engineering students should have some familiarity with this subject, even if their work doesn't often require it. In Chapter 19, we show how complex variable theory can be used to evaluate real integrals, to sum series into closed forms, to solve boundary value problems, and to solve fluid-flow problems. The final two chapters discuss probability theory and mathematical statistics. In particular, we discuss confidence intervals, goodness-of-fit tests, and regression and correlation in the last chapter.

No one can learn this material (or anything else in the sciences or in engineering, for that matter) without doing lots of problems. For this reason, I have provided at least 15 to 20 problems at the end of each section. These problems sometimes serve to fill in gaps or to extend the material presented in the section, but they are most often used to illustrate applications of the material. In all, there are almost 3000 problems in the book, and I have provided answers to many of them at the back of the book.

A number of powerful commercial computer packages are available nowadays that can be used to solve many of the problems in this book. These programs not only provide numerical answers, but they can also perform algebraic manipulations, and for that reason they are called computer algebra systems (CAS). Some of the prominent CAS are MatLab, Maple, Mathematica, and MathCad. I happen to know and use Mathematica, and I have presented examples of one-line Mathematica commands throughout the book that can be used to solve given problems. These commands are just meant to provide examples of the utility of any CAS, and there are a number of problems that ask you to "use any CAS to solve . . . " These programs are so available and user-friendly that you might wonder at times "Why do I need to learn all the stuff in this book when I could use a CAS to solve my problem?" I think that anyone with experience would agree when I say that these programs are a wonderfully useful supplement to the material in this book, but are no substitute for it. During the writing of this book, I found countless examples where a thoughtless use of a CAS would lead you astray. Furthermore, many problems are such that you need to apply mathematical knowledge to get them into a form that the CAS can handle. In spite of the friendliness of CAS, as in most things, you have to know what you're doing first in order to use them with confidence.

A singular feature of the book is the inclusion of biographies at the beginning of each chapter. Many of the mathematicians that we refer to were rather colorful characters, and I personally find it enjoyable reading about them. I wish to thank my publisher for encouraging me to include them and my wife, Carole, for researching the material and for writing every one of them. Each one could easily have been several pages long, and it was difficult to cut them down to one page. We both wish to acknowledge a terrific website at the University of St. Andrews in Scotland (www-history.mcs.st-and.ac.uk, and yes, the www- is correct) that lists hundreds of biographies of famous mathematicians, as well as other mathematical subjects.

You read in many prefaces that "this book could not have been written and produced without the help of many people," and it is definitely true. I am particularly grateful to my reviewers, Dennis DeTurck of the University of Pennsylvania, Scott Feller of Wabash College, David Wunsch of the University of Massachusetts at Lowell, Mervin Hanson of Humboldt State University, and Heather Cox of the California Institute of Technology, who slogged through first drafts of all the chapters and who made many great suggestions. I am also grateful to my son, Allan, of the Johns Hopkins University Applied Physics Laboratory, who contributed a great deal to Chapters 21 and 22. I also wish to thank Christine Taylor and her staff at Wilsted & Taylor Publishing Services for coordinating the entire project, especially Caroline Roberts and Melody Lacina for correcting all my spelling errors. I thank as well Bob Ishi for designing his usual beautiful-looking and inviting book, Jane Ellis for dealing with many of the production details and for procuring all the photographs and likenesses for the biographies, Mervin Hanson for rendering over 700 figures in Mathematica and keeping them all straight in spite of countless alterations, John Murdzek for very helpful copyediting, Paul Anagnostopoulos for composing the entire book, and my publisher Bruce Armbruster and his wife and associate, Kathy, for being the best publishers around and good friends in addition. Finally, I wish to thank my wife, Carole, for preparing the manuscript in TEX, for reading many of the chapters, and for being my best critic, in general.

There are bound to be both typographical and conceptual errors in a book of this breadth and length, and I would appreciate your letting me know about them so that they can be corrected in subsequent printings. I also would welcome general comments, questions, and suggestions either at mquarrie@mcn.org or through the University Science Books website, www.uscibooks.com, where any ancillary material or notices will be posted.

Donald A. McQuarrie

CHAPTER 1

Functions of a Single Variable

This first chapter is a review of a number of topics from your beginning calculus course. It assumes that you haven't forgotten how to differentiate and that you have access to a table of integrals such as the *CRC Standard Mathematical Tables and Formulas*. In addition, there are a number of computer programs such as Mathematica, Matlab, Maple, and MathCad that have sophisticated numerical *and* symbolic capabilities. We'll refer to these programs as *computer algebra systems* (CAS) and illustrate their use a number of times throughout the book. Any student of applied mathematics should become comfortable with one of these programs.

Some mathematical methods books start off with more advanced topics, assuming that you already know or are at least familiar with the material in this chapter. Some of you will find this chapter to be fairly easy, while others will have to work through some of the problems to get back up to speed. Every one of you can become quite proficient at mathematics at the level presented throughout the book, however, by doing lots of problems and thereby gaining experience and confidence. The chapter starts with a definition of the idea of a function of a single variable and then goes on to discuss limiting processes and limits. We will introduce the ϵ–δ notation, a concise and precise notation that is worth the effort to learn. The logical topic after studying limits is that of continuity, which is defined through a limiting process. Then we go on to define derivatives of functions of a single variable and then we take up integrals, again defined by a limiting process. We spend some amount of time on what are called improper integrals, integrals whose limits are infinite or whose integrands are unbounded (blow up) somewhere in the range of integration. The last section deals with the notion of the uniform convergence of integrals, which may be new to many of you. It's in this section that we'll learn about the properties of a function $F(x)$ if it is defined by

$$F(x) = \int_0^\infty f(x, t)dt$$

Notice that we are integrating over t, so that the resulting integral is a function of x. We might ask under what conditions is $F(x)$ a continuous function of x; or when can we differentiate with respect to x under the integral sign to write the derivative of $F(x)$ as

$$F'(x) = \int_0^\infty \frac{\partial f(x,t)}{\partial x} dt$$

Most of the examples and problems in this chapter are what you might call "just math problems". The material in this chapter and the next few chapters is background for later chapters, where many physically motivated problems are discussed.

1.1 Functions

Recall from calculus that a function is a rule that relates one number, x, to another, y. We express this relationship by writing $y = f(x)$, where f represents the function. The set of values of x for which $f(x)$ is defined is called the *domain* of the function and the set of all values of y produced from all the x is called the *range* of f. If only one value of y is produced from each value of x, then the function is said to be *single-valued*. If more than one value of y is produced from a value of x, then f is said to be *multiple-valued*. We will show later that a multiple-valued function can be viewed as a collection of single-valued functions, called *branches*, and so we will assume that all our functions are single-valued. Some authors even require a function to be single-valued, but we'll adopt the somewhat more liberal definition given above.

Let's look at some examples. Consider the relation $y = x^2$, or $y = f(x) = x^2$, for values of x given by $-2 \le x \le 2$. In this case, f is single-valued because each value of x leads to only one value of y. Note that the domain of f is the interval $[-2, 2]$ and the range of f is $[0, 4]$. The notation $[-2, 2]$ corresponds to the *closed interval*, $-2 \le x \le 2$. We call the interval closed because it includes its endpoints, -2 and 2. The notation for the corresponding *open interval*, $-2 < x < 2$, is $(-2, 2)$. If the domain of $f(x) = x^2$ had been the open interval $(-2, 2)$, then the range of f would have been $[0, 4)$. Now consider $y^2 = x$, where $0 < x \le 1$. (We denote the interval for x by $(0,1]$.) Solving for y, we obtain $y = \pm\sqrt{x}$, showing that there are two values of y for each value of x. We can view this relationship as corresponding to two single-valued functions, $y = f_1(x) = \sqrt{x}$ and $y = f_2(x) = -\sqrt{x}$. Note that f_1 and f_2 have the same domains, but completely different ranges, $(0,1]$ and $[-1, 0)$.

Strictly speaking, a function is denoted by f and the value obtained when f is applied to x is denoted by $y = f(x)$. However, it is common practice to call $f(x)$ a function, and a "function of x", in particular. We even write $y = y(x)$ to indicate that the value y results when the rule for sending x into y is applied to x. This notation is very common and very convenient. In any case, x is called the *independent variable* and y is called the *dependent variable*.

There are two broad classes of functions, *algebraic functions* and *transcendental functions*. An algebraic function, $y(x)$, is a solution to the polynomial equation

$$p_0(x)y^n + p_1(x)y^{n-1} + \cdots + p_{n-1}(x)y + p_n(x) = 0$$

where the $p_j(x)$s are polynomials in x. If $y(x)$ can be expressed as the ratio of two polynomials, then it is a *rational algebraic function*; otherwise it is an *irrational algebraic function*. For example, $y = (x^2 + 2)/(x - 1)$ is a rational algebraic function and $y = (x^2 + 2)/\sqrt{x - 1}$ is an irrational algebraic function.

Functions that are not algebraic functions are called transcendental functions. Exponential functions, logarithmic functions, trigonometric functions (Figures 1.1 through 1.3), and hyperbolic functions (Figures 1.4 through 1.6) are examples of transcendental functions. Recall that the hyperbolic functions are

$$
\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} & \cosh x &= \frac{e^x + e^{-x}}{2} \\
\operatorname{csch} x &= \frac{2}{e^x - e^{-x}} & \operatorname{sech} x &= \frac{2}{e^x + e^{-x}} \\
\tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} & \coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}}
\end{aligned}
\tag{1}
$$

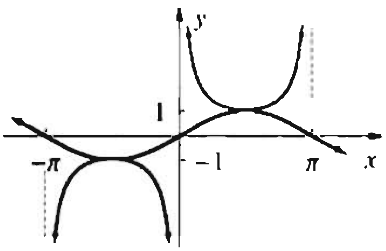

Figure 1.1
The trigonometric functions $\sin x$ and $\csc x = 1/\sin x$ (color) plotted against x. The asymptotes of $\csc x$ are shown as the colored dashed lines and the y axis.

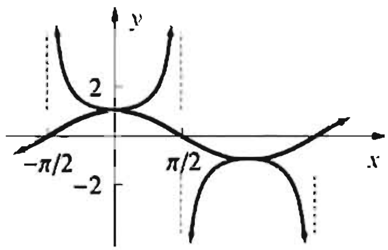

Figure 1.2
The trigonometric functions $\cos x$ and $\sec x = 1/\cos x$ (color) plotted against x. The asymptotes of $\sec x$ are shown as the colored dashed lines.

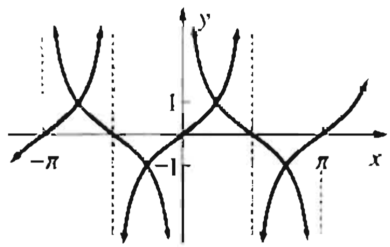

Figure 1.3
The trigonometric functions $\tan x$ and $\cot x = 1/\tan x$ (color) plotted against x. The asymptotes of $\tan x$ are shown as the dashed lines at $x = -\pi/2$ and $\pi/2$. The asymptotes of $\cot x$ are shown as the colored dashed lines and the y axis.

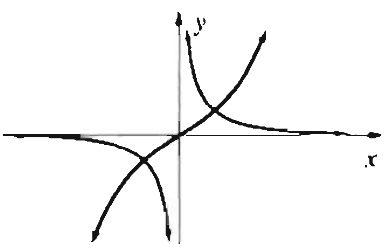

Figure 1.4
The hyperbolic functions $\sinh x$ and $\operatorname{csch} x = 1/\sinh x$ (color) plotted against x.

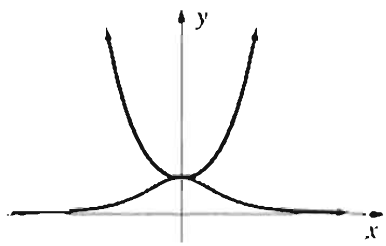

Figure 1.5
The hyperbolic functions $\cosh x$ and $\operatorname{sech} x = 1/\cosh x$ (color) plotted against x.

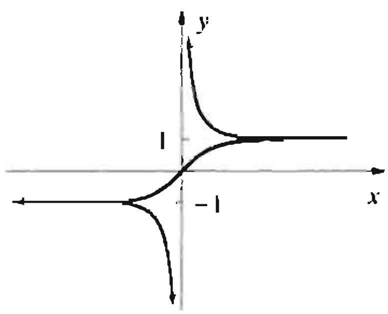

Figure 1.6
The hyperbolic functions $\tanh x$ and $\coth x = 1/\tanh x$ (color) plotted against x.

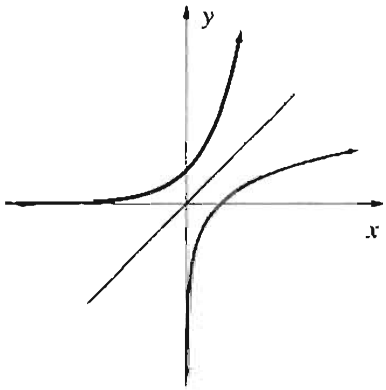

Figure 1.7
The exponential e^x and the logarithmic function $\ln x$ (color) plotted against x. Note that the two function are symmetric about the line $y = x$.

The exponential function e^x and the logarithmic function $\ln x$ bear a special relation to each other. If $y = y(x) = e^x$, then $x = x(y) = \ln y$. The two functions, $y(x) = e^x$ and $x(y) = \ln y$ are *inverses* of each other; the function e^x sends x into $y = e^x$ and the function $x = \ln y$ sends y back into x. Using $y = e^x$ and $x = \ln y$, we see that $e^{\ln y} = y$ and that $x = \ln e^x$. The graphs of $y = e^x$ and $x = \ln y$ are shown in Figure 1.7. Note that one graph can be obtained from the other by simply interchanging the x and y axes, or what amounts to the same thing, by flipping either curve about the line $y = x$.

Note that the correspondence between x and y in Figure 1.7 is one-to-one, in that each value of x corresponds to exactly one value of y and each value of y corresponds to exactly one value of x. We express such a correspondence by

$$y = f(x) \quad \text{and} \quad x = f^{-1}(y)$$

where f^{-1} is the *inverse function* of f. For example, if $y = f(x) = 2x - 3$, then $x = f^{-1}(y) = (3 + y)/2$. [Figure 1.8 shows the relation between $y = f(x)$ and $x = f^{-1}(x)$.] The function $y = f(x) = x^2$ does not have a unique inverse because $x = \pm\sqrt{y}$. The inverse function in this case is double-valued, and its two branches are $\sqrt{y}$ and $-\sqrt{y}$. We can choose to work with either one of these branches so long as we keep the restriction $x \geq 0$ or $x \leq 0$ in mind. Problem 2 has you show that the graphs of $y(x) = x^2$ and $y(x) = \sqrt{x}$ are symmetric about the line $y = x$.

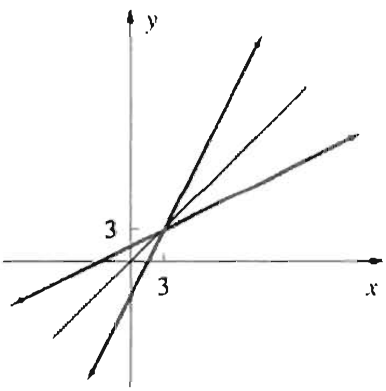

Figure 1.8
The relation between the function $y = 2x - 3$ and its inverse $x = (3 + y)/2$ [plotted as $(3 + x)/2$)]. Note that the two functions are symmetric about the line $y = x$, just as in Figure 1.7.

A class of important inverse functions are the inverse trigonometric functions. Consider $x = \sin y$. We assign an angle y in radians and calculate a unique value of x. But as Figure 1.9 shows, $\sin y$ is periodic in y with a period 2π; in other words, $\sin y$ repeats itself every 2π units along the y axis, or in an equation, $\sin y = \sin(y + 2\pi n)$ where $n = 0, \pm 1, \pm 2, \ldots$. Consequently, the inverse function, which we write as $y = \sin^{-1} x$, is hardly unique, in the sense that many values of y (actually, an infinite number in this case) correspond to the same value of x, as shown in Figure 1.10. For example, if $x = \sqrt{2}/2$, then $y = \pi/4$, $\pi/4 + 2\pi$, $\pi/4 + 4\pi$, or generally $\pi/4 + 2\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$. Thus, the inverse function $y = \sin^{-1} x$ has an infinite number of branches, one for each value of n. As Figure 1.10 shows, the value of y will be determined uniquely from the value of x if we restrict y to the values $-\pi/2 \leq y \leq \pi/2$. Thus, we can write the inverse of $x = \sin y$ as $y = \sin^{-1} x$ with the restriction $-\pi/2 \leq y \leq \pi/2$. We call this function the *principal branch* of $\sin^{-1} x$ and we call the value of y the *principal value* of $\sin^{-1} x$ (Figure 1.11). The other inverse trigonometric functions have similar restrictions, and their principal branches are

$$\begin{aligned} y &= \sin^{-1} x & -\pi/2 \leq y \leq \pi/2 \\ y &= \cos^{-1} x & 0 \leq y \leq \pi \\ y &= \tan^{-1} x & -\pi/2 \leq y \leq \pi/2 \end{aligned} \tag{2}$$

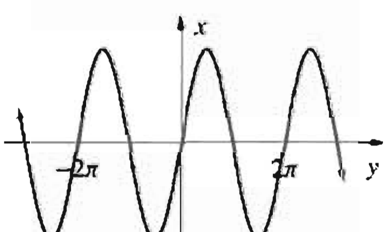

Figure 1.9
The function $x = \sin y$ is a periodic function of y with a period 2π.

(See Figures 1.12 and 1.13.) We shall see in Section 4 that the apparently arbitrary domains for the inverse trigonometric functions in Equation 2 result in simple formulas for their derivatives.

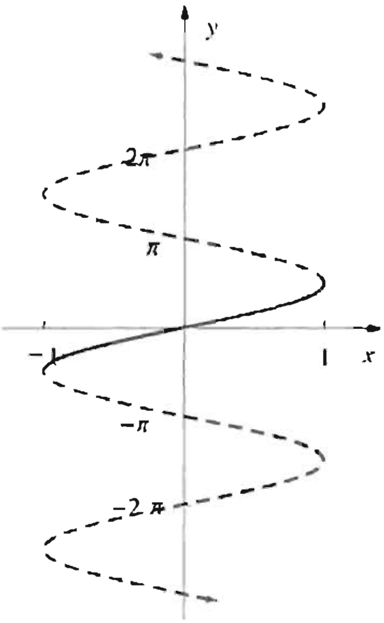

Figure 1.10
The function $\sin^{-1} x$ is a multiple-valued function of x. Note that $\sin x$ and $\sin^{-1} x$ can be obtained from one another by interchanging the x and y axes. The principal branch is the solid line.

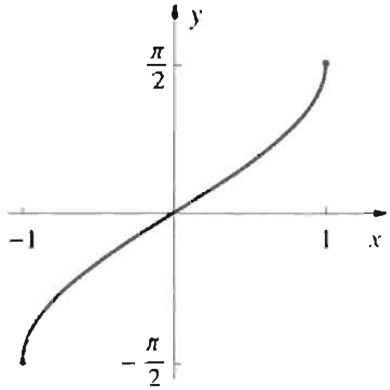

Figure 1.11
The principal branch of $y = \sin^{-1} x$.

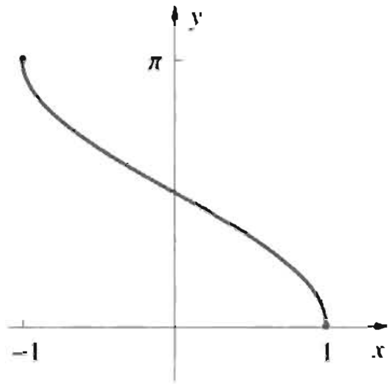

Figure 1.12
The principal branch of $y = \cos^{-1} x$.

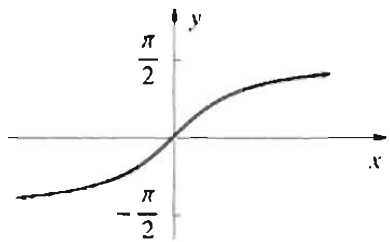

Figure 1.13
The principal branch of $y = \tan^{-1} x$.

Example 1:
Show that

$$\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$$

if $x < 1$.

SOLUTION: Let $\alpha = \sin^{-1} x$, so that $x = \sin \alpha$. Figure 1.14 illustrates that $\sin \alpha = x/1$. Then

$$\tan \alpha = \frac{x}{\sqrt{1-x^2}}$$

and so

$$\alpha = \sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$$

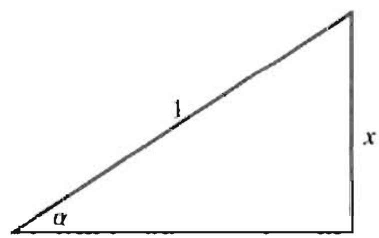

Figure 1.14
The right triangle used in Example 1.

Another common notation for the inverse trigonometric functions is $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$. If they are restricted to their principal values, then they are often denoted by $\text{Arcsin}\,(x)$, $\text{Arccos}\,(x)$, and $\text{Arctan}\,(x)$.

Using the definitions of the hyperbolic functions given by Equations 1, we can derive explicit expressions for the inverses. Consider $y = \sinh x = (e^x - e^{-x})/2$. If we let $z = e^x$, then we have $y = (z - 1/z)/2$. We can rearrange this expression into $z^2 - 2yz - 1 = 0$, and solving for $z = e^x$ gives

$$z = e^x = y \pm \sqrt{y^2 + 1}$$

Taking the logarithm gives

$$x = \sinh^{-1} y = \ln\left(y \pm \sqrt{y^2 + 1}\right)$$

We reject the negative sign above because the argument of the logarithm must be positive. Interchanging y and x for conformity gives us

$$y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad \text{all } x$$

Example 2:
Derive an explicit expression for $\cosh^{-1} x$.

SOLUTION: Let $z = e^x$ in Equation 1 for $\cosh x$ and rearrange to obtain $z^2 - 2yz + 1 = 0$. Solving for $z = e^x$ gives

$$z = e^x = y \pm \sqrt{y^2 - 1} \qquad y \geq 1$$

Choosing the $+$ sign as the principal value and taking logarithms gives

$$x = \cosh^{-1} y = \ln\left(y + \sqrt{y^2 - 1}\right) \qquad y \geq 1$$

Interchanging y and x, we obtain

$$y = \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geq 1$$

Note that the restriction $x \geq 1$ in the expression for $\cosh^{-1} x$ is related to the fact that $\cosh u \geq 1$ for any value of u. (See Figure 1.15.)

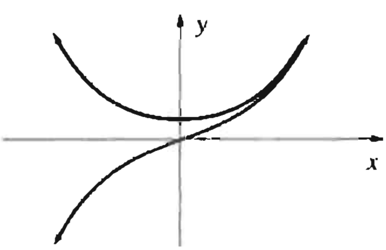

Figure 1.15
The functions $\sinh x$ (color) and $\cosh x$ plotted against x. Note that $\cosh x$ is symmetric and that $\sinh x$ is antisymmetric about the y axis.

Expressions for other inverse hyperbolic functions can be found in many mathematical tables, such as the *CRC Standard Mathematical Tables*.

Figure 1.15 shows graphs of $\sinh x$ and $\cosh x$. Note that $\cosh x$ is symmetric about the y axis and that $\sinh x$ changes sign when it is reflected through the y axis. Analytically, these properties are expressed by $\cosh(-x) = \cosh(x)$ and

$\sinh(-x) = -\sinh(x)$. Generally, a function with the property that $f(-x) = f(x)$ is called an *even function* of x and one with the property that $f(-x) = -f(x)$ is called an *odd function* of x. Not all functions are even or odd, but any function can be written as a sum of an even function and an odd function by writing

$$f(x) = \left[\frac{f(x) + f(-x)}{2}\right] + \left[\frac{f(x) - f(-x)}{2}\right]$$

We shall see later that a recognition of the parity of a function can be very useful when integrating.

Example 3:
Prove that $\sinh x$ is an odd function of x.

SOLUTION:

$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2} = -\frac{e^{x} - e^{-x}}{2} = -\sinh x$$

1.1 Problems

1. Determine the maximum domains of the real-valued functions

   (a) $y = \sqrt{16 - x^2}$ (b) $y = \dfrac{1}{x^2 + 6}$

   (c) $y = \ln x$ (d) $y = \dfrac{1}{x - 1}$

2. Show that the graphs of $y(x) = x^2$ and $y = \sqrt{x}$ are symmetric about the line $y = x$.

3. Plot the functions (a) $y = |x|$, (b) $y = -|x|$, and (c) $y = 1 - |x|$ for $-3 \le x \le 3$.

4. The Heaviside step function, $H(x)$, is defined by $H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$. Plot the function $y(x) = H(x) - H(x - 1)$.

5. Plot the function defined by $f(x) = 2 - 6(x - 1)H(x - 1) + 4(x + 4)H(x - 3)$, where $H(x)$ is defined in Problem 4.

6. Plot the function defined by

$$f(t) = t - 2(t - 2)H(t - 2) + 2(t - 4)H(t - 4) - 2(t - 6)H(t - 6) + \cdots$$

7. Plot $y(x) = x/|x|$ for $-10 < x < 10$.

8. Plot $y(x) = x - |x|$ for $-10 < x < 10$.

9. Consider the points $(-c, 0)$ and $(c, 0)$. Derive an equation for the set of all points (x, y) such that the sum of the distances from (x, y) to $(-c, 0)$ and to $(c, 0)$ is a constant $= 2a$. Do you recognize the equation? What is a called?

10. How would the resulting equation in Problem 9 change if the center were changed to $(2, -1)$ instead of at $(0, 0)$?

11. Consider a vertical line L at $x = -p$ and the point $(p, 0)$. Now derive an equation for the set of all points equidistant from the line L and the point $(p, 0)$. Do you recognize this curve?

12. Derive the relations $\sin^{-1} x + \cos^{-1} x = \pi/2$ and $\tan^{-1} x + \cot^{-1} x = \pi/2$. *Hint*: Use Figure 1.16 and $\alpha + \beta = \pi/2$ for the first part.

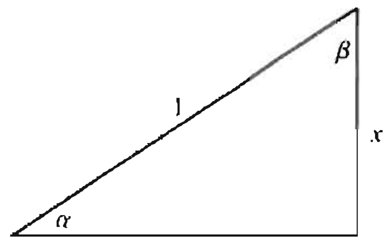

Figure 1.16
The geometry for Problem 12.

13. Show that $\cot^{-1} x = \tan^{-1}(1/x) = \pi/2 - \cot^{-1}(1/x)$. *Hint*: Use Figure 1.17 and $\alpha + \beta = \pi/2$ for the first part.

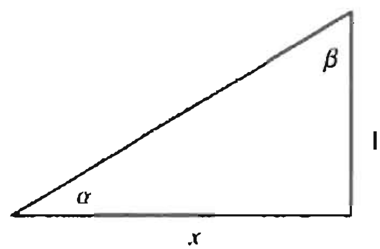

Figure 1.17
The geometry for Problem 13.

14. Show that $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \qquad |x| < 1.$

15. Which of the following functions is periodic? What are their periods?

(a) $\tan 2x$ (b) $|\cos x|$ (c) $\dfrac{\sin x}{x}$

16. Classify each of the following functions as even, odd, or neither.

(a) $\tanh x$ (b) $e^x \sin x$ (c) $\dfrac{e^x}{(e^x + 1)^2}$ (d) $\cos x + \sin x$

17. Show that the equation $\tanh x = e^{-2x}$ is equivalent to $2x = \sinh^{-1} 1$.

18. Prove the triangle inequality, $|a + b| \le |a| + |b|$. *Hint*: Add the two inequalities $-|a| \le a \le |a|$ and $-|b| \le b \le |b|$.

19. Determine $f(f(x))$ if $f(x) = x^2 + 1$.

20. Find $g(f(y))$ if $g(x) = 1/(1+x)$ and $f(y) = 1/y^2$.

21. Show that $y = f(x) = (x+1)/(x-1)$ is its own inverse; in other words, show that $x = (y+1)/(y-1)$.

22. Find the values of x for which

(a) $x^2 + 2 < 3x$ (b) $-2 < \dfrac{3x+1}{x-1} < 2$

23. Plot the function $f(x) = \sum_{n=0}^{\infty} H(x-n)$.

24. Plot the function $f(x) = \sum_{n=0}^{\infty} (-1)^n H(x-n)$.

1.2 Limits

In this section, we are going to discuss the idea of a limit, but first we need to define a neighborhood. The set of all points x such that $a - \delta < x < a + \delta$ where $\delta > 0$ is called a *δ-neighborhood* of the point a. The interval $a - \delta < x < a + \delta$ can be written as $|x - a| < \delta$ (Problem 1). If the point a is excluded, then we have the set of all points x such that $0 < |x - a| < \delta$, which is called a *deleted δ-neighborhood* of a. Now let $f(x)$ be defined and single-valued for all x in a deleted δ-neighborhood of a. We say that $f(x)$ has the limit l as x approaches a if for any positive number ϵ, however small, we can find some positive number δ such that

$$|f(x) - l| < \epsilon \qquad \text{if} \quad 0 < |x - a| < \delta$$

and write

$$\lim_{x\to a} f(x) = l \qquad \text{or} \qquad f(x) \to l \qquad \text{as} \quad x \to a \tag{1}$$

In other words, we can make $f(x)$ as close as we wish to l so long as we choose x sufficiently close to a, *but not equal to a itself*. In fact, $f(x)$ need not even be defined at a. The value of δ may depend upon the value of ϵ, so we will often write $\delta(\epsilon)$.

Let's use this ϵ-δ notation to show that $\lim_{x\to 1}(3x+2) = 5$. Having chosen an ϵ, however small, we must find a δ such that $|(3x+2) - 5| < \epsilon$ whenever $0 < |x - 1| < \delta$. First note that $|(3x+2) - 5| = 3|x - 1|$. Now if we take δ to be $\epsilon/3$, then $|(3x+2) - 5| = 3|x - 1| < \epsilon$ if $0 < |x - 1| < \delta$.

Example 1:
Use the ϵ-δ notation to show that $\lim_{x\to 0} x\sin(1/x) = 0$.

SOLUTION: The limit l is equal to 0, and so we wish to show that

$$\left| x\sin\frac{1}{x} - 0 \right| < \epsilon$$

if $0 < |x - 0| < \delta$. The value of $|\sin(1/x)| \le 1$ for all $x \ne 0$, and so

$$\left| x\sin\frac{1}{x} \right| = |x| \left| \sin\frac{1}{x} \right| \le |x| < \delta$$

where the last inequality follows from $0 < |x - 0| < \delta$. Now if we just let $\delta = \epsilon$, we get

$$\left| x\sin\frac{1}{x} \right| < \epsilon \quad \text{if} \quad |x| < \delta = \epsilon$$

and so $\lim_{x\to 0} x\sin(1/x) = 0$ as was to be proved (see Figure 1.18). Note that $f(x) = x\sin(1/x)$ is not even defined at $x = 0$.

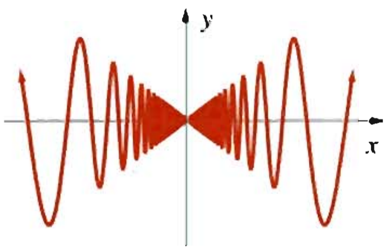

Figure 1.18
The function $f(x) = x\sin(1/x)$ plotted against x for small values of x.

It is easy to prove that limits have the following properties:

$$\lim_{x\to a}[\alpha f(x) + \beta g(x)] = \alpha \lim_{x\to a} f(x) + \beta \lim_{x\to a} g(x)$$

$$\lim_{x\to a} f(x)g(x) = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad \left(\lim_{x\to a} g(x) \ne 0\right) \tag{2}$$

$$\lim_{x\to a} x^r = a^r \qquad (a > 0)$$

where α, β, and r are any real numbers.

Although the ϵ–δ notation provides a formal definition of a limit, we shall usually determine limits more directly. Let's look at $\lim_{x\to 0} \dfrac{\sqrt{x+16} - 4}{x}$. This limit is of the form 0/0, so we certainly cannot let $x = 0$. Figure 1.19 shows $f(x) = (\sqrt{x+16} - 4)/x$ plotted against x, and it seems to go to a finite limit (1/8) as $x \to 0$.

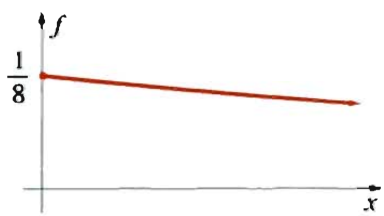

Figure 1.19
The function $f(x) = (\sqrt{x+16} - 4)/x$ plotted against x for small values of x.

To see that this is indeed so, multiply and divide by $\sqrt{x+16} + 4$ to obtain

$$\lim_{x\to 0} \frac{\sqrt{x+16} - 4}{x} \cdot \frac{\sqrt{x+16} + 4}{\sqrt{x+16} + 4}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+16} + 4)} = \lim_{x\to 0} \frac{1}{\sqrt{x+16} + 4} = \frac{1}{8}$$

where we realize that $x + 16 \to 16$ as $x \to 0$.

Example 2:

Investigate $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x+3} - \frac{1}{3}\right)$.

SOLUTION: The function $f(x)$ is plotted in Figure 1.20.

$$\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x+3} - \frac{1}{3}\right) = \lim_{x\to 0} \frac{-x}{3x(x+3)} = \lim_{x\to 0} \frac{-1}{3(x+3)} = -\frac{1}{9}$$

as Figure 1.20 suggests.

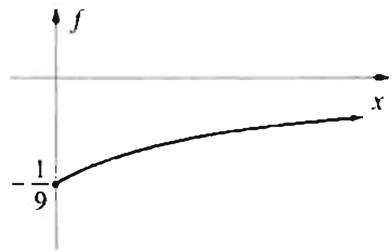

Figure 1.20
The function $f(x) = [1/(x+3) - 1/3]/x$ plotted against x for small values of x.

We often wish to find the limit of a function $f(x)$ as $x \to \infty$. What we mean in this case is that there exists a number N such that $|f(x) - l| < \epsilon$ whenever $x > N$. Clearly, functions like $1/x$, $1/x^2$, etc. have the limit 0 as $x \to \infty$. How about something like

$$f(x) = \frac{2x^2 + 6x - 1}{3x^2 - 2x + 7}$$

We rewrite $f(x)$ as

$$f(x) = \frac{2x^2}{3x^2}\left[\frac{1 + \frac{3}{x} - \frac{1}{2x^2}}{1 - \frac{2}{3x} + \frac{7}{3x^2}}\right] = \frac{2}{3}\left[\frac{1 + \frac{3}{x} - \frac{1}{2x^2}}{1 - \frac{2}{3x} + \frac{7}{3x^2}}\right]$$

and use the fact that $1/x$ and $1/x^2$ go to zero to obtain $\lim_{x\to\infty} f(x) = 2/3$. A more demanding example is given in the following Example.

Example 3:

Show that $\lim_{x\to\infty} (x - \sqrt{x^2 + a}) = 0$, where a is a constant.

SOLUTION: Multiply and divide by $x + \sqrt{x^2 + a}$ to get

$$\lim_{x\to\infty} \frac{-a}{x + \sqrt{x^2 + a}} = 0$$

Example 4:

Find $\lim_{x\to\infty} x(x - \sqrt{x^2 - a})$, where a is a constant.

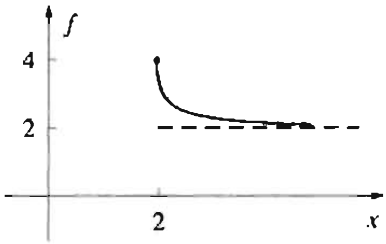

Figure 1.21
The function $f(x) = x(x - \sqrt{x^2 - 4})$ plotted against x. The graph (color) begins at the point (2, 4) and the asymptote is shown as a gray dashed line.

SOLUTION: Multiply and divide by $x + \sqrt{x^2 - a}$ as in the previous Example to get

$$\lim_{x\to\infty} \frac{ax}{x + \sqrt{x^2 - a}} = \lim_{x\to\infty} \frac{ax}{x + x\sqrt{1 - \frac{a}{x^2}}} = \lim_{x\to\infty} \frac{ax}{2x} = \frac{a}{2}$$

Figure 1.21 shows the graph of $x(x - \sqrt{x^2 - 4})$ plotted against x. Notice that it approaches 2 as x gets large.

A limit that we shall use fairly often is

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{3}$$

This limit has a nice geometric proof, which is developed in Problem 2. Notice once again that $\sin x/x$ is not defined at $x = 0$, nor need it be. You may remember that limits of the type 0/0 (like the one above), ∞/∞, and some others like these are called *indeterminate forms* and can be readily evaluated by l'Hôpital's rule. We shall derive l'Hôpital's rule in Section 6.

In many applications we are interested in the limit of a function at some point a when x approaches a from one side only. If $f(x)$ has the limit l when x approaches a through positive values of $x - a$ (in other words, from the right), we write

$$\lim_{x\to a+} f(x) = l_+ \tag{4}$$

We will often use an equivalent notation, which reads

$$\lim_{x\to a+} f(x) = \lim_{\epsilon\to 0} f(a + \epsilon) \tag{5}$$

where $\epsilon > 0$. This limit is called the *right-hand limit* of $f(x)$ at a, and is sometimes denoted by $f(a+)$. Of course, we can have left-hand limits also, in which case

$$\lim_{\epsilon\to 0} f(a - \epsilon) = l_- = f(a-) \tag{6}$$

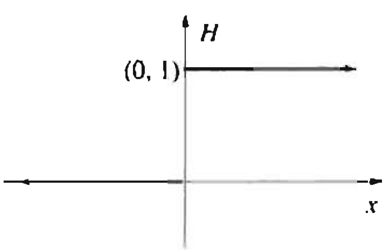

Figure 1.22
The Heaviside step function. $H(x) = 0$ when $x < 0$ and $H(x) = 1$ when $x > 0$.

A good example of a function that has different right-hand and left-hand limits is the Heaviside step function (Figure 1.22), defined by

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{7}$$

In this case, $\lim_{\epsilon\to 0} H(-\epsilon) = 0$ and $\lim_{\epsilon\to 0} H(\epsilon) = 1$. As you might have guessed already, a function will have different right-hand and left-hand limits at a point where it is discontinuous.

The next Example illustrates different right-hand and left-hand limits *and* infinite limits. If $|f(x)| > N$, where N is a positive number, however large, as $\epsilon \to 0$, then we say that $|f(x)| \to \infty$ as $\epsilon \to 0$.

Example 5:
Investigate the behavior of $f(x) = (x+2)/(x-1)$ near the point $x = 1$.

SOLUTION: It is clear that $\lim_{x\to 1} f(x)$ does not exist. But let's look at the two one-sided limits.

$$\lim_{x\to 1-} \frac{x+2}{x-1} = \lim_{\epsilon\to 0} \frac{1-\epsilon+2}{1-\epsilon-1} = -\infty$$

$$\lim_{x\to 1+} \frac{x+2}{x-1} = \lim_{\epsilon\to 0} \frac{1+\epsilon+2}{1+\epsilon-1} = \infty$$

Figure 1.23 shows $f(x)$.

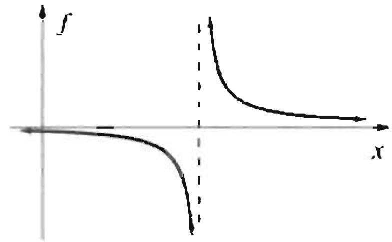

Figure 1.23
The function $f(x) = (x+2)/(x-1)$ plotted against x near the point $x = 1$. The asymptote of $f(x)$ is indicated by the vertical dashed line.

You should be aware that a number of the commercially available computer programs, such as Mathematica, Maple, Matlab, and MathCad, can evaluate limits. For example, the one-line command in Mathematica

```
Limit [ ( ( 3+x )^2 - 9 ) / x,  x→0 ]
```

gives 6. These programs not only carry out numerical calculations but can perform algebraic manipulations as well. Consequently, they are often referred to as Computer Algebra Systems (CAS). We shall refer to them collectively as CAS. To encourage you to learn how to use one of these CAS, Problems 15 through 20 ask you to use any one of them to evaluate some limits.

1.2 Problems

1. Show that $a - \delta < x < a + \delta$ can be written as $|x - a| < \delta$.
2. Use Figure 1.24 to prove that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

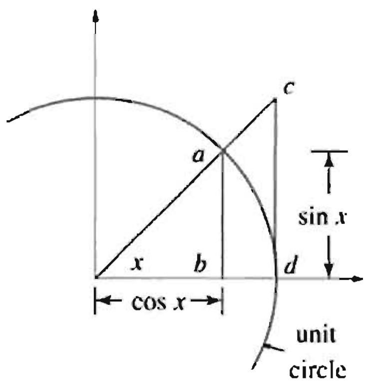

Figure 1.24
Geometry associated with the proof that $\lim_{x\to 0} (\sin x)/x = 1$.

3. Use the result of Problem 2 to show that $\dfrac{1-\cos x}{x} \to 0$ as $x \to 0$. Notice that this ratio is of the limiting form 0/0.

4. Find the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$ (b) $\lim\limits_{x\to 0} \dfrac{\sin x}{x^{1/2}}$

(c) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{x^2}$ (d) $\lim\limits_{x\to 0} \dfrac{\tan 2x}{3x}$

5. Find the following limits:

(a) $\dfrac{\sin 2x}{\sin x}$ as $x \to 0$ (b) $\dfrac{1+\cos \pi x}{\tan^2 \pi x}$ as $x \to 1$

(c) $\dfrac{x^2-9}{x+3}$ as $x \to -3$ (d) $\dfrac{1-\cos x}{x^2}$ as $x \to 0$

6. Find the limit of $[f(x+h) - f(x)]/h$ as $h \to 0$ for the following functions:

(a) $f(x) = x^2$ (b) $f(x) = 1/x$

(c) $f(x) = \sin x$ (d) $f(x) = \cos x$

7. Find the limits of the following functions as $x \to \infty$:

(a) $\sqrt{x+a} - \sqrt{x+b}$ (b) $(\sqrt{x+1} - \sqrt{x})\sqrt{x+\frac{1}{2}}$

8. Find the limit of $\dfrac{\sqrt{x+2}-\sqrt{2}}{x}$ as $x \to 0$.

9. Determine the one-sided limits of $f(x) = x/|x|$.

10. Evaluate $\lim\limits_{x\to 0+} \dfrac{1}{1+e^{-1/x}}$.

11. Given that $u_n = \sqrt{12 + u_{n-1}}$ has a limit, find its value.

12. Suppose that $f(x) \le g(x) \le h(x)$ for all values of x in some deleted neighborhood of a. If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, prove that $\lim\limits_{x\to a} g(x) = L$. Some authors call this result the *squeeze law*.

13. Prove that $\lim\limits_{x\to a} f(x)g(x) = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right]$.

14. The function $y = 1/(1-x)^2$ is unbounded as $x \to 1$. Calculate the value of δ such that $y > 10^6$ if $|x-1| < \delta$.

The next 6 problems involve using any CAS to evaluate limits. Use any CAS to find the following limits:

15. $\lim\limits_{x\to 1} \left(\dfrac{x-1}{\sqrt{x}-1}\right)$ **16.** $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-3x+2}$ **17.** $\lim\limits_{x\to \infty} \dfrac{2x^2+xe^{-x}}{3x^2-2x+1}$

18. $\lim\limits_{x\to 2+} \dfrac{x^2-4}{(x-2)^2}$ **19.** $\lim\limits_{x\to 2-} \dfrac{x^2-4}{(x-2)^2}$ **20.** $\lim\limits_{x\to 0+} \dfrac{(x^3-1)|x|}{x}$

1.3 Continuity

A function $f(x)$ is continuous at a if the following three conditions hold:

1. $f(x)$ is defined at a
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

We can summarize these three conditions by writing

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right) = f(a)$$

If $f(x)$ does not satisfy these conditions, then $f(x)$ is said to be discontinuous at a. For example, the function $1/x$ is discontinuous at $x = 0$ because $f(x)$ is not defined at $x = 0$, nor does $\lim_{x \to 0} f(x)$ exist. The function $f(x) = (x^2 - 1)/(x - 1)$ is discontinuous at $x = 1$ because $f(1)$ is not defined, but using the fact that $x^2 - 1 = (x + 1)(x - 1)$, we see that $\lim_{x \to 1} f(x) = \lim_{x \to 1}(x + 1) = 2$, so that condition 2 holds. In this case we say that $x = 1$ is a *removable discontinuity*.

A more formal definition of continuity is that $f(x)$ is continuous at $x = a$ if it is possible to find a δ, which may depend upon both ϵ and a, such that $|f(x) - f(a)| < \epsilon$, however small, if $|x - a| < \delta$. Using this δ-ϵ definition, it is easy to show using Equation 2.2 that if $f(x)$ and $g(x)$ are continuous at $x = a$, so are $f(x) + g(x)$, $f(x)g(x)$, and $f(x)/g(x)$ provided $g(a) \neq 0$.

Intuitively, a discontinuity is a jump in the graph of the function. For example, the Heaviside step function (Figure 1.22) has a jump discontinuity at $x = 0$, and the function defined by

$$f(x) = \begin{cases} x^2 + 1 & -1 \leq x \leq 1 \quad \text{but} \quad x \neq 0 \\ 0 & x = 0 \end{cases}$$

is discontinuous at $x = 0$ (Figure 1.25).

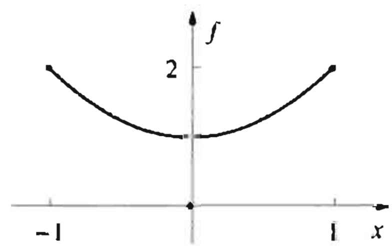

Figure 1.25
The discontinuous function $f(x) = x^2 + 1$ for $-1 \leq x \leq 1$ but $x \neq 0$ and $f(x) = 0$ for $x = 0$ plotted against x.

Another type of discontinuity is displayed by the function $1/(1 - x)^2$ at $x = 1$ (Figure 1.26). We say that $1/(1 - x)^2$ has an infinite discontinuity at $x = 1$.

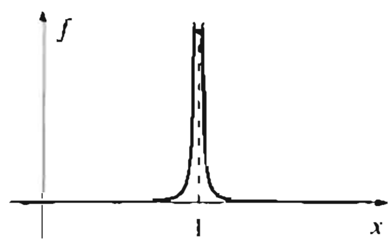

Figure 1.26
The function $f(x) = 1/(1 - x)^2$ plotted against x near the point $x = 1$.

Just as we have right-hand and left-hand limits, we have continuity from the right and continuity from the left. For example, we say that $f(x)$ is continuous from the right at a if $\lim_{x \to a+} f(x) = f(a)$, or more simply, if $f(a+) = f(a)$. A function is continuous at $x = a$ if it is continuous both from the right and from the left at $x = a$.

We say that a function is continuous in an interval if it is continuous at all points in the interval. If the interval is closed, then if $f(x)$ is continuous, it must be continuous at the end points a and b; in other words, it must be continuous from the left at b and continuous from the right at a.

Example 1:
Is $f(x) = |x| - x$ continuous for $-\infty < x < \infty$?

SOLUTION: $f(x) = 2x$ when $x \leq 0$ and $f(x) = 0$ when $x \geq 0$. Thus $f(x)$ is continuous over the entire x axis (Figure 1.27).

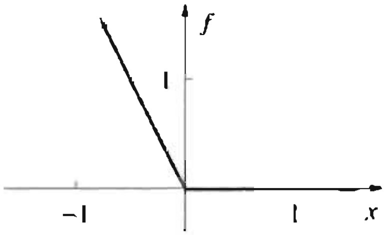

Figure 1.27
The function $f(x) = |x| - x$ plotted against x.

Example 2:
Is $f(x) = 1/x$ continuous in the interval $[0, 1]$? In $(0, 1]$?

SOLUTION: Because $x = 0$ is included in the interval $[0, 1]$, $f(x)$ is not continuous in that interval because $f(x)$ is not defined at $x = 0$. The only possible troublesome region in the half-open interval is near $x = 0$. To investigate this region, let $r > 0$ be as small as you wish. Then, we need to find a $\delta = \delta(\epsilon, r)$ such that

$$\left| \frac{1}{r} - \frac{1}{r+\delta} \right| < \epsilon$$

Solving this inequality for δ gives $\delta < \epsilon r^2/(1 - \epsilon r) \approx \epsilon r^2$, so $f(x)$ is continuous on the half-open interval $(0, 1]$. Note that δ depends upon both ϵ and r in this case.

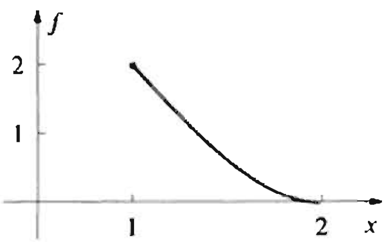

Figure 1.28
The function $f(x) = x^3 - 3x^2 + 4$ defined on the half-open interval $[1, 2)$.

There are a number of properties of continuous functions that we use frequently (and intuitively). One property is given by the *extreme value theorem*:

> *If $f(x)$ is continuous in the closed interval $[a, b]$, then it has a maximum value and a minimum value in the interval.*

Note that the interval must be closed for the extreme value theorem to apply. For example, consider $f(x) = x^3 - 3x^2 + 4$ defined on the half-open interval $[1, 2)$ (Figure 1.28). This function attains a maximum value at $x = 1$, but attains no minimum value since $x = 2$, the only possibility for yielding a minimum value of $f(x)$ according to Figure 1.28, is not included in the interval. No matter how close we get to $x = 2$, there are smaller values of $f(x)$ as x gets closer to 2. The necessity of the continuity of the function can be seen by considering the function (Figure 1.29)

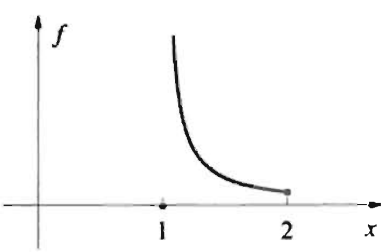

Figure 1.29
The function $f(x) = 1/(x - 1)$ when $1 < x \leq 2$ and $f(x) = 0$ when $x = 1$ plotted against x.

$$f(x) = \begin{cases} \dfrac{1}{x-1} & 1 < x \leq 2 \\ 0 & x = 1 \end{cases}$$

This function is not continuous on the closed interval $[1, 2]$ because the $\lim_{x \to 1+} f(x)$ does not exist. It attains minimum values at $x = 1$ and at $x = 2$, but attains no maximum value. No matter how closely we approach $x = 1$ from the right, there are larger values of $f(x)$ as x gets closer to 1.

Another useful property of continuous functions is given by the *intermediate value theorem*:

> *If $f(x)$ is continuous in the closed interval $[\,a, b\,]$, then $f(x)$ assumes every intermediate value between $f(a)$ and $f(b)$. So if η is a number such that $f(a) \le \eta \le f(b)$, then there is at least one point c in $[a, b]$ such that $f(c) = \eta$.*

A consequence of this theorem says that if $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ will be zero for at least one value of x in $[\,a, b\,]$.

To see that the function in the intermediate value theorem must be continuous, note that the Heaviside step function attains *no* intermediate values between $y = 0$ and $y = 1$ (Figure 1.22).

Example 3:
Show that $f(x) = x^2 + x - 1$ has at least one zero [a point at which $f(x) = 0$] in the interval $[\,0, 1\,]$.

SOLUTION: $f(x)$ is continuous in $[\,0, 1\,]$ and $f(1) = 1$ and $f(0) = -1$. Therefore, $f(x)$ must cross the x axis at least once in $[\,0, 1\,]$ (Figure 1.30).

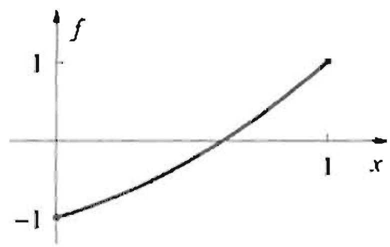

Figure 1.30
The function $f(x) = x^2 + x - 1$ plotted against x in the interval $[\,0, 1\,]$.

The Heaviside step function is an example of a function that is not continuous over the whole interval $(-\infty, \infty)$, but *is* continuous over the two subintervals $(-\infty, 0)$ and $(0, \infty)$. Suppose that a function $f(x)$ is not continuous over an entire interval $[\,a, b\,]$, but the interval can be divided into a finite number of subintervals where $f(x)$ is continuous over each subinterval and finite at the end points of each subinterval. Then $f(x)$ is said to be *sectionally continuous* or *piecewise continuous* over $[\,a, b\,]$. The Heaviside step function is piecewise continuous over the x axis. Another example of a piecewise continuous function is

$$f(x) = \begin{cases} 0 & x < -a \\ -V_0 & -a < x < a \\ 0 & x > a \end{cases}$$

which is the potential used for a quantum-mechanical particle in a finite well (Figure 1.31).

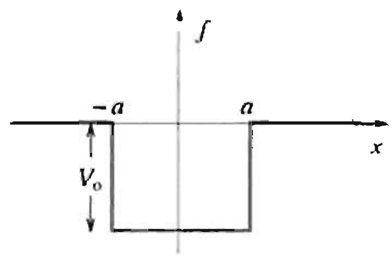

Figure 1.31
The function $f(x) = 0$ for $x < -a$, $f(x) = -V_0$ for $-a < x < a$, and $f(x) = 0$ for $x > a$.

We shall discuss one last brief topic before leaving this section. Go back to Example 2 and notice that the value of δ depended upon both ϵ and r. Let $f(x)$ be continuous in an interval. If it is possible to find a value of δ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ for each point of the interval and where δ is *independent* of a, then $f(x)$ is said to be *uniformly continuous* in the interval.

Example 2 shows that $f(x) = 1/x$ is not uniformly continuous in the half-open interval (0,1). (It is not even continuous in the closed interval [0, 1].) Clearly if a function is uniformly continuous, then it is continuous.

1.3 Problems

1. Use the ϵ-δ notation to prove that $f(x) = x^2$ is continuous at $x = 2$.
2. Is $f(x) = (x^4 + x^3 - 3x + 2x - 1)/(x - 1)$ continuous at $x = 1$?
3. Prove that the equation $2x^5 + 2x + 1 = 0$ has at least one solution between 1 and -1.
4. Prove that $\cos x = x$ has a solution between 0 and $\pi/2$.
5. Graph the function $f(x) = |x - 1|/(x - 1)$. Find

   (a) $\lim_{x\to 1-} f(x)$ (b) $\lim_{x\to 1+} f(x)$ (c) $\lim_{x\to 1} f(x)$
6. At what points is $f(x) = x \csc x$ discontinuous?
7. Prove that $f(x) = x^2$ is uniformly continuous in (0,1).
8. Show that $\sin x$ and $\cos x$ are continuous functions for all values of x. *Hint*: Use the relation $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$.
9. Discuss the difference in behavior of the two functions $f(x) = 1/(2 - x)$ and $g(x) = 1/(2 - x)^2$ at their points of discontinuity.
10. The function $f(x) = (x^3 - 1)/(x^2 - 1)$ is not defined at $x = 1$. Is the point $x = 1$ a removable discontinuity? What value must $f(x)$ be assigned at $x = 1$ in order to make it continuous there?
11. Show that the equation $x3^x = 2$ has at least one solution in the interval $0 < x < 1$.
12. Find α and β such that $f(x)$ given below is continuous for $0 < x < 2\pi$.

$$f(x) = \begin{cases} -\sin x & 0 < x < \pi/2 \\ \alpha\sin x + \beta & \pi/2 < x < 3\pi/2 \\ \left(x - \dfrac{3\pi}{2}\right)^2 & 3\pi/2 < x < 2\pi \end{cases}$$

13. Show that if $g(x)$ is continous at $x = a$ and if f is continuous at $g(a)$, then $f(g(x))$ is continuous at $x = a$.
14. Prove that if $f(x)$ and $g(x)$ are continuous at $x = a$, then so is $f(x)g(x)$. *Hint*: Use Equation 2.2.
15. Prove that if $f(x)$ and $g(x)$ are continuous at $x = a$, then so is $f(x)/g(x)$, provided $g(a) \neq 0$. *Hint*: Use Equation 2.2.

1.4 Differentiation

Recall that the derivative of a function $f(x)$ at a point a is the slope of the straight line tangent to $f(x)$ at a. The derivative of $f(x)$ at a point a is defined by the limiting process

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

Another commonly used notation is

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{y(a+\Delta x) - y(a)}{\Delta x} \tag{2}$$

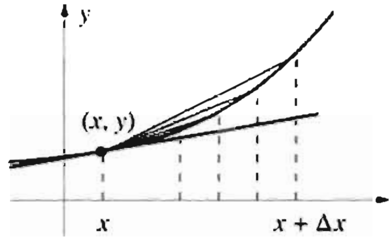

Figure 1.32
An illustration of the limiting process in the definition of the derivative of $y(x)$. As $\Delta x \to 0$, $[y(x+\Delta x) - y(x)]/\Delta x$ approaches the tangent line at the point (x, y).

Figure 1.32 illustrates this limiting process. In the beginning of your calculus course, you learned how to differentiate a number of functions using the above formulas. For example, if $y(x) = x^2 + 2$, then

$$\begin{aligned} f'(x) = \frac{dy}{dx} &= \lim_{\Delta x\to 0} \frac{(x+\Delta x)^2 + 2 - x^2 - 2}{\Delta x} \\ &= \lim_{\Delta x\to 0} (2x + \Delta x) = 2x \end{aligned}$$

A little more challenging is $y(x) = \cos x$:

$$\begin{aligned} \frac{d\cos x}{dx} &= \lim_{\Delta x\to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \end{aligned}$$

or

$$\frac{d\cos x}{dx} = \lim_{\Delta x\to 0} \cos x \left(\frac{\cos \Delta x - 1}{\Delta x}\right) - \sin x \lim_{\Delta x\to 0} \frac{\sin \Delta x}{\Delta x}$$

Problem 2.3 shows that $(\cos \Delta x - 1)/\Delta x \to 0$ as $\Delta x \to 0$, and so

$$\frac{d\cos x}{dx} = -\sin x \lim_{\Delta x\to 0} \frac{\sin \Delta x}{\Delta x}$$

But we saw in the previous section that this limit is equal to one, so we have

$$\frac{d\cos x}{dx} = -\sin x$$

Typically, in a calculus course, you use Equation 1 or 2 to differentiate a variety of functions and then use these results and some general rules such as $(uv)' = uv' + u'v$ and $(u/v)' = (vu' - uv')/v^2$ to differentiate just about anything. Mathematical tables have several pages of derivative formulas and most of the CAS can differentiate symbolically.

Example 1:
Differentiate $y = f(x) = x^2 e^{-x} \cos x$.

SOLUTION: Use $y' = (uvw)' = u'vw + uv'w + uvw'$:

$$y'(x) = 2xe^{-x} \cos x - x^2 e^{-x} \cos x - x^2 e^{-x} \sin x$$
$$= xe^{-x}[\,2 \cos x - x(\cos x + \sin x)]$$

It is often convenient to think of differentiation as an operation on a function $f(x)$. An operator is a symbol that tells you to do something to whatever function follows the symbol. For example, we can consider df/dx to be the d/dx operator acting upon $f(x)$. If we denote the differentiation operator by $\hat{D} = d/dx$, then we can write

$$\hat{D} f(x) = \frac{df}{dx}$$

You learn in calculus that if c_1 and c_2 are constants, then

$$\hat{D}[c_1 f_1(x) + c_2 f_2(x)] = c_1 \hat{D} f(x) + c_2 \hat{D} f_2(x) \tag{3}$$

An operator with this property is said to be a *linear operator*. Differentiation is a *linear* operation.

Just as we define left-hand and right-hand limits, we can define left-hand and right-hand derivatives by

$$f'(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$
$$f'(b) = \lim_{h \to 0-} \frac{f(b+h) - f(b)}{h}$$

We can say that a function has a derivative at some point $x = c$ if its right-hand and left-hand derivatives are equal at $x = c$. A function that has a derivative at all points in an interval is said to be differentiable in that interval. If the interval is closed, $[\,a, b\,]$, then not only must $f'(x)$ exist for all x in the open interval (a, b), but the end point derivatives must exist also.

Example 2:
Suppose that $f(x) = (\sinh x)/x$ in the interval $(0, 1\,]$ and that $f(x) = ax + b$ in the interval $[\,1, 2\,)$. Find the two constants a and b such that $f(x)$ and $f'(x)$ are continuous at $x = 1$. Plot the resulting function.

SOLUTION: Continuity of $f(x)$ at $x = 1$ gives $\sinh 1 = a + b$ and continuity of $f'(x)$ at $x = 1$ gives $\cosh 1 - \sinh 1 = a$. Solving for a and b gives $a = \cosh 1 - \sinh 1 = 1/e$ and $b = 2\sinh 1 - \cosh 1 = (e^2 - 3)/2e$. The resulting function

$$f(x) = \begin{cases} \dfrac{\sinh x}{x} & 0 < x \le 1 \\ \dfrac{x}{e} + \dfrac{e^2 - 3}{2e} & 1 \le x < 2 \end{cases}$$

is plotted in Figure 1.33.

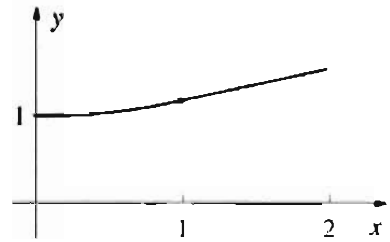

Figure 1.33
The function determined in Example 2 plotted against x.

You also learn to differentiate composite functions in a calculus class. A composite function is a function of a function. If $y = f(u)$ and $u = g(x)$, then $y = f(g(x))$ is a composite function of x. Recall that the derivative of y with respect to x is given by the *chain rule*.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

For example, if $y(x) = \sin(x^2 + 2)$, then we have $y = f(u) = \sin u$ and $u = x^2 + 2$, so

$$\frac{dy}{dx} = (\cos u)2x = 2x\cos(x^2 + 2)$$

Example 3:
Find dy/dx if $y(x) = e^{-(x^2+a^2)^{1/2}}$ where a is a constant.

SOLUTION: We use the chain rule with $y = e^{-u}$ and $u = (x^2 + a^2)^{1/2}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = (-e^{-u})\left[\frac{x}{(x^2 + a^2)^{1/2}}\right] \\ &= -\frac{xe^{-(x^2+a^2)^{1/2}}}{(x^2 + a^2)^{1/2}}\end{aligned}$$

We don't have to stop at first derivatives, do we? We can take sequential derivatives to form second derivatives, third derivatives, and so on. For example, if $y(x) = x^2e^x$, then

$$y'(x) = \frac{dy}{dx} = \hat{D}_x(x^2e^x) = (2x + x^2)e^x$$

$$y''(x) = \frac{d^2y}{dx^2} = \hat{D}_x^2(x^2e^x) = \hat{D}_x[(2x + x^2)e^x] = (2 + 4x + x^2)e^x$$

where we have used the operator notation for a derivative by writing D_x for d/dx. Note that the notation $\hat{D}_x^2$ means $\hat{D}_x$ applied *sequentially*, so that $\hat{D}_x^2 f(x) = \hat{D}_x[\hat{D}_x(f(x))]$.

Frequently a function $y(x)$ is defined *implicitly* through a relation such as $f(x, y) = 0$, and it may not be possible or convenient to solve explicitly for $y(x)$. For example, the implicit function of y and x might be

$$f(x, y) = \frac{1}{x^3} - \frac{1}{y^2} + xy - 1 = 0$$

We can differentiate this equation with respect to x using the chain rule to obtain

$$-\frac{3}{x^4} + \frac{2}{y^3}\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

and solve for dy/dx:

$$\frac{dy}{dx} = \frac{3 - x^4 y}{x^4}\,\frac{y^3}{xy^3 + 2}$$

Usually the result will contain both y and x explicitly, but this is no problem. If we wanted to know the derivative at $x = 1$, we would substitute $x = 1$ into $f(x, y) = 0$ to find y (in this case $y = 1$) and then evaluate dy/dx at the point $(1, 1)$ to get $dy/dx = 2/3$.

Example 4:
Find the value of dy/dx at the point $(0, 1)$ if $y(x) = e^y \sin xy$.

SOLUTION: Differentiating implicitly gives

$$y' = y'e^y \sin xy + xy'e^y \cos xy + ye^y \cos xy$$

At $(0, 1)$, $y' = e$.

We said in Section 1 that the apparently arbitrary choices for the domains of the inverse trigonometric functions were made so that their derivatives have simple formulas. Let's see why this is so. Consider $y = \sin^{-1} x$, where $-\pi/2 \le y \le \pi/2$ (Figure 1.11). To find the derivative of $y = \sin^{-1} x$, we write $\sin y = x$ and then, differentiate implicitly with respect to x to obtain $y' \cos y = 1$, or

$$y' = \frac{1}{\cos y} = \pm\frac{1}{\sqrt{1 - \sin^2 y}} = \pm\frac{1}{\sqrt{1 - x^2}}$$

Now $\cos y \ge 0$ when $-\pi/2 \le y \le \pi/2$, and so we use the positive sign for y' over the entire range of x and write

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad |x| < 1$$

Example 5:
Show that

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \qquad |x| < 1$$

SOLUTION: We start with $y = \cos^{-1}x$, where $0 \le y \le \pi$ (Figure 1.12). Differentiating $x = \cos y$ implicitly with respect to x gives $-y' \sin y = 1$, or

$$y' = -\frac{1}{\sin y} = \pm\frac{1}{\sqrt{1-x^2}}$$

Now $\sin y \ge 0$ when $0 \le y \le \pi$ and so we use the negative sign for y' over the entire range of x and write

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \qquad |x| < 1$$

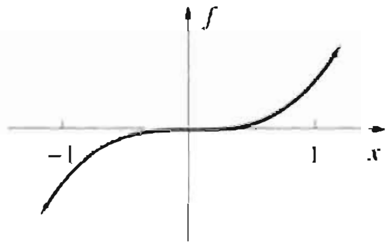

Figure 1.34
The function $f(x) = x^3$ plotted against x.

One of the principal applications of derivatives is to find the local maximum and minimum values (extrema) of a function in an interval. Points at which $f'(x) = 0$ are called *critical points* of $f(x)$. Although the condition $f'(c) = 0$ is used to locate extrema, it does not guarantee that $f(x)$ have a local extremum there. The simplest illustration of this is $f(x) = x^3$; $f'(0) = 0$, but $f(x)$ is not an extremum at $x = 0$ (see Figure 1.34).

Let $f(x)$ be continuous on the open interval (a, b) and let $f'(x)$ exist and be continuous in (a, b). If $f'(x) > 0$ in (a, c) and $f'(x) < 0$ in (c, b), then $f(x)$ is concave downward at c. A function $f(x)$ is concave downward at $x = c$ if the graph of $f(x)$ in the neighborhood of c lies below the tangent line at $x = c$ (Figure 1.35a). On the other hand, if $f'(x) < 0$ in (a, c) and $f'(x) > 0$ in (c, b), then $f(x)$ is concave upward at c. A function $f(x)$ is concave upward at $x = c$ if the graph of $f(x)$ in the neighborhood of c lies above the tangent line at $x = c$ (Figure 1.35b). The type of concavity is related to the sign of the second derivative, and so we have the second derivative test to determine if a critical point is a local extremum or not: If $f'(c) = 0$ and $f''(c)$ exists, then

1. if $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$
2. if $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$
3. if $f''(c) = 0$, then *no* conclusion can be drawn without further analysis

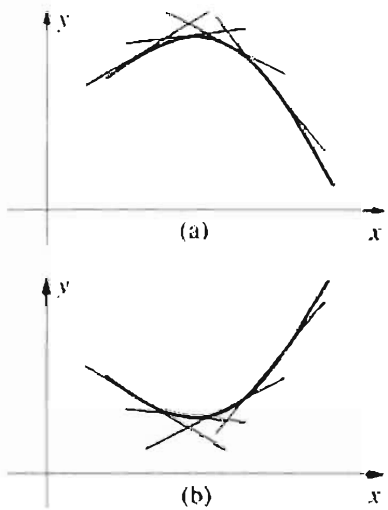

Figure 1.35
(a) The graph of a concave downward function. (b) The graph of a concave upward function.

A point $x = c$ is called an *inflection point* if $f(x)$ is concave upward on one side of c and concave downward on the other side. Consequently, $f''(x) = 0$ at an inflection point. It is not necessary that $f'(x) = 0$ at an inflection point. For example, the graph of $f(x) = (x-1)^3 + 2(x-1) + 1$ has an inflection point at $x = 1$ [$f''(1) = 0$] even though $f'(1) = 2$ there (Figure 1.36). Incidently, it seems that many students think that the condition $f''(c) = 0$ implies an inflection point at $x = c$, but this is not so. All three functions, $f(x) = x^4$, $g(x) = x^3$, $h(x) = -x^4$ have first and second derivatives that are equal to zero at $x = 0$, yet $f(x)$ has a relative minimum, $g(x)$ has an inflection point, and $h(x)$ has a relative maximum at $x = 0$. (See Figure 1.37.) If $f''(x) = 0$ at a critical point, you must investigate higher order derivatives to determine its nature (see Section 6.8).

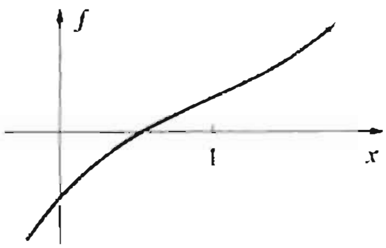

Figure 1.36
The function $f(x) = (x-1)^3 + 2(x-1) + 1$ plotted against x.

As another twist to consider, let's look at $f(x) = x^{2/3}$ defined on the interval $(-1, 1]$. In this case, $f'(x) = 2/(3x^{1/3})$, which diverges to ∞ as $x \to 0$ through positive values and to $-\infty$ as $x \to 0$ through negative values. Thus, $f'(x)$ is not defined at $x = 0$, yet $f(x)$ has a minimum value there (see Figure 1.38). This illustrates the fact that $f'(x)$ must exist and be continuous throughout the interval in order to use the above criteria.

We have assumed that $f(x)$ is defined over an open interval. If $f(x)$ is defined over a closed interval, you must also check the end points of the interval. Consider the function $f(x) = 2x^3 + 3x^2 - 12x - 5$ defined over the closed interval $[-3, 3]$. It has two critical points, one at $x = 1$ with $f''(1) = 18 > 0$ and one at $x = -2$ with $f''(-2) = -18 < 0$. Thus, there is a local minimum at $x = 1$ [with $f(1) = -12$] and a local maximum at $x = -2$ [with $f(-2) = 15$]. Although we find a local maximum at $x = -2$, it is not an *absolute* maximum because $f(x) = 40$ at its endpoint $x = 3$ (see Figure 1.39). The message here is that if $f(x)$ is defined over a closed interval, then you must examine the behavior of $f(x)$ not only at its critical points, but at its end points as well.

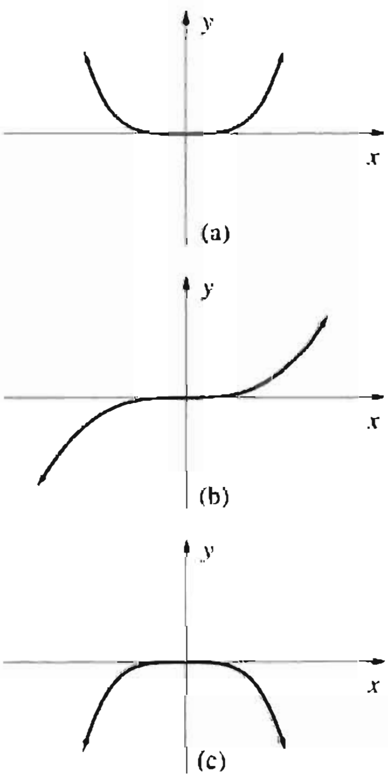

Figure 1.37
The functions (a) $f(x) = x^4$, (b) $g(x) = x^3$, and (c) $h(x) = -x^4$ plotted against x.

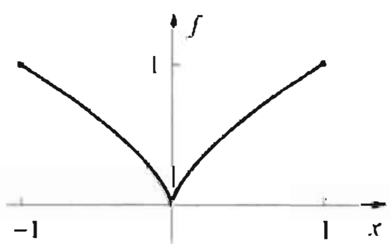

Figure 1.38
The function $f(x) = x^{2/3}$ defined on the closed interval $[-1, 1]$ plotted against x.

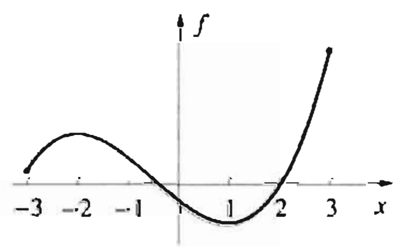

Figure 1.39
The function $f(x) = 2x^3 + 3x^2 - 12x - 5$ defined on the closed interval $[-3, 3]$ plotted against x.

Example 6:
Find the local extrema and the inflection points of $f(x) = x^2(1-x)^2$ over the entire x axis.

SOLUTION: The equation

$$f'(x) = 2x(1-x)^2 - 2x^2(1-x) = 2x(1-x)(1-2x) = 0$$

shows that there are critical points at $x = 0$, $x = 1/2$, and $x = 1$. The second derivative is

$$f''(x) = 12x^2 - 12x + 2$$

The fact that $f''(0) > 0$ and $f''(1) > 0$ and that $f''(1/2) < 0$ tells us that the critical points $x = 0$ and $x = 1$ are local minima and that $x = 1/2$ is a local maximum. The inflection points are given by $f''(x) = 0$, or at $x = (3 \pm \sqrt{3})/6$ (Figure 1.40).

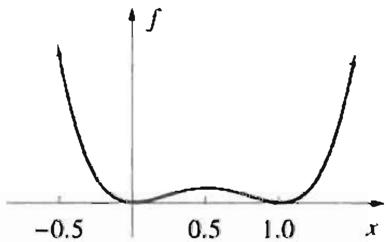

Figure 1.40
The function $f(x) = x^2(1-x)^2$ plotted against x.

Note from Figure 1.40 that $f(x) = x^2(1-x)^2$ is symmetric about the vertical line at $x = 1/2$. To see that this is so analytically, let $\xi = x - 1/2$, so that the function now reads $f(\xi) = (\frac{1}{2} + \xi)^2(\frac{1}{2} - \xi)^2$, which is an even function in ξ.

1.4 Problems

1. Differentiate

   (a) $(2+x)e^{-x^2}$; (b) $\dfrac{\sin x}{x}$; (c) $x^2 \tan 2x$

2. Differentiate

   (a) $(x-1)^{-3/2}$; (b) $\sqrt{x^2 - 3x + 1}$; (c) a^x

3. Differentiate

   (a) $\tan^{-1}(e^{-x})$; (b) $\ln(\sec x + \tan x)$; (c) $x^{\sin x}$

4. The tangent line to a curve at some point (a, b) has the slope $m = (dy/dx)_{x=a} = f'(a)$. Show that the slope of the line perpendicular to the curve at (a, b) is equal to $-1/m$.

5. Does $f(x) = |x|$ have a derivative at $x = 0$?

6. Prove that $f(x) = x^3$ is differentiable in the closed interval $[\,0, 1\,]$.

7. The graph corresponding to $2x^2 - 2xy + y^2 = 4$ is an ellipse whose major axis makes an angle with respect to the x axis. Plot the function. Show that the slopes of the tangent lines to this curve at the two points where it crosses the x axis are the same. In other words, show that the tangent lines at $(\pm\sqrt{2}, 0)$ are parallel. Do the same for the tangent lines at $(0, \pm 2)$, where the ellipse crosses the y axis.

8. Find the local extrema and the inflection points of $f(x) = 3x^4 - 4x^3 - 24x^2 + 48x - 20$ over the entire x axis.

9. Determine the minimum value of $f(x) = 1 + x^{2/3}$.

10. Prove that $f(x) = (x - 1)^2(x - 2)^2$ is symmetric about the vertical line located at $x = 3/2$.

11. Prove that if $y = f(x)$ and $x = g(y)$, then

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

Hint: Differentiate $f(g(y))$ as a composite function.

12. Show that $\dfrac{d}{dx}u^v = vu^{v-1}\dfrac{du}{dx} + (\ln u)u^v\dfrac{dv}{dx}$. *Hint*: Let $y = u^v$ and differentiate $\ln y$.

13. Use the result of the previous problem to find $\dfrac{d}{dx}x^x$.

14. Show that $\alpha = \beta$ in Figure 1.41. This property is known as the *reflection property of a parabola* and is the basis for a parabolic lens; incoming light is focused at the point F, the focus of the parabola. The equation of the parabola is $y^2 = 4px$, where p is the distance OF in Figure 1.41.

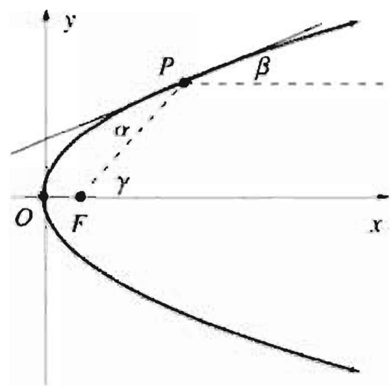

Figure 1.41
Illustration of the reflection property of a parabola. (See Problem 14.)

15. Notice that $f'(a)$ is defined by Equation 1. Normally we evaluate $f'(a)$ by finding $f'(x)$ and then letting $x = a$, but this procedure is not quite the same as using Equation 1. Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The derviative of $f(x)$ at $x = 0$ is, by *definition*,

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x}$$

Show that $f'(0) = 0$ in this case. Now determine $f'(x)$ and then let $x \to 0$ and show that the limit of $f'(x)$ as $x \to 0$ does not exist because $\lim_{x \to 0} \cos(1/x)$ does not exist. Therefore, we see that in this case, $f'(0) \neq \lim_{x \to 0} f'(x)$. The problem here is that $f'(x)$ is not continuous at $x = 0$. The functions that we deal with in physical problems almost always have continuous derivatives, but it's good to keep Equation 1 in mind.

16. The height of a body shot vertically upward is given as a function of time by $h(t) = 40t - 32t^2$. How high will it go?

17. Show that the rectangle of largest possible area for a given perimeter is a square.

18. Which points on the curve $xy^2 = 1$ are closest to the origin?

19. Two particles are moving in a plane according to the parametric equations $(3t, 4t^3 - 6t + 1)$ and $(3t + 1, 4t^3 - 8t + 2)$. How close do they come to each other?

20. Find the largest possible area of an isosceles triangle inscribed in a circle of radius r.

21. The blackbody radiation law is given by

$$\rho(\lambda, t) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda k_B T} - 1}$$

where $\rho(\lambda, t)d\lambda$ is the energy between λ and $\lambda + d\lambda$, λ is the wavelength of the radiation, h is the Planck constant, k_B is the Boltzmann constant, c is the speed of light, and T is the kelvin temperature. The Wien displacement law says that $\lambda_{max}T$ = constant where λ_{max} is the value of λ at which $\rho(\lambda, t)$ is a maximum. Derive the Wien displacement law from the blackbody radiation law. Show that "constant" $= hc/4.965k_B$.

22. Prove that if $f(x)$ has a derivative at $x = a$, then it must be continuous there.

23. It is easy to prove (Problem 22) that if $f(x)$ has a derivative at $x = a$, then it must be continuous at $x = a$. The converse is not necessarily true, however. For example, show that the function

$$y(x) = \begin{cases} x \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous at $x = 0$ but has no derivative there. Actually, mathematicians have constructed functions that are continuous at all points in an interval but differentiable at no points. (See page 573.) Fortunately, such pathological functions do not normally arise in physical applications.

24. Derive the formula for the derivative of the product of two functions.

25. Verify the second derivative conditions for a local extremum.

The next 4 problems have you use any CAS to differentiate functions symbolically.

26. Differentiate $f(x) = e^{-x}x^2 \sin^3(3x^2 + 2)$.

27. Evaluate $f'(1)$ from the previous problem.

28. Find the second derivative of $f(x)$ from Problem 26.

29. Evaluate $f''(1)$ for the previous problem.

1.5 Differentials

Calculus books emphasize that the expression dy/dx is a single quantity and *not* the ratio of dy over dx. Yet we later use expressions such as $dy = y'(x)dx$ with abandon. The use of $dy = y'(x)dx$ is justified, however, if we think of it as the *differential* of y. We also use differentials to estimate errors in measurements, as we shall see.

The derivative of $y = y(x)$ is given by

$$y'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{1}$$

where $\Delta y = y(x + \Delta x) - y(x)$. For small values of Δx then, we expect $y'(x)$ to be close to $\Delta y / \Delta x$, or

$$\Delta y \approx y'(x)\Delta x \qquad (\text{small } \Delta x) \tag{2}$$

We can rewrite this as

$$y(x + \Delta x) \approx y(x) + y'(x)\Delta x \tag{3}$$

Let's use this expression to estimate $\sin(0.10)$. In this case, we take $y(x) = \sin x$, $x = 0$, $\Delta x = 0.10$, and $y'(x) = \cos x$ in Equation 3 to get

$$\sin(0.10) \approx \sin 0 + (\cos 0)(0.10) = 0.10$$

The actual value is 0.09983 to four places.

Example 1:
Suppose that we estimate the volume of a sphere by measuring its circumference and find it to be 162 cm with an uncertainty of 0.20 cm. Estimate the uncertainty in the volume of the sphere.

SOLUTION: The volume is $4\pi r^3/3$ and the circumference is $C = 2\pi r$. Thus,

$$V = \frac{4\pi r^3}{3} = \frac{C^3}{6\pi^2}$$

Using Equation 3, we write

$$\text{uncertainty in } V = \Delta V = \left(\frac{dV}{dC}\right) \Delta C$$

So that

$$\Delta V = \left(\frac{C^2}{2\pi^2}\right) \Delta C = \frac{(162 \text{ cm})^2}{2\pi^2} (0.20 \text{ cm}) = 266 \text{ cm}^3$$

or about a 0.4% uncertainty in the sense that $\Delta V / V = (266 \text{ cm}^3 / 71800 \text{ cm}^3) = 0.0037$.

The increments Δy and Δx and the tangent line at x are shown in Figure 1.42. The quantity $\Delta y = y(x + \Delta x) - y(x)$ is the change in y that results if we continue along the curve and dy is the change in y that results if we continue along the tangent line from x to $x + \Delta x$. This change in y is the *differential* of y and is given by

$$dy = y'(x)\Delta x \tag{4}$$

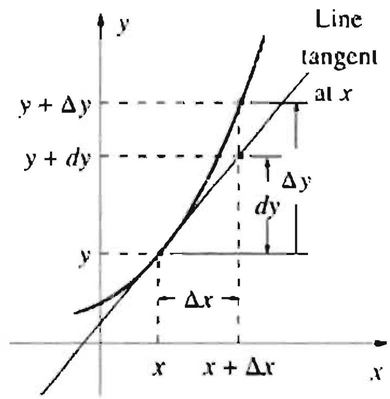

Figure 1.42
An illustration of the difference between dy and Δy.

Figure 1.42 suggests that the smaller the value of Δx, the closer Δy and dy become. To see that this is so, let $\Delta x = x - x_0$ and write Equation 1 as

$$\lim_{x \to x_0} \frac{\Delta y - y'(x_0)(x - x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\Delta y - y'(x_0)\Delta x}{x - x_0} = \lim_{x \to x_0} \frac{\Delta y - dy}{x - x_0} = 0 \tag{5}$$

Equation 5 implies that the difference between Δy and dy in Figure 1.42 not only goes to zero as $x \to x_0$, but goes to zero faster than $\Delta x = (x - x_0)$ goes to zero [for example, as $(\Delta x)^2$] because Δx appears in the denominator. To see what we mean by this, define ϵ by

$$\epsilon = \frac{\Delta y}{\Delta x} - y'(x_0) \tag{6}$$

If we substitute Equation 6 into Equation 5, we see that

$$\lim_{x \to x_0} \frac{\Delta y - y'(x)\Delta x}{\Delta x} = \lim_{x \to x_0} \frac{\epsilon\, \Delta x}{\Delta x} = \lim_{\Delta x \to 0} \epsilon = 0 \tag{7}$$

Thus we see that $\epsilon \to 0$ as $\Delta x \to 0$. Now multiply Equation 6 by Δx and use Equation 4 to write

$$\Delta y = dy + \epsilon\, \Delta x \tag{8}$$

Thus we see that $\Delta y \to dy$ even faster than Δx [more like $(\Delta x)^2$] as $\Delta x \to 0$.

Example 2:

Derive an expression for $\epsilon = \dfrac{\Delta y}{\Delta x} - \dfrac{dy}{dx}$ if $y = x^3 + x$ and show that $\epsilon \to 0$ as $\Delta x \to 0$.

SOLUTION:

$$\Delta y = y(x + \Delta x) - y(x) = (x + \Delta x)^3 + (x + \Delta x) - x^3 - x$$
$$= (3x^2 + 1)\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + 1 + 3x\Delta x + (\Delta x)^2$$

$$\epsilon = \frac{\Delta y}{\Delta x} - \frac{dy}{dx} = 3x\Delta x + (\Delta x)^2$$

Because $\Delta y = dy + \epsilon\, \Delta x$ and $\epsilon \to 0$ as $\Delta x \to 0$, Equation 2 can be written as

$$dy \approx y'(x)\Delta x + \epsilon\, \Delta x \tag{9}$$

which shows that

$$dy = y'(x)\, dx \tag{10}$$

is an excellent approximation in the sense that the correction term, $\epsilon\, dx$ goes to zero faster than Δx or dx go to zero.

Leibnitz, co-founder of calculus with Newton, introduced differential notation and considered the slope of the tangent line to be the ratio of the infinitesimal increments dy and dx, as shown in Figure 1.42. Realize, however, that dy and dx are *not* the limits of Δy and Δx as $\Delta x \to 0$, since these limits are necessarily zero, whereas dy and dx, as we have introduced them above, are not necessarily zero. Occasionally you'll see written that $\Delta y \approx y'(x)\Delta x$ becomes $dy = y'(x)dx$ when Δx becomes "infinitesimally small", where the quotation marks emphasize that the term "infinitesimally small" is vague. The early development of calculus was fraught with examples where certain quantities were "small" at one stage, zero at another, and then "small" again at a later stage in the same discussion. These vagaries were eventually ironed out by the introduction of limits as we use them today.

The introduction of differentials allows us to express derivatives in what is called differential notation

$$d(uv) = udv + vdu \qquad d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$

and so on.

Example 3:
Use differential notation to find dy/dx for $xy - 2x^2 - 1/y^2 = 6$.

SOLUTION: We have

$$xdy + ydx - 4xdx + \frac{2dy}{y^3} = 0$$

or

$$\frac{dy}{dx} = \frac{4x - y}{x + \dfrac{2}{y^3}} = \frac{4xy^3 - y^4}{xy^3 + 2}$$

1.5 Problems

1. Estimate $\sqrt[3]{120}$ from $\sqrt{125} = 5$.
2. The sides of a cube are observed to be 8.00 ± 0.02 cm. Estimate the error in its volume.
3. Derive an expression for $\epsilon = \dfrac{\Delta y}{\Delta x} - \dfrac{dy}{dx}$ if $y = x^3 + x^2 + 6x - 3$. Show that $\epsilon \to 0$ as $\Delta x \to 0$.
4. Derive an expression for $\epsilon = \dfrac{\Delta y}{\Delta x} - \dfrac{dy}{dx}$ if $y = 1/(x + a)$. Show that $\epsilon \to 0$ as $\Delta x \to 0$.
5. Find Δy and dy for $y = y(x) = x^2 - x$ at $x = 10.0$ and $\Delta x = 0.10$. Sketch a figure similar to Figure 1.42.
6. Find Δy and dy for $y = y(x) = x^{1/2}$ at $x = 4.00$ and $\Delta x = 0.35$. Calculate the absolute and relative errors in replacing Δy by dy.
7. Estimate the change in $\cos\theta$ if θ is changed from $25.00°$ to $25.20°$.
8. Find dy if $y =$

   (a) $\dfrac{x^{1/2}}{4}$ (b) $\dfrac{1}{4x^4}$

   (c) $\tan^2 x$ (d) $\dfrac{1}{2x^{1/2}}$
9. Find dy if $y =$

   (a) $(x^2 - 2)^{1/3}$ (b) $\sin\sqrt{x}$

   (c) $e^{\cos x}$
10. Does $y = (1 + 2\ln x)/(x - \ln x)$ satisfy the relation $(xy - y - 2)dx + (x^2 - x\ln x)dy = 0$?
11. Use differentials to find dy/dx for

   (a) $x^2 \sin y = \frac{1}{2}$ (b) $x^3 + y^2 = 6xy$ (c) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

1.6 Mean Value Theorems

There is a theorem concerning differentiable functions that is used often in both pure and applied mathematics. The theorem is called the *mean value theorem for derviatives*. Before we discuss this theorem, we shall discuss a somewhat simpler theorem called *Rolle's theorem*. Rolle's theorem says that

> *If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and if $f(a) = f(b) = 0$, then there must be at least one point ξ in (a, b) such that $f'(\xi) = 0$.*

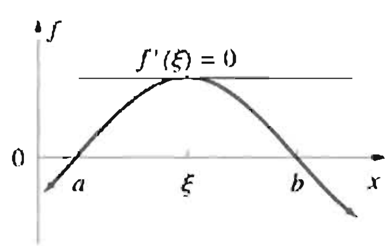

Figure 1.43
An illustration of Rolle's theorem.

(Figure 1.43) In other words, if the graph of a continuous function $f(x)$ intersects the x axis at $x = a$ and $x = b$, and if the function is differentiable between a and

b, then there is at least one point between a and b such that the derivative of $f(x)$ is zero. (The proof of Rolle's theorem is sketched in Problem 13.)

Now on to the mean value theorem for derivatives.

If $f(x)$ is continuous in the closed interval $[a, b]$ and differentiable in the open interval (a, b), then there is a point ξ in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \qquad a < \xi < b \tag{1}$$

Note that we can express Equation 1 in the form

$$f(x) = f(a) + f'(\xi)(x - a) \qquad a < \xi < x \tag{2}$$

Problem 14 sketches the proof of this theorem.

Equation 1 has a nice physical interpretation. Suppose that $f(b) - f(a)$ represents the distance between two points and that $b - a$ represents the time it takes to travel from a to b. Then $[f(b) - f(a)]/(b - a)$ represents the average speed for that trip. The mean value theorem (Equation 1) says that if you average $100 \text{ km} \cdot \text{hr}^{-1}$, say, then at some point during your trip, your instantaneous speed must be $100 \text{ km} \cdot \text{hr}^{-1}$.

An immediate consequence of the mean value theorem is that if $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then $f(x)$ is an increasing function in $[a, b]$ if $f'(x) > 0$ for all x in (a, b). To prove this, let x be in $[x_1, x_2]$ with $x_2 > x_1$ and use Equation 2 with a replaced with x_1 and x with x_2 to get

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$$

where $x_1 \le \xi \le x_2$. Since $x_2 > x_1$ and $f'(\xi) > 0$, we see that $f(x_2) > f(x_1)$. Conversely, if $f'(x) < 0$, then $f(x)$ is a decreasing function in $[a, b]$.

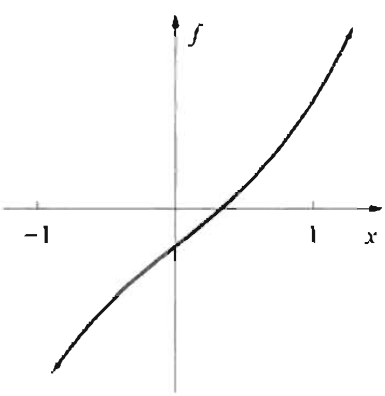

Figure 1.44
The function $f(x) = x^3 + 3x - 1$ plotted against x.

Example 1:
Show that $f(x) = x^3 + 3x - 1$ has exactly one (real) zero in the interval $[-1, 1]$.

SOLUTION: Since $f(x)$ is equal to -5 at $x = -1$ and $+3$ at $x = 1$, there is at least one zero in $[-1, 1]$. Because

$$f'(x) = 3x^2 + 3 > 0$$

for all x, $f(x)$ is an increasing function on the entire x axis and so cannot have more than one zero (Figure 1.44).

There is an extension of the mean value theorem that we can use to derive the formula for a Taylor series with a remainder. This *higher-order mean value theorem* says that

If $f(x)$ and its first n derivatives are continuous in $[a, b]$ and if $f^{(n+1)}(x)$ exists on (a, b), then there is at least one point ξ in $[a, b]$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \cdots$$
$$+ \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1} \quad (3)$$

The proof of this theorem is outlined in Problem 17. The mean value theorem is just Equation 3 with $n = 0$.

If we let $b = x$ in Equation 3, we get

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$
$$+ \cdots + \frac{f^n(a)}{n!}(x-a)^n + \frac{f^{n+1}(\xi)}{(n+1)!}(x-a)^{n+1} \quad (4)$$

where $a < \xi < x$. Equation 4 is the formula of a Taylor series with remainder, where the remainder is

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \quad (5)$$

We shall use this formula in later chapters.

Example 2:
Use Equation 4 to calculate the value of ln 1.200 to four-place accuracy.

SOLUTION: Let $f(x) = \ln x$ and $a = 1$ in Equation 4:

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$
$$+ \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \frac{(-1)^n}{n+1}(\xi - 1)^{n+1}$$

where $1 < \xi < 1.200$. We want the magnitude of the remainder term to be ≤ 0.00005. This term will be greatest if we choose $\xi = 1.200$ and so $|R_n| = (0.20)^{n+1}/(n+1)$ will equal 6.40×10^{-5} if $n = 4$; 1.077×10^{-5} if $n = 5$; and 1.83×10^{-6} if $n = 6$. Thus we will be assured four-place accuracy if we choose $n = 5$. This gives ln 1.200 = 0.18233 compared to the tabulated value 0.18232.

You probably remember l'Hôpital's rule from your calculus course. You used it to evaluate the limits of indeterminate forms such as 0/0, $0 \cdot \infty$, and ∞/∞. l'Hôpital's rule says that if $f(x)$ and $g(x)$ both approach zero or both approach $\pm\infty$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} \tag{6}$$

provided the right-hand limit exists. l'Hôpital's rule is included in this section because its proof follows immediately from a version of the mean value theorem (Problems 15 and 16).

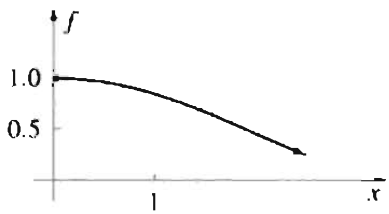

Figure 1.45
The behavior of the function $f(x) = (\sin x)/x$ as $x \to 0$.

For example, $\lim\limits_{x \to 0} \sin x/x$ is of the indeterminate form 0/0. l'Hôpital's rule tells us that

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

which agrees with our geometric proof in Section 2. Figure 1.45 shows the behavior of $\sin x/x$ as $x \to 0$. How about $\lim\limits_{x \to 0+} x \ln x$? This limit occurs fairly often in the physical chemistry of electrolyte solutions, such as aqueous solutions of sodium chloride. In this case, we have the indeterminate form $-0 \cdot \infty$, so let's look at

$$\lim_{x \to 0+} \frac{\ln x}{1/x} = \lim_{x \to 0+} \frac{1/x}{-1/x^2} = -\lim_{x \to 0+} x = 0$$

Figure 1.46 shows the behavior of $x \ln x$ as $x \to 0$.

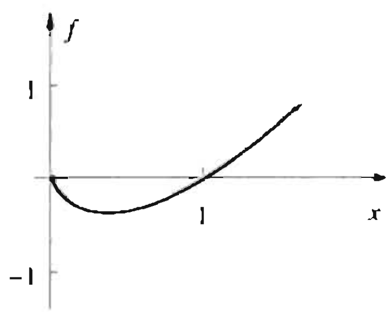

Figure 1.46
The behavior of the function $f(x) = x \ln x$ as $x \to 0$.

Example 3:
Determine $\lim\limits_{x \to \infty} xe^{-x}$.

SOLUTION: This expression is of the indeterminate form $\infty \cdot 0$. It becomes an ∞/∞ form by writing it as

$$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$$

The result of Example 3 is a special case of the limit of $x^n e^{-x}$ as $x \to \infty$, where n is any integer. We can easily find this general limit using mathematical induction. We know from Example 3 that $x^n e^{-x} \to 0$ as $x \to \infty$ when $n = 1$. When using mathematical induction, we assume that if a statement is true for some value of $n \geq 1$, then it must be true for $n + 1$ also. So

$$\lim_{x \to \infty} \frac{x^{n+1}}{e^x} = \lim_{x \to \infty} \frac{(n+1)x^n}{e^x} = (n+1) \lim_{x \to \infty} \frac{x^n}{e^x} = 0 \tag{7}$$

But we know that $\lim x^n e^{-x} = 0$ as $x \to \infty$ for $n = 1$, so it must be true for $n = 2$, $n = 3$, and so on. This result, which is worth remembering, says that $e^{-x} \to 0$ faster than any power of x as $x \to \infty$.

Another limit worth remembering is

$$\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0 \tag{8}$$

for any $\alpha > 0$. This limit says that $\ln x \to \infty$ more slowly than any positive power of x, no matter how small (Problem 7); or equivalently, that $x^\alpha \ln x \to 0$ as $x \to 0$ for any $\alpha > 0$ (Problem 8).

Other indeterminate forms such as 0^0, ∞^0, and 1^∞ can often be handled by taking the logarithm and manipulating the result into the standard indeterminate forms $0/0$ or ∞/∞. For example, consider $\lim\limits_{x\to 0+} x^x$. Let $y = x^x$, and then look at $\lim\limits_{x\to 0+} \ln y = \lim\limits_{x\to 0+} x \ln x = 0$; and so $\lim\limits_{x\to 0+} y = 1$, or finally $\lim\limits_{x\to 0+} x^x = 1$.

Example 4:
Determine $\lim\limits_{n\to\infty} \sqrt[n]{p}$, where $p > 0$.

SOLUTION: We'll let $y = \sqrt[n]{p}$, take logarithms, and treat n as a continuous variable.

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{n} \ln p = 0$$

So $\lim\limits_{n\to\infty} y = \lim\limits_{n\to\infty} \sqrt[n]{p} = 1$. Figure 1.47 shows $\sqrt[n]{2}$ plotted against n. Problem 12 has you show that $\sqrt[n]{n} \to 1$ as $n \to \infty$.

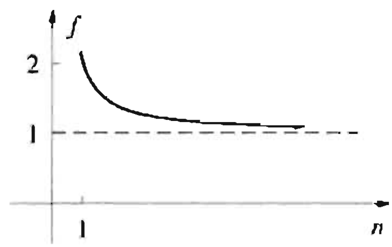

Figure 1.47
The function $f(x) = \sqrt[n]{2}$ plotted against n. The asymptote is shown as a dashed line.

What if you apply l'Hôpital's rule and you still get an indeterminate form? Simply apply it successively until you no longer obtain an indeterminate form. For example,

$$\lim_{x\to 1} \frac{2x^3 - 9x^2 + 12x - 5}{2x^3 - 18x^2 + 30x - 14} = \lim_{x\to 1} \frac{6x^2 - 18x + 12}{6x^2 - 36x + 30}$$
$$= \lim_{x\to 1} \frac{12x - 18}{12x - 36} = \frac{1}{4}$$

(See Problem 11, however.)

1.6 Problems

1. Use Equation 4 to calculate $\sin(\pi/4)$ to four-place accuracy. *Hint*: Realize that $|\sin x| \le 1$ and that $|\cos x| \le 1$.
2. Use Equation 4 to calculate the value of e to five-place accuracy. *Hint*: Use the fact that $e \approx 3$.

3. Argue that $f(x) = x^3 + px + q$ has one real root if $p > 0$.

4. Use l'Hôpital's rule to determine the following limits:

(a) $\lim_{x\to 0} \frac{\sin^2 x}{x}$ (b) $\lim_{x\to 0} \frac{1-\cos x}{x}$

(c) $\lim_{x\to 0} \frac{e^x - 1 - x}{x}$ (d) $\lim_{x\to \pi/2} \frac{1+\cos 2x}{1-\sin 2x}$

5. Use l'Hôpital's rule to determine the following limits:

(a) $\lim_{x\to 0} \frac{e^x - 1}{x}$ (b) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

(c) $\lim_{x\to 1} \frac{\ln x}{1-x}$ (d) $\lim_{x\to \infty} (\sqrt{x^2+2x} - x)$

6. Use l'Hôpital's rule to determine the following limits:

(a) $\lim_{x\to 0} (\csc x - \cot x)$ (b) $\lim_{x\to 0} \frac{1-\cos^2 x}{x^2}$

(c) $\lim_{x\to 0^+} \frac{\ln \sin x}{\ln \tan x}$ (d) $\lim_{x\to 0} \frac{\ln(1+x)}{x}$

7. Show that for every $\alpha > 0$, $\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0$.

8. Show that for every $\alpha > 0$, $x^\alpha \ln x \to 0$ as $x \to 0$.

9. Determine $\lim_{x\to\infty} x^{1/x}$.

10. Determine $\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x}$.

11. At the end of the section, we found the limit of $(2x^3 - 9x^2 + 12x - 5)/(2x^3 - 18x^2 + 30x - 14)$ as $x \to 1$ and got 1/4 as an answer. What's wrong with the following?

$$\lim_{x\to 1} \frac{2x^3 - 9x^2 + 12x - 5}{2x^3 - 18x^2 + 30x - 14} = \lim_{x\to 1} \frac{6x^2 - 18x + 12}{6x^2 - 36x + 30}$$
$$= \lim_{x\to 1} \frac{12x - 18}{12x - 36} = \lim_{x\to 1} \frac{12}{12} = 1$$

12. Show that $\sqrt[n]{n} \to 1$ as $n \to \infty$.

13. We'll prove Rolle's theorem in this problem. Use the extreme value theorem to argue that $f(x)$ must have a maximum and a minimum value on $[a, b]$. Now argue that if $f(x)$ has any positive values at all, then there is at least one point ξ in (a, b) where $f(x)$ is a maximum, so that $f'(\xi) = 0$. Similarly, argue that if $f(x)$ has a negative value, then there is at least one point η in (a, b) where $f(x)$ is a minimum, so that $f'(\eta) = 0$. Finally, show that if $f(x)$ has no positive or negative values in (a, b), then $f(x)$ is identically zero on $[a, b]$, and so $f'(x) = 0$ for all x on (a, b).

14. Figure 1.48 illustrates the mean value theorem for derivatives. To prove this theorem, we consider the function $F(x)$, which is the difference between $f(x)$ and the straight line connecting the points $(a, f(a))$ and $(b, f(b))$, as shown in Figure 1.48. Show that the equation for $F(x)$ is

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

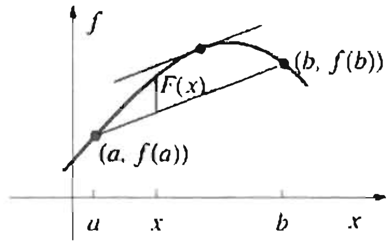

Figure 1.48
An aid to the proof of the mean value theorem of derivatives.

Note that $F(a) = F(b) = 0$, as implied by Figure 1.48 and that $F(x)$ satisfies the criteria of Rolle's theorem. Using the fact that there is at least one point c in (a, b) at which $F'(x) = 0$, show that $f(b) - f(a) = f'(c)(b - a)$. Note that we obtain Rolle's theorem if $f(b) = f(a) = 0$.

15. The standard proof of l'Hôpital's rule is based on what is called the *extended mean value theorem of derivatives*. It says that if $f(x)$ and $g(x)$ are continuous in $[\,a, b\,]$ and differentiable in (a, b), then there is at least one point c in (a, b) for which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

where we assume that $g(a) \neq g(b)$ and $f'(x)$ and $g'(x)$ are not simultaneously zero. Note that this theorem reduces to the mean value theorem if $g(x) = x$. Prove the extended mean value theorem by starting with

$$F(x) = f(x) - f(b) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(b)]$$

16. Use the result of Problem 15 to prove l'Hôpital's rule for the form 0/0. Assume that $f(x)$ and $g(x)$ are differentiable, that $g(x) \neq 0$ in (a, b), and that $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0$.

17. We shall outline the proof of Equation 3, the higher-order mean value theorem, in this problem. Let a constant k be defined by

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + k(b - a)^{n+1}$$

Now apply Rolle's theorem to

$$F(x) = f(x) - f(b) + f'(x)(b - x) + \frac{f''(x)}{2!}(b - x)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(b - x)^n + k(b - x)^{n+1}$$

and show that $k = f^{(n+1)}(\xi)/(n + 1)!$.

1.7 Integration

The idea of an integral was originally developed to calculate the area bounded by given curves, but nowadays an integral is defined by a limiting process. Consider the situation in Figure 1.49. The interval $[\,a, b\,]$ is subdivided into n sub-intervals,

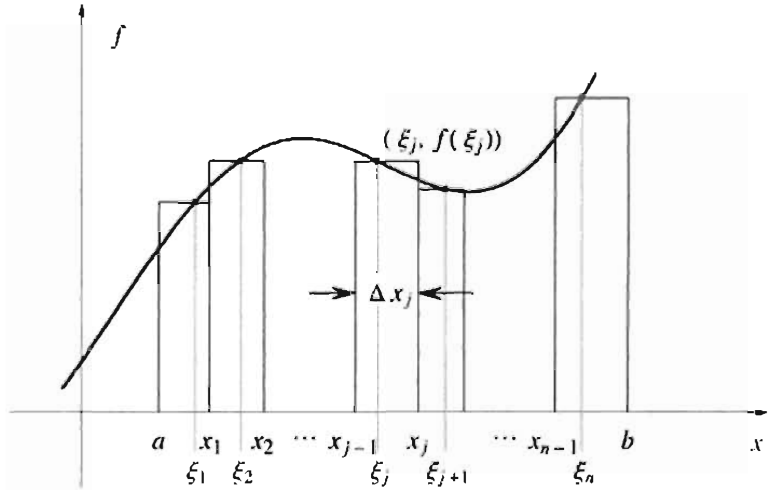

Figure 1.49
The construction associated with a Riemann sum.

(a, x_1), (x_1, x_2), $\ldots$, (x_{n-1}, b) and the point, ξ_j, is located anywhere within the jth interval for $j = 1, 2, \ldots, n$. We now form the sum

$$S_n = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} f(\xi_j)\Delta x_j$$

where $x_0 = a$, $x_n = b$, and $\Delta x_j = x_j - x_{j-1}$. Geometrically this sum represents the sum of the areas of the rectangles in Figure 1.49 and is called a *Riemann sum*. If we subdivide $[a, b]$ into more and more subintervals with smaller and smaller widths, we eventually reach a limit, called the *Riemann integral* of $f(x)$, denoted by

$$\int_a^b f(x)dx = \lim_{|L|\to 0} \sum_{j=1}^{n} f(\xi_j)\Delta x_j \tag{1}$$

where $|L|$ is the width of the largest subinterval. Of course, we are assuming here that the above limit exists. In the ϵ-δ notation, we have

$$\left| \int_a^b f(x)dx - \sum_{j=1}^{n} f(\xi_j)\Delta x_j \right| < \epsilon \qquad \text{whenever} \qquad |L| < \delta \tag{2}$$

This limit exists if $f(x)$ is continuous (or even piecewise continuous) in $[a, b]$. Geometrically, the integral of $f(x)$ from a to b represents the area between $f(x)$ and the x axis and the vertical lines at $x = a$ and $x = b$ if $f(x)$ is positive everywhere between a and b. Otherwise, it represents the net area, with areas above the x axis treated as positive and areas below the x axis treated as negative.

We certainly don't use Equations 1 or 2 to evaluate integrals. As you know, and we'll show below, integration and differentiation are inverse operations, so we'll evaluate integrals by working backwards from tables of differentiation formulas. In fact, the great achievement of Newton and Leibnitz in their development of calculus in the late 1600's was to appreciate the inverse relationship between differentiation and integration.

Prior to that time, the formulas of the slopes of many curves had been derived and the areas bounded by many curves had been determined, usually by ingenious, if not even quirky, methods, but the inverse relation between finding the slope of the tangent lines to curves and areas bounded by these curves was not recognized. A good example of the determination of areas bounded by given curves is due to Archimedes, who determined the area under the parabola, $y = x^2$, almost two thousand years before the formal development of calculus. We know this area to be

$$A = \int_a^b x^2 dx = \left[\frac{x^3}{3} \right]_a^b = \frac{b^3 - a^3}{3}$$

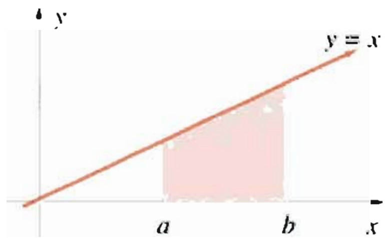

Figure 1.50
The integral $\int_a^b x\,dx$ is given by the shaded area.

but let's spend a page or two discussing this "pre-calculus" calculation. Let's start with the simple example, $y = x$. We don't even need calculus to determine the area under this curve from a to b because it is the difference between the areas of the two triangles in Figure 1.50. Of course, the integral of $y = x$ from a to b gives $[\,x^2/2\,]_a^b = (b^2 - a^2)/2$, which is the same thing. As a preparation for Archimedes's calculation for $y = x^2$, let's determine the integral of $f(x) = x$ using a limiting process. For convenience only, let's take the n subintervals in Equation 1 to be equal and given by $\Delta x = (b - a)/n$ and take ξ_j to be the extreme right-hand value within each interval. Then S_n becomes

$$\begin{aligned} S_n &= \sum_{j=1}^{n} \xi_j \Delta x = \sum_{j=1}^{n} (a + j\Delta x)\Delta x \\ &= a \sum_{j=1}^{n} \Delta x + \sum_{j=1}^{n} j(\Delta x)^2 \\ &= an\Delta x + (\Delta x)^2 \sum_{j=1}^{n} j \end{aligned}$$

The above summation over j is the sum of the first n integers, which is equal to $n(n+1)/2$ (Problem 12). This result was well known in Archimedes' time, and so

$$S_n = na\Delta x + \frac{n(n+1)}{2}(\Delta x)^2$$

But $\Delta x = (b - a)/n$, so

$$S_n = a(b-a) + \frac{(b-a)^2}{2} + \frac{(b-a)^2}{2n}$$

and

$$\int_a^b x\,dx = \lim_{n\to\infty} S_n = \frac{b^2 - a^2}{2}$$

For the case of the area under the curve $y = x^2$, we have

$$S_n = \sum_{j=1}^{n} (a + j\Delta x)^2 \Delta x = na^2 \Delta x + 2a(\Delta x)^2 \sum_{j=1}^{n} j + (\Delta x)^3 \sum_{j=1}^{n} j^2$$

The sum of the squares of the first n integers is equal to $n(n+1)(2n+1)/6$, again well-known to the ancient Greek mathematicians (see Problem 13). Therefore,

$$S_n = a^2(b-a) + \frac{a(b-a)^2 n(n+1)}{n^2} + \frac{(b-a)^3}{6} \frac{n(n+1)(2n+1)}{n^3}$$

$$\rightarrow \frac{b^3 - a^3}{3} \qquad \text{as} \qquad n \rightarrow \infty$$

or

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$$

There are a few properties of definite integrals that we should point out here. Most follow from the definition given in Equation 1. If c_1 and c_2 are constants, then

$$\int_a^b [c_1 f_1(x) + c_2 f_2(x)]\,dx = c_1 \int_a^b f_1(x)\,dx + c_2 \int_a^b f_2(x)\,dx \tag{3}$$

Equation 3 is similar to Equation 4.3 for differentiation. Just as we define a differential operator in Section 4, we define an integral operator by

$$\hat{I} f(x) = \int_a^b f(x)\,dx$$

Equation 3 says that integration is a linear operation.

$$\hat{I} [c_1 f_1(x) + c_2 f_2(x)] = c_1 \hat{I} f_1(x) + c_2 \hat{I} f_2(x) \tag{4}$$

Some other properties that follow from Equation 1 are

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{5}$$

provided $f(x)$ is integrable in $[a, c]$ and $[c, b]$; in other words, provided the limit defining the two integrals exists. Also, if $m \le f(x) \le M$ in $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a) \tag{6}$$

If $f(x) \le g(x)$ in $[a, b]$, then (Problem 15)

$$\int_a^b f(u)du \le \int_a^b g(z)dz \tag{7}$$

and (Problem 18)

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx \tag{8}$$

Example 1:

Prove that $\lim\limits_{n\to\infty} \int_0^{2\pi} \dfrac{\cos nx}{x^2+n^2}dx = 0$.

SOLUTION: We use Equation 8 to write

$$\left|\int_0^{2\pi} \frac{\cos nx}{x^2+n^2}dx\right| \le \int_0^{2\pi} \left|\frac{\cos nx}{x^2+n^2}\right| dx \le \int_0^{2\pi} \frac{dx}{x^2+n^2} \le \int_0^{2\pi} \frac{dx}{n^2} = \frac{2\pi}{n^2}$$

which $\to 0$ as $n \to \infty$. In going from the second term to the third term, we used the fact that $|\cos nx| \le 1$.

Suppose that $f(x)$ is piecewise continuous in $[\,a, b\,]$ with a jump discontinuity at $x = c$. Then

$$I = \int_a^b f(x)dx = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)\,dx \lim_{\epsilon\to 0} \int_{c+\epsilon}^b f(x)\,dx \tag{9}$$

Example 2:

Find the area between the x axis and (Figure 1.51)

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x < 1 \\ 2 & 1 \le x < 2 \\ 0 & x \ge 2 \end{cases}$$

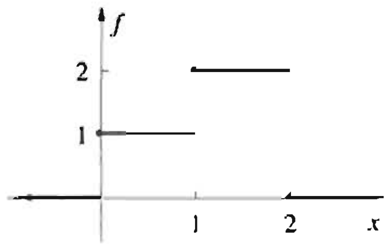

Figure 1.51
The function $f(x) = 0$ for $x < 0$; $f(x) = 1$ for $0 \le x < 1$; $f(x) = 2$ for $1 \le x < 2$; and $f(x) = 0$ for $x \ge 2$.

SOLUTION:

$$I = \lim_{\epsilon\to 0} \int_0^{1-\epsilon} dx + \lim_{\epsilon,\epsilon'\to 0} \int_{1+\epsilon}^{2-\epsilon'} 2dx$$

$$= \lim_{\epsilon\to 0}(1-\epsilon) + 2 \lim_{\epsilon,\epsilon'\to 0}(2-\epsilon'-1+\epsilon) = 3$$

We have two jump discontinuities in Example 2, but the generalization of Equation 9 to more than one jump discontinuity is apparent. In fact, $f(x)$ can have an infinite number of jump discontinuities if it is a countably infinite number. Also, the values of $f(x)$ at the end points of each interval in Example 2 are irrelevant. Whether we write $f(x) = 1$ for $0 \le x < 1$, or $0 < x < 1$, or $0 \le x \le 1$, or $0 < x \le 1$ makes no difference at all.

We presented the mean value theorem for derivatives in Section 6. The very name of the theorem hints that there is a *mean value theorem for integrals*. If $f(x)$ is continuous in $[a, b]$, then there is a point c in $[a, b]$ such that

$$\int_a^b f(x)dx = (b - a)f(c) \tag{10}$$

To see that this is so, let m and M be the minimum and maximum values of $f(x)$ in $[a, b]$ and use Equation 6 to write

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$$

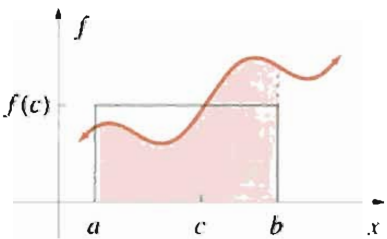

Figure 1.52
A pictorial representation of the mean value theorem of integration, Equation 10. The area within the solid rectangle equals the shaded area under the colored curve.

Since $f(x)$ is continuous, it takes on all values between m and M. So there must be some point c in $[a, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c) \tag{11}$$

Equation 11 says that the area $\int_a^b f(x)dx$ is equal to the area of the rectangle defined by $x = a$, $x = b$, $y = 0$, and $y = f(c)$. Figure 1.52 gives a pictorial representation of Equation 10.

There is a generalization of the mean value theorem for integrals that says that if $f(x)$, $g(x)$, and $g'(x)$ are continuous on $[a, b]$ and $g(x)$ does not change sign in $[a, b]$, then there is a point c in $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx \tag{12}$$

Notice that this reduces to Equation 10 if $g(x) = 1$.

The integrals that we have been discussing so far have been between a and b and are definite integrals. If the upper limit, b, is replaced by x, then the integral defines a function of x and is called an *indefinite integral*. We write this as

$$F(x) = \int_a^x f(u)du \tag{13}$$

Notice, incidentally, that we are using u and not x as our variable of integration so that we can distinguish between the variable of integration and the upper limit. The designation of the integration variable is arbitrary and is what we call a dummy variable. In other words,

$$F(x) = \int_a^x f(u)du = \int_a^x f(z)dz = \int_a^x f(t)dt$$

are completely equivalent. However, writing

$$F(x) = \int_a^x f(x)dx$$

is poor practice and should be avoided because the x in the upper limit and the x in the integrand represent different quantities.

The *fundamental theorem of calculus* says that if $f(x)$ is continuous in the interval $[a, b]$ and if

$$F(x) = \int_a^x f(u)du \tag{14}$$

then $F(x)$ is an *antiderivative* of $f(x)$; in other words, $F'(x) = f(x)$ in the interval (a, b). The fundamental theorem of calculus gives us the inverse relation between differentiation and integration. The utility of this theorem cannot be overstated and was essentially unrecognized before Newton and Leibnitz.

To obtain $f(x)$ from Equation 14, we differentiate with respect to the upper limit, x. Write

$$F(x + \Delta x) - F(x) = \int_a^{x+\Delta x} f(u)du - \int_a^x f(u)du = \int_x^{x+\Delta x} f(u)du$$

Assuming that $f(x)$ is continuous in the interval $(x, x + \Delta x)$, we can use the mean value theorem of integration to write

$$F(x + \Delta x) - F(x) = f(\xi)\Delta x$$

If we divide by Δx and then take the limit $\Delta x \to 0$, we obtain

$$\frac{dF}{dx} = F'(x) = f(x) \tag{15}$$

Equations 14 and 15 summarize the fundamental theorem of calculus.

Incidentally, we can generalize the above differentiation to the case where both limits are functions of x. If

$$G(x) = \int_{u(x)}^{v(x)} f(t)\,dt$$

then (Problem 9)

$$G'(x) = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx} \tag{16}$$

Equation 16 is called *Leibnitz's rule*. Of course, we are assuming that $v(x)$ and $u(x)$ are suitably differentiable.

Example 3:
Suppose that

$$I(x) = \int_x^{x^2} \frac{dz}{z}$$

Evaluate $I'(x)$.

SOLUTION: Using Equation 16,

$$I'(x) = \frac{1}{x^2} \cdot 2x - \frac{1}{x} = \frac{1}{x}$$

In your calculus course, you probably spent weeks learning how to evaluate integrals. We start with a fairly complete table of derivative formulas and use the fundamental theorem of calculus to find antiderivatives. There is a bewildering array of techniques or tricks that can be used to manipulate expressions into forms that can be recognized as antiderivatives. For example, we have integration by parts (see Problem 2), trigonometric substitutions (Problem 3), partial fractions, and "miscellaneous" substitutions. With enough practice, most students can become pretty proficient in these techniques, but integration is still somewhat of an art.

Although our derivative formulas allow us to evaluate derivatives of any (differentiable) functions, there are many integrals that can not be expressed in terms of known functions. We'll see in Chapter 3 that a number of well-known functions are actually *defined* in terms of integrals. For example, we'll see that the error function, erf (x), is *defined* by

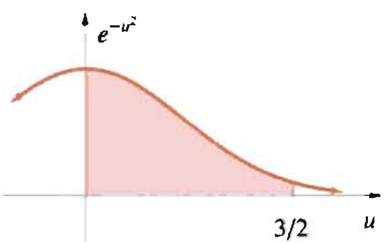

Figure 1.53
The area between the u-axis and the graph of e^{-u^2} from 0 to 3/2 is equal to $\int_0^{3/2} e^{-u^2} du$.

$$\text{erf}\,(x) = \frac{2}{\sqrt{2}} \int_0^x e^{-u^2} du$$

To evaluate erf (1.5), for example, we determine the area between the x axis and the graph of e^{-u^2} from 0 to 1.5 as shown in Figure 1.53. There are many numerical routines to do this. You may remember learning the trapezoidal rule or Simpson's rule in your calculus course. Basically, these methods provide numerical approximations for areas. We're not going to discuss numerical methods here because there are many computer packages available nowadays that use much more sophisticated routines than Simpson's rule. For example, the commercially available computer programs, Mathematica, MathCad, Maple, and Matlab, can be used to evaluate integrals numerically. The one-line command in Mathematica

```
Integrate [ Exp [ -x ^ 2 ], {x, 0, 3/2} ] // N
```

gives

$$\int_0^{3/2} e^{-u^2} du = 0.856\,188$$

Certainly owning a good table of integrals goes a long way to being able to evaluate many integrals. The *CRC Standard Mathematical Tables and Formulae*, which is a standard reference, lists over 50 pages of integrals. The most comprehensive tables (over 1000 pages!) are the *Tables of Integrals, Series, and Products* by Gradshteyn and Ryzhik, which are indispensable for anyone who works in applied mathematics. In addition to these standard references, many of the CAS can be used to integrate *symbolically*, meaning that they provide analytic expressions for integrals. For example, the one line in Mathematica

```
Integrate [ x^3 · Cos [ a · x ], x ]
```

gives the indefinite integral

$$\int x^3 \cos ax\, dx = \frac{3(a^2x^2 - 2)\cos ax}{a^4} + \frac{x(a^2x^2 - 6)\sin ax}{a^3}$$

and

```
Integrate [ x · Log [ a · x + b ], {x, 0, 1} ]
```

gives the definite integral

$$\int_0^1 x \ln(ax + b)\, dx = \frac{2ab - a^2 + 2b^2 \ln b + 2(a^2 - b^2)\ln(a + b)}{4a^2}$$

Other CAS, such as Maple and Matlab, have the same capabilities. Almost every academic science or engineering department or industrial research laboratory owns a license for at least one of these CAS. Learning to use any one of these programs will not only save you hours of algebra, along with its concomitant errors, but will also allow you to concentrate on the central aspects of a problem rather than on drudgery.

1.7 Problems

1. Derive the formula for integration by parts.
2. Use integration by parts to evaluate

   (a) $\int x e^{-x}\, dx$ (b) $\int x \sin x\, dx$ (c) $\int \frac{\ln x}{x}$ (d) $\int \ln x\, dx$
3. Use trigonometric substitution to evaluate

   (a) $\int \frac{dx}{\sqrt{a^2 - x^2}}$ (b) $\int_0^a \frac{dx}{\sqrt{a^2 + x^2}}$ (c) $\int_0^1 \frac{dx}{(1 + x^2)^2}$ (d) $\int_0^a (a^2 - x^2)^{1/2}\, dx$
4. Use hyperbolic substitution to evaluate

   (a) $\int \frac{dx}{\sqrt{x^2 - a^2}}$ (b) $\int_0^1 (x^2 + 1)^{1/2} dx$ (c) $\int_0^1 \frac{dx}{\sqrt{1 + x^2}}$ (d) $\int_0^{1/2} \frac{dx}{1 - x^2}$

5. Find the area bounded by $y = 2x$ and $y = x^2$ from $x = 0$ to 2.

6. Find the area between the curve $f(x) = \sqrt{a^2 - x^2}$, $-a \le x \le a$, and the x axis.

7. Find the arc length of $y = x^2$ from $x = 0$ to 2.

8. Find the volume of a right circular cone of base radius r and height h.

9. Prove Leibnitz's rule (Equation 16).

10. Show that $\int_{-A}^{A} f(x)dx = 2\int_0^A f(x)\,dx$ if $f(x) = f(-x)$ and that $\int_{-A}^{A} f(x)\,dx = 0$ if $f(x) = -f(x)$.

11. Show that the area of an ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, is πab.

12. Prove that the sum of the first n integers is $n(n+1)/2$. *Hint*: List the n integers sequentially on one line, and under that line list them backwards. Do you see the trick here?

13. In this problem, we'll prove that the sum of the first n squares is $n(n+1)(2n+1)/6$. Begin with the formula $(\nu+1)^3 - \nu^3 = 3\nu^2 + 3\nu + 1$ and sum from $\nu = 1$ to n to obtain $(n+1)^3 = 3S_2 + 3S_1 + n + 1$ where $S_1 = \sum_{j=1}^{n} j$ and $S_2 = \sum_{j=1}^{n} j^2$. Now show that $S_2 = n(n+1)(2n+1)/6$.

14. Can you generalize Problem 13 to find S_3?

15. Prove the statement of Equation 7. *Hint*: Use Equation 1.

16. Show that the sum of the first n odd integers is n^2. *Hint*: Evaluate $\sum_{j=1}^{n}(2j-1)$ using the fact that $\sum_{j=1}^{n} j = n(n+1)/2$.

17. We'll prove the triangle inequality, $|x+y| \le |x| + |y|$, in this problem. The triangle inequality is used often in applied mathematics. Add the inequalities $-|x| \le x \le |x|$ and $-|y| \le y \le |y|$ and then take the absolute value to obtain $|x+y| \le |x| + |y|$.

18. Prove the statement of Equation 8. *Hint*: Use Equation 1 and the triangle inequality, $|a+b| \le |a| + |b|$. (See the previous problem.)

19. Consider Figure 1.54, which shows the hyperbola $x^2 - y^2 = a^2$. Show that the area of the shaded region is $A = a^2 \ln \dfrac{x_0 + y_0}{a}$. This area divided by a^2 is a *hyperbolic radian*, which we denote by u. Show that $\cosh u = \dfrac{x}{a}$ and $\sinh u = \dfrac{y}{a}$. *Hint*: Use the fact that $x_0^2 - y_0^2 = (x_0 + y_0)(x_0 - y_0) = a^2$.

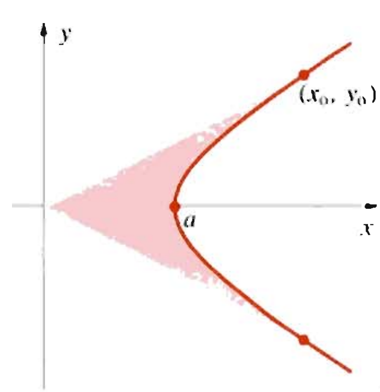

Figure 1.54
An illustration of a hyperbolic radian.

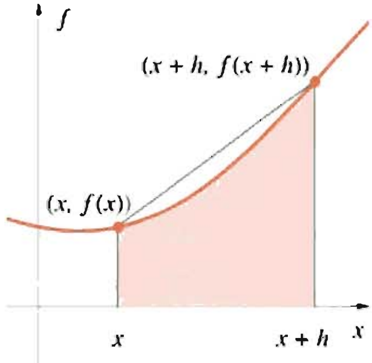

Figure 1.55
A pictorial aid for Problem 20.

20. This problem offers another proof of the fundamental theorem of calculus. Use Figure 1.55 to show that $F(x+h) - F(x) \approx \frac{h}{2}\left[f(x+h) + f(x)\right]$ for small values of h and that $F'(x) = f(x)$.

21. Evaluate $\lim_{x \to 0} \dfrac{\int_0^x \sin u^2 du}{x^3}$.

22. Show that $\displaystyle\int_0^{\pi/2} \frac{d\theta}{a^2 + b^2 \cos^2\theta} = \frac{\pi}{2|a|(a^2+b^2)^{1/2}}$. *Hint*: First write $\cos^2\theta = 1/\sec^2\theta$, notice that $d\tan\theta = \sec^2\theta\, d\theta$, and then use the identity $\sec^2\theta = 1 + \tan^2\theta$.

Use any CAS to evaluate the integrals in Problems 23 through 27.

23. $\displaystyle\int x^2 e^{-x} \cos\, dx$

24. $\displaystyle\int \frac{(x^2 - x + 1)^{1/2}}{x} dx$

25. $\displaystyle\int x^3 \ln(ax + b)\, dx$

26. $\displaystyle\int \frac{x^2 + 2x - 4}{x^2 + 3x + 2} dx$

27. $\displaystyle\int \frac{x^3}{(ax^2 + b)^{1/2}} dx$

1.8 Improper Integrals

There are two types of integrals that are called *improper integrals*. The first type, Type 1, consists of integrals with one or more infinite limits, which are defined by

$$\int_a^\infty f(x)dx = \lim_{c \to \infty} \int_a^c f(x)dx \tag{1}$$

$$\int_{-\infty}^b f(x)dx = \lim_{c \to -\infty} \int_c^b f(x)dx \tag{2}$$

and

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{c\to-\infty}\int_{c}^{z} f(x)dx + \lim_{d\to\infty}\int_{z}^{d} f(x)dx \tag{3}$$

In Equation 3, z is any convenient point that you are free to choose.

The second type of improper integral, Type 2, consists of integrals where the integrands are unbounded at one or more points in the region of integration $[a, b]$. Three cases can occur: $f(x)$ is continuous everywhere in $[a, b]$ except at a, in which case

$$\int_{a}^{b} f(x)dx = \lim_{\epsilon\to 0}\int_{a+\epsilon}^{b} f(x)dx\ ; \tag{4}$$

$f(x)$ is continuous everywhere in $[a, b]$ except at b, in which case

$$\int_{a}^{b} f(x)dx = \lim_{\epsilon\to 0}\int_{a}^{b-\epsilon} f(x)dx\ ; \tag{5}$$

and $f(x)$ is continuous everywhere in $[a, b]$ except at some point c in (a, b), in which case

$$\int_{a}^{b} f(x)dx = \lim_{\epsilon_1\to 0}\int_{a}^{c-\epsilon_1} f(x)dx + \lim_{\epsilon_2\to 0}\int_{c+\epsilon_2}^{b} f(x)dx \tag{6}$$

provided that both limits on the right exist.

We can also have improper integrals that are a combination of the above types, where the integration limits are infinite *and* the integrand is unbounded. When the limiting process defining an improper integral exists, the integral is said to converge, or to be *convergent*. Otherwise, the integral is said to diverge, or to be *divergent*.

Example 1:

Examine the convergence of $\displaystyle\int_{1}^{\infty}\frac{dx}{x^p}$ as a function of p.

SOLUTION:

$$\int_{1}^{\infty}\frac{dx}{x^p} = \lim_{b\to\infty}\int_{1}^{b}\frac{dx}{x^p} = \left[\lim_{b\to\infty}\frac{1}{1-p}\frac{1}{x^{p-1}}\right]_{1}^{b}$$

If $p > 1$, then we have

$$\lim_{b\to\infty}\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right) = \frac{1}{1-p}(0-1) = \frac{1}{1-p}$$

If $p < 1$, then

$$\lim_{b\to\infty}\frac{1}{1-p}(b^{1-p} - 1) = \infty$$

If $p = 1$, then

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln b = \infty$$

So we see that $\int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \le 1$. This result is worth remembering.

Example 2:

Examine the convergence of $\int_a^\infty e^{-sx}dx$ as a function of s.

SOLUTION:

$$\int_a^\infty e^{-sx}dx = \lim_{b\to\infty} \int_a^b e^{-sx}dx$$

$$= \lim_{b\to\infty} \frac{e^{-sa} - e^{-sb}}{s} = \frac{e^{-sa}}{s} \qquad (s > 0)$$

but equals ∞ if $s = 0$ or if $s < 0$. Thus the integral converges if $s > 0$ and diverges if $s \le 0$.

Note that Equation 3 does *not* say that

$$\int_{-\infty}^\infty f(x)dx = \lim_{c\to\infty} \int_{-c}^c f(x)dx \qquad \text{(not true)}$$

For example, consider $I = \int_{-\infty}^\infty \frac{udu}{1+u^2}$. We choose $z = 0$ in Equation 3, and consider the two integrals

$$\lim_{a\to\infty} \int_{-a}^0 \frac{udu}{1+u^2} \qquad \text{and} \qquad \lim_{b\to\infty} \int_0^b \frac{udu}{1+u^2}$$

Let $u = -x$ in the first integral to obtain

$$-\lim_{a\to\infty} \int_0^a \frac{xdx}{1+x^2} = -\lim_{a\to\infty} \frac{1}{2} \ln(1+a^2) = -\infty$$

so the integral $\int_{-\infty}^\infty \frac{udu}{1+u^2}$ diverges. Now let's look at the integral from 0 to ∞.

$$\lim_{b\to\infty} \int_0^b \frac{udu}{1+u^2} = \lim_{b\to\infty} \frac{1}{2} \ln(1+b^2) = \infty$$

Thus, *both* contributions to I diverge. *If* we had used

$$\lim_{a\to\infty}\int_{-a}^{a}\frac{u\,du}{1+u^2}$$

then we would have obtained a value of zero $[\frac{1}{2}\ln(1+a^2)-\frac{1}{2}\ln(1+a^2)]$ for the integral.

It is useful to have some tools to determine easily if an integral converges or diverges without having to evaluate it. If $f(x)$ and $g(x)$ are bounded and $0\le f(x)\le g(x)$ for $x\ge a$, then $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges. Conversely, if $f(x)\ge g(x)\ge 0$ for $x\ge 0$, then $\int_a^\infty f(x)dx$ diverges if $\int_a^\infty g(x)dx$ diverges. This test is called the *comparison test* (Problem 22).

To see how to use the comparison text, let's investigate the behavior of

$$I=\int_0^\infty \frac{e^{-x}}{(1+x)^2}dx$$

Let $f(x)=e^{-x}/(1+x)^2$ and $g(x)=e^{-x}$. Then

$$f(x)=\frac{e^{-x}}{(1+x)^2}\le e^{-x}=g(x)\qquad \text{for}\quad 0\le x<\infty$$

Using the fact that

$$\int_0^\infty g(x)\,dx=\int_0^\infty e^{-x}dx=1$$

we conclude that I converges. In fact, its numerical value is 0.403 653 (Problem 23).

Example 3:
Show that the integral

$$\int_1^\infty \frac{\sin^2 x}{x^3}dx$$

converges.

SOLUTION: We let $f(x)=\sin^2 x/x^3$ and $g(x)=1/x^3$. Then because $\sin^2 x\le 1$ for all x, we have $f(x)\le g(x)$ for all x. Using the fact that

$$\int_1^\infty \frac{dx}{x^3}=\frac{1}{4}$$

we conclude that the integral in question converges. In fact, its numerical value is 0.385 705 (Problem 24).

We also have a *ratio comparison test* for an integral such as

$$I = \int_a^\infty f(x)\,dx$$

where $f(x) \geq 0$ for $a \leq x < \infty$. Now find a function $g(x) \geq 0$ for $a \leq x < \infty$ for which the integral $\int_a^\infty g(x)\,dx$ is readily evaluated. Then the test for the convergence of $I = \int_a^\infty f(x)\,dx$ rests upon the limit

$$K = \lim_{x\to\infty} \frac{f(x)}{g(x)}$$

Three cases arise:

1. If $K \neq 0$ and is finite, then $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ either both converge or both diverge.
2. If $K = 0$ and $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
3. If $K = \infty$ and $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

Given an integral $\int_a^\infty f(x)\,dx$, it is often easy to choose an appropriate function $g(x)$, as the next example shows. Let's investigate the convergence of

$$I = \int_1^\infty f(x)\,dx = \int_1^\infty \frac{x^2}{(x^6+1)^{1/2}}dx$$

The ratio K involves the limit of $f(x)$ as $x \to \infty$. In this case, $f(x) \to 1/x$ as $x \to \infty$, so we'll choose $g(x) = 1/x$, in which case $K = 1$. But we know (Example 1) that $\int_1^\infty dx/x$ diverges, and so our test tells us that $\int_1^\infty \frac{x^2}{(x^6+1)^{1/2}}dx$ diverges also. The nice feature of this test is that you need only the behavior of $f(x)$ at large values of x, which is often fairly easy to see.

Example 4:
Use the ratio comparison test to show that the integral

$$I = \int_0^\infty \frac{x^2}{x^4+x^2+1}dx = \int_0^\infty f(x)\,dx$$

converges.

SOLUTION: Because $f(x)$ becomes $1/x^2$ as x gets large, we choose $g(x) = 1/x^2$. Because $\int_0^\infty x^{-2}dx$ converges, we conclude that $I = \int_0^\infty \frac{x^2}{x^4+x^2+1}dx$ converges.

Another useful test, which is closely related to the previous test and is based on the result in Example 1, is the following:
If $\lim_{x\to\infty} x^p f(x) = K$, then

1. $\int_a^\infty f(x)dx$ converges if $p > 1$ and K is finite, and
2. $\int_a^\infty f(x)dx$ diverges if $p \le 1$ and $K \neq 0$.

We shall refer to this test as the *p test for Type 1 improper integrals*. This test is particularly easy to apply.

Example 5:
Investigate the convergence of $\displaystyle\int_1^\infty \frac{\ln x}{x+a}dx$.

SOLUTION:

$$\lim_{x\to\infty} x \cdot \frac{\ln x}{x+a} = \infty$$

Thus, $p = 1$ and $K \neq 0$ in the p test and the integral diverges.

If $\int_a^\infty |f(x)|dx$ converges, then $\int_a^\infty f(x)dx$ is said to be *absolutely convergent*. If $\int_a^\infty |f(x)|dx$ diverges, but $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty f(x)dx$ is said to be *conditionally convergent*. If $\int_a^\infty f(x)dx$ is only conditionally convergent, it converges because of cancellation of positive and negative contributions to the integral. Clearly, if an integral is absolutely convergent, then it is convergent. An example of an integral that is conditionally convergent is $\int_1^\infty (\sin x)/x dx$.

So far we have discussed convergence only for Type 1 improper integrals, those whose limits are infinite. What about Type 2 improper integrals, those that are improper because the integrand is unbounded in $[a, b]$? The tests for convergence are not very different from the ones above. For simplicity, we'll just state the p test for Type 2 improper integrals for the case in which $f(x)$ is unbounded at a, but the other cases are essentially the same. Let $\lim_{x\to a+} (x - a)^p f(x) = K$. Then

1. $\displaystyle\int_a^b f(x)\,dx$ converges if $p < 1$ and K is finite
2. $\displaystyle\int_a^b f(x)\,dx$ diverges if $p \ge 1$ and $K \neq 0$

Example 6:

Investigate the convergence of $\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}}$.

SOLUTION: This integral is unbounded at both limits, so let's write it as

$$\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}} = \int_1^2 \frac{dx}{\sqrt{(x-1)(3-x)}} + \int_2^3 \frac{dx}{\sqrt{(x-1)(3-x)}}$$

and work on each one separately. Because

$$\lim_{x \to 1+} (x-1)^{1/2} \frac{1}{\sqrt{(x-1)(3-x)}} = K = \frac{1}{\sqrt{2}} < 1$$

and

$$\lim_{x \to 3-} (3-x)^{1/2} \frac{1}{\sqrt{(x-1)(3-x)}} = K = \frac{1}{\sqrt{2}} < 1$$

both integrals converge.

1.8 Problems

1. Evaluate $\int_0^\infty \frac{dx}{x^2+1}$.

2. The square-well potential for the interaction of two spherically symmetric molecules separated by a distance r is given by (see Figure 1.56)

$$u(r) = \begin{cases} \infty & r < \sigma \\ -\varepsilon & \sigma < r < \lambda\sigma \\ 0 & r > \lambda\sigma \end{cases}$$

where σ, λ, and ε are constants that are characteristic of the molecule. The second virial coefficient of imperfect gas theory is

$$B(T) = -2\pi \int_0^\infty [e^{-u(r)/k_B T} - 1] r^2 dr$$

where k_B is the Boltzmann constant and T is the kelvin temperature. Derive an expression for $B(T)$ for the square-well potential.

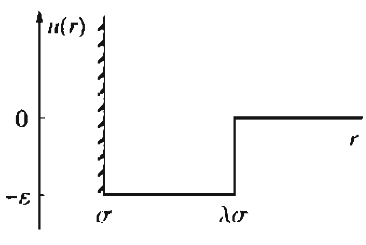

Figure 1.56
The square-well potential for the interaction of two spherically symmetric molecules.

3. Evaluate $\int_{-\infty}^{\infty} \operatorname{sech} x \, dx$.

4. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

5. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

6. Evaluate $\int_0^{\pi/2} \sec x \, dx$.

7. Use the comparison test to show that $\int_1^{\infty} \frac{dx}{x^2 + x^{1/2}}$ converges.

8. Use the comparison test to show that $\int_1^{\infty} \frac{\sqrt{x}}{1+x} dx$ diverges.

9. Use the comparison test to show that $\int_1^{\infty} \frac{dx}{1+x}$ diverges.

10. Use the comparison test to show that $\int_1^{\infty} \frac{1+\cos x}{x^3+4} dx$ converges.

11. Use the ratio comparison test to show that $\int_1^{\infty} \frac{x}{(x^4+1)^{1/2}} dx$ diverges.

12. Use the ratio comparison test to show that $\int_1^{\infty} \frac{x}{(x^6+1)^{1/2}} dx$ converges.

13. Use the p test for Type 1 improper integrals to show that $\int_1^{\infty} \frac{x^2+1}{(x^6+1)^{1/2}} dx$ diverges.

14. Use the p test for Type 1 improper integrals to show that $\int_1^{\infty} \frac{x^2 \, dx}{x^4+1}$ converges.

15. Use the p test for Type 2 improper integrals to show that $\int_4^{\infty} \frac{dx}{x^3\sqrt{(x-2)(x-4)}} dx$ converges.

16. Use the p test for Type 2 improper integrals to show that $\int_1^{\infty} \frac{x^2+2}{(x^3-1)^{1/3}} dx$ diverges.

17. Show that $\int_a^b \frac{dx}{(x-b)^p}$ converges if $p < 1$ and diverges if $p \geq 1$.

18. Show that $\int_0^{\infty} e^{-x^2+6x} dx$ converges.

19. Show that $\int_0^{\pi} \frac{\sin x}{x} dx$ converges.

20. Determine if $\int_0^{\infty} \frac{\sin x}{x^2} dx$ converges.

21. Show that $\int_0^{\infty} \frac{\sinh ax}{\sinh \pi x} dx$ converges if $0 < a \leq \pi$ and diverges if $a \geq \pi$.

22. Prove the comparison test for improper integrals of Type 1.

Use any CAS to evaluate the integrals in Problems 23 through 27.

23. $\displaystyle\int_0^\infty \frac{e^{-x}}{(1+x)^2}\,dx$

24. $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^3}\,dx$

25. $\displaystyle\int_1^\infty \frac{\ln x}{(1+x)^2}\,dx$

26. $\displaystyle\int_0^\infty e^{-a^2x^2}\cos bx\,dx$

27. $\displaystyle\int_0^\infty e^{-a^2x^2-b^2/x^2}dx$

1.9 Uniform Convergence of Integrals

Integrals that arise in physical applications are often functions of one or more parameters. For example, later on we'll be using Laplace transforms to solve differential equations. A Laplace transform is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

where s is a parameter that we can vary and manipulate.

We can express the general situation as

$$F(x) = \int_a^\infty f(x,t)dt \qquad x_1 \le x \le x_2 \tag{1}$$

When, for instance, can we determine $\lim\limits_{x\to x_0} F(x)$ by writing

$$\lim_{x\to x_0} F(x) = \lim_{x\to x_0}\int_a^\infty f(x,t)dt = \int_a^\infty \lim_{x\to x_0} f(x,t)dt\ ? \tag{2}$$

Or, when can we find the derivative of $F(x)$ by differentiating $f(x,t)$ under the integral sign, as in

$$F'(x) = \int_a^\infty \frac{\partial f(x,t)}{\partial x}dt\ ? \tag{3}$$

Since $F(x)$, being an integral, is defined through a limiting process and since the derivative of $F(x)$ involves another limiting process, we must wonder just when these limiting processes can be interchanged.

Before we go on, we should mention a nice analogy concerning the interchange of limits that is due to the 20th-century Russian mathematician, Vladimir Arnold. Suppose we have a container of water and that there is a hole of radius r

in the bottom of the container. Now consider the limit

$$\lim_{t\to\infty,\, r\to 0} \{\text{amount of water in the container}\}$$

If $r \to 0$ before $t \to \infty$, then there will be water left in the container. If, on the other hand, $t \to \infty$ before $r \to 0$, then there will be no water left. This example shows that the order in which we take limits is crucial and that obtaining the same result upon interchanging them is in no way assured.

To address the questions illustrated by Equations 2 and 3, we first define the notion of the *uniform convergence of improper integrals*. First, let's suppose that $F(x)$ given by Equation 1 converges for each x in $[x_1, x_2]$. In other words, suppose that

$$\left| F(x) - \int_a^b f(x,t)\, dt \right| < \epsilon \qquad \text{whenever} \quad b > N(\epsilon, x) \tag{4}$$

where N is a number that depends upon ϵ and x with $x_1 \le x \le x_2$. Equation 4 is the formal way of expressing that $F(x)$ converges for each x in $[x_1, x_2]$. Let's suppose, now, that Equation 4 is satisfied for a number $N(\epsilon)$ that depends only upon ϵ and not upon x. In other words, suppose that

$$\left| F(x) - \int_a^b f(x,t)dt \right| < \epsilon \quad \text{whenever} \quad b > N(\epsilon) \quad \text{and} \quad x_1 \le x \le x_2$$

where $N(\epsilon)$ is *independent* of x. In this case, we say that $F(x)$ converges uniformly in $[x_1, x_2]$. For example, the integral

$$F(x) = \int_0^\infty e^{-xt} dt = \frac{1}{x}$$

converges *uniformly* to $1/x$ for $x \ge 1$ because

$$\left| \frac{1}{x} - \int_0^b e^{-xt} dt \right| = \frac{e^{-xb}}{x} \le e^{-b}$$

and e^{-b} will be $< \epsilon$ if we choose N to be $\ln 1/\epsilon$, independent of x for $x \ge 1$.

On the other hand, the integral

$$G(x) = \int_0^\infty x e^{-xt} dt$$

converges for $x \ge 0$, but *does not* converge uniformly for $x \ge 0$. To see why this is so, first note that

$$G(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

and that

$$\left| G(x) - \int_0^b xe^{-xt}dt \right| = \begin{cases} 0 & x = 0 \\ e^{-xb} & x > 0 \end{cases}$$

The integral is uniformly convergent for $x \geq x_1 > 0$ because $e^{-xb} \leq e^{-x_1 b} < \epsilon$ if we choose $b > N = 1/x_1 \ln(1/\epsilon)$. As $x_1 \to 0$, however, $N \to \infty$ and so the integral is not uniformly convergent for $x \geq 0$.

The following theorems tell us why uniform convergence is so important:

1. *If $f(x,t)$ is continuous for $t \geq a$ and $x_1 \leq x \leq x_2$, and if $\int_a^\infty f(x,t)dt$ converges uniformly to $F(x)$ in $[x_1, x_2]$, then $F(x)$ is continuous in the interval $[x_1, x_2]$. This result allows us to write*

$$\lim_{x \to x_0} F(x) = \lim_{x \to x_0} \int_a^\infty f(x,t)dt = \int_a^\infty \lim_{x \to x_0} f(x,t)dt \tag{5}$$

2. *If $f(x,t)$ satisfies the above conditions, then*

$$\int_{x_1}^{x_2} F(x)dx = \int_{x_1}^{x_2} \left[\int_a^\infty f(x,t)dt \right] dx = \int_a^\infty \left[\int_{x_1}^{x_2} f(x,t)dx \right] dt \tag{6}$$

In other words, we can interchange the order of integration.

3. *If $f(x,t)$ and $\partial f/\partial x$ are continuous for $t \geq a$ and $x_1 \leq x \leq x_2$, if $\int_a^\infty f(x,t)dt$ converges to $F(x)$ in $[x_1, x_2]$, and if $\int_a^\infty \partial f/\partial x\, dt$ converges uniformly in $[x_1, x_2]$, then*

$$F'(x) = \int_a^\infty \frac{\partial f(x,t)}{\partial x} dt$$

Note that this last result requires more of $f(x,t)$ than do the first two theorems.

Example 1:
Show that

$$\lim_{x \to 0} \int_0^\infty xe^{-xt}dt \neq \int_0^\infty \lim_{x \to 0} xe^{-xt}dt$$

and explain why.

SOLUTION:

$$\lim_{x \to 0} \int_0^\infty xe^{-xt}dt = \lim_{x \to 0} 1 = 1$$

and

$$\int_0^\infty \lim_{x \to 0} xe^{-xt}dt = 0$$

The two limits are not necessarily equal because, as we showed above, the integral is not uniformly convergent for $x \geq 0$, so we should not expect that $F(x)$ will be continuous for $x \geq 0$.

To effectively use the above theorems, we need a simple test for the uniform convergence of an integral. Let $f(x, t)$ be continuous for $t \ge a$ and $x_1 \le x \le x_2$. Now if you can find a function $M(t)$ continuous for $t \ge a$, such that $|f(x, t)| \le M(t)$ for $t \ge a$ and $x_1 \le x \le x_2$, and if $\int_a^\infty M(t)dt$ converges, then $\int_a^\infty f(x, t)dt$ converges uniformly for $x_1 \le x \le x_2$. This test is known as the *Weierstrass M test*. The proof is fairly easy. Let $F(x) = \int_a^\infty f(x, t)dt$. Then,

$$\left|F(x) - \int_a^b f(x, t)dt\right| = \left|\int_a^\infty f(x, t)dt - \int_a^b f(x, t)dt\right| = \left|\int_b^\infty f(x, t)\,dt\right|$$
$$\le \int_b^\infty |f(x, t)|\,dt \le \int_b^\infty M(t)dt$$

This last integral goes to zero as $b \to \infty$, independent of x, so the theorem is proved.

Example 2:

Use the Weierstrass M test to show that $F(x) = \displaystyle\int_0^\infty e^{-xt^2}dt$ is uniformly convergent for $x \ge \alpha > 0$.

SOLUTION: If we choose $M(t) = e^{-\alpha t^2}$, then $f(x, t) = e^{-xt^2} \le e^{-\alpha t^2}$ for $t \ge 0$ and $x \ge \alpha$. Because $\int_0^\infty e^{-\alpha t^2}dt = (\pi/4\alpha)^{1/2}$ is finite for $\alpha > 0$, $F(x) = \int_0^\infty e^{-xt^2}dt$ converges uniformly for $x \ge \alpha > 0$.

Example 3:

Show that $G(x) = \displaystyle\int_0^\infty t^2e^{-xt^2}dt$ is uniformly convergent for $x \ge \alpha > 0$.

SOLUTION: Choose $M(t) = t^2e^{-\alpha t}$ because $f(x, t) = t^2e^{-xt^2} \le t^2e^{-\alpha t^2}$ for $t > 0$ and $x \ge \alpha$. Because $\int_0^\infty t^2e^{-\alpha t^2}dt = \pi^{1/2}/4\alpha^{3/2}$ is finite for $\alpha > 0$, $G(x) = \int_0^\infty t^2e^{-xt^2}dt$ converges uniformly for $x \ge \alpha > 0$.

Example 4:

Evaluate $F'(x)$ in Example 2 by differentiating under the integral sign and then integrating and compare your result to the one you get by differentiating $F(x)$ after first evaluating the integral.

SOLUTION: Differentiating under the integral sign and then integrating gives

$$F'(x) = \int_0^\infty \frac{\partial e^{-xt^2}}{\partial x} dt = -\int_0^\infty t^2 e^{-xt^2} dt = -\frac{\pi^{1/2}}{4}\frac{1}{x^{3/2}}$$

If we evaluate $F(x)$ first, we obtain $F(x) = (\pi/4x)^{1/2}$, which gives $F'(x) = -\pi^{1/2}/4x^{3/2}$, in agreement with the first result. We obtain the same result because e^{-xt^2} and $t^2e^{-xt^2}$ are continuous functions of x and t and $\int_0^\infty t^2 e^{-xt^2} dt$ converges uniformly for $x > 0$. (See the previous Example.)

Before we leave this section (and this chapter) we should mention some corresponding results for integrals with finite limits.

1. *If $f(x, t)$ is continuous in the rectangle $a \le t \le b$, $x_1 \le x \le x_2$, then $F(x) = \int_a^b f(x, t)dt$ is continuous for $x_1 \le x \le x_2$. This theorem allows us to write*

$$\lim_{x \to x_0} \int_a^b f(x, t)dt = \int_a^b \lim_{x \to x_0} f(x, t)dt \qquad x_1 \le x \le x_2$$

2. *Under the conditions of the previous theorem, we have*

$$\int_{x_1}^{x_2} F(x)dx = \int_{x_1}^{x_2} \left[\int_a^b f(x, t)dt\right] dx = \int_a^b \left[\int_{x_1}^{x_2} f(x, t)dx\right] dt$$

3. *If $f(x, t)$ and $\partial f(x, t)/\partial x$ are continuous in the rectangle $a \le t \le b$, $x_1 \le x \le x_2$, then*

$$F'(x) = \int_a^b \frac{\partial f(x, t)}{\partial x} dt$$

You can easily test these three theorems with $F(x) = \int_0^1 e^{-xt} dt$ (Problem 10).

1.9 Problems

1. Show that $\displaystyle\int_0^\infty e^{-xt} dt$ is uniformly convergent for $x \ge \alpha > 0$.

2. Show that $\displaystyle\int_0^\infty t^n e^{-xt} dt$ is uniformly convergent for $x \ge \alpha > 0$ where n a positive integer.

3. We showed that $\displaystyle\int_0^\infty te^{-xt} dt = 1/x$ is uniformly convergent for $x \ge \alpha > 0$ in Problem 1. Show that $\displaystyle\int_0^\infty t^n e^{-xt} dt = \frac{n!}{x^{n+1}}$.

4. Prove that $\displaystyle\int_0^\infty e^{-xt} \cos t \, dt$ converges uniformly (and absolutely) for $a \le x \le b$, where $0 < a < b$.

5. Example 2 shows that $\int_0^\infty e^{-xt^2}dt = \left(\frac{\pi}{4x}\right)^{1/2}$ is uniformly convergent for $x \geq \alpha > 0$. Show that $\int_0^\infty t^{2n}e^{-\alpha t^2}dt = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}\alpha^n}\left(\frac{\pi}{\alpha}\right)^{1/2}$.

6. Show that $\int_0^\infty \frac{\cos \alpha x}{x^2+1}$ is uniformly convergent for all real values of α.

7. There is a standard trick to evaluate $\int_0^\infty e^{-x^2}\cos\alpha x\,dx$. Differentiate with respect to α, integrate by parts, and notice the result. The answer is $(\pi^{1/2}/2)\,e^{-\alpha^2/4}$.

8. Show that $\int_0^\infty e^{-\alpha x}\cos x\,dx$ is a continuous function of α for $\alpha > 0$.

9. Does $\int_0^\infty \frac{dx}{1+x^2} = \lim_{\alpha\to 0}\int_0^\infty \frac{\cos\alpha x\,dx}{1+x^2}$?

10. Verify the last three theorems at the end of the section with $F(x) = \int_0^1 e^{-xt}dt$ and $0 \leq x \leq 1$.

11. Show that $\int_0^\infty(\sin x)/x\,dx = \pi/2$ by writing

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^\infty \sin x\left(\int_0^\infty e^{-xt}dt\right)dx$$

and then interchanging orders of integration.

12. Show that $\int_0^\infty e^{-ax}(\sin x)/x\,dx = \cot^{-1}a$ by writing

$$\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx = \int_0^\infty e^{-ax}\left(\int_0^1 \cos xz\,dz\right)dx$$

and interchanging orders of integration.

13. Evaluate the integral in the previous problem by differentiating with respect to a.

14. Show that $I(a,b) = \int_0^\infty e^{-a^2x^2-b^2/x^2}dx = \frac{\pi^{1/2}}{2a}e^{-2ab}$ by differentiating with respect to b, then letting $x = b/az$, and then integrating with respect to b.

15. Given that $\int_0^\infty e^{-u}\cos xu\,du = \frac{1}{1+x^2}$, show that $\int_0^\infty ue^{-u}\sin xu\,du = \frac{2x}{(1+x^2)^2}$.

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CHAPTER 2

Infinite Series

Infinite series play an important role in applied mathematics. The use of infinite series is a standard method for solving certain types of ordinary differential equations and many of the famous functions of applied mathematics are defined in terms of infinite series. Bessel functions and Legendre functions are just two examples. Integrals can often be evaluated by first expanding the integrands in a Taylor series and integrating term by term. Also, you may have heard of solving partial differential equations by the method of separation of variables. This frequently used method (and one that we shall study in detail in later chapters) uses various types of Fourier series, which are infinite series in sines and cosines.

In Section 1, we discuss infinite sequences and the ideas of convergence and divergence. We discuss the convergence and divergence of infinite series in Section 2, and then present a number of tests to determine whether or not an infinite series converges. Infinite series whose successive terms have alternating signs are called alternating series, the topic of Section 4. In this section, we'll learn about the difference between an absolutely convergent series and a conditionally convergent series. Then, in Section 5, we'll introduce the important idea of a uniformly convergent series. Uniformly convergent series have the convenient property that they can be manipulated pretty much like polynomials under fairly general conditions. The most commonly occuring infinite series in physical problems are power series, which are of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$ where c and the a_n are constants. It turns out that if a power series converges for values of x on $[a, b]$, then it converges uniformly *within* the interval $[a, b]$. Power series occur frequently because well-behaved functions, like those that occur in physical problems, can be expressed as power series using Taylor's formula. We'll develop Taylor series in Section 7 and then present a number of applications of Taylor series in Section 8. In the final section, we'll discuss asymptotic series, which are quite different from power series in the sense that asymptotic series are useful for large values of x and, in fact, become better approximations as x increases.

2.1 Infinite Sequences

An infinite sequence is a function whose domain is the set of positive integers. We designate an infinite sequence by $s_1, s_2, s_3, \ldots$, where s_n is called the nth term, or simply by $\{s_n\}$. For example, the integers themselves, 1, 2, 3, ... form an infinite sequence, as do their reciprocals, 1, 1/2, 1/3, ... Usually the nth term will be given as an explicit function of n, such as $1/n$ or $(-1)^n/n^3$.

Let's consider the sequence $\left\{\left(1-\frac{1}{n}\right)\right\}$. You can see that $s_n \to 1$ as $n \to \infty$ in this case. We say that the limit of an infinite sequence is l if $\lim_{n\to\infty} s_n = l$. In terms of ϵ and δ, the sequence s_n converges to the limit l if for any ϵ, however small, there is an integer n_0 such that

$$|s_n - l| < \epsilon \qquad n > n_0 \tag{1}$$

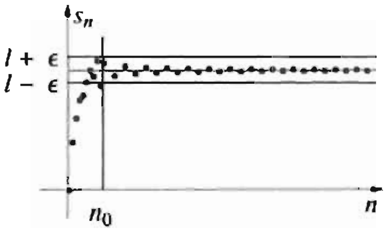

Figure 2.1
An illustration of the convergence of a sequence.

Pictorially, Equation 1 means that if we plot s_n against n, then all the points for $n > n_0$ will lie within the band $l \pm \epsilon$ (Figure 2.1). If there is such a value of l, then the sequence converges; otherwise, it diverges. For example, $\{(-1/2)^n\}$ converges to the limit 0, but $\{n\}$ diverges because $\lim_{n\to\infty} n = \infty$. The sequence $\{(-1)^n\}$ also diverges since it does not approach a limit l; instead it oscillates between +1 and −1. Thus, a sequence doesn't necessarily have to be unbounded in order to diverge. Nevertheless, every convergent infinite sequence must be bounded.

We say that a sequence is *bounded from above* if there is a number c such that $s_n \le c$ for all n, and *bounded from below* if $s_n \ge b$ for all n. A bounded sequence is bounded from above and from below.

Example 1:

Show that the sequence $\left\{\dfrac{n}{n+1}\right\}$ is bounded from above and from below.

SOLUTION: The sequence is $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$. The terms of the sequence are increasing monotonically because

$$s_{n+1} - s_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{n^2+n+1}{(n+1)(n+2)} > 0$$

The limit

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

so the terms are bounded from above by 1. Furthermore,

$$\frac{n}{n+1} \ge \frac{1}{2} \qquad \text{when} \quad n \ge 1$$

so 1/2 is a lower bound. Thus, $1/2 \le s_n \le 1$ for all n (Figure 2.2).

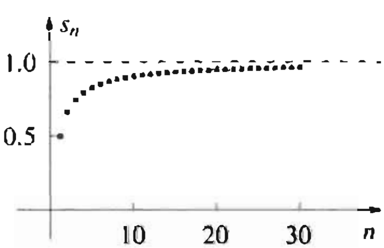

Figure 2.2
A plot of $n/(1+n)$ against n for $n \ge 1$. The limiting value is shown as a dashed line.

A sequence is said to be *nondecreasing* if $s_{n+1} \geq s_n$ for all n. It is *monotonically increasing* if $s_{n+1} > s_n$ for all n, *nonincreasing* if $s_{n-1} \leq s_n$ for all n and *monotonically decreasing* if $s_{n+1} < s_n$ for all n. Clearly, every bounded monotonic sequence is convergent. Although every convergent sequence is bounded, the converse is not true; there are bounded sequences [for example, $\{(-1)^n\}$] that are not convergent.

You should realize that the convergence or divergence of a sequence is not affected by adding or deleting a finite number of terms of the sequence. Convergence depends upon the large n behavior of the sequence, or on the far "tail" of the sequence. The criterion given in Equation 1 explicitly shows that convergence depends only upon the behavior of $\{s_n\}$ for $n > n_0$.

Example 2:

Determine whether the sequence $\left\{\dfrac{\ln n}{n}\right\}$ is increasing or decreasing for large n. Determine its limit as $n \to \infty$.

SOLUTION: Let $f(x) = \dfrac{\ln x}{x}$. Now $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x \geq e$, so the sequence $\left\{\dfrac{\ln n}{n}\right\}$ decreases for $n \geq 3$. Using l'Hôpital's rule, we see that its limit is zero as $n \to \infty$.

2.1 Problems

1. Show that $\lim\limits_{n \to \infty} \dfrac{3n^2 - 6n + 2}{n^2 + 1} = 3$.
2. Show that $\lim\limits_{n \to \infty} \dfrac{\sqrt{n}}{n+1} = 0$.
3. Show that $\lim\limits_{n \to \infty} e^{1/n} = 1$.
4. Show that $\lim\limits_{n \to \infty} \dfrac{\cos n}{n} = 0$.
5. Show that $\lim\limits_{n \to \infty} \sqrt[n]{a} = 1$ for $a > 0$.
6. Show that $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$.
7. Show that $n!/5^n$ is increasing for $n > 5$.
8. Show that $\dfrac{2^n}{n!} \to 0$ as $n \to \infty$. Can you show that this result generalizes to $\dfrac{x^n}{n!} \to 0$ as $n \to \infty$ for any value of x?
9. A simple method for showing whether a sequence is increasing or decreasing is to use a continuous function $f(x)$ such that $f(n) = s_n$ for $n = 1, 2, \ldots$ and show that either $f'(x) > 0$ for $x \geq 1$ (an increasing sequence)

or that $f'(x) < 0$ for $x \geq 1$ (a decreasing sequence). Determine if the following sequences are increasing or decreasing:

(a) $\left\{\dfrac{4n-1}{6n+2}\right\}$ (b) $\left\{\dfrac{\ln n}{n}\right\}$ (c) $\left\{\dfrac{n+2}{n}\right\}$ (d) $\left\{\ln \dfrac{n+1}{n}\right\}$

10. Suppose that s_n is defined by the recursion formula $s_{n+1} = \dfrac{1}{2}\left(s_n + \dfrac{A}{s_n}\right)$. Show that if $\lim_{n\to\infty} s_n = l$ exists, then $l = A^{1/2}$.

11. The recursion formula $F_{n+1} = F_n + F_{n-1}$ defines the Fibonacci sequence, where each term is the sum of the two preceding terms. The sequence is 1, 1, 2, 3, 5, 8, 13, 21, ... The $\lim_{n\to\infty} F_n$ does not exist in this case, but assume that $\lim_{n\to\infty} \dfrac{F_{n+1}}{F_n}$ does exist. Show that this limit is equal to $\frac{1}{2}(1+\sqrt{5})$.

12. Which of the following sequences converge?

(a) $s_n = \dfrac{\cosh n}{\sinh n}$ (b) $s_n = \left(\dfrac{2}{n}\right)^{1/n}$ (c) $s_n = \left(1 + \dfrac{1}{n}\right)^n$ (d) $s_n = \left(\dfrac{n-1}{n+1}\right)^n$

13. Determine $\lim_{n\to\infty} \dfrac{n!}{n^n}$.

14. Use the ϵ-δ notation to prove that $\left\{\dfrac{1}{n}\right\}$ converges to 0.

15. If $\lim_{n\to\infty} s_n = a$ and $\lim_{n\to\infty} t_n = b$, then prove that $\lim_{n\to\infty} s_n t_n = ab$ and $\lim_{n\to\infty} \dfrac{s_n}{t_n} = \dfrac{a}{b}$ provided that $t_n \neq 0$ and $b \neq 0$.

2.2 Convergence and Divergence of Infinite Series

An *infinite series* is an expression of the form

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots$$

The partial sums of this series are

$$S_1 = u_1 \qquad S_2 = u_1 + u_2 \qquad S_3 = u_1 + u_2 + u_3$$

and the nth partial sum is

$$S_n = \sum_{j=1}^{n} u_j$$

If the sequence of partial sums converges, then the series is said to converge, or to be convergent. Otherwise, the series diverges, or is divergent. If

$$\lim_{n\to\infty} S_n = S$$

then S is called the sum of the infinite series.

The standard example of an infinite series is the geometric series, whose nth partial sum is

$$S_n = \sum_{j=1}^{n} r^{j-1} = 1 + r + r^2 + \cdots + r^{n-1}$$

Note that $u_{n+1} = u_n r$. It turns out that it is possible (and easy) to obtain a closed form expression for S_n. Multiply S_n by r and subtract the result from S_n to get

$$S_n - rS_n = 1 - r^n$$

or

$$S_n = \frac{1 - r^n}{1 - r} \tag{1}$$

It's easy to see here that

$$\lim_{n\to\infty} S_n = \begin{cases} \dfrac{1}{1-r} & |r| < 1 \\ \infty & |r| > 1 \end{cases}$$

Note that $S_n = n \to \infty$ as $n \to \infty$ if $r = 1$ and oscillates between 1 and 0 if $r = -1$. Thus, the geometric series converges if $|r| < 1$ and diverges if $|r| \geq 1$. Figure 2.3 shows the partial sums plotted against n for $r = 1/2$. The geometric series is often written as

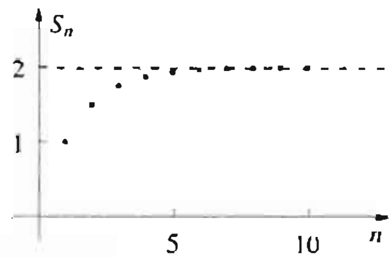

Figure 2.3
A plot of $2[1 - (1/2)^n]$, the partial sums of the geometric series for $r = 1/2$, against n. The limiting value is shown as a dashed line.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x| < 1 \tag{2}$$

Note that we start the summation with an $n = 0$ term in this case.

Example 1:
The partition function of a diatomic molecule modeled as a quantum-mechanical harmonic oscillator is

$$q(T) = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})h\nu/k_BT}$$

where h is the Planck constant, ν is the frequency of the oscillator, k_B is the Boltzmann constant, and T is the kelvin temperature. Express $q(T)$ in closed form.

SOLUTION: Factor $e^{-h\nu/2k_BT}$ from the sum and let $r = e^{-h\nu/k_BT} < 1$. Then, using the geometric series

$$q(T) = e^{-h\nu/2k_BT} \sum_{n=0}^{\infty} r^n$$

$$= \frac{e^{-h\nu/2k_BT}}{1 - e^{-h\nu/k_BT}}$$

We can use the geometric series to show that any recurring decimal expression must be a rational number. Consider, for example, the decimal $a = 0.0909090909\cdots$. We can write a in the form of a geometric series

$$a = \frac{9}{10^2} + \frac{9}{10^4} + \frac{9}{10^6} + \cdots$$

$$= 9\sum_{n=1}^{\infty}\frac{1}{10^{2n}} = 9\sum_{n=1}^{\infty}\frac{1}{(100)^n} = \frac{9}{100}\sum_{n=0}^{\infty}\frac{1}{(100)^n} = \frac{9/100}{1 - 1/100} = \frac{9}{99} = \frac{1}{11}$$

Example 2:
Show that $a = 0.083333\cdots$ is a rational number.

SOLUTION: We write a as

$$a = 0.0800\cdots + 0.003333\cdots$$

$$= \frac{8}{100} + 3\sum_{n=0}^{\infty}\frac{1}{10^{n+3}}$$

$$= \frac{4}{50} + \frac{3}{1000}\left(\frac{1}{1 - 1/10}\right) = \frac{1}{12}$$

It is easy to prove that if two series converge, then their sum and difference both converge. In fact, if

$$S = \sum_{n=1}^{\infty} s_n \qquad \text{and} \qquad T = \sum_{n=1}^{\infty} t_n$$

both converge, then

$$c_1S + c_2T = \sum_{n=1}^{\infty}(c_1s_n + c_2t_n)$$

where c_1 and c_2 are constants. Also if $\sum u_n$ converges, then *necessarily* $u_n \to 0$ as $n \to \infty$. To see this, simply write

$$S_n = \sum_{j=1}^{n} u_j \qquad \text{and} \qquad S_{n-1} = \sum_{j=1}^{n-1} u_j$$

or $S_n - S_{n-1} = u_n$. But $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} S_{n-1} = S$, so

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} 0 = 0$$

It is interesting and important to note that the requirement that $u_n \to 0$ as $n \to \infty$ is a *necessary* condition for convergence, but it is *not* sufficient. The classic example of a series whose nth term goes to zero, but for which the series does not converge, is the *harmonic series*

$$S = \sum_{n=1}^{\infty} \frac{1}{n}$$

Problem 5 leads you through the standard proof that $S_n \to \infty$ as $n \to \infty$.

As in the case of infinite sequences, the convergence or lack of convergence of an infinite series is not affected by the addition or deletion of a finite number of terms. For example, if we were to add 100 terms (which add up to c) to the beginning of a series, then its partial sums would be $S_n + c$ instead of S_n, and the sum, S_+, of the augmented series would be

$$S_+ = \lim_{n\to\infty}(S_n + c) = S + c$$

2.2 Problems

1. Evaluate the sum of the series $S = \sum_{n=0}^{\infty} \frac{1}{3^n}$.

2. Evaluate the sum of the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$.

3. Show that 0.142 857 142 857 $\cdots$ is a rational number.

4. Express the recurring decimal, 0.27272727 $\cdots$ as a fraction.

5. Here is a standard proof that the harmonic series diverges. Show that

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2} \quad S_4 > 1 + \frac{2}{2} \quad S_8 > 1 + \frac{3}{2} \quad S_{16} > 1 + \frac{4}{2}$$

and so on, and then argue that S_n is unbounded as $n \to \infty$.

6. Evaluate the series $S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. *Hint:* Use partial fractions.

7. Derive an expression for the nth partial sum of $S = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \cdots$ and show that $S = 1/2$. *Hint:* Use partial fractions.

8. Evaluate the series $S = \sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$.

9. Evaluate the series $S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$. *Hint*: Use partial fractions.

10. Find an expression for the general term of the series $S = \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \cdots$ and show that the series diverges.

11. Evaluate the series $S = \frac{1}{2^5} + \frac{1}{2^7} + \frac{1}{2^9} + \frac{1}{2^{11}} + \cdots$.

12. Find all the values of x for which each of the following series converges:

(a) $\sum_{n=0}^{\infty} (2x)^n$ (b) $\sum_{n=0}^{\infty} (x-1)^n$

(c) $\sum_{n=0}^{\infty} \left(\frac{2x-1}{3}\right)^n$ (d) $\sum_{n=0}^{\infty} e^{nx}$

13. Show that the series $\sum_{n=0}^{\infty} (\sqrt{n+1} - \sqrt{n})$ diverges even though $\lim_{n\to\infty} u_n = 0$.

14. Does $\sum_{n=1}^{\infty} \sin \frac{n\pi}{2}$ converge?

2.3 Tests for Convergence

A series that consists of positive terms only is called a *positive series*. In this section we present several tests for the convergence of positive series. In the next section, we will discuss *alternating series*, which are series whose successive terms alternate in sign. The convergence properties of positive series and alternating series are somewhat different, so we will treat them separately.

The *integral test* compares a positive series to an integral of a positive function $f(x)$ such that $f(n) > u_n$ for all $n \geq 1$. The convergence of the integral will then assure the convergence of the series. The advantage of the integral test is that it is usually easier to evaluate an integral than it is to sum a series into a closed form.

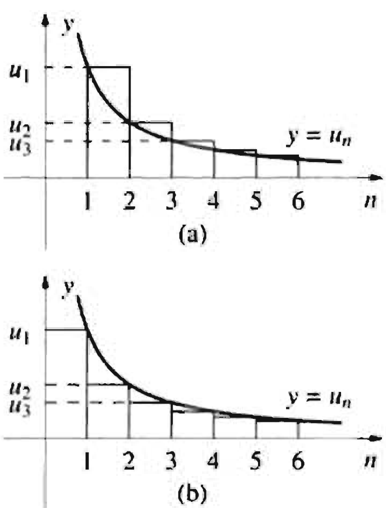

Figure 2.4
A geometric aid to the proof of the integral test.

Integral Test: *Let $\sum u_n$ be a positive series and let $f(x)$ be a continuous, positive, decreasing function such that $f(n) = u_n$ for $n = 1, 2, \ldots$. Then, $\sum_n u_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.*

The proof of the integral test is illustrated in Figures 2.4a and 2.4b. We see in Figure 2.4a that the area under the curve from 1 to N is less than the total area of the rectangles, which is $u_1 + u_2 + \cdots + u_N$. Thus, we have

$$\int_1^N f(x)dx < u_1 + u_2 + \cdots + u_N = S_N \tag{1}$$

If $\sum u_n$ converges, then the sequence of partial sums will be bounded, and so $\int_1^N f(x)dx$ will converge as $N \to \infty$. Conversely, Figure 2.4b shows that

$$\int_1^N f(x)dx > u_2 + u_3 + \cdots + u_N = S_N - u_1$$

or

$$S_N < \int_1^N f(x)dx + u_1 \tag{2}$$

So, if $\int_1^N f(x)dx$ converges, then $S_N < \int_1^\infty f(x)dx + u_1$ and so the sequence S_N will be bounded and the series $\sum u_n$ converges.

Actually, the integral test doesn't have to start with the $n = 1$ term. It's the "long tail" behavior of a series that determines whether it converges or diverges, so we can start with any finite term, say, M, and investigate

$$I_M = \int_M^\infty f(x)dx$$

We'll usually take $M = 1$, however.

Let's consider the series $\sum_{n=1}^\infty \frac{1}{n^2}$. The function $f(x) = 1/x^2$ equals $1/n^2$ when x is an integer. Furthermore, $f(x)$ is continuous and monotonically decreasing. Now

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = 1$$

so the series $\sum_{n=1}^\infty \frac{1}{n^2}$ converges. Unfortunately, neither the integral test, nor any other test, tells us what the sum is equal to; they simply tell us whether the series converges or diverges.

Example 1:
Investigate the convergence of the series

$$S(p) = \sum_{n=1}^\infty \frac{1}{n^p}$$

as a function of p. We shall call $S(p)$ the *p series*.

SOLUTION: Take $f(x) = 1/x^p$ to be the test function for the integral test.

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty} \left[-\frac{-x^{1-p}}{1-p}\right]_1^b = \lim_{b\to\infty} \left(\frac{b^{1-p} - 1}{1-p}\right)$$

If $p > 1$, then $\lim_{b\to\infty} (b^{1-p} - 1)/(1-p) = 1/(p-1)$, and so the p series converges. If $p < 1$, then $\lim_{b\to\infty} (b^{1-p} - 1)/(1-p) = \infty$, and so the p series diverges. We already know that the series diverges for $p = 1$, the harmonic series, but if we didn't, we would see so anyway because $f(x) = 1/x$ in this case and

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \Big[\ln x \Big]_1^b = \infty$$

We can use the integral test to determine the error that occurs when we approximate the sum of a series by its partial sums. The error R_N incurred by using S_N is (Problem 3)

$$\int_{N+1}^\infty f(x)dx \le R_N \le \int_N^\infty f(x)dx \tag{3}$$

If we sum $u_n = 1/n^3$ to ten terms, the error will lie between

$$\int_{11}^\infty \frac{dx}{x^3} \le R_N \le \int_{10}^\infty \frac{dx}{x^3}$$

or between 0.00413 and 0.00500.

Equation 3 can be used to determine how many terms we need to use to achieve a given accuracy. For example, to achieve an accuracy of ± 0.0002 in $\sum_{n=1}^\infty \frac{1}{n^4}$, we need to take 12 terms. To achieve the same accuracy for $\sum_{n=1}^\infty \frac{\ln n}{n^2}$, we need to take over 100 000 terms (Problem 5)! We say that the first sum converges much faster than the second. In fact, we might say that the first one converges rather quickly (see Figure 2.5), and that the second one converges very slowly. Obviously, rapidly converging series are more useful than very slowly converging series for numerical work. There are, however, a number of numerical methods for accelerating the rate of convergence of infinite series.

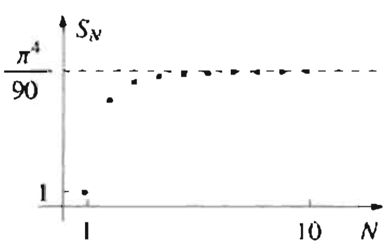

Figure 2.5

The partial sums $S_N = \sum_{n=1}^N 1/n^4$ plotted against N. The limiting value is shown as a dashed line.

The geometric series and the p series are frequently used to test for convergence of other series by a test called the *comparison test*. Before we discuss this test, we display the geometric series and the p series for reference.

$$\text{geometric series:} \quad \sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad |x| < 1 \tag{4}$$

$$p \text{ series:} \quad \sum_{n=1}^\infty \frac{1}{n^p} \quad \begin{array}{ll} \text{converges} & p > 1 \\ \text{diverges} & p \le 1 \end{array} \tag{5}$$

The Comparison Test: *Let $\sum u_n$ and $\sum v_n$ be two positive series with $u_n \le v_n$ for all $n > N$. Then $\sum u_n$ converges if $\sum v_n$ converges, and $\sum v_n$ diverges if $\sum u_n$ diverges.*

(See Problem 14 for an outline of the proof.)

Does the series $\sum_{n=1}^{\infty} \frac{1}{n^3+3}$ converge? Well, $\frac{1}{n^3+3} < \frac{1}{n^3}$ and we know that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p series with $p = 3$), and so $\sum_{n=1}^{\infty} \frac{1}{n^3+3}$ converges. What if the series started with the $n = 0$ term? We wouldn't be able to apply the above inequality for the $n = 0$ term. Does this make any difference? All we have to do is to write the $n = 0$ term explicitly and then consider the series $\frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{n^3+3}$. The point is that we can apply the comparison test starting with any term because the convergence of an infinite series is determined by the large-n tail of the series, so the first finite number of terms has no effect on the convergence or divergence of an infinite series.

Example 2:
Examine the convergence of

$$S = \sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

SOLUTION:

$$\frac{1}{n!} = \frac{1}{n(n-1)(n-2)\cdots 3\cdot 2} \le \frac{1}{2^{n-1}} \quad \text{for} \quad n \ge 1$$

So

$$S \le \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

But

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

and so $S = \sum_{n=1}^{\infty} \frac{1}{n!}$ converges. We'll see in Section 7 that $S = e - 1$.

Another useful test for the convergence of a positive series is the

Limit Comparison Test: *If $\sum u_n$ and $\sum v_n$ are two positive series such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ with $0 < l < \infty$, then $\sum u_n$ and $\sum v_n$ either both converge or both diverge. If $l = 0$ and $\sum v_n$ converges, then $\sum u_n$ converges. If $l = \infty$ and $\sum v_n$ diverges, then $\sum u_n$ diverges.*

Let's use the limit comparison test to test the convergence of $\sum u_n$ where $u_n = n/(n^2+1)$. For large n, u_n behaves as $1/n$, so let's use $v_n = 1/n$ in the theorem. In this case,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

We know, however, that $\sum v_n$ diverges (the harmonic series), so $\sum u_n$ diverges as well.

If $u_n = n/(n^3+1)$ instead of $n/(n^2+1)$, then u_n behaves as $1/n^2$ for large n and we would choose $v_n = 1/n^2$, in which case

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$$

But $\sum v_n$ converges (the p series with $p = 2$) and so $\sum u_n$ converges.

The two examples that we have just discussed lead us to another useful test (we'll call it the *p test*).

***p* Test:** *If* $\lim_{n\to\infty} n^p u_n = l$, *then* $\sum u_n$ *converges if* $p > 1$ *(and* l *is finite, even* 0*), or* $\sum u_n$ *diverges if* $p \leq 1$ *and* $l \neq 0$ *(but* l *may be infinite).*

Example 3:
Test the series

$$S = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

for convergence.

SOLUTION: We'll multiply $\ln n/n^2$ by n^p and see if there is a value of $p > 1$ that gives us a finite limit. If we choose $p = 3/2$, for example, we have

$$\lim_{n\to\infty} n^{3/2} \cdot \frac{\ln n}{n^2} = \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} = 0$$

(by l'Hôpital's rule, for instance). Therefore,

$$S = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

converges. We chose $p = 3/2$ for concreteness above, but note that we could have chosen any value such that $1 < p < 2$.

2.3 Problems

1. Does the series $S = \sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge?

2. Does the series $S = \sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converge?

3. Derive Equation 3.

4. How many terms do we need to take to achieve an accuracy of ± 0.001 for $\sum_{n=1}^{\infty} \frac{1}{n^3}$?

5. Show that you need 15 terms to achieve an accuracy of ± 0.0001 for the sum $\sum_{n=1}^{\infty} \frac{1}{n^3}$, but that you need more than 125 000 terms to achieve the same accuracy for $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

6. Show that the series $S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ converges.

7. Write out a few terms in the series in the previous problem and show that $S = 1/2$. This type of series is called a *telescoping series* because most of its terms cancel pairwise. *Hint:* Use partial fractions.

8. Can we conclude that the series $S = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right)$ diverges because $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n+2}$ diverge? Why not?

9. Write out a few terms of the series in the previous problem and show that $S = 3/2$.

10. Test each of the following series for convergence:

    (a) $\sum_{n=0}^{\infty} n e^{-n^2}$ (b) $\sum_{n=3}^{\infty} \frac{1}{n^2+3}$

    (c) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$

11. Does $\sum_{n=1}^{\infty} \frac{(\ln n)^5}{n^3}$ converge?

12. Does $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ converge? What about $\sum_{n=1}^{\infty} \frac{1}{n+n^{3/2}}$?

13. Does $\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$ converge?

14. Prove the comparison test. (*Hint:* Appeal to the fact that a bounded monotonic series converges.)

2.4 Alternating Series

An *alternating series* is an infinite series whose successive terms alternate in sign. The general formula for an alternating series is

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} v_n = v_1 - v_2 + v_3 - v_4 + \cdots \tag{1}$$

where $v_n \geq 0$. We have a very convenient convergence test for alternating series that is due to Leibnitz.

If $0 < v_{n+1} \leq v_n$ *and* $\lim_{n\to\infty} v_n = 0$, *then* S *converges.*

(See Problem 17.) For example, the *alternating harmonic series*

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{2}$$

converges according to the Leibnitz test.

Convergent alternating series have a property that allows you to estimate their sums easily. If S is the sum of the (alternating) series and S_N is its nth partial sum, then the remainder after N terms is $R_N = S - S_N$. It turns out that $|R_N| < v_{N+1}$; in other words, the magnitude of the error that occurs from approximating S by using just the first N terms is not greater than the magnitude of the first omitted term (Problem 18).

Let's illustrate this property using the alternating series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

which we show is equal to $1 - e^{-1}$ in Example 2.7-2. Table 2.1 lists the values of v_{N+1}, S_N, and $|R_N|$ for $N = 1$ through 6. Notice that $|R_N| < v_{N+1}$.

Table 2.1
The first six partial sums and remainders for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} = 1 - e^{-1} = 0.63212$.

N	v_{N+1}	S_N	$R_N = S - S_N$	$\lvert R_N \rvert$
1	0.5000	1.0000	−0.3679	0.3679
2	0.1667	0.5000	+0.1321	0.1321
3	0.0417	0.6667	−0.0345	0.0345
4	0.0083	0.6250	+0.0071	0.0071
5	0.0014	0.6333	−0.0012	0.0012
6	0.0002	0.6319	+0.0002	0.0002

Let's see how many terms we should use in the above series to get five-place accuracy. We'll require the first term omitted to be $< 5 \times 10^{-6}$ in order to assure five-place accuracy. Therefore, we choose $N = 8$ since $1/9! = 2.8 \times 10^{-6}$. If we add the first eight terms, we get $S_8 = 0.632\,1181$. The correct value to seven places is 0.632 1206.

Example 1:
How many terms should we use to calculate

$$S = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

to four-place accuracy?

SOLUTION: We want the first omitted term to be less than 1×10^{-4}, so we want $1/(2k+1)^3$ to be less than 10^{-4}, which will be so if $k > 10$. Choosing $k = 11$ gives $S_{10} = 0.968\,993$. We'll learn in Section 3.7 that $S = \pi^3/32 = 0.968\,946\cdots$.

You should be aware that many of the CAS that are available can be used to evaluate series such as the one in Example 1. For example, the one instruction line in Mathematica

```
NSum[ (-1)^n / (2 • n + 1)^3, {n, 0, 100} ]
```

evaluates the sum numerically from $n = 0$ to $n = 100$ (0.968946).

Another property of alternating series that is closely related to the fact that $|R_N| \le v_{N+1}$ is that the value of the series S always lies in the closed interval $[S_N, S_{N+1}]$. The proof that $|R_N| \le v_{N+1}$, which we relegated to Problem 17, has as a by-product that successive values of R_N have opposite signs (if they don't happen to equal zero). So, if $S = S_N + R_N = S_{N+1} + R_{N+1}$, and $R_N \ge 0$ and $R_{N+1} \le 0$, then $S \ge S_N$ and $S \le S_{N+1}$, or

$$S_N \le S \le S_{N+1} \qquad (R_N \ge 0) \tag{3}$$

Similarly, if $R_N \le 0$ and $R_{N+1} \ge 0$, then

$$S_N \ge S \ge S_{N+1} \qquad (R_N \le 0) \tag{4}$$

Either way, we have the result that S lies within the closed interval $[S_N, S_{N+1}]$. Table 2.1 shows that Equations 3 and 4 are satisfied (up to $N = 6$ at least) and the partial sums for Example 1 are plotted against N in Figure 2.6. The "exact" value is $0.968\,946\cdots$ and this value lies between any two consecutive partial sums.

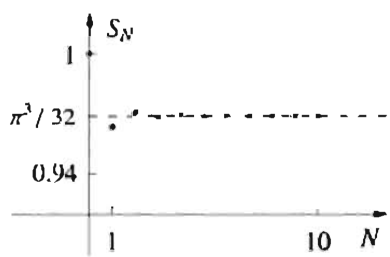

Figure 2.6

The partial sums $S_N = \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)^3}$ plotted against N. The limiting value is shown as a dashed line. Note the scale on the vertical axis.

The harmonic series diverges, but according to the Leibnitz test, the alternating harmonic series converges. A series $\sum u_n$ is called *absolutely convergent* if $\sum |u_n|$ converges. If $\sum u_n$ converges, but $\sum |u_n|$ does not converge, then the series $\sum u_n$ is called *conditionally convergent*. The alternating harmonic series is an example of a conditionally convergent series. Surely, if a series converges absolutely, then it converges conditionally.

Let's consider the series

$$S = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

This (alternating) series is convergent because $(-1)^n/\ln n$ decreases monotonically with increasing n and the nth term goes to zero as $n \to \infty$. It is not absolutely convergent, however, because

$$S_1 = \sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

diverges (use the p test with $p = 1/2$, Problem 8).

Example 2:
Test the following series for absolute convergence and conditional convergence.

$$S = \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$$

SOLUTION: Notice that $\cos n\pi = (-1)^n$, so S is an alternating series. Since

$$\left| \frac{\cos n\pi}{n^2} \right| = \left| \frac{1}{n^2} \right|$$

the series is absolutely convergent by comparison with a p series with $p = 2$. Thus, S is convergent.

We now present several tests for the absolute convergence of a series. Of course, these tests can also be used to test for convergence of a positive series.

The Ratio Test: *Let $\sum u_n$ be any series and let*

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \rho$$

Then $\sum u_n$ converges absolutely if $\rho < 1$, diverges if $\rho > 1$, and the test is inconclusive if $\rho = 1$.

(See Problem 11.) We'll use the series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n}$$

to illustrate the ratio test.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{2^n n}{2^{n+1}(n+1)}\right| = \lim_{n\to\infty} \left|\frac{1}{2\left(1+\frac{1}{n}\right)}\right| = \frac{1}{2} = \rho < 1$$

so the series converges absolutely.

How about the series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1} \tag{5}$$

In this case,

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{n^2+1}{(n+1)^2+1}\right| = 1 = \rho$$

so the ratio test fails us. There is an extension of the ratio test, however, that is often useful when $\rho = 1$.

Raabe's Test: *Let*

$$\lim_{n\to\infty} \left[n\left(1 - \left|\frac{u_{n+1}}{u_n}\right|\right)\right] = l \tag{6}$$

The series $\sum u_n$ converges absolutely if $l > 1$ and either diverges or converges conditionally if $l < 1$. As with the simple ratio test, the test tells us nothing if $l = 1$.

We'll apply Raabe's test to the series in Equation 5, where the ratio test failed us. Applying Equation 6:

$$\lim_{n\to\infty} \left[n\left(1 - \left|\frac{u_{n+1}}{u_n}\right|\right)\right] = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+2} = l = 2$$

Therefore, the series in Equation 5 converges absolutely.

Example 3:

Investigate the convergence of

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3}{(n+1)!}$$

SOLUTION: The ratio test gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3}{n^3}\cdot\frac{(n+1)!}{(n+2)!}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 \cdot \lim_{n\to\infty}\frac{1}{n+2} = 0$$

so the series is absolutely convergent.

Example 4:
Test the following series for convergence:

$$S = \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$$

SOLUTION: The ratio test gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2n+1}{2(n+2)}\right| = 1$$

so the ratio test is inconclusive in this case. Raabe's test gives

$$\lim_{n\to\infty} n\left(1-\frac{2n+1}{2(n+2)}\right) = \lim_{n\to\infty}\frac{3n}{2(n+2)} = \frac{3}{2} > 1$$

Therefore, we see that the series converges absolutely.

The above tests for convergence give us *sufficient* conditions for the absolute convergence of a series; in other words, when the conditions are satisfied, the series definitely converges absolutely. The tests, however, do *not* give us *necessary* conditions; that is, there are absolutely convergent series which do not satisfy the above conditions.

Before leaving this section, we point out a somewhat disconcerting property of conditionally convergent series that you should keep in mind. Consider the alternating harmonic series, which we know is conditionally convergent. Write it as

$$S = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots$$

Multiply S by 1/2 and write the result as

$$\frac{1}{2}S = \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots$$

and add this to S above to get

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

This last series is actually just a rearrangement of the series for S, so we are led to the absurd result that $S = \frac{3}{2}S$. The message here is that you cannot rearrange the terms in a conditionally convergent series without changing its value. In fact, if $\sum u_n$ is conditionally convergent, it is possible to rearrange $\sum u_n$ to have *any* sum or to even destroy its convergence. Absolutely convergent series, on the other hand, are fairly well behaved in the sense that rearrangement does not alter the value of the sum of the series.

2.4 Problems

1. Construct a table like Table 2.1 for the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$. (We'll see in Section 3.7 that $S = 7\pi^4/720$.) Show that $|R_N| < u_{N+1}$ and that Equations 3 and 4 are satisfied.

2. Construct a table like Table 2.1 for the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. (We'll see in Section 3.7 that $S = \pi^2/12$.) Show that $|R_N| < u_{N+1}$ and that Equations 3 and 4 are satisfied.

3. How many terms should we use to have an error of ± 0.0001 for $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{4^n}$?

4. How many terms should we use to have an error of ± 0.00001 for $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$? We'll see in Section 3.7 that the exact value of S is $7\pi^4/720$.

5. Use any CAS to construct a figure like Figure 2.6 for $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{2^n}$. How many terms should you take to have an error of $< 10^{-4}$?

6. Use any CAS to construct a figure like Figure 2.6 for S given in Problem 2.

7. Use any CAS to construct a figure like Figure 2.6 for S given in Problem 4.

8. Show that $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

9. Use the integral test to test the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ for convergence.

10. Use the integral test to test the series $\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$ for convergence.

11. In this problem we shall walk through the proof of the ratio test. Assume that $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = r < 1$. Choose t such that $r < t < 1$, so that $\left| \frac{u_{n+1}}{u_n} \right| \leq t$ if $n > N$. Now argue that $|u_{n+k}| \leq |u_n| t^k$ and appeal to the geometric series. The proofs for the case $r > 1$ and $r = 1$ are similar.

12. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$ (b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+1}}$ (c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(n+1)^2}$

(d) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{2^n}$ (e) $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$ (f) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^3}{(n^2+2)^{3/2}}$

13. Investigate the convergence of the series $S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$.

14. Test the series $S = \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$ for convergence.

15. There is another test for convergence called the *Root Test*: Let $r = \lim_{n\to\infty} \sqrt[n]{|u_n|}$.

If $r < 1$, then $\sum u_n$ converges (absolutely).
If $r > 1$, then $\sum u_n$ diverges.
If $r = 1$, then the root test is inconclusive.

(In applying the root test, it is often helpful to remember that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.) Use the root test to discuss the convergence of the series

$$S(x) = \sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + \cdots$$

16. Use the root test to see if $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n$ converges.

17. Prove Leibnitz's alternating series theorem. *Hint*: First show that $S_{2n} \geq 0$ and that $S_{2n} \leq v_1$, and then argue that $\lim_{n\to\infty} S_{2n} = S$, for example. Then show that $\lim_{n\to\infty} S_{2n+1} = S$ also.

18. We'll prove that $|R_N| \leq v_{N+1}$ in this problem. First consider the error made in stopping after $2N$ terms:

$$R_{2N} = (v_{2N+1} - v_{2N+2}) + (v_{2N+3} - v_{2N+4}) + (v_{2N+5} - v_{2N+6}) + \cdots$$

Show that $R_{2N} \geq 0$. Now write

$$R_{2N} = v_{2N+1} - (v_{2N+2} - v_{2N+3}) - (v_{2N+4} - v_{2N+5}) - (v_{2N+6} - v_{2N+7}) - \cdots$$

Show that $0 \leq R_{2N} \leq v_{2N+1}$, the magnitude of the first omitted term. Using a similar argument, show that the error made by stopping after $2N+1$ terms is $-v_{2N+1} < R_{2N+1} \leq 0$. (Note that R_{2N} and R_{2N+1} have opposite signs if they are not equal to zero.) Combine these two results to show $|R_N| \leq v_{N+1}$.

19. Apply the result of the previous problem to the alternating series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + 0 + 0 + 0 + 0 + \cdots$$

20. Test the series $S = \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{1}{n}$ for convergence.

2.5 Uniform Convergence

Up to this point, most of the series that we have discussed are series whose terms are constants. Now let's consider series whose terms are functions. For example, we might have

$$S(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

or

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 + x}$$

The definition of convergence of such series follows immediately from our definition of convergence for series of constant terms. The series

$$S(x) = \sum_{n=1}^{\infty} u_n(x)$$

is said to converge to $S(x)$ in $[a, b]$ if

$$|S(x) - S_n(x)| < \epsilon \qquad \text{for} \qquad n > N(\epsilon, x) \tag{1}$$

for all x in $[a, b]$. In other words, the sequence of partial sums $\{S_n(x)\}$ converges in $[a, b]$. Note that the value of N may depend upon both ϵ and x. If Equation 1 is satisfied, we write

$$S(x) = \lim_{n \to \infty} S_n(x) \tag{2}$$

for each value of x.

Example 1:
Test the following series for convergence:

$$S(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n(n+1)}$$

SOLUTION: We'll use the ratio test:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \left| (x-2)\frac{n(n+1)}{(n+1)(n+2)} \right| = |x-2|$$

Thus, the series converges if $|x-2| < 1$, or if $1 < x < 3$. The series is convergent at both end points since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, so we write $1 \le x \le 3$. We'll include the points $x = 1$ and $x = 3$ in the interval of convergence.

We will show in Example 3 that the series in Example 1 converges to $1 + \left(\frac{3-x}{x-2}\right)\ln(3-x)$ for $1 \le x \le 3$.

Now consider the series

$$S(x) = \sum_{j=0}^{\infty} \frac{x^2}{(1+x^2)^j} = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots \tag{3}$$

If $x = 0$, then $S_n(x) = 0$ for all n and $S(x) = \lim_{n\to\infty} S_n(x) = 0$. If $x \neq 0$, then $1/(1+x^2) < 1$, and the sum is a geometric series with

$$S(x) = x^2\left(\frac{1}{1-\dfrac{1}{1+x^2}}\right) = 1 + x^2 \qquad x \neq 0$$

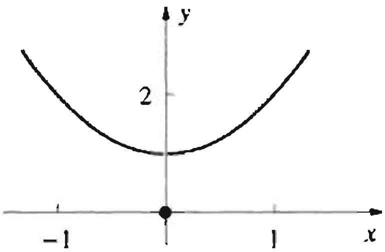

Figure 2.7
A plot of the function $S(x)$ defined by Equation 3.

Note that $\lim_{x\to 0} S(x) = 1$, so $S(x)$ has a discontinuity at $x = 0$ in this case (Figure 2.7).

In Example 1, $S(x)$ is a continuous function of x, but the function represented by Equation 3 is discontinuous. The terms of both series are continuous, so why is $S(x)$ continuous in one case but not in the other? The answer to this question lies in the idea of *uniform convergence*. We say that an infinite series is uniformly convergent if

$$|S(x) - S_n(x)| < \epsilon \qquad \text{for} \quad n > N(\epsilon) \tag{4}$$

where N depends upon ϵ, but is *independent of* x. Let's look at Equation 3:

$$S(x) = \sum_{j=0}^{\infty} \frac{x^2}{(1+x^2)^j}$$

Recall that $S(x) = 0$ when $x = 0$ and $S(x) = 1 + x^2$ otherwise. Let's consider the case where $x \neq 0$. Then the nth partial sum is

$$S_n(x) = x^2\left[\frac{1-\left(\dfrac{1}{1+x^2}\right)^n}{1-\dfrac{1}{1+x^2}}\right] = (1+x^2)\left[1-\left(\frac{1}{1+x^2}\right)^n\right] \quad x \neq 0$$

and

$$|S(x) - S_n(x)| = \left|\left(\frac{1}{1+x^2}\right)^{n-1}\right| \qquad x \neq 0$$

Now let's see if we can satisfy Equation 4. The difference $|S(x) - S_n(x)|$ will be $< \epsilon$ if we choose $[1/(1+x^2)]^{N-1} < \epsilon$, or $N > \dfrac{|\ln\epsilon|}{\ln(1+x^2)}$. Notice, however, that N becomes unbounded as $x \to 0$, so the series is *not* uniformly convergent at $x = 0$, the point at which $S(x)$ is discontinuous.

Suppose that $\lim_{n\to\infty} S_n(x) = S(x)$ uniformly in $[a, b]$. Then, given $\epsilon > 0$, there is an N such that if $n > N$, $|S(x) - S_n(x)| < \epsilon$ for $a \le x \le b$. In other words, $S(x) - \epsilon < S_n(x) < S(x) + \epsilon$ for $a \le x \le b$ if $n > N$. Thus, $S_n(x)$ lies within the band shown in Figure 2.8.

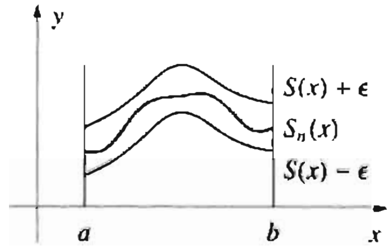

Figure 2.8
An illustration of uniform convergence.

The series in Equation 3 does not conform to the picture in Figure 2.8. The partial sums of Equation 3 for $n = 20$ and 100 are plotted in Figure 2.9 along with $S(x)$. Note that the discontinuity in $S(x)$ at $x = 0$ forces the partial sums to reach down from the curve $1 + x^2$ to reach the value $S(0) = 0$. None of these partial sums can lie within a band $S(x) \pm \epsilon$ because of the behavior near the origin. They all break off from the parabola $S(x) = 1 + x^2$, and hence from the band $S(x) \pm \epsilon$, as x approaches zero.

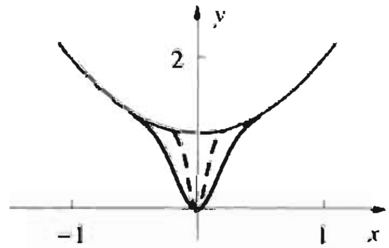

Figure 2.9
The partial sums of Equation 3 for $n = 20$ and 100 plotted against x.

What about the series in Example 1, where $S(x)$ is a continuous function for $1 \le x \le 3$? In order to address this question, we need a useful method to establish the uniform convergence of a series. The definition given in Equation 4 is often awkward to implement. The following test is much more useful:

The Weierstrass M Test: *If a sequence of constants M_n can be found such that*

$$|u_n(x)| \le M_n$$

for x in $[a, b]$, and M_n converges, then $\sum u_n(x)$ is uniformly and absolutely convergent for x in $[a, b]$.

Example 2:
Use the Weierstrass M test to show that the series in Example 1 is uniformly convergent.

SOLUTION: Choose $M_n = \dfrac{1}{n^2}$. Then

$$\left|\frac{(x-2)^n}{n(n+1)}\right| < \frac{1}{n^2} \qquad 1 \le x \le 3$$

Then $\sum M_n$ is a p series with $p = 2$ and thus converges. Therefore, the series in Example 1 is uniformly convergent for $1 \le x \le 3$.

We're hinting around that the series $S(x)$ in Example 1 is a continuous function of x because the series is uniformly convergent. Well, it's true. Uniformly convergent series have a number of nice properties, and continuity is one of them.

1. *If all the terms in $S(x) = \sum_{n=1}^{\infty} u_n(x)$ are continuous in $[a, b]$ and $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent in $[a, b]$, then $S(x)$ is continuous in $[a, b]$. In other words,*

$$\lim_{x\to a}\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x\to a} u_n(x) = \sum_{n=1}^{\infty} u_n(a) \tag{5}$$

Thus, a uniformly convergent series of continuous functions is a continuous function. Not only is $S(x)$ continuous if a uniformly convergent series consists of terms that are continuous, but the converse is true: if $S(x)$ is discontinuous, then the series is not uniformly convergent.

Two other useful properties of uniformly convergent series are given in the following two theorems:

2. *If all the terms in* $S(x) = \sum_{n=1}^{\infty} u_n(x)$ *are continuous in* $[a, b]$ *and* $\sum_{n=1}^{\infty} u_n(x)$ *is uniformly convergent in* $[a, b]$, *then*

$$\int_a^b S(x)dx = \sum_{n=1}^{\infty}\int_a^b u_n(x)dx \tag{6}$$

or

$$\int_a^b \left[\sum_{n=1}^{\infty} u_n(x)\right] dx = \sum_{n=1}^{\infty}\int_a^b u_n(x)dx \tag{7}$$

In other words, a uniformly continuous series can be integrated term by term.

3. *If* $\sum_{n=1}^{\infty} u_n(x)$ *converges to* $S(x)$ *and all the terms in* $S(x) = \sum_{n=1}^{\infty} u_n(x)$ *are differentiable in* $[a, b]$ *and if* $\sum_{n=1}^{\infty} u_n'(x)$ *converges uniformly in* $[a, b]$, *then*

$$S'(x) = \sum_{n=1}^{\infty} u_n'(x) \tag{8}$$

or

$$\frac{d}{dx}\left[\sum_{n=1}^{\infty} u_n(x)\right] = \sum_{n=1}^{\infty}\frac{du_n(x)}{dx} \tag{9}$$

In other words, the series can be differentiated term by term. Notice that the conditions in **3** are more stringent than those in **1** and **2**.

Example 3:

Start with the geometric series for $1/(1-x)$ and integrate twice term by term to derive an expression for

$$S(x) = \sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$$

Compare this series to the one in Example 1.

SOLUTION: Start with

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

Integrate both sides of this equation from 0 to x ($|x| < 1$) to obtain

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad |x| < 1$$

Integrate both sides once again

$$-\int_0^x \ln(1-t)dt = \int_1^{1-x} \ln u\, du = \Big[u \ln u - u\Big]_1^{1-x}$$

$$= x + (1-x)\ln(1-x) = \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots$$

$$= x\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = xS(x)$$

or

$$S(x) = 1 + \left(\frac{1-x}{x}\right)\ln(1-x) \qquad |x| < 1$$

The series in Example 1 is expressed in terms of $x-2$ rather than x, so substitute $x-2$ for x in the above expression to get

$$1 + \frac{3-x}{x-2}\ln(3-x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n(n+1)} \qquad |x-2| < 1$$

The great advantage of uniform convergence is that uniformly convergent series can be manipulated pretty much like polynomials. The most common series in which the terms are functions of x are *power series*, which are series of the form

$$S(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \tag{10}$$

where the a_n are constants. We are going to study power series in the next two sections, where we shall prove that *all* convergent power series are uniformly convergent, and so we can manipulate power series just as we manipulate polynomials.

2.5 Problems

1. For what values of x do the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n2^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}}$

2. For what values of x do the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} n!x^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(x-1)^n}{2^n}$

3. We'll learn in Chapter 12 that the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\,n!}$ defines a Bessel function, which we denote by $J_0(x)$. For what values of x does $J_0(x)$ converge?

4. Use Equation 4 to show that $S(x) = \displaystyle\sum_{n=0}^{\infty} x^n$ converges uniformly to $1/(1-x)$ for $|x| \le a < 1$.

5. We'll prove the Weierstrass M test in this problem. Let $S(x) = \displaystyle\sum_{n=1}^{\infty} u_n(x)$ and $S_N(x) = \displaystyle\sum_{n=1}^{N} u_n(x)$. Now show that

$$|R_N(x)| = |S(x) - S_N(x)| \le \sum_{n=N+1}^{\infty} |u_n(x)| \le \sum_{n=N+1}^{\infty} M_n$$

and that given an ϵ, however small, there is a number N_0 such that $|R_N(x)| \le \epsilon$ if $N > N_0$, independent of x.

6. Show that the following series are uniformly convergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ $|x| \le 1$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$ $|x| < \infty$

7. The nature of the Weierstrass M test might lead you to believe that if a series converges uniformly, then it is absolutely convergent. Show that $S(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + x^2}$ is uniformly convergent for all x, but only conditionally convergent.

8. Prove that if $S_n(x)$ is continuous in $[a, b]$, and if $S(x) = \lim_{n\to\infty} S_n(x)$ is uniformly convergent in $[a, b]$, then $S(x)$ is continuous in $[a, b]$.

9. Show that $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is continuous for all x.

10. We shall see in Chapter 15 that the series

$$S(x) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0, \pi \\ -1 & \pi < x < 2\pi \end{cases}$$

Is the series uniformly convergent?

11. First prove that the series $y(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(ax)^n}{n!}$, where $-\infty < x < \infty$, converges uniformly for x in any finite interval. Then show that $y'(x) = ay(x)$ and integrate to get $y(x) = e^{ax}$.

12. Differentiate the geometric series term by term and derive a power series for $(1-x)^{-2}$. Justify your steps.

13. Integrate the series for $1/(1+x^2)$ term by term to obtain a power series for $\tan^{-1} x$.

14. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ converge uniformly?

15. Consider the series whose partial sums are $S_n(x) = nxe^{-nx^2}$ for $n \geq 1$ and $0 \leq x \leq 1$. Show that

$$\int_0^x \left(\lim_{n\to\infty} S_n(u)\right) du \neq \lim_{n\to\infty} \int_0^x S_n(u)\, du$$

Why are these two expressions not equal to each other?

16. Can you prove the assertion that you make in the previous problem? *Hint*: Investigate the behavior of $|S(x) - S_n(x)|$ at $x = 1/\sqrt{n}$.

2.6 Power Series

An infinite series of the form

$$S(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots \tag{1}$$

is called a *power series* in x about the point c. A common special case is the series about $c = 0$

$$S(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Let's apply the ratio test to determine the convergence of $S(x)$ in Equation 1.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| = |x-c| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = l|x-c|$$

where $l = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. Therefore, we see that the series converges absolutely for $|x-c| < 1/l = R$ and diverges (in the absolute sense) for $|x-c| > 1/l = R$. The range of x for which the series converges, $c - R < x < c + R$, is called the *interval of convergence* of the series and R is called its *radius of convergence*.

Example 1:
Find the interval of convergence and the radius of convergence of

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$$

SOLUTION:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x| \lim_{n\to\infty}\left|\frac{n \cdot 2^n}{(n+1)2^{n+1}}\right| = \frac{|x|}{2}$$

The series converges absolutely if $|x| < 2$ and diverges if $|x| > 2$. When $x = 2$, we have the harmonic series (divergent) and when $x = -2$, we have the alternating harmonic series (conditionally convergent). Thus, the interval of convergence is $[-2, 2)$ and the radius of convergence is $R = 2$.

The radius of convergence of a power series may equal zero or infinity, in which case it converges for only one value of x or for all values of x, respectively. An example of the first case is

$$S(x) = \sum_{n=1}^{\infty} n!(x-1)^n$$

which converges for only $x = 1$ (Problem 1) and an example of the second case is

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

which converges for every value of x (Problem 1).

We see then that there are three possible situations for a power series $S(x) = \sum_{n=1}^{\infty} a_n(x-c)^n$, namely

1. It converges for all values of x; or
2. It converges for values of x in an open interval $(c - R, c + R)$, but not outside the closed interval $[c - R, c + R]$; or
3. It converges only for $x = c$.

In case 1, the interval of convergence is $(-\infty, \infty)$; in case 2, it is $(c - R, c + R)$ and possibly one or both of its endpoints; and in case 3, it is only the point $x = c$. The radius of convergence in each case is ∞, R, and 0, respectively.

A central theorem for power series is the following:

If a power series $\sum a_n(x-c)^n$ converges for $x = \xi$, then it converges absolutely in the interval $|x| < |\xi|$ and uniformly in the interval $|x| \leq \eta < |\xi|$, where η is some positive number.

In other words, a power series converges uniformly and absolutely in any interval that lies entirely *within* its interval of convergence. (The proof of this theorem is developed in Problem 14.)

As a direct consequence of the above theorem, we have

1. *If* $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *converges in the interval* $(c-R, c+R)$, *then* $f(x)$ *is continuous in the interval* $(c-R, c+R)$.

2. *If* $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *converges in the interval* $(c-R, c+R)$, *then*

$$\int_c^x f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1}$$

converges in the interval $(c-R, c+R)$.

3. *If* $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *converges in the interval* $(c-R, c+R)$, *then*

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

converges in the interval $(c-R, c+R)$.

For example, the geometric series converges uniformly for $|t| < 1$.

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + \cdots \qquad |t| < 1$$

If we integrate both sides of this equation from $t = 0$ to $t = x$ with $|x| < 1$, we obtain

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad -1 \le x < 1 \tag{2}$$

We wrote $-1 \le x < 1$ here because if we investigate the end points $x = \pm 1$, separately, we see that this series converges for $x = -1$ (alternating harmonic series) and diverges for $x = 1$ (harmonic series). We can also differentiate the geometric series term by term to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \cdots \qquad -1 < x < 1 \tag{3}$$

In this case, no end point is included in the interval of convergence (Problem 13).

We can also derive a power series expression for $\tan^{-1} x$. Let's start with the standard integral

$$\int_0^x \frac{du}{1+u^2} = \tan^{-1} x$$

Now expand the integrand as a geometric series to write

$$\frac{1}{1+u^2} = 1 - u^2 + u^4 - u^6 + \cdots \qquad |u| < 1$$

and then integrate term by term to get

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad |x| < 1 \tag{4}$$

Example 2:
Derive a closed expression for

$$f(x) = \sum_{n=0}^{\infty} n x^n$$

SOLUTION: We first note that the radius of convergence of this series is $R = 1$ since

$$\lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = |x| \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = |x|$$

Now notice that $f(x)$ is similar to a geometric series, but with n in front of x^n. You can arrive at this form, however, by differentiating x^n to get nx^{n-1} and then multiplying by x. So if we start with

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad |x| < 1$$

and differentiate both sides and then multiply by x, we get

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 \cdots = \sum_{n=1}^{\infty} nx^n = \sum_{n=0}^{\infty} nx^n \qquad |x| < 1 \tag{5}$$

This series is used to calculate the average energy of a quantum-mechanical harmonic oscillator (Problem 12).

In this section we have derived power series for $\ln(1-x)$, $(1-x)^{-2}$, and $\tan^{-1} x$, all of which were derived from the geometric series. In the next section, we shall study Taylor series, which furnishes us with a general method to derive power series for any (suitably well-behaved) function.

2.6 Problems

1. Prove that $\sum_{n=1}^{\infty} n!(x-1)^n$ converges for only $x = 1$ and that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

2. Determine the interval of convergence of

   (a) $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$ (b) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

3. Determine the interval of convergence of

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$

4. Determine the interval of convergence of

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ (b) $\displaystyle\sum_{n=1}^{\infty} n^2(x+1)^n$

5. Substitute u^2 for x in the geometric series $\displaystyle\sum_{n=0}^{\infty} x^n$ and then integrate from 0 to x to obtain a power series expansion for $\tanh^{-1} x$.

6. Evaluate $\displaystyle\int_0^{\pi/4} \frac{dx}{1+x^4}$ to four decimal-place accuracy.

7. Show that $\displaystyle\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=0}^{\infty} n(n+1)x^{n-1} \qquad |x| < 1.$

8. Show that $\displaystyle\frac{1}{2}\ln\frac{1+x}{1-x} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \qquad |x| < 1.$

9. Consider the two power series $S(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $C(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$. Show that $S'(x) = C(x)$ and that $C'(x) = -S'(x)$. Do these results suggest anything to you? Can you show (at least for the first few terms) that $S^2(x) + C^2(x) = 1$?

10. Use Equation 4 to calculate the value of π to six decimal places.

11. Evaluate $\displaystyle\sum_{n=1}^{\infty} n^2 x^n$ in closed form.

12. The energy of a quantum-mechanical harmonic oscillator is given by $\varepsilon_n = (n + \frac{1}{2})h\nu$, $n = 0, 1, 2, \ldots,$ where h is the Planck constant and ν is the fundamental frequency of the oscillator. The average vibrational energy of a harmonic oscillator in an ideal gas is given by

$$\varepsilon_{\text{vib}} = (1 - e^{-h\nu/k_BT})\sum_{n=0}^{\infty} \varepsilon_n e^{-nh\nu/k_BT}$$

where k_B is the Boltzmann constant and T is the kelvin temperature. Show that

$$\varepsilon_{\text{vib}} = \frac{h\nu}{2} + \frac{h\nu e^{-h\nu/k_BT}}{(1 - e^{-h\nu/k_BT})}$$

13. Show that the interval of convergence of Equation 3 is $|x| < 1$.

14. In this problem, we'll prove that a power series converges uniformly and absolutely in any interval that lies entirely within its interval of convergence, $(c - R, c + R)$. First show that

$$|a_n| < \frac{1}{|R - c|^n} \qquad \text{for} \quad n > N$$

Use this result to show that

$$\sum_{n=N+1}^{\infty} |a_n(x-c)^n| = \sum_{n=N+1}^{\infty} |a_n| \, |x-c|^n < \sum_{n=N+1}^{\infty} \frac{|x-c|^n}{|R-c|^n}$$

Why does this series converge absolutely? To prove uniform convergence, use the Weierstrass *M* test and let $M = |\eta - c|^n/|R-c|^n$, where $|x| \leq \eta < R$.

2.7 Taylor Series

In Section 1.5, we formulated an extension of the mean value theorem, which resulted in the expression

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots \\ + f^{n}(a)\frac{(x-a)^n}{n!} + f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!} \end{aligned} \tag{1}$$

In Equation 1 we assume that $f(x)$ and its first n derivatives are continuous on $[a, x]$ and that $f^{(n+1)}(\xi)$ is continuous on $(a, x]$. Equation 1 is known as *Taylor's formula with remainder*. The remainder term is given by

$$R_n(x, \xi) = f(x) - \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} = f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!} \tag{2}$$

In Equation 2, the notation $f^{(0)}(a)$ means the function itself; i.e. $f(a)$. If $R_n(x, \xi) \to 0$ as $n \to \infty$, then

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)\frac{(x-a)^k}{k!} \tag{3}$$

is called the *Taylor series* or *Taylor expansion* of $f(x)$ about $x = a$. If $a = 0$ in Equation 3, the series is called the *Maclaurin series* or *Maclaurin expansion* of $f(x)$.

Let's consider $f(x) = \sin x$ as an example. For $f(x) = \sin x$, all the derivatives of $f(x)$ are continuous for all values of x. Furthermore,

$$|R_n(x, \xi)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \right| \leq \frac{|x-a|^{n+1}}{(n+1)!}$$

because $|f^{(n+1)}(\xi)| = |\sin \xi|$ or $|\cos \xi|$, both of which are ≤ 1. But $\lim_{n\to\infty} \dfrac{|x-a|^{n+1}}{(n+1)!} = 0$ (Problem 1) and so $\lim_{n\to\infty} |R_n(x, \xi)| = 0$. The Maclaurin series for $\sin x$ is

$$\sin x = \sin 0 + (\cos 0)x - (\sin 0)\frac{x^2}{2!} - (\cos 0)\frac{x^3}{3!} + \cdots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{4}$$

Not only does the series on the right converge for all x, but it is equal to $\sin x$ for all x. Similarly, the Maclaurin series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{5}$$

Example 1:
Derive the Maclaurin expansion of $f(x) = e^x$.

SOLUTION: All the derivatives of e^x are continuous and equal to e^x. The remainder formula is

$$R_n(x, \xi) = \frac{e^{\xi} x^{n+1}}{(n+1)!}$$

where ξ lies between 0 and x. Certainly $0 \leq e^{\xi} \leq e^{|x|}$, so $R_n(x) \to 0$ as $n \to \infty$ for all values of x. Thus,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{6}$$

which converges for all values of x.

Example 2:
Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} = 1 - e^{-1}$$

SOLUTION: Start with

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n!}$$

Let $x = 1$ and solve for the summation to get the equation in the statement of the Example.

The series 4 through 6 are worth remembering. All mathematical handbooks, such as *The CRC Standard Mathematical Tables and Formulas*, have lists of Taylor expansions or Maclaurin expansions of functions. Also, many of the CAS can derive Taylor series with a one-line command. For example, the command in Mathematica

```
Series[ (1 -Exp[ -x^2 ] ) / ( x Sin[ x^2+3 x ] ), {x, 0, 5} ]
```

gives the Maclaurin expansion of $(1 - e^{-x^2})/x \sin(x^2 + 3x)$ up to fifth order in x:

$$\frac{1 - e^{-x^2}}{x \sin(x^2 + 3x)} = \frac{1}{3} - \frac{x}{9} + \frac{10x^2}{27} + \frac{17x^3}{81} + \frac{3073x^4}{9720} + \frac{12479x^5}{29160} + O(x^6)$$

One of the most useful expansions in applied mathematics is the *binomial series*. Recall that the *binomial expansion* is

$$(1 + x)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k$$

where n is a positive integer. For example

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$$

and

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

(see Problem 2). When n is a positive integer, $(1 + x)^n$ consists of $n + 1$ terms, starting with $x^0 = 1$ and ending with x^n. When n is not an integer, however, the expansion of $(1 + x)^\alpha$ (α not an integer) is an infinite series. To derive the binomial series, we use

$$\begin{aligned}
f(x) &= (1 + x)^\alpha \\
f'(x) &= \alpha(1 + x)^{\alpha-1} \\
f''(x) &= \alpha(\alpha - 1)(1 + x)^{\alpha-2} \\
&\cdots \\
f^{(n)}(x) &= \underbrace{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}_{n \text{ terms}}(1 + x)^{\alpha-n}
\end{aligned}$$

Problem 5 has you show that the remainder term $R_n(x, \xi) \to 0$ for $|x| < 1$ as $n \to \infty$, so we can substitute $f^{(n)}(0) = \alpha(\alpha - 1)\cdots(\alpha - n + 1)$ into Equation 3 with $a = 0$ to get

$$\begin{aligned}
(1 + x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!}x^3 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n \qquad |x| < 1 \qquad (7)
\end{aligned}$$

When α is a positive integer, the binomial series terminates and becomes the binomial expansion (Problem 4). To determine the interval of convergence of the binomial series, we look at

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)x^{n+1}}{(n+1)!}\cdot\frac{n!}{\alpha(\alpha-1)\cdots(\alpha-n+1)x^x}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(\alpha-n)x}{n+1}\right| = |x|$$

Therefore, the binomial series converges absolutely for $|x| < 1$. The behavior at the end points $x = \pm 1$ depends upon the value of α.

Example 3:
Starting with

$$\sinh^{-1} x = \int_0^x \frac{du}{(1+u^2)^{1/2}}$$

derive a series expression for $\sinh^{-1} x$.

SOLUTION: The binomial series with x replaced by u^2 is (we'll let $\alpha = -1/2$ later)

$$(1+u^2)^\alpha = 1 + \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}u^{2n}$$

Integrating both sides from 0 to x, letting $\alpha = -1/2$ yields

$$\sinh^{-1} x = 1 + \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!(2n+1)}x^{2n+1}$$

$$= 1 - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots$$

We should point out that not all functions have a Maclaurin expansion. For example, $f(x) = \ln x$ has no Maclaurin expansion because none of the derivatives of $\ln x$ exist at $x = 0$. The function $\ln x$ does have Taylor expansions about points other than $x = 0$, however. For example, if we expand $\ln x$ about the point $x = 1$, we obtain (Problem 6)

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots \qquad 0 < x < 2$$

Before we leave this section, let's consider the Maclaurin expansion of $f(x) = 1/(1+x^2)$

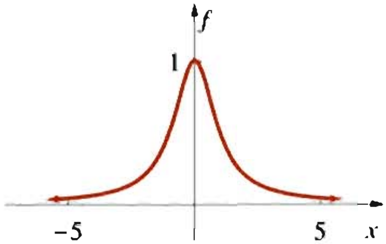

Figure 2.10
The function $f(x) = 1/(1+x^2)$ plotted against x.

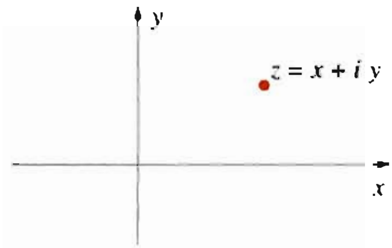

Figure 2.11
A point $z = x + iy$ in the complex plane.

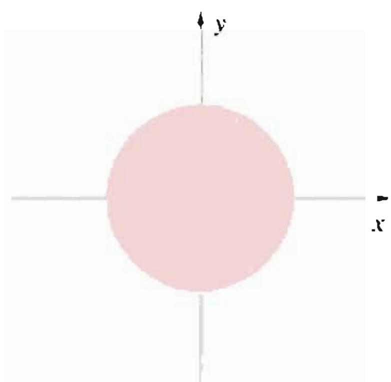

Figure 2.12
The region $|z| < 1$ in the complex plane.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^{n+1}x^{2n} \qquad |x| < 1$$

We obtained this expansion in the previous section by replacing u in $1/(1-u)$ by $-x^2$ and then using the geometric series. The above series diverges at the end points $x = \pm 1$, yet $1/(1+x^2)$ seems to be perfectly well-behaved at $x = \pm 1$ (Figure 2.10). Where did this restriction come from?

In Chapter 4, we're going to study complex numbers, the complex plane, and the behavior of a few functions where we'll allow the argument to take on complex values. You probably remember, however, that a complex number is of the form $x + iy$, where x and y are real numbers and $i^2 = -1$. We can plot complex numbers in a coordinate system where x represents the horizontal axis and y the vertical axis. Figure 2.11 shows the point $z = x + iy$ plotted in this coordinate system called the *complex plane*. If we vary x and y, z becomes variable and is called a *complex variable*, and we can investigate the behavior of functions of z. In particular, we can replace x in $f(x) = 1/(1+x^2)$ by z and write

$$\begin{aligned} f(z) &= \frac{1}{1+z^2} \\ &= 1 - z^2 + z^4 + z^6 - z^8 + \cdots \qquad |z| < 1 \end{aligned}$$

where by $|z|$ we mean the distance of the point (x, y) to the origin of the coordinate system, or $(x^2 + y^2)^{1/2}$. The region corresponding to $|z| < 1$ in the complex plane is the region within a unit circle centered at the origin (Figure 2.12). Now, if we investigate the behavior of $f(z)$ as a function of z within the region $|z| < 1$, letting x and y vary within that region, we see that the denominator of $f(z)$ equals zero at $z = \pm i$. Thus, the region of convergence of $f(z) = 1/(1+z^2)$ is restricted to within the unit circle in the complex plane and even if we consider $f(z)$ only along the x-axis, where it equals $1/(1+x^2)$, the values of x are restricted to lie within the unit circle.

Thus, we see that the behavior of a real function $f(x)$ is actually influenced by how its corresponding complex function $f(z)$ behaves as z varies in the complex plane. The calculus of functions of a complex variable z is an extremely rich subject, and one that has many applications in applied mathematics (Chapters 18 and 19).

2.7 Problems

1. Show that $\dfrac{x^n}{n!} \to 0$ as $n \to \infty$ for all x.

2. The coefficients of the expansion of $(1+x)^n$ can be arranged in the following form:

n									
0					1				
1				1		1			
2			1		2		1		
3		1		3		3		1	
4	1		4		6		4		1

Do you see a pattern in going from one row to the next? The triangular arrangement here is called Pascal's triangle.

3. Prove that $\displaystyle\sum_{k=0}^{N} \frac{N!}{k!(N-k)!} = 2^N$.

4. Show that the binomial series truncates if α is an integer.

5. Prove that $|R_n| \to 0$ as $n \to \infty$ for the binomial series, provided $|x| < 1$.

6. Show that $\ln x = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$ for $0 < x < 2$. Use this result to show that the alternating harmonic series is equal to $\ln 2$.

7. Assuming that Equation 6 is valid for imaginary numbers, show that $e^{ix} = \cos x + i \sin x$.

8. Use Equation 6 to calculate the value of e to six decimal places. (Assume initially that $e \approx 3$.)

9. Find the Maclaurin expansion of xe^{-x^2}. (*Hint*: Do *not* use Equation 3.)

10. Find the Maclaurin expansion of $\sqrt{1+x}$.

11. Prove that if $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n (x-c)^n$, then this series is the Taylor series for $f(x)$, even if it is obtained without using Equation 3, as we did for $\tan^{-1} x$ and $\ln(1-x)$ in the previous section. (*Hint*: Derive an expression for the a_n.)

12. Show that the derivative of Equation 7 gives $\alpha(1+x)^{\alpha-1}$.

13. Show that

$$(1+x)^{-\alpha} = 1 - \alpha x + \frac{\alpha(\alpha+1)}{2!}x^2 - \frac{\alpha(\alpha+1)(\alpha+2)}{3!}x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha+n-1)!x^n}{n!(\alpha-1)!}$$

Remember that $0! = 1$. (We will discuss factorials in the next chapter if you are not familiar with them.)

14. Show that $(1-x^2)^{-1/2} = \displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$.

15. Show that

(a) $1 + x^2 - \dfrac{x^4}{3!} + \dfrac{x^6}{5!} - \dfrac{x^8}{7!} + \cdots = 1 + x \sin x$

(b) $1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \cdots = \dfrac{\sin x}{x}$

(c) $\dfrac{1}{2} - \dfrac{x^2}{4!} + \dfrac{x^4}{6!} - \dfrac{x^6}{8!} + \cdots = \dfrac{1 - \cos x}{x^2}$

16. Consider two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$. It is straightforward to show that if we denote their product by $\sum_{n=0}^{\infty} c_n x^n$, then

$$\begin{aligned} c_n &= a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 \\ &= \sum_{k=0}^{n} a_k b_{n-k} \end{aligned}$$

Use this result to show that $e^{(z_1+z_2)} = e^{z_1} e^{z_2}$.

17. Show that $\displaystyle\int_0^{\infty} e^{-x^2} \cos \alpha x \, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$ by expanding $\cos \alpha x$ in a Maclaurin series and integrating term by term.

18. In this problem, we'll show that even if all the derivatives of a function exist and are continuous at some point, the function still may not have a Taylor expansion about that point. A classic example of this behavior is given by $f(x) = e^{-1/x^2}$, $x \neq 0$, and $f(x) = 0$, $x = 0$ (Figure 2.13). Show that the Maclaurin expansion of e^{-1/x^2} is $e^{-1/x^2} = 0 + 0 + \cdots + 0 + R_n$ and that the remainder is equal to the function itself for all values of n and does not vanish, except for $x = 0$.

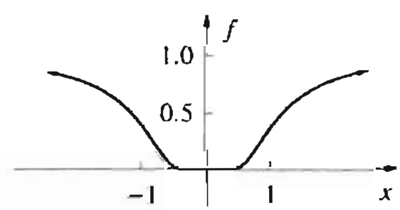

Figure 2.13
The function $f(x) = e^{-1/x^2}$ plotted against x.

19. Use any CAS to find the Maclaurin expansion of $\cos(x \sin t)$ to fifth order in t.

20. Use any CAS to find the Maclaurin expansion of $\tanh(x^3 - x)$ to fifth order in x.

21. Use any CAS to find the Taylor series of $x^3 \ln x$ about the point $x = 1$ to sixth order in $(x - 1)$.

22. Use any CAS to find the Taylor series of $x^3 \sec x$ about the point $x = \pi$ to fifth order in $(x - \pi)$.

2.8 Applications of Taylor Series

Taylor series have numerous applications in practice. For example, suppose we want to know if the integral

$$I = \int_0^1 \frac{\sin x}{x^2} dx$$

is finite. The region around $x = 0$ is problematic because the denominator approaches zero. Let's expand $\sin x$ about $x = 0$ and write I as

$$\begin{aligned} I &= \int_0^1 \frac{1}{x^2}\left(x - \frac{x^3}{3} + \frac{x^5}{5!} + \cdots\right) dx \\ &= \int_0^1 \frac{dx}{x} - \int_0^1 \frac{x}{3!} dx + \int_0^1 \frac{x^3}{5!} dx + \cdots \end{aligned}$$

All the integrals beyond the first are finite, but the first integral diverges because

$$\int_0^1 \frac{dx}{x} = \Big[\ln n \Big]_0^1 = \infty$$

and so I itself diverges.

We can also use Taylor series to evaluate integrals. The integral

$$I = \int_0^\infty \frac{x^3}{e^x - 1} dx \tag{1}$$

occurs in the theory of blackbody radiation. This integral does not appear in the *CRC Mathematical Tables* nor in Gradshteyn and Ryzhik (see the references at the end of Chapter 1). We can evaluate it by multiplying the numerator and denominator of the integrand by e^{-x} and expanding the denominator in powers of e^x using the geometric series.

$$I = \int_0^\infty x^3 e^{-x} \left[\sum_{n=0}^{\infty} (e^{-x})^n \right] dx$$

The series here is uniformly convergent for $x > 0$ and so we can interchange orders of summation and integration to write

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \int_0^\infty x^3 e^{-x(n+1)} dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)^4} \int_0^\infty u^3 e^{-u} du \\ &= 6 \sum_{n=0}^{\infty} \frac{1}{(n+1)^4} = 6 \sum_{n=1}^{\infty} \frac{1}{n^4} \end{aligned}$$

The summation here is in most handbooks (see also Section 3.7) and is equal to $\pi^4/90$. Thus, we see that $I = \pi^4/15$.

It turns out that the integral

$$I(x) = \int_0^x e^{-u^2} du$$

which occurs in the kinetic theory of gases and in many other areas, cannot be evaluated in terms of elementary functions. We can obtain a useful power series for $I(x)$ by expanding e^{-u^2} in a power series in u and then integrating term by term

$$I(x) = \int_0^x \left[1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \frac{u^8}{4!} + \cdots\right] du$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} + \cdots$$

or

$$I(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)} \tag{2}$$

This is an alternating series, so the error we incur by truncating is less in magnitude than the first term neglected. If we keep four terms to evaluate $I(1/2)$, the error will be less than $(1/2)^9/(9 \cdot 4!) = 9 \times 10^{-6}$. We get $I(1/2) = 0.46127$ compared to the accepted "exact" value of 0.46128.

Example 1:

Use the Taylor series to evaluate $\int_0^1 \frac{\sin x}{x} dx$ to four-place accuracy.

SOLUTION:

$$\int_0^1 \frac{\sin x}{x} dx = \int_0^1 \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} + \cdots\right] dx$$

$$= 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \frac{1}{7 \cdot 7!} + \frac{1}{9 \cdot 9!} + \cdots$$

If we truncate this alternating series after the third term, we have an accuracy of $1/(7 \cdot 7!) = 3 \times 10^{-5}$. So, if we use just three terms, $\int_0^1 \frac{\sin x}{x} dx = 0.94611$ compared to the accepted value of 0.94608.

It is also of interest in the theory of blackbody radiation to know the behavior of the integrand in Equation 1 for small values of x. The integrand is of the form 0/0 at $x = 0$, but if we replace e^x by $1 + x + x^2/2! + \cdots$, we see that

$$\frac{x^3}{e^x - 1} = \frac{x^3}{x + \frac{x^2}{2!} + \cdots} = \frac{x^2}{1 + \frac{x}{2} + \cdots} = x^2 - \frac{x^3}{2} + \cdots$$

This is probably a good place to introduce a frequently used notation that is very useful when working with power series. We write $f(x) = O[g(x)]$ as $x \to a$ if $f(x)/g(x)$ is bounded as $x \to a$; in other words, $f(x) = O[g(x)]$ as $x \to a$ if $|f(x)| \leq M|g(x)|$ as $x \to a$, where M is a constant. Usually $g(x)$ will be a power of x. For example, we write $f(x) = 1 - \cos x = O(x^2)$ as $x \to 0$ because $\cos x = 1 - x^2/2! + x^4/4! + \cdots$ as $x \to 0$; and so $(1 - \cos x)/x^2 \to 1/2$ as $x \to 0$. We will use this notation in a bookkeeping sense to keep track of powers of x that we neglect when truncating series expansions.

Example 2:
One statistical-mechanical theory of solutions of strong electrolytes (such as an aqueous solution of sodium chloride) gives the energy of the solution as

$$E(\kappa, R) = -\frac{x^2 + x - x(1 + 2x)^{1/2}}{4\pi\beta R^3}$$

where $\beta = 1/k_B T$, R is the average radius of the positive and negative ions, and $x = \kappa R$, where κ is a known parameter. Show that E goes as κ^3 as $\kappa \to 0$.

SOLUTION: We need to write E as a power series in x. Using the binomial series

$$\begin{aligned}
-(4\pi\beta R^3)E &= x^2 + x - x(1 + 2x)^{1/2} \\
&= x^2 + x - x\left[1 + \frac{2x}{2} - \frac{(2x)^2}{8} + O(x^3)\right] \\
&= x^2 + x - x - x^2 + \frac{x^3}{2} + O(x^4) \\
&= \frac{x^3}{2} + O(x^4)
\end{aligned}$$

or

$$E = -\frac{\kappa^3}{8\pi\beta} + O(\kappa^4)$$

Another example from electrolyte solutions is the following: One theory expresses the osmotic pressure of a solution of a strong electrolyte in terms of a function σ given by

$$\sigma = \frac{3}{x^3}\left[1 + x - \frac{1}{1+x} - 2\ln(1+x)\right]$$

where once again $x = \kappa R$. The small-kappa (dilute solution) behavior of σ is given by

$$\begin{aligned}\sigma &= \frac{3}{x^3}\Big\{1 + x - [1 - x + x^2 - x^3 + x^4 + O(x^5)] \\ &\qquad -2\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)\right]\Big\} \\ &= \frac{3}{x^3}\left[\frac{x^3}{3} - \frac{x^4}{4} + O(x^5)\right] = 1 - \frac{3}{2}x + O(x^2)\end{aligned}$$

Note that we keep track of the powers of x in the expansions of $1/(1+x)$ and $\ln(1+x)$ by writing $O(x^5)$ to indicate the first power of x neglected in each expansion. Without this notation to remind us, it would be easy to make the mistake of neglecting some power of x in one term but not the other. This is particularly important when a number of lower powers cancel, as they do (the x^0, x^1, and x^2 powers cancel) in the above two cases.

Sometimes power series expressions are a little easier to use than l'Hôpital's rule to evaluate indeterminate expressions such as 0/0. For example, let's evaluate

$$\lim_{x\to 0} \frac{x - \sin x}{x^2 \tan x}$$

using the series expansion for $\sin x$ in the previous section and

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

(This expansion may be found in any handbook.) We get

$$\begin{aligned}\lim_{x\to 0} \frac{x - \sin x}{x^2 \tan x} &= \lim_{x\to 0} \frac{x - \left[x - \frac{x^3}{6} + O(x^5)\right]}{x^2[x + O(x^3)]} \\ &= \frac{1}{6}\end{aligned}$$

Example 3:
Use Taylor series to evaluate

$$\lim_{x\to\infty} x^{1/2}(\sqrt{x+3} - \sqrt{x+1})$$

SOLUTION: First let $x = 1/u$ and look at the limit $u \to 0$.

$$\lim_{u\to 0}\frac{(1+3u)^{1/2}-(1+u)^{1/2}}{u} = \lim_{u\to 0}\frac{1}{u}\left\{1+\frac{3u}{2}+O(u^2)-\left[1+\frac{u}{2}+O(u^2)\right]\right\}$$
$$= \lim_{u\to 0}\frac{u}{u} = 1$$

To end this section, we'll use Taylor's formula to give a simple proof of the second-derivative criteria for an extremum. Suppose that a is a critical point so that $f'(a) = 0$. Then Equation 7.1 with $n = 1$ is

$$f(x) - f(a) = \frac{1}{2}f''(\xi)(x-a)^2$$

where $a < \xi < x$. If x is close enough to a so that $f''(\xi)$ has the same sign as $f''(a)$, then we see that $f(x) - f(a)$ and $f''(\xi)$ have the same sign. If $f''(\xi) < 0$, then $f(x) < f(a)$ and $f(a)$ is a maximum, and if $f''(\xi) > 0$, then $f(x) > f(a)$ and $f(a)$ is a minimum.

2.8 Problems

1. Does $\displaystyle\int_0^1 \frac{e^{-x}-1+x}{x^2}dx$ converge?
2. Use any CAS to evaluate the integral in the previous problem.
3. Does $\displaystyle\int_0^1 \frac{\tan^{-1}x}{x^2}dx$ converge?
4. What happens when you try to evaluate the integral in the previous problem with a CAS?
5. Expand $(1+x^4)^{1/2}$ in a binomial series and evaluate $\displaystyle\int_0^{1/2}(1+x^4)^{1/2}dx$ to four decimal places.
6. Evaluate the integral in the previous problem using any CAS.
7. The integral

$$K(k) = \int_0^{\pi/2}\frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}}$$

called a complete elliptic integral of the first kind, arises in a study of a pendulum of arbitrary amplitude (Section 3.5). The integral cannot be expressed in terms of elementary functions, but derive the first few terms of an expansion of K in powers of k.

8. Use any CAS to do the previous problem.
9. Evaluate $\displaystyle\int_0^1 \sin\sqrt{x}\,dx$ to five-decimal place accuracy.
10. Use any CAS to do the previous problem.
11. Evaluate $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\dfrac{4}{5!}+\cdots = \displaystyle\sum_{n=2}^{\infty}\frac{n-1}{n!}$ in closed form.

12. Derive a power series for $f(x) = \int_0^x \frac{u\,du}{1+u^4}$.

13. Find $f^{(10)}(0)$ for $\sin\sqrt{x}/\sqrt{x}$. *Hint*: Do *not* differentiate the expression ten times.

14. Use any CAS to find $f'(0)$ for the previous problem. Do you run into any difficulty? Is $f'(0)$ finite?

15. Show that $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}e^{-b^2/4a^2}$ by expanding $\cos bx$ in a Taylor series about $x = 0$ and integrating term by term. Use the fact that

$$\int_0^\infty x^{2n}e^{-a^2x^2}dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^{2n+1}}\sqrt{\pi}$$

(Problem 1.9.5).

16. Show that $\int_0^\infty e^{-ax}\sinh bx\,dx = \frac{b}{a^2-b^2}$ $|b| < a$ by expanding $\sinh bx$ in a Taylor series at $x = 0$ and integrating term by term.

17. We defined a p series in Section 3 by $S(p) = \sum_{n=1}^{\infty}\frac{1}{n^p}$. Show that

$$S(p) = \frac{1}{(p-1)!}\int_0^\infty \frac{x^{p-1}dx}{e^x-1}$$

where for now $p = 2, 3, \ldots$. (We'll generalize this formula in the next chapter, where we'll learn how to handle factorials for non-integral values of p.)

18. The theory of aqueous solutions of strong electrolytes expresses the free energy of the solution in terms of a quantity τ defined by

$$\tau = \frac{3}{(\kappa R)^3}\left\{\ln(1+\kappa R) - \kappa R + \frac{\kappa^2R^2}{2}\right\}$$

where R is the average radius of the ions and κ is a known parameter. Show that

$$\tau \to 1 - \frac{3}{4}\kappa R + \frac{3}{5}(\kappa R)^2 + O[(\kappa R)^3] \qquad \text{as} \qquad \kappa R \to 0$$

19. Use any CAS to carry out the previous problem to $O[(\kappa R)^5]$.

20. Show that $\frac{2}{\pi}\int_0^{\pi/2}\cos(x\sin t)\,dt = 1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)$.

21. Use any CAS to deterine the series expansion in the previous problem through $O(x^{10})$.

22. Use Taylor series to derive l'Hôpital's rule.

23. For the chemical reaction A + B → C, the concentrations of A and B as a function of time are given by $kt = \frac{1}{A_0 - B_0}\ln\frac{A(t)B_0}{A_0B(t)}$, where A_0 and B_0 are the initial concentrations and k is a constant. Find the limiting expression if $A_0 = B_0$. *Hint*: $A(t) = B(t)$ if $A_0 = B_0$.

2.9 Asymptotic Expansions

Taylor series are most useful for small values of $(x - c)$ because the terms usually decrease steadily and only a few terms need be used to achieve a predetermined accuracy. Frequently, however, we would like to know the behavior of a function for large values of x. For example, we might want to know how a system behaves as the number of particles increases, or how it behaves at long times as it approaches equilibrium.

For example, the following integral arises in a quantum-mechanical treatment of a hydrogen molecule:

$$E_1(x) = \int_x^\infty \frac{e^{-z}}{z} dz \tag{1}$$

and we are often interested in the value of this integral for large values of x. Integrate $E_1(x)$ repeatedly by parts (with "u" $= 1/z$" and "dv" $= -e^{-z}dz$) to get an expansion in inverse powers of x:

$$E_1(x) = \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3}\right) + 4!\int_x^\infty \frac{e^{-z}}{z^5} dz \tag{2}$$

Equation 1 is an identity; there is no approximation yet, but there will be if we neglect the integral on the right side. Let's look at the error involved if we do drop it. Since $z > x$ throughout the range of integration, we have

$$4!\int_x^\infty \frac{e^{-z}}{z^5} dz < \frac{4!}{x^5}\int_x^\infty e^{-z}\, dz = \frac{4! e^{-x}}{x^5}$$

If $x = 5$, the error involved is 5.2×10^{-5}. Thus we can use

$$E_1(x) = \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3}\right) \qquad x > 5 \tag{3}$$

to achieve four-decimal accuracy. Equation 3 gives $E_1(5) = 0.00112 \pm 0.00005$, or $0.00110 < E_1(5) < 0.00121$, compared to the "exact" value of 0.00115.

It might be tempting to continue the integration by parts in Equation 1 and write

$$E_1(x) = \frac{e^{-x}}{x}\sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^n}$$

but this series diverges for all x. The ratio test yields

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{n+1}{x}\right| > 1$$

for any value of x. Nevertheless, we obtained an excellent numerical approximation to $E_1(x)$ by using the truncated version, Equation 3.

To see what's going on here, write Equation 2 more generally as

$$E_1(x) = e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots \frac{(-1)^{n+1}(n-1)!}{x^n}\right]$$

$$+(-1)^n n! \int_x^\infty \frac{e^{-z}}{z^{n+1}} dz \tag{4}$$

$$= S_n(x) + R_n(x)$$

where $S_n(x)$ denotes the sum up through the $1/x^n$ term and $R_n(x)$ denotes the integral, which represents the sum of the series after the $1/x^n$ term, or the remainder. In this case

$$|R_n(x)| < \frac{n!}{x^{n+1}} \int_x^\infty e^{-z} dz = \frac{n! e^{-x}}{x^{n+1}}$$

and

$$\frac{|R_n(x)|}{|S_n(x)|} < \frac{n!}{x^n - x^{n-1} + 2!x^{n-2} + \cdots (-1)^{n+1}(n-1)!x} \to 0 \quad \text{as} \quad x \to \infty \tag{5}$$

for a *fixed value of n*. The ratio of the error to the terms kept tends to zero as $x \to 0$; in other words, the approximation improves as x gets larger. If we divide both sides of Equation 4 by $S_n(x)$, we see that

$$\frac{E_1(x)}{S_n(x)} = 1 + \frac{R_n(x)}{S_n(x)} \to 1 \quad \text{as} \quad x \to \infty \tag{6}$$

for a fixed value of n. We express Equation 6 by writing

$$E_1(x) \sim e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots \frac{(-1)^{n+1}(n-1)!}{x^n}\right]$$

where the $\sim$ sign means *is asymptotic to*.

Generally, we say that

$$f(x) \sim \sum_{j=0}^{n} \frac{a_j}{x^j} \tag{7}$$

is an *asymptotic expansion* of $f(x)$ if for each n

$$x^n\left[f(x) - \sum_{j=0}^{n} \frac{a_j}{x^j}\right] \to 0 \qquad \text{as} \quad x \to \infty \tag{8}$$

The term in brackets here is $R_n(x)$, the remainder after $1/x^n$ term. Thus we can write Equation 8 as

$$\lim_{x\to\infty} x^n[f(x) - S_n(x)] = \lim_{x\to\infty} x^n R_n(x) = 0 \tag{9}$$

It is common practice to write

$$f(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{x^n} \tag{10}$$

and to call this the *asymptotic series* of $f(x)$. We must be alert to the fact, however, that you must truncate the series at some value of n. This subject provides a good example where we can *not* interchange the orders of summation and integration in Equation 1.

The asymptotic series of $E_1(x)$ is

$$E_1(x) \sim \frac{e^{-x}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^n} \tag{11}$$

This series satisfies the requirement of Equation 9, since $|R_n(x)| = \dfrac{n!e^{-x}}{x^{n+1}}$ and $|x^n R_n(x)| \to 0$ as $x \to \infty$ for each fixed n. It is convenient to rewrite Equation 11 as

$$xe^x E_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^n} \tag{12}$$

Table 2.2 lists the successive terms and the corresponding successive partial sums of $xe^x E_1(x)$ for $x = 5$. (See also Figures 2.14 and 2.15.) Notice that the terms first decrease to 0.0384 and then increase. This should come as no surprise because Equation 11 *is* a divergent series.

Since Equation 11 is an alternating series, its actual sum lies between any two consecutive partial sums. If we scan down Table 2.2, we see that the tightest pair of partial sums is the pair (0.8704, 0.8320), so we expect the "exact" value (which is 0.85211) to lie between these values.

A similar calculation for $x = 10$ shows that the successive terms decrease until $n = 10$, and then increase after that. The tightest pair of partial sums is (0.915 46, 0.915 82), and the "exact" value is 0.915 63. For $x = 20$, the successive terms decrease until $n = 20$ and give essentially the "exact" value, 0.954 370 91 (Problem 6).

It's possible to *estimate* the value of n at which the terms in Equation 12 have a minimum value. For a fixed value of x, we wish to determine the minimum value of $n!/x^n$. To do this, first take the logarithm of $y = n!/x^n$ to get

$$\ln y = \ln n! - n \ln x = \sum_{j=1}^{n} \ln j - n \ln x$$

We can find the minimum of $y = n!/x^n$ by finding the minimum of $\ln y$ because $\ln y$ is a monotonic function of y. Now, approximate $\sum_{j=1}^{n} \ln j$ by $\int_1^n \ln z \, dz$ and differentiate $\ln y$ with respect to n as if it were a continuous variable, then set the result to zero, yielding

$$\frac{d \ln y}{dn} \approx \ln n - \ln x = 0$$

or $n \approx x$. This result is in accord with our above calculations for $x = 5$, 10, and 20.

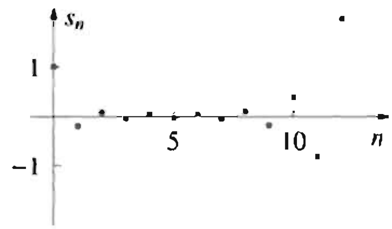

Figure 2.14
The successive terms of the asymptotic series $xe^x E_1(x)$ given by Equation 12 for $x = 5$ plotted against n.

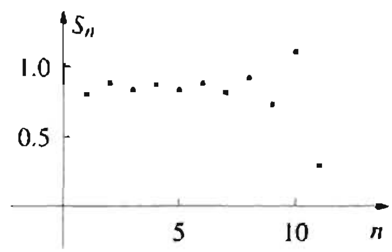

Figure 2.15
The successive partial sums of the asymptotic series $xe^x E_1(x)$ given by Equation 12 for $x = 5$ plotted against n.

Table 2.2
The successive terms and the partial sums of $xe^x E_1(x)$ for $x = 5$.

n	successive terms	partial sum
0	1.0000	1.0000
1	−0.2000	0.8000
2	0.0800	0.8800
3	−0.0480	0.8320
4	0.0384	0.8704
5	−0.0384	0.8320
6	0.0461	0.8781
7	−0.0645	0.8136
8	0.1032	0.9168
9	−0.1858	0.7310
10	0.3716	1.1026
11	−0.8175	0.2851
12	1.9620	2.2471
13	−5.1012	−2.8541
14	14.2833	11.4290

Example 1:
Determine the asymptotic behavior of

$$E_2(x) = \int_1^\infty \frac{e^{-xu}}{u^2}\,du$$

SOLUTION: First let $z = xu$ to get $E_2(x)$ into the form

$$E_2(x) = x\int_x^\infty \frac{e^{-z}}{z^2}\,dz$$

Now integrate by parts repeatedly, always using "dv" $= e^{-z}dz$ to get

$$\frac{E_2(x)}{x} = e^{-x}\left[\frac{1}{x^2} - \frac{2!}{x^3} + \frac{3!}{x^4} + \cdots + (-1)^n\frac{(n-1)!}{x^n}\right]$$
$$+ (-1)^{n+1}n!\int_x^\infty \frac{e^{-z}}{z^{n+1}}\,dz$$
$$= S_n(x) + R_n(x)$$

Replace z^{n+1} by x^{n+1} in the remainder term and write

$$|R_n(x)| < \frac{n!}{x^{n+1}}\int_x^\infty e^{-z}dz = \frac{n!e^{-x}}{x^{n+1}}$$

and so we see that $|x^n R_n(x)| \to 0$ as $x \to \infty$, and so Equation 9 is satisfied.

Therefore, we can write

$$E_2(x) \sim e^{-x} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}$$

and in particular that

$$E_2(x) \sim \frac{e^{-x}}{x} \qquad \text{as} \qquad x \to \infty$$

In the next chapter, we will discuss some functions that are defined by their integral expressions. One of them is called the error function, which is a central function in statistics and one that occurs in the kinetic theory of gases. An integral that is closely related to the error function is

$$I(x) = \int_x^{\infty} e^{-u^2} du$$

The asymptotic formula for $I(x)$ is obtained by repeated integration by parts (Problem 1):

$$I(x) = \frac{e^{-x^2}}{2} \left[\frac{1}{x} - \frac{1}{2x^3} + \frac{1 \cdot 3}{2^2 x^5} + \cdots + (-1)^n \frac{n(n-\frac{1}{2})(n-\frac{3}{2})\cdots(\frac{1}{2})}{x^{2n+1}} \right]$$
$$+ (-1)^{n+1} 2 \left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) \cdots \left(\frac{1}{2}\right) \int_x^{\infty} \frac{e^{-z^2}}{z^{2n+1}} dz \tag{13}$$

The corresponding asymptotic series is

$$I(x) \sim \frac{e^{-x^2}}{2x} \left[1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{2^2 x^4} + \cdots \right]$$
$$\sim \frac{e^{-x^2}}{2x} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (n-\frac{1}{2})(n-\frac{3}{2})\cdots(\frac{1}{2})}{x^{2n}} \right]$$

or

$$2xe^{x^2} I(x) \sim 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (n-\frac{1}{2})(n-\frac{3}{2})\cdots(\frac{1}{2})}{x^{2n}} \tag{14}$$

Figure 2.16 shows the successive partial sums $S_n(x)$ of this series plotted against n for $x = \sqrt{8}$; they are also listed in Table 2.3.

If we scan down Table 2.3, we see that the tightest pair of partial sums is the pair (0.94638, 0.94684), so we expect that the "exact" value of $2xe^{x^2} I(x)$ lies between these values. The "exact value" is 0.94661.

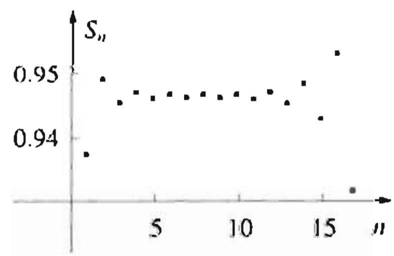

Figure 2.16
The successive partial sums of the asymptotic series $2xe^{x^2} I(x)$ given by Equation 14 for $x = \sqrt{8}$ plotted against n. Note the scale on the vertical axis.

Table 2.3
The successive partial sums of $2xe^{x^2}I(x)$ given by Equation 14 for $x = \sqrt{8}$.

n	S_n	n	S_n
0	1.000 00	10	0.946 94
1	0.937 50	11	0.946 16
2	0.949 22	12	0.947 28
3	0.945 56	13	0.945 52
4	0.947 16	14	0.948 48
5	0.946 26	15	0.943 12
6	0.946 88	16	0.953 52
7	0.946 38	17	0.932 06
8	0.946 84	18	0.979 00
9	0.946 34		

The asymptotic series that we have discussed in this section are divergent asymptotic series. Not all asymptotic series diverge; some converge for large values of x. But, divergent or not, asymptotic series can be used to obtain good approximations to functions for large values of x.

2.9 Problems

1. Derive Equation 13.

2. Show that $f(x) = \displaystyle\int_0^\infty \frac{e^{-xt}dt}{t^2+1} \sim \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}}$.

3. Show that $g(x) = \displaystyle\int_0^\infty \frac{te^{-xt}dt}{t^2+1} \sim \frac{1}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}}$.

4. The following integral arises in a treatment of the radiation emitted from a linear antenna: $C(x) = \displaystyle\int_x^\infty \frac{\cos u}{u}du$. Derive an asymptotic series for $C(x)$ in terms of $f(x)$ and $g(x)$ as defined in Problems 2 and 3.

5. Derive an asymptotic series for $\displaystyle\int_0^\infty e^{-xu}\sin u\, du$ by letting $z = xu$, then by using the Maclaurin expansion of $\sin(z/x)$, and then by integrating term by term. Compare your result to $\displaystyle\int_0^\infty e^{-xu}\sin u\, du = \frac{1}{1+x^2}$.

6. Calculate the successive terms and the first 25 partial sums of Equation 12 for $x = 20$ and show that the successive terms decrease until $n = 20$ and give the result 0.954 3709.

7. Verify the entries in Table 2.3.

8. We argued just before Example 1 that the value of n at which successive terms in Equation 11 have a minimum value is approximately equal to x. Table 2.3 shows, however, that this value of n is approximately x^2 ($n = 8$ and $x = \sqrt{8}$) for Equation 14. Argue that this should be the case for Equation 14. *Hint*: Write $(n - \frac{1}{2})(n - \frac{3}{2}) \cdots (\frac{1}{2})$ as $\dfrac{(2n-1)(2n-3)\cdots(1)}{2^n}$ and then use the relation $\{(1)(3)\cdots(2n-3)(2n-1)\} \times [2 \cdot 4 \cdot 6 \cdots 2n] = (2n)!$.

9. We can derive an asymptotic expansion of $F(s) = \int_0^\infty f(x)e^{-sx}dx$ for large values of s by integrating by parts repeatedly. Show that

$$F(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \cdots + \frac{f^{(n)}(0)}{s^{n+1}} + \frac{1}{s^{n+1}} \int_0^\infty f^{(n+1)}(x)e^{-sx}dx$$

Use this result to show that

$$F(s) = \int_0^\infty e^{-sx} \cos x \, dx = \frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^5} - \frac{1}{s^5} \int_0^\infty e^{-sx} \sin x \, dx$$

Show that the remainder term (the integral on the right) is less than s^{-6} and calculate $F(10)$. Show that $F(s) = s/(1+s^2)$ and that $F(10) = 0.090\,09900\cdots$. Compare your results.

10. Use the method in the previous problem to calculate $F(s) = \int_0^\infty e^{-sx}\frac{\sin x}{x}dx$ for $s = 10$ and estimate the error. Given that $F(s) = \tan^{-1}(1/s)$, compare your result to the "exact" answer.

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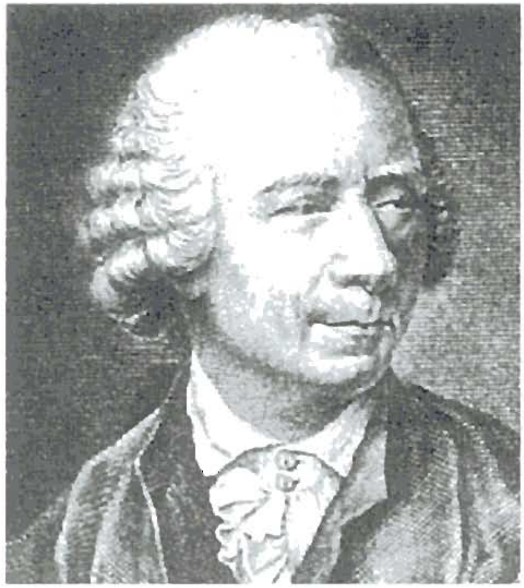

Leonhard Euler (1707–1783), one of the most outstanding and certainly most prolific mathematicians of all time, was born on April 15, 1707, in Basel, Switzerland. His father was a Protestant minister who had studied mathematics with the Bernoullis. Euler entered the University of Basel at age 14 to prepare for the ministry, but he soon discovered his talent for mathematics. Although he switched from religious studies to mathematics, he remained devout throughout his life. He published many papers on a variety of mathematical topics when he was a student. Upon completion of his university studies in 1726, he was offered a position at the St. Petersburg Academy of Science in Russia, in part due to the efforts of the Bernoulli family. Here he was surrounded by gifted scientists, and Euler was able to study every facet of mathematics, both pure and applied. In 1734, he married Katharina Gsell, a local Swiss woman. The marriage was long and harmonious, producing 13 children, of whom only 5 survived past infancy. In 1735, he suffered from a serious fever, resulting in the loss of most of his vision in his right eye; however, his productivity was not hindered. Due to political turmoil in Russia, Euler accepted a position at the Berlin Academy of Science in 1741 at the invitation of Frederick the Great. During his 25 years in Berlin, he published close to 400 articles, including a popular science book, *Letters to a German Princess*, his most widely read book. Personal differences with Frederick led to his return to St. Petersburg in 1766. In 1771, he lost his wife and then the sight in his good eye. In spite of these tragedies, he continued his prodigious mathematical output, aided by his two sons and other assistants. He died of a brain hemorrhage on September 18, 1783. After his death, the Academy at St. Petersburg continued to publish his unpublished but finished work for almost 40 years. There is no field of mathematics, both pure and applied, to which Euler did not make important contributions.

CHAPTER 3

Functions Defined As Integrals

In applied mathematics, we frequently need to evaluate integrals such as $\int_0^x e^{-u^2}\,du$ or $\int_0^x \frac{\sin u}{u}\,du$ for various values of x, but it turns out that it is not possible to express either of these integrals in terms of any of the functions that we study in calculus. There is no simple function whose derivative is e^{-x^2} or $(\sin x)/x$. We say that the above integrals cannot be expressed in terms of elementary functions, an admittedly somewhat vague term. This situation occurs often in applied problems, and we use such integrals to *define* new (non-elementary) functions, which if they occur frequently enough, become as well accepted as the "elementary" functions. We shall also see in Chapter 12 that differential equations can serve to define new functions, such as Bessel functions and Legendre functions.

In this chapter, we will discuss a number of functions that are defined in one way or another through integrals. In the first two sections, we discuss the gamma function, and its close relative, the beta function. We'll see that a gamma function generalizes our idea of a factorial, in that it equals a factorial for integer values of x, but is defined (and useful) for non-integer values. In Section 3, we discuss the error function, a central function of statistics, and one that occurs in a wide variety of physical problems. In the next two sections, we discuss the exponential integral and then elliptic integrals, and in Section 6, we discuss the Dirac delta function, a powerful manipulative tool used in a number of quantum-mechanical problems. In the last section, we discuss Bernoulli numbers and Bernoulli polynomials. These aren't really functions defined by integrals, but they are used in dealing with summations, which are the discrete versions of integrals. The standard reference for the material of this chapter (and much more) is Abramowitz and Stegun. (See the references at the end of the chapter.)

3.1 The Gamma Function

The expression $n!$, equal to $1 \cdot 2 \cdot 3 \cdots n$, occurs when we enumerate permutations and combinations of things, such as the number of ways that N molecules can be distributed over n molecular quantum states. In the 1700s, Euler introduced a function that yields $n!$ when n is a positive integer, but is well-defined when n isn't a positive integer. The function is called the *gamma function* and is defined by the integral expression

$$\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz \qquad x > 0 \tag{1}$$

Notice that the integrand is a function of x and z and that the resulting integral is a function of x. If x is a positive integer greater than or equal to 2, we can integrate $\Gamma(x)$ by parts. Letting $e^{-z}dz$ be "dv" and z^{x-1} be "u", we obtain

$$\Gamma(x) = \left[-z^{x-1}e^{-z} \right]_0^\infty + (x-1)\int_0^\infty z^{x-2}e^{-z}dz = (x-1)\int_0^\infty z^{x-2}e^{-z}dz$$

If we compare the last integral here to $\Gamma(x)$ in Equation 1, we see that it is equal to $\Gamma(x-1)$, so we have

$$\Gamma(x) = (x-1)\Gamma(x-1) \tag{2}$$

We can now substitute $\Gamma(x-1) = (x-2)\Gamma(x-2)$ and so on into Equation 2 to get

$$\Gamma(x) = (x-1)(x-2)\cdots\Gamma(1) \tag{3}$$

where

$$\Gamma(1) = \int_0^\infty e^{-z}dz = 1$$

Therefore, Equation 3 reads

$$\Gamma(x) = (x-1)(x-2)\cdots(1) = (x-1)! \qquad x = 2,\ 3,\ \cdots \tag{4}$$

Up to this point, Equation 4 is restricted to integer values of $x \geq 2$, but we can use it to *define* factorials for other values of x. If we let $x = 1$ in Equation 4, we have $\Gamma(1) = 0!$. But Equation 1 is perfectly well-defined for $x = 1$, and yields $\Gamma(1) = 1$. Therefore, we can say that $0! = \Gamma(1) = 1$, a relation that you may have come across before. The fact that $0! = 1$ shouldn't bother you; the familiar relation $n! = 1 \cdot 2 \cdot 3 \cdots n$ is true only for *positive* integers.

We can use Equation 4 to define a factorial for non-integers. This extension, or generalization, of a factorial turns out to be very convenient in practice. We can let $x = 1/2$ in Equation 4 to write $\Gamma(1/2) = (-1/2)!$. Now we can use Equation 1 to evaluate $\Gamma(1/2)$. Letting $z = u^2$, Equation 1 becomes

$$\Gamma(1/2) = \int_0^\infty z^{-1/2}e^{-z}dz = 2\int_0^\infty e^{-u^2}du = \sqrt{\pi}$$

so that $\Gamma(1/2) = (-1/2)! = \sqrt{\pi}$ (Problem 10). What about $\Gamma(3/2) = (1/2)!$? Simply use Equation 2 to write

$$\Gamma(3/2) = (1/2)\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

We can continue this process and write (Problem 8)

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}\Gamma(1/2) \qquad n \geq 1 \tag{5}$$

for any integer value of n.

You might be wondering why we would want to consider factorials of numbers other than positive integers. The definitions of many functions that occur naturally in physical problems involve quantities such as $\Gamma(1/2)$ and $\Gamma(1/3)$. For example, we'll study Bessel functions in Chapter 12, and the definition of these functions involves gamma functions. The following Example gives another reason.

Example 1:
Evaluate $\int_0^\infty x e^{-ax^4} dx$ in terms of a gamma function ($a > 0$).

SOLUTION: Let $u = ax^4$; then $x = (u/a)^{1/4}$, $dx = \frac{1}{4a^{1/4}}u^{-3/4}du$, and

$$\int_0^\infty x e^{-ax^4} dx = \frac{1}{4a^{1/2}}\int_0^\infty \frac{e^{-u}}{u^{1/2}}\, du = \frac{\Gamma(1/2)}{4a^{1/2}} = \frac{1}{4}\left(\frac{\pi}{a}\right)^{1/2}$$

The definition of $\Gamma(x)$ given by Equation 1 restricts x to values greater than zero (Problem 9), but we can use Equation 2 in the form

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \tag{6}$$

to *extend* our definition of $\Gamma(x)$ to values of x less than zero. For example, let's use Equation 6 to calculate $\Gamma(-1/2)$. (We'll start abandoning the factorial notation now.) Equation 6 gives

$$\Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\sqrt{\pi}$$

Using a combination of Equations 1 and 6, we can calculate $\Gamma(x)$ for all values of x, other than 0 and negative integers (see below).

Example 2:
Calculate $\Gamma(-5/2)$.

SOLUTION: We use Equation 6:

$$\Gamma(-5/2) = \frac{\Gamma(-\frac{3}{2})}{(-\frac{5}{2})} = \left(-\frac{2}{5}\right)\frac{\Gamma(-\frac{1}{2})}{(-\frac{3}{2})} = \left(\frac{4}{15}\right)\frac{\Gamma(\frac{1}{2})}{(-\frac{1}{2})} = -\frac{8}{15}\sqrt{\pi}$$

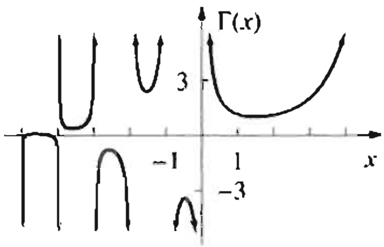

Figure 3.1
The gamma function $\Gamma(x)$ plotted against x.

There are some special values of x that we have avoided up to now. According to both Equation 1 (if we let $0 \to 0+$) and Equation 6, $\Gamma(0)$ is not defined. Furthermore, Equation 6 gives us $\Gamma(-1) = \Gamma(0)/(-1)$, which is also not defined. Figure 3.1 shows $\Gamma(x)$ plotted against x. Notice that $\Gamma(x)$ has a vertical asymptote when x is a negative integer. Thus, $\Gamma(x)$ is not defined when $x = 0$ or a negative number. The graph of $\Gamma(x)$ is given by Equation 1 for $x > 0$ and Equation 6 for $x < 0$.

Example 3:
Show that $\lim_{x\to 0-} \Gamma(x) = -\infty$ and that

$\lim_{x\to -1-} \Gamma(x) = \lim_{\epsilon\to 0} \Gamma(-1-\epsilon) = +\infty$ where $\epsilon > 0$.

SOLUTION: We use Equation 6:

$$\lim_{x\to 0-} \Gamma(x) = \lim_{x\to 0-} \frac{\Gamma(x+1)}{x} = \Gamma(1) \lim_{x\to 0-} \frac{1}{x} = -\infty$$

Similarly, for $\epsilon > 0$,

$$\lim_{\epsilon\to 0} \Gamma(-1-\epsilon) = \lim_{\epsilon\to 0} \frac{\Gamma(-\epsilon)}{-1-\epsilon} = \infty$$

The reasoning that we use in Example 3 can be used to verify the behavior of $\Gamma(x)$ in Figure 3.1 at all the negative integers.

Using Equation 6, it's possible to reduce the evaluation of $\Gamma(x)$ for any (non-negative integral) value of x to the evaluation of $\Gamma(x)$ for $1 \le x \le 2$. For example, $\Gamma(6.13) = (5.13)\Gamma(5.13) = (5.13)(4.13)(3.13)(2.13)(1.13)\Gamma(1.13) = (159.6)\,\Gamma(1.13)$. Therefore, mathematical tables list numerical values of $\Gamma(x)$ for values of x only between 1.00 and 2.00 (Table 3.1).

Table 3.1
Some values of $\Gamma(x)$ for $1 \leq x \leq 2$.

x	1.00	1.10	1.20	1.30	1.40	
$\Gamma(x)$	1.000 00	0.951 35	0.918 17	0.897 47	0.887 26	
x	1.50	1.60	1.70	1.80	1.90	2.00
$\Gamma(x)$	0.886 23	0.893 52	0.908 64	0.931 38	0.961 77	1.000 00

Example 4:
Use Table 3.1 to determine the value of $\Gamma(-0.800)$.

SOLUTION: Use Equation 6:

$$\Gamma(-0.800) = \frac{\Gamma(0.200)}{(-0.800)} = \frac{\Gamma(1.200)}{(-0.800)(0.200)} = \frac{0.918\ 17}{(-0.160)} = -5.738$$

There are a couple of useful formulas involving $\Gamma(x)$ that we shall present but not prove here (see the next section, however). These two formulas are

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{7}$$

and

$$\Gamma(2x) = \pi^{-1/2} 2^{2x-1} \Gamma(x) \Gamma\left(x + \tfrac{1}{2}\right) \tag{8}$$

Equation 7 is called the *reflection formula* and Equation 8 is called the *duplication formula* for the gamma function. We shall use these formulas in later chapters.

Example 5:
Use Equation 7 to show that $\Gamma(1/2) = \sqrt{\pi}$.

SOLUTION: Let $x = 1/2$ in Equation 7 to obtain $\Gamma^2(1/2) = \pi/\sin(\pi/2) = \pi$, or $\Gamma(1/2) = \sqrt{\pi}$.

In many applications of the gamma function (particularly in statistical mechanics), we need values of $x!$ for large values of x (such as the Avogadro constant). We can start with Equation 1 and derive an asymptotic series for $\Gamma(x+1) = x!$. The derivation for the complete expansion is fairly involved, but we can derive the most important factors here. Start with

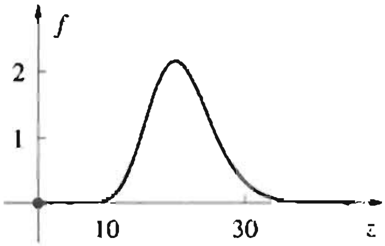

Figure 3.2
The function $f(z) = e^{x \ln z - z}$, which is the integrand of Equation 9, plotted against z for $x = 20$.

$$\Gamma(x+1) = \int_0^\infty z^x e^{-z} dz \tag{9}$$

Figure 3.2 shows the integrand of Equation 9 plotted against z for $x = 20$. You can see that the integrand peaks sharply at $x = 20$. Generally, the integrand looks like a Gaussian curve and peaks sharply about its maximum value, which occurs at $z = x$ (Problem 11). This suggests that we write the integrand as an exponential. To do this, let $z^x = e^{x \ln z}$ and write

$$\Gamma(x+1) = \int_0^\infty e^{x \ln z - z} dz \tag{10}$$

Because the integrand peaks sharply at $z = x$, only values of z near x will contribute to the integral. Because $\exp(x \ln z - z)$ is a monotonic function of $x \ln z - z$, we can work with the simpler $x \ln z - z$ instead of $\exp(x \ln z - z)$. Let's expand $x \ln z - z$ about $z = x$ to get (Problem 12)

$$x \ln z - z = x \ln x - x - \frac{(z-x)^2}{2x} + \frac{(z-x)^3}{3x^2} + \cdots$$

Because x is large and only terms where $z \approx x$ are important, we neglect terms in $(z-x)^3$ and higher, giving

$$x \ln z - z \approx x \ln x - x - \frac{(z-x)^2}{2x}$$

Substitute this back into $\Gamma(x+1)$ to get

$$\begin{aligned}\Gamma(x+1) &\approx \int_0^\infty e^{x \ln x - x - (z-x)^2/2x} dz \\ &\approx e^{x \ln x - x} \int_0^\infty e^{-(z-x)^2/2x} dz\end{aligned}$$

Let $(z-x)^2/2x = u^2$, so that $\Gamma(x+1)$ becomes

$$\Gamma(x+1) \approx (2x)^{1/2} x^x e^{-x} \int_{-(x/2)^{1/2}}^\infty e^{-u^2} du$$

Because x is large and the integrand falls off rapidly about $u = 0$, we can extend the lower limit here to $-\infty$ to finally get

$$\Gamma(x+1) = x! \sim (2\pi x)^{1/2} x^x e^{-x} \tag{11}$$

or

$$\ln x! \sim x \ln x - x + \frac{1}{2} \ln(2\pi x) \tag{12}$$

Equations 11 and 12 are known as *Stirling's approximation*, and are used frequently in statistical mechanics. Equation 11 is actually the first term of an asymptotic series for $\Gamma(x+1)$; that's why we used $\sim$ in Equations 11 and 12. The series itself is

$$\Gamma(x+1) = x! \sim (2\pi x)^{1/2} x^x e^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right) \tag{13}$$

Table 3.2 compares Stirling's approximation, Equation 12, to the exact result for $n = 5$, 10, 15, 20, and 50. Even for $n = 10$, it is in error by only 0.05%.

Table 3.2
A numerical evaluation of Stirling's approximation, Equation 12.

n	$\ln n!$	Equation 12	$(\ln n! - \text{Equation 12})/\ln n!$
5	4.7875	4.7708	1.7×10^{-2}
10	15.1044	15.0961	5.5×10^{-4}
15	27.8993	27.8937	2.0×10^{-4}
20	42.3356	42.3315	9.8×10^{-5}
50	148.4778	148.4761	1.1×10^{-5}

3.1 Problems

1. Evaluate $\int_0^\infty e^{-au} u^{3/2} du$.
2. Evaluate $3\Gamma(5/4)/2\Gamma(1/4)$.
3. Evaluate $\int_0^\infty z^{1/2} e^{-z^3} dz$ in terms of a gamma function.
4. Evaluate $\int_0^\infty x^5 e^{-x^4} dx$ in terms of a gamma function.
5. Evaluate $\int_0^1 (\ln x)^n dx$.
6. Evaluate $\int_0^1 \frac{dx}{\sqrt{x \ln(1/x)}}$.
7. Show that $\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)}$ for $-n < x < -n+1$.
8. Derive Equation 5.
9. Show that Equation 1 diverges as $x \to 0+$.
10. In this problem we will use a trick to prove that $I = \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$. First square I and write it as

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$$

Now convert to plane polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) and use the fact that $dxdy = rdrd\theta$ to show that $I^2 = \pi/4$, or that $I = \sqrt{\pi}/2$.

11. Show that $f(x) = \exp(x\ln z - z)$ has a maximum at $z = x$. *Hint*: Work with $x\ln z - z$ instead of $\exp(x\ln z - z)$.

12. Show that the Taylor expansion of $f(z) = x\ln z - z$ about $z = x$ is equal to $x\ln x - x - \dfrac{(z-x)^2}{2x} + \dfrac{(z-x)^3}{3x^2} + \cdots$.

13. Use Table 3.1 to evaluate $\Gamma(4.30)$.

14. Show that the binomial series can be written as $(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \dfrac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)} x^n$.

15. A factorial notation that is sometimes used is $n!! = n(n-2)(n-4)\cdots$. Evaluate (a) $10!!$ and (b) $7!!$.

16. Show that $(2n)!! = 2^n n!$ and that $(2n+1)!! = 2^n\Gamma(n+3/2)/\pi^{1/2}$.

17. Evaluate $\int_0^\infty x^m e^{-x^n} dx$, where m and n are positive integers, in terms of a gamma function.

18. Show that $(1-x)^{-1/2} = \sum_{n=0}^{\infty} \dfrac{(2n)!}{(n!)^2}\left(\dfrac{x}{4}\right)^n$.

19. Use Equation 7 to show that $\Gamma\left(n+\dfrac{1}{2}\right)\Gamma\left(\dfrac{1}{2}-n\right) = \dfrac{\pi}{\cos n\pi}$.

20. Show that $\int_0^\infty x^{2n} e^{-ax^2} dx = \Gamma\left(n+\dfrac{1}{2}\right)/2a^{n+1/2}$.

21. Show that $\int_0^\infty e^{-at^2}\cos 2xt\, dt = (\pi/4a)^{1/2} e^{-x^2/a}$ by expanding $\cos 2xt$ in a Maclaurin series and then integrating term by term.

22. The logarithm function $\ln x$ can be *defined* by the integral $\ln x = \int_1^x \dfrac{du}{u}$. Show that $\ln ab = \ln a + \ln b$ and that $\ln a^b = b\ln a$.

3.2 The Beta Function

In this section we shall discuss another useful function introduced by Euler. It is the *beta function*, defined by

$$B(x,y) = \int_0^1 z^{x-1}(1-z)^{y-1}dz \qquad 0 < x,\ 0 < y \tag{1}$$

By letting $1 - z = u$, we find that $B(x,y) = B(y,x)$ (Problem 1); in other words, the beta function is a symmetric function in x and y.

It turns out that $B(x, y)$ is related to the gamma function by the relation

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2}$$

which also shows that $B(x, y) = B(y, x)$. The proof of Equation 2 is outlined in Problem 3.

The integral in Equation 1 occurs in a variational treatment of the quantum-mechanical problem of a particle in a box. In that case, x and y take on the integral values 2, 3, ...

Example 1:

Evaluate the integral $\int_0^1 z^4(1-z)^4 dz$.

SOLUTION: We simply use Equation 2 with $x = y = 5$ to get

$$\int_0^1 z^4(1-z)^4 dz = \frac{\Gamma(5)\Gamma(5)}{\Gamma(10)} = \frac{(4!)^2}{9!} = \frac{4 \cdot 3 \cdot 2}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5} = \frac{1}{630}$$

We could have evaluated the integral in Example 1 by expanding $(1-z)^4$ using the binomial formula, but that approach gets tedious pretty quickly.

The beta function can be transformed into other useful forms by a simple change of variable. If we let $z = \sin^2\theta$ in Equation 1, then (Problem 2)

$$B(x, y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1} d\theta \qquad 0 < x,\ 0 < y \tag{3}$$

Letting $t = z/(1-z)$ in Equation 1 gives (Problem 4)

$$B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt \tag{4}$$

If we let $x + y = 1$, Equation 4 becomes (after using Equation 2 above and Equation 7 of the previous section)

$$B(x, 1-x) = \int_0^\infty \frac{t^{x-1}}{1+t} dt = \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad 0 < x < 1 \tag{5}$$

(We'll evaluate this integral another way in Chapter 19.)

Equations 3 through 5 can be used to evaluate a host of integrals. These are best illustrated by examples.

Example 2:

Evaluate $\int_0^{\pi/2} (\tan x)^{1/3} dx$.

SOLUTION: Let $\tan x = \sin x / \cos x$ and use Equation 3 with $2x - 1 = 1/3$ and $2y - 1 = -1/3$:

$$\int_0^{\pi/2} (\sin\theta)^{1/3}(\cos\theta)^{-1/3} d\theta = \frac{1}{2}\frac{\Gamma(2/3)\Gamma(1/3)}{\Gamma(1)} = \frac{1}{2}\Gamma(2/3)\Gamma(1/3)$$

Example 3:

Evaluate $\int_0^2 u^3(4-u^2)^{3/2} du$.

SOLUTION: First let $u^2 = 4z$, so that $(4-u^2)^{3/2} = 8(1-z)^{3/2}$, $u^3 = 8z^{3/2}$, and $du = dz/z^{1/2}$. Then

$$\begin{aligned}\int_0^2 u^3(4-u^2)^{3/2} du &= 64\int_0^1 z(1-z)^{3/2} dz \\ &= 64\frac{\Gamma(2)\Gamma(5/2)}{\Gamma(9/2)} = \frac{256}{35}\end{aligned}$$

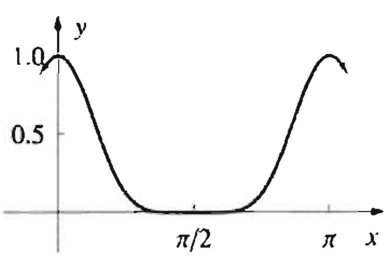

Figure 3.3
The function $y(\theta) = \cos^6\theta$ is symmetric about $\theta = \pi/2$.

Example 4:

Evaluate $\int_0^{\pi} \cos^6\theta\, d\theta$.

SOLUTION: We could use Equation 3 if the limits were 0 to $\pi/2$. Note, however, that $\cos^6\theta$ is symmetric about $\theta = \pi/2$ (Figure 3.3), so that $\int_0^{\pi} = 2\int_0^{\pi/2}$. Now we use Equation 3 to write

$$\int_0^{\pi} \cos^6\theta\, d\theta = 2\int_0^{\pi/2} \cos^6\theta\, d\theta = 2\cdot\frac{1}{2}\frac{\Gamma(1/2)\Gamma(7/2)}{\Gamma(4)} = \frac{5\pi}{16}$$

Example 5:

Evaluate $\int_0^{\infty} \frac{du}{1+u^4}$.

SOLUTION: Let $z = u^4$, $du = dz/4z^{3/4}$, and write

$$\int_0^\infty \frac{du}{1+u^4} = \frac{1}{4}\int_0^\infty \frac{z^{-3/4}}{1+z}dz$$

$$= \frac{1}{4}\frac{\pi}{\sin(\pi/4)} = \frac{\pi}{2^{3/2}}$$

We used Equation 5 to evaluate the last integral.

3.2 Problems

1. Show that $B(x, y) = B(y, x)$.
2. Derive Equation 3.
3. (This problem involves polar coordinates.) You can derive Equation 2 in the following way. Start with $\Gamma(m) = \int_0^\infty z^{m-1}e^{-z}dz$ and let $z = x^2$. Do the same for $\Gamma(n)$. Now form $\Gamma(m)\Gamma(n)$ as a double integral, transform to polar coordinates, and use Equation 3.
4. Derive Equation 4.
5. Evaluate $\displaystyle\int_0^{2\pi} \cos^6\theta d\theta$.
6. Evaluate $\displaystyle\int_0^1 \sqrt[3]{u(1-u)}\, du$.
7. Evaluate $\displaystyle\int_0^1 \left(1-\frac{1}{x}\right)^{2/3} dx$.
8. Evaluate $\displaystyle\int_0^2 u^2\sqrt[3]{8-u^3}\, du$.
9. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{e^{2u}}{1+e^{3u}}\, du$.
10. Evaluate $\displaystyle\int_0^a \frac{dx}{\sqrt{a^6-x^6}}$.
11. Evaluate $\displaystyle\int_0^{\pi/2} \sin^{1/3}2\theta\, d\theta$.
12. Compute the area bounded by the curve $x^{2/3} + y^{2/3} = 1$. *Hint*: Plot this expression first.
13. We shall derive the duplication formula for the gamma function (Equation 1.8) in this problem. First use Equation 3 to write

$$I_1 = \int_0^{\pi/2} \sin^{2x} u\, du = \frac{1}{2}B(x+\tfrac{1}{2}, \tfrac{1}{2}) = \frac{\Gamma(x+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(x+1)}$$

Now consider the integral $I_2 = \int_0^{\pi/2} \sin^{2x} 2u\, du$. Use the relation $\sin 2u = 2 \sin u \cos u$ to show that

$$I_2 = 2^{2x-1} \frac{\Gamma(x + \frac{1}{2})\Gamma(x + \frac{1}{2})}{\Gamma(2x + 1)}$$

Now let $2u = z$ in I_2 and show that $I_2 = I_1$. Finally equate the expressions for I_1 and I_2 above to get

$$\Gamma(2x) = \pi^{-1/2} 2^{2x-1} \Gamma(x)\Gamma(x + \tfrac{1}{2})$$

3.3 The Error Function

One of the most commonly occurring and important integrals that cannot be expressed in terms of elementary functions is of the form $\int_0^x e^{-u^2} du$. This type of integral occurs so frequently that the function of x that it defines is a standard function of applied mathematics. We *define* the *error function* by the integral

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \qquad -\infty < x < \infty \tag{1}$$

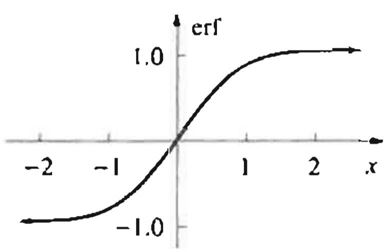

Figure 3.4
The error function $\operatorname{erf}(x)$ plotted against x.

Even though we cannot express erf (x) in terms of simpler functions, it is a perfectly well-defined function of x and can be evaluated by numerical integration. Almost any mathematical handbook will have tables of $\operatorname{erf}(x)$ and any CAS will have the error function as a standard function.

The $2/\sqrt{\pi}$ factor in Equation 1 is chosen so that $\operatorname{erf}(\infty) = 1$. Furthermore, it's easy to show that (Problem 1)

$$\operatorname{erf}(-x) = -\operatorname{erf}(x) \tag{2}$$

so that $\operatorname{erf}(-\infty) = -1$. Figure 3.4 shows $\operatorname{erf}(x)$ plotted against x and Table 3.3 lists $\operatorname{erf}(x)$ for a few (positive) values of x.

Table 3.3
Values of $\operatorname{erf}(x)$ for a few positive values of x.

x	$\operatorname{erf}(x)$	x	$\operatorname{erf}(x)$
0.00	0.00000	1.20	0.91031
0.20	0.22270	1.40	0.95228
0.40	0.42839	1.60	0.97635
0.60	0.60386	1.80	0.98909
0.80	0.74210	2.00	0.99532
1.00	0.84270		

Example 1:
The error function plays an important role in the kinetic theory of gases. The fraction of molecules that have a component (x, y, or z) of velocity between v and $v + dv$ is given by

$$f(v)\,dv = \left(\frac{m}{2\pi k_B T}\right)^{1/2} e^{-mv^2/2k_B T}\,dv$$

where m is the mass of a molecule, T is the kelvin temperature, and k_B is the Boltzmann constant. Calculate the fraction of molecules with $-(2k_B T/m)^{1/2} \le v \le (2k_B T/m)^{1/2} = v_0$.

SOLUTION: The fraction is given by

$$F = \int_{-v_0}^{v_0} f(v)\,dv = 2\int_0^{v_0} f(v)\,dv$$

because the integrand is an even function of v. Now let $u = (m/2k_B T)^{1/2} v$ to write

$$F = \frac{2}{\sqrt{\pi}}\int_0^1 e^{-u^2}du = \text{erf}(1) = 0.842\ 70$$

Thus, about 84% of the molecules have a magnitude of a component of velocity that is less than $(2k_B T/m)^{1/2}$.

Suppose we want to calculate the fraction of molecules in a gas whose x-component of velocity is greater in magnitude than v_0 (perhaps to determine the fraction of particularly energetic molecules). We would calculate the following (see Figure 3.5):

$$F = \int_{-\infty}^{-v_0} f(v)\,dv + \int_{v_0}^{\infty} f(v)\,dv = 2\left(\frac{m}{2\pi k_B T}\right)^{1/2}\int_{|v_0|}^{\infty} e^{-mv^2/2k_B T}\,dv$$

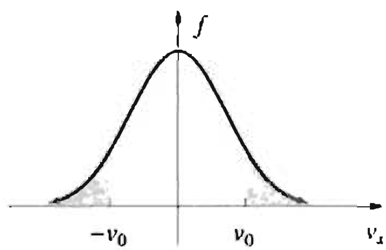

Figure 3.5
The shaded area represents the fraction of molecules in a gas with an x-component of the velocity that exceeds some value v_0 in magnitude.

If $v_0 = (2k_B T/m)^{1/2}$, as in Example 1, then

$$F = \frac{2}{\sqrt{\pi}}\int_1^{\infty} e^{-u^2}du$$

which is a special case of

$$F(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}du$$

Equation 1 says that $F(x) = 1 - \text{erf}(x)$. Although $F(x) = 1 - \text{erf}(x)$, this integral occurs frequently enough that it is used to define the *complementary error function*, erfc(x),

$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \tag{3}$$

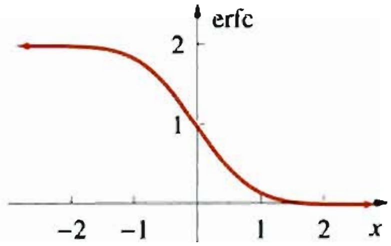

Figure 3.6
The complementary error function $\text{erfc}(x)$ plotted against x.

Note that $\text{erfc}(-\infty) = 2$, $\text{erfc}(0) = 1$, and $\text{erfc}(\infty) = 0$. (See Figure 3.6.)

The error function also occurs when we solve certain differential equations. In Chapter 17, for example, we'll learn how to solve the diffusion equation, which describes how a substance diffuses through a solution. Suppose we have a long narrow cylindrical tube ($-\infty < x < \infty$) and we prepare the solution initially such that the concentration of diffusing species, $c(x)$, is (see Figure 3.7)

$$c(x) = \begin{cases} 0 & x < 0 \\ c_0 & x > 0 \end{cases} \qquad t = 0$$

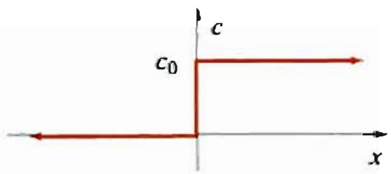

Figure 3.7
The initial concentration profile $c(x) = 0$ for $x < 0$ and $c(x) = c_0$ for $x > 0$ in a long cylindrical tube.

We expect that as time evolves, the concentration profile will smooth out and eventually become uniform, as shown in Figure 3.8. When we solve this problem in Chapter 17, the concentration profile will be given by

$$c(x, t) = \frac{c_0}{2}\left[1 + \text{erf}\left(\frac{x}{2D\sqrt{t}}\right)\right] \tag{4}$$

where D is the diffusion constant, which characterizes how rapidly the substance diffuses. Note that when $t = 0$, $\text{erf}(x/2D\sqrt{t}) = \text{erf}(-\infty) = -1$ when $x < 0$ and $\text{erf}(\infty) = +1$ when $x > 0$, so that

$$c(x, 0) = \begin{cases} 0 & x < 0 \\ c_0 & x > 0 \end{cases}$$

in agreement with the given initial conditions. Also, as $t \to \infty$, $\text{erf}(x/2D\sqrt{t}) \to 0$, so that $c(x, t) \to$ a uniform concentration of $c_0/2$ along the entire tube. For intermediate values of t, Equation 4 gives the concentration profiles shown in Figure 3.8.

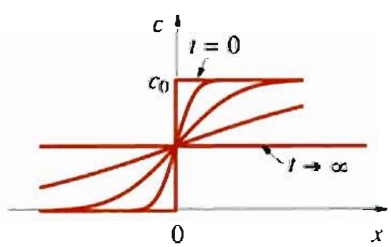

Figure 3.8
The concentration profiles in a long cylindrical tube, where the initial concentration profile was $c(x) = 0$ for $x < 0$ and $c(x) = c_0$ for $x > 0$.

The error function, or very closely related functions, plays a prominent role in statistics. The Gaussian distribution or normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty \tag{5}$$

describes the distribution of errors in many types of experimental observations (Chapter 22). In Equation 5, μ is the mean, or the average, of the observations and σ is the standard deviation of the observations. In other words,

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx \tag{6}$$

and

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx \tag{7}$$

(Problem 5).

Equation 5 is the famous bell-shaped curve of statistics, and is plotted in Figure 3.9. Two important features of the curves in Figure 3.9 are that the curves are centered at $x = \mu$ and that the widths of the curves are controlled by the value of σ; the smaller σ is, the narrower and more highly peaked the curves are.

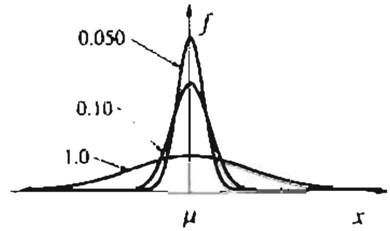

Figure 3.9
The Gaussian distribution, Equation 5, plotted against x for $\sigma = 1.0, 0.10$, and 0.050.

Equation 5 is used to predict the chance that an experimental observation will lie within a certain interval about $x = \mu$. For example, we might want to know the chance that an observation will lie within one standard deviation of μ; this is given by

$$I = \int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx$$

Let $(x - \mu)/(\sqrt{2}\,\sigma) = u$ to write

$$\begin{aligned} I &= \left(\frac{1}{\pi}\right)^{1/2} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} e^{-u^2} du = \left(\frac{4}{\pi}\right)^{1/2} \int_{0}^{1/\sqrt{2}} e^{-u^2} du \\ &= \operatorname{erf}(1/\sqrt{2}) = 0.683 \end{aligned}$$

If you've had an introduction to statistics, you might remember that the corresponding values for an observation being within two or three standard deviations of the mean are 95.5% and 99.7%, respectively (Problem 6).

A word of caution: The functions tabulated in most statistics sources are not erf (x), as we have defined it in Equation 1. The function usually tabulated is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \tag{8}$$

which is related to the error function by (Problem 7)

$$\Phi(x) = \frac{1}{2}[1 + \operatorname{erf}(x/\sqrt{2})] \qquad x \geq 0 \tag{9}$$

The error function has a useful Maclaurin expansion for small values of x and an asymptotic expansion for large values of x. The Maclaurin expansion of erf(x) is (Problem 9)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \tag{10}$$

and we showed in Section 2.9 (in slightly different notation) that the asymptotic series of erfc (x) is

$$\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}\,x} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n}\right] \tag{11}$$

Example 2:
Use Equation 10 to evaluate erf(1/2) to four-place accuracy.

SOLUTION: Equation 10 is an alternating series, so the truncation error will be less than the magnitude of the first term dropped. We want four-place accuracy, so we set

$$\frac{(1/2)^{2n+1}}{n!(2n+1)} < 10^{-4}$$

This will be so for $n = 4$, so we have

$$\begin{aligned}\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}}\left[\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{10}\left(\frac{1}{2}\right)^5 - \frac{1}{42}\left(\frac{1}{2}\right)^7\right] \\ &= 0.52049\cdots\end{aligned}$$

The "exact" value to five places is 0.52050.

A number of definite integrals can be expressed in terms of the error function or its related functions. For example (see Problems 11 through 14),

$$\int_0^\infty \frac{e^{-at}\,dt}{\sqrt{t+x^2}} = \left(\frac{\pi}{a}\right)^{1/2} e^{ax^2} \operatorname{erfc}(\sqrt{a}\,x)$$

and

$$\int_0^\infty e^{-(au^2+2bu+c)}du = \left(\frac{\pi}{4a}\right)^{1/2} e^{(b^2-ac)/a} \operatorname{erfc}\left(\frac{b}{\sqrt{a}}\right)$$

3.3 Problems

1. Prove that $\operatorname{erf}(-x) = -\operatorname{erf}(x)$.
2. Determine the value of $\int_0^{2\sqrt{2}} e^{-z^2}dz$.
3. Referring to Example 1, show that Prob $\{-v_{x0} \le v_x \le v_{x0}\} = \operatorname{erf}[(m/2k_BT)^{1/2}v_{x0}]$.
4. The probability that the x-component of the velocity of a gas molecule exceeds a value of v_{x0} is given by

$$\text{Prob}\{v_x > |v_{x0}|\} = \left(\frac{m}{2\pi k_BT}\right)^{1/2} 2\int_{v_{x0}}^\infty e^{-mv_x^2/2k_BT}dv_x$$

   Show that this probability is given by $\operatorname{erfc}(u_0)$, where $u_0 = (m/2k_BT)^{1/2}v_{x0}$.
5. Verify Equations 6 and 7.

6. Using Equation 5, show that the chance that an experimental observation will lie within two standard deviations from the mean is 95.5%.

7. Derive Equation 9.

8. The kinetic theory of gases provides the following formula for the probability that the *speed* of a gas molecule exceeds a value c_0:

$$\text{Prob}\,\{c \geq c_0\} = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\int_{c_0}^{\infty} c^2 e^{-mc^2/2k_BT}\,dc$$

where $c = (v_x^2 + v_y^2 + v_z^2)^{1/2} \geq 0$ is the speed of the molecule. Show that

$$\text{Prob}\,\{c \geq c_0\} = \frac{2}{\sqrt{\pi}}\left\{\left(\frac{m}{2k_BT}\right)^{1/2} c_0 e^{-mc_0/2k_BT} + \frac{\sqrt{\pi}}{2}\,\text{erfc}\left[\left(\frac{m}{2k_BT}\right)^{1/2} c_0\right]\right\}$$

9. Derive the first few terms of the Maclaurin series for erf (x).

10. Use Equation 10 to calculate erf (1) to three-place accuracy. The "exact" value is 0.84270 to five places.

11. Show that $\displaystyle\int_0^\infty e^{-(ax^2+2bx+c)}dx = \left(\frac{\pi}{4a}\right)^{1/2} e^{(b^2-ac)/a}\,\text{erfc}\left(\frac{b}{\sqrt{a}}\right)$.

12. Show that $\displaystyle\int_0^\infty \frac{e^{-at}dt}{\sqrt{t+x^2}} = \left(\frac{\pi}{a}\right)^{1/2} e^{ax^2}\,\text{erfc}(\sqrt{a}x)$.

13. Show that $\displaystyle\int_0^\infty \frac{e^{-at}dt}{t^{1/2}(t+x)} = \frac{\pi}{\sqrt{x}}e^{ax}\,\text{erfc}(\sqrt{ax})$.

14. Show that $\displaystyle\int_0^\infty \frac{\cos u}{\sqrt{u}}du = \int_0^\infty \frac{\sin u}{\sqrt{u}}du = \left(\frac{\pi}{2}\right)^{1/2}$ and use this result to show that

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}$$

Hint: Substitute $\displaystyle\frac{1}{\sqrt{u}} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-ut}}{\sqrt{t}}dt$ into the first integrals and exchange orders of integration.

15. Show that $\displaystyle\int_0^\infty e^{-ax-x^2/4}dx = \pi^{1/2}e^{a^2}\,\text{erfc}(a)$.

16. Show that $\displaystyle\int_x^\infty \frac{e^{-u^2}}{u^2}du = \frac{e^{-x^2}}{x} - \pi^{1/2}\,\text{erfc}(x)$, $\quad x > 0$.

17. Two functions that are closely related to the error function are the *Fresnel sine integral* and the *Fresnel cosine integral* (in the notation of Abramowitz and Stegun):

$$S(x) = \int_0^x \sin\left(\frac{\pi u^2}{2}\right) du \tag{12}$$

and

$$C(x) = \int_0^x \cos\left(\frac{\pi u^2}{2}\right) du \tag{13}$$

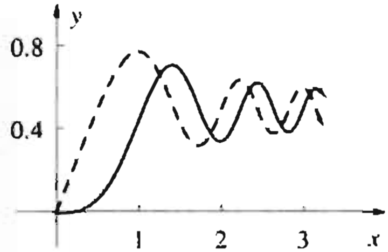

Figure 3.10
The Fresnel sine integral, $S(x)$, defined by Equation 12 (solid) and the Fresnel cosine integral, $C(x)$, defined by Equation 13 (dashed) plotted against x.

Figure 3.10 shows $S(x)$ and $C(x)$ plotted against x. As the names might imply, these integrals arise in the study of the diffraction of radiation. Show that

$$C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(\pi/2)^{2n}}{(2n)!(4n+1)} x^{4n+1}$$

and

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(\pi/2)^{2n+1}}{(2n+1)!(4n+3)} x^{4n+3}$$

18. Starting with $\int_0^{\infty} e^{-a^2x^2}dx = \frac{\pi^{1/2}}{2a}$, let $a = (1-i)/2^{1/2}$ and separate the result into real and imaginary parts to verify the integrals in Problem 14.

19. Differentiate $I(a, b) = \int_0^{\infty} e^{-bu^2} \cos au \, du = \left(\frac{\pi}{4b}\right)^{1/2} e^{-a^2/4b}$ with respect to a and then integrate with respect to b from b to ∞ to show that

$$\int_0^{\infty} \frac{e^{-bu^2} \sin au}{u} du = \frac{\pi}{2} \operatorname{erf}\left(\frac{a}{2b^{1/2}}\right).$$

20. The complementary error function also occurs in the quantum-mechanical discussion of a harmonic oscillator. Recall that a harmonic oscillator can be used as a model for the oscillatory behavior of two masses connected by a spring (a model of a diatomic molecule). If the relaxed length of the spring is l, then the two masses oscillate sinusoidally about l and the potential energy of the system is given by $V(x) = \frac{k}{2}(x-l)^2$, where k is the force constant of the spring. Show that the maximum displacement of the two masses is given by $x_{\max} = l + \left(\frac{2E}{k}\right)^{1/2}$, where E is the total energy. The quantity $x_{\max}$ is called the *classically-allowed amplitude*. One of the many strange results of quantum mechanics is that there is a nonzero probability that the displacement of the oscillator will exceed its classically-allowed amplitude, even though it doesn't have sufficient energy. For a quantum-mechanical harmonic oscillator in its lowest energy state, this probability is given by

$$\text{Prob} = 2\left(\frac{\alpha}{\pi}\right)^{1/2} \int_{\alpha^{-1/2}}^{\infty} e^{-\alpha x^2} dx$$

where α is a constant that is characteristic of the oscillator. Show that this probability is given by $\text{Prob} = \operatorname{erfc}(1) = 1 - \operatorname{erf}(1) = 0.15730$. Thus, we see that there is almost a 16% chance that the displacement will exceed its classically-allowed amplitude.

3.4 The Exponential Integral

Another type of integral that occurs fairly often in physical problems has the general form

$$E_n(x) = \int_1^\infty \frac{e^{-xz}}{z^n}\,dz \qquad x > 0, \quad n = 0, 1, \ldots \tag{1}$$

In particular, E_1 arises in a quantum-mechanical treatment of the electronic energy levels of a hydrogen molecule. Figure 3.11 shows graphs of $E_n(x)$ plotted against x for various values of n.

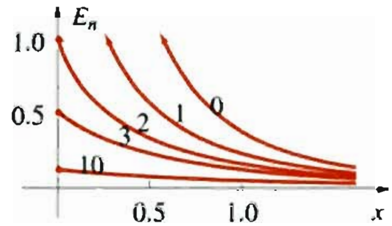

Figure 3.11
The functions $E_n(x)$ defined by Equation 1 plotted against x.

Some special values of $E_n(x)$ are

$$E_n(0) = \frac{1}{n-1} \qquad n \geq 2$$

while $E_0(0)$ and $E_1(0)$ diverge at $x = 0$. In fact,

$$E_0(x) = \frac{e^{-x}}{x} \tag{2}$$

so it diverges as $1/x$ as $x \to 0$.

We can derive a recurrence relation between $E_{n+1}(x)$ and $E_n(x)$ (Problem 1)

$$E_{n+1}(x) = \frac{1}{n}[e^{-x} - xE_n(x)] \tag{3}$$

by integrating Equation 1 by parts. We can also derive the relation

$$\frac{dE_n(x)}{dx} = -E_{n-1}(x) \tag{4}$$

by differentiating with respect to x under the integral sign in Equation 1 (Problem 11).

Some authors call $E_1(x)$ the *exponential integral* and others call the closely related integral

$$Ei(x) = \int_{-\infty}^{x} \frac{e^u}{u}\,du \qquad x > 0 \tag{5}$$

the exponential integral. Figure 3.12 shows $Ei(x)$ plotted against x. Note that $Ei(x) \to -\infty$ as $x \to 0$ and $\to \infty$ as $x \to \infty$. Both $E_1(x)$ and $Ei(x)$ are well tabulated and are built into most CAS. The two functions are related to each other, but we must wait until the next chapter, when we study complex numbers, to see what the relationship is.

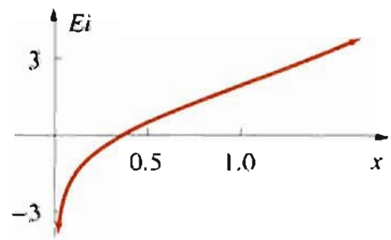

Figure 3.12
The function $Ei(x)$ defined by Equation 5 plotted against x.

The series expansion of $E_1(x)$ introduces us to a fundamental numerical constant called *Euler's constant*. If we denote this constant by γ, then

$$E_1(x) = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n\,n!} \qquad x > 0 \tag{6}$$

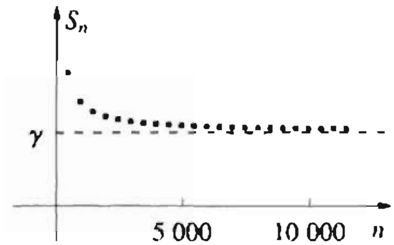

Figure 3.13
The function $\sum_{k=1}^{n} \frac{1}{k} - \ln n$ plotted against n. The limiting value is equal to Euler's constant.

where

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = 0.577\,215 \tag{7}$$

The limit in Equation 7 is of the indeterminate form $\infty - \infty$, but the limit does exist. Figure 3.13 shows the right side of Equation 7 plotted against n. Note that Equation 7 suggests that the harmonic series diverges as $\ln n$. The series expansion of $Ei(x)$ is somewhat similar to that of $E_1(x)$:

$$Ei(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n\, n!} \tag{8}$$

Note that $E_1(x)$ is almost equal to $-Ei(-x)$, except for the $\ln x$ term. We say "almost" because $\ln(-x)$ is undefined for positive values of x, so the "almost" relation is meaningless for positive values of x. We'll see in the next chapter, however, that we *can* give a meaning to $\ln(-x)$ if we allow x to take on complex values.

In Section 2.9, we showed that the asymptotic series for $E_1(x)$ (in different notation) is

$$E_1(x) \sim \frac{e^{-x}}{x} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n n!}{x^n} \right] \tag{9}$$

This large x behavior can be seen in Figure 3.11. The asymptotic expansion for $E_n(x)$ is (Problem 4)

$$E_n(x) \sim \frac{e^{-x}}{x} \left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} - \frac{n(n+1)(n+2)}{x^3} + \cdots \right] \tag{10}$$

A number of definite integrals can be expressed in terms of $E_1(x)$ and $Ei(x)$.

Example 1:
Show that

$$\int_0^{\infty} \frac{e^{-at}}{b+t} dt = e^{ab} E_1(ab)$$

SOLUTION: Let $b + t = u$ and write

$$\int_0^{\infty} \frac{e^{-at}}{b+t} dt = e^{ab} \int_b^{\infty} \frac{e^{-au}}{u} du$$

Then let $u = bz$ to obtain

$$e^{ab} \int_b^{\infty} \frac{e^{-au}}{u} du = e^{ab} \int_1^{\infty} \frac{e^{-abz}}{z} dz = e^{ab} E_1(ab)$$

Example 2:
Show that

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \ln \frac{b}{a}$$

SOLUTION:

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \lim_{x\to 0}\left[\int_x^\infty \frac{e^{-at}}{t} dt - \int_x^\infty \frac{e^{-bt}}{t} dt\right]$$
$$= \lim_{x\to 0}[E_1(ax) - E_1(bx)]$$

Now use the series expansion for $E_1(x)$ given by Equation 6 to write

$$\lim_{x\to 0}[E_1(ax) - E_1(bx)] = \lim_{x\to 0}[-\ln(ax) + \ln(bx) + O(x)]$$
$$= \ln \frac{b}{a}$$

Two other integrals that are usually discussed together with the exponential integral [either $E_1(x)$ or $E_i(x)$] are the *sine integral* and the *cosine integral*, defined by

$$Si(x) = \int_0^x \frac{\sin u}{u} du \qquad x > 0 \tag{11}$$

and

$$Ci(x) = -\int_x^\infty \frac{\cos u}{u} du \qquad x > 0 \tag{12}$$

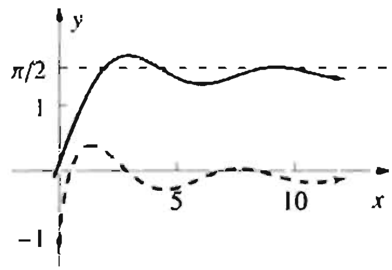

Figure 3.14
The sine integral, $Si(x)$ (solid), defined by Equation 11, and the cosine integral, $Ci(x)$ (dashed), defined by Equation 12, plotted against x.

The graphs of these two functions are shown in Figure 3.14.

The series expansions of $Si(x)$ and $Ci(x)$ are (Problem 5)

$$Si(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \tag{13}$$

$$Ci(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)(2n)!} \tag{14}$$

Example 3:
Show that

$$\lim_{x\to\infty} Si(x) = \int_0^\infty \frac{\sin u}{u} du = \frac{\pi}{2}$$

SOLUTION: Let $\frac{1}{u} = \int_0^\infty e^{-uz} dz$ and interchange orders of integration to get

$$\int_0^\infty \frac{\sin u}{u} du = \int_0^\infty \sin u \left[\int_0^\infty e^{-uz} dz \right] du$$
$$= \int_0^\infty \left[\int_0^\infty e^{-uz} \sin u \, du \right] dz$$
$$= \int_0^\infty \frac{dz}{1+z^2} = \left[\tan^{-1} z \right]_0^\infty = \frac{\pi}{2}$$

3.4 Problems

1. Derive Equations 3 and 4. (See also Problem 11.)
2. Figure 3.11 suggests that $E_n(x) > E_{n+1}(x)$. Prove it.
3. Show that $Ei(x)$ can also be written as $Ei(x) = -\int_{-x}^\infty \frac{e^{-t}}{t} dt$.
4. Derive an asymptotic expansion of $E_n(x)$.
5. Derive the series expansion of $Si(x)$, Equation 13.
6. Given that $\lim_{\epsilon \to 0} \int_\epsilon^\infty e^{-x} \ln x \, dx = -\gamma$, show that $\int_0^x \frac{1-e^{-t}}{t} dt = \gamma + \ln x + E_1(x)$.
7. Given that $\lim_{\epsilon \to 0} \int_\epsilon^\infty e^{-x} \ln x \, dx = -\gamma$, show that $\int_0^x \frac{e^t - 1}{t} dt = Ei(x) - \ln x - \gamma$.
8. Given that $\lim_{\epsilon \to 0} \int_\epsilon^\infty e^{-x} \ln x \, dx = -\gamma$, show that $\int_0^x \frac{\cos u - 1}{u} du = Ci(x) - \gamma - \ln x$.
9. Show that $\int_0^\infty e^{-at} Ci(t) \, dt = -\frac{1}{2a} \ln(1 + a^2)$.
10. The following integral occurs in the theory of the conductance of solutions of strong electrolytes:

$$S_n = R^{n-2} \int_R^\infty \frac{e^{-n\kappa r}}{r^{n-1}} dr \qquad n = 1, 2, \ldots$$

where R is the average radius of the ions and κ (> 0) is a known parameter. First show that S_n is a function of only κR. Now show that $S_1(\kappa R) = \frac{e^{-\kappa R}}{\kappa R}$ and $S_2(\kappa R) = E_1(2\kappa R)$.

11. Justify differentiating under the integral sign in Equation 1 to derive Equation 4.
12. Another expression for Euler's constant is $\gamma = -\lim_{\epsilon \to 0} \int_\epsilon^\infty e^{-x} \ln x \, dx$. Use a CAS to show that $\gamma = 0.5772$ to four places.
13. Show that $\int_0^\infty e^{-ax} E_1(x) \, dx = \frac{1}{a} \ln(1 + a)$.

14. To derive Equation 6, you need to use the relation $\int_\epsilon^\infty e^{-x} \ln x \, dx = \int_\epsilon^\infty \frac{e^{-x}}{x} dx + \ln \epsilon$. Verify this relation.

15. Use the relation in the previous problem to derive Equation 6.

3.5 Elliptic Integrals

We'll start this section with a discussion of a physical problem that leads naturally to an elliptic integral. Consider a pendulum consisting of a mass m attached to the end of a rigid, massless rod that is suspended from a fixed point (Figure 3.15). Assuming that the oscillatory motion of the mass takes place in a single plane, the kinetic energy is given by

$$\mathcal{K} = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{ml^2}{2}\left(\frac{d\theta}{dt}\right)^2 \tag{1}$$

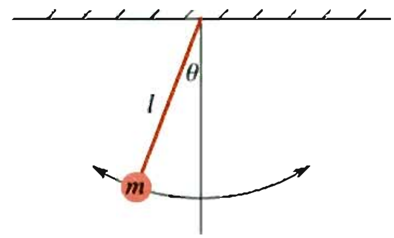

Figure 3.15
The geometry of the pendulum discussed in Equations 1 through 6.

where s is the distance along the arc, l is the length of the rod, and θ is the angle that the rod makes with the vertical. The potential energy of the mass is mg times its height above its vertical position, or

$$V(\theta) = mg(l - l\cos\theta) = mgl(1 - \cos\theta) \tag{2}$$

The total energy of the mass is $\mathcal{E} = \mathcal{K} + V$, or

$$\frac{ml^2}{2}\left(\frac{d\theta}{dt}\right)^2 + mgl(1 - \cos\theta) = \mathcal{E} \tag{3}$$

We can determine $\mathcal{E}$ from an initial condition, such as when the mass is started from rest at $\theta = \theta_0$, so that $d\theta/dt = 0$ at that time. The initial total energy $\mathcal{E}_0$ is

$$\mathcal{E}_0 = mgl(1 - \cos\theta_0)$$

The energy is conserved (Problem 2), so $\mathcal{E} = \mathcal{E}_0$ in Equation 3, and we have

$$\frac{ml^2}{2}\left(\frac{d\theta}{dt}\right)^2 + mgl(1 - \cos\theta) = mgl(1 - \cos\theta_0)$$

at any time, or

$$\frac{d\theta}{dt} = \left(\frac{2g}{l}\right)^{1/2}(\cos\theta - \cos\theta_0)^{1/2} \tag{4}$$

The mass will oscillate from θ_0 to $-\theta_0$ and back to θ_0 repeatedly, and one-half of the period of oscillation is the time that it takes to go from θ_0 to $-\theta_0$ (or from $-\theta_0$ to θ_0). If we solve Equation 4 for dt and integrate from $-\theta_0$ to θ_0, then we have that $\tau/2$, one-half the period of oscillation, is given by

$$\frac{\tau}{2} = \left(\frac{l}{2g}\right)^{1/2} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{(\cos\theta - \cos\theta_0)^{1/2}}$$

or

$$\tau = \left(\frac{2l}{g}\right)^{1/2} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{(\cos\theta - \cos\theta_0)^{1/2}} = \left(\frac{8l}{g}\right)^{1/2} \int_{0}^{\theta_0} \frac{d\theta}{(\cos\theta - \cos\theta_0)^{1/2}} \tag{5}$$

where in the last step we have used the fact that the integrand is an even function of θ because $\cos(-\theta) = \cos(\theta)$.

Equation 5 is a little awkward to use numerically in general because of the singularity at $\theta = \theta_0$. It is customary (and very convenient) to transform the integral in Equation 5 so that Equation 5 becomes (Problem 4)

$$\tau = 4\left(\frac{l}{g}\right)^{1/2} \int_{0}^{\pi/2} \frac{du}{(1 - k^2\sin^2 u)^{1/2}} \tag{6}$$

with $k = \sin(\theta_0/2)$.

We can investigate Equation 5 for small values of θ_0, where the pendulum undergoes harmonic motion. If θ_0 is small enough [and consequently $k = \sin(\theta_0/2)$ is also small enough] that we can neglect $k^2\sin^2 u$ in the denominator, then $\tau_0 = 2\pi(l/g)^{1/2}$, the period of a simple harmonic pendulum. Otherwise, τ represents the period of a pendulum of arbitrary amplitude.

The integral in Equation 6 cannot be evaluated in terms of elementary functions (unless $k = 0$) and is called a *complete elliptic integral of the first kind*. It is usually written as

$$K(k) = K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \qquad 0 \le k < 1 \tag{7}$$

The "complete" in the name implies that the upper limit is $\pi/2$. If the upper limit is arbitrary, then we have an *incomplete elliptic integral of the first kind*:

$$F(k,\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \qquad 0 \le k < 1 \tag{8}$$

As you may have guessed from the name, there are also complete and incomplete elliptic integrals of the second kind, namely

$$E(k) = E = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\theta}\, d\theta \qquad 0 \le k \le 1 \tag{9}$$

and

$$E(k,\phi) = \int_0^{\phi} \sqrt{1 - k^2\sin^2\theta}\, d\theta \qquad 0 \le k \le 1 \tag{10}$$

All these elliptic integrals are well tabulated and Figures 3.16 and 3.17 show $K(k)$ and $E(k)$ plotted against k. Some special values of $K(k)$ and $E(k)$ are (Problem 5)

$$K(0) = \frac{\pi}{2}; \quad K(1) = \infty; \quad E(0) = \frac{\pi}{2}; \quad E(1) = 1 \tag{11}$$

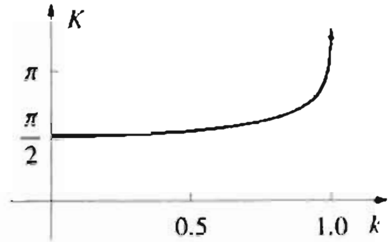

Figure 3.16
The complete elliptic integral of the first kind, $K(k)$, defined by Equation 7, plotted against k.

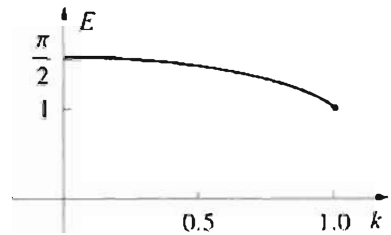

Figure 3.17
The complete elliptic integral of the second kind, $E(k)$, defined by Equation 9, plotted against k.

Example 1:
Evaluate $F(0, \phi)$, $E(0, \phi)$, $F(1, \phi)$, and $E(1, \phi)$.

SOLUTION: Equations 8 and 10, with $k = 0$ and $k = 1$, yield

$$F(0, \phi) = \int_0^{\phi} d\theta = \phi$$

$$E(0, \phi) = \int_0^{\phi} d\theta = \phi$$

$$F(1, \phi) = \int_0^{\phi} \frac{d\theta}{\cos\theta} = \Big[\ln(\sec\theta + \tan\theta) \Big]_0^{\phi}$$

$$= \ln(\sec\phi + \tan\phi)$$

$$E(1, \phi) = \int_0^{\phi} \cos\theta \, d\theta = \sin\phi$$

Note that we obtain Equations 11 if $\phi = \pi/2$.

Returning to the pendulum problem that we used to introduce this section, we see that the ratio of the period τ, given by Equation 6, to the period of a simple harmonic pendulum, $\tau_0 = 2\pi (l/g)^{1/2}$, is given by

$$\frac{\tau}{\tau_0} = \frac{2}{\pi} K(k) \tag{12}$$

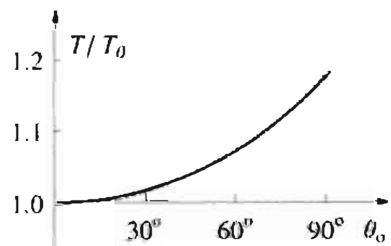

Figure 3.18
The ratio of the periods of a general oscillator and a harmonic oscillator plotted against θ_0, the initial angle in Figure 3.15.

where $k = \sin(\theta_0/2)$. Equation 12 is plotted against θ_0 in Figure 3.18, showing the deviation from simple harmonic motion as θ_0 increases.

We can derive series expansions of the elliptic integrals by using the binomial series in each case to expand in powers of k^2. For example,

$$K(k) = \int_0^{\pi/2} d\theta \left(1 + \frac{1}{2} k^2 \sin^2\theta + \frac{1 \cdot 3}{2^2\, 2!} k^4 \sin^4\theta + \frac{1 \cdot 3 \cdot 5}{2^3\, 3!} k^6 \sin^6\theta + \cdots \right)$$

Using

$$\int_0^{\pi/2} \sin^{2n}\theta \, d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \left(\frac{\pi}{2} \right) \qquad n = 1, 2, \ldots$$

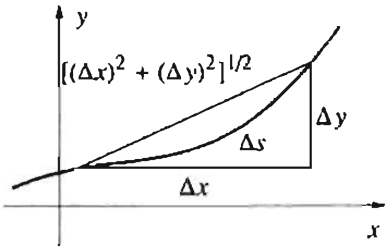

Figure 3.19
The geometry associated with the calculation of arc length.

we have

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right] \tag{13}$$

Similarly, (Problem 8)

$$E(k) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} + \cdots\right] \tag{14}$$

The elliptic integral of the second kind arises naturally when we calculate the length of the arcs of certain curves. Figure 3.19 shows the geometry associated with the calculation of the length of the arc of a curve. If the two ends of the arc are given by $x = x_1$ and $x = x_2$, then l, the length of the arc, is given by

$$l = \int_{x=x_1}^{x=x_2} ds = \int_{x=x_1}^{x=x_2} [(dx)^2 + (dy)^2]^{1/2} \tag{15}$$

If x and y are given parametrically by $x = x(t)$ and $y = y(t)$, then Equation 15 becomes

$$l = \int_{t_1}^{t_2} \{[x'(t)]^2 + [y'(t)]^2\}^{1/2} dt \tag{16}$$

or, if y is given as a function of x, we can use the differential form $dy = f'(x)dx$ to write Equation 15 as

$$l = \int_{x_1}^{x_2} \{1 + [f'(x)]^2\}^{1/2} dx \tag{17}$$

Let's use these equations to calculate the arc lengths of some curves. For example, let's use Equation 16 to calculate the circumference of a circle. The equation of a circle of radius a, $x^2 + y^2 = a^2$, can be expressed parametrically by the two equations, $x = a\cos\theta$ and $y = a\sin\theta$, where $0 \le \theta \le 2\pi$. Substitute these relations into Equation 16 with $t = \theta$ to get

$$l = \int_0^{2\pi} (a^2 \sin^2\theta + a^2\cos^2\theta)^{1/2} d\theta = 2\pi a$$

That was easy. Now let's calculate the circumference of an ellipse.

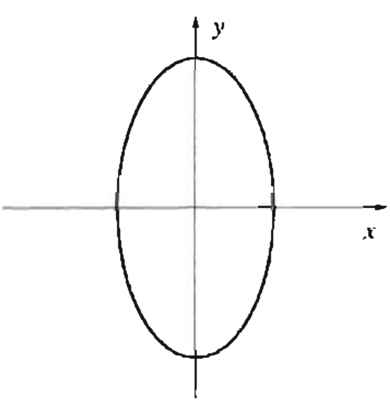

Figure 3.20
An ellipse with the major axis along the y axis.

Example 2:
An ellipse can be described by the parametric equations $x = a\cos\theta$ and $y = b\sin\theta$ for $0 \le \theta \le 2\pi$. (Figure 3.20 shows an ellipse for the case in which $b > a$.) Calculate the circumference of the ellipse.

SOLUTION: Using Equation 16 with $t = \theta$,

$$\begin{aligned} I &= \int_0^{2\pi} [a^2 \sin^2\theta + b^2\cos^2\theta]^{1/2} d\theta \\ &= \int_0^{2\pi} [b^2 - (b^2 - a^2)\sin^2\theta]^{1/2} d\theta \\ &= b\int_0^{2\pi} \left[1 - \left(\frac{b^2-a^2}{b^2}\right)\sin^2\theta\right]^{1/2} d\theta \end{aligned}$$

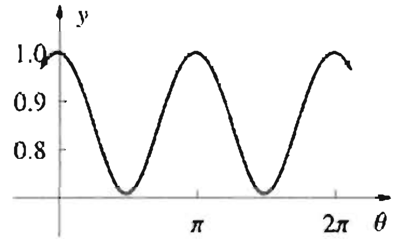

Figure 3.21
An illustration of the fact that $y(\theta) = (1 - k^2\sin^2\theta)^{1/2}$ is periodic with a period π. In this figure $k^2 = 0.50$.

This integral would be of the form of Equation 9 if the upper limit were $\pi/2$ instead of 2π. We can get an upper limit of $\pi/2$ by either calculating 1/4 of the circumference ($0 \le \theta \le \pi/2$) and then multiplying by 4, or by realizing that the areas in the intervals $(0, \pi/2)$, $(\pi/2, \pi)$, $(\pi, 3\pi/2)$, and $(3\pi/2, 2\pi)$ are the same. (See Figure 3.21.) Either way, we get

$$I = 4bE\left(\sqrt{b^2 - a^2}\,/b\right) = 4bE(e)$$

where $e = \sqrt{b^2 - a^2}\,/b$ is the eccentricity of the ellipse. Note that $I = 2\pi b$ if $a = b$.

Suppose that $a > b$ in Example 2. In that case, we have

$$I = 4b\int_0^{\pi/2} \left[1 + \left(\frac{a^2-b^2}{b^2}\right)\sin^2\theta\right]^{1/2} d\theta \tag{18}$$

This integral is not of the form of Equation 9, but we can transform it into the form of Equation 9 by first using the relation $\sin^2\theta = 1 - \cos^2\theta$ to get

$$I = 4b\int_0^{\pi/2} \left[\frac{a^2}{b^2} - \left(\frac{a^2-b^2}{b^2}\right)\cos^2\theta\right]^{1/2} d\theta \tag{19}$$

To transform $\cos^2\theta$ to $\sin^2\theta$, simply let $\theta = \pi/2 - \phi$ [because $\cos(\frac{\pi}{2} - \phi) = \sin\phi$]. Equation 19 becomes

$$\begin{aligned} I &= 4b\int_0^{\pi/2} \left[\frac{a^2}{b^2} - \left(\frac{a^2-b^2}{b^2}\right)\sin^2\phi\right]^{1/2} d\phi \\ &= 4a\int_0^{\pi/2} \left[1 - \left(\frac{a^2-b^2}{a^2}\right)\sin^2\phi\right]^{1/2} d\phi \\ &= 4aE\left(\sqrt{a^2-b^2}/a\right) = 4aE(e) \end{aligned} \tag{20}$$

where the eccentricity $e = \sqrt{a^2 - b^2}/a$ in this case. Equation 20 is exactly the answer obtained in Example 2 with a and b interchanged.

We can use such manipulation to evaluate many integrals involving square roots of trigonometric functions in terms of elliptic integrals.

Example 3:
Evaluate the integral

$$I = \int_0^{\pi/2} (1 + 3\sin^2\theta)^{1/2} d\theta$$

in terms of a complete elliptic integral of the second kind.

SOLUTION: Following the same transformation as above, let $\sin^2\theta = 1 - \cos^2\theta$ to get

$$I = \int_0^{\pi/2} (4 - 3\cos^2\theta)^{1/2} d\theta$$

Now transform $\cos^2\theta$ to $\sin^2\theta$ by letting $\theta = \pi/2 - \phi$ to obtain

$$I = \int_0^{\pi/2} (4 - 3\sin^2\theta)^{1/2} d\theta = 2\int_0^{\pi/2} \left(1 - \frac{3}{4}\sin^2\phi\right)^{1/2} d\phi$$
$$= 2E(\sqrt{3}/2) = 2.42211\cdots$$

(See Problem 11.)

Many types of integrals involving square roots of trigonometric functions can be evaluated in terms of elliptic integrals. We won't illustrate any more here, but the reference to Spiegel at the end of the chapter has many different examples.

We can also use elliptic integrals to evaluate integrals involving square roots of polynomials. Let $x = \sin\theta$ in Equations 8 and 10 to get (Problem 15)

$$F(k, z) = \int_0^z \frac{dx}{(1 - x^2)^{1/2}(1 - k^2x^2)^{1/2}} \tag{21}$$

and

$$E(k, z) = \int_0^z \left(\frac{1 - k^2x^2}{1 - x^2}\right)^{1/2} dx \tag{22}$$

where $z = \sin\phi$. These become complete elliptic integrals if $z = \sin(\pi/2) = 1$. It turns out that integrals of the type

$$I = \int_a^b \frac{dx}{\sqrt{P(x)}}$$

where $P(x)$ is a third or fourth degree polynomial with real zeros [i.e. real roots of $P(x) = 0$] can be transformed into elliptic integrals. This is a vast subject, but we'll give just one example here.

Example 4:
Evaluate

$$I = \int_0^\infty \frac{dx}{\sqrt{(1+x^2)(1+2x^2)}}$$

in terms of an elliptic integral.

SOLUTION: Let $x = \tan\theta$ to obtain

$$I = \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sqrt{(1+\tan^2\theta)(1+2\tan^2\theta)}}$$

$$= \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2\theta + 2\sin^2\theta}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2\theta}}$$

This last integral is similar to the one in Example 3. Using the same substitutions gives

$$I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{2-\sin^2\theta}} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$$

$$= \frac{1}{\sqrt{2}} K(1/\sqrt{2}) = 1.31103\cdots$$

(See Problem 13.)

Once again, the reference to Spiegel presents many examples and the reference to Hancock is an entire book on the subject.

3.5 Problems

1. Derive the equation of motion ($F = ma$) of the pendulum shown in Figure 3.15.

2. Problem 1 shows that the equation of motion of the pendulum shown in Figure 3.15 is $ml\dfrac{d^2\theta}{dt^2} = -mg\sin\theta$. Show that energy is conserved for this system. *Hint*: Multiply both sides by $d\theta/dt$, and use the relation $\dfrac{d}{dt}\left(\dfrac{d\theta}{dt}\right)^2 = 2\dfrac{d\theta}{dt}\dfrac{d^2\theta}{dt^2}$ and then integrate both sides.

3. Let θ be restricted to small angles in Equation 4 and show that the motion is cosinusoidal with a period $\tau = 2\pi(l/g)^{1/2}$.

4. We'll derive Equation 6 in this problem. First use the trigonometric identity $\cos\theta = 1 - 2\sin^2(\theta/2)$ to write Equation 5 as

$$\tau = \left(\frac{4l}{g}\right)^{1/2} \int_0^{\theta_0} \frac{d\theta}{[\,\sin^2(\theta_0/2) - \sin^2(\theta/2)\,]^{1/2}}$$

Now transform the θ_0 upper limit in this equation to $\pi/2$ by defining a new integration variable u by the relation $\sin(\theta/2) = \sin u \sin(\theta_0/2)$ and derive Equation 6. (Note that $\sin u = 1$ or $u = \pi/2$ when $\theta = \theta_0$.)

5. Show that $K(0) = \pi/2$, $E(0) = \pi/2$, $K(1) = \infty$, and $E(1) = 1$.

6. Calculate the arc length of the curve $y = \cos\theta$ from 0 to 2π.

7. Use Example 2 to calculate the circumference of the ellipse whose equation is $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$.

8. Show that $E(k) = \dfrac{\pi}{2}\left[1 - \left(\dfrac{1}{2}\right)k^2 + \left(\dfrac{1\cdot 3}{2\cdot 4}\right)\dfrac{k^4}{3} - \left(\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\dfrac{k^6}{5} + \cdots\right]$.

9. Evaluate $I = \displaystyle\int_0^{\pi/2} \frac{d\theta}{(3 - \cos\theta)^{1/2}}$ in terms of elliptic integrals.

10. The answer to the previous problem is $K(1/\sqrt{2}) - F(1/\sqrt{2}, \pi/4)$. Use either tables or a CAS to assign a numerical value to this result.

11. Use either tables or a CAS to verify that the answer to Example 3 is $2.42211\cdots$.

12. Use a CAS to evaluate the integral in Example 3 directly and compare your answer to that in the previous problem.

13. Use either tables or a CAS to verify that the answer to Example 4 is $1.31103\cdots$.

14. Use a CAS to evaluate the integral in Example 4 directly and compare your answer to that in the previous problem.

15. Derive Equations 21 and 22.

16. Evaluate $I = \displaystyle\int_0^3 \frac{dx}{\sqrt{(16 - x^2)(9 - x^2)}}$ in terms of elliptic integrals. *Hint*: Let $x = 3\sin\theta$.

17. Evaluate $I = \displaystyle\int_2^\infty \frac{dx}{(x^2 - 4)^{1/2}(x^2 + 1)^{1/2}}$ in terms of elliptic integrals. *Hint*: Let $x = 2\sec\theta$.

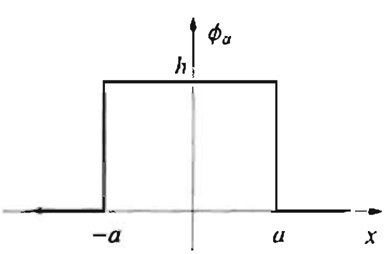

Figure 3.22
The function $\phi_a(x)$ defined by Equation 1 plotted against x.

3.6 The Dirac Delta Function

In this section we will introduce a famous function that is not a function. Consider the function defined by (Figure 3.22)

$$\phi_a(x) = \begin{cases} 0 & x < -a \\ h & -a < x < a \\ 0 & x > a \end{cases} \tag{1}$$

where h and a are related such that

$$\int_{-\infty}^{\infty} \phi_a(x)dx = 2ah = 1 \tag{2}$$

Now let's form the integral

$$I = \int_{-\infty}^{\infty} \phi_a(x) f(x) dx$$

where $f(x)$ is a continuous function. Because of the definition of $\phi_a(x)$,

$$I = \int_{-a}^{a} \phi_a(x) f(x) dx$$

Because $f(x)$ is continuous, we can use the mean value theorem for integrals and Equation 2 to write

$$I = f(\xi) \int_{-a}^{a} \phi_a(x) dx = f(\xi)$$

where $-a \leq \xi \leq a$. As $a \to 0$, $\xi \to 0$ and I becomes

$$\lim_{a\to 0} I = \lim_{a\to 0} f(\xi) \int_{-a}^{a} \phi_a(x)\, dx = \lim_{a\to 0} f(\xi) \int_{-a}^{a} \frac{dx}{2a} = f(0) \tag{3}$$

By multiplying $f(x)$ by $\phi_a(x)$ and then letting $a \to 0$ and $h \to \infty$, such that $2ah = 1$, we have sifted out the value of $f(x)$ at $x = 0$.

We can use a construction like $\phi_a(x)$ to sift out $f(x)$ at any value of x. To isolate $f(x_0)$, let

$$\phi_{a0}(x) = \begin{cases} 0 & x < x_0 - a \\ h & x_0 - a < x < x_0 + a \\ 0 & x_0 + a < x \end{cases} \tag{4}$$

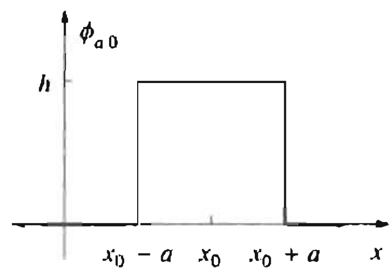

Figure 3.23
The function $\phi_{a0}(x)$ defined by Equation 4 plotted against x.

with $2ah = 1$ (Figure 3.23). Equation 4 simply defines $\phi_a(x)$ to be centered at x_0 rather than at $x = 0$. Clearly

$$\lim_{a\to 0} \int_{-\infty}^{\infty} \phi_{a0}(x) f(x) dx = f(x_0) \tag{5}$$

In the limiting process in Equation 5, $\phi_{a0}(x)$ is getting increasingly narrow and increasingly tall such that the area of the rectangle $2ah = 1$. We denote this limiting "function" by $\delta(x - x_0)$, and write

$$\delta(x - x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases} \tag{6}$$

always keeping in mind that Equation 6 is a shorthand notation for $\phi_{a0}(x)$ as $a \to 0$ and $h \to \infty$ such that $2ah = 1$. Physically, Equation 6 represents a spike at $x = x_0$

or an impulsive force if x represents time. In terms of $\delta(x - x_0)$, Equations 2 and 5 read

$$\int_{-\infty}^{\infty} \delta(x - x_0)dx = 1 \tag{7}$$

and

$$\int_{-\infty}^{\infty} \delta(x - x_0)f(x)dx = f(x_0) \tag{8}$$

where $f(x)$ is a continuous function. Equations 7 and 8 serve to define the *Dirac delta function*, which was introduced by the British theoretical physicist Paul Dirac in 1927. Equation 8 illustrates the *sifting property* of the delta function.

Example 1:

Evaluate $I = \int_{-\infty}^{\infty} \delta(x - x_0)e^{-x^2}dx$.

SOLUTION: The function e^{-x^2} is continuous, so we simply replace x by x_0 in e^{-x^2} to get

$$I = e^{-x_0^2}$$

Another function that can act as a delta function in a limiting process is the Gaussian distribution function:

$$p_\sigma(x) = \frac{1}{(2\pi\sigma^2)^{1/2}}e^{-(x-x_0)^2/2\sigma^2} \tag{9}$$

which we discussed in Section 3. Figure 3.24 shows $p_\sigma(x)$ plotted against x for several values of σ. The function is centered at $x = x_0$ and the width about x_0 is governed by the value of σ: the smaller σ is, the narrower and more peaked is $p_\sigma(x)$. The factor of $1/(2\pi\sigma^2)^{1/2}$ in front assures that

$$\int_{-\infty}^{\infty} p_\sigma(x)dx = 1$$

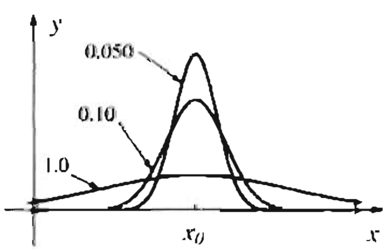

Figure 3.24
The Gaussian distribution in Equation 9 plotted against x for $\sigma = 1.0$, 0.10, and 0.050 (see Figure 3.9).

The curves in Figure 3.24 get both narrower and taller with decreasing values of σ because the areas under the curves are all the same (equal to 1).

If we multiply a continuous function $f(x)$ by $p_\sigma(x)$ in Equation 9 and let $\sigma \to 0$, then $p_\sigma(x)$ approaches $\delta(x - x_0)$:

$$\lim_{\sigma\to 0}(2\pi\sigma^2)^{-1/2}\int_{-\infty}^{\infty} f(x)e^{-(x-x_0)^2/2\sigma^2}dx = f(x_0)$$

Example 2:
Evaluate

$$I(\sigma) = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} e^{-(x-x_0)^2/2\sigma^2} \cos x \, dx$$

explicitly and then let $\sigma \to 0$ to show that $\lim_{\sigma \to 0} I(\sigma) = \cos x_0$.

SOLUTION:

$$I(\sigma) = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2} \cos(z + x_0) \, dz$$

$$= (2\pi\sigma^2)^{-1/2} \left[\cos x_0 \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2} \cos z \, dz - \sin x_0 \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2} \sin z \, dz \right]$$

The first integral is equal to $2\left(\frac{\pi}{2}\right)^{1/2} \sigma e^{-\sigma^2/2}$ and the second integral is equal to zero because the integrand is an odd function of x. Thus,

$$I(\sigma) = e^{-\sigma^2/2} \cos x_0$$

and so

$$\lim_{\sigma \to 0} I(\sigma) = \cos x_0$$

We can present Equations 4 and 9 in a more general way. From Equation 4, define

$$\delta_n(x) = \begin{cases} 0 & x < -\frac{a}{n} \\ \frac{n}{2a} & -\frac{a}{n} < x < \frac{a}{n} \\ 0 & x > \frac{a}{n} \end{cases} \tag{10}$$

and from Equation 9, define

$$\delta_n(x - x_0) = \frac{n}{\sqrt{\pi}} e^{-n^2(x-x_0^2)} \qquad -\infty < x < \infty \tag{11}$$

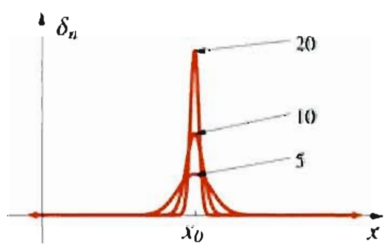

Figure 3.25
The delta sequence in Equation 11 plotted against x for $n = 5$, 10, and 20.

The graphs of the sequences $\delta_n(x - x_0)$ in each case get taller and narrower as n increases (Figure 3.25). In both cases, $\int_{-\infty}^{\infty} \delta_n(x - x_0) dx = 1$ for any value of n, and

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \delta_n(x - x_0) \, f(x) \, dx = f(x_0)$$

for any continuous function $f(x)$. Consequently, we say that Equations 10 and 11 are *delta sequences* in the sense that $\delta_n(x - x_0) \to \delta(x - x_0)$ as $n \to \infty$. They both have the property that

$$\int_{-\infty}^{\infty} \delta_n(x - x_0)\, dx = 1 \tag{12}$$

and that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x - x_0) f(x)\, dx = f(x_0) \tag{13}$$

Another delta sequence is (Problem 4)

$$\delta_n(x - x_0) = \frac{n}{\pi[1 + n^2(x - x_0)^2]} \qquad -\infty < x < \infty \tag{14}$$

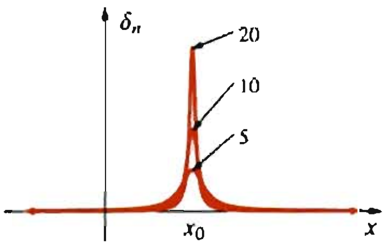

Figure 3.26
The delta sequence in Equation 14 plotted against x for $n = 5$, 10, and 20.

Equation 14 is plotted against $x - x_0$ for various values of n in Figure 3.26.

The delta function is not a function in the strict sense, but has meaning only if it occurs multiplying a continuous function under an integral sign. For example, we can assign a meaning to $x\delta(x)$ by multiplying by a continuous function $f(x)$ and integrating to get

$$\int_{-\infty}^{\infty} f(x)\, x\, \delta(x)\, dx = 0 \cdot f(0) \cdot 1 = 0 \tag{15}$$

and so one often sees the expression

$$x\delta(x) = 0 \tag{16}$$

Keep in mind, however, that Equation 16 is a shorthand notation for Equation 15. Similarly, we write

$$x\delta'(x) = -\delta(x) \tag{17}$$

and

$$\delta(ax) = \frac{1}{a}\delta(x) \tag{18}$$

where, once again, you must be aware of the meaning of these relations (Problem 1).

3.6 Problems

1. Write out the actual meanings of Equations 17 and 18.
2. Show that $x\delta'(x) = -\delta(x)$.
3. Show that $\delta(ax) = a^{-1}\delta(x)$.

4. Show that Equation 14 is a delta sequence; in other words, show that $\int_{-\infty}^{\infty} \delta_n(x - x_0)\, dx = 1$ and that $\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x - x_0)\, f(x)\, dx = f(x_0)$ if $f(x)$ is continuous.

5. Use the delta sequence in Equation 14 to prove that $\int_{-\infty}^{x} \delta(u)du = H(x)$, where $H(x)$ is the Heaviside step function.

6. Multiply $\delta_n(x)$ in Equation 14 (with $x_0 = 0$) by e^{-x^2}, integrate from $-\infty$ to ∞, and then let $n \to \infty$ to show that $\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x)e^{-x^2}dx = 1$. *Hint*: You need to use $\int_0^{\infty} \frac{e^{-at^2}}{t^2 + x^2}dt = \frac{\pi}{2x}e^{ax^2}\, \text{erfc}(\sqrt{a}x)$.

7. Notice that the delta sequences in Equations 10–12 are of the form $p(nx)$, where $\int_{-\infty}^{\infty} p(x)dx = 1$. Show that $\int_{-\infty}^{\infty} f(x)p[n(x - x_0)]\,dx = f(x_0)$ if $f(x)$ is continuous.

8. Evaluate $I(\sigma) = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} e^{-(x-x_0)^2/2\sigma^2} \sin x\, dx$ explicitly and then let $\sigma \to 0$ to show that $\lim_{\sigma\to 0} I(\sigma) = \sin x_0$.

3.7 Bernoulli Numbers and Bernoulli Polynomials

In this section, we will first introduce the *Riemann zeta function*, which is not defined by an integral but by a summation. If we denote the Riemann zeta function by $\zeta(s)$, then

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad s > 1 \tag{1}$$

We know from Chapter 2 that $\zeta(s)$ diverges for $s \le 1$, being the harmonic series when $s = 1$. We also used the fact that $\zeta(4) = \pi^4/90$ in Section 2.8. The zeta function has been a wellspring of rich and extensive mathematics when s is allowed to take on complex values, but we shall restrict s to a positive integer, n. We will use $\zeta(n)$ simply to discuss the sums of reciprocal powers of integers, which arise fairly frequently in applied mathematics. We shall evaluate $\zeta(2)$ and $\zeta(4)$ with the aid of Fourier series in Chapter 15:

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

and

$$\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

Table 3.4
The values of $\zeta(n)$, $\lambda(n)$, $\eta(n)$, and $\beta(n)$ (Equations 1 through 4) for $n = 1, 2, \ldots, 10$.

n	$\zeta(n)$	$\lambda(n)$	$\eta(n)$	$\beta(n)$
1	∞	∞	0.693 14	0.785 39
2	1.6449	1.2337	0.822 47	0.915 97
3	1.2020	1.0518	0.901 54	0.968 95
4	1.0823	1.0147	0.947 03	0.988 95
5	1.0369	1.0045	0.972 12	0.996 16
6	1.0173	1.0014	0.985 55	0.998 68
7	1.0083	1.0005	0.992 59	0.999 55
8	1.0041	1.0002	0.996 23	0.999 85
9	1.0020	1.0001	0.998 09	0.999 95
10	1.0010	1.0000	0.999 03	0.999 98

Abramowitz and Stegun (see references at the end of the chapter) define the following auxilary functions involving sums of reciprocal powers of integers:

$$\eta(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \tag{2}$$

$$\lambda(n) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n} \tag{3}$$

and

$$\beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n} \tag{4}$$

The four functions $\zeta(n)$, $\lambda(n)$, $\eta(n)$, and $\beta(n)$ are tabulated in Table 3.4. Note that all of them approach unity very rapidly as n increases.

Example 1:
Show that $\lambda(n) = (1 - 2^{-n})\zeta(n)$.

SOLUTION:

$$\begin{aligned}(1 - 2^{-n})\zeta(n) &= \sum_{k=1}^{\infty} \frac{1}{k^n} - \sum_{k=1}^{\infty} \frac{1}{(2k)^n} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n} + \sum_{k=1}^{\infty} \frac{1}{(2k)^n} - \sum_{k=1}^{\infty} \frac{1}{(2k)^n} \\ &= \lambda(n)\end{aligned}$$

Some special values of Equations 2 through 4 are

$$\eta(2) = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

$$\eta(4) = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots = \frac{7\pi^4}{720}$$

$$\lambda(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

$$\lambda(4) = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

$$\beta(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

$$\beta(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$$

Note that $\eta(1)$ and $\beta(1)$ are conditionally convergent series and recall from Section 2.7 that $\eta(1) = \ln 2$.

We occasionally need expressions for the sums of positive powers of the integers. We showed in Section 1.7 that the sum of the first n integers is given by

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{5}$$

and that the sum of the first n squares is given by

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{6}$$

We derived these two formulas in Section 1.7 by what you might call tricks, but there is actually an organized, straightforward way to derive the general formula for the sum for any integer power. There is an area of mathematics called the *calculus of finite differences*, which deals with sums instead of integrals and differences instead of derivatives. Just as we deal with differential equations in ordinary calculus, we deal with difference equations in the calculus of finite differences (a common type of difference equation is a recurrence relation, where the nth term of a sequence is expressed in terms of the preceding terms). It would take us too much space to develop the tools of the calculus of finite differences here, but we will finish this chapter with some useful results from that field.

If we expand the function

$$f(x,t) = \frac{te^{tx}}{e^t - 1}$$

as a power series in t, the coefficients will be functions of x. In particular,

$$\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}=B_0(x)+B_1(x)t+B_2(x)\frac{t^2}{2!}+\cdots \tag{7}$$

where the $B_n(x)$'s are called *Bernoulli polynomials* and are (Problem 2)

$$\begin{aligned}&B_0(x)=1;\ B_1(x)=x-\frac{1}{2};\ B_2(x)=x^2-x+\frac{1}{6}\\&B_3(x)=x^3-\frac{3x^2}{2}+\frac{x}{2};\ B_4(x)=x^4-2x^3+x^2-\frac{1}{30}\end{aligned} \tag{8}$$

and so on. Equation 7 is called a *generating function* for the Bernoulli polynomials.

Example 2:
Show that $B_n'(x)=nB_{n-1}(x)$.

SOLUTION: Differentiate both sides of Equation 7 with respect to x.

$$\begin{aligned}\frac{t^2e^{tx}}{e^t-1}&=\sum_{n=0}^{\infty}B_n'(x)\frac{t^n}{n!}=B_0'(x)+\sum_{n=1}^{\infty}B_n'(x)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}B_{n+1}'(x)\frac{t^{n+1}}{(n+1)!}\end{aligned}$$

where we used the fact that $B_0(x)=1$. But

$$\frac{t^2e^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^{n+1}}{n!}$$

Equating equal powers of t^{n+1} in the two sums gives

$$\frac{B_n(x)}{n!}=\frac{B_{n+1}'(x)}{(n+1)!}$$

or $B_{n+1}'(x)=(n+1)B_n(x)$, or $B_n'(x)=nB_{n-1}(x)$ for $n\geq 0$.

The calculus of finite differences gives us a general formula for the sum of the mth power of the first n integers in terms of Bernoulli polynomials:

$$\sum_{k=1}^{n}k^m=\frac{B_{m+1}(n+1)-B_{m+1}(0)}{m+1} \tag{9}$$

Note that Equation 9 looks something like an integration formula between the limits of 0 and $n+1$, except in this case, it's a summation formula. Let's use this formula to derive Equations 5 and 6. In the case of Equation 5, $m = 1$, and so

$$S_1 = \sum_{k=1}^{n} k = \frac{B_2(n+1) - B_2(0)}{2}$$

$$= \frac{(n+1)^2 - (n+1)}{2} = \frac{n(n+1)}{2}$$

and for $m = 2$

$$S_2 = \sum_{k=1}^{n} k^2 = \frac{B_3(n+1) - B_3(0)}{3} = \frac{(n+1)^3 - \frac{3}{2}(n+1)^2 + \frac{n+1}{2}}{3}$$

$$= \frac{(n+1)}{3}(n^2 + 2n + 1 - \tfrac{3}{2}n - \tfrac{3}{2} + \tfrac{1}{2})$$

$$= \frac{(n+1)}{3}\left(n^2 + \frac{n}{2}\right) = \frac{n(n+1)(2n+1)}{6}$$

When $x = 0$ in Equation 7, the results are called *Bernoulli numbers* (Problem 7):

$$B_0 = 1; \qquad B_1 = -\frac{1}{2}; \qquad B_2 = \frac{1}{6}; \qquad B_4 = -\frac{1}{30}; \qquad B_6 = \frac{1}{42}$$

and

$$B_{2n+1} = 0 \qquad n \geq 1 \tag{10}$$

Bernoulli numbers occur in a number of various formulas. For example, the Riemann zeta function can be expressed in terms of Bernoulli numbers for even values of its argument:

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}| \tag{11}$$

If $n = 1$, we have $\zeta(2) = (2\pi)^2 B_2/4 = \pi^2/6$, and if $n = 2$, $\zeta(4) = (2\pi)^4|B_4|/2 \cdot 4! = \pi^4/90$.

Bernoulli numbers also occur in the series expansions of certain functions. For example,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!} + \cdots$$

and

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \cdots + \frac{(-1)^{n+1}(2^{2n}-2)}{(2n)!}B_{2n}x^{2n-1} + \cdots$$

They also occur in the Euler-Maclaurin summation formula, which we'll see below is a scheme for approximating a summation by an integral. For example,

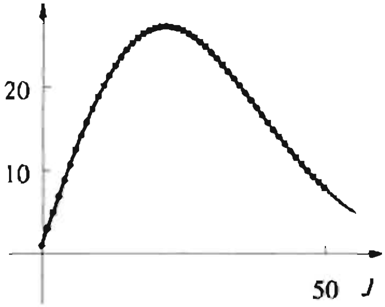

Figure 3.27
The terms of the series in Equation 12 and the continuous function $(2J+1)e^{-\alpha J(J+1)}$ plotted against J for $\alpha = 0.001$.

the series

$$S = \sum_{J=0}^{\infty} (2J+1)e^{-\alpha J(J+1)} \tag{12}$$

where α is a parameter occurs in the statistical mechanical treatment of rotating molecules. If α is small, then a plot of the successive terms in this series is essentially the same as a plot of $(2J+1)e^{-\alpha J(J+1)}$, with J being treated as a continuous variable (Figure 3.27). Figure 3.27 suggests that we can approximate the summation in Equation 12 by an integral, and write

$$S \approx \int_0^{\infty} (2J+1)e^{-\alpha J(J+1)} dJ \tag{13}$$

If we let $x = J(J+1)$, then $dx = (2J+1)dJ$ and S becomes $S \approx 1/\alpha$. If $\alpha = 0.001$ (a fairly typical value), then Equations 12 and 13 differ by less than 0.5%.

Because it is usually easier to integrate than it is to find a closed expression for a summation, being able to approximate a summation by an integral is very convenient. It is possible to express a summation as an integral plus correction terms. The series

$$\sum_{x=1}^{n} f(x) = \int_1^n f(x)dx + \frac{1}{2}[f(n) + f(1)] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(n) - f^{(2k-1)}(1)] \tag{14}$$

is called the *Euler-Maclaurin summation formula*. Note that it expresses a summation as an integral plus a sum of correction terms. The first few terms of Equation 14 are

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)dx + \frac{1}{2}[f(n) + f(1)] + \frac{1}{12}[f^{(1)}(n) - f^{(1)}(1)]$$
$$- \frac{1}{720}[f^{(3)}(n) - f^{(3)}(1)] + \frac{1}{30\,240}[f^{(5)}(n) - f^{(5)}(1)] + \cdots \tag{15}$$

where we assume that all the derivatives of $f(x)$ are well behaved.

Let's apply Equation 15 to the summation

$$S = \sum_{k=1}^{n} \frac{1}{k^2}$$

The various terms in Equation 15 are

$$\int_1^n f(x)\,dx = \int_0^n \frac{dx}{x^2} = 1 - \frac{1}{n}$$

$$\frac{1}{2}[f(n) + f(1)] = \frac{1}{2}\left(\frac{1}{n^2} + 1\right)$$

$$\frac{1}{12}\left[f^{(1)}(n) - f^{(1)}(1)\right] = -\frac{1}{6}\left(\frac{1}{n^3} - 1\right)$$

$$-\frac{1}{720}\left[f^{(3)}(n) - f^{(3)}(1)\right] = \frac{1}{30}\left(\frac{1}{n^5} - 1\right)$$

and so on to obtain

$$\sum_{k=1}^{n}\frac{1}{k^2} = \left(1 - \frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n^2} + 1\right) - \frac{1}{6}\left(\frac{1}{n^3} - 1\right)$$

$$+ \frac{1}{30}\left(\frac{1}{n^5} - 1\right) - \frac{1}{42}\left(\frac{1}{n^7} - 1\right) + \cdots$$

which gives 1.5620, compared to the exact value of 1.5498 (to four places) for $n = 10$, a difference of less than 1%.

Example 3:
Use the Euler-Maclaurin expansion to calculate $\ln n!$ for $n = 10$.

SOLUTION: First write $\ln n!$ as $\sum_{x=1}^{n} \ln x$ and then apply Equation 15:

$$\int_1^n \ln x \, dx = \left[x \ln x - x\right]_1^n = n \ln n - n + 1$$

$$\frac{1}{2}[f(n) + f(1)] = \frac{1}{2}\ln n$$

$$\frac{1}{12}\left[f^{(1)}(n) - f^{(1)}(1)\right] = \frac{1}{12}\left(\frac{1}{n} - 1\right)$$

$$-\frac{1}{720}\left[f^{(3)}(n) - f^{(3)}(1)\right] = -\frac{1}{360}\left(\frac{1}{n^3} - 1\right)$$

and so on to obtain

$$\sum_{x=1}^{n} \ln x = n \ln n - n + 1 + \frac{1}{2}\ln n$$

$$+ \frac{1}{12}\left(\frac{1}{n} - 1\right) - \frac{1}{360}\left(\frac{1}{n^3} - 1\right) + \frac{1}{1260}\left(\frac{1}{n^5} - 1\right) - \frac{1}{1680}\left(\frac{1}{n^7} - 1\right)$$

This formula gives 15.0992 versus the exact value of 15.10441 (to five places) for $n = 10$, a difference of 0.03%.

Before we leave this section, we should point out one fact about Bernoulli numbers that should be kept in mind. We gave the values of the B_n's up to $n = 6$, and they appear to decrease with increasing n (actually $2n$ since $B_{2n+1} = 0$ for $n \geq 1$). If you look at a table of Bernoulli numbers, however, you'll see that $B_8 = -1/30$, $B_{10} = 5/66$, $B_{12} = -691/2730$, and that they get larger and larger in absolute value as n increases. In fact, it's possible to show that the values of $|B_{2n}|$ satisfy the bounds

$$\frac{2(2n)!}{(2\pi)^{2n}}\left(\frac{2^{2n}}{2^{2n}-2}\right) > |B_{2n}| > \frac{2(2n)!}{(2\pi)^{2n}} \tag{16}$$

The factor $2^{2n}/(2^{2n}-2)$ goes to unity as n increases, and is equal to 1.000002 when n is only 10. Thus, Equation 16 shows that

$$|B_{2n}| \to \frac{2(2n)!}{(2\pi)^{2n}} \tag{17}$$

For example, Equation 16 gives $|B_{20}| = 529.1237$ versus the exact value of 529.1242 (to four places). This behavior of the Bernoulli numbers does not endanger the usefulness of the Euler-Maclaurin expansion, however, because the coefficients in Equation 15 involve the ratio $B_{2k}/(2k)!$, which Equation 17 shows goes as $2/(2\pi)^{2k}$.

3.7 Problems

1. Show that $\eta(n) = (1 - 2^{1-n})\zeta(n)$.
2. Use Equation 7 to derive expressions for the first few Bernoulli polynomials.
3. Show that $B_n(x+1) - B_n(x) = nx^{n-1}$.
4. Show that $B_n(1-x) = (-1)^n B_n(x)$.
5. Show that $\displaystyle\int_a^x B_n(u)du = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1}$.
6. Show that $(-1)^n B_n(-x) = B_n(x) + nx^{n-1}$.
7. Use Equation 7 with $x = 0$ to derive the first few Bernoulli numbers.
8. Use Equation 9 to derive a formula for $S_3 = \displaystyle\sum_{k=1}^{n} k^3$. Compare your result to the one in the *CRC Mathematical Tables*.
9. Use Equation 17 to show that $\zeta(2n) \to 1$ as $n \to \infty$.
10. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$.
11. Use the Euler-Maclaurin expansion to evaluate $S_m = \displaystyle\sum_{k=1}^{n} k^m$ for $m = 1, 2$, and 3. Compare your result for S_3 to the one that you get in Problem 8.
12. Derive an Euler-Maclaurin expansion for $\displaystyle\sum_{x=1}^{n} x^{-1/2}$ and use it to evaluate the sum for $n = 10$ and $n = 50$.

13. Use a CAS to evaluate the sum in the previous problem and compare your answer to the one obtained using the Euler-Maclaurin expansion.

14. Consider the sum $S = \sum_{k=0}^{\infty} e^{-\alpha k} = \dfrac{1}{1 - e^{-\alpha}}$ (geometric series). Expand S as a power series in α. Then apply the Euler-Maclaurin summation formula to the sum and compare your result.

15. One of the most famous applications of the Euler-Maclaurin summation formula to a physical problem is its application to the rotational partition function of a rigid rotator (a good model for a rotating diatomic molecule). The rotational partition function is $q = \sum_{J=0}^{\infty} (2J + 1)e^{-\Theta J(J+1)/T}$, where T is the kelvin temperature and Θ is a parameter that is characteristic of the molecule. Apply the Euler-Maclaurin summation formula to this sum and derive

$$q = \frac{T}{\Theta}\left[1 + \frac{1}{3}\left(\frac{\Theta}{T}\right) + \frac{1}{15}\left(\frac{\Theta}{T}\right)^2 + \cdots\right]$$

Typically, Θ/T is small, so this turns out to be a very useful expansion.

16. Use the defining expression for the Bernoulli numbers, $\dfrac{x}{e^x - 1} = \sum_{n=0}^{\infty} \dfrac{B_n x^n}{n!}$, to show that $\cot x = \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n}}{(2n)!} B_{2n} x^{2n-1}$.

17. Use a CAS to verify that the series in Equation 12 and the integral in Equation 13 differ by less than 0.05% for $\alpha = 0.001$.

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CHAPTER 4

Complex Numbers and Complex Functions

Complex numbers are usually introduced by considering a quadratic equation of the type $x^2 - x + 1 = 0$, where the quadratic formula gives

$$x = \frac{1}{2} \pm \frac{\sqrt{-3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

where $i = \sqrt{-1}$ is the imaginary unit. A number of the form $a + ib$, where a and b are real numbers, is called a *complex number*. If $a = 0$, then $x = ib$ is called an *imaginary number*. The message here is that we must introduce imaginary numbers in order to be able to solve quadratic equations in general. It shouldn't be surprising that initially there was a great resistance to the introduction of complex numbers and that it took many years for them to be accepted as legitimate members of our number system. The very name "imaginary number" seems to convey a certain degree of mysticism to these numbers.

If complex numbers had arisen only with quadratic equations, then it might have been easy to reject them by asserting that the equation $x^2 - x + 1 = 0$ has no solutions. After all, we're probably comfortable saying that $\sin x = 2$ has no solution for real values of x. Historically, it was in the study of the solutions to cubic equations that imaginary numbers were most puzzling. Consider the cubic equation $x^3 + 2x^2 - x - 2 = 0$. You can verify by inspection that this equation has three real roots, ± 1 and -2. Yet when you solve this equation using the standard (fairly messy) formula for calculating the three roots, square roots of negative numbers occur at several intermediate steps. In fact, the very requirement that there are three real, distinct roots leads to the square root of a negative number. The final results are the three real roots, so it is apparent that the occurrence of imaginary numbers doesn't invalidate any of the formulas. Eventually mathematicians not only came to tolerate imaginary numbers, but to embrace them fully.

You might wonder if more complicated polynomial equations (such as 17th degree equations) require the introduction of types of numbers "beyond" complex numbers. It turns out that they do *not*. There is a remarkable theorem of algebra called nothing less than the *fundamental theorem of algebra* that says that every Nth degree polynomial equation, $a_N x^N + a_{N-1} x^{N-1} + \cdots + a_1 x + a_0 = 0$, even with complex numbers as coefficients, has exactly N roots over the complex numbers. In other words, complex numbers are sufficient to satisfy *any* polynomial equation.

In this chapter we shall first review the algebraic rules for complex numbers and then consider complex numbers as variables and define functions of complex variables. The study of complex variables is one of the richest areas of mathematical analysis and has countless physical applications. We shall learn only the basic properties of complex functions in this chapter, and wait until Chapter 18 to exploit them fully.

4.1 Complex Numbers and the Complex Plane

We denote a complex number by $c = a + ib$ where a and b are real numbers and i is called the imaginary unit. The imaginary unit has the property that $i^2 = -1$, and is sometimes written as $i = \sqrt{-1}$. (Because an electric current is usually denoted by i, electrical engineers often denote complex numbers by $a + j\,b$.) The real numbers a and b are called the *real* and *imaginary parts*, respectively, of c, and we write

$$\operatorname{Re} c = a \qquad \operatorname{Im} c = b \tag{1}$$

The real numbers are clearly a subset of the complex numbers because $c = a$, a real number, if $b = 0$. The complex number $c = 0 + i0$ corresponds to 0. If $a = 0$ and $b \neq 0$, then $c = ib$ is called an *imaginary number*, or a *purely imaginary number*. Two complex numbers are equal if and only if their real and imaginary parts are equal; i.e.

$$c_1 = c_2 \quad \Longleftrightarrow \quad a_1 = a_2 \quad \text{and} \quad b_1 = b_2$$

where $\Longleftrightarrow$ means "if and only if."

We can do arithmetic with complex numbers by using the following rules: We can multiply a complex number by a real number β according to

$$\beta c = \beta a + i\beta b \tag{2}$$

The addition of two complex numbers is given by

$$c = c_1 + c_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) \tag{3}$$

For subtraction, simply add $c_2 = -a_2 - ib_2$ to c_1. To form the product of two complex numbers, we multiply them as we would multiply two binomials, using the relation $i^2 = -1$. Thus,

$$\begin{aligned} c_1c_2 = (a_1 + ib_1)(a_2 + ib_2) &= a_1a_2 + i^2b_1b_2 + i(a_1b_2 + b_1a_2) \\ &= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2) \end{aligned} \tag{4}$$

Finally, division is obtained in the following manner:

$$\begin{aligned}\frac{c_1}{c_2} = \frac{a_1 + ib_1}{a_2 + ib_2} &= \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} \\ &= \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + i\left(\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}\right)\end{aligned} \tag{5}$$

An important quantity associated with a complex number c is its *complex conjugate*, defined by

$$c^* = a - ib \tag{6}$$

(Some authors use $\bar{c}$ to denote a complex conjugate.) Note that $cc^* = (a + ib)(a - ib) = a^2 + b^2$ is an intrinsically positive quantity (unless $a = b = 0$). We call $\sqrt{cc^*}$ the *modulus*, or *absolute value*, or the *magnitude* of c and write

$$|c| = |a + ib| = (cc^*)^{1/2} = \sqrt{a^2 + b^2} \tag{7}$$

Equation 7 also tells us that

$$\frac{1}{c} = \frac{c^*}{|c|^2} \tag{8}$$

Complex numbers cannot be ordered, in the sense that the inequality $c_1 < c_2$ has no meaning. Nevertheless, the absolute values of complex numbers, being real numbers, can be ordered. Thus, for example, $|c| < 1$ means that c is such that $\sqrt{a^2 + b^2} < 1$.

Example 1:
Given $c_1 = -1 + i$ and $c_2 = 2 - 3i$, find $|c_1 c_2|$ and $|c_1/c_2|$.

SOLUTION: First find $c_1 c_2$ and c_1/c_2:

$$c_1 c_2 = (-1 + i)(2 - 3i) = (-2 + 3) + (2 + 3)i = 1 + 5i$$

$$\frac{c_1}{c_2} = \frac{(-1 + i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = -\frac{5}{13} - \frac{i}{13}$$

Then

$$|c_1 c_2| = \sqrt{1 + 25} = \sqrt{26}$$

$$\left|\frac{c_1}{c_2}\right| = \frac{\sqrt{26}}{13}$$

The rules that we have presented for manipulating complex numbers are the ones that are usually presented in introductory or elementary discussions. In more advanced discussions, it is desirable to treat complex numbers as ordered pairs of real numbers (a, b) that satisfy certain algebraic rules. For example, we say that $(a_1, b_1) = (a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$. Similarly, if α is a real number, then $\alpha(a, b) = (\alpha a, \alpha b)$. Addition and subtraction satisfy the rule $(a_1, b_1) \pm (a_2, b_2) = (a_1 \pm a_2, b_1 \pm b_2)$ and multiplication satisfies the apparently complicated rule $(a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)$. If you compare these rules to Equations 2 through 4, you see that they are the very same. The advantage of the more abstract ordered pair approach is that it is based entirely on real numbers (the imaginary unit doesn't appear in any of the rules) and that it can be used to develop an axiomatic foundation of complex numbers.

The fact that a complex number is an ordered pair of real numbers suggests that we can represent a complex number as a point in a cartesian coordinate system. From now on, we'll use the standard notation $z = x + iy$. We'll let the horizontal axis represent the real part of z and the vertical axis represent the imaginary part of z, as shown in Figure 4.1. The horizontal axis is called the *real axis* and the vertical axis is called the *imaginary axis* (although the real number y is plotted along that axis). The xy-plane is called the *complex plane*. The distance of the point z to the origin is the modulus of z because $|z| = \sqrt{x^2 + y^2}$ is equal to this distance.

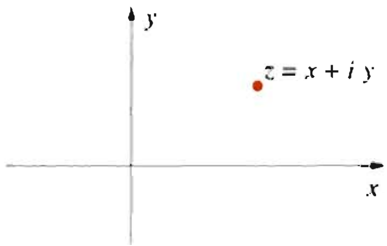

Figure 4.1
The complex plane. The real part of $z = x + iy$ is plotted along the horizontal axis and the imaginary part is plotted along the vertical axis.

Example 2:
Determine the curve in the complex plane that is described by $|z - 1| = 2$.

SOLUTION:

$$|z - 1| = |(x - 1) + iy| = [(x - 1)^2 + y^2]^{1/2}$$

and so $|z - 1| = 2$ corresponds to

$$(x - 1)^2 + y^2 = 4$$

which is a circle of radius 2 centered at $x = 1$, $y = 0$ (Figure 4.2).

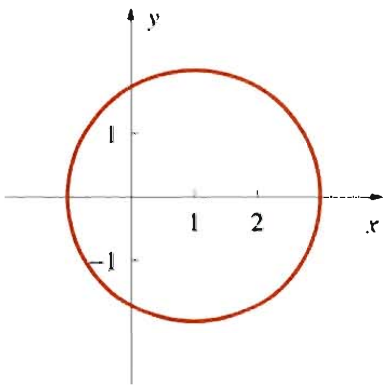

Figure 4.2
The graph of $|z - 1| = 2$, or $(x - 1)^2 + y^2 = 4$, in the complex plane.

Example 3:
Determine the set of points in the complex plane that is described by $|z - 1| \leq 2$.

SOLUTION: In this case, we have

$$(x - 1)^2 + y^2 \leq 4$$

which is the entire region bounded by the circle $(x - 1)^2 + y^2 = 4$, including the circle itself (Figure 4.3).

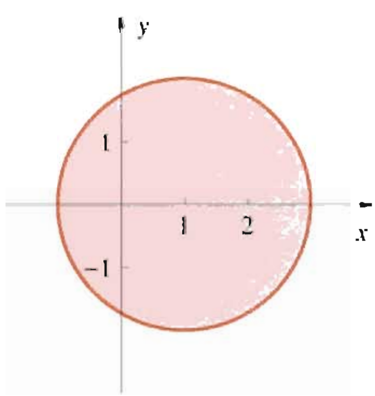

Figure 4.3
The region in the complex plane described by $|z - 1| \leq 2$, or $(x - 1)^2 + y^2 \leq 4$.

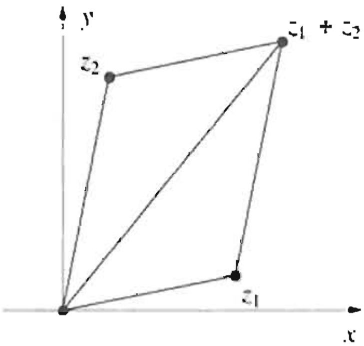

Figure 4.4
A geometrical interpretation of the addition of two complex numbers, z_1 and z_2.

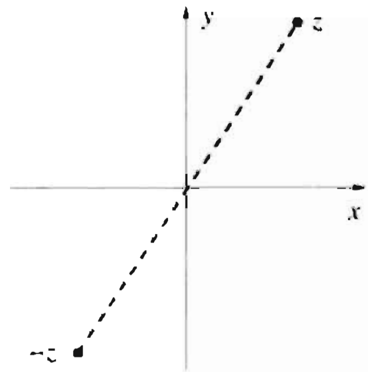

Figure 4.5
The negative of a complex number is its reflection through the origin.

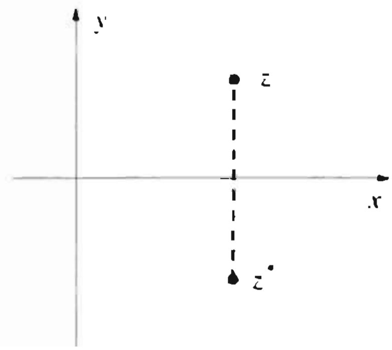

Figure 4.6
The complex conjugate of a complex number is its reflection through the x axis.

The addition of two numbers in the complex plane has a nice geometrical interpretation. Figure 4.4 illustrates the addition of $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. The point $z_1 + z_2$ completes the parallelogram whose legs are z_1 and z_2. We say that the addition of z_1 and z_2 satisfies the *parallelogram law*. Two other operations have simple geometric interpretations: The negative of a complex number is its reflection through the origin (Figure 4.5) and the complex conjugate is a reflection through the x axis (Figure 4.6).

If we refer to Figure 4.7, we see that we can represent a complex number z in *polar form*, by letting r be the distance of z from the origin and θ be the angle that a line from the origin to z makes with the real axis. Thus, we have

$$x = r\cos\theta \qquad y = r\sin\theta \tag{9}$$

and

$$z = r(\cos\theta + i\sin\theta) \tag{10}$$

where

$$r = (x^2 + y^2)^{1/2} \tag{11}$$

and

$$\tan\theta = \frac{y}{x} \tag{12}$$

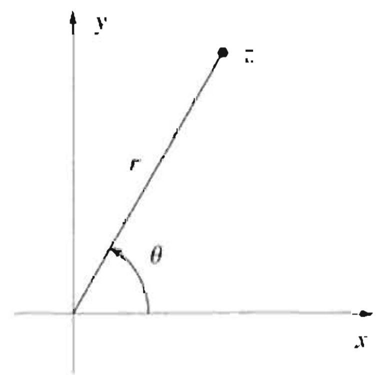

Figure 4.7
The polar form of z locates the point z by specifying r and θ.

The angle θ is called the *argument* or the *amplitude* of z and r is its modulus. We often denote these two quantities by $\theta = \arg z$ and $r = |z|$. Equation 10 is called the *polar form* of z (Figure 4.7). Equation 10 implies that $z^*(r, \theta) = z(r, -\theta)$, the reflection of z through the x axis.

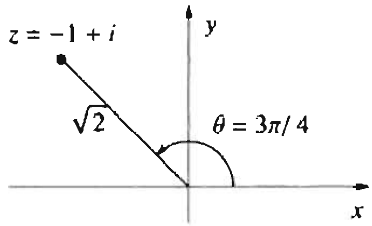

Figure 4.8
The complex number $z = -1 + i$ in polar form.

Example 4:
Express $z = -1 + i$ in polar form.

SOLUTION: The modulus of z is $\sqrt{2}$ and $\tan\theta = \dfrac{1}{-1} = -1$ or $\theta = 3\pi/4$. Thus, the polar form of z is

$$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

(Figure 4.8).

The polar form of z provides a geometrical interpretation of the product of two complex numbers. Write

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

Thus $z = z_1 z_2$ is obtained from z_1 and z_2 by multiplying the moduli and adding the two polar angles.

4.1 Problems

1. Prove that (a) $(z_1 z_2)^* = z_1^* z_2^*$, (b) $(1/z)^* = 1/z^*$, and (c) $(z^n)^* = (z^*)^n$ where n is a positive integer.
2. Show that $\text{Re}\,(z) = (z + z^*)/2$ and that $\text{Im}\,(z) = (z - z^*)/2i$.
3. Show that $|z_1 z_2| = |z_1||z_2|$.
4. Does $\dfrac{1}{|z|} = \left|\dfrac{1}{z}\right|$?
5. Determine the curve described by $|z - 2| + |z + 2| = 5$. What type of curve is it?
6. Determine the region in the complex plane described by $1 < |z + i| \le 3$.
7. Determine the region in the complex plane described by $\pi/4 \le \arg z \le \pi/2$.
8. Determine the region in the complex plane described by $|z - 2i| < |z + i|$.
9. Determine the region in the complex plane described by $\text{Re}\,(z + i) \ge 2$.
10. Evaluate

    (a) $\text{Re}\,\dfrac{3+i}{1-i}$ (b) $\left|\dfrac{(1-i)^3}{(3+2i)^2}\right|$ (c) $\displaystyle\sum_{k=0}^{5} i^k$ (d) $\text{Im}\left(\dfrac{1}{3+i} - \dfrac{1}{3-i}\right)$

11. Express the following complex numbers in polar form:

    (a) $2 - 2i$ (b) -1 (c) $-2 - 2\sqrt{3}\,i$ (d) $-3 + \sqrt{3}\,i$

12. Express the following polar forms of complex numbers in cartesian forms:

(a) $2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ (b) $\sqrt{3}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

(c) $\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$ (d) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

13. Leibnitz was aware of the relation $\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}$ but didn't know what to make of it. Prove that this relation is correct.

14. Describe the set of complex numbers $\{z\}$ that satisfy the expression $z=\eta z_1+(1-\eta)z_2$ where $0\le\eta\le 1$ and z_1 and z_2 are fixed.

15. Prove that $|z_1+z_2|\le|z_1|+|z_2|$. This is known as the *triangle inequality* because $|z_1+z_2|$, $|z_1|$, and $|z_2|$ represent the lengths of the legs of a triangle.

16. Show that $(i/2)^{1/2}=(1+i)/2$.

4.2 Functions of a Complex Variable

If we have a prescription from which we can calculate a complex number w from the complex number z, then we say that w is a function of z and we write $w=f(z)$. If w is specified uniquely by z, then $w=f(z)$ is said to be a *single-valued function*. Otherwise, $w=f(z)$ is *multiple-valued* or *many valued*. We shall see later in this section that a multiple-valued function may be considered to consist of a set of single-valued functions, so we shall restrict our functions to be single-valued (see Section 1.1).

Because $z=x+iy$, w will be complex and also a function of x and y. It is customary to write w in the form

$$w=u(x,y)+iv(x,y) \tag{1}$$

For example, if $w=f(z)=z^2$, then

$$w=z^2=x^2+2ixy-y^2$$

so that

$$u(x,y)=x^2-y^2 \qquad \text{and} \qquad v(x,y)=2xy$$

Example 1:
Suppose that $w=f(z)=1/(1+z)$. Determine $u(x,y)$ and $v(x,y)$.

SOLUTION:

$$w=f(z)=\frac{1}{1+z}=\frac{1}{(x+1)+iy}=\frac{(x+1)-iy}{(x+1)^2+y^2}$$

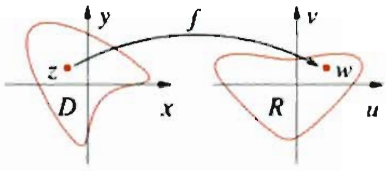

Figure 4.9
An illustration of a mapping from the z-plane to the w-plane. D represents the domain of w and R represents the range of w.

and so

$$u(x, y) = \frac{x+1}{(x+1)^2 + y^2} \qquad \text{and} \qquad v(x, y) = -\frac{y}{(x+1)^2 + y^2}$$

We can visualize a function of a complex variable as a transformation or a mapping of a set of points in the z-plane to a set of points in the w-plane (Figure 4.9). The set of points in the z-plane for which w is defined is called the *domain* of w, and the set of corresponding points in the w-plane is called the *range* of w. The domain and range of w are illustrated in Figure 4.9.

Consider the function $w = f(z) = z^2$, where $0 < x < \infty$ and $0 < y < \infty$. The domain of f in this case is the first quadrant of the z-plane, excluding the x and y axes. We saw above that $w = u(x, y) + iv(x, y)$ with $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Because $0 < x < \infty$ and $0 < y < \infty$, $-\infty < u < \infty$ and $0 < v < \infty$, so that the range of w is the entire upper half of the w-plane, excluding the u axis, as shown in Figure 4.10.

Example 2:
Determine the range of $w = f(z) = iz$ for $0 < x < a$ and $0 < y < a$. Map the four lines that are boundaries for the domain in the z-plane into curves in the w-plane.

SOLUTION: $f(z) = iz$ gives $u(x, y) = -y$ and $v(x, y) = x$. Because $0 < x < a$ and $0 < y < a$, then $-a < u < 0$ and $0 < v < a$. The range is shown in Figure 4.11. The boundaries of the domain are numbered in

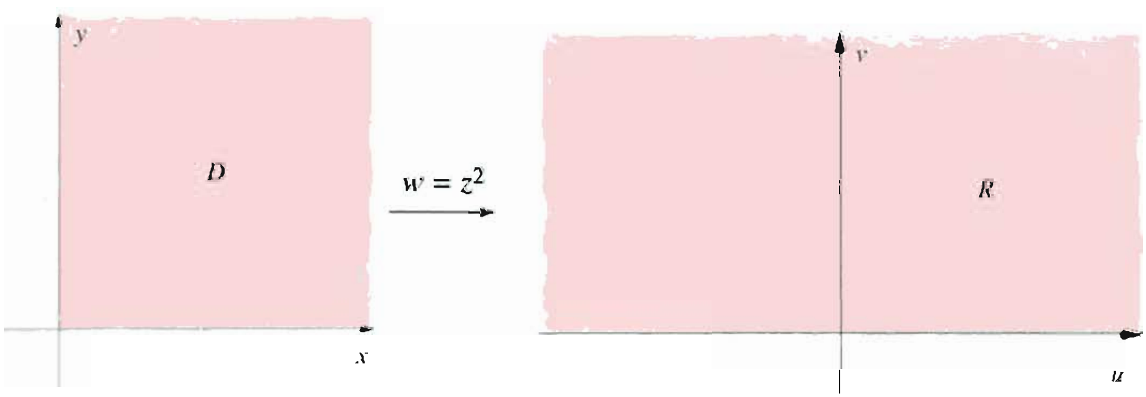

Figure 4.10
The domain and the range of the mapping $w = f(z) = z^2$ for $0 < x < \infty$ and $0 < y < \infty$.

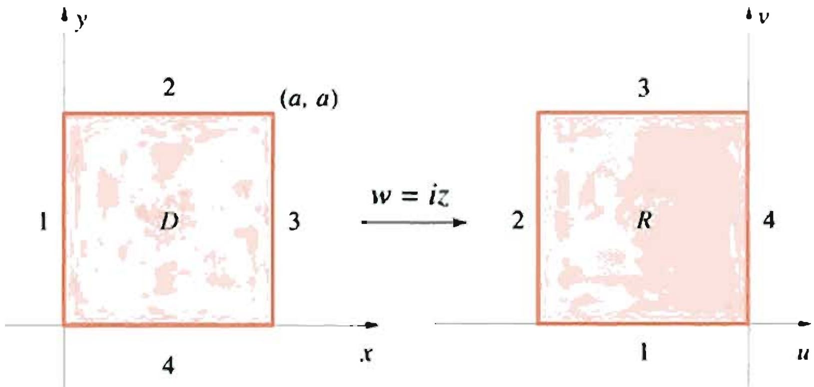

Figure 4.11
The domain and the range of the mapping $w = f(z) = iz$ for $0 < x < a$ and $0 < y < a$.

Figure 4.11. Curve 1, in the z-plane, ($x = 0$ and $0 < y < a$) maps into $v = 0$ and $-a < u < 0$, which is curve 1 in the w-plane. Curve 2 in the z-plane ($y = a$ and $0 < x < a$) maps into $u = -a$ and $0 < v < a$, which is curve 2 in the w-plane. Curves 3 and 4 are also shown in the figure. Note that the mapping corresponds to a counterclockwise rotation of 90°.

Let's consider $f(z) = z^2$ with $-\infty < x < \infty$ and $-\infty < y < \infty$. In this case, the range of

$$u(x, y) = x^2 - y^2 \qquad \text{and} \qquad v(x, y) = 2xy$$

is the entire uv-plane. The curves

$$u(x, y) = x^2 - y^2 = u_0 \qquad \text{and} \qquad v(x, y) = 2xy = v_0$$

define two families of curves in an xy-plane (Figure 4.12). We'll now show that these two families of curves are orthogonal to each other. Consider y to be a function of x and then differentiate $u(x, y)$ and $v(x, y)$ with respect to x.

$$2x - 2y\frac{dy}{dx} = 0 \qquad 2y + 2x\frac{dy}{dx} = 0$$

solve for dy/dx in each case and multiply the two slopes together to get -1. Recall that the result -1 implies that the two curves are orthogonal.

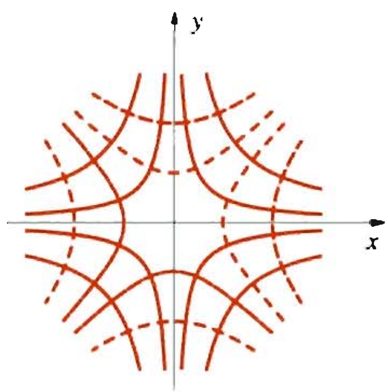

Figure 4.12
The two families of curves, $x^2 - y^2 = u_0$ (dashed) and $2xy = v_0$ (solid).

Example 3:
Show that the two families of curves $u(x, y) = u_0$ and $v(x, y) = v_0$ associated with $f(z) = 1/z$ are orthogonal.

SOLUTION:

$$f(z) = \frac{1}{z} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$$

and so the two families of curves are

$$u(x, y) = \frac{x}{x^2 + y^2} = u_0 \qquad \text{and} \qquad v(x, y) = -\frac{y}{x^2 + y^2} = v_0$$

Differentiate $u(x, y)$ and $v(x, y)$ with respect to x to obtain

$$\frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2}\frac{dy}{dx} = 0$$

and

$$\frac{2xy}{(x^2 + y^2)^2} - \frac{1}{(x^2 + y^2)}\frac{dy}{dx} + \frac{2y^2}{(x^2 + y^2)^2}\frac{dy}{dx} = 0$$

After a little algebra, the two equations give

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

and so we see that their product is -1.

Lest you think that the families $u(x, y) = u_0$ and $v(x, y) = v_0$ are always orthogonal, do Problem 15. It is true, however, that if $f(z)$ is differentiable with respect to the complex variable z, then the $u(x, y) = u_0$ and $v(x, y) = v_0$ families of curves will be orthogonal. We have to wait until Chapter 18 before we define just what we mean by "differentiable" with respect to the complex variable z and why the families of curves $u(x, y) = u_0$ and $v(x, y) = v_0$ are orthogonal in some cases but not in others.

4.2 Problems

1. Express each of the following functions in the form $w = u(x, y) + iv(x, y)$:

(a) z^3 **(b)** $1/(1 - z)^2$ **(c)** z^*/z

2. Express each of the following functions in the form $w = u(x, y) + iv(x, y)$:

(a) $\dfrac{1}{z^2 + i}$ **(b)** $\dfrac{1}{z}$ **(c)** $\left|\dfrac{1}{z}\right|$

3. Determine $w = f(z)$ if $u(x, y) = x/(x^2 + y^2)$ and $v(x, y) = -iy/(x^2 + y^2)$.

4. Determine $w = f(z)$ if $u(x, y) = 1/(x^2 + y^2)$ and $v(x, y) = 0$.

5. Find the values of w that correspond to the points $z = 1 + i$ and $z = 1 - i$ under the mapping $w = f(z) = 1/z$.

6. Find the values of w that correspond to the points $z = 2 + i$ and $z = 1 - i$ under the mapping $w = f(z) = z^2$.

7. Determine how the line that connects the two points in the z-plane in the previous problem is mapped into the w-plane. *Hint*: Take the equation of the line to be $z = \eta z_1 + (1 - \eta)z_2$ where $0 \le \eta \le 1$ and z_1 and z_2 are fixed.

8. Determine how the line $x = 1$ maps into the w-plane under the mapping $w = f(z) = 1/z$.

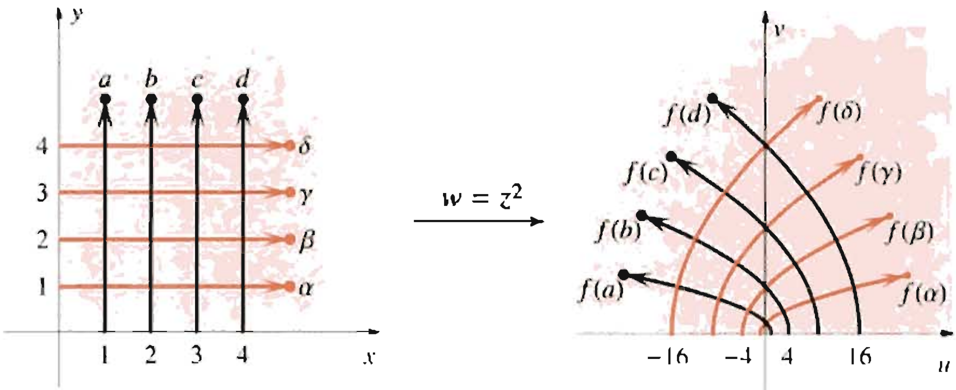

Figure 4.13
The lines in the z-plane that are to be mapped into lines in the w-plane in Problem 16 are shown on the left. The images in the w-plane are shown on the right.

9. Determine the range of $w = f(z) = 1/z$ for the domain $|z| < 2$.
10. Determine the range of $w = f(z) = z^2$ for the domain $0 \le x \le 1, 0 \le y \le 1$.
11. Determine the range of $w = f(z) = |z - 1| + 2i$ if $0 < x < \infty$ and $-\infty < y < \infty$.
12. Determine the range of $f(z) = z^2$ for $1 < x < 2$ and $1 < y < 3$. Map the four lines that are boundaries for the domain in the z-plane into the w-plane.
13. Determine the range of $f(z) = iz + 2$ for $0 < x < \infty$ and $0 < y < \infty$. Map the lines that are boundaries for the domain in the z-plane into the w-plane.
14. Show that the two families of curves, $u(x, y) = u_0$ and $v(x, y) = v_0$, associated with $f(z) = iz$ are orthogonal.
15. Show that the two families of curves, $u(x, y) = u_0$ and $v(x, y) = v_0$, associated with $f(z) = |z| + i(y - x)$ are *not* orthogonal.
16. Consider $f(z) = z^2$ for $0 < x < \infty$ and $0 < y < \infty$. Map the lines shown in Figure 4.13 into the w-plane. Show that the lines in the w-plane are orthogonal. Notice that the 90° angle intersections in the z-plane are mapped into 90° intersections in the w-plane.

4.3 Euler's Formula and the Polar Form of Complex Numbers

We learned in Section 2.7 that the series expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad -\infty < x < \infty \tag{1}$$

We can extend the definition of e^x to complex values of x by *defining* e^z by the series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{2}$$

It is common in mathematics to extend the definitions of functions in this manner. Equation 2 certainly reduces to Equation 1 when z is real. Problem 15 has you show that Equation 2 leads to the relation $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

Letting z be purely imaginary in Equation 2 (i.e. $z = iy$), we find that

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots\right) \end{aligned}$$

The series in parentheses are $\cos y$ and $\sin y$, however, so we arrive at the formula

$$e^{iy} = \cos y + i \sin y \tag{3}$$

Equation 3 is called *Euler's formula*, and is extremely useful in dealing with complex numbers. If we let $y = \pi$, Equation 3 becomes

$$e^{i\pi} + 1 = 0 \tag{4}$$

one of the most remarkable formulas in all of mathematics. Equation 4 contains the imaginary unit, i, the two most famous transcendental numbers, e and π (a transcendental number is a number which cannot be expressed as the root of a polynomial equation with rational coefficients), and the two most fundamental numbers of the real number system, 0 and 1. The physicist Richard Feynman, used to refer to Equation 4 as a "jewel."

We can also express Euler's formula as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) \tag{5}$$

From Equation 5, the real and imaginary parts of e^z are

$$\begin{aligned} e^z &= e^x \cos y + i e^x \sin y \\ &= u(x, y) + i v(x, y) \end{aligned}$$

It's easy to show that the two families $u(x, y) = u_0$ and $v(x, y) = v_0$ are orthogonal (Problem 14).

Euler's formula provides a convenient polar representation of complex numbers. We learned in Section 1 that we can write

$$z = x + iy = r \cos\theta + i\, r \sin\theta \tag{6}$$

where the modulus $r = (x^2 + y^2)^{1/2}$ is the distance from the origin to the point (x, y) and $\theta = \arg z$ is given by $\tan\theta = y/x$. Using Equation 3, we have

$$z = r(\cos\theta + i \sin\theta) = re^{i\theta} \tag{7}$$

which we call the *polar form* of z. Note that $|z| = (zz^*)^{1/2} = (re^{i\theta}re^{-i\theta})^{1/2} = r$.

Example 1:
Express $z = -1 + i$ in polar form (Figure 4.8).

SOLUTION: $r = \sqrt{2}$ and $\tan\theta = 1/(-1) = -1$, so $\theta = 3\pi/4$. Thus,

$$z = \sqrt{2}\, e^{3\pi i/4}$$

In Example 1, we took the angle θ to be $3\pi/4$. We could also have taken θ to be $3\pi/4 + 2\pi$, or even $3\pi/4 + 2\pi n$ where n is any integer. If we restrict the values of θ to the interval $0 \le \theta < 2\pi$, then θ is called the *principal argument* of z and is denoted by θ_0 or Arg z. We'll denote general values of θ by arg z and write

$$\theta = \arg z = \operatorname{Arg} z + 2\pi n = \theta_0 + 2\pi n \tag{8}$$

for $n = 0, \pm 1, \pm 2, \ldots$.

We can illustrate the restriction $0 \le \theta < 2\pi$ for $\theta(z) = \operatorname{Arg} z$ graphically as shown in Figure 4.14. The two closely spaced lines depict a *branch cut*. Suppose that $\theta = 0$ on the upper line. Then, as θ increases in a counterclockwise direction, it reaches the lower line, where θ approaches, but does not equal, 2π. Theta must cross the lower line in order to equal 2π, which it does along the upper line. The significance of the branch cut is to emphasize that the function $\theta(z)$ changes as the branch cut is crossed. For example, as long as $0 \le \theta < 2\pi$, then $\theta(z) = \operatorname{Arg} z$. However, according to Equation 8, if θ starts at some value $\theta_0 = \operatorname{Arg} z$ (where $0 \le \theta < 2\pi$) and increases by 2π, and consequently, crosses the branch cut in Figure 4.14, then

$$\theta = \arg z = \operatorname{Arg} z + 2\pi$$

is no longer equal to Arg z. The branch cut serves to specify the function $\theta(z)$. Whenever θ crosses the branch cut, the value of n in Equation 8 changes and $\theta(z)$ changes to a new branch. We depict the branch cut as two closely spaced lines for pictorial reasons, but in reality it is one line, and is sometimes depicted as one heavy line. The upper line in Figure 4.14 corresponds to the top part of the branch cut and the lower line corresponds to the bottom part of the branch cut. In either pictorial representation, whenever the branch cut is crossed, $\theta(z)$ changes to a new branch.

The little circle around the origin in Figure 4.14 reflects the fact that arg z is undefined at $z = 0$, and serves to exclude the origin. The origin in this case is called a *branch point*. We shall have more to say about branch cuts and branch points in Section 6.

The multiple-valued nature of arg z is due to the fact that $\tan^{-1}(y/x)$ is a multiple-valued function. You also have to be careful in computing the value of θ from $\theta = \tan^{-1}(y/x)$ because it can give two different values of θ even if $0 \le \theta < 2\pi$. For example, in Example 1 we chose $\theta = 3\pi/4$ because Figure 4.8

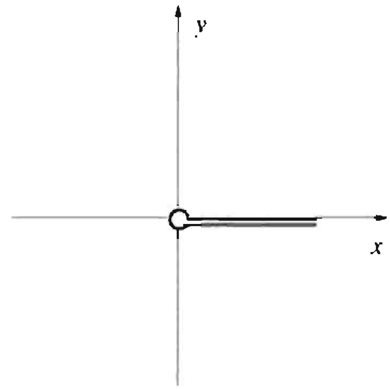

Figure 4.14
An illustration of a branch cut for the function $\theta(z) = \arg z$.

showed that $z = -1 + i$ lies in the second quadrant. However, $\tan^{-1}(-1)$ also gives $\theta = 7\pi/4$, which would be the correct value of θ for $z = 1 - i$. This ambiguity causes no problem if you keep in mind just where the point z lies in the complex plane.

Example 2:
Determine the polar representations of $z_1 = 1 + i$ and $z_2 = -1 - i$.

SOLUTION: In both cases, $r = \sqrt{2}$. The point $z = 1 + i$ lies in the first quadrant, so $\tan^{-1}(1) = \pi/4$, and

$$z_1 = \sqrt{2}\, e^{i\pi/4}$$

The point $z = -1 - i$ lies in the third quadrant, so $\tan^{-1}(1) = 5\pi/4$, and

$$z_2 = \sqrt{2}\, e^{5\pi i/4}$$

Both z_1 and z_2 have the same value of y/x.

Multiplying and dividing complex numbers is easy in polar form:

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

For example, the product of z_1 and z_2 in Example 2 is $z_1 z_2 = 2e^{6\pi i/4} = 2e^{3\pi i/2} = -2i$ and their ratio is $z_1/z_2 = e^{-i\pi} = -1$.

We can also use Euler's formula to derive the formulas (Problem 4)

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{9}$$

These formulas are often useful when evaluating integrals involving $\sin x$ or $\cos x$.

Example 3:
Evaluate

$$I = \int_0^\infty e^{-\alpha t} \sin t \, dt \qquad (\alpha > 0)$$

by using Equation 9.

SOLUTION:

$$I = \frac{1}{2i}\int_0^\infty e^{-(\alpha-i)t}\,dt - \frac{1}{2i}\int_0^\infty e^{-(\alpha+i)t}\,dt$$
$$= \frac{1}{2i}\left(\frac{1}{\alpha-i} - \frac{1}{\alpha+i}\right) = \frac{2i}{2i(\alpha^2+1)} = \frac{1}{\alpha^2+1}$$

We can evaluate I in Example 3 another way. Because $e^{it} = \cos t + i \sin t$, we can write I as

$$I = \int_0^\infty e^{-\alpha t}\sin t\,dt = \text{Im}\int_0^\infty e^{-(\alpha-i)t}\,dt$$
$$= \text{Im}\left(\frac{1}{\alpha-i}\right) = \frac{1}{\alpha^2+1}$$

This procedure gives us

$$\int_0^\infty e^{-\alpha t}\cos t\,dt = \text{Re}\int_0^\infty e^{-(\alpha-i)t}\,dt = \text{Re}\left(\frac{1}{\alpha-i}\right) = \frac{\alpha}{\alpha^2+1}$$

as a by-product.

Equation 7 also gives us a formula known as *de Moivre's formula*. Taking the nth power of both sides of Equation 7 gives

$$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta) \tag{10}$$

or

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{11}$$

We can use de Moivre's formula to derive multiple angle formulas for the trigonometric functions. Letting $n = 2$ in Equation 11 gives

$$\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta = \cos 2\theta + i\sin 2\theta$$

Equating the real and imaginary parts of this relation gives

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \sin 2\theta = 2\cos\theta\sin\theta \tag{12}$$

Of course, we can use Equation 11 to derive formulas for $\cos 3\theta$ and $\sin 3\theta$, and so on (Problem 7).

Equation 7 is also useful for evaluating powers of complex numbers. For example, consider $(1+i)^3$. In this case, $r = \sqrt{2}$ and $\theta_0 = \pi/4$, so $(1+i)^3 = 2^{3/2}e^{3\pi i/4}$. Figure 4.15 shows z and z^3 plotted in the complex plane. Note that the magnitude of z is cubed and its principal argument is multiplied by 3.

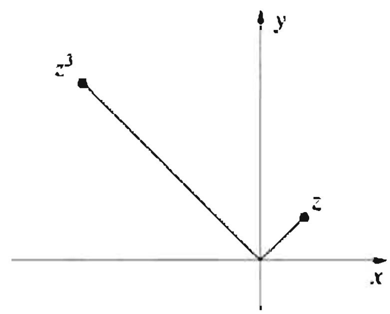

Figure 4.15
The relation between z and z^3 in the complex plane for $z = 1 + i$.

Example 4:
Evaluate $z = (1 - i)^6$.

SOLUTION: First see that $r = \sqrt{2}$ and that $\theta = 7\pi/4$. Thus,

$$z^6 = (\sqrt{2})^6 e^{i6 \cdot 7\pi/4} = 8\, e^{42\pi i/4} = 8\, e^{21\pi i/2} = 8\, e^{\pi i/2}\, e^{10\pi i}$$

$$= 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 8i$$

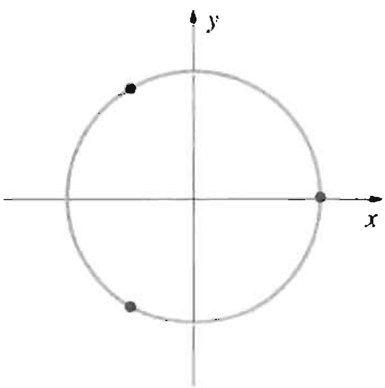

Figure 4.16
The three cube roots of unity plotted in the complex plane.

The equation $x^2 = 1$ has two distinct roots, $x = \pm 1$. The equation $x^N = 1$ has N distinct roots, called the Nth roots of unity. As we shall see, some of these roots turn out to be complex, so we'll write the equation as $z^N = 1$. Now let $z = re^{i\theta}$ to obtain $r^N e^{iN\theta} = 1$. The value of r is unity and so we have

$$e^{iN\theta} = \cos N\theta + i\sin N\theta = 1$$

This equation says that $\cos N\theta = 1$ and $\sin N\theta = 0$; this is so only if $N\theta = 2\pi n$, where $n = 0,\ 1,\ 2,\ \ldots,\ N-1$. (If $n \geq N$, $\cos N\theta$ and $\sin N\theta$ just repeat themselves.) Thus we see that the Nth roots of unity are given by

$$z_n = e^{2\pi i n/N} \qquad n = 0,\ 1,\ 2,\ \ldots,\ N-1 \tag{13}$$

For example, if $N = 2$, we obtain

$$z_0 = e^0 = 1 \qquad \text{and} \qquad z_1 = e^{i\pi} = -1$$

If $N = 3$, we get

$$z_0 = e^0 = 1 \qquad\qquad z_1 = e^{2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

and

$$z_2 = e^{4\pi i/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

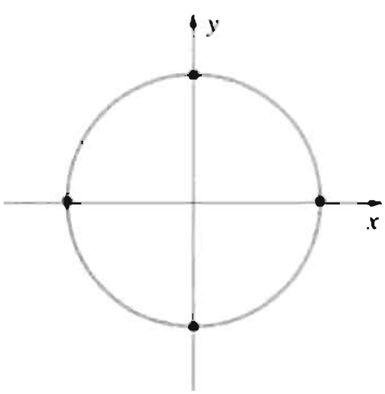

Figure 4.17
The four 4th roots of unity plotted in the complex plane.

Figure 4.16 shows these three roots plotted in the complex plane. Note that all of them lie on a unit circle centered at the origin because they all have unit modulus. One root ($z = 1$) lies along the x axis and the other two are symmetrically distributed about the origin. Figure 4.17 shows the four roots of $z^4 = 1$ (Problem 10). Generally the Nth roots of unity lie on the unit circle centered at the origin; one of the points lies along the x axis; and the others are symmetrically distributed about the origin.

We can also find the Nth roots of any complex number. We simply express z in polar form, and then take the Nth root as we did above. In this case, we start with

$$z = re^{i(\theta_0+2\pi n)} \qquad n = 0,\ 1,\ 2,\ \ldots \tag{14}$$

where $\theta_0 = \text{Arg } z$, the principal argument of z. The following Example shows how to find the three cube roots of $z = 1 + i$.

Example 5:
Find the three cube roots of $z = 1 + i$.

SOLUTION: First write z in polar form. $\text{Arg } z = \pi/4$, and so

$$z = 1 + i = \sqrt{2}\, e^{i(\theta_0+2\pi n)} = \sqrt{2}\, e^{i(\pi/4+2\pi n)}$$

Now take the cube root:

$$z^{1/3} = 2^{1/6} e^{i(\pi/12+2\pi n/3)}$$

The argument of $z^{1/3}$ will lie in the interval $0 \le \theta < 2\pi$ if $n = 0$, 1, and 2. Therefore,

$$z^{1/3} = 2^{1/6} e^{i\pi/12},\ 2^{1/6} e^{i9\pi/12},\ 2^{1/6} e^{i17\pi/12}$$

These three roots are shown in Figure 4.18. They are symmetrically distributed on a circle of radius $2^{1/6}$, but none of them is directed along the positive x axis, as in the case of the Nth roots of unity.

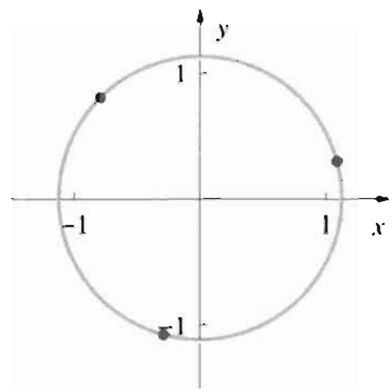

Figure 4.18
The three cube roots of $z = 1 + i$ plotted in the complex plane. The radius of the circle in this case is $2^{1/6}$.

4.3 Problems

1. Determine r and the principal argument for

   (a) $6i$ **(b)** $4 - \sqrt{2}i$ **(c)** $-1 - 2i$ **(d)** $\pi + ei$

2. Express the following polar forms in cartesian form:

   (a) $e^{i\pi/4}$ **(b)** $6\, e^{2\pi i/3}$ **(c)** $e^{-(\pi/4)i+\ln 2}$ **(d)** $e^{-2\pi i} + e^{4\pi i}$

3. This problem gives a simple derivation of Euler's formula. Start with $f(\theta) = \cos\theta + i\sin\theta$. Differentiate with respect to θ and show that $f'(\theta) = if(\theta)$. Now integrate to get Euler's formula.

4. Derive Equations 9.

5. Evaluate $\int_0^\infty e^{-\alpha t}\cos t\, dt$ ($\alpha > 0$) using Euler's formula.

6. Use Euler's formula to show that (n and m are integers) $\int_0^{2\pi} \sin nx \sin mx\, dx = \int_0^{2\pi} \cos nx \cos mx\, dx = \int_0^{2\pi} \sin nx \cos mx\, dx = 0$ if $m \neq n$.

7. Use Equation 11 to derive formulas for $\cos 3\theta$ and $\sin 3\theta$.

8. Evaluate $\left(\dfrac{1-i}{1+i}\right)^8$.

9. Evaluate $(2-i)^{10}$.

10. Find the four 4th roots of unity; in other words, find the four solutions to $z^4 - 1 = 0$.

11. Find the six 6th roots of $z^6 = 64$. Plot the six roots in the complex plane.

12. Find the four 4th roots of i.

13. Find the three cube roots of $\sqrt{3} - i$.

14. Starting with $e^z = e^x \cos y + ie^x \sin y = u(x, y) + iv(x, y)$, show that the two families of curves $u(x, y) = u_0$ and $v(x, y) = v_0$ are orthogonal.

15. Use Equation 2 to show that $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

16. Show that the region $-\infty < x < \infty$, $0 \le y \le a$ in the z-plane maps into the upper half plane in the w-plane under the transformation $w = e^{\pi z/a}$.

17. Starting with $\displaystyle\int_0^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a}$, let $a = (1-i)/\sqrt{2}$ to show that

$$\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \left(\frac{\pi}{8}\right)^{1/2}$$

4.4 Trigonometric and Hyperbolic Functions

In the previous section, we derived the formulas

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{1}$$

and

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{2}$$

where x is a real number. Equations 1 and 2 suggest that we define the sine and cosine of complex numbers by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{3}$$

and

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{4}$$

We can show that these definitions are consistent with all our standard formulas for the trigonometric functions. For example, Equations 3 and 4 say that

$$\sin(-z) = -\sin z$$

and that

$$\cos(-z) = \cos z$$

Furthermore,

$$\sin^2 z + \cos^2 z = -\frac{(e^{iz} - e^{-iz})^2}{4} + \frac{(e^{iz} + e^{-iz})^2}{4}$$
$$= \frac{e^{2iz} + 2 + e^{-2iz} - e^{2iz} + 2 - e^{-2iz}}{4} = 1$$

Example 1:
Use Equations 3 and 4 to show that $\sin 2z = 2 \sin z \cos z$.

SOLUTION:

$$2 \sin z \cos z = \frac{(e^{iz} - e^{-iz})(e^{iz} + e^{-iz})}{2i}$$
$$= \frac{e^{2iz} - e^{-2iz}}{2i} = \sin 2z$$

We can extend the definitions of the hyperbolic functions to complex variables by defining

$$\sinh z = \frac{e^z - e^{-z}}{2} \tag{5}$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \tag{6}$$

Once again, these definitions reduce to those in Section 1.1 for the hyperbolic functions of real variables. Just as Equations 3 and 4 satisfy all the trigonometric identities for real variables, Equations 5 and 6 satisfy all the identities involving hyperbolic functions. For example,

$$\cosh^2 z - \sinh^2 z = \frac{(e^z + e^{-z})^2}{4} - \frac{(e^z - e^{-z})^2}{4} = 1$$

and

$$\sinh 2z = \frac{e^{2z} - e^{-2z}}{2} = 2\frac{(e^z + e^{-z})}{2}\frac{(e^z - e^{-z})}{2}$$
$$= 2 \cosh z \sinh z$$

There is a close relationship between the trigonometric and hyperbolic functions when we define them in terms of complex variables. If we substitute iz for z

in Equations 3 and 4, we obtain

$$\sin iz = \frac{e^{-z} - e^{z}}{2i} = i\frac{e^{z} - e^{-z}}{2} = i \sinh z \tag{7}$$

and

$$\cos iz = \frac{e^{-z} + e^{z}}{2} = \cosh z \tag{8}$$

Furthermore, if we substitute iz for z in the right-hand sides of Equations 5 and 6, we have

$$\sinh iz = i \sin z \tag{9}$$

and

$$\cosh iz = \cos z \tag{10}$$

Equation 7 shows that if z is real, then

$$\sin ix = i \sinh x \tag{11}$$

Thus, although $\sin x$ is our ordinary sine function along the real axis, in the complex plane it behaves as $\sinh x$ along the imaginary axis. To see how $\sin z$ behaves over the entire complex plane, consider

$$\begin{aligned}
\sin z = \sin(x + iy) &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i} \\
&= \frac{1}{2i}\left[e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\right] \\
&= \sin x \left(\frac{e^{y} + e^{-y}}{2}\right) + i \cos x \left(\frac{e^{y} - e^{-y}}{2}\right) \\
&= \sin x \cosh y + i \cos x \sinh y
\end{aligned}$$

We can also derive this result by using the trigonometric formula for $\sin(\alpha + \beta)$:

$$\begin{aligned}
\sin z = \sin(x + iy) &= \sin x \cos iy + \cos x \sin iy \\
&= \sin x \cosh y + i \cos x \sinh y
\end{aligned} \tag{14}$$

When $y = 0$, $\sin z = \sin x$, and when $x = 0$, $\sin z = i \sinh y$.

Example 2:
Derive an expression for the real and imaginary parts of $\cos z$.

SOLUTION: Use $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ to write

$$\cos z = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy$$
$$= \cos x \cosh y - i \sin x \sinh y \qquad (15)$$

When $y = 0$, we have $\cos z = \cos x$ and when $x = 0$, we have $\cos z = \cosh y$ in Equation 15. Figures 4.19 and 4.20 show the real and imaginary parts of $\sin z$ plotted in the complex plane.

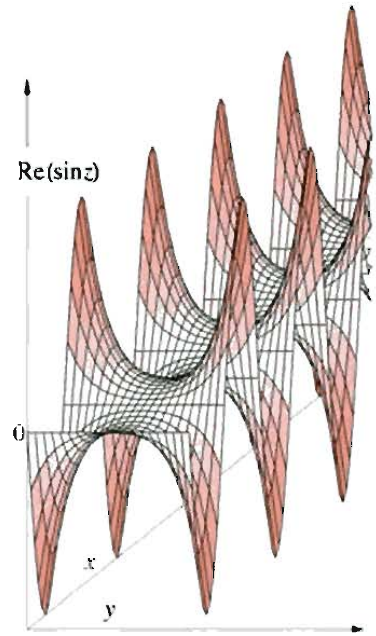

Figure 4.19
The real part of $\sin z$ plotted against x and y. Note the periodicity in the x direction and the exponential growth in the y direction.

Example 3:
Evaluate $\cos(\pi - i)$.

SOLUTION: Use Equation 15 with $x = \pi$ and $y = -1$:

$$\cos(\pi - i) = \cos \pi \cosh(-1) - i \sin \pi \sinh(-1)$$
$$= -\cosh(-1) = -\cosh 1$$

Note that Im $(\cos z) = 0$ along the vertical lines that are integer multiples of π.

In Section 2, we viewed $w = f(z)$ as a mapping from a region of the z-plane to a region of the w-plane. Let's see how the domain $-\pi/2 \le x \le \pi/2$, $0 \le y < \infty$ maps into the w-plane when $w = f(z) = \cos z$. The domain in the z-plane is shown in Figure 4.21. For $w = f(z) = \cos z$,

$$u(x, y) = \cos x \cosh y \qquad v(x, y) = -\sin x \sinh y$$

Along the vertical line at $x = \pi/2$ in the z-plane,

$$u(\tfrac{\pi}{2}, y) = 0 \qquad v(\tfrac{\pi}{2}, y) = -\sinh y \qquad 0 \le y < \infty$$

Therefore, the vertical line at $x = \pi/2$ maps into the negative v axis in the w-plane. Along the vertical line at $x = -\pi/2$ in the z-plane,

$$u(-\tfrac{\pi}{2}, y) = 0 \qquad v(-\tfrac{\pi}{2}, y) = \sinh y \qquad 0 \le y < \infty$$

and so the vertical line at $x = -\pi/2$ maps into the positive v axis. Now, let's look at points within the domain. Because $0 \le \cos x \le 1$ for $-\pi/2 \le x \le \pi/2$ and $\cosh y \ge 1$ for all values of y, we see that $u(x, y) \ge 0$ everywhere within the domain, and so the domain in Figure 4.21 gets mapped into the entire right-half plane in the w-plane.

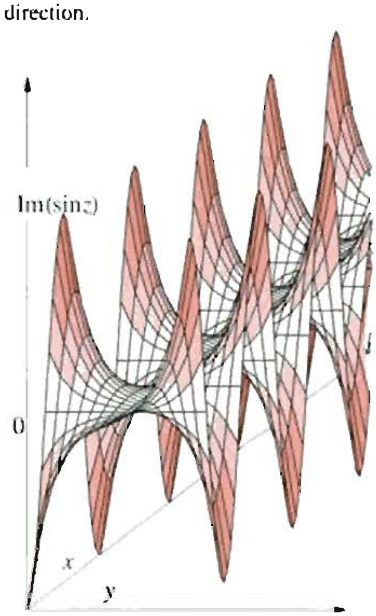

Figure 4.20
The imaginary part of $\sin z$ plotted against x and y. Note the periodicity in the x direction and the exponential growth in the y direction.

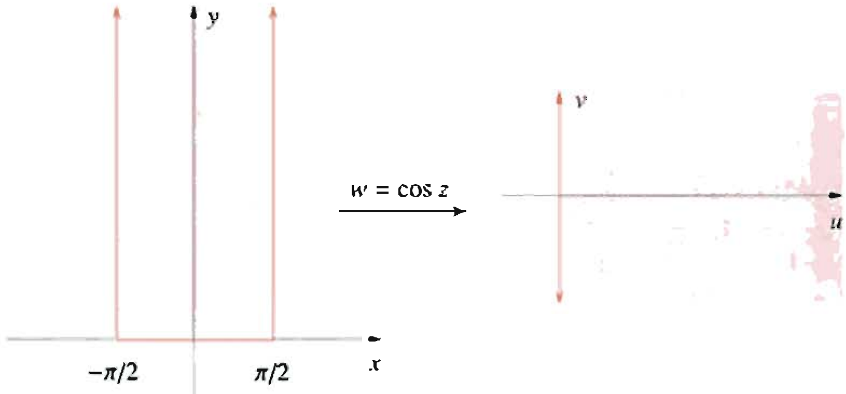

Figure 4.21
The mapping of the region $-\pi/2 \le x \le \pi/2$, $0 \le y < \infty$ in the z-plane into $w = f(z) = \cos z$ in the w-plane.

4.4 Problems

1. Use Equations 3 and 4 to show that $\cos 2z = 2\cos^2 z - 1$.
2. Use Equations 3 and 4 to show that $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.
3. Express $\cosh z$ in the form $u(x, y) + iv(x, y)$.
4. Express $\sinh z$ in the form $u(x, y) + iv(x, y)$.
5. Use Equations 5 and 6 to show that $\cosh 2z = \cosh^2 z + \sinh^2 z$.
6. Use Equations 5 and 6 to show that $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ and $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$.
7. Use Equations 5 and 6 to show that $\tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} + O(z^7)$.
8. Show that $\sinh(z + 2\pi i) = \sinh z$ and that $\cosh(z + 2\pi i) = \cosh z$. Interpret this result.
9. Evaluate $\cos(\frac{\pi}{2} - 2i)$.
10. Evaluate $\sinh(2 + i\pi)$.
11. Show that

    (a) $\tanh u = -i \tan iu$ (b) $\coth u = i \cot iu$ (c) $\cos iu = \cosh u$ (d) $\sinh iu = i \sin u$

12. Show that

    (a) $\operatorname{sech} iu = \sec u$ (b) $\coth iu = -i \cot u$ (c) $\operatorname{sech} u = \sec iu$ (d) $\sin iu = i \sinh u$

13. Map the region $-\pi/2 \le x \le \pi/2$, $0 \le y < \infty$ in the z-plane into the w-plane under the transformation $w = \sin z$.
14. Map the region $0 \le x \le \pi/2$, $0 \le y < \infty$ in the z-plane into the region in the w-plane under the transformation $w = \sin z$.
15. Show that $e^{nz} = (\cosh z + \sinh z)^n = \cosh nz + \sinh nz$ and that $e^{-nz} = (\cosh z - \sinh z)^n = \cosh nz - \sinh nz$.
16. Use the relation in the previous problem to show that $\sinh 2u = 2 \sinh u \cosh u$ and that $\cosh 2u = 2\cosh^2 u - 1 = 1 + 2\sinh^2 u$.

17. Show that the region $0 \le x < \infty, 0 \le y \le a$ in the z-plane maps into the upper half plane in the w-plane under the tranformation $w = \cosh \pi z/a$.

18. Show that the region $0 \le x \le a, 0 \le y < \infty$ in the z-plane maps into the first quadrant in the w-plane under the transformation $w = \sin \pi z/2a$.

19. Use the defining expression for Bernoulli numbers (Equations 3.7.7 with $x = 0$) to show that
$\cot u = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} B_{2n} u^{2n-1}$.

20. Show that the family of curves $u(x, y) = u_0$ and $v(x, y) = v_0$ associated with $f(z) = \cos z$ are orthogonal.

21. Use a CAS to produce figures like Figures 19 and 20 for $\cos z$.

4.5 The Logarithms of Complex Numbers

We define the logarithm of a complex variable as the inverse of the exponential function e^z, and write

$$w = \ln z \qquad z \neq 0 \tag{1}$$

We have the restriction that $z \neq 0$ because e^w cannot equal zero for any finite value of w. If we let $z = re^{i\theta}$ and $w = u + iv$, then we find that

$$u = \ln r \qquad \text{and} \qquad v = \theta \tag{2}$$

where $\ln r$ is the usual natural logarithm of a real number. But $e^{i\theta} = e^{i\theta_0 + 2\pi ni}$ where θ_0 is the principal argument of z $(0 \le \theta_0 < 2\pi)$ and $n = 0, \pm 1, \pm 2, \ldots$. So we rewrite v as

$$v = \theta_0 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots \tag{3}$$

and

$$\ln z = \ln r + i(\theta_0 + 2\pi n) \qquad n = 0, \pm 1, \pm 2, \ldots \tag{4}$$

Equation 4 shows that $\ln z$ has a different value for each value of n, or that $\ln z$ has an infinite number of values. We can also write Equation 4 in terms of $u(x, y)$ and $v(x, y)$:

$$\ln z = \ln(x^2 + y^2)^{1/2} + i \tan^{-1} \frac{y}{x} \tag{5}$$

where $\tan^{-1}(y/x) = \theta_0 + 2\pi n$ when $n = 0, \pm 1, \pm 2, \ldots$.

Example 1:
Determine the value of $\ln(3 + i\sqrt{3})$.

SOLUTION: First express $3 + i\sqrt{3}$ in polar form:

$$3 + \sqrt{3}i = (12)^{1/2} e^{i(\frac{\pi}{6} + 2\pi n)} \qquad n = 0, \pm 1, \pm 2, \ldots$$

and so

$$\ln(3 + \sqrt{3}\,i) = \frac{1}{2}\ln 12 + i\left(\frac{\pi}{6} + 2\pi n\right)$$
$$= 1.242\ldots + \frac{i\pi}{6},\ 1.242\ldots + \frac{13\pi i}{6},$$
$$1.242\ldots - \frac{11\pi i}{6},\ 1.242\ldots + \frac{25\pi i}{6}$$

and so on.

Before the development of functions of a complex variable, mathematicians of the 18th century were uncertain of the meaning that should be given to logarithms of negative real numbers. To determine $\ln(-1)$, we write -1 as $e^{i\pi}$ and write

$$\ln(-1) = i\pi$$

a purely imaginary number. Because -1 is also equal to $e^{i(\pi+2\pi n)}$ for $n = 0, \pm 1, \cdots$, we have in general that

$$\ln(-1) = i(\pi + 2\pi n) = \pi i,\ 3\pi i,\ -\pi i,\ \ldots \tag{6}$$

which are all purely imaginary numbers.

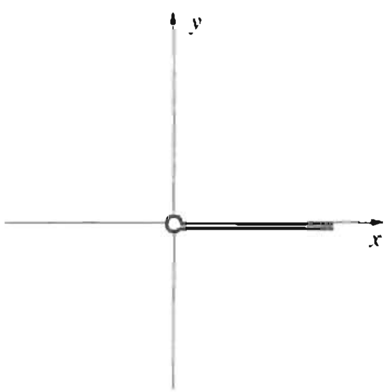

Figure 4.22
The branch cut for $\ln z$ in the complex plane that restricts $\arg z$ to the values $0 \le \arg z < 2\pi$. The origin is also cut out because $\ln z$ is not defined at $z = 0$.

Even though $\ln z$ has an infinite number of values, we can render it a single-valued function by restricting the value of θ. Just how we choose to do this is rather arbitrary, but there are two commonly used choices. One of these restricts θ to the values $0 \le \theta_0 < 2\pi$, where θ_0 is the principal argument of $\ln z$. We can illustrate this choice graphically in Figure 4.22 by drawing a branch cut along the positive x axis as we did for $\arg z$ in Figure 4.14. In this case, the origin is a branch point because $\ln z$ is not defined for $z = 0$.

If the value of θ is restricted to the interval $[0, 2\pi)$, then $\ln z$ is called the *principal value* of $\ln z$ and is denoted $\text{Ln}\, z$. (Unfortunately, there is no general consensus on the notation, $\text{Ln}\, z$ and $\ln z$. Some authors use $\ln z$ for the principal value of $\ln z$, while others use $\text{Log}\, z$ and $\log z$.) Thus, we write

$$\text{Ln}\, z = \ln r + i\,\text{Arg}\, z \tag{7}$$

where $\operatorname{Arg} z = \theta_0$, $0 \le \theta_0 < 2\pi$. This single-valued function is called a *branch* of $\ln z$. If θ increases and crosses the branch cut shown in Figure 4.22, then

$$\ln z = \ln r + i(\theta_0 + 2\pi) = \ln r + i\theta \qquad 2\pi \le \theta < 4\pi \tag{8}$$

and we have another branch of $\ln z$. The function described by Equation 8 is one of an infinite number of branches of $\ln z$. Each time θ crosses a branch cut, $\ln z$ goes from one branch to another.

Example 2:
Evaluate $\ln z_1 z_2$ and $\operatorname{Ln} z_1 z_2$, if $z_1 = -1 - i$ and $z_2 = -1 - i\sqrt{3}/3$.

SOLUTION: The polar forms of z_1 and z_2 are

$$z_1 = 2^{1/2} e^{i(5\pi/4 + 2\pi n)} \qquad \text{and} \qquad z_2 = (4/3)^{1/2} e^{i(7\pi/6 + 2\pi k)}$$

where n and $k = 0, \pm 1, \pm 2, \ldots$. Note that $\operatorname{Arg} z_2 = 7\pi/6$ because z_2 lies in the third quadrant. (This is an example of a case where a careless use of a hand calculator will give $\theta = \pi/6$.) The product of z_1 and z_2 is

$$\begin{aligned} z_1 z_2 &= (8/3)^{1/2} e^{i(29\pi/12 + 2\pi n + 2\pi k)} \\ &= (8/3)^{1/2} e^{i(5\pi/12 + 2\pi m)} \end{aligned}$$

where $m = n + k + 1 = 0, \pm 1, \pm 2, \ldots$. Therefore,

$$\begin{aligned} \ln z_1 z_2 &= \frac{1}{2} \ln \frac{8}{3} + i\left(\frac{5\pi}{12} + 2\pi m\right) \\ &= 0.4904 + \frac{5\pi i}{12};\ 0.4904 + \frac{29\pi i}{12}; \\ &\quad 0.4904 - \frac{19\pi i}{12};\ 0.4904 + \frac{53\pi i}{12} \end{aligned}$$

and so on. For $\operatorname{Ln} z_1 z_2$, we choose only the principal argument of $\ln z_1 z_2$, or $\theta_0 = 5\pi/12$, and write

$$\operatorname{Ln} z_1 z_2 = 0.4904 + \frac{5\pi i}{12}$$

Note that we can move from one branch to another when we evaluate $\ln z_1 z_2$ or $\operatorname{Ln}(z_1 z_2)$. Therefore, although $\ln z_1 z_2$ will always equal $\ln z_1 + \ln z_2$, $\operatorname{Ln} z_1 z_2$ will not equal $\operatorname{Ln} z_1 + \operatorname{Ln} z_2$ unless $0 \le \operatorname{Arg} z_1 + \operatorname{Arg} z_2 < 2\pi$. Another commonly used branch cut for $\ln z$ is shown in Figure 4.23. In this case, θ is restricted to the values $-\pi < \theta_0 \le \pi$. You can use either branch cut, as long as you are consistent.

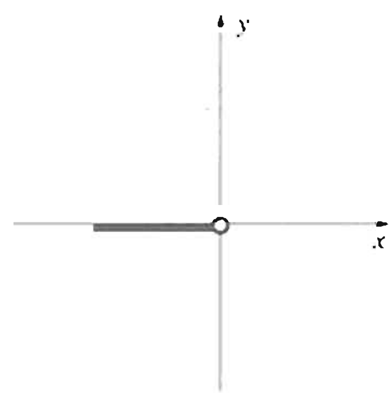

Figure 4.23
The branch cut in the complex plane that restricts $\arg z$ to the values $-\pi < \arg z \le \pi$. The origin is also cut out because $\ln z$ is not defined at $z = 0$.

4.5 Problems

1. Determine $\ln(-i)$ and $\mathrm{Ln}\,(-i)$.
2. Find $\ln z$ and $\mathrm{Ln}\, z$ for (a) $e^{i\pi/3}$ and (b) $-1+i\sqrt{3}$.
3. Find $\mathrm{Ln}\,[\,\mathrm{Ln}\,(1+i\sqrt{3})]$.
4. Find $\mathrm{Ln}\sinh\dfrac{i\pi}{2}$ and $\ln\sinh\dfrac{i\pi}{2}$.
5. Solve the following equations:

   (a) $\ln z = -1$ (b) $\mathrm{Ln}\, z = -1$

   (c) $\ln z = 0$ (d) $\mathrm{Ln}\, z = 0$
6. Does $\mathrm{Ln}\, z_1 z_2 = \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2$ if $z_1 = -1+i$ and $z_2 = -1$?
7. Show that $\mathrm{Ln}\, z = u + iv$ where $u = \frac{1}{2}\ln(x^2+y^2)$ and $v = \tan^{-1}\dfrac{y}{x}$. Show that the families of curves $u(x, y) = u_0$ and $v(x, y) = v_0$ are orthogonal.
8. Show that $\sin^{-1} z = -i\ln(iz \pm \sqrt{1-z^2})$. *Hint*: Solve $(e^{ix} - e^{-ix})/2i = z$ for x.
9. Use the result of the previous problem to solve the equation $\sin z = 2$. Note that this equation has no solution if z is real.
10. In the previous problem, you showed that $\sin^{-1} 2 = \dfrac{\pi}{2} - i\ \ln(2 \pm \sqrt{3})$. Now show that
    $$\sin\left[\frac{\pi}{2} - i\ \ln(2\pm\sqrt{3})\right] = 2.$$
11. Show that $\mathrm{Ln}\, z = \ln r + i\,\theta_0$ gives $e^{\mathrm{Ln}\, z} = z$.
12. Find all the values of $\ln e^z$, where $z = 1 + \pi\, i$.

4.6 Powers of Complex Numbers

We can use $\ln z$ to evaluate general powers of complex numbers, and some of these results may be surprising to you. Let

$$w = z^c \tag{1}$$

where both z and c may be complex. We can express z^c in terms of $\ln z$ by writing

$$w = z^c = e^{c\ln z} = e^{c[\ln r + i\theta_0 + 2\pi k i]} \tag{2}$$

where $k = 0, \ \pm 1, \ \pm 2, \ldots$. The properties of w as a complex number differ markedly depending upon whether c is a (real) integer, a (real) rational number, a (real) irrational number, or a complex number, so let's look at each case in turn.

We discussed the cases where $c = n$ and $c = 1/n$ (n an integer) in Section 3, but we include them here for completeness. If c is an integer, then Equation 2 gives

$$\begin{aligned} w = z^n &= e^{n[\ln r + i(\theta_0 + 2\pi k)]} \\ &= e^{n \ln r + i(n\theta_0 + 2\pi k n)} \\ &= r^n e^{in\theta_0} e^{2\pi n k i} = r^n e^{in\theta_0} \end{aligned} \tag{3}$$

since $e^{i2\pi nk} = 1$.

If $c = 1/n$, so that we are evaluating the nth root of z, then

$$z^{1/n} = e^{\frac{1}{n}\ln r + i\left(\frac{\theta_0}{n} + 2\pi \frac{k}{n}\right)} = r^{1/n} e^{i\theta_0/n} e^{i2\pi k/n} \qquad k = 0,\ \pm 1,\ \pm 2,\ \ldots \tag{4}$$

In this case $z^{1/n}$ is multiple-valued. For $\theta = \theta_0 + 2\pi k/n$, k takes on the values 0, 1, 2, 3, . . . , $n - 1$ before θ goes through a complete cycle of 2π. These n values or n roots of z are distributed uniformly on a circle of radius $r^{1/n}$ centered at the origin in the complex plane.

Example 1:
Find the four 4th roots of $z = 1 + i$.

SOLUTION: The polar form of z is $2^{1/2} e^{i(\pi/4 + 2\pi k)}$ with $k = 0, \pm 1, \pm 2, \ldots$, so the 4th roots of z are $z_k = 2^{1/8} e^{i(\pi/16 + \pi k/2)}$ with $k = 0, 1, 2,$ and 3. The four roots are

$$z_1 = 2^{1/8}\left(\cos\frac{\pi}{16} + i \sin\frac{\pi}{16}\right)$$

$$z_2 = 2^{1/8}\left(\cos\frac{9\pi}{16} + i \sin\frac{9\pi}{16}\right)$$

$$z_3 = 2^{1/8}\left(\cos\frac{17\pi}{16} + i \sin\frac{17\pi}{16}\right)$$

$$z_4 = 2^{1/8}\left(\cos\frac{25\pi}{16} + i \sin\frac{25\pi}{16}\right)$$

Figure 4.24 shows these four roots plotted in the complex plane.

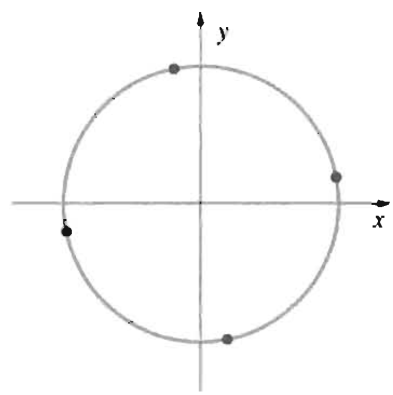

Figure 4.24
The four 4th roots of $z = 1 + i$ plotted in the complex plane. The radius of the circle is $2^{1/8}$.

Now let c be a rational number and express it as $c = m/n$, where n and m are integers. A straightforward application of Equation 2 gives

$$z^{m/n} = r^{m/n} e^{i(m/n)(\theta_0 + 2\pi k)} = r^{m/n}\left[\cos\frac{m}{n}(\theta_0 + 2\pi k) + i \sin\frac{m}{n}(\theta_0 + 2\pi k)\right] \tag{5}$$

where $k = 0, 1, \ldots, n - 1$ (Problem 3). Note that $z^{m/n}$ is n-valued, just like $z^{1/n}$.

Example 2:
Evaluate $(1+i)^{2/3}$.

SOLUTION: The polar form of z is $z = 1 + i = \sqrt{2}\, e^{i(\pi/4+2\pi k)}$ with $k = 0, \pm 1, \pm 2, \ldots$, and so

$$z^{2/3} = (\sqrt{2})^{2/3} \exp\left[i\frac{2}{3}\left(\frac{\pi}{4} + 2\pi k\right)\right] = 2^{1/3} \exp\left[i\left(\frac{\pi}{6} + \frac{4\pi}{3}k\right)\right]$$

where $k = 0$, 1, and 2. The three distinct values of $z^{2/3}$ are

$$\begin{aligned} z_0 &= 2^{1/3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\ &= \frac{3^{1/2}}{2^{2/3}} + \frac{i}{2^{2/3}} \\ z_1 &= 2^{1/3}\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right) \\ &= -2^{1/3}i \\ z_2 &= 2^{1/3}\left(\cos\frac{17\pi}{6} + i\sin\frac{17\pi}{6}\right) \\ &= -\frac{3^{1/2}}{2^{2/3}} + \frac{i}{2^{2/3}} \end{aligned}$$

These three values are plotted in Figure 4.25. Note that they are symmetrically distributed about the origin and are the squares of the three roots that we found in Example 4.2-5.

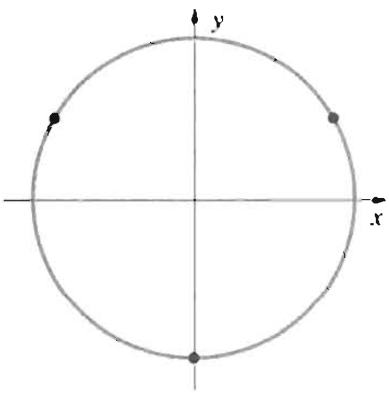

Figure 4.25
The three values of $(1+i)^{2/3}$ plotted in the complex plane. The radius of the circle is $2^{1/3}$.

Problem 4 has you show that z^c has an infinite number of values if c is an irrational number because θ will never repeat as $k = 0, \pm 1$, and so on. Thus, an expression such as $(1+i)^{\sqrt{2}}$ has an infinite number of values

$$(1+i)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}} e^{i\sqrt{2}(\pi/4+2\pi k)} \qquad k = 0, \pm 1, \pm 2, \ldots$$

Thus, the values of $(1+i)^{\sqrt{2}}$ will be densely distributed in the complex plane on the circle of radius $\sqrt{2}^{\sqrt{2}}$ centered at the origin.

Lastly, z^c is also infinite-valued if c is complex, even if a and b are integers. Let $c = a + ib$ in Equation 2 to obtain

$$\begin{aligned} z^c &= e^{(a+ib)[\ln r + i(\theta_0+2\pi k)]} \\ &= e^{a\ln r - b(\theta_0+2\pi k)} e^{i[b\ln r + a(\theta_0+2\pi k)]} \\ &= e^{a\ln r - b(\theta_0+2\pi k)}\{\cos[b\ln r + a(\theta_0+2\pi k)] + i\sin[b\ln r + a(\theta_0+2\pi k)]\} \quad (6) \end{aligned}$$

where $k = 0, \pm 1, \pm 2, \ldots$. Equation 6 looks rather messy, but let's use it to evaluate 1^i.

Example 3:
Evaluate 1^i.

SOLUTION: Substitute $r = 1$, $\theta_0 = 0$, $a = 0$, and $b = 1$ directly into Equation 6 to obtain

$$1^i = e^{-2\pi k} \qquad k = 0, \pm 1, \pm 2, \ldots$$

Thus, 1^i is not necessarily equal to 1; only for $k = 0$ is $1^i = 1$. In fact, 1 raised to any complex power will have an infinite number of values.

Before we finish this chapter, let's look at the transformation $w = z^{1/2}$ from the z-plane to the w-plane. In polar form, we have $w = r^{1/2}e^{i\theta/2}$. When $\theta = 0$, $z = r$ and $w = r^{1/2}$. As θ increases (moving in a counter-clockwise direction), z varies as $re^{i\theta}$ and w varies as $r^{1/2}e^{i\theta/2}$. When θ has made one complete revolution in the z-plane, $w = r^{1/2}e^{i2\pi/2} = r^{1/2}e^{i\pi} = -r^{1/2}$, and so θ has made only one half of a revolution in the w-plane. Then as θ makes a second revolution in the z-plane, θ goes from π to 2π in the w-plane. This behavior occurs because $w = z^{1/2}$ is a double-valued function. We can make w single-valued by restricting θ in the z-plane to the values $0 \le \theta < 2\pi$. As we did in the previous section for Ln z, we can indicate this restriction on θ by a branch cut along the positive x axis in the z-plane (Figure 4.26). As long as θ does not cross the branch cut, w is a single-valued function; it is the principal branch of the double-valued function $w = z^{1/2}$. If θ crosses the branch cut, so that $2\pi \le \theta < 4\pi$, then w represents the second branch of $w = z^{1/2}$. The two branches

$$w_1 = r^{1/2}e^{i\theta/2} \qquad 0 \le \theta < 2\pi$$

and

$$w_2 = r^{1/2}e^{i\theta/2} \qquad 2\pi \le \theta < 4\pi$$

correspond to the two square roots of z, $\pm z^{1/2}$. Further revolutions of θ simply repeat the values of w_1 and w_2.

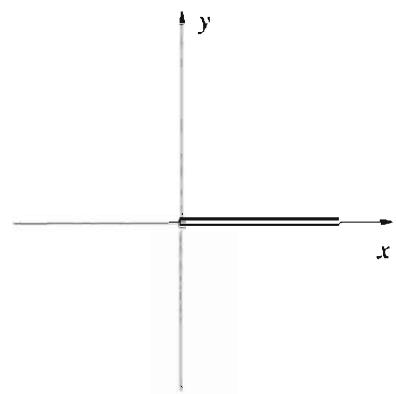

Figure 4.26
The branch cut in the z-plane for the function $w = f(z) = z^{1/2}$. This branch cut restricts w to be single-valued.

4.6 Problems

1. Determine all the values of $(1 - i)^{2/3}$.
2. Determine all the values of $(1 - i)^{3/2}$.

3. Argue that $k = 0, 1, 2, \ldots, n-1$ in Equation 5.

4. Argue that there are an infinite number of values of z^c if c is an irrational number.

5. Determine all the values of $(1+i)^i$.

6. Determine all the values of $(1+i)^{1+i}$.

7. Discuss the transformation $w = z^{1/3}$ from the z-plane to the w-plane, indicating the appropriate branch in the z-plane.

8. Discuss the transformation $w = z^{1/2}$ using a branch cut along the negative x axis in the w-plane.

9. Evaluate i^i.

10. Evaluate 1^π.

11. Show that $(i/2)^{1/2} = (1+i)/2$.

12. Map the region $\operatorname{Arg} z = \pi/m$ (with $m \geq 1/2$) in the z-plane into the w-plane under the transformation $w = z^m$.

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The Geometric Interpretation of the Complex Numbers

Before the 19th century, imaginary and complex numbers posed a problem for mathematicans, although they arose naturally in solving algebraic equations and proved useful in a number of applications. The interpretation of complex numbers was very elusive, but at the end of the 18th century, two obscure men independently offered a beautiful geometric interpretation.

The first to do so was Caspar Wessel (1745-1818), a surveyor. He was born in Norway, but he and his two older brothers attended the University of Copenhagen because there were no Norwegian universities at this time. After one year, Wessel left the University for financial reasons to work as an assistant to his brother, who was a surveyor with the Royal Danish Academy. After 15 years working as a surveyor on the topographical survey of Denmark, he received sabbatical leave with full pay to complete his law degree. After obtaining his law degree, he returned to his surveyor position. Wessel developed his mathematical skills in order to solve some of the difficult problems in geographical surveying. By 1796, he had produced the first accurate map of Denmark. In 1797, Wessel presented his only mathematical paper to the Royal Danish Academy, in which he described the geometric interpretation of complex numbers and the addition and multiplication of vectors as we know them today. Unfortunately, the significance of his work was not recognized, and the paper was not translated from Danish for a wider distribution. Wessel's work was unknown until it was discovered in 1895.

The second one was Jean-Robert Argand (1768-1822), an accountant and bookkeeper in Paris. Very little is known about his background and education. How he came to be interested in complex numbers is also unknown. Argand first published his geometric interpretation of complex numbers in 1806 in a book that he had printed at his own expense. Surprisingly, he did not even put his name on it. His work became known in a rather strange manner. Legendre received a copy of the book and wrote favorably about it to François Français, a fellow mathematican. When Français died, his brother, Jacques, also a mathematician, discovered Legendre's letter and the book. In 1813, he published a paper based on Argand's ideas and asked that the author of the book come forward to receive the recognition that was his due. Argand responded to Français's request and submitted a refined version of his work.

In 1831, Gauss published a geometric interpretation of complex numbers leading to a general acceptance of what is now known as an Argand diagram. Interestingly, Gauss was also involved in survey work in some of the same regions that Wessel surveyed.

CHAPTER 5

Vectors

Many quantities in the physical sciences are *vectors* because they have both a magnitude and a direction associated with them. Examples are velocity, force, angular momentum, and the electric or magnetic field at some point. Contrast these quantities to temperature, density, or time, which have magnitude only, and are called *scalars*.

We begin the chapter with a discussion of vectors in two dimensions because their properties are easy to visualize and the results are readily extended to three (or more) dimensions. In Section 2, we discuss vectors that are functions of a single variable, such as time. We'll show how a vector function of time describes the trajectory of a particle, and then show how its acceleration can be resolved into components that are tangential and perpendicular to its trajectory. We discuss vectors in three dimensions in Section 3 and three-dimensional vector functions in Section 4, with a number of applications to classical mechanics. In the final section, Section 5, we apply these vector methods to the properties of lines and planes in three dimensions. Section 5 nicely illustrates how simple and powerful vector methods are in solving geometrical and physical problems. We'll solve a number of problems in this section that would be much more tedious without vector methods.

5.1 Vectors in Two Dimensions

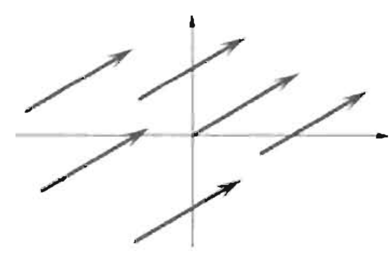


Figure 5.1
All the vectors in this figure are equal because they have the same length and same direction.

Two-dimensional vectors can be defined as ordered pairs of real numbers (a, b) that obey certain algebraic rules that we shall develop below. The numbers a and b are called *components* of the vector. The vector (a, b) can be represented geometrically by a directed line segment (arrow) from the origin of a coordinate system to the point (a, b). The length of the arrow represents the magnitude of the vector and the direction of the arrow represents the direction of the vector. Vectors that have the same length and the same direction are equal. Thus, all the vectors shown in Figure 5.1 are equal. It makes no difference where the tail of the vector is located, although we often locate it at the origin of a coordinate system for convenience. Vectors are usually denoted by bold face type and scalars by italic type. The magnitude (or length) of a vector is denoted by $a = |\mathbf{a}|$. Two vectors, $\mathbf{u} = (u_1, u_2)$

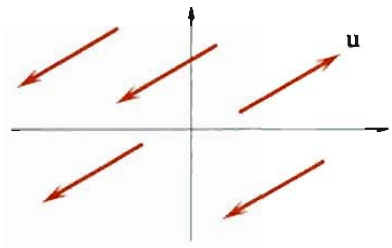


Figure 5.2
The vector $-\mathbf{u}$ points in the opposite direction of $\mathbf{u}$. All the vectors pointing downward in the figure are equal to $-\mathbf{u}$.

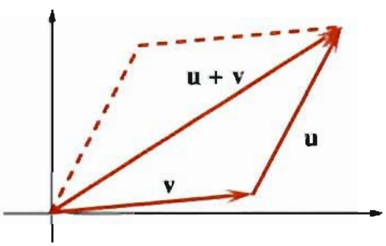


Figure 5.3
An illustration of the parallelogram law of vector addition.

and $\mathbf{v} = (v_1, v_2)$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$. Geometrically, this means that the two vectors have the same magnitude and the same direction, but can be located anywhere, as in Figure 5.1.

We can multiply a vector $\mathbf{v}$ by a scalar c, the result being

$$c\,\mathbf{v} = (cv_1, cv_2) \tag{1}$$

Geometrically, the magnitude of $\mathbf{v}$ is changed by this operation. If $c > 0$, the length of $\mathbf{v}$ is scaled by a factor of c; if $c < 0$, the direction of $\mathbf{v}$ is reversed as well (Figure 5.2). If $c = 0$, we have what we call the *zero vector*, $\mathbf{0} = (0, 0)$.

The addition of two vectors is defined by the relation

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2) \tag{2}$$

In other words, we add vectors by adding their components. The addition of two vectors has the simple geometric interpretation illustrated in Figure 5.3. Place the two vectors tail-to-head and then draw the resultant vector as shown in the figure. This procedure is sometimes called the *parallelogram law of vector addition*. Either Equation 2 or the parallelogram law shows that vector addition is commutative. The subtraction of two vectors is defined by

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2) \tag{3}$$

The direction of $\mathbf{u} - \mathbf{v}$ is shown in Figure 5.4. Note that it is directed from the tip of $\mathbf{v}$ to the tip of $\mathbf{u}$. This can be seen most easily by writing $\mathbf{w} = \mathbf{u} - \mathbf{v}$, in which case $\mathbf{u} = \mathbf{v} + \mathbf{w}$, as shown in Figure 5.4.

Consider the two-dimensional cartesian coordinate system shown in Figure 5.5. The two vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ have unit length (they are *unit vectors*) and point along the x and y axes, respectively. Every two-dimensional vector can be expressed as an additive combination of $\mathbf{i}$ and $\mathbf{j}$; in other words, any

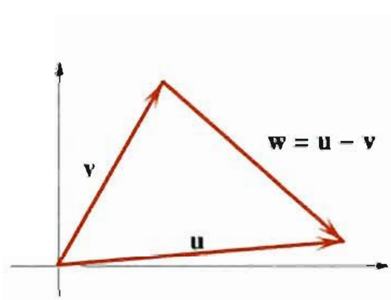


Figure 5.4
An illustration of the subtraction of two vectors. Note that $\mathbf{w} = \mathbf{u} - \mathbf{v}$ points from the tip of $\mathbf{v}$ to the tip of $\mathbf{u}$.

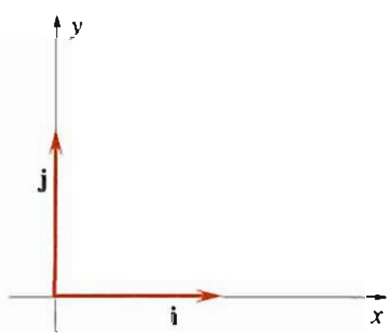


Figure 5.5
The unit vectors of a two-dimensional cartesian coordinate system.

vector **u** is given by

$$\mathbf{u} = (u_1, u_2) = (u_1, 0) + (0, u_2) = u_1(1, 0) + u_2(0, 1) = u_1\mathbf{i} + u_2\mathbf{j} \quad (4)$$

Equation 4 represents the resolution of **u** into a horizontal component and a vertical component. Because u_1 and u_2 are the components of **u** in the x and y directions, respectively, **u** is usually written as $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j}$. The length of **u**, or the *norm* of **u**, in this representation is given by

$$u = |\mathbf{u}| = (u_x^2 + u_y^2)^{1/2} \quad (5)$$

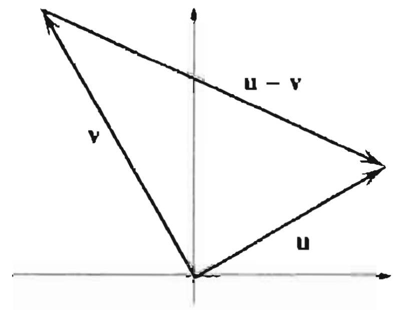


Figure 5.6
The vectors **u**, **v**, and **u** − **v**, where **u** and **v** are perpendicular to each other.

Figure 5.6 shows the vectors **u**, **v**, and **u** − **v**, where **u** and **v** are perpendicular to each other. Pythagoras tells us that

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 = |\mathbf{u} - \mathbf{v}|^2$$

or that (from Equation 5)

$$u_x^2 + u_y^2 + v_x^2 + v_y^2 = (u_x - v_x)^2 + (u_y - v_y)^2$$

or that

$$u_xv_x + u_yv_y = 0 \quad (6)$$

Equation 6 gives us the condition that must hold if two vectors are perpendicular to each other. We can define the *dot product*, an *inner product*, or a *scalar product* of two vectors by

$$\mathbf{u} \cdot \mathbf{v} = u_xv_x + u_yv_y \quad (7)$$

Equation 7 shows that

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \quad (8)$$

and that

$$\mathbf{u} \cdot \mathbf{u} = u_x^2 + u_y^2 = |\mathbf{u}|^2 = u^2 \quad (9)$$

Thus, the dot product is a commutative operation. Equation 6 implies that

$$\mathbf{u} \cdot \mathbf{v} = 0 \qquad \text{if} \qquad \mathbf{u} \perp \mathbf{v} \quad (10)$$

Equation 10 shows that the vector $\mathbf{u} = \mathbf{0}$ is perpendicular to all vectors. Vectors that are perpendicular to each other are said to be *orthogonal*. Thus, the condition $\mathbf{u} \cdot \mathbf{v} = 0$ implies that **u** and **v** are orthogonal. Note that $\mathbf{u} \cdot \mathbf{v} = 0$ does *not* necessarily imply that either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. For example, $\mathbf{u} \cdot \mathbf{v} = 0$ if $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$.

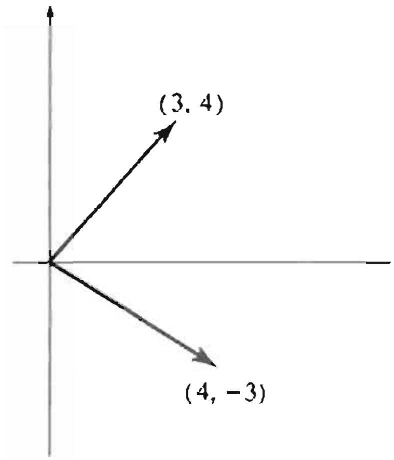


Figure 5.7
The two vectors (3,4) and (4,−3).

Example 1:
Show that the two vectors $\mathbf{u} = (3, 4)$ and $\mathbf{v} = (4, -3)$ are orthogonal. Draw these two vectors in a cartesian coordinate system.

SOLUTION: The dot product of $\mathbf{u}$ and $\mathbf{v}$ is equal to

$$\mathbf{u} \cdot \mathbf{v} = (3)(4) + (4)(-3) = 0$$

and so the two vectors are orthogonal. Figure 5.7 shows these two vectors.

Example 2:
Prove that the line from the apex of an isosceles triangle that bisects its base is perpendicular to the base.

SOLUTION: Figure 5.8 shows that

$$\mathbf{m} + \frac{1}{2}(\mathbf{u} - \mathbf{v}) = \mathbf{u}$$

$$\mathbf{m} - \frac{1}{2}(\mathbf{u} - \mathbf{v}) = \mathbf{v}$$

or that $\mathbf{m} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$. The base is represented by $\mathbf{u} - \mathbf{v}$, and

$$\mathbf{m} \cdot (\mathbf{u} - \mathbf{v}) = \frac{1}{2}(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$$

$$= \frac{1}{2}(|\mathbf{u}|^2 - |\mathbf{v}|^2) = 0$$

since $|\mathbf{u}| = |\mathbf{v}|$. Thus, we see that $\mathbf{m}$ is perpendicular to the base of the triangle.

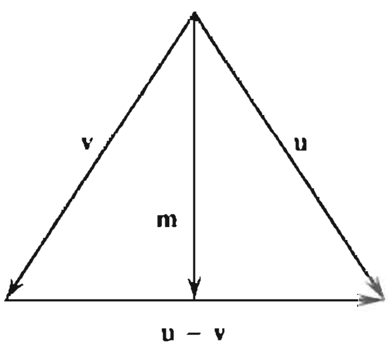


Figure 5.8
A pictorial aid to Example 2.

We can use the law of cosines to derive another expression for the dot product of two vectors. Refer back to Figure 5.6 and write Equation 9 as

$$|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = u^2 + v^2 - 2\mathbf{u} \cdot \mathbf{v} \tag{11}$$

Now the law of cosines says that

$$|\mathbf{u} - \mathbf{v}|^2 = u^2 + v^2 - 2uv\cos\theta \tag{12}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and where $\mathbf{u}$ and $\mathbf{v}$ are arranged tail-to-tail as in Figure 5.6. Comparing Equations 11 and 12, we see that

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \tag{13}$$

Equation 13 is a standard definition of the dot product of two vectors. Either Equation 7 or 13 shows that the dot product of two vectors is a scalar quantity. Sometimes the dot product is called the *scalar product* because of this. Of course, Equations 7 and 13 are equivalent to each other.

Example 3:
Given that $\mathbf{u} = \mathbf{i} + 2\,\mathbf{j}$ and $\mathbf{w} = 2\,\mathbf{i} + \mathbf{j}$, determine the angle ϕ in Figure 5.9.

SOLUTION: The angle θ in Equation 13 is the angle between $\mathbf{u}$ and $\mathbf{v}$ when they are arranged tail-to-tail. Thus, the angle θ, given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{uv}$$

is equal to $\pi - \phi$, as shown in Figure 5.9. Using $\mathbf{v} = \mathbf{w} - \mathbf{u} = \mathbf{i} - \mathbf{j}$ in Equation 13, we have

$$\cos\theta = \frac{-1}{\sqrt{10}}, \qquad \text{or} \qquad \theta = 108.4^\circ$$

or $\phi = 71.6^\circ$.

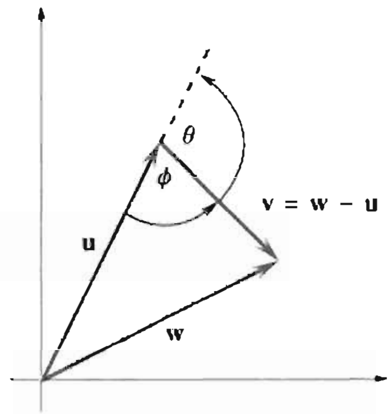


Figure 5.9
The geometry associated with Example 3.

Equation 13 gives the following for the unit vectors $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{i}\cdot\mathbf{i} = |\mathbf{i}|^2\cos 0 = 1 = \mathbf{j}\cdot\mathbf{j}$$

and

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = |\mathbf{i}|\,|\mathbf{j}|\cos\frac{\pi}{2} = 0$$

so

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (u_x\,\mathbf{i} + u_y\,\mathbf{j})\cdot(v_x\,\mathbf{i} + v_y\,\mathbf{j})\\ &= u_x v_x + u_y v_y + 0 + 0 = u_x v_x + u_y v_y\end{aligned}$$

in agreement with Equation 7. It's fairly easy (Problem 15) to show that the dot product satisfies the distributive law:

$$\mathbf{u}\cdot(\mathbf{v} + \mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} \tag{14}$$

Example 4:
Show that the vector $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ is perpendicular to the straight line expressed by $ax + by + c = 0$.

SOLUTION: Let (x_1, y_1) and (x_2, y_2) be two points on the line. Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$. Subtract these two equations to

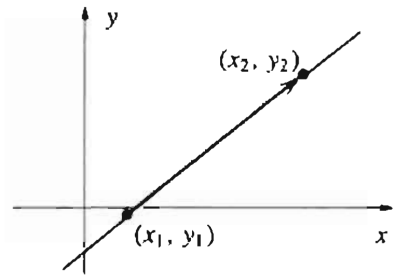


Figure 5.10
If (x_1, y_1) and (x_2, y_2) are two points on the line expressed by $ax + by + c = 0$, then the vector $(x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}$ is coincident with the line.

get

$$a(x_2 - x_1) + b(y_2 - y_1) = 0$$

But

$$a(x_2 - x_1) + b(y_2 - y_1) = (a\,\mathbf{i} + b\,\mathbf{j}) \cdot [(x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}] = 0$$

The vector $(x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}$ is parallel to the straight line (Figure 5.10), so we see that $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ is perpendicular to it.

5.1 Problems

1. Let $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-1, 1)$. Find $u = |\mathbf{u}|$, $v = |\mathbf{v}|$, $\mathbf{u} + \mathbf{v}$, $2\mathbf{u} - 3\mathbf{v}$, and $|\mathbf{u} + \mathbf{v}|$.

2. Solve $a(1, 1) + b(-1, 0) = (1, 0)$ for a and b.

3. Find the unit vector in the same direction as

(a) $(3, -1)$ **(b)** $2\,\mathbf{i} + 3\,\mathbf{j}$ **(c)** $\mathbf{i} + \mathbf{j}$

4. Given that both $\mathbf{u} = 2\,\mathbf{i} + 3\,\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 2\,\mathbf{j}$ start at the origin, as shown in Figure 5.11, calculate the distance between the heads of $\mathbf{u}$ and $\mathbf{v}$.

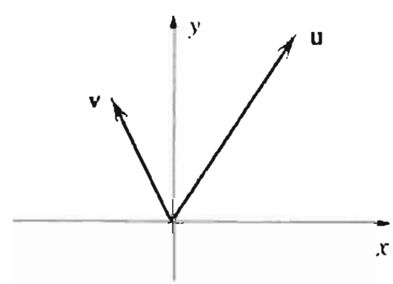


Figure 5.11
The vectors used in Problem 4.

5. Determine the angle between the following pairs of vectors:

(a) $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (-1, 3)$ **(b)** $\mathbf{u} = 3\,\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{j}$

6. Determine the angle between the following pair of vectors:

(a) $\mathbf{u} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j}$ **(b)** $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$

7. Find all the vectors that are perpendicular to $\mathbf{u} = \mathbf{i} - \mathbf{j}$, but are twice as long as $\mathbf{u}$.

8. What angles do the following vectors make with the positive x axis?

(a) $\mathbf{i} + \mathbf{j}$ **(b)** $-\mathbf{i} + \mathbf{j}$ **(c)** $-\mathbf{i} - \mathbf{j}$

9. Consider the triangle sketched in Figure 5.12. Prove that the line joining the midpoints of the sides u and v is one-half the length of the third side and parallel to it.

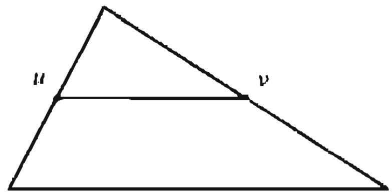


Figure 5.12
A pictorial aid to Problem 9.

10. Prove that the diagonals of a parallelogram bisect each other.

11. Resolve $\mathbf{u}$ into components that are parallel and perpendicular to any other nonzero vector $\mathbf{v}$.

12. Find the perpendicular distance from the point (4, 3) to the line $x + 2y - 4 = 0$.

13. Consider a vector $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$. Rotate $\mathbf{u}$ 90° in a counterclockwise direction to obtain $\mathbf{u}_{90}$, and show that $\mathbf{u}_{90} = -u_2\,\mathbf{i} + u_1\,\mathbf{j}$.

14. Show that $\mathbf{u}\cdot\mathbf{v}$ is the length of the projection of $\mathbf{u}$ onto $\mathbf{v}$ times the length of $\mathbf{v}$.

15. Prove the distributive law geometrically for the dot product of two vectors.

5.2 Vector Functions in Two Dimensions

The vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, directed from the origin to a point (x, y), is called a *position vector*. If the point (x, y) moves with time [in other words, if x and y are functions of time, $x(t)$ and $y(t)$], then $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ is a (vector) function of time. The function $\mathbf{r}(t)$ traces out a curve in the plane as t varies. We can denote a point on this curve by $\mathbf{r}(x, y) = \mathbf{r}(x(t), y(t)) = \mathbf{r}(t)$. We can consider a two-dimensional vector function to be an ordered pair of real-valued functions $(x(t), y(t))$. Most of the concepts associated with real-valued functions carry over to the case of vector functions. For example,

$$\lim_{t\to a} \mathbf{r}(t) = \lim_{t\to a} x(t)\,\mathbf{i} + \lim_{t\to a} y(t)\,\mathbf{j} \tag{1}$$

and we say that $\mathbf{r}(t)$ is continuous at a if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a) = x(a)\,\mathbf{i} + y(a)\,\mathbf{j}$. The continuity of $\mathbf{r}(t)$ depends upon the continuity of its components. We define the derivative of a vector function by

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{\Delta t\to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \tag{2}$$

provided this limit exists. If $\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, we can find $\mathbf{r}'(t)$ by differentiating $\mathbf{r}(t)$ to obtain

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} \tag{3}$$

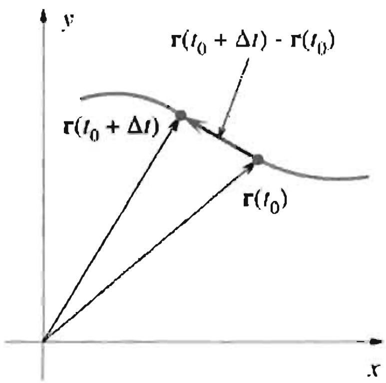


Figure 5.13
An illustration that $\mathbf{r}'(t)$ is tangent to the curve generated by $\mathbf{r}$.

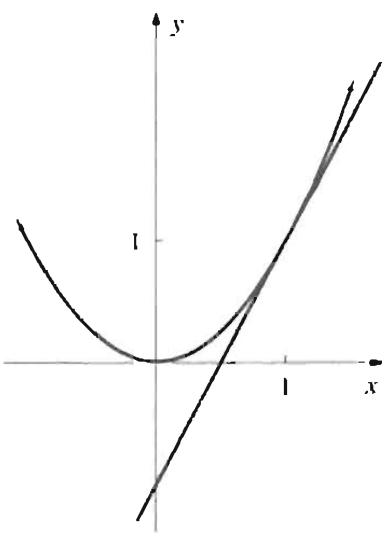


Figure 5.14
The parametric curve $x = t$, $y = t^2$, and the tangent line at the point $t = 1$.

(Note that no derivatives involving $\mathbf{i}$ and $\mathbf{j}$ appear in Equation 3 because neither their magnitudes nor their directions change with time. Their time derivatives are equal to zero.)

Let's look at Equation 2 pictorially, as shown in Figure 5.13. Figure 5.13 shows the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}(t_0 + \Delta t)$, and the difference $\mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0)$. As $\Delta t \to 0$, $|\,\mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0)|/\Delta t$ becomes tangent to the curve $\mathbf{r}(t)$ at $t = t_0$ and is called the *tangent* to the curve at t_0. The vector $\mathbf{r}(t_0) + \mathbf{r}'(t_0)(t - t_0)$ defines the *tangent line* to the curve $\mathbf{r}(t)$ at t_0.

Example 1:
Find the equation of the tangent line at the point $t = 1$ to the curve described parametrically by $x = t$, $y = t^2$.

SOLUTION: The tangent vector at the point $t = 1$ is given by

$$\mathbf{r}'(1) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} = \mathbf{i} + 2\,\mathbf{j}$$

and the equation of the tangent line is given by

$$\mathbf{r}(1) + \mathbf{r}'(1)(t - 1) = \mathbf{i} + \mathbf{j} + (\mathbf{i} + 2\,\mathbf{j})(t - 1) = t\,\mathbf{i} + (2t - 1)\,\mathbf{j}$$

or by $x = t$, $y = 2t - 1$ (Figure 5.14).

If $\mathbf{r}(t)$ describes the position of a particle, then Equation 3 gives us the velocity of the particle. The magnitude of the velocity is the speed of the particle:

$$v = |\mathbf{v}| = |\mathbf{r}'(t)| = (v_x^2 + v_y^2)^{1/2} = \{[x'(t)]^2 + [y'(t)]^2\}^{1/2} \tag{4}$$

The time derivative of the velocity gives us the acceleration of the particle:

$$\mathbf{a} = \mathbf{v}'(t) = \mathbf{r}''(t) = \frac{d^2x}{dt^2}\,\mathbf{i} + \frac{d^2y}{dt^2}\,\mathbf{j} = a_x\,\mathbf{i} + a_y\,\mathbf{j} \tag{5}$$

Example 2:
Describe the motion given by

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$$

Find the velocity, the speed, the acceleration, and the magnitude of the acceleration when $t = 2\pi/3$.

SOLUTION: The parametric equations of the motion are $x(t) = 2\cos t$ and $y(t) = \sin t$. Thus, $x^2 = 4\cos^2 t$, $y^2 = \sin^2 t$, and $x^2 + 4y^2 = 4$. The motion traces out an ellipse in a counterclockwise direction as t increases.

starting at (2, 0) and completing a cycle for $t = 2\pi, 4\pi, \ldots$. Equations 3 and 5 give

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j}$$

$$v = (4\sin^2 t + \cos^2 t)^{1/2} = (4 - 3\cos^2 t)^{1/2}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-2\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j}$$

$$a = (4\cos^2 t + \sin^2 t)^{1/2} = (4 - 3\sin^2 t)^{1/2}$$

At $t = 2\pi/3$, $\mathbf{v} = -(3)^{1/2}\,\mathbf{i} - (1/2)\,\mathbf{j}$, $\mathbf{a} = \mathbf{i} - (3^{1/2}/2)\,\mathbf{j}$, $v = \sqrt{13}/2$, and $a = \sqrt{7}/2$.

The distance travelled along the curve traced by $\mathbf{r}(t)$ from t_0 to t_1 is called the *arc length* of the curve. As Figure 5.15 suggests, the arc length is given by $ds^2 = dx^2 + dy^2$, or (see also Equation 3.5.16)

$$s = \int_{t_0}^{t_1} \{[x'(t)]^2 + [y'(t)]^2\}^{1/2} dt = \int_{t_0}^{t_1} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)]^{1/2} dt = \int_{t_0}^{t_1} v(t)\, dt \quad (6)$$

Note that this equation is equivalent to $v = ds/dt$

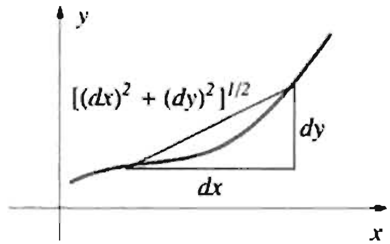


Figure 5.15
A geometric aid to the calculation of the arc length of a curve. Note that $ds^2 = dx^2 + dy^2$.

Example 3:
Calculate the arc length of the curve traced by $\mathbf{r}(t) = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j}$ from $t = 0$ to 2π.

SOLUTION: First determine

$$\mathbf{r}'(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\,\mathbf{j}$$

and

$$v^2(t) = \mathbf{r}'(t) \cdot \mathbf{r}'(t) = 1 + t^2$$

Therefore,

$$s = \int_0^{2\pi} [\,1 + t^2\,]^{1/2} dt = \left[\frac{1}{2}t(1+t^2)^{1/2} + \ln\left(t + \sqrt{1+t^2}\right)\right]_0^{2\pi}$$

$$= \pi(1 + 4\pi^2)^{1/2} + \ln\left(2\pi + \sqrt{1 + 4\pi^2}\right)$$

$$= \pi(1 + 4\pi)^{1/2} + \frac{1}{2}\text{Arcsinh}\, 2\pi = 21.2563\ldots$$

The velocity vector $\mathbf{v}(t)$ is tangent to the curve generated by $\mathbf{r}(t)$. If we denote the arc length along the curve by s, then the *unit tangent vector* is given by

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} \tag{7}$$

where we have used the fact that $v = ds/dt$ (Equation 6). We can write Equation 7 as

$$\mathbf{v}(t) = v(t)\,\mathbf{T}(t) = \frac{ds}{dt}\,\mathbf{T}(t) \tag{8}$$

The acceleration is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{T}\frac{d^2s}{dt^2} + \frac{d\mathbf{T}}{dt}\frac{ds}{dt} = \mathbf{T}\frac{d^2s}{dt^2} + \frac{d\mathbf{T}}{ds}\left(\frac{ds}{dt}\right)^2 \tag{9}$$

It turns out that the magnitude of the vector $d\mathbf{T}/ds$ has an important geometric interpretation. To see what this interpretation is, start with $\mathbf{T}(t)\cdot\mathbf{T}(t) = 1$ (see Equation 7). It is easy to show (Problem 3) that for two arbitrary, differentiable vector functions $\mathbf{u}(t)$ and $\mathbf{w}(t)$,

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{w}) = \mathbf{u}\cdot\frac{d\mathbf{w}}{dt} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{w} \tag{10}$$

Differentiate $\mathbf{T}(t)\cdot\mathbf{T}(t) = 1$ with respect to t to obtain

$$\begin{aligned}\frac{d}{dt}(\mathbf{T}(t)\cdot\mathbf{T}(t)) &= \mathbf{T}\cdot\frac{d\mathbf{T}}{dt} + \frac{d\mathbf{T}}{dt}\cdot\mathbf{T}\\ &= 2\mathbf{T}\cdot\frac{d\mathbf{T}}{dt} = 0\end{aligned} \tag{11}$$

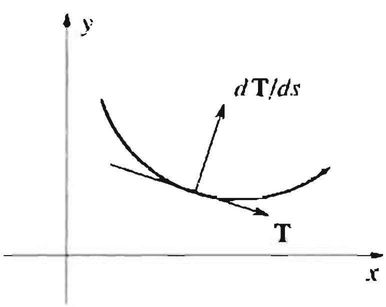


Figure 5.16
The vectors $\mathbf{T}(s)$ and $d\mathbf{T}/ds$ are perpendicular to each other. $d\mathbf{T}/ds$ measures the rate of change of the direction of $\mathbf{T}(s)$ as a function of the arc length s.

Now, using the chain rule of differentiation,

$$\begin{aligned}\frac{d}{dt}(\mathbf{T}\cdot\mathbf{T}) &= \frac{ds}{dt}\frac{d}{ds}(\mathbf{T}\cdot\mathbf{T}) = v(t)\frac{d}{ds}(\mathbf{T}\cdot\mathbf{T})\\ &= 2v(t)\mathbf{T}\cdot\frac{d\mathbf{T}}{ds} = 0\end{aligned} \tag{12}$$

and so we see that the vectors $\mathbf{T}(s)$ and $d\mathbf{T}/ds$ are perpendicular to each other (Figure 5.16).

As the point $\mathbf{r} = (x, y)$ moves along the curve shown in Figure 5.16, the magnitude of $\mathbf{T}$ remains constant; therefore, $d\mathbf{T}/ds$ measures the rate of change of the direction of $\mathbf{T}$. If θ is the angle that $\mathbf{T}$ makes with the x axis, then $\mathbf{T} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Differentiating both sides with respect to s gives

$$\frac{d\mathbf{T}}{ds} = (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\frac{d\theta}{ds}$$

Taking the absolute value of both sides yields

$$\left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\theta}{ds}\right| \tag{13}$$

The rate of change of θ with s is called the *curvature* κ of a curve described parametrically by $x(t)$ and $y(t)$. The curvature tells us how rapidly the direction of the tangent line is changing as we move along the curve. A large value of κ means that the curve is bending sharply, while a small value means that the curve is bending slowly. The reciprocal of κ is called the *radius of curvature*, which we denote by ρ. The radius of curvature has a nice geometric interpretation. Let a curve have a given radius of curvature ρ at the point (x, y). We define the *osculating circle* to the curve at (x, y) to be the circle of radius ρ passing through (x, y) having the same tangent to the curve at (x, y) and whose center lies on the concave side of the curve (Figure 5.17). If the curve is bending slowly, then the radius of the osculating circle will be large, whereas if the curve is bending rapidly, then the radius of the osculating circle will be small.

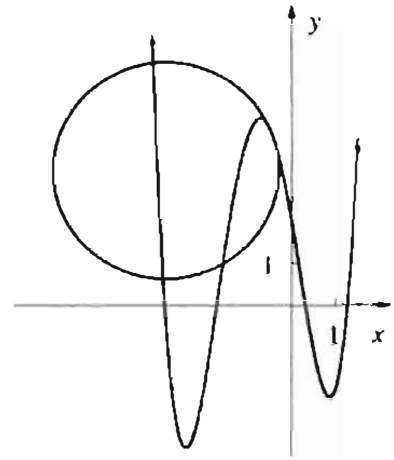


Figure 5.17
An illustration of an osculating circle.

We can derive an expression for κ in terms of the (nonparametric) equation of the curve, $y = y(x)$. Start with

$$\frac{dy}{dx} = \tan\theta$$

Then

$$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d\theta}{ds}\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{d\tan\theta}{d\theta}\frac{d\theta}{ds}$$

$$= \sec^2\theta\frac{d\theta}{ds} = (1+\tan^2\theta)\frac{d\theta}{ds} = \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d\theta}{ds} \tag{14}$$

But

$$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{dx}{ds}\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dx}{ds}\frac{d^2y}{dx^2} \tag{15}$$

and (Problem 8)

$$\frac{dx}{ds} = \frac{1}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}}$$

because

$$ds = [(dx)^2 + (dy)^2]^{1/2} = dx\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

Finally, after equating Equations 14 and 15, we see that

$$\kappa = \left|\frac{d\theta}{ds}\right| = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{\left|\dfrac{d^2y}{dx^2}\right|}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \tag{16}$$

We took the absolute value of d^2y/dx^2 in Equation 16 because we have taken κ to be a positive quantity.

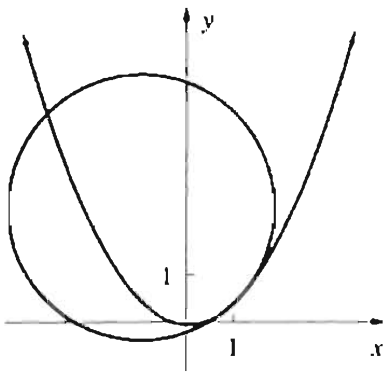


Figure 5.18
The osculating circle calculated in Example 4.

Example 4:
Find the curvature of the parabola $y = x^2/2$ at (a) $x = 0$, (b) $x = 1$, and (c) $x = 2$, and draw the osculating circle at $x = 1$.

SOLUTION: Equation 16 gives us

$$\kappa = \frac{1}{[1 + x^2]^{3/2}}$$

The curvature is equal to 1 at $x = 0$, 0.354 at $x = 1$, and 0.0894 at $x = 2$. The radius of curvature at $x = 1$ is equal to 2.83 and the osculating circle at $x = 1$ is shown in Figure 5.18.

Let's now return to Equation 9 for $\mathbf{a}$:

$$\mathbf{a} = \mathbf{T}\frac{d^2s}{dt^2} + \frac{d\mathbf{T}}{ds}\left(\frac{ds}{dt}\right)^2 \tag{17}$$

If we let $\mathbf{N}$ be a unit vector in the direction of $d\mathbf{T}/ds$, then we can write

$$\frac{d\mathbf{T}}{ds} = \left|\frac{d\mathbf{T}}{ds}\right|\mathbf{N} = \kappa\mathbf{N} \tag{18}$$

and Equation 17 becomes

$$\mathbf{a} = \mathbf{T}\frac{dv}{dt} + \kappa v^2\mathbf{N} \tag{19}$$

Equation 19 resolves the acceleration vector into components tangential and perpendicular to the trajectory of the particle. The tangential component

$$a_{\mathrm{T}} = \frac{dv}{dt}$$

is the rate of change of the speed of the particle and the normal component

$$a_{\mathrm{N}} = \kappa v^2$$

is a measure of the rate of change of the *direction* of the motion of the particle.

Example 5:
The radius vector of a mass moving in a circular orbit of radius R with a uniform angular speed of ω radians per second is given by

$$\mathbf{r} = (R\cos\omega t)\,\mathbf{i} + (R\sin\omega t)\,\mathbf{j}$$

Determine $\mathbf{v}$, v, a_{T}, and a_{N} and interpret the result for a_{N}.

SOLUTION:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-R\omega\sin\omega t)\,\mathbf{i} + (R\omega\cos\omega t)\,\mathbf{j}$$

and

$$v = (R^2\omega^2\sin^2\omega t + R^2\omega^2\cos^2\omega t)^{1/2} = R\omega$$

(This result for v also follows from differentiating the arc length $s = R\theta$ with respect to time and using $\omega = d\theta/dt$.) Note that $\mathbf{r}\cdot\mathbf{v} = 0$ for circular motion. The tangential and normal components of the acceleration are given by

$$a_{\mathrm{T}} = \frac{dv}{dt} = 0$$

and

$$a_{\mathrm{N}} = \kappa v^2$$

Problem 16 has you show that $\kappa = 1/R$ for a circle, so we find that

$$a_{\mathrm{N}} = \frac{v^2}{R} = R\omega^2$$

which is called *centripetal acceleration*. If the moving particle has mass m, then a force $ma_{\mathrm{N}} = mv^2/R$ directed toward the origin is required to maintain the circular motion.

5.2 Problems

1. Determine the equation of the tangent line at $t = 1$ to the curve described by the parametric equations $x = 2t^2 + 1$, $y = t^3 + 2t$.

2. Determine the equation of the tangent line at $t = \pi/4$ to the curve described by the parametric equations $x = \cos^3 t$, $y = \sin^3 t$.

3. Show that $\dfrac{d}{dt}[f(t)\mathbf{w}(t)] = \dfrac{df}{dt}\mathbf{w}(t) + f(t)\dfrac{d\mathbf{w}}{dt}$ and that $\dfrac{d}{dt}(\mathbf{u}\cdot\mathbf{w}) = \dfrac{d\mathbf{u}}{dt}\cdot\mathbf{w} + \mathbf{u}\cdot\dfrac{d\mathbf{w}}{dt}$.

4. We can integrate vector functions by integrating the components. For example, if $\mathbf{v}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, then $\int_a^b \mathbf{v}(u)\,du = \mathbf{i}\int_a^b x(u)\,du + \mathbf{j}\int_a^b y(u)\,du$. Suppose that a particle has an acceleration $\mathbf{a}(t) = 3t\,\mathbf{i} - 2t^2\,\mathbf{j}$ and initial values $\mathbf{r}(0) = 2\,\mathbf{i}$ and $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$. Find its position and velocity at any time.

5. Prove that $\mathbf{a}(u)$ and $d\mathbf{a}/du$ are perpendicular if $|\mathbf{a}| = c =$ constant, provided neither $\mathbf{a}$ nor $d\mathbf{a}/du$ is equal to zero.

6. Determine the arc length of the curve described by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$ from $t = 0$ to $t = 1$.

7. Determine the arc length of the curve described by $\mathbf{r}(t) = t\,\mathbf{i} + \dfrac{t^2}{\sqrt{2}}\,\mathbf{j}$ from $t = 0$ to $t = 1$.

8. Show that if a curve is described nonparametrically by $y = y(x)$, then the arc length is given by $s = \int_a^b \{1 + [y'(x)]^2\}^{1/2} dx$.

9. Find $\mathbf{T}$ for $\mathbf{r}(u) = (\cos^2 u)\,\mathbf{i} + (\sin^2 u)\,\mathbf{j}$. Why is $\mathbf{T}$ a constant vector?

10. Show that $dx/ds = \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{-1/2}$.

11. Determine dx/ds and dy/ds on the parabola $y = x^2$.

12. Determine the curvature of a straight line.

13. Determine the curvature of the ellipse $x^2/16 + y^2/9 = 1$ at the point $(0, 3)$.

14. Determine the value of x at which $y = \ln x$ has its greatest curvature.

15. The center of the osculating circle, called the *center of curvature*, is given by $\gamma = \mathbf{r} + \rho\,\mathbf{N}$, where $\mathbf{r}$ is the position vector to the point (x, y) and $\mathbf{N}$ is a unit normal vector to the curve $y = f(x)$ (Figure 5.19). Determine the radius of curvature and the center of curvature of the parabola $y = 2x^2 + 1$ at the point $(1, 1)$.

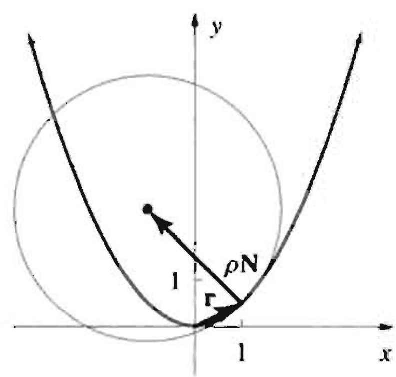


Figure 5.19
An illustration of the center of curvature.

16. Show that $\rho = 1/\kappa = R$ for a circle of radius R.

17. Find the tangent vector and the normal vector to the parabola $y = 2x^2 + 1$ at the point $(1, 1)$.

18. Find the tangent vector and the normal vector to the curve described by $\mathbf{r}(t) = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j}$ at $t = \pi$.

19. Find the tangential and normal components of acceleration at $t = 1/4$ for a particle moving according to $\mathbf{r}(t) = (2\cos 2\pi t)\,\mathbf{i} + (\sin 2\pi t)\,\mathbf{j}$.

20. Find the tangential and normal components of acceleration at $t = 0$ for a particle moving according to $\mathbf{r}(t) = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j}$.

21. The parametric equations of a cycloid are $x(u) = a(u - \sin u)$ and $y(u) = a(1 - \cos u)$. Calculate the length of one arc of a cycloid.

22. Calculate the area under one arc of a cycloid (see the previous problem).

5.3 Vectors in Three Dimensions

Three-dimensional vectors can be treated as ordered triplets of three numbers and obey rules very similar to those obeyed by two-dimensional vectors. We represent three-dimensional vectors by arrows and the geometric interpretation of the addition and subtraction of these vectors follows a parallelogram rule just as it does in two dimensions. We define unit vectors **i**, **j**, and **k** along the x, y, and z axes of a cartesian coordinate system and express three-dimensional vectors as

$$\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k} \tag{1}$$

In terms of ordered triplets of real numbers,

$$\mathbf{i} = (1, 0, 0), \qquad \mathbf{j} = (0, 1, 0), \qquad \text{and} \qquad \mathbf{k} = (0, 0, 1)$$

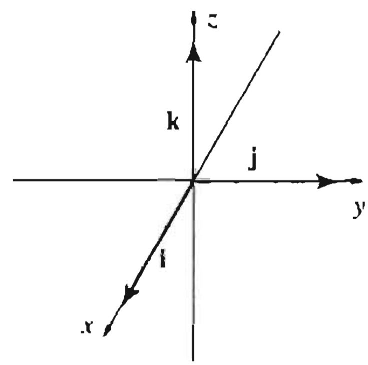


Figure 5.20
The unit vectors of a three-dimensional cartesian coordinate system.

Figure 5.20 shows a cartesian coordinate system and the three unit vectors **i**, **j**, and **k**. The coordinate system shown is said to be a *right-handed coordinate system* because as the x axis is rotated toward the y axis through the smaller angle between them (90° in this case), the positive z axis is in the direction of advance of a right-handed screw. A right-handed coordinate system also obeys the *right-hand rule*: if you curl the fingers of your right hand pointing in the customary direction of a counterclockwise rotation in the xy-plane, then your thumb will point in the same direction as the positive z axis (Figure 5.21). It is customary to use right-handed coordinate systems and we shall do so throughout the book.

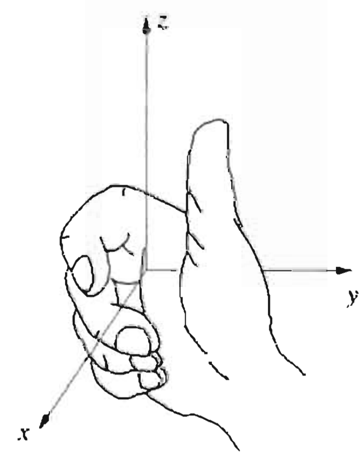


Figure 5.21
An illustration of the right-hand rule.

We can define the dot product of two vectors by

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = uv\cos\theta \tag{2}$$

where the length of **u** is given by

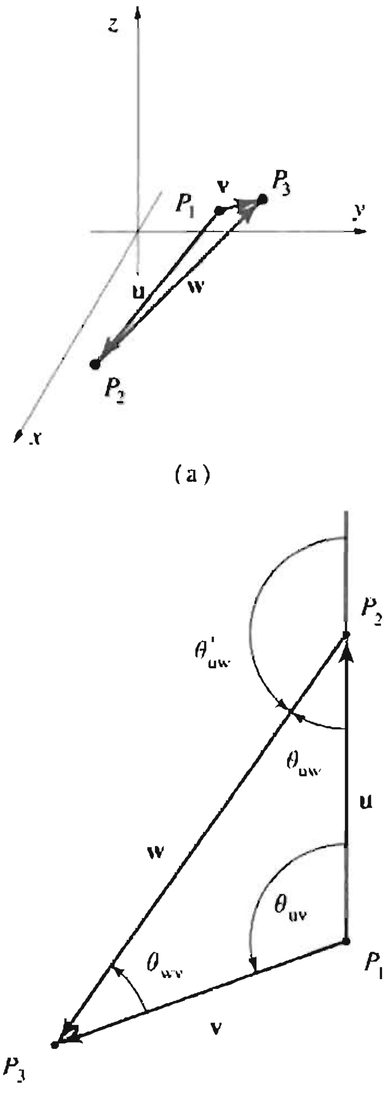
$$u = (\mathbf{u} \cdot \mathbf{u})^{1/2} \tag{3}$$

As in the two-dimensional case, θ is the angle between **u** and **v**, where **u** and **v** are arranged tail-to-tail. According to Equation 2, the dot products of the unit vectors **i**, **j**, and **k** are

$$\begin{gathered} \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \\ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0 \end{gathered} \tag{4}$$

For $\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k}$ and $\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$, we have (Problem 1)

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z \tag{5}$$



(b)

Figure 5.22
(a) The vectors **u**, **v**, and **w** in Example 2. (b) The vectors **u**, **v**, and **w** in Example 2 viewed normal to the plane of the triangle that they form.

and

$$u = (u_x^2 + u_y^2 + u_z^2)^{1/2} \tag{6}$$

Equation 6 is a three-dimensional version of the Pythagoras theorem. Equation 5 can also serve as the definition of the dot product of two vectors.

Example 1:
Find the angle between the two vectors $\mathbf{u} = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v} = 2\,\mathbf{i} - 3\,\mathbf{j} - \mathbf{k}$.

SOLUTION: According to Equation 5.

$$\mathbf{u} \cdot \mathbf{v} = (1)(2) + (2)(-3) + (3)(-1) = -7$$

$$u = (1 + 4 - 9)^{1/2} = \sqrt{14}$$

$$v = (4 + 9 + 1)^{1/2} = \sqrt{14}$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{uv} = -\frac{1}{2} \quad \text{or} \quad \theta = \frac{2\pi}{3} = 120^\circ$$

Example 2:
Determine the angles in the triangle formed by the three vertices, $P_1 = (2, 2, 2)$, $P_2 = (3, 1, 1)$, and $P_3 = (3, 3, 3)$.

SOLUTION: One side of the triangle is given by the vector associated with $P_2 - P_1$, or

$$\mathbf{u} = (3 - 2)\,\mathbf{i} + (1 - 2)\,\mathbf{j} + (1 - 2)\,\mathbf{k} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

and another by $P_3 - P_1$:

$$\mathbf{v} = (3 - 2)\,\mathbf{i} + (3 - 2)\,\mathbf{j} + (3 - 2)\,\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

The third side is given by $P_3 - P_2$, or

$$\mathbf{w} = (3 - 3)\,\mathbf{i} + (3 - 1)\,\mathbf{j} + (3 - 1)\,\mathbf{k} = 2\,\mathbf{j} + 2\,\mathbf{k}$$

Note that $\mathbf{w} = \mathbf{v} - \mathbf{u}$. The triangle is pictured in Figure 5.22a. As Figure 5.22b shows, the angles between **u** and **v** (θ_{uv}) and between **v** and **w** (θ_{vw}) are given by Equation 2. Thus we have

$$\cos\theta_{uv} = \frac{\mathbf{u} \cdot \mathbf{v}}{uv} = -\frac{1}{3} \qquad \theta_{uv} = 109.47^\circ$$

$$\cos\theta_{vw} = \frac{\mathbf{u} \cdot \mathbf{w}}{uw} = \frac{4}{\sqrt{24}} \qquad \theta_{vw} = 35.26^\circ$$

To calculate θ_{uw}, note that $\mathbf{u} \cdot \mathbf{w}/uw$ gives the complement of the interior angle between **w** and **u** (see Figure 5.22b). Therefore,

$$\cos\theta'_{uw} = \frac{\mathbf{u}\cdot\mathbf{w}}{uw} = -\frac{4}{\sqrt{24}} \qquad \theta'_{uw} = 144.74^\circ$$

and so $\theta_{uw} = 180^\circ - 144.74^\circ = 35.26^\circ$. Note that the sum of the three angles is 180°.

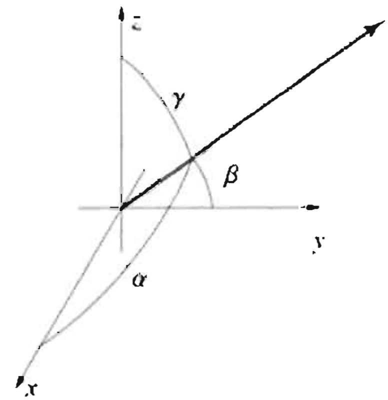


Figure 5.23
The direction angles of a vector **u**.

The *direction angles* of a vector **u** are the angles α, β, and γ that the vector makes with each of the coordinate axes (Figure 5.23). The cosines of these angles are called the *direction cosines* of **u**. For $\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k}$, $\mathbf{i}\cdot\mathbf{u} = u\cos\alpha$, $\mathbf{j}\cdot\mathbf{u} = u\cos\beta$, and $\mathbf{k}\cdot\mathbf{u} = u\cos\gamma$, or

$$\cos\alpha = \frac{\mathbf{i}\cdot\mathbf{u}}{u} = \frac{u_x}{u}; \quad \cos\beta = \frac{\mathbf{j}\cdot\mathbf{u}}{u} = \frac{u_y}{u}; \quad \cos\gamma = \frac{\mathbf{k}\cdot\mathbf{u}}{u} = \frac{u_z}{u} \tag{7}$$

Equation 7 shows that the direction cosines are the components of a unit vector in the same direction as **u**; or

$$\frac{\mathbf{u}}{u} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} \tag{8}$$

Equation 8 gives

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{9}$$

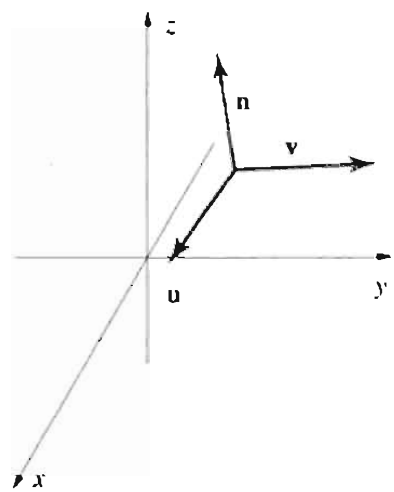


Figure 5.24
The unit vector **n** in the direction $\mathbf{u}\times\mathbf{v}$.

We have defined the dot product of two vectors, which is a scalar quantity. There is another useful definition of the product of two vectors that results in a vector quantity. We define the *cross product* or the *vector product* of **u** and **v** by

$$\mathbf{u}\times\mathbf{v} = |\mathbf{u}||\mathbf{v}|\sin\theta\,\mathbf{n} \tag{10}$$

where θ is the angle ($< 180^\circ$) between **u** and **v** (arranged tail-to-tail as before) and **n** is a unit vector perpendicular to the plane formed by **u** and **v**, and thus perpendicular to both **u** and **v**. The direction of **n** is the same that a right-handed screw would advance if **u** were rotated toward **v** through their smaller angle (Figure 5.24).

Because $\sin\theta = 0$ when $\theta = 0$, Equation 10 says that

$$\mathbf{u}\times\mathbf{u} = \mathbf{0} \tag{11}$$

We also have

$$\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u} \tag{12}$$

Equation 11 says that the cross product of parallel vectors is equal to zero. Equation 12 results because the direction of **n** is reversed if we interchange **u** and **v**.

It is useful to apply Equations 11 and 12 to the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. (Refer to Figure 5.20.)

$$
\begin{gathered}
\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0 \\
\mathbf{i}\times\mathbf{j}=\mathbf{k} \qquad \mathbf{j}\times\mathbf{i}=-\mathbf{k} \\
\mathbf{i}\times\mathbf{k}=-\mathbf{j} \qquad \mathbf{k}\times\mathbf{i}=\mathbf{j} \\
\mathbf{j}\times\mathbf{k}=\mathbf{i} \qquad \mathbf{k}\times\mathbf{j}=-\mathbf{i}
\end{gathered}
\tag{13}
$$

The distributive law for the cross product of two vectors is more lengthy to prove than for the dot product of two vectors, so we simply state without proof (see the references at the end of the chapter) that

$$\mathbf{u}\times(\mathbf{v}+\mathbf{w})=\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$$

The cross product of $\mathbf{u}$ and $\mathbf{v}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is given by

$$
\begin{aligned}
\mathbf{u}\times\mathbf{v} &= (u_x\,\mathbf{i}+u_y\,\mathbf{j}+u_z\,\mathbf{k})\times(v_x\,\mathbf{i}+v_y\,\mathbf{j}+v_z\,\mathbf{k}) \\
&= u_xv_x\,\mathbf{i}\times\mathbf{i}+u_xv_y\,\mathbf{i}\times\mathbf{j}+u_xv_z\mathbf{i}\times\mathbf{k} \\
&\quad + u_yv_x\,\mathbf{j}\times\mathbf{i}+u_yv_y\,\mathbf{j}\times\mathbf{j}+u_yv_z\,\mathbf{j}\times\mathbf{k} \\
&\quad + u_zv_x\,\mathbf{k}\times\mathbf{i}+u_zv_y\,\mathbf{k}\times\mathbf{j}+u_zv_z\,\mathbf{k}\times\mathbf{k} \\
&= (u_yv_z-u_zv_y)\,\mathbf{i}+(u_zv_x-u_xv_z)\,\mathbf{j}+(u_xv_y-u_yv_x)\,\mathbf{k}
\end{aligned}
\tag{14}
$$

Can you see a pattern in the terms of Equation 14?

Example 3:
Show that $\mathbf{u}\times\mathbf{v}$ given by Equation 14 is perpendicular to $\mathbf{u}$.

SOLUTION: We wish to show that $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=0$. Using Equation 14, we get

$$
\begin{aligned}
\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) &= u_x(u_yv_z-u_zv_y)+u_y(u_zv_x-u_xv_z)+u_z(u_xv_y-u_yv_x) \\
&= (u_xu_yv_z-u_zu_xv_y)+(u_zu_yv_x-u_xu_yv_z)+(u_xu_zv_y-u_zu_yv_x) \\
&= 0
\end{aligned}
$$

The product $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})=0$, also. (Can you see this without carrying out the algebra?)

Equation 14 can be rewritten concisely in terms of a determinant as

$$
\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}
\tag{15}
$$

We haven't studied determinants yet (we will in Chapter 9), but you might recognize that Equation 14 results by expanding the determinant in Equation 15 in terms of cofactors of the first row. Recall that

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \mathbf{i}\,(u_y v_z - u_z v_y) + \mathbf{j}\,(u_z v_x - u_x v_z) + \mathbf{k}\,(u_x v_y - u_y v_x) \quad (16)$$

Example 4:
Evaluate $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = 3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{v} = 2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

SOLUTION: Equation 16 gives

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{k}$$

$$= -3\,\mathbf{i} + 7\,\mathbf{j} + 8\,\mathbf{k}$$

Note that $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Cross products occur in a number of physical problems. For example, if a particle is moving about a fixed center with momentum $\mathbf{p}$, then $\mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the position vector of the particle from the fixed center, is equal to the angular momentum of the particle about the fixed center. Similarly, if the particle is acted upon by a force $\mathbf{F}$, then $\mathbf{r} \times \mathbf{F}$ is the torque acting on the particle. Another example is provided by the force acting on a charged particle moving in a magnetic field. In this case, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, where q is the charge on the particle, $\mathbf{v}$ its velocity, and $\mathbf{B}$ the strength of the magnetic field.

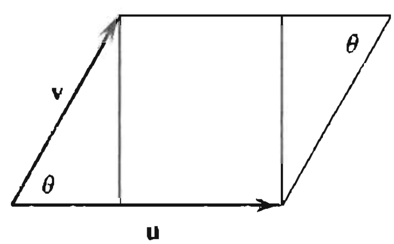


Figure 5.25
An illustration of the geometric interpretation of $\mathbf{u} \times \mathbf{v}$.

The magnitude of the cross product of two vectors has a useful geometric interpretation. Figure 5.25 shows a parallelogram whose sides are the vectors $\mathbf{u}$ and $\mathbf{v}$. The area of this parallelogram is equal to the areas of the two triangles plus the area of the rectangle, or

$$A = 2\left(\frac{1}{2}\right)(v\cos\theta)(v\sin\theta) + (v\sin\theta)(u - v\cos\theta) = uv\sin\theta \quad (17)$$

Notice that Equation 17 says that the area is equal to the length of one side times the height. But $uv\sin\theta$ is simply the magnitude of $\mathbf{u} \times \mathbf{v}$, so we can write

$$A = |\mathbf{u} \times \mathbf{v}| = uv\sin\theta \quad (18)$$

Equation 18 can also be used to calculate the area of a triangle formed by vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} - \mathbf{u}$ (Problem 17).

The cross product results in a vector, so now let's consider the *triple scalar product* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. Using Equation 14, we have

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_x(v_y w_z - v_z w_y) + u_y(v_z w_x - v_x w_z) + u_z(v_x w_y - v_y w_x)$$

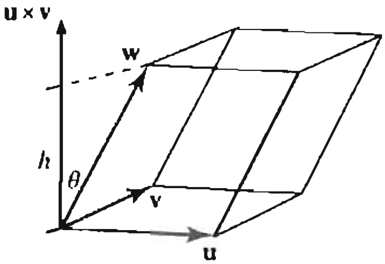


Figure 5.26
A parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

This expression looks just like Equation 14 with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ replaced by u_x, u_y, and u_z, however, so $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ can be expressed in the form

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \tag{19}$$

which is just Equation 15 with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ replaced by u_x, u_y, and u_z.

Just as $|\mathbf{v} \times \mathbf{w}|$ is the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$, $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (Figure 5.26). To see that this is so, note that $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = u|\mathbf{v} \times \mathbf{w}| \cos\theta$ and that $|\mathbf{v} \times \mathbf{w}|$ is the area of the base of the parallelepiped shown in Figure 5.26 and $u \cos\theta$ is its height.

Example 5:
Show that the three vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = 2\,\mathbf{j} + \mathbf{k}$, and $\mathbf{w} = 2\,\mathbf{i} + 4\,\mathbf{j} - \mathbf{k}$ are coplanar.

SOLUTION: It suffices to show that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$, for then the volume of the corresponding parallepiped will be zero.

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 1 \\ 2 & 4 & -1 \end{vmatrix} = -6\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}$$

and

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) - 6 + 2 + 4 = 0$$

We could also have used Equation 19 directly.

We summarize here some general properties of vector products:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \tag{20}$$

$$(c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) = c\mathbf{u} \times \mathbf{v} \tag{21}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \tag{22}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \tag{23}$$

The proofs of all these relations are straightforward.

Equation 23 is called a triple scalar product. We also can have a *triple vector product*, defined by $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$. It is straightforward, albeit somewhat lengthy, to show that (Problem 13)

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \tag{24}$$

Notice that Equation 24 says that the triple cross product is an additive combination of $\mathbf{v}$ and $\mathbf{w}$ of the form $\alpha\, \mathbf{v} - \beta\, \mathbf{w}$ where $\alpha = (\mathbf{u} \cdot \mathbf{w})$ and $\beta = (\mathbf{u} \cdot \mathbf{v})$.

5.3 Problems

1. Use Equations 4 to derive Equation 5.
2. Find the angle between the vectors $\mathbf{u} = 2\,\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v} = 2\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$.
3. The point $(1, -1, 2)$ is connected to the points $(2, 0, 1)$ and $(3, 1, -1)$. Find the angle between the lines connecting these points.
4. Find the direction angles of the vector $\mathbf{u} = \mathbf{i} + \mathbf{j}$.
5. Let $\mathbf{u} = (\cos\alpha_1, \cos\beta_1, \cos\gamma_1)$ and $\mathbf{v} = (\cos\alpha_2, \cos\beta_2, \cos\gamma_2)$. Show that the angle between $\mathbf{u}$ and $\mathbf{v}$ is given by $\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2$.
6. Find the area of a parallelogram with vertices $(0, 0, 0)$, $(1, 1, 1)$, and $(2, 3, 5)$.
7. Determine the interior angle of a regular tetrahedron. *Hint*: See Figure 5.27.

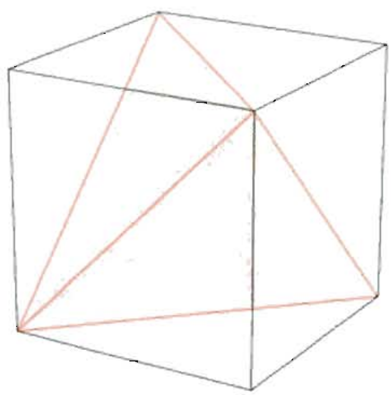


Figure 5.27
The relation between a tetrahedron and a cube.

8. Determine the direction cosines of

   (a) $\mathbf{v} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$ (b) $\mathbf{v} = 2\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}$

9. Use vector methods to derive the law of sines of plane trigonometry. *Hint*: Start with $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$, the condition that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a triangle, and then take cross products.
10. Show that the points $P_1 = (1, -1, 1)$, $P_2 = (2, 0, 1)$, $P_3 = (2, 2, 0)$, and $P_4 = (3, 3, 0)$ are coplanar.
11. Show that the volume of a regular tetrahedron is given by $\sqrt{2}a^3/12$, where a is the length of an edge. Refer to Figure 5.27 for the geometry. *Hint*: The volume of the tetrahedron is 1/6 the volume of the parallelepiped formed by the three vectors forming the tetrahedron.

12. Show that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. Interpret this result.

13. Verify Equation 24.

14. Prove the *Cauchy-Schwartz inequality*, $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}||\mathbf{v}|$.

15. Use the result of the previous problem to prove the *triangle inequality*, $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$. *Hint*: Use the Cauchy-Schwartz inequality from Problem 14.

16. Derive a relation between the cross product of two vectors and the area of the triangle formed by these vectors and their difference.

17. Use the result that you derived in the previous problem to calculate the area of a triangle with vertices $P_1 = (1, -1, 1)$, $P_2 = (0, 2, 1)$, and $P_3 = (3, 1, 2)$.

18. Is $\mathbf{u} \times (\mathbf{v} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{v}$?

5.4 Vector Functions in Three Dimensions

In Section 2, we discussed vector functions in two dimensions. Many of the results of that section carry over to here. The vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, directed from the origin to a point (x, y, z), is called a *position vector*. If the point (x, y, z) moves with time, then $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is a vector function of time. The function $\mathbf{r}(t)$ traces out a curve in space (a *space curve*) as t varies. We can denote a point on this space curve by $\mathbf{r}(x, y, z) = \mathbf{r}(x(t), y(t), z(t)) = \mathbf{r}(t)$. The velocity of the point is given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k} \tag{1}$$

As in the case of two dimensions, $\mathbf{r}'(t) = \mathbf{v}(t)$ is tangent to the curve described by $\mathbf{r}(t)$. The speed of a moving point is given by

$$v = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2} \tag{2}$$

The formulas for the derivatives of quantities such as $f(t)\mathbf{u}(t)$ and $\mathbf{u}(t) \cdot \mathbf{v}(t)$ are the same as for two dimensions:

$$\frac{d}{dt} f(t)\mathbf{u}(t) = \frac{df}{dt}\mathbf{u}(t) + f(t)\frac{d\mathbf{u}}{dt} \tag{3}$$

and

$$\frac{d}{dt}\mathbf{u}(t) \cdot \mathbf{v}(t) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} \tag{4}$$

Formulas that are new to this section are those involving the cross product of two vectors since the cross product is a three-dimensional quantity. It is easy to show that (Problem 1)

$$\frac{d}{dt}\mathbf{u}(t) \times \mathbf{w}(t) = \frac{d\mathbf{u}}{dt} \times \mathbf{w} + \mathbf{u} \times \frac{d\mathbf{w}}{dt} \tag{5}$$

and that

$$\begin{aligned}\frac{d}{dt}[\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})] &= \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \frac{d}{dt}(\mathbf{v} \times \mathbf{w}) \\ &= \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \left(\frac{d\mathbf{v}}{dt} \times \mathbf{w}\right) + \mathbf{u} \cdot \left(\mathbf{v} \times \frac{d\mathbf{w}}{dt}\right) \end{aligned} \tag{6}$$

Here is a nice application of Equation 5. Newton's equations of motion for a body of (constant) mass m are, in vector form,

$$m\frac{d^2\mathbf{r}}{dt} = \mathbf{F}(x, y, z) \tag{7}$$

Multiply both sides of Equation 7 by $\mathbf{r}\times$ to obtain

$$m\,\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{r} \times \mathbf{F} \tag{8}$$

The left side of this equation can be obtained from

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) &= \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} \\ &= \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\end{aligned}$$

because $d\mathbf{r}/dt \times d\mathbf{r}/dt = \mathbf{0}$. Thus Equation 8 may be written as

$$\frac{d}{dt}\left(\mathbf{r} \times m\frac{d\mathbf{r}}{dt}\right) = \mathbf{r} \times \mathbf{F} \tag{9}$$

The left side here is the time derivative of the *angular momentum*:

$$\mathbf{L} = \mathbf{r} \times m\frac{d\mathbf{r}}{dt}$$

and the right side is the *torque*, $\mathbf{N} = \mathbf{r} \times \mathbf{F}$, so that Equation 8 becomes

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} \tag{10}$$

Now suppose that the force acting upon m is a *central force*; that is, suppose that $\mathbf{F}$ is directed along $\mathbf{r}$. Then $\mathbf{r} \times \mathbf{F} = \mathbf{0}$ and

$$\frac{d\mathbf{L}}{dt} = 0 \quad \text{(central force)} \tag{11}$$

Equation 11 says that the angular momentum is a constant of motion in this case.

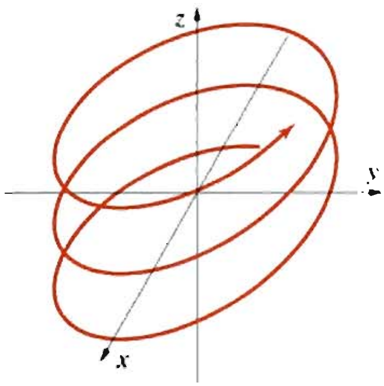


Figure 5.28
The helix described by $x = a\cos qBt/m$, $y = a\sin qBt/m$, $z = bt$.

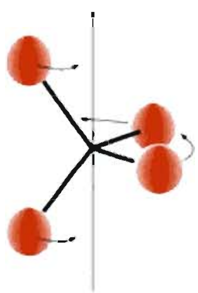


Figure 5.29
A rigid body represented by a collection of masses rigidly separated from each other.

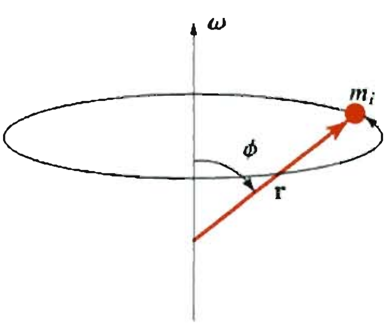


Figure 5.30
The rotation of mass m_i about an axis of rotation $\boldsymbol{\omega}$.

But, since $\mathbf{L}$ is a vector quantity, both its magnitude *and* its direction are constants. Consequently, the path of the particle will remain in a single plane.

Example 1:
The force acting on a particle of charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is $\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$. Determine the motion if $\mathbf{B} = B\mathbf{k}$, where B is a constant.

SOLUTION: Newton's equations of motion are

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{v} \times \mathbf{B} = q\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix}$$

$$= qv_yB\mathbf{i} - qv_xB\mathbf{j}$$

or

$$m\frac{dv_x}{dt} = qv_yB \qquad m\frac{dv_y}{dt} = -qv_xB \qquad m\frac{dv_z}{dt} = 0$$

We'll learn how to solve these equations for $\mathbf{v}(t)$ and subsequently $\mathbf{r}(t)$ in Chapter 11, and a solution is (see also Problem 5 for a reminder of how to solve these equations)

$$x(t) = a\cos\frac{qBt}{m} \qquad y(t) = a\sin\frac{qBt}{m} \qquad z(t) = bt$$

where a and b are constants that depend upon the initial values of $\mathbf{r}(t)$ and $\mathbf{v}(t)$. Thus, the trajectory of the particle is a helix with a uniform speed in the z direction (Figure 5.28). The frequency with which the particle revolves in the xy-direction, qB/m, is called the *Larmour frequency*, or the *cyclotron frequency*.

Now let's consider a rigid body that is rotating about an axis passing through the center of mass, and let's represent the rigid body by a collection of masses rigidly separated from each other, as shown in Figure 5.29. We'll focus on the motion of the ith mass and then sum over all of them at a later stage. Figure 5.30 shows this mass rotating about the axis of rotation, which we denote by $\boldsymbol{\omega}$. The position vector $\mathbf{r}_i$ of this mass is referenced to the center of mass of the rigid body. In a time interval Δt, the mass rotates through an angle $\Delta\theta$ to a new position specified by $\mathbf{r} + \Delta\mathbf{r}$. Now, if Δt is sufficiently small, the magnitude of $\Delta\mathbf{r}$ will approximate the circular arc length $R\Delta\theta$, where $R = r\sin\phi$ and where ϕ is the angle between $\mathbf{r}$ and the axis of rotation $\boldsymbol{\omega}$ (see Figure 5.30). If we let $\mathbf{n}$ be a unit vector along the axis $\boldsymbol{\omega}$ in the direction such that $\Delta\theta > 0$ corresponds to a right-handed system, then we can write

$$\Delta\mathbf{r} \approx \Delta\theta\,\mathbf{n} \times \mathbf{r}$$

Note that $|\Delta\mathbf{r}| \approx r \sin\phi\, \Delta\theta$, as we stated above. As $\Delta t \to 0$, this equation, upon introducing differentials, becomes

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\theta}{dt}\,\mathbf{n} \times \mathbf{r} = \omega\,\mathbf{n} \times \mathbf{r} \tag{12}$$

where ω is equal to $d\theta/dt$. This result suggests that we define an *angular velocity vector* by

$$\boldsymbol{\omega} = \omega\,\mathbf{n} \tag{13}$$

and write Equation 12 as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} \tag{14}$$

The angular momentum of the rigid body consisting of n such masses is given by

$$\mathbf{L} = \sum_{i=1}^{n} m_i(\mathbf{r}_i \times \mathbf{v}_i) = \sum_{i=1}^{n} m_i\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \tag{15}$$

where we used Equation 14 for $\mathbf{v}_i$. Notice that $\boldsymbol{\omega}$ is not subscripted because we are discussing a rigid body, where all the masses rotate with the same angular velocity. We can now derive the equation of motion of the rigid body by differentiating $\mathbf{L}$ with respect to t.

$$\frac{d\mathbf{L}}{dt} = \sum_{i=1}^{n} m_i[\mathbf{v}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) + \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{v}_i)] \tag{16}$$

Using Equation 14, we see that the first term on the right side vanishes. Problem 6 has you show that $\sum_{i=1}^{n} \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{v}_i) = \boldsymbol{\omega} \times \mathbf{L}$ in this case, so Equation 16 becomes

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\omega} \times \mathbf{L} \tag{17}$$

which is a fundamental equation of motion for a rigid body rotating at a constant angular velocity $\boldsymbol{\omega}$. Equation 17 is essentially "$F = ma$" for such a system.

Example 2:
Use Equation 17 to show that $\mathbf{L}$ precesses about the z axis with an angular frequency ω if $\boldsymbol{\omega} = \omega\mathbf{k}$.

SOLUTION:

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\omega} \times \mathbf{L} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ L_x & L_y & L_z \end{vmatrix} = -\omega L_y\,\mathbf{i} + \omega L_x\,\mathbf{j}$$

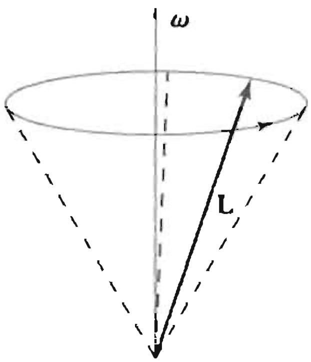


Figure 5.31
An illustration of the precession of the angular momentum vector **L** about the rotation axis ω.

or

$$\frac{dL_x}{dt} = -\omega L_y \qquad \frac{dL_y}{dt} = \omega L_x \qquad \frac{dL_z}{dt} = 0$$

These are similar to the equations that we obtained in Example 1. Certainly, $L_x(t) = a\cos\omega t$, $L_y(t) = a\sin\omega t$, and $L_z(t) = b$, where a and b are constants that could be determined from some initial conditions, satisfy these equations (Problem 5). Thus, we see that L_z remains constant while **L** traces out a circular motion of angular frequency ω about the z axis (the ω axis in Figure 5.31). The angle that **L** makes with ω is given by $L_z/|\mathbf{L}|$.

The trajectory that a particle traces out with time is described by a position vector $\mathbf{r}(t)$ and a velocity vector $\mathbf{v}(t) = d\mathbf{r}/dt$. The vector $\mathbf{v}(t)$ is tangent to the curve at t and the *unit tangent vector* is given by

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} \tag{18}$$

or

$$\mathbf{v}(t) = v(t)\mathbf{T}(t) \tag{19}$$

We can also express $\mathbf{T}(t)$ by (Problem 8)

$$\mathbf{T}(t) = \frac{d\mathbf{r}}{ds} \tag{20}$$

The speed of the particle is also given by

$$v = \frac{ds}{dt} \tag{21}$$

where s is the arc length of the curve, given by

$$s = \int_a^b v(t)dt = \int_a^b \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2} dt \tag{22}$$

Example 3:
Determine the arc length of one cycle of the helix described by

$$\mathbf{r}(t) = b\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + ct\,\mathbf{k}$$

SOLUTION: One cycle occurs when t goes from 0 to 2π (or when t goes from t_0 to $t_0 + 2\pi$), so

$$s = \int_0^{2\pi} (b^2\sin^2 t + b^2\cos^2 t + c^2)^{1/2} dt = 2\pi(b^2 + c^2)^{1/2}$$

As in Section 2, we define the curvature κ by

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \frac{1}{v}\left|\frac{d\mathbf{T}}{dt}\right| \tag{23}$$

and the *principal normal vector* **N** by

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \tag{24}$$

Because **T** is a unit vector, **T** and **N** are perpendicular. (See Equation 12 of Section 2.)

Example 4:
Calculate the curvature of the helical curve described in Example 3.

SOLUTION: We start with

$$\mathbf{v}(t) = -b\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + c\,\mathbf{k}$$

and

$$v(t) = |\mathbf{v}(t)| = (b^2 + c^2)^{1/2}$$

Thus,

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{-b\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + c\,\mathbf{k}}{(b^2 + c^2)^{1/2}}$$

and

$$\frac{d\mathbf{T}}{dt} = \frac{-b\cos t\,\mathbf{i} - b\sin t\,\mathbf{j}}{(b^2 + c^2)^{1/2}}$$

Finally, using Equation 23,

$$\kappa = \frac{1}{v(t)}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{b}{b^2 + c^2}$$

Notice that the curvature is uniform.

The acceleration of a particle is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{25}$$

We can resolve **a** into components that are tangent and normal to the trajectory by using Equations 19 through 22. The result is formally the same as for the two-dimensional case (Problem 14):

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2 \mathbf{N} \tag{26}$$

The tangential component of **a** corresponds to the change in the length of **v** while the normal component corresponds to the change in the direction of **v**. The normal component is called the *centripetal acceleration*.

Example 5:
Determine a_{T} and a_{N} for the trajectory given in Examples 3 and 4.

SOLUTION: $v(t) = (b^2 + c^2)^{1/2}$ and so $dv/dt = 0$. Therefore,

$$a_{\mathrm{T}} = \frac{dv}{dt} = 0 \qquad a_{\mathrm{N}} = \kappa(b^2 + c^2) = b$$

For a space curve traced out by $\mathbf{r}(t)$, $\mathbf{T} = d\mathbf{r}/ds$ is a unit tangent vector and **N** (the principal normal vector) is a unit vector perpendicular to **T**. It seems natural to introduce a third unit vector perpendicular to both **T** and **N** to constitute a set of mutually orthogonal unit vectors attached to the curve. We define the *binormal vector* **B** by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} \tag{27}$$

so that **T**, **N**, and **B** form a right-handed coordinate system (see Figure 5.32 and Problem 22). As $\mathbf{r}(t)$ traces out the path of the curve, the **T**, **N**, **B** coordinate system moves along the curve. Although the magnitudes of **T**, **N**, and **B** stay fixed (they're unit vectors), their orientation varies with time.

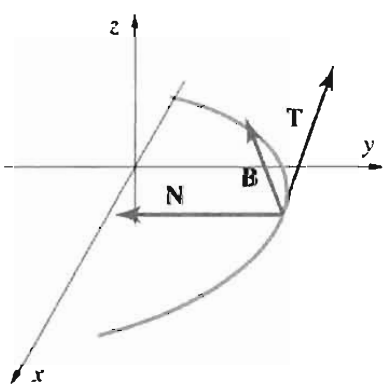


Figure 5.32
The (right-handed) cartesian coordinate system formed by the unit tangent vector **T**, the principal normal vector **N**, and the binormal vector **B**.

Equation 24 says that $d\mathbf{T}/ds = \kappa\mathbf{N}$. Now let's see how **N** and **B** vary with s. Differentiating Equation 27 gives

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds} \tag{28}$$

The first term on the right equals zero because, according to Equation 24, **N** and $d\mathbf{T}/ds$ are parallel. Equation 28 tells us that $d\mathbf{B}/ds$ is perpendicular to **T**. But **B** is a unit vector, so $d\mathbf{B}/ds$ is also perpendicular to **B**. Thus, we see that $d\mathbf{B}/ds$ is perpendicular to both **B** and **T**, and so must be parallel to **N**. (See Figure 5.32.) Therefore, we write

$$\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N} \tag{29}$$

where $\tau(s)$ is called the *torsion of the curve*. The negative sign in Equation 29 is just a convention. It turns out that the shape of a space curve is uniquely determined from a knowledge of κ and τ as functions of the arc length s.

Equations 24 and 29 tell us how $\mathbf{T}$ and $\mathbf{B}$ vary with s. To see how $\mathbf{N}$ varies with s, we differentiate $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ to obtain

$$\begin{aligned}\frac{d\mathbf{N}}{ds} &= \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} = -\tau\mathbf{N} \times \mathbf{T} + \kappa\mathbf{B} \times \mathbf{N} \\ &= \tau(s)\mathbf{B} - \kappa(s)\mathbf{T} \qquad (30)\end{aligned}$$

Equations 24, 28, and 30 are called the *Frenet-Serret equations* and are fundamental equations of differential geometry, which is the study of space curves and surfaces.

Example 6:
Calculate $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, and τ for the helical curve described by

$$\mathbf{r}(t) = b\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + ct\,\mathbf{k}$$

SOLUTION: We start with

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -b\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + c\,\mathbf{k}$$

and so

$$v = \frac{ds}{dt} = (b^2 + c^2)^{1/2}$$

Therefore,

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{-b\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + c\,\mathbf{k}}{(b^2 + c^2)^{1/2}}$$

and

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{-b\cos t\,\mathbf{i} - b\sin t\,\mathbf{j}}{(b^2 + c^2)} = \kappa\mathbf{N}$$

where

$$\mathbf{N} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

and

$$\kappa = \frac{b}{(b^2 + c^2)^{1/2}}$$

Finally,

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{(b^2 + c^2)^{1/2}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -b\sin t & b\cos t & c \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{1}{(b^2 + c^2)^{1/2}}(c\sin t\,\mathbf{i} - c\cos t\,\mathbf{j} + b\,\mathbf{k})$$

and

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt}\frac{dt}{ds} = \frac{c\cos t\,\mathbf{i} + c\sin t\,\mathbf{j}}{(b^2 + c^2)} = \tau\mathbf{N}$$

where

$$\tau = -\frac{c}{b^2 + c^2}$$

5.4 Problems

1. Prove that $\dfrac{d(\mathbf{u} \times \mathbf{v})}{dt} = \dfrac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \dfrac{d\mathbf{v}}{dt}$.

2. Show that if v = constant, then $\mathbf{v}$ and $d\mathbf{v}/dt$ are perpendicular (unless $\mathbf{v}$ or $d\mathbf{v}/dt = 0$).

3. Given $\mathbf{a}(t) = t\,\mathbf{j} + t^2\,\mathbf{k}$, $\mathbf{r}(0) = 0$, and $\mathbf{v}(0) = 3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$, find $\mathbf{r}(t)$ and $\mathbf{v}(t)$.

4. Start with $m_1 \dfrac{d^2\mathbf{r}_1}{dt^2} = \mathbf{F}_{12}$ and $m_2 \dfrac{d^2\mathbf{r}_2}{dt^2} = \mathbf{F}_{21}$, and derive the equation $\dfrac{m_1 m_2}{m_1 + m_2}\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{F}_{21}$, where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. Interpret this (important) result. *Hint*: Use Newton's third law, which tells us that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

5. Solve the equations for v_x and v_y in Example 1 by differentiating them with respect to time to obtain two equations of the form $\dfrac{d^2u}{dt^2} + \alpha^2 u = 0$, where $u = v_x$ or v_y and $\alpha^2 = qB/m$. Then show that $u = A \cos \alpha t$ and $u = B \sin \alpha t$, where A and B are constants, satisfy this equation.

6. Show that the right side of Equation 16 is equal to $\boldsymbol{\omega} \times \mathbf{L}$. *Hint*: Use the fact that $\mathbf{r} \cdot \mathbf{v} = 0$ and $\boldsymbol{\omega} \cdot \mathbf{v} = 0$ for circular motion.

7. Let's look at the motion of a charged particle in a magnetic field from a different point of view than we did in Example 1. You might recall that a charged particle moving with a velocity $\mathbf{v}$ in a closed orbit produces a magnetic moment $\boldsymbol{\mu}$ given by $\boldsymbol{\mu} = \dfrac{q(\mathbf{r} \times \mathbf{v})}{2}$, where q is the charge. Show that $\mu = iA$, where i is the associated electric current and A is the area if the orbit is circular. Express $\boldsymbol{\mu}$ in terms of the angular momentum of the particle and describe the motion of $\boldsymbol{\mu}$ and the charged particle in a static magnetic field.

8. Show that the unit tangent vector $\mathbf{T}$ is equal to $d\mathbf{r}/ds$.

9. Use Equation 18 to find the unit tangent vector to the curve traced out by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$.

10. Use Equation 20 to derive the result of the previous problem.

11. Use Equation 18 to find the unit tangent vector to the curve traced out by the helix described by $\mathbf{r}(t) = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j} + bt\,\mathbf{k}$.

12. Use Equation 20 to derive the result of the previous problem.

13. Prove that $\mathbf{T}$ and $\mathbf{N}$ are perpendicular.

14. Derive Equation 26.

15. Start with Equation 26 and show that $\kappa = |\mathbf{v} \times \mathbf{a}|/v^3$.

16. Use the formula for κ in the previous problem to find the curvature of the curve traced by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$.

17. Use the formula for κ in Problem 15 to find the curvature of the curve traced by the helix described by $\mathbf{r}(t) = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j} + bt\,\mathbf{k}$.

18. The trajectory of a particle is described by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$. Find the tangential and normal components of the acceleration $\mathbf{a}$ at $t = 1$.

19. The trajectory of a particle is described by the helix $\mathbf{r}(t) = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j} + bt\,\mathbf{k}$. Find the tangential and normal components of the acceleration $\mathbf{a}$ at $t = 2\pi$.

20. Use Equation 22 to calculate the arc length from $t = 0$ to $t = 2$ of the curve with parametric equations $x(t) = a_x t + b_x$, $y(t) = a_y t + b_y$, $z(t) = a_z t + b_z$, where the a's are constants. Does your answer make sense?

21. Calculate the curvature of the line given in the previous problem. Does your answer make sense?

22. Show that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and $\mathbf{T} = \mathbf{N} \times \mathbf{B}$. (See Figure 5.32.)

23. Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, and τ for the curve traced out by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{2}{3}t^3\,\mathbf{k}$.

24. Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, and τ for the space curve traced out by $\mathbf{r}(t) = \mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k}$. Explain your result for τ.

25. A nucleus with a nonzero spin has a magnetic moment $\boldsymbol{\mu}$ that is proportional to the nuclear spin angular momentum $\mathbf{I}$. The proportionality constant γ in the relation $\boldsymbol{\mu} = \gamma\mathbf{I}$ is called the magnetogyric ratio. We can describe the precession of $\boldsymbol{\mu}$ classically in the following way. A magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ experiences a torque $\boldsymbol{\mu} \times \mathbf{B}$ and so the corresponding equation of motion is $\dfrac{d\mathbf{I}}{dt} = \boldsymbol{\mu} \times \mathbf{B}$. Using the relation $\boldsymbol{\mu} = \gamma\,\mathbf{I}$, we have $\dfrac{d\boldsymbol{\mu}}{dt} = \gamma\boldsymbol{\mu} \times \mathbf{B}$. Show that if $\mathbf{B}$ is directed along the z direction, then the direction of $\boldsymbol{\mu} \times \mathbf{B}$ is such that the tip of the vector $\boldsymbol{\mu}$ travels along the circle of the cone shown in Figure 5.33, resulting in precessional motion.

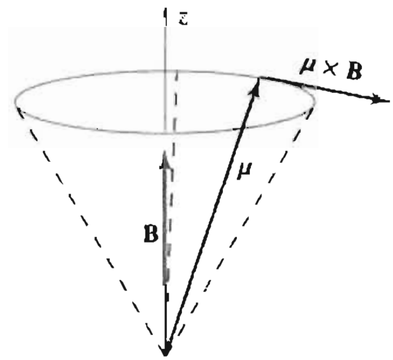


Figure 5.33
The precession of the tip of the magnetic moment vector $\boldsymbol{\mu}$ along the circle of a cone.

5.5 Lines and Planes in Space

We can use the vector formalism that we have developed to easily solve a number of problems in three-dimensional analytic geometry that would be fairly difficult to solve without using vectors. These calculations are best illustrated by example, so this section will contain a fair number of Examples.

Let's start off by describing a straight line in three-dimensional space. There are several ways to do this. For example, we can specify the line by saying that it passes through some point $\mathbf{r}_0(x_0, y_0, z_0)$ and is parallel to some vector $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$. If $\mathbf{r}(x, y, z)$ is any other point on the line, then the vector $\mathbf{r} - \mathbf{r}_0$ is parallel to $\mathbf{v}$ (see Figure 5.34) and we can write the equation of the straight line as

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{v} \tag{1}$$

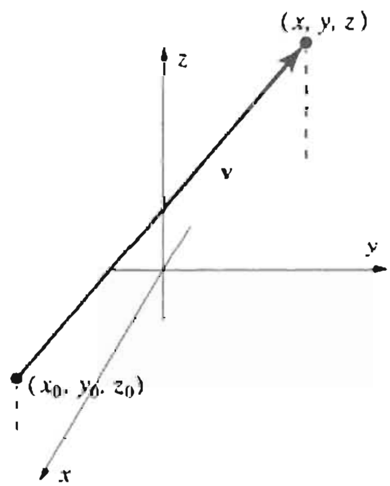


Figure 5.34
The line defined by the two points (x, y, z) and (x_0, y_0, z_0).

where t is a parameter that can take on any real value. Equation 1, which actually represents three equations, is called the *parametric form* of a straight line. The three separate parametric equations are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct \tag{2}$$

The numbers (a, b, c) are called the *direction numbers* of the line and $a/|\mathbf{v}|$, $b/|\mathbf{v}|$, and $c/|\mathbf{v}|$ are the direction cosines of the line.

The following Example shows how to obtain the equations of a line if it is specified by two points.

Example 1:
Determine the equation of the straight line that passes through the two points $\mathbf{r}_1(1, -1, 2)$ and $\mathbf{r}_2(-2, 3, 1)$.

SOLUTION: The line is parallel to the vector

$$\begin{aligned}\mathbf{r}_2 - \mathbf{r}_1 &= (-2 - 1)\,\mathbf{i} + (3 + 1)\,\mathbf{j} + (1 - 2)\,\mathbf{k} \\ &= -3\,\mathbf{i} + 4\,\mathbf{j} - \mathbf{k}\end{aligned}$$

Choosing the point $\mathbf{r}_1$, Equations 2 give

$$x = 1 - 3t \qquad y = -1 + 4t \qquad z = 2 - t$$

If we had chosen the point $\mathbf{r}_2$, instead, we would have obtained

$$x = -2 - 3t \qquad y = 3 + 4t \qquad z = 1 - t$$

The parametric equations of a line are not unique. (The second set is obtained from the first set by replacing t by $t + 1$.)

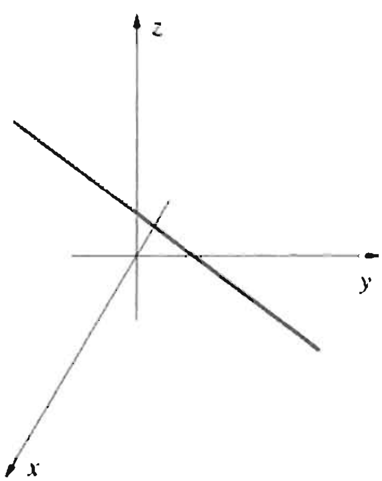


Figure 5.35
The straight line described by the parametric equations $x = 1 + 2t$, $y = 6 - 2t$, $z = -2 + 4t$.

The parametric equations give us an easy way to see where a line passes through the xy-plane, the xz-plane, and the yz-plane (the three coordinate planes). Consider the line described by

$$x = 1 + 2t \qquad y = 6 - 2t \qquad z = -2 + 4t$$

This line passes through the yz-plane when $x = 0$, or when $t = -1/2$. The values of y and z at $t = -1/2$ are $y = 7$ and $z = -4$. Similarly, the line passes through the xy-plane at $x = 2$, $y = 5$ (when $t = 1/2$) and through the xz-plane at $x = 7$, $z = 10$ (when $t = 3$) (Figure 5.35).

We can express the equation(s) of a straight line in what is called the *point-direction form*. Provided a, b, and c are not zero, we can solve for t in each case in Equations 2 and write

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{3}$$

If one of the a, b, or c happens to be zero (let's say $a = 0$), then Equations 2 yield

$$x = x_0; \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

instead of Equations 3. The equation $x = x_0$ means that the line lies in the plane perpendicular to the x axis at $x = x_0$.

Example 2:
Determine both the parametric form and the point-direction form of the line that passes through the points $\mathbf{r}_1(2, 1, 1)$ and $\mathbf{r}_2(-1, 0, 1)$.

SOLUTION: The line is parallel to the vector

$$\begin{aligned}\mathbf{r}_2 - \mathbf{r}_1 &= (-1 - 2)\,\mathbf{i} + (0 - 1)\,\mathbf{j} + (1 - 1)\,\mathbf{k} \\ &= -3\,\mathbf{i} - \mathbf{j} + 0\,\mathbf{k}\end{aligned}$$

The parametric equations are (choosing the point specified by $\mathbf{r}_1$)

$$x = 2 - 3t \qquad y = 1 - t \qquad z = 1$$

and the equations in the point-direction form are

$$\frac{x - 2}{-3} = \frac{y - 1}{-1}; \qquad z = 1$$

or simply

$$\frac{x - 2}{3} = \frac{y - 1}{1}; \qquad z = 1$$

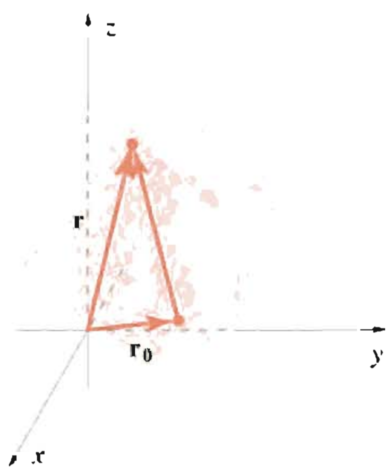


Figure 5.36
The vector $\mathbf{r} - \mathbf{r}_0$ lying in a plane.

Geometrically, a plane is determined by a point that lies in the plane and a vector normal to the plane. Let the point be $\mathbf{r}_0(x_0, y_0, z_0)$ and let $\mathbf{v} = A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$ be the vector normal to the plane. If $\mathbf{r}(x, y, z)$ is any other point in the plane, then $\mathbf{r} - \mathbf{r}_0$ lies in the plane (see Figure 5.36). Consequently, $\mathbf{r} - \mathbf{r}_0$ and $\mathbf{v}$ are perpendicular, and so

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{v} = 0 \tag{4}$$

is the equation of the plane. Writing $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{v}$ in terms of rectangular coordinates gives

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{5}$$

or

$$Ax + By + Cz + D = 0 \tag{6}$$

where $D = -(Ax_0 + By_0 + Cz_0)$. The next Example shows that we can also specify a plane by three points that lie in the plane.

Example 3:
Determine the equation of a plane through the three points $\mathbf{r}_1(1, 1, 7)$, $\mathbf{r}_2(-2, 2, 3)$, and $\mathbf{r}_3(1, -1, 6)$.

SOLUTION: The two vectors

$$\begin{aligned}\mathbf{r}_2 - \mathbf{r}_1 &= (-2-1)\,\mathbf{i} + (2-1)\,\mathbf{j} + (3-7)\,\mathbf{k}\\ &= -3\,\mathbf{i} + \mathbf{j} - 4\,\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\mathbf{r}_3 - \mathbf{r}_1 &= (1-1)\,\mathbf{i} + (-1-1)\,\mathbf{j} + (6-7)\,\mathbf{k}\\ &= -2\,\mathbf{j} - \mathbf{k}\end{aligned}$$

lie in the plane. Using the fact that the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane formed by $\mathbf{u}$ and $\mathbf{v}$, we see that a vector that is perpendicular to $\mathbf{r}_2 - \mathbf{r}_1$ and to $\mathbf{r}_3 - \mathbf{r}_1$ is given by

$$\mathbf{v} = (\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -4 \\ 0 & -2 & -1 \end{vmatrix}$$

$$= -9\,\mathbf{i} - 3\,\mathbf{j} + 6\,\mathbf{k}$$

Thus, according to Equation 5, the equation of the plane is

$$-9(x-1) - 3(y-1) + 6(z-7) = 0$$

or

$$3x + y - 2z + 10 = 0$$

Example 4:
Determine the equation of the line through the point $\mathbf{r}(2, -1, 3)$ that is perpendicular to the plane $x - y + 2z = 4$.

SOLUTION: According to Equation 6, the vector $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$ is normal to the plane and is thus parallel to the desired line. The parametric equations of the line are $\mathbf{r} + \mathbf{v}\,t$ in vector form, or

$$x = 2 + t \qquad y = -1 - t \qquad z = 3 + 2t$$

The point-direction form is given by

$$\frac{x-2}{1} = \frac{y+1}{-1} = \frac{z-3}{2}$$

Vector methods also provide us with a straightforward way to find the perpendicular distance of a point P to a line. Let $\mathbf{r}$ be a vector from P to any point on the line, and let $\mathbf{v}$ be a vector parallel to the line. Then Figure 5.37 shows that the perpendicular distance from P to the line is given by

$$d = |r \sin\theta| = \left|\frac{\mathbf{r}\times\mathbf{v}}{v}\right| \tag{7}$$

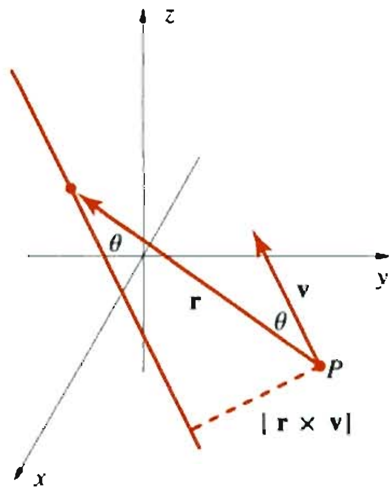


Figure 5.37
The perpendicular distance from a point P to a given line.

Example 5:
Find the perpendicular distance from the point $\mathbf{r}_1(1, 2, -1)$ to the line joining the points $\mathbf{r}_2(1, 0, 0)$ and $\mathbf{r}_3(1, -1, 2)$.

SOLUTION: The line is parallel to the vector

$$\begin{aligned}\mathbf{v} = \mathbf{r}_3 - \mathbf{r}_2 &= (1-1)\,\mathbf{i} + (-1-0)\,\mathbf{j} + (2-0)\,\mathbf{k}\\ &= -\mathbf{j} + 2\,\mathbf{k}\end{aligned}$$

Now choose $\mathbf{r}$ in Equation 7 to be $\mathbf{r}_2 - \mathbf{r}_1$

$$\begin{aligned}\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 &= (1-1)\,\mathbf{i} + (0-2)\,\mathbf{j} + (0+1)\,\mathbf{k}\\ &= -2\,\mathbf{j} + \mathbf{k}\end{aligned}$$

Then

$$\mathbf{r}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & -2 & 1\\ 0 & -1 & 2\end{vmatrix} = -3\,\mathbf{i}$$

and

$$d = \left|\frac{\mathbf{r}\times\mathbf{v}}{v}\right| = \left|\frac{3}{\sqrt{5}}\right| = \frac{3}{\sqrt{5}}$$

Of course, we obtain the same result if we had chosen $\mathbf{r} = \mathbf{r}_3 - \mathbf{r}_1$ instead of $\mathbf{r}_2 - \mathbf{r}_1$ (Problem 11).

Vector methods provide us with a straightforward method to find the perpendicular distance of a point P to a plane. Let $\mathbf{r}$ be a vector from P to any point in the plane, and let $\mathbf{v}$ be a vector normal to the plane. Then Figure 5.38 shows that the perpendicular distance d from P to the plane is equal to

$$d = |r\cos(\pi - \theta)| = |r\cos\theta| = \left|\frac{\mathbf{r}\cdot\mathbf{v}}{v}\right| \tag{8}$$

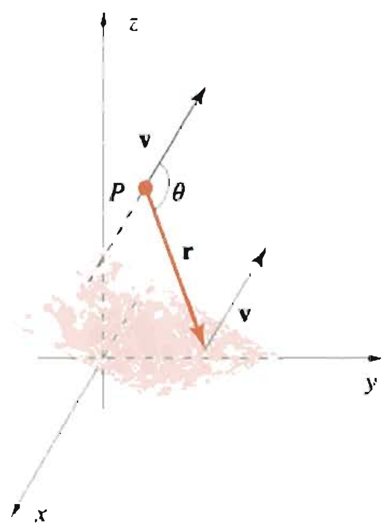


Figure 5.38
The perpendicular distance from a point P to a given plane.

Example 6:
Find the perpendicular distance from the point $(1, 2, 3)$ to the plane described by the equation $3x - 2y + 5z = 10$.

SOLUTION: According to Equation 6, the vector normal to the plane is given by

$$\mathbf{v} = 3\,\mathbf{i} - 2\,\mathbf{j} + 5\,\mathbf{k}$$

Let $x = y = 1$ and solve $3x - 2y + 5z = 10$ for z to see that the point $(1, 1, 9/5)$ lies in the plane. Thus, the vector $\mathbf{r}$ in Equation 8 is given by

$$\mathbf{r} = (1-1)\,\mathbf{i} + (1-2)\,\mathbf{j} + \left(\frac{9}{5} - 3\right)\,\mathbf{k} = -\mathbf{j} - \frac{6}{5}\,\mathbf{k}$$

and Equation 8 gives

$$d = \left|\frac{2-6}{\sqrt{38}}\right| = \frac{4}{\sqrt{38}}$$

Once again the result is independent of the point that we choose in the plane (Problem 13).

We can use vector methods to find the tangent line to a curve and the tangent plane to a surface. Let's look for the tangent line first. Consider a curve in three-dimensional space (*a space curve*) described parametrically by $x(t)$, $y(t)$, and $z(t)$. The position vector is $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ and $d\mathbf{r}/dt$ is the vector tangent to the curve traced out by $\mathbf{r}(t)$ as t varies. The vector $\mathbf{r}'(t)$ is called the *tangent vector* of the curve at the point $x(t)$, $y(t)$, $z(t)$ provided $\mathbf{r}'(t) \neq 0$. The *tangent line* is the straight line passing through the point $\mathbf{r}(t_0) = x(t_0)\,\mathbf{i} + y(t_0)\,\mathbf{j} + z(t_0)\,\mathbf{k}$ and parallel to

$$\mathbf{r}'(t_0) = \left(\frac{dx}{dt}\right)_{t_0}\mathbf{i} + \left(\frac{dy}{dt}\right)_{t_0}\mathbf{j} + \left(\frac{dz}{dt}\right)_{t_0}\mathbf{k}$$

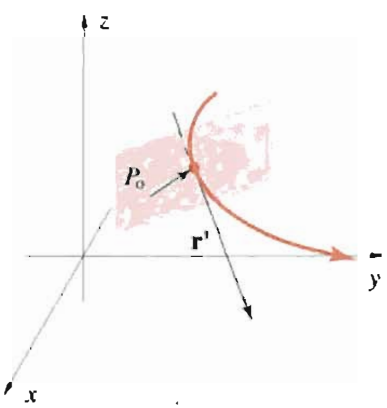


Figure 5.39
A space curve, its tangent vector $\mathbf{r}'(t)$, and its normal plane.

According to Equations 2, the tangent line is described by the vector equation

$$\mathbf{r} - \mathbf{r}(t_0) = t\left(\frac{d\mathbf{r}}{dt}\right)_{t_0} \tag{9}$$

or by the parametric equations

$$x - x_0 = t\left(\frac{dx}{dt}\right)_{t_0} \qquad y - y_0 = t\left(\frac{dy}{dt}\right)_{t_0} \qquad z - z_0 = t\left(\frac{dz}{dt}\right)_{t_0} \tag{10}$$

where t can take on any real value.

The *normal plane* to the curve at the point $P_0 = (x_0, y_0, z_0)$ passes through P_0 and is perpendicular to the tangent line (Figure 5.39). According to Equation 5, if $\mathbf{v} = A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$ is a vector that is normal to a plane, then the equation for the plane is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. Using Equation 9 or 10, we see that the equation for the normal plane is

$$(x - x_0)\left(\frac{dx}{dt}\right)_{t_0} + (y - y_0)\left(\frac{dy}{dt}\right)_{t_0} + (z - z_0)\left(\frac{dz}{dt}\right)_{t_0} = 0 \qquad (11)$$

Example 7:
Determine the tangent line and the normal plane at $\theta = 2\pi$ to the curve whose parametric equations are

$$x(\theta) = b\cos\theta \qquad y(\theta) = b\sin\theta \qquad z(\theta) = c\theta$$

SOLUTION: $\mathbf{r}(\theta) = b\cos\theta\,\mathbf{i} + b\sin\theta\,\mathbf{j} + c\theta\,\mathbf{k}$ and $\mathbf{r}'(\theta) = -b\sin\theta\,\mathbf{i} + b\cos\theta\,\mathbf{j} + c\,\mathbf{k}$. At $\theta = 2\pi$, $\mathbf{r}_0 = b\,\mathbf{i} + 2\pi c\,\mathbf{k}$, $\mathbf{r}'_0 = b\,\mathbf{j} + c\,\mathbf{k}$, and so Equations 10 give

$$x = b \qquad y = bt \qquad z = 2\pi c + ct$$

The normal plane is described by $by + c(z - 2\pi c) = 0$.

5.5 Problems

1. Determine the parametric equations of the straight line that passes through the points (2, 0, 3) and (−1, 2, −1).
2. Determine the point-direction equations of the straight line that passes through the point (2, 1 − 3) and is parallel to $\mathbf{v} = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$. Does the point (0, 1, −6) lie on this line? What about the point (4, 0, 0)?
3. Determine the equation of the plane that passes through the point (1, −1, 3) and is parallel to the xy-plane.
4. Determine the equation of the plane that passes through the point (3, 1, 2) and is parallel to the plane described by $3x - 2y + 4z = 5$.
5. Determine the equation of the plane that passes through the points (1, 2, 1), (−1, 0, 1), and (3, 1, 2).
6. Determine the equation of the plane that contains the two vectors $\mathbf{u} = 2\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$.
7. Define the angle between two planes to be the angle between their normal vectors. Find the angle between the planes described by $x - y + 2z = 3$ and $3x + y + 2z = 7$.
8. Determine the equation of the line of intersection of the two planes in Problem 7. *Hint*: Find one point on the line and a vector parallel to the line.
9. Calculate the perpendicular distance from the point (3, −1, −1) to the line joining (0, 2, 1) and (−2, −1, −1).
10. Find the perpendicular distance from the point (−2, 1, 1) to the plane described by $3x - 2y + 4z = 6$.
11. Calculate d in Example 5 using $\mathbf{r} = \mathbf{r}_3 - \mathbf{r}_1$.
12. Why is Equation 7 independent of the choice of the point on the line?
13. Why is the result in Example 6 independent of the choice of the point in the plane?
14. Show that the perpendicular distance from the point (x_0, y_0, z_0) to the plane described by $Ax + By + cz + D = 0$ is given by $d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{(A^2 + B^2 + C^2)^{1/2}}$.

15. Show that the angle between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z = D_2$ is given by $\cos\theta = \dfrac{A_1A_2 + B_1B_2 + C_1C_2}{(A_1^2 + B_1^2 + C_1^2)^{1/2}(A_2^2 + B_2^2 + C_2^2)^{1/2}}$.

16. Determine the equations of the tangent line and the normal plane to the space curve described by $x = t + 1$, $y = 2t^2 - 1$, $z = 2t^3$, where the curves cross the yz-plane.

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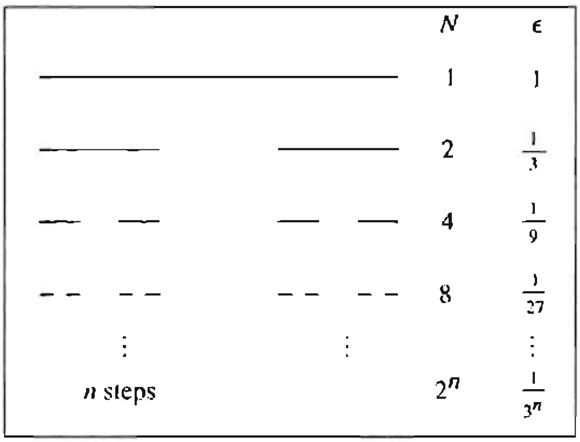
The Concept of Fractal Dimension

Let's consider what we mean by dimension. Suppose we have a continuous curve of unit length. The number of little straight-line segments of length ϵ that it will take to "cover" the curve is $N(\epsilon) = 1/\epsilon$. Similarly, if we have a unit area, then the number of little squares of area ϵ^2 that it will take to "cover" the area is $N(\epsilon) = 1/\epsilon^2$. You can see that the dimension d in each case is given by ϵ^d, which we can formalize by defining the dimension by

$$d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}$$

Now let's consider the following figure:



We start with a unit line segment and remove the middle third. Then we remove the middle third from each of the two remaining segments. We continue this process, producing in the limit an infinite number of separate pieces, each of zero length. This limiting result is known as a Cantor set. The figure shows N, the number of segments, and ϵ, the length of each one, at each stage of subdivision. The dimension of the Cantor set is less than one, but perhaps greater than zero. Using our formal definition, we have

$$d = \lim_{n \to \infty} \frac{\ln 2^n}{\ln 3^n} = \frac{\ln 2}{\ln 3} = 0.6309$$

Thus, we find that the dimension of a Cantor set is a noninteger. Sets with noninteger dimension are called *fractals*. The Cantor set is just one of many sets that are fractals, which play a key role in nonlinear dynamics.

CHAPTER 6

Functions of Several Variables

In Chapter 1, we reviewed the calculus of functions of a single variable. As the title says, in this chapter we will discuss functions of more than one variable. Many physical quantities depend upon more than one variable. For example, the pressure of a fixed quantity of a gas depends upon the temperature and the volume. The temperature of a body may vary from point to point and so depend upon the three spatial variables, x, y, and z. Even just a cursory look at any book on thermodynamics shows that the formulas of thermodynamics abound in partial derivatives.

Many of the concepts of a function of a single variable, such as limits of functions and continuity, carry over to functions of two or more variables, except that the domains are regions in two or more dimensions rather than intervals on the x axis. A significant difference between functions of a single variable and those of several variables is that you can form partial derivatives of functions of several variables, which leads to mixed higher partial derivatives and a variety of chain rules. After discussing these topics in Sections 3 and 4, we go on to study differentials of functions of several variables in Section 5 and then directional derivatives in Section 6. The study of directional derivatives leads to the gradient vector of a function $f(x, y, z)$. We shall see that the gradient vector of $f(x, y, z)$ has the physical interpretation of pointing in the direction of the most rapid change in $f(x, y, z)$. Because of this property, the gradient vector occurs in the description of a number of physical phenomena such as the direction of heat flow or the direction of diffusive flow. After discussing the extension of Taylor's formula to functions of several variables, we discuss the important topic of the maxima and minima of functions of several variables in Section 8. The investigation of maxima and minima of functions of even two variables is a little more involved than for functions of a single variable. The conditions that $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ at an extremum are necessary conditions (as is $df/dx = 0$ for a single variable), but they are not sufficient. We'll learn in Section 8 that the nature of a critical point (the point where first partial derivatives equal zero) depends upon the relative values of the four different second partial derivatives, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$, $\partial^2 f/\partial x\partial y$, and $\partial^2 f/\partial y\partial x$ (although usually $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$). In Section 9, we discuss Lagrange's method of undetermined multipliers, which is used to maximize or minimize functions of several variables when the variables are related to each other

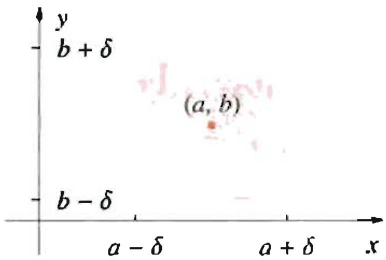


Figure 6.1
A rectangular δ neighborhood defined by $|x-a|<\delta$, $|y-b|<\delta$.

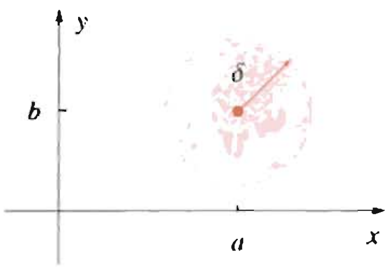


Figure 6.2
The circular δ neighborhood defined by $(x-a)^2+(y-b)^2<\delta^2$.

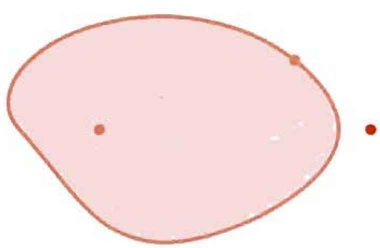


Figure 6.3
An example of an interior point, a boundary point, and an exterior point of a set.

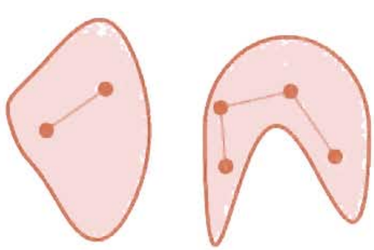


Figure 6.4
An illustration of connected sets.

by one or more constraints. Finally, in Section 10, we discuss multiple integrals. In most cases throughout the chapter, we shall restrict ourselves to functions of two or three variables, but most of the results are readily extended to any number of variables.

6.1 Functions

We define a function of more than one variable much the same as we define a function of one variable. We say that z is a function of x and y if there is a rule that allows us to determine one or more values of z for a given pair (x, y). We shall restrict ourselves to *real* functions in this chapter. In this case, the domain is a set of pairs of numbers (x, y) and the range is a set of real numbers z. We denote this relationship by $z = f(x, y)$, or $z = z(x, y)$. In this second expression, z on the left denotes the value of z and z on the right denotes the function. This slight ambiguity causes little problem in practice and is common notation.

If $z = f(x, y)$, we say that z is a *dependent variable* and the x and y are *independent variables*. As with functions of a single variable, z is *single-valued* if one value of z results from each pair (x, y) and *multiple-valued* if more than one value of z results from a pair (x, y). As with functions of a single variable, a multiple-valued function can be viewed as a collection of single-valued functions. As usual, we shall mostly restrict ourselves to single-valued functions.

Because $f(x, y)$ is a mapping from a set of ordered pairs of points (x, y) to a set of single points, we need to define a few terms that describe sets of pairs of points. First we define a *rectangular δ neighborhood* of the point (a, b) as the set of all points such that $|x-a|<\delta$ and $|y-b|<\delta$, where $\delta>0$ (Figure 6.1).

If the point (a, b) is excluded, so that $0<|x-a|<\delta$ and $0<|y-ba|<\delta$, then we have a *deleted rectangular δ neighborhood* of (a, b). We can also have a *circular δ neighborhood*, defined as the set of all points such that $(x-a)^2+(y-b)^2<\delta^2$ (Figure 6.2). If $0<(x-a)^2+(y-b)^2<\delta^2$, we have a *deleted circular δ neighborhood* of (a, b).

A point (a, b) is called an *interior point* of a set S if there exists a neighborhood of (a, b) that lies entirely within S. If *every* neighborhood of (a, b) contains not only points in S but also points not in S, then (a, b) is called a *boundary point*. A point (a, b) not in S is called an *exterior point* of S if there exists a neighborhood of (a, b) in which none of the points belong to S. Figure 6.3 illustrates this difference between an interior point, a boundary point, and an exterior point.

If a set consists entirely of interior points, then it is called an *open set*. For example, the set of all points such that $x^2+y^2<a^2$ is an open set. An open set contains no boundary points. Conversely, a closed set contains all of its boundary points. Finally, a set is said to be *connected* if any two points in S can be connected by a piecewise smooth curve lying entirely within S (Figure 6.4). An open connected set is called a *domain*, or an *open region*. Thus, the annulus $a<x^2+y^2<b$ is a domain. A *region* is a domain containing some or all of its

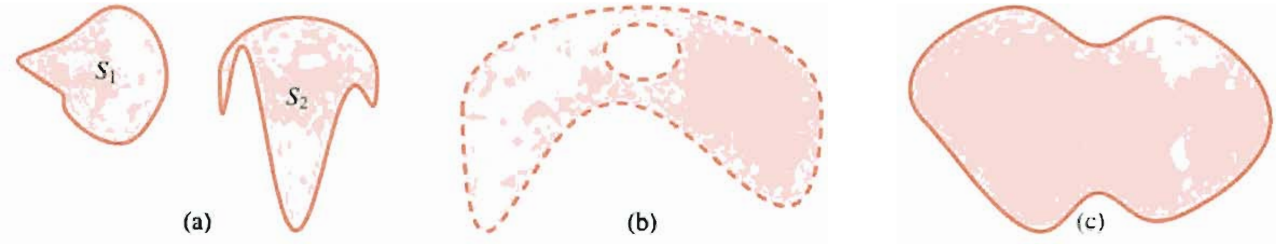


Figure 6.5
(a) A non-connected set. The set of points within the two sets S_1 and S_2 does not constitute a region.
(b) An open region, or a domain. (The dashed lines signify that the boundary points are not included in S.)
(c) A closed region.

boundary points. If it contains all of its boundary points, it is called a *closed region.* Figure 6.5 summarizes these definitions.

Example 1:
Describe the set $S\{(x, y)\}$ such that

$$2 < x^2 + y^2 < 9$$

in terms of the above definitions. Is the set connected? Is it a domain?

SOLUTION: The set is shown in Figure 6.6. It is connected because we can join any two points in S by a piecewise smooth curve lying completely within S. It is a domain, or an open region, because it does not include its two boundaries.

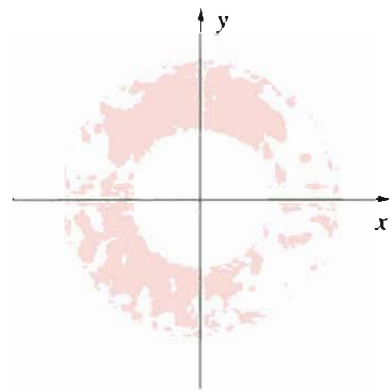


Figure 6.6
The set of all points such that $2 < x^2 + y^2 < 9$.

Example 2:
Describe the set $S = \{(x, y)\}$ such that $1 \le x^2 + y^2 < 2$ in terms of the above definitions. Is the set connected? Is it a domain?

SOLUTION: The set is shown in Figure 6.7. It is connected because we can join any two points in S by a piecewise smooth curve lying completely within S. It is not a domain because it contains some of its boundary points ($x^2 + y^2 = 1$), but it is a region.

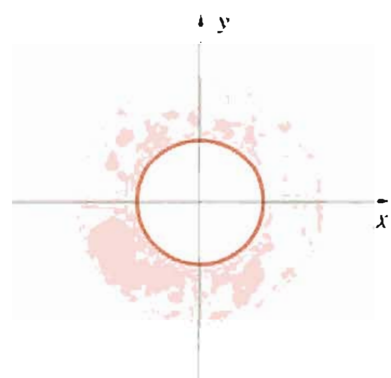


Figure 6.7
The set of all points such that $1 \le x^2 + y^2 < 2$.

We can represent the functions that we shall encounter in this book by plotting $z = f(x, y)$ in a three-dimensional cartesian coordinate system. The graph of $z = f(x, y)$ is the set of all points with coordinates (x, y, z) that satisfy the equation $z = f(x, y)$. The graph of a function of two variables is called a *surface.*

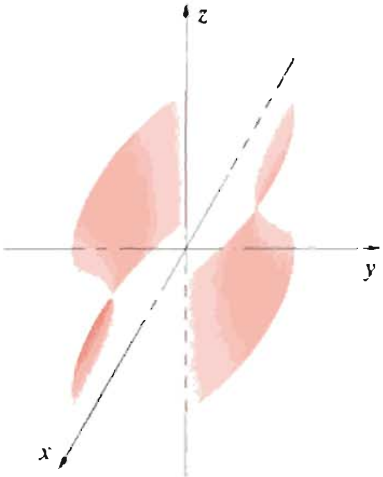


Figure 6.19
The surface of a hyperboloid of one sheet described by $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

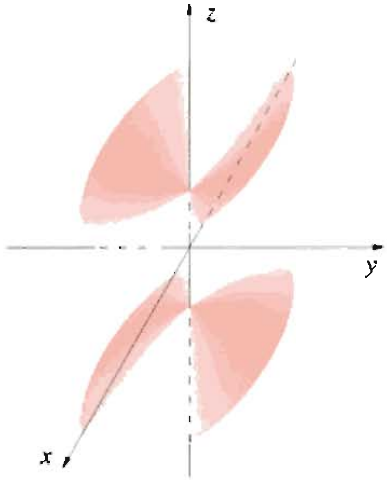


Figure 6.20
The surface of a hyperboloid of two sheets described by $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{c^2}=1$.

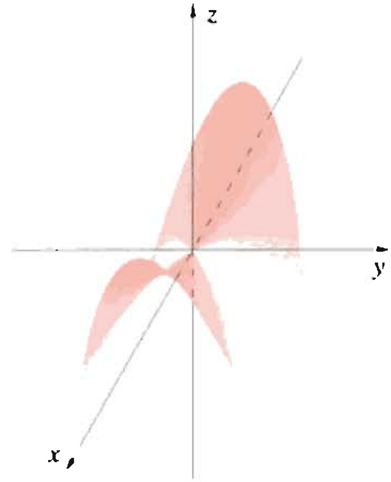


Figure 6.21
The surface of a hyperbolic paraboloid of one sheet described by $cz=\frac{x^2}{a^2}-\frac{y^2}{b^2}$.

(Figure 6.18)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2} \qquad \text{(an elliptic cone)} \qquad (5)$$

(Figure 6.19)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad \text{(a hyperboloid of one sheet)} \qquad (6)$$

(Figure 6.20)

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{(a hyperboloid of two sheets)} \qquad (7)$$

(Figure 6.21)

$$cz=\frac{x^2}{a^2}-\frac{y^2}{b^2} \qquad \text{(a hyperbolic paraboloid)} \qquad (8)$$

In this section, we have limited ourselves to functions of two variables for concreteness. Generally we can have functions of any number of variables and write $z=f(x_1, x_2, \ldots, x_n)$. Then $z=f(x_1, x_2, \ldots, x_n)$ represents a mapping from an n-dimensional space to a one-dimensional space.

6.1 Problems

1. Calculate the distance between the points $(1, -1, 2)$ and $(3, 2, 6)$ in a three-dimensional space.
2. Determine the domain associated with the real function $\sin^{-1}\dfrac{x-y}{x+y}$.
3. Is the set of points for which $\sin x = 0$ a connected set?
4. Is the set of points for $1 \le x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 5$ a domain; a region?
5. The equation $x + 2y + z = 4$ is the equation of a plane. Let x, y, and z equal zero in turn to sketch the planar surface.
6. Describe the planar surface given by $x + y = 0$.
7. Equation 2 is the equation of an ellipsoid centered at the origin. Write down the equation of an ellipsoid centered at $(1, -1, 0)$.
8. Sketch the level curves in the xy-plane for the surface in the previous problem.
9. Sketch some level curves of $f(x, y) = y - 2x^3$.
10. Sketch some level curves of $f(x, y) = x^2 + y^2 - 6x + 2y$.
11. Identify the surface in three-dimensional space that is described by the equation $x^2 - 2x + y^2 + z^2 + 4z = 8$.
12. Identify the surface in three-dimensional space that is described by the equation $4x^2 + 16y^2 = 32$.
13. Identify the surface in three-dimensional space represented by $x^2 + 2x + 4y^2 = 8z - 17$.
14. Starting from Equation 5, sketch the surface that it describes.
15. Starting with Equation 6, sketch the curve that it describes.
16. Determine the distance between the points $(1, 0, -1, 2, 3)$ and $(2, 1, -1, 0, 2)$ in a five-dimensional space. *Hint*: Generalize the Pythagorean theorem.
17. Explain why there is a gap between the two sheets in Figure 6.20.
18. Use any CAS to plot the surface $xe^{-(x^2+y^2)^{1/2}}$. (This function is a hydrogen p orbital.)
19. Plot the level curves for the function in the previous problem.
20. Use any CAS to plot the surface $xye^{-(x^2+y^2)^{1/2}}$. (This function is a hydrogen d orbital.)
21. Plot the level curves for the function in the previous problem.
22. Use any CAS to plot the surface $\sin(xy/10)$.

6.2 Limits and Continuity

The definition of the limit of a function of several variables is similar to that of a function of a single variable. We say that

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l \tag{1}$$

if for any positive number ϵ, we can find a positive number δ such that $|f(x, y) - l| < \epsilon$ whenever $0 < (x - a)^2 + (y - b)^2 < \delta^2$. The value of δ depends upon the value of ϵ and may depend upon the point (a, b). The function need not be defined at (a, b) for the limit to exist.

For example, suppose that $f(x, y) = 3x^2 - y^2$. It appears that the limit of $f(x, y)$ as $x \to 1$ and $y \to 1$ is 2. We can prove that this limit is 2 by finding a δ such that

$$|f(x, y) - 2| = |3x^2 - y^2 - 2| < \epsilon \tag{2}$$

whenever

$$0 < (x - 1)^2 + (y - 1)^2 < \delta^2 \tag{3}$$

We can satisfy Equation 3 by writing $0 < (x - 1)^2 < \delta^2/2$ and $0 < (y - 1)^2 < \delta^2/2$, or by writing $x = 1 \pm \delta/\sqrt{2}$ and $y = 1 \pm \delta/\sqrt{2}$. Substitute these equations into Equation 2 to obtain

$$\left| 3\left(1 \pm \frac{\delta}{\sqrt{2}}\right)^2 - \left(1 \pm \frac{\delta}{\sqrt{2}}\right)^2 - 2 \right| < \epsilon$$

Solving this expression for δ gives $\delta < \epsilon/2^{3/2}$ for small values of ϵ (Problem 1).

The limit in Equation 1 is carried out in the xy-plane. Just as the limit of a function of a single variable must be unique, the limit of $f(x, y)$ must be unique. This means that the limit in Equation 1 must be *independent* of the path along which x and y approach the point (a, b) in the xy-plane (Figure 6.22). Let's reconsider the limit of $f(x, y) = 3x^2 - y^2$ as $(x, y) \to (1, 1)$. The family of straight lines that pass through the point $(1, 1)$ is given by $y = m(x - 1) + 1$, where m is the slope (Problem 2). If we substitute $y = m(x - 1) + 1$ into $f(x, y) = 3x^2 - y^2$, we have

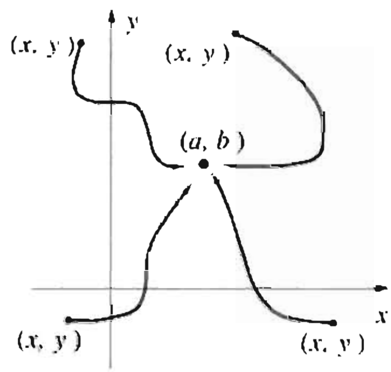


Figure 6.22
If the limit of $f(x, y)$ exists at the point (a, b), it must have the same value independent of the path along which (x, y) approaches (a, b).

$$f(x, y) = 3x^2 - m^2(x - 1)^2 - 2m(x - 1) - 1$$

Now let $x = 1 + \epsilon$ and let $\epsilon \to 0$ to get $\lim\limits_{(x,y)\to(1,1)} f(x, y) = 2$, *independent of m*.

This exercise is not *sufficient* to prove that the limit exists, however, because the limit must be independent of the *path* along which (x, y) approaches $(1, 2)$. It is certainly necessary that the limit be independent of any straight line path, but a rigorous approach consists of Equations 2 and 3 above.

Example 1:
Determine

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

where

$$f(x, y) = \begin{cases} \dfrac{x-y}{x+y} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

SOLUTION: The family of straight lines that approach (0, 0) is $y = mx$. Substitute this into $f(x, y)$ to get

$$f(x, y) = \frac{1-m}{1+m}$$

The limit here depends upon m, the direction in which x and y approach the origin, so the limit does not exist (it is not unique).

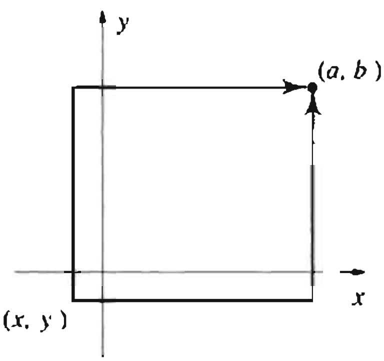


Figure 6.23
An illustration of the sequential limits in Equation 4.

We can also take the limit $(x, y) \to (a, b)$ in a sequential fashion by (Figure 6.23)

$$\lim_{x\to a}\left[\lim_{y\to b} f(x, y)\right] \quad \text{or} \quad \lim_{y\to b}\left[\lim_{x\to a} f(x, y)\right] \tag{4}$$

The *sequential limits* of $f(x, y) = 3x^2 - y^2$ as $(x, y) \to (1, 1)$ are

$$\lim_{x\to 1}\left[\lim_{y\to 1}(3x^2 - y^2)\right] = \lim_{x\to 1}(3x^2 - 1) = 2$$

and

$$\lim_{y\to 1}\left[\lim_{x\to 1}(3x^2 - y^2)\right] = \lim_{y\to 1}(3 - y^2) = 2$$

The two sequential limits come out to be the same in this case because the limit exists.

Example 2:
Determine the sequential limits of the function in Example 1.

SOLUTION:

$$\lim_{x\to 0}\left[\lim_{y\to 0}\frac{x-y}{x+y}\right] = \lim_{x\to 0}(1) = 1$$

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{x-y}{x+y}\right] = \lim_{x\to 0}(-1) = -1$$

The function has no limit as $(x, y) \to (0, 0)$ because the two limits differ.

If the sequential limits differ, the limit does not exist. If the sequential limits are the same, however, the limit may or may not exist. Thus, the equality of the sequential limits is a necessary but not sufficient condition for the existence of a limit. Sequential limits constitute only *two* directions, and the limits must be equal for *all* directions (see Problem 12).

Just as for a function of a single variable, the limits of functions of several variables have the following properties:

$$\lim_{(x,y)\to(a,b)}[\alpha f(x,y)+\beta g(x,y)]=\alpha\lim_{(x,y)\to(a,b)}f(x,y)+\beta\lim_{(x,y)\to(a,b)}g(x,y) \tag{5}$$

$$\lim_{(x,y)\to(a,b)}f(x,y)g(x,y)=\lim_{(x,y)\to(a,b)}f(x,y)\cdot\lim_{(x,y)\to(a,b)}g(x,y) \tag{6}$$

$$\lim_{(x,y)\to(a,b)}\frac{f(x,y)}{g(x,y)}=\frac{\lim_{(x,y)\to(a,b)}f(x,y)}{\lim_{(x,y)\to(a,b)}g(x,y)} \tag{7}$$

where α and β are constants in Equation 5 and provided $g(x,y)\neq 0$ in Equation 7. Equations 5 through 7 essentially tell us that if $f(x,y)$ and $g(x,y)$ are continuous at (a,b), then so are $\alpha f(x,y)+\beta g(x,y)$, $f(x,y)g(x,y)$, and $f(x,y)/g(x,y)$, provided $g(x,y)\neq 0$.

The limiting processes discussed so far in this section lead directly to the definition of the continuity of a function of several variables. The function $f(x,y)$ is said to be continuous at (a,b) if $f(x,y)\to f(a,b)$ as $(x,y)\to(a,b)$. In terms of δ and ϵ, $f(x,y)$ is continuous at (a,b) if for any positive number ϵ, we can determine a positive number δ such that

$$|f(x,y)-f(a,b)|<\epsilon$$

whenever

$$(x-a)^2+(y-b)^2<\delta^2 \tag{8}$$

As usual, the value of δ will depend upon the value of ϵ and may depend upon (a,b).

As with functions of a single variable, the following three conditions must hold for $f(x,y)$ to be continuous at (a,b):

$$\begin{aligned}&1.\ f(x,y)\text{ is defined at }(a,b)\\&2.\ \lim_{(x,y)\to(a,b)}f(x,y)\text{ exists}\\&3.\ \lim_{(x,y)\to(a,b)}f(x,y)=f(a,b)\end{aligned} \tag{9}$$

We can summarize these three conditions by writing

$$\lim_{(x,y)\to(a,b)}f(x,y)=f\left(\lim_{(x,y)\to(a,b)}(x,y)\right)=f(a,b) \tag{10}$$

If $f(x, y)$ does not satisfy these conditions, then $f(x, y)$ is said to be discontinuous at (a, b). For example, the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is discontinuous at $(0, 0)$ because the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist.

Example 3:
Show that

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \dfrac{1}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous at $(0, 0)$.

SOLUTION: To determine the limit, $\lim\limits_{(x,y)\to(0,0)} f(x, y)$, let $x^2 + y^2 = r^2$ and let $r^2 \to 0$. Because $|\sin 1/(x^2 + y^2)| \leq 1$,

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2) \sin \frac{1}{x^2 + y^2} = 0$$

independent of the manner in which (x, y) approaches $(0, 0)$. The three conditions of Equations 9 are satisfied, so $f(x, y)$ is continuous at $(0, 0)$.

We defined uniform continuity of a function of a single variable in Section 1.3. In an entirely analogous manner, we say that $f(x, y)$ is uniformly continuous in a region R if to an arbitrary $\epsilon > 0$ there corresponds a number δ such that for all points (x_1, y_1) and (x_2, y_2) in R

$$|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$$

This occurs whenever $(x_1 - x_2)^2 + (y_1 - y_2)^2 < \delta^2$. The key requirement of the *uniformity* of the continuity is that the same value of δ works for all the points in R. We state without proof that a function which is continuous in a closed region is uniformly continuous in that region.

Although we have limited our discussion to functions of only two variables in this section, the results are readily extended to more than two variables by a slight change in notation. For example, a function of three variables (u, v, w) is continuous at (a, b, c) if

$$\lim_{(u,v,w)\to(a,b,c)} f(u, v, w) = f\left(\lim_{(u,v,w)\to(a,b,c)} (u, v, w)\right) = f(a, b, c)$$

Example 4:
Does the limit

$$f(x, y, z) = \frac{x + 2y - 3z}{2x - y}$$

as $(x, y, z) \to (0, 0, 0)$ exist?

SOLUTION: The limit must be independent of the path along which (x, y, z) approaches $(0, 0, 0)$. If we let $z \to 0$ and then $x \to 0$, we obtain -2 as the limit. If we let $z \to 0$ and then $y \to 0$, we obtain $1/2$. The limit is not unique, and so does not exist.

6.2 Problems

1. Show that $\left|3\left(1 \pm \frac{\delta}{\sqrt{2}}\right)^2 - \left(1 \pm \frac{\delta}{\sqrt{2}}\right)^2 - 2\right| < \epsilon$ gives $\delta < \epsilon/2^{3/2}$ for small values of ϵ.
2. Verify that $y = m(x - 1) + 1$ is a family of straight lines passing through the point $(1, 1)$.
3. Use the ϵ-δ notation to show that $\lim_{(x,y)\to(1,2)} (x + 2y) = 5$.
4. Use the ϵ-δ notation to show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = 0$.
5. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^4}$ does not exist.
6. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + y^2} = 0$. *Hint*: Use $0 \le (x^2 - y)^2$ to show that $2x^2|y| \le x^4 + y^2$.
7. Evaluate the following limits, if they exist:

   (a) $\lim_{(x,y)\to(2,\pi)} x^2 \sin \frac{y}{x}$ (b) $\lim_{(x,y)\to(a,b)} \frac{x^2 + y^2}{xy}$
8. Evaluate the following limits, if they exist:

   (a) $\lim_{(x,y)\to(0,0)} \frac{2x - y}{x + y}$ (b) $\lim_{(x,y)\to(0,0)} e^{-1/xy}$
9. Evaluate the following limits, if they exist:

   (a) $\lim_{(x,y)\to(0,1)} \tan^{-1} \frac{y}{x}$ (b) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$
10. Evaluate the following limits, if they exist:

    (a) $\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$ (b) $\lim_{(x,y)\to(0,0)} \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & x \ne y \\ 0 & x = y \end{cases}$
11. Take sequential limits of the following and determine if the limit exists:

    (a) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$ (b) $\lim_{(x,y)\to(-1,2)} \frac{x - 2y}{x^2 + y}$

12. Show that the sequential limits of $\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$ are equal. Now show that the limit is not independent of m if $y = mx$. To what values of m do the sequential limits correspond?

13. Is $f(x, y) = xy^2/(x^2 + y^2)$ continuous at $(0, 0)$?

14. Is $f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ continuous at $(0, 0)$?

15. Is $f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^4} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ continuous at $(0, 0)$?

16. Which of the following functions is continuous at $(0, 0)$?

(a) $x^2 + y^2$ **(b)** $xy + 6$ **(c)** $\frac{xy}{x^2 + y^2}$ **(d)** $\frac{x}{2x + y}$

6.3 Partial Derivatives

The evaluation of a partial derivative of a function of several variables is similar to the evaluation of the derivative of a function of a single variable. Suppose we have a function $f(x, y)$. Then, the partial derivative of $f(x, y)$ with respect to x is defined by

$$f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{1}$$

when this limit exists. The partial derivative of $f(x, y)$ with respect to y is defined in a similar manner:

$$f_y = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{2}$$

Example 1:
Use Equations 1 and 2 to determine f_x and f_y if $f(x, y) = 2x^3 + 6xy + y^2$.

SOLUTION: Using Equation 1,

$$\begin{aligned} f_x &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2(x + \Delta x)^3 + 6(x + \Delta x)y + y^2 - 2x^3 - 6xy - y^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{6x^2 \Delta x + 6x(\Delta x)^2 + 2(\Delta x)^3 + 6y\Delta x}{\Delta x} \\ &= 6x^2 + 6y \end{aligned}$$

Using Equation 2.

$$\begin{aligned}
f_y &= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{2x^3 + 6x(y + \Delta y) + (y + \Delta y)^2 - 2x^3 - 6xy - y^2}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{6x\Delta y + 2y\Delta y + (\Delta y)^2}{\Delta y} \\
&= 6x + 2y
\end{aligned}$$

Example 2:
Use Equations 1 and 2 to evaluate f_x and f_y at the point (2, 1) if $f(x, y) = 2x^3 + 6xy + y^2$.

SOLUTION: Using Equation 1, we have

$$\begin{aligned}
f_x(2, 1) &= \lim_{\Delta x \to 0} \frac{f(2 + \Delta x, 1) - f(2, 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2(2 + \Delta x)^3 + 6(2 + \Delta x) + 1 - 29}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{30\Delta x + 12(\Delta x)^2 + 2(\Delta x)^3}{\Delta x} = 30
\end{aligned}$$

Using Equation 2, we have

$$\begin{aligned}
f_y(2, 1) &= \lim_{\Delta y \to 0} \frac{f(2, 1 + \Delta y) - f(2, 1)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{16 + 12(1 + \Delta y) + (1 + \Delta y)^2 - 29}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{14\Delta y + (\Delta y)^2}{\Delta y} = 14
\end{aligned}$$

Notice that we obtain the same result by using

$$f_x = 6x^2 + 6y \qquad \text{and} \qquad f_y = 6x + 2y$$

and letting $x = 2$ and $y = 1$. As we pointed out in Section 1.4, this procedure is valid only if f_x and f_y are continuous functions of x and y, which is most often the case in physical problems. (See, however, Problem 14.)

Equations 1 and 2 say that you can determine partial derivatives by differentiating with respect to one variable while keeping the other one as a constant. For example, if $f(x, y) = e^x \sin xy$, then

$$f_x = \frac{\partial f}{\partial x} = e^x \sin xy + ye^x \cos xy$$

and

$$f_y = \frac{\partial f}{\partial y} = xe^x \cos xy$$

Sometimes we write $(\partial f/\partial x)_y$ to emphasize that y is held constant when we take the derivative of f with respect to x. Usually, however, it will be clear from the context which variables are held constant.

Example 3:
The van der Waals equation is an approximate equation for the pressure of a gas as a function of its temperature and volume. The van der Waals equation for one mole of a gas is

$$P = \frac{RT}{V - b} - \frac{a}{V^2} \tag{3}$$

where R is the molar gas constant and a and b are constants that are characteristic of the particular gas. Determine $(\partial P/\partial T)_V$ and $(\partial P/\partial V)_T$.

SOLUTION:

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{R}{V - b}$$

and

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{RT}{(V - b)^2} + \frac{2a}{V^3}$$

Physically, these equations govern how the pressure varies as we change the temperature at constant volume, and how the pressure changes as we change the volume at constant temperature. Notice that $(\partial P/\partial V)_T$ is a function of both T and V.

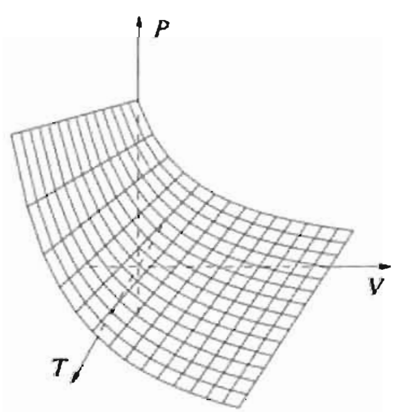


Figure 6.24
The pressure surface for the van der Waals equation.

Partial derivatives have a nice geometric interpretation. Figure 6.24 shows the surface corresponding to Equation 3. Let's choose some fixed temperature T_0. The condition $T_0 =$ constant is described by a plane parallel to the PV-plane and intersecting the T axis at T_0. Then, the partial derivative $(\partial P/\partial V)_{T_0}$ is the slope of the tangent line to the surface in the $T_0 =$ constant plane. Figure 6.25 shows the intersection of the $T_0 = 500$ K plane and the pressure surface in Figure 6.24. The slope of the tangent line is $(\partial P/\partial V)_{T_0=500\text{ K}}$. Similarly, the partial derivative $(\partial P/\partial T)_V$ is the slope of the tangent line to the surface in Figure 6.24 in a plane that is parallel to the PT-plane.

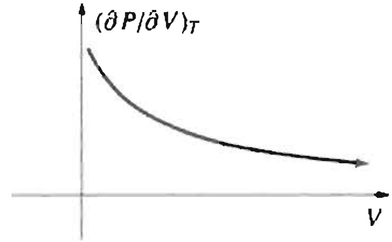


Figure 6.25
The partial derivative $(\partial P/\partial V)_T$ at $T = 500$ K for the van der Waals equation is the slope of the P-V cross section shown above.

The tangent lines to a surface can be used to define a tangent plane. Suppose we have a surface described by $z = f(x, y)$. Then $\partial f/\partial x$ at a point $P = (a, b, f(a, b))$ on the surface is the slope of the tangent to the line formed by the surface with the plane $y = b$, as shown in Figure 6.26a. Figure 6.26a shows a vector of the form

$$\mathbf{u} = \mathbf{i} + f_x(a, b)\,\mathbf{k}$$

Similarly, as Figure 26b shows, the tangent line to the intersection of the surface and the plane $x = a$ at the point (a, b) can be expressed as

$$\mathbf{v} = \mathbf{j} + f_y(a, b)\,\mathbf{k}$$

These two vectors define a plane called the *tangent plane* to the surface $z = f(x, y)$ at the point $P = (a, b, f(a, b))$. The *unit normal vector* to the surface at P is given by

$$\begin{aligned}\mathbf{n} = \frac{\mathbf{u}\times\mathbf{v}}{|\,\mathbf{u}\times\mathbf{v}\,|} &= \frac{1}{|\mathbf{u}\times\mathbf{v}|}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & f_x(a,b)\\ 0 & 1 & f_y(a,b)\end{vmatrix}\\ &= \frac{f_x(a,b)\,\mathbf{i} + f_y(a,b)\,\mathbf{j} - \mathbf{k}}{[1 + f_x^2(a,b) + f_y(a,b)]^{1/2}}\end{aligned} \qquad (4)$$

Note that the direction of $\mathbf{n}$ is not specified by Equation 4; in other words, $\pm\mathbf{n}$ are both unit normal vectors.

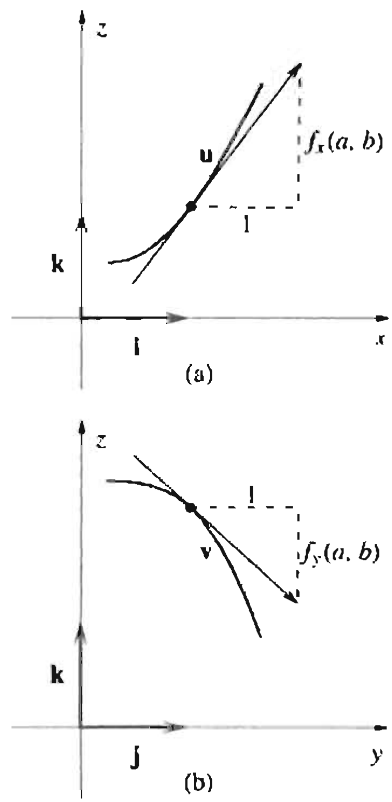


Figure 6.26
(a) The tangent line to the intersection of the surface $z = f(x, y)$ in the plane $y = b$ at the point $P = (a, b, f(a, b))$ is parallel to the vector $\mathbf{u} = \mathbf{i} + f_x(a, b)\,\mathbf{k}$.
(b) The tangent line to the intersection of the surface $z = f(x, y)$ in the plane $x = a$ at the point $P = (a, b, f(a, b))$ is parallel to the vector $\mathbf{v} = \mathbf{j} + f_y(a, b)\,\mathbf{k}$.

Example 4:
Determine the vector normal to the spherical surface $x^2 + y^2 + z^2 = a^2$ at the point $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$.

SOLUTION: Start with

$$z = f(x, y) = (a^2 - x^2 - y^2)^{1/2}$$

Then

$$f_x\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = f_y\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = -1$$

and so

$$\mathbf{n} = \frac{-\mathbf{i} - \mathbf{j} - \mathbf{k}}{\sqrt{3}}$$

This vector points inward from the spherical surface at the point $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$. The outward pointing vector is given by (Figure 6.27)

$$\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$$

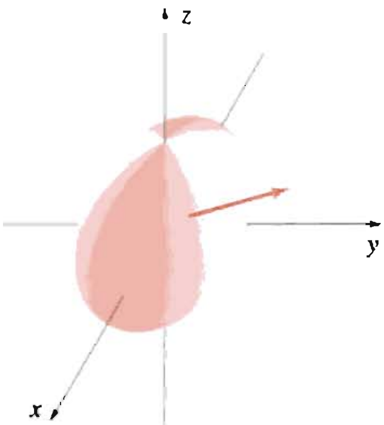


Figure 6.27
The outward unit normal vector to the spherical surface $x^2 + y^2 + z^2 = a^2$ at the point $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$.

The tangent plane to the surface at a point $P = (a, b, f(a, b))$ is normal to $\mathbf{n}$. If we let $\mathbf{r}_P$ be a vector to the point P and $\mathbf{r}$ be a vector to any other point in the tangent plane, then $\mathbf{r} - \mathbf{r}_P$ will lie in the tangent plane and thus,

$$(\mathbf{r} - \mathbf{r}_P) \cdot \mathbf{n} = 0$$

where $\mathbf{n}$ is the unit normal vector at the point P. Using Equation 4 for $\mathbf{n}$, we see that the equation of the tangent plane at the point P on the surface is given by

$$(x - a)f_x(a, b) + (y - b)f_y(a, b) - [z - f(a, b)] = 0$$

or by

$$z = (x - a)f_x(a, b) + (y - b)f_y(a, b) + f(a, b) \tag{5}$$

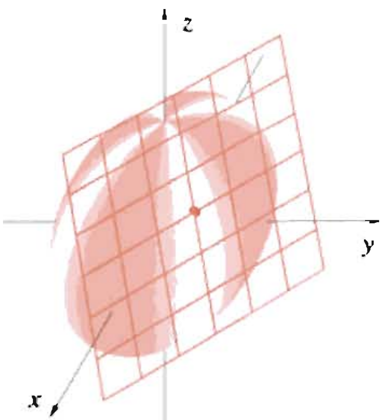


Figure 6.28
The tangent plane to the spherical surface $x^2 + y^2 + z^2 = a^2$ at the point $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$.

Example 5:
Find the tangent plane to the spherical surface described by $x^2 + y^2 + z^2 = a^2$ at the point $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$.

SOLUTION: We found in Example 4 that

$$f_x(a/\sqrt{3}, a/\sqrt{3}) = f_y(a/\sqrt{3}, a/\sqrt{3}) = -1$$

so the equation of the tangent plane at $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$ is (Figure 6.28)

$$x + y + z = \sqrt{3}\,a$$

You should realize that f_x and f_y themselves can be functions of x and y. For example, if $f(x, y) = y^2e^x$, then $f_x = y^2e^x$ and $f_y = 2ye^x$. Therefore, we can form partial derivatives of f_x and f_y just as we did for $f(x, y)$. The second partial derivatives of $f(x, y)$ are

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x^2} = f_{xx} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial y^2} = f_{yy} \end{aligned} \tag{6}$$

Example 6:
Show that $V(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

everywhere except for the origin (0, 0, 0). This equation is called Laplace's equation. Among other things, Laplace's equation determines the electrostatic potential $V(x, y, z)$ in a charge-free region. Notice that $V(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} = 1/r$, the Coulomb potential (within a multiplicative constant) due to a charge situated at the origin.

SOLUTION:

$$V_x = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$V_{xx} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

Because $V(x, y, z)$ is symmetric in x, y, and z, we can obtain V_{yy} by interchanging x and y in V_{xx} and we can obtain V_{zz} by interchanging x and z. Therefore,

$$\begin{aligned} V_{xx} + V_{yy} + V_{zz} &= \frac{(2x^2 - y^2 - z^2) + (2y^2 - x^2 - z^2) + (2z^2 - x^2 - y^2)}{(x^2 + y^2 + z^2)^{5/2}} \\ &= 0 \end{aligned}$$

The types of derivatives in Equation 6 are similar to the second derivative of a function of a single variable, but functions of more than one variable admit *mixed partial second derivatives* as well:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \end{aligned} \tag{7}$$

Example 7:
Find f_{xx}, f_{yy}, f_{xy}, and f_{yx} if $f(x, y) = xy^2 + e^{x^2 y}$.

SOLUTION:

$$f_x = y^2 + 2xye^{x^2y} \qquad f_y = 2xy + x^2e^{x^2y}$$

$$f_{xx} = (2y + 4x^2y^2)e^{x^2y} \qquad f_{yy} = 2x + x^4e^{x^2y}$$

$$f_{xy} = 2y + 2x(1 + x^2y)e^{x^2y} \qquad f_{yx} = 2y + 2x(1 + x^2y)e^{x^2y}$$

Notice that $f_{xy} = f_{yx}$ in Example 7. Do you think that it is a coincidence? Let's evaluate $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for

$$f(x, y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \tag{8}$$

To evaluate $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$, we use Equations 1 and 8, which in this case give

$$f_{xy}(0, 0) = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y}$$

and

$$f_{yx}(0, 0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x}$$

so we need $f_x(x, y)$ and $f_y(x, y)$ for $(x, y) \neq (0, 0)$. These turn out to be

$$f_x(x, y) = \frac{4x^2y^3 + x^4y - y^5}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}$$

Therefore,

$$f_{xy}(0, 0) = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-\Delta y}{\Delta y} = -1$$

$$f_{yx}(0, 0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

We see then that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ in this case. What's the difference between the behaviors of $f(x, y)$ in this example and $f(x, y)$ in Example 7? We'll answer this question with a theorem:

If f_{xy} and f_{yx} are continuous at a point (a, b), then $f_{xy} = f_{yx}$ at (a, b); otherwise f_{xy} and f_{yx} may not be equal.

It turns out that f_{xy} and f_{yx} are continuous for all x and y and so $f_{xy} = f_{yx}$ in Example 7, but that neither f_{xy} nor f_{yx} is continuous at $(0, 0)$ for the function defined by Equation 8 (Problem 14). Fortunately, most of the functions that we deal with in physical applications are continuous.

The equality of mixed partial second derivatives is used often in thermodynamics. For example, thermodynamics tells us that the entropy (S) and the pressure (P) of a substance can be expressed as partial derivatives of a function called the Helmholtz energy, $A = A(V, T)$, which is a function of the volume and the temperature:

$$S = -\left(\frac{\partial A}{\partial T}\right)_V \qquad P = -\left(\frac{\partial A}{\partial V}\right)_T \tag{9}$$

Using the relation

$$\begin{aligned}\frac{\partial^2 A}{\partial V \partial T} &= \left[\frac{\partial}{\partial V}\left(\frac{\partial A}{\partial T}\right)_V\right]_S = \left[\frac{\partial}{\partial T}\left(\frac{\partial A}{\partial V}\right)_T\right]_V \\ &= \frac{\partial^2 A}{\partial T \partial V}\end{aligned}$$

we see that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V \tag{10}$$

Equation 10 is known as a Maxwell relation in thermodynamics. The derivation of Equation 10 is a typical thermodynamic manipulation. Equation 10 is an important and useful equation because it allows us to calcuate the entropy of a substance (which is not a directly measurable quantity) in terms of the pressure-volume-temperature (P-V-T) dependence of the substance, which is readily measurable.

6.3 Problems

1. Determine all the partial derivatives up to second order of $f(x, y) =$

   (a) $xe^y + y$ **(b)** $y \sin x + x^2$

2. Determine all the partial derivatives up to second order of $f(x, y) =$

   (a) $\tan^{-1} \frac{y}{x}$ **(b)** $e^{-(x^2+y^2)}$

3. Show that $f_{xy} = f_{yx}$ for (a) $f(x, y) = x^2 e^{-y^2}$ and (b) $e^{-y} \cos xy$.

4. Show that $f_{xxy} = f_{xyx} = f_{yxx}$ for $f(x, y) =$

   (a) $x^4y^2 + xy$ **(b)** $e^{-(x+y)}$

5. Show that if (a) $f(x, y) = \ln(x^2 + y^2)$ with $(x, y) \neq (0, 0)$, then $f_{xx} + f_{yy} = 0$, and if (b) $f(x, y, z) = \ln(x^2 + y^2 + z^2)$ with $(x, y, z) \neq (0, 0, 0)$, then $xf_{yz} = yf_{zx} = zf_{xy}$.

6. Show that $xf_x + yf_y = 2f$ if $f(x, y) = xy \tan(y/x)$, $x \neq 0$.

7. Show that $c(x, t) = (4\pi Dt)^{-1/2}e^{-x^2/4Dt}$ satisfies the equation $\dfrac{\partial c}{\partial t} = D\dfrac{\partial^2 c}{\partial x^2}$.

8. Show that $c(x, t) = \dfrac{1}{2}\left[1 + \text{erf}\left(\dfrac{x}{2\sqrt{Dt}}\right)\right]$ satisfies the equation $\dfrac{\partial c}{\partial t} = D\dfrac{\partial^2 c}{\partial x^2}$.

9. Show that $f(x, y) = \sin ax \sinh ay$ satisfies Laplace's equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

10. Show that $Y(\theta, \phi) = e^{\pm i\phi} \sin\theta\cos\theta$ satisfies the equation
$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} + 6\sin^2\theta Y(\theta, \phi) = 0.$$

11. Find the equation of the tangent plane to the surface $z = xy + 3y^2$ at the point $(1, 1, 4)$.

12. Find the equation of the tangent plane to the surface $z = x^2/(x + y)$ at the point $(2, 2, 1)$.

13. Find the equation of the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

14. Show that $f_{xy} = \dfrac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3} = f_{yx}$ for $(x, y) \neq (0, 0)$ for the function defined by Equation 8, but that f_{xy} and f_{yx} are not continuous at $(0, 0)$.

15. Show that $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ is an example of a function where both first partial derivatives exist at $(0, 0)$ but that $f(x, y)$ is discontinuous there. Contrary to the case of functions of a single variable, the existence of first derivatives is not sufficient to guarantee continuity.

The next five problems involve applications of partial derivatives to thermodynamics.

16. The isothermal compressibility, κ_T, of a substance is defined as $\kappa_T = -\dfrac{1}{V}\left(\dfrac{\partial V}{\partial P}\right)_T$. Obtain an expression for the isothermal compressibility of an ideal gas ($PV = RT$) in terms of P.

The coefficient of thermal expansion, α, of a substance is defined as $\alpha = \dfrac{1}{V}\left(\dfrac{\partial V}{\partial T}\right)_P$. Obtain an expression for the coefficient of thermal expansion of an ideal gas ($PV = RT$) in terms of T.

17. Given that $U = k_B T^2\left(\dfrac{\partial \ln Q}{\partial T}\right)_{N,V}$, where $Q(N, V, T) = \dfrac{1}{N!}\left(\dfrac{2\pi m k_B T}{h^2}\right)^{3N/2} V^N$ and k_B, m, and h are constants, determine U as a function of T.

18. The thermodynamic equation $\left(\dfrac{\partial U}{\partial V}\right)_T = T\left(\dfrac{\partial P}{\partial T}\right)_V - P$ shows how the energy U of a system varies with the volume in terms of pressure, volume, and (kelvin) temperature of the system. Evaluate $(\partial U/\partial V)_T$ for an ideal gas ($PV = RT$) and for a van der Waals gas $\left[\left(P + \dfrac{a}{V^2}\right)(V - b) = RT\right]$, where a and b are constants.

19. Given that the heat capacity at constant volume is defined by $C_V = \left(\dfrac{\partial U}{\partial T}\right)_V$ and given the expression in Problem 18, derive the equation $\left(\dfrac{\partial C_V}{\partial V}\right)_T = T\left(\dfrac{\partial^2 P}{\partial T^2}\right)_V$.

20. Thermodynamics tells us that the difference between the heat capacity at constant pressure and the heat capacity at constant volume is given by $C_P - C_V = T\left(\dfrac{\partial P}{\partial T}\right)_V\left(\dfrac{\partial V}{\partial T}\right)_P$. Show that $C_P - C_V = R$ for an ideal gas.

6.4 Chain Rules for Partial Differentiation

Suppose that $u = f(x, y)$ where x and y are functions of a single variable t. Then the composite function $f(x(t), y(t)) = U(t)$ is a function of a single variable t and we shall prove that

$$\frac{dU}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \tag{1}$$

provided $\partial u/\partial x$ and $\partial u/\partial y$ are continuous and $x(t)$ and $y(t)$ are differentiable. Equation 1 is called the *chain rule of partial differentiation*. Before we prove Equation 1, we should say a few words about the notation. First note that we wrote $f(x(t), y(t)) = U(t)$ instead of simply $u(t)$. We did this to emphasize that $u = f(x, y)$ and $U = f(x(t), y(t))$ are actually different functions. For example, if $u = x^2y$ and $x = te^{-t}$ and $y = e^{-2t}$, then $U = t^2e^{-4t}$. Many authors don't make this distinction, nor shall we most of the time, but it is occasionally helpful to keep the distinction in mind.

Equation 1 is fairly easy to prove. We assume that $x(t)$ and $y(t)$ have finite derivatives so that a change in t produces changes Δx and Δy which tend to zero as $\Delta t \to 0$. The changes Δx and Δy produce a change in u, and so we write

$$\begin{aligned}\Delta u &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]\end{aligned} \tag{2}$$

where we have simply added and subtracted $f(x, y + \Delta y)$ to the first line. If f_x exists and is finite in a region of the xy-plane containing the point $(x(t), y(t))$, then the mean value theorem for derivatives (Section 1.6) says that

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = f_x(\xi, y + \Delta y)\Delta x \tag{3}$$

where ξ lies in the open interval Δx. Similarly, if f_y exists and is finite in a similar region, then

$$f(x, y + \Delta y) - f(x, y) = f_y(x, \eta)\Delta y \tag{4}$$

where η lies in the open interval Δy. Combining Equations 2 through 4 and dividing through by Δt gives

$$\frac{\Delta u}{\Delta t} = f_x(\xi, y + \Delta y)\frac{\Delta x}{\Delta t} + f_y(x, \eta)\frac{\Delta y}{\Delta t} \tag{5}$$

Now as $\Delta t \to 0$, $\Delta u/\Delta t$, $\Delta x/\Delta t$, and $\Delta y/\Delta t$ tend to du/dt, dx/dt, and dy/dt, and if f_x and f_y are continuous, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \tag{6}$$

Example 1:
Use Equation 6 to evaluate du/dt if $u = x^2y + xy^2$ and $x(t) = te^{-t}$ and $y = e^{-t}$.

SOLUTION:

$$\frac{\partial u}{\partial x} = 2xy + y^2 \qquad \frac{\partial u}{\partial y} = x^2 + 2xy$$

$$\frac{dx}{dt} = (1-t)e^{-t} \qquad \frac{dy}{dt} = -e^{-t}$$

and so

$$\begin{aligned}\frac{du}{dt} &= (2xy + y^2)(1-t)e^{-t} + (x^2 + 2xy)(-e^{-t}) \\ &= [(1+2t)(1-t) - t(t+2)]\,e^{-3t} \\ &= (1 - t - 3t^2)e^{-3t}\end{aligned}$$

Of course you get the same result by substituting $x(t)$ and $y(t)$ into $u(x, y)$ and then differentiating with respect to t (Problem 1).

Equation 6 is readily extended to a function of more than two independent variables. If $u = u(x_1, x_2, \ldots, x_n)$ and $x_j = x_j(t)$ for $j = 1$ to n, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt} \tag{7}$$

Suppose now that $u = u(x, y)$ and that $y = y(x)$, so that u is actually a function of a single variable x. This is just a specific case of Equation 6 with $t = x$, and so we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \tag{8}$$

In Equation 8, du/dx represents the *total derivative* of u with respect to x while $(\partial u/\partial x)$ represents the partial derivative of u with respect to x.

Example 2:
Use Equation 8 to evaluate du/dx if $u = y \sin x$ and $y = xe^{-x}$.

SOLUTION:

$$\frac{\partial u}{\partial x} = y \cos x \qquad \frac{\partial u}{\partial y} = \sin x$$

$$\frac{du}{dx} = y\cos x + (\sin x)(1-x)e^{-x}$$

$$= xe^{-x}\cos x + (1-x)e^{-x}\sin x$$

$$= e^{-x}\sin x + xe^{-x}(\cos x - \sin x)$$

Of course you get the same result by substituting $y = xe^{-x}$ into $u = y\sin x$ and then differentiating with respect to x.

Now suppose that $u = u(x, y)$ and that $x = x(s, t)$ and $y = y(s, t)$. In this case u is a function of two variables, s and t, or $u(x, y) = U(s, t)$. We can extend Equation 6 to write

$$\frac{\partial U}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \tag{9a}$$

and

$$\frac{\partial U}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \tag{9b}$$

We hold t constant everywhere in Equation 9a and s constant everywhere in Equation 9b, so Equations 9 are essentially equivalent to Equation 6. We made a notational distinction between $u(x(s, t), y(s, t))$ and $U(s, t)$ in Equations 9a and 9b, but you'll usually see them written as

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \tag{10a}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \tag{10b}$$

When u is differentiated with respect to s or t, then u is regarded to be a function of s and t and the other variable is held constant during the partial differentiation. When u is differentiated with respect to x or y, then u is regarded to be a function of x and y and the other variable is held constant during partial differentiation.

Example 3:
If $u(x, y) = ye^{-x} + xy$ and $x(s, t) = s^2 t$ and $y(s, t) = e^{-s} + t$, evaluate $\partial u/\partial s$ and $\partial u/\partial t$.

SOLUTION:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= (-ye^{-x} + y)(2st) + (e^{-x} + x)(-e^{-s})$$

$$= 2st(e^{-s} + t)(1 - e^{-s^2t}) - (e^{-s^2t} + s^2t)e^{-s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= (-ye^{-x} + y)s^2 + (e^{-x} + x)$$

$$= (e^{-s^2t} + s^2t) + s^2(e^{-s} + t)(1 - e^{-s^2t})$$

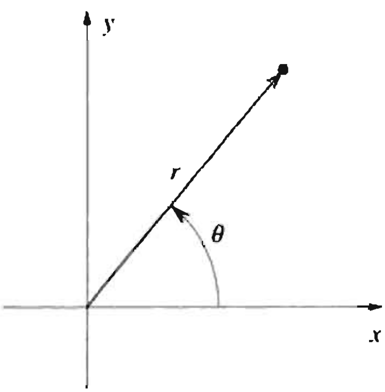


Figure 6.29
A point in a plane may be specified by its distance from the origin (r) and the angle that a line from the origin to the point makes with the x axis (θ). The quantities r and θ are called polar coordinates.

Example 4:
The expression

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is called the Laplacian operator (in two dimensions) and occurs throughout applied mathematics. The Laplacian operator above is expressed in cartesian coordinates. We're going to study various other coordinate systems in Chapter 8, but you might remember from other courses that it is sometimes convenient to use *polar coordinates*, where x and y are expressed in terms of r and θ in Figure 6.29.

$$x = r\cos\theta \qquad y = r\sin\theta$$

Take the special case where $r = a =$ constant and express ∇^2 in terms of θ.

SOLUTION: We first let ∇^2 operate on a function $f(x, y)$ and write

$$\nabla^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Now, using Equations 10 gives

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial \theta}\right)_r \left(\frac{\partial \theta}{\partial x}\right)_y$$

But $\theta = \tan^{-1}(y/x)$, so $(\partial\theta/\partial x)_y = -\sin\theta/a$, and

$$\left(\frac{\partial f}{\partial x}\right)_y = -\frac{\sin\theta}{a}\left(\frac{\partial f}{\partial \theta}\right)_r$$

Using Equations 10 again, we write

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)_y = \left[\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)_y\right]\left(\frac{\partial \theta}{\partial x}\right)_y$$

$$= -\frac{\sin\theta}{a}\frac{\partial}{\partial\theta}\left[-\frac{\sin\theta}{a}\frac{\partial f}{\partial\theta}\right]$$

$$= \frac{\sin^2\theta}{a^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\sin\theta\cos\theta}{a^2}\frac{\partial f}{\partial\theta}$$

The result for $\partial^2 f/\partial y^2$ is obtained in a similar manner and yields

$$\frac{\partial^2 f}{\partial y^2} = \frac{\cos^2\theta}{a^2}\frac{\partial^2 f}{\partial\theta^2} - \frac{\sin\theta\cos\theta}{a^2}\frac{\partial f}{\partial\theta}$$

and so we find that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \longrightarrow \frac{1}{a^2}\frac{\partial^2 f}{\partial\theta^2}$$

We can express this result in operator form:

$$\nabla^2 = \frac{1}{a^2}\frac{\partial^2}{\partial\theta^2}$$

Equations 10 are easily extended to any number of variables. If $u = u(x_1, x_2, \dots, x_n)$ and $x_j = x_j(s_1, s_2, \dots, s_m)$, then u is a function of $s_1, s_2, \dots, s_m$ and

$$\frac{\partial u}{\partial s_k} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial s_k} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial s_k} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial s_k} \tag{11}$$

Because $k = 1, 2, \dots, m$, Equation 11 consists of m equations.

There is a theorem called *Euler's theorem* that is extremely useful in thermodynamics and a number of other fields. First we define a *homogeneous function of degree p* as one that has the property that

$$f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^p f(x_1, x_2, \dots, x_n) \tag{12}$$

where λ is a parameter. For example, $f(x, y, z) = x^2z + yz^2 + xyz$ is homogeneous of degree 3 since

$$f(\lambda x, \lambda y, \lambda z) = (\lambda x)^2(\lambda z) + (\lambda y)(\lambda z)^2 + (\lambda x)(\lambda y)(\lambda z)$$
$$= \lambda^3(x^2z + yz^2 + xyz) = \lambda^3 f(x, y, z)$$

Not every independent variable has to appear in Equation 12. The function $f(x, y, z, w)$ given by

$$f(x, y, z, w) = xy\sin z + \frac{x^3}{y}e^{-w^2}$$

is homogeneous of degree 2 in the independent variables x and y because

$$f(\lambda x, \lambda y) = \lambda^2 f(x, y) \tag{13}$$

The independent variables z and w are simply suppressed in Equation 13.

Euler's theorem says that

If $f(\lambda x, \lambda y) = \lambda^p f(x, y)$, *then*

$$pf(x, y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \tag{14}$$

The proof of Euler's theorem is easy. Start with

$$f(\lambda x, \lambda y) = \lambda^p f(x, y)$$

and let $u = \lambda x$ and $v = \lambda y$. Now differentiate both sides of

$$f(\lambda x, \lambda y) = \lambda^p f(x, y)$$

with respect to λ to get

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial \lambda} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \lambda} = p\lambda^{p-1} f(x, y)$$

But $\partial u/\partial \lambda = x$ and $\partial v/\partial \lambda = y$, so

$$p\lambda^{p-1} f(x, y) = x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}$$

Because this equation is true for *any* value of λ, it is true for $\lambda = 1$ and so $u = x$ and $v = y$ and

$$pf(x, y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

We'll illustrate Equation 12 with a "mathematical" Example for those who have an aversion to thermodynamics for one reason or another, but Problems 15 through 18 illustrate some thermodynamic applications of Euler's theorem.

Example 5:

Show that $f = xy \sin z + \dfrac{x^3}{y} e^{-w^2}$ satisfies Equation 12.

SOLUTION: The function f is homogeneous of degree 2 in the two independent variables x and y:

$$\frac{\partial f}{\partial x} = y \sin z + \frac{3x^2}{y} e^{-w^2}$$

$$\frac{\partial f}{\partial y} = x \sin z - \frac{x^3}{y^2} e^{-w^2}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = xy \sin z + \frac{3x^3}{y}e^{-w^2} + xy \sin z - \frac{x^3}{y}e^{-w^2}$$
$$= 2f$$

Note that $\sin z$ and e^{-w^2} play the role of multiplicative constants in each term of $f(x, y, z, w)$.

6.4 Problems

1. Verify that you get the same result for Example 1 if you substitute $x(t)$ and $y(t)$ into u and then differentiate with respect to t.
2. Use the chain rule to evaluate du/dt if $u(x, y) = \cos(x^2 + 2y)$ where $x(t) = t$ and $y(t) = t^2$.
3. Use the chain rule to evaluate du/dt if $u(x, y) = ye^{-x^2}$ where $x(t) = t^2$ and $y(t) = t^4$.
4. Use the chain rule to evaluate df/dt if $f(x, y) = e^{-xy}$ where $x(t) = \sin t$ and $y(t) = \cos t$.
5. Use the chain rule to evaluate du/dt if $u(x, y, z) = x^2 + ze^y$ where $x(t) = t$, $y(t) = t^2$, and $z(t) = t^3$.
6. Use Equation 8 to evaluate du/dx if $u(x, y) = y \cos xy$ and $y = e^{-x}$. Verify your result by substituting $y = e^{-x}$ into u first and then differentiating.
7. Evaluate $\partial u/\partial r$ if $u(x, y) = x^2 - xy + y^2$ and $x = r\cos\theta$ and $y = r\sin\theta$.
8. Evaluate $\partial u/\partial r$ if $u = (x^2 + y^2 + z^2)^{1/2}$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$.
9. Evaluate $\partial u/\partial s$ if $u(x, y) = e^{x+y}$, $x = te^s$, and $y = \sin s$.
10. Determine $\partial u/\partial s$ at $(s, t) = (1, -1)$ if $u(x, y) = xy - y^2$, $x = se^{-t}$, and $y = st^2$.
11. Suppose that $u = f(x, y, s, t)$, where $x = x(s, t)$ and $y = y(s, t)$. Show that $\dfrac{\partial u}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial f}{\partial s}$ and that $\dfrac{\partial u}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial f}{\partial t}$.
12. Show that $u(x + ct) + u(x - ct)$, where x and t are variables and c is a constant, satisfies $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$.
13. Consider the equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$, where c is a constant. Define new variables $\xi = x - ct$ and $\eta = x + ct$ and now show that the above equation becomes $\dfrac{\partial^2 u}{\partial\xi\,\partial\eta} = 0$.
14. If $u = u(x, y)$ and $x = r\cos\theta$ and $y = r\sin\theta$, then show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

The applicability of Euler's theorem to thermodynamics rests upon the concept of an extensive thermodynamic quantity, *which is a quantity that is directly proportional to the size of the system. Extensive properties are volume, mass, number of moles, energy, and entropy. An intensive thermodynamic quantity, such as* *temperature or pressure, is independent of the size of the system. Functions of extensive variables are homogeneous of degree one. For example, the volume of a system depends upon the number of moles of each constituent, and so*

$$V(\lambda n_1, \lambda n_2, \ldots, \lambda n_n) = \lambda V(n_1, n_2, \ldots, n_n) \tag{15}$$

Physically, if we double the amount of each constituent, then Equation 15 says that we double the volume of the system. The next four problems are applications of Euler's theorem to thermodynamics.

15. Let Y be any extensive property. Use Euler's theorem to prove that $Y(n_1, n_2, \ldots, T, P) = \sum n_j \overline{Y}_j$, where $\overline{Y}_j = (\partial Y/\partial n_j)_{T,P,n_{k \neq j}}$.

16. The thermodynamic energy (U) of a system can be expressed as a function of the entropy (S), the volume (V), and the number of moles (n). Use Euler's theorem to derive $U = S\left(\frac{\partial U}{\partial S}\right)_{V,n} + V\left(\frac{\partial U}{\partial V}\right)_{S,n} + n\left(\frac{\partial U}{\partial n}\right)_{S,V}$. Do you recognize the resulting equation?

17. The Helmholtz energy (A) of a system can be expressed as a function of the temperature (T), the volume (V), and the number of moles (n). Apply Euler's theorem to $A = A(T, V, n)$. Do you recognize the resulting equation?

18. Apply Euler's theorem to $V = V(T, P, n_1, n_2)$.

6.5 Differentials and the Total Differential

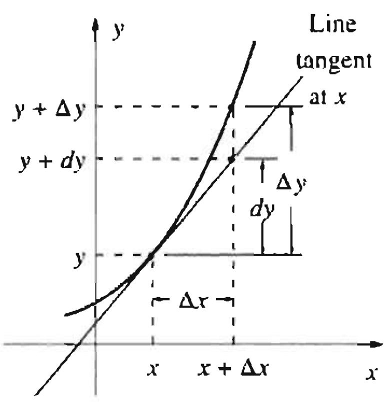


Figure 6.30
An illustration of the difference between dy and Δy.

In Section 1.5, we defined the differential of a function of a single variable. We shall extend these ideas to a function of several variables in this section. Recall from Section 1.5 that if dy and Δy are defined as in Figure 6.30 (see also Figure 1.42), then the difference between dy and Δy goes as

$$\Delta y = dy + \epsilon\, \Delta x \tag{1}$$

where $\epsilon \to 0$ as $\Delta x \to 0$. In other words, the difference between dy and Δy goes to zero faster than Δx as $\Delta x \to 0$. Because of this,

$$dy = y'(x)\, dx \tag{2}$$

is an excellent approximation. We can readily extend these ideas to functions of more than one variable.

Let $f(x, y)$ be a function of the independent variables x and y. Now change x by Δx and y by Δy and let Δf be the corresponding change in $f(x, y)$, then

$$\begin{aligned}\Delta f &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y)\end{aligned} \tag{3}$$

Because $y + \Delta y$ is held constant in the first two terms in Equation 3 and x is held constant in the second two terms, we can now use the mean value theorem for derivatives of a single variable in Equation 3 and write

$$\Delta f = f_x(\xi, y + \Delta y)\Delta x + f_y(x, \eta)\Delta y \tag{4}$$

where $x < \xi < x + \Delta x$ and $y < \eta < y + \Delta y$. Assuming that f_x and f_y are continuous, Equation 4 becomes

$$\Delta f = f_x(x, y)\Delta x + f_y(x, y)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y = df + \epsilon_1 \Delta x + \epsilon_2 \Delta y \tag{5}$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$, as both $\Delta x \to 0$ and $\Delta y \to 0$. Equation 5 is the two-variable analog of Equation 1. It shows that $\Delta f \to df$ even faster than $\Delta x \to 0$ or $\Delta y \to 0$. Thus, we see that

$$df = f_x dx + f_y dy \tag{6}$$

is an excellent approximation as Δx and Δy approach zero. Equation 6 is the two-variable analog of Equation 2. We call df the *total differential of f*.

Example 1:
Determine ϵ_1 and ϵ_2 in Equation 5 or 6 and show that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$ if $f(x, y) = x^3 y + xy^2$.

SOLUTION:

$$\begin{aligned}
\Delta f &= f(x + \Delta x, y + \Delta y) - f(x, y) \\
&= (x + \Delta x)^3(y + \Delta y) + (x + \Delta x)(y + \Delta y)^2 - x^3 y - xy^2 \\
&= (3x^2 y + y^2)\Delta x + (x^3 + 2xy)\Delta y \\
&\quad + [3xy\Delta x + (3x^2 + 2y)\Delta y + y(\Delta x)^2]\Delta x \\
&\quad + [x\Delta y + 3x(\Delta x)^2 + (\Delta x)(\Delta y) + (\Delta x)^3]\Delta y \\
&= f_x \Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y
\end{aligned}$$

where $\epsilon_1 = 3xy\Delta x + (3x^2 + 2y)\Delta y + y(\Delta x)^2$ and $\epsilon_2 = x\Delta y + 3x(\Delta x)^2 + (\Delta x)(\Delta y) + (\Delta x)^3$. Both $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

If $f(x, y)$ is such that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$, we say that $f(x, y)$ is differentiable at the point (x, y). Contrary to what you might gather from Equation 1, the existence of f_x and f_y is not sufficient to guarantee that $f(x, y)$ is differentiable; f_x and f_y must also be continuous.

For the rest of this section, we shall use the pressure of a gas instead of $f(x, y)$ to illustrate differentials. Generally, the pressure of a gas depends upon the (kelvin) temperature T and the molar volume V (the volume per mole), and so we write $P(V, T)$. Recall that one mole of all gases at sufficiently low pressure obeys the ideal-gas equation of state:

$$PV = RT \tag{7}$$

where R is a constant, whose value is $R = 0.0821$ liter · atmospheres per kelvin per mole. At higher pressures, where deviations from ideal behavior occur, the van der Waals equation is often used:

$$P = \frac{RT}{V - b} - \frac{a}{V^2} \tag{8}$$

where a and b are constants that are characteristic of the particular gas. We shall use these two equations as examples of functions of two independent variables.

As Equations 7 and 8 imply, the pressure of a gas depends upon its temperature and volume. The total differential of the pressure for a fixed amount of gas is given by

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV \tag{9}$$

We have subscripted the partial derivatives in Equation 9 to emphasize which variable is varied and which is held constant. Physically, Equation 9 says that it is an excellent approximation to calculate the total change in the pressure of a gas due to a change in both the temperature and the volume in two steps – that due to the temperature change keeping the molar volume fixed at its initial value and then that due to the change in the molar volume keeping the temperature fixed at its final value.

Example 2:
Evaluate the total differential of the pressure for an ideal gas.

SOLUTION: Using Equation 9, we see that

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV$$

with $\left(\frac{\partial P}{\partial T}\right)_V = \frac{R}{V}$ and $\left(\frac{\partial P}{\partial V}\right)_T = -\frac{RT}{V^2}$ so

$$dP = \frac{R}{V}dT - \frac{RT}{V^2}dV \tag{10}$$

We can use Equation 10 to estimate the change in pressure when both the temperature and volume of an ideal gas are changed slightly. We write Equation 10 for finite ΔT and ΔV as

$$\Delta P \approx \frac{R}{V}\Delta T - \frac{RT}{V^2}\Delta V \tag{11}$$

Let's use this equation to estimate the change in pressure of one mole of an ideal gas if the temperature is changed from 273.15 K to 274.00 K and the volume is changed from 10.00 L to 9.90 L:

$$\begin{aligned}\Delta P &\approx \frac{R}{V}\Delta T - \frac{RT}{V^2}\Delta V \\ &\approx \frac{(0.0821\ \text{L}\cdot\text{atm}\cdot\text{K}^{-1}\cdot\text{mol}^{-1})}{(10.00\ \text{L}\cdot\text{mol}^{-1})}(0.85\ \text{K}) \\ &\quad - \frac{(0.0821\ \text{L}\cdot\text{atm}\cdot\text{K}^{-1}\cdot\text{mol}^{-1})(273.15\ \text{K})}{(10.00\ \text{L}\cdot\text{mol}^{-1})^2}(-0.10\ \text{L}\cdot\text{mol}^{-1}) \\ &\approx 0.0294\ \text{atm}\end{aligned}$$

For comparison, in this particularly simple case, we can calculate the exact change in P from

$$\begin{aligned}\Delta P &= \frac{RT_2}{V_2} - \frac{RT_1}{V_1} \\ &= (0.0821\ \text{L}\cdot\text{atm}\cdot\text{K}^{-1}\cdot\text{mol}^{-1})\left(\frac{274.00\ \text{K}}{9.90\ \text{L}\cdot\text{mol}^{-1}} - \frac{273.15\ \text{K}}{10.00\ \text{L}\cdot\text{mol}^{-1}}\right) \\ &= 0.0297\ \text{atm}\end{aligned}$$

You can see that Equation 11 gives us a good estimate of ΔP.

The total differential of the pressure of a gas described by the van der Waals equation is (Problem 5)

$$dP = \frac{R}{V-b}dT + \left[\frac{2a}{V^3} - \frac{RT}{(V-b)^2}\right]dV \tag{12}$$

We know that this expression is the total differential of P given by the van der Waals equation, but suppose we are given an arbitrary expression, say

$$\frac{RT}{V-b}dT + \left[\frac{RT}{(V-b)^2} - \frac{a}{TV^2}\right]dV \tag{13}$$

and are asked to determine the equation of state $P = P(T, V)$ that leads to Equation 13. In fact, a simpler question is to ask if there even is a function $P(T, V)$ whose total differential is given by Equation 13. How can we tell?

If there is such a function $P(T, V)$, then its total differential is

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV$$

Using the fact that mixed second partial derivatives are equal (provided they are continuous functions of T and V), we have the requirement that

$$\left(\frac{\partial^2 P}{\partial V \partial T}\right) = \left[\frac{\partial}{\partial V}\left(\frac{\partial P}{\partial T}\right)_V\right]_T$$

and

$$\left(\frac{\partial^2 P}{\partial T \partial V}\right) = \left[\frac{\partial}{\partial T}\left(\frac{\partial P}{\partial V}\right)_T\right]_V$$

must be equal. If we apply this requirement to Equation 13, we find that

$$\frac{\partial}{\partial T}\left[\frac{RT}{(V-b)^2} - \frac{a}{TV^2}\right] = \frac{R}{(V-b)^2} + \frac{a}{T^2V^2}$$

and

$$\frac{\partial}{\partial V}\left(\frac{RT}{V-b}\right) = -\frac{RT}{(V-b)^2}$$

Thus, we see that the cross-derivatives are not equal, so the expression given by Equation 13 is not the differential of any function $P(T, V)$. The differential given by Equation 13 is called an *inexact differential*.

We can obtain an example of an *exact differential* simply by explicitly differentiating any function $P(T, V)$, such as we did for the van der Waals equation to obtain Equation 12. The mixed second derivatives in Equation 12 are equal, as they must be for an exact differential.

Example 3:
Is

$$\left[\frac{R}{V-\beta} + \frac{\alpha}{2T^{3/2}V(V+\beta)}\right]dT + \left[-\frac{RT}{(V-\beta)^2} + \frac{\alpha(2V+\beta)}{T^{1/2}V^2(V+\beta)^2}\right]dV \tag{14}$$

where α and β are constants, an exact differential?

SOLUTION: We evaluate the two derivatives

$$\left\{\frac{\partial}{\partial V}\left[\frac{R}{V-\beta} + \frac{\alpha}{2T^{3/2}V(V+\beta)}\right]\right\}_T = -\frac{R}{(V-\beta)^2} - \frac{\alpha(2V+\beta)}{2T^{3/2}V^2(V+\beta)^2}$$

and

$$\left\{\frac{\partial}{\partial T}\left[-\frac{RT}{(V-\beta)^2} + \frac{\alpha(2V+\beta)}{T^{1/2}V^2(V+\beta)^2}\right]\right\} = -\frac{R}{(V-\beta)^2} - \frac{\alpha(2V+\beta)}{2T^{3/2}V^2(V+\beta)^2}$$

These derivatives are equal, so Equation 14 represents an exact differential.

We know how to tell if an expression is the total differential of some function, but is it possible to determine what the function is? Let's go back to Equation 10, which is the total differential of the pressure given by the ideal gas equation of state:

$$dP = \frac{R}{V}dT - \frac{RT}{V^2}dV$$

Because Equation 10 is an exact differential, we know that

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{R}{V} \quad \text{and} \quad \left(\frac{\partial P}{\partial V}\right)_T = -\frac{RT}{V^2} \tag{15}$$

Integrate the first of these two equations with respect to T, treating V as a constant, to get

$$P = \frac{RT}{V} + f(V) \tag{16}$$

where $f(V)$ is an arbitrary function of V that occurs because we integrated with respect to T, but with V constant. The function $f(V)$ corresponds to the constant of integration in "ordinary" integration. Now take the partial derivatives of Equation 16 with respect to V to obtain

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{RT}{V^2} + \frac{df(V)}{dV}$$

Equate this result to the second of Equations 15 and get

$$\frac{df}{dV} = 0$$

or

$$f(V) = \text{constant}$$

Substitute this result into Equation 16 to get

$$P = \frac{RT}{V} + \text{constant}$$

We can determine that the value of the constant here must equal zero because we know that the pressure of a gas approaches zero as the volume increases without bound. Thus we see that

$$P = \frac{RT}{V}$$

Example 4:
Determine P from the exact differential

$$dP = \frac{R}{V-b}\,dT + \left[\frac{2a}{V^3} - \frac{RT}{(V-b)^2}\right] dV$$

where a and b are constants.

SOLUTION: Integrate

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{R}{V-b}$$

"partially" to obtain

$$P = \frac{RT}{V-b} + f(V)$$

where $f(V)$ is an arbitrary function of V to be determined. Now differentiate P partially with respect to V and use the above expression for dP to write

$$-\frac{RT}{(V-b)^2} + \frac{df}{dV} = \frac{2a}{V^3} - \frac{RT}{(V-b)^2}$$

or

$$\frac{df}{dV} = \frac{2a}{V^3}$$

Integration gives

$$f(V) = -\frac{a}{V^2} + \text{constant}$$

So

$$P = \frac{RT}{V-b} - \frac{a}{V^2} + \text{constant}$$

But $P \to 0$ as $V \to \infty$, so "constant"$= 0$, and we have the van der Waals equation.

6.5 Problems

1. Determine ϵ_1 and ϵ_2 in Equation 5 and show that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$ and that $\Delta y \to dy$ if $f(x, y) = x \sin y + x^2 e^y$.
2. Show that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ in Equation 5 as $\Delta x \to 0$ and $\Delta y \to 0$ and that $\Delta y \to dy$ if $f(x, y) = e^x \cos xy^2$.
3. Determine the total differential of (a) $f(x, y) = x^2 \sin y$ and (b) $g(u, v) = (u^3 + v)e^v$.
4. Determine the total differential of (a) $f(x, y, z) = x^2 y + y^2 x + z^2 x$ and (b) $f(x, y, z) = \dfrac{xyz}{1+z^2}$.

5. Determine dP if (van der Waals equation for one mole) $P = \dfrac{RT}{V-b} - \dfrac{a}{V^2}$.

6. Let $z(x, y)$ satisfy the relation $F(x, y, z) = 0$ where F is a continuously differentiable function. Show that $\dfrac{\partial z}{\partial x} = -\dfrac{\partial F/\partial x}{\partial F/\partial z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{\partial F/\partial y}{\partial F/\partial z}$, where $\partial F/\partial z \neq 0$.

7. Use the result of the previous problem to find $\partial z/\partial x$ and $\partial z/\partial y$ at the point (1, 1, 1) for $x^3 + xy^2z + z^5 = 3$.

8. Use the result of Problem 6 to determine dz implicitly for

   (a) $x^2yz = 1$ **(b)** $z^5 + 6xz^2 + 8y^2z = 3$

9. The volume of an ellipsoid is $V = \frac{4}{3}\pi abc$, where a, b, and c are the lengths of the semiaxes. Calculate the uncertainty in V if $a = b = 10.0 \pm 0.050$ and $c = 8.00 \pm 0.050$. Calculate the relative uncertainty.

10. The surface tension of a liquid may be determined by observing how high the liquid will rise in a capillary tube. For the case of water in a glass capillary, the surface tension γ is given by $\gamma = \frac{1}{2}\rho g r h$, where ρ is the density of water ($0.998\ \text{kg}\cdot\text{dm}^{-3}$), g is the acceleration due to gravity ($9.81\ \text{m}\cdot\text{s}^{-2}$), r is the radius of the capillary tube, and h is the height that the water rises. Calculate the uncertainty in γ if $h = (42 \pm 0.15)$ mm and $r = (0.35 \pm 0.010)$ mm.

11. Is $\pi r^2 dh + 2\pi rh dr$ an exact differential? Is $\pi r dh + \pi h^2 dr$?

12. Is $dx = C_V(T)dT + \dfrac{nRT}{V}dV$ an exact or inexact differential? The quantity $C_V(T)$ is simply an arbitrary function of T. Is dx/T an exact differential?

13. Is $(2xy + y^2)dx + (x^2 + 2xy)dy$ an exact differential? If it is, determine $f(x, y)$ to within an additive constant.

14. Given that $df = 2x \sin y\, dx + (x^2 \cos y + e^y)dy$ is an exact differential, determine $f(x, y)$ to within an additive constant.

6.6 The Directional Derivative and the Gradient

Consider some quantity such as temperature or electrostatic potential that varies in space, which we express by writing $w = f(x, y, z)$. The various partial derivatives of $f(x, y, z)$ show how w changes as we change one of the independent variables while the other two are held constant. For example, $(\partial f/\partial x)_{y,z}$ expresses how w varies in a plane perpendicular to the x axis. We wish to generalize these ideas and be able to determine how $w = f(x, y, z)$ varies in an arbitrary direction, rather than just in the x, y, and z directions separately. Consider some point (x_0, y_0, z_0) and a unit vector emanating from that point (Figure 6.31).

The unit vector is specified by its three direction cosines, $a = \cos\alpha$, $b = \cos\beta$, and $c = \cos\gamma$, and we write the unit vector as $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$. The parametric equations of the straight line coincident with $\mathbf{u}$ are

$$\begin{aligned} x &= x_0 + as \\ y &= y_0 + bs \\ z &= z_0 + cs \end{aligned} \tag{1}$$

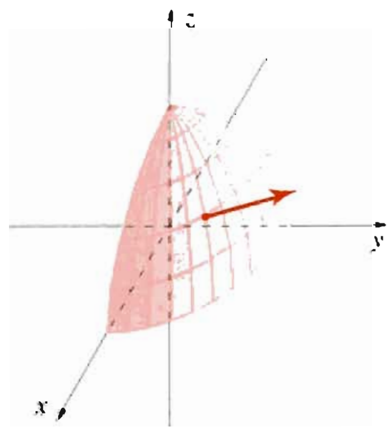


Figure 6.31
A unit vector emanating from a point on a surface $z = f(x, y)$. The unit vector is specified by its three direction cosines, $a = \cos\alpha$, $b = \cos\beta$, and $c = \cos\gamma$.

where $-\infty \leq s < \infty$. The potential now is a function of a single variable s, and the derivative of $f(x, y, z)$ in the direction of $\mathbf{u}$ is given by the chain rule:

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \tag{2}$$

where all the derivatives in Equation 2 are evaluated at the point (x_0, y_0, z_0). The right side of Equation 2 can be written as the dot product of two vectors:

$$\frac{df}{ds} = \nabla f \cdot \mathbf{u} \tag{3}$$

where

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} \tag{4}$$

is evaluated at the point (x_0, y_0, z_0) and where

$$\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k} \tag{5}$$

The derivative df/ds is called the *directional derivative* of $w = f(x, y, z)$ in the direction $\mathbf{u}$. Note that if $\mathbf{u} = \mathbf{i}$, for example, then the directional derivative is simply $(\partial f/\partial x)_{y,z}$. Thus, the directional derivative is an extension of partial derivatives such as $(\partial f/\partial x)_{y,z}$ to an arbitrary direction. The vector in Equation 4 occurs frequently in physical problems and is called the *gradient vector of* $f(x, y, z)$ or simply the *gradient of* $f(x, y, z)$. The gradient of f is often denoted by grad f.

Example 1:
Find the gradient of $f(x, y, z) = x^2 - yz + xz^2$ at the point $(1, 1, 1)$.

SOLUTION: We use Equation 4:

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} = (2x + z^2)\,\mathbf{i} - z\,\mathbf{j} + (2xz - y)\,\mathbf{k}$$

At the point $(1, 1, 1)$, we have grad $f = 3\,\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Example 2:
Find the directional derivative of $f(x, y, z) = xy^2z^3$ at the point $(3, 2, 1)$ in the direction of the vector $\mathbf{v} = -2\,\mathbf{i} - \mathbf{j} + \mathbf{k}$.

SOLUTION: The partial derivatives of $f(x, y, z)$ are

$$f_x = \frac{\partial f}{\partial x} = y^2z^3; \qquad f_y = \frac{\partial f}{\partial y} = 2xyz^3; \qquad f_z = \frac{\partial f}{\partial z} = 3xy^2z^2$$

and so

$$\text{grad } f = \nabla f = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

The unit vector **u** in the same direction as **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-2\,\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{6}}$$

Using Equation 3, the directional derivative is

$$\frac{df}{ds} = \nabla f \cdot \mathbf{u} = \frac{-2y^2z^3 - 2xyz^3 + 3xy^2z^2}{\sqrt{6}}$$

$$= \frac{-8 - 12 + 36}{\sqrt{6}} = \frac{16}{\sqrt{6}} \quad \text{at } (3, 2, 1)$$

Example 3:
Find the directional derivative of $\phi(x, y, z) = x^3 + 2xy^2 + yz^2$ from the point $P_1 = (1, 2, 1)$ toward the point $P_2 = (-1, 0, 1)$.

SOLUTION:

$$\phi_x(1, 2, 1) = 11; \qquad \phi_y(1, 2, 1) = 9; \qquad \phi_z(1, 2, 1) = 4$$

The vector from P_1 to P_2 is

$$\mathbf{v} = (-1 - 1)\,\mathbf{i} + (0 - 2)\,\mathbf{j} + (1 - 1)\,\mathbf{k} = -2\,\mathbf{i} - 2\,\mathbf{j}$$

and the unit vector in the direction of **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{1}{\sqrt{2}}\,\mathbf{i} - \frac{1}{\sqrt{2}}\,\mathbf{j}$$

Therefore,

$$\frac{d\phi}{ds} = \nabla\phi \cdot \mathbf{u} = (11\,\mathbf{i} + 9\,\mathbf{j} + 4\,\mathbf{k}) \cdot \left(\frac{-\mathbf{i} - \mathbf{j}}{\sqrt{2}}\right)$$

$$= -\frac{11}{\sqrt{2}} - \frac{9}{\sqrt{2}} = -\frac{20}{\sqrt{2}} \quad \text{at } (1, 2, 1)$$

The gradient vector ∇f has an important interpretation. If θ is the angle between ∇f and **u**, then Equation 3 says that

$$\frac{df}{ds} = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta \tag{6}$$

because $|\mathbf{u}| = 1$. The maximum value of $\cos\theta = 1$, and this occurs when $\mathbf{u} = \nabla f/|\nabla f|$. Thus, we see that the maximum value of the directional derivative occurs when $\mathbf{u}$ points in the same direction as ∇f, or that ∇f points in the direction in which $f(x, y, z)$ increases most rapidly, and the rate of increase is given by the magnitude of ∇f. Similarly, $-\nabla f$ points in the direction in which $f(x, y, z)$ decreases most rapidly.

Example 4:

Suppose that the temperature T throughout a body varies as $T(x, y, z) = 100 + xyz$. Find the maximum rate of increase in temperature (with respect to distance) at the point $(1, 1, 1)$ and the direction in which it occurs. What's the direction of the most rapid rate of decrease in temperature?

SOLUTION: The maximum rate of increase in temperature at the point $(1, 1, 1)$ is in the direction of the gradient at that point

$$\begin{aligned}\nabla T &= yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}\\ &= \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{at } (1, 1, 1)\end{aligned}$$

The magnitude of the increase is given by $|\nabla T| = \sqrt{3}$. The direction of the most rapid rate of decrease in temperature is given by $-\nabla T$, or $-\mathbf{i} - \mathbf{j} - \mathbf{k}$.

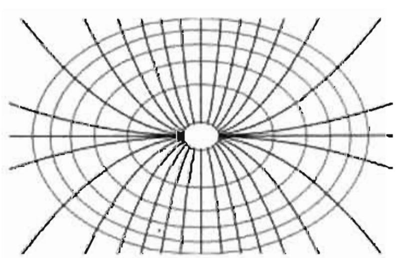


Figure 6.32
A set of level curves (color) for the surface $z = f(x, y)$ and the path of ∇f, which follows the direction of steepest descent (black).

We can give a nice physical interpretation to the gradient of a function. Recall that the level curves of $z = f(x, y)$ are the curves $z =$ constant in the xy-plane (Figure 6.32). If we consider these curves to be expressed by the parametric equations $x(t)$, $y(t)$, then the total differential of $f(x, y) = c$ at point $(x_0, y_0) = (x(t_0), y(t_0))$ is

$$df = \left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} \left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} \left(\frac{dy}{dt}\right)_{t_0} = 0$$

which says that

$$\nabla f = \left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} \mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} \mathbf{j}$$

and

$$\mathbf{v} = \left(\frac{dx}{dt}\right)_{t_0} \mathbf{i} + \left(\frac{dy}{dt}\right)_{t_0} \mathbf{j}$$

are orthogonal at the point (x_0, y_0, z_0), provided $\nabla f \neq \mathbf{0}$. The vector $\mathbf{v} = d\mathbf{r}/dt$ is tangent to the level curve $f(x, y) = c$, however, so ∇f is normal to the level curve. The path traced out by ∇f in Figure 6.32 is normal to each level curve that it crosses and follows the direction of steepest descent. For a set of equipotential curves, $-\nabla f$ represents the corresponding electric field and denotes the path that a charged particle will follow.

Example 5:

The electrostatic potential produced by a dipole moment μ located at the origin and directed along the x axis is given by

$$\phi(x, y, z) = \frac{\mu x}{(x^2 + y^2 + z^2)^{3/2}} \qquad (x, y, z \neq 0)$$

Derive an expression for the electric field **E** associated with this potential.

SOLUTION:

$$\mathbf{E} = -\nabla\phi = -\frac{\partial\phi}{\partial x}\,\mathbf{i} - \frac{\partial\phi}{\partial y}\,\mathbf{j} - \frac{\partial\phi}{\partial z}\,\mathbf{k}$$

$$= \frac{2x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}\,\mathbf{i} + \frac{3xy}{(x^2 + y^2 + z^2)^{5/2}}\,\mathbf{j} + \frac{3xz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

Figure 6.33 shows a set of equipotentials (color) and the force field (black) given by **E**.

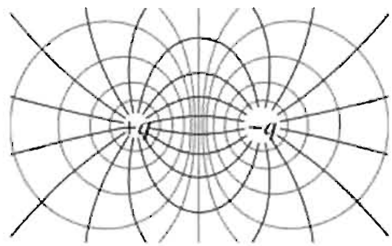


Figure 6.33
The equipotentials (color) and the electric field (black) of an electric dipole formed by equal and opposite charges.

Before we finish this section, we will show how to find a vector normal to a surface at some given point on the surface. We're going to see in the next chapter that the flow of a substance or the flow of energy as heat through a small surface area can be expressed in terms of a unit vector that is normal to the surface area. Let the surface be given by $f(x, y, z) = c$, which we may consider to be a special case of $F(x, y, z) = 0$. By an extension of our argument that ∇f is normal to the level curves $f(x, y) =$ constant, we find that $\nabla f(x, y, z)$ is normal to the surface $F(x, y, z) = 0$. The unit normal vector **n** to the surface is given by

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} \tag{7}$$

Let's use Equation 7 to determine the unit vector that is normal to the spherical surface described by $x^2 + y^2 + z^2 = a^2$ at the point $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$. The normal vector is given by

$$\begin{aligned}\nabla f &= \nabla(x^2 + y^2 + z^2 - a^2)\\ &= 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}\\ &= \frac{2a}{\sqrt{3}}\,\mathbf{i} + \frac{2a}{\sqrt{3}}\,\mathbf{j} + \frac{2a}{\sqrt{3}}\,\mathbf{k}\end{aligned}$$

and the unit normal vector $\mathbf{n}$ is

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$$

in agreement with our result in Example 6.3.4 (Figure 6.27).

Example 6:
Find the equation of the tangent plane to the spherical surface described by $x^2 + y^2 + z^2 = a^2$ at the point $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$.

SOLUTION: If $\mathbf{r}_0$ is a vector from the origin to the point (x_0, y_0, z_0) and $\mathbf{r}$ is a vector to any point in the tangent plane, then $\mathbf{r} - \mathbf{r}_0$ lies in the tangent plane. The tangent plane (x_0, y_0, z_0) is normal to the gradient at that point, so we have

$$\nabla F(x_0, y_0, z_0) \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \tag{8}$$

The gradient of $x^2 + y^2 + z^2 = a^2$ at $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$ is $(2a/\sqrt{3})(\mathbf{i} + \mathbf{j} + \mathbf{k})$, so Equation 8 tells us that the equation of the tangent plane is given by

$$\frac{2a}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) \cdot \left[\left(x - \frac{a}{\sqrt{3}}\right)\mathbf{i} + \left(y - \frac{a}{\sqrt{3}}\right)\mathbf{j} + \left(z - \frac{a}{\sqrt{3}}\right)\mathbf{k}\right] = 0$$

or

$$x + y + z - \sqrt{3}\,a = 0$$

in agreement with the result that we obtained in Example 6.3.5 (Figure 6.28).

6.6 Problems

1. Determine $\nabla\phi$ for $\phi(x, y) = x^2y + 3xy^3$.
2. Determine $\nabla\phi$ for $\phi(x, y, z) = \sin xy + xye^z$.
3. Find the directional derivative of $\phi(x, y) = e^x \cos y$ at the point $(0, 0)$ in the direction of $\mathbf{u} = \mathbf{i}$.
4. Find the directional derivative of $\phi(x, y) = \sin x \cos y$ at the point $(\pi/3, -2\pi/3)$ in the direction of $\mathbf{v} = \mathbf{i} - \mathbf{j}$.
5. Find the directional derivative of $\phi(x, y, z) = xyz$ at the point $(1, -1, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
6. Find the largest directional derivative of $\phi(x, y, z) = x^3 + y^2 + z$ at the point $(0, 0, 0)$.
7. Determine the largest directional derivative of $\phi(x, y, z) = (x^2 + y^2)e^z$ at the point $(1, 1, 0)$.
8. Find the region where $\nabla\phi$ is parallel to the xy-plane if $\phi(x, y, z) = x^2 + y^2 + z^2 + 2xy + 3yz + 8xz$.
9. Use gradients to find the angle between the two families of curves $xy = c_1$ and $x^2 - y^2 = c_2$.

10. Suppose the temperature throughout a region in space varies as $T(x, y, z) = x^2 + yz$. Find the direction of the most rapid increase in temperature.

11. Suppose the electrostatic potential is given by $\phi(x, y, z) = 100 - x^2 - y^2 - z^2$. In what direction does it increase most rapidly from the point $(3, -4, 5)$?

12. Suppose that the surface of a hill can be described by $z = 1000e^{-(2x^2+y^2)/200}$. If you are standing at the point $x = 5$, $y = 10$, in which direction should you go in order to descend most rapidly? What is your rate of descent?

13. Show that the paths of steepest ascent for the surface described by $z = \frac{1}{2}x^2 + y^2$ are of the form $y = kx^2$.

14. The equipotential surfaces about a charge q located at the origin are $\phi(x, y, z) = \dfrac{q}{4\pi\epsilon_0(x^2 + y^2 + z^2)^{1/2}}$, where ϵ_0 is the permittivity of free space. Show that the lines of force are radial lines emanating from the origin.

15. The electrostatic potential due to parallel lines of charge of linear densities λ and $-\lambda$ is given by $\phi(x, y, z) = \dfrac{\lambda}{2\pi\epsilon_0} \ln \dfrac{r_2}{r_1}$, where r_j is the perpendicular distance from the line of charge to the point (x, y, z). Sketch the equipotential surfaces and the lines of force.

16. Determine the unit normal vector to the surface of the ellipsoid described by $2x^2 + y^2 + z^2 = 4$ at the point $(1, 1, 1)$.

17. Determine the equation of the tangent plane for the previous problem.

18. Determine the unit normal vector to the surface described by $xyz = 1$ at the point $(-1, 1, -1)$.

19. Determine the angle with which the line $\mathbf{r}(t) = (t, t, t)$ intersects the surface described by $x^2 + y^2 + 2z^2 = 1$ in the first octant.

20. Find the equation of the tangent line to the surface described by $x^2 - 3xy + y^2 = -1$ at the point $(1, 2)$.

21. Find the equation of the tangent line to the surface described by $y + \sin xy = 1$ at the point $(0, 1)$.

6.7 Taylor's Formula in Several Variables

Recall from Section 1.6 that if $f(x)$ is a function of a single variable x and if $f(x)$ has continuous derivatives $f'(x)$, $f''(x)$, ..., $f^{(n+1)}$ in a closed interval $\alpha \le x \le \beta$, then Taylor's formula with a remainder term is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + R_n \tag{1}$$

where R_n is the remainder term

$$R_n = \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \tag{2}$$

where $0 < \xi < x$.

Now consider a function $f(x, y)$ of two variables all of whose first $n + 1$ partial derivatives are continuous in a closed region of the xy-plane. Let $F(t)$ be defined as $f(x, y) = f(a + ht, b + kt)$, where a, b, h, and k are constants.

Applying Equation 1 to $F(t)$ gives

$$F(t) = F(0) + tF'(0) + \frac{t^2}{2!}F''(0) + \cdots + \frac{t^n}{n!}F^{(n)}(0)$$
$$+ \frac{t^{n+1}}{(n+1)!}F^{(n+1)}(\theta) \tag{3}$$

where $0 < \theta < t$. We can now evaluate the $F^{(j)}(0)$ in Equation 3. First of all, $F(0) = f(a, b)$. Using $x = a + ht$ and $y = b + kt$, the chain rule gives us

$$F'(t) = \frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = hf_x + kf_y \tag{4}$$

It is convenient to express Equation 4 in operator notation:

$$F'(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) \tag{5}$$

in which the operator $(h\partial/\partial x + k\partial/\partial y)$ acts on $f(x, y)$. At $t = 0$, Equation 5 becomes

$$F'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y)\bigg|_{(x,y)=(a,b)} \tag{6}$$

We can evaluate $F''(0)$ in Equation 3 in a similar manner:

$$F''(t) = \frac{dF'(t)}{dt} = \frac{d}{dt}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) = \frac{d}{dt}(hf_x + kf_y)$$
$$= h\frac{df_x}{dt} + k\frac{df_y}{dt}$$

We use the chain rule to evaluate df_x/dt and df_y/dt and obtain

$$F''(t) = h\left(\frac{\partial f_x}{\partial x}\frac{dx}{dt} + \frac{\partial f_x}{\partial y}\frac{dy}{dt}\right) + k\left(\frac{\partial f_y}{\partial x}\frac{dx}{dt} + \frac{\partial f_y}{\partial y}\frac{dy}{dt}\right)$$
$$= (h^2 f_{xx} + hkf_{xy}) + (hkf_{yx} + k^2 f_{yy})$$
$$= h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy} \tag{7}$$

We can write Equation 7 in operator notation by writing

$$F''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) \tag{8}$$

In Equation 8, the square on the operator in parentheses means that the operator $h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}$ acts *sequentially* on $f(x, y)$. In other words,

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x,y)$$

(Problems 1 through 7 provide you a little practice with operator notation if it is completely new to you.) If we let $t=0$ in Equation 8, we have

$$F''(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)\bigg|_{(x,y)=(a,b)} \tag{9}$$

A repeated application of this procedure shows that (Problem 8)

$$F^{(n)}(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x,y)\bigg|_{(x,y)=(a,b)} \tag{10}$$

If we substitute these results for the derivatives of $F(t)$ at $t=0$ into Equation 3, we obtain

$$\begin{aligned} f(a+ht,b+kt) &= f(a,b)+t\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x,y)\bigg|_{(x,y)=(a,b)} \\ &\quad+\frac{t^2}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)\bigg|_{(x,y)=(a,b)} \\ &\quad+\cdots+\frac{t^n}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x,y)\bigg|_{(x,y)=(a,b)} \\ &\quad+\frac{t^{n+1}}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x,y)\bigg|_{(x,y)=(a+\theta h,b+\theta k)} \end{aligned} \tag{11}$$

where $0<\theta<1$. Letting $t=1$ in Equation 11 finally gives *Taylor's formula for a function of two variables*:

$$\begin{aligned} f(a+h,b+k) &= f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x,y)\bigg|_{(x,y)=(a,b)} \\ &\quad+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)\bigg|_{(x,y)=(a,b)} \\ &\quad+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x,y)\bigg|_{(x,y)=(a,b)}+R_n \end{aligned} \tag{12}$$

where the remainder is

$$R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x,y)\bigg|_{(x,y)=(a+\theta h,b+\theta k)} \tag{13}$$

where $0<\theta<1$. In other words, the remainder is evaluated somewhere along the line segment that connects (a,b) to $(a+h,b+k)$.

If we let $n = 0$ in Equation 12, we have

$$f(a+h, b+k) = f(a, b) + hf_x(a+\theta h) + kf_y(b+\theta k) \tag{14}$$

where $0 < \theta < 1$. Equation 14 is the *mean value theorem* for a function of several variables and is a direct extension of Equation 1.4.1.

Equation 12 up to quadratic terms is

$$\begin{aligned} f(x, y) = f(a, b) &+ (x-a)f_x(a, b) + (y-b)f_y(a, b) \\ &+ \frac{(x-a)^2}{2} f_{xx}(a, b) + (x-a)(y-b)f_{xy}(a, b) \\ &+ \frac{(y-b)^2}{2} f_{yy}(a, b) \end{aligned} \tag{15}$$

where we have used $x = a + h$ and $y = b + k$.

Example 1:
Expand e^{xy} about $(0, 0)$ up to quadratic terms.

SOLUTION: We'll use Equation 15.

$$f_x(0, 0) = f_y(0, 0) = f_{xx}(0, 0) = f_{yy}(0, 0) = 0$$

and

$$f_{xy}(0, 0) = 1$$

Therefore,

$$e^{xy} = 1 + xy \ + \text{third order terms and higher}$$

Note that we can obtain the same result more easily by just using $e^z = 1 + z + z^2/2 + \cdots$. This is not the case for the next Example, however.

Example 2:
Expand $\ln(x^2 + y^2)$ about $(1, 1)$ up to quadratic terms.

SOLUTION: The necessary derivatives are

$$f_x = \frac{2x}{x^2+y^2} \quad \text{and} \quad f_x(1, 1) = 1$$

$$f_y = \frac{2y}{x^2+y^2} \quad \text{and} \quad f_y(1, 1) = 1$$

$$f_{xx} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2} \quad \text{and} \quad f_{xx}(1, 1) = 0$$

$$f_{yy} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} \quad \text{and} \quad f_{yy}(1, 1) = 0$$

$$f_{xy} = -\frac{4xy}{(x^2 + y^2)^2} \quad \text{and} \quad f_{xy}(1, 1) = -1$$

Therefore, using Equation 15, we find that

$$\begin{aligned}\ln(x^2 + y^2) &= \ln 2 + (x - 1) + (y - 1) - (x - 1)(y - 1) + \cdots \\ &= \ln 2 - 2 + 2x + 2y - xy \ + \text{third order terms and higher}\end{aligned}$$

Example 3:
Expand $\sin xy$ in a Taylor expansion around $a = \pi/2$ and $b = 1$ up to quadratic terms.

SOLUTION: Let $h = x - a$ and $k = y - b$ with $a = \pi/2$ and $b = 1$ in Equation 15:

$$f(x, y) = \sin xy; \qquad f\left(\frac{\pi}{2}, 1\right) = 1$$

$$f_x = -y \cos xy; \qquad f_x\left(\frac{\pi}{2}, 1\right) = 0$$

$$f_y = -x \cos xy; \qquad f_y\left(\frac{\pi}{2}, 1\right) = 0$$

$$f_{xx} = -y^2 \sin xy; \qquad f_{xx}\left(\frac{\pi}{2}, 1\right) = -1$$

$$f_{yy} = -x^2 \sin xy; \qquad f_{yy}\left(\frac{\pi}{2}, 1\right) = -\left(\frac{\pi}{2}\right)^2$$

$$f_{xy} = -xy \sin xy; \qquad f_{xy}\left(\frac{\pi}{2}, 1\right) = -\frac{\pi}{2}$$

$$\sin xy = 1 - \frac{(x - \pi/2)^2}{2} - \frac{\pi^2 (y - 1)^2}{8} - \frac{\pi}{2}\left(x - \frac{\pi}{2}\right)(y - 1) + \cdots$$

You should be aware that most of the CAS can derive Taylor series in more than one variable. For example, the command

```
Series [ Exp [ Sin [ (x+y) ] ], { x, 0, 3 }, { y, 0, 3 } ]
```

in Mathematica gives the Taylor series of $e^{\sin(x+y)}$ about the point (0, 0) up to third-order terms in x and y:

$$e^{\sin(x-y)} = 1 + y + \frac{y^2}{2} + O(y^4) + \left(1 + y - \frac{y^3}{3}\right)x$$
$$+ \left(\frac{1}{2} - \frac{3y^2}{4} - \frac{2y^2}{3}\right)x^2 + \left(-\frac{y}{2} - \frac{2y^2}{3} - \frac{y^2}{12}\right)x^3 + O(x^4)$$

and the line

```
Series [ Sin [ x • y ], { x, Pi / 2, 2 }, { y, 1, 2 } ]
```

gives the result in Example 3.

6.7 Problems

1. Let $\hat{A}$ be some given operator. Perform the following operations:

   (a) $\hat{A}(2x)$ $\quad \hat{A} = \dfrac{d^2}{dx^2}$ (b) $\hat{A}(x^2)$ $\quad \hat{A} = \dfrac{d^2}{dx^2} + \dfrac{d}{dx} + 3$

   (c) $\hat{A}(xy^2)$ $\quad \hat{A} = \dfrac{\partial}{\partial x} + 2\dfrac{\partial}{\partial y}$ (d) $\hat{A}(\sin xy)$ $\quad \hat{A} = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$

2. An operator is said to be linear if $\hat{A}[c_1 f_1(x) + c_2 f_2(x)] = c_1\hat{A}f_1(x) + c_2\hat{A}f_2(x)$, where c_1 and c_2 are (possibly complex) constants and where all the indicated operators are well defined. Otherwise, $\hat{A}$ is said to be nonlinear. Determine whether the following operators are linear or nonlinear:

   (a) integration (b) differentiation

   (c) square root (d) take complex conjugate

3. The operator $\hat{A}^2 f(x)$ means that $\hat{A}$ acts sequently on $f(x)$; in other words, $\hat{A}^2 f(x) = \hat{A}[\hat{A}f(x)]$. Write out the operator $\hat{A}^2$ for $\hat{A} =$

   (a) $\dfrac{d^2}{dx^2}$ (b) $\dfrac{d}{dx} + x$

   Hint: Be sure to include $f(x)$ in $\hat{A}^2 f(x)$ before carrying out the operations.

4. Evaluate $\nabla^2(\cos ax)(\cos by)(\cos cz)$, where the operator $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

5. Two operators are said to *commute* if $\hat{A}\hat{B}f(x) = \hat{B}\hat{A}f(x)$ for an arbitrary function $f(x)$. (As usual, we assume that the indicated operators are well defined.) Determine whether or not the following pairs of operators commute:

   (a) $\dfrac{d}{dx}$; $\dfrac{d^2}{dx^2} + 2\dfrac{d}{dx}$ (b) multiply by x; $\dfrac{d}{dx}$ (c) $\dfrac{\partial}{\partial x}$; $\dfrac{\partial}{\partial y}$

   Hint: Be sure to include $f(x)$ in $\hat{A}\hat{B}f(x)$ and $\hat{B}\hat{A}f(x)$, where $f(x)$ is a suitably arbitrary function.

6. Evaluate (a) $\left(1 - \dfrac{d}{dx}\right)^2 \sin x$ and (b) $\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^3 xy^3$, where h and k are constants.

7. In ordinary algebra, $(P+Q)(P-Q) = P^2 - Q^2$. Expand $(\hat{P}+\hat{Q})(\hat{P}-\hat{Q})$ where the supercarats denote that $\hat{P}$ and $\hat{Q}$ are operators. Under what conditions do we find the same result for operator algebra as we do for "ordinary" algebra?

8. Show that $F^{(3)}(t) = \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^3 f(x, y)$ in the Taylor expansion of $f(x, y)$.

9. Expand $x^2 - xy + y^2$ in a Taylor series about $(1, 1)$ up to quadratic terms and check your result by algebra.

10. Expand e^{xy} about $(1, 1)$ through quadratic terms.

11. Expand $f(x, y) = \sin xy$ about $(0, 0)$ up to quadratic terms.

12. Expand $f(x, y) = \sin xy$ about $(1, 1)$ up to quadratic terms.

13. Expand $f(x, y) = \sin x \cos y$ about $(0, 0)$ up to third-order terms and compare your result to the product of the Maclaurin expansions of $\sin x$ and $\cos y$.

14. Use any CAS to verify your result in Problem 10.

15. Use any CAS to verify your result in Problem 11.

16. Use any CAS to verify your result in Problem 12.

17. Use any CAS to verify your result in Problem 13.

6.8 Maxima and Minima

In Section 1.4, we discussed how to find maximum or minimum values of a function of a single variable over some interval. The *necessary* condition that $f(x)$ have an extremum at $x = a$ is that $f'(a) = 0$, assuming that $f(x)$ is differentiable in a neighborhood of $x = a$. Recall that later in Section 2.7, we used Taylor's formula to give a simple, direct proof of the *sufficient* conditions. Briefly, if

$$f'(a) = f''(a) = \cdots = f^{(2n-1)}(a) = 0$$

and

$$f^{(2n)}(a) \neq 0 \tag{1}$$

and if $f^{(2n)}(x)$ is continuous in some δ neighborhood of a, then

1. $f(x)$ has a local maximum value at $x = a$ if $f^{(2n)}(a) < 0$.
2. $f(x)$ has a local minimum value at $x = a$ if $f^{(2n)} > 0$.

The proof rests upon Taylor's formula, which for this case $[f'(a) = f''(a) = \cdots = f^{(2n-1)}(a) = 0$ and $f^{(2n)}(a) \neq 0]$ is

$$f(x) = f(a) + \frac{f^{(2n)}(\xi)}{(2n)!}(x-a)^{2n} \tag{2}$$

for $|h| < \delta$. But this statement simply says that $f(x, b)$ is a local maximum at $x = a$ where $f(x, y)$ is considered to be a function of x with y held constant at $y = b$. Thus, the condition that a function of a single variable be a maximum at $x = a$, namely $f'(x) = 0$ at $x = a$, becomes

$$\frac{\partial f}{\partial x} = 0 \quad \text{at } x = a,\ y = b \tag{3}$$

The same argument with $h = 0$ instead of $k = 0$ gives

$$\frac{\partial f}{\partial y} = 0 \quad \text{at } x = a,\ y = b \tag{4}$$

Furthermore, the same argument applies to the case in which $f(x, y)$ is a local minimum at (a, b), so Equations 3 and 4 must be satisfied for $f(x, y)$ to have a local extremum at (a, b). The point (a, b) is said to be a *critical point* of $f(x, y)$.

Just as in the case of a function of a single variable, however, Equations 3 and 4 are *necessary conditions*, but not sufficient conditions that $f(x, y)$ be a local extremum at (a, b). A good example of the fact that Equations 3 and 4 are necessary but not sufficient conditions that $f(x, y)$ be an extremum at (a, b) is provided by $f(x, y) = x^2 - y^2$. We see that $f_x = 2x$ and $f_y = -2y$ are both equal to zero at the point $(0, 0)$, so Equations 3 and 4 are satisfied. Yet, considered as a function of x with y held constant at $y = 0$, $f(x, 0)$ has a minimum at $(0, 0)$ because $f_{xx} = 2 > 0$ at $(0, 0)$, while considered as a function of y with x held constant at $x = 0$, $f(0, y)$ has a maximum at $(0, 0)$ because $f_{yy} = -2 < 0$ at $(0, 0)$. Thus, the surface $z = f(x, y)$ has a maximum in the yz-plane and a minimum in the xz-plane at the origin, as shown in Figure 6.37. The critical point in this case is called a *saddle point* for obvious reasons.

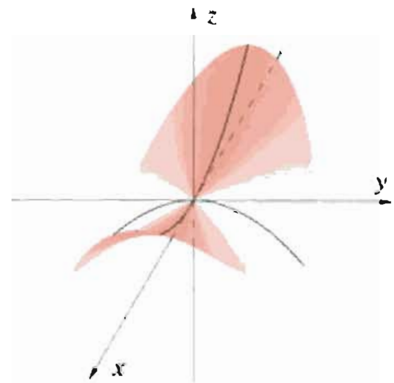


Figure 6.37
The saddle point of $z = x^2 - y^2$ at the point $(0, 0)$.

Example 2:
Find the critical points for

$$f(x, y) = x^3 + y^3 - x - 6y + 10$$

SOLUTION: The equations for the critical points are

$$\frac{\partial f}{\partial x} = 3x^2 - 1 = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 3y^2 - 6 = 0$$

which yields the critical points $x = \pm 1/\sqrt{3}$ and $y = \pm\sqrt{2}$. The critical points are at

$$\left(\frac{1}{\sqrt{3}}, \sqrt{2}\right),\ \left(-\frac{1}{\sqrt{3}}, \sqrt{2}\right),\ \left(\frac{1}{\sqrt{3}}, -\sqrt{2}\right),\ \text{and}\ \left(-\frac{1}{\sqrt{3}}, -\sqrt{2}\right)$$

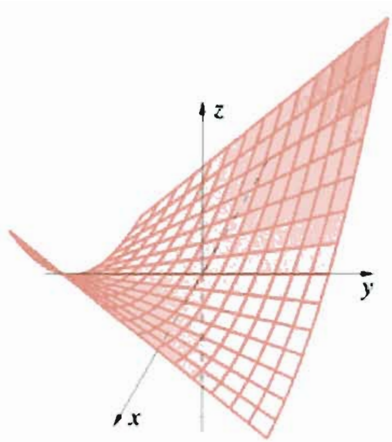


Figure 6.43
The behavior of $z = \frac{1}{2}x^2 - xy$ around its critical point at (0,0).

Example 4:
Investigate the critical points of

$$f(x, y) = \frac{1}{2}x^2 - xy$$

SOLUTION: The critical points are given by

$$f_x = x - y = 0 \qquad f_y = -x = 0$$

and so we see that there is a critical point at (0, 0). The second partial derivatives evaluated at the critical point are

$$f_{xx} = 1 > 0; \qquad f_{yy} = 0; \quad f_{xy} = -1 < 0; \qquad D < 0$$

Therefore, the critical point is a saddle point (Figure 6.43).

6.8 Problems

The first 6 problems review maxima and minima for functions of a single variable.

1. Test $f(x) = 1 - |x|$ for a maximum value. Why doesn't the first derivative test apply?
2. Test $f(x) = 1 - x^4$ for extrema.
3. Test $f(x) = x^x$ for an extremum in the interval (0, 1]. What about in the open interval (0, 1)?
4. Test $f(x) = \int_0^x (t^2 - 1)^3 dt$ for an extremum in the interval $(-2, 2)$. (Do not evaluate the integral.)
5. Test $f(x) = x(1 - e^x)\sin x$ for an extremum at $x = 0$.
6. Show that if $f^{(k)}(a) = 0$ for $k = 2, \ldots, 2n$ and $f^{(2n+1)}(a) \neq 0$, then $f(x)$ has an inflection point at $x = a$. *Hint*: Use the fact that $f(x) - [f(a) + f'(a)(x - a)]$ is the difference between $f(x)$ and its tangent line at a.
7. Find all the critical points of (a) $f(x, y) = xy^2(3x + 6y - 2)$ and (b) $f(x, y) = x^4 + y^4 - 2(x - y)^2$.
8. Find all the critical points of (a) $f(x, y) = x^3 - 4x^2 - xy - y^2$ and (b) $f(x, y) = (y - x^2)(2 - x - y)$.
9. Classify all the critical points of the two functions in Problem 7.
10. Classify all the critical points of the two functions in Problem 8.
11. Classify all the critical points of (a) $f(x, y) = xy$ and (b) $f(x, y) = x^2 + 2xy + 2y^2 + 4x$.
12. Classify all the critical points of (a) $f(x, y) = 3x^2 + 6xy + 2y^3 + 12x - 24y$ and (b) $f(x, y) = x^2 + y^2 + 4x - 2y + 3$.
13. Derive Equation 6.
14. Prove that $f(a, b)$ is a local minimum if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$.
15. The sum of three numbers is 150. What is the maximum value of their product?
16. Determine the shape and maximum volume of a rectangular parallelepiped whose total surface area is A.

6.9 The Method of Lagrange Multipliers

The final problem of the previous section asks you to determine the shape and maximum volume of a rectangular parallelepiped whose total surface area is fixed at some value A. Therefore, you have to maximize $V = abc$ with $A = 2(ab + bc + ac)$ fixed. We can use the equation for A to eliminate a from the equation of V and then minimize:

$$V = \frac{bc}{b+c}\left(\frac{A}{2} - bc\right)$$

Setting both $\partial V/\partial b$ and $\partial V/\partial c$ equal to zero gives $b = c = (A/6)^{1/2}$, and then substituting this result back into the equation for A gives $a = b = c = (A/6)^{1/2}$. Thus the rectangular shape that produces the maximum value for a fixed surface area is a cube.

This problem involves maximizing V with the *constraint* that $A =$ constant. The occurrence of the constraint means that a, b, and c are not independent, and so we used the expression for A to eliminate one of them in terms of the others and then treat V as a function of *two* independent variables. In this case, it was easy to solve the expression for A for a in terms of b and c; however, it may not always be so. In this section, we will present a general, rather powerful, method to maximize (or minimize) a function with one or more constraints. This method is due to Lagrange and is called the *method of Lagrange multipliers*.

Suppose we wish to find an extremum of the function $f(x_1, x_2, \ldots, x_n)$ of n variables, where the n variables must satisfy some auxiliary condition (a constraint)

$$g(x_1, x_2, \ldots, x_n) = \text{constant} \tag{1}$$

If $f(x_1, x_2, \ldots, x_n)$ is an extremum at some point $P = (x_{10}, x_{20}, \ldots, x_{n0})$, then

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n = 0 \tag{2}$$

at that point. If there were no constraint, $x_1, x_2, \ldots, x_n$ would be independent, and so we could vary $dx_1, dx_2, \ldots, dx_n$ in Equation 2 independently. For example, we could let them all be zero except for dx_k, say, and find that $\partial f/\partial x_k = 0$ for $k = 1, 2, \ldots, n$, giving us the necessary condition for an unconstrained extremum. Because of Equation 1, however, $x_1, x_2, \ldots, x_n$ are *not* independent, so the dx_j cannot be varied independently, and $\partial f/\partial x_k = 0$ for $k = 1, 2, \ldots, n$ is *not* the condition for an extremum in this case.

In the method of Lagrange multipliers, we multiply the total differential of g,

$$dg = \frac{\partial g}{\partial x_1}dx_1 + \frac{\partial g}{\partial x_2}dx_2 + \cdots + \frac{\partial g}{\partial x_n}dx_n = 0 \tag{3}$$

by a parameter (a multiplier) λ and subtract the result from Equation 2 to obtain

$$\left(\frac{\partial f}{\partial x_1} - \lambda\frac{\partial g}{\partial x_1}\right)dx_1 + \left(\frac{\partial f}{\partial x_2} - \lambda\frac{\partial g}{\partial x_2}\right)dx_2 + \cdots + \left(\frac{\partial f}{\partial x_n} - \lambda\frac{\partial g}{\partial x_n}\right)dx_n = 0 \tag{4}$$

Equation 3 may be viewed as a relation between the n differentials dx_j, so only $n - 1$ of the dx_j in Equation 4 are independent. We have not specified λ yet, and are free to choose it as we wish. Let's choose dx_n to be the differential that depends upon all the others through Equation 3. Then we'll choose λ to eliminate the term in dx_n in Equation 4 by letting λ be given by

$$\frac{\partial f}{\partial x_n} - \lambda\frac{\partial g}{\partial x_n} = 0 \tag{5}$$

at the extremum. Now the remaining differentials in Equation 4 *are* independent, so make the usual argument and find that

$$\frac{\partial f}{\partial x_j} - \lambda\frac{\partial g}{\partial x_j} = 0 \qquad j = 1, 2, \ldots, n-1 \tag{6}$$

at the extremum.

Equations 5 and 6 may now be combined to write

$$\frac{\partial f}{\partial x_j} - \lambda\frac{\partial g}{\partial x_j} = 0 \qquad j = 1, 2, \ldots, n \tag{7}$$

Equation 1 and Equations 7 constitute a set of $n + 1$ equations in $n + 1$ unknowns, $(x_{10}, x_{20}, \ldots, x_{n0})$ and λ. We will illustrate the use of Equations 7 with several Examples.

Example 1:
Find the shape and the maximum volume of the paralellepiped, whose total surface area is A, using the method of Lagrange multipliers.

SOLUTION: We have

$$V = abc \qquad \text{with} \qquad g = ab + bc + ac = A/2 = \text{constant}$$

Equations 7 become

$$\frac{\partial V}{\partial a} - \lambda\frac{\partial g}{\partial a} = bc - \lambda(b + c) = 0$$

$$\frac{\partial V}{\partial b} - \lambda\frac{\partial g}{\partial b} = ac - \lambda(a + c) = 0$$

$$\frac{\partial V}{\partial c} - \lambda\frac{\partial g}{\partial c} = ab - \lambda(a + b) = 0$$

It's easy to solve each of these equations for λ and then find that $a = b = c$. Substituting this result into the equation for A gives $a = b = c = (A/6)^{1/2}$. Thus, the shape is a cube and its volume is $V = (A/6)^{3/2}$.

Example 2:
The equation

$$x + y + z = 1$$

represents a plane that cuts the axes at the points (1, 0, 0), (0, 1, 0), and (0, 0, 1) (see Figure 6.44). Find the shortest distance from the origin to this plane.

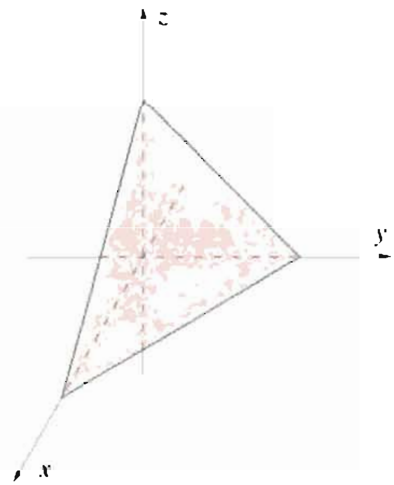


Figure 6.44
The plane $x + y + z = 1$ cuts the axes at the points (1,0,0), (0,1,0), and (0,0,1).

SOLUTION: We shall minimize $d^2 = x^2 + y^2 + z^2$ with the constraint $g = x + y + z = 1$. Equations 7 become

$$\frac{\partial d^2}{\partial x} - \lambda \frac{\partial g}{\partial x} = 2x - \lambda = 0$$

$$\frac{\partial d^2}{\partial y} - \lambda \frac{\partial g}{\partial y} = 2y - \lambda = 0$$

$$\frac{\partial d^2}{\partial z} - \lambda \frac{\partial g}{\partial z} = 2z - \lambda = 0$$

which yields $x = y = z$. Substituting this result into $x + y + z = 1$ gives $x = y = z = 1/3$, and $d = 1/\sqrt{3}$.

Example 3:
Determine the shape of a triangle of maximum area for a fixed perimeter l.

SOLUTION: Start with Heron's formula for the area of a triangle with sides a, b, and c:

$$A = [s(s-a)(s-b)(s-c)]^{1/2}$$

where $s = (a + b + c)/2$. (This formula is in almost any mathematical handbook.) We want to maximize A, subject to $a + b + c = l = 2s =$ constant. Equations 7 give

$$\frac{\partial A}{\partial a} - \lambda \frac{\partial l}{\partial a} = -\frac{[s(s-b)(s-c)]^{1/2}}{2(s-a)^{1/2}} - \lambda = 0$$

$$\frac{\partial A}{\partial b} - \lambda\frac{\partial l}{\partial b} = -\frac{[s(s-a)(s-c)]^{1/2}}{2(s-b)^{1/2}} - \lambda = 0$$

$$\frac{\partial A}{\partial c} - \lambda\frac{\partial l}{\partial c} = -\frac{[s(s-a)(s-b)]^{1/2}}{2(s-c)^{1/2}} - \lambda = 0$$

These three equations give

$$\lambda = \frac{A}{2(s-a)} = \frac{A}{2(s-b)} = \frac{A}{2(s-c)}$$

Therefore, $a = b = c$ and the triangle is an equilateral triangle.

One nice feature of this method is that it can be readily generalized to more than one constraint. If we wish to find an extremum of $f(x_1, x_2, \ldots, x_n)$ subject to two constraints

$$\begin{aligned} g(x_1, x_2, \ldots, x_n) &= \text{constant} \\ h(x_1, x_2, \ldots, x_n) &= \text{constant} \end{aligned} \tag{8}$$

the resulting equations are

$$\frac{\partial f}{\partial x_j} - \lambda\frac{\partial g}{\partial x_j} - \mu\frac{\partial h}{\partial x_j} = 0 \qquad j = 1, 2, \ldots, n \tag{9}$$

where both λ and μ are Lagrange multipliers.

Example 4:

One of the most well known applications of the method of Lagrange multipliers is in statistical mechanics. Consider a gas consisting of N molecules. Each of these molecules can be in a quantum-mechanical state j with energy E_j with $j = 1, 2, \ldots, M$ (M may be infinite). If there are N_j molecules in the state j, then the fraction of molecules in state j, $f_j = N_j/N$, is given by the following procedure. Maximize

$$W(N_1, N_2, \ldots, N_M) = \frac{N!}{N_1!N_2!\cdots N_M!} \tag{10}$$

with respect to each N_j with

$$\sum_j N_j = N = \text{constant} \tag{11}$$

$$\sum_j N_j E_j = \text{another constant} \tag{12}$$

Show that this procedure leads to

$$f_j = \frac{N_j}{N} = \frac{e^{-\mu E_j}}{\sum_{j=1}^{M} e^{-\mu E_j}}$$

where μ is one of the Lagrange multipliers.

SOLUTION: To maximize $W(N_1, N_2, \ldots, N_M)$, it is convenient to take the logarithm of W and then use Stirling's approximation for $N_j!$ (Section 3.1). This procedure is valid because $\ln x$ is a monotonically increasing function of x. Taking logarithms, we have

$$\ln W = \ln N! - \sum_{j=1}^{M} N_j! = N \ln N - N - \sum_{j=1}^{M} N_j \ln N_j + \sum_{j=1}^{M} N_j$$

The two constraints are

$$g = \sum_{j=1}^{M} N_j = N \qquad \text{and} \qquad h = \sum_{j=1}^{M} N_j E_j = \text{constant}$$

Equations 9 give

$$\frac{\partial \ln W}{\partial N_j} - \lambda \frac{\partial g}{\partial N_j} - \mu \frac{\partial h}{\partial N_j} = -\ln N_j - 1 + 1 - \lambda - \mu E_j = 0$$

Solving for N_j gives

$$N_j = e^{-\lambda} e^{-\mu E_j}$$

We can eliminate λ by summing both sides of this expression over N_j and using Equation 11:

$$N = e^{-\lambda} \sum_{j=1}^{M} e^{-\mu E_j}$$

Eliminating e^{λ}, we have

$$\frac{N_j}{N} = \frac{e^{-\mu E_j}}{\sum_{j=1}^{M} e^{-\mu E_j}}$$

It turns out that $\mu = 1/k_{BT}$, where k_B is the Boltzmann constant and T is the kelvin temperature, and the resulting expression for N_j/N is called the **Boltzmann distribution**.

You may have noticed in the examples that although we found an extremum of $f(x_1, \ldots, x_n)$, we glossed over whether a result was a maximum or a minimum. The determination of the actual nature of an extremum requires quite a bit more effort. Usually in physically or geometrically motivated problems it's clear whether an extremum is a maximum or a minimum, but you must be careful about it.

6.9 Problems

1. Determine the maximum distance from the origin to the ellipse described by $3x^2 + 3y^2 + 4xy = 2$ (Figure 6.45).

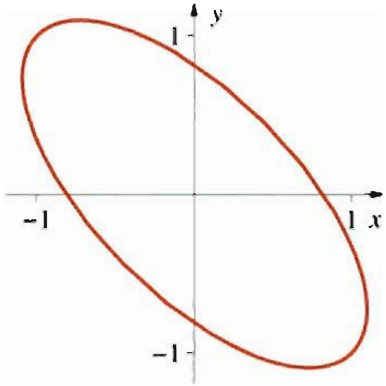


Figure 6.45
The graph of the ellipse described by $3x^2 + 3y^2 + 4xy = 2$.

2. Find the rectangle of maximum area that can be inscribed in the ellipse described by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (Figure 6.46).

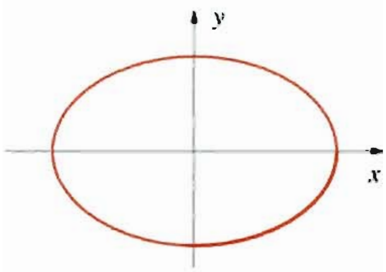


Figure 6.46
The graph of the ellipse described by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

3. Generalize the previous problem to a rectangular parallelepiped inscribed in an ellipsoid described by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

4. Find the rectangle of fixed perimeter l that has the shortest diagonal.

5. Find the maximum value of $ax + by + cz$ on the surface of a sphere of radius R.

6. Find the rectangular parallelepiped of fixed total edge length with maximal surface area.

7. Find the maximum volume of a right cylinder with *total* surface area A (including top and bottom).

8. Find the maximum volume of a right cylinder cone of fixed lateral surface area A. *Hint*: $V = \frac{1}{3}\pi R^2 h$ and $A = \pi R s$. (See Figure 6.47.)

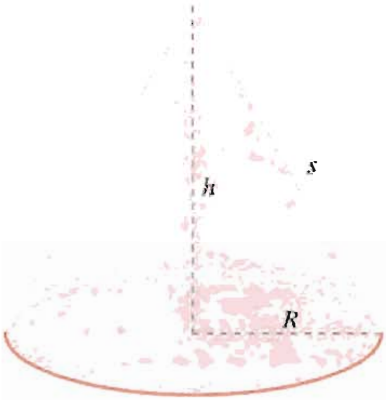


Figure 6.47
A right cylinder cone showing the quantities R, h, and s.

9. Determine the minimum distance from the origin to the plane described by $3x + 2y + z = 6$.

10. Determine the minimum distance from the origin to the plane described by $ax + by + cz = 1$. Check your result against the previous problem.

11. Find the shortest distance from the point $(3, -2, 1)$ to the plane described by $2x - 3y + z = 1$.

12. Find the shortest distance from the origin to the surface described by $x^2 + y^2 - 2xz = 4$.

13. Find the shortest distance from the origin to the line formed from the intersection of the two planes described by $x + y + z = 1$ and $x + 2y + 3z = 6$.

14. Find the point on the plane described by $3x - 2z = 0$ such that the sum of the squared distances from the points $P_1 = (1, 1, 1)$ and $P_2 = (2, 3, 4)$ is a minimum.

15. Find the maximum value of z on the ellipsoid described by $3x^2 + 5y^2 + 2z^2 - 10xy + 2xz = 10$.

16. The two planes $2x + y - z = 1$ and $x - y + z = 2$ intersect to form a line in space. Find the shortest distance from the origin to this line.

17. Determine the coordinates of the points on the spherical surface described by $x^2 + y^2 + z^2 = 1$ that are closest to and farthest from the point $(1, 3, 4)$.

18. Maximize the product $x_1 x_2 \cdots x_n$, subject to $x_1 + x_2 + \cdots + x_n = a$, where the x_j's and a are positive.

6.10 Multiple Integrals

If $f(x, y)$ is defined in some closed region R in the xy-plane, the *double integral* of $f(x, y)$ is denoted by

$$I = \iint_R f(x, y)\,dA \tag{1}$$

where dA is an element of area in the xy-plane. Just as the single integral $\int_a^b f(x)\,dx$ has a geometric interpretation of being equal to the net area between $f(x)$ and the x axis over the interval (a, b), a double integral has a geometric interpretation of being the net volume between the surface $f(x, y)$ and the xy-plane.

Equation 1 is the limit of a Riemann sum. The region R is subdivided into n subregions ΔR_i of area ΔA_i. We then form the sum

$$I_n = \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i \tag{2}$$

where the point (x_i, y_i) lies in ΔR_i. If $f(x, y)$ is continuous in R, then the limit of Equation 2 as the number of subregions increases and the size of each one decreases is equal to Equation 1.

Consider the region shown in Figure 6.48, where any line parallel to the y axis crosses the boundary of R at two points at the most. Suppose that the top boundary

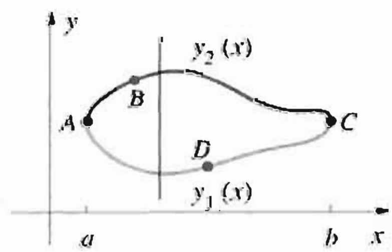


Figure 6.48
An illustration of a two-dimensional region R where any line parallel to the y axis crosses the boundary of R at two points at the most.

of R (ABC in the figure) is described by $y_2(x)$ and the lower boundary (ADC) by $y_1(x)$, where $y_1(x)$ and $y_2(x)$ are continuous in $a \le x \le b$. In this case, we can evaluate the integral in Equation 1 by letting $dA = dxdy$ and integrating over x and y in turn. Consider a vertical strip of width dx in Figure 6.48. The contribution to I of this strip between the curves ABC and ADB is given by

$$d\sigma(x) = dx \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy$$

Note that $d\sigma(x)$ is a function of x. We can find I over the region in Figure 6.48 by adding the areas of all the vertical strips between $x = a$ and $x = b$, which amounts to integrating $d\sigma(x)$ over x from a to b, or in an equation

$$I = \int_a^b \left\{ \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy \right\} dx \qquad (3)$$

Equation 3 is called an *iterated integral*. As we said above, the y integration in brackets produces a function of x, which is then integrated between the limits $a \le x \le b$.

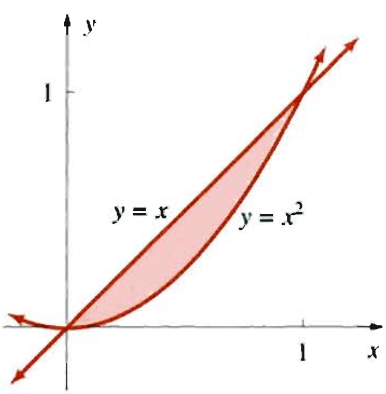


Figure 6.49
The shaded region whose area is determined in Example 1. It is the area bounded by $y = x$ (upper) and $y = x^2$ (lower) between $x = 0$ and $x = 1$.

Example 1:
Use Equation 3 to evaluate the area of the shaded region shown in Figure 6.49.

SOLUTION: In this case $f(x, y) = 1$ in Equation 3. The shaded region is bounded by $y = x$ (upper) and $y = x^2$ (lower).

Equation 3 gives

$$I = \int_0^1 \left\{ \int_{x^2}^{x} dy \right\} dx$$

$$= \int_0^1 (x - x^2) dx = \frac{1}{6}$$

Example 2:
Use Equation 3 to evaluate the integral

$$I = \iint_R x^2 y\, dxdy$$

where R is the region in the first quadrant bounded by $y = x^2$ and $y = 2$ (Figure 6.50).

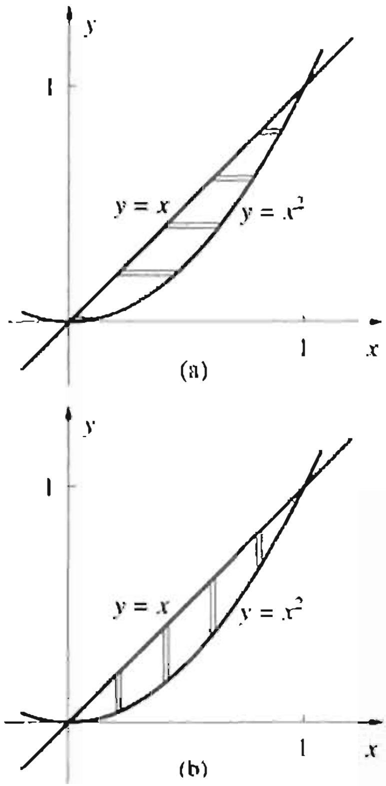


Figure 6.53
The region in the first quadrant bounded by the curves $y = x^2$ and $y = x$. (a) The integration is over x first (y to $y^{1/2}$) and then over y (0 to 1). (b) The integration is over y first (x^2 to x) and then over x (0 to 1).

Let's use Equation 4 to evaluate the integral

$$I = \iint_R x\, dx dy$$

where R is the region in the first quadrant bounded by the curves $y = x^2$ and $y = x$ (Figure 6.53a). In this case, the limits of the x integration are $x = y$ to $x = y^{1/2}$ and the limits of the y integration are 0 to 1. Therefore,

$$I = \int_0^1 \left\{ \int_y^{y^{1/2}} x\, dx \right\} dy = \int_0^1 \left[\frac{y}{2} - \frac{y^2}{2} \right] dy = \frac{1}{12}$$

We could also have used Equation 3 to evaluate I. In this case, the limits of the y integration are x^2 to x and the limits of the x integration are 0 to 1 (Figure 6.53b), and so

$$I = \int_0^1 \left\{ x \int_{x^2}^{x} dy \right\} dx = \int_0^1 x(x - x^2)\, dx = \frac{1}{12}$$

In the first case, we find the areas of horizontal strips and then add them up in the vertical direction (Figure 6.53a) and in the second case, we find the areas of vertical strips and then add them up in the horizontal direction (Figure 6.53b).

In Equations 3 and 4, we used curly brackets to emphasize which variable is integrated first. This is not standard notation. Equation 3 is often written as

$$I = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y) dy\, dx \tag{5}$$

with the understanding that the inner integration is performed first.

Another notation, which is commonly used in physics, is to write Equation 3 as

$$I = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy\, f(x, y) \tag{6}$$

where the y integration is thought of as an operator that acts on $f(x, y)$ that produces a function of x followed by the x integration as an operator. The order of the two operators is from left to right as usual. For example, if $f(x, y) = xy$ and $y_2(x) = 2x$ and $y_1(x) = x$ in Equation 6, then

$$\begin{aligned} I &= \int_a^b dx\, x \int_x^{2x} dy\, y \\ &= \int_a^b dx\, x \left(\frac{4x^2 - x^2}{2} \right) = \frac{3}{2} \int_a^b dx\, x^3 = \frac{3}{8}(b^4 - a^4) \end{aligned}$$

The key point here is to realize that you perform the integrations sequentially from right to left; you wait for the y integration to produce its result *before* you integrate over x. This notation is very convenient and well worth using.

It is often beneficial to reverse the order of integration in a double integral. A double integral such as

$$I = \int_0^x du \int_0^u dt\, v(t) \tag{7}$$

occurs in the statistical mechanics of fluids. Let's reverse the order of integration and integrate over u first. It is usually helpful to draw a picture illustrating the integration region. Figure 6.54a shows this region for Equation 7. We integrate over t from 0 to the line $t = u$ for some arbitrary value of u (the horizontal strip in Figure 6.54a) and then over u from 0 to x (we add up the horizontal strips). Figure 6.54b illustrates the reverse order. We integrate over u from t to x (the vertical strip in Figure 6.54b) and then over t from 0 to x (add up the vertical strips). Thus,

$$I = \int_0^x du \int_0^u dt\, v(t) = \int_0^x dt\, v(t) \int_t^x du = \int_0^x dt (x - t) v(t) \tag{8}$$

We were able to reduce Equation 7 to a single integral by reversing the order of integration. Example 4 provides another example where it is beneficial to reverse the order of integration.

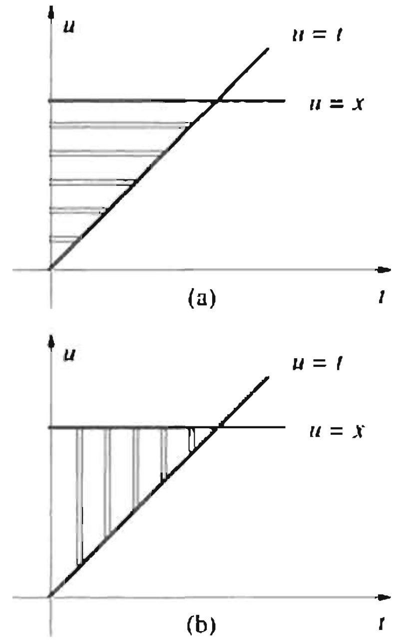


Figure 6.54
Pictorial aids to the evaluation of the integral in Equation 7 by reversing the orders of integration. In (a), we integrate over t first, and in (b), we integrate over u first.

Example 4:
Recall from Section 3.4 that the exponential integral $E_1(t)$ is defined by

$$E_1(t) = \int_1^{\infty} \frac{e^{-tz}}{z} dz = \int_t^{\infty} \frac{e^{-u}}{u} du$$

Show that

$$\int_0^x dt\, E_1(t) = 1 - e^{-x} + x E_1(x)$$

SOLUTION: First we write

$$\int_0^x dt\, E_1(t) = \int_0^x dt \int_t^{\infty} du \frac{e^{-u}}{u}$$

The integration scheme here is presented in Figure 6.55a. We first integrate over u from t to ∞ (the horizontal strips) and then over t from 0 to x (adding up the horizontal strips). The reversed integration scheme is shown in Figure 6.55b. In this case, we integrate over t from 0 to u if $u < x$ and then from 0 to x if $u \geq x$. Then we add up all the vertical strips by integrating over u from 0 to ∞. Using Figure 6.55, we have

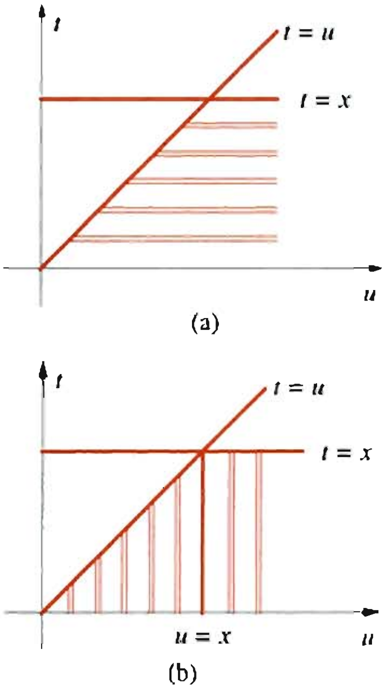


Figure 6.55
Pictorial aids to the evaluation of $\int_0^x dt\, E_1(t)$ in Example 4. In (a), we first integrate over u from t to ∞ (horizontal strips) and then over t from 0 to x (adding up the horizontal strips). In (b), we first integrate over t from 0 to u if $u < x$ and from 0 to x if $u \geq x$ (vertical strips) and then we integrate over u from 0 to ∞.

$$\int_0^x dt\, E_1(t) = \int_0^x du\, \frac{e^{-u}}{u} \int_0^u dt + \int_x^\infty du\, \frac{e^{-u}}{u} \int_0^x dt$$
$$= \int_0^x du\, e^{-u} + x \int_x^\infty du\, \frac{e^{-u}}{u}$$
$$= 1 - e^{-x} + x E_1(x)$$

Equations 3 and 4 are readily extended to three dimensions. For example,

$$I = \int_a^b \left\{ \int_{y_1(x)}^{y_2(x)} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z) dz \right] dy \right\} dx \tag{9}$$

or in operator notation

$$I = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{g_1(x,y)}^{g_2(x,y)} dz\, f(x, y, z) \tag{10}$$

Notice that in either Equation 9 or 10, the z integration yields a function of x and y, then the y integration yields a function x, and finally the x integration between a and b yields I. Equations 9 and 10 are best illustrated with an Example.

Example 5:
The three-dimensional region bounded by the planes $x = 0$, $y = 0$, and $z = 0$ and the plane $x + y + z = a$ is shown in Figure 6.56. Evaluate

$$I = \iiint_R xyz\, dx dy dz$$

over that region.

SOLUTION: Integrate over x first. The limits of integration are 0 to $a - y - z$ (Figure 6.56). This gives

$$I = \iint dz dy \int_0^{a-y-z} xyz\, dx = \iint dz dy\, \frac{yz(a - y - z)^2}{2}$$

Now integrate over y; the integration limits are 0 to $a - z$.

$$I = \frac{1}{2} \int_0^a dz\, z \int_0^{a-z} dy\, [y(a - z)^2 - 2y^2(a - z) + y^3] = \frac{1}{2} \int_0^a dz\, z \frac{(a - z)^4}{12}$$
$$= \frac{a^6}{24} B(2, 5) = \frac{a^6}{24} \frac{\Gamma(2)\Gamma(5)}{\Gamma(7)} = \frac{a^6}{720}$$

where $B(2, 5)$ is a beta function (Section 3.1).

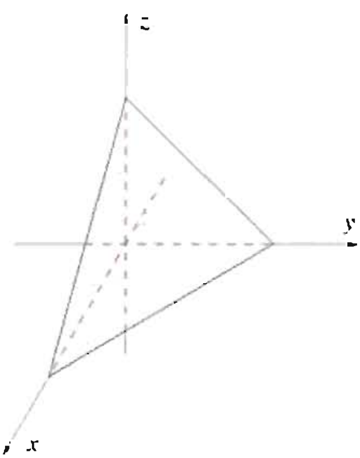


Figure 6.56
The region bounded by the plane $x + y + z = a$ and the three coordinate planes $x = 0$, $y = 0$, and $z = 0$.

7. Calculate the volume of the region bounded by $x = 1$, $x = 2$; $y = 0$, $y = x^2$; and $z = 0$, $z = 1/x$ (see Figure 6.58).

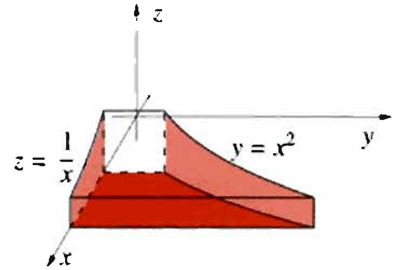


Figure 6.58
An illustration of the volume to be determined in Problem 7.

8. Evaluate the integral $\iiint x^3y^2z\,dxdydz$ over the same volume as in Problem 7.

9. Find the center of mass of the solid with boundary planes $x = 0$, $y = 0$, $z = 0$, and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, if it has a uniform density $\rho = 1$ (Figure 6.59). (See Problem 3 for the definition of center of mass coordinates.)

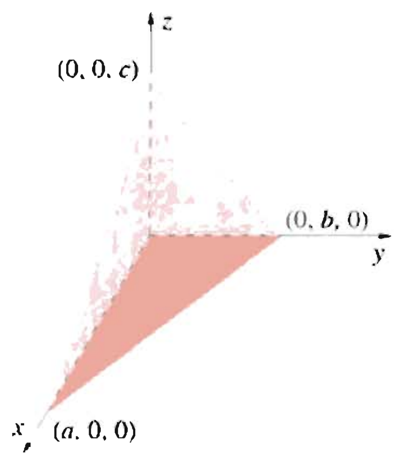


Figure 6.59
An illustration of the body whose center of mass is to be determined in Problem 9.

10. The moment of inertia of a solid body about the z axis is given by
$I_z = \frac{1}{M}\iiint(x^2 + y^2)\rho(x, y, z)\,dxdydz$. Calculate I_z for the body in Problem 9.

11. If $\sigma(x, y)$ is the mass density per unit area in a two-dimensional sheet, the total mass of the sheet is given by $M = \iint_R \sigma(x, y)dxdy$. Calculate the total mass of a circular sheet of radius a if $\sigma(x, y) = x^2y^2$.

12. Re-do Problem 1 by reversing the order of the integration.

13. Evaluate the following integrals by reversing the order of integration:

(a) $\int_0^1 dy \int_0^{\cos^{-1} y} dx\ \sec x$ (b) $\int_0^1 dy \int_y^1 dx\ \frac{ye^x}{x}$

14. Show that $\int_0^1 dy \int_y^1 dx\ ye^{x^3} = \frac{1}{6}(e - 1)$.

15. Show that the moment of inertia of a uniform right circular cylinder of radius R and height h about its longitudinal axis is equal to $R^2/2$.

16. Show that $\int_0^\infty e^{-at}\operatorname{erf}(bt)dt = \frac{e^{a^2/4b^2}}{a}\operatorname{erfc}\frac{a}{2b}$. *Hint*: You need the integral (Problem 3.3.11) $\int_0^\infty e^{-(at^2+2bt+c)}dt = \left(\frac{\pi}{4a}\right)^{1/2} e^{(b^2-ac)/a}\operatorname{erfc}\frac{b}{\sqrt{a}}$.

17. Show that $\int_0^x C(u)du = xC(x) - \frac{1}{\pi}\sin\frac{\pi x^2}{2}$, where $C(u)$ is the Fresnel integral (Section 3.3) $C(u) = \int_0^u dz\cos\frac{\pi z^2}{2}$.

18. Show that $\int_0^{\pi/2} dx \int_x^{\pi/2} \frac{\sin u}{u} du = 1$.

19. Show that $\int_0^\infty e^{-st}\operatorname{erf}\{(at)^{1/2}\}\,dt = \frac{1}{s}\left(\frac{a}{s+a}\right)^{1/2}$

20. Use any CAS to evaluate $\int_0^1 dx \int_0^{x^2} dy\, e^{y/x}$.

21. Use any CAS to evaluate $\int_0^2 dy \int_0^y dx\, \sqrt{x^2+y^2}$.

22. Use any CAS to evaluate the integral in Problem 13a.

23. Use any CAS to evaluate the integral in Problem 11.

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CHAPTER 7

Vector Calculus

The central theme of this chapter is that of a vector field, which is a vector function defined at each point (x, y, z) in a region of space. For example, the electric field throughout some region or the velocity of a fluid at each point in the fluid can be represented as a vector field. Generally, a vector function $\mathbf{v} = \mathbf{v}(x, y, z)$ represents a vector field. Vector fields play prominent roles in all areas of the physical sciences and engineering. The fundamental laws of classical mechanics, electricity and magnetism, fluid mechanics, elasticity, heat flow, and other areas are expressed in terms of vector fields. The vector field methods that we develop in this chapter allow us to express these laws in a compact form and to derive many useful relations from them in a precise, straightforward manner. We shall first introduce the gradient, the divergence, and the curl operations, which are the three central differential operator quantities involving vector fields. Then, in Section 2, we discuss line integrals and show how they are used to calculate the work done in mechanical systems. We finish this section with a discussion of Green's theorem in the plane, which is used widely in mathematics, as we shall see in later chapters. After discussing surface integrals in Section 3, we go on to discuss the divergence theorem, which relates a surface integral to a volume integral, and Stokes's theorem, which relates a line integral to a surface integral. Many authors refer to these two theorems as the "big" theorems of vector analysis. We shall see that both of these theorems have a great variety of applications to physical problems.

7.1 Vector Fields

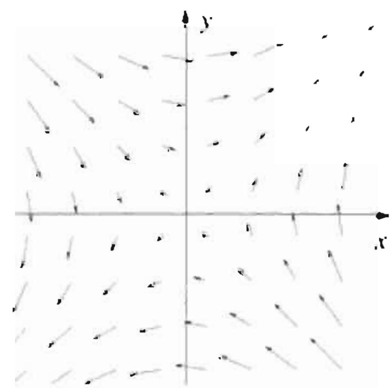

Figure 7.1
A pictorial representation of the vector field described by $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$.

As we said in the introduction, a vector field is a vector function defined at each point (x, y, z) in a region of space. Consequently, we can associate a vector with each point of the region, which we can represent graphically as we do in Figure 7.1 for the two-dimensional vector function $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$. This vector field represents fluid flow in toward the origin from the second and fourth quadrants being forced out into the first and third quadrants.

In the previous chapter, we learned that the gradient of a scalar function $\phi = \phi(x, y, z)$ is defined by

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\,\mathbf{i} + \frac{\partial\phi}{\partial y}\,\mathbf{j} + \frac{\partial\phi}{\partial z}\,\mathbf{k} \tag{1}$$

The gradient arose naturally when we discussed the directional derivative of $\phi(x, y, z)$, and we learned that the maximum rate of increase of ϕ is in the direction of $\nabla\phi$ and that its magnitude is given by $|\nabla\phi|$. We also saw that if $f(x, y, z) = c$ describes a surface, then ∇f is normal to the surface, provided $\nabla f \neq \mathbf{0}$. Of course we are assuming that $f(x, y, z)$ is differentiable, for otherwise ∇f would not exist. The gradient occurs frequently in physical applications. In classical mechanics, if $V(x, y, z)$ represents the potential energy, then the corresponding force field is given by

$$\mathbf{F}(x, y, z) = -\nabla V(x, y, z)$$

In electricity and magnetism, if $V(x, y, z)$ represents the electrostatic potential, then the corresponding electric field intensity is given by

$$\mathbf{E}(x, y, z) = -\nabla V(x, y, z) = -\text{grad } V(x, y, z)$$

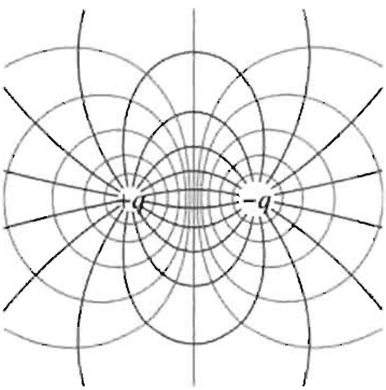

Figure 7.2
The equipotentials (color) and the electric field lines (black) due to an electric dipole situated at the origin and oriented in the x direction.

For example, the electrostatic potential due to an electric dipole located at the origin and oriented in the x direction is

$$V(x, y, z) = \frac{\mu x}{4\pi\varepsilon_0(x^2 + y^2 + z^2)} \tag{2}$$

where μ is the magnitude of the dipole moment and ε_0 is the permittivity of free space. The level curves of $V(x, y, z)$ in the xy-plane are shown in Figure 7.2.

The gradient of $V(x, y, z)$ is

$$\nabla V = \frac{\mu}{4\pi\varepsilon_0}\left[\frac{y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^2}\,\mathbf{i} - \frac{2xy}{(x^2 + y^2 + z^2)^2}\,\mathbf{j} - \frac{2xz}{(x^2 + y^2 + z^2)^2}\,\mathbf{k}\right]$$

Figure 7.2 also shows the vector field associated with ∇V in the xy-plane. Note that ∇V is perpendicular to the level curves of $V(x, y, z)$, as we learned in the previous chapter.

The gradient is also related to heat flow. Let the temperature of a substance vary according to $T(x, y, z)$. If the temperature is not uniform, energy as heat will flow from regions of higher temperature to regions of lower temperature in the direction of the maximum decrease of temperature. If $\mathbf{q}$ is the flow of energy as heat through a unit surface area (perpendicular to $\mathbf{q}$) per unit time, then *Fourier's law of heat flow* says that

$$\mathbf{q}(x, y, z) = -\kappa\nabla T(x, y, z) = -\kappa \text{ grad } T(x, y, z) \tag{3}$$

where κ is called the thermal conductivity of the substance.

The diffusion of a substance due to a concentration gradient is governed by an equation very similar to Equation 3. If $\mathbf{J}$ is the diffusive flow rate of a substance through a unit area (perpendicular to $\mathbf{J}$) per unit time, then $\mathbf{J}$ is often given to a good approximation by

$$\mathbf{J}(x, y, z) = -D\nabla c(x, y, z) = -D \operatorname{grad} c(x, y, z) \tag{4}$$

where $c(x, y, z)$ is the concentration of the substance and D is called the diffusion coefficient, whose value depends upon both the substance that is diffusing and the medium through which it is diffusing. Equation 4 is known as *Fick's law of diffusion*.

The vectorial quantities $\mathbf{q}$ and $\mathbf{J}$ in Equations 3 and 4 are called *fluxes*. Consider the situation in Figure 7.3, which shows a small area element dS whose orientation is specified by an outward unit normal vector $\mathbf{n}$. If $\mathbf{q}$ and $\mathbf{n}$ are pointing in the same direction, then the flow rate of energy as heat across the surface dS is given by $\mathbf{q}\,dS$, and has units of energy per unit time. If, on the other hand, $\mathbf{q}$ and $\mathbf{n}$ are not pointing in the same direction, then the flow rate of energy as heat across the surface dS is given by $\mathbf{q}\cdot\mathbf{n}\,dS$. For example, if $\mathbf{q}$ is parallel to dS (perpendicular to $\mathbf{n}$), then there is no flow across dS. We can define a vectorial surface element (it has orientation and area) by $d\mathbf{S} = \mathbf{n}\,dS$ and write

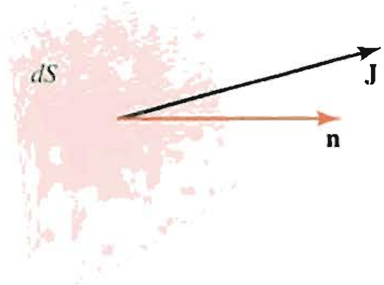

Figure 7.3
An area element dS whose orientation is specified by the normal vector $\mathbf{n}$. The vector $\mathbf{J}$ is a flux across $d\mathbf{S} = \mathbf{n}dS$.

$$\text{flow rate across } dS = \mathbf{J}\cdot d\mathbf{S} \tag{5}$$

We shall use Equation 5 several times in this chapter.

It is often convenient to view

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{6}$$

as a vector operator, called the *del operator*. Just as we can operate with ∇ on a scalar field, we can operate with ∇ on a vector field $\mathbf{A}(x, y, z)$ by taking the dot product $\nabla\cdot\mathbf{A}$. This quantity is a scalar field called the *divergence of* $\mathbf{A}$ and is written as

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{7}$$

(Note that A_x, A_y, and A_z in Equation 7 are the components of $\mathbf{A}$, and not partial derivatives.)

Example 1:
Determine the expression for div $\mathbf{F}$ if $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz^2\,\mathbf{j} + x^2yz\,\mathbf{k}$.

SOLUTION:

$$\begin{aligned}\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\\ &= y + z^2 + x^2y\end{aligned}$$

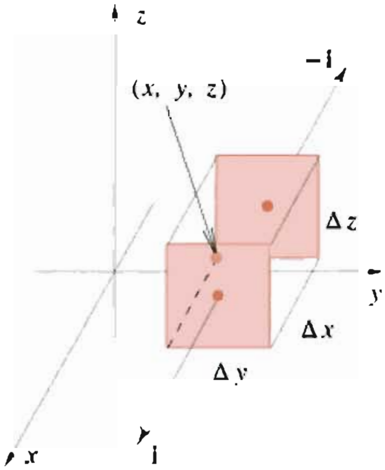

Figure 7.4
The geometry used for a derivation of the divergence theorem.

The divergence has an important physical interpretation which makes it useful for physical problems. Let's consider the total flow rate of $\mathbf{A}$ per unit volume out of the small rectangular region of volume $\Delta V = \Delta x \Delta y \Delta z$ shown in Figure 7.4. For example, $\mathbf{A}$ might be a flux of matter, given by $\mathbf{J} = \rho\,\mathbf{u}$, where ρ is the mass density and $\mathbf{u}$ is the velocity of the flow. (Note that J has units of mass per unit area per unit time.) The normal unit vector facing out of the front face (located at $x + \Delta x$) in Figure 7.4 is $\mathbf{n} = \mathbf{i}$ and the area of the face is $\Delta y \Delta z$. We can write this vectorial area as $\Delta\mathbf{S} = \Delta y \Delta z \mathbf{i}$, and thus according to Equation 5, the flow rate of $\mathbf{A}$ out of the front face is

$$\begin{aligned}\begin{matrix}\text{flow rate of } \mathbf{A} \text{ out of}\\ \text{the front face in Figure 7.4}\end{matrix} &= \int \mathbf{A}\cdot d\mathbf{S} = \int A_x(x+\Delta x, y, z)\,dydz\\ &= A_x(x+\Delta x, y_1, z_1)\Delta y \Delta z\end{aligned}$$

where, according to the mean value theorem of integration, $(x + \Delta x, y_1, z_1)$ is a point on the front surface. In other words, we have used the relation

$$\iint_{S=\Delta y \Delta z} f(y,z)dydz = f(y_1, z_1)\Delta y \Delta z$$

for a continuous function $f(y, z)$, where $y \le y_1 \le y + \Delta y$ and $z \le z_1 \le z + \Delta z$.

For small values of Δx, we can write $A_x(x + \Delta x, y, z)$ as $A_x(x, y, z) + \dfrac{\partial A_x}{\partial x}\Delta x + \cdots$ and so

$$\begin{matrix}\text{flow rate of } \mathbf{A} \text{ out of}\\ \text{the front face in Figure 7.4}\end{matrix} = \left[A_x(x,y,z) + \frac{\partial A_x}{\partial x}\Delta x\right]\Delta y \Delta z$$

where $\partial A_x/\partial x$ depends upon x, y, and z.

Similarly, the normal unit vector facing out of the back face (located at x) in Figure 7.4 is $\mathbf{n} = -\mathbf{i}$. Therefore, the flow rate of $\mathbf{A}$ out of the back face is

$$\begin{aligned}\begin{matrix}\text{flow rate of } \mathbf{A} \text{ out of}\\ \text{the back face in Figure 7.4}\end{matrix} &= \int \mathbf{A}\cdot d\mathbf{S} = -\int A_x(x, y, z)\,dydz\\ &= -A_x(x, y_1, z_1)\Delta y \Delta z\end{aligned}$$

where, once again, $y \le y_1 \le y + \Delta y$ and $z \le z_1 \le z + \Delta z$. The net flow rate of $\mathbf{A}$ through the two faces perpendicular to the x axis in Figure 7.4 is given by $(\partial A_x/\partial x)\Delta x \Delta y \Delta z$. If we let Δx, Δy, and Δz approach zero, then

$$\begin{matrix}\text{net flow rate of } \mathbf{A} \text{ out of the two surfaces}\\ \text{that are perpendicular to the } x \text{ axis in Figure 7.4}\end{matrix} = \frac{\partial A_x}{\partial x}\,dxdydz$$

We obtain similar expressions for the flow rate through the other faces in Figure 7.4, and so the total flow rate of $\mathbf{A}$ out of $dV = dxdydz$ is given by

$$\begin{aligned}\text{net flow rate of } \mathbf{A} \text{ out of} \\ \text{the rectangular region in Figure 7.4} &= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dx\,dy\,dz \\ &= \text{div}\ \mathbf{A}\ dx\,dy\,dz\end{aligned} \tag{8}$$

Thus, we see that the divergence of **A** represents the flow rate per unit volume out of a given region in space in the limit in which the volume approaches zero.

For the case where **A** is the flow of some substance out of a small volume V, then the flux is given by $\mathbf{J} = \rho\mathbf{u}$, and Equation 8 says that

$$\text{net flow rate of mass per unit volume out of } V = \text{div}\ \mathbf{J} = \text{div}\ (\rho\mathbf{u}) \tag{9}$$

Note that div **J** has units of mass per unit volume per unit time, or density per unit time. The net flow rate of mass per unit volume out of V must be equal to the rate of change of density within V, and so we have

$$\frac{\partial\rho}{\partial t} = -\text{div}(\rho\mathbf{u}) \tag{10}$$

where the minus sign simply says that if the density increases with time, then the outward flow must be negative or, in other words, inward. Equation 9 is called the *continuity equation* and is one of the most fundamental equations of the physical sciences; it is nothing less than an expression of conservation of mass in differential form.

There is a continuity equation for each constituent in a multiconstituent system. If c_k is the concentration of the kth constituent, then Equation 10 reads

$$\frac{\partial c_k}{\partial t} + \text{div}\ (\mathbf{J}_k) = 0 \tag{11}$$

Equation 11 is an expression of the conservation of the mass of the kth constituent, or the number of particles of the kth constituent. Fick's law (Equation 4) says that the flux of the kth constituent can be approximated by

$$\mathbf{J}_k = -D_k\ \text{grad}\ c_k \tag{12}$$

where D_k is the diffusion constant of the kth constituent. Substitute this approximation into Equation 11 (which is exact) to obtain

$$\frac{\partial c_k}{\partial t} = D_k\ \text{div grad}\ c_k \tag{13}$$

assuming that D_k does not vary with position. It turns out that div grad can be expressed in a fairly simple form. Problem 11 has you show that

$$\text{div grad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{14}$$

The operator expression in Equation 14 occurs frequently in physical problems and is called the *Laplacian operator* and is denoted by ∇^2. Equation 14 becomes

$$\nabla^2\phi = \text{div grad }\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{15}$$

Using this notation, Equation 13 becomes

$$\frac{\partial c_k}{\partial t} = D_k\,\nabla^2 c_k \tag{16}$$

Equation 16 is called the *diffusion equation* and models how the concentration of a substance varies as a function of space and time starting from some initial distribution. It is a partial differential equation in the unknown $c_k(x, y, z, t)$. We shall learn how to solve partial differential equations in Chapter 16, but in the following Example we investigate one particular solution of Equation 16.

Example 2:
Consider the diffusion of a substance along a long thin cylinder. In this case, the diffusion occurs along only one direction and can be described by a one-dimensional version of Equation 16:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \qquad -\infty < x < \infty$$

where we have dropped the k subscript. Show that

$$c(x, t) = \frac{c_0}{2(\pi Dt)^{1/2}}e^{-x^2/4Dt}$$

is a solution to Equation 16. Interpret this solution physically.

SOLUTION: The spatial derivatives of $c(x, t)$ are

$$\frac{\partial c}{\partial x} = -\frac{c_0 x e^{-x^2/4Dt}}{(4Dt)(\pi Dt)^{1/2}}$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{c_0 x^2 e^{-x^2/4Dt}}{(8D^2t^2)(\pi Dt)^{1/2}} - \frac{c_0 e^{-x^2/4Dt}}{(4Dt)(\pi Dt)^{1/2}}$$

and the time derivative is

$$\frac{\partial c}{\partial t} = \frac{c_0 x^2 e^{-x^2/4Dt}}{(8Dt^2)(\pi Dt)^{1/2}} - \frac{c_0 e^{-x^2/4Dt}}{4(\pi Dt)^{1/2}t}$$

Therefore, we find that

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

The solution is plotted in Figure 7.5.

Initially, all the diffusing substance is located at the origin, in the form of a spike $[c(x, t) \to c_0\delta(x)$ as $t \to 0]$. Then, as time increases, the substance

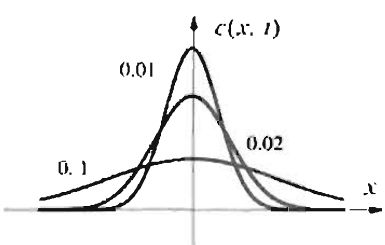

Figure 7.5
The solution to the diffusion equation given in Example 2 plotted for various values of Dt.

spreads out in both directions from the origin, and eventually becomes uniformly distributed.

Forming the dot product of ∇ with a vector $\mathbf{A}$ gives us the divergence of $\mathbf{A}$. We can also form the cross product of ∇ with $\mathbf{A}$ to give what we call the *curl of* $\mathbf{A}$, written as

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A}$$
$$= \mathbf{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{17}$$

We can write this expression formally in determinantal form

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{18}$$

Example 3:
Find curl $\mathbf{v}$ if $\mathbf{v} = xz\,\mathbf{i} + xy^2\,\mathbf{j} + y^2z\,\mathbf{k}$.

SOLUTION:

$$\text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xy^2 & y^2z \end{vmatrix} = \mathbf{i}\,(2yz - 0) + \mathbf{j}\,(x - 0) + \mathbf{k}\,(y^2 - 0)$$
$$= 2yz\,\mathbf{i} + x\,\mathbf{j} + y^2\,\mathbf{k}$$

If $\mathbf{A} = \text{grad } \phi$, we call $\phi(x, y, z)$ the *scalar potential* of the vector field $\mathbf{A}(x, y, z)$, and if $\mathbf{A} = \text{curl } \mathbf{u}$, we call $\mathbf{u}(x, y, z)$ the *vector potential* of $\mathbf{A}(x, y, z)$. We'll learn how to determine the scalar potential of $\mathbf{A}(x, y, z)$ in the next section and we'll learn how to determine the vector potential of $\mathbf{A}(x, y, z)$ in Section 5.

Just as the divergence has a nice physical interpretation, so does the curl. We saw in Chapter 5 that the velocity at a point $\mathbf{r}$ in a rigid body rotating with a rotational vector $\boldsymbol{\omega}$ is given by (Equation 5.4.14)

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

(Figure 7.6). For simplicity, let $\boldsymbol{\omega}$ be directed along the z axis, so that $\boldsymbol{\omega} = \omega\mathbf{k}$ and

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$$

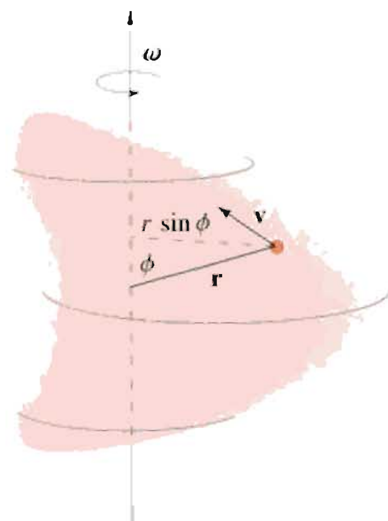

Figure 7.6
The rotation of a rigid body about the ω axis, illustrating that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

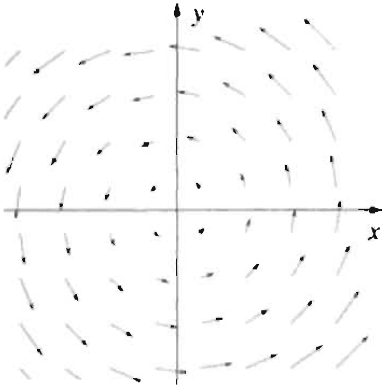

Figure 7.7
A pictorial representation of the vector field described by $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$.

Therefore,

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\,\mathbf{k} = 2\boldsymbol{\omega}$$

Thus, curl **v** is twice the angular velocity of the rotating body. Figure 7.7 shows the vector field $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$. You can see that this field represents circulatory motion about the origin.

We illustrated the physical meaning of curl **v** by considering a rotating rigid body, but the curl plays a prominent role in fluid dynamics also. Generally, if **v** is the velocity field, or the flow lines, of a fluid, then curl **v**, called the *vorticity vector of the fluid*, points in the direction about which a vortex motion takes place and is a measure of the angular velocity of the flow. The flow of a fluid in a velocity field in which curl $\mathbf{v} = \mathbf{0}$ is called *irrotational flow*. Irrotational flow has the following special property: Suppose that **v** is given by the gradient of a "velocity potential", ψ, so that $\mathbf{v} = \nabla\psi = \text{grad } \psi$ at any point in the fluid. Then

$$\text{curl } \mathbf{v} = \text{curl grad } \psi$$

But it is easy to show that

$$\text{curl grad } \psi = \mathbf{0} \tag{19}$$

for any function that has continuous second partial derivatives (Problem 12). Equation 19 tells us that the flow must be irrotational for there to be a (scalar) velocity potential whose gradient gives the velocity at any point in the fluid. Because of this important property, fluid dynamics texts spend a fair amount of time on irrotational flow.

There is another relation similar to Equation 19 that you should know: namely, (Problem 13)

$$\text{div curl } \mathbf{v} = 0 \tag{20}$$

where the components of **v** have continuous second partial derivatives. Equation 20 is consistent with the physical interpretations of the divergence and the curl. The curl represents circular motion, and so Equation 20 says that the fluid is contained within a volume enclosing the motion (no flow in or out of the volume).

To summarize the key results of this section, we have the three operations with the del operator, ∇:

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\,\mathbf{i} + \frac{\partial\phi}{\partial y}\,\mathbf{j} + \frac{\partial\phi}{\partial z}\,\mathbf{k} \tag{21}$$

$$\text{div } \mathbf{v} = \nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{22}$$

and

$$\text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \tag{23}$$

and the three "combination" relations:

$$\text{div grad } \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{24}$$

$$\text{curl grad } \phi = \mathbf{0} \tag{25}$$

and

$$\text{div curl } \mathbf{v} = 0 \tag{26}$$

There is also a number of other relations that can be proved straightforwardly from the definitions (Problem 14), such as

$$\text{div } \phi\mathbf{v} = \nabla \cdot \phi\mathbf{v} = \phi\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla\phi \tag{27}$$

$$\text{div } (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v} \tag{28}$$

$$\text{div } (\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi \tag{29}$$

$$\text{curl } \phi\mathbf{v} = \nabla\phi \times \mathbf{v} + \phi \text{ curl } \mathbf{v} \tag{30}$$

Most of the CAS can be used to calculate the gradient, divergence, and curl. Problems 20 and 21 ask you to use any CAS for such calculations.

7.1 Problems

1. Recall from the previous chapter that if $f(x, y, z) = c$ describes a surface, then ∇f is normal to the surface. Find the unit normal vector to the surface of the elliptic paraboloid described by $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.
2. Find the unit normal vector to the surface of the circular cone described by $x^2 + y^2 = 2z^2$ at the point $(1, 1, 1)$.
3. Find div $\mathbf{A}$ and curl $\mathbf{A}$ for $\mathbf{A} = xy^2\,\mathbf{i} + 2xyz\,\mathbf{j} - x^2z\,\mathbf{k}$.
4. Determine div $\mathbf{A}$ and curl $\mathbf{A}$ for $\mathbf{A} = (x - \cos yz)\,\mathbf{i} + (y - \cos xz)\,\mathbf{j} + (z - \cos xy)\,\mathbf{k}$.

For the next five problems, take $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

5. Show that $\nabla \cdot \mathbf{r} = 3$ and that $\nabla \times \mathbf{r} = \mathbf{0}$ (provided $\mathbf{r} \neq \mathbf{0}$).
6. Show that div $(\mathbf{r}/r^3) = 0$ (provided $\mathbf{r} \neq \mathbf{0}$).
7. Show that grad $(1/r) = -\mathbf{r}/r^3$ (provided $\mathbf{r} \neq \mathbf{0}$).
8. Show that $\nabla^2 \left(\dfrac{1}{r}\right) = 0$ (provided $r \neq 0$).
9. Show that $\nabla \times [f(r)\,\mathbf{r}] = 0$.

10. If the density of a fluid remains constant, the flow is said to be *incompressible*.

(a) Is the flow incompressible if $\mathbf{v} = y\,\mathbf{i} - x\,\mathbf{j}$?
(b) What if $\mathbf{v} = 2x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$?
(c) Is either flow irrotational?

11. Prove that div grad $= \nabla^2$.

12. Prove that curl grad $\psi = 0$ where ψ has continuous second partial derivatives.

13. Prove that div curl $\mathbf{v} = 0$ where the components of $\mathbf{v}$ have continuous second partial derivatives.

14. (a) Prove that div $\phi\mathbf{v} = \phi\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla\phi$.
(b) Use this result to evaluate div $\phi\mathbf{v}$ if $\phi = xy$ and $\mathbf{v} = y^2\,\mathbf{i} + xz\,\mathbf{k}$.
(c) Evaluate it by applying div directly to $\phi\mathbf{v}$ and compare your result.

15. Prove that div $(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$.

16. Consider a point P surrounded by a region of volume V with a surface S. Let dS be a small area element of S with a normal outward vector $\mathbf{n}$, as shown in Figure 7.8. A more fundamental definition of the gradient of ϕ at the point P is

$$\text{grad}\,\phi = \lim_{V \to 0} \left(\frac{\iint_S \mathbf{n}\,\phi\,dS}{V} \right)$$

This definition is independent of the coordinate system used. Show that this definition is equivalent to Equation 1 if we use a cartesian coordinate system. *Hint*: Use an argument like the one we used with Figure 7.4 to derive Equation 8.

17. Referring to Figure 7.8 as we did in Problem 16, we can state a more fundamental definition of div $\mathbf{v}$ by

$$\text{div}\,\mathbf{v} = \lim_{V \to 0} \left(\frac{\iint_S \mathbf{n} \cdot \mathbf{v}\,dS}{V} \right)$$

This definition is independent of the coordinate system used. Following Problem 16, show that this definition is equivalent to Equation 7 if we use a cartesian coordinate system. *Hint*: Use an argument like the one we used with Figure 7.4 to derive Equation 8.

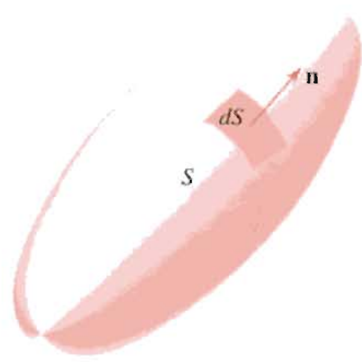

Figure 7.8
The geometry associated with the fundamental definition of the gradient presented in Problem 16.

18. Referring to Figure 7.8 as we did in Problems 16 and 17, we can state a more fundamental definition of the curl by

$$\text{curl } \mathbf{v} = \lim_{V \to 0} \left(\frac{\iint_S \mathbf{n} \times \mathbf{v}\, dS}{V} \right)$$

This definition is independent of the coordinate system used. Following Problems 16 and 17, show that this definition is equivalent to Equation 17 if we use a cartesian coordinate system.

19. Show that curl curl $\mathbf{v}$ = grad div $\mathbf{v} - \nabla^2\mathbf{v}$.

20. Use any CAS to calculate grad f if $f(x, y, z) = (z^2 - x^2)e^{-y^2}$.

21. Use any CAS to calculate div $\mathbf{u}$ and curl $\mathbf{u}$ if $\mathbf{u} = x^2 e^{-z^2}\,\mathbf{i} - xyz^2\,\mathbf{j} + e^{-(x^2+y^2+z^2)}\,\mathbf{k}$.

7.2 Line Integrals

The work done by a force $\mathbf{F}$ on a body that undergoes a displacement $d\mathbf{r}$ is $dW = \mathbf{F} \cdot d\mathbf{r}$. Now suppose that the body moves from point a to point b along some curve C. The total work involved is given by

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_x dx + \int_C F_y dy + \int_C F_z dz \tag{1}$$

where the C under the integral signs emphasizes that the integration is carried out along the curve C. The integral in Equation 1 is called a *line integral* or a *path integral* because it is carried out along a given path. We can describe the curve parametrically by a position vector $\mathbf{r}(u) = x(u)\,\mathbf{i} + y(u)\,\mathbf{j} + z(u)\,\mathbf{k}$, which we will always require to be piecewise smooth; in other words, the curve consists of a finite number of segments each of which has a unique tangent at each point and whose direction varies continuously as the parameter u varies. This requirement allows us to write

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{du}\, du \tag{2}$$

where $\mathbf{r}(a)$ is the initial point and $\mathbf{r}(b)$ is the final point on the integration path. The path dependence of the integral is reflected in the factor $d\mathbf{r}/du$.

Example 1:
Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = (x + y)\,\mathbf{i} - y\,\mathbf{j}$ and $x = 1 - u$, $y = u^2$, from $u = 0$ to $u = 1$.

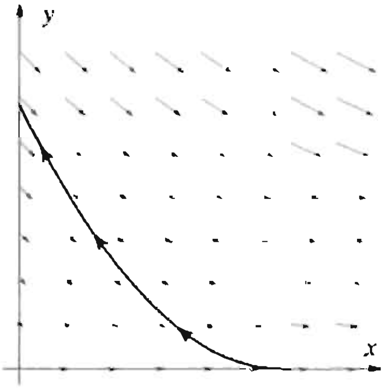

Figure 7.9
A pictorial representation of the force field and the path of integration in Example 1.

SOLUTION: Using Equation 2.

$$\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{du}\, du = \int_0^1 [x(u) + y(u)]x'(u)\, du - \int_0^1 y(u)y'(u)\, du$$

$$= \int_0^1 (1 - u + u^2)(-du) - 2\int_0^1 u^3 du = -\frac{4}{3}$$

Figure 7.9 shows why the answer is negative.

The path of integration is going against the force field, so $\mathbf{F} \cdot d\mathbf{r} < 0$ and the integral is negative.

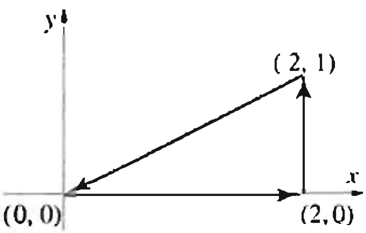

Figure 7.10
A triangular path used for the evaluation of $\int_C \mathbf{A} \cdot d\mathbf{r}$, where $\mathbf{A} = (x - y)\,\mathbf{i} + (x + y)\,\mathbf{j}$.

Let's evaluate $\int \mathbf{A} \cdot d\mathbf{r}$ around the triangular path shown in Figure 7.10 if $\mathbf{A} = (x - y)\,\mathbf{i} + (x + y)\,\mathbf{j}$. We'll calculate the integral along each of the three paths in Figure 7.10 and then add the results. Along path 1, $y = 0$ and $dy = 0$, so

$$\int_{C_1} \mathbf{A} \cdot d\mathbf{r} = \int_0^2 x\, dx = 2$$

Along path 2, $x = 2$ and $dx = 0$, so

$$\int_{C_2} \mathbf{A} \cdot d\mathbf{r} = \int_0^1 (2 + y)\, dy = \frac{5}{2}$$

Along path 3, $y = x/2$, so

$$\int_{C_3} \mathbf{A} \cdot d\mathbf{r} = \int [(x - y)dx + (x + y)dy]$$

Letting $y = x/2$ and $dy = dx/2$, we have

$$\int_{C_3} \mathbf{A} \cdot d\mathbf{r} = \int_2^0 \left(\frac{x}{2} + \frac{3x}{4}\right) dx = -\frac{5}{2}$$

(You should verify right now that you obtain the same result if you let $x = 2y$ and $dx = 2dy$ and then integrate over y from 1 to 0.) The integral around the triangular path shown in Figure 7.10 is equal to $2 + 5/2 - 5/2 = 2$. We can express this result symbolically by writing $\oint \mathbf{A} \cdot d\mathbf{r} = 2$, where the circle on the integral sign emphasizes that the path is closed. Furthermore, unless we state otherwise, we shall always traverse a closed path in a counterclockwise direction, as indicated by the arrows in Figure 7.10. The area enclosed by the closed path always lies to the left as you traverse the path. Note that $\oint \mathbf{A} \cdot d\mathbf{r}$ in this case is *not* zero. The path integral around a closed path need not equal zero. It may equal zero, however, as the next Example shows.

Example 2:
Evaluate $\int \mathbf{A} \cdot d\mathbf{r}$ around the closed path shown in Figure 7.10 if $\mathbf{A} = (y - x)\,\mathbf{i} + (x + y)\,\mathbf{j}$.

SOLUTION:

$$\int_{C_1} \mathbf{A} \cdot d\mathbf{r} = -\int_0^2 x\,dx = -2$$

$$\int_{C_2} \mathbf{A} \cdot d\mathbf{r} = \int_0^1 (2 + y)\,dy = \frac{5}{2}$$

$$\int_{C_3} \mathbf{A} \cdot d\mathbf{r} = \int [(y - x)dx + (x + y)dy]$$

$$= \int_2^0 \left(-\frac{x}{2} + \frac{3x}{4}\right) dx = -\frac{1}{2}$$

The sum of the three path integrals gives $\int \mathbf{A} \cdot d\mathbf{r} = 0$. The integral around the triangular path in Figure 7.10 is equal to zero in this case.

Before we examine why the integral around a closed path was equal to zero in one case but not in the other, let's notice one other property of line integrals. If we reverse the directions of the three paths in Figure 7.10, then the values of all the integrals that we obtained would reverse sign. Thus, the value of a line integral depends upon the direction that we take along the path. We can express this result symbolically by writing

$$\int_{-C} \mathbf{A} \cdot d\mathbf{r} = -\int_{C} \mathbf{A} \cdot d\mathbf{r} \tag{3}$$

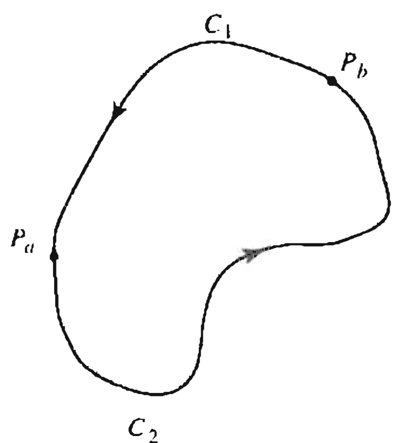

Figure 7.11
An aid to the proof that if $\int \mathbf{A} \cdot d\mathbf{r} = 0$ for all closed paths in a domain D, then $\int \mathbf{A} \cdot d\mathbf{r}$ is path independent.

We'll now prove that if $\int \mathbf{A} \cdot d\mathbf{r} = 0$ for all closed paths in a domain D, then the value of $\int \mathbf{A} \cdot d\mathbf{r}$ depends *only* upon the end points P_a and P_b and is independent of the curve connecting P_a and P_b. Figure 7.11 shows an arbitrary closed curve C and two arbitrary points, P_a and P_b, which break C into two (arbitrary) curves, C_1 and C_2. Then

$$\oint \mathbf{A} \cdot d\mathbf{r} = 0 = \int_{C_1} \mathbf{A} \cdot d\mathbf{r} + \int_{C_2} \mathbf{A} \cdot d\mathbf{r}$$

Using Equation 3,

$$\int_{C_2} \mathbf{A} \cdot d\mathbf{r} = -\int_{-C_2} \mathbf{A} \cdot d\mathbf{r}$$

so

$$\int_{C_1} \mathbf{A}\cdot d\mathbf{r} = \int_{-C_2} \mathbf{A}\cdot d\mathbf{r} \tag{4}$$

But C_1 and $-C_2$ are two arbitrary paths from P_a to P_b, so Equation 4 says that the line integral from P_a to P_b is independent of the path.

Let's apply this result by evaluating $\int \mathbf{A}\cdot d\mathbf{r}$ from (0, 0) to (2, 1) along the parabola $y = x^2/4$ for **A** given in Example 2:

$$\begin{aligned}\int_C \mathbf{A}\cdot d\mathbf{r} &= \int [(y-x)dx + (x+y)dy\,] \\ &= \int_0^2 \left[\left(\frac{x^2}{4} - x\right) + \left(x + \frac{x^2}{4}\right)\frac{x}{2}\right] dx \\ &= \frac{1}{2}\end{aligned}$$

which is the value of $\int \mathbf{A}\cdot d\mathbf{r}$ for the sum of paths 1 and 2 in Example 2. Furthermore, it is also the negative of the result for path 3.

A line integral $\int \mathbf{A}\cdot d\mathbf{r}$ between two points P_a and P_b is said to be *path independent* if the value of $\int \mathbf{A}\cdot d\mathbf{r}$ does not depend upon the path connecting the two end points. Its value depends only upon the end points, and we express this by writing

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_a^b \mathbf{A}\cdot d\mathbf{r} \qquad \text{(path independent)} \tag{5}$$

It is easy to prove the converse of the above theorem; namely, that if $\int \mathbf{A}\cdot d\mathbf{r}$ is path independent, then

$$\oint \mathbf{A}\cdot d\mathbf{r} = 0 \qquad \text{(path independent)} \tag{6}$$

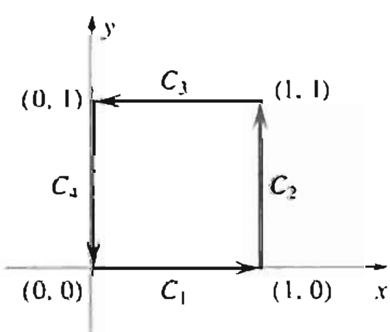

Figure 7.12
The closed path used to evaluate $\oint \mathbf{A}\cdot d\mathbf{r}$ in Example 3.

Example 3:
Evaluate $\oint \mathbf{A}\cdot d\mathbf{r}$ around the closed path shown in Figure 7.12 if $\mathbf{A} = (y^2 + 2xy)\,\mathbf{i} + (x^2 + 2xy)\,\mathbf{j}$.

SOLUTION: Along path 1, $y = 0$ and $dy = 0$, so

$$\int_{C_1} \mathbf{A}\cdot d\mathbf{r} = 0$$

Along path 2, $x = 1$ and $dx = 0$, so

$$\int_{C_2} \mathbf{A}\cdot d\mathbf{r} = \int_0^1 (1 + 2y)dy = 2$$

Along path 3, $y = 1$ and $dy = 0$, so

$$\int_{C_3} \mathbf{A} \cdot d\mathbf{r} = \int_1^0 (1 + 2x)dx = -2$$

Along path 4, $x = 0$ and $dx = 0$, so

$$\int_{C_4} \mathbf{A} \cdot d\mathbf{r} = 0$$

Therefore, $\oint \mathbf{A} \cdot d\mathbf{r} = 0$.

We haven't yet addressed the question of why $\oint \mathbf{A} \cdot d\mathbf{r} = 0$ in some cases but not in others, but we have seen that an equivalent question is why $\int \mathbf{A} \cdot d\mathbf{r}$ is path independent in some cases but not in others. Notice that if $\mathbf{A} \cdot d\mathbf{r} = d\phi$, where $\phi = \phi(x, y, z)$, then

$$\int \mathbf{A} \cdot d\mathbf{r} = \int d\phi = \phi(b) - \phi(a) \tag{7}$$

Now, $\mathbf{A} \cdot d\mathbf{r}$ would equal $d\phi$ if

$$\mathbf{A} \cdot d\mathbf{r} = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

But this will occur if

$$\mathbf{A} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} \tag{8}$$

or if $\mathbf{A} = \text{grad}\, \phi$. Thus, we see that it is sufficient that $\int \mathbf{A} \cdot d\mathbf{r}$ be path independent if $\mathbf{A} = \text{grad}\, \phi$. It turns out that this is a necessary condition as well. We can formalize this result by the following theorem:

> *Let* $\mathbf{A}$ *be continuous in a domain D. Then, the line integral* $\int \mathbf{A} \cdot d\mathbf{r}$ *is path independent in D if and only if* $\mathbf{A} = \text{grad}\, \phi$ *for some function* $\phi(x, y, z)$ *defined in D.*

How can we determine if a given vector function $\mathbf{A}$ can be written as the gradient of a scalar function $\phi(x, y, z)$? Equivalently, how can we determine if $\mathbf{A} \cdot d\mathbf{r}$ is an exact differential $\mathbf{A} \cdot d\mathbf{r} = d\phi$? It's tempting to say that if $\mathbf{A}$ can be written in the form of Equation 8, then

$$A_x = \frac{\partial \phi}{\partial x}; \qquad A_y = \frac{\partial \phi}{\partial y}; \qquad A_z = \frac{\partial \phi}{\partial z}$$

and so

$$\begin{aligned}
\frac{\partial A_x}{\partial y} &= \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial A_y}{\partial x} \\
\frac{\partial A_y}{\partial z} &= \frac{\partial^2 \phi}{\partial z \partial y} = \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial A_z}{\partial y} \qquad (9) \\
\frac{\partial A_z}{\partial x} &= \frac{\partial^2 \phi}{\partial x \partial z} = \frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial A_x}{\partial z}
\end{aligned}$$

wherever the mixed second partial derivatives are continuous. However, Equations 9 are necessary conditions that $\mathbf{A} \cdot d\mathbf{r} = d\phi$, but they are not sufficient.

The classic example of a case where the mixed second partial derivatives are equal, yet $\mathbf{A} \cdot d\mathbf{r}$ is not an exact differential, is for

$$\mathbf{A} = -\frac{y\,\mathbf{i}}{x^2 + y^2} + \frac{x\,\mathbf{j}}{x^2 + y^2} \qquad (x, y) \neq (0, 0) \qquad (10)$$

It's easy to show that

$$\frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

but let's calculate $\oint \mathbf{A} \cdot d\mathbf{r}$ around a unit circle centered at the origin. If $\mathbf{A} \cdot d\mathbf{r}$ is an exact differential, then the result should equal zero. The integral to evaluate is

$$I = \oint \left(-\frac{y\,dx}{x^2 + y^2} + \frac{x\,dy}{x^2 + y^2} \right) \qquad (11)$$

Let $x^2 + y^2 = 1$, $x = \cos\theta$, and $y = \cos\theta$ ($0 \leq \theta \leq 2\pi$). Then

$$I = \int_0^{2\pi} (\sin^2\theta\, d\theta + \cos^2\theta\, d\theta) = \int_0^{2\pi} d\theta = 2\pi$$

The result is *not* equal to zero, and so $\mathbf{A} \cdot d\mathbf{r}$ is not an exact differential.

The problem here is that $\mathbf{A}$ and its partial derivatives are not defined at the origin, and the integration path encloses the origin. If we integrate $\mathbf{A} \cdot d\mathbf{r}$ around a path that does not enclose the origin, then we do get zero for the result.

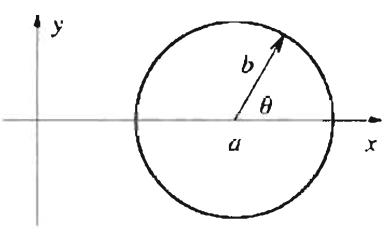

Figure 7.13
A circle of radius b centered at the point $(a, 0)$, described by the parametric equations $x = a + b\cos\theta$ and $y = b\sin\theta$ for $0 \leq \theta \leq 2\pi$.

Example 4:
Evaluate the integral in Equation 11 around a circle of radius b centered at the point $(a, 0)$ (Figure 7.13).

SOLUTION: The circle shown in Figure 7.13 can be described parametrically by the equations

$$x(\theta) = a + b\cos\theta, \qquad y(\theta) = b\sin\theta, \quad 0 \leq \theta \leq 2\pi$$

Substituting $x(\theta)$ and $y(\theta)$ into Equation 11 gives

$$\begin{aligned} I &= \int_0^{2\pi} \frac{b^2 + ab\cos\theta}{a^2 + b^2 + 2ab\cos\theta} d\theta \\ &= b^2 \int_0^{2\pi} \frac{d\theta}{a^2 + b^2 + 2ab\cos\theta} + ab \int_0^{2\pi} \frac{\cos\theta\, d\theta}{a^2 + b^2 + 2ab\cos\theta} \end{aligned} \tag{12}$$

Figure 7.14 shows that these integrals are symmetric about $\theta = \pi$, so we can write them as 2 times the integrals from 0 to π.

These integrals are in the *CRC Mathematical Tables* (see also Problem 1.7.22) and give

$$I = \pi - 2\left[\tan^{-1}\left\{\left(\frac{a^2 - b^2}{a^2 + b^2}\right)\tan\frac{\theta}{2}\right\}\right]_0^{\pi}$$

Now, if $a > b$, then we get $\pi - 2(\pi/2) = 0$, and if $a < b$, then we get $\pi + 2(\pi/2) = 2\pi$. Thus, $I = 0$ if the integration path does not enclose the origin, but $I = 2\pi$ if it does.

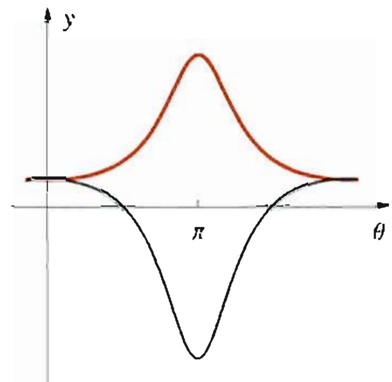

Figure 7.14
The functions $1/(a^2 + b^2 + 2ab\cos\theta)$ (color) and $\cos\theta/(a^2 + b^2 + 2ab\cos\theta)$ (black) plotted against θ ($a \neq b$).

Generally, $\mathbf{A} \cdot d\mathbf{r}$ in the integrand of Equation 11 *is* an exact differential in any region that does not include the origin. However, it is not an exact differential in a region that does include the origin. These results lead us to the following definition: A region R is said to be *simply connected* if any closed curve lying within R can be continuously shrunk to a point without leaving the region. Roughly speaking, a simply connected region has no holes in it. Figure 7.15 illustrates some various types of regions.

Figure 7.15
An illustration of (a) a simply connected region and (b) a region that is not simply connected.

We can now state the following result:

If the components of $\mathbf{A}$ *are continuous and have continuous first partial derivatives in a simply connected region* R, *then* $\mathbf{A} \cdot d\mathbf{r}$ *will be an exact differential if and only if Equations 9 are satisfied. Furthermore,* $\oint \mathbf{A} \cdot d\mathbf{r} = 0$ *around any simple (nonintersecting) closed path lying within* R.

Let's use Equations 9 to see that $\mathbf{A}$ in Example 2 is the gradient of some scalar function. In that (two-dimensional) case, $A_x = y - x$ and $A_y = x + y$ in a simply connected region, and

$$\frac{\partial A_x}{\partial y} = 1 = \frac{\partial A_y}{\partial x}$$

Therefore, $\mathbf{A}$ *is* equal to the gradient of a scalar function. We can even determine this function $\phi(x, y)$ by writing

$$A_x = \frac{\partial\phi}{\partial x} = y - x \qquad \text{and} \qquad A_y = \frac{\partial\phi}{\partial y} = x + y$$

Integrate $\partial\phi/\partial x$ "partially" with respect to x with y "constant" to obtain $\phi(x, y) = xy - x^2/2 + f(y)$, where $f(y)$ is the "constant" of integration. Now form $\partial\phi/\partial y$ and equate the result to $\partial\phi/\partial y = A_y = x + y$ to get

$$\frac{\partial \phi}{\partial y} = x + \frac{df}{dy} = x + y$$

or $df/dy = y$. Integration gives us $f(y) = y^2/2 + c$, where c is a constant. Thus, we see that $\phi(x, y) = xy - x^2/2 + y^2/2 + c$.

Example 5:
Is the integral $\int \mathbf{A} \cdot d\mathbf{r}$ path independent if $\mathbf{A} = (2x - y)\,\mathbf{i} + (x + y)\,\mathbf{j}$? Evaluate $\oint \mathbf{A} \cdot d\mathbf{r}$ around a unit circle centered at the origin. What is the value of the integral if we take a clockwise direction?

SOLUTION:

$$\frac{\partial A_x}{\partial y} = -1 \neq \frac{\partial A_y}{\partial x} = 1$$

so the integral is not path independent. This means that integrals around closed paths will generally not be equal to zero.

To evaluate $\oint \mathbf{A} \cdot d\mathbf{r}$ in a counterclockwise direction around a unit circle centered at the origin, let $x = \cos u$ and $y = \sin u$ and write

$$\oint \mathbf{A} \cdot d\mathbf{r} = \int_0^{2\pi} [(2\cos u - \sin u)(-\sin u) + (\cos u + \sin u)\cos u]\,du$$

$$= \int_0^{2\pi} (1 - \cos u \sin u)\,du = 2\pi - 0 = 2\pi$$

The value of $\oint \mathbf{A} \cdot d\mathbf{r}$ in a clockwise direction is -2π.

Equations 9 give the necessary conditions that $\mathbf{A} = \text{grad}\,\phi$. Do these conditions look familiar? If you compare them to Equation 17 of the previous section, you'll see that they are similar to the terms in the definition of curl $\mathbf{A}$, and in fact, Equations 9 say that curl $\mathbf{A} = 0$. Thus, Equations 9 are equivalent to the identity curl grad $\phi = 0$.

A necessary and sufficient condition that $\mathbf{A}$ *be path independent for any path lying within a simply connected region* R *is that* curl $\mathbf{A} = \mathbf{0}$ *in* R.

For the vector $\mathbf{A}$ in Example 3, curl $\mathbf{A} = \mathbf{0}$, but in Example 4, curl $\mathbf{A} = 2\,\mathbf{k} \neq \mathbf{0}$.

If the vector $\mathbf{F}$ is a force, then $\int \mathbf{F} \cdot d\mathbf{r}$ is the work done in going from a to b along some path C. The work will depend upon the path unless $\mathbf{F} = -\text{grad}\,\phi$ (the minus sign here is just a convention), in which case $\mathbf{F}$ is said to be *conservative* (or a *conservative force field*) and $\phi(x, y, z)$ the *potential*. To see why $\mathbf{F}(x, y, z)$ is called conservative, consider Newton's equations of motion

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} = -\nabla\phi \tag{13}$$

Take the dot product of both sides of Equation 13 by $d\mathbf{r}$ and integrate. The left side becomes

$$m\int \frac{d^2\mathbf{r}}{dt^2}\cdot d\mathbf{r} = m\int \frac{d^2\mathbf{r}}{dt^2}\cdot\frac{d\mathbf{r}}{dt}dt = \frac{m}{2}\int \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}\right)dt$$
$$= \frac{mv^2}{2} + \text{constant}$$

The left side of Equation 13 becomes

$$-\int \nabla\phi\cdot d\mathbf{r} = -\phi + \text{constant}$$

Equating the results of the two sides of Equation 13 gives

$$\frac{mv^2}{2} + \phi(x, y, z) = \text{constant}$$

or that the total energy is conserved. Thus, a conservative force field implies that energy is conserved.

Although we have used an example from classical mechanics to introduce the idea of a conservative field, any vector field $\mathbf{v}(x, y, z)$, which can be expressed as the gradient of a scalar field $\phi(x, y, z)$, is called a *conservative vector field* and ϕ is called the *scalar potential*.

Before we finish this section, we shall discuss an important relation between line integrals and surface integrals in a plane. Let R be a closed region in the xy-plane and let the closed curve C be the boundary of R. The arrows on C in Figure 7.16 indicate a counterclockwise direction on C, where the region R is always on the left as we go around C.

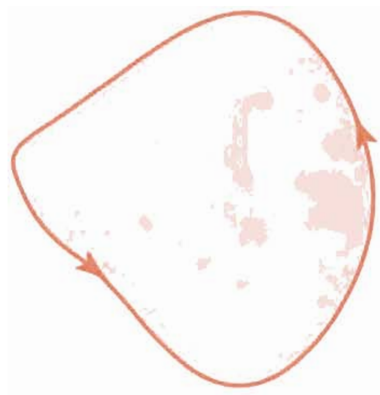

Figure 7.16
A closed region in the xy-plane and its boundary curve.

If C is described by the parametric equations $x(t)$ and $y(t)$ and if $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$, then the line integral of $\mathbf{F}$ around C is given by

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C [\,P\,dx + Q\,dy\,] \tag{14}$$

As usual, the integral in Equation 14 is taken in a counterclockwise direction. We now state *Green's theorem in the plane:*

If R is a simply connected region in the xy-plane bounded by a piecewise smooth curve C, and if $P(x, y)$ and $Q(x, y)$ are continuous with continuous first partial derivatives in an open region containing R, then

$$\oint_C [\,P\,dx + Q\,dy\,] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{15}$$

(Problem 15 takes you through a simplified proof of this theorem.) Equation 15 will be used a number of times in later chapters.

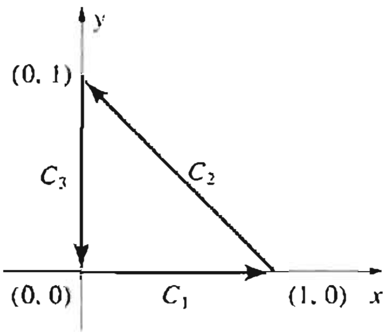

Figure 7.17
A triangular region in the xy-plane with vertices at the points (0, 0), (0, 1), and (1, 0).

Let's verify Green's theorem in the plane for $P = x^2 + y^2$ and $Q = x + 2$ over the boundary curve of a triangle with vertices at the points (0, 0), (0, 1), and (1, 0) (Figure 7.17). Using $y = 1 - x$ along the segment 2 in Figure 7.17, we have

$$\oint [P\,dx + Q\,dy] = \int_0^1 x^2 dx + \int_1^0 [x^2 + (1-x)^2 - (x+2)]\,dx + \int_1^0 2\,dy$$

$$= \frac{1}{3} + \frac{11}{6} - 2 = \frac{1}{6}$$

Now,

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R (1 - 2y)\,dxdy$$

$$= \frac{1}{2} - 2\int_0^1 dx \int_0^{1-x} y\,dy = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

We finish this section by expressing Green's theorem in the plane in vector notation. We can write $P(x, y)\,dx + Q(x, y)\,dy$ as

$$P\,dx + Q\,dy = (P\,\mathbf{i} + Q\,\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j}) = \mathbf{A} \cdot d\mathbf{r}$$

where $\mathbf{A} = P\,\mathbf{i} + Q\,\mathbf{j}$. Now form the curl of $\mathbf{A}$:

$$\text{curl}\,\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Equation 14 becomes

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_R (\text{curl}\,\mathbf{A}) \cdot \mathbf{k}\,dxdy \tag{16}$$

Equation 16 is the planar version of Stokes's theorem, which we shall discuss in Section 5.

7.2 Problems

1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the parametric curve $x = u^2$, $y = u$ from $u = 0$ to $u = 1$ if $\mathbf{F} = -y\,\mathbf{i} + x^2\,\mathbf{j}$.

2. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ from (0, 0) to (1, 2) along the parabola $y = 2x^2$ if $\mathbf{F} = xy\,\mathbf{i} - y^2\,\mathbf{j}$.

3. Evaluate $\oint \mathbf{A} \cdot d\mathbf{r}$ counterclockwise around a unit circle centered at the origin if $\mathbf{A} = y\,\mathbf{i} - x\,\mathbf{j}$.

4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ from (−1, −1) to (1, 1) along the curve $x = y^3$ if $\mathbf{F} = y^2\,\mathbf{i} + x\,\mathbf{j}$.

5. Evaluate $\int_C \mathbf{A} \cdot d\mathbf{r}$ along the straight line (0, 0, 0) to (1, 1, 3) if $\mathbf{A} = x^2y\,\mathbf{i} + y^2z\,\mathbf{j} + z^2x\,\mathbf{k}$.

6. Evaluate $\oint \mathbf{F} \cdot d\mathbf{r}$ counterclockwise around a unit circle centered at the origin if $\mathbf{F} = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$.

7. Evaluate $\oint \mathbf{A} \cdot d\mathbf{r}$ counterclockwise around a unit circle centered at the origin if $\mathbf{A} = xy\,\mathbf{i} + y^2\,\mathbf{j}$. Is the vector field $\mathbf{A}$ conservative? Now evaluate the integral counterclockwise around the square whose corners are (0, 0), (1, 0), (1, 1), and (0, 1).

8. Does $\mathbf{A}(x, y) = (3x^2 + 2y^2)\,\mathbf{i} + (4xy + 6y^2)\,\mathbf{j}$ represent a conservative force field? If so, determine the potential ϕ in $\mathbf{A} = \text{grad}\,\phi$.

9. Find a potential function corresponding to $\mathbf{F} = (x\,\mathbf{i} + y\,\mathbf{j})/(x^2 + y^2)^{3/2}$.

10. Does $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ represent a conservative force field? If so, determine ϕ in $\mathbf{F} = \text{grad}\,\phi$.

11. Does $\mathbf{F}(x, y, z) = 2xyz\,\mathbf{i} + x^2z\,\mathbf{j} + (x^2y + 4z)\,\mathbf{k}$ represent a conservative force field? If so, determine ϕ in $\mathbf{F} = \text{grad}\,\phi$.

12. In fluid dynamics, the quantity $\kappa = \oint \mathbf{v} \cdot d\mathbf{r}$, where $\mathbf{v}$ is the velocity vector of the fluid, is called the *circulation*, and is used to describe the character of the fluid flow. The circulation depends upon the integration path as well as the vector field. Determine κ counterclockwise around a unit circle centered at the origin if

    (a) $\mathbf{v} = v_0\,\mathbf{i}$ (b) $\mathbf{v} = 2xy\,\mathbf{i} + (x^2 - y^2)\,\mathbf{j}$

13. Consider the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = -\dfrac{y}{x^2 + y^2}\,\mathbf{i} + \dfrac{x}{x^2 + y^2}\,\mathbf{j}$. Is this line integral path independent? In what domain? Do you think that $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ if the closed integration path encloses the origin?

14. Consider the vector $\mathbf{F} = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$. Does $\mathbf{F}(x, y)$ represent a conservative force field? Now calculate the work along a unit circle centered at the origin. What happened? Why?

15. We shall prove Green's theorem in the plane for the special case in which the closed region R and its boundary curve C have the property that any straight line parallel to the coordinate axes cuts C in at most two places (see Figure 7.18).

    Let the upper curve ABC in Figure 7.18 be described by $y_2(x)$ and the lower curve ADC be described by $y_1(x)$. Now show that

$$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy\,\frac{\partial P}{\partial y} = \int_a^b dx\,[\,P(x, y_2(x)) - P(x, y_1(x))]$$

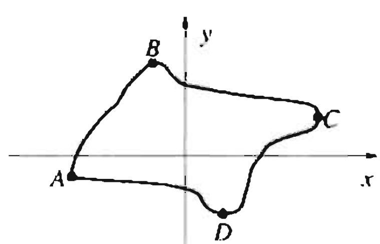

Figure 7.18
The region used in Problem 15 to prove Green's theorem in the plane.

and that this result is equal to $\oint_C P\,dx$. Similarly, let the right boundary curve BCD be described by $x_2(y)$ and the left boundary curve BAD be described by $x_1(y)$ and show that

$$\iint_R \frac{\partial Q}{\partial x}\,dx = \oint_C Q\,dy$$

and hence prove Equation 15.

16. Verify Green's theorem in the plane for $P = y - x^2$ and $Q = 2x + y^2$ over the boundary shown in Figure 7.19.

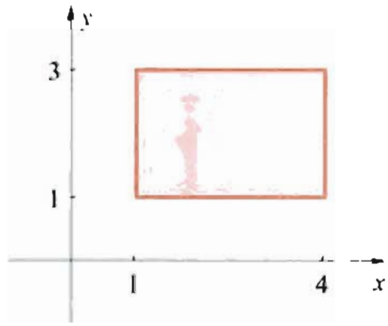

Figure 7.19
The region and boundary curve used in Problem 16.

17. Use Green's theorem in the plane to show that the area bounded by a simple closed curve C (a closed curve that does not cross itself) is given by $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$.

18. Use the result of the previous problem to find the area enclosed by the ellipse described by $x^2/a^2 + y^2/b^2 = 1$. *Hint*: Use the parametric equations $x = a\cos\theta$ and $y = b\sin\theta$ for $0 \le \theta \le 2\pi$.

19. Given that ϕ and ψ are continuous functions of x and y with continuous first partial derivatives, use Green's theorem in the plane to derive the relation

$$\iint_R \left(\phi\frac{\partial v}{\partial x} + \psi\frac{\partial v}{\partial y}\right) dx\,dy = \oint_C v(\phi\,dy - \psi\,dx) - \iint_R v\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) dx\,dy$$

Compare this formula to the formula for integration by parts. *Hint*: Set $P = v\phi$ and $Q = -v\psi$ in Equation 15.

7.3 Surface Integrals

We have discussed line integrals in the previous section and in this section we shall discuss surface integrals. Before doing so, however, we shall discuss surfaces a little more thoroughly than we have previously. Recall that a surface in three-dimensional space can be represented parametrically by

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \tag{1}$$

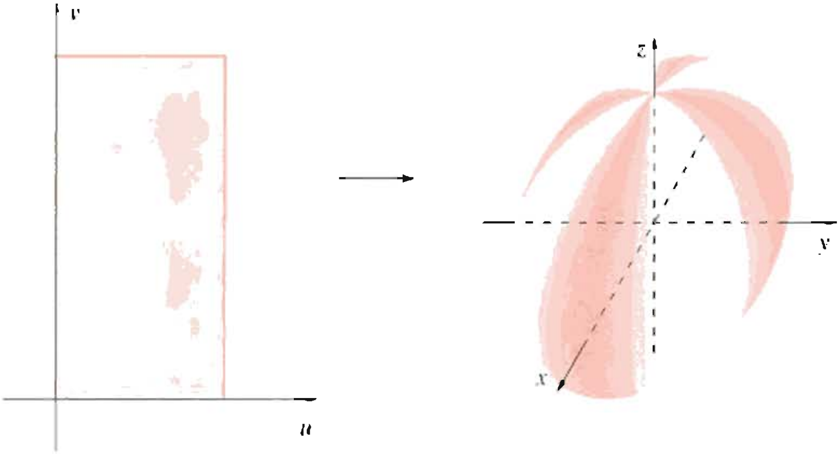

Figure 7.20
A mapping from the uv-plane to x, y, z space. In particular, this mapping is from $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$ to a sphere by the mapping $x = a \sin\theta \cos\phi$, $y = a \sin\theta \sin\phi$, and $z = a \cos\theta$.

As u and v vary over a region in the uv-plane, the tip of $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ maps out a surface in x, y, z space. Figure 7.20 illustrates a mapping from the uv-plane to x, y, z space.

We say that $\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$ describes a *smooth surface* if each of the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ and each of their first partial derivatives are continuous over the region of u and v, and if

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}$$

and

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j} + \frac{\partial z}{\partial v}\,\mathbf{k}$$

are nonzero and nonparallel over the domain of u and v. Recall that $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ are tangent to the surface, and so

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0} \tag{2}$$

is normal to the surface. This last condition assures that the surface has a tangent plane at all points (x, y, z) and is the analog of a space curve having a nonzero derivative everywhere. We say that a surface is *piecewise smooth* if it consists of a finite number of smooth surfaces. For example, we say that a cube is piecewise smooth because it consists of six smooth surfaces.

Example 1:
Describe the surface defined by $\mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + (1 - u - v)\,\mathbf{k}$. Evaluate $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}$ for this surface and interpret the result.

There is one other variation of the formula for surface area that you commonly see, particularly if $z = f(x, y)$ does not change sign as x and y vary over some region R. In that case, R is the projection of S onto the xy-plane, as shown in Figure 7.27.

Let γ be the angle between the upward-directed normal to a surface element ΔS and the z axis, so that $\cos\gamma = |\mathbf{n}\cdot\mathbf{k}|$. Then ΔS and its projection onto the xy-plane, ΔR, are approximately related by $\Delta R = \Delta S\cos\gamma$, or $\Delta S = \Delta R\sec\gamma$. This relation between ΔS and ΔR becomes exact in the limit, in which case the surface area can be written as

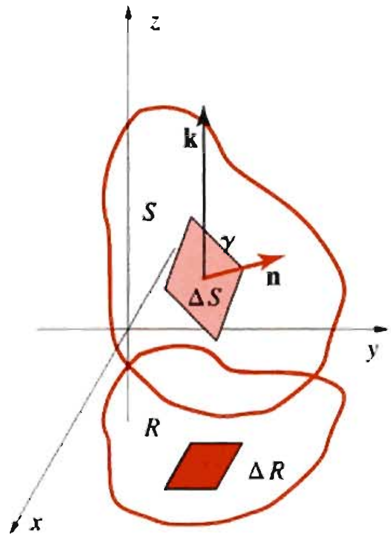

Figure 7.27
The projection of a surface element ΔS onto the xy-plane.

$$A = \iint_{\sigma} dS = \iint_{R} \sec\gamma\, dxdy = \iint_{R} \frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} \tag{5}$$

Equation 5 is equivalent to Equation 4. To see this, note that the normal vector to the surface $z = h(x, y)$ is the gradient of $\phi(x, y, z) = z - h(x, y) = 0$, which is

$$\nabla\phi = -\frac{\partial h}{\partial x}\mathbf{i} - \frac{\partial h}{\partial y}\mathbf{j} + \mathbf{k}$$

and the unit normal vector is

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-\dfrac{\partial h}{\partial x}\mathbf{i} - \dfrac{\partial h}{\partial y}\mathbf{j} + \mathbf{k}}{\left[1 + \left(\dfrac{\partial h}{\partial x}\right)^2 + \left(\dfrac{\partial h}{\partial y}\right)^2\right]^{1/2}}$$

But $\cos\gamma = \mathbf{n}\cdot\mathbf{k}$, so we have $\sec\gamma = 1/\mathbf{n}\cdot\mathbf{k}$, or

$$\sec\gamma = \left[1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right]^{1/2}$$

which shows the equivalence of Equations 4 and 5.

So far we have considered only the calculation of surface areas. Integrals of the type

$$\iint_{S} f(x, y, z)\, dS$$

occur frequently in physical problems. To evaluate these types of integrals, we simply use the generalization of Equation 3:

$$\iint_{S} f(x(u, v), y(u, v), z(u, v)) \left|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right| dudv \tag{6}$$

if the surface is given in parametric form, or we use

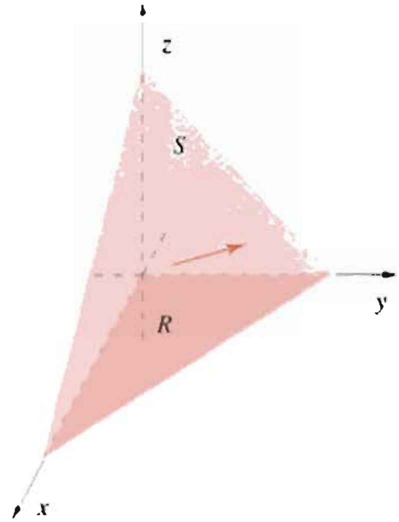

Figure 7.28
The plane, S, decribed by $x + y + z = 1$ and its projection, R, onto the xy-plane.

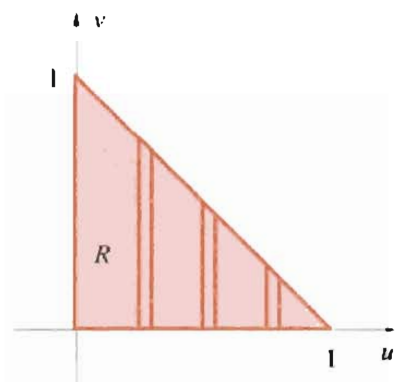

Figure 7.29
The integration region R in Example 4.

$$\iint_R f(x, y, z)\left[1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right]^{1/2} dx dy = \iint_R f(x, y, z) \sec\gamma \, dx dy \tag{7}$$

if the surface is described by an equation of the form $z = h(x, y)$.

Example 4:
Evaluate the surface integral

$$I = \iint_S xyz \, dS$$

over the plane $x + y + z = 1$ in the first octant.

SOLUTION: The plane and its projection onto the xy-plane are shown in Figure 7.28. If we use Equation 6, we parametrize the plane by $x = u$, $y = v$, and $z = 1 - u - v$, so that

$$\mathbf{r} = u\,\mathbf{i} + v\,\mathbf{j} + (1 - u - v)\,\mathbf{k}$$

and

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| = |\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{3}$$

Equation 6 gives

$$I = \sqrt{3} \iint_R uv(1 - u - v) du dv$$

where $0 < u < 1$ and $0 < v < 1 - u$ (see Figure 7.29). Thus, $I = \sqrt{3}/120$ (Problem 15). Of course, we would have obtained the same result if we had used Equation 7 instead of Equation 6 (Problem 16).

Recall that a flux $\mathbf{J}$ is the flow rate of a substance through a unit area perpendicular to $\mathbf{J}$. Consider the flux of a substance (or even the flux of some kind of vector field, such as an electric field intensity) through a surface element dS whose normal outward unit vector is $\mathbf{n}$ (see Figure 7.30).

Then the flow rate through dS is given by $\mathbf{J} \cdot \mathbf{n}\, dS$, and the total flow rate out of a volume bounded by the surface S is given by

$$I = \iint_S \mathbf{J} \cdot \mathbf{n}\, dS \tag{8}$$

If we represent the surface $d\mathbf{S} = \mathbf{n}\, dS$, then Equation 8 takes the form

$$I = \iint_S \mathbf{F} \cdot d\mathbf{S} \tag{9}$$

Surface integrals of the form of Equation 8 or 9 occur frequently in physical problems.

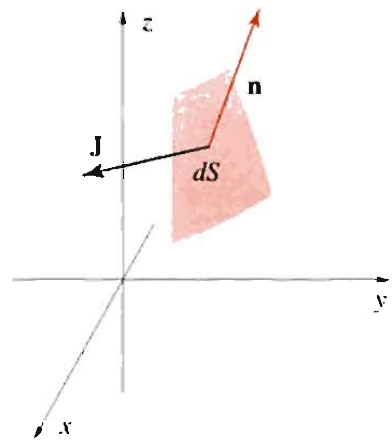

Figure 7.30
The flux through a directed surface element $d\mathbf{S} = \mathbf{n}\, dS$.

Example 5:
Let the flux of a fluid be given by $\mathbf{J} = \rho v_0 \mathbf{k}$, where ρ is the mass density of the fluid and $v_0 \mathbf{k}$ is its velocity. Calculate the flow rate of fluid (mass per unit time) through a hemispherical surface of radius a (Figure 7.31).

SOLUTION: The equation of the surface is $z = (a^2 - x^2 - y^2)^{1/2}$, and so $\mathbf{n}$ is given by $\nabla\phi/|\nabla\phi|$, where $\phi = z - (a^2 - x^2 - y^2)^{1/2}$. Therefore,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \frac{x}{z}\mathbf{i} + \frac{y}{z}\mathbf{j} + \mathbf{k}$$

and

$$|\nabla\phi| = \left(\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1\right)^{1/2} = \frac{(x^2+y^2+z^2)^{1/2}}{z} = \frac{a}{z}$$

Therefore,

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$$

and $\mathbf{J}\cdot\mathbf{n} = \rho v_0 \mathbf{k}\cdot\mathbf{n} = \rho v_0 z/a$. Equation 8 gives us

$$\begin{aligned} I &= \iint_S \frac{\rho v_0}{a} z\, dS \\ &= \frac{\rho v_0}{a}\iint_R z\left[1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right]^{1/2} dx\,dy \\ &= \frac{\rho v_0}{a}\iint_R z\left(1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}\right)^{1/2} dx\,dy \\ &= \frac{\rho v_0}{a}\iint_R z\frac{a}{z}\,dx\,dy = \rho v_0 \iint_R dx\,dy \\ &= \rho v_0 \pi a^2 \end{aligned}$$

We see that the total flow through the hemispherical cap is just the upward flux times the area of the projection of the cap onto the xy-plane.

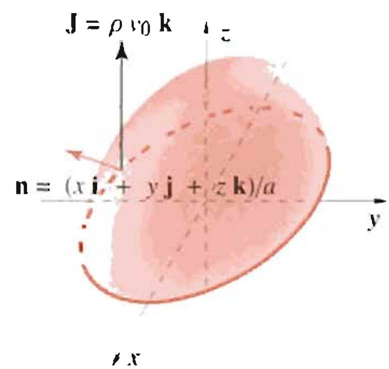

Figure 7.31
The flux of liquid through a hemispherical dome of radius a.

7.3 Problems

1. Express the following surfaces in parametric form:

(a) $y = z^2$ (b) $x + 2y - z = 2$

(c) $x^2 + y^2 = z^2$ (d) $x^2 + y^2 = z$

2. Sketch the surfaces given in Problem 1.

3. The vector $\mathbf{r}(u, v) = e^u\,\mathbf{i} + v\,\mathbf{j} + (e^{2u} + v^2)\,\mathbf{k}$ represents the circular paraboloid $x^2 + y^2 = z$. Does it represent the entire paraboloid?

4. Use Equation 3 to determine the area of the cylindrical surface $x^2 + z^2 = 1$ from $y = 0$ to $y = 1$.

5. Determine the area of the part of the plane $z = 2x + 3y$ that lies inside the elliptical cylinder described by $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

6. Determine the area of the plane $2x + 3y + z = 6$ that lies in the first octant.

7. Find the surface area of the torus described by the parametric equations, $x(\theta, \phi) = (a + b\cos\phi)\cos\theta$, $y(\theta, \phi) = (a + b\cos\phi)\sin\theta$, and $z(\theta, \phi) = b\sin\theta$, with $0 < \theta < 2\pi$ and $0 < \phi < 2\pi$ (see Figure 7.32).

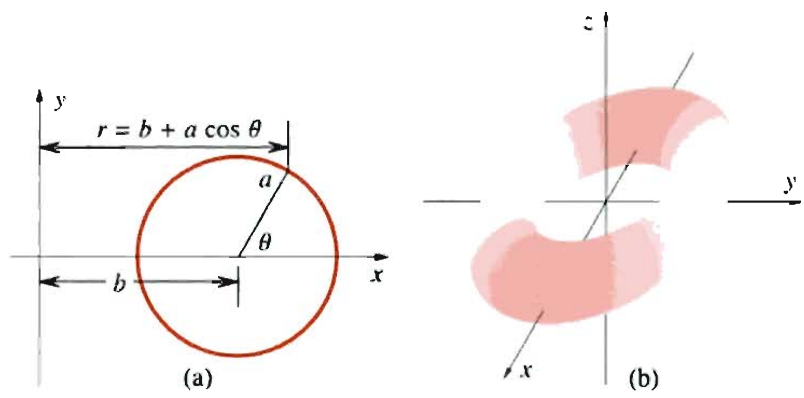

Figure 7.32
(a) The parametric equations that generate a torus. (b) The torus results from rotating the above circle about the y axis in (a).

8. Show that $\left|\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}\right| = \left[1 + \left(\dfrac{\partial h}{\partial x}\right)^2 + \left(\dfrac{\partial h}{\partial y}\right)^2\right]^{1/2}$ if $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + h(x, y)\,\mathbf{k}$.

9. Evaluate the surface integral $\displaystyle\iint_S \left(z + 2x + \frac{4}{3}y\right) dS$ over that part of the plane $6x + 4y + 3z = 12$ that lies in the first octant.

10. Evaluate the surface integral $\displaystyle\iint_S x\,dS$ over that part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.

11. Evaluate the surface integral $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$ over the triangular surface bounded by the points (2, 0, 0), (0, 2, 0), and (0, 0, 2), where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $\mathbf{n}$ is the outward normal unit vector from the triangular surface.

12. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ over the cube $0 < x < 1$, $0 < y < 1$, $0 < z < 1$, where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{n}$ is the outward normal unit vector from each face of the cube.

13. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ over the surface of a unit sphere centered at the origin, where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $\mathbf{n}$ is the outward normal unit vector of the sphere.

14. Evaluate the surface integral $\iint_S (6x + z - y^2)\, dS$ over the surface defined by $\mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + u\,\mathbf{k}$ and for $0 < u < 1$ and $0 < v < 1$. What is the surface?

15. Finish the calculation in Example 4.

16. Use Equation 7 to evaluate the surface integral in Example 4.

7.4 The Divergence Theorem

In Section 1, we saw that the divergence of a vector represents the flow rate of some quantity out of a given region in space, and in the previous section, we learned that a surface integral of the type

$$\int_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = \int_\sigma \mathbf{F} \cdot d\mathbf{S} \tag{1}$$

represents the flow rate of the quantity represented by $\mathbf{F}$ through a surface S. Therefore, it may not be surprising that there is a relation between div $\mathbf{F}$ and a surface integral like the one in Equation 1. This relation is given by the *divergence theorem*:

If S is a piecewise smooth surface enclosing a three-dimensional region V, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV \tag{2}$$

where $\mathbf{n}$ is the outward unit normal vector to S, i.e., the unit normal vector pointing away from V.

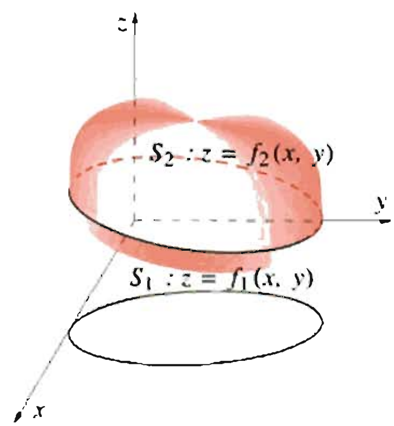

Figure 7.33
A closed surface consisting of an upper surface S_2 described by $z = f_2(x, y)$, and a lower surface S_1 described by $z = f_1(x, y)$. The projection of the surface onto the xy-plane is shown by the closed curve.

The divergence theorem was obtained independently by Gauss and Ostrogradsky, but is referred to as Gauss's theorem in the Western literature.

Given the physical meanings of the two sides of Equation 2, the divergence theorem is almost self-evident, but we shall give an outline of its mathematical proof. Let the surface S be such that it can be decomposed into an upper surface S_2 described by $z = f_2(x, y)$, and a lower surface S_1 described by $z = f_1(x, y)$ (Figure 7.33). If $\mathbf{F}(x, y, z) = F_x(x, y, z)\,\mathbf{i} + F_y(x, y, z)\,\mathbf{j} + F_z(x, y, z)\,\mathbf{k}$, then

The left side of Equation 2 is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\text{top part}} z\,dS + \iint_{\text{curved part}} \mathbf{F}\cdot\mathbf{n}\,dS - \iint_{\text{bottom part}} z\,dS$$

$$= (4\pi)(4) + \iint_{\text{curved part}} \mathbf{F}\cdot\mathbf{n}\,dS + 0$$

where we used the fact that $z = 4$ on the top surface and $z = 0$ on the bottom surface. For the remaining integral, we use $\phi = x^2 + y^2 - 4$ to get

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{(4x^2 + 4y^2)^{1/2}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{2}$$

and $\mathbf{F}\cdot\mathbf{n} = (x^2 + y^2)/2 = 4/2 = 2$. Therefore,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 2(2\pi\cdot 2)4 + 16\pi = 48\pi$$

The divergence theorem is used to derive many of the equations of physics and engineering that involve some sort of conservation condition, such as mass balance or energy balance. Let $\rho(x, y, z, t)$ be the mass density of a fluid at the point (x, y, z) at time t, and let V be an arbitrary fixed volume located within the fluid and let S be the boundary of V (Figure 7.36). The total mass within V is

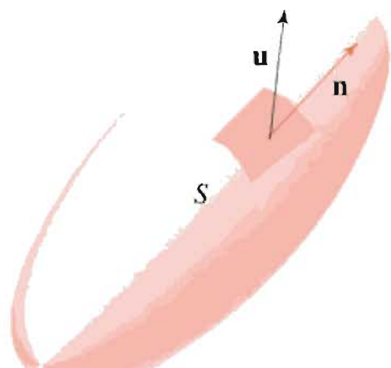

Figure 7.36
An arbitrary fixed volume V located within a fluid; S is the boundary of V.

$$M = \iiint_V \rho\,dV$$

and the rate of change of the mass within V is

$$\frac{dM}{dt} = \iiint_V \frac{\partial\rho}{\partial t}\,dV \tag{7}$$

since V is fixed in space.

Now, because of the conservation of mass, the rate of change of mass within V must be balanced by the net rate at which mass flows through the surface S. This rate is given by

$$\frac{dM}{dt} = -\iint_S \rho\mathbf{u}\cdot\mathbf{n}\,dS \tag{8}$$

where $\mathbf{n}$ is the outward unit normal vector at dS and $\mathbf{u}(x, y, z, t)$ is the velocity of the fluid. The negative sign on the right of Equation 8 accounts for the fact that $dM/dt < 0$ if the net flow is outward. Equating dM/dt from Equations 7 and 8 gives

$$\iiint_V \frac{\partial \rho}{\partial t} dV = -\iint_S \rho \mathbf{u} \cdot \mathbf{n}\, dS \tag{9}$$

Apply the divergence theorem, Equation 2, to the right of Equation 9 to obtain

$$\iiint_V \frac{\partial \rho}{\partial t} dV = -\iiint_V \operatorname{div}(\rho \mathbf{u})\, dV$$

or

$$\iiint_V \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) \right] dV = 0 \tag{10}$$

The volume V in Equation 10 is arbitrary, however, so the integrand must equal zero, or

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{11}$$

Equation 11, which expresses the conservation of mass in differential form, is called the *continuity equation*. We actually derived this equation in Section 1 for the special case of a rectangular parallelpiped, but the derivation here is more general.

Example 2:

Let $T(x, y, z, t)$ denote the temperature of a homogeneous body at the point (x, y, z) at the time t. Fourier's law of heat flow says that $\mathbf{q}$, the flux of energy as heat, is given by

$$\mathbf{q} = -\kappa \operatorname{grad} T$$

where κ is the thermal conductivity of the body. Derive the *heat equation*

$$\frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T$$

where $\alpha^2 = \kappa / c_V \rho$, where c_V is the specific heat of the fluid (heat capacity per unit mass) and ρ is the mass density.

SOLUTION: We're going to express the conservation of energy in differential form using the divergence theorem, but before doing so, we must express the energy in terms of the temperature T. The product of the specific heat c_V and the mass density ρ is the heat capacity per unit volume. If we let u be the energy density within the volume V, then

$$u = c_V \rho T + \text{constant}$$

where the constant simply sets the zero of energy. Now, the rate of change of energy U within an arbitrary volume V is given by

$$\frac{dU}{dt} = \iiint_V \frac{\partial u}{\partial t}\,dV = \iiint_V \frac{\partial u}{\partial T}\frac{\partial T}{\partial t}\,dV = c_V\rho \iiint_V \frac{\partial T}{\partial t}\,dV$$

where we used the above equation to replace $\partial u/\partial T$ by $c_V\rho$. The flow rate of energy across the surface S due to a temperature gradient is given by

$$\iint_S \mathbf{q}\cdot\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{q}\,dV$$

where we have used the divergence theorem, Equation 2. Equating the two rates of change of energy gives

$$c_V\rho \iiint_V \frac{\partial T}{\partial t}\,dV = -\iint_S \mathbf{q}\cdot\mathbf{n}\,dS = -\iiint_V \operatorname{div}\mathbf{q}\,dV$$

where once again the negative sign accounts for the fact that $\mathbf{n}$ is a unit normal *outward* vector. Because the volume V is arbitrary, we have

$$c_V\rho\frac{\partial T}{\partial t} = -\operatorname{div}\mathbf{q}$$

Fourier's law of heat flow says that $\mathbf{q} = -\kappa \operatorname{grad} T$, however, so the right side of this equation is $\kappa \operatorname{div}\operatorname{grad} T = \kappa\nabla^2 T$ and we have the heat equation:

$$c_V\rho\frac{\partial T}{\partial t} = \kappa\nabla^2 T$$

The heat equation models the temperature distribution throughout a homogeneous body. Notice that the heat equation is essentially the same as the diffusion equation (Equation 15 of Section 1). The reason they are so similar is that the diffusion of a substance or the flow of energy as heat have similar molecular descriptions. Like the diffusion equation, the heat equation is a partial differential equation, which is the subject of Chapter 16.

The divergence theorem plays an important role in electrostatics. Recall that the electric field intensity produced by a charge q located at the origin of a coordinate system is given by Coulomb's law,

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r \tag{12}$$

where $\mathbf{e}_r$ is the unit vector $(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})/r$ and ε_0 is the permittivity of a vacuum. Let's consider the flux of $\mathbf{E}$ through a closed surface surrounding the charge. The

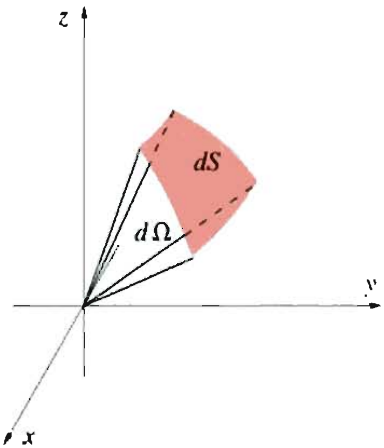

Figure 7.37
A solid angle element $d\Omega$.

total flux Φ_E will be given by

$$\Phi_E = \iint \mathbf{E}\cdot\mathbf{n}\,dS = \frac{q}{4\pi\epsilon_0}\iint\limits_S \frac{\mathbf{e}_r\cdot\mathbf{n}}{r^2}\,dS$$

Before going on, let's review the idea of a solid angle. Let S be an area on a unit sphere centered at the origin. All the rays starting at the origin and passing through S form a cone, which is called the *solid angle* Ω. We say that Ω is subtended by S. The units of solid angles are *steradians*, just as the units of planar angles are radians. Figure 7.37 shows the solid angle $d\Omega$ subtended by dS.

Just as the arc length ds on a circle is related to the angle $d\theta$ (in radians) that it subtends by $ds = r\,d\theta$, where r is the radius of the circle, dS is related to the solid angle $d\Omega$ (in steradians) that it subtends by $dS = r^2 d\Omega$. For example, if $r = a =$ constant, then the total surface area of the sphere is $S = 4\pi a^2$, so that a complete solid angle is 4π, just as a complete angle for a circle is 2π.

Unless S happens to be a sphere, $\mathbf{e}_r$ and $\mathbf{n}$ will not be parallel, but $\mathbf{e}_r\cdot\mathbf{n}\,dS$ will be the projection of dS onto the sphere of radius a (Figure 7.38), so that

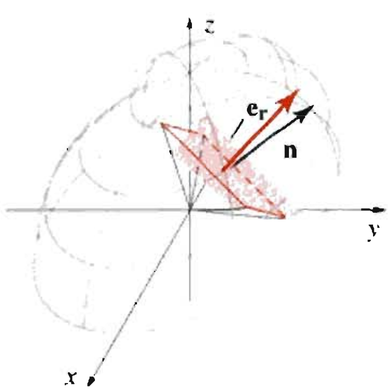

Figure 7.38
The projection of $\mathbf{e}_r\cdot\mathbf{n}\,dS$ onto the surface of a sphere of radius a.

$$\iint\limits_S \mathbf{E}\cdot\mathbf{n}\,dS = \frac{q}{4\pi\varepsilon_0}\iint\limits_S \frac{\mathbf{e}_r\cdot\mathbf{n}}{r^2}\,dS = \frac{q}{4\pi\varepsilon_0}\iint\limits_S d\Omega = \frac{q}{\varepsilon_0} \tag{13}$$

Equation 13 is a fundamental equation of electrostatics called *Gauss's law*. We can transform Equation 13 into another well-known equation by writing

$$q = \int \rho_c\,dV \tag{14}$$

where ρ_c is the charge density within V. Applying the divergence theorem to Equations 13 and 14 gives

$$\operatorname{div}\mathbf{E} = \frac{\rho_c}{\varepsilon_0} \tag{15}$$

which is one of Maxwell's equations.

If we define the electrostatic potential by $\nabla\phi = -\mathbf{E}$, then Equation 15 becomes *Poisson's equation*,

$$\nabla^2\phi = -\frac{\rho_c}{\varepsilon_0} \tag{16}$$

which gives the electrostatic potential $\phi(x, y, z)$ due to a charge distribution $\rho_c(x, y, z)$. Lastly, for a charged free region, Equation 16 becomes *Laplace's equation*,

$$\nabla^2\phi = 0$$

one of the fundamental equations of physics.

7.4 Problems

The first four problems are meant as a review of triple integration.

1. Calculate the volume of the solid body bounded by the three coordinate planes ($x = 0$, $y = 0$, and $z = 0$) and $x + y + z = a$ for $a > 0$.

2. Evaluate the integral $I = \iiint_V dxdydz\,(x^2 + y^2 + z^2)$ over the same volume as in Problem 1.

3. Evaluate the integral $\iiint dxdydz\,xyz$ over the volume of a tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0), and (0, 0, 1) (Figure 7.39).

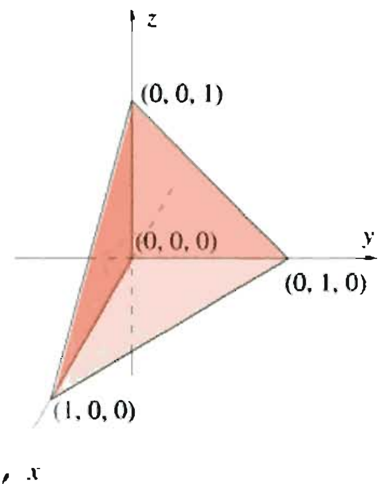

Figure 7.39
A tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0), and (0, 0, 1).

4. Evaluate the integral $I = \iiint dxdydz\,x^2y^2z^2$ over the volume of an octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$.

5. Evaluate the surface integral $I = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ if $\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + z^2\,\mathbf{k}$, where S is the surface of the sphere described by $x^2 + y^2 + z^2 = 9$, and $\mathbf{n}$ is the outward unit normal vector.

6. Verify the divergence theorem if $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the surface of the cube bounded by the three coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$.

7. Evaluate the surface integral $I = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ over the surface of the sphere described by $x^2 + y^2 + z^2 = 16$ if $\mathbf{F} = xy^2\,\mathbf{i} + yz^2\,\mathbf{j} + zx^2\,\mathbf{k}$.

8. Verify the divergence theorem if $\mathbf{F} = xy^2\,\mathbf{i} + x^2y\,\mathbf{j} + y\,\mathbf{k}$ and S is the right circular cylinder described by $x^2 + y^2 = 1$, $-1 < z < 1$.

9. Use the divergence theorem to evaluate the surface integral $I = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 = 4$ between $-1 < z < 1$.

10. In Problem 17 of Section 1, we introduced the definition of div **v** given by div $\mathbf{v} = \lim_{V \to 0} \dfrac{\iint_S \mathbf{n} \cdot \mathbf{v}\, dS}{V}$. Use the divergence theorem to verify this relation. *Hint:* Use a generalization of the mean value theorem of integration to three dimensions.

11. If f and g are scalar functions such that $\mathbf{F} = f\nabla g$ has continuous first partial derivatives in some region R, show that $\iiint_V (f\nabla^2 g + \nabla f \cdot \nabla g)\,dV = \iint_S f\dfrac{\partial g}{\partial n}\,dS$, where $\partial g/\partial n$ denotes the directional derivative of g in the direction of the outer normal to S, which bounds V. This relation is known as *Green's first identity*.

12. Using the result of Problem 11, prove *Green's second identity*,

$$\iiint_V (f\nabla^2 g - g\nabla^2 f)\,dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS.$$

13. Use Green's first identity to show that $\iiint_V |\nabla f|^2 dV = \iint_S \dfrac{\partial f}{\partial n}\,dS$.

14. Use the divergence theorem to show that $\iiint_V \nabla f\, dV = \iint_S f\mathbf{n}\, dS$. Note that the result here is a vector. *Hint:* Consider the vector $\mathbf{F} = F\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.

15. Use the result of the previous problem to show that $\iint_S \mathbf{n}\, dS = 0$, where S is any piecewise smooth closed surface and $\mathbf{n}$ is the outward unit normal vector.

16. Use Equations 13 and 14 to show that if $\rho_c(x, y, z)$ is spherically symmetric, then the electric field **E** that is observed at any point that is outside the charge density is the same as that due to a charge $q = \int \rho_c dv$ located at the origin.

17. Show that the result of Problem 16 also applies to the gravitational force due to a mass distribution.

18. Show that the divergence theorem in two dimensions is of the form $\iint_R \operatorname{div} \mathbf{v}\, dS = \int_C \mathbf{n} \cdot \mathbf{v}\, ds$, where $\mathbf{v} = \mathbf{i}\, v_x + \mathbf{j}\, v_y$. *Hint:* Consider a slab-like volume V of height h with base R in the z direction, and then apply the divergence theorem to **v** in V.

19. Show that $\iint_S \mathbf{B} \cdot \mathbf{n}\, dS = 0$ for any closed surface if $\mathbf{B} = \operatorname{curl} \mathbf{A}$.

7.5 Stokes's Theorem

The divergence theorem relates a surface integral to a volume integral. There also is a theorem, called *Stokes's theorem*, that relates a line integral to a surface integral. We've already encountered a relationship between a line integral and a surface integral. At the end of Section 2, we presented Green's theorem in the plane in vector notation:

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_R (\operatorname{curl} \mathbf{A}) \cdot \mathbf{k}\, dx\, dy \tag{1}$$

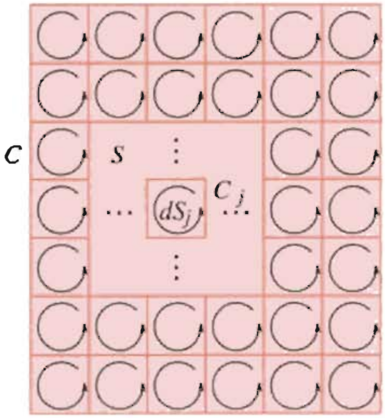

Figure 7.43
A surface S and a boundary curve C, where the surface is partitioned into a mesh of surfaces dS_j with boundary curves C_j.

region containing S, then

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dS \tag{2}$$

where C is traversed in the direction with respect to **n** *in accord with the right-hand rule.*

The proof of Stokes's theorem is fairly involved but we'll give a suggestive physical argument of Stokes's theorem. Figure 7.43 shows a surface S and a boundary curve C, where the surface is partitioned into a mesh of surfaces dS_j with boundary curves C_j. Just for simplicity at this point, let's evaluate $\oint \mathbf{v} \cdot d\mathbf{r}$ for a small rectangle in the xy-plane. First we'll determine the contribution from sides 1 and 3 in Figure 7.44.

$$\int_1 \mathbf{v} \cdot d\mathbf{r} + \int_3 \mathbf{v} \cdot d\mathbf{r} = \int_x^{x+\Delta x} v_x(x', y)dx' + \int_{x+\Delta x}^{x} v_x(x', y + \Delta y)dx'$$

$$= -\int_x^{x+\Delta x} \{v_x(x', y + \Delta y) - v_x(x', y)\}dx'$$

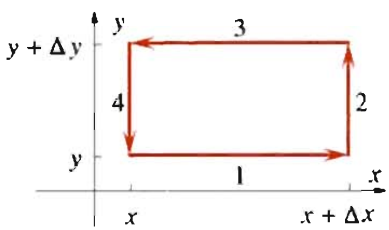

Figure 7.44
A pictorial aid to the evaluation of $\oint \mathbf{v} \cdot d\mathbf{r}$ around one of the surfaces dS_j in Figure 7.43.

Now $v_x(x, y + \Delta y) - v_x(x, y) = \dfrac{\partial v_x}{\partial y}\Delta y + [(\Delta y)^2]$ and so we can write

$$\int_{1+3} \mathbf{v} \cdot d\mathbf{r} = \frac{\partial v_x}{\partial y} dx dy$$

Similarly, the integrals over sides 2 and 4 in Figure 7.44 give

$$\int_{2+4} \mathbf{v} \cdot d\mathbf{r} = -\frac{\partial v_y}{\partial x} dx dy$$

and so

$$\oint_{C_j} \mathbf{v} \cdot d\mathbf{r} = \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) dx dy \tag{3}$$

The right-hand side of Equation 3 is the z-component of $\nabla \times \mathbf{v}\, dxdy$, and so we can write Equation 3 as

$$\oint_{C_j} \mathbf{v} \cdot d\mathbf{r} = (\nabla \times \mathbf{v}) \cdot \mathbf{k}\, dxdy \tag{4}$$

Generally dS_j in Figure 7.43 will not lie in the xy-plane, and so we replace **k** by **n** and $dxdy$ by dS_j in Equation 4 to give

$$\oint_{C_j} \mathbf{v} \cdot d\mathbf{r} = (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dS_j \tag{5}$$

We're almost finished. If we sum over all the area elements in Figure 7.43, we see that *all* the interior line integrals vanish because the integrals along the borders of adjacent area elements are in opposite directions along the same line. The only contribution to the sum of the line integrals on the left of Equation 5 are due to the curve C that encloses S. Therefore, Equation 5 becomes Equation 2, our statement of Stokes's theorem.

Example 1:
Verify Stokes's theorem by evaluating both sides of Equation 2 where $\mathbf{v} = -y\,\mathbf{i} + x\,\mathbf{j} + \mathbf{k}$. S is the hemisphere described by $z = (4 - x^2 - y^2)^{1/2}$, and C is a circle of radius 2 centered at the origin in the xy-plane (Figure 7.45).

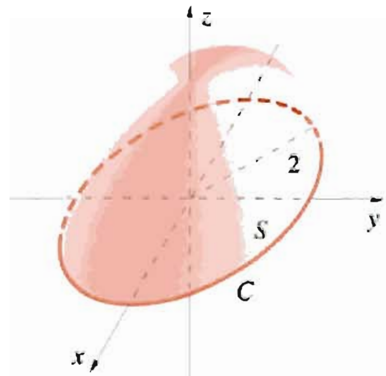

Figure 7.45
The surface and its boundary curve that are used in Example 1.

SOLUTION: The outward unit normal vector to S is given in terms of $\phi(x, y, z) = z - (4 - x^2 - y^2)^{1/2}$ by

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\dfrac{x}{z}\,\mathbf{i} + \dfrac{y}{z}\,\mathbf{j} + \mathbf{k}}{\left(\dfrac{x^2}{z^2} + \dfrac{y^2}{z^2} + 1\right)^{1/2}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{2}$$

and so the right-hand rule tells us to traverse the circle in a counterclockwise direction. Let the circle be described by the parametric equations $x(\theta) = 2\cos\theta$ and $y(\theta) = 2\sin\theta$, $0 \le \theta \le 2\pi$. Then, $\mathbf{r}(\theta) = 2\cos\theta\,\mathbf{i} + 2\sin\theta\,\mathbf{j}$ and $\mathbf{v}(\theta) = -2\sin\theta\,\mathbf{i} + 2\cos\theta\,\mathbf{j} + \mathbf{k}$, and so

$$\oint \mathbf{v}\cdot d\mathbf{r} = \int_0^{2\pi} d\theta\,[(-2\sin\theta)(-2\sin\theta) + (2\cos\theta)(2\cos\theta)]$$

$$= 4\int_0^{2\pi} d\theta = 8\pi$$

Now, $\nabla \times \mathbf{v} = 2\,\mathbf{k}$, and so

$$\iint_S (\nabla \times \mathbf{v})\cdot\mathbf{n}\,dS = \iint_S z\,dS$$

$$= \iint_R \frac{z}{|\mathbf{n}\cdot\mathbf{k}|}\,dx\,dy = 2\iint_R dx\,dy = 8\pi$$

where we have used Equation 5 of Section 3.

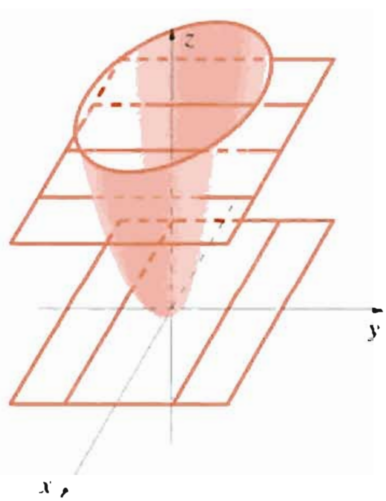

Figure 7.46
The circular paraboloid $z = x^2 + y^2$ and the two planes $z = 0$ and $z = 1$ used in Example 2.

Example 2:
Verify Stokes's theorem for $\mathbf{v} = y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$, where S is the circular paraboloid described by $z = x^2 + y^2$ with $0 \le z \le 1$ (Figure 7.46).

SOLUTION: Figure 7.46 shows that the boundary curve is the circle described by $x^2 + y^2 = 1$ and $z = 1$. We can describe this circle parametrically by the position vector

$$\mathbf{r}(u) = \cos u\, \mathbf{i} + \sin u\, \mathbf{j} + \mathbf{k}$$

for $0 \leq u \leq 2\pi$. The left side of Equation 2 gives

$$\oint \mathbf{v} \cdot d\mathbf{r} = \int_0^{2\pi} du\, [-\sin^3 u + \cos^2 u] = \pi$$

The unit normal vector is given by $\mathbf{n} = \nabla f / |\nabla f|$ where $f = z - x^2 - y^2$, or

$$\mathbf{n} = \frac{-2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}}{(1 + 4x^2 + 4y^2)^{1/2}}$$

and the curl $\mathbf{v}$ is given by

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x & z^2 \end{vmatrix} = (1 - 2y)\,\mathbf{k}$$

Therefore, the right side of Equation 2 becomes

$$\iint_S \frac{(1-2y)\,dS}{(1+4x^2+4y^2)^{1/2}} = \iint_R \frac{(1-2y)\,dxdy}{(1+4x^2+4y^2)^{1/2}(\mathbf{n}\cdot\mathbf{k})}$$
$$= \iint_R (1-2y)\,dxdy$$

where we have used Equation 7.3.5. The above integral is

$$\iint_R (1-2y)\,dxdy = \int_{-1}^{1} dx \int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} dy\,(1-2y) = \pi$$

Stokes's theorem says that the surface integral of curl $\mathbf{F} \cdot \mathbf{n}$ over a capping surface S is equal to the line integral of $\mathbf{F}$ around the boundary curve of S. Notice that this implies that $\iint_S \mathbf{A} \cdot \mathbf{n}\, dS$ is independent of the surface bounded by C, provided $\mathbf{A} = \text{curl } \mathbf{F}$ and S and $\mathbf{n}$ satisfy the requirements of Stokes's theorem.

Let's go back to Example 1, where the boundary curve is a circle of radius 2 centered at the origin and the capping surface is a hemisphere. We integrated curl $\mathbf{F} \cdot \mathbf{n}$ over the surface of the hemisphere in Example 1, but we could just as well have integrated over the unit disk itself (see Figure 7.41). If we do that, $\mathbf{n} = \mathbf{k}$ and we have

$$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dx\,dy = 2\iint dx\,dy = 2(4\pi) = 8\pi$$

in agreement with the result in Example 1.

Example 3:

Use Stokes's theorem to evaluate the surface integral $\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$.

where $\mathbf{F} = 3y\,\mathbf{i} - 2x\,\mathbf{j} + xy\,\mathbf{k}$ and S is the hemispherical surface described by $x^2 + y^2 + z^2 = 4$ with $z \geq 0$ (Figure 7.45).

SOLUTION: Stokes's theorem allows us to use any capping surface, so let's use the disk of radius 2 given by $x^2 + y^2 \leq 4$ and $z = 0$. In this case, $\mathbf{n} = \mathbf{k}$, and using curl $\mathbf{F} = x^2\,\mathbf{i} - yz\,\mathbf{j} - 5\,\mathbf{k}$, we have

$$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = -5\iint dx\,dy = -5(4\pi) = -20\pi$$

Problem 16 has you show that you obtain the same result using the hemispherical surface.

According to Stokes's theorem, if $\nabla\times\mathbf{v} = 0$ in some region containing S, then $\oint \mathbf{v}\cdot d\mathbf{r} = 0$ for all simple closed curves in that region. In that case, $\mathbf{v}$ is the gradient of a scalar potential ϕ, or $\mathbf{v} = \nabla\phi$. This is such an important result in physics and engineering that we should be sure to be aware of the strict conditions involved. Thus we state the following theorem:

If $\mathbf{v}(x, y, z)$ is a vector field with continuous first-order partial derivatives in a simply connected region R, then $\nabla\times\mathbf{v} = 0$ if and only if $\mathbf{v} = \nabla\phi$ for $\phi(x, y, z)$ defined in R.

The operative phrase here is "simply connected region" (see Section 2).

A vector field $\mathbf{F}$ for which $\nabla\times\mathbf{F} = \mathbf{0}$ is called *irrotational*. Let's use Stokes's theorem to see why this is so. Consider a circular disk of radius a centered at a point P with a boundary curve C (Figure 7.47). Then Stokes's theorem says that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS \tag{6}$$

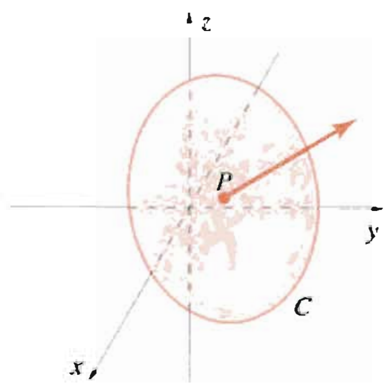

Figure 7.47
A circular disk centered at a point P with a boundary curve C.

where $\mathbf{n}$ is perpendicular to the disk in Figure 7.47. Using the mean value theorem of integration on the right side of Equation 6, we can write

$$\oint_C \mathbf{F}\cdot\mathbf{n} = \pi a^2[\text{curl}\,\mathbf{F}\cdot\mathbf{n}]_\zeta$$

where ζ is some point in the disk. If we divide by πa^2 and let $a \to 0$, then we have

$$[\operatorname{curl} \mathbf{F} \cdot \mathbf{n}]_P = \lim_{a \to 0} \frac{1}{\pi a^2} \oint_C \mathbf{F} \cdot d\mathbf{r} \tag{7}$$

The right side of Equation 7 has a nice physical interpretation. Suppose that $\mathbf{F}$ represents (steady-state) fluid flow, so that $\mathbf{F} = \rho \mathbf{v}$, where ρ is the mass density of the fluid and $\mathbf{v}$ is its velocity at any point. Then $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the flow rate of fluid around the curve C. Thus, the quantity $[\operatorname{curl} \mathbf{F} \cdot \mathbf{n}]_P$, which represents the flow rate of fluid around the point P, is maximal when curl $\mathbf{F}$ and $\mathbf{n}$ point in the same direction. Therefore, the axis about which the fluid rotates most rapidly is in the same direction as curl $\mathbf{F}$.

Stokes's theorem tells us that the vector field is irrotational if and only if it is a conservative field; in other words, curl $\mathbf{F} = \mathbf{0}$ if and only if $\mathbf{F} = \nabla\phi$, where the potential function ϕ is defined in a simply connected region R.

Stokes's theorem is often used to derive other results. For example, a wire carrying an electric current generates a magnetic field. The relation between the current and the magnetic field intensity is given by *Ampere's law*, which says that

$$\oint_C \mathbf{H} \cdot d\mathbf{r} = I \tag{8}$$

where $\mathbf{H}$ is the magnetic intensity, C is a closed curve enclosing the wire, and I is the steady current crossing any surface bounded by C. If $\mathbf{J}$ is the steady current crossing a unit area perpendicular to $\mathbf{J}$ (the flux of electrical charge), then the current crossing S is given by

$$I = \iint_S \mathbf{J} \cdot \mathbf{n}\, dS \tag{9}$$

Applying Stokes's theorem to Equation 8 gives

$$I = \iint_S (\nabla \times \mathbf{H}) \cdot \mathbf{n}\, dS \tag{10}$$

Because the surface S in Equations 9 and 10 is arbitrary, the integrals must be equal, or

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{11}$$

which is one of Maxwell's equations for the case where $\mathbf{H}$ and $\mathbf{J}$ do not vary with time.

We can summarize much of this section by saying that if $\mathbf{v}$ has continuous first-order partial derivatives in a simply connected region R, then the following five statements are equivalent:

1. $\nabla \times \mathbf{v} = \mathbf{0}$ at every point in R.
2. $\oint \mathbf{v} \cdot d\mathbf{r} = 0$.
3. $\int_a^b \mathbf{v} \cdot d\mathbf{r}$ is path independent.
4. $\mathbf{v} \cdot d\mathbf{r}$ is an exact differential.
5. $\mathbf{v} = \text{grad}\,\phi$, where ϕ is a scalar potential.

7.5 Problems

1. Verify Stokes's theorem if $\mathbf{v} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ is taken over the hemispherical surface $x^2 + y^2 + z^2 = 1$, $z > 0$.
2. Verify Stokes's theorem if $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ and S is the surface of the circular paraboloid described by $z = 1 - x^2 - y^2$ with $z \geq 0$.
3. Evaluate $\iint \mathbf{r} \cdot \mathbf{n}\, dS$ where S is the surface of a sphere of radius 1 centered at the origin.
4. Verify Stokes's theorem if $\mathbf{v} = (y - z + 2)\,\mathbf{i} + (yz + 4)\,\mathbf{j} - xz\,\mathbf{k}$ is taken over a cube bounded by planes $x = 0$, $y = 0$, $z = 0$, $x = a$, $y = a$, $z = a$, excluding the face on the plane $z = 0$.
5. Use Stokes's theorem to evaluate the integral $\iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dS$ over the surface of the circular paraboloid $z = 16 - (x^2 + y^2)$ above the xy-plane if $\mathbf{v} = (x + y - 2)\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$.
6. Verify Stokes's theorem if $\mathbf{F} = (2y + z)\,\mathbf{i} + (x - z)\,\mathbf{j} + (y - x)\,\mathbf{k}$ and S is the plane $x + y + z = 1$ in the first octant.
7. Verify Stokes's theorem if $\mathbf{F} = 2x\,\mathbf{i} + 3y^2\,\mathbf{j} + xz\,\mathbf{k}$ and S is the surface of the circular paraboloid $x^2 + y^2 = 2 - z$ with $x^2 + y^2 \leq 1$.
8. Use Stokes's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$ and C is the unit circle $x^2 + y^2 = 1$ in the $z = 2$ plane.
9. Use Stokes's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = y^2\,\mathbf{i} - x^2\,\mathbf{j} - (y + z)\,\mathbf{k}$ and C is the triangle formed by the points $(0, 0, 0)$, $(1, 0, 0)$, and $(1, 1, 0)$.
10. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} + xy\,\mathbf{j} - xz\,\mathbf{k}$ and C is the circle described by $x^2 + y^2 = 4$ in the xy-plane.
11. Use Stokes's theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = x^2y^3\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$ around the circle described by $x^2 + y^2 = a^2$ in the xy-plane.
12. Use Stokes's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (z - 2y)\,\mathbf{i} + (3x - 4y)\,\mathbf{j} + (z + 3y)\,\mathbf{k}$ and C is the boundary of the triangle formed by the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.
13. Verify Stokes's theorem if $\mathbf{F} = y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ for the surface described by $z = x^2 + y^2$, $z \leq 1$.
14. Use Stokes's theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (x + y)\,\mathbf{i} + (z - 2x + y)\,\mathbf{j} + (y - z)\,\mathbf{k}$ and C is the unit circle $x^2 + y^2 = 1$ in the plane $z = 5$.
15. Use Stokes's theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (y - x)\,\mathbf{i} + (x - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$ and C is the part of the plane $x + 2y + z = 2$ in the first octant.

16. Carry out Example 3 using the hemispherical surface and show that you obtain the same result.

17. Use Stokes's theorem to show that curl grad $\phi = 0$.

18. Apply Stokes's theorem to the vector field $\mathbf{v} = -y\,\mathbf{i} + x\,\mathbf{j}$ and a surface in the xy-plane bounded by a smooth simple closed curve C to show that the area bounded by C is given by $A = \frac{1}{2}\oint(x\,dy - y\,dx)$.

19. Use the formula derived in Problem 18 to determine the area of an ellipse described parametrically by $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$.

20. Show that $\oint \mathbf{v}\cdot d\mathbf{r} = 2\pi$ for $\mathbf{v}$ given in Example 2 for the closed path around the square whose vertices are $(-1,-1)$, $(1,-1)$, $(1,1)$, and $(-1,1)$.

21. Faraday's law of electricity and magnetism says that the electromotive force around a closed current-carrying loop is equal to the negative of the rate of change of the magnetic flux through the loop. In terms of an equation, Faraday's law is $\int_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}\,dS$ for any fixed surface in the field. Use Stokes's theorem to derive one of Maxwell's equations, $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$.

22. Use Stokes's theorem to show that $\nabla\times(\nabla f) = 0$.

23. Derive Green's theorem in a plane from Stokes's theorem.

24. We're going to derive a two-dimensional version of the divergence theorem in this problem. Let $\mathbf{v}$ be a two-dimensional field with components v_x and v_y and let $\mathbf{F} = -v_y\,\mathbf{i} + v_x\,\mathbf{j}$. Show that $\iint_R \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dxdy = \iint_R \nabla\cdot\mathbf{v}\,dxdy$. Now apply Green's theorem in the plane (Equation 7.2.16) to the field $\mathbf{F}$ to obtain

$$\iint_R \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dxdy = \oint_C[-v_y dx + v_x dy] = \oint_C \mathbf{v}\cdot\left(\frac{dy}{ds}\,\mathbf{i} - \frac{dx}{ds}\,\mathbf{j}\right)ds$$
$$= \oint_C \mathbf{v}\cdot\mathbf{n}\,ds$$

The two-dimensional divergence theorem is $\iint_R \nabla\cdot\mathbf{v}\,dxdy = \oint_C \mathbf{v}\cdot\mathbf{n}\,ds$.

25. In this problem, we'll show how to find a vector $\mathbf{w}$ such that $\mathbf{v} = \text{curl}\,\mathbf{w}$ if we are given $\mathbf{v}$. In other words, we'll determine a vector potential for $\mathbf{v}$. First show that

$$v_x = \frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z}, \qquad v_y = \frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x}, \qquad v_z = \frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} \tag{12}$$

Now take $w_x = 0$ (arbitrarily) and show that $w_z = -\int_{x_0}^{x} v_y dx' + \phi(y,z)$ and $w_y = \int_{x_0}^{x} v_z dx' + \psi(y,z)$, where ϕ and ψ are arbitrary differentiable functions of y and z. Now substitute these results into the first of Equations 12 to obtain

$$v_x = -\int_{x_0}^{x}\left(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)dx' + \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial z}$$

Show that div $\mathbf{v} = 0$ if $\mathbf{v} = \text{curl } \mathbf{w}$ and use this result to derive $v_x(x_0, y, z) = \dfrac{\partial \phi}{\partial y} - \dfrac{\partial \psi}{\partial z}$. This equation can be satisfied by letting $\psi = 0$, in which case we have
$\phi(y, z) = \displaystyle\int_{y_0}^{y} v_x(x_0, y', z)\, dy'$. Show that one possible solution for $\mathbf{w}$ is then

$$\mathbf{w} = \left[\int_{x_0}^{x} v_z(x', y, z)\, dx'\right] \mathbf{j} + \left[-\int_{x_0}^{x} v_y(x', y, z)\, dx' + \int_{y_0}^{y} v_x(x_0, y', z)\, dy'\right] \mathbf{k}$$

26. Show that the solution $\mathbf{w}$ to the equation $\mathbf{v} = \text{curl } \mathbf{w}$ for a given vector field $\mathbf{v}$ is given by $\mathbf{w}$ in the previous problem plus a term ∇u, where u is an arbitrary scalar function with continuous second partial derivatives.

27. Use the result of Problem 25 to find $\mathbf{w}(x, y, z)$ such that $\mathbf{v} = (z - y)\,\mathbf{i} + (x - z)\,\mathbf{j} + (y - x)\,\mathbf{k} = \text{curl } \mathbf{w}$.

28. Use the result of Problem 25 to find $\mathbf{w}(x, y, z)$ such that $\mathbf{v} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k} = \text{curl } \mathbf{w}$.

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CHAPTER 8

Curvilinear Coordinates

Although cartesian, or rectangular, coordinates are the first ones we learn to use and use most often, they are not always the most convenient. For example, if the system has a natural center of symmetry, as in the case of an atom with its massive nucleus at its center, it is much more convenient to use spherical coordinates, which are constructed with exactly such systems in mind. There are many examples of problems that become much easier by using the appropriate choice of coordinate system. Usually the symmetry of the system of interest will suggest which of a number of available coordinate systems to use. In this chapter, we'll study plane polar coordinates, cylindrical coordinates, and spherical coordinates and learn how to express vector quantities, such as the gradient, the divergence, and the curl, in these coordinate systems. We'll see how we can unify the results for these various coordinate systems by introducing the idea of a metric coefficient. In Section 4, we'll learn how to convert differential volume elements from one coordinate system to another by means of Jacobian determinants and, in Section 5, we'll generalize most of our previous results into one set of equations by introducing curvilinear coordinates. In the last section, we'll introduce two other coordinate systems involving spheroidal coordinates to give a little experience in lesser-used coordinate systems.

8.1 Plane Polar Coordinates

Instead of locating a point in a plane by the two coordinates (x, y), we can equally well locate it by specifying its distance r from the origin, and the angle θ with which the line from the origin to the point makes with the positive x axis (Figure 8.1). The coordinates r and θ are called *polar coordinates*. We shall restrict r to $r \geq 0$ and allow θ to take on any value, although often θ will vary from 0 to 2π. You can see from Figure 8.1 that the relation between rectangular coordinates and polar coordinates is given by

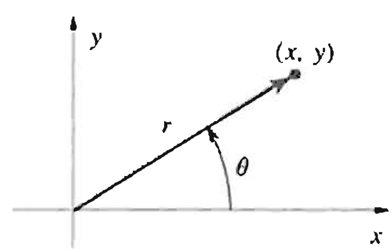

Figure 8.1
The specification of the location of a point in a plane by polar coordinates, (r, θ).

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{1}$$

and

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \theta = \tan^{-1}\frac{y}{x} \tag{2}$$

Thus the point (1, 1) in rectangular coordinates becomes the point $(\sqrt{2}, \pi/4)$ in polar coordinates. When calculating θ from the arctangent formula in Equation 2, you must bear in mind in which quadrant the point lies. Using Equation 2 blindly for the point $(x = -1, y = -1)$ gives $\theta = \pi/4$, but realize that $\tan 5\pi/4 = \tan(225°) = 1$, also.

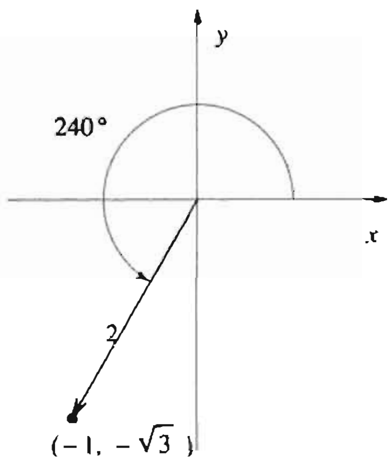

Figure 8.2
The point $x = -1$, $y = -\sqrt{3}$.

Example 1:
Equations 1 give

$$\theta = \cos^{-1}\frac{x}{r} \qquad \text{and} \qquad \theta = \sin^{-1}\frac{y}{r}$$

and Equation 2 says that

$$\theta = \tan^{-1}\frac{y}{x}$$

Use each of these relations to calculate θ for the point $(x = -1, y = -\sqrt{3})$ (Figure 8.2).

SOLUTION: In this case, $r = (x^2 + y^2)^{1/2} = 2$. Using a hand calculator, you'll find that

$$\theta = \cos^{-1}\left(\frac{-1}{2}\right) = 120°$$

$$\theta = \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -60°$$

and

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = 60°$$

none of which is correct! The point lies in the third quadrant, and the correct answer is $180° + \cos^{-1}(1/2) = 240°$.

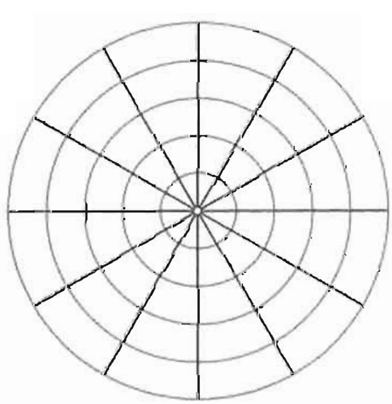

Figure 8.3
A polar grid of coordinates for plotting functions expressed in polar coordinates.

Many of the famous classical curves of mathematics are conveniently expressed in polar form. Figure 8.3 shows a polar grid of coordinates for plotting functions expressed in polar coordinates. Because lines of constant r (circles centered at the origin) and lines of constant θ (straight lines emanating from the origin) intersect at right angles, the polar coordinate system is said to be *orthogonal*. A circle of radius a centered at the origin is described by $r = a$. This result follows immediately from the equation $x^2 + y^2 = a^2$. What about a circle centered somewhere else, say at $(x = a,\ y = 0)$? In that case, we have $(x - a)^2 + y^2 = a^2$ as the equation in rectangular coordinates, or $(r\cos\theta - a)^2 + r^2\sin^2\theta = a^2$, or $r^2 = 4a^2\cos^2\theta$. You should verify by plotting that $r^2 = 4a^2\cos^2\theta$ is indeed the equation of a circle of radius a centered at $(a, 0)$ and that $r^2 = 4a^2 \sin^2 \theta$ is a circle of radius a centered at $(0, a)$.

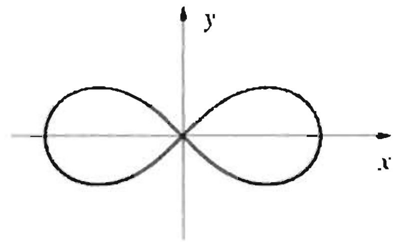

Figure 8.4
The lemniscate of Bernoulli, $r^2 = \cos 2\theta$.

Example 2:
Plot the equation $r^2 = \cos 2\theta$ for $0 \le \theta < 2\pi$. The resulting curve is called the *Lemniscate of Bernoulli*.

SOLUTION: Let's first make a small table of values of r (remember that we restrict r to $r \ge 0$):

θ	0°	15°	30°	45°
r	+1.00	+0.93	+0.71	0

This gives the upper half of the right lobe in Figure 8.4. In the open interval (45°, 135°), $r^2 < 0$, and so there are no real values of r in this range. From 135° to 225°, we obtain the entire left lobe in Figure 8.4. In the open interval (225°, 315°), $r^2 < 0$ once again, and finally we obtain the lower half of the right lobe in Figure 8.4 for $315° \le \theta < 360°$.

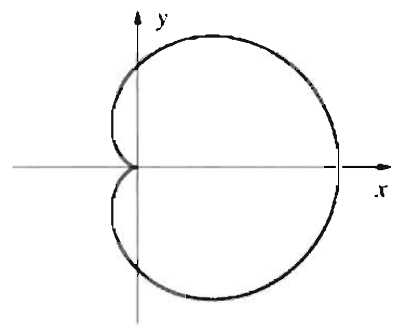

Figure 8.5
A cardioid, $r = a(\cos\theta + 1)$.

If you look at a handbook such as the *CRC Standard Mathematical Tables*, you'll see a number of beautiful curves with names such as cardioid (Figure 8.5) or the folium of Descartes expressed in polar form. Better yet, the School of Mathematics at the University of St. Andrews has an outstanding tutorial website (www-history.mcs.st-and.ac.uk/history/) with a "Famous Curves Index," which gives interactive access to almost 100 "famous curves", with names such as Freeth's nephroid (Figure 8.6), Fermat's spiral, and the conchoid of de Sluze.

It's easy to determine the slope of a curve that is expressed in polar coordinates, $r = r(\theta)$. In this case $x = r(\theta)\cos\theta$ and $y = r(\theta)\sin\theta$, and using

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$$

we see that the slope is given by

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{r'(\theta)\sin\theta + r\cos\theta}{r'(\theta)\cos\theta - r\sin\theta} \tag{3}$$

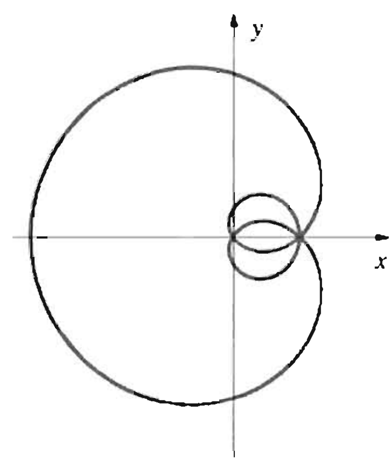

Figure 8.6
Freeth's nephroid. The polar equation is $r = a(1 + 2\sin\theta/2)$. (In this plot, r is allowed to take on negative values.)

Example 3:
Determine the slope of the cardioid described by $r(\theta) = 1 + \cos\theta$ (Figure 8.5) at any point.

SOLUTION: Using Equation 3,

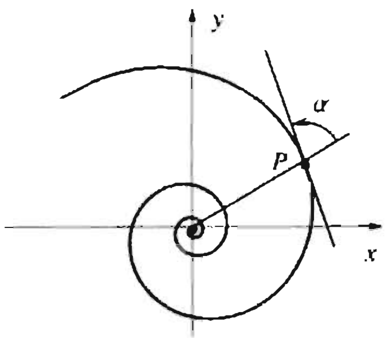

Figure 8.7
A logarithmic spiral, $r = e^{\beta\theta}$.

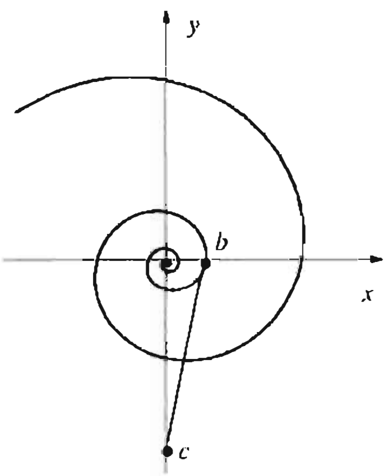

Figure 8.8
The tangent line of a logarithmic spiral at $\theta = 2\pi n$ (n = integer).

$$\frac{dy}{dx} = \frac{(-\sin\theta)\sin\theta + \cos\theta + \cos^2\theta}{(-\sin\theta)\cos\theta - \sin\theta - \sin\theta\cos\theta}$$
$$= -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$$

For example, the slope at $\pi/4$ is -0.414 (see Figure 8.5).

Figure 8.7 shows a curve described by $r = e^{\beta\theta}$ ($-\infty < \theta < \infty$), called a *logarithmic spiral*. This curve has the unique property that the angle between the extended radial line OP and the tangent to the curve at P is a constant. This angle is denoted by α in Figure 8.7. Let's prove this property of a logarithmic spiral. First, we need an equation for the angle α. Problem 5 helps you prove that

$$\cot\alpha = \frac{1}{r}\frac{dr}{d\theta} \tag{4}$$

If we substitute $r = e^{\beta\theta}$ into Equation 4, we find that $\cot\alpha = \beta =$ constant. Furthermore, because $(1/r)(dr/d\theta) =$ constant, we see that the logarithmic spiral is the *only* curve with this property.

A logarithmic spiral has another interesting property. The tangent line at $\theta = 2\pi n$ (n = integer) is shown in Figure 8.8. It turns out that the length of this line between where it cuts the x and y axes (bc) is equal to the length of the spiral from the origin (where $\theta \to -\infty$) to $\theta = 2\pi n$. To prove this result, we need a formula for the arc length of a curve that is described by polar coordinates. This is not difficult. Simply write $ds^2 = dx^2 + dy^2$ and use Equations 2 to obtain

$$ds^2 = dx^2 + dy^2 = (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2$$
$$= dr^2 + r^2 d\theta^2$$

or

$$ds = \left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]^{1/2} d\theta \tag{5}$$

Therefore, the arc length of a logarithmic spiral from the origin (where $\theta \to -\infty$) to $\theta = 2\pi n$ is given by

$$s = \int_{-\infty}^{2\pi n} (\beta^2 e^{2\beta\theta} + e^{2\beta\theta})^{1/2} d\theta$$
$$= (1+\beta^2)^{1/2} \int_{-\infty}^{2\pi n} e^{\beta\theta} d\theta = \frac{(1+\beta^2)^{1/2}}{\beta} e^{2\pi\beta n}$$

Problem 8 has you show that this is equal to the length of the line bc in Figure 8.8.

Example 4:
Determine the length of the perimeter of the cardioid described by $r = 1 + \cos\theta$.

SOLUTION: We use Equation 5:

$$s = \int_0^{2\pi} (\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta)^{1/2} d\theta$$

$$= \sqrt{2}\int_0^{2\pi} (1+\cos\theta)^{1/2} d\theta = 2\sqrt{2}\int_0^{\pi} (1+\cos\theta)^{1/2} d\theta$$

$$= 4\int_0^{\pi} \cos\frac{\theta}{2} d\theta = 8\int_0^{\pi/2} \cos u\, du = 8$$

Now that we know how to calculate slopes and arc lengths of curves that are described by polar coordinates, let's calculate the area of the cardioid given by $r(\theta) = 1 + \cos\theta$. In rectangular coordinates, the differential area element is $dxdy$, but it is *not* $drd\theta$ in polar coordinates. We'll determine just what it is both geometrically and analytically. Figure 8.9 shows an area element generated by varying r by Δr and θ by $\Delta\theta$. If Δr and $\Delta\theta$ are small (which we anticipate), then the area of the area element is $(\Delta r)(r\Delta\theta)$, or $rdrd\theta$ in differential form. Thus, we see that $dA = rdrd\theta$. The area of a figure described by $r = r(\theta)$ is given by

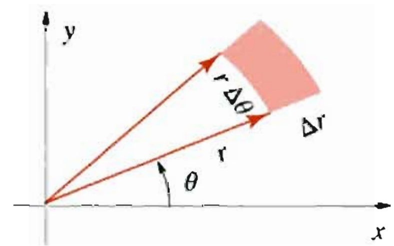

Figure 8.9
A differential area element in polar coordinates generated by varying r by Δr and θ by $\Delta\theta$.

$$A = \int d\theta \int_0^{r(\theta)} dr\, r = \frac{1}{2}\int r^2(\theta)\, d\theta$$

For the cardioid $r(\theta) = 1 + \cos\theta$ (Figure 8.5),

$$A = \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 d\theta = \frac{1}{2}\left(2\pi + \frac{2\pi}{2}\right) = \frac{3\pi}{2}$$

Example 5:
Determine the area of one lobe of the lemniscate of Bernoulli shown in Figure 8.4.

SOLUTION: The equation of the curve in polar coordinates is $r^2 = \cos 2\theta$. The area of one lobe is given by

$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta = \int_0^{\pi/4} \cos^2 2\theta\, d\theta = \frac{\pi}{8}$$

Before we finish this section, we'll show how to evaluate the integral $I = \int_0^\infty e^{-ax^2}dx$ using polar coordinates. Write I^2 as

$$I^2 = \int_0^\infty e^{-ax^2}dx \int_0^\infty e^{-ay^2}dy = \iint\limits_{0\,0}^{\infty\,\infty} dx\,dy\, e^{-a(x^2+y^2)}$$

Now convert to polar coordinates, realizing that the area of integration is the first quadrant, or $0 \le r < \infty$ and $0 \le \theta \le \pi/2$. Thus,

$$I^2 = \int_0^{\pi/2} d\theta \int_0^\infty e^{-ar^2} r\, dr = \frac{\pi}{2}\int_0^\infty \frac{e^{-au}}{2} du = \frac{\pi}{4a}$$

and so $I = (\pi/4a)^{1/2}$. This is a standard trick for evaluating this integral.

8.1 Problems

1. Plot each of the following curves in polar coordinates. In each case, $0 \le \theta < 2\pi$. Recall that we have elected to allow r to take on only non-negative values ($r \ge 0$).

   (a) $r = 2\sin\theta$ (b) $r = 2\cos\theta + \sin\theta$

   (c) $r = \sin\theta\cos^2\theta$ (d) $r = \dfrac{1}{\sin\theta + \cos\theta}$ $-\dfrac{\pi}{6} \le \theta \le \dfrac{2\pi}{3}$

   (e) $r = 1 + 2\cos\theta$ (f) $r = a\cos 2\theta$

2. Some authors (particularly of calculus texts) allow r to take on negative values. Plot the curves in Problem 1, allowing r to take on negative values.

3. Find the slope of $r = \sin 2\theta$ at $\theta = \pi/4$ and $5\pi/4$. (This represents a two-leaved rose if $r \ge 0$, and a four-leaved rose if r is allowed to assume negative values.)

4. Find the slope of $\sqrt{2}\, r\cos\left(\theta - \dfrac{\pi}{4}\right) = 1, -\dfrac{\pi}{6} < \theta < \dfrac{2\pi}{3}$ at any value of θ in the interval $\left(-\dfrac{\pi}{6}, \dfrac{2\pi}{3}\right)$.

5. Derive Equation 4. *Hint*: First show that the angle α in Figure 8.7 is equal to $\gamma - \theta$, where slope $= \tan\gamma$ and then use the relation $dy/dx = \tan\gamma$ along with Equation 3.

6. Determine the angle α between the extended radial line OP and the tangent to the curve at P for

   (a) $r = e^{\theta/2}$ at any θ (b) $r = 1 + \sin\theta$ at $\theta = \pi/2$

   (c) $r = \sin 3\theta$ at $\theta = \pi/12$

7. Calculate the arc length of the perimeter of the limaçon of Pascal described by $r = a + b\cos\theta$ with $a > b$. (Refer to Section 3.5.)

8. Show that the length of the line bc in Figure 8.8 is equal to $(1 + \beta^2)^{1/2} r^{2\pi\beta n}/\beta$.

9. Determine the area bounded by the Leminscate of Bernoulli (Example 2).

10. Determine the area of the limaçon of Pascal described by $r = a + b\cos\theta$ with $a > b$.

11. Calculate the area enclosed by the two-leaved rose $r = 2\cos 2\theta$ ($r \geq 0$).

12. Determine the area between the curve given by $r^2 = 4\sin^2\theta$ and the circle given by $r = 1$. (See Figure 8.10.)

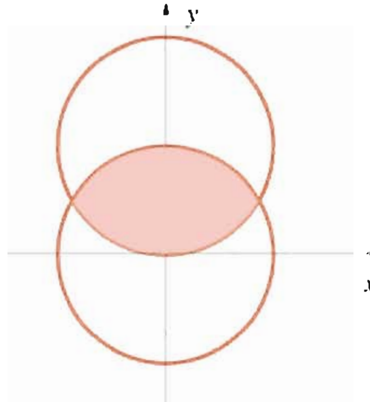

Figure 8.10
The area between the curve given by $r^2 = 4\sin^2\theta$ and the circle given by $r = 1$.

13. Determine the volume of the paraboloid given by $z = 9 - x^2 - y^2$ above the xy-plane.

14. A circular disk of radius a has a density that varies as $\rho = e^{-2(x^2+y^2)^{1/2}}$. Calculate the mass of the disk.

15. Use polar coordinates to calculate the volume of a hemisphere of radius a.

16. Show that the formula $A = \frac{1}{2}\int(x\,dy - y\,dx)$ becomes $A = \frac{1}{2}\int r^2 d\theta$ in polar coordinates.

8.2 Vectors in Plane Polar Coordinates

In Chapter 5, we expressed the location of a particle in two dimensions by the vector $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$. The velocity and the acceleration of the particle are given by

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} \tag{1}$$

and

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\,\mathbf{i} + \frac{d^2y}{dt^2}\,\mathbf{j} \tag{2}$$

Let's look at the corresponding expressions in plane polar coordinates, and write

$$\mathbf{r}(t) = r(t)\,\mathbf{e}_r \tag{3}$$

where $\mathbf{e}_r$ is a unit vector in the direction of $\mathbf{r}$. Before we differentiate this expression with respect to t, let's realize that unlike $\mathbf{i}$ and $\mathbf{j}$ in Equation 1, the direction of $\mathbf{e}_r$ changes as $\mathbf{r}(t)$ varies with time (Figure 8.11). Thus, the time derivative of Equation 3 is

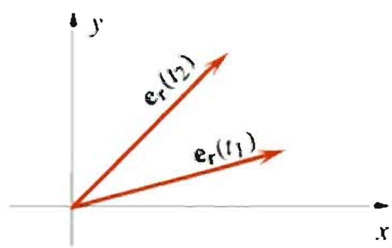

Figure 8.11
The unit vector $\mathbf{e}_r = \mathbf{e}_r(t)$ in polar coordinates at two different times.

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\,\mathbf{e}_r + r(t)\frac{d\mathbf{e}_r}{dt} \tag{4}$$

Although its length is always unity, Figure 8.11 shows that $\mathbf{e}_r$ changes with time because its *direction* changes with time if θ changes with time. Thus, we see that $\mathbf{e}_r = \mathbf{e}_r(\theta)$. The time derivative of $\mathbf{e}_r$ is equal to

$$\frac{d\mathbf{e}_r}{dt} = \frac{d\theta}{dt}\frac{d\mathbf{e}_r}{d\theta} \tag{5}$$

We'll encounter a number of examples where the unit vectors in coordinate systems other than rectangular coordinate systems depend upon the coordinates, so we're going to have to learn how to evaluate derivatives like $d\mathbf{e}_r/d\theta$. Perhaps the simplest way is to express the unit vector $\mathbf{e}_r$ in terms of the cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$:

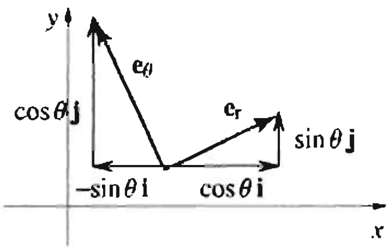

Figure 8.12
The polar coordinate unit vectors $\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and $\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$ expressed in terms of the cartesian coordinate unit vectors $\mathbf{i}$ and $\mathbf{j}$.

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

Then

$$\frac{d\mathbf{e}_r}{d\theta} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

This unit vector is perpendicular to $\mathbf{e}_r$ and is shown in Figure 8.12. If we define $\mathbf{e}_\theta$ to be a unit vector perpendicular to $\mathbf{e}_r$ and in the direction of increasing θ, then we can identify $d\mathbf{e}_r/d\theta$ with $\mathbf{e}_\theta$ and write

$$\frac{d\mathbf{e}_r}{d\theta} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} = \mathbf{e}_\theta \tag{6}$$

Equation 4 tells us that

$$\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r + r\frac{d\theta}{dt}\,\mathbf{e}_\theta \tag{7}$$

The two unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ constitute an orthogonal pair of unit vectors ($\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$) and any vector in the plane can be written as

$$\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta \tag{8}$$

Taking the dot product of $\mathbf{u}$ with $\mathbf{e}_r$ and $\mathbf{e}_\theta$ shows that $u_r = \mathbf{u}\cdot\mathbf{e}_r$ and $u_\theta = \mathbf{u}\cdot\mathbf{e}_\theta$, so Equation 8 can be written as

$$\mathbf{u} = (\mathbf{u}\cdot\mathbf{e}_r)\,\mathbf{e}_r + (\mathbf{u}\cdot\mathbf{e}_\theta)\,\mathbf{e}_\theta \tag{9}$$

The two vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are said to form a *basis* and $u_r = \mathbf{u}\cdot\mathbf{e}_r = u_r(r,\theta)$ and $u_\theta = \mathbf{u}\cdot\mathbf{e}_\theta = u_\theta(r,\theta)$ are called the *components of* $\mathbf{u}$ in this basis. The vectors $\mathbf{i}$ and $\mathbf{j}$ also form a basis in two-dimensions since $\mathbf{u}$ can be expressed as $\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j}$. It so happens that *any* pair of non-colinear vectors can form a basis in two dimensions, but each coordinate system has a "natural" basis which is most convenient.

The acceleration of a particle is given by the time derivative of Equation 7:

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\,\mathbf{e}_r + \frac{dr}{dt}\frac{d\mathbf{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\,\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\,\mathbf{e}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{e}_\theta}{dt} \tag{10}$$

We need to evaluate $d\mathbf{e}_\theta/dt$, or $d\mathbf{e}_\theta/d\theta$ because $d\mathbf{e}_\theta/dt = (d\mathbf{e}_\theta/d\theta)(d\theta/dt)$. Problem 1 has you show that

$$\frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r \tag{11}$$

Using Equations 6 and 11, Equation 10 becomes

$$\mathbf{a}(t) = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta \tag{12}$$

which may be familiar to you if you've had a course in classical mechanics.

Example 1:
A force that can be expressed as $\mathbf{F} = f(r)\,\mathbf{e}_r$ is called a *central force*. Show that the angular momentum J ($mr^2 d\theta/dt$ in this case) is conserved.

SOLUTION: Newton's equations, $m\mathbf{a} = \mathbf{F}$, read

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = f(r)$$

and

$$m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = 0$$

But this equation can be written as

$$\frac{1}{r}\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = 0$$

which gives $J = mr^2 d\theta/dt =$ constant.

So far we have shown that $d\mathbf{e}_r/d\theta = \mathbf{e}_\theta$ and that $d\mathbf{e}_\theta/d\theta = -\mathbf{e}_r$. Figure 8.13 shows pictorially that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are independent of r because varying r simply moves both unit vectors along r without changing their directions. Therefore, we have that $d\mathbf{e}_r/dr = \mathbf{0}$ and $d\mathbf{e}_\theta/dr = \mathbf{0}$. These results are summarized in Table 8.1 at the end of this section.

We can use plane polar coordinates to introduce the idea of a *scale factor*, or a *metric coefficient*. If we start with $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, then you can see that $\partial\mathbf{r}/\partial r$ lies along $\mathbf{e}_r$ (actually, it's even equal to $\mathbf{e}_r$, but it's not necessary for what follows) and $\partial\mathbf{r}/\partial\theta$ lies along $\mathbf{e}_\theta$. We define the proportionality factors h_r and h_θ, called scale factors or metric coefficients, by

$$h_r\,\mathbf{e}_r = \frac{\partial\mathbf{r}}{\partial r} \qquad \text{and} \qquad h_\theta\,\mathbf{e}_\theta = \frac{\partial\mathbf{r}}{\partial\theta} \tag{13}$$

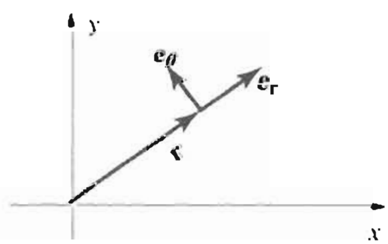

Figure 8.13
A pictorial argument that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are independent of r.

so that $h_r = |\partial \mathbf{r}/\partial r|$ and $h_\theta = |\partial \mathbf{r}/\partial \theta|$. We can determine h_r by differentiating $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ with respect to r and then taking its absolute value.

$$h_r = \left[\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2\right]^{1/2} = (\cos^2\theta + \sin^2\theta)^{1/2} = 1 \tag{14}$$

Similarly, differentiating $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ with respect to θ gives

$$h_\theta = \left[\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2\right]^{1/2} = (r^2\sin^2\theta + r^2\cos^2\theta)^{1/2} = r \tag{15}$$

Note that Equation 13 allows us to write

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial r}dr + \frac{\partial \mathbf{r}}{\partial \theta}d\theta = h_r dr\,\mathbf{e}_r + h_\theta d\theta\,\mathbf{e}_\theta \tag{16}$$

and

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = h_r^2 dr^2 + h_\theta^2 d\theta^2 \tag{17}$$

which in the case of plane polar coordinates gives $ds^2 = dr^2 + r^2 d\theta^2$.

We're now ready to derive expressions for the gradient, divergence, and the Laplacian operator in polar coordinates (remember that the curl is essentially a three-dimensional quantity). Start with

$$\text{grad } f = \nabla f = (\nabla f)_r\,\mathbf{e}_r + (\nabla f)_\theta\,\mathbf{e}_\theta \tag{18}$$

which defines the components of grad f in polar coordinates. Substitute this expression for ∇f into

$$df = \nabla f \cdot d\mathbf{r} \tag{19}$$

with $d\mathbf{r}$ given by (see Equation 16)

$$\begin{aligned} d\mathbf{r} &= h_r dr\,\mathbf{e}_r + h_\theta d\theta\,\mathbf{e}_\theta \\ &= \mathbf{e}_r dr + r d\theta\,\mathbf{e}_\theta \end{aligned} \tag{20}$$

to obtain

$$\begin{aligned} df &= [\,(\nabla f)_r\mathbf{e}_r + (\nabla f)_\theta\mathbf{e}_\theta] \cdot [\,\mathbf{e}_r\,dr + r\,d\theta\,\mathbf{e}_\theta] \\ &= (\nabla f)_r\mathbf{e}_r \cdot dr\,\mathbf{e}_r + (\nabla f)_\theta\,\mathbf{e}_\theta \cdot r d\theta\,\mathbf{e}_\theta \\ &= (\nabla f)_r dr + (\nabla f)_\theta r d\theta \end{aligned} \tag{21}$$

If we compare Equation 21 with

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta$$

we see that

$$(\nabla f)_r = \frac{\partial f}{\partial r} \qquad \text{and} \qquad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

or that

$$\nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta \tag{22}$$

or

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} \tag{23}$$

in operator notation.

Notice also that we can express Equation 22 in terms of h_r and h_θ by writing

$$\nabla = \frac{\mathbf{e}_r}{h_r}\frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{h_\theta}\frac{\partial}{\partial \theta} \tag{24}$$

We'll see in Section 5 that this is a general result. Equation 22 can also be derived in other ways (Problem 12).

Example 2:
Evaluate ∇f where $f(r, \theta) = r^2 \sin\theta$.

SOLUTION: We simply use Equation 22:

$$\begin{aligned}\nabla f &= \mathbf{e}_r \frac{\partial f}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial f}{\partial \theta} \\ &= 2r\sin\theta\,\mathbf{e}_r + r\cos\theta\,\mathbf{e}_\theta\end{aligned}$$

We can find an expression for div $\mathbf{u}$ in polar coordinates by writing

$$\operatorname{div}\mathbf{u} = \nabla\cdot\mathbf{u} = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta}\right)\cdot(u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta)$$

and using Table 8.1 for the derivatives of $\mathbf{e}_r$ and $\mathbf{e}_\theta$. The final result is

$$\operatorname{div}\mathbf{u} = \nabla\cdot\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \tag{25}$$

Example 3:
Evaluate div $\mathbf{u}$ if $\mathbf{u} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta$.

SOLUTION: Use Equation 25:

$$\nabla\cdot\mathbf{u} = \frac{\cos\theta - \cos\theta}{r} = 0$$

Finally, let's find the expression for the Laplacian operator in polar coordinates. We'll do this by using $\nabla^2 f = \nabla \cdot \nabla f$ and the above results.

$$\nabla^2 f = \nabla \cdot \nabla f = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta}\right) \cdot \left(\mathbf{e}_r \frac{\partial f}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial f}{\partial \theta}\right)$$

$$= \mathbf{e}_r \cdot \mathbf{e}_r \frac{\partial^2 f}{\partial r^2} + \mathbf{e}_r \cdot \frac{\partial}{\partial r}\left(\frac{\mathbf{e}_\theta}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\mathbf{e}_\theta}{r} \cdot \frac{\partial}{\partial \theta}\left(\mathbf{e}_r \frac{\partial f}{\partial r}\right) + \frac{\mathbf{e}_\theta \cdot \mathbf{e}_\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

$$= \frac{\partial^2 f}{\partial r^2} + \mathbf{e}_r \cdot \left(-\frac{\mathbf{e}_\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\mathbf{e}_\theta}{r}\frac{\partial^2 f}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial f}{\partial \theta}\frac{\partial \mathbf{e}_\theta}{\partial r}\right)$$

$$+ \frac{\mathbf{e}_\theta}{r} \cdot \left(\frac{\partial^2 f}{\partial \theta \partial r}\mathbf{e}_r + \frac{\partial f}{\partial r}\frac{\partial \mathbf{e}_r}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

or

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \tag{26}$$

or, in operator form,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{27}$$

Example 4:
The equation $\nabla^2 f = 0$ is called *Laplace's equation*. Show that $f(r, \theta) = r^n \sin n\theta$, where n is any positive or negative integer ($r \neq 0$ if $n < 0$), satisfies Laplace's equation.

SOLUTION: We use Equation 27 to write

$$\begin{aligned}\nabla^2 f &= n(n-1)r^{n-2}\sin n\theta + nr^{n-2}\sin n\theta - n^2 r^{n-2}\sin \theta \\ &= 0\end{aligned}$$

Table 8.1 summarizes the results of this section.

We've spent some amount of time on plane polar coordinates because it's pedagogically useful to illustrate some general methods in two dimensions first. You'll see in the sections that follow that it will be relatively easy to derive expressions involving the ∇ operator in any coordinate system using the methods that we have developed here.

Table 8.1
Some useful formulas in plane polar coordinates.

$h_r = 1$	$h_\theta = r$	$\text{grad } f = \nabla f = \dfrac{\partial f}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\mathbf{e}_\theta$
$\dfrac{\partial \mathbf{e}_r}{\partial r} = 0$	$\dfrac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta$	$\text{div } \mathbf{u} = \nabla \cdot \mathbf{u} = \dfrac{1}{r}\dfrac{\partial (r u_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}$
$\dfrac{\partial \mathbf{e}_\theta}{\partial r} = 0$	$\dfrac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r$	$\nabla^2 f = \dfrac{\partial^2 f}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial f}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial \theta^2}$

8.2 Problems

1. Show that $d\mathbf{e}_\theta/d\theta = -\mathbf{e}_r$.
2. Determine $d\mathbf{u}/dt$ if $\mathbf{u} = \sin\theta\, \mathbf{e}_r - r^2\theta\, \mathbf{e}_\theta$ and $r = t^2$ and $\theta = 2t$.
3. Show that $h_r = 1$ and $h_\theta = r$.
4. Newton's equations in rectangular coordinates for a particle moving in a plane under the influence of a coulombic potential are $m\dfrac{d^2x}{dt^2} = -\dfrac{kx}{(x^2+y^2)^{3/2}}$ and $m\dfrac{d^2y}{dt^2} = -\dfrac{ky}{(x^2+y^2)^{3/2}}$. Express these equations in polar coordinates.
5. Derive the equations in Problem 4 starting with a coulombic potential.
6. The trajectory of a particle in the xy-plane is described by $x = t$, $y = t^2$. Derive an equation for its velocity in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$.
7. Evaluate grad f if $f(r,\theta) = r^2 - a^2\cos\theta$ where a is a constant.
8. Evaluate div $\mathbf{u}$ if $\mathbf{u} = r\cos\theta\, \mathbf{e}_r - r\sin\theta\, \mathbf{e}_\theta$.
9. Show that $f(r) = \ln r$ $(r \neq 0)$ satisfies Laplace's equation.
10. Show that $\nabla \cdot \mathbf{u}$ given by Equation 25 satisfies the equation
$$\nabla \cdot \mathbf{u} = \frac{1}{h_r h_\theta}\left[\frac{\partial}{\partial r}(h_\theta u_r) + \frac{\partial}{\partial \theta}(h_r u_\theta)\right].$$
11. Show that $\nabla^2 f$ given by Equation 26 satisfies the equation
$$\nabla^2 f = \frac{1}{h_r h_\theta}\left[\frac{\partial}{\partial r}\left(\frac{h_\theta}{h_r}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{h_r}{h_\theta}\frac{\partial f}{\partial \theta}\right)\right].$$
12. We can derive Equation 22 in a more pedestrian way than we did in using Equations 18 and 19. First write $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j}$ and then show that
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} \text{ and}$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}.$$
Figure 8.12 shows that $\mathbf{e}_r = \cos\, \mathbf{i} + \sin\theta\, \mathbf{j}$ and $\mathbf{e}_\theta = \cos(\frac{\pi}{2}+\theta)\,\mathbf{i} + \sin(\frac{\pi}{2}+\theta)\,\mathbf{j} = -\sin\theta\, \mathbf{i} + \cos\theta\, \mathbf{j}$ (note that $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$). Solve these two equations for $\mathbf{i}$ and $\mathbf{j}$ and substitute the result and the above equations into $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j}$ to get Equation 22.

13. This problem asks you to convert $\nabla^2 f$ from rectangular coordinates to polar coordinates with r fixed by repeated application of the chain rule of partial differentiation. Start with

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{1}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{2}$$

and show that

$$\frac{\partial f}{\partial x} = -\frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} \qquad \frac{\partial f}{\partial y} = \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} \quad (r \text{ fixed})$$

Now apply equations 1 and 2 again to show that

$$\frac{\partial^2 f}{\partial x^2} = \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} \quad (r \text{ fixed})$$

Similarly, show that

$$\frac{\partial^2 f}{\partial y^2} = -\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} \quad (r \text{ fixed})$$

and so

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \quad (r \text{ fixed})$$

14. Generalize Problem 13 to the case in which r is not fixed.

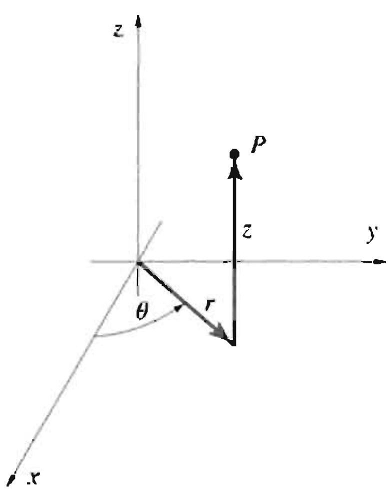

Figure 8.14
A point specified by the cylindrical coordinates r, θ, and z.

8.3 Cylindrical Coordinates

A direct extension of polar coordinates to three dimensions is to use r and θ to locate the projection of a point onto the xy-plane and z to locate the vertical distance of the point from the xy-plane (Figure 8.14). The three coordinates, r, θ, and z, are called *cylindrical coordinates*. A surface given by r = constant is a right circular cylinder; a surface given by θ = constant is a half-plane with its edge being the z axis; and a surface given by z = constant is a plane perpendicular to the z axis (Figure 8.15). Each of these three surfaces is perpendicular to the other two, and so they are mutually orthogonal.

The curve formed by the intersection of a θ = constant surface and a z = constant surface is called an *r curve*. The curve formed by the intersection of a r = constant surface and a z = constant surface is called a *θ curve*; and the curve formed by the intersection of a r = constant surface and the θ = constant surface is called a *z curve* (Figure 8.16).

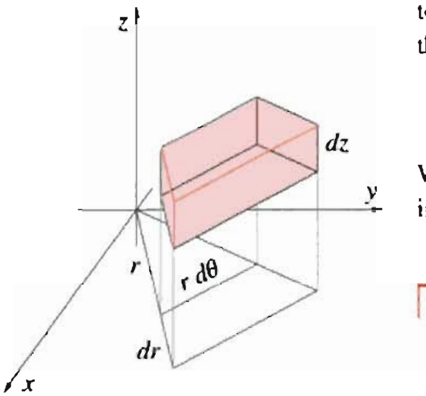

Figure 8.18
A pictorial argument that the volume element in cylindrical coordinates is given by $dV = r dr d\theta dz$.

to Figure 8.18, which shows such a volume element. You can see from this figure that

$$dV = r dr\, d\theta\, dz \tag{4}$$

We'll discuss a general analytical method to find the appropriate volume elements in Section 5, but we'll just use geometric arguments for now.

Example 3:
Evaluate the integral:

$$I = \iiint xyz\, dxdydz$$

over the region $x \geq 0$, $y \geq 0$, $0 \leq z < b$, and $x^2 + y^2 \leq a^2$.

SOLUTION: The conditions $x \geq 0$, $y \geq 0$ imply that $0 \leq \theta < \pi/2$, so

$$\begin{aligned} I &= \iiint (r^2 \cos\theta \sin\theta) zr\, drd\theta dz \\ &= \int_0^b dz\, z \int_0^{\pi/2} d\theta\, \cos\theta \sin\theta \int_0^a dr\, r^3 \\ &= \frac{b^2}{2} \cdot \frac{1}{2} \cdot \frac{a^4}{4} = \frac{a^4 b^2}{16} \end{aligned}$$

Notice that the triple integral in Example 3 factored into a product of three single integrals because the limits are constants rather than involving the other coordinates. This will often happen if you employ the "right" or the "natural" coordinate system for the problem. Problem 4 asks you to evaluate the above integral using rectangular coordinates and you'll see how some of the limits involve the other coordinates.

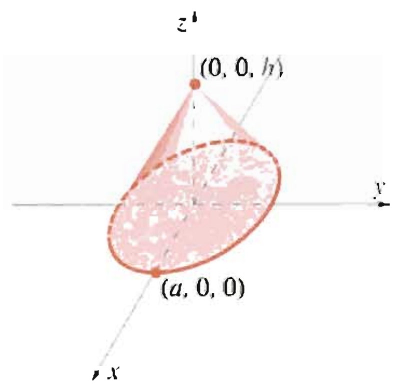

Figure 8.19
A right circular cone with base $x^2 + y^2 \leq a^2$ and apex at $(0, 0, h)$.

We'll evaluate another integral using cylindrical coordinates. Figure 8.19 shows a right circular cone with base $x^2 + y^2 \leq a^2$ and apex at $(0, 0, h)$. Let's find the volume of this cone. The upper limit of the z integration depends upon the value of $r = (x^2 + y^2)^{1/2}$. The surface of the cone is given by the equation $z = h - \dfrac{h}{a} r$, and so

$$V = \int_0^{2\pi} d\theta \int_0^a dr\, r \int_0^{h - \frac{h}{a} r} dz = \frac{1}{3}\pi a^2 h$$

The limits on z involve r here, but the integral is *much* easier to evaluate in cylindrical coordinates than in rectangular coordinates (Problem 5).

Figure 8.15 shows the surfaces $r = \text{constant}$, $\theta = \text{constant}$, and $z = \text{constant}$ for cylindrical coordinates. The surfaces are mutually orthogonal at any point of intersection, and cylindrical coordinates are another example of an orthogonal coordinate system. We can prove analytically that a cylindrical coordinate system is orthogonal, and in the process introduce the unit vectors in this system. The position vector of any point is

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + z\,\mathbf{k} \tag{5}$$

The vector $\partial\mathbf{r}/\partial r$ is tangent to the r curve in Figure 8.16. Similarly, $\partial\mathbf{r}/\partial\theta$ is tangent to the θ curve and $\partial\mathbf{r}/\partial z$ is tangent to the z curve. Thus, $\partial\mathbf{r}/\partial r$, $\partial\mathbf{r}/\partial\theta$, and $\partial\mathbf{r}/\partial z$ form a set of mutually orthogonal vectors. They aren't necessarily unit vectors, but if we divide each one by its length, then we have

$$\mathbf{e}_r = \frac{\partial\mathbf{r}/\partial r}{|\partial\mathbf{r}/\partial r|}, \qquad \mathbf{e}_\theta = \frac{\partial\mathbf{r}/\partial\theta}{|\partial\mathbf{r}/\partial\theta|}, \qquad \mathbf{e}_z = \frac{\partial\mathbf{r}/\partial z}{|\partial\mathbf{r}/\partial z|} \tag{6}$$

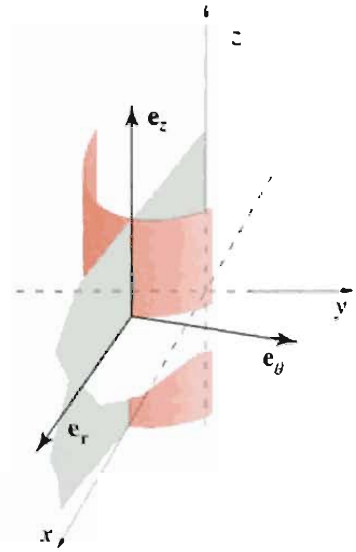

Figure 8.20
The unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ of a cylindrical coordinate system. Note that they form a right-handed coordinate system.

Substituting Equation 5 into Equations 6 shows that $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ are indeed mutually orthogonal unit vectors (Figure 8.20) (Problem 6). It's easy to see from Equation 5 that $h_r = 1$, $h_\theta = r$, and $h_z = |\partial\mathbf{r}/\partial z| = 1$. The metric coefficients for a cylindrical coordinate system are given in Table 8.2 at the end of the section. Note that ds^2 (Equation 2) can be written as $ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_z^2 dz^2$ (Problem 7).

The difference between the operator ∇ in plane polar coordinates and cylindrical coordinates is just the $\mathbf{k}\,\partial/\partial z$ term, and so we can write

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z \frac{\partial}{\partial z} \tag{7}$$

in cylindrical coordinates. (In this case, $\mathbf{e}_z = \mathbf{k}$.)

Example 4:
Evaluate grad f if $f(r, \theta, z) = z\cos\theta/r$.

SOLUTION:

$$\text{grad } f = \nabla f = -\frac{z\cos\theta}{r^2}\,\mathbf{e}_r - \frac{z\sin\theta}{r^2}\,\mathbf{e}_\theta + \frac{\cos\theta}{r}\,\mathbf{e}_z$$

We can obtain expressions for div $\mathbf{u}$ and $\nabla^2 f$ in cylindrical coordinates by the same method that we used in the previous section. The only new factor is that $\mathbf{e}_z$ is a constant vector, so that $\partial\mathbf{e}_z/\partial r = \partial\mathbf{e}_z/\partial\theta = \partial\mathbf{e}_z/\partial z = \mathbf{0}$. Furthermore, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ do not vary with z, so $\partial\mathbf{e}_r/\partial z = \partial\mathbf{e}_\theta/\partial z = \mathbf{0}$. You can see this pictorially from Figure 8.20 or analytically (Problem 8). The results for div $\mathbf{u}$ and $\nabla^2 f$ are (Problems 9 and 10)

$$\operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \tag{8}$$

and

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{9}$$

We didn't consider curl $\mathbf{u} = \nabla \times \mathbf{u}$ in the previous section because the curl is essentially a three-dimensional operation. In this case,

$$\begin{aligned}
\nabla \times \mathbf{u} &= \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \times (u_r\, \mathbf{e}_r + u_\theta\, \mathbf{e}_\theta + u_z\, \mathbf{e}_z) \\
&= \mathbf{e}_r \times \left(\frac{\partial u_\theta}{\partial r}\, \mathbf{e}_\theta + \frac{\partial u_z}{\partial r}\, \mathbf{e}_z\right) \\
&\quad + \frac{\mathbf{e}_\theta}{r} \times \left(\frac{\partial u_r}{\partial \theta}\, \mathbf{e}_r - u_\theta\, \mathbf{e}_r + \frac{\partial u_z}{\partial \theta}\, \mathbf{e}_z\right) \\
&\quad + \mathbf{e}_z \times \left(\frac{\partial u_r}{\partial z}\, \mathbf{e}_r + \frac{\partial u_\theta}{\partial z}\, \mathbf{e}_\theta\right)
\end{aligned} \tag{10}$$

Equation 10 is an intermediate result. To arrive at Equation 10, we have used the known expressions for the derivatives of the unit vectors with respect to the coordinates and $\mathbf{e}_r \times \mathbf{e}_r = \mathbf{e}_\theta \times \mathbf{e}_\theta = \mathbf{e}_z \times \mathbf{e}_z = \mathbf{0}$ (Problem 11). Referring to Figure 8.20, we see that (Problem 12)

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_z; \qquad \mathbf{e}_z \times \mathbf{e}_r = \mathbf{e}_\theta; \qquad \mathbf{e}_\theta \times \mathbf{e}_z = \mathbf{e}_r \tag{11}$$

Using Equations 11 in Equation 10 gives (Problem 13)

$$\nabla \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) \mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right) \mathbf{e}_z \tag{12}$$

Example 5:
Evaluate div $\mathbf{u}$ and curl $\mathbf{u}$ if $\mathbf{u}(r, \theta, z) = z^2\, \mathbf{e}_r + r^2 \sin\theta\, \mathbf{e}_\theta + zr\, \mathbf{e}_z$.

SOLUTION: Equation 8 gives

$$\operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u} = \frac{z^2}{r} + r\cos\theta + r$$

and Equation 12 gives

$$\operatorname{curl} \mathbf{u} = \nabla \times \mathbf{u} = z\, \mathbf{e}_\theta + 3r\sin\theta\, \mathbf{e}_z$$

Problems 20 and 21 ask you to use a CAS to evaluate the gradient, divergence, and curl in cylindrical coordinates.

Let's finish this section with a warning regarding the curl in cylindrical coordinates. Although

$$\text{curl}\,\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} \tag{13}$$

in rectangular coordinates, there is no obvious corresponding determinantal expression in curvilinear coordinates. The correct formula for cylindrical coordinates is

$$\text{curl}\,\mathbf{u} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ u_r & ru_\theta & u_z \end{vmatrix} \tag{14}$$

which can be obtained by working Equation 12 backwards (Problem 17).

The key formulas of this section are summarized in Table 8.2.

Table 8.2
Some useful formulas in cylindrical coordinates.

$$h_r = 1 \qquad h_\theta = r \qquad h_z = 1$$

$$\frac{\partial \mathbf{e}_r}{\partial r} = \mathbf{0} \qquad \frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \qquad \frac{\partial \mathbf{e}_r}{\partial z} = \mathbf{0}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0} \qquad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \qquad \frac{\partial \mathbf{e}_\theta}{\partial z} = \mathbf{0}$$

$$\frac{\partial \mathbf{e}_z}{\partial r} = \mathbf{0} \qquad \frac{\partial \mathbf{e}_z}{\partial \theta} = \mathbf{0} \qquad \frac{\partial \mathbf{e}_z}{\partial z} = \mathbf{0}$$

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{e}_z$$

$$\text{div}\,\mathbf{u} = \nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\mathbf{e}_z$$

$$= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ u_r & ru_\theta & u_z \end{vmatrix}$$

8.3 Problems

1. Describe the graph that each of the following equations represents in cylindrical coordinates:

   (a) $r = a$ (b) $\theta = \dfrac{\pi}{2}$

   (c) $r = \text{constant}$ (d) $z = r^2$

2. Use Equations 1 to derive Equation 2.
3. Determine the arc length of a space curve given by $r = at$, $\theta = bt$, $z = ct$, from $t = 0$ to $t = 1$.
4. Evaluate the integral in Example 3 using rectangular coordinates.
5. Use rectangular coordinates to calculate the volume of a right circular cone with base $x^2 + y^2 \le a^2$ and height h. (Use a CAS to evaluate the integral.)
6. Substitute Equation 5 into Equation 6 to show that $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ constitute a set of mutually orthogonal unit vectors.
7. Show that $ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_z^2 dz^2$, where $h_r^2 = \left(\dfrac{\partial x}{\partial r}\right)^2 + \left(\dfrac{\partial y}{\partial r}\right)^2 + \left(\dfrac{\partial z}{\partial r}\right)^2 = 1$, $h_\theta^2 = \left(\dfrac{\partial x}{\partial \theta}\right)^2 + \left(\dfrac{\partial y}{\partial \theta}\right)^2 + \left(\dfrac{\partial z}{\partial \theta}\right)^2 = r^2$, and $h_z^2 = \left(\dfrac{\partial x}{\partial z}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + \left(\dfrac{\partial z}{\partial z}\right)^2 = 1$.
8. Use the definition $\mathbf{e}_z = \dfrac{1}{h_z}\dfrac{\partial \mathbf{r}}{\partial z}$ to show that $\mathbf{e}_z$ is a constant vector.
9. Show that div $\mathbf{u}$ is given by Equation 8.
10. Show that $\nabla^2 f$ is given by Equation 9.
11. Derive the intermediate result given by Equation 10.
12. Show that $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_z$, $\mathbf{e}_z \times \mathbf{e}_r = \mathbf{e}_\theta$, and that $\mathbf{e}_\theta \times \mathbf{e}_z = \mathbf{e}_r$.
13. Finish the derivation of Equation 12.
14. Show that $f(r, z) = (r^2 + z^2)^{-1/2}$ $(r^2 + z^2 \neq 0)$ is a solution to Laplace's equation.
15. Express the velocity of the space curve $[r(t), \theta(t), z(t)]$ in terms of $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$.
16. Here's a fairly easy way to express grad f in cylindrical coordinates. Write $df = \dfrac{\partial f}{\partial r}dr + \dfrac{\partial f}{\partial \theta}d\theta + \dfrac{\partial f}{\partial z}dz$. Now use the fact that $df = (\nabla f) \cdot d\mathbf{r}$ to determine the components of ∇f.
17. Show that Equations 12 and 14 are equivalent.
18. Express the vector $\mathbf{u} = 2z\,\mathbf{i} - x\,\mathbf{j} + y\,\mathbf{k}$ in cylindrical coordinates.
19. Determine $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$ if $\mathbf{u} = \mathbf{e}_r + \cos\theta\,\mathbf{e}_z$.
20. Use a CAS to evaluate ∇f if $f(r, \theta, z) = (r^2 - z^2)e^{rz}\cos^2\theta$.
21. Use a CAS to evaluate div $\mathbf{u}$ and curl $\mathbf{u}$ if $\mathbf{u}(r, \theta, z) = r^2\cos\theta\,\mathbf{e}_r - rz^2\sin^2\theta\,\mathbf{e}_\theta + e^z\,\mathbf{e}_k$.

$$V = \int_0^a r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{4\pi a^3}{3}$$

Notice that the triple integral for V factors into three separate single integrals.

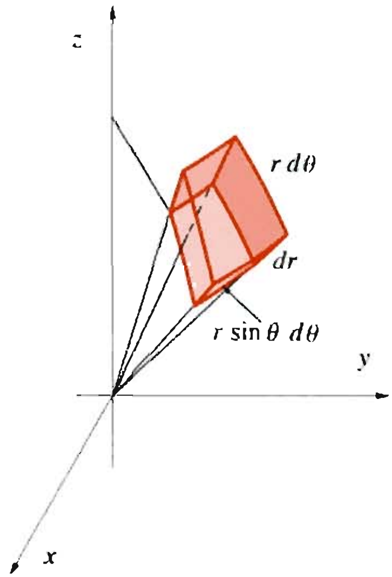

Figure 8.26
A pictorial argument that the volume element in spherical coordinates is given by $dV = r^2 \sin\theta\, dr d\theta d\phi$.

Example 2:
Evaluate

$$I = \iiint_V z^2 dx dy dz$$

over the spherical region of radius a centered at the origin.

SOLUTION:

$$I = \int_0^a r^4 dr \int_o^{\pi} \cos^2\theta \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{a^5}{5} \cdot \frac{2}{3} \cdot 2\pi = \frac{4\pi a^5}{15}$$

The evaluation of this integral would have been quite a bit messier in rectangular coordinates (Problem 4).

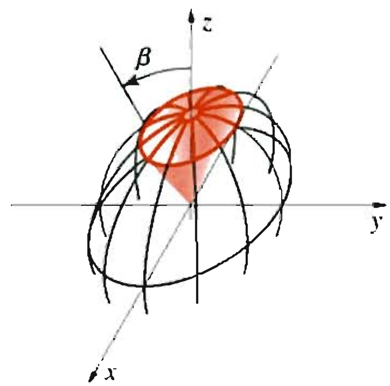

Figure 8.27
The volume to be determined in Example 3.

Example 3:
Determine the volume of the part of a unit sphere centered at the origin that lies within the right circular cone with its apex at the origin and making an angle β with the positive z axis (Figure 8.27).

SOLUTION:

$$V = \int_0^1 r^2 dr \int_0^{\beta} \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{2\pi}{3}(1 - \cos\beta)$$

Note that $V = 0$ if $\beta = 0$, and $V = 4\pi/3$ if $\beta = \pi$.

Figure 8.26 can also be used to convince yourself that an element of surface area on the surface of a sphere of radius a centered at the origin is

$$dS = a^2 \sin\theta\, d\theta d\phi \tag{5}$$

Equation 5 tells us that the area of the polar cap with $\theta \le \beta$ is

$$S = a^2 \int_0^{\beta} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 2\pi a^2(1 - \cos\beta)$$

If $\beta = \pi$, we obtain the surface area of a sphere ($4\pi a^2$).

$$ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_\phi^2 d\phi^2$$
$$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

as in Equation 2.

We can derive a formula for the gradient in spherical coordinates by letting $\nabla f = (\nabla f)_r \mathbf{e}_r + (\nabla f)_\theta \mathbf{e}_\theta + (\nabla f)_\phi \mathbf{e}_\phi$ and comparing the following two expressions for df:

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi$$

and

$$df = \nabla f \cdot d\mathbf{r} = (\nabla f)_r \mathbf{e}_r \cdot dr\mathbf{e}_r + (\nabla f)_\theta \mathbf{e}_\theta \cdot r d\theta\, \mathbf{e}_\theta + (\nabla f)_\phi \mathbf{e}_\phi \cdot r \sin\theta d\phi\, \mathbf{e}_\phi$$
$$= (\nabla f)_r dr + (\nabla f)_\theta r d\theta + (\nabla f)_\phi r \sin\theta d\phi$$

where we have used Equation 11 for $d\mathbf{r}$. Comparing the two expressions for df, we see that

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \qquad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}; \qquad (\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Thus,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \tag{12}$$

or

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} + \frac{\mathbf{e}_\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{13}$$

in operator form.

Example 4:
Evaluate grad f if $f(r, \theta, \phi) = r^2 \sin\phi$.

SOLUTION: Using Equation 12, we obtain

$$\text{grad } f = \nabla f = 2r\sin\phi\, \mathbf{e}_r + \frac{r\cos\phi}{\sin\theta}\mathbf{e}_\phi$$

To derive expressions for the divergence and curl, we have to see how the unit vectors, $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ vary with r, θ, and ϕ. We'll do this by first expressing $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. According to Equations 1,

$$\mathbf{e}_r = \sin\theta\cos\phi\, \mathbf{i} + \sin\theta\sin\phi\, \mathbf{j} + \cos\theta\, \mathbf{k} \tag{14}$$

Figure 8.28 shows that $\mathbf{e}_\theta = \partial \mathbf{e}_r/\partial\theta$, in which case we have

$$\mathbf{e}_\theta = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k} \tag{15}$$

Notice that $\mathbf{e}_\theta$ can also be obtained by adding $\pi/2$ to θ (Problem 11). Figure 8.25 shows that $\mathbf{e}_\phi = \mathbf{e}_r \times \mathbf{e}_\theta$, and so we find that

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \tag{16}$$

None of these unit vectors depends upon r, so we have immediately that

$$\frac{\partial \mathbf{e}_r}{\partial r} = \frac{\partial \mathbf{e}_\theta}{\partial r} = \frac{\partial \mathbf{e}_\phi}{\partial r} = \mathbf{0} \tag{17}$$

You can also see this result from Figure 8.28; varying r simply moves the entire set of unit vectors along r without changing their directions. We'll leave the details to Problem 13 just to save space, but it is straightforward (and fairly brief) to show that

$$\begin{aligned}
\frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta & \frac{\partial \mathbf{e}_r}{\partial \phi} &= \sin\theta\,\mathbf{e}_\phi \\
\frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -\mathbf{e}_r & \frac{\partial \mathbf{e}_\theta}{\partial \phi} &= \cos\theta\,\mathbf{e}_\phi \\
\frac{\partial \mathbf{e}_\phi}{\partial \theta} &= \mathbf{0} & \frac{\partial \mathbf{e}_\phi}{\partial \phi} &= -\sin\theta\,\mathbf{e}_r - \cos\theta\,\mathbf{e}_\theta
\end{aligned} \tag{18}$$

These results are summarized in Table 8.3 at the end of the section.

We're now ready to find the spherical coordinate expressions for div $\mathbf{u}$.

$$\begin{aligned}
\operatorname{div}\mathbf{u} = \nabla\cdot\mathbf{u} &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\mathbf{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot(u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta + u_\phi\,\mathbf{e}_\phi) \\
&= \mathbf{e}_r\cdot\left(\frac{\partial u_r}{\partial r}\mathbf{e}_r + u_r\frac{\partial \mathbf{e}_r}{\partial r} + \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta + u_\theta\frac{\partial \mathbf{e}_\theta}{\partial r} + \frac{\partial u_\phi}{\partial r}\mathbf{e}_\phi + u_\phi\frac{\partial \mathbf{e}_\phi}{\partial r}\right) \\
&\quad + \frac{\mathbf{e}_\theta}{r}\cdot\left(\frac{\partial u_r}{\partial \theta}\mathbf{e}_r + u_r\frac{\partial \mathbf{e}_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \theta}\mathbf{e}_\theta + u_\theta\frac{\partial \mathbf{e}_\theta}{\partial \theta} + \frac{\partial u_\phi}{\partial \theta}\mathbf{e}_\phi + u_\phi\frac{\partial \mathbf{e}_\phi}{\partial \theta}\right) \\
&\quad + \frac{\mathbf{e}_\phi}{r\sin\theta}\cdot\left(\frac{\partial u_r}{\partial \phi}\mathbf{e}_r + u_r\frac{\partial \mathbf{e}_r}{\partial \phi} + \frac{\partial u_\theta}{\partial \phi}\mathbf{e}_\theta + u_\theta\frac{\partial \mathbf{e}_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \phi}\mathbf{e}_\phi + u_\phi\frac{\partial \mathbf{e}_\phi}{\partial \phi}\right)
\end{aligned}$$

Use Equations 17 and 18 to get

$$\begin{aligned}
\operatorname{div}\mathbf{u} &= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} + \frac{u_\theta\cos\theta}{r\sin\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} \\
&= \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}
\end{aligned} \tag{19}$$

To find ∇^2, substitute $u_r = \partial f/\partial r$, $u_\theta = (1/r)(\partial f/\partial \theta)$, and $u_\phi = (1/r\sin\theta)\partial f\partial\phi$ from Equation 12 into Equation 19:

$$\nabla^2 f = \text{div grad } f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{20}$$

Example 5:
Show that $f(r, \theta, \phi) = r\sin\theta\cos\phi$ satisfies Laplace's equation.

SOLUTION:

$$\begin{aligned}
\nabla^2 f &= \frac{2\sin\theta\cos\phi}{r} + \frac{r\cos\phi(\cos^2\theta - \sin^2\theta)}{r^2\sin\theta} - \frac{r\sin\theta\cos\phi}{r^2\sin^2\theta} \\
&= \frac{\cos\phi}{r}\left(2\sin\theta + \frac{\cos^2\theta - \sin^2\theta}{\sin\theta} - \frac{\sin\theta}{\sin^2\theta}\right) \\
&= \frac{\sin\theta\cos\phi}{r\sin^2\theta}(2\sin^2\theta + \cos^2\theta - \sin^2\theta - 1) = 0
\end{aligned}$$

We'll leave the derivation of curl **u** to the problems; the result is (Problem 15)

$$\begin{aligned}
\text{curl } \mathbf{u} = \nabla \times \mathbf{u} = &\ \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, u_\phi) - \frac{\partial u_\theta}{\partial\phi}\right]\mathbf{e}_r \\
&+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial(ru_\phi)}{\partial r}\right]\mathbf{e}_\theta \\
&+ \frac{1}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right]\mathbf{e}_\phi
\end{aligned} \tag{21}$$

or

$$\text{curl } \mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ u_r & ru_\theta & r\sin\theta u_\phi \end{vmatrix} \tag{22}$$

Example 6:
Evaluate div **u** and curl **u** for $\mathbf{u} = r\,\mathbf{e}_r$.

SOLUTION: Equation 19 gives

$$\text{div } \mathbf{u} = \nabla \cdot \mathbf{u} = 3$$

and Equation 21 or 22 gives

$$\text{curl } \mathbf{u} = \nabla \times \mathbf{u} = \mathbf{0}$$

Table 8.3
Some useful formulas in spherical coordinates.

$h_r = 1$	$h_\theta = r$	$h_\phi = r \sin \phi$
$\dfrac{\partial \mathbf{e}_r}{\partial r} = \mathbf{0}$	$\dfrac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta$	$\dfrac{\partial \mathbf{e}_r}{\partial \phi} = \sin\theta\, \mathbf{e}_\phi$
$\dfrac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0}$	$\dfrac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r$	$\dfrac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos\theta\, \mathbf{e}_\phi$
$\dfrac{\partial \mathbf{e}_\phi}{\partial r} = \mathbf{0}$	$\dfrac{\partial \mathbf{e}_\phi}{\partial \theta} = \mathbf{0}$	$\dfrac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin\theta\, \mathbf{e}_r - \cos\theta\, \mathbf{e}_\theta$

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$

$$\text{div } \mathbf{u} = \nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$\text{curl } \mathbf{u} = \nabla \times \mathbf{u} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, u_\phi) - \frac{\partial u_\theta}{\partial\phi}\right]\mathbf{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial(r u_\phi)}{\partial r}\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right]\mathbf{e}_\phi$$

Problems 22 and 23 ask you to use a CAS to evaluate the gradient, divergence, and curl in spherical coordinates.

We summarize the key formulas of this section in Table 8.3.

8.4 Problems

1. What is the relation between geographical latitude (α) and longitude (β) and the angles θ and ϕ?
2. Use Equation 2 to calculate the arc length of the equator of a sphere of radius a.
3. Show that $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$ in spherical coordinates.
4. Evaluate the integral in Example 2 using cartesian coordinates.
5. Evaluate the integral $I = \iiint x^2\, dxdydz$ over a sphere of radius a centered at the origin and compare your result to the one obtained in Example 2. Why are the answers the same?
6. Evaluate the integral $I = \iiint r^2\, dxdydz$ over a sphere of radius a centered at the origin and compare your result to those obtained in Example 2 and Problem 5. Why is the answer to this problem three times the answer to Problem 5 and Example 2?
7. Use spherical coordinates to evaluate the integral $I = \iiint e^{-2r}\cos^2\theta\, dV$ over all space.

8. Consider a sphere of unit radius so that $dS = \sin\theta\, d\theta d\phi$. We call the region connecting the origin with dS a *solid angle*, $d\Omega$ (see Figure 8.29). We express this by writing $d\Omega = \sin\theta\, d\theta d\phi$. Notice that a complete solid angle $\Omega = 4\pi$ (called 4π *steradians*) just as a complete circle $= 2\pi$ radians. Evaluate $\iint\limits_S \sin^2\theta \cos^2\theta\, d\Omega$ over the surface of a sphere of unit radius.

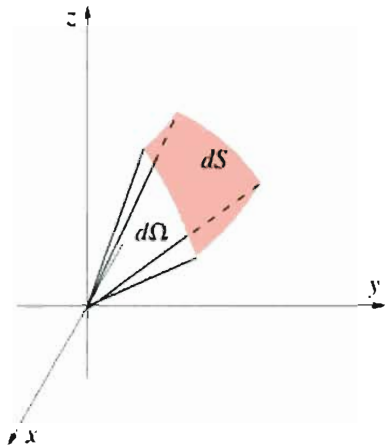

Figure 8.29
An illustration of a solid angle $d\Omega$.

9. Starting with $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$, show that $\partial\mathbf{r}/\partial r$, $\partial\mathbf{r}/\partial\theta$, and $\partial\mathbf{r}/\partial\phi$ are mutually orthogonal.

10. Show that $h_\theta = r$ and $h_\phi = r\sin\theta$.

11. Show that $\mathbf{e}_\theta$ can be obtained from $\mathbf{e}_r$ by adding $\pi/2$ to θ.

12. Evaluate grad f if $f = r\cos\phi$.

13. Derive Equations 18 by starting with Equations 14 through 16.

14. Show that $f(r, \theta, \phi) = r^2\sin^2\theta\sin 2\phi$ satisfies Laplace's equation.

15. Derive Equation 21.

16. Show that Equation 22 is equivalent to Equation 21.

17. Use the metric coefficients $h_r = 1$, $h_\theta = r$, and $h_\phi = r\sin\theta$ and the entry in Table 8.3 to show that

$$\text{div}\,\mathbf{u} = \nabla\cdot\mathbf{u} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3u_1) + \frac{\partial}{\partial u_2}(h_3h_1u_2) + \frac{\partial}{\partial u_3}(h_1h_2u_3)\right]$$

if $u_1 = r$, $u_2 = \theta$, and $u_3 = \phi$.

18. Use the metric coefficients $h_r = 1$, $h_\theta = r$, and $h_\phi = r\sin\theta$ and the entry in Table 8.3 to show that

$$\text{curl}\,\mathbf{u} = \nabla\times\mathbf{u} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1u_1 & h_2u_2 & h_3u_3 \end{vmatrix}$$

if $u_1 = r$, $u_2 = \theta$, and $u_3 = \phi$.

19. An integral that occurs in a number of applications (and one that we will encounter in Chapter 17) is $F(k) = \iiint f(r)e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}$. Show that $F(k) = \iiint_{-\infty}^{\infty} f(r)\frac{r\sin kr}{k}d\mathbf{r}$ by letting $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$ and using a spherical coordinate system with $\mathbf{k}$ along the "z axis." The notation $d\mathbf{r}$ means $r^2\sin\theta dr d\theta d\phi$.

20. Show that the great circle distance between two points of latitude and longitude (α_1, β_2) and (α_2, β_2) is given by $d = R\cos^{-1}[\cos(\beta_1 - \beta_2)\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2]$.

21. Evaluate the integral $\iiint(ax + by + cz)^{2n}dxdydz$ over the region $x^2 + y^2 + z^2 \le R^2$. *Hint*: Recognize that $ax + by + cz$ is $\mathbf{r}\cdot\mathbf{v}$ where $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and take the "z axis" to be directed along $\mathbf{v}$.

22. Use any CAS to find grad f if $f = e^{-r^2}\cos^2\theta\sin\phi$.

23. Use any CAS to find div $\mathbf{u}$ and curl $\mathbf{u}$ if $\mathbf{u} = e^{-r^2}\cos\theta\,\mathbf{e}_r - r^2\sin\phi\,\mathbf{e}_\theta + \cos\theta\cos\phi\,\mathbf{e}_\phi$.

8.5 Curvilinear Coordinates

In the previous sections of this chapter, we deduced the form of the differential volume elements in cylindrical coordinates and spherical coordinates geometrically with the aid of Figures 8.18 and 8.26. We found that $dA = rdrd\theta dz$ in cylindrical coordinates and that $dA = r^2\sin\theta drd\theta d\phi$ in spherical coordinates. We can actually derive these relations analytically using vector methods. We can handle both cylindrical coordinates and spherical coordinates simultaneously by considering three general orthogonal coordinates, u_1, u_2, and u_3, where $u_1 = r$, $u_2 = \theta$, and $u_3 = z$ in the case of cylindrical coordinates and $u_1 = r$, $u_2 = \theta$, and $u_3 = \phi$ in the case of spherical coordinates. We say that u_1, u_2, and u_3 are orthogonal coordinates if the surfaces $u_1 =$ constant, $u_2 =$ constant, and $u_3 =$ constant intersect at right angles. Let

$$x = x(u_1, u_2, u_3); \qquad y = y(u_1, u_2, u_3); \qquad z = z(u_1, u_2, u_3) \tag{1}$$

be the equations that relate the x, y, z coordinates to the u_1, u_2, u_3 coordinates. (Compare these to Equations 1 of the previous two sections.) If the functions $x(u_1, u_2, u_3)$, $y(u_1, u_2, u_3)$, and $z(u_1, u_2, u_3)$ are continuous, have continuous first partial derivatives, and have single-valued inverses, then Equations 1 represent a one-to-one correspondence between points in the x, y, z coordinate system and the u_1, u_2, u_3 coordinate system. The position vector $\mathbf{r}$ in either coordinate system is given by

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = x(u_1, u_2, u_3)\,\mathbf{i} + y(u_1, u_2, u_3)\,\mathbf{j} + z(u_1, u_2, u_3)\,\mathbf{k} \tag{2}$$

Now consider a differential volume element in the xyz coordinate system, where $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$. We learned in Section 5.3 that the volume of a parallelepiped formed by three non-coplanar vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is given by

$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$, so that the volume element in the xyz coordinate system is given by $dV = |dx\,\mathbf{i} \cdot (dy\,\mathbf{j} \times dz\,\mathbf{k})| = |\mathbf{i} \cdot (\mathbf{j} \times \mathbf{k})|\,dxdydz$. But $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually orthogonal unit vectors, so $dV = dxdydz$. Now consider a differential volume element in spherical coordinates, where $d\mathbf{r} = dr\,\mathbf{e}_r + rd\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathbf{e}_\phi$. Recall that $\partial\mathbf{r}/\partial r$ is tangent to the r curve and that $\partial\mathbf{r}/\partial\theta$ is tangent to the θ curve, and $\partial\mathbf{r}/\partial\phi$ is tangent to the ϕ curve. The differential volume element in spherical coordinates is given by $dV = |dr\,\mathbf{e}_r \cdot (rd\theta\,\mathbf{e}_\theta \times r\sin\theta\,\mathbf{e}_\phi)| = r^2\sin\theta\,drd\theta d\phi|\,\mathbf{e}_r \cdot (\mathbf{e}_\theta \times \mathbf{e}_\phi)|$. But $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ are orthogonal unit vectors, so $|\mathbf{e}_r \cdot (\mathbf{e}_\theta \times \mathbf{e}_\phi)| = 1$ and $dV = r^2\sin\theta\,drd\theta d\phi$.

Now let's consider the corresponding differential volume element in terms of u_1, u_2, u_3, which we will call *curvilinear coordinates*. The vector $\partial\mathbf{r}/\partial u_1$ will be tangent to the u_1 curve, which is the curve that is formed by the intersection of the $u_2 =$ constant and $u_3 =$ constant surfaces (Figure 8.30). Similarly, $\partial\mathbf{r}/\partial u_2$ is tangent to the u_2 curve and $\partial\mathbf{r}/\partial u_3$ is tangent to the u_3 curve. Thus, the three vectors $\partial\mathbf{r}/\partial u_1$, $\partial\mathbf{r}/\partial u_2$, and $\partial\mathbf{r}/\partial u_3$ are mutually orthogonal (Figure 8.31). They are not necessarily unit vectors, but we can form unit vectors by writing

$$\mathbf{e}_1 = \frac{\partial\mathbf{r}/\partial u_1}{|\partial\mathbf{r}/\partial u_1|}; \qquad \mathbf{e}_2 = \frac{\partial\mathbf{r}/\partial u_2}{|\partial\mathbf{r}/\partial u_2|}; \qquad \mathbf{e}_3 = \frac{\partial\mathbf{r}/\partial u_3}{|\partial\mathbf{r}/\partial u_3|} \tag{3}$$

or

$$\frac{\partial\mathbf{r}}{\partial u_1} = h_1\,\mathbf{e}_1; \qquad \frac{\partial\mathbf{r}}{\partial u_2} = h_2\,\mathbf{e}_2; \qquad \frac{\partial\mathbf{r}}{\partial u_3} = h_3\,\mathbf{e}_3 \tag{4}$$

where $h_j = |\partial\mathbf{r}/\partial u_j|$ for $j = 1, 2, 3$. Using Equation 3, we can write

$$d\mathbf{r} = h_1\,du_1\,\mathbf{e}_1 + h_2\,du_2\,\mathbf{e}_2 + h_3\,du_3\,\mathbf{e}_3 \tag{5}$$

Example 1:
Show that

$$h_j = \left[\left(\frac{\partial x}{\partial u_j}\right)^2 + \left(\frac{\partial y}{\partial u_j}\right)^2 + \left(\frac{\partial z}{\partial u_j}\right)^2\right]^{1/2}$$

SOLUTION:

$$h_j = \left|\frac{\partial\mathbf{r}}{\partial u_j}\right| = \left|\frac{\partial x}{\partial u_j}\,\mathbf{i} + \frac{\partial y}{\partial u_j}\,\mathbf{j} + \frac{\partial z}{\partial u_j}\,\mathbf{k}\right|$$

$$= \left[\left(\frac{\partial x}{\partial u_j}\right)^2 + \left(\frac{\partial y}{\partial u_j}\right)^2 + \left(\frac{\partial z}{\partial u_j}\right)^2\right]^{1/2}$$

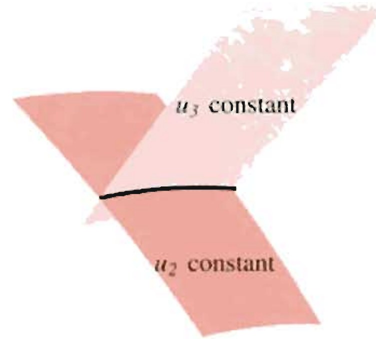

Figure 8.30
The intersection of two coordinate surfaces produces a coordinate curve.

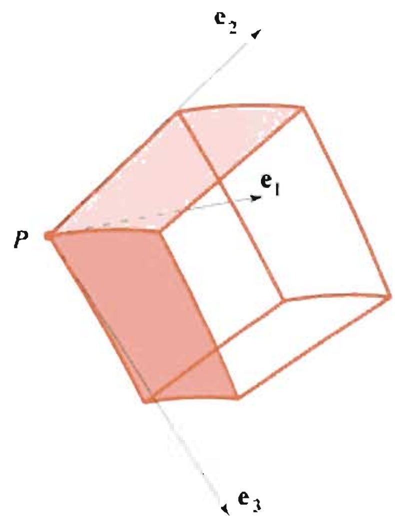

Figure 8.31
The coordinate surfaces of an orthogonal coordinate system are mutually perpendicular at any point of intersection.

The differential volume element in the u_1, u_2, u_3 coordinate system is given by

$$dV = |h_1\,du_1\,\mathbf{e}_1 \cdot (h_2\,du_2\,\mathbf{e}_2 \times h_3\,du_3\,\mathbf{e}_3)| = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)|h_1 h_2 h_3\,du_1\,du_2\,du_3 \quad (6)$$

But $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are mutually orthogonal unit vectors, so $|\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)| = 1$ and Equation 6 becomes

$$dV = h_1 h_2 h_3\,du_1\,du_2\,du_3 \quad (7)$$

which is our desired result.

Example 2:
Show that Equation 7 is consistent with our earlier results $dV = r\,drd\theta d\phi$ for cylindrical coordinates and $dV = r^2 \sin\theta\,drd\theta d\phi$ for spherical coordinates.

SOLUTION: Recall that $h_r = 1$, $h_\theta = r$, and $h_z = 1$ for cylindrical coordinates and that $h_r = 1$, $h_\theta = r$, and $h_\phi = r\sin\theta$ for spherical coordinates. Therefore,

$$h_1 h_2 h_3\,du_1\,du_2\,du_3 = r\,drd\theta dz$$

and

$$h_1 h_2 h_3\,du_1\,du_2\,du_3 = r^2 \sin\theta\,drd\theta d\phi$$

for cylindrical coordinates and spherical coordinates, respectively.

Equation 7 is often written in the form

$$dV = \left|\frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3}\right| du_1\,du_2\,du_3 \quad (8)$$

Recall from Section 5.3 that the expression in brackets in Equation 8 (called a *scalar vector product*) can be written in determinantal form (Problem 7)

$$\frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} = \begin{vmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} & \dfrac{\partial z}{\partial u_1} \\ \dfrac{\partial x}{\partial u_2} & \dfrac{\partial y}{\partial u_2} & \dfrac{\partial z}{\partial u_2} \\ \dfrac{\partial x}{\partial u_3} & \dfrac{\partial y}{\partial u_3} & \dfrac{\partial z}{\partial u_3} \end{vmatrix} \quad (9)$$

(We haven't discussed determinants "officially" so far (see Chapter 9), but we assume that you remember how to expand a 3×3 determinant. That's all you'll need to know here.)

The type of determinant in Equation 9 occurs fairly often and is called a *Jacobian determinant*, or usually just a *Jacobian*. In particular, the Jacobian in Equation 9 is said to be the Jacobian of x, y, z with respect to u_1, u_2, u_3 and is written as $\partial(x, y, z)/\partial(u_1, u_2, u_3)$ so that Equation 8 becomes

$$dV = \left| \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \right| du_1\, du_2\, du_3 \tag{10}$$

Example 3:
Evaluate the Jacobian for polar coordinates $x = r \cos\theta$ and $y = r \sin\theta$.

SOLUTION: We want to evaluate $\partial(x, y)/\partial(r, \theta)$.

$$\begin{aligned} \frac{\partial(x, y)}{\partial(r, \theta)} &= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \\ &= r(\cos^2\theta + \sin^2\theta) = r \end{aligned}$$

Therefore, $dV = r\, dr d\theta$ in polar coordinates.

We can also derive expressions for differential area elements in curvilinear coordinates. Let's consider the area element involving the u_2 and u_3 coordinates on the $u_1 = \text{constant}$ surface. For example, u_2 and u_3 would be θ and ϕ on the surface of a sphere of fixed radius in spherical coordinates. Figure 8.31 shows that

$$\begin{aligned} dA_{23} &= |(h_2\, du_2\, \mathbf{e}_2) \times (h_3\, du_3\, \mathbf{e}_3)| \\ &= h_2 h_3 |\, \mathbf{e}_2 \times \mathbf{e}_3\, |\, du_2\, du_3 = h_2 h_3\, du_2\, du_3 \end{aligned} \tag{11}$$

since $\mathbf{e}_2$ and $\mathbf{e}_3$ are mutually orthogonal unit vectors. The area elements on other surfaces are similar to Equation 11.

Example 4:
Use Equation 11 to derive an expression for the differential area element on the surface of a sphere.

SOLUTION: According to Equation 11, we use

$$dA = h_\theta h_\phi\, d\theta d\phi = r^2 \sin\theta\, d\theta d\phi$$

in agreement with Equation 5 in the previous section.

We've derived equations for arc length of a space curve in terms of the $\mathbf{e}_j$ and h_j in previous sections. To do this in general, start with

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 = h_1 \mathbf{e}_1\, du_1 + h_2 \mathbf{e}_2\, du_2 + h_3 \mathbf{e}_3\, du_3 \quad (12)$$

and form

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2 \quad (13)$$

If u_1, u_2, and u_3 depend parametrically on some variable t, then the arc length is given by

$$l = \int_{t_1}^{t_2} \left[h_1^2 \left(\frac{du_1}{dt} \right)^2 + h_2^2 \left(\frac{du_2}{dt} \right)^2 + h_3^2 \left(\frac{du_3}{dt} \right)^2 \right]^{1/2} dt \quad (14)$$

We can derive an expression for the gradient in curvilinear coordinates by recalling that

$$df = \text{grad } f \cdot d\mathbf{r} = \nabla f \cdot d\mathbf{r} \quad (15)$$

If $f = f(u_1, u_2, u_3)$, then

$$df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3 \quad (16)$$

and using Equation 12 for $d\mathbf{r}$ gives

$$\begin{aligned} df &= \nabla f \cdot d\mathbf{r} \\ &= [(\nabla f)_{u_1} \mathbf{e}_1 + (\nabla f)_{u_2} \mathbf{e}_2 + (\nabla f)_{u_3} \mathbf{e}_3] \cdot (h_1 \mathbf{e}_1\, du_1 + h_2 \mathbf{e}_2\, du_2 + h_3 \mathbf{e}_3\, du_3) \\ &= (\nabla f)_{u_1} h_1\, du_1 + (\nabla f)_{u_2} h_2\, du_2 + (\nabla f)_{u_3} h_3\, du_3 \quad (17) \end{aligned}$$

Comparing Equations 16 and 17, we see that

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \mathbf{e}_3 \quad (18)$$

or

$$\nabla = \frac{1}{h_1} \mathbf{e}_1 \frac{\partial}{\partial u_1} + \frac{1}{h_2} \mathbf{e}_2 \frac{\partial}{\partial u_2} + \frac{1}{h_3} \mathbf{e}_3 \frac{\partial}{\partial u_3} \quad (19)$$

in operator form.

Example 5:
Show how Equation 18 is consistent with our previous results in cylindrical and spherical coordinates.

SOLUTION: For cylindrical coordinates, $u_1 = r$, $u_2 = \theta$, and $u_3 = z$, and $h_r = 1$, $h_\theta = r$, and $h_z = 1$, so Equation 18 gives

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_k$$

Similarly, $u_1 = r$, $u_2 = \theta$, and $u_3 = \phi$, and $h_r = 1$, $h_\theta = r$, and $h_\phi = r\sin\theta$, so

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$

We can derive an expression for the divergence by evaluating $\nabla \cdot \mathbf{v}$ using Equation 19. When you do this, you must remember that the unit vectors $\mathbf{e}_j$ are *not* fixed in space, and so may depend upon u_1, u_2, and u_3. We've evaluated the derivatives $\partial\mathbf{e}_j/\partial u_k$ for cylindrical and spherical coordinates in previous sections, but Problem 17 helps you show that

$$\frac{\partial \mathbf{e}_i}{\partial u_i} = -\sum_{k\neq i}\frac{1}{h_k}\frac{\partial h_i}{\partial u_k}\mathbf{e}_k \qquad \begin{matrix} i = 1, 2, 3 \\ k = 1, 2, 3 \end{matrix} \tag{20}$$

and

$$\frac{\partial \mathbf{e}_i}{\partial u_j} = \frac{\mathbf{e}_j}{h_i}\frac{\partial h_j}{\partial u_i} \qquad i \neq j \qquad \begin{matrix} i = 1, 2, 3 \\ j = 1, 2, 3 \end{matrix} \tag{21}$$

in general. Thus, the various derivatives of the $\mathbf{e}_j$ depend upon the metric coefficients (as does everything else). The following Example shows how to use Equations 20 and 21.

Example 6:
Use Equations 20 and 21 to derive the results in Table 8.2.

SOLUTION: First note that $h_r = 1$, $h_\theta = r$, and $h_z = 1$. From Equation 20

$$\frac{\partial \mathbf{e}_r}{\partial r} = -\frac{1}{h_\theta}\frac{\partial h_r}{\partial \theta}\mathbf{e}_\theta - \frac{1}{h_z}\frac{\partial h_r}{\partial z}\mathbf{e}_z$$
$$= 0$$
$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\frac{1}{h_r}\frac{\partial h_\theta}{\partial r}\mathbf{e}_r - \frac{1}{h_z}\frac{\partial h_\theta}{\partial z}\mathbf{e}_z$$
$$= -\mathbf{e}_r$$
$$\frac{\partial \mathbf{e}_z}{\partial z} = -\frac{1}{h_r}\frac{\partial h_z}{\partial r}\mathbf{e}_r - \frac{1}{h_\theta}\frac{\partial h_z}{\partial \theta}\mathbf{e}_\theta$$
$$= 0$$

Using Equation 21,

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \frac{\mathbf{e}_\theta}{h_r}\frac{\partial h_\theta}{\partial r} = \mathbf{e}_\theta$$

$$\frac{\partial \mathbf{e}_r}{\partial z} = \frac{\mathbf{e}_z}{h_r}\frac{\partial h_z}{\partial r}\,\mathbf{e}_z = \mathbf{0}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial r} = \frac{\mathbf{e}_r}{h_\theta}\frac{\partial h_r}{\partial \theta} = \mathbf{0}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial z} = \frac{\mathbf{e}_z}{h_\theta}\frac{\partial h_z}{\partial \theta} = \mathbf{0}$$

$$\frac{\partial \mathbf{e}_z}{\partial r} = \frac{\mathbf{e}_r}{h_z}\frac{\partial h_r}{\partial z} = \mathbf{0}$$

$$\frac{\partial \mathbf{e}_z}{\partial \theta} = \frac{\mathbf{e}_\theta}{h_z}\frac{\partial h_\theta}{\partial z} = \mathbf{0}$$

Problem 12 asks you to do this for spherical coordinates.

Using Equations 20 and 21 along with div $\mathbf{v} = \nabla \cdot \mathbf{v}$ gives (Problem 10)

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 v_1) + \frac{\partial}{\partial u_2}(h_1 h_3 v_2) + \frac{\partial}{\partial u_3}(h_1 h_2 v_3)\right] \tag{22}$$

We can also derive Equation 22 by starting with the definition

$$\operatorname{div} \mathbf{v} = \lim_{\Delta V \to 0} \frac{\oint \mathbf{v} \cdot \mathbf{n}\, dS}{\Delta V} \tag{23}$$

and considering the flux in and out of the volume ΔV (Problem 11). Similarly, it turns out that (Problem 13)

$$\begin{aligned}
\operatorname{curl} \mathbf{v} &= \nabla \times \mathbf{v} \\
&= \frac{1}{h_1 h_2 h_3}\left\{\mathbf{e}_1 h_1\left[\frac{\partial(h_3 v_3)}{\partial u_2} - \frac{\partial(h_2 v_2)}{\partial u_3}\right] + \mathbf{e}_2 h_2\left[\frac{\partial(h_1 v_1)}{\partial u_3} - \frac{\partial(h_3 v_3)}{\partial u_1}\right]\right. \\
&\qquad \left. + \ \mathbf{e}_3 h_3\left[\frac{\partial(h_2 v_2)}{\partial u_1} - \frac{\partial(h_1 v_1)}{\partial u_2}\right]\right\} \\
&= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}
\end{aligned} \tag{24}$$

Finally, we can derive an expression for the Laplacian operator in curvilinear coordinates by using $\nabla^2 f = \operatorname{div} \operatorname{grad} f = \nabla \cdot \nabla f$ and Equations 18 and 22 (Problem 14):

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right] \tag{25}$$

Example 7:
Use Equation 25 to derive Equation 20 of Section 4.

SOLUTION: Recall that $h_r = 1$, $h_\theta = r$, and $h_\phi = r\sin\theta$.

$$\nabla^2 f = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\phi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

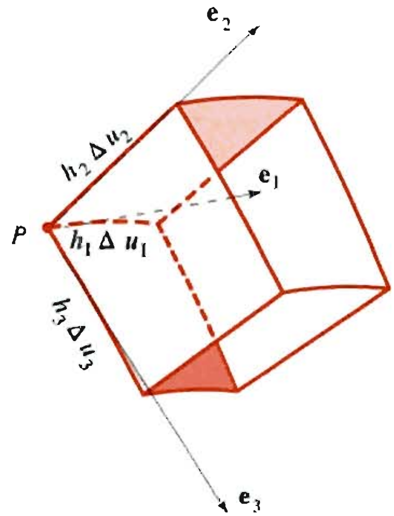

Figure 8.32
The volume formed by a set of intersecting orthogonal coordinate surfaces and used to help derive an expression for div $\mathbf{v}$ in curvilinear coordinates in Problem 11.

8.5 Problems

1. Describe the coordinate surfaces and the coordinate curves in cylindrical coordinates.
2. Describe the coordinate surfaces and the coordinate curves in spherical coordinates.
3. Show that cylindrical coordinates form an orthogonal coordinate system by showing that the $\partial\mathbf{r}/\partial u_j$ are orthogonal.
4. Show that spherical coordinates form an orthogonal coordinate system by showing that the $\partial\mathbf{r}/\partial u_j$ are orthogonal.
5. Express the vector $\mathbf{v} = xy\,\mathbf{i} + z\,\mathbf{k}$ in a cylindrical coordinate system.
6. Show that $\mathbf{i}\cdot(\mathbf{j}\times\mathbf{k}) = 1$.
7. Show that $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$ can be expressed in terms of a determinant.
8. Evaluate $\partial(x, y, z)/\partial(r, \theta, z)$ for cylindrical coordinates.
9. Evaluate $\partial(x, y, z)/\partial(r, \theta, \phi)$ for spherical coordinates.
10. Derive Equation 22 using $\nabla\cdot\mathbf{v}$ and Equations 20 and 21.
11. We can derive an expression for the divergence in curvilinear coordinates by starting with the definition div $\mathbf{v} = \lim\limits_{V\to 0}\dfrac{\int_S \mathbf{v}\cdot\mathbf{n}\,dS}{V}$, where the surface S encloses the volume V. Let V be formed by a set of intersecting orthogonal coordinate surfaces, as in Figure 8.32, and calculate the flow of $\mathbf{v}$ through all the surfaces to derive an expression for div $\mathbf{v}$.
12. Use Equations 20 and 21 to verify the corresponding entries in Table 8.3.

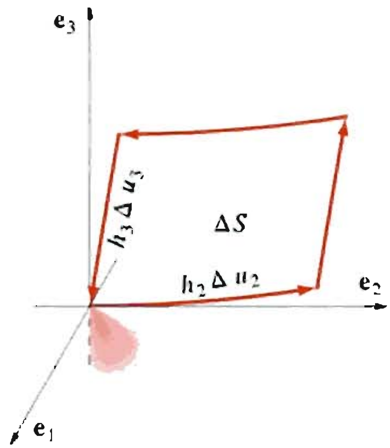

Figure 8.33
The u_1 = constant curvilinear surface that is used to derive an expression for curl $\mathbf{v}$ in curvilinear coordinates in Problem 13.

13. We can derive an expression for the curl in curvilinear coordinates by starting with the definition curl $\mathbf{v} = \lim\limits_{S\to 0} \dfrac{\oint \mathbf{v}\cdot d\mathbf{s}}{S}$, where the curve s surrounds the area S. Consider the $u_1 = c_1$ coordinate surface in Figure 8.33. Evaluate $\oint \mathbf{v}\cdot d\mathbf{s}$ around the path in Figure 8.33 to derive an expression for curl $\mathbf{v}$.

14. Derive Equation 25 using $\nabla^2 f = \text{div grad } f = \nabla\cdot\nabla f$ and Equations 18 and 22.

15. Use Equation 11 to calculate the surface area of a unit sphere.

16. Use Equation 11 to calculate the total surface area of a right circular cone of slant height s with a base of radius R.

17. This problem helps you derive Equations 20 and 21. Start with Equations 4 and use the fact that $\partial^2\mathbf{r}/\partial u_i\partial u_j = \partial^2\mathbf{r}/\partial u_j\partial u_i$ to get

$$\mathbf{e}_j\frac{\partial h_j}{\partial u_i} + h_j\frac{\partial \mathbf{e}_j}{\partial u_i} = \mathbf{e}_i\frac{\partial h_i}{\partial u_j} + h_i\frac{\partial \mathbf{e}_i}{\partial u_j} \tag{1}$$

Let $\mathbf{E}_{ij} = \partial\mathbf{e}_i/\partial u_j$ so that (1) can be written as

$$\mathbf{e}_j\frac{\partial h_j}{\partial u_i} + h_j\,\mathbf{E}_{ji} = \mathbf{e}_i\frac{\partial h_i}{\partial u_j} + h_i\,\mathbf{E}_{ij} \tag{2}$$

with no restrictions on i, j, or k. Now differentiate $\mathbf{e}_i\cdot\mathbf{e}_k = \delta_{ik}$ (Kronecker delta) and show that

$$\mathbf{E}_{ij}\cdot\mathbf{e}_k = -\mathbf{E}_{kj}\cdot\mathbf{e}_i \tag{3}$$

with no restrictions on i, j, or k. Use (3) to show that

$$\mathbf{E}_{ij}\cdot\mathbf{e}_i = 0 \tag{4}$$

Convince yourself that (4) represents nine equations. Operate on (2) with $\mathbf{e}_j$ $(j \neq i)$ and use (4) to get

$$\mathbf{E}_{ij}\cdot\mathbf{e}_j = \frac{1}{h_i}\frac{\partial h_j}{\partial u_i} \qquad (j\neq i) \tag{5}$$

Similarly, operate on (2) with $\mathbf{e}_k$ ($k \neq i, \neq j$) to get

$$h_i \mathbf{E}_{ij} \cdot \mathbf{e}_k = h_j \mathbf{E}_{ji} \cdot \mathbf{e}_k \qquad (k \neq i, \neq j) \tag{6}$$

Now substitute (3) into (6) to get

$$h_j \mathbf{E}_{ji} \cdot \mathbf{e}_k = -h_i \mathbf{E}_{kj} \cdot \mathbf{e}_i \quad i \neq j \neq k \tag{7}$$

Let $i, j, k = (1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$ and show that $\mathbf{E}_{ij} \cdot \mathbf{e}_k = 0$ for $i \neq j \neq k$. Using these results, show that

$$\begin{aligned} \frac{\partial \mathbf{e}_1}{\partial u_2} &= \mathbf{E}_{12} = (\mathbf{E}_{12} \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{E}_{12} \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\mathbf{E}_{12} \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\ &= \frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\,\mathbf{e}_2 \end{aligned}$$

and $\dfrac{\partial \mathbf{e}_1}{\partial u_3} = \dfrac{1}{h_1}\dfrac{\partial h_3}{\partial u_1}\,\mathbf{e}_3$ and $\dfrac{\partial \mathbf{e}_2}{\partial u_1} = \dfrac{1}{h_2}\dfrac{\partial h_1}{\partial u_2}\,\mathbf{e}_1$, or generally

$$\frac{\partial \mathbf{e}_i}{\partial u_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial u_i}\,\mathbf{e}_j \qquad i \neq j \tag{8}$$

Now show that

$$\begin{aligned} \frac{\partial \mathbf{e}_1}{\partial u_1} &= (\mathbf{E}_{11} \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{E}_{11} \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\mathbf{E}_{11} \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\ &= -(\mathbf{E}_{21} \cdot \mathbf{e}_1)\,\mathbf{e}_2 - (\mathbf{E}_{31} \cdot \mathbf{e}_1)\,\mathbf{e}_3 \\ &= -\frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\,\mathbf{e}_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial u_3}\,\mathbf{e}_3 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \mathbf{e}_2}{\partial u_2} &= -\frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\,\mathbf{e}_1 - \frac{1}{h_3}\frac{\partial h_2}{\partial u_3}\,\mathbf{e}_3 \\ \frac{\partial \mathbf{e}_3}{\partial u_3} &= -\frac{1}{h_1}\frac{\partial h_3}{\partial u_1}\,\mathbf{e}_1 - \frac{1}{h_2}\frac{\partial h_3}{\partial u_2}\,\mathbf{e}_2 \end{aligned}$$

8.6 Some Other Coordinate Systems

The only coordinate systems that we have discussed so far are plane polar coordinates, cylindrical coordinates, and spherical coordinates. If a physical system has a certain degree of symmetry, then choosing the "right" coordinate system to describe the system can greatly simplify the problem. For example, spherical coordinates are natural coordinates in dealing with systems that have a center of symmetry. There are a number of well studied coordinate systems, each of which is appropriate for problems with certain symmetry. In this section, we shall briefly introduce a few other coordinate systems to give an example of what's available.

A prolate spheroid is obtained by rotating the ellipse shown in Figure 8.34 about its long axis. Suppose we want to determine the electrostatic potential or the electrostatic field about an isolated metallic prolate spheroid whose surface is at a potential V. The natural coordinate to use in this case is a *prolate spheroidal coordinate system*. We shall also see that this same coordinate system can be used to calculate the properties of a molecular hydrogen ion, H_2^+, which consists of two (massive) nuclei separated by a distance R with one electron interacting with them. Figure 8.35 shows the geometry for this system.

We can generate a prolate spheroidal coordinate system by rotating a family of ellipses and hyperbolas having the same foci (confocal) about the major axis of the ellipse, as shown in Figure 8.36. The distance between the two foci is $2a$ and η, θ, and ϕ are the prolate spheroidal coordinates. The relations between x, y, z and η, θ, ϕ are

$$\begin{aligned} x &= a \sinh \eta \sin \theta \cos \phi \\ y &= a \sinh \eta \sin \theta \sin \phi \\ z &= a \cosh \eta \cos \theta \end{aligned} \tag{1}$$

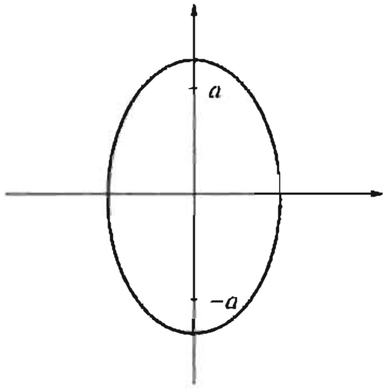

Figure 8.34
A prolate spheroid is obtained by rotating the ellipse in the figure about its long axis.

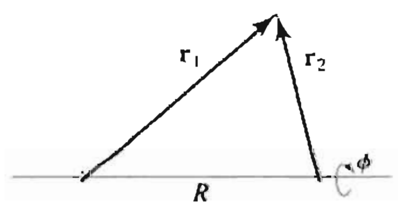

Figure 8.35
The geometry associated with a hydrogen molecular ion, H_2^+. The two nuclei are separated by a distance $R = 2a$ and the electron is located at the point (x, y, z).

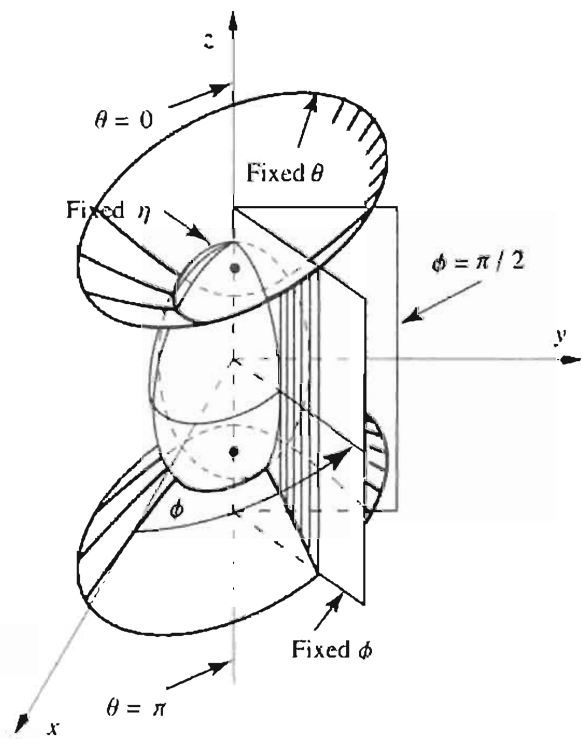

Figure 8.36
A prolate spheroidal coordinate system. The coordinate surfaces are prolate spheroids given by η = constant, hyperboloids of two sheets given by θ = constant, and planes containing the z axis given by ϕ = constant.

Let's look at each prolate spheroidal coordinate surface in turn. The surfaces $\eta =$ constant are prolate spheroids

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad c > b$$

where $b = a \sinh \eta$ and $c = a \cosh \eta$, and η varies from 0 to ∞. The surfaces $\theta =$ constant are hyperboloids of two sheets. The angle θ is the angle between the asymptotic cone of a hyperboloid and the z axis as shown in Figure 8.36, and varies from 0 to π: $0 \le \theta < \pi/2$ describes the upper hyperboloid in Figure 8.36 and $\pi/2 < \theta \le \pi$ describes the lower hyperboloid. When $\theta = \pi/2$, the two hyperboloids degenerate into the xy-plane. The angle ϕ simply represents the angle about the z axis and varies from 0 to 2π. The $\phi =$ constant surfaces are half planes containing the z axis. Equations 2 summarize the full ranges of the three coordinates:

$$0 \le \eta < \infty; \qquad 0 \le \theta \le \pi; \qquad 0 \le \phi \le 2\pi \tag{2}$$

As with any new coordinate system, it takes a little practice and experience to become comfortable with it.

Example 1:
Show that prolate spheroidal coordinates are an orthogonal coordinate system.

SOLUTION: Write $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ in terms of η, θ, ϕ, using Equations 1.

$$\mathbf{r}(\eta, \theta, \phi) = a \sinh \eta \sin \theta \cos \phi \,\mathbf{i} + a \sinh \eta \sin \theta \sin \phi \,\mathbf{j} + a \cosh \eta \cos \theta \,\mathbf{k}$$

Then,

$$\frac{\partial \mathbf{r}}{\partial \eta} = a \cosh \eta \sin \theta \cos \phi \,\mathbf{i} + a \cosh \eta \sin \theta \sin \phi \,\mathbf{j} + a \sinh \eta \cos \theta \,\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = a \sinh \eta \cos \theta \cos \phi \,\mathbf{i} + a \sinh \eta \cos \theta \sin \phi \,\mathbf{j} - a \cosh \eta \sin \theta \,\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = -a \sinh \eta \sin \theta \sin \phi \,\mathbf{i} + a \sinh \eta \sin \theta \cos \phi \,\mathbf{j} + 0\,\mathbf{k}$$

and

$$\frac{\partial \mathbf{r}}{\partial \eta} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = \frac{\partial \mathbf{r}}{\partial \eta} \cdot \frac{\partial \mathbf{r}}{\partial \phi} = \frac{\partial \mathbf{r}}{\partial \theta} \cdot \frac{\partial \mathbf{r}}{\partial \phi} = 0$$

We can use Equations 1 to evaluate the scale factors h_η, h_θ, and h_ϕ. For example,

$$\begin{aligned}
h_\eta &= \left[\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2\right]^{1/2} \\
&= (a^2\cosh^2\eta\sin^2\theta\cos^2\phi + a^2\cosh^2\eta\sin^2\theta\sin^2\phi + a^2\sinh^2\eta\cos^2\theta)^{1/2} \\
&= (a^2\cosh^2\eta\sin^2\theta + a^2\sinh^2\eta\cos^2\theta)^{1/2} \\
&= a\,[(1+\sinh^2\eta)\sin^2\theta + \sinh^2\eta(1-\sin^2\theta)]^{1/2} \\
&= a(\sinh^2\eta + \sin^2\theta)^{1/2} \qquad (3)
\end{aligned}$$

Similarly (Problem 1),

$$h_\theta = h_\eta \qquad \text{and} \qquad h_\phi = a\sinh\sin\theta \qquad (4)$$

We can use the scale factors to calculate the volume of a prolate spheroid.

Example 2:
Use prolate spheroidal coordinates to calculate the volume of a prolate spheroid.

SOLUTION:

$$\begin{aligned}
V &= \int_0^{\eta_0} d\eta \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, h_\eta h_\theta h_\phi \\
&= 2\pi a^3 \int_0^{\eta_0} d\eta\ \sinh^3\eta \int_0^{\pi} d\theta\ \sin\theta + 2\pi a^3 \int_0^{\eta_0} d\eta\ \sinh\eta \int_0^{\pi} d\theta\ \sin^3\theta \\
&= 4\pi a^3\left(\frac{\cosh^3\eta_0}{3} - \cosh\eta_0 + \frac{2}{3}\right) + \frac{8\pi a^3}{3}(\cosh\eta_0 - 1) \\
&= \frac{4\pi a^3}{3}\sinh^2\eta_0\cosh\eta_0
\end{aligned}$$

But $b = a\sinh\eta_0$ and $c = a\cosh\eta_0$, so

$$V = \frac{4\pi}{3}b^2c$$

Problem 2 asks you to calculate the surface area of a prolate spheroid in a similar way.

Laplace's equation in prolate spheroidal coordinates is (Problem 3)

$$\nabla^2 f = \frac{1}{a^2(\sinh^2\eta + \sin^2\theta)}\left(\frac{\partial^2 f}{\partial\eta^2} + \coth\eta\frac{\partial f}{\partial\eta} + \frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{a^2\sinh^2\eta\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = 0 \tag{5}$$

This equation may not look too inviting, but it reduces to a fairly simple equation under certain (realistic) conditions. Suppose we want to determine the electrostatic field about an isolated metallic prolate spheroid $\eta = \eta_0$, which is at a fixed potential V_0. (Recall that Laplace's equation governs the electrostatic potential throughout a charged-free region.) In this case, the potential will depend upon only η and so Equation 5 will become

$$\nabla^2 V = \frac{d^2V}{d\eta^2} + \coth\eta\frac{dV}{d\eta} = 0 \tag{6}$$

We haven't studied differential equations yet, but surely Equation 6 is a lot simpler than Equation 5, and this simplification will occur only in a spheroidal coordinate system. The solution to Equation 6 is

$$V(\eta) = V_0\frac{\ln\tanh(\eta/2)}{\ln\tanh(\eta_0/2)} \tag{7}$$

(Equation 6 is actually easy to solve and Problem 5 helps you solve it.) Figure 8.37 shows that the equipotential surfaces are spheroidal surfaces $\eta = c_1$.

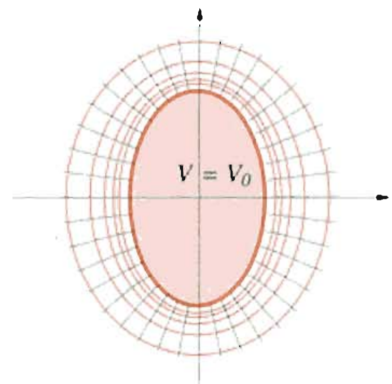

Figure 8.37
The equipotential surfaces (color) and the corresponding electric field (black) about a prolate spheroid held at a fixed potential.

There is another way of expressing prolate spheroidal coordinates that is used in molecular quantum mechanics. Consider two nuclei separated by a distance $R = 2a$ and an electron located at a point (x, y, z), as in Figure 8.35. Let r_1 and r_2 be the distance of the electron from each nucleus. Then $r_1 + r_2 =$ constant maps out a prolate spheroid and $r_1 - r_2 =$ constant maps out the hyperboloids in Figure 8.36. To see this analytically, we use

$$r_1 = [(z+a)^2 + x^2 + y^2]^{1/2}$$

and

$$r_2 = [(z-a)^2 + x^2 + y^2]^{1/2}$$

and Equations 1 to get (Problem 6)

$$\lambda = \frac{r_1 + r_2}{2a} = \cosh\eta \quad \text{and} \quad \mu = \frac{r_1 - r_2}{2a} = \cos\theta \quad \text{and} \quad \phi = \phi \tag{8}$$

The two coordinates λ and μ and ϕ, the angle about the z axis as in Figure 8.36, constitute an orthogonal coordinate system equivalent to the prolate spheroidal coordinates η, θ, ϕ (Problem 7). This coordinate system, which is sometimes called a bipolar coordinate system, is shown in Figure 8.38.

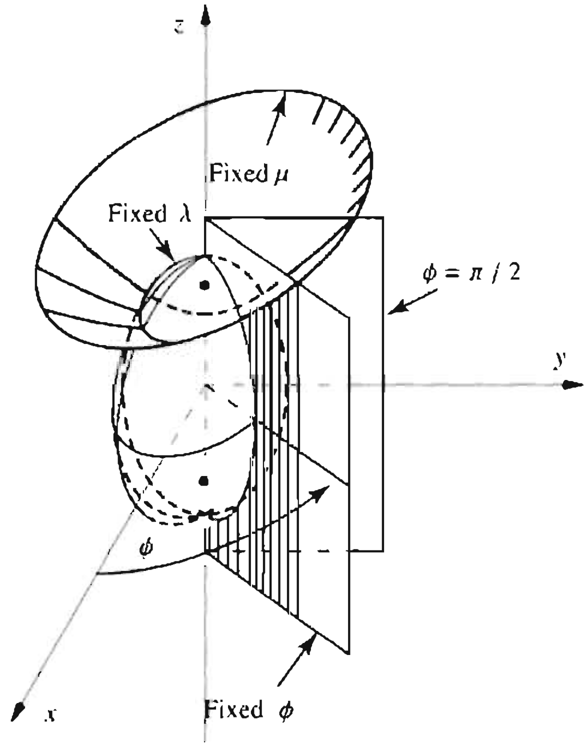

Figure 8.38
A bipolar coordinate system, where $\lambda = (r_1 + r_2)/2a$, $\mu = (r_1 - r_2)/2a$, and ϕ is the angle about the interfocal axis.

Because $\lambda = \cosh\eta$ and $0 \le \eta < \infty$, then $1 \le \lambda < \infty$ (as you can also see pictorially from Figure 8.38). Also, because $\mu = \cos\theta$, then $-1 \le \mu \le 1$. Of course, ϕ, being the same as the ϕ in Figure 8.38, varies from 0 to 2π. Thus, we have

$$1 \le \lambda < \infty \qquad -1 \le \mu \le 1 \qquad 0 \le \phi \le 2\pi \tag{9}$$

Notice that Equations 8 say that λ = constant (η = constant) surfaces are prolate spheroids and μ = constant (θ = constant) surfaces are hyperboloids. Problem 8 has you show that

$$h_\lambda = \frac{a(\lambda^2 - \mu^2)^{1/2}}{(\lambda^2 - 1)^{1/2}}; \qquad h_\mu = \frac{a(\lambda^2 - \mu^2)^{1/2}}{(1 - \mu^2)^{1/2}}; \qquad h_\phi = a(\lambda^2 - 1)^{1/2}(1 - \mu^2)^{1/2} \tag{10}$$

Example 3:
Use Equations 10 to calculate the volume of a prolate spheroid.

SOLUTION:

$$V = a^3 \int_0^{2\pi} d\phi \int_1^{\lambda_0} d\lambda \int_{-1}^{1} d\mu\, (\lambda^2 - \mu^2)$$

$$= 4\pi a^3 \int_1^{\lambda_0} \left(\lambda^2 - \frac{1}{3}\right) d\lambda = \frac{4\pi a^3}{3}\lambda_0(\lambda_0^2 - 1)$$

$$= \frac{4\pi a^3}{3} \cdot \frac{c}{a} \cdot \frac{b^2}{a^2} = \frac{4\pi c b^2}{3}$$

because $\lambda_0 = \cosh \eta_0 = c/a$ and $\lambda_0^2 - 1 = \sinh^2 \eta_0 = b^2/a^2$.

An integral that occurs in a quantum-mechanical treatment of a hydrogen molecule is

$$I = \frac{1}{\pi} \iiint\limits_{\text{all space}} e^{-r_1} e^{-r_2} dV \tag{11}$$

where r_1 and r_2 are as depicted in Figure 8.35. This integral is fairly awkward to evaluate (see Problem 10) unless we use the bipolar coordinates λ, μ, ϕ, in which case it's simple. In terms of λ, μ, and ϕ, I becomes

$$I = \frac{2\pi a^3}{\pi} \int_1^{\infty} d\lambda \int_{-1}^{1} d\mu \, e^{-2a\lambda}(\lambda^2 - \mu^2)$$

$$= 4a^3 \int_1^{\infty} d\lambda e^{-2a\lambda} \left(\lambda^2 - \frac{1}{3}\right)$$

$$= 4\left(\frac{R}{2}\right)^3 e^{-R} \left(\frac{2}{3R} + \frac{2}{R^2} + \frac{2}{R^3}\right)$$

$$= e^{-R}\left(1 + R + \frac{1}{3}R^2\right) \tag{12}$$

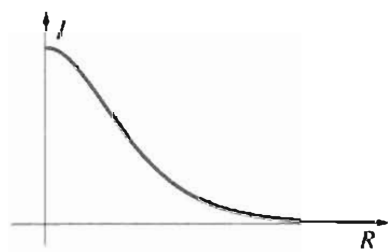

Figure 8.39
The overlap integral, Equation 12, which is a measure of the overlap of the wave function centered on one nucleus with the wave function centered on the other nucleus.

Physically, I is a measure of the overlap of the wave function centered on one nucleus with the wave function centered on the other nucleus and is called an *overlap integral* (Figure 8.39).

Another coordinate system that is similar to the one we have discussed here is an *oblate spheroidal coordinate system* (Figure 8.40). The three oblate spheroidal coordinates are related to x, y, z by

$$x = a \cosh \eta \sin \theta \cos \phi$$

$$y = a \cosh \eta \sin \theta \sin \phi \tag{13}$$

$$z = a \sinh \eta \cos \theta$$

We can generate an oblate spheroidal coordinate system by rotating an orthogonal family of confocal ellipses and hyperbolas about the minor axis of the ellipse (compare to prolate spheroidal coordinates). As in the case of prolate spheroidal coordinates, $2a$ is the distance between the foci. The η = constant surfaces are

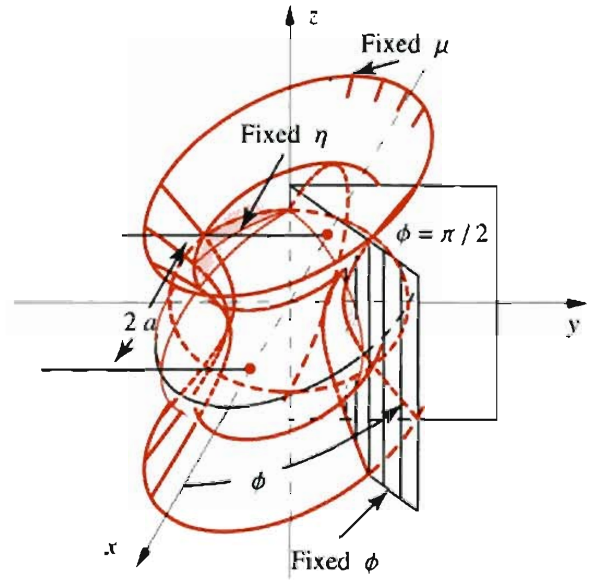

Figure 8.40
An oblate spheroidal coordinate system.

oblate spheroids

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad b > c$$

where $b = a \cosh \eta$ and $c = a \sinh \eta$. The surfaces θ = constant are hyperboloids of one sheet and ϕ is the angle about the z axis. The coordinates range from

$$0 \leq \eta < \infty; \qquad 0 \leq \theta \leq \pi; \qquad 0 \leq \phi \leq 2\pi \tag{14}$$

We leave the discussion of this coordinate system to a set of problems (Problems 12–16).

8.6 Problems

1. Show that $h_\theta = a(\sinh^2 \eta + \sin^2 \theta)^{1/2}$ and $h_\phi = a \sinh \eta \sin \theta$ for prolate spheroidal coordinates.
2. Use prolate spheroidal coordinates to calculate the surface area of a prolate spheroid.
3. Show that Laplace's equation in prolate spheroidal coordinates is

$$\nabla^2 f = \frac{1}{a^2(\sinh^2 \eta + \sin^2 \theta)} \left(\frac{\partial^2 f}{\partial \eta^2} + \coth \eta \frac{\partial f}{\partial \eta} + \frac{\partial^2 f}{\partial \theta^2} + \cot \theta \frac{\partial f}{\partial \theta} \right)$$

$$+ \frac{1}{a^2 \sinh^2 \eta \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} = 0$$

4. Show that $f(\eta, \theta, \phi) = \sinh \eta \sin \theta \sin \phi$ is a solution to Laplace's equation in prolate spheroidal coordinates.
5. Here's how to solve Equation 6. First let $g = dV/d\eta$ and then separate variables and integrate to get g. Then

integrate again to get V. You can determine the two integration constants by using the fact that $V(\eta) = 0$ as $\eta \to \infty$ and that $V(\eta_0) = V_0$.

6. Start with the equations $r_1 = [(a+z)^2 + x^2 + y^2]^{1/2}$ and $r_2 = [(a-z)^2 + x^2 + y^2]^{1/2}$ to derive Equations 8.
7. Show that λ, μ, ϕ in Equations 8 form an orthogonal coordinate system.
8. Show that $h_\lambda = \dfrac{a(\lambda^2-\mu^2)^{1/2}}{(\lambda^2-1)^{1/2}}$; $h_\mu = \dfrac{a(\lambda^2-\mu^2)^{1/2}}{(1-\mu^2)^{1/2}}$; and $h_\phi = a(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2}$ for the λ, μ, ϕ coordinate system defined by Equations 8.
9. Use the prolate spheroidal coordinates λ, μ, ϕ to calculate the surface area of a prolate spheroid.
10. Show that the integral in Equation 11 is given by
$I(R) = \dfrac{1}{\pi}\displaystyle\int_0^\infty dr_1\, e^{-r_1} r_1^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta\, \sin\theta e^{-r_2}$ using spherical coordinates centered on nucleus A, as shown in Figure 8.41. To evaluate this integral, use the law of cosines to express r_2 in terms of r_1, θ, and R.

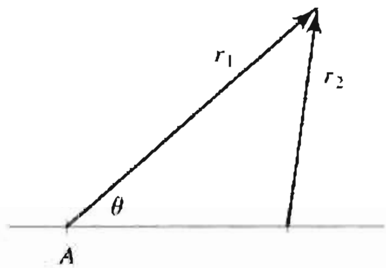

Figure 8.41
The geometry used in Problem 10 to evaluate the integral in Equation 11.

11. Another integral that occurs in a quantum-mechanical treatment of a hydrogen atom is $J = \displaystyle\iiint \frac{e^{-2r_1}}{r_2} dV$, where the integration is over all space. Use the λ, μ, ϕ bipolar coordinate system to show that $J = \dfrac{1}{R} - e^{-2R}\left(1 + \dfrac{1}{R}\right)$, where R is the distance between the nuclei.
12. Show that the oblate spheroidal coordinate system is orthogonal.
13. Determine the scale factors h_η, h_μ, and h_ϕ for oblate spheroidal coordinates.
14. Use the scale factors that you determined in the previous problem to calculate the volume of an oblate spheroid.
15. Use the scale factors that you determined in Problem 13 to calculate the surface area of an oblate spheroid. Show that your answer reduces to that of a sphere if $b = c$.
16. Show that Laplace's equation in oblate spheroidal coordinates reduces to $\dfrac{d^2f}{d\eta^2} + \tanh\eta \dfrac{df}{d\eta} = 0$ if f depends only upon η. Can you give an example of a physical problem where this would be the case?

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CHAPTER 9

Linear Algebra and Vector Spaces

Although we learned how to solve simultaneous linear algebraic equations in high school, the fundamental theory behind the manipulations we learned constitutes the study of linear algebra, one of the most sophisticated and beautiful fields of mathematics. We can only touch on some of the more relevant topics for our purposes in this chapter. In Section 1, we introduce determinants and show how they can be used to solve simultaneous linear algebraic equations by Cramer's rule. As elegant as Cramer's rule may be, it is not well suited computationally, and in Section 2, we use Gaussian elimination to not only solve n linear equations in n unknowns, but other cases as well. The key quantity in Section 2 is the augmented matrix. In Section 3, we discuss matrices more fully, and learn how to multiply matrices together and to find their inverses, among other things. Section 4 deals with the idea of the rank of a matrix, one of the most important quantities for determining the nature of the solutions to sets of linear algebraic equations. Closely related to rank is the concept of linear independence of a set of vectors, which leads naturally to Section 5, where we introduce and discuss abstract vector spaces. After presenting the axioms of a vector space, we define a basis of a vector space and its dimension. When we define the operation of an inner product between the vectors in a vector space, we then have what is called an inner product space, which is the subject of Section 6. Introducing an inner product allows us to discuss the lengths of abstract vectors, the angle between them, the distance between them, orthogonality, and a number of other geometric quantities. Finally, in Section 7, we generalize the results of the previous two sections to include complex inner product spaces, which play a particularly important role in quantum mechanics.

9.1 Determinants

We frequently encounter simultaneous algebraic equations in physical applications. Let's start off with just two equations in two unknowns:

$$\begin{aligned} a_{11}x + a_{12}y &= h_1 \\ a_{21}x + a_{22}y &= h_2 \end{aligned} \tag{1}$$

If we multiply the first of these equations by a_{22} and the second by a_{12} and then subtract, we obtain

$$(a_{11}a_{22} - a_{12}a_{21})x = h_1 a_{22} - h_2 a_{12}$$

or

$$x = \frac{a_{22}h_1 - a_{12}h_2}{a_{11}a_{22} - a_{12}a_{21}} \tag{2}$$

Similarly, if we multiply the first equation by a_{21} and the second by a_{11} and then subtract, we get

$$y = \frac{a_{21}h_1 - a_{11}h_2}{a_{11}a_{22} - a_{12}a_{21}} \tag{3}$$

Notice that the denominators in both Equations 2 and 3 are the same. If we had started with three equations instead of two as in Equation 1, the denominators would have come out to be $a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$.

We represent $a_{11}a_{22} - a_{12}a_{21}$ and the corresponding expression for three simultaneous equations by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{4}$$

and

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \end{matrix} \tag{5}$$

The quantities introduced in Equations 4 and 5 are called a 2×2 *determinant* and a 3×3 *determinant*, respectively. The reason for introducing this notation is that it readily generalizes to n equations in n unknowns. An $n \times n$ determinant, called an *nth order determinant*, is a square array of n^2 elements arranged in n rows and n columns. We'll represent a determinant consisting of elements a_{ij} by $|\mathrm{A}|$. Note that the element a_{ij} occurs in the ith row and the jth column of $|\mathrm{A}|$. As of now, Equations 4 and 5 simply introduce symbols for the right-hand sides of these equations, but we shall develop convenient procedures for evaluating any size determinant.

Let's rearrange the right side of Equation 5 in the following way:

$$|\mathrm{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{matrix} \tag{6}$$

Notice that each term in parentheses in Equation 6 is equal to the 2×2 determinant that is obtained by striking out the row and the column of the factor in front of each set of parentheses. Furthermore, these factors are the members of the first row of $|A|$. Thus, Equation 6 shows us that we can evaluate a 3×3 determinant in terms of three 2×2 determinants.

We can express Equation 6 in a systematic way by first introducing the *minor* of an element of a $n \times n$ determinant $|A|$. The minor M_{ij} of an element a_{ij} is a $(n-1) \times (n-1)$ determinant obtained by deleting the ith row and the jth column. We now define the *cofactor* A_{ij} of a_{ij} by $A_{ij} = (-1)^{i+j} M_{ij}$. For example, A_{12}, the cofactor of a_{12}, in Equation 6 is

$$A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

The introduction of cofactors allows us to write Equation 6 as

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \tag{7}$$

Equation 7 represents $|A|$ as an *expansion in cofactors*. In particular, it is an expansion in cofactors about the first row of $|A|$. You can start with Equation 5 to show that $|A|$ can be expressed in an expansion of cofactors about *any* row or *any* column (Problem 5).

Example 1:
Expand

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & -2 & 1 \end{vmatrix}$$

in an expansion in cofactors about the second row and about the third column of $|A|$.

SOLUTION: We use Equation 7:

$$|A| = -0(-1+2) + 3(2-2) - (-1)(-4-2) = -2$$

$$|A| = 1(0-6) - (-1)(-4+2) + 1(6-0) = -2$$

We shall show below that Equation 7 readily generalizes to determinants of any order.

Example 2:
The *determinantal equation*

$$\begin{vmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{vmatrix} = 0$$

occurs in a quantum-mechanical treatment of a butadiene molecule. Expand this determinantal equation into a quartic equation for x.

SOLUTION: Expand about the first row of elements to obtain

$$x\begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & x & 1 \\ 0 & 1 & x \end{vmatrix} = 0$$

Now expand about the first row of each of the 3 × 3 determinants to obtain

$$(x)(x)\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} - (x)(1)\begin{vmatrix} 1 & 1 \\ 0 & x \end{vmatrix} - (1)\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} - (-1)\begin{vmatrix} 0 & 1 \\ 0 & x \end{vmatrix} = 0$$

or

$$x^2(x^2 - 1) - x(x) - (1)(x^2 - 1) = 0$$

or

$$x^4 - 3x^2 + 1 = 0$$

Note that because we can choose any row or column to expand the determinant, it is easiest to take the one with the most zeroes.

Up to now we have used 3 × 3 determinants to illustrate how to evaluate determinants. We're going to discuss a number of general properties of determinants below, and so we need to discuss determinants more generally at this point. First, consider the product of elements of an $n \times n$ determinant

$$a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n}$$

where the j's are distinct and take on the values 1 through n. Note that there is only one element from each row and one element from each column in this product, and that the first subscripts are in their "natural order." Also note that there are $n!$ possible products because j_1 can take on one of n values, j_2 one of $n - 1$ values, and so on. Now consider the process of interchanging the elements in the above product successively until the second set of subscripts is in its "natural order." This will require either an even or an odd number of interchanges. For example,

$$a_{12}\, a_{21}\, a_{33} \longrightarrow a_{21}\, a_{12}\, a_{33}$$
$$a_{12}\, a_{23}\, a_{31} \longrightarrow a_{31}\, a_{23}\, a_{12} \longrightarrow a_{31}\, a_{12}\, a_{23}$$

require one and two interchanges, respectively. Now define the symbol

$$\epsilon_{j_1 j_2 j_3 \cdots j_n} = \pm 1$$

depending upon whether it takes an even or an odd number of interchanges to order the n j's into their "natural order." For example, $\epsilon_{12} = +1$, $\epsilon_{21} = -1$, $\epsilon_{132} = -1$, and so on. Finally, the general definition of an $n \times n$ determinant is

$$|A| = \sum \epsilon_{j_1 j_2 j_3 \cdots j_n} a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n} \tag{8}$$

where the summation is over all the $n!$ permutations of the $j_1, j_2, \cdots, j_n$.

Equation 8 is the formal definition of a determinant. It may not be familiar to you and may be even a little formidable, but we won't have to use it to evaluate a determinant. Its primary use is to prove some important general properties of determinants. First, let's use Equation 8 to evaluate a 2×2 determinant. For a 2×2 determinant, Equation 8 gives

$$|A| = \sum_{j_1} \sum_{j_2} \epsilon_{j_1 j_2} a_{1j_1} a_{2j_2}$$

In these summations, j_1 and j_2 take on the values 1 and 2, but there are no terms with $j_1 = j_2$ because the j's are distinct. Therefore,

$$|A| = \epsilon_{12} a_{11} a_{22} + \epsilon_{21} a_{12} a_{21}$$

Using the fact that $\epsilon_{12} = +1$ and $\epsilon_{21} = -1$, we have

$$|A| = a_{11} a_{22} - a_{12} a_{21}$$

in agreement with Equation 4. Problem 11 has you show that Equation 8 yields Equation 5 for 3×3 determinant.

The real utility of Equation 8 is that we can use it to derive general results for determinants. For example, suppose that we interchange two adjacent rows, r and $r + 1$. If we denote the resulting determinant by $|A_{r\leftrightarrow r+1}|$, then

$$|A_{r\leftrightarrow r+1}| = \sum \epsilon_{j_1 j_2 j_3 \cdots j_n} a_{1j_1} a_{2j_2} \cdots a_{(r+1)j_{r+1}} a_{rj_r} \cdots a_{nj_n}$$

We can get the order of the first subscripts back into natural order by one permutation, however, so $|A_{r\leftrightarrow r+1}|$ differs from $|A|$ by one inversion. Therefore we have that $|A_{r\leftrightarrow r+1}| = -|A|$. To analyze the interchange of two adjacent rows is fairly easy. Suppose now that we want to interchange two rows separated by one row. For concreteness, let these rows be rows 2, 3, and 4. We can interchange rows 2 and 4 by the process $(2, 3, 4) \rightarrow (2, 4, 3) \rightarrow (4, 2, 3) \rightarrow (4, 3, 2)$, which requires three steps. This is a general result and so we can write $|A_{r\leftrightarrow r+2}| = -|A|$. Problem 12 has you show that it requires $2k - 1$ steps to interchange rows r and $r + k$, so that we see that

$$|A_{r\leftrightarrow r+k}| = (-1)^{2k-1}|A| = -|A| \tag{9}$$

or that $|A|$ changes sign when ***any*** two rows are exchanged. (This is property 3 below.)

Equation 9 says that $|A_{r\leftrightarrow r+k}| = -|A|$ even if the two rows are identical. If the two rows are identical, however, then the value of $|A|$ cannot change. The only way that $|A| = -|A|$ is if $|A| = 0$. Thus, we see that $|A| = 0$ if two rows (or two columns) are the same. This is property 2 below. The rest of the properties listed below can be proved using Equation 8 in a similar manner.

Example 3:
Show that

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 3 & 1 & 6 \end{vmatrix} = 0$$

SOLUTION: Expand the cofactors about the first column to obtain

$$|A| = 1(-6) - 0(10) + 3(+2) = 0$$

Notice that the third column is twice the first column. We'll see below that $|A| = 0$ because of this.

Some properties of determinants that are useful to know are:

1. The value of a determinant is unchanged if the rows are made into columns in the same order; in other words, first row becomes first column, second row becomes second column, and so on. For example,

$$\begin{vmatrix} 1 & 2 & 5 \\ -1 & 0 & -1 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 5 & -1 & 2 \end{vmatrix} = -6$$

2. If any two rows or columns are the same, the value of the determinant is zero. For example,

$$\begin{vmatrix} 4 & 2 & 4 \\ -1 & 0 & -1 \\ 3 & 1 & 3 \end{vmatrix} = 0$$

3. If any two rows or columns are interchanged, the sign of the determinant is changed. For example,

$$\begin{vmatrix} 3 & 1 & -1 \\ -6 & 4 & 5 \\ 1 & 2 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & -1 \\ 4 & -6 & 5 \\ 2 & 1 & 2 \end{vmatrix}$$

4. If every element in a single row or column is multiplied by a factor k, the value of the determinant is multiplied by k (Problem 13). For example,

$$\begin{vmatrix} 6 & 8 \\ -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 20$$

5. If any row or column is written as the sum or difference of two or more terms, the determinant can be written as the sum or difference of two or more determinants

according to (Problem 14)

$$\begin{vmatrix} a_{11} \pm a'_{11} & a_{12} & a_{13} \\ a_{21} \pm a'_{21} & a_{22} & a_{23} \\ a_{31} \pm a'_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \pm \begin{vmatrix} a'_{11} & a_{12} & a_{13} \\ a'_{21} & a_{22} & a_{23} \\ a'_{31} & a_{32} & a_{33} \end{vmatrix}$$

For example,

$$\begin{vmatrix} 3 & 3 \\ 2 & 6 \end{vmatrix} = \begin{vmatrix} 2+1 & 3 \\ -2+4 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -2 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

6. The value of a determinant is unchanged if one row or column is added or subtracted to another, as in

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+a_{12} & a_{12} & a_{13} \\ a_{21}+a_{22} & a_{22} & a_{23} \\ a_{31}+a_{32} & a_{32} & a_{33} \end{vmatrix}$$

For example

$$\begin{vmatrix} 1 & -1 & 3 \\ 4 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 3 \\ 4 & 0 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

where we added column 2 to column 1. This procedure may be repeated n times to obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+na_{12} & a_{12} & a_{13} \\ a_{21}+na_{22} & a_{22} & a_{23} \\ a_{31}+na_{32} & a_{32} & a_{33} \end{vmatrix} \tag{10}$$

This result is easy to prove:

$$\begin{vmatrix} a_{11}+na_{12} & a_{12} & a_{13} \\ a_{21}+na_{22} & a_{22} & a_{23} \\ a_{31}+na_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + n \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + 0$$

where we used Rule 5 to write the first line. The second determinant on the right side equals zero because two columns are the same.

Example 4:
Show that the value of

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix}$$

is unchanged if we add 2 times row 2 to row 3.

SOLUTION: First of all, $|A| = 3(-1) - (-1)(-7) + 2(-9) = -28$. Now

$$\begin{vmatrix} 3 & -1 & 2 \\ -2 & 1 & 1 \\ -3 & 6 & 5 \end{vmatrix} = 3(-1) - (-1)(-7) + 2(-9) = -28$$

Simultaneous linear algebraic equations can be solved in terms of determinants. For simplicity, we will consider only a pair of equations, but the final result is easy to generalize. Consider the two equations

$$\begin{aligned} a_{11}x + a_{12}y &= h_1 \\ a_{21}x + a_{22}y &= h_2 \end{aligned} \tag{11}$$

The determinant of the coefficients of x and y is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

According to Rule 4,

$$\begin{vmatrix} a_{11}x & a_{12} \\ a_{21}x & a_{22} \end{vmatrix} = x\,|A|$$

Furthermore, according to Rule 6,

$$\begin{vmatrix} a_{11}x + a_{12}y & a_{12} \\ a_{21}x + a_{22}y & a_{22} \end{vmatrix} = x\,|A| \tag{12}$$

If we substitute Equation 11 into Equation 12, then we have

$$\begin{vmatrix} h_1 & a_{12} \\ h_2 & a_{22} \end{vmatrix} = x\,|A|$$

Solving for x gives

$$x = \frac{\begin{vmatrix} h_1 & a_{12} \\ h_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \tag{13}$$

Similarly, we get

$$y = \frac{\begin{vmatrix} a_{11} & h_1 \\ a_{21} & h_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \tag{14}$$

Notice that Equations 13 and 14 are identical to Equations 2 and 3. The solution for x and y in terms of determinants is called *Cramer's rule*. Note that the determinant in the numerator is obtained by replacing the column in |A| that is associated with the unknown quantity and replacing it with the column associated with the right sides of Equations 11. We shall show after the next Example that this result is readily extended to more than two simultaneous equations.

Example 5:
Use Cramer's rule to solve the equations

$$x + y + z = 2$$

$$2x - y - z = 1$$

$$x + 2y - z = -3$$

SOLUTION: The extension of Equations 13 and 14 is

$$x = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ -3 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{vmatrix}} = \frac{9}{9} = 1$$

Similarly,

$$y = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & -3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{vmatrix}} = \frac{-9}{9} = -1$$

and

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{vmatrix}} = \frac{18}{9} = 2$$

Before we finish this section, let's discuss the expansion of a determinant in terms of its cofactors and how it can be used to derive Cramer's rule for n simultaneous equations. Equation 7 says that

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

where A_{ij} is the cofactor of a_{ij}. We derived Equation 7 from the expansion of a 3×3 determinant, but it is a general result in the sense that if $|\mathrm{A}|$ is an $n \times n$ determinant, then

$$|\mathrm{A}| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

or, in summation notation,

$$|\mathrm{A}| = \sum_{j=1}^{n} a_{1j}A_{1j} \tag{15}$$

Equation 15 can be derived directly from Equation 8, but the general proof is a little long. (See "Lipschutz" in the References.)

Equation 15 represents an expansion of $|\mathrm{A}|$ about its first row. More generally, we can expand about the ith row of $|\mathrm{A}|$, and so we also have

$$|\mathrm{A}| = \sum_{j=1}^{n} a_{ij}A_{ij} \tag{16}$$

If we had expanded about the ith column, Equation 16 would read

$$|\mathrm{A}| = \sum_{k=1}^{n} a_{ki}A_{ki} \tag{17}$$

Now if we replace the ith column by some other column, say the jth column, then $|\mathrm{A}| = 0$ because $|\mathrm{A}|$ now has two identical columns. The cofactors in Equation 17 don't change, however, so we have the result

$$\sum_{k=1}^{n} a_{kj}A_{ki} = 0 \qquad i \neq j \tag{18}$$

Notice that we can combine Equations 17 and 18 to read

$$\sum_{k=1}^{n} a_{kj}A_{ki} = |\mathrm{A}|\, \delta_{ij} \tag{19}$$

where δ_{ij}, the *Kronecker delta*

$$\delta_{ij} = \begin{matrix} 1 & \qquad i = j \\ 0 & \qquad i \neq j \end{matrix} \tag{20}$$

We will refer to Equation 19 several times in later chapters.

We can use Equation 19 to derive Cramer's rule. Start with the set of equations

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = h_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = h_2 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = h_n \end{matrix} \tag{21}$$

Multiply the first equation by A_{11}, the second by A_{21}, and so on, and then add to obtain

$$\sum_{j=1}^{n} a_{j1}A_{j1}x_1 + \sum_{j=1}^{n} a_{j2}A_{j1}x_2 + \cdots + \sum_{j=1}^{n} a_{jn}A_{j1}x_n = \sum_{j=1}^{n} A_{j1}h_j \qquad (22)$$

According to Equation 19, all the terms on the left side except the first one vanish, and so the left side is equal to $|\mathrm{A}|\,x_1$. According to Equation 17, the right side is the original determinant $|\mathrm{A}|$ but with the first column replaced by $h_1, h_2, \ldots, h_n$. Thus, we see that

$$|\mathrm{A}|\,x_1 = \begin{vmatrix} h_1 & a_{12} & \cdots & a_{1n} \\ h_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad (23)$$

Similar equations for x_2 through x_n can be obtained by multiplying Equation 21 by other cofactors (Problem 19).

9.1 Problems

1. Evaluate the determinant $|\mathrm{A}| = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix}$. Add column 2 to column 1 to get $\begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ and evaluate it.

   Compare your result with the value of $|\mathrm{A}|$. Now add row 2 to row 1 of $|\mathrm{A}|$ to get $\begin{vmatrix} 1 & 4 & 3 \\ -1 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ and evaluate it.

   Compare your result with the value of $|\mathrm{A}|$ above.

2. Interchange columns 1 and 3 in $|\mathrm{A}|$ in Problem 1 and evaluate the resulting determinant. Compare your result with the value of $|\mathrm{A}|$. Interchange rows 1 and 2 and do the same.

3. Evaluate the determinant $|\mathrm{A}| = \begin{vmatrix} 1 & 6 & 1 \\ -2 & 4 & -2 \\ 1 & -3 & 1 \end{vmatrix}$. Can you determine its value by inspection? What about

   $|\mathrm{A}| = \begin{vmatrix} 2 & 6 & 1 \\ -4 & 4 & -2 \\ 2 & -3 & 1 \end{vmatrix}$?

4. Starting with $|\mathrm{A}|$ in Problem 1, add two times the third row to the second row and evaluate the resulting determinant.

5. Use Equation 5 to derive an expansion in cofactors about the third column.

6. Evaluate $|\mathrm{A}| = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$.

7. Find the values of x that satisfy the determinantal equation, $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 0 & 0 \\ 1 & 0 & x & 0 \\ 1 & 0 & 0 & x \end{vmatrix} = 0.$

8. Find the values of x that satisfy the determinantal equation, $\begin{vmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{vmatrix} = 0.$

9. Show that $\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$

10. Evaluate (a) ϵ_{13245}, (b) ϵ_{32145}, and (c) ϵ_{54321}.

11. Show that Equation 8 yields Equation 5 for a 3×3 determinant.

12. Show that it requires $2k - 1$ steps to interchange rows r and $r + k$ in a determinant.

13. Use Equation 8 to prove property 4.

14. Use Equation 8 to prove property 5.

15. Solve the following set of equations using Cramer's rule:

$$x + y = 2$$
$$3x - 2y = 5$$

16. Solve the following set of equations using Cramer's rule:

$$x + 2y + 3z = -5$$
$$-x - 3y + z = -14$$
$$2x + y + z = 1$$

17. Use Cramer's rule to solve

$$x + 2y = 3$$
$$2x + 4y = 1$$

What goes wrong here? Why?

18. Verify Equation 19 for the determinant in Equation 5 for $i = 2$ and $j = 1$.

19. Derive an equation for x_2 starting with Equation 21.

20. Use any CAS to evaluate $\begin{vmatrix} 2 & 5 & 1 \\ 3 & 1 & 2 \\ -2 & 1 & 0 \end{vmatrix}$.

21. Use any CAS to evaluate $\begin{vmatrix} 1 & 0 & 3 & -2 \\ 6 & 1 & -1 & 3 \\ 2 & 0 & 1 & 1 \\ 4 & 3 & 2 & 5 \end{vmatrix}$.

9.2 Gaussian Elimination

Although Cramer's rule provides a systematic, compact approach to solving simultaneous linear algebraic equations, it is not a convenient computational procedure because of the necessity of evaluating numerous determinants. Nor does Cramer's rule apply if the number of equations does not equal the number of unknowns. In this section, we shall present an alternative method of solving simultaneous equations that is not only computationally convenient, but is not limited to $n \times n$ systems. Before we present this method, however, we shall discuss some general ideas about systems of linear algebraic equations.

Let's start off again with two equations in two unknowns.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Geometrically we have three possibilities: 1. the graphs of the two straight lines intersect and we have a unique solution; 2. the lines are parallel and we have no solution; and 3. the lines coincide and we have an infinite number of solutions (Figure 9.1). An example of the first case is

$$2x_1 + x_2 = 3$$
$$x_1 - 3x_2 = -2$$

with $x_1 = 1$ and $x_2 = 1$ as its unique solution. An example of the second case is

$$2x_1 + x_2 = 3$$
$$2x_1 + x_2 = 5$$

These two lines are parallel and have no point in common. An example of the third case is

$$2x_1 + x_2 = 3$$
$$4x_1 + 2x_2 = 6$$

These two lines actually coincide and the solution can be written as $x_2 = 3 - 2x_1$, where x_1 can take on any value.

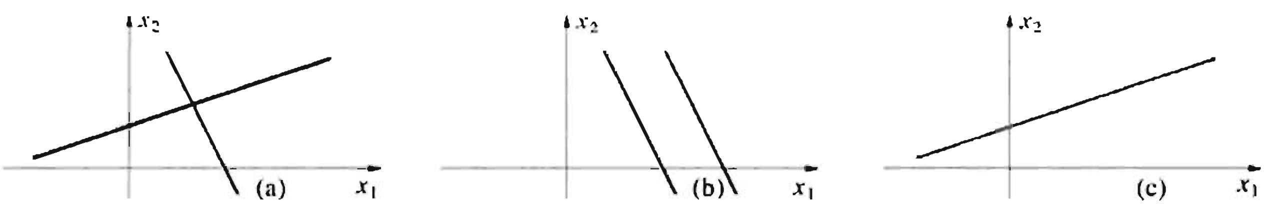

Figure 9.1
The three geometric possibilities of two linear equations in two unknowns, x_1 and x_2. (a) The colored line ($2x_1 + x_2 = 3$) and the black line ($x_1 - 3x_2 = -2$) have a unique point of intersection. (b) The colored line ($2x_1 + x_2 = 3$) and the black line ($2x_1 - x_2 = 5$) are parallel and have no point of intersection. (c) The two lines ($2x_1 - x_2 = 1$) and ($4x_1 - 2x_2 = 2$) superimpose, and so there is an infinite number of solutions.

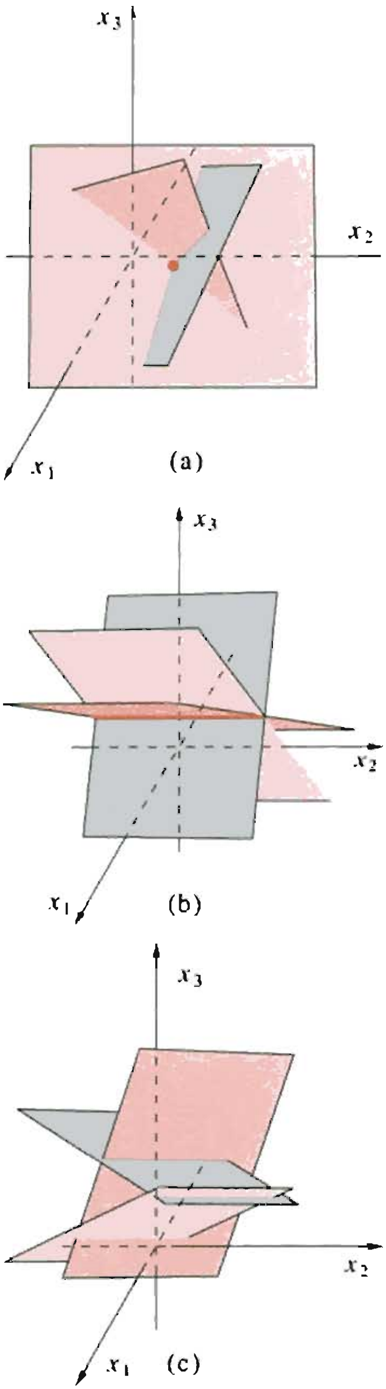

Figure 9.2
The three geometric possibilities of the graphs of three linear algebraic equations: (a) a unique solution; (b) an infinite number of solutions; and (c) no solution.

The geometric interpretation for a 3×3 set of equations involves planes. If the three planes intersect at one point, there is a unique solution (Figure 9.2a). If the three planes intersect as shown in Figure 9.2b, there is an infinite number of solutions. And if the three planes have no common point of intersection, as in Figure 9.2c, there is no solution.

Let's now consider a general $n \times n$ system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= h_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= h_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= h_n \end{aligned} \tag{1}$$

If all the h_j in Equations 1 are equal to zero, the system of equations is called *homogeneous*. If at least one $h_j \neq 0$, the system is called *nonhomogeneous*. We may re-express Equations 1 as

$$\mathsf{AX} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = \mathsf{H} \tag{2}$$

if we agree that the rth equation in Equations 1 is formed by multiplying each element of the rth row of A by the corresponding element of X, adding the results, and then equating the sum to the rth element in H. For example, the second line in Equations 1 is given by

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = h_2 \tag{3}$$

Thus, we can write Equations 1 as AX = H, where A is called the *coefficient matrix*, X is called the column vector of unknowns, and H is called the constant vector of the system. Note that the left side of Equation 3 can be viewed as the dot product of the rth row vector of A and the column vector X.

The quantity A in Equation 2 is an $n \times n$ matrix, which is an array of elements that obeys certain algebraic rules such as Equation 3. We shall discuss matrices and their algebraic rules in some detail in the next section. The important point here is that a matrix is *not* equal to a single number. However, we can associate a determinant with a matrix and write

$$\det \mathsf{A} = |\mathsf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{4}$$

which is a single number.

Clearly, the coefficient matrix A must have a lot to say about the existence and nature of the solutions to Equations 1. Cramer's rule contains |A| in the denominators of the equations for the unknowns, so |A| cannot equal zero if there exists a unique solution to a nonhomogeneous system. If $|A| = 0$, then A is said to be *singular*; if $|A| \neq 0$, then A is said to be *nonsingular*. In fact, we have the following theorem, which we shall prove later:

The $n \times n$ system $AX = H$ *has a unique solution if and only if* A *is nonsingular.*

If $H = 0$, that is, if the system is homogeneous, then $x_1 = x_2 = \cdots = x_n = 0$ (called the *trivial solution*) is always a solution. But the above theorem says that a solution is unique if A is nonsingular, so if A is nonsingular, there is *only* a trivial solution to a homogeneous system. To have a nontrivial solution to an $n \times n$ set of homogeneous equations, the coefficient matrix must be singular.

We shall now spend the rest of this section actually finding solutions to systems of linear equations, even if the coefficient matrix is not square. Let's consider the equations

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 4 \\ 2x_1 - 2x_2 - x_3 &= 1 \\ -2x_1 + 4x_2 + x_3 &= 1 \end{aligned} \tag{5}$$

The coefficient matrix and the constant vector are

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & -2 & -1 \\ -2 & 4 & 1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

We now form a new matrix, called the *augmented matrix*, by adjoining H to A so that it is the last column

$$A|H = \left(\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 2 & -2 & -1 & 1 \\ -2 & 4 & 1 & 1 \end{array}\right) \tag{6}$$

Clearly this matrix contains *all* the information in Equations 5, and is just a succinct expression of them. Just as we may multiply any of the equations in Equations 5 by a nonzero constant without jeopardizing the solutions, we may multiply any row of A|H without altering its content. Similarly, we may interchange any two rows of either Equations 5 or A|H and replace any row by the sum of that row and a constant times another row. These three operations are called *elementary (row) operations*:

1. We may multiply any row by a nonzero constant.
2. We may interchange any pair of rows.
3. We may replace any row by the sum of that row and a constant times another row.

The key point is that these elementary operations produce an *equivalent system*, that is, a system with the same solution as the original system. Matrices that differ by a set of elementary operations are said to be *equivalent*.

We are now going to manipulate A|H by elementary operations so that there are zeros in the lower left positions of A|H. Add -1 times row 1 to row 2 and add row 1 to row 3 in Equation 5 to get

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 0 & -3 & -4 & -3 \\ 0 & 5 & 4 & 5 \end{array}\right)$$

Now add 5/3 times row 2 to row 3 to get

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 0 & -3 & -4 & -3 \\ 0 & 0 & -8/3 & 0 \end{array}\right)$$

Write out the corresponding system of equations:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 4 \\ -3x_2 - 4x_3 &= -3 \\ -8x_3/3 &= 0 \end{aligned}$$

and work your way from bottom to top to find $x_3 = 0$, $x_2 = 1$, and $x_1 = 3/2$.

This procedure is called *Gaussian elimination* and the final form of A|H is said to be in *echelon form*. The following Examples provide two other applications of Gaussian elimination.

Example 1:
Solve the equations

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ 2x_1 - x_2 + 3x_3 &= 5 \\ 3x_1 + 2x_2 - 2x_3 &= 5 \end{aligned}$$

SOLUTION: The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 3 & 5 \\ 3 & 2 & -2 & 5 \end{array}\right)$$

Add -2 times row 1 to row 2 and -3 times row 1 to row 3 to obtain

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 5 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right)$$

To avoid introducing fractions, interchange rows 2 and 3 and then add -3 times the new row 2 to the new row 3 to get

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right)$$

The corresponding set of equations is

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ -x_2 + x_3 &= -1 \\ 2x_3 &= 4 \end{aligned}$$

Solving these equations from top to bottom gives $x_3 = 2$, $x_2 = 3$, and $x_1 = 1$.

Example 2:
Solve the equations

$$\begin{aligned} x_1 + x_2 + x_3 &= -2 \\ x_1 - x_2 + x_3 &= 2 \\ -x_1 + x_2 - x_3 &= -2 \end{aligned}$$

SOLUTION: The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & -2 \end{array}\right)$$

Add -1 times row 1 to row 2 and add row 1 to row 3 to obtain

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & 4 \\ 0 & 2 & 0 & -4 \end{array}\right)$$

Now add row 2 to row 3 to get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

In this case, the corresponding set of equations is

$$\begin{aligned} x_1 + x_2 + x_3 &= -2 \\ -2x_2 \qquad &= 4 \\ 0x_3 &= 0 \end{aligned}$$

The solutions are x_3 = arbitrary, $x_2 = -2$, and $x_1 = -x_3$, so the solution is not unique. Note that $|A| = 0$ in this case, so we should not expect a unique solution.

Example 3:
Solve the equations

$$2x_1 - x_3 = -1$$
$$3x_1 + 2x_2 = 4$$
$$4x_2 + 3x_3 = 6$$

SOLUTION: The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & -1 \\ 3 & 2 & 0 & 4 \\ 0 & 4 & 3 & 6 \end{array}\right)$$

Add $-3/2$ times row 1 to row 2 to obtain

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & -1 \\ 0 & 2 & 3/2 & 11/2 \\ 0 & 4 & 3 & 6 \end{array}\right)$$

Now add -2 times row 2 to row 3 to get

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & -1 \\ 0 & 2 & 3/2 & 11/2 \\ 0 & 0 & 0 & -10/2 \end{array}\right)$$

This last line says that $-10/2 = 0$, meaning that there is no solution to the above equations. They are inconsistent.

Up to now we have considered only $n \times n$ systems of equations. Suppose we have a system with more equations than unknowns. (Such systems are called *overdetermined*.) For example, consider

$$\begin{aligned} x_1 + x_2 &= 4 \\ 3x_1 - 4x_2 &= 9 \\ 5x_1 - 2x_2 &= 17 \end{aligned} \tag{7}$$

The augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & -4 & 9 \\ 5 & -2 & 17 \end{array}\right)$$

Multiplying row 1 by -3 and adding to row 2, and then multiplying row 1 by -5 and adding to row 3 gives

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -7 & -3 \\ 0 & -7 & -3 \end{array}\right)$$

Now multiply row 2 by -1 and add to row 3:

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -7 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

The corresponding algebraic equations are

$$\begin{aligned} x_1 + x_2 &= 4 \\ -7x_2 &= -3 \end{aligned}$$

and the solution is $x_2 = 3/7$ and $x_1 = 25/7$. The coefficient matrix of the final set of equations is $\begin{pmatrix} 1 & 1 \\ 0 & -7 \end{pmatrix}$ and is nonsingular; thus the solution is unique.

Example 4:
Solve the equations

$$\begin{aligned} 3x_1 - 2x_2 &= 2 \\ -6x_1 + 4x_2 &= -4 \\ -3x_1 + 2x_2 &= 2 \end{aligned}$$

SOLUTION: The augmented matrix is

$$\left(\begin{array}{cc|c} 3 & -2 & 2 \\ -6 & 4 & -4 \\ -3 & 2 & 2 \end{array}\right)$$

Adding 2 times row 1 to row 2, then adding row 1 to row 3, and then interchanging the resultant rows 2 and 3 gives

$$\left(\begin{array}{cc|c} 3 & -2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

The second line here claims that $0 = 4$, so there is no solution.

Example 5:
Solve the equations

$$\begin{aligned} x_1 - 2x_2 &= 3 \\ 2x_1 - 4x_2 &= 6 \\ -3x_1 + 6x_2 &= -9 \end{aligned}$$

SOLUTION: The augmented matrix is

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -3 & 6 & -9 \end{array}\right)$$

Add -2 times row 1 to row 2 and 3 times row 1 to row 3 to get

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The corresponding algebraic equations are $x_1 - 2x_2 = 3$, or $x_1 = 2x_2 + 3$. Thus, there is an infinite number of solutions.

In summary, if we have more equations than unknowns, then there are three possible outcomes: 1. there is a unique solution; 2. there is no solution; and 3. there is an infinite number of solutions. In each case, Gaussian elimination leads us to the correct result.

If we have more unknowns than equations, as in

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 3x_2 + 2x_3 + 4x_4 &= 0 \\ 2x_1 \qquad\quad + x_3 - x_4 &= 0 \end{aligned} \tag{8}$$

then the system is called *underdetermined*. The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 0 \\ 2 & 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

where we have written $\sim$ to indicate that the first matrix is equivalent to the second; that is, it can be manipulated into the second by elementary operations. The corresponding algebraic equations are

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_2 + x_3 + 3x_4 &= 0 \end{aligned}$$

Solving for x_1 and x_2 in terms of x_3 and x_4, we have $x_1 = (x_4 - x_3)/2$ and $x_2 = -(x_3 + 3x_4)/2$. Thus, there is an infinite number of solutions in this case.

Example 6:
Solve the equations

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 - 2x_3 &= 2 \\ 4x_1 + 5x_2 + 3x_3 \qquad &= 7 \end{aligned}$$

SOLUTION: The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right)$$

The last line corresponds to $0 = -5$, so there are no solutions.

There are only two possibilities when there are more unknowns than equations. Either there is an infinite number of solutions, or there are no solutions. Either way, Gaussian elimination will give the correct answer.

Before we finish this section, we should point out that any CAS can be used to solve simultaneous linear equations (Problems 14 through 16).

9.2 Problems

1. Solve the equations

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 \\ 2x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

2. Solve the equations

$$\begin{aligned} 2x + 5y + z &= 5 \\ x + 4y + 2z &= 1 \\ 4x + 10y - z &= 1 \end{aligned}$$

3. Solve the equations

$$\begin{aligned} x + y &= 1 \\ x + z &= 1 \\ 2x + y + z &= 0 \end{aligned}$$

4. Solve the equations

$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 &= -2 \\ x_1 - x_2 - x_3 + x_4 &= 1 \\ x_1 - 4x_2 - 2x_3 + 2x_4 &= 6 \\ 4x_1 + x_2 - 3x_3 + 3x_4 &= -1 \end{aligned}$$

5. Solve the equations

$$\begin{aligned} x + 2y - 6z &= 2 \\ x + 4y + 4z &= 1 \\ 3x + 10y + 2z &= -1 \end{aligned}$$

6. Solve the equations

$$\begin{aligned} x_1 + 2x_3 - x_4 &= 3 \\ x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 - 2x_4 &= -1 \\ -x_3 + 4x_4 &= 13 \\ 2x_1 - x_3 + 3x_4 &= 11 \end{aligned}$$

7. Solve the equations

$$\begin{aligned} 2x_1 - 4x_2 + x_3 - 3x_4 &= 6 \\ x_1 - 2x_2 + 3x_3 + 6x_4 &= 2 \end{aligned}$$

8. Solve the equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 0 \\ -x_1 + 2x_3 + x_4 &= 0 \\ 2x_1 + x_2 - 2x_4 &= 0 \end{aligned}$$

9. Solve the equations

$$x_1 - 2x_2 + x_3 - x_4 + 2x_5 = -7$$
$$x_2 + x_3 + 2x_4 - x_5 = 5$$
$$x_1 - x_2 - 2x_3 + 2x_4 + 2x_5 = -1$$

10. For what values of λ will the following equations have a unique solution?

$$x + y = \lambda x$$
$$-x + y = \lambda y$$

11. For what values of λ will the following equations have a unique solution?

$$x + y + z = 6$$
$$x + \lambda y + \lambda z = 2$$

12. For what values of λ will the equations in the previous problem have an infinite number of solutions?

13. For what values of λ will the following equations have a unique solution?

$$5x + \lambda y = 4$$
$$4x + 3y = 3$$
$$\lambda x - 6y = 3$$

14. Use any CAS to solve the equations in Problems 2 through 4.

15. Use any CAS to solve the equations in Problems 5 through 7.

16. Use any CAS to solve the equations in Problems 8 and 9.

9.3 Matrices

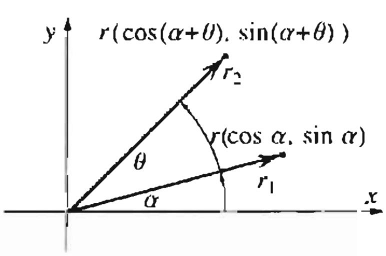

Figure 9.3
A pictorial representation of the rotation of the vector $\mathbf{r}_1$ through an angle θ in a counterclockwise direction. The result is the vector $\mathbf{r}_2$.

Up to now we have used matrices only as a representation of the coefficients in systems of linear algebraic equations. The utility of matrices far exceeds that use, however, and in this section we shall present some of the basic properties of matrices. Then, in Chapter 10, we will discuss a number of important physical applications of matrices.

Many physical operations such as magnification, rotation, and reflection through a plane can be represented mathematically by quantities called matrices. Consider the lower of the two vectors shown in Figure 9.3. The x and y components of the vector are given by $x_1 = r\cos\alpha$ and $y_1 = r\sin\alpha$, where r is the length of $\mathbf{r}_1$. Now, let's rotate the vector counterclockwise through an angle θ, so that $x_2 = r\cos(\alpha+\theta)$ and $y_2 = r\sin(\alpha+\theta)$ (see Figure 9.3). Using trigonometric formulas, we can write

$$x_2 = r\cos(\alpha+\theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$$
$$y_2 = r\sin(\alpha+\theta) = r\cos\alpha\sin\theta + r\sin\alpha\cos\theta$$

or

$$\begin{aligned} x_2 &= x_1 \cos\theta - y_1 \sin\theta \\ y_2 &= x_1 \sin\theta + y_1 \cos\theta \end{aligned} \tag{1}$$

We can display the set of coefficients here in the form

$$\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

We have expressed R in the form of a *matrix*, which is an array of numbers (or functions in this case) that obey a certain set of rules, called *matrix algebra*. Unlike determinants, matrices do not have to be square arrays. The matrix R in Equation 2 corresponds to a rotation of a vector through an angle θ.

Example 1:
Show that the matrices

$$\mathsf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

correspond to reflections of a vector through the x axis and y axis, respectively.

SOLUTION: If we reflect the vector $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}$ through the x axis, we obtain the vector $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} = x_1\mathbf{i} - y_1\mathbf{j}$ (Figure 9.4a). Thus, we can write

$$x_2 = x_1 = x_1 + 0y_1$$

$$y_2 = -y_1 = 0x_1 - y_1$$

The set of coefficients can be expressed as

$$\mathsf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so we see that the matrix A corresponds to a reflection of a vector through the x axis. Similarly, for a reflection through the y axis

$$x_2 = -x_1 = -x_1 + 0y_1$$

$$y_2 = y_1 = 0x_1 + y_1$$

so that $\mathsf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to a reflection through the y axis (Figure 9.4b).

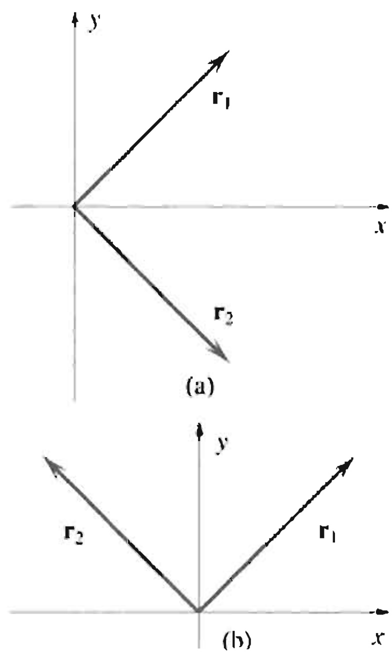

Figure 9.4
A pictorial representation of the reflection of a vector through (a) the x axis and (b) the y axis.

We shall see that matrices usually correspond to physical transformations.

The entries in a matrix A are called its *matrix elements* and are denoted by a_{ij} where, as in the case of determinants, i designates the row and j designates the column. Two matrices, A and B, are equal if and only if they are of the same dimension (that is, have the same number of rows and the same number of columns), and if and only if $a_{ij} = b_{ij}$ for all i and j. In other words, equal matrices are identical. Matrices can be added or subtracted only if they have the same number of rows and columns, in which case the elements of the resultant matrix are given by $a_{ij} + b_{ij}$. Thus, if

$$\mathsf{A} = \begin{pmatrix} -3 & 6 & 4 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} 2 & 1 & 1 \\ -6 & 4 & 3 \end{pmatrix}$$

then

$$\mathsf{C} = \mathsf{A} + \mathsf{B} = \begin{pmatrix} -1 & 7 & 5 \\ -5 & 4 & 5 \end{pmatrix}$$

If we write

$$\mathsf{A} + \mathsf{A} = 2\mathsf{A} = \begin{pmatrix} -6 & 12 & 8 \\ 2 & 0 & 4 \end{pmatrix}$$

we see that scalar multiplication of a matrix means that each element is multiplied by the scalar. Thus,

$$c\,\mathsf{M} = \begin{pmatrix} cM_{11} & cM_{12} \\ cM_{21} & cM_{22} \end{pmatrix} \tag{3}$$

Example 2:
Using the matrices A and B above, form the matrix $\mathsf{D} = 3\,\mathsf{A} - 2\,\mathsf{B}$.

SOLUTION:

$$\mathsf{D} = 3\begin{pmatrix} -3 & 6 & 4 \\ 1 & 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 2 & 1 & 1 \\ -6 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 18 & 12 \\ 3 & 0 & 6 \end{pmatrix} - \begin{pmatrix} 4 & 2 & 2 \\ -12 & 8 & 6 \end{pmatrix} = \begin{pmatrix} -13 & 16 & 10 \\ 15 & -8 & 0 \end{pmatrix}$$

One of the most important operations involving matrices is matrix multiplication. For simplicity, we will discuss the multiplication of square matrices first. Consider some linear transformation of (x_1, y_1) into (x_2, y_2):

$$\begin{aligned} x_2 &= a_{11}\,x_1 + a_{12}\,y_1 \\ y_2 &= a_{21}\,x_1 + a_{22}\,y_1 \end{aligned} \tag{4}$$

represented by the matrix equation

$$\mathsf{AV}_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \mathsf{V}_2 \tag{5}$$

Now let's transform $(x_2,\ y_2)$ into $(x_3,\ y_3)$:

$$\begin{aligned} x_3 &= b_{11}\,x_2 + b_{12}\,y_2 \\ y_3 &= b_{21}\,x_2 + b_{22}\,y_2 \end{aligned} \tag{6}$$

represented by the matrix equation

$$\mathsf{BV}_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \mathsf{V}_3 \tag{7}$$

Let the transformation of $(x_1,\ y_1)$ directly into $(x_3,\ y_3)$ be given by

$$\begin{aligned} x_3 &= c_{11}\,x_1 + c_{12}\,y_1 \\ y_3 &= c_{21}\,x_1 + c_{22}\,y_1 \end{aligned} \tag{8}$$

represented by the matrix equation

$$\mathsf{CV}_1 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \mathsf{V}_3 \tag{9}$$

Symbolically, we can write that

$$\mathsf{V}_3 = \mathsf{CV}_1 = \mathsf{BV}_2 = \mathsf{BAV}_1$$

or that

$$\mathsf{C} = \mathsf{BA}$$

Let's find the relation between the elements of C and those of A and B. Substitute Equations 4 into 6 to obtain

$$\begin{aligned} x_3 &= b_{11}(a_{11}\,x_1 + a_{12}\,y_1) + b_{12}(a_{21}\,x_1 + a_{22}\,y_1) \\ y_3 &= b_{21}(a_{11}\,x_1 + a_{12}\,y_1) + b_{22}(a_{21}\,x_1 + a_{22}\,y_1) \end{aligned} \tag{10}$$

or

$$\begin{aligned} x_3 &= (b_{11}\,a_{11} + b_{12}\,a_{21})x_1 + (b_{11}\,a_{12} + b_{12}\,a_{22})y_1 \\ y_3 &= (b_{21}\,a_{11} + b_{22}\,a_{21})x_1 + (b_{21}\,a_{12} + b_{22}\,a_{22})y_1 \end{aligned}$$

Thus, we see that

$$\mathsf{C} = \mathsf{BA} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11}\,a_{11} + b_{12}\,a_{21} & b_{11}\,a_{12} + b_{12}\,a_{22} \\ b_{21}\,a_{11} + b_{22}\,a_{21} & b_{21}\,a_{12} + b_{22}\,a_{22} \end{pmatrix} \tag{11}$$

This result may look complicated, but it has a nice pattern which we will illustrate two ways. Mathematically, the ijth element of C is given by the formula

$$c_{ij} = \sum_k b_{ik}\, a_{kj} \tag{12}$$

Note that the right side of Equation 12 is the dot product of a row of B into a column of A. For example,

$$c_{11} = \sum_k b_{1k}\, a_{k1} = b_{11}\, a_{11} + b_{12}\, a_{21}$$

as in Equation 11. A more pictorial way is to notice that any element in C can be obtained by multiplying elements in any row in B by the corresponding elements in any column in A, adding them, and then placing them in C where the row and column intersect. For example, c_{11} is obtained by multiplying the elements of row 1 of B with the elements of column 1 of A, or by the scheme

$$\begin{array}{c} \\ \rightarrow \end{array}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{array}{c} \downarrow \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array} = \begin{pmatrix} b_{11}\, a_{11} + b_{12}\, a_{21} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

and c_{12} by

$$\begin{array}{c} \\ \rightarrow \end{array}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{array}{c} \qquad\downarrow \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array} = \begin{pmatrix} \cdot & b_{11}\, a_{12} + b_{12}\, a_{22} \\ \cdot & \cdot \end{pmatrix}$$

Example 3:
Find C = BA if

$$\mathsf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \mathsf{A} = \begin{pmatrix} -3 & 0 & -1 \\ 1 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

SOLUTION:

$$\begin{aligned} \mathsf{C} &= \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 & -1 \\ 1 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -3+2+1 & 0+8+1 & -1+0+1 \\ -9+0-1 & 0+0-1 & -3+0-1 \\ 3-1+2 & 0-4+2 & 1+0+2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 9 & 0 \\ -10 & -1 & -4 \\ 4 & -2 & 3 \end{pmatrix} \end{aligned}$$

Example 4:
The matrix R given by Equation 2 represents a rotation through the angle θ. Show that R^2 represents a rotation through an angle 2θ.

SOLUTION:

$$R^2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{pmatrix}$$

Using standard trigonometric identities, we get

$$R^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

which represents rotation through an angle 2θ.

Matrices do not have to be square to be multiplied together, but either Equation 11 or the pictorial method illustrated above suggests that the number of columns of B must be equal to the number of rows of A. When this is so, A and B are said to be *compatible*. We call a matrix having n rows and m columns an $n \times m$ matrix. Thus, an $n \times m$ matrix can multiply into only an $m \times p$ matrix and produces an $n \times p$ matrix.

For example, the product of a 2×3 matrix and a 3×3 matrix produces a 2×3 matrix:

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ -2 & 6 & -2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -4 & 10 \\ -3 & 17 & -4 \end{pmatrix}$$

An important aspect of matrix multiplication is that BA does not usually equal AB. For example, if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

so $AB \neq BA$. If it does happen that $AB = BA$, then A and B are said to *commute*.

Example 5:
Do the matrices A and B commute if

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

SOLUTION:

$$AB = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

so they do not commute.

Another property of matrix multiplication that differs from ordinary scalar multiplication is that the equation

$$AB = 0$$

where 0 is the *zero matrix* or the *null matrix* (all elements equal to zero) does *not* imply that A or B necessarily is a zero matrix. For example,

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Although matrices correspond to transformations and should not be confused with determinants, we can associate a determinant with a square matrix. In fact, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

then

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{13}$$

If $\det A \neq 0$, then A is said to be *nonsingular*. Conversely, if $\det A = 0$, then A is said to be *singular*. A useful property of the determinants of matrices is that

$$\det AB = (\det A)(\det B) \tag{14}$$

(See Problem 21). Of course, A and B must both be square matrices and of the same dimension.

Example 6:
Given

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 3 \\ 2 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

Show that $|AB| = |A|\,|B|$.

SOLUTION: First calculate AB:

$$AB = \begin{pmatrix} 4 & 0 & -1 \\ 1 & 1 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -6 & 3 & 0 \\ 6 & 4 & 1 \\ -4 & 1 & 0 \end{pmatrix}$$

Then $|A| = -2$, $|B| = 3$, and $|AB| = -6$.

The determinant of a matrix A is used in the construction of the inverse of A, which we define below.

A transformation that leaves (x_1, y_1) unaltered is called the identity transformation, and the corresponding matrix is called the *identity matrix* or the *unit matrix*. All the elements of the identity matrix are equal to zero, except those along the main diagonal, which equal one:

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

A unit matrix is necessarily a square matrix. The elements of I are δ_{ij}, the Kronecker delta, which equals one when $i = j$ and zero when $i \neq j$. The unit matrix has the property that

$$IA = AI \tag{15}$$

The unit matrix is an example of a *diagonal matrix*. The only nonzero elements of a diagonal matrix are along its main diagonal. Generally, the elements on the main diagonal of a matrix are called *diagonal elements* and the others are called *off-diagonal elements*. Thus, we can say that all the off-diagonal elements of a diagonal matrix are zero. Diagonal matrices are necessarily square matrices. Also, any two $n \times n$ diagonal matrices commute with each other.

If $BA = AB = I$, then B is said to be the *inverse* of A, and is denoted by A^{-1}. Thus, A^{-1} has the property that

$$AA^{-1} = A^{-1}A = I \tag{16}$$

If A represents some transformation, then A^{-1} undoes that transformation and restores the original state. The fact that $AA^{-1} = A^{-1}A$ implies that A must be a square matrix. It should be clear on physical grounds that the inverse of R in Equation 2 is a rotation through the angle $-\theta$. Thus, we write

$$\mathrm{R}^{-1}(\theta) = \mathrm{R}(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{17}$$

which is obtained from R by replacing θ by $-\theta$. It's easy to show that $\mathrm{R}(\theta)\mathrm{R}^{-1}(\theta) = \mathrm{R}^{-1}(\theta)\mathrm{R}(\theta) = \mathrm{I}$.

We found the inverse of $\mathrm{R}(\theta)$ in Equation 2 by a physical argument, but how do we find the inverse of a (square) matrix in general? It turns out that we essentially derived the formula for the inverse of A in Section 1. Equation 19 of that section is

$$\sum_{k=1}^{n} \left(\frac{A_{ki}}{|\mathrm{A}|}\right) a_{kj} = \delta_{ij} \tag{18}$$

The quantities A_{ki} are the cofactors of the a_{ki} of A, and the right side of Equation 18 are the elements of a unit matrix.

Let's first define a *matrix of cofactors* of A by

$$\mathrm{A}_{\mathrm{cof}} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \tag{19}$$

Now we define the *transpose* A^{T} of a matrix A to be the matrix that is obtained by interchanging the rows and columns of A. In terms of the matrix elements of a general matrix (a_{ij}), we have $a_{ij}^{\mathrm{T}} = a_{ji}$. For example, if

$$\mathrm{A} = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 1 & 2 \\ -2 & -1 & -3 \end{pmatrix}, \quad \text{then} \quad \mathrm{A}^{\mathrm{T}} = \begin{pmatrix} 3 & -1 & -2 \\ 0 & 1 & -1 \\ 1 & 2 & -3 \end{pmatrix}$$

Notice that we can also form A^{T} from A by simply flipping A about its main diagonal.

We can now write the term in parentheses in Equation 18 as the ikth element of $\mathrm{A}_{\mathrm{cof}}^{\mathrm{T}}/|\mathrm{A}|$, so that Equation 18 becomes, in matrix notation,

$$\frac{\mathrm{A}_{\mathrm{cof}}^{\mathrm{T}}}{|\mathrm{A}|}\mathrm{A} = \mathrm{I} \tag{20}$$

where I is a unit matrix. Thus, we see that if $\det \mathrm{A} \neq 0$, then the inverse of A is given by

$$\mathrm{A}^{-1} = \frac{\mathrm{A}_{\mathrm{cof}}^{\mathrm{T}}}{|\mathrm{A}|} \tag{21}$$

Some authors call $\mathrm{A}_{\mathrm{cof}}^{\mathrm{T}}$ the *adjoint of* A, written as adj (A). One clear implication of Equation 21 is that A must be nonsingular. Singular matrices do not have inverses.

Equation 21 may look awkward to use, but it's pretty straightforward. Let's find the inverse of

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

The determinant of A is equal to -4 and the matrix of cofactors is

$$A_{\text{cof}} = \begin{pmatrix} -2 & 2 & -2 \\ 0 & 0 & -4 \\ -1 & -1 & 1 \end{pmatrix}$$

Using Equation 21, we have

$$A^{-1} = -\frac{1}{4}\begin{pmatrix} -2 & 0 & -1 \\ 2 & 0 & -1 \\ -2 & -4 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 1 & -\frac{1}{4} \end{pmatrix}$$

It's readily verified that $A^{-1}A = AA^{-1} = I$.

Example 7:
Find the inverse of

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 3 \\ 0 & -2 & 1 \end{pmatrix}$$

SOLUTION: $\det A = 16$ and

$$A_{\text{cof}} = \begin{pmatrix} 7 & 1 & 2 \\ -2 & 2 & 4 \\ -1 & -7 & 2 \end{pmatrix}$$

and so

$$A^{-1} = \frac{1}{16}\begin{pmatrix} 7 & -2 & -1 \\ 1 & 2 & -7 \\ 2 & 4 & 2 \end{pmatrix}$$

An example of a matrix that has no inverse, which occurs in a number of physical applications, is a matrix that corresponds to a projection of a vector onto a coordinate axis. For example, the matrix

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

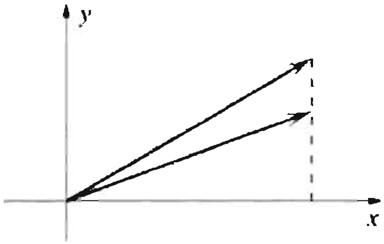

Figure 9.5
An illustration of why a matrix corresponding to a projection does not have an inverse. Both vectors have the same projection onto the x axis.

projects the vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ onto the x axis; that is,

$$\mathsf{P}\,\mathbf{r} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Mathematically, P has no inverse because $|\mathsf{P}| = 0$. Physically, P has no inverse because any vector $\mathbf{r}$ with an x component x_0 will yield the same result, as you can see in Figure 9.5.

Finally, we mention that any CAS can readily find inverses of matrices (Problems 23 and 24).

9.3 Problems

1. Given the two matrices $\mathsf{A} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} -1 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, form the matrices $\mathsf{C} = 2\mathsf{A} - 3\mathsf{B}$ and $\mathsf{D} = 6\mathsf{B} - \mathsf{A}$.

2. Given the three matrices $\mathsf{A} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathsf{B} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\mathsf{C} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, show that $\mathsf{A}^2 + \mathsf{B}^2 + \mathsf{C}^2 = \frac{3}{4}\mathsf{I}$, where I is a unit matrix. Also show that

$$\begin{aligned} \mathsf{AB} - \mathsf{BA} &= i\mathsf{C} \\ \mathsf{BC} - \mathsf{CB} &= i\mathsf{A} \\ \mathsf{CA} - \mathsf{AC} &= i\mathsf{B} \end{aligned}$$

3. Given the matrices $\mathsf{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\mathsf{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, and $\mathsf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, show that

$$\begin{aligned} \mathsf{AB} - \mathsf{BA} &= i\mathsf{C} \\ \mathsf{BC} - \mathsf{CB} &= i\mathsf{A} \\ \mathsf{CA} - \mathsf{AC} &= i\mathsf{B} \\ \mathsf{A}^2 + \mathsf{B}^2 + \mathsf{C}^2 &= 2\mathsf{I} \end{aligned}$$

where I is a unit matrix.

4. Does $(\mathsf{A} + \mathsf{B})^2$ always equal $\mathsf{A}^2 + 2\mathsf{AB} + \mathsf{B}^2$? Does $(\mathsf{AB})^2 = \mathsf{A}^2\mathsf{B}^2$?

5. A three-dimensional rotation of a vector about the z axis can be represented by the matrix $\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Show that $\det \mathsf{R} = |\mathsf{R}| = 1$ and that $\mathsf{R}^{-1} = \mathsf{R}(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

6. Show that (a) $(\mathsf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathsf{A}$; (b) $(\mathsf{A} + \mathsf{B})^{\mathsf{T}} = \mathsf{A}^{\mathsf{T}} + \mathsf{B}^{\mathsf{T}}$; (c) $(\alpha\mathsf{A})^{\mathsf{T}} = \alpha\mathsf{A}^{\mathsf{T}}$; (d) $(\mathsf{AB})^{\mathsf{T}} = \mathsf{B}^{\mathsf{T}}\mathsf{A}^{\mathsf{T}}$.

7. Given the matrices $C_3 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, $\sigma_v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_v' = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, and $\sigma_v'' = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, show that $\sigma_v C_3 = \sigma_v''$, $C_3\sigma_v = \sigma_v'$, $\sigma_v''\sigma_v' = C_3$, and $C_3\sigma_v'' = \sigma_v$. Calculate the determinant associated with each matrix.

8. If $A^T = A^{-1}$, then A is said to be *orthogonal*. Which of the matrices in Problem 7 are orthogonal?

9. Find the matrix of cofactors, A_{cof}, and the adjoint of A, adj (A), for

(a) $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

10. Verify that $(AB)^T = B^TA^T$ if $A = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$.

11. Prove that A^{-1} is unique.

12. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. (a) Find a nonzero matrix B such that AB = 0. Does BA = 0? (b) Can you find B such that AB = 0 if $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$?

13. Prove that (a) $(A^{-1})^{-1} = A$; (b) $(A^T)^{-1} = (A^{-1})^T$.

14. Prove that $\det(A^{-1}) = (\det A)^{-1}$. *Hint*: Use the relation $\det AB = (\det A)(\det B)$.

15. Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

16. Use $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ to verify the relations in Problems 13 through 15.

17. Use $A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ to verify the relations in Problems 13 through 15.

18. Find the inverse of (a) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, and (c) $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$.

19. Solve the equations

$$x + y = 3$$
$$4x - 3y = 5$$

by writing them as AX = H and then $X = A^{-1}H$.

20. Show that two $n \times n$ diagonal matrices commute. Does an $n \times n$ diagonal matrix necessarily commute with any $n \times n$ matrix?

21. The general proof that $\det(AB) = (\det A)(\det B)$ is fairly long and so we shall not prove it here. Nevertheless, verify that it is true for 2×2 matrices.

22. A matrix that satisfies the relation $A^2 = A$ is called *idempotent*. Show that if A has an inverse, then it must be the identity matrix. Argue that a projection matrix must be idempotent. Does a projection matrix have an inverse?

23. Use any CAS to find the inverse of $\begin{pmatrix} 2 & 5 & 1 \\ 3 & 1 & 2 \\ -2 & 1 & 0 \end{pmatrix}$.

24. Use any CAS to find the inverse of $\begin{pmatrix} 1 & 0 & 3 & -2 \\ 6 & 1 & -1 & 3 \\ 2 & 0 & 1 & 1 \\ 4 & 3 & 2 & 5 \end{pmatrix}$.

9.4 Rank of a Matrix

In Section 2, we used Gaussian elimination to solve sets of linear algebraic equations and saw that we could have a unique solution, an infinity of solutions, or no solutions. Gaussian elimination leads directly to the correct result in each case, but it would be nice to have a general theory that tells us beforehand what to expect about the solutions. Surely, the nature of the solutions depends upon some property of the coefficient matrix and/or the augmented matrix since they describe the system of equations completely. This property is the *rank* of a matrix, which is the subject of this section.

There are several equivalent definitions of rank. One definition of rank is expressed in terms of square submatrices of A. A *square submatrix* of A is any square matrix obtained from A by deleting a certain number of rows and columns. If A happens to be square, then A is a submatrix of itself, obtained by deleting no rows and no columns. For example, consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

The possible square submatrices of A are the 2×2 matrices

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

and the 1×1 matrices (1), (2), (3), and (4).

The rank, $r(A)$, of A is the order of the largest square submatrix of A whose determinant is not equal to zero. The rank of the matrix A, above, is 2. The rank of

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

is also 2 because the submatrix $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ is nonsingular, even though the other two 2×2 submatrices are singular.

Example 1:
Determine the rank of

$$A = \begin{pmatrix} 2 & 1 & 0 & 5 \\ 3 & 6 & 1 & 1 \\ 5 & 7 & 1 & 8 \end{pmatrix}$$

SOLUTION: The largest that $r(A)$ can be is 3 since the largest possible square submatrix of A is 3×3, and there are four 3×3 square submatrices. The determinant of the 3×3 submatrix obtained by striking out the fourth column is zero, but the determinants of the other three 3×3 submatrices are not equal to zero. Therefore, $r(A) = 3$.

Another, perhaps more convenient but nevertheless equivalent, definition of rank is the number of nonzero rows in the matrix after it has been transformed into echelon form by elementary row operations. Since we are going to base the defintion of rank on the echelon form of a matrix, we should give a formal definition of what we mean by echelon form. We say that a matrix is in echelon if

1. All rows consisting of all zeros appear at the bottom.
2. If the first nonzero element of a row appears in column c, then all the elements in column c in lower rows are zero.
3. The first nonzero element of any nonzero row appears to the right of the first nonzero element in any higher row.

All the final versions of the augmented matrices in Section 2 were in echelon form.

Let's determine the rank of the matrix in Example 1 by this method. In obvious notation,

$$\begin{pmatrix} 2 & 1 & 0 & 5 \\ 3 & 6 & 1 & 1 \\ 5 & 7 & 1 & 8 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 & 5 \\ 0 & 9/2 & 1 & -13/2 \\ 0 & 9/2 & 1 & -9/2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 & 5 \\ 0 & 9/2 & 1 & -13/2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

There are three nonzero rows, so $r(A) = 3$.

Example 2:
Determine the rank of

$$A = \begin{pmatrix} 3 & 2 & 1 & -4 & 1 \\ 2 & 3 & 0 & -1 & -1 \\ 1 & -6 & 3 & -8 & 7 \end{pmatrix}$$

SOLUTION: Rearrange the rows so that the left-most column reads 1,2,3 (this avoids introducing fractions). Now add -2 times row 1 to row 2 and

-3 times row 1 to row 3 to obtain

$$\begin{pmatrix} 1 & -6 & 3 & -8 & 7 \\ 0 & 15 & -6 & 15 & -15 \\ 0 & 20 & -8 & 20 & -20 \end{pmatrix}$$

The last two rows are a constant multiple of each other, so if we multiply row 2 by $-4/3$ and add the result to row 3, we get

$$\begin{pmatrix} 1 & -6 & 3 & -8 & 7 \\ 0 & 15 & -6 & 15 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two nonzero rows, so the rank of A is 2.

There are ten 3×3 submatrices of the matrix in Example 2 (can you show this?), so you would have to evaluate ten 3×3 determinants just to find out the rank of A is not equal to 3. This result suggests that the row echelon method is usually much easier to apply than our first definition of the rank of a matrix. Nevertheless, the definition of rank in terms of the largest nonsingular square submatrix of A is a standard definition.

There is another definition of rank that we will present here for completeness. We say that the m nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are *linearly dependent* if there exist constants $c_1, c_2, \ldots, c_m$ not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = 0 \tag{1}$$

Linear dependence means that one of the vectors can be written as a linear combination of the others. If Equation 1 is satisfied only if all the $c_j = 0$, then the vectors are said to be *linearly independent*. In three dimensions, the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are linearly independent, but any other vector in three dimensions can be written as a linear combination of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ ($\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$).

We now define the rank of a matrix in terms of linear independence of vectors. Recall from Chapter 5 that a vector can be represented by an ordered n-tuple of numbers, $(v_1, v_2, \ldots, v_n)$, where we can think of the v_j as the components of $\mathbf{v}$ in some coordinate system. We think of the rows of the matrix A as vectors. If A is $n \times m$, then we have n m-dimensional vectors constituting the rows of A. The rank of A is the maximum number of linearly independent vectors that can be formed from these row vectors. In practice, this fundamental definition of rank isn't that useful because it often isn't easy to use Equation 1 to determine if a set of vectors is linearly independent or not. In fact, the easiest way to determine if a set of vectors is linearly independent or linearly dependent is to use the echelon matrix procedure to determine the rank of A, and hence the number of linearly independent vectors. Nevertheless, this definition is useful in theoretical discussions. Of course, the three definitions of rank that we have presented are equivalent.

To see more clearly the relation between rank and the number of linearly independent rows of a matrix, consider the following matrix in echelon form:

$$A = \begin{pmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row 1 cannot possibly be a linear combination of rows 2 and 3 because they have zeros in their first entries and so any linear combination of rows 2 and 3 must have a zero in its first position. Similarly, row 2 cannot be a multiple of row 3 because it has a zero in its first and second entries. Working from Row 3 upwards now, notice that no row can be a linear combination of higher rows because of the positions of the leading zeros in each row. Thus, there are three linearly independent vectors in this rank 3 matrix.

Example 3:
Determine whether the three vectors $(3, 2, 1, -4, 1)$, $(2, 3, 0, -1, -1)$, and $(1, -6, 3, -8, 7)$ are linearly independent.

SOLUTION:

$$A = \begin{pmatrix} 1 & -6 & 3 & -8 & 7 \\ 2 & 3 & 0 & -1 & -1 \\ 3 & 2 & 1 & -4 & 1 \end{pmatrix}$$

(We have arranged the rows in this way to avoid fractions.) This is the same matrix as in Example 2, where we determined that the rank of A is 2. Therefore, only two of the three vectors are linearly independent.

We now present a theorem on the existence of solutions to a set of m linear algebraic equations in n unknowns in terms of rank.

Let A *be the* $m \times n$ *coefficient matrix of the set of* m *linear algebraic equations* AX = H *and let* A|H *be the* $m \times (n + 1)$ *augmented matrix of the system. If*

1. $r(\mathrm{A}) = r(\mathrm{A|H}) = n$, *there is a unique solution.*
2. $r(\mathrm{A}) = r(\mathrm{A|H}) < n$, *there are infinitely many solutions, expressible in terms of* $n - r(A)$ *parameters.*
3. $r(\mathrm{A}) < r(\mathrm{A|H})$, *there are no solutions.*

This theorem summarizes all the possible cases for all linear systems, homogeneous or nonhomogeneous. Let's go back and examine each of the cases in

Section 2 in terms of the ranks of A and A|H. For Equations 5,

$$\mathsf{A} = \begin{pmatrix} 2 & 1 & 3 \\ 2 & -2 & -1 \\ -2 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathsf{A|H} = \left(\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 2 & -2 & -1 & 1 \\ -2 & 4 & 1 & 1 \end{array}\right)$$

Both $r(\mathsf{A})$ and $r(\mathsf{A|H}) = 3$, so Equations 5 have a unique solution.

For Example 1,

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 3 & 2 & -2 \end{pmatrix} \quad \text{and} \quad \mathsf{A|H} = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 3 & 5 \\ 3 & 2 & -2 & 5 \end{array}\right)$$

In this case, $r(\mathsf{A}) = r(\mathsf{A|H}) = 3$, and so the solution is unique.

For Example 2,

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathsf{A|H} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & -2 \end{array}\right)$$

In this case, $r(\mathsf{A}) = r(\mathsf{A|H}) = 2 < 3$, and so there are infinitely many solutions, expressible in terms of one parameter.

For Equation 7,

$$\mathsf{A} = \begin{pmatrix} 1 & 1 \\ 3 & -4 \\ 5 & -2 \end{pmatrix} \quad \text{and} \quad \mathsf{A|H} = \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & -4 & 9 \\ 5 & -2 & 17 \end{array}\right)$$

In this case, $r(\mathsf{A}) = r(\mathsf{A|H}) = 2$, and so the solution is unique. The rest of the cases are left to the Problems.

Before leaving this section, we shall present a theorem regarding homogeneous sets of linear algebraic equations. Even though the above theorem applies to both homogeneous and nonhomogeneous systems, homogeneous systems occur quite often in physical problems, so we'll present the implications of the above general theorem to homogeneous systems.

> *The $m \times n$ **homogeneous** system $\mathsf{AX} = 0$ always has a trivial solution, $x_1 = x_2 = \cdots = x_n = 0$. [It is always consistent because $r(\mathsf{A}) = r(\mathsf{A|H})$.]*

If $r(\mathsf{A}) = n$, then the trivial solution is the only solution. If $r(\mathsf{A}) < n$, then the general theorem assures the existence of non-trivial solutions. In particular, these non-trivial solutions constitute an $[n - r(A)]$-parameter family of solutions.

An $n \times n$ homogeneous system has a property that we shall emphasize here by setting it off:

> *The $n \times n$ homogeneous system of linear algebraic equations $\mathsf{AX} = 0$ has a non-trivial solution if and only if $\det \mathsf{A} = 0$.*

This last theorem follows directly from everything above, but it is important enough to emphasize.

Example 4:
Determine the values of x such that the equations

$$\begin{aligned} xc_1 + c_2 \qquad\qquad &= 0 \\ c_1 + xc_2 + c_3 \qquad &= 0 \\ c_2 + xc_3 + c_4 &= 0 \\ c_3 + xc_4 &= 0 \end{aligned}$$

have non-trivial solutions for the c_j. (This set of equations occurs in a quantum-mechanical calculation for a butadiene molecule.)

SOLUTION: To assure a non-trivial solution, the determinant of the coefficient matrix must vanish.

$$\begin{vmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{vmatrix} = 0$$

Expanding in cofactors about the first row gives

$$x(x^3 - 2x) - (x^2 - 1) = 0$$

or $x^4 - 3x^2 + 1 = 0$, or $x^2 = (3 \pm \sqrt{5})/2$, or $x = \pm 1.61804$ and ± 0.61804.

9.4 Problems

Use the concept of rank to investigate the nature of the solutions in Problems 1 through 12.

1. Example 4 of Section 2.
2. Equations 8 of Section 2.
3. Example 5 of Section 2.
4. Problem 1 of Section 2.
5. Problem 2 of Section 2.
6. Problem 3 of Section 2.
7. Problem 4 of Section 2.
8. Problem 5 of Section 2.
9. Problem 6 of Section 2.
10. Problem 7 of Section 2.
11. Problem 8 of Section 2.
12. Problem 9 of Section 2.

13. For what values of x will the following equations have non-trivial solutions?

$$\begin{aligned} xc_1 + c_2 \qquad\quad + c_4 &= 0 \\ c_1 + xc_2 + c_3 \qquad &= 0 \\ c_2 + xc_3 + c_4 &= 0 \\ c_1 \qquad + c_3 + xc_4 &= 0 \end{aligned}$$

14. Determine the values of x for which the following equations will have a non-trivial solution:

$$\begin{aligned} xc_1 + c_2 + c_3 + c_4 &= 0 \\ c_1 + xc_2 \qquad + c_4 &= 0 \\ c_1 \qquad + xc_3 + c_4 &= 0 \\ c_1 + c_2 + c_3 + xc_4 &= 0 \end{aligned}$$

9.5 Vector Spaces

Although we didn't point it out explicitly in the previous section, matrices obey the following algebraic rules:

1. $\mathsf{A} + \mathsf{B} = \mathsf{B} + \mathsf{A}$
2. $\mathsf{A} + (\mathsf{B} + \mathsf{C}) = (\mathsf{A} + \mathsf{B}) + \mathsf{C}$
3. $a(\mathsf{A} + \mathsf{B}) = a\mathsf{A} + a\mathsf{B}$
4. $(a + b)\mathsf{A} = a\mathsf{A} + b\mathsf{A}$
5. $a(b\mathsf{A}) = (ab)\mathsf{A}$

where a and b are scalars. It so happens that many other mathematical quantities obey the same set of rules. For example, complex numbers, vectors, and functions obey these rules. There is a mathematical formalism that treats all these quantities in an abstract unified manner and allows us to see the similarities between them.

We define a *vector space* V to be a set of objects (which we'll call *vectors*) for which addition and multiplication by a scalar, either real or complex, are defined and satisfy the following requirements:

1. Two vectors, $\mathbf{x}$ and $\mathbf{y}$, may be added to give another vector, $\mathbf{x} + \mathbf{y}$, which is also in V. (We say that the set is *closed* under addition.)
2. Addition is commutative; in other words, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
3. Addition is associative; in other words, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}) = \mathbf{x} + \mathbf{y} + \mathbf{z}$.
4. There exists in V a unique *zero vector*, $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ for any $\mathbf{x}$ in V.
5. For every $\mathbf{x}$ in V, there is an additive inverse $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
6. Any vector $\mathbf{x}$ may be multiplied by a scalar c such that $c\mathbf{x}$ is in V. In other words, the set is closed under scalar multiplication.

7. Scalar multiplication is associative; in other words, for any two numbers a and b, $a(b\mathbf{x}) = (ab)\mathbf{x}$.
8. Scalar multiplication is distributive over addition; in other words,

$$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y} \text{ and } (a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}.$$

9. For the unit scalar, $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in V.

Properties 1 through 9 are called the *axioms of a vector space*. If the scalars are real numbers, V is called a *real vector space*; if they are complex, V is called a *complex vector space*.

The geometric vectors that we discussed in Chapter 5 satisfy all the above properties of a vector space, and form what is called a Euclidean vector space, in particular. The elements or members of a vector space, however, need not be geometric vectors. For example, the set of all nth order polynomials, P_n, with real or complex coefficients, forms a vector space, as long as we consider mth order polynomials ($m < n$) to be nth order polynomials with certain zero coefficients. Another vector space consists of all n-tuples of real numbers, where the sum of $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ is defined as $(u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$ and the product of an n-tuple by a scalar is defined as $a(u_1, u_2, \ldots, u_n) = (au_1, au_2, \ldots, au_n)$ (Problem 1). (We'll use ordered n-tuples fairly often to illustrate the properties of vector spaces, so we'll designate the space of all ordered n-tuples of real numbers by R^n and that of complex numbers by C^n.)

Example 1:
Show that the set of functions whose first derivatives are continuous in $[a, b]$ and satisfy

$$\frac{df}{dx} + 2f(x) = 0$$

form a vector space.

SOLUTION: To show that the set is closed under addition (1), let f and g be two elements of V (in other words, both f and g satisfy the above equation). Then,

$$\begin{aligned}\frac{d}{dx}(f + g) + 2(f + g) &= \frac{df}{dx} + \frac{dg}{dx} + 2f + 2g \\ &= \left(\frac{df}{dx} + 2f\right) + \left(\frac{dg}{dx} + 2g\right) = 0 + 0 = 0\end{aligned}$$

To show that the set is closed under scalar multiplication (6), consider

$$\frac{d}{dx}(af) + 2(af) = a\left(\frac{df}{dx} + 2f\right) = 0$$

The other axioms are satisfied by any continuous function.

Example 1 suggests that the set of solutions to any linear homogeneous differential equation forms a vector space (Chapter 11).

It often happens that a subset of the vectors in V forms a vector space with respect to the same addition and multiplication operations as V. In such a case, the set of vectors is said to form a *subspace* of V. A simple geometric example of a subspace is the xy-plane of a three-dimensional Euclidean space. The set of all vectors that lie in the xy-plane forms a vector space. To see another example of a subspace, consider the vector space R^n made up of ordered n-tuples of real numbers $(u_1, u_2, \ldots, u_n)$. The set of n-tuples $(a, a, \ldots, a)$ forms a subspace of R^n (Problem 5).

An important concept associated with vector spaces is the linear independence and linear dependence of vectors. We touched upon this idea in the previous section, but we shall study it more thoroughly here. Let $\{\mathbf{v}_j;\ j = 1, 2, \ldots, n\}$ be a set of nonzero vectors from a vector space V. We say that the set of vectors is *linearly independent* if the only way that

$$\sum_{j=1}^{n} c_j \mathbf{v}_j = 0$$

is for each and every $c_j = 0$. If the vectors are not independent, then they are *linearly dependent*. There are several convenient ways to determine if a set of vectors is independent or not. Let's test the three vectors $(1, 0, 0)$, $(1, -1, 1)$, and $(1, 2, -1)$ for linear independence. Is there a set of c_j, not all zero, such that

$$\sum_{j=1}^{n} c_j \mathbf{v}_j = (c_1, 0, 0) + (c_2, -c_2, c_2) + (c_3, 2c_3, -c_3) = (0, 0, 0)$$

or is there a nontrivial solution to the equations

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_2 + 2c_3 &= 0 \\ c_2 - c_3 &= 0 \end{aligned} \tag{1}$$

The determinant of the matrix of coefficients is nonzero, so the only solution is the solution $c_1 = c_2 = c_3 = 0$, and so the three vectors are linearly independent. We could also have arranged the three vectors as the rows of a matrix and then transformed it into echelon form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

which shows that the three vectors are linearly independent.

It's easy to see from the echelon form approach that a set of m n-tuples *must* be linearly dependent if $m > n$ because the bottom $m - n$ rows will always be zeros. For example, consider the three vectors (1, 1), (1, −1), and (−1, 1). Then,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}$$

and so only two of the vectors are linearly independent. We can also see that the rank of A can equal 2 at the most.

Example 2:
How many linearly independent vectors are there in the set {(1, 1, 0, 1), (−1, −1, 0, −1), (1, 0, 1, 1), (−1, 0, −1, −1)}?

SOLUTION:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where we placed the two zero rows at the bottom. Thus, there are two linearly independent vectors in the set.

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in a vector space V and if any other vector $\mathbf{u}$ in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ so that

$$\mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n$$

where the c_j are constants, then we say that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *span* V. For example, the three unit vectors **i**, **j**, and **k** span the three-dimensional space R^3, as does any three non-coplanar (linearly independent) vectors. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V are linearly independent and span V, then the set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is called a *basis* or *basis set* for V. The unit vectors **i**, **j**, and **k**, or any three non-coplanar vectors, form a basis in R^3. The number of vectors in a basis is defined to be the *dimension* of the vector space. The dimension of a vector space V is equal to the maximum number of linearly independent vectors in V.

Suppose that $\{\mathbf{v}_j;\ j = 1, 2, \ldots, n\}$ is a set of linearly independent vectors in an n-dimensional vector space V. If the set composed of the $\mathbf{v}_j$ and any other (non-zero) vector $\mathbf{u}$ in V is linearly dependent, then the set $\{\mathbf{v}_j;\ j = 1, 2, \ldots, n\}$ is said to be *maximal*. Thus, the maximum number of linearly independent vectors in V is n. Because $\mathbf{u}$ and the $\mathbf{v}_j$ are linearly dependent, we can write

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n + c_u \mathbf{u} = 0$$

where the c's are not all zero. If $c_u = 0$, then the set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is linearly dependent, contrary to our assertion. Therefore, $c_u \neq 0$ and we can write

$$\mathbf{u} = -\frac{c_1}{c_u}\mathbf{v}_1 - \frac{c_2}{c_u}\mathbf{v}_2 - \cdots - \frac{c_n}{c_u}\mathbf{v}_n$$

We see, then, that $\mathbf{u}$ can be written as a linear combination of the set of linearly independent vectors, and so the $\mathbf{v}_j$ constitute a basis.

Example 3:
Show that the vectors (1, 0, 0), (0, 1, 0), and (0, 0, 1) form a basis in R^3. What familiar vectors do they correspond to?

SOLUTION: They form a basis in R^3 because any vector $\mathbf{u} = (x, y, z)$ in R^3 can be written as

$$\mathbf{u} = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

These vectors correspond to the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Example 4:
Show that the vector space V of ordered n-tuples is an n-dimensional space.

SOLUTION: The n vectors $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ constitute a linearly independent set of vectors in V. Furthermore, they span V because any other vector in V can be written as a linear combination of these vectors according to

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) = u_1(1, 0, \ldots, 0) + u_2(0, 1, \ldots, 0) + \cdots + u_n(0, 0, \ldots, 1)$$

Thus, the n vectors constitute a basis and the dimension of V is n.

Suppose that $\{\mathbf{v}_j;\ j = 1, 2, \ldots, n\}$ is a basis of V. Then any vector $\mathbf{u}$ of V can be written as a linear combination of the $\mathbf{v}_j$:

$$\mathbf{u} = \sum_{j=1}^{n} u_j \mathbf{v}_j$$

We say that u_j is the jth coordinate of $\mathbf{u}$ with respect to the given basis set. Problem 19 asks you to show that the coordinates of $\mathbf{u}$ with respect to a given basis in a given basis are unique.

When we study differential equations in Chapter 11, we'll see that the set of solutions to an nth order linear homogeneous differential equation forms a vector space. Consequently, it's not unusual to inquire about the linear independence of a set of functions over some interval I. In other words, we ask if the only way that

$$\sum_{j=1}^{n} c_j f_j(x) = 0 \tag{2}$$

is for $c_j = 0$ for $j = 1, 2, \ldots, n$. If that's the case, we say that the n functions $f_j(x)$, $j = 1, 2, \ldots, n$ are linearly independent. Otherwise, they are linearly dependent.

There is a convenient way to test for the linear independence of a set of functions. Start with Equation 2:

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \tag{3}$$

If the $f_j(x)$ are differentiable up to $(n-1)$th order over the interval I, differentiate Equation 3 $n-1$ times to obtain

$$\begin{array}{ccccccc} c_1 f_1'(x) & + c_2 f_2'(x) + & \cdots & + c_n f_n'(x) = 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_1 f_1^{(n-1)}(x) & + c_2 f_2^{(n-1)}(x) + & \cdots & + c_n f_n^{(n-1)}(x) = 0 \end{array} \tag{4}$$

The coefficient matrix of these equations is

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix} \tag{5}$$

The determinant $W(x)$ in Equation 5 is called the *Wronskian* of the functions $f_j(x)$. If $W(x) \neq 0$ at any point in the interval I, then the $f_j(x)$ are linearly independent.

Example 5:
Test the functions $\sin x$ and $\cos x$ for linear independence over the interval $-\infty < x < \infty$.

SOLUTION: The Wronskian is

$$W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

for all x, and so $\sin x$ and $\cos x$ are linearly independent for all values of x.

Unfortunately, the converse of the above result is not true. If $W(x) = 0$, the $f_j(x)$ may or may not be linearly independent, as the next Example shows. (See Problem 18 also.)

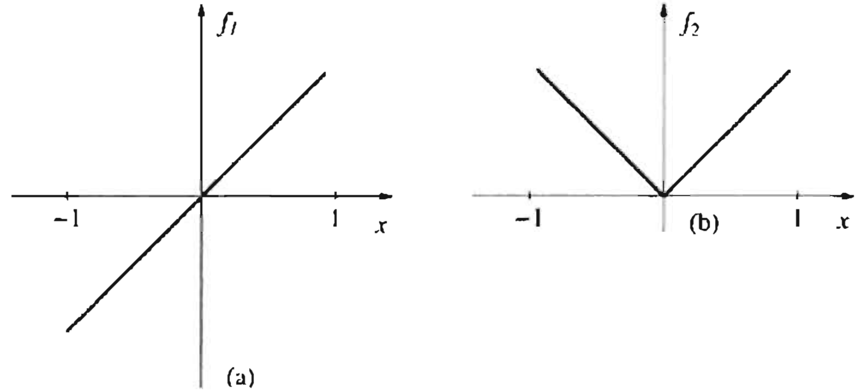

Figure 9.6
The functions $f_1(x) = x$ and $f_2(x) = |x|$ in Example 6 are linearly independent over the interval $-1 \le x \le 1$.

Example 6:
Show that $f_1(x) = x$ and $f_2(x) = |x|$ are linearly independent over the interval $-1 \le x \le 1$, but linearly dependent over the interval $0 \le x \le 1$.

SOLUTION: For the interval $[-1, 1]$,

$$W(x) = \begin{cases} \begin{vmatrix} x & x \\ 1 & -1 \end{vmatrix} = -2x & x \le 0 \\ \begin{vmatrix} x & x \\ 1 & 1 \end{vmatrix} = 0 & x \ge 0 \end{cases}$$

$W(x) \ne 0$ for $-1 \le x \le 0$, so $f_1(x)$ and $f_2(x)$ are linearly independent over the interval $[-1, 1]$ (Figure 9.6).

For the interval $[0, 1]$

$$W(x) = \begin{vmatrix} x & x \\ 1 & 1 \end{vmatrix} = 0 \qquad 0 \le x \le 1$$

and so this test does not tell us anything. However, $f_1(x)$ and $f_2(x)$ are identical over the interval $[0, 1]$, so they are linearly dependent (Figure 9.7).

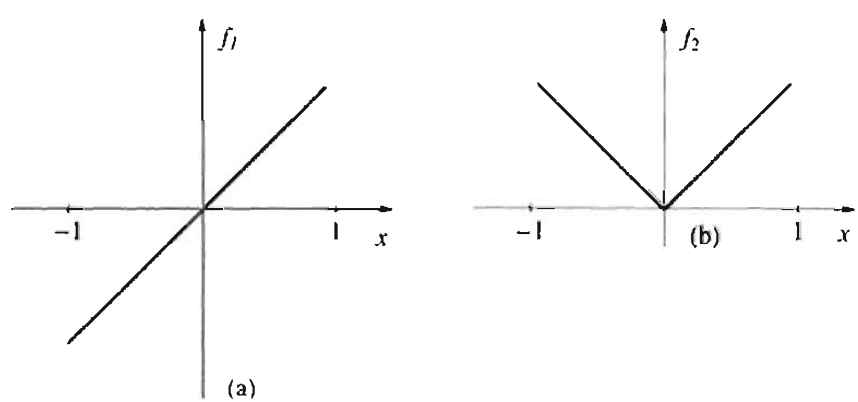

Figure 9.7
The functions $f_1(x) = x$ and $f_2(x) = |x|$ in Example 6 are linearly dependent over the interval $0 \le x \le 1$.

One final comment before we leave this section. It's easy to show (Problem 17) that if a set of differentiable functions $f_1(x)$, $f_2(x)$, . . . , $f_n(x)$ is linearly dependent on an interval I, then the Wronskian of these functions vanishes over the entire interval.

9.5 Problems

1. Show that the set of all ordered n-tuples of real numbers forms a vector space if addition of two n-tuples $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ is defined as $(u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$ and scalar multiplication is defined by $c(u_1, u_2, \ldots, u_n) = (cu_1, cu_2, \ldots, cu_n)$.

2. Show that the set of all two-dimensional geometric vectors forms a vector space.

3. Show that the set of all polynomials of degree less than or equal to 3 forms a vector space. What is its dimension?

4. Show that the set of functions that are continuous in the interval (a, b) forms a vector space.

5. Show that the set of n-tuples $(a, a, \ldots, a)$ is a subspace of R^n.

6. Let S be a subset of R^3 spanned by $(1, 1, 0)$ and $(1, 0, 1)$. Is S a subspace of V?

7. Test the following vectors for linear independence: $(0, 1, 0, 0)$, $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, and $(0, 0, 0, 1)$.

8. Test the following vectors for linear independence: $(1, 1, 1)$, $(1, -1, 1)$, and $(-1, 1, -1)$.

9. Is the vector $(1, 0, 2)$ in the set spanned by $(1, 1, 1)$, $(1, -1, -1)$, $(3, 1, 1)$?

10. Show that the set of vectors $\{(1, 1, 1, 1), (1, -1, 1, -1), (1, 2, 3, 4), (1, 0, 2, 0)\}$ is a basis for R^4.

11. Show that $(1, 1, 0)$ and $(1, 0, 1)$ are linearly independent in R^3 and find a third linearly independent vector.

12. Find the coordinates of $(1, 2, 3)$ with respect to the basis $(1, 1, 0)$, $(1, 0, 1)$, $(1, 1, 1)$ in R^3.

13. Use the Wronskian to test the three functions 1, $\sin x$, and $\cos x$ for linear independence.

14. Use the Wronskian to test the three functions e^x, e^{-x}, and $\sinh x$ for linear independence.

15. Evaluate the Wronskian of $f_1(x) = x^2$ and $f_2(x) = |x|x$ over the interval $[-1, 1]$. Are the functions linearly independent over the interval $[-1, 1]$? What about over the open interval $(0, 1]$?

16. Use the Wronskian to test for the linear independence of $f_1(x) = 1$, $f_2(x) = \sin^2 x$, and $f_3(x) = \cos^2 x$ for all x. If $W = 0$, can you check for linear independence by any other method?

17. We'll prove that if a set of $(n - 1)$ times differentiable functions $f_1(x)$, $f_2(x)$, . . . , $f_n(x)$ in an interval I is linearly dependent, their Wronskian vanishes identically on I. Argue that there must be nonzero constants in the expression $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for every x in I. Now form the set of n equations in the $f_j(x)$ and their first $n - 1$ derivatives. Why must the Wronskian equal zero?

18. This problem shows that the Wronskian of linearly independent functions may equal zero. First show that $f_1(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0 \end{cases}$ and $f_2(x) = \begin{cases} -x^2 & x < 0 \\ 0 & x \geq 0 \end{cases}$ are linearly independent over $(-\infty, \infty)$. Now show that $W(x) = 0$.

19. Prove that the coordinates of a vector in a given basis are unique.

20. Show that the coefficient matrix of Equations 1 is made up of the three vectors in question arranged as columns. Using this observation, test the vectors $(1, -1, 1, -1)$, $(2, 3, -4, 1)$, and $(0, -5, 6, -3)$ for linear independence in R^4.

9.6 Inner Product Spaces

The idea of a vector space generalizes the spaces of two- and three-dimensional geometric vectors that we discussed in Chapter 5. In those spaces we used a dot product to define lengths of vectors and the angle between two vectors. These concepts are so useful that it is desirable to introduce them into our general vector spaces. We shall now introduce a definition of an inner product for the vectors of a vector space.

A vector space is called an *inner product space* if in addition to the nine requirements that we listed in the previous section, there is a rule that associates with any two vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number, written as $\langle \mathbf{u}, \mathbf{v} \rangle$ [some authors use $(\mathbf{u}, \mathbf{v})$], that satisfies for all vectors in V

1.
$$\langle a\,\mathbf{u}_1 + b\,\mathbf{u}_2, \mathbf{u}_3 \rangle = a\langle \mathbf{u}_1, \mathbf{u}_3 \rangle + b\langle \mathbf{u}_2, \mathbf{u}_3 \rangle \tag{1}$$
where a and b are scalars. (The inner product is a linear property.)

2.
$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \tag{2}$$
(The inner product is commutative.)

3.
$$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0; \qquad \text{and} \qquad \langle \mathbf{u}, \mathbf{u} \rangle = 0 \qquad \text{if and only if} \qquad \mathbf{u} = 0 \tag{3}$$
(This property is known as *positive definiteness*.)

Problem 1 has you prove that the dot product that we defined for geometric vectors is an inner product, so that the Euclidian space of two- or three-dimensional geometric vectors with a defined dot product is an inner product space.

Example 1:
Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Show that the product defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + \cdots + u_n v_n$$

in the vector space R^n is an inner product.

SOLUTION: We shall verify each of the above three properties in turn:

1. $\langle a\,\mathbf{u} + b\,\mathbf{v}, \mathbf{w} \rangle = (au_1 + bv_1)w_1 + (au_2 + bv_2)w_2 + \cdots + (au_n + bv_n)w_n$
$= au_1w_1 + au_2w_2 + \cdots + au_nw_n + bv_1w_1 + \cdots + bv_nw_n$
$= a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$

2. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + \cdots + u_nv_n = v_1u_1 + \cdots + v_nu_n = \langle \mathbf{v}, \mathbf{u} \rangle$

3. $\langle \mathbf{u}, \mathbf{u} \rangle = u_1^2 + \cdots + u_n^2 > 0$ unless $u_1 = u_2 = \cdots = u_n = 0$.

Example 2:
Let V be the vector space of real-valued functions that are continuous on the interval $[\alpha, \beta]$. Show that

$$\langle f, g\rangle = \int_\alpha^\beta f(x)g(x)dx$$

is an inner product.

SOLUTION:

1. $\langle af_1 + bf_2, g\rangle = \int_\alpha^\beta (af_1 + bf_2)g\,dx = a\langle f_1, g\rangle + b\langle f_2, g\rangle$
2. $\langle f, g\rangle = \int_\alpha^\beta f(x)g(x)dx = \int_\alpha^\beta g(x)f(x)dx = \langle g, f\rangle$
3. $\langle f, f\rangle = \int_\alpha^\beta f^2(x)dx \geq 0$ and is equal to zero if and only if $f(x) = 0$ in $[\alpha, \beta]$.

Motivated by geometric vectors, we define the length of a vector in V by

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u}\rangle^{1/2} \tag{4}$$

We also call $\|\mathbf{u}\|$ the *norm* of $\mathbf{u}$. For the case of R^n, the norm is given by

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u}\rangle^{1/2} = (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2} \tag{5}$$

The inner product satisfies an important inequality called the *Schwarz inequality*:

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\| \tag{6}$$

The proof of the Schwarz inequality goes as follows: Start with $\langle \mathbf{u} + \lambda\mathbf{v}, \mathbf{u} + \lambda\mathbf{v}\rangle \geq 0$, where λ is an arbitrary constant. Expand this inner product to write

$$\lambda^2\langle \mathbf{v}, \mathbf{v}\rangle + 2\lambda\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{u}\rangle \geq 0 \tag{7}$$

This inequality must be true for any value of λ, so we choose $\lambda = -\langle \mathbf{u}, \mathbf{v}\rangle/\langle \mathbf{v}, \mathbf{v}\rangle$. Substituting this choice of λ into Equation 7 gives

$$\langle \mathbf{u}, \mathbf{v}\rangle^2 \leq \langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle$$

The square root of this result gives Equation 6. Notice from Equation 6 that we can define the angle between $\mathbf{u}$ and $\mathbf{v}$ by $\cos\theta = \langle \mathbf{u}, \mathbf{v}\rangle/\|\mathbf{u}\|\,\|\mathbf{v}\|$, where $0 \leq \theta \leq \pi$ because $-1 \leq \langle \mathbf{u}, \mathbf{v}\rangle/\|\mathbf{u}\|\,\|\mathbf{v}\| \leq +1$.

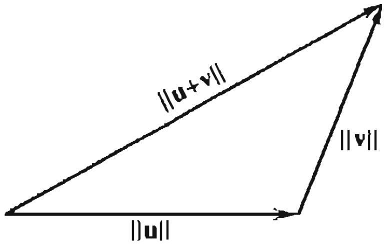

Figure 9.8
An illustration of the triangle inequality presented in Equation 10.

The norm in a vector space V satisfies the following properties:

$$\|\mathbf{v}\| \geq 0; \qquad \|\mathbf{v}\| = 0 \quad \text{if and only if} \quad \mathbf{v} = 0 \tag{8}$$

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\| \tag{9}$$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \tag{10}$$

Equation 10 is called the *triangle inequality* (Figure 9.8). It can be readily proved using the Schwarz inequality (Problem 4).

Example 3:
Verify the Schwarz inequality for $\mathbf{u} = (2, 1, -1, 2, 0)$ and $\mathbf{v} = (-1, 0, 1, 2, -2)$.

SOLUTION: We use Equation 6:

$$\langle \mathbf{u}, \mathbf{v}\rangle = -2 + 0 - 1 + 4 + 0 = 1$$

$$\langle \mathbf{u}, \mathbf{u}\rangle = 4 + 1 + 1 + 4 + 0 = 10$$

$$\langle \mathbf{v}, \mathbf{v}\rangle = 1 + 0 + 1 + 4 + 4 = 10$$

The inequality reads $1 \leq 10$ in this case.

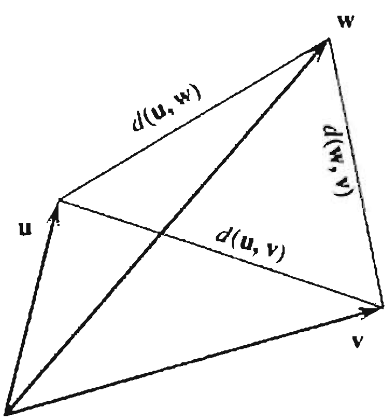

Figure 9.9
An illustration of the triangle inequality presented in Equation 13.

If $\mathbf{u}$ and $\mathbf{v}$ represent geometric vectors from an origin to points given by the tips of $\mathbf{u}$ and $\mathbf{v}$, then $|\mathbf{u} - \mathbf{v}|$ is the geometric distance between these points. We define the distance between vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V by $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$, which you can show satisfies the following conditions (Problem 8):

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| \geq 0, \ \text{which} \ = 0 \quad \text{if and only if} \quad \mathbf{u} = \mathbf{v} \tag{11}$$

$$= d(\mathbf{v}, \mathbf{u}) \tag{12}$$

$$\leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v}) \tag{13}$$

where $\mathbf{w}$ is a third vector (Figure 9.9). Equation 13 is another form of the triangle inequality.

If $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, $\mathbf{u}$ and $\mathbf{v}$ are said to be *orthogonal*. For two- and three-dimensional geometric vectors, orthogonality means that the vectors are perpendicular to each other, but orthogonality is more general than that. Using the definition of the inner product of two functions as given in Example 2, we say the two functions, $f(x)$ and $g(x)$, are orthogonal over $[a, b]$ if

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx = 0$$

Suppose that $\mathbf{v}_j$ for $j = 1, 2, \ldots, n$ is a set of orthogonal vectors in a vector space V, then we can express orthogonality by writing

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \|\mathbf{v}_j\|^2 \delta_{ij}$$

where δ_{ij} is the Kronecker delta. If the lengths of all the vectors are made to be unity by dividing each one by its length $\|\mathbf{v}_j\|$, then the new set is called *orthonormal*. An orthonormal set of vectors satisfies

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij} \tag{14}$$

Example 4:
Show that the geometric vectors $\mathbf{v}_1 = (\mathbf{i} + \mathbf{j})/\sqrt{2}$, $\mathbf{v}_2 = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, and $\mathbf{v}_3 = \mathbf{k}$ form an orthonormal set of vectors.

SOLUTION:

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \frac{(\mathbf{i}+\mathbf{j}) \cdot (\mathbf{i}+\mathbf{j})}{2} = 1 \qquad \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \frac{(\mathbf{i}-\mathbf{j}) \cdot (\mathbf{i}-\mathbf{j})}{2} = 1 \qquad \langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \frac{1}{2}(\mathbf{i}+\mathbf{j}) \cdot (\mathbf{i}-\mathbf{j}) = 0 \qquad \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \cdot \mathbf{k} = 0$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j}) \cdot \mathbf{k} = 0$$

Therefore, $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$.

It's easy to show that an orthonormal set of vectors is linearly independent. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an orthonormal set, and form

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = 0 \tag{15}$$

where the c_j are to be determined. Now form the inner product of Equation 15 with each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in turn, and find that $c_j = 0$ for $j = 1, 2, \ldots, n$. Thus, the set of vectors is linearly independent.

We shall now show that every n-dimensional vector space V has an orthonormal basis by actually constructing one. Let $\mathbf{v}_j$ for $j = 1, 2, \ldots, n$ be any set of (nonzero) linearly independent vectors in V. Start with $\mathbf{v}_1$ and call it $\mathbf{u}_1$. Now take the second vector in the new set to be a linear combination of $\mathbf{v}_2$ and $\mathbf{u}_1$:

$$\mathbf{u}_2 = \mathbf{v}_2 + a_1 \mathbf{u}_1$$

such that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$. This condition gives $0 = \langle \mathbf{u}_1, \mathbf{v}_2 \rangle + a_1 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle$ and so

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{u}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 \tag{16}$$

Now take the third member of the orthogonal set to be

$$\mathbf{u}_3 = \mathbf{v}_3 + b_1 \mathbf{u}_1 + b_2 \mathbf{u}_2$$

and require that $\langle \mathbf{u}_3, \mathbf{u}_1 \rangle = 0$ and $\langle \mathbf{u}_3, \mathbf{u}_2 \rangle = 0$. This gives (Problem 15)

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 \tag{17}$$

The general pattern is evident now, and

$$\mathbf{u}_j = \mathbf{v}_j - \sum_{i=1}^{j-1} \frac{\langle \mathbf{v}_j, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i \tag{18}$$

The $\mathbf{u}_j$ do not form an orthonormal set because they are not normalized, but it is easy to normalize each one simply by dividing by its length. This procedure for generating an orthonormal basis from a general basis is called *Gram-Schmidt orthogonalization*.

Example 5:
The three functions $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = x^2$ form a basis for the vector space of all polynomials of degree equal to or less than 2. Using the definition of an inner product given in Example 2, find an orthonormal basis over the interval $[-1, 1]$.

SOLUTION: We start with $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, and $\mathbf{v}_3 = x^2$. Take $\mathbf{u}_1 = 1$ and write

$$\langle \mathbf{u}_1, \mathbf{v}_2 \rangle = \int_{-1}^{1} x\, dx = 0; \qquad \langle \mathbf{u}_1, \mathbf{u}_1 \rangle = \int_{-1}^{1} dx = 2$$

Using Equation 16, we find that $\mathbf{u}_2 = x$. Equation 17 requires that we evaluate

$$\langle \mathbf{u}_1, \mathbf{v}_3 \rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$\langle \mathbf{u}_2, \mathbf{v}_3 \rangle = \int_{-1}^{1} x^3 dx = 0; \qquad \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

which gives $\mathbf{u}_3 = x^2 - \frac{1}{3}$. We can normalize $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$:

$$\langle \mathbf{u}_1, \mathbf{u}_1 \rangle = 2 \qquad \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3} \quad \text{and} \quad \langle \mathbf{u}_3, \mathbf{u}_3 \rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3} \right)^2 dx = \frac{8}{45}$$

and so the three orthonormal vectors are

$$\phi_1(x) = \left(\frac{1}{2} \right)^{1/2}; \quad \phi_2(x) = \left(\frac{3}{2} \right)^{1/2} x; \quad \phi_3(x) = \left(\frac{45}{8} \right)^{1/2} \left(x^2 - \frac{1}{3} \right)$$

9.6 Problems

1. Show that the dot product that we defined in Chapter 5 for geometric vectors is an inner product.
2. Show that the two geometric vectors $\mathbf{u} = \mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ satisfy the Schwarz inequality.
3. Show that the two functions $f_1(x) = 1 + x$ and $f_2(x) = x$ over the interval $[\,0, 1\,]$ satisfy the Schwarz inequality given the definition of the inner product in Example 2.
4. Prove the triangle inequality (Equation 10).
5. Show that $\|\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{u}_2\|^2 + \cdots + \|\mathbf{u}_n\|^2$ if the $\mathbf{u}_j$ are orthogonal. This is the Pythagorean theorem in n-dimensions.
6. Let V be the vector space of real functions that are continuous on $[-\pi, \pi]$. Using the inner product defined in Example 2, show that $1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots$ form an orthogonal subset of V.
7. Suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is an orthogonal basis of V. Show that
$$\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u}_1\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\,\mathbf{u}_1 + \frac{\langle \mathbf{v}, \mathbf{u}_2\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\,\mathbf{u}_2 + \cdots + \frac{\langle \mathbf{v}, \mathbf{u}_n\rangle}{\langle \mathbf{u}_n, \mathbf{u}_n\rangle}\,\mathbf{u}_n.$$
8. Show that the distance function $d(\mathbf{u}, \mathbf{v})$ satisfies Equations 11 to 13.
9. Using the inner product defined in Example 1, show that the norm of an ordered n-tuple of real numbers is $\|(u_1, u_2, \ldots, u_n)\| = (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}$.
10. We defined the so-called Euclidean norm in Problem 9. We could also define the norm of an ordered n-tuple of real numbers by $\|\mathbf{u}\| = |u_1| + |u_2| + \cdots + |u_n|$. Show that this definition satisfies Equations 8 through 10.
11. Show that if we define the norm of an ordered n-tuple of real numbers by $\|\mathbf{u}\| = \max\,(|u_j|)$, then this definition satisfies Equations 8 through 10.
12. Calculate the norms of the vectors $(1, -2, 3)$ and $(2, 4, -1)$ according to the definition given in Problem 10.
13. Calculate the norms of the vectors $(1, -2, 3)$ and $(2, 4, -1)$ according to the definition given in Problem 11.
14. Calculate the distance functions of the vectors in Problem 12 using the three definitions of a norm that we have presented.
15. Derive Equation 17.
16. Construct an orthonormal basis from the three vectors $(1, -1, 0)$, $(1, 1, 0)$, and $(0, 1, 1)$.
17. The functions $f_1(x) = 1$, $f_2(x) = \sin 2x$, and $f_3(x) = \cos 2x$ over $[-\pi, \pi]$ are a basis for a three-dimensional vector space. Construct an orthonormal set from these three vectors.

9.7 Complex Inner Product Spaces

In our discussion of vector spaces, we have tacitly assumed that the scalars and vectors are real-valued quantities. It turns out that quantum mechanics can be formulated in terms of complex vector spaces with complex inner products, so in this brief section, we shall extend the results of the previous sections to include complex numbers.

The central notion of linear independence is not altered if we are dealing with a complex vector space instead of a real one. Nowhere in the previous section did we specify that the vectors or the set of constants in the definition of linear independence had to be real. Let's determine whether the vectors $(1, i, -1)$, $(1+i, 0, 1-i)$, and $(i, -1, -i)$ are linearly independent or linearly dependent. We form a matrix with the vectors as rows and then transform the matrix into echelon form:

$$\begin{pmatrix} 1 & i & -1 \\ 1+i & 0 & 1-i \\ i & -1 & -i \end{pmatrix} \sim \begin{pmatrix} 1 & i & -1 \\ 0 & 1-i & 1+i \\ 0 & 0 & 0 \end{pmatrix}$$

Thus we see that the vectors are not linearly independent. (See also Problem 1.)

The primary difference between a real and a complex inner product space is in our definition of an inner product. If we allow the scalars and vectors to be complex, Equations 1 through 3 of the previous section become

$$\langle \mathbf{u}|a\,\mathbf{v}_1 + b\,\mathbf{v}_2\rangle = a\langle \mathbf{u}|\mathbf{v}_1\rangle + b\langle \mathbf{u}|\mathbf{v}_2\rangle \tag{1}$$

$$\langle \mathbf{u}|\mathbf{v}\rangle = \langle \mathbf{v}|\mathbf{u}\rangle^* \tag{2}$$

$$\langle \mathbf{u}|\mathbf{u}\rangle \geq 0 \qquad \text{and} \qquad \langle \mathbf{u}|\mathbf{u}\rangle = 0 \quad \text{if and only if} \quad \mathbf{u} = 0 \tag{3}$$

We are using a vertical line rather than a comma to separate the two vectors enclosed by the angular brackets in Equations 1 through 3 to distinguish between real and complex product spaces. (This is standard notation in quantum mechanics.)

If we take the complex conjugate of Equation 1 and use Equation 2, we obtain

$$\langle a\,\mathbf{v}_1 + b\,\mathbf{v}_2|\mathbf{u}\rangle = a^*\langle \mathbf{v}_1|\mathbf{u}\rangle + b^*\langle \mathbf{v}_2|\mathbf{u}\rangle \tag{4}$$

In particular, if we let $\mathbf{v}_2 = 0$, then we have

$$\langle \mathbf{u}|a\,\mathbf{v}\rangle = a\langle \mathbf{u}|\mathbf{v}\rangle \tag{5}$$

from Equation 1 and

$$\langle a\,\mathbf{u}|\mathbf{v}\rangle = a^*\langle \mathbf{u}|\mathbf{v}\rangle \tag{6}$$

from Equation 4. Note that these two equations say that a scalar comes out of the inner product "as is" from the second position, but as its complex conjugate from the first position. This is standard notation in quantum mechanics texts, but not in all mathematics texts. Some mathematics texts define a complex inner product such that $\langle \mathbf{u}|a\,\mathbf{v}\rangle = a^*\langle \mathbf{u}|\mathbf{v}\rangle$ and $\langle a\,\mathbf{u}|\mathbf{v}\rangle = a\langle \mathbf{u}|\mathbf{v}\rangle$.

If $\mathbf{u}$ and $\mathbf{v}$ are ordered n-tuples of complex numbers, then we can define $\langle \mathbf{u}|\mathbf{v}\rangle$ by

$$\langle \mathbf{u}|\mathbf{v}\rangle = u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n \tag{7}$$

and the length of $\mathbf{u}$, $\|\mathbf{u}\|$ by

$$\|\mathbf{u}\| = \langle \mathbf{u}|\mathbf{u}\rangle = u_1^* u_1 + u_2^* u_2 + \cdots + u_n^* u_n \geq 0 \tag{8}$$

Equation 7 is sometimes called a *Hermitian inner product*. Problem 9 has you show that this definition satisfies Equations 1 through 3.

Example 1:
Given $\mathbf{u} = (1+i, 3, 4-i)$ and $\mathbf{v} = (3-4i, 1+i, 2i)$, find $\langle \mathbf{u}|\mathbf{v}\rangle$, $\langle \mathbf{v}|\mathbf{u}\rangle$, $\|\mathbf{u}\|$, and $\|\mathbf{v}\|$.

SOLUTION:

$$\langle \mathbf{u}|\mathbf{v}\rangle = (1-i)(3-4i) + 3(1+i) + (4+i)(2i) = 4i$$

$$\langle \mathbf{v}|\mathbf{u}\rangle = (3+4i)(1+i) + 3(1-i) - 2i(4-i) = -4i = \langle \mathbf{u}|\mathbf{v}\rangle^*$$

$$\|\mathbf{u}\| = [(1+i)(1-i) + 9 + (4+i)(4-i)]^{1/2} = \sqrt{28}$$

$$\|\mathbf{v}\| = [(3-4i)(3-4i) + (1+i)(1-i) + (2i)(-2i)]^{1/2} = \sqrt{31}$$

If a set of vectors $\mathbf{v}_j$ for $j = 1, 2, \ldots, n$ satisfies the condition $\langle \mathbf{v}_i|\mathbf{v}_j\rangle = \delta_{ij}$, we say that the set is orthonormal.

Example 2:
Show that the three 3-tuples $\mathbf{u}_1 = (1, i, 1+i)$, $\mathbf{u}_2 = (0, 1-i, i)$, and $\mathbf{u}_3 = (3i-3, 1+i, 2)$ form an orthogonal set. How would you make them orthonormal?

SOLUTION:

$$\langle \mathbf{u}_1|\mathbf{u}_2\rangle = 0 - i(1-i) + (1-i)i = 0$$

$$\langle \mathbf{u}_1|\mathbf{u}_3\rangle = (3i-3) - i(1+i) + 2(1-i) = 0$$

$$\langle \mathbf{u}_2|\mathbf{u}_3\rangle = 0 + (1+i)(1+i) - 2i = 0$$

To make them orthonormal, divide each one by its norm:

$$\|\mathbf{u}_1\| = \langle \mathbf{u}_1|\mathbf{u}_2\rangle^{1/2} = [(1)(1) - i(i) + (1-i)(1+i)]^{1/2} = 2$$

$$\|\mathbf{u}_2\| = \langle \mathbf{u}_2|\mathbf{u}_2\rangle^{1/2} = [(1+i)(1-i) + (-i)(i)]^{1/2} = 3^{1/2}$$

$$\|\mathbf{u}_3\| = \langle \mathbf{u}_3|\mathbf{u}_3\rangle^{1/2} = [(-3i-3)(3i-3) + (1-i)(1+i) + 4]^{1/2} = (24)^{1/2}$$

Example 3:
Show that the set of functions

$$\phi_m(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta} \qquad 0 \le \theta \le 2\pi$$

for $m = 0, \pm 1, \pm 2, \ldots$ is orthonormal if we define the complex inner product by

$$\langle \phi_m | \phi_n \rangle = \int_0^{2\pi} \phi_m^*(\theta)\phi_n(\theta)\, d\theta$$

SOLUTION:

$$\langle \phi_m | \phi_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} e^{in\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} d\theta$$

The integral here equals 1 if $m = n$ and 0 if $m \neq n$. If $m \neq n$, then it is an integral over complete cycles of the complex exponential function.

The Schwarz inequality takes on the same form for a complex vector space:

$$|\langle \mathbf{u} | \mathbf{v} \rangle| \le \|\mathbf{u}\|\, \|\mathbf{v}\| \tag{9}$$

and its proof parallels the one for a real vector space (Problem 11).

The Gram-Schmidt procedure is also valid for complex vector spaces. Let's construct an orthonormal basis from the two vectors $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (i, -1)$. We start with $\mathbf{u}_1 = \mathbf{v}_1 = (-1, 1)$ and write

$$\mathbf{u}_2 = \mathbf{v}_2 + a\, \mathbf{u}_1$$

Form the inner product with $\mathbf{u}_1$ from the right (see Problem 13) to get $\langle \mathbf{u}_2 | \mathbf{u}_1 \rangle = 0 = \langle \mathbf{v}_2 | \mathbf{u}_1 \rangle + \langle a\, \mathbf{u}_1 | \mathbf{u}_1 \rangle$, which in this case gives

$$0 = i - 1 + a^* \langle \mathbf{u}_1 | \mathbf{u}_1 \rangle = i - 1 + 2a^*$$

or $a = (1 + i)/2$. Thus, the two orthogonal vectors are $\mathbf{u}_1 = (-1, 1)$ and

$$\mathbf{u}_2 = (i, -1) + \frac{1+i}{2}(-1, 1) = \left(\frac{i-1}{2}, \frac{i-1}{2} \right) = \frac{1}{2}(i - 1, i - 1)$$

The two orthonormal vectors are

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(-1, 1) \qquad \text{and} \qquad \mathbf{u}_2 = \frac{1}{2}(i - 1, i - 1)$$

9.7 Problems

1. Show that the three vectors $(1, i, -1)$, $(1+i, 0, 1-i)$, and $(i, -1, -i)$ are not linearly independent by expressing one of them as a linear combination of the others.
2. Determine if the vectors $(1, 1, -i)$, $(0, i, i)$, and $(0, 1, -1)$ are linearly independent.
3. Determine if the vectors $(i, 0, 0)$, $(i, i, 0)$, and (i, i, i) are linearly independent.
4. Suppose that $\langle \mathbf{u} \mid \mathbf{v} \rangle = 2 + i$. Determine (a) $\langle (1-i)\mathbf{u} \mid \mathbf{v} \rangle$ and (b) $\langle \mathbf{u} \mid 2i\, \mathbf{v} \rangle$.
5. Show that the projection of a geometric vector $\mathbf{u}$ onto another geometric vector $\mathbf{v}$ is given by $\dfrac{\langle \mathbf{u}|\mathbf{v}\rangle \mathbf{v}}{\langle \mathbf{v}|\mathbf{v}\rangle}$. Calculate the projection of $\mathbf{u} = (1-i, 2i)$ onto $\mathbf{v} = (3+2i, -2+i)$ in a complex inner product space.
6. Determine if the four matrices $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are linearly independent. These matrices are called the Pauli spin matrices.
7. Verify Equation 2 explicitly for (a) $\mathbf{u} = (1+i, 1)$; $\mathbf{v} = (-i, -1)$ and (b) $\mathbf{u} = (3, -i, 2i)$; $\mathbf{v} = (1, 3i, -1)$.
8. Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -2)$. Verify Equations 5 and 6 explicitly for $a = i$.
9. Show that the inner product defined by Equation 7 satisfies Equations 1 through 3.
10. Find the inner products and the lengths of the vectors in Problem 2.
11. Prove the Schwarz inequality for a complex vector space.
12. The two vectors $(1, i)$ and $(1, 1)$ are the basis of a two-dimensional complex vector space. Construct a pair of orthogonal vectors in this space.
13. Re-do the derivation at the end of the section by forming the inner product of $\mathbf{u}_1$ from the left.
14. We derive *Bessel's inequality* in this problem. Let $\boldsymbol{\phi}_j$, $j = 1, 2, \ldots, n$ be an orthonormal set of vectors in V and let $\mathbf{v}$ be any vector in V. If

$$\mathbf{v} = c_1 \boldsymbol{\phi}_1 + c_2 \boldsymbol{\phi}_j + \cdots + c_n \boldsymbol{\phi}_n \tag{10}$$

show that $c_j = \langle \mathbf{v}|\boldsymbol{\phi}_j\rangle$. Now form $\mathbf{u} = \mathbf{v} - \sum_{j=1}^{r} c_j \boldsymbol{\phi}_j$, where $r \le n$, and show that $\|\mathbf{v}\|^2 \ge \sum_{j=1}^{r} c_j^2$.

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CHAPTER 10

Matrices and Eigenvalue Problems

In the previous chapter, we learned that a matrix is a rectangular array of elements that obeys a certain set of algebraic rules, called matrix algebra. For example, we learned to add and subtract matrices and to multiply them together. We saw that we can associate a matrix with an operation or a transformation such as a rotation of a vector through some angle θ about some axis or the reflection of a vector through a plane. Matrix multiplication corresponds to applying two or more transformations in succession. The idea of undoing a transformation leads to the idea of the inverse of a matrix, so that if S represents some transformation, then S^{-1} represents the effect of undoing that transformation.

We shall continue our discussion of matrices in this chapter. Section 1 deals with orthogonal transformations. A matrix that represents a rotation or a reflection is called an orthogonal matrix and satisfies the convenient relation $A^{-1} = A^{T}$. This relation implies that the rows (columns) of an orthogonal matrix are orthogonal. In Section 2, we introduce one of the most important matrix equations, those that have the form $A\mathbf{x} = \lambda\mathbf{x}$, where λ is a scalar and $\mathbf{x}$ is a (nonzero) column vector. Values of λ that satisfy this condition are called eigenvalues, and the corresponding vectors $\mathbf{x}$ are called eigenvectors. We'll learn how to determine the set of eigenvalues and eigenvectors of a given matrix (this is called an eigenvalue problem). Many diverse physical problems can be formulated as eigenvalue problems, and we discuss several illustrative applications in Section 3. We shall see that the values of the elements of a transformation matrix A depend upon the particular basis being used. If we change the basis, then we change the elements of A. It is often possible that if we choose the basis appropriately, A will be a diagonal matrix, which is very convenient computationally. In Section 5, we learn how to determine the basis for which A is diagonal, a process called diagonalization. Many mathematical and physical problems involve the diagonalization of matrices, and we discuss a number of these problems in the final section.

10.1 Orthogonal and Unitary Transformations

In Section 9.3, we showed that the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

corresponds to the rotation of a two-dimensional vector through an angle θ in a counterclockwise direction. In other words, $R\mathbf{u} = \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are shown in

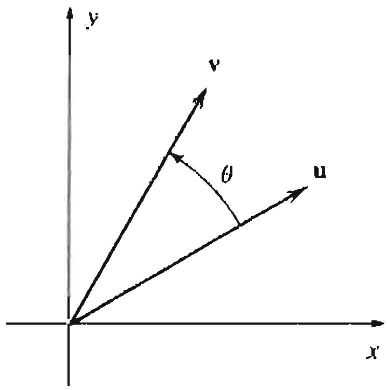


Figure 10.1
The rotation of a vector counterclockwise through an angle θ.

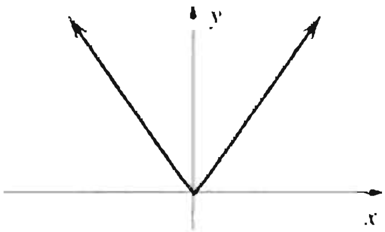


Figure 10.2
The reflection of a vector through the y axis.

Figure 10.1. We say that R transforms **u** into **v**. Similarly, the matrix

$$\mathsf{S}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

corresponds to reflecting a vector **u** through the y axis because $\mathsf{S}\mathbf{u}=(-u_x, u_y)^{\mathsf{T}}$ (Figure 10.2). (We write the column vector with components $-u_x$ and u_y as $(-u_x, u_y)^{\mathsf{T}}$ for convenience in typesetting.)

The matrices R and S represent not only transformations, but they represent *linear* transformations. A *linear transformation* A on the vectors of a vector space V is a function that takes one vector in V into another vector in V such that for all **u** and **v** in V

$$\mathsf{A}(\alpha\mathbf{u}+\beta\mathbf{v})=\alpha\mathsf{A}\mathbf{u}+\beta\mathsf{A}\mathbf{v} \tag{3}$$

where α and β are scalars. Let's show that R given by Equation 1 is a linear transformation. Let $\mathbf{u}=u_x\,\mathbf{i}+u_y\,\mathbf{j}$ and $\mathbf{v}=v_x\,\mathbf{i}+v_y\,\mathbf{j}$. Then,

$$\begin{aligned}
\mathsf{R}(\alpha\mathbf{u}+\beta\mathbf{v}) &= \mathsf{R}\left[\,(\alpha u_x+\beta v_x)\,\mathbf{i}+(\alpha u_y+\beta v_y)\,\mathbf{j}\,\right] \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \alpha u_x+\beta v_x \\ \alpha u_y+\beta v_y \end{pmatrix} \\
&= \begin{pmatrix} (\alpha u_x+\beta v_x)\cos\theta-(\alpha u_y+\beta v_y)\sin\theta \\ (\alpha u_x+\beta v_x)\sin\theta+(\alpha u_y+\beta v_y)\cos\theta \end{pmatrix}
\end{aligned} \tag{4}$$

Now $\mathsf{R}\mathbf{u}$ and $\mathsf{R}\mathbf{v}$ are given by

$$\mathsf{R}\mathbf{u}=\begin{pmatrix} \cos\theta\, u_x-\sin\theta\, u_y \\ \sin\theta\, u_x+\cos\theta\, u_y \end{pmatrix} \quad \text{and} \quad \mathsf{R}\mathbf{v}=\begin{pmatrix} \cos\theta\, v_x-\sin\theta\, v_y \\ \sin\theta\, v_x+\cos\theta\, v_y \end{pmatrix}$$

Thus,

$$\alpha\mathsf{R}\mathbf{u}+\beta\mathsf{R}\mathbf{v}=\begin{pmatrix} (\alpha u_x+\beta v_x)\cos\theta-(\alpha u_y+\beta v_y)\sin\theta \\ (\alpha u_x+\beta v_x)\sin\theta+(\alpha u_y+\beta v_y)\cos\theta \end{pmatrix} \tag{5}$$

in agreement with Equation 4.

Example 1:
Show that

$$\mathsf{A}(\alpha\mathbf{u}+\beta\mathbf{v})=\alpha\mathsf{A}\mathbf{u}+\beta\mathsf{A}\mathbf{v}$$

if

$$\mathsf{A}=\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

SOLUTION: The expression $\mathsf{A}(\alpha\mathbf{u}+\beta\mathbf{v})$ in matrix form is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha u_x + \beta v_x \\ \alpha u_y + \beta v_y \\ \alpha u_z + \beta v_z \end{pmatrix} = \begin{pmatrix} \alpha u_x + \beta v_x - (\alpha u_z + \beta v_z) \\ \alpha u_y + \beta v_y + \alpha u_z + \beta v_z \\ 2(\alpha u_x + \beta v_x) - (\alpha u_y + \beta v_y) \end{pmatrix}$$

$$= \alpha \begin{pmatrix} u_x - u_z \\ u_y + u_z \\ 2u_x - u_y \end{pmatrix} + \beta \begin{pmatrix} v_x - v_z \\ v_y + v_z \\ 2v_x - v_y \end{pmatrix}$$

$$= \alpha \mathsf{A}\mathbf{u} + \beta \mathsf{A}\mathbf{v}$$

If we introduce a basis set into our vector spaces, then we can investigate the result of the transformation in terms of the basis vectors. For concreteness (and simplicity), we will choose the so-called *standard basis* in $V = \mathrm{R}^n$. By a standard basis, we mean a set of basis vectors $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)^{\mathsf{T}}$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)^{\mathsf{T}}$, $\mathbf{e}_3 = (0, 0, 1, \ldots, 0)^{\mathsf{T}}$, and so on up to $\mathbf{e}_n = (0, 0, 0, \ldots, 1)^{\mathsf{T}}$, where n is the dimensionality of V. Note that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ correspond to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in three-dimensional Euclidian space. We shall restrict our discussion to transformations $\mathbf{v} = \mathsf{A}\mathbf{u}$ such that $\mathbf{v}$ is in the same vector space as $\mathbf{u}$. Because $\mathbf{v} = \mathsf{A}\mathbf{u}$ is in V, we can write

$$\mathbf{u} = \sum_{j=1}^{n} u_j \mathbf{e}_j \qquad \text{and} \qquad \mathbf{v} = \sum_{j=1}^{n} v_j \mathbf{e}_j$$

Furthermore,

$$\mathbf{v} = \mathsf{A}\mathbf{u} = \mathsf{A} \sum_{j=1}^{n} u_j \mathbf{e}_j = \sum_{j=1}^{n} u_j \mathsf{A}\mathbf{e}_j \tag{6}$$

But the vector $\mathsf{A}\mathbf{e}_j$ can be expressed as a linear combination of the $\{\mathbf{e}_i\}$, so

$$\mathsf{A}\mathbf{e}_j = \sum_{i=1}^{n} a_{ij} \mathbf{e}_i \tag{7}$$

In Equation 7, a_{ij} is the ith component of the vector $\mathsf{A}\mathbf{e}_j$ in the standard basis $\{\mathbf{e}_i\}$. Substituting Equation 7 into Equation 6 gives

$$\mathbf{v} = \sum_{j=1}^{n} u_j \sum_{i=1}^{n} a_{ij} \mathbf{e}_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} u_j \right) \mathbf{e}_i$$

Thus, the ith component of $\mathbf{v}$ in the standard basis is given by

$$v_i = \sum_{j=1}^{n} a_{ij} u_j \tag{8}$$

Let's write out Equation 8 in full so we can view the components more clearly.

$$\begin{aligned} v_1 &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\ v_2 &= a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n \\ \vdots\ & \quad\ \vdots \qquad\quad \vdots \qquad \ddots \qquad \vdots \\ v_n &= a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n \end{aligned} \tag{9}$$

Notice that if $\mathbf{u} = \mathbf{e}_1 = (1, 0, \ldots, 0)^\mathsf{T}$, then $\mathbf{v} = (a_{11}, a_{21}, \ldots, a_{n1})^\mathsf{T}$; if $\mathbf{u} = \mathbf{e}_2 = (0, 1, 0, \ldots, 0)^\mathsf{T}$, then $\mathbf{v} = (a_{12}, a_{22}, \ldots, a_{n2})^\mathsf{T}$, and so forth. Thus, the jth column of A is given by $\mathsf{A}\mathbf{e}_j$. In other words, the matrix A is completely determined by how it transforms the vectors of the standard basis.

Example 2:
Determine the matrix A if $\mathsf{A}\mathbf{e}_1 = (1, 2, -1)^\mathsf{T}$, $\mathsf{A}\mathbf{e}_2 = (0, 1, 1)^\mathsf{T}$, and $\mathsf{A}\mathbf{e}_3 = (1, 1, 0)^\mathsf{T}$. Show that A is nonsingular.

SOLUTION: The jth column of A is given by $\mathsf{A}\mathbf{e}_j$, so

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

The determinant of $\mathsf{A} = 2$, so A is nonsingular.

The linear transformations that we have considered in this section transform given vectors into other vectors. For example, the matrix described by Equation 1 corresponds to a rotation of a vector through an angle θ in a counterclockwise direction. Since $\mathsf{R}\mathbf{u}$ corresponds to a rotation of $\mathbf{u}$, we should expect that the length of $\mathbf{u}$ does not get altered under R (Problem 7). Let $\mathbf{v} = \mathsf{A}\mathbf{u}$ be a linear transformation of $\mathbf{u}$ that preserves its length. Then

$$\langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathsf{A}\mathbf{u}, \mathsf{A}\mathbf{u} \rangle = (\mathsf{A}\mathbf{u})^\mathsf{T}(\mathsf{A}\mathbf{u}) = \mathbf{u}^\mathsf{T}\mathsf{A}^\mathsf{T}\mathsf{A}\mathbf{u} \tag{10}$$

If $\mathbf{v}$ and $\mathbf{u}$ are to have the same length, then we must have $\mathsf{A}^\mathsf{T}\mathsf{A} = \mathsf{I}$, or that $\mathsf{A}^\mathsf{T} = \mathsf{A}^{-1}$. A matrix with this property is said to be *orthogonal*, and $\mathbf{v} = \mathsf{A}\mathbf{u}$ is said to be an *orthogonal transformation*. We can state this important result as follows:

A linear transformation preserves lengths if and only if its matrix is orthogonal.

Example 3:
Show that R given by Equation 1 is an orthogonal matrix.

SOLUTION: We need to show that $R^T = R^{-1}$ or that $RR^T = I$.

$$R^TR = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It is also true that $RR^T = I$.

Not only are the lengths of vectors preserved under orthogonal transformations, but the angles between vectors are preserved as well. Recall that the angle between two vectors in a vector space is defined by $\cos\theta = \dfrac{\langle \mathbf{x}, \mathbf{y}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle^{1/2}\langle \mathbf{y}, \mathbf{y}\rangle^{1/2}}$. To prove that the angle is preserved under an orthogonal transformation, we simply need to show that $\langle A\mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$. Now

$$\langle A\mathbf{x}, A\mathbf{y}\rangle = (A\mathbf{x})^TA\mathbf{y} = \mathbf{x}^TA^TA\mathbf{y} = \mathbf{x}^T\mathbf{y} = \langle \mathbf{x}, \mathbf{y}\rangle \tag{11}$$

Furthermore, the distances between the ends of vectors are also preserved by orthogonal transformations (Problem 8).

The orthogonality expression

$$A^TA = AA^T = I \tag{12}$$

implies that the rows of A^T and the columns of A are orthonormal. The rows of A^T, however, are the columns of A, so Equation 12 implies that the column vectors (also the row vectors) of an orthogonal matrix are orthonormal. To see this analytically, insert the fact that $a_{ij}^T = a_{ji}$ into $A^TA = I$ to get

$$\sum_{j=1}^{n} a_{ij}^T a_{jk} = \delta_{ik} = \sum_{j=1}^{n} a_{ji}a_{jk} \tag{13}$$

Equation 13 says that the ith and kth columns of A are orthonormal vectors. If we use $A\,A^T = I$ instead of $A^TA = I$, we find that the ith and kth rows of A are orthonormal vectors (Problem 9).

Example 4:
First show that

$$A = \frac{1}{9}\begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix}$$

is orthogonal and then show explicitly that its rows (columns) are orthonormal vectors.

SOLUTION:

$$A^{T}A = \frac{1}{81}\begin{pmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{pmatrix}\begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix} = \frac{1}{81}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix}$$

$$AA^{T} = \frac{1}{81}\begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{pmatrix} = \frac{1}{81}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix}$$

The columns are normalized because

$$\frac{1}{81}(1+16+64) = 1, \quad \frac{1}{81}(64+16+1) = 1, \quad \text{and} \quad \frac{1}{81}(16+49+16) = 1$$

Column 1 and column 2 are orthogonal vectors because

$$(1, 4, 8)\begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix} = 8 - 16 + 8 = 0$$

Similarly, columns 1 and 3 and columns 2 and 3 are orthogonal. For rows 2 and rows 3,

$$(4, -4, -7)\begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} = 32 - 4 - 28 = 0$$

Rows 1 and 2 and rows 1 and 3 are also orthogonal.

Using the fact that det (AB) = det (A) det (B), it is easy to show that the determinant of an orthogonal matrix is equal to ± 1 (Problem 11). If det (A) = 1, then A corresponds to a pure rotation, and if det (A) = -1, A corresponds to a rotation *and* an inversion through a plane.

Up to this point, we have considered only matrices whose elements are real. We frequently deal with vector spaces over the field of complex numbers in quantum mechanics, and so we shall end this section with an extension of our results to matrices whose elements are complex. Recall from Section 9.7 that we defined an inner product in a complex vector space by

$$\langle \mathbf{u}, \mathbf{v} \rangle = (\mathbf{u}^{*T})\mathbf{v} = u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n \tag{14}$$

so that the length of a vector is given by

$$\|\mathbf{u}\| = (u_1^* u_1 + u_2^* u_2 + \cdots + u_n^* u_n)^{1/2} \tag{15}$$

A linear transformation that preserves lengths must satisfy

$$\langle A\mathbf{u}, A\mathbf{u} \rangle = (A^*\mathbf{u}^*)^T(A\mathbf{u}) = \mathbf{u}^{*T}(A^*)^T A\mathbf{u} = \mathbf{u}^{*T}\mathbf{u} = \langle \mathbf{u}, \mathbf{u} \rangle$$

or

$$(A^*)^T = A^{-1} \tag{16}$$

A matrix that satisfies Equation 16 is said to be *unitary*. We shall denote $(A^*)^T$ by $A^\dagger$ (called the *Hermitian conjugate* of A), so that Equation 16 becomes

$$A^\dagger A = AA^\dagger = I \tag{17}$$

or $A^{-1} = A^\dagger$. Note that a unitary matrix is the analog of an orthogonal matrix in a complex vector space. The analog of Equation 12 is $A^\dagger A = AA^\dagger = I$. Equation 17 shows that the rows (columns) of a unitary matrix are orthonormal. Using the fact that $a_{ij}^\dagger = a_{ji}^*$, Equation 17 becomes

$$\sum_{j=1}^{n} a_{ij}^\dagger a_{jk} = \sum_{j=1}^{n} a_{ji}^* a_{jk} = \delta_{ik}$$

which shows that the rows of $A^\dagger$ are orthonormal. The second expression in Equation 17 shows that the columns are orthonormal. Problem 14 has you show that the determinant of a unitary matrix is of absolute value 1.

Example 5:
Show that

$$A = \frac{1}{5}\begin{pmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{pmatrix}$$

is unitary.

SOLUTION:

$$A^\dagger = (A^*)^T = \frac{1}{5}\begin{pmatrix} -1-2i & 2+4i \\ -4+2i & -2+i \end{pmatrix}$$

and

$$A^\dagger A = \frac{1}{25}\begin{pmatrix} -1-2i & 2+4i \\ -4+2i & -2+i \end{pmatrix}\begin{pmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

$$AA^\dagger = \frac{1}{25}\begin{pmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{pmatrix}\begin{pmatrix} -1-2i & 2+4i \\ -4+2i & -2+i \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

Note also that the rows and columns of A are orthonormal and that $\det A = (20 - 15i)/25$, which is of unit magnitude.

10.1 Problems

1. Show that the matrix S given by Equation 2 represents a reflection of $\mathbf{u}$ through the y axis.
2. Construct a 2×2 matrix that represents the reflection of a two-dimensional vector through the origin. (This operation is called *inversion*.)
3. Determine the geometric result of the following matrices acting on a two-dimensional vector $\mathbf{u}$:

   (a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (c) $\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
4. Prove that a nonsingular transformation transforms linearly independent vectors into linearly independent vectors. *Hint*: Assume the contrary.
5. Determine the matrix A if $\mathsf{A}\mathbf{e}_1 = (0, 1, 1)^{\mathsf{T}}$, $\mathsf{A}\mathbf{e}_2 = (1, 0, 1)^{\mathsf{T}}$, and $\mathsf{A}\mathbf{e}_3 = (1, 1, 0)^{\mathsf{T}}$. Show that A is nonsingular.
6. Suppose the linear transformation $\mathbf{v} = \mathsf{A}\mathbf{u}$ transforms $\mathbf{u}_1 = (1, 0, 1)$ into $\mathbf{v}_1 = (2, 1, -1)$, $\mathbf{u}_2 = (0, 1, 1)$ into $\mathbf{v}_2 = (1, -1, 0)$ and $\mathbf{u}_3 = (1, 1, 0)$ into $\mathbf{v}_3 = (0, -1, 1)$. Find the matrix that corresponds to this linear transformation.
7. Show that $\mathbf{u}$ and $\mathsf{R}\mathbf{u}$ have the same length if R is given by Equation 1.
8. Show that the distance between two points is preserved under an orthogonal transformation; in other words, show that orthogonal transformations preserve distances.
9. Show that the rows of an orthogonal matrix are orthogonal vectors.
10. Prove that the product of two orthogonal matrices is orthogonal.
11. Using the fact that $\det(\mathsf{AB}) = \det(\mathsf{A})\det(\mathsf{B})$, show that the determinant of an orthogonal matrix is equal to ± 1.
12. Show that $\mathsf{A} = \dfrac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$ is orthogonal.
13. Show that the columns of a unitary matrix are orthogonal.
14. Show that the determinant of a unitary matrix is of absolute value unity. *Hint*: You need the relation $\det(\mathsf{AB}) = \det(\mathsf{A})\det(\mathsf{B})$.
15. Show that $\mathsf{A} = \dfrac{1}{6}\begin{pmatrix} 2-4i & 4i \\ -4i & -2-4i \end{pmatrix}$ is unitary.
16. Show that the rows and columns of the unitary matrix in Problem 15 are orthonormal.
17. Show that $\mathsf{A} = \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -i \\ 1 & -1 & 0 & i \\ 0 & -i & i & 1 \end{pmatrix}$ is unitary.

10.2 Eigenvalues and Eigenvectors

Most of the linear transformations that we have discussed have been of the form $\mathsf{A}\mathbf{u} = \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are vectors in the same vector space, be it real or complex. It turns out that linear transformations of the form

$$A\mathbf{x} = \lambda \mathbf{x} \tag{1}$$

play an important role in a wide variety of physical applications. In Equation 1, A is a given square matrix, λ is a scalar that we do not know beforehand, and $\mathbf{x}$ is a vector to be determined. Certainly $\mathbf{x} = \mathbf{0}$ satisfies Equation 1, but we wish to find solutions other than trivial solutions. Values of λ that satisfy Equation 1 are called *eigenvalues* and nontrivial vectors that satisfy Equation 1 are called *eigenvectors*. The problem of determining the sets of λ and $\mathbf{x}$ that satisfy Equation 1 is called an *eigenvalue problem*.

We can rewrite Equation 1 in the form

$$(A - \lambda I)\,\mathbf{x} = 0 \tag{2}$$

where we have inserted the unit matrix I so that the factor in parentheses is a well-defined matrix. If A is an $n \times n$ matrix, Equation 2 represents a system of n simultaneous linear algebraic equations in n unknowns, $x_1, x_2, \ldots, x_n$. We know from Chapter 9 that the condition that there be a nontrivial solution to Equations 2 is that

$$\det(A - \lambda I) = |A - \lambda I| = 0 \tag{3}$$

Upon expanding this $n \times n$ determinant, you'll find that it yields an nth degree polynomial in (the unknown) λ. The polynomial equation given by Equation 3 is called the *characteristic equation* or the *secular equation*. The n solutions to the characteristic equation give n values of λ for which Equation 1 will be satisfied.

Example 1:
Find the eigenvalues of the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 4 & 2 \end{pmatrix}$$

SOLUTION: The characteristic equation is

$$\begin{vmatrix} -1-\lambda & 1 \\ 4 & 2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = 0$$

or $\lambda = 3$ and -2. Thus, this 2×2 eigenvalue problem yields two eigenvalues.

Once the eigenvalues are determined, we can determine the corresponding eigenvectors. If we substitute $\lambda = 3$ into $A\mathbf{x} = \lambda\mathbf{x}$, we have

$$-x + y = 3x$$
$$4x + 2y = 3y$$

or

$$-4x + y = 0$$
$$4x - y = 0$$

These two equations are essentially the same, and yield $y = 4x$. Thus, the eigenvalue corresponding to $\lambda = 3$ is $(a, 4a)^T$, where a is any nonzero constant. For $\lambda = -2$, we find that $A\mathbf{x} = -2\,\mathbf{x}$ gives

$$x + y = 0$$
$$4x + 4y = 0$$

and so the corresponding eigenvector is $(b, -b)^T$, where b is any nonzero constant. It shouldn't be surprising that both eigenvectors are determined only to within an arbitrary factor because you can see from Equation 1 that if $\mathbf{x}$ is a solution, so is any multiple of $\mathbf{x}$. Thus, Equation 1 can determine $\mathbf{x}$ only to within a multiplicative constant. Notice that the two eigenvectors $(a, 4a)$ and $(b, -b)$ are linearly independent.

Example 2:
Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

SOLUTION: The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = 0$$

or $(1-\lambda) = \pm 2i$, or $\lambda = 1 \mp 2i$. For $\lambda = 1 + 2i$, we have

$$2ix + 2y = 0$$
$$2x - 2iy = 0$$

or $x = iy$. Thus, the corresponding eigenvector is $(ia, a)^T$. For $\lambda = 1 - 2i$, we have

$$2ix - 2y = 0$$
$$2x + 2iy = 0$$

or $x = -iy$. The corresponding eigenvector is $(-ib, b)^T$.

Example 3:
Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

SOLUTION: The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = (3-\lambda)(2-3\lambda+\lambda^2) = 0$$

and so $\lambda = 1, 2$, and 3. For $\lambda = 1$, $A\mathbf{x} = \mathbf{x}$ gives

$$\begin{aligned} z &= 0 \\ x + y + z &= 0 \\ x + y + z &= 0 \end{aligned}$$

or $\mathbf{x}_1 = (a, -a, 0)^{\mathsf{T}}$. For $\lambda = 2$, we get

$$\begin{aligned} x \qquad\quad + z &= 0 \\ x \qquad\quad + z &= 0 \\ 2x + 2y + z &= 0 \end{aligned}$$

or $\mathbf{x}_2 = (-2b, b, 2b)^{\mathsf{T}}$. For $\lambda = 3$, we get $\mathbf{x}_3 = (c, -c, -2c)^{\mathsf{T}}$.

If the eigenvalues of a matrix are distinct, we say that they are *non-degenerate*. If an eigenvalue repeats, we say that it is *degenerate*, and if it repeats k times, we say that it is *k-fold degenerate*. All the eigenvalues that we have found so far are non-degenerate.

You may not have noticed it, but the product of the eigenvalues in each case above is equal to the determinant of A. To see why this is so, write the characteristic polynomial as $(\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n eigenvalues. This product is equal to

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \tag{4}$$

If we let $\lambda = 0$, we see that $\lambda_1\lambda_2 \ldots \lambda_n = \det(A)$. This result tells us that A is a singular matrix if and only if at least one of its eigenvalues is equal to zero.

The above calculations of eigenvalues also suggest that the sum of the eigenvalues of A is equal to the sum of the diagonal elements of A. In Section 4, we'll prove that this is true in general. The sum of the diagonal elements of a matrix A is called the *trace of* A, and so we can write

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \mathrm{Tr}\, A \tag{5}$$

Another observation from the four calculations that we did above is that the eigenvectors corresponding to different eigenvalues are linearly independent (Problem 7). We can formalize this observation with the following theorem:

If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are the eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent.

Problem 8 works you through a proof of this theorem.

If the eigenvalues aren't distinct, then there is no assurance that the eigenvectors will be linearly independent. They may or may not be. Problem 3 treats a 2×2 matrix with equal eigenvalues for which there is only one linearly independent eigenvector, whereas Problem 4 treats a 3×3 matrix with a degenerate eigenvalue with three linearly independent eigenvectors.

There is a certain class of matrices, however, for which the eigenvectors are linearly independent even if the eigenvalues are not all distinct. In a real vector space, these matrices are *symmetric*, meaning that

$$\mathsf{A}^{\mathsf{T}} = \mathsf{A} \quad \text{(symmetric matrix)} \tag{6}$$

In terms of the elements, $a_{ij} = a_{ji}$. The corresponding property for a complex vector space is that

$$(\mathsf{A}^{\mathsf{T}})^* = \mathsf{A}^{\dagger} = \mathsf{A} \quad \text{(Hermitian matrix)} \tag{7}$$

or that A equals its Hermitian conjugate. In terms of the elements, $a_{ij} = a_{ij}^{\dagger} = a_{ji}^*$. Matrices that satisfy Equation 7 are called *Hermitian*. A Hermitian matrix is the complex vector space analog of a symmetric matrix in a real vector space. It turns out that symmetric and Hermitian matrices arise frequently in applied problems. In fact, the formalism of quantum mechanics is based upon Hermitian matrices, and Hermitian operators in general.

Example 4:
Show that the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & i & 1-i \\ -i & 0 & -1+i \\ 1+i & -1-i & 3 \end{pmatrix}$$

is Hermitian.

SOLUTION: The complex conjugate of A^{T} is

$$\mathsf{A}^{\dagger} = \begin{pmatrix} 1 & i & 1-i \\ -i & 0 & -1+i \\ 1+i & -1-i & 3 \end{pmatrix} = \mathsf{A}$$

and so A is Hermitian. Also note that $a_{ij}^{\dagger} = a_{ij} = a_{ji}^*$.

Symmetric and Hermitian matrices have a number of important and useful properties. Since a symmetric matrix is a special case of a Hermitian matrix if its elements happen to be real, we shall illustrate these properties for Hermitian matrices.

The eigenvalues of a Hermitian matrix are real.

The proof of this is fairly straightforward. Let A be a Hermitian matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$. Then

$$\mathsf{A}\mathbf{x} = \lambda\mathbf{x} \qquad \text{and} \qquad \mathsf{A}^*\mathbf{x}^* = \lambda^*\mathbf{x}^*$$

Multiply the first equation from the left by $(\mathbf{x}^*)^{\mathsf{T}} = \mathbf{x}^\dagger$, to get

$$\mathbf{x}^\dagger\mathsf{A}\mathbf{x} = \lambda\,\mathbf{x}^\dagger\mathbf{x} \tag{8}$$

Now, transpose the second equation

$$(\mathsf{A}^*\mathbf{x}^*)^{\mathsf{T}} = (\mathbf{x}^*)^{\mathsf{T}}(\mathsf{A}^*)^{\mathsf{T}} = \mathbf{x}^\dagger\mathsf{A}^\dagger = \lambda^*\mathbf{x}^\dagger$$

and then multiply the result from the right by $\mathbf{x}$ to obtain

$$\mathbf{x}^\dagger\mathsf{A}^\dagger\mathbf{x} = \lambda^*\mathbf{x}^\dagger\mathbf{x} \tag{9}$$

Now subtract Equations 8 and 9 to get

$$\mathbf{x}^\dagger(\mathsf{A} - \mathsf{A}^\dagger)\,\mathbf{x} = (\lambda - \lambda^*)\,\mathbf{x}^\dagger\mathbf{x}$$

But $\mathsf{A} = \mathsf{A}^\dagger$ because A is Hermitian, and $\mathbf{x}^\dagger\mathbf{x} > 0$ since $\mathbf{x} \neq \mathbf{0}$, so $\lambda = \lambda^*$, or, in other words, λ is real. In quantum mechanics, measurable quantities correspond to eigenvalues of matrices that are required to be Hermitian matrices because measurable quantities must be real.

The proof that we just presented to prove that the eigenvalues of a Hermitian matrix are real can be slightly modified to prove that

The eigenvectors corresponding to distinct eigenvalues of a Hermitian matrix are orthogonal.

To prove this, let A be a Hermitian matrix with distinct eigenvalues λ and μ and corresponding eigenvectors $\mathbf{x}$ and $\mathbf{y}$, respectively, so that

$$\mathsf{A}\mathbf{x} = \lambda\mathbf{x} \qquad \text{and} \qquad \mathsf{A}^*\mathbf{y}^* = \mu^*\mathbf{y}^* = \mu\,\mathbf{y}^*$$

Multiply the first equation from the left by $\mathbf{y}^{\dagger}$, then transpose the second equation and multiply it into $\mathbf{x}$ to get

$$\mathbf{y}^{\dagger}\mathsf{A}\mathbf{x} = \lambda\,\mathbf{y}^{\dagger}\mathbf{x} \qquad \text{and} \qquad \mathbf{y}^{\dagger}\mathsf{A}^{\dagger}\mathbf{x} = \mu\,\mathbf{y}^{\dagger}\mathbf{x}$$

Subtract these two equations and recognize that $\mathsf{A} = \mathsf{A}^{\dagger}$, because A is Hermitian, to arrive at

$$(\lambda - \mu)\,\mathbf{y}^{\dagger}\mathbf{x} = 0$$

Because $\lambda \neq \mu$, we have $\mathbf{y}^{\dagger}\mathbf{x} = 0$, meaning that the eigenvalues of distinct eigenvalues are orthogonal.

Example 5:
Find the eigenvalues and eigenvectors of

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and show that the eigenvectors are mutually orthogonal.

SOLUTION: The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - (1-\lambda) = (1-\lambda)(\lambda^2 - 2\lambda) = 0$$

or $\lambda = 0$, 2, and 1. For $\lambda = 0$, we have

$$\begin{aligned} x \quad\quad + z &= 0 \\ y \quad\; &= 0 \\ x \quad\quad + z &= 0 \end{aligned}$$

or $\mathbf{x}_0 = (a, 0, -a)^{\mathsf{T}}$. For $\lambda = 1$ and 2, we get $\mathbf{x}_1 = (0, b, 0)^{\mathsf{T}}$ and $\mathbf{x}_2 = (c, 0, c)^{\mathsf{T}}$, respectively.

$$\mathbf{x}_0^{\mathsf{T}}\mathbf{x}_1 = (a \;\; 0 \;\; -a)\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = 0; \quad \mathbf{x}_0^{\mathsf{T}}\mathbf{x}_2 = (a \;\; 0 \;\; -a)\begin{pmatrix} c \\ 0 \\ c \end{pmatrix} = 0$$

and

$$\mathbf{x}_1^{\mathsf{T}}\mathbf{x}_2 = (0 \;\; b \;\; 0)\begin{pmatrix} c \\ 0 \\ c \end{pmatrix} = 0$$

Even if the eigenvalues of a Hermitian matrix are not distinct, we can construct an orthogonal set of eigenvectors. Suppose that the eigenvalue λ_1 is k-fold degenerate. In this case, we have

$$A\mathbf{x}_i = \lambda_1 \mathbf{x}_i \qquad \text{for} \quad i = 1, 2, \ldots, k \tag{10}$$

Although we shall not prove it here, if λ is a k-fold degenerate eigenvalue of a Hermitian matrix, then there are always k linearly independent eigenvectors corresponding to λ. Because of this, we can use the Gram-Schmidt orthogonalization procedure that we introduced in Section 9.6 to construct k orthogonal eigenvectors from the k linearly independent eigenvectors in Equation 10. Consequently, we can say that

An $n \times n$ Hermitian matrix has n mutually orthogonal eigenvalues.

Example 6:
Find an orthogonal set of eigenvectors of

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

SOLUTION: The characteristic equation is $(2 - \lambda)(\lambda^2 - 6\lambda + 8) = 0$ or $\lambda = 2, 2,$ and 4. The eigenvector corresponding to $\lambda = 4$ is $\mathbf{x}_1 = (a, a, 0)^{\mathsf{T}}$. For $\lambda = 2$, we get $x = -y$ and $z = z$. If we let $x = \alpha$ and $z = \beta$, the eigenvector can be written as

$$\begin{pmatrix} \alpha \\ -\alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Therefore, $\mathbf{x}_2 = (1,\ -1,\ 0)^{\mathsf{T}}$ and $\mathbf{x}_3 = (0,\ 0,\ 1)^{\mathsf{T}}$ are (linearly independent) eigenvectors corresponding to $\lambda = 2$, as you can show by direct calculation. It turns out that $\mathbf{x}_2$ and $\mathbf{x}_3$ as we have written them here are orthogonal to $\mathbf{x}_1$ and orthogonal to each other.

As is often the case, we conclude this section by pointing out that any CAS can be used to calculate the eigenvalues and eigenvectors of a matrix. For example, the one-line command in Mathematica,

```
Eigensystem [ { {1,0,1}, {0,1,0}, {1,0,1} } ]
```

gives the results of Example 5. Problems 17 through 19 ask you to use any CAS to determine the eigenvalues and eigenvectors of some matrices.

10.2 Problems

1. Determine the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. How many linearly independent eigenvectors are there?

2. Determine the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$. How many linearly independent eigenvectors are there?

3. Determine the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$. Show that there is only one linearly independent eigenvector.

4. Determine the eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & -2 & -1 \\ 3 & -4 & -3 \\ 2 & -4 & 0 \end{pmatrix}$. Show that there are three linearly independent eigenvectors.

5. Show that the eigenvalues obtained in Problems 1 through 4 satisfy Equations 4 and 5.

6. Show that Equation 4 is true for a general 3×3 matrix. Do you see any pattern?

7. Show that the eigenvectors in Examples 2 and 3 are linearly independent.

8. Prove that if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are the eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent. *Hint*: Suppose that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$ for some nonzero c_js. Show that $(\mathsf{A} - \lambda_2\mathsf{I})(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n)$ is missing the term $c_2\mathbf{x}_2$. Now operate with $(\mathsf{A} - \lambda_3\mathsf{I})(\mathsf{A} - \lambda_4\mathsf{I}) \cdots (\mathsf{A} - \lambda_n\mathsf{I})$ and show that the result is $c_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_n)\mathbf{x}_1 = \mathbf{0}$. Now argue that because all the λ's are distinct and $\mathbf{x}_i \neq \mathbf{0}$, then $c_1 = 0$. Derive a similar result for all the $\mathbf{x}_j$ and show that $c_j = 0$ for $j = 1, 2, \ldots, n$.

9. Show that the eigenvalues of A^{T} are the same as those of A.

10. Show that if A is a real matrix with a complex eigenvalue λ and eigenvector $\mathbf{x}$, then the complex conjugates λ^* and $\mathbf{x}^*$ are also eigenvalues and eigenvectors.

11. Show that the eigenvalues of a unitary matrix are of absolute value one.

12. A matrix is called skew-Hermitian if $\mathsf{A} = -(\mathsf{A}^*)^{\mathsf{T}} = -\mathsf{A}^{\dagger}$. Show that the eigenvalues of a skew-Hermitian matrix are pure imaginary.

13. Show that the eigenvalues of A^{-1} are λ_j^{-1}.

14. Show that $\mathsf{A}^n\mathbf{x} = \lambda^n\mathbf{x}$ if $\mathsf{A}\mathbf{x} = \lambda\mathbf{x}$.

15. We can formally define a function of a matrix by a Maclaurin expansion. If a function has a Maclaurin expansion $f(x) = \sum_{n=0}^{\infty} a_n x^n$, which converges for $|x| < R$, then the matrix series $f(\mathsf{A}) = \sum_{n=0}^{\infty} a_n \mathsf{A}^n$ converges if each of the eigenvalues of A is less than R. Show that if $\mathsf{A}\,\mathbf{x} = \lambda\mathbf{x}$, then $f(\mathsf{A})\,\mathbf{x} = f(\lambda)\,\mathbf{x}$.

16. The *Cayley-Hamilton theorem* says that every square matrix satisfies its own characteristic equation. Verify this theorem for the matrices given in Problems 1 and 3.

17. Use any CAS to determine the eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

18. Use any CAS to determine the eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

19. Use any CAS to determine the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & i/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 2 & 0 \\ i/\sqrt{2} & 0 & 2 \end{pmatrix}.$$

10.3 Some Applied Eigenvalue Problems

In this section, we shall present a few examples of eigenvalue problems that arise in physical systems. A couple of these will involve differential equations (which we don't discuss formally until the next chapter), but only in an elementary way.

A wide variety of applications of eigenvalue problems involve systems of simultaneous first-order linear differential equations with constant coefficients. For example, a sequence of radioactive decays from a parent isotope through subsequent generations

$$A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow \cdots \longrightarrow X$$

or the time dependence of the occupation of energy levels of the molecules in a lasing material, or a study of the stability of equilibrium systems, or calculations of currents in electrical networks lead to sets of differential equations of the form

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned} \tag{1}$$

where the a_{ij}s are constants. We can write this set of equations in matrix form:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{2}$$

Let's consider a two-dimensional case first:

$$\begin{aligned} \dot{x}_1 &= x_1 + 2x_2 \\ \dot{x}_2 &= 2x_1 + x_2 \end{aligned} \tag{3}$$

where the overdots are standard notation for time derivatives. In the one-dimensional case, $dx/dt = ax$, the solution is simply $x(t) = x(0)e^{at}$. Therefore, let's try a solution of the form $x_j(t) = u_j e^{\lambda t}$. If we substitute this into Equation 3, we obtain

$$u_1\lambda = u_1 + 2u_2$$
$$u_2\lambda = 2u_1 + u_2$$

or

$$\begin{aligned} u_1(1-\lambda) + 2u_2 &= 0 \\ 2u_1 + u_2(1-\lambda) &= 0 \end{aligned} \tag{4}$$

To have a nontrivial solution for u_1 and u_2, the determinant of the coefficients must equal zero, so that we have

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

Note that this is just the determinantal characteristic equation of the coefficient matrix A in Equation 3. The eigenvalues are $\lambda = -1$ and $\lambda = 3$. To determine u_1 and u_2 associated with each value of λ, we set $\lambda = -1$ in Equation 4 to find that $u_1 = -u_2$, or that $\mathbf{u} = (1, -1)^{\mathsf{T}}$. Similarly, for $\lambda = 3$, we have $\mathbf{u} = (1, 1)^{\mathsf{T}}$. Therefore,

$$\mathbf{v}_1 = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

are both solutions to Equation 3. We'll learn in the next chapter that the general solution is a linear combination of $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$, or

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \tag{5}$$

We can determine the values of c_1 and c_2 by employing the initial values of $x_1(t)$ and $x_2(t)$. Suppose that $x_1(0) = 1$ and $x_2(0) = 0$. Then Equation 5 reads

$$x_1(0) = c_1 + c_2 = 1$$
$$x_2(0) = -c_1 + c_2 = 0$$

or $c_1 = c_2 = 1/2$. Thus, Equation 5 becomes

$$\mathbf{x}(t) = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

or, in terms of $x_1(t)$ and $x_2(t)$,

$$x_1(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t}$$
$$x_2(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{3t}$$

It is a simple matter to show that $x_1(t)$ and $x_2(t)$ satisfy Equation 3 *and* the initial conditions.

In general, we substitute $\mathbf{x}(t) = \mathbf{u}e^{\lambda t}$ into Equation 2 to obtain

$$(\mathsf{A} - \lambda \mathsf{I})\,\mathbf{u} = 0 \tag{6}$$

Let the eigenvalues and the corresponding eigenvectors of A be $\lambda_1, \ldots, \lambda_n$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n$, so that

$$\mathsf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_j \qquad j = 1, 2, \ldots, n \tag{7}$$

Substituting $\mathbf{x}(t) = \mathbf{u}_j e^{\lambda_j t}$ into Equation 2 shows that this is a solution to Equation 2. The general solution to Equation 2 can be written as a linear combination of the individual solutions $\mathbf{u}_j e^{\lambda_j t}$:

$$\mathbf{x}(t) = c_1\mathbf{u}_1 e^{\lambda_1 t} + c_2\mathbf{u}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{u}_n e^{\lambda_n t} \tag{8}$$

To prove that this is indeed a solution to Equation 2, we substitute it into Equation 2 and show that the two sides are the same (Problem 1). Equation 8 expresses the solution to Equation 2 in terms of the eigenvalues and eigenvectors of A. The c_j in Equation 8 depend upon the initial conditions $x_1(0), x_2(0), \ldots, x_n(0)$.

Example 1:
Solve the system of equations

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= 3x_1 + 7x_2 - 9x_3 \\ \dot{x}_3 &= 2x_2 - x_3 \end{aligned}$$

SOLUTION: Let $x_j(x) = u_j e^{\lambda t}$ to obtain

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 3 & 7-\lambda & -9 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0$$

(See Equation 6.) The three eigenvalues are $\lambda = 1$, 2, and 3 and the corresponding eigenvectors are

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \qquad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$$

According to Equation 8, the general solution is

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} e^{3t}$$

Example 2:
The radioactive decay of a certain nucleus satisfies the equations

$$\frac{dA}{dt} = -k_1 A$$

$$\frac{dB}{dt} = k_1 A - k_2 B$$

$$\frac{dC}{dt} = k_2 B$$

Solve this set of equations for $k_1 = 2$ and $k_2 = 1$ with the initial conditions $A(0) = A_0$ and $B(0) = C(0) = 0$.

SOLUTION: We write these equations in matrix notation

$$\frac{d\mathbf{x}}{dt} = \mathsf{M}\,\mathbf{x} = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{x}$$

The eigenvalues and corresponding eigenvectors of M are $\lambda_1 = -2$, $\lambda_2 = -1$, $\lambda_3 = 0$ and $\mathbf{u}_1 = (1, -2, 1)^{\mathsf{T}}$, $\mathbf{u}_2 = (0, -1, 1)^{\mathsf{T}}$, $\mathbf{u}_3 = (0, 0, 1)^{\mathsf{T}}$. Thus, the general solution is

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Using $A(0) = A_0$ and $B(0) = C(0) = 0$ gives $c_1 = A_0$, $c_2 = -2A_0$, and $c_3 = A_0$, so that

$$A(t) = A_0 e^{-2t}$$

$$B(t) = 2A_0(e^{-t} - e^{-2t})$$

$$C(t) = A_0(1 - 2e^{-t} + e^{-2t})$$

Figure 10.3 shows $A(t)$, $B(t)$, and $C(t)$ plotted against t. Can you explain the shape of the curve $B(t)$?

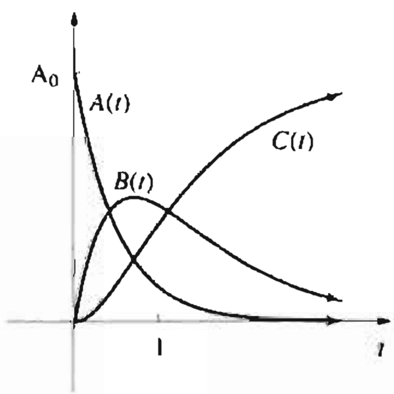


Figure 10.3
The solutions to the radioactive decay rate equations in Example 2.

The examples that we have considered so far have had real eigenvalues. What if the eigenvalues come out to be complex?

Example 3:
Solve the system of equations

$$\dot{x}_1 = -3x_1 - 2x_2$$

$$\dot{x}_2 = 2x_2 - x_3$$

SOLUTION: Let $\mathbf{x} = \mathbf{u}\, e^{\lambda t}$ to obtain

$$\begin{pmatrix} -3-\lambda & -2 \\ 2 & -1-\lambda \end{pmatrix} \mathbf{u} = 0$$

The eigenvalues and corresponding eigenvectors are $-2 \pm i\sqrt{3}$ and $\mathbf{u}_1 = ((-1 - i\sqrt{3})/2, 1)$, $\mathbf{u}_2 = ((-1 + i\sqrt{3})/2, 1)$. Proceeding formally, the solution is

$$\mathbf{x} = c_1 \mathbf{u}_1 e^{(-2+i\sqrt{3})t} + c_2 \mathbf{u}_2 e^{(-2-i\sqrt{3})t}$$

We can use Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$) to write $\mathbf{x}$ as

$$\begin{aligned}\mathbf{x} &= (c_1\mathbf{u}_1 + c_2\mathbf{u}_2)e^{-2t}\cos\sqrt{3}\,t + i(c_1\mathbf{u}_1 - c_2\mathbf{u}_2)e^{-2t}\sin\sqrt{3}\,t \\ &= \mathbf{a}\, e^{-2t}\cos\sqrt{3}\,t + \mathbf{b}\, e^{-2t}\sin\sqrt{3}\,t\end{aligned}$$

Thus, we see that complex roots lead to exponentially damped harmonic behavior (Figure 10.4). We'll discuss this type of behavior in more detail in the next chapter.

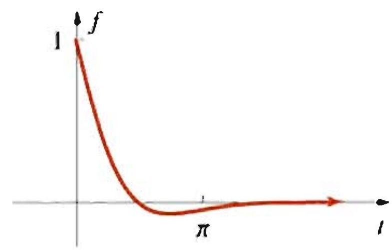


Figure 10.4
An illustration of exponentially damped harmonic behavior. The function $e^{-t}\cos t$ is plotted against t.

Although we illustrated this approach with 2×2 and 3×3 systems, the size of the system is really irrelevant, particularly with the CAS that are currently available. (See Problem 21.)

Vibrating or oscillating mechanical systems can be formulated as eigenvalue problems. Our first example will consist of two particles of mass m connected by three identical springs of relaxed length l and constrained to move horizontally as shown in Figure 10.5. We shall displace the masses from their equilibrium positions, then let them go and investigate their subsequent motion. If x_1 and x_2 denote small displacements of the two particles from their equilibrium positions, then the potential energy of the system is given by

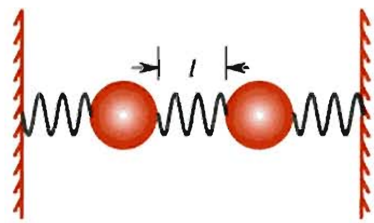


Figure 10.5
Two particles of mass m connected by three identical springs of relaxed length l and constrained to move longitudinally.

$$V = \frac{k}{2}x_1^2 + \frac{k}{2}(x_2 - x_1)^2 + \frac{k}{2}x_2^2 \tag{9}$$

where k is the force constant of each spring. Newton's equation for each particle is

$$\begin{aligned} m\frac{d^2x_1}{dt^2} &= -\frac{\partial V}{\partial x_1} = k(x_2 - 2x_1) \\ m\frac{d^2x_2}{dt^2} &= -\frac{\partial V}{\partial x_2} = k(x_1 - 2x_2) \end{aligned} \tag{10}$$

Write Equation 10 in matrix notation:

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathsf{A}\mathbf{x} \tag{11}$$

where

$$\mathsf{A} = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} \tag{12}$$

We know that this system oscillates in time, so we'll assume that $\mathbf{x}(t) = \mathbf{u}\, e^{i\omega t}$. Substituting this expresion into Equation 11 gives

$$-m\omega^2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Thus, ω^2 turns out to be given by

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0 \tag{13}$$

and so we find two *characteristic frequencies*

$$\omega_1 = \left(\frac{k}{m}\right)^{1/2} \qquad \text{and} \qquad \omega_2 = \left(\frac{3k}{m}\right)^{1/2} \tag{14}$$

with corresponding eigenvectors

$$\mathbf{u}_1 = \begin{pmatrix} a \\ a \end{pmatrix} \qquad \text{and} \qquad \mathbf{u}_2 = \begin{pmatrix} a \\ -a \end{pmatrix} \tag{15}$$

where a is an arbitrary (possibly complex) constant. The motion of the two masses is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} a \\ a \end{pmatrix} e^{i\omega_1 t} + c_2 \begin{pmatrix} a \\ -a \end{pmatrix} e^{i\omega_2 t} \tag{16}$$

or

$$\begin{aligned} x_1(t) &= b_1 e^{i\omega_1 t} + b_2 e^{i\omega_2 t} \\ x_2(t) &= b_1 e^{i\omega_1 t} - b_2 e^{i\omega_2 t} \end{aligned} \tag{17}$$

where $b_1 = ac_1$ and $b_2 = ac_2$.

Equations 17 say that the masses will vibrate in phase if $b_2 = 0$ ($b_1 \neq 0$) and 180° out of phase if $b_1 = 0$ ($b_2 \neq 0$). To prove these last statements, write $b_1 = A_1 e^{i\phi_1}$, so that the real part of the solutions associated with frequency ω_1 are

$$x_1(t) = A_1 \cos(\omega_1 t - \phi_1) \qquad \text{and} \qquad x_2(t) = A_1 \cos(\omega_1 t - \phi_1) \tag{18}$$

Equation 18 shows that the two masses vibrate back and forth in unison. Similarly, write $b_2 = A_2 e^{i\phi_2}$ and find that the real part of the solutions associated with frequency ω_2 are

$$x_1(t) = A_2 \cos(\omega_2 t - \phi_2)$$
$$x_2(t) = -A_2 \cos(\omega_2 t - \phi_2) = A_2 \cos(\omega_2 t - \phi_2 + \pi) \tag{19}$$

Thus, in this case, the two masses vibrate in opposite directions.

Equations 18 and 19 are called the *normal modes* of vibration of the two-mass system. If the initial conditions are just so, the system will vibrate either in phase with a frequency ω_1 or 180° out of phase with frequency ω_2 (Figure 10.6) (Problem 10). Usually, however, the motion of the system will be described by a superposition of the two normal modes according to

$$x_1(t) = A_1 \cos(\omega_1 t - \phi_1) + A_2 \cos(\omega_2 t - \phi_2)$$

and

$$x_2(t) = A_1 \cos(\omega_1 t - \phi_1) - A_2 \cos(\omega_2 t - \phi_2)$$

The four constants, A_1, A_2, ϕ_1, and ϕ_2, can be determined from the values of $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$, and $\dot{x}_2(0)$.

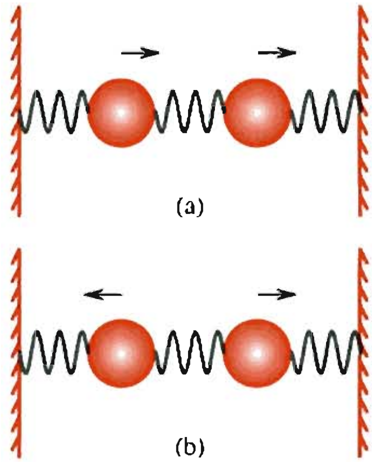


Figure 10.6
An illustration of the normal modes of the system shown in Figure 10.5.

Example 4:
Figure 10.7 shows two pendula of length l constrained to move in a single plane coupled by a harmonic spring with force constant k. Problem 9 has you show that the equations of motion for small displacements from the vertical position are

$$m\frac{d^2s_1}{dt^2} = -\frac{\partial V}{\partial s_1} = -\frac{mgs_1}{l} + k(s_2 - s_1)$$

and

$$m\frac{d^2s_2}{dt^2} = -\frac{\partial V}{\partial s_2} = -\frac{mgs_2}{l} - k(s_2 - s_1)$$

where s_1 and s_2 measure the distances of the masses along their arc of motion from their vertical positions. Solve for the characteristic frequencies and the normal modes of this system.

SOLUTION: The equations of motion in matrix form are

$$m\frac{d^2\mathbf{s}}{dt^2} = \mathsf{A}\,\mathbf{s} = \begin{pmatrix} -\left(\frac{mg}{l} + k\right) & k \\ k & -\left(\frac{mg}{l} + k\right) \end{pmatrix} \mathbf{s}$$

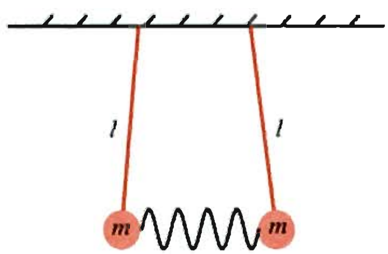


Figure 10.7
Two pendula of length l coupled by a harmonic spring and constrained to move in a single plane.

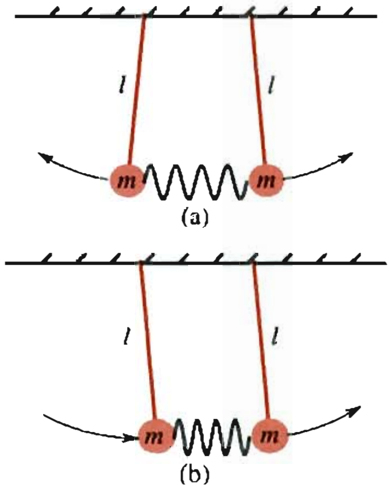


Figure 10.8
An illustration of the two normal modes of the coupled pendula shown in Figure 10.7.

Substitute $\mathbf{s} = \mathbf{u}e^{i\omega t}$ into these equations to get the eigenvalue equation

$$\begin{vmatrix} \frac{mg}{l} + k - m\omega^2 & -k \\ -k & \frac{mg}{l} + k - m\omega^2 \end{vmatrix} = 0$$

which gives $\omega_1 = (g/l)^{1/2}$ and $\omega_2 = (g/l)^{1/2}(1 + 2kl/mg)^{1/2}$.

For $\omega_1^2 = g/l$, the eigenvector is given by

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

or $u_1 = u_2$. For $\omega_2^2 = g(1 + 2kl/mg)/l$, we get $u_1 = -u_2$. Thus, the two normal modes for this system are

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = c_1 \begin{pmatrix} a \\ a \end{pmatrix} e^{i\omega_1 t} + c_2 \begin{pmatrix} a \\ -a \end{pmatrix} e^{i\omega_2 t}$$

The motion corresponding to ω_1 is in phase and that corresponding to ω_2 is 180° out of phase (Figure 10.8).

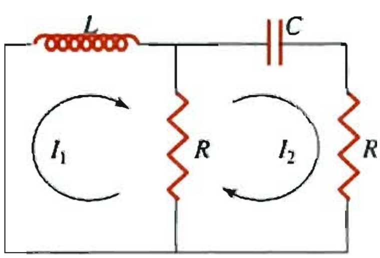


Figure 10.9
The electrical circuit described by Equation 20.

Eigenvalue problems also arise in the analysis of electrical circuits. Consider the circuit shown in Figure 10.9. Using the fact that the sum of the voltage drops around each closed loop is equal to zero, we have

$$\begin{aligned} L\frac{dI_1}{dt} + R(I_1 - I_2) &= 0 \\ \frac{1}{C}\int I_2\,dt + RI_2 + R(I_2 - I_1) &= 0 \end{aligned} \tag{20}$$

Differentiating the second equation with repsect to time gives the pair of simultaneous linear equations

$$\begin{aligned} L\frac{dI_1}{dt} + R(I_1 - I_2) &= 0 \\ I_2 + CR\frac{dI_2}{dt} + CR\left(\frac{dI_2}{dt} - \frac{dI_1}{dt}\right) &= 0 \end{aligned} \tag{21}$$

Before we go on, we can make these equations look cleaner if we choose to measure time in units of CR. If we let $\tau = t/CR$, then these equations become

$$\begin{aligned} \frac{L}{CR^2}\frac{dI_1}{d\tau} + I_1 - I_2 &= 0 \\ I_2 + 2\frac{dI_2}{d\tau} - \frac{dI_1}{d\tau} &= 0 \end{aligned} \tag{22}$$

Notice that Equations 22 contain only one apparent parameter (the CR is absorbed into τ). We can write Equations 22 in matrix notation as follows:

$$\begin{pmatrix} L/CR^2 & 0 \\ -1 & 2 \end{pmatrix} \dot{\mathbf{I}} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \mathbf{I} = 0 \tag{23}$$

where the dot represents differentiation with respect to $\tau = t/RC$. If we let $\mathbf{I} = \mathbf{u}\, e^{\lambda t}$, we obtain

$$\begin{pmatrix} \alpha\lambda & 0 \\ -\lambda & 2\lambda \end{pmatrix} \mathbf{u} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \mathbf{u} = 0$$

or

$$\begin{pmatrix} \alpha\lambda + 1 & -1 \\ -\lambda & 1 + 2\lambda \end{pmatrix} \mathbf{u} = 0 \tag{24}$$

where $\alpha = L/CR^2$. For concreteness, take $C = 20$ microfarads, $L = 100$ millihenrys, and $R = 50$ ohms, so that $\alpha = 2$. In that case, we have the condition

$$\begin{pmatrix} 2\lambda + 1 & -1 \\ -\lambda & 1 + 2\lambda \end{pmatrix} \mathbf{u} = 0 \tag{25}$$

which gives $\lambda_{\pm} = (-3 \pm i\sqrt{7})/8$, so that $\mathbf{I} = \mathbf{u}\, e^{-3\tau/8} e^{\pm i\sqrt{7}\,\tau/8}$. Thus, we see that the current will undergo damped harmonic motion.

Notice that the equation of the eigenvalues in this case is not of the form $|A - \lambda I| = 0$ because the coefficient matrix of $\dot{I}_1$ and $\dot{I}_2$ is not diagonal. We could have manipulated Equation 23 into the form $\dot{\mathbf{I}} = A\mathbf{I}$ by multiplying by the inverse of the matrix multiplying $\dot{\mathbf{I}}$ in Equation 23. Of course, we must end up with the same result (Problem 19).

We have presented just two different examples of the application of eigenvalue problems in this section, but they occur frequently in various physical problems.

10.3 Problems

1. Show that Equation 8 is a solution to Equation 2.
2. Solve the simultaneous equations
$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -3x_2$$
by assuming that $x_j(t) = u_j e^{\lambda t}$.
3. Solve the simultaneous equations
$$\dot{x}_1 = x_1 + x_2$$
$$\dot{x}_2 = 4x_1 + x_2$$
4. Solve the simultaneous equations
$$\dot{x}_1 = 6x_1 + 8x_2$$
$$\dot{x}_2 = -3x_1 - 4x_2$$

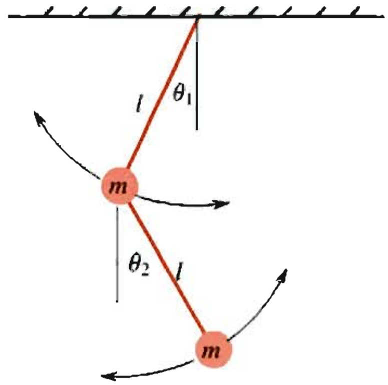


Figure 10.11
A double pendulum. One pendulum is suspended from the other and both are constrained to move in the same vertical plane.

18. Show that the total energy associated with the normal mode of lowest frequency of the system shown in Figure 10.10 is conserved.

19. Convert Equation 23 into the form $\dot{\mathbf{I}} = \mathsf{A}\mathbf{I}$ and show that the eigenvalues of A are the same as those we obtain from Equation 24.

20. Figure 10.11 shows a double pendulum, which consists of one pendulum suspended freely from another, with both constrained to swing in the same vertical plane. The equations of motion for small amplitude are

$$\frac{d^2\theta_1}{dt^2} + M\frac{d^2\theta_2}{dt^2} + \frac{g}{l}\theta_1 = 0$$

$$\frac{d^2\theta_1}{dt^2} + \frac{d^2\theta_2}{dt^2} + \frac{g}{l}\theta_2 = 0$$

where $M = m_2/(m_1 + m_2)$ and l is the length of each pendulum. Show that the two characteristic frequencies of this system are given by $\omega^2 = g/l(1 \pm \sqrt{M}\,)$.

21. Use any CAS to solve the simulataneous equations

$$\dot{x}_1 = -x_1 + 2x_2 + 2x_4 + x_5$$

$$\dot{x}_2 = 2x_1 - x_2 - 3x_3 - 2x_4 + x_5$$

$$\dot{x}_3 = -3x_2 + 2x_3 - x_4 - 2x_5$$

$$\dot{x}_4 = 2x_1 + 2x_2 - x_3 + 2x_4$$

$$\dot{x}_5 = x_1 + x_2 - 2x_3 - x_5$$

10.4 Change of Basis

Up to this point, our linear transformations have acted upon vectors to produce new vectors, all with respect to the same basis. For concreteness and simplicity, we have used the standard basis in R^n, $\mathbf{e}_1 = (1, 0, \ldots, 0)^{\mathsf{T}}$, $\mathbf{e}_2 = (0, 1, \ldots, 0)^{\mathsf{T}}$, $\ldots$, $\mathbf{e}_n = (0, 0, \ldots, 1)^T$ throughout. It is often convenient to think of the linear transformation as acting on the basis to produce a new basis. Consider the *quadratic form*

$$f(x, y) = x^2 + 4xy + y^2 \tag{1}$$

A quadratic form in n variables is an expression of the form

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \tag{2}$$

containing both square terms and cross products of the n variables. We can express the quadratic form in Equation 1 in matrix form:

$$f(x, y) = \mathbf{x}^{\mathsf{T}} \mathsf{A} \mathbf{x} = (x \;\; y) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

as you can verify by direct calculation. If we plot the equation

$$f(x, y) = (x \;\; y) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + 4xy + y^2 = 1 \tag{4}$$

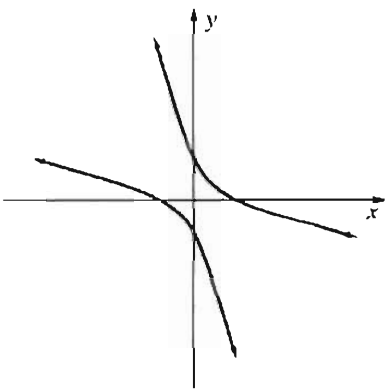


Figure 10.12
The hyperbola described by the equation $x^2 + 4xy + y^2 = 1$.

we see that it represents a hyperbola (Figure 10.12). The occurrence of the cross term $4xy$ tilts the hyperbola at an angle with respect to the x and y axes. It would appear from Figure 10.12 that if we were to rotate the axes through some yet undetermined angle, then the plot of the hyperbola would look like the one in Figure 10.13.

We'll refer to Figure 10.14 to derive the transformation of coordinate axes that results when we rotate the x, y axes in a counterclockwise direction through an angle θ. If we denote the fixed vector by $\mathbf{r}'$ when we express it in the rotated (the primed) coordinates, then according to Figure 10.14,

$$\begin{aligned}
\mathbf{r}' &= x'\,\mathbf{i}' + y'\,\mathbf{j}' \\
&= r\cos(\alpha - \theta)\,\mathbf{i}' + r\sin(\alpha - \theta)\,\mathbf{j}' \\
&= r\cos\alpha\cos\theta\,\mathbf{i}' + r\sin\alpha\sin\theta\,\mathbf{i}' \\
&\qquad + r\sin\alpha\cos\theta\,\mathbf{j}' - r\cos\alpha\sin\theta\,\mathbf{j}' \\
&= x\cos\theta\,\mathbf{i}' + y\sin\theta\,\mathbf{i}' + y\cos\theta\,\mathbf{j}' - x\sin\theta\,\mathbf{j}'
\end{aligned}$$

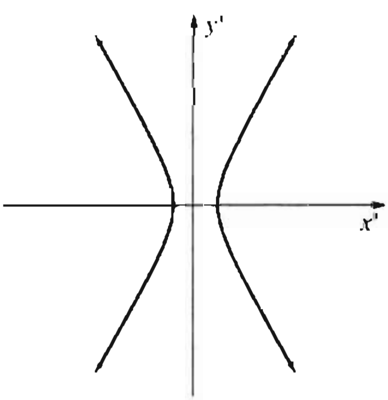


Figure 10.13
The hyperbola that results from rotating the x, y axes in Figure 10.12.

where we have used the fact that $x = r\cos\alpha$ and $y = r\sin\alpha$. Thus, we have

$$x' = x\cos\theta + y\sin\theta$$

$$y' = y\cos\theta - x\sin\theta$$

We can express the relation between x', y', and x, y in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

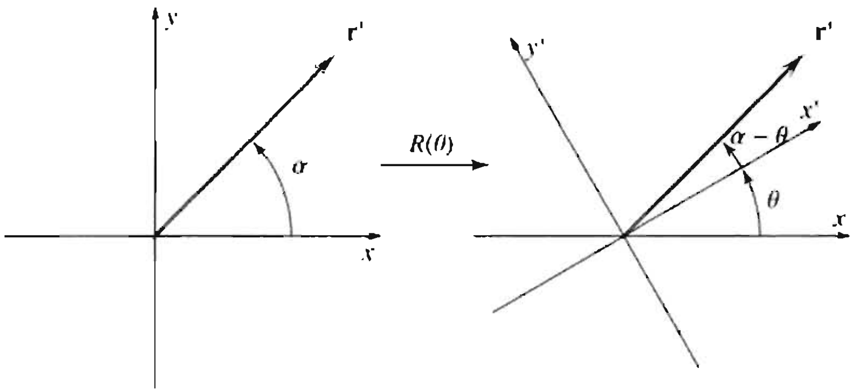


Figure 10.14
An illustration of rotating the x, y axes counterclockwise through an angle $+\theta$. The vector $\mathbf{r}'$ is fixed.

Notice that the matrix in this case is $\mathsf{R}(-\theta)$ from Section 9.3, where $\mathsf{R}(\theta)$ is the matrix that rotates $\mathbf{r}$ keeping the axes fixed. Rotating $\mathbf{r}$ through an angle $+\theta$ is equivalent to rotating the axes through an angle $-\theta$. Thus, Equation 5 can be written as $\mathbf{x}' = \mathsf{R}(-\theta)\,\mathbf{x}$. But $\mathsf{R}(-\theta) = \mathsf{R}^{-1}(\theta)$, and $\mathsf{R}^{-1}(\theta) = \mathsf{R}^{\mathsf{T}}(\theta)$ because $\mathsf{R}(\theta)$ is orthogonal. Solving Equation 5 for $\mathbf{x}$ in terms of $\mathbf{x}'$ gives $\mathbf{x} = \mathsf{R}(\theta)\mathbf{x}'$. If we substitute this result into

$$f(x, y) = \mathbf{x}^{\mathsf{T}}\mathsf{A}\,\mathbf{x} = 1$$

we have

$$g(x', y') = (\mathsf{R}\mathbf{x}')^{\mathsf{T}}\mathsf{A}\,(\mathsf{R}\mathbf{x}') = \mathbf{x}'^{\mathsf{T}}\mathsf{R}^{\mathsf{T}}\mathsf{A}\mathsf{R}\mathbf{x}' \tag{6}$$

$$= (x'\ y')\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= (x'\ y')\begin{pmatrix} 1+4\cos\theta\sin\theta & 2\cos^2\theta - 2\sin^2\theta \\ 2\cos^2\theta - 2\sin\theta & 1-4\cos\theta\sin\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= (x'\ y')\begin{pmatrix} 1+2\sin 2\theta & 2\cos 2\theta \\ 2\cos 2\theta & 1-2\sin 2\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{x}'^{\mathsf{T}}\mathsf{A}'\,\mathbf{x}'$$

$$= x'^2 + 2x'^2\sin 2\theta + 4x'y'\cos 2\theta + y'^2 - 2y'^2\sin 2\theta = 1 \tag{7}$$

Note that we use the notation $g(x', y')$ to represent $f(x, y)$ when x and y are eliminated in terms of x' and y'.

If we eliminate the cross term in Equation 7, then $g(x', y')$ will be of the form $g(x', y') = ax'^2 + by'^2$, which is the standard form of a conic section. We can eliminate the cross term in Equation 7 by choosing θ such that $\cos 2\theta = 0$, or by choosing $\theta = \pi/4$. In this case, Equation 7 becomes

$$g(x', y') = 3x'^2 - y'^2 = 1 \tag{8}$$

It's easy to see now that $g(x', y') = 1$ is a hyperbola, whereas the nature of $x^2 + 4xy + y^2 = 1$ is not at all clear. The equation $g(x', y') = 3x'^2 - y'^2 = 1$ is

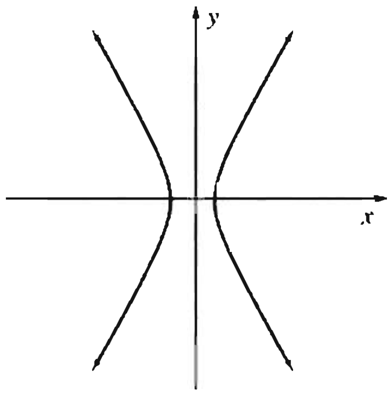


Figure 10.15
The hyperbola described by the equation $f(x, y) = 3x^2 - y^2 = 1$.

plotted in Figure 10.15, and you can see that it is simply the same curve as in Figure 10.11 but plotted in a coordinate system that has been rotated by 45°.

The relation between the original coordinates and the new coordinates is given by Equation 5 with $\theta = \pi/4$, or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\begin{aligned} x' &= \frac{1}{\sqrt{2}}(x + y) \\ y' &= \frac{1}{\sqrt{2}}(-x + y) \end{aligned} \tag{9}$$

If we substitute Equations 9 into Equation 8, you get Equation 4. We see that the quadratic form in Equation 4 has only quadratic terms (no cross terms) if it is expressed in terms of (x', y') instead of (x, y). We will develop a more efficient way of determining coordinates like (x', y') in the next section, but the message here is that it is often desirable to transform coordinate systems, or basis sets into new basis sets.

We can summarize the above procedure by writing $\mathbf{x} = \mathsf{R}\mathbf{x}'$, where

$$\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Now substitute $\mathbf{x} = \mathsf{R}\mathbf{x}'$ into

$$f(x, y) = \mathbf{x}^{\mathsf{T}}\mathsf{A}\mathbf{x} \tag{10}$$

to obtain

$$g(x', y') = (\mathsf{R}\mathbf{x}')^{\mathsf{T}}\mathsf{A}\,(\mathsf{R}\mathbf{x}') = \mathbf{x}'^{\mathsf{T}}\mathsf{R}^{\mathsf{T}}\mathsf{A}\,\mathsf{R}\mathbf{x}' = 1 \tag{11}$$

Generally, $g(x'y')$ will contain a cross term whose coefficient depends upon θ, as in Equation 7. Now choose θ so that there are no cross terms in $g(x', y')$. This amounts to choosing θ such that $\mathsf{R}^{\mathsf{T}}\mathsf{A}\,\mathsf{R}$ is diagonal.

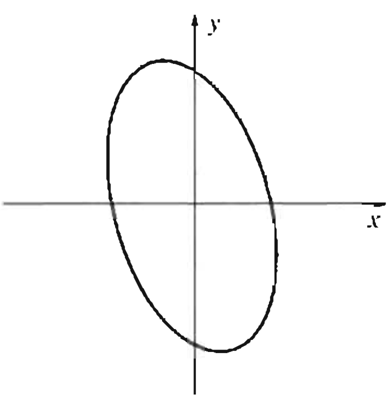


Figure 10.16
A plot of $9x^2 + 2\sqrt{3}\,xy + 3y^2 = 1$, showing that its graph is an ellipse whose major axis is not aligned with either coordinate axis.

Example 1:
Determine the rotation of coordinate axes that eliminates the cross term in the equation

$$9x^2 + 2\sqrt{3}\,xy + 3y^2 = 1$$

(See Figure 10.16.)

SOLUTION: First write the equation in matrix form:

$$(\,x \;\; y\,)\begin{pmatrix} 9 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{x}^{\mathsf{T}}\mathsf{A}\mathbf{x}$$

Now use Equation 6, where

$$\mathsf{A}' = \mathsf{R}^{\mathsf{T}}\mathsf{A}\mathsf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 9 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 9\cos^2\theta + 3\sin^2\theta + 2\sqrt{3}\sin\theta\cos\theta & \sqrt{3}\cos^2\theta - \sqrt{3}\sin^2\theta - 6\sin\theta\cos\theta \\ \sqrt{3}\cos^2\theta - \sqrt{3}\sin^2\theta - 6\sin\theta\cos\theta & 9\cos^2\theta + 3\sin^2\theta - 2\sqrt{3}\sin\theta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 3 + 6\cos^2\theta + \sqrt{3}\sin 2\theta & \sqrt{3}\cos 2\theta - 3\sin 2\theta \\ \sqrt{3}\cos 2\theta - 3\sin 2\theta & 3 + 6\sin^2\theta - \sqrt{3}\sin 2\theta \end{pmatrix}$$

Set $\sqrt{3}\cos 2\theta - 3\sin 2\theta = 0$, or $\tan 2\theta = \sqrt{3}/3$, to see that $2\theta = \pi/6$, or $\theta = \pi/12$. Substituting $\theta = \pi/12$ into A' gives

$$\mathsf{A}' = \begin{pmatrix} 9.4641 & 0 \\ 0 & 2.5359 \end{pmatrix}$$

so that

$$g(x', y') = 9.4641x'^2 + 2.5359y'^2 = 1$$

This equation is plotted in Figure 10.17. The transformation that produces this result is given by Equation 5 with $\theta = \pi/12$.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0.9659 & 0.2588 \\ -0.2588 & 0.9659 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$x' = 0.9659x + 0.2588y$$

$$y' = -0.2588x + 0.9659y$$

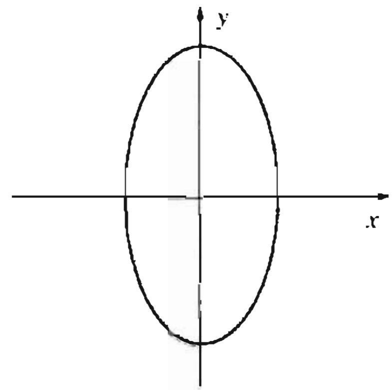


Figure 10.17
The figure in Figure 10.16 rotated through 15° so that its axes are aligned with the x axis and the y axis.

We'll now spend the rest of this section discussing general linear transformations of basis sets, or coordinate systems. There are many instances where it is convenient to analyze a problem in one basis, but to report the results in another basis. For example, in studying the dynamics of a rotating body, it is most convenient to use a coordinate system that is rotating with the body, but you might want to express your result in the coordinate system of a fixed observer. Or, if you are analyzing the dynamics of the collisions between particles, it is conveninet to use center-of-mass coordinates, but you would report your results in laboratory-fixed coordinates.

The only basis set that we have dealt with so far is the standard basis, $\mathbf{e}_1 = (1, 0, \ldots, 0)^{\mathsf{T}}, \mathbf{e}_2 = (0, 1, \ldots, 0)^{\mathsf{T}}, \cdots, \mathbf{e}_n = (0, 0, \ldots, 1)^{\mathsf{T}}$ in $V = \mathsf{R}^n$. Now let

$\{\mathbf{z}_i\}$ be another basis in V. This new basis can be expanded in terms of the standard basis

$$\mathbf{z}_i = \sum_{j=1}^{n} z_{ji}\mathbf{e}_j \qquad i = 1, 2, \ldots, n \tag{12}$$

A vector $\mathbf{u}$ can be expressed in this basis as

$$\mathbf{u}_z = \sum_{i=1}^{n} u_{zi}\mathbf{z}_i \tag{13}$$

where we subscript $\mathbf{u}$ with a z to emphasize that we are expanding $\mathbf{u}$ in the z-basis. We can express $\mathbf{u}$ expanded in the standard basis $\{\mathbf{e}_j\}$ as

$$\mathbf{u}_E = \sum_{j=1}^{n} u_{Ej}\mathbf{e}_j \tag{14}$$

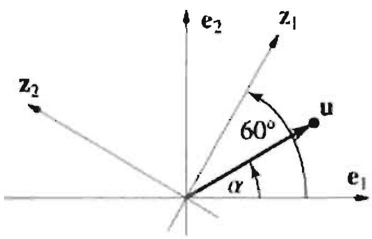


Figure 10.18
The same vector in two coordinate systems (bases) that differ by a rotation through $+60°$.

Realize that $\mathbf{u}_E$ and $\mathbf{u}_z$ represent the very same vector, $\mathbf{u}$, expressed in terms of different bases. For example, Figure 10.18 shows the same vector in two coordinate systems that differ from each other by a rotation through $+60°$. In Figure 10.18, the components of $\mathbf{u}$ in the $\{\mathbf{z}_j\}$ basis are related to the components of $\mathbf{u}$ in the $\{\mathbf{e}_j\}$ basis by

$$\begin{pmatrix} u_{z1} \\ u_{z2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_{E1} \\ u_{E2} \end{pmatrix}$$

(Problem 6).

We'd like to know in general how the components of $\mathbf{u}_z$ are related to those of $\mathbf{u}_E$. Substitute Equation 12 into Equation 13 to obtain

$$\mathbf{u}_E = \sum_{i=1}^{n} u_{zi} \sum_{j=1}^{n} z_{ji}\,\mathbf{e}_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} z_{ji}u_{zi} \right) \mathbf{e}_j \tag{15}$$

If we compare this result with Equation 14, we see that

$$u_{Ej} = \sum_{i=1}^{n} z_{ji}u_{zi} \tag{16}$$

We can express this result in matrix notation by writing

$$\mathbf{u}_E = \mathsf{Z}\,\mathbf{u}_z \tag{17}$$

By referring to Equation 12, you can see that the ith column of Z consists of the components of $\mathbf{z}_i$ in the standard basis. We can express Equation 17 in a more symmetric form by writing $\mathsf{I}\,\mathbf{u}_E = \mathsf{Z}\,\mathbf{u}_z$, where I is the unit matrix.

Example 2:
Suppose we have a basis $\mathbf{z}_1 = (1, 0, 0)^T$, $\mathbf{z}_2 = (1, 1, 0)^T$, and $\mathbf{z}_3 = (1, 1, 1)^T$. Determine $\mathbf{u}_z$ if $\mathbf{u}_E = (1, 2, 3)^T$.

SOLUTION: The matrix Z in Equation 17 consists of the column vectors $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$, so

$$Z = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

According to Equation 17, $\mathbf{u}_z = Z^{-1}\mathbf{u}_E$, and so to find $\mathbf{u}_z$ in terms of $\mathbf{u}_E$, we need Z^{-1}, which is

$$Z^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$\mathbf{u}_z = Z^{-1}\mathbf{u}_E = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

Let's check this result. We've just shown that

$$\mathbf{u}_z = -\mathbf{z}_1 - \mathbf{z}_2 + 3\,\mathbf{z}_3$$

Now substitute $\mathbf{z}_1 = \mathbf{e}_1$, $\mathbf{z}_2 = \mathbf{e}_1 + \mathbf{e}_2$, and $\mathbf{z}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ into this result to obtain

$$\mathbf{u}_E = \mathbf{e}_1 + 2\,\mathbf{e}_2 + 3\,\mathbf{e}_3 = (1, 2, 3)$$

Example 3:
Show that $\mathbf{u}_E$ and $\mathbf{u}_z$ in Example 2 have the same length.

SOLUTION: Because $\mathbf{u}_E$ is expressed in terms of an orthonormal basis,

$$\begin{aligned}\|\mathbf{u}_E\|^2 &= \mathbf{u}_E \cdot \mathbf{u}_E = (\mathbf{e}_1 + 2\,\mathbf{e}_2 + 3\,\mathbf{e}_3) \cdot (\mathbf{e}_1 + 2\,\mathbf{e}_2 + 3\,\mathbf{e}_3) \\ &= 1^2 + 2^2 + 3^2 = 14\end{aligned}$$

so that the (Euclidian) length of $\mathbf{u}_E$ is $(14)^{1/2}$. Note, however, that the z basis is *not* orthogonal. Nevertheless,

$$
\begin{aligned}
|\mathbf{u}_z|^2 = \mathbf{u}_z \cdot \mathbf{u}_z &= (-\mathbf{z}_1 - \mathbf{z}_2 + 3\,\mathbf{z}_3) \cdot (-\mathbf{z}_1 - \mathbf{z}_2 + 3\,\mathbf{z}_3) \\
&= \mathbf{z}_1 \cdot \mathbf{z}_1 + \mathbf{z}_2 \cdot \mathbf{z}_2 + 9\,\mathbf{z}_3 \cdot \mathbf{z}_3 \\
&\qquad + 2\,\mathbf{z}_1 \cdot \mathbf{z}_2 - 6\,\mathbf{z}_1 \cdot \mathbf{z}_3 - 6\,\mathbf{z}_2 \cdot \mathbf{z}_3 \\
&= 1 + 2 + 9(3) + 2(1) - 6(1) - 6(2) \\
&= 14
\end{aligned}
$$

so that the (Euclidean) length of $\mathbf{u}_z$ is $(14)^{1/2}$. Of course, $\mathbf{u}_E$ and $\mathbf{u}_z$ must have the same length because they are the same vector expressed in different coordinate systems.

We can generalize Equation 17 to the case where $\mathbf{u}$ is expressed in terms of two (arbitrary) coordinate systems z and w, instead of in the standard basis E and one (arbitrary) coordinate system. If $\{\mathbf{w}_j\}$ is yet another basis in V, then Equation 17 becomes

$$\mathbf{u}_E = \mathsf{Z}\mathbf{u}_z = \mathsf{W}\mathbf{u}_w \tag{18}$$

where the ith column of W consists of the components of $\mathbf{w}_i$ in the standard basis. Thus, according to Equation 18, if $\{\mathbf{z}_i\}$ and $\{\mathbf{w}_j\}$ are any two basis sets in V, then

$$\mathbf{u}_z = \mathsf{Z}^{-1}\mathsf{W}\mathbf{u}_w = \mathsf{P}\mathbf{u}_w \tag{19}$$

where $\mathsf{P} = \mathsf{Z}^{-1}\,\mathsf{W}$ is a linear transformation that relates $\mathbf{u}_z$ to $\mathbf{u}_w$.

One last topic: Suppose we have a transformation matrix A relative to some given basis, and we wish to determine the form of A relative to some other basis. In other words, suppose we have A in a basis $\{\mathbf{z}_j\}$, so that

$$\mathbf{v}_z = \mathsf{A}_z\,\mathbf{u}_z$$

(the z subscripts here are just for emphasis) and we introduce a new basis $\{\mathbf{w}_j\}$ related to $\{\mathbf{z}_j\}$ by

$$\mathbf{v}_z = \mathsf{P}\mathbf{v}_w \qquad \text{and} \qquad \mathbf{u}_z = \mathsf{P}\mathbf{u}_w \tag{20}$$

as in Equation 19. Then,

$$\mathbf{v}_w = \mathsf{P}^{-1}\,\mathbf{v}_z = \mathsf{P}^{-1}\mathsf{A}_z\mathbf{u}_z = \mathsf{P}^{-1}\mathsf{A}_z\mathsf{P}\mathbf{u}_w = \mathsf{A}_w\mathbf{u}_w \tag{21}$$

where $\mathsf{A}_w = \mathsf{P}^{-1}\mathsf{A}_z\mathsf{P}$. Therefore, we see that

$$\mathsf{A}_w = \mathsf{P}^{-1}\mathsf{A}_z\mathsf{P} \tag{22}$$

where P is defined through Equation 19. Thus, if we can construct a matrix in one basis (such as the standard basis), we can construct the matrix in another basis if we know the linear transformation that relates the two basis sets. The transformation depicted in Equation 22 is called a *similarity transformation* and two matrices that are related by a similarity transformation are said to be *similar*. We shall see in the next section that similarity transformations play a central role in transforming matrices into diagonal form.

Example 4:
Given the representation of a matrix

$$A_E = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

in the standard basis, determine A_Z in terms of the basis in Example 2.

SOLUTION: We first write

$$\mathbf{v}_E = A_E \mathbf{u}_E$$

Equation 18 gives us $\mathbf{u}_E = Z\,\mathbf{u}_Z$ and $\mathbf{v}_E = Z\,\mathbf{v}_Z$. Therefore,

$$\mathbf{v}_Z = Z^{-1}\mathbf{v}_E = Z^{-1}A_E\mathbf{u}_E = Z^{-1}A_E Z\,\mathbf{u}_Z$$

Using Z and Z^{-1} from Example 2, we have

$$\begin{aligned} A_Z &= Z^{-1}A_E Z \\ &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -2 & -2 \\ 3 & 3 & 3 \\ -1 & 0 & 1 \end{pmatrix} \end{aligned}$$

Notice that the determinants of A_E and A_Z are equal (Problem 23).

In the next section, we'll learn how to transform quadratic forms into canonical form by using the eigenvalues and eigenvectors of A.

10.4 Problems

1. Determine the rotation of coordinate axes that eliminates the cross term in the equation $f(x, y) = 2x^2 + 2xy + 2y^2 = 1$.
2. Determine the rotation of coordinate axes that eliminates the cross term in the quadratic form $f(x, y) = x^2 - 3xy - y^2$.

3. Show that if $f(x, y) = (x\ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, then a rotation of the axes through an angle that satisfies the relation $\tan 2\theta = \dfrac{2b}{a-c}$ will eliminate the cross term in $f(x, y)$.

4. Determine the rotation of coordinate axes that eliminates the cross term in the equation $3x^2 + 6xy - 4y^2 = 1$.

5. Determine the rotation of coordinate axes that eliminates the cross term in the equation $3x^2 + 2\sqrt{3}\,xy + y^2 = 1$.

6. Show that the components of the vectors in Figure 10.18 are related by

$$\begin{pmatrix} u_{z1} \\ u_{z2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} u_{E1} \\ u_{E2} \end{pmatrix}.$$

7. Suppose that we have a basis set $\mathbf{z}_1 = (1, 0, 2)^T$, $\mathbf{z}_2 = (-1, 1, 2)^T$, and $\mathbf{z}_3 = (0, 1, 0)^T$ with $\mathbf{u}_z = (-1, 2, 1)^T$. Find $\mathbf{u}_E$.

8. Show that $\mathbf{u}_z$ and $\mathbf{u}_E$ in the previous problem have the same (Euclidian) length.

9. Referring to Problem 7, calculate $\mathbf{u}_z$ given that $\mathbf{u}_E = (-3, 3, 2)^T$.

10. Show that $\mathbf{u}_z$ and $\mathbf{u}_E$ in the previous problem have the same (Euclidian) length.

11. Suppose we have a basis $\mathbf{z}_1 = (2, -4)^T$ and $\mathbf{z}_2 = (3, 8)^T$. Given $\mathbf{u}_E = (1, 1)^T$, find $\mathbf{u}_z$.

12. Show that $\mathbf{u}_z$ and $\mathbf{u}_E$ in the previous problem have the same (Euclidian) length.

13. Consider the two basis sets $\mathbf{e}_1 = (1, 0)^T$, $\mathbf{e}_2 = (0, 1)^T$, and $\mathbf{z}_1 = (2, 1)^T$, $\mathbf{z}_2 = (-3, 4)^T$. (a) Find the matrix that will express $\mathbf{u}_z$ in the basis set $\mathbf{e}_1, \mathbf{e}_2$. (b) Find the matrix that will express $\mathbf{u}_E$ in the basis set $\mathbf{z}_1, \mathbf{z}_2$.

14. Suppose that $\mathbf{u}_E = (1, 2, 1)^T$. Find $\mathbf{u}$ relative to the basis set $\mathbf{z}_1 = (1, 1, 0)^T$, $\mathbf{z}_2 = (1, 0, 1)^T$, and $\mathbf{z}_3 = (1, 1, 1)^T$.

15. Consider two basis sets, $\mathbf{z}_1 = (1, 1, 0)^T$, $\mathbf{z}_2 = (1, 0, 1)^T$, and $\mathbf{z}_3 = (1, 1, 1)^T$, and $\mathbf{w}_1 = (1, 0, 0)^T$, $\mathbf{w}_2 = (1, 1, 0)^T$, and $\mathbf{w}_3 = (1, 1, 1)^T$. Determine the matrix that transforms $\mathbf{u}$ in the z basis to $\mathbf{u}$ in the w basis.

16. Find the coordinates of $\mathbf{u}_E = (1, 1, 0)^T$ relative to the basis $\mathbf{z}_1 = (1, 1, 2)^T$, $\mathbf{z}_2 = (2, 2, 1)^T$, and $\mathbf{z}_3 = (1, 2, 2)^T$.

17. A certain matrix A transforms $\mathbf{u}_1 = (1, 0, 1)^T$ into $(2, 3, -1)^T$, $\mathbf{u}_2 = (1, -1, 1)^T$ into $(3, 0, -2)^T$, and $\mathbf{u}_3 = (1, 2, -1)^T$ into $(-2, 7, -1)^T$. Determine the representation of A in the standard basis.

18. Let $\mathsf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{pmatrix}$ be a matrix expressed in the standard basis and let $\mathbf{w}_1 = (1, 0, 0)^T$, $\mathbf{w}_2 = (0, 1, 2)^T$, and $\mathbf{w}_3 = (0, 0, 1)^T$ be another basis. If $\mathbf{u}_E = (3, 0, 2)^T$, find the vector $\mathsf{A}\,\mathbf{u}_E$ in the w basis.

19. Referring to Problem 18, find the transformation $\mathbf{u}_w = \mathsf{B}\,\mathbf{u}_w$ corresponding to $\mathbf{v}_E = \mathsf{A}\,\mathbf{u}_E$.

20. Consider two basis sets of an n-dimensional vector space $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$. If A is a matrix representation in the z basis, show that its representation in the w basis is $\mathsf{A}_w = \mathsf{W}^{-1}\mathsf{Z}\,\mathsf{A}_z\mathsf{Z}^{-1}\mathsf{W}$.

21. If $\mathsf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ is a matrix representation in the z basis of Problem 15, determine the corresponding matrix representation in the w basis of that problem. Use the result of the previous problem.

22. If $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ is a matrix representation in the basis $\mathbf{w}_1 = (0, -1, 2)^T$, $\mathbf{w}_2 = (4, 1, 0)^T$, and $\mathbf{w}_3 = (-2, 0, 4)^T$, find the corresponding matrix in the z basis $\mathbf{z}_1 = (1, -1, 1)^T$, $\mathbf{z}_2 = (1, 0, -1)^T$, and $\mathbf{z}_3 = (1, 2, 1)^T$.

23. Show that the determinants of A_E and A_z in Example 4 are equal to 3.

10.5 Diagonalization of Matrices

Consider the eigenvalue problem $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$ in which A is $n \times n$ and $j = 1, 2, \ldots, n$. It turns out that a matrix whose columns are the eigenvectors of A has a very useful property. Let

$$S = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) \tag{1}$$

where the jth column of S is $\mathbf{u}_j$. More explicitly,

$$S = \begin{pmatrix} u_{11} & u_{21} & u_{31} & \cdots & u_{n1} \\ u_{12} & u_{22} & u_{32} & \cdots & u_{n2} \\ u_{13} & u_{23} & u_{33} & \cdots & u_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{1n} & u_{2n} & u_{3n} & \cdots & u_{nn} \end{pmatrix} \tag{2}$$

Now notice that

$$\begin{aligned} AS &= (A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_n) = (\lambda_1\mathbf{u}_1, \lambda_2\mathbf{u}_2, \ldots, \lambda_n\mathbf{u}_n) \\ &= S\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} = SD \end{aligned} \tag{3}$$

(Problems 1 and 2). The matrix D is a diagonal matrix whose elements are the eigenvalues of A.

If the n eigenvectors of A are linearly independent, then S is non-singular. Therefore, S^{-1} exists and we can multiply Equation 3 from the left by S^{-1} to obtain

$$D = S^{-1}AS \tag{4}$$

Thus, we see that $S^{-1}AS$ yields a diagonal matrix whose elements are the eigenvalues of A.

Example 1:
Diagonalize

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$$

SOLUTION: The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$ and the corresponding eigenvectors are

$$\mathbf{u}_1 = \begin{pmatrix} a \\ a \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{pmatrix} 2b \\ b \end{pmatrix}$$

where a and b are arbitrary constants, which for convenience we'll let be equal to one. Therefore,

$$\mathsf{S} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

and

$$\mathsf{S}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

and

$$\mathsf{S}^{-1}\mathsf{A}\mathsf{S} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} = \mathsf{D}$$

Notice that the elements of D are the eigenvalues of A.

As we said at the end of the previous section, two matrices A and B that are related by a relation such as

$$\mathsf{B} = \mathsf{S}^{-1}\mathsf{A}\mathsf{S} \tag{5}$$

are said to be *similar* and Equation 5 is said to be a *similarity transformation*. A matrix S, whose columns are the eigenvectors of A, is called a *modal matrix of* A. We shall now prove that similar matrices have the same characteristic equation, and consequently the same eigenvalues. If A and B are two similar matrices, then

$$\mathsf{B} - \lambda\,\mathsf{I} = \mathsf{S}^{-1}\mathsf{A}\mathsf{S} - \lambda\,\mathsf{I} = \mathsf{S}^{-1}(\mathsf{A} - \lambda\,\mathsf{I})\mathsf{S}$$

Take the determinant of these matrices and use the property that det AB = (det A)(det B) to obtain

$$|\mathsf{B} - \lambda\,\mathsf{I}| = |\mathsf{S}\mathsf{S}^{-1}|\,|\mathsf{A} - \lambda\,\mathsf{I}| = |\mathsf{A} - \lambda\,\mathsf{I}| \tag{6}$$

Thus, A and B have the same characteristic equation. Write the characteristic equation of B and A as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = \lambda^n - \beta_1\lambda^{n-1} + \beta_2\lambda^{n-2} + \cdots + (-1)^n\beta_n \tag{7}$$

Multiply out the left side of this equation and compare like powers of λ to obtain

$$
\begin{aligned}
\beta_1 &= \lambda_1 + \lambda_2 + \cdots + \lambda_n \\
\beta_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n \\
&\ \vdots \\
\beta_n &= \lambda_1\lambda_2 \cdots \lambda_n
\end{aligned}
\tag{8}
$$

The βs do not change under a similarity transformation and are said to be *invariants* of the similarity transformation. Of particular interest are β_1 and β_n, which are the trace of a matrix (the sum of its main diagonal terms) and its determinant, respectively.

Example 2:
Show that Tr (AB) = Tr (BA).

SOLUTION:

$$
\begin{aligned}
\text{Tr (AB)} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \text{Tr (BA)}
\end{aligned}
$$

Example 3:
Show explicitly that the trace of a matrix is invariant under a similarity transformation.

SOLUTION: We wish to show that Tr (S^{-1}A S) = Tr A. Using the fact that Tr (AB) = Tr (BA) from Example 2, we have

$$
\text{Tr }(S^{-1}AS) = \text{Tr }(SS^{-1}A) = \text{Tr }(A)
$$

We're going to see a number of applications of the diagonalization of matrices in the remainder of this chapter. As our first application, consider the rate equations

$$
\begin{aligned}
\frac{dx_1}{dt} &= -3x_1 + 2x_2 \\
\frac{dx_2}{dt} &= -x_1
\end{aligned}
$$

which we write in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x} \tag{9}$$

In the one-dimensional case, the solution to Equation 9 is $x(t) = x(0)e^{at}$. It is tempting to write the solution to Equation 9 in general as

$$\mathbf{x}(t) = e^{\mathsf{A}t}\mathbf{x}(0) \tag{10}$$

which would be useful if we can find a meaningful expression for $e^{\mathsf{A}t}$.

We can actually define a function of a matrix through its Maclaurin expansion. If $f(x)$ has a Maclaurin expansion

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

which converges for $|x| < R$, then the matrix series

$$f(\mathsf{A}) = \sum_{n=0}^{\infty} c_n \mathsf{A}^n \tag{11}$$

converges and is a well-defined function of A, provided that A is square and each of its eigenvalues has absolute value less than R. Now, if $\mathsf{D} = \mathsf{S}^{-1}\mathsf{A}\mathsf{S}$ where D is diagonal, then $\mathsf{A} = \mathsf{S}\mathsf{D}\mathsf{S}^{-1}$ and A^n is given by

$$\mathsf{A}^n = \underbrace{(\mathsf{S}\mathsf{D}\mathsf{S}^{-1})(\mathsf{S}\mathsf{D}\mathsf{S}^{-1})\cdots(\mathsf{S}\mathsf{D}\mathsf{S}^{-1})}_{n \text{ times}} = \mathsf{S}\mathsf{D}^n\mathsf{S}^{-1} \tag{12}$$

Because D is diagonal, D^n is simply each diagonal element raised to the nth power (Problem 20).

Let's apply this result to the matrix in Equation 9. We determined S and S^{-1} for A in Example 1 and found that

$$\mathsf{D} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

Using Equation 12, we can write

$$\begin{aligned} e^{\mathsf{A}t} &= \sum_{n=0}^{\infty} \frac{(\mathsf{A}t)^n}{n!} = \mathsf{S}\left\{\sum_{n=0}^{\infty} \frac{(\mathsf{D}t)^n}{n!}\right\}\mathsf{S}^{-1} = \mathsf{S}e^{\mathsf{D}t}\mathsf{S}^{-1} \\ &= \mathsf{S}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\mathsf{S}^{-1} = \begin{pmatrix} 2e^{-2t} - e^{-t} & 2e^{-t} & 2e^{-2t} \\ e^{-2t} - e^{-t} & 2e^{-t} - e^{-2t} \end{pmatrix} \end{aligned}$$

and so

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 2e^{-2t} - e^{-t} & 2e^{-t} - 2e^{-2t} \\ e^{-2t} - e^{-t} & 2e^{-t} - e^{-2t} \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

$$= \begin{pmatrix} (2e^{-2t} - e^{-t})x_{10} + (2e^{-t} - 2e^{-2t})x_{20} \\ (e^{-2t} - e^{-t})x_{10} + (2e^{-t} - e^{-2t})x_{20} \end{pmatrix}$$

is the solution to Equation 9. Although Equation 9 is just a 2×2 system, our method of solution is easily extended to $n \times n$ systems. Most CAS have routines to find e^{A} (Problem 21).

Realize that the eigenvectors of A must be linearly independent in order to diagonalize A. We saw in Section 2 that if an $n \times n$ matrix has n distinct eigenvalues, then its n eigenvectors are linearly independent. Consequently, we can say that

If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable by a similarity transformation $\mathsf{S}^{-1}\mathsf{AS}$, *where the columns of* S *are the eigenvectors of* A.

The eigenvectors of a matrix with degenerate eigenvalues may or may not be linearly independent. For example, the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix} \tag{13}$$

has eigenvalues $\lambda = 1, 2$, and 2, and corresponding eigenvectors $(-1, -1, 1)^{\mathsf{T}}$, $(2, 1, 0)^{\mathsf{T}}$, and $(0, 0, 0)^{\mathsf{T}}$. These eigenvectors are not linearly independent, so A in Equation 13 cannot be diagonalized by a similarity transformation. Another way to see this is to realize that if the eigenvectors are not linearly independent, then $|\mathsf{S}| = 0$ and S^{-1} does not exist.

We saw in Section 3 that the eigenvectors of a symmetric matrix (or a Hermitian matrix in complex vector space) are not only linearly independent, but can be made to be orthogonal, even if the matrix has repeated roots. Consequently, we can state that

A symmetric or a Hermitian matrix A *is diagonalizable by a similarity transformation.*

Furthermore, if we normalize the eigenvectors of A and then construct the modal matrix of A (which we'll call a *normalized modal matrix*), then

$$\mathsf{S}^{-1} = \mathsf{S}^{\mathsf{T}} \tag{14}$$

In other words, S, the normalized modal matrix of A, is orthogonal.

Example 4:
First show that the normalized modal matrix S of

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

is orthogonal, and then diagonalize A by a similarity transformation, $S^{-1}AS = S^TAS$.

SOLUTION: The characteristic equation of A is $\lambda^2 - 25 = 0$, or $\lambda_1 = 5$ and $\lambda_2 = -5$. The corresponding normalized eigenvectors are $\mathbf{u}_1 = (2, 1)^T/\sqrt{5}$ and $\mathbf{u}_2 = (-1, 2)^T/\sqrt{5}$. The normalized modal matrix of A is

$$S = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

and it is easy to show that $S^TS = SS^T = I$. Note that $S^T \neq S^{-1}$ if we had not normalized the eigenvectors of A.

Let's now diagonalize A:

$$D = S^TAS = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$
$$= \frac{1}{5}\begin{pmatrix} 25 & 0 \\ 0 & -25 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Note that $\lambda_1 + \lambda_2 = \text{Tr } A$ and that $\lambda_1\lambda_2 = \det A$.

Symmetric matrices are diagonalized by similarity transformations $S^{-1}AS = S^TAS$, where S is an orthogonal matrix. Hermitian matrices, which are the analog of symmetric matrices in complex vector spaces, are diagonalized by similarity transformation of the form $S^{-1}AS = S^\dagger AS$, where S is a unitary matrix.

Example 5:
First show that the normalized modal matrix of

$$A = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$$

is unitary, and then diagonalize A by the transformation $S^{-1}AS = S^\dagger AS$.

SOLUTION: The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 3$. The corresponding eigenvectors are $\mathbf{u}_1 = a(-1-i, 1)^T$ and $\mathbf{u}_2 = b(1+i, 2)^T$. We normalize these according to $\mathbf{u}_1^\dagger\mathbf{u}_1 = 1$ and $\mathbf{u}_2^\dagger\mathbf{u}_2 = 1$, so we have $\mathbf{u}_1 = (-1-i, 1)^T/\sqrt{3}$ and $\mathbf{u}_2 = (1+i, 2)^T/\sqrt{6}$. Therefore,

$$S = \begin{pmatrix} -\dfrac{1+i}{\sqrt{3}} & \dfrac{1+i}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{2}{\sqrt{6}} \end{pmatrix} \quad \text{and} \quad S^{\dagger} = \begin{pmatrix} -\dfrac{1-i}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{1-i}{\sqrt{6}} & \dfrac{2}{\sqrt{6}} \end{pmatrix}$$

and

$$D = S^{\dagger}AS = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

Note that Tr D = Tr A and det D = det A.

Our last topic of this section is the following: It is often important in quantum mechanics to diagonalize two matrices A and B by the same similarity transformation. What are the special properties of A and B that allow this to be done? We'll now show that if two matrices can be diagonalized by the same similarity transformation, they necessarily commute. Let

$$D_1 = S^{-1}AS \qquad \text{and} \qquad D_2 = S^{-1}BS$$

Then, using the fact that $D_1D_2 = D_2D_1$, we have

$$(S^{-1}AS)(S^{-1}BS) = S^{-1}ABS = (S^{-1}BS)(S^{-1}AS) = S^{-1}BAS$$

Multiply from the left by S and from the right by S^{-1} and find that AB = BA. Although we shall not prove it here, the converse of this statement is also true: If two matrices commute, they can be diagonalized by the same similarity transformation.

Example 6:
Show that

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

commute and diagonalize both with the same similarity transformation.

SOLUTION:

$$AB = \begin{pmatrix} 8 & 7 \\ 7 & 8 \end{pmatrix} = BA$$

The eigenvalues and corresponding eigenvectors of A are $\lambda_1 = 1$ and $\lambda_2 = 3$ and $\mathbf{u}_1 = (1, -1)^{\mathsf{T}}/\sqrt{2}$ and $\mathbf{u}_2 = (1, 1)^{\mathsf{T}}/\sqrt{2}$. Those of B are $\lambda_1 = 1$ and $\lambda_2 = 5$ and $\mathbf{u}_1 = (1, -1)^{\mathsf{T}}/\sqrt{2}$ and $\mathbf{u}_2 = (1, 1)^{\mathsf{T}}/\sqrt{2}$. Thus,

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

in both cases and

$$D_1 = S^{-1}AS = S^TAS = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

and

$$D_2 = S^TBS = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

10.5 Problems

1. Substitute Equation 2 into Equation 3 to show that $AS = SD$.
2. Here is another proof of Equation 3. First show that $(AS)_{ij} = \lambda_j u_{ji}$. Now use the fact that $D_{ij} = \lambda_j \delta_{ij}$ to show that $(SD)_{ij} = \lambda_j u_{ji}$.
3. Diagonalize $A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$.
4. Diagonalize $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.
5. Diagonalize $A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.
6. Which of the following matrices is diagonalizable?

   (a) $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
7. Diagonalize $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}$.
8. Diagonalize $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
9. Diagonalize $A = \begin{pmatrix} -2 & 3+3i \\ 3-3i & 1 \end{pmatrix}$.
10. Diagonalize $A = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$.
11. Diagonalize $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.
12. Find e^A if $A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$.

13. Find sin A if $A = \begin{pmatrix} \pi & 1 \\ 0 & 2\pi \end{pmatrix}$

14. Under what conditions does $e^{A+B} = e^A e^B$?

15. Find e^{At} if $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

16. Solve the kinetic system

$$\dot{x}_1 = x_1 + 2x_2$$
$$\dot{x}_2 = 2x_1 + x_2$$

that we solved in Section 3, but by evaluating e^{At}.

17. Show that your answer to the previous problem is the same as the result that we obtained in Section 3.

18. Suppose that $S = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ is the modal matrix of $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$. Show that $SS^T = S^TS \neq I$ unless S is a *normalized* modal matrix.

19. Show that the commuting matrices $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ can both be diagonalized by the same similarity transformation and show the similarity transformation.

20. Show that if D is diagonal, then D^n is diagonal and its elements are the elements of D raised to the nth power.

21. Use any CAS to find e^A if $A = \begin{pmatrix} 3 & -2 & -1 \\ 3 & -4 & -3 \\ 2 & -4 & 0 \end{pmatrix}$.

10.6 Quadratic Forms

The expression $ax_1^2 + bx_1x_2 + cx_2^2$, which consists of squares of variables or products of two variables, is called a *quadratic form*. The general expression of a quadratic form is

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \tag{1}$$

where the a_{ij} are real constants. Quadratic forms occur in a wide variety of applications. For example, consider the simple one-dimensional system shown in Figure 10.19. Let the equilibrium separation of each pair of neighboring masses be l. Then the potential energy of the system depends upon the displacements of the masses about the equilibrium positions, x_1, x_2, x_3, and x_4. For small displacements, we can expand $V(x_1, x_2, x_3, x_4)$ in a Taylor series about the equilibrium positions ($x_1 = x_2 = x_3 = x_4 = 0$) to obtain

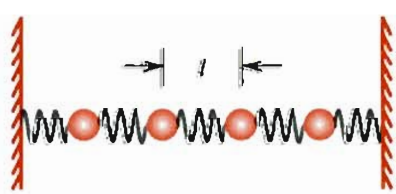


Figure 10.19
Four masses connected by springs. The motion is constrained to be only longitudinal.

$$V(x_1, x_2, x_3, x_4) = V(0, 0, 0, 0) + \sum_{j=1}^{4} \left(\frac{\partial V}{\partial x_j}\right)_0 x_j + \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_0 x_i x_j + \cdots$$

where the zero subscript means that the derivatives are evaluated at $x_1 = x_2 = x_3 = x_4 = 0$. The first term is just a constant, which we can let be equal to zero, and the second terms vanish because the $(\partial V/\partial x_j)_0$ are forces, which equal zero at the equilibrium position $x_1 = x_2 = x_3 = x_4 = 0$. Consequently, if we neglect cubic terms in the x_i, then

$$V(x_1, x_2, x_3, x_4) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} k_{ij}x_ix_j$$

is a quadratic form, which reflects Hooke's law for this system. There are many examples in physical applications where a quadratic form occurs because we expand some quantity in a multidimensional Taylor series and truncate after the quadratic terms.

We can write Equation 1 in matrix notation:

$$Q = \mathbf{x}^{\mathsf{T}}\mathsf{A}\mathbf{x} \tag{2}$$

For example, if $Q = 3x_1^2 + 8x_1x_2 - 3x_2^2$, then we can express Q as

$$Q = (x_1\ x_2)\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3}$$

Notice that we split the cross term $8x_1x_2$ into two equal terms when we wrote Equation 3. This makes A a symmetric matrix, which we know from the previous section has two linearly independent eigenvectors. We shall always write Equation 2 with A expressed as a symmetric matrix (Problem 2).

We now wish to express $x_1, x_2, \ldots, x_n$ as a linear combination of new coordinates $x'_1, x'_2, \ldots, x'_n$ such that Q becomes a sum of only squares of the x'_j, there being no cross terms. In other words, we want to express Q in *canonical form*. Let this transformation be denoted by

$$\mathbf{x} = \mathsf{S}\mathbf{x}' \tag{4}$$

where S is the $n \times n$ normalized modal matrix of A. Substitute Equation 4 into Equation 2 to obtain

$$Q = (\mathsf{S}\mathbf{x}')^{\mathsf{T}}\mathsf{A}(\mathsf{S}\mathbf{x}') = \mathbf{x}'^{\mathsf{T}}\mathsf{S}^{\mathsf{T}}\mathsf{A}\mathsf{S}\mathbf{x}' = \mathbf{x}'^{\mathsf{T}}\mathsf{A}'\mathbf{x}' \tag{5}$$

which will be in canonical form because the matrix A′ is diagonal. We know from the previous section that if the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then Q will be in the form

$$Q = \lambda_1 x_1'^2 + \lambda_2 x_2'^2 + \cdots + \lambda_n x_n'^2 \tag{6}$$

This procedure is similar to the rotations that we carried out in Section 4 but is more general.

Let's apply these results to Equation 3. The eigenvalues of A are $\lambda = \pm 5$. Thus, the canonical form of Equation 3 is $Q = 5x_1'^2 - 5x_2'^2$. The normalized eigenvectors of A are $\mathbf{u}_1 = (2, 1)^T/\sqrt{5}$ and $\mathbf{u}_2 = (-1, 2)^T/\sqrt{5}$ and so the matrix S in Equation 4, the normalized modal matrix of A, is

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \tag{7}$$

Therefore,

$$\mathbf{x}' = S^{-1}\mathbf{x} = S^T\mathbf{x} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

$$\begin{aligned} x_1' &= \frac{1}{\sqrt{5}}(2x_1 + x_2) \\ x_2' &= \frac{1}{\sqrt{5}}(-x_1 + 2x_2) \end{aligned} \tag{8}$$

If you substitute Equations 8 into $Q = 5x_1'^2 - 5x_2'^2$, you get Equation 3, as you should expect.

Example 1:

What type of conic section (ellipse or hyperbola) is described by the equation $3x^2 + 8xy + 3y^2 = 1$?

SOLUTION: The symmetric matrix of the quadratic form is

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors of A are $\lambda = -1, 7$ and $\mathbf{u}_1 = (-1, 1)^T$ and $\mathbf{u}_2 = (1, 1)^T$. The normalized modal matrix is

$$S = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$S^TAS = \begin{pmatrix} -1 & 0 \\ 0 & 7 \end{pmatrix}$$

Thus, the transformed equation is $g(x', y') = -x'^2 + 7y'^2 = 1$, or that of a hyperbola (Figure 10.20).

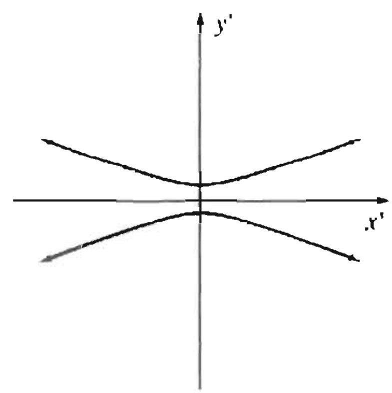


Figure 10.20
A plot of the hyperbola $-x'^2 + 7y'^2 = 1$ determined in Example 1.

Example 2:
Reduce the quadratic form

$$f(x, y, z) = 2x^2 + 3y^2 + 23z^2 + 72xz + 1 = 0$$

to canonical form. Describe the resulting quadric surface.

SOLUTION: The matrix A in $\mathbf{x}^T A \mathbf{x}$ is

$$A = \begin{pmatrix} 2 & 0 & 36 \\ 0 & 3 & 0 \\ 36 & 0 & 23 \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors of A are $\lambda = -25$, 3, and 50, and $(-4, 0, 3)^T$, $(0, 1, 0)^T$, and $(3, 0, 4)^T$. The normalized modal matrix of A is

$$S = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

Then

$$S^T A S = \begin{pmatrix} -25 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 50 \end{pmatrix}$$

and the canonical form of $f(x, y, z)$ is

$$g(x', y', z') = -25x'^2 + 3y'^2 + 50z'^2 + 1 = 0$$

which is a hyperboloid of two sheets (Figure 10.21).

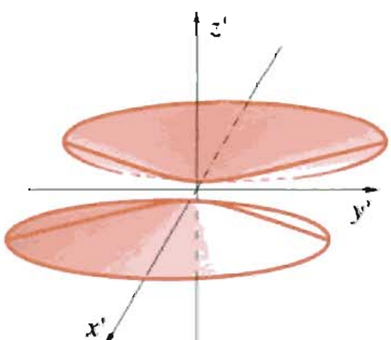


Figure 10.21
A plot of the hyperboloid of two sheets determined in Example 2.

Diagonalizing a quadratic form allows us to see more clearly the type of curve or surface that it describes. It's not obvious that the quadratic form in Example 1 is a hyperbola or that the quadratic form in Example 2 is a hyperboloid of two sheets until they are expressed in canonical form.

The more general expression

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0 \tag{9}$$

is called a quadratic equation, and $ax^2 + 2bxy + cy^2$ is called the *associated quadratic form*. Equation 9 represents a conic section whose major axes do not coincide with the x and y axes if $b \neq 0$, $d \neq 0$, or $e \neq 0$. For example, consider the quadratic equation

$$x^2 - 2x + 3y^2 + 12y + 12 = 0 \tag{10}$$

The linear terms in x and y show that the graph of this equation is not centered at the origin (Figure 10.22). The lack of cross terms shows that the major axes of the graph are parallel to the x and y axes. To express Equation 10 as a sum of squared terms, we complete the squares of the x and y terms to get $(x-1)^2+3(y+2)^2=1$, which is the equation of an ellipse centered at $x=1$ and $y=-2$ (Figure 10.22). We'll call the equation $(x-1)^2+3(y+2)^2=1$ the *standard form* of Equation 10.

To convert a quadratic equation such as

$$9x^2-4xy+6y^2-10x-20y=5 \tag{11}$$

into standard form, first diagonalize the quadratic part (the associated quadratic form) and then eliminate linear terms by completing the square.

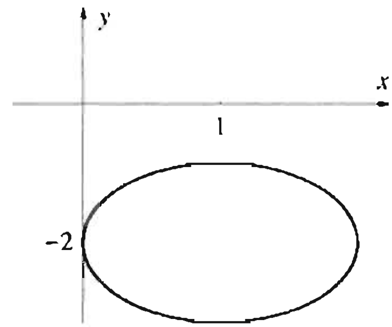


Figure 10.22
A graph of the equation $x^2-2x+3y^2+12y+12=0$.

Example 3:
Express Equation 11 in standard form (Figure 10.23).

SOLUTION: The associated quadratic form is $9x^2-4xy+6y^2$. The eigenvalues and eigenvectors of the symmetric coefficient matrix are $\lambda_1=5$, $\lambda_2=10$, $\mathbf{u}_1=(1,2)^T/\sqrt{5}$, and $\mathbf{u}_2=(-2,1)^T/\sqrt{5}$. The matrix that rotates the graph of Equation 10 into canonical form according to $\mathbf{x}=S\,\mathbf{x}'$ (Equation 4) is

$$S=\frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$$

or

$$x=\frac{1}{\sqrt{5}}(x'-2y')$$

$$y=\frac{1}{\sqrt{5}}(2x'+y')$$

Substituting x and y into the quadratic equation gives

$$5x^2+10y^2-10\sqrt{5}x=5$$

Completing the square by adding 5 to both sides gives

$$(x-\sqrt{5})^2+2y^2=6$$

Thus, the quadratic equation represents an ellipse centered at $x=\sqrt{5}$, $y=0$ (Figure 10.24). Notice that we have rotated *and* shifted the ellipse in Figure 10.23.

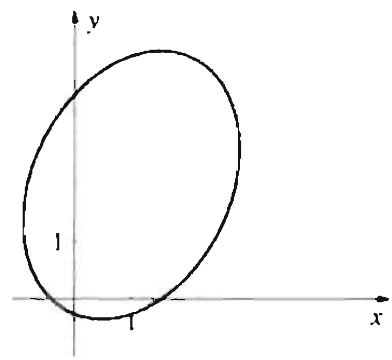


Figure 10.23
A graph of the equation $9x^2-4xy+6y^2-10x-20y=5$ used in Example 3.

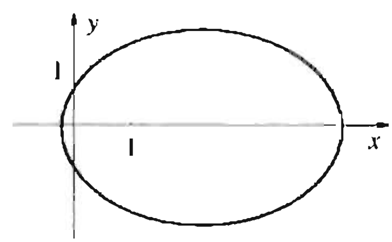


Figure 10.24
The graph of the ellipse described by $(x-\sqrt{5})^2+2y^2=6$.

There are a number of physical applications where quadratic forms occur. Recall from Section 6.5 that the rotation of a rigid body may be viewed as the rotation

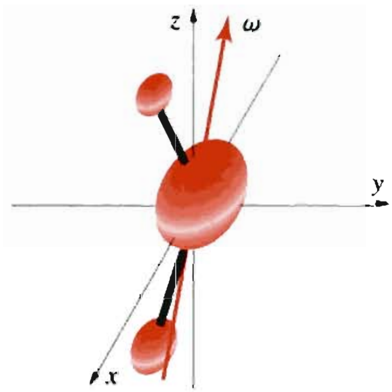


Figure 10.25
A rigid body rotating about an axis passing through the center of mass of the body.

with an angular velocity $\boldsymbol{\omega}$ about an axis passing through some origin, which for simplicity, we take to be the center of mass (see Figure 10.25). Equation 5.4.15 says that the total angular momentum of a collection of masses m_i located by the position vectors $\mathbf{r}_i$ is given by

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times (m_i \mathbf{v}_i) = \sum_{i=1}^{n} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \tag{12}$$

If the triple cross product in Equation 12 is written out (Problem 9), Equation 12 becomes

$$\begin{aligned} \mathbf{L} = \mathbf{i}\,&(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) \\ &+ \mathbf{j}\,(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) \\ &+ \mathbf{k}\,(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) \end{aligned} \tag{13}$$

where

$$\begin{aligned} I_{xx} &= \sum_{i=1}^{n} m_i(y_i^2 + z_i^2) \quad & I_{yy} &= \sum_{i=1}^{n} m_i(x_i^2 + z_i^2) \quad & I_{zz} &= \sum_{i=1}^{n} m_i(x_i^2 + y_i^2) \\ I_{xy} &= -\sum_{i=1}^{n} m_i x_i y_i \quad & I_{xz} &= -\sum_{i=1}^{n} m_i x_i z_i \quad & I_{yz} &= -\sum_{i=1}^{n} m_i y_i z_i \end{aligned} \tag{14}$$

Equation 13 can be written in matrix notation by writing

$$\mathbf{L} = \mathsf{I}\boldsymbol{\omega} \tag{15}$$

where the elements of the moment of inertia matrix I (not to be confused with the identity matrix) are given by Equations 14. In Problem 16, we show that the kinetic energy T of the rotating body is given by

$$T = \frac{1}{2}\boldsymbol{\omega}^{\mathsf{T}}\mathsf{I}\,\boldsymbol{\omega} \tag{16}$$

The axes in Figure 10.25 were chosen arbitrarily, and that's why there are nonzero non-diagonal elements in I. It should be clear from our discussion of quadratic forms, however, that we can find an orthogonal transformation of the axes in Figure 10.25 such that I becomes diagonal. The axes in which I is diagonal are called *principal axes* and the diagonal elements of I are called the *principal moments of inertia* of the body.

Another important application where quadratic forms arise is in the theory of the thermodynamics of irreversible processes. Consider an abiabatically insulated system, which is described thermodynamically by a set of state variables. The entropy S of the system is a maximum when these state variables assume their equilibrium values, say $A_1^\circ, A_2^\circ, \ldots, A_n^\circ$. Suppose now that the system undergoes a fluctuation from its equilibrium state, where the state variables have (nonequilibrium) values $A_1, A_2, \ldots, A_n$. Then the deviation of the entropy of the system from its equilibrium value is a function of the quantities $\{\alpha_j = A_j - A_j^\circ\}$, and if the fluctuations are small so that the $\{\alpha_j\}$ are small, then we can write ΔS as

$$\Delta S = -\frac{k_B}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\alpha_i\alpha_j$$

where k_B is the Boltzmann constant. There are no linear terms here because S is a maximum when the $\alpha_j = 0$. Using the relation between the entropy of a system and its disorder, it turns out the probability of observing the values $\{\alpha_i\}$ is given by

$$\begin{aligned} p(\alpha_1, \alpha_2, \ldots, \alpha_n) &\propto e^{-\Delta S/k_B} \\ &= \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\alpha_i\alpha_j\right) \end{aligned}$$

All the central results of the thermodynamics of irreversible processes can be derived from this basic relation, which is a multivariate Gaussian distribution.

Recall (Section 3.3) that a Gaussian distribution is of the form

$$f(x)dx = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-x^2/2\sigma^2} dx \qquad -\infty < x < \infty \tag{17}$$

A general multivariate Gaussian distribution is of the form

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= ce^{-\frac{1}{2}(a_{11}^2 x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{n-1n}x_{n-1}x_n + a_{nn}x_n^2)} \\ &= ce^{-\frac{1}{2}\mathbf{x}^\mathsf{T}\mathsf{A}\mathbf{x}} \qquad -\infty < x_j < \infty \qquad j = 1, 2, \ldots, n \end{aligned} \tag{18}$$

where c is a normalization constant. Multivariate Gaussian distributions occur in statistics, in fluctuation theory in statistical mechanics, in the theory of nonequilibrium thermodynamics, and in a number of other areas. (See Chapter 21.)

We often need to evaluate integrals involving $f(x_1, x_2, \ldots, x_n)$. For example, the determination of the statistical correlation between the variables $x_1, x_2, \ldots, x_n$ requires that we evaluate integrals like

$$M_{ij} = \iint_{-\infty}^{\infty} x_i x_j f(x_1, x_2, \ldots, x_n) dx_1 \ldots dx_n \tag{19}$$

The cross terms in the exponential make these integrals difficult to evaluate. However, if we write the quadratic form in the exponential in canonical form, then $f(x_1, \ldots, x_n)$ becomes just a product of independent factors and integrals like Equation 19 can be readily evaluated.

We'll evaluate the normalization constant c in Equation 18 first. The normalization constant satisfies the relation

$$c\int_{-\infty}^{\infty}\cdots\int f(x_1,x_2,\ldots,x_n)dx_1dx_2\ldots dx_n$$
$$= c\int_{-\infty}^{\infty}\cdots\int e^{-\frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathsf{A}\mathbf{x}}dx_1dx_2\ldots dx_n = 1 \tag{20}$$

Write $\mathbf{x} = \mathsf{S}\mathbf{x}'$ or $\mathbf{x}' = \mathsf{S}^{-1}\mathbf{x}$. Then, $\mathbf{x}^{\mathsf{T}}\mathsf{A}\mathbf{x} = \mathbf{x}'^{\mathsf{T}}\mathsf{S}^{\mathsf{T}}\mathsf{A}\mathsf{S}\mathbf{x}' = \mathbf{x}'^{\mathsf{T}}\mathsf{D}\mathbf{x}'$, where

$$\mathsf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{21}$$

Therefore, $\mathbf{x}^{\mathsf{T}}\mathsf{A}\mathbf{x} = \mathbf{x}'^{\mathsf{T}}\mathsf{D}\mathbf{x}' = \lambda_1 x_1'^2 + \lambda_2 x_2'^2 + \cdots + \lambda_n x_n'^2$ and $f(x_1, x_2, \ldots, x_n) = g(x_1', x_2', \ldots, x_n')$. Note that all the eigenvalues of A must be greater than zero in order that integrals involving $g(x_1', x_2', \ldots, x_n')$ converge. We convert the "volume element" $dx_1dx_2\cdots dx_n$ to $dx_1'dx_2'\ldots dx_n'$ by introducing the Jacobian determinant

$$J\begin{pmatrix}\mathbf{x}\\ \mathbf{x}'\end{pmatrix} = \begin{vmatrix} \dfrac{\partial x_1}{\partial x_1'} & \dfrac{\partial x_1}{\partial x_2'} & \cdots & \dfrac{\partial x_1}{\partial x_n'} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial x_n}{\partial x_1'} & \dfrac{\partial x_n}{\partial x_2'} & \cdots & \dfrac{\partial x_n}{\partial x_n'} \end{vmatrix} \tag{22}$$

But $\mathbf{x} = \mathsf{S}\mathbf{x}'$, so this determinant is equal to $|\mathsf{S}|$ (Problem 19). Therefore,

$$J\begin{pmatrix}\mathbf{x}\\ \mathbf{x}'\end{pmatrix} = |\mathsf{S}| = 1$$

because S is orthogonal.

Equation 20 now reads

$$c\int_{-\infty}^{\infty}\cdots\int g(x_1', x_2', \ldots, x_n')dx_1'dx_2'\ldots dx_n'$$
$$= c\int_{-\infty}^{\infty}\cdots\int e^{-\frac{1}{2}(\lambda_1 x_1'^2+\lambda_2 x_2'^2+\cdots+\lambda_n x_n'^2)}dx_1'dx_2'\ldots dx_n' = 1$$

This n-fold integral is actually the product of integrals, so we can write

$$c\int_{-\infty}^{\infty} dx_1 e^{-\lambda_1 x_1^2/2} \int_{-\infty}^{\infty} dx_2 e^{-\lambda_2 x_2^2/2} \cdots \int_{-\infty}^{\infty} dx_n e^{-\lambda_n x_n^2/2}$$

$$= c\left(\frac{2\pi}{\lambda_1}\right)^{1/2}\left(\frac{2\pi}{\lambda_2}\right)^{1/2}\cdots\left(\frac{2\pi}{\lambda_n}\right)^{1/2}$$

$$= c\frac{(2\pi)^{n/2}}{(\lambda_1\lambda_2\cdots\lambda_n)^{1/2}} = 1 \tag{23}$$

But $\lambda_1\lambda_2\cdots\lambda_n = \det \mathsf{A}$, so we finally have that

$$c = \frac{(\det \mathsf{A})^{1/2}}{(2\pi)^{n/2}} \tag{24}$$

Example 4:
Evaluate the integral

$$I = \iint_{-\infty}^{\infty} dx dy e^{-\frac{1}{2}(3x^2-2xy+3y^2)}$$

by diagonalizing the quadratic form explicitly and in doing so verify that the Jacobian determinant in Equation 22 is equal to one and that your final result agrees with $(2\pi)^{n/2}/(\det \mathsf{A})^{1/2}$ (with $n = 2$).

SOLUTION: The matrix A is

$$\mathsf{A} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

The eigenvalues of A are $\lambda = 2$ and 4, and its corresponding normalized eigenvectors are $(1, 1)^{\mathsf{T}}/\sqrt{2}$ and $(-1, 1)^{\mathsf{T}}/\sqrt{2}$. Its normalized modal matrix is

$$\mathsf{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

or

$$x = \frac{1}{\sqrt{2}}(x' - y') \qquad \text{and} \qquad y = \frac{1}{\sqrt{2}}(x' + y') \tag{25}$$

The quadratic form becomes $2x'^2 + 4y'^2$ when expressed in terms of the primed variables.

To evaluate the Jacobian determinant, we differentiate Equation 25 to obtain

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial x'} & \dfrac{\partial x}{\partial y'} \\ \dfrac{\partial y}{\partial x'} & \dfrac{\partial y}{\partial y'} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{vmatrix} = 1$$

We'll now determine I from

$$\begin{aligned} I &= \iint\limits_{-\infty}^{\infty} dx'dy' e^{-\frac{1}{2}(2x'^2+4y'^2)} \\ &= \int_{-\infty}^{\infty} dx'\, e^{-x'^2} \int_{-\infty}^{\infty} dy'\, e^{-2y'^2} = \pi^{1/2}\frac{\pi^{1/2}}{2^{1/2}} \\ &= \frac{\pi}{2^{1/2}} = \frac{2\pi}{8^{1/2}} = \frac{2\pi}{|\mathsf{A}|^{1/2}} \end{aligned}$$

For our final manipulation of multidimensional Gaussian integrals, let's evaluate M_{ij} in Equation 19. We're going to see that M_{ij} is the ijth element of A^{-1}. First, write M_{ij} as

$$M_{ij} = \frac{(\det \mathsf{A})^{1/2}}{(2\pi)^{n/2}} \int\limits_{-\infty}^{\infty}\cdots\int x_i x_j e^{-\frac{1}{2}(\lambda_1 x_1'^2+\lambda_2 x_2'^2+\cdots\lambda_n x_n'^2)} dx_1'dx_2'\ldots dx_n' \quad (26)$$

and use the relation $\mathbf{x} = \mathsf{S}\,\mathbf{x}'$ to write

$$x_i = \sum_{k=1}^{n} s_{ik}x_k' \qquad \text{and} \qquad x_j = \sum_{l=1}^{n} s_{jl}x_l' \quad (27)$$

Substitute Equations 27 into Equation 26 to get

$$M_{ij} = \frac{(\det \mathsf{A})^{1/2}}{(2\pi)^{n/2}} \sum_{k=1}^{n}\sum_{l=1}^{n} s_{ik}s_{jl} \int\limits_{-\infty}^{\infty}\cdots\int x_i' x_j' e^{-\frac{1}{2}(\lambda_1 x_1'^2+\lambda_2 x_2'^2+\cdots\lambda_n x_n'^2)} dx_1'dx_2'\ldots dx_n' \quad (28)$$

The only terms that are nonzero in this double summation are the terms in which $k = l$ because terms such as

$$\iint\limits_{-\infty}^{\infty} xye^{-\frac{1}{2}(\lambda_1 x^2+\lambda_2 y^2)}dxdy = \int_{\infty}^{\infty} xe^{-\lambda_1 x^2/2}dx \int_{\infty}^{\infty} ye^{-\lambda_2 y^2/2}dy = 0$$

Thus, Equation 28 becomes

$$M_{ij} = \frac{(\det A)^{1/2}}{(2\pi)^{n/2}} \sum_{k=1}^{n}\sum_{l=1}^{n} s_{ik}s_{jl}\delta_{kl} \int_{-\infty}^{\infty}\cdots\int x_l'^2 e^{-\frac{1}{2}(\lambda_1 x_1'^2+\lambda_2 x_2'^2+\cdots\lambda_n x_n'^2)}dx_1'dx_2'\ldots dx_n'$$

$$= \frac{(\det A)^{1/2}}{(2\pi)^{n/2}} \sum_{l=1}^{n} s_{il}s_{jl}\left(\frac{2\pi}{\lambda_1}\right)^{1/2}\left(\frac{2\pi}{\lambda_2}\right)^{1/2}\cdots\left(\frac{2\pi}{\lambda_l^3}\right)^{1/2}\cdots\left(\frac{2\pi}{\lambda_n}\right)^{1/2}$$

$$= \sum_{l=1}^{n}\frac{s_{il}s_{jl}}{\lambda_l} \tag{29}$$

We'll now show that this summation is equal to the ijth element of A^{-1}. Start with

$$S^TAS = D$$

Take the inverse of both sides and use the relation $(AB)^{-1} = B^{-1}A^{-1}$ and the fact that $S^T = S^{-1}$ to obtain

$$S^{-1}A^{-1}(S^T)^{-1} = S^TA^{-1}S = D^{-1}$$

Now multiply from the left by S and from the right by S^T to get

$$A^{-1} = SD^{-1}S^T \tag{30}$$

The ijth element of A^{-1} according to Equation 30 is

$$(A^{-1})_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n} s_{ik}D_{kl}^{-1}s_{lj}^T$$

But $D_{kl} = \delta_{kl}/\lambda_l$, so we have

$$(A)_{ij} = \sum_{l=1}^{n}\frac{s_{il}s_{lj}^T}{\lambda_l} \tag{31}$$

If we realize that $s_{lj}^T = s_{jl}$, then Equation 31 is the same as Equation 29, so we have $M_{ij} = (A^{-1})_{ij}$, or

$$M = A^{-1} \tag{32}$$

in matrix notation. The quantities M_{ij} are called *covariances* and M is called the *covariance matrix*. The values of the covariances are a measure of the statistical correlation between the x_i.

Because of the relation $A = M^{-1}$ given by Equation 32 and the relation det (A) = 1/det (A^{-1}), the multivariate Gaussian distribution is often written as

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}(\det \mathsf{M})^{1/2}} e^{-\frac{1}{2}\mathbf{x}^T\mathsf{M}^{-1}\mathbf{x}} \tag{33}$$

Example 5:
Use Equation 33 to write out a Gaussian distribution in two variables. Use the notation

$$\langle x^2\rangle = \sigma_x^2\,; \qquad \langle y^2\rangle = \sigma_y^2\,; \qquad \langle xy\rangle = r\sigma_x\sigma_y$$

SOLUTION: The covariance matrix is given by

$$\mathsf{M} = \begin{pmatrix} \sigma_x^2 & r\sigma_x\sigma_y \\ r\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}$$

$$\det \mathsf{M} = \sigma_x^2\sigma_y^2(1 - r^2)$$

$$\mathsf{M}^{-1} = \frac{1}{\sigma_x^2\sigma_y^2(1 - r^2)} \begin{pmatrix} \sigma_y^2 & -r\sigma_x\sigma_y \\ -r\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}$$

$$\frac{1}{2}\mathbf{x}^T\mathsf{M}^{-1}\mathbf{x} = \frac{1}{2(1 - r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)$$

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y(1 - r^2)^{1/2}} \exp\left[-\frac{1}{2(1 - r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right]$$

Notice that $f(x, y) = p(x)p(y)$ when $r = 0$, when x and y are statistically independent or uncorrelated. Because of this, r^2 is called the *correlation coefficient*. Also note that r must satisfy $-1 \le r \le 1$ (Problem 13).

Another important application of the diagonalization of a quadratic form is the following. Recall from Section 6.8 that we investigated the nature of the extremum of a function of two variables at a point (a, b) by considering Taylor's formula,

$$f(a + h, b + k) = f(a, b) + \frac{1}{2}[h^2 f_{xx}(\xi, \eta) + 2hk f_{xy}(\xi, \eta) + k^2 f_{yy}(\xi, \eta)] \tag{34}$$

where $a - h < \xi < a + h$ and $b - k < \eta < b + k$. The nature of the extremum depends upon the sign of the quadratic form in Equation 34 for the various values of h and k about the point (a, b). If we convert this quadratic form to canonical form, then we have

$$Q = \lambda_1 h'^2 + \lambda_2 k'^2 \tag{35}$$

where the sign of Q now depends entirely upon the signs of λ_1 and λ_2.

A quadratic form is said to be *positive definite* if $Q = \mathbf{x}^T A\mathbf{x} > 0$ for all values of $\mathbf{x} \neq 0$. It is called *positive semi-definite* if $\mathbf{x}^T A\mathbf{x} \geq 0$; *negative definite* if $\mathbf{x}^T A\mathbf{x} < 0$; *negative semi-definite* if $\mathbf{x}^T A\mathbf{x} \leq 0$; and *indefinite* if $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values. You can see from Equation 35 that $\mathbf{x}^T A\mathbf{x}$ is positive definite if and only if $\lambda_1 > 0$ and $\lambda_2 > 0$ and negative definite if and only if $\lambda_1 < 0$ and $\lambda_2 < 0$. Thus, the behavior of the sign of $Q = \mathbf{x}^T A\mathbf{x}$ and the quadratic form in Equation 34 depend upon the eigenvalues of

$$A = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

which are

$$\lambda_{\pm} = \frac{f_{xx} + f_{yy}}{2} \pm \frac{1}{2}[(f_{xx} + f_{yy})^2 - 4(f_{xx}f_{yy} - f_{xy}^2)]^{1/2} \qquad (36)$$

You can see from Equation 36 that both eigenvalues will be positive if $f_{xx} + f_{yy} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$, or equivalently, if either $f_{xx} > 0$ or $f_{yy} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ (because if $f_{xx} > 0$, then $f_{yy} > 0$ if $f_{xx}f_{yy} - f_{xy}^2 > 0$). If this is so, then $f(a + h, b + k) > f(a, b)$ and $f(a, b)$ is a local minimum.

Similarly, if $f_{xx} + f_{yy} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$, or equivalently, if either $f_{xx} < 0$ or $f_{yy} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ (because if $f_{xx} < 0$, then $f_{yy} < 0$ if $f_{xx}f_{yy} - f_{xy}^2 > 0$), both eigenvalues will be negative. In this case, $f(a + h, b + k) < f(a, b)$ and $f(a, b)$ will be a local maximum. If the eigenvalues have opposite signs, then $f(a, b)$ is a saddle point.

We derived these results for a function of two variables in Chapter 6 without the use of the theory of quadratic forms, but for the case of n variables, the characterization of an extremum in terms of the eigenvalues of A is indispensable.

10.6 Problems

1. Express the following quadratic forms in matrix notation $\mathbf{x}^T A\mathbf{x}$, where A is a symmetric matrix:

   (a) $3x_1^2 - x_1x_2 + 2x_1x_3 + x_3^2$ (b) $x_1^2 - 3x_1x_2 + 6x_2^2$ (c) $6x_1x_3 + x_2^2$

2. Show that writing a quadratic form in terms of a symmetric matrix is equivalent to replacing the original matrix A by the symmetric matrix $(A + A^T)/2$. Now show that $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T[(A + A^T)/2]\mathbf{x}$.
3. Find the standard form of the equation $5x^2 + 4xy + 2y^2 = 1$. Find the orthonormal basis which yields this standard form.
4. Identify the graph of $2x^2 + y^2 - 4xy - 4yz = 4$.
5. Identify the type of surface described by $3x^2 + 6y^2 + 3z^2 - 6xy - 2yz = 1$.
6. Identify the type of surface described by $2x^2 + 5y^2 + 5z^2 + 4xy - 4xz - 8yz = 10$.
7. Express $3x^2 - 8xy - 12y^2 - 30x - 64y = 0$ in standard form.
8. Identify the type of surface described by $2xy + 2xz + 2yz = 1$.

9. Derive Equation 13 from Equation 12.

10. Show that the probability distribution in Equation 17 is normalized; in other words, show that $\int_{-\infty}^{\infty} f(x)dx = 1$ and that $\langle x \rangle = 0$ and $\langle x^2 \rangle = \sigma^2$, where $\langle x^n \rangle = \int_{-\infty}^{\infty} x^n f(x)dx$. The quantity $\langle x \rangle$ is the mean, or the average, of $f(x)$, and $\langle x^2 \rangle$ is its variance [the square of the standard deviation of $f(x)$].

11. If $\langle x \rangle = \mu \neq 0$, Equation 17 becomes $f(x)dx = (2\pi\sigma^2)^{-1/2}e^{-(x-\mu)^2/2\sigma^2}dx$ for $-\infty < x < \infty$. Show that $\langle x \rangle = \mu$ as inferred. Now show that $\langle (x-\mu)^2 \rangle = \sigma^2$. Interpret $\langle (x-\mu)^2 \rangle$ physically. It is called the *second central moment*.

12. Use the relation det (AB) = det (A) det (B) to show that det $\mathsf{A}^{-1} = 1/\det \mathsf{A}$.

13. In this problem, we prove that the correlation coefficient r defined in Example 5 must satisfy $-1 \le r \le 1$. First show that $\langle(\alpha x + \beta y)^2\rangle = \alpha^2\sigma_x^2 + 2\alpha\beta r\sigma_x\sigma_y + \beta^2\sigma_y^2 \ge 0$. Now take $\alpha = \sigma_y^2$ and $\beta = -r\sigma_x\sigma_y$ and show that $r^2 \le 1$, or $-1 \le r \le 1$.

14. In this problem, we show that

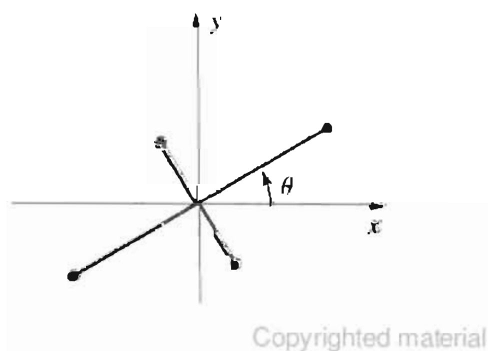
$$\int_{-\infty}^{\infty}\cdots\int e^{i\mathbf{t}^\mathsf{T}\mathbf{x}-\frac{1}{2}\mathbf{x}^\mathsf{T}\mathsf{A}\mathbf{x}}dx_1dx_2\ldots dx_n = \frac{(2\pi)^{n/2}}{(\det \mathsf{A})^{1/2}}e^{-\frac{1}{2}\mathbf{t}^\mathsf{T}\mathsf{A}^{-1}\mathbf{t}} \tag{1}$$

Let S be the normalized modal matrix of A, so that $\mathsf{S}^\mathsf{T}\mathsf{A}\mathsf{S} = \mathsf{D}$ and $\mathbf{x}^\mathsf{T}\mathsf{A}\mathbf{x} = \mathbf{x}'^\mathsf{T}\mathsf{D}\mathbf{x}'$, where $\mathbf{x} = \mathsf{S}\mathbf{x}'$. (The quadratic form is now diagonalized.) Now let $\mathbf{u} = \mathsf{S}^{-1}\mathbf{t} = \mathsf{S}^\mathsf{T}\mathbf{t}$ so that $\mathbf{t}^\mathsf{T}\mathbf{x} = (\mathsf{S}\mathbf{u})^\mathsf{T}\mathbf{x} = \mathbf{u}^\mathsf{T}\mathsf{S}^\mathsf{T}\mathbf{x} = \mathbf{u}^\mathsf{T}\mathbf{x}'$. Thus, we have $\int_{-\infty}^{\infty}\cdots\int e^{i\mathbf{u}^\mathsf{T}\mathbf{x}'-\frac{1}{2}\mathbf{x}'^\mathsf{T}\mathsf{D}\mathbf{x}'}dx_1'dx_2'\ldots dx_n' = \prod_{j=1}^{n}\int_{-\infty}^{\infty} e^{iu_jx_j'}e^{-\frac{1}{2}\lambda_jx_j'^2}dx_j'$, where the λ_j are the eigenvalues of A. Now use the fact that (see the next problem for a proof) $\int_{-\infty}^{\infty} e^{iu_jx_j'}e^{-\frac{1}{2}\lambda_jx_j'^2}dx_j' = \left(\frac{2\pi}{\lambda_j}\right)^{1/2}e^{-u_j^2/2\lambda_j}$, and use $\mathbf{u}^\mathsf{T}\mathsf{D}^{-1}\mathbf{u} = \mathbf{t}^\mathsf{T}\mathsf{S}\mathsf{D}^{-1}\mathsf{S}^\mathsf{T}\mathbf{t} = \mathbf{t}^\mathsf{T}\mathsf{A}^{-1}\mathbf{t}$ to verify equation 1.

15. Show that $I = \int_{-\infty}^{\infty} e^{itx-\frac{1}{2}hx^2}dx = \left(\frac{2\pi}{h}\right)^{1/2}e^{-t^2/2h}$ by expanding e^{itx} and integrating term by term.

16. We derive Equation 16 in this problem. First, write the kinetic energy as $T = \sum m_j\dot{r}_j^2/2$ where $\dot{\mathbf{r}}_j = d\mathbf{r}_j/dt$. Now use $\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j$ to get $T = \frac{1}{2}\sum_{j=1}^{n} m_j(\boldsymbol{\omega}\times\mathbf{r}_j)\cdot(\boldsymbol{\omega}\times\mathbf{r}_j)$. Now use the scalar triple product formula $\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3) = (\mathbf{v}_1\times\mathbf{v}_2)\cdot\mathbf{v}_3 = \mathbf{v}_2\cdot(\mathbf{v}_3\times\mathbf{v}_1)$ to write $T = \frac{1}{2}\boldsymbol{\omega}\cdot\sum_{j=1}^{n} m_j\mathbf{r}_j\times(\boldsymbol{\omega}\times\mathbf{r}_j) = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathsf{I}\cdot\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}^\mathsf{T}\mathsf{I}\boldsymbol{\omega}$.

17. This problem illustrates how the principal moments of inertia can be obtained as an eigenvalue problem. We will work in two dimensions for simplicity. Consider the "molecule" represented here.



where all the masses are unit masses and the long and short bond lengths are 2 and 1, respectively. Show that

$$I_{xx} = 2\cos^2\theta + 8\sin^2\theta$$

$$I_{yy} = 8\cos^2\theta + 2\sin^2\theta$$

$$I_{xy} = -6\cos\theta\sin\theta$$

The fact that $I_{xy} \neq 0$ indicates that these I_{ij} are not the principal moments of inertia. Now solve the secular determinantal equation for λ,

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} \\ I_{xy} & I_{yy} - \lambda \end{vmatrix} = 0$$

and compare your result with the values of I_{xx} and I_{yy} that you would obtain if you align the "molecule" and the coordinate system such that $\theta = 90°$. What does this comparison tell you? What are the values of I_{xx} and I_{yy} if $\theta = 0°$?

18. Which of the following quadratic forms is positive definite?

(a) $x^2 - 4xy + 5y^2$ (b) $x^2 + 6xy + 3y^2$ (c) $2x^2 - 12xy + 5y^2$

19. Show that the Jacobian determinant in Equation 22 is equal to $|\mathsf{S}|$.

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CHAPTER 11

Ordinary Differential Equations

Many scientific laws can be expressed in terms of differential equations. In fact, differential equations are the most common and most useful means of formulating these laws. A differential equation is an equation involving derivatives of an unknown function that depends upon one or more independent variables. If the unknown function depends upon only one independent variable, then the equation is called an *ordinary differential equation*. Ordinary differential equations necessarily involve ordinary derivatives. Examples of ordinary differential equations are

$$(a) \quad \frac{dy}{dx} = 2y^2 \qquad (b) \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = e^x \tag{1}$$

$$(a) \quad \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^3y}{dx^3} \qquad (b) \quad (x^2 + y^2)\frac{dy}{dx} = xy \tag{2}$$

In each case, there is only one independent variable, x, and one dependent variable, y. If the unknown function depends upon more than one independent variable, then its equation is called a *partial differential equation*. Partial differential equations necessarily involve partial derivatives. Examples are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

We shall not discuss partial differential equations until Chapter 16.

There are several terms that we often use when discussing differential equations. We say that $y(x)$ is a *solution* of an ordinary differential equation if it satisfies the equation identically over some interval (α, β) of x when it and its derivatives are substituted into the equation. The *order* of a differential equation is the order of the highest derivative that occurs in the equation. Equations 1a and 2b are first-order equations. 1b is a second-order equation, and 2a is a third-order equation. The *degree* of a differential equation is the power of its highest-order derivative when the equation is written as a polynomial in all the derivatives involved. Equations 1a, 1b, and 2b are easily seen to be first-degree equations and Equation 2a is second-degree because it takes the form

$$k^2\left(\frac{d^3y}{dx^3}\right)^2 = 1 + 3\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{dy}{dx}\right)^4 + \left(\frac{dy}{dx}\right)^6$$

when it is written as a polynomial in its derivatives.

One reason that we classify differential equations according to order and degree is because the properties of their solutions and the methods that we use to determine the solutions often depend upon this classification. The titles of the sections of this chapter specify the types of equations that are discussed. Section 1 deals with differential equations of first order and first degree and Section 2 deals with *linear* first-order equations. A differential equation is said to be linear if the dependent variable (the function to be determined) and all its derivatives occur only to the first power and there are no cross terms. Equation 1b is the only linear differential equation of Equations 1 and 2. Equation 1a is nonlinear because of the y^2 term; Equation 2a is nonlinear because of the $(dy/dx)^2$ term; and Equation 2b is nonlinear because of the $y^2\, dy/dx$ term. Sections 3 and 4 deal exclusively with linear differential equations. Section 6 deals with systems of first-order differential equations, such as the simultaneous equations

$$\frac{dx}{dt} = x + 4y$$

$$\frac{dy}{dt} = -2x + y$$

In this case, we have two differential equations to solve simultaneously.

11.1 Differential Equations of First Order and First Degree

A first-order differential equation can be written in the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

If $f(x, y) = -M(x, y)/N(x, y)$, then Equation 1 can be written in terms of differentials:

$$M(x, y)dx + N(x, y)dy = 0 \tag{2}$$

In Equation 1, we say that x is the independent variable and that y, which depends upon x, is the dependent variable. In Equation 2, we don't necessarily distinguish between the independent variable and the dependent variable. In either case, a solution is a relation between x and y that satisfies the differential equation. For example, the function $y(x) = c/(1 - cx)$, where c is a constant, satisfies the differential equation

$$\frac{dy}{dx} = y^2 \tag{3}$$

if $x \neq 1/c$. You can see that this is so by differentiating $y(x) = c/(1 - cx)$. It turns out that *every* solution to Equation 3 is of the form $y(x) = c/(1 - cx)$, and so we say that $y(x) = c/(1 - cx)$ is a *general solution* of Equation 3. If we are given additional information, such as $y = 1$ when $x = 1$, then we can determine the constant c, which in this case would be $c = 1/2$. The solution $y(x) = 1/(2 - x)$ is called a *particular solution* of Equation 3.

Example 1:
Show that $y(x) = ce^x - 2 - 2x - x^2$, where c is a constant, is a solution of

$$\frac{dy}{dx} = y + x^2$$

Find a particular solution if $y = 1$ when $x = 0$.

SOLUTION: Differentiating $y(x)$ gives

$$\frac{dy}{dx} = ce^x - 2 - 2x = y + x^2$$

Because $y(0) = 1$, $c = 3$, and so the particular solution is $y = 3e^x - 2 - 2x - x^2$.

Before we go on, we should address the question of whether a given differential equation even has a solution. For example, the equation $(y')^2 + y^2 + 1 = 0$ doesn't even have a real solution because the left side is necessarily positive. The following important theorem, which is proved in most textbooks on differential equations, addresses not only the existence of a solution, but its uniqueness as well.

Consider the first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

along with the auxiliary condition $y(x_0) = y_0$. If $f(x, y)$ and $\partial f/\partial y$ are real, finite, single-valued, and continuous in some region surrounding the point (x_0, y_0), then there is one and only one solution to the above equation in an interval $-h \leq x_0 \leq h$ lying within the region.

Applying this theorem to the equation in Example 1 shows that the solution is unique for all values of x.

Example 2:
The functions $y_1(x) = 0$ and $y_2(x) = x^3$ are solutions to

$$y' = 3y^{2/3} \qquad y(0) = 0$$

How do you reconcile this with the above theorem?

SOLUTION: Because $\partial f/\partial y = 2/y^{1/3}$ is not continuous along the line $y = 0$, there is no guarantee that the solution is unique in any region containing the line $y = 0$. In fact, there are infinitely many solutions because

$$y_k(x) = \begin{cases} 0 & -\infty < x \le k \\ (x-k)^3 & k < x < \infty \end{cases}$$

is a solution for any value of $k \ge 0$. Figure 11.1 shows this family of curves.

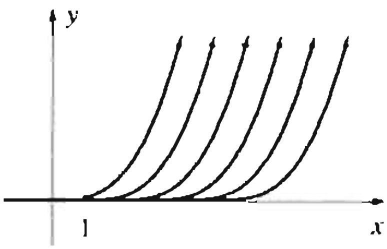

Figure 11.1
A family of solutions of $y' = 3y^{2/3}$ with $y(0) = 0$ given in Example 2.

We should mention at this point that the continuity of $f(x, y)$ and $\partial f/\partial y$ is a sufficient but not necessary condition. In particular, the condition on the derivative can be substantially relaxed, but the continuity of $\partial f/\partial y$ will cover all the cases we shall consider.

Let's go back to Equation 2. If $M(x, y)$ and $N(x, y)$ are such that Equation 2 can be written as

$$f(x)dx + g(y)dy = 0 \tag{4}$$

then we can simply integrate both sides of Equation 4 to obtain a solution to the differential equation. In this case, we say that Equation 2 is *separable*. A separable differential equation is the easiest to solve because it readily reduces to a problem of integration. For example, the differential equation

$$\frac{dy}{dx} = \frac{xy}{y+1}$$

is separable because it can be written as

$$x\,dx = \left(\frac{y+1}{y}\right)dy$$

which can be integrated to give $\dfrac{x^2}{2} = y + \ln y + c$ as the solution.

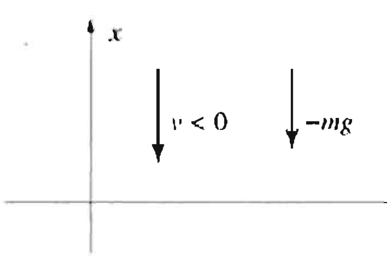

Figure 11.2
The coordinate system used in Example 3. The positive x axis is directed upward, so that $v = dx/dt$ is negative for a falling body.

Example 3:
Consider the vertical motion of a body of mass m subject to a gravitational force mg and a resistive force proportional to its speed. If we let $v = dx/dt$ and take the positive x axis to be directed upward as in Figure 11.2, then the

equation of motion of the body is

$$m\frac{dv}{dt} = -\gamma v - mg$$

where v is its velocity. Solve this equation with the initial condition $v = v_0$.

SOLUTION: This differential equation is separable, so

$$\frac{m\,dv}{\gamma v + mg} = -dt$$

or

$$\frac{m}{\gamma}\ln(\gamma v + mg) = -t + c$$

or

$$\gamma v + mg = Ae^{-\gamma t/m}$$

where $A = e^{\gamma c/m}$. Letting $v = v_0$ at $t = 0$ gives $A = \gamma v_0 + mg$, or

$$v(t) = \left(v_0 + \frac{mg}{\gamma}\right)e^{-\gamma t/m} - \frac{mg}{\gamma}$$

Note that this solution says that a body falling through a resistive medium approaches a constant velocity, $-mg/\gamma$, called the *terminal velocity*. Note also that we can obtain this result by letting $dv/dt = 0$ in the original equation.

Another type of first-order differential equation that is easy to solve is an *exact differential equation*. An exact differential equation is the total derivative of $F(x, y) = c$, or

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = M(x, y)dx + N(x, y)dy = 0 \tag{5}$$

in which case the equality of the mixed second partial derivatives of $F(x, y)$ gives the criterion

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} \tag{6}$$

If Equation 6 is satisfied, then the solution is given by

$$F(x, y) = \int \frac{\partial F}{\partial x}dx + g(y) = \int \frac{\partial F}{\partial y}dy + h(x) \tag{7}$$

where the integrations are "partial" integrations and where $h(x)$ and $g(y)$ are arbitrary "functions of integration." In other words, y is held fixed in the first integral and x is held fixed in the second. For example, consider the differential equation

$$(2y + 2)dx + 2x\, dy = 0$$

Equation 6 is satisfied because $\partial(2y + 2)/\partial y = \partial(2x)/\partial x$, and so

$$F(x, y) = \int (2y + 2)dx + g(y)$$

Now y is a constant in this integral, so we have

$$F(x, y) = 2xy + 2x + g(y)$$

We can determine $g(y)$ by substituting this result into $\partial F/\partial y = N(x, y) = 2x$ to obtain

$$\frac{\partial F}{\partial y} = 2x + \frac{dg}{dy} = 2x$$

from which we have $g(y) = c$, where c is a constant. The solution, then, is

$$F(x, y) = 2xy + 2x = A$$

where A is another constant.

Example 4:
Solve the differential equation

$$\frac{dy}{dx} = \frac{a^2 - 2xy - y^2}{(x + y)^2}$$

SOLUTION: First write the equation in the form

$$(a^2 - 2xy - y^2)dx - (x + y)^2 dy = 0$$

Equation 6 is satisfied because

$$\frac{\partial(a^2 - 2xy - y^2)}{\partial y} = -2x - 2y = -\frac{\partial(x + y)^2}{\partial x}$$

and so we see that the differential equation is exact. Therefore,

$$F(x, y) = \int (a^2 - 2xy - y^2)dx + g(y)$$

$$= a^2x - x^2y - xy^2 + g(y)$$

Now use the fact that

$$\frac{\partial F}{\partial y} = -x^2 - 2xy + \frac{dg}{dy} = N(x, y) = -(x + y)^2 = -x^2 - 2xy - y^2$$

to obtain $g(y) = -\dfrac{y^3}{3} + c$. Therefore,

$$F(x, y) = a^2x - x^2y - xy^2 - \frac{y^3}{3} + c$$

The total derivative of $F(x, y) = -c$ is exactly the original differential equation.

If an equation is not exact, it can sometimes be turned into an exact equation by multiplying it by an appropriate function of x and y. Such a function, if it exists, is called an *integrating factor*. For example, if $M(x, y)$ and $N(x, y)$ are sums of products of powers of x and y, then a possible integrating factor is $x^\alpha y^\beta$. Let's consider the differential equation

$$(1 - xy)\frac{dy}{dx} + y^2 + 3xy^3 = 0 \tag{8}$$

where $M(x, y) = y^2 + 3xy^3$ and $N(x, y) = 1 - xy$. Equation 8 is not exact as it stands, but if we multiply it by $x^\alpha y^\beta$ and then set $\partial M/\partial y = \partial N/\partial x$, we find that $\alpha = 0$ and $\beta = -3$ (Problem 22). Equation 8 becomes

$$\left(3x + \frac{1}{y}\right) dx + \left(\frac{1}{y^3} - \frac{x}{y^2}\right) dy = 0 \tag{9}$$

which is now exact. Using the procedure that we developed above for exact equations, we find that the solution to Equation 8 or 9 is (Problem 23)

$$f(x, y) = \frac{3x^2}{2} + \frac{x}{y} - \frac{1}{2y^2} + c \tag{10}$$

as you can verify by direct substitution.

Textbooks on differential equations develop procedures for finding integrating factors for certain types of differential equations, but even then finding an integrating factor can be a matter of good fortune. Nevertheless, integrating factors play an important role in some theoretical developments of differential equations, as we shall see in the next section.

Recall from Section 6.4 that a homogeneous function of degree n is a function that satisfies the relation

$$f(\lambda x, \lambda y) = \lambda^n f(x, y) \tag{11}$$

A differential equation $M(x, y)dx + N(x, y)dy = 0$ is said to be homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree. Problem 18 has you prove that the substitution $u = y/x$ transforms a homogeneous differential equation into one whose variables are separable. For example, the equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

is homogeneous. The substitution $y = wx$ yields

$$x\frac{dw}{dx} + w = \frac{1 + w^2}{2w}$$

which yields

$$-\ln(1 - w^2) = \ln x + c_1$$

or

$$y(x) = \pm(x^2 - c_2 x)^{1/2}$$

Example 5:
In Chapter 13, we shall encounter equations of the form

$$\frac{dy}{dx} = \frac{2y}{x - y}$$

Find the general solution to this equation.

SOLUTION: Both $2y$ and $x - y$ are homogeneous functions of degree one, so we let $y = wx$ to get

$$x\frac{dw}{dx} = \frac{w(1 + w)}{1 - w}$$

or

$$\frac{(1 - w)dw}{w(1 + w)} = \frac{dx}{x}$$

Upon integration, this gives

$$(x + y)^2 = cy$$

11.1 Problems

1. Show that the expression on the left is a solution to the differential equation on the right.

(a) $y = x^2 + cx$ $\qquad x\dfrac{dy}{dx} = x^2 + y$

(b) $y = c_1 \cos x + c_2 \sin x$ $\qquad \dfrac{d^2y}{dx^2} + y = 0$

(c) $y = c_1 e^{2x} + c_2 e^{-3x}$ $\qquad \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$

(d) $y = cx + c^2$ $\qquad y = x\dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2$

2. Solve the following differential equations:

(a) $\dfrac{dy}{dx} = -\dfrac{y}{x-1}$ $\quad$ (b) $a\left(x\dfrac{dy}{dx} + 2y\right) = xy\dfrac{dy}{dx}$ $\quad$ (c) $\dfrac{ds}{dt} = 15 - 16s$; $s = 0$ when $t = 0$

3. Solve $x^2\dfrac{dy}{dx} = y^2\dfrac{dx}{dy}$.

4. A copper pellet at a temperature of 200°C is dropped into a large bucket of water at 20°C. (The bucket is so large that the temperature of the water doesn't change.) After six minutes, the temperature of the pellet is 100°C. How much longer will it take for the temperature of the pellet to reach 25°C? Use Newton's law of cooling, which says in this case that $\dfrac{dT}{dt} = -k(T - 20)$, where T is the temperature in degrees Celsius and k is an empirical constant.

5. A simple equation describing population growth is $\dfrac{dx}{dt} = ax - bx^2$, where $x(t)$ is the population size and a and b are empirical constants. Derive the solution to this equation, called the *logistic curve*. Plot this curve and explain its behavior. Show that $x(t) \to a/b$ for all $x(0)$ [except for $x(0) = 0$].

6. Determine which of the following equations is exact and solve the ones that are:

(a) $2xy\,dx + (x^2 + y^2)dy = 0$ $\quad$ (b) $(2x + y)dx + (y - x)dy = 0$

(c) $(4 + xy^2)dx + yx^2dy = 0$ $\quad$ (d) $(\sin x + y)dx + (x - 2\cos y)dy = 0$

7. Sometimes a simple substitution turns an opaque differential equation into one that is easy to solve. Can you find such substitutions for the following differential equations?

(a) $x\,dy + y\,dx = xy^3dx$ $\quad$ (b) $\dfrac{dy}{dx} = \dfrac{x - y}{x + y}$ $\quad$ (c) $\dfrac{dy}{dx} = \dfrac{x + y - 1}{x + y + 1}$ $\quad$ (d) $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}$

8. The speed of a body falling through air experiences a resistance that is a function of its speed. Newton's equation can be written as $m\dfrac{dv}{dt} = mg - f(v)$. In many cases, $f(v)$ is found to be proportional to the speed. Derive an equation for $v(t)$ assuming that $f(v) = \alpha v$ and that the body is initially released from rest, so that $v(0) = 0$.

9. Derive and solve a differential equation that describes the curve such that the intercept of its tangent line at any point (x, y) with the y axis is equal to $2xy^2$.

10. Determine the equations of the curves for which the normals (lines that are perpendicular to the tangent lines) at any point pass through the origin.

11. Determine the family of curves that are orthogonal to the family of rectangular hyperbolas described by $xy = c$.

12. Determine a family of curves that is orthogonal to the family of cardioids, $r = c(1 + \sin\theta)$. *Hint*: Use Equation 8.1.3.

13. Find the equation of a curve that has the property that every point (x, y) on the curve is equidistant from the origin and the intersection of the x axis with the normal to the curve at that point.

14. Consider the flow of a liquid through a hole in the bottom of a container. If $h(t)$ is the height of the liquid above the hole, then the velocity of the liquid emerging from the hole will be given by $v = c(2gh)^{1/2}$, where g

is the acceleration of gravity (9.81 $\text{m} \cdot \text{s}^{-2}$) and c is an empirical constant, which is about 0.6 in many cases. The rate of change of the volume of liquid in the tank is $\frac{dV}{dt} = -av = -ac(2gh)^{1/2}$, where a is the cross-sectional area of the hole. Using the fact that $V = \int_0^h A(h)dh$, where $A(h)$ is the cross-sectional area of the tank at a height h, show that $\frac{dh}{dt} = -\frac{ac(2gh)^{1/2}}{A(h)}$. This result is known as Torricelli's law.

15. This problem relies upon the previous problem. A hemispherical tank of radius 1.00 meters is full of liquid. How long will it take all the fluid to flow through a hole of cross-sectional area 1.00 cm^2?

16. A tank intially contains 1000 liters of an aqueous salt solution of concentration 100 grams per liter. If pure water enters the tank at 5 liters per minute and the solution flows out at the same rate, calculate the concentration of salt in the solution in the tank after one hour. Assume that the solution in the tank is well stirred, so that solution is kept uniform throughout.

17. Repeat the previous problem if a salt solution of 5 grams per liter is used to flush out the tank.

18. Show that the substitution $u = y/x$ transforms a homogeneous differential equation into a differential equation whose variables are separable. *Hint*: Let $\lambda = 1/x$ in the definition of a homogeneous function to write $\frac{1}{x^n} f(x, y) = f\left(\frac{x}{x}, \frac{y}{x}\right) = \phi\left(\frac{y}{x}\right)$.

19. Use the method outlined in the previous problem to solve the following differential equations:

(a) $\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$ (b) $\frac{dy}{dx} = \frac{xy}{x^2 - y^2}$

20. Show that a differential equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{\alpha x + \beta x + \gamma}$ with $a\beta - \alpha b \neq 0$ can be reduced to a homogeneous differential equation by the substitution $x = u + x_0$ and $y = v + y_0$ where x_0 and y_0 are given by

$$ax_0 + by_0 + c = 0$$
$$\alpha x_0 + \beta y_0 + \gamma = 0$$

21. Use the method of the previous problem to solve $\frac{dy}{dx} = \frac{2x + y - 4}{x - y + 1}$.

22. Evaluate α and β such that Equation 8 becomes exact if it is multiplied by $x^\alpha y^\beta$.

23. Show that Equation 10 is a solution to Equation 8.

11.2 Linear First-Order Differential Equations

There is an important class of differential equations that we didn't mention in Section 1. A differential equation is said to be *linear* if every term containing the dependent variable is raised to the first power and if, in addition, no term contains a product of the dependent variable and any of its derivatives. Thus,

$$\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x)$$

is linear, whereas

$$y \frac{dy}{dx} + x = 0$$

and

$$\frac{dy}{dx} + xy^2 = 0$$

are nonlinear. It turns out that linear differential equations are much easier to solve than nonlinear differential equations. Furthermore, many natural laws can be expressed by linear differential equations to a high degree of accuracy. Consequently, linear differential equations occur frequently in applied problems.

The general form of a first-order linear differential equation is

$$\frac{dy}{dx} + p(x)y = q(x) \qquad a < x < b \tag{1}$$

where $p(x)$ and $q(x)$ are known functions. We can solve this equation in general by finding an integrating factor, which in this case is fairly straightforward. First write Equation 1 as

$$[\, p(x)y - q(x)\,]dx + dy = 0$$

Now multiply by $\mu(x)$, which we hope turns out to be an integrating factor:

$$[\, \mu(x)p(x)y - \mu(x)q(x)\,]dx + \mu(x)dy = 0 \tag{2}$$

If Equation 2 is to be exact, then we must have that

$$\frac{\partial \mu}{\partial x} = \frac{\partial}{\partial y}[\, \mu(x)p(x)y - \mu(x)q(x)\,] = p(x)\mu(x)$$

or

$$\frac{d \ln \mu(x)}{dx} = p(x)$$

Solving for $\mu(x)$ gives $\mu(x) = e^{\int p(x)\,dx}$. If we multiply Equation 1 by this integrating factor, then we obtain

$$\frac{dy}{dx} e^{\int p(x)\,dx} + p(x)ye^{\int p(x)\,dx} = q(x)e^{\int p(x)\,dx}$$

Notice that the left side of this result is the derivative of $ye^{\int p(x)\,dx}$ with respect to x, and so we have

$$\frac{d}{dx}\left[ye^{\int p(x)\,dx} \right] = q(x)e^{\int p(x)\,dx}$$

or

$$y(x)e^{\int p(x)\,dx} = \int q(x)e^{\int p(x)\,dx}dx + c \tag{3}$$

Solving Equation 3 for $y(x)$ gives

$$y(x) = e^{-\int p(x)\,dx} \int q(x)e^{\int p(x)\,dx}dx + ce^{-\int p(x)\,dx} \tag{4}$$

The existence-uniqueness theorem for first-order differential equations presented in the previous section tells us that Equation 4 is the general solution to Equation 1 over any interval in which $p(x)$ and $q(x)$ are continuous.

Let's use this result to solve

$$x\frac{dy}{dx} + 2y = x^3$$

First divide by x to put this equation into the form of Equation 1:

$$\frac{dy}{dx} + \frac{2}{x}y = x^2$$

Thus, the integrating factor is $\mu(x) = e^{\int p(x)dx} = e^{\int \frac{2}{x}dx} = e^{\ln x^2} = x^2$. Equation 4 gives

$$\begin{aligned} y(x) &= \frac{1}{x^2}\int x^4 dx + \frac{c}{x^2} \\ &= \frac{x^3}{5} + \frac{c}{x^2} \end{aligned}$$

as you can verify by direct substitution.

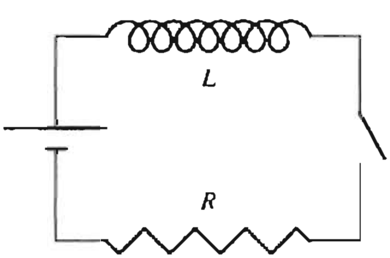

Figure 11.3
An electrical circuit consisting of a resistance R and an inductance L.

Example 1:
Figure 11.3 shows an RL circuit, which is an electrical circuit containing a resistance R and an inductance L. The voltage drop is Ri across the resistance and $L\,di/dt$ across the inductance, where i is the current. If $E(t)$ is the driving voltage, then Kirchoff's law for electrical circuits gives

$$L\frac{di}{dt} + Ri = E(t)$$

Solve this equation if $E(t) = E_0 =$ constant.

SOLUTION: First write the equation in standard form:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}$$

Then "$\int p\,dx$" $= Rt/L$, and Equation 3 gives

$$i(t)e^{Rt/L} = \int \frac{E_0}{L} e^{Rt/L} dt + c$$

$$= \frac{E_0}{R} e^{Rt/L} + c$$

Let $i(0) = i_0$, so $c = i_0 - E_0/R$, and

$$i(t) = \frac{E_0}{R} + \left(i_0 - \frac{E_0}{R}\right) e^{-Rt/L}$$

Figure 11.4 shows that $i(t)$ becomes steady at E_0/R when $t \gg L/R$.

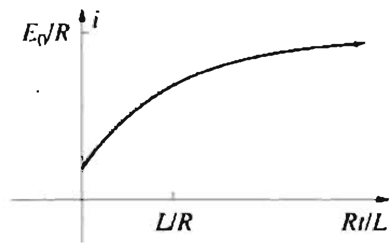

Figure 11.4
The current in Example 1 plotted against Rt/L.

Example 2:
Find the general solution of

$$(\sin 2\theta - 2r \cos \theta) d\theta = 2dr$$

SOLUTION: First write the equation as

$$\frac{dr}{d\theta} + r \cos \theta = \frac{1}{2} \sin 2\theta$$

which is in the form of Equation 1. Now "$\int p\,dx$" $= \int \cos\theta\, d\theta = \sin\theta$ and

$$r(\theta)e^{\sin\theta} = \frac{1}{2} \int \sin 2\theta\, e^{\sin\theta} d\theta + c$$

$$= \int \sin\theta \cos\theta\, e^{\sin\theta} d\theta + c$$

Integrate by parts, letting "u" $= \sin\theta$ and "dv" $= e^{\sin\theta} \cos\theta\, d\theta$, to get

$$r(\theta)e^{\sin\theta} = \sin\theta\, e^{\sin\theta} - \int e^{\sin\theta} d(\sin\theta) + c$$

$$= \sin\theta e^{\sin\theta} - e^{\sin\theta} + c$$

And so

$$r(\theta) = \sin\theta - 1 + ce^{-\sin\theta}$$

There is a slight extension of Equation 1, which is always discussed along with Equation 1. The equation is

$$\frac{dy}{dx} + p(x)y = q(x)y^n \tag{5}$$

and is called *Bernoulli's equation*. When $n = 0$ or 1, Equation 5 is just a first-order linear differential equation. When $n \neq 0$ or 1, even though it is nonlinear, it can be reduced to the first-order linear equation

$$\frac{du}{dx} + (1 - n)p(x)u = (1 - n)q(x) \tag{6}$$

by the substitution $u = y^{1-n}$ (Problem 10).

Example 3:
Solve

$$\frac{dy}{dx} + \frac{1}{x}y + x^3y^2 = 0$$

with the condition $y(1) = 1$.

SOLUTION: Comparing this equation with Equation 5, we see that this is a Bernoulli equation with $n = 2$, so we let $u = y^{1-n} = 1/y$. The above equation then becomes

$$\frac{du}{dx} - \frac{1}{x}u = x^3$$

The solution to this first-order linear equation is

$$\frac{u}{x} = \frac{x^3}{3} + c$$

or

$$y(x) = \frac{3}{3cx + x^4}$$

(Problem 11 asks you to verify that this is indeed a solution to the original differential equation.) The condition $y(1) = 1$ gives $c = 2/3$, so the particular solution we are seeking is

$$y(x) = \frac{3}{2x + x^4}$$

11.2 Problems

1. Find the general solutions of

(a) $\frac{dy}{dx} = x^2 - 3x^2y$ (b) $\frac{dy}{dx} + \frac{2}{x}y = x^2 + 2$

2. Find the general solutions of

(a) $\dfrac{dx}{dy} = 2y - \dfrac{3x}{y}$ (b) $t\dfrac{ds}{dt} = (3t+1)s + t^3e^{3t}$

3. Find the solutions of

(a) $x\dfrac{dy}{dx} + y = 2x \quad y = 2,\ x = 2$ (b) $\dfrac{dy}{dx} + (\tan x)\, y = \cos^2 x \quad y = -1,\ x = 0$

4. Find a continuous solution of $\dfrac{dy}{dx} + 2xy = f(x)$, where $f(x) = \begin{cases} x & 0 \le x < 2 \\ 0 & x \ge 2 \end{cases}$ and $y(0) = 2$.

5. Find the general solutions of

(a) $(x + y^2)\dfrac{dy}{dx} = 1$ (b) $\dfrac{di}{dt} = 3i - 5\sin t$

6. Derive an expression for $i(t)$ in Example 1 for the square potential.

$$E(t) = \begin{cases} 0 & t < 0 \\ E_0 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}.$$

Let $i_0 = 0$ and assume that the current is a continuous function of time.

7. Derive an expression for $i(t)$ in Example 1 if $E(t) = \begin{cases} 0 & t < 0 \\ E_0 \sin \omega t & t > 0 \end{cases}$. Assume that $i(0) = 0$.

8. Consider the two-step kinetic process

$$\mathrm{A} \xrightarrow{k_1} \mathrm{B}$$

$$\mathrm{B} \xrightarrow{k_2} \mathrm{C}$$

This kinetic process might represent radioactive decay or a chemical reaction. The differential equations describing this scheme are

$$\frac{d\mathrm{A}}{dt} = -k_1\mathrm{A}$$

$$\frac{d\mathrm{B}}{dt} = k_1\mathrm{A} - k_2\mathrm{B}$$

$$\frac{d\mathrm{C}}{dt} = k_2\mathrm{B}$$

where k_1 and k_2 are called rate constants. Solve the first equation for A and substitute the result into the second equation and then solve the resulting first-order linear differential equation for B. Plot the result for various ratios of k_1 and k_2. Assume that $\mathrm{A}(0) = \mathrm{A}_0$ and $\mathrm{B}_0 = 0$.

9. This problem presents another method of solving Equation 1. If $q(x) = 0$ in Equation 1, then Equation 1 is said to be *homogeneous*. If Equation 1 were homogeneous, it would be easy to solve by separation of variables to give $y(x) = Ae^{-\int p\,dx}$. Assume then that the solution to Equation 1 (the inhomogeneous equation) has the same form, but with $A = u(x)$, and determine $u(x)$ and hence the solution to Equation 1.

10. Verify that the substitution $u = y^{1-n}$ into Equation 5 yields Equation 6.

11. Verify that the solution found in Example 3 is indeed a solution to the original equation.

12. Find the general solutions of

(a) $x\dfrac{dy}{dx} + y = 3x^3y^2$ (b) $y\dfrac{dx}{dy} + 2x = 2x^3y^2$

13. Find the solution of $\dfrac{dy}{dx} - y + xe^{-2x}y^3 = 0$ with $y(0) = 1$.

14. Here are some first-order differential equations. Find their general solutions.

(a) $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{3xy^2}$ (b) $\dfrac{dy}{dx} = \dfrac{x + y}{3x + 3y - 4}$

15. Here are some first-order differential equations. Find their general solutions.

(a) $\dfrac{du}{dt} = 4t - 2tu$ (b) $\dfrac{dy}{dx} + \dfrac{y}{3} = \dfrac{(1 - 2x)y^4}{3}$

16. In chemical kinetics and other types of rate processes, you frequently encounter the scheme $\mathrm{A} \rightleftharpoons \mathrm{B}$ representing the interconversion of two species, A and B. The rate equation for this interconversion can be written as $\dfrac{d\mathrm{A}}{dt} = -k_1\mathrm{A} + k_2\mathrm{B}$, where k_1 and k_2 are called rate constants. By conservation of mass, $\mathrm{A}(t) + \mathrm{B}(t) = \mathrm{A}_0 + \mathrm{B}_0$, where $\mathrm{A}_0 = \mathrm{A}(0)$ and $\mathrm{B}_0 = \mathrm{B}(0)$. Solve the above equation for $\mathrm{A}(t)$ and $\mathrm{B}(t)$.

17. Another kinetic scheme involves the sequential decay, $\mathrm{A} \xrightarrow{k_1} \mathrm{B} \xrightarrow{k_2} \mathrm{C}$. The simultaneous equations that describe this scheme are $\dfrac{d\mathrm{A}}{dt} = -k_1\mathrm{A}$, $\dfrac{d\mathrm{B}}{dt} = k_1\mathrm{A} - k_2\mathrm{B}$, and $\dfrac{d\mathrm{C}}{dt} = k_2\mathrm{B}$. Solve the first equation with the initial condition $\mathrm{A}(0) = \mathrm{A}_0$ and substitute the result into the second equation to obtain $\dfrac{d\mathrm{B}}{dt} = k_1\mathrm{A}_0e^{-k_1t} - k_2\mathrm{B}$. Solve this equation with the initial condition $\mathrm{B}(0) = 0$. Plot your result for various values of k_1 and k_2 and interpret the resulting curves.

18. Solve the equation $y' = 1 + 2xy$ in terms of an error function.

19. Determine the equation of the curve which passes through the point $(1, 2)$ whose slope at any point (x, y) is $2 - y/x$.

20. A container contains 100 liters of a salt solution whose concentration is 200 grams per liter. A salt solution of concentration 2 grams per liter is added to the container at a rate of 10 liters per minute and the efflux from the container is 5 liters per minute. Calculate the minimum amount of salt in the container and when it will occur. Assume that the solution is stirred vigorously so that it is maintained at a uniform concentration.

21. Solve the equation $x^2\dfrac{dy}{dx} + xy = \sin x$ along with the condition $y = 2$ when $x = 1$. *Hint*: Leave your answer in terms of a definite integral.

11.3 Homogeneous Linear Differential Equations with Constant Coefficients

In the previous section, we found the general solution of a first-order linear differential equation. This is not possible for general higher-order linear differential equations. We'll see in this section, however, that we can solve higher-order linear differential equations if the coefficients are constants. Fortunately, a great many of the differential equations that occur in physical applications have constant coefficients. We'll discuss some properties of general higher-order linear differential equations first, and then spend the rest of this section discussing those with constant coefficients.

A general nth order linear differential equation can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{1}$$

where we shall always assume that the $a_j(x)$ are continuous functions of x on some interval (α, β). Note that all the terms involving y or its derivatives occur only to the first power and that there are no cross terms. If $f(x) = 0$, Equation 1 is said to be *homogeneous*; otherwise it is *nonhomogeneous*. It is sometimes convenient to write Equation 1 in the abbreviated form,

$$\mathcal{L}y = f(x) \tag{2}$$

where $\mathcal{L}$ is the linear differential operator

$$\mathcal{L} = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x) \tag{3}$$

An important property of a homogeneous linear differential equation, $\mathcal{L}y = 0$, is that if $y(x)$ is a solution, then so is $cy(x)$ where c is a constant. Furthermore, if $y_1(x)$ and $y_2(x)$ are solutions to $\mathcal{L}y = 0$, then so is $c_1y_1(x) + c_2y_2(x)$, or

$$\mathcal{L}[c_1y_1(x) + c_2y_2(x)] = c_1\mathcal{L}y_1(x) + c_2\mathcal{L}y_2(x) = 0 \tag{4}$$

because $\mathcal{L}y_1(x) = 0$ and $\mathcal{L}y_2(x) = 0$. We can continue this process and say that if $y_1(x), y_2(x), \ldots, y_n(x)$ (n can be any positive integer) are solutions to the homogeneous equation, then so is the linear combination $c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$.

We introduced the idea of a vector space in Section 9.5. Recall that a vector space is a set of quantities that satisfy a set of requirements, laid down toward the end of Section 9.5. It turns out that the set of solutions of an nth order homogeneous linear differential equation forms a vector space, called the *solution space*. This statement follows directly from Equation 4, for if $y_1(x)$ and $y_2(x)$ are any two solutions to $\mathcal{L}y = 0$, then so is the linear combination $c_1y_1(x) + c_2y_2(x)$. Furthermore, the dimension of the solution space of an nth order homogeneous linear differential equation is n. Thus, there must be n linearly independent solutions to $\mathcal{L}y = 0$, and a general solution is of the form

$$y(x) = \sum_{i=1}^{n} c_i y_i(x) \tag{5}$$

if the $y_i(x)$ are linearly independent.

Recall from Section 9.5 that a set of functions $f_j(x)$, $= 1, 2, \ldots, n$ is said to be linearly independent in an interval (α, β) if

$$\sum_{j=1}^{n} c_j f_j = 0 \qquad \alpha \le x \le \beta \tag{6}$$

implies that all the $c_j = 0$. We presented a convenient test for linear independence in terms of the Wronskian determinant. Differentiating Equation 6 $n - 1$ times gives us n simultaneous equations in the c_j, which can be written in matrix form as

$$\begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0 \tag{7}$$

If the $f_j(x)$ are linearly independent, then $\mathbf{c} = (c_1, c_2, \ldots, c_n)^{\mathsf{T}} = \mathbf{0}$ is the unique solution to Equation 7. But this will be so only if the determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \neq 0 \tag{8}$$

for some value of x in the interval (α, β). The determinant given by Equation 8 is called the Wronskian determinant, and Equation 8 is a test for linear independence.

Example 1:
Three solutions to

$$y''' - 3y'' + 3y' - y = 0$$

are $y_1(x) = e^x$, $y_2(x) = xe^x$, and $y_3(x) = x^2e^x$. Are they linearly independent?

SOLUTION: The coefficients of the differential equation are continuous for all values of x (they're constants) and the Wronskian determinant is equal to

$$\begin{aligned} W &= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} \\ &= 2e^{3x} \neq 0 \end{aligned}$$

Therefore, the three solutions are linearly independent over the entire x axis.

We'll now spend the rest of the section discussing homogeneous linear differential equations with constant coefficients. These are best discussed by means of examples. Let's start with

$$y''(x) + y'(x) - 6y(x) = 0 \tag{9}$$

This equation will be satisfied by a function whose derivatives are multiples of itself. The function $e^{\alpha x}$ (where α is a constant) is such a function. If we substitute $y = e^{\alpha x}$ into Equation 9, we get

$$(\alpha^2 + \alpha - 6)e^{x} = 0 \tag{10}$$

The factor $e^{x} \neq 0$, so Equation 10 tells us that

$$\alpha^2 + \alpha - 6 = 0$$

or that $\alpha = 2$ and -3. Two solutions to Equation 9 are e^{2x} and e^{-3x} and the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}$$

Equation 10 is called the *auxiliary equation* of Equation 9.

Example 2:
Determine the solution of

$$y'' + y' - 2y = 0$$

subject to the conditions $y(0) = 0$ and $y'(0) = 6$.

SOLUTION: The auxiliary equation is $\alpha^2 + \alpha - 2 = 0$, which gives $\alpha = 1$ and -2. The general solution is

$$y(x) = c_1 e^{x} + c_2 e^{-2x}$$

Applying the conditions $y(0) = 0$ and $y'(0) = 6$ gives $c_1 + c_2 = 0$ and $c_1 - 2c_2 = 6$, or $c_1 = 2$ and $c_2 = -2$. Thus, the particular solution is

$$y(x) = 2e^{x} - 2e^{-2x}$$

If we attempt to solve

$$y''(x) - 2y'(x) + y(x) = 0 \tag{11}$$

the auxiliary equation, $\alpha^2 - 2\alpha + 1 = 0$, gives us only one distinct root $\alpha = 1$. Thus $y(x) = e^{x}$ is one solution, but we need to find another linearly independent solution in order to have a general solution. It is not uncommon to have one solution in hand and to search for another. This can be done very nicely by a method called *reduction of order*. The solution that we have in hand is $y(x) = ce^x$. To find a second solution, we *assume* that

$$y(x) = u(x)e^x \tag{12}$$

where $u(x)$ is to be determined. Substitute Equation 12 into Equation 11 to obtain

$$u''(x)e^x = 0$$

The factor $e^x \neq 0$, so $u''(x) = 0$. This gives us $u(x) = c_1x + c_2$, which substituted into Equation 12 gives

$$y(x) = (c_1x + c_2)e^x = c_1xe^x + c_2e^x \tag{13}$$

The two functions e^x and xe^x are linearly independent (see Example 1), and so Equation 13 represents the general solution to Equation 12.

Although we have introduced the method of reduction by a specific example, the method is general.

Example 3:
Find the general solution of

$$y'''(x) - 3y'(x) + 2y(x) = 0$$

SOLUTION: The auxiliary equation is $\alpha^3 - 3\alpha + 2 = 0$. You can see by inspection that $\alpha = 1$ is a solution. Dividing $\alpha^3 - 3\alpha + 2$ by $\alpha - 1$ gives $\alpha^2 + \alpha - 2$, and so the other two values of α are $\alpha = 1$ and $\alpha = -2$. The root $\alpha = 1$ is repeated, so substitute $y(x) = u(x)e^x$ into the above differential equation to obtain

$$(u''' + 3u'')e^x = 0$$

or $u''' + 3u'' = 0$. We can solve this equation by first letting $u'' = z$ to get $z' + 3z = 0$. Now we integrate to get

$$z = u'' = e^{-3x}$$

Integrate twice more to get

$$u(x) = c_1x + c_2 + \frac{e^{-3x}}{9}$$

The complete solution is

$$
\begin{aligned}
y(x) &= u(x)e^{x} + c_3 e^{-2x} \\
&= (c_1 x + c_2)e^{x} + \left(c_3 + \frac{1}{9}\right) e^{-2x} \\
&= (c_1 x + c_2)e^{x} + c_4 e^{-2x}
\end{aligned}
$$

where we have simply replaced $c_3 + 1/9$ by another constant, c_4.

Example 3 shows us that our method of solving homogeneous linear differential equations with constant coefficients is certainly not limited to second order equations. Furthermore, we can use reduction of order to find the "companion" solution(s) of the one solution obtained from a repeated root of the auxiliary equation. If a root α occurs n times, then the solution associated with that root will be of the form

$$y(x) = (c_1 + c_2 x + c_3 x^2 + \cdots + c_{n-1} x^{n-1})e^{\alpha x} \tag{14}$$

Example 4:
Find the general solution of

$$y''' - 3y'' + 3y' - y = 0$$

SOLUTION: The auxiliary equation is $\alpha^3 - 3\alpha^2 + 3\alpha - 1 = 0$, or $(\alpha - 1)^3 = 0$, so that we have a triple root of $\alpha = 1$. According to Equation 14, the general solution is

$$y(x) = (c_0 + c_1 x + c_2 x^2)e^{x}$$

as you can verify by direct substitution.

So far, the roots of the auxiliary equation have been real. Let's consider the equation

$$x''(t) + x(t) = 0 \tag{15}$$

The auxiliary equation is $\alpha^2 + 1 = 0$, so $\alpha = \pm i$. The general solution in this case is

$$x(t) = c_1 e^{it} + c_2 e^{-it} \tag{16}$$

We can use Euler's formula, $e^{it} = \cos t + i \sin t$, to write Equation 16 as

$$x(t) = c_3 \cos t + c_4 \sin t \tag{17}$$

where $c_3 = c_1 + c_2$ and $c_4 = i(c_1 - c_2)$. Figure 11.5 shows Equation 17 plotted for various values of c_1 and c_2. The reason that the plots are harmonic (sinusoidal

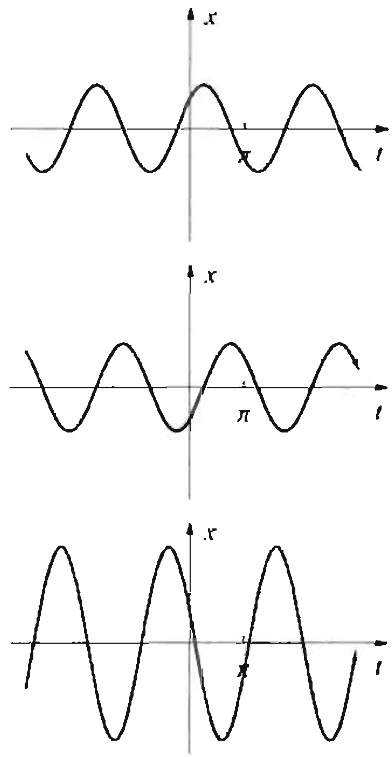

Figure 11.5
The function $x(t) = c_1 \cos t + c_2 \sin t$ plotted against t for various values of c_1 and c_2. Note that the motion is harmonic in all cases.

or cosinusoidal) is because $x(t)$ can be written as $A\cos(\omega t + \phi)$, where $A = (c_3^2 + c_4^2)^{1/2}$ and $\phi = \tan^{-1}(-c_4/c_3)$ (Problem 18). Thus, we see that Equation 15 has oscillatory solutions.

Let's consider the following case, where the roots of the auxiliary equation are a complex conjugate pair:

$$x''(t) + 2x'(t) + 2x(t) = 0$$

The auxiliary equation is $\alpha^2 + 2\alpha + 2 = 0$, so $\alpha = -1 \pm i$. The general solution is

$$\begin{aligned} x(t) &= c_1 e^{-t} e^{it} + c_2 e^{-t} e^{-it} \\ &= e^{-t}(c_3 \cos t + c_4 \sin t) \end{aligned}$$

where $c_3 = c_1 + c_2$ and $c_4 = i(c_1 - c_2)$. Problem 18 helps you show this equation can be written as

$$x(t) = Ae^{-t}\cos(t + \phi) \tag{18}$$

where $A = (c_3^2 + c_4^2)^{1/2}$ and $\phi = \tan^{-1}(-c_4/c_3)$. Thus, in this case, the solution displays damped harmonic behavior (Figure 11.6).

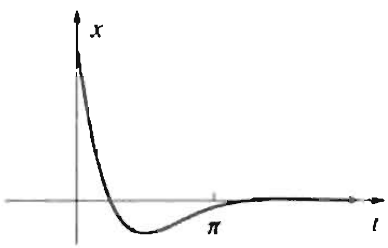

Figure 11.6
The function $x(t)$ given by Equation 18 plotted against t.

Example 5:
Solve the equation

$$\frac{d^2x}{dt^2} + \omega^2 x(t) = 0$$

subject to the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$. What do these initial conditions represent physically?

SOLUTION: The auxiliary equation yields $\alpha = \pm i\,\omega$, and so the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

The initial conditions require that $c_1 = A$ and $c_2 = 0$, so the particular solution we are seeking is

$$x(t) = A \cos \omega t$$

This solution oscillates cosinusoidally in time, with an amplitude A and a frequency of ω radians per second or $\nu = \omega/2\pi$ cycles per second. The differential equation is that of a harmonic oscillator and the initial conditions depict displacing the oscillator to its amplitude and then just letting go.

The equation in Example 5 occurs in a variety of physical applications. It represents the motion of a mass connected to a spring that obeys Hooke's law, a pendulum swinging through small angles, the electric current in a circuit containing an inductance and capacitance, and numerous others. Let's use the case of a pendulum swinging in a fixed plane for concreteness. We'll express the equation of motion in terms of the arc length $s(t) = l\theta(t)$, where θ is the angle that the pendulum makes with respect to the vertical (Figure 11.7).

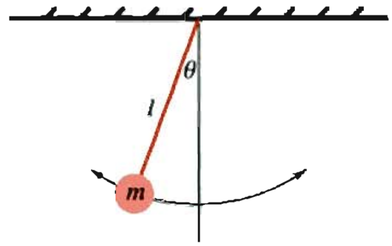

Figure 11.7
A pendulum oscillating in a single plane. The pendulum support is rigid and mass-less, and supports a mass m.

The potential energy is given by (see also Section 3.5)

$$V(\theta) = mg(l - l\cos\theta)$$

and the force is given by

$$f = -\frac{\partial V}{\partial s} = -\frac{1}{l}\frac{\partial V}{\partial \theta} = -mg\sin\theta$$

which becomes $f = -mg\theta$ for small values of θ. The momentum of the supported mass is $m\dot{s} = ml\dot{\theta}$, so Newton's equation is

$$ml\frac{d^2\theta}{dt^2} = -mg\theta$$

or

$$\frac{d^2\theta}{dt^2} + \omega_0^2\theta = 0 \tag{19}$$

where $\omega_0 = (g/l)^{1/2}$ is the natural frequency of the pendulum. The solution to Equation 19 with the initial condition $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$ is $\theta(t) = \theta_0\cos\omega_0 t$. (See Example 5.) Physically, this solution depicts the back and forth motion of the pendulum.

We can introduce frictional resistance in a fairly simple way by saying that the frictional force is proportional to but opposite the motion, $ds/dt = l\,d\theta/dt$. Equation 19 then becomes

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0^2\theta = 0 \tag{20}$$

where γ is a frictional coefficient. Let's solve Equation 20 under the initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$. The auxiliary equation is $\alpha^2 + \gamma\alpha + \omega_0^2 = 0$, and yields

$$\alpha = -\frac{\gamma}{2} \pm \frac{1}{2}(\gamma^2 - 4\omega_0^2)^{1/2} \tag{21}$$

You can see that the motion of the pendulum depends upon the relative values of γ^2 and $4\omega_0^2$. If $4\omega_0^2 > \gamma^2$, α is a complex conjugate pair and the solution to Equation 20 is

$$\theta(t) = e^{-\gamma t/2}(c_1\cos\omega t + c_2\sin\omega t)$$

where $\omega = (4\omega_0^2 - \gamma^2)^{1/2}/2$. Applying the initial conditions, $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, gives (Problem 16)

$$\theta(t) = e^{-\gamma t/2}\left(\theta_0 \cos \omega t + \frac{\gamma\theta_0}{2\omega} \sin \omega t\right) \tag{22}$$

Using the results of Problem 17, Equation 22 can be written as (Problem 19)

$$\theta(t) = \theta_0 e^{-\gamma t/2}\left(1 + \frac{\gamma^2}{4\omega^2}\right)^{1/2} \sin(\omega t + \phi)$$

or

$$\theta(t) = \frac{\theta_0 e^{-\delta\omega_0 t}}{(1-\delta^2)^{1/2}} \sin[(1-\delta^2)^{1/2}\omega_0 t + \phi] \tag{23}$$

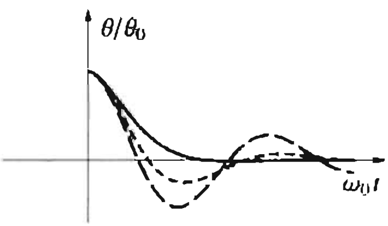

Figure 11.8
Equation 23 plotted against $\omega_0 t$ for several values of $\delta = \gamma/2\omega_0 = 0.20$ (long dashed), 0.40 (short dashed), and 0.80 (solid).

where $\delta = \gamma/2\omega_0$ and $\phi = \tan^{-1}[(1-\delta^2)^{1/2}/\delta]$. Equation 23 reduces to $\theta(t) = \theta_0 \cos \omega_0 t$, as $\gamma \to 0$ (Problem 19).

Equation 23 is plotted against $\omega_0 t$ in Figure 11.8 for several values of $\delta = \gamma/2\omega_0 < 1$. Note that the motion represents damped oscillations and that $\theta(t) \to 0$ as $t \to \infty$ in all cases, which represents the pendulum having stopped and hanging straight down. The motion in this case is called *underdamped*.

As γ approaches $2\omega_0$, the oscillations given by Equation 23 become weaker and weaker, until they disappear altogether when $\gamma = 2\omega_0$. As $\gamma \to 2\omega_0$, Equation 23 becomes

$$\theta(t) = \theta_0 e^{-\omega_0 t}(1 + \omega_0 t) \tag{24}$$

You can see this by taking the limit, $\delta \to 1$, in Equation 23 (Problem 20) or by going back to Equation 21, where $\alpha = -\gamma/2$. Being a repeated root, the general solution to Equation 20 is

$$\omega(t) = (c_1 + c_2 t)e^{-\gamma t/2}$$

and Equation 24 is the particular solution for $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$. The motion corresponding to the case where $\gamma = 2\omega_0$ is called *critically damped* because it represents the point where the oscillations no longer occur as γ increases.

For $\gamma^2 > 4\omega_0^2$, Equation 21 yields two real values of α, and the general solution to Equation 20 is

$$\theta(t) = e^{-\gamma t/2}[\, c_1 e^{(\gamma^2 - 4\omega_0^2)^{1/2} t/2} + c_2 e^{-(\gamma^2 - 4\omega_0^2)^{1/2} t/2}\,]$$

or

$$\theta(t) = e^{-\delta\omega_0 t}\{c_3 \cosh[(\delta^2 - 1)^{1/2}\omega_0 t] + c_4 \sinh[(\delta^2 - 1)^{1/2}\omega_0 t]\} \tag{25}$$

where $c_3 = c_1 + c_2$ and $c_4 = c_1 - c_2$. The particular solution for $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$ is (Problem 21)

$$\theta(t) = \frac{\theta_0 e^{-\delta\omega_0 t}}{(\delta^2 - 1)^{1/2}}\{\delta \sinh[(\delta^2 - 1)^{1/2}\omega_0 t] + (\delta^2 - 1)^{1/2} \cosh[(\delta^2 - 1)^{1/2}\omega_0 t]\} \tag{26}$$

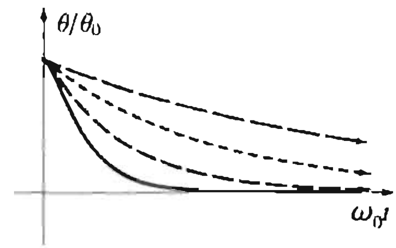

Figure 11.9
Equation 26 plotted against $\omega_0 t$ for several values of $\delta = \gamma/2\omega_0 = 1.1$ (solid), 2.0 (long dashed), 4.0 (short dashed), and 8.0 (dash-dot).

Equation 26 is plotted against $\omega_0 t$ for several values of $\delta > 1$ in Figure 11.9. The pendulum simply approaches its equilibrium position $\theta = 0$ monotonically. There are no oscillations in this case, and the motion is called *overdamped*. Using the limits $\sinh x \to x$ and $\cosh x \to 1$ as $x \to 0$, you can readily show that Equation 26 reduces to Equation 24 as $\delta \to 1$. Furthermore, you can show that Equation 26 becomes Equation 23 if $\delta < 1$ (Problem 22).

11.3 Problems

1. Are $1, x, x^2$, and x^3 linearly independent? What about $1, 1+x, 1+x^2, 1+x^3$?
2. Are e^x, $\sinh x$, and $\cosh x$ linearly independent?
3. Are $1+x$, $1-x$, and x^2 linearly independent?
4. Find the general solution of

   (a) $y''(x) - y'(x) - 2y(x) = 0$ (b) $y''(x) - 6y'(x) + 9y(x) = 0$ (c) $y''(x) + 4y'(x) + y(x) = 0$
5. Find the general solution of

   (a) $y''(x) - 4y(x) = 0$ (b) $y''(x) + 2y'(x) + 4y(x) = 0$ (c) $y''(x) + 9y(x) = 0$
6. Find the general solution of

   (a) $y''(x) + 6y'(x) = 0$ (b) $y''(x) - 4y'(x) + 3y(x) = 0$ (c) $y''(x) + 3y(x) = 0$
7. Find the particular solutions to the equations in Problem 4 if $y(0) = 1$ and $y'(0) = 0$.
8. Find the particular solutions to the equations in Problem 4 if $y(0) = 0$ and $y'(0) = 1$.
9. Solve

   (a) $y''(x) - 4y(x) = 0$ $y(0) = 2$, $y'(0) = 4$
   (b) $y''(x) - 5y'(x) + 6y(x) = 0$ $y(0) = -1$, $y'(0) = 0$
   (c) $y'(x) - 2y(x) = 0$ $y(0) = 2$
10. Find the general solution of $y'''(x) - 2y''(x) - y'(x) + 2y(x) = 0$.
11. Find the general solution of $y'''(x) - 6y''(x) + 12y'(x) - 8y(x) = 0$.
12. Given that $y = x^2$ satisfies $x^2y''(x) + xy'(x) - 4y(x) = 0$, use reduction of order to find a second solution.
13. Given that $y = x$ satisfies $x^2y''(x) - xy'(x) + y(x) = 0$, use reduction of order to find a second solution.
14. In this problem, we will derive a general result using the method of reduction of order. Consider the second-order equation $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$, where $a_2(x)$, $a_1(x)$, and $a_0(x)$ are continuous in the

open interval (α, β). Suppose we know one solution, $y_1(x)$. Assume that a second solution will be of the form $y_2(x) = u(x)y_1(x)$, where $u(x)$ is to be determined. Show that $u(x)$ satisfies

$$a_2(x)y_1(x)u''(x) + [\,2a_2(x)y_1'(x) + a_1(x)y_1(x)\,]u'(x) = 0$$

Note that this result is a linear first-order differential equation in $u'(x)$. Let $v(x) = u'(x)$ to write $v'(x) + \left[\dfrac{2y_1'(x)}{y_1(x)} + \dfrac{a_1(x)}{a_2(x)}\right] v(x) = 0$. Show that

$$v(x) = u'(x) = \frac{c_1}{y_1^2(x)} \exp\left[-\int \frac{a_1(x)}{a_2(x)}dx\right].$$

where c_1 is a constant, which we can take equal to 1. (Why?) Now show that

$$u(x) = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_2(x)}dx\right]}{y_1^2(x)}dx + c_2.$$

where we can let $c_2 = 0$ (why?). The second solution then is

$$y_2(x) = y_1(x) \int \frac{\exp\left[-\int \frac{a_1(x)}{a_2(x)}dx\right]}{y_1^2(x)}dx.$$

15. Use the general result of Problem 14 to re-do Problems 12 and 13.

16. Derive Equation 22.

17. Show that $A \cos t + B \sin t$ can be written as $C \sin(t + \phi)$, where $C = (A^2 + B^2)^{1/2}$ and $\phi = \tan^{-1}(A/B)$. *Hint*: Work backwards from $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$.

18. Show that $A \cos t + B \sin t$ can be written as $C \cos(t + \psi)$, where $C = (A^2 + B^2)^{1/2}$ and $\psi = \tan^{-1}(-B/A)$. *Hint*: Work backwards from $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$.

19. Derive Equation 23 from Equation 22 and show that it reduces to $\theta(t) = \theta_0 \cos\omega t$ as $\gamma \to 0$.

20. Show that Equation 23 reduces to Equation 24 as $\delta \to 1$ $(\gamma \to 2\omega)$.

21. Derive Equation 26 from Equation 25.

22. Show that Equation 26 becomes Equation 24 as $\delta \to 1$ and Equation 23 if $\delta < 1$.

23. In this problem, we will show that the Wronskian determinant of the two solutions of $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$ can be written as $W = ce^{-\int p\,dx}$, where $p = a_1(x)/a_2(x)$ and c is a constant. This result is known as *Abel's formula*. Start with $a_2y_1'' + a_1y_1' + a_0y_1 = 0$ and $a_2y_2'' + a_1y_2' + a_0y_2 = 0$. Multiply the first of these equations by y_2 and the second by y_1, then subtract, and show that the result can be written as $a_2\dfrac{dW}{dx} + a_1W = 0$, where $W = y_1y_2' - y_1'y_2$. Now obtain Abel's formula.

24. Use the result of the previous problem to argue that $W = 0$ either everywhere in the interval (α, β) where $a_1(x)/a_2(x)$ and $a_0(x)/a_2(x)$ are continuous or nowhere in the interval. Thus, if $W \neq 0$ at some point in (α, β), then $W \neq 0$ everywhere in (α, β).

11.4 Nonhomogeneous Linear Differential Equations with Constant Coefficients

In this section, we shall consider nonhomogeneous linear differential equations,

$$\mathcal{L}y = f(x) \tag{1}$$

where $f(x) \neq 0$ and

$$\mathcal{L} = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x) \tag{2}$$

The method of finding the general solution to Equation 1 is based upon the following theorem:

If $y_p(x)$ is any solution to Equation 1, then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x) = y_c(x) + y_p(x) \tag{3}$$

where $y_c(x)$ is the general solution to the homogeneous equation $\mathcal{L}y = 0$.

In Equation 3, $y_c(x)$ is called the *complementary solution* and $y_p(x)$ is called a *particular solution*.

The proof of the above statement is fairly short. Given $y_p(x)$, let $y_{p1}(x)$ be any other solution of Equation 1. Because $\mathcal{L}$ is a linear operator,

$$\mathcal{L}(y_p - y_{p1}) = \mathcal{L}y_p - \mathcal{L}y_{p1} = f(x) - f(x) = 0 \tag{4}$$

Thus, $y_p - y_{p1}$ is a solution of the homogeneous equation (Equation 1 with $f(x) = 0$), and so $y_p - y_{p1} = y_c$, or

$$y_{p1} = y_c + y_p$$

Because $y_{p1}(x)$ is any other solution to Equation 1, all solutions can be written as Equation 3.

Equation 4 says that the difference between any two particular solutions is a solution to the homogenous equation. Consequently, two particular solutions to Equation 1 may be quite different. For example, both $y_p(x) = 3x - \sin 2x$ and $3x + 4\cos 2x$ are solutions to

$$y''(x) + 4y(x) = 12x \tag{5}$$

They differ by $4\cos 2x + \sin 2x$, which is a solution to $y''(x) + 4y(x) = 0$, whose general solution is $y_c = c_1\cos 2x + c_2\sin 2x$. The general solution to Equation 5 is

$$y(x) = c_1\cos 2x + c_2\sin 2x + 3x \tag{6}$$

It may be fine that $y_p(x) = 3x$ is a particular solution of Equation 5, but how do we find a particular solution in the first place? There are two methods that are presented in all texts on differential equations. These methods are called the *method of undetermined coefficients* and the method of *variation of parameters*. The method of undetermined coefficients is not only limited to linear differential equations with constant coefficients, but the function $f(x)$ in Equation 1 must be of a certain form, which we will learn about later. Its advantage is that the method is fairly easy to use. The method of variation of parameters is not limited to linear differential equations with constant coefficients nor is $f(x)$ restricted to any particular form, but the method is fairly tedious to use. Consequently, we shall discuss the method of variation of parameters only briefly. The references at the end of the chapter discuss both methods in detail.

We're going to use an almost empirical approach and introduce the method of undetermined coefficients with a few examples and then develop it into a formal procedure. Let's consider the nonhomogeneous equation

$$y'' - 2y' + y = 2x \tag{7}$$

We can see by inspection that a particular solution is of the form

$$y_p(x) = \alpha + \beta x \tag{8}$$

because if we substitute Equation 8 into Equation 7, we obtain

$$-2\beta + \alpha + \beta x = 2x$$

Equating coefficients of like powers of x on the two sides of this equation gives $\alpha - 2\beta = 0$ and $\beta = 2$, and so we see that

$$y_p(x) = 4 + 2x$$

which you can verify by direct substitution into Equation 7. If $f(x) = x^3$ instead of $2x$, then the differential equation is

$$y'' - 2y' + y = x^3$$

we would try

$$y_p(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$$

Substituting this into Equation 7 gives

$$2\gamma + 6\delta x - 2\beta - 4\gamma x - 6\delta x^2 + \alpha + \beta x + \gamma x^2 + \delta x^3 = x^3$$

Equating coefficients of like powers of x on the two sides gives $\alpha = 24$, $\beta = 18$, $\gamma = 6$, and $\delta = 1$, and so

$$y_p(x) = 24 + 18x + 6x^2 + x^3$$

as you can verify by direct substitution.

Generally, if $f(x)$ is a polynomial of degree n, we assume that $y_p(x)$ is a polynomial of degree n and determine its parameters by substituting it into the nonhomogeneous equation and equating the coefficients of like powers of x on each side of the resulting equation.

Example 1:
Solve the differential equation

$$y'' + 3y' + 2y = 6 + x^2$$

SOLUTION: We assume that $y_p(x) = \alpha + \beta x + \gamma x^2$. Substituting $y_p(x)$ into the differential equation gives

$$2\gamma + 3\beta + 6\gamma x + 2\alpha + 2\beta x + 2\gamma x^2 = 6 + x^2$$

from which we find $\alpha = 19/4$, $\beta = -3/2$, and $\gamma = 1/2$, or

$$y_p(x) = \frac{19}{4} - \frac{3}{2}x + \frac{1}{2}x^2$$

The complementary solution is

$$y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$$

and so the complete solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{19}{4} - \frac{3}{2}x + \frac{1}{2}x^2$$

Now let's consider the equation

$$y'' + 3y' + 2y = 3e^{2x}$$

Again, by inspection, you can see that $y_p(x) = \alpha e^{2x}$. Substituting this into the differential equation gives

$$4\alpha e^{2x} + 6\alpha e^{2x} + 2\alpha e^{2x} = 3e^{2x}$$

or that $\alpha = 1/4$. Thus, we see that a particular solution is $y_p(x) = e^{2x}/4$, as you can verify by direct substitution.

Example 2:
Find a particular solution of

$$y'' + 3y' + 2y = 12xe^{2x}$$

SOLUTION: You will find that $y_p(x) = \alpha e^{2x}$ will not work because we need terms of the form xe^{2x} on the left side of the above equation. Let's try $y_p(x)$ of the form

$$y_p(x) = (\alpha + \beta x)e^{2x}$$

Substituting this into the differential equation gives

$$(4\alpha + 4\beta)e^{2x} + 4\beta xe^{2x} + (6\alpha + 3\beta)e^{2x} + 6\beta xe^{2x} + 2\alpha e^{2x} + 2\beta xe^{2x}$$
$$= 12xe^{2x}$$

Equating coefficients of e^{2x} and xe^{2x} on the two sides gives $\alpha = -7/12$ and $\beta = 1$, or

$$y_p(x) = \left(x - \frac{7}{12}\right)e^{2x}$$

as you can verify by direct substitution.

Generally, if $f(x)$ is a polynomial of degree n times an exponential, then $y_p(x)$ will be of the same form. (See Table 11.1.)

Let's look at another case, and find the particular solution of

$$y'' + 3y' + 2y = 10 \sin x \tag{9}$$

Notice that if we assume that $y_p(x) = \sin x$, then we'll generate terms involving $\cos x$ when we substitute it into Equation 9. So let's try

$$y_p(x) = \alpha \cos x + \beta \sin x$$

Substituting this guess into Equation 9 gives

$$(\alpha + 3\beta) \cos x + (\beta - 3\alpha) \sin x = 10 \sin x$$

Equating coefficients of like terms gives

$$y_p(x) = \sin x - 3 \cos x$$

as you can verify by direct substitution.

Example 3:
Find a particular solution of

$$y'' + 3y' + 2y = 10x \cos x$$

SOLUTION: Using Example 2 as a guide, we'll try

$$y_p(x) = (\alpha + \beta x) \cos x + (\gamma + \delta x) \sin x$$

Substituting this into the differential equation gives

$$(\alpha + 3\beta + 3\gamma + 2\delta) \cos x + (\gamma - 3\alpha + 3\delta - 2\beta) \sin x + (\beta + 3\delta)x \cos x$$
$$+ (\delta - 3\beta)x \sin x = 10x \cos x$$

from which we find $\alpha = 6/15$, $\beta = 1$, $\gamma = -17/15$, and $\delta = 3$. Therefore,

$$y_p(x) = \left(\frac{6}{5} + x\right) \cos x + \left(3x - \frac{17}{5}\right) \sin x$$

Let's look at one last case, and find the particular solution of

$$y'' + 3y' + 2y = \tan x \tag{10}$$

Certainly assuming that $y_p(x) = \alpha \tan x$ will not work because this will yield $2\alpha \sec^2 x \tan x + 3\alpha \sec^2 x + 2\alpha \tan x$ on the left side of Equation 10. If we try $y_p(x) = \alpha \sec^2 x \tan x + \beta \sec^2 x + \gamma \tan x$, then it gets even worse. The problem here is that $f(x) = \tan x$ generates more and more terms as it is repeatedly differentiated. The limitations on $f(x)$ that we alluded to in the beginning of this section is that repeated differentiation of $f(x)$ must yield only a finite number of terms. All the successful cases that we tried above are of this nature. Table 11.1 summarizes the forms of $f(x)$ and of $y_p(x)$ for which the method of undetermined coefficients is successful. Note that the last entry in Table 11.1 includes all the others as special cases.

Before we apply the method of undetermined coefficients to some mechanical and electrical systems, we must discuss an important modification that is required if $f(x)$ happens to be a solution to the homogeneous equation. Consider the equation

$$y'' + 3y' + 2y = 2e^{-x} \tag{11}$$

If we assume that $y_p(x) = \alpha e^{-x}$, then we obtain

$$\alpha e^{-x} - 3\alpha e^{-x} + 2\alpha e^{-x} = 0 \neq 2e^{-x}$$

The problem here is that αe^{-x} is a solution to the homogeneous equation. We need $y_p(x)$ to be of the form such that its derivatives yield e^{-x} along with some other terms. Let's try $y_p(x) = \alpha x e^{-x}$. Substituting this guess into Equation 11 gives

Table 11.1
Forms of $y_p(x)$ to use for various forms of $f(x)$.

$f(x)$	$y_p(x)$
$c_0 + c_1x + \cdots + c_nx^n$	$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$
$(c_0 + c_1x + \cdots + c_nx^n)e^{rx}$	$(\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n)e^{rx}$
$(c_0 + c_1x + \cdots + c_nx^n)\sin\omega x$ or $(c_0 + c_1x + \cdots + c_nx^n)\cos\omega x$	$(\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n)\cos\omega x + (\beta_0 + \beta_1 x + \cdots + \beta_n x^n)\sin\omega x$
$(c_0 + c_1x + \cdots + c_nx^n)e^{rx}\sin\omega x$ or $(c_0 + c_1x + \cdots + c_nx^n)e^{rx}\cos\omega x$	$(\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n)e^{rx}\cos\omega x + (\beta_0 + \beta_1 x + \cdots + \beta_n x^n)e^{rx}\sin\omega x$

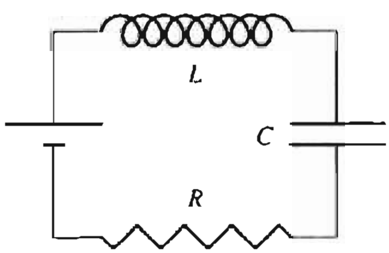

Figure 11.10
An electrical circuit containing a resistance, R, an inductance, L, and a capacitance, C.

$\alpha = 2$, so we see that $2xe^{-x}$ is a particular solution of Equation 11. Generally, if a term of the assumed form for $y_p(x)$ in Table 11.1 is the same as a term in the complementary solution, then all the terms in the entry in Table 11.1 must be multiplied by x, and sometimes even a higher power of x. We're going to use this result when we discuss an oscillator in resonance with no damping (see Equation 24).

The method of undetermined coefficients is applicable to a number of problems involving mechanical oscillators and electrical circuits. We'll use an RLC circuit as our primary example (Figure 11.10). We can model an electrical circuit as having components that resist an electric current (a resistance, R), a change in the current (an inductance, L), or a change in voltage (a capacitance, C) (Figure 11.10). The voltage drop across each component is given by $V_R = iR$, $V_l = L\,di/dt$, and $V_C = \int i(t)\,dt/C$, where $i(t)$ is the current. According to Kircholf's law, the driving voltage of the circuit is equal to the sum of the voltage drops across each component, and so we write

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int^t i(t')\,dt' = V$$

Differentiating with respect to time gives

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dV}{dt} \tag{12}$$

Before we find a particular solution of Equation 12, let's investigate the properties of the complementary solutions, or the solutions of

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0 \tag{13}$$

The auxiliary equation associated with Equation 13 is $L\alpha^2 + R\alpha + \dfrac{1}{C} = 0$, whose solutions are

$$\alpha = \frac{-R \pm (R^2 - 4L/C)^{1/2}}{2L} \tag{14}$$

If $R^2 > 4L/C$, both values of α are real and the solution to Equation 13 is of the form

$$i(t) = c_1 e^{-(a+b)t} + c_2 e^{-(a-b)t} \tag{15}$$

where $a = R/2L$ and $b = (R^2 - 4L/C)^{1/2}/2L = R(1 - 4L/R^2C)^{1/2}/2L$. Figure 11.11 shows $i(t)$ plotted against $Rt/2L$ for $4L/R^2C = 1/2$ for various values of c_1 and c_2. The graphs cut the horizontal axis one time at most and correspond to *overdamping*. Figure 11.12 shows $i(t)$ for the *critically damped* case, where $R^2 = 4L/C$ and $i(t) = (c_1 + c_2 t)e^{-Rt/2L}$. The curves are similar to those of overdamping.

If $R^2 < 4L/C$, then the values of α in Equation 14 are complex conjugates, which we write as $-a \pm i\omega_0$, where $\omega_0^2 = (4L/C - R^2)/2L$. The solution to Equation 13 in this case is given by

$$i(t) = e^{-Rt/2L}(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t) \tag{16}$$

Equation 16 corresponds to *underdamping*, as shown in Figure 11.13. Thus, we see that RLC circuits provide examples of overdamping, critical damping, and underdamping, just as we saw in Section 3 for mechanical oscillators.

Now let's go back to Equation 12, the nonhomogeneous equation. Assume that $V(t) = E_0 \cos \omega t$, so that Equation 12 reads

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = -E_0\omega \sin \omega t \tag{17}$$

We'll consider the case where $R = 0$ first and then include the effect of resistance later. Equation 17 with $R = 0$ can be written as

$$\frac{d^2 i}{dt^2} + \omega_0^2 i = -\frac{E_0\omega}{L}\omega \sin \omega t \tag{18}$$

where $\omega_0^2 = 1/LC$. Equation 18 corresponds to a harmonically driven harmonic oscillator with no resistance.

Let's consider the case $\omega \neq \omega_0$ first. If $\omega \neq \omega_0$, Table 11.1 tells us to use $i_p(t) = \alpha \cos \omega t + \beta \sin \omega t$, in which case we easily find that $\alpha = 0$ and $\beta = -E_0\omega/L(\omega_0^2 - \omega^2)$, or

$$i_p(t) = -\frac{E_0\omega}{L(\omega_0^2 - \omega^2)} \sin \omega t \qquad \omega_0 \neq \omega$$

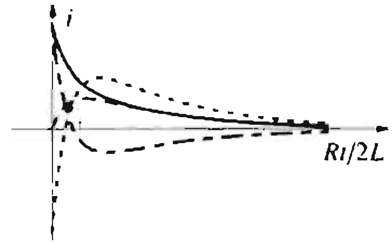

Figure 11.11
Plots of $i(t)$ given by Equation 15 against $Rt/2L$ for $4L/R^2C = 1/2$ for various values of c_1 and c_2. The behavior in this case is called overdamping.

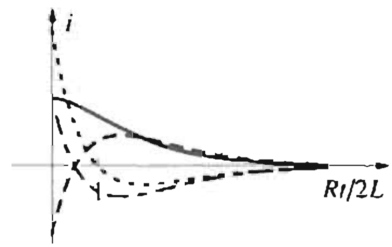

Figure 11.12
Plots of $i(t)$ given by Equation 15 against $Rt/2L$ for $4L/R^2C = 1.0$ for various values of c_1 and c_2. The behavior in this case is called critical damping.

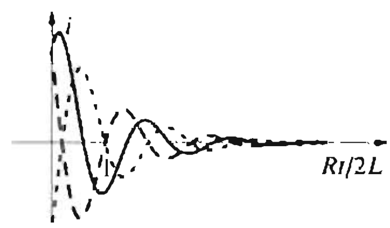

Figure 11.13
Plots of $i(t)$ given by Equation 16 against $Rt/2L$ for $L\omega_0/R = 2.0$ for various values of c_1 and c_2. The behavior in this case is called underdamping.

The complete solution is the sum of $i_c(t)$ and $i_p(t)$, or

$$i(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t - \frac{E_0 \omega}{L(\omega_0^2 - \omega^2)} \sin \omega t \qquad \omega_0 \neq \omega \tag{19}$$

We can use Equation 19 to illustrate the phenomenon of beats. Suppose initially that the circuit is in a quiescent state, with $i = 0$ and $di/dt = 0$, and then the driving voltage is imposed. In this case

$$c_1 = 0 \qquad \text{and} \qquad c_2 = \frac{E_0 \omega^2}{L\omega_0(\omega_0^2 - \omega^2)}$$

and so

$$i(t) = \frac{E_0 \omega}{L\omega_0(\omega_0^2 - \omega^2)}(\omega \sin \omega_0 t - \omega_0 \sin \omega t) \qquad \omega_0 \neq \omega \tag{20}$$

Now if we let $\omega \approx \omega_0$ and write $\omega = \omega_0 + \epsilon$, where ϵ is small, then Equation 20 becomes

$$i(t) \approx -\frac{E_0}{2L\epsilon}\{\sin \omega_0 t - \sin(\omega_0 + \epsilon)t\} \tag{21}$$

We can use the trigonometric formula for $\sin \alpha - \sin \beta$,

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

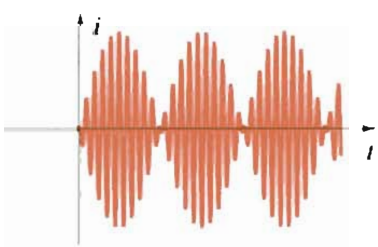

Figure 11.14
Equation 22 plotted against t for $\epsilon = 0.10$ and $\omega_0 = 1.0$, illustrating what is called *amplitude-modulation*.

to write Equation 21 as

$$i(t) \approx \frac{E_0}{L\epsilon} \sin \frac{\epsilon t}{2} \cdot \cos \omega_0 t \tag{22}$$

Equation 22 is plotted in Figure 11.14. Because ϵ is small, the period of $\sin(\epsilon t/2)$ is large, and $x(t)$ oscillates with a frequency ω_0 (wavelength $2\pi/\omega_0$) with an amplitude that oscillates with a frequency $\epsilon/2$ (wavelength $4\pi/\epsilon$). Equation 22 illustrates an example of the phenomenon of *beats*, which occurs when the driving force is close to the natural frequency of the system, or when a system is driven by two nearly equal frequencies. Figure 11.14 illustrates what is called an *amplitude-modulated wave*, or *amplitude modulation* (the "AM" on a radio dial).

If $\omega_0 = \omega$ in Equation 18, then we must assume that $i_p(t)$ is of the form

$$(\alpha + \beta t) \cos \omega_0 t + (\gamma + \delta t) \sin \omega_0 t \tag{23}$$

When Equation 23 is substituted into Equation 18, we find that $i_p(t) = -A(t \cos \omega_0 t)/2\omega_0$, or that

$$i(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{E_0}{2L} t \cos \omega_0 t \tag{24}$$

When $\omega = \omega_0$, the system is said to be in *resonance*. In this case, the displacement increases with time until the circuit undergoes failure (Figure 11.15). The frequency $\omega_0 = \sqrt{1/LC}$ is called *resonant angular frequency*.

Let's consider the more realistic case in which there is a resistance. The differential equation to solve is

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = -E_0\omega \sin \omega t \tag{25}$$

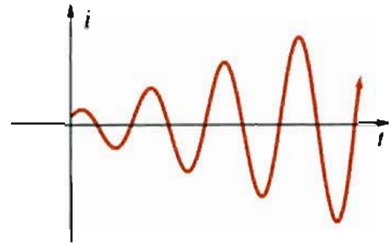

Figure 11.15
An illustration of resonance in an LC circuit. The frequency ω_0 is called the resonance frequency.

The particular solution to this equation is (Problem 10)

$$i_p(t) = \frac{RE_0}{Z^2}\cos \omega t + \frac{XE_0}{Z^2}\sin \omega t \tag{26}$$

where

$$X = \omega L - \frac{1}{\omega C} \tag{27}$$

is called the *reactance* of the circuit and

$$Z = (R^2 + X^2)^{1/2} \tag{28}$$

is called the *impedance*. We can write Equation 26 as (Problem 11)

$$i_p(t) = \frac{E_0}{Z}\cos(\omega t + \phi) \tag{29}$$

where the phase angle ϕ is given by

$$\phi = \tan^{-1}\left(-\frac{X}{R}\right) = -\tan^{-1}\left(\frac{X}{R}\right) \tag{30}$$

The complete solution is the sum of $i_c(t)$ (Equation 16) and $i_p(t)$ (Equation 29). Because of the factor $e^{-\gamma t/2}$ in $i_c(t)$, this part of the complete solution dies out for large values of time, and the solution is then given by Equation 29. The solution $i_c(t)$ is called the *transient solution* and $i_p(t)$ is called the *steady-state solution*. Thus, after the steady state has been reached, the system oscillates with the frequency of the driving force, albeit with a phase factor ϕ, whose magnitude depends upon the resistance. The amplitude of the oscillation is given by

$$a = \frac{E_0}{Z} = \frac{E_0}{\left[R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2\right]^{1/2}} \tag{31}$$

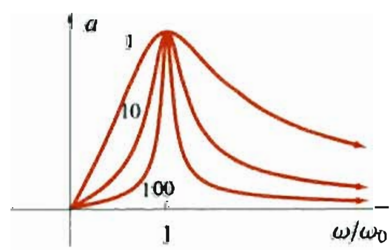

Figure 11.16
The amplitude of the steady-state current in an RLC circuit plotted against ω/ω_0 for various values of $L^2\omega_0^2/R^2$, where $\omega_0^2 = 1/LC$.

The ratio Ra/E_0 is plotted against ω/ω_0, where $\omega_0 = (1/LC)^{1/2}$, for various values of $L^2\omega_0^2/R^2$ (Problem 12) in Figure 11.16. The value of ω that produces the greatest magnitude is called the *resonance frequency*. Note that the amplitude remains finite, unlike in Figure 11.15, where R is equal to zero. Typically, the experimental quantity that is observed is the absorption of energy from electromagnetic radiation by an atomic oscillator, the resonance behavior of a cavity in a laser, or the amplification of a signal in an electric circuit, among many others.

Example 4:
It's possible to determine the steady-state solution to Equation 25 directly. We know physically that at steady state the circuit sustains oscillations with a frequency equal to the driving frequency but with a phase angle ϕ. Therefore, substitute $i(t) = a\cos(\omega t + \phi)$ into Equation 25 and determine a and ϕ.

SOLUTION: Substituting $i(t) = a\cos(\omega t + \phi)$ into Equation 25 gives

$$-L\omega^2 a\cos(\omega t + \phi) - R\omega a\sin(\omega t + \phi) + \frac{a}{C}\cos(\omega t + \phi) = -E_0\omega\sin\omega t$$

Now use $\sin(\omega t + \phi) = \sin\omega t\cos\phi + \cos\omega t\sin\phi$ and $\cos(\omega t + \phi) = \cos\omega t\cos\phi - \sin\omega t\sin\phi$ to get

$$\left(-L\omega^2 a\cos\phi - R\omega a\sin\phi + \frac{a}{C}\cos\phi\right)\cos\omega t$$
$$+\left(L\omega^2 a\sin\phi - R\omega a\cos\phi - \frac{a}{C}\sin\phi\right)\sin\omega t = -E_0\omega\sin\omega t$$

from which we find

$$-a\omega X\cos\phi - R\omega a\sin\phi = 0$$

and

$$a\omega X\sin\phi - R\omega a\cos\phi = -E_0\omega\sin\omega t$$

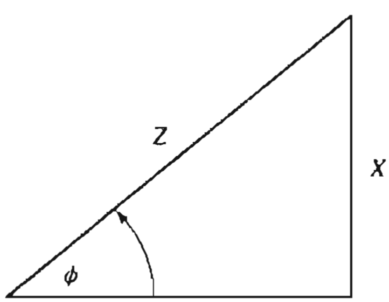

Figure 11.17
A geometric aid to the determination of Equation 32.

The first equation gives $\phi = \tan^{-1}(-X/R) = -\tan^{-1}(X/R)$. We can determine $\cos\phi$ and $\sin\phi$ from this expression for $\tan\phi$ by using Figure 11.17, where we construct a right triangle such that $\tan\phi = X/R$. The hypotenuse is equal to $Z = (R^2 + X^2)^{1/2}$, and so

$$\cos(-\phi) = \frac{R}{Z} \qquad \text{and} \qquad \sin(-\phi) = -\sin\phi = \frac{X}{Z} \tag{32}$$

Substituting into the second of the above two equations gives $a = E_0/Z$, in agreement with Equation 29.

There is actually an easier way to do Example 4. This method uses complex exponentials rather than sines and cosines. Its efficacy results from the fact that all derivatives of $e^{i\omega t}$ produce $e^{i\omega t}$, whereas derivatives of sines and cosines alternate between the two. Let $V(t) = E_0 e^{i\omega t}$, so that $\text{Re}\{V\} = E_0\cos\omega t$. The response of the RLC circuit to $V(t)$ can be expressed as $i(t) = Ie^{i\omega(t+\phi)}$, where I and ϕ are to be determined. The response to the actual voltage, Re $\{V(t)\}$, is Re $\{i(t)\}$. We write $i(t)$ in the notation

$$i(t) = Ie^{i\phi}e^{i\omega t} = \mathbf{I}\, e^{i\omega t} \tag{33}$$

where $\mathbf{I} = Ie^{i\phi}$ is called a *phasor*. Solving Equation 25 is equivalent to determining $\mathbf{I}$. Substitute $V(t) = E_0e^{i\omega t}$ and Equation 33 into Equation 12 to obtain

$$-L\omega^2\mathbf{I} + i\omega R\mathbf{I} + \frac{1}{C}\mathbf{I} = i\omega E_0$$

Solving for $\mathbf{I}$ gives

$$\mathbf{I} = \frac{iE_0}{iR - X} = \frac{E_0}{R + iX} = \frac{E_0}{Z^2}(R - iX)$$

The magnitude of $\mathbf{I}$ is E_0/Z and its argument is $\phi = \tan^{-1}(-X/R)$, so that the real part of $\mathbf{I}$ is

$$i(t) = \frac{E_0}{Z}\cos[\omega t - \tan^{-1}(X/R)]$$

in agreement with Equation 29 and Example 4. The use of phasors is readily extended to networks and essentially reduces calculations involving a-c networks to the algebra of complex numbers. Electrical engineers have taken such calculations to a fine art.

We'll conclude this section by briefly presenting an alternative method for solving nonhomogeneous linear differential equations. This method, called the *method of variation of parameters*, is not limited to linear differential equations with constant coefficients nor must the repeated differentiation of $f(x)$ yield a finite number of terms. We shall illustrate the method of variation of parameters with the second-order equation

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x) \tag{34}$$

Suppose we know that the general solution to the homogeneous equation is

$$y(x) = c_1y_1(x) + c_2y_2(x) \tag{35}$$

The method of variation of parameters assumes that the solution to Equation 34 is of the form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{36}$$

Recall that we made similar assumptions when we solved a linear first-order equation in Section 2 and when we introduced the method of reduction of order in the previous section. Substitution of Equation 36 into Equation 34 yields (Problem 25)

$$\begin{aligned} &u_1(a_2y_1''+a_1y_1'+a_0y_1)+u_2(a_2y_2''+a_1y_2'+a_0y_2)\\ &\quad+\ a_2(u_1'y_1'+u_2'y_2')+a_2\frac{d}{dx}(u_1'y_1+u_2'y_2)+a_1(u_1'y_1+u_2'y_2)\\ &\quad=f(x) \end{aligned} \tag{37}$$

Because y_1 and y_2 are solutions to the homogeneous equation, the first two terms in Equation 37 vanish. Equation 37 now becomes

$$a_2(u_1'y_1'+u_2'y_2')+a_2\frac{d}{dx}(u_1'y_1+u_2'y_2)+a_1(u_1'y_1+u_2'y_2)=f(x) \tag{38}$$

Furthermore, because we have two functions, $u_1(x)$ and $u_2(x)$, to be determined, we need to impose two conditions on them. The first is that Equation 36 must satisfy Equation 34, which leads to Equation 37 in the first place. The second is clearly suggested by looking at Equation 38, which simplifies greatly if we say that

$$u_1'y_1+u_2'y_2=0 \tag{39}$$

because then the second and third terms in Equation 38 vanish, and Equation 38 becomes simply

$$a_2(u_1'y_1'+u_2'y_2')=f(x) \tag{40}$$

Equations 39 and 40 constitute two simultaneous equations for u_1' and u_2'. We can solve these equations using Cramer's rule to get

$$u_1'=-\frac{y_2f(x)}{W(y_1,y_2)a_2(x)} \tag{41}$$

and

$$u_2'=\frac{y_1f(x)}{W(y_1,y_2)a_2(x)} \tag{42}$$

where $W(y_1, y_2)$ is the Wronskian determinant of y_1 and y_2. Because y_1 and y_2 are linearly independent, $W \neq 0$ in the interval in which $a_2(x)$, $a_1(x)$, and $a_0(x)$ are continuous. Integrating Equations 41 and 42 gives us $u_1(x)$ and $u_2(x)$, and therefore a particular solution to Equation 34.

We'll illustrate the method of variation of parameters with two Examples.

Example 5:
Find the general solution of

$$y''(x)+4y(x)=\tan 2x \qquad 0\le x<\pi/4$$

SOLUTION: The solution to the homogeneous equation is

$$y_c(x) = c_1 \sin 2x + c_2 \cos 2x$$

and $W(y_1, y_2) = -2$. Using Equations 41 and 42 with $y_1(x) = \sin 2x$ and $y_2 = \cos 2x$ gives us

$$u_1'(x) = \frac{1}{2} \cos 2x \tan 2x = \frac{1}{2} \sin 2x$$

$$u_2'(x) = -\frac{1}{2} \sin 2x \tan 2x$$

So

$$u_1(x) = -\frac{1}{4} \cos 2x$$

$$u_2(x) = \frac{1}{4} \sin 2x - \frac{1}{4} \ln(\sec 2x + \tan 2x)$$

and

$$y_p(x) = -\frac{1}{4} \cos 2x \ln(\sec 2x + \tan 2x) \qquad 0 \leq x < \pi/4$$

The general solution is $y(x) = y_c(x) + y_p(x)$. Why do you think that there is a restriction that $x < \pi/4$?

Example 6:
Find the general solution of

$$y'' + 2y' + y = 2x^{-2}e^{-x}$$

SOLUTION: The complementary solution is

$$y_c(x) = c_1 e^{-x} + c_2 x e^{-x}$$

So, using Equations 41 and 42 with $y_1 = e^{-x}$ and $y_2 = xe^{-x}$, $W(y_1, y_2) = e^{-2x}$, and

$$u_1(x) = -\int \frac{(xe^{-x})(2x^{-2}e^{-x})}{e^{-2x}} dx = -2 \ln x$$

$$u_2(x) = \int \frac{(e^{-x})(2x^{-2}e^{-x})}{e^{-2x}} dx = -\frac{2}{x}$$

The particular solution that we obtain from this is

$$y_p(x) = -2e^{-x} \ln x - 2e^{-x}$$

The general solution is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} - 2e^{-x} \ln x$$

Notice that neither Example can be handled by the method of undetermined coefficients.

11.4 Problems

1. Explain how both $\frac{3}{4}+\frac{1}{2}x$ and $e^x+\frac{3}{4}+\frac{1}{2}x$ can be particular solutions of $y''-3y'+2y=x$.
2. Given that $x-1$ is a solution of $y''+3y'+2y=2x+1$, find the general solution.
3. Given that $x+2$, $x+\sin x$, and $x+1-\sin x$ are solutions to a certain nonhomogeneous linear second-order differential equation, find the general solution.
4. Use the method of undetermined coefficients to find a particular solution of

   (a) $y''-2y'+2y=xe^x$ (b) $y''-y=x^2$
5. Use the method of undetermined coefficients to find a particular solution of

   (a) $y''+y=x+e^{-x}$ (b) $y''-y=2x^3-x$
6. Use the method of undetermined coefficients to find a particular solution of

   (a) $y''+y=e^x$ (b) $y''-3y'+2y=x^3$
7. Use the method of undetermined coefficients to find a particular solution of

   (a) $y''(x)+y(x)=\sin x$ (b) $y''(x)-y(x)=e^x$
8. Find the general solution of $y''+y'-6y=x$.
9. Find the general solution of $y''+4y=6\cos 3x+20\sin 3x$.
10. Show that Equation 26 is a particular solution of Equation 25.
11. Show that Equation 26 can be written as Equation 29.
12. Show that Equation 31 can be written in the form
$$\frac{Ra}{E_0}=\frac{1}{\left[1+\dfrac{L^2\omega_0^2}{R^2}\left(\dfrac{\omega}{\omega_0}-\dfrac{\omega_0}{\omega}\right)^2\right]^{1/2}},\text{ where }\omega_0=1/LC.$$

The next ten problems deal with various mechanical systems.

13. Consider a frictionless harmonic oscillator (with $m=1$) driven by an external force $f(t)=A\sin\omega t$, so that $\frac{d^2x}{dt^2}+\omega_0^2x=A\sin\omega t$. Show that the particular solution for $\omega\neq\omega_0$ is $x_p(t)=\frac{A}{\omega_0^2-\omega^2}\sin\omega t$.
14. Suppose that the oscillator in the previous problem is initially in its quiescent state ($x(0)=0$, $\dot{x}(0)=0$) and then the driving force $f(t)=A\sin\omega t$ is imposed. Show that the resulting complete solution is $x(t)=\frac{A}{\omega_0(\omega_0^2-\omega^2)}(\omega_0\sin\omega t-\omega\sin\omega_0 t)$.
15. Use the result of the previous problem to illustrate the phenomenon of beats. *Hint*: Let $\omega\approx\omega_0$ and use the trigonometric relation $\sin\alpha-\sin\beta=2[\cos(\alpha+\beta)/2][\sin(\alpha-\beta)/2]$.

16. Show that the solution to the equation in Problem 13 for $\omega = \omega_0$ is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega_0 t - \frac{A}{2\omega_0} t \cos \omega_0 t$$

Use this result to illustrate the phenomenon of resonance.

17. Consider a driven harmonic oscillator with a damping term $\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = A \sin \Omega t$, where we have set $m = 1$ for convenience. Show that the general solution is

$$x(t) = e^{-\gamma t/2}(c_1 \cos \omega t + c_2 \sin \omega t) - \frac{\gamma \Omega A \cos \Omega t + (\Omega^2 - \omega_0^2) A \sin \omega t}{\gamma^2 \Omega^2 + (\Omega^2 - \omega_0^2)^2}$$

where $\omega^2 = \omega_0^2 - \gamma^2/4 > 0$.

18. Show that the steady-state solution for the previous problem is $x_{ss}(t) = \frac{A \sin(\Omega t + \phi)}{[\gamma^2\Omega^2 + (\Omega^2 - \omega_0^2)^2]^{1/2}}$, where $\phi = \tan^{-1}[\gamma\Omega/(\Omega^2 - \omega_0^2)]$.

19. Use the result of the previous problem to illustrate resonance with a damping factor.

20. Start with the differential equation in Problem 17, multiply by dx/dt, and derive

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{\omega_0^2 x^2}{2}\right] + \gamma \left(\frac{dx}{dt}\right)^2 = \frac{dx}{dt} A \sin \Omega t$$

Give a physical interpretation to each of these terms. (Remember that we have set the mass = 1.) Show that

$$\left[\gamma \left(\frac{dx_{ss}}{dt}\right)^2\right]_{\text{ave}} = \frac{1}{(2\pi/\Omega)} \int_0^{2\pi/\Omega} \gamma \left(\frac{dx_{ss}}{dt}\right)^2 dt$$

$$= \frac{\gamma A^2 \Omega^2}{2[(\omega_0^2 - \Omega^2)^2 + \gamma^2 \Omega^2]}$$

in the steady state. Show that this quantity has a maximum at $\Omega = \omega_0$. What does this mean physically?

21. Referring to the previous problem, the average rate at which the driving force is doing work is given by $\left[\left(\frac{dx_{ss}}{dt}\right) A \sin \Omega t\right]_{\text{ave}}$. Show that this is equal to the average rate at which energy is dissipated.

22. A simple yet useful model for the absorption of electromagnetic radiation by an electron in an atom is to assume that the electron is bound elastically by the nucleus and so obeys the equation $m\frac{d^2x}{dt^2} + m\gamma \frac{dx}{dt} + m\omega_0^2 x = Ee \cos \omega t$, where m is the mass of the electron, e is its charge, ω_0 is the "natural" frequency of the elastically bound electron, E is the amplitude of the radiation, and γ is the *radiation-damping* term. The nature of the energy dissipated through γ is a continual reradiation of some of the absorbed energy. Show that the average rate of absorption of energy is given by (see the previous problem) $I(\omega) = \frac{e^2 E^2 \omega^2 \gamma}{2m[\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2]}$. Plot $I(\omega)$ against ω for small values of γ and discuss your result.

23. Show that a solution to $Ly = f_1(x) + f_2(x)$ is $y_{p1}(x) + y_{p2}(x)$, where $y_{p1}(x)$ is a solution to $Ly = f_1(x)$ and $y_{p2}(x)$ is a solution to $Ly = f_2(x)$.

24. Show that the particular solution to $y''(x) + y(x) = f(x)$ can be expressed as $y_p(x) = \int_0^x f(z)\sin(x - z)dz$.

25. Derive Equation 37.

26. Use the method of variation of parameters to solve $y''(x) - 2y'(x) = e^x \sin x$.

27. Use the method of variation of parameters to solve $y''(x) + y(x) = \csc x$.

28. Use the method of variation of parameters to solve $y''(x) + 2y'(x) + y(x) = \dfrac{e^{-x}}{x}$.

29. Use the method of variation of parameters to solve $y''(x) + 2y'(x) + y(x) = e^{-x}\ln x, x \neq 0$.

30. Use a CAS to verify your solutions to Problems 26 through 29.

11.5 Some Other Types of Higher-Order Differential Equations

Up to this point, almost all the higher-order differential equations that we have solved have had constant coefficients. There is no general procedure for solving higher-order linear differential equations with variable coefficients as there is for first-order equations (Section 2), but there are a few special cases that occur frequently enough in applications that it is worth discussing them here. We'll discuss three types of higher-order equations in this section:

1. The Euler, or Cauchy, equation which is of the form

$$a_n x^n y^{(n)}(x) + a_{n-1}x^{n-1}y^{n-1}(x) + \cdots + a_1 x y'(x) + a_0 y(x) = 0 \tag{1}$$

   where the a_js are constants. Note that the power of x in each term is the same as the order of the derivative.

2. Equations, even nonlinear ones, in which the dependent variable, y, is missing. An example of this case is

$$y''(x) = [y'(x)]^3 + y'(x) \tag{2}$$

3. Equations, even nonlinear ones, in which the independent variable, say x, is missing. An example is

$$yy'' - 2(y')^2 + 2y' = 0 \tag{3}$$

We'll learn how to solve each of these types of equations in turn.

Let's look at the Euler-Cauchy equation

$$x^2y''(x) + 4xy'(x) + 2y(x) = 0 \tag{4}$$

From the nature of each term, it is clear that the substitution $y(x) = x^m$ might yield a solution. Substituting $y(x) = x^m$ into Equation 4 gives

$$m(m-1) + 4m + 2 = 0$$

or $m = -2$ and -1. So, the general solution to Equation 4 is $y(x) = c_1x^{-2} + c_2x^{-1}$.

Generally, if the second-order Euler-Cauchy equation is of the form

$$a_2x^2y''(x) + a_1xy'(x) + a_0y(x) = 0 \tag{5}$$

then the equation for m (which we'll call the auxiliary equation) is

$$a_2m^2 + (a_1 - a_2)m + a_0 = 0 \tag{6}$$

If the two roots of Equation 6 are distinct, then we obtain two linearly independent solutions, as we did above.

The following example shows that Euler's equation can be transformed into one with constant coefficients.

Example 1:
Show that the substitution $x = e^z$ transforms Equation 5 into an equation with constant coefficients.

SOLUTION: We use the chain rule to write

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{dz}{dx}\frac{d^2y}{dz^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

Substituting these results into Equation 5 gives

$$a_2\frac{d^2y}{dz^2} + (a_1 - a_2)\frac{dy}{dz} + a_0y = 0 \tag{7}$$

Example 2:
Use the result of Example 1 to solve Equation 4.

SOLUTION: The transformed equation is

$$y''(z) + 3y'(z) + 2y(z) = 0$$

The auxiliary equation is $\alpha^2 + 3\alpha + 2 = 0$, which gives $\alpha = -1$ and $\alpha = -2$. The general solution is $c_1 e^{-2z} + c_2 e^{-z}$, or $c_1 x^{-2} + c_2 x^{-1}$, in agreement with our above result.

Problems 3 through 6 have you find solutions to Equation 5 when the roots of Equation 5 are repeated or when they are complex conjugates.

Now let's consider differential equations in which the dependent variable y is missing, such as Equation 2:

$$y''(x) = [y'(x)]^3 + y'(x)$$

In this case, we simply let $y' = p(x)$ and the equation reduces to a first-order equation in $p(x)$. Equation 2 becomes

$$\frac{dp}{dx} = p(p^2 + 1)$$

or

$$\ln \frac{p^2}{1 + p^2} = 2x + c_1$$

Solving for p gives

$$dy = \left(\frac{c_2 e^{2x}}{1 - c_2 e^{2x}} \right)^{1/2} dx$$

where $c_2 = e^{c_1}$. Let $c_2 e^{2x} = u$ and integrate

$$\begin{aligned} y(x) &= \sin^{-1}(u^{1/2}) + c_3 \\ &= \sin^{-1}(c_4 e^x) + c_3 \end{aligned}$$

where $c_4 = c_2^{1/2}$.

Example 3:

The differential equation for an ideal cable that is suspended between two horizontal points and hanging under its own weight is

$$\frac{d^2 y}{dx^2} = \alpha \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2}$$

where α is a constant that depends upon the tension in the cable and its weight. Solve this equation for $y(0) = 1/\alpha$ and $dy/dx = 0$ at $x = 0$.

SOLUTION: The dependent variable is missing, so we let $y' = p$ and write

$$\frac{dp}{dx} = \alpha(1 + p^2)^{1/2}$$

The solution to this equation is

$$\ln [p + (1 + p^2)^{1/2}] = \alpha x + c$$

or

$$p + (1 + p^2)^{1/2} = a e^{\alpha x}$$

The condition $p = 0$ when $x = 0$ gives us $a = 1$. Now solve for p to obtain

$$p = \frac{e^{2\alpha x} - 1}{2e^{\alpha x}} = \sinh \alpha x$$

Another integration gives

$$y(x) = \frac{1}{\alpha} \cosh \alpha x + c$$

If $y(0) = 1/\alpha$, then $c = 0$ and we have

$$y(x) = \frac{1}{\alpha} \cosh \alpha x$$

The curve is called a *catenary* and is plotted in Figure 11.18.

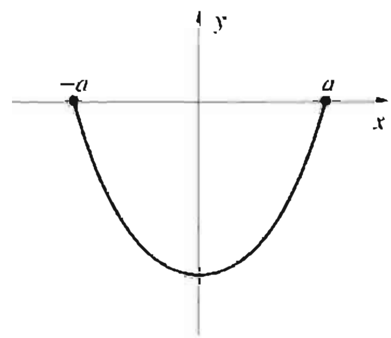

Figure 11.18
A catenary, $y(x) = \frac{1}{\alpha} \cosh \alpha x$, is the shape of an ideal cable suspended at both ends.

The next type of equations that we shall discuss are those in which the independent variable is missing. Equation 3 is an example, since it does not contain x. We solve these by a kind of reduction of order, where we think of y as the independent variable, and p as a function of y. If we let $y' = p$ and write y'' as $\frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$, then (Equation 3)

$$yy'' - 2(y')^2 + 2y' = 0$$

becomes

$$p\left(y\frac{dp}{dy} - 2p + 2\right) = 0$$

Certainly, $p = 0$, or $y(x) = c$, is a solution to Equation 3. Another solution is given by

$$\frac{dp}{p - 1} = 2\frac{dy}{y}$$

or $\ln(p - 1) = \ln y^2 + c_1$, or $p = 1 + c_2 y^2$ (where $c_1 = \ln c_2$). Thus, we have

$$\frac{dy}{dx} = 1 + c_2 y^2$$

which gives upon integration

$$\frac{1}{\sqrt{c_2}} \tan^{-1} \sqrt{c_2}\, y = x + c_3 \qquad c_2 > 0$$

or

$$y(x) = \frac{1}{\sqrt{c_2}} \tan(\sqrt{c_2}\, x + c_4) \qquad c_2 > 0$$

Example 4:
The equation

$$\frac{d^2\phi}{dx^2} = \sinh \phi$$

called the Poisson-Boltzmann equation, occurs frequently in colloidal science and biophysics. It's a fairly good approximation for the electrostatic potential ϕ at a distance x from a charge surface that is in contact with an electrolyte solution. Solve this equation under the condition that $\phi(x) \to 0$ (and $d\phi/dx \to 0$ as $x \to \infty$) and that $\phi(0) = \phi_0$.

SOLUTION: The equation is missing the independent variable x, so we let $d^2\phi/dx^2 = p\, dp/d\phi$ to get

$$p^2 = 2 \cosh \phi + c_1$$

or

$$\frac{d\phi}{dx} = \pm(2 \cosh \phi + c_1)^{1/2}$$

Using the fact that both $\phi \to 0$ and $d\phi/dx \to 0$ as $x \to \infty$, we find that $c_1 = -2$, or that

$$\frac{d\phi}{dx} = \pm(2 \cosh \phi - 2)^{1/2} = \pm 2 \sinh \frac{\phi}{2}$$

Integrate again, using $\phi(0) = \phi_0$, to obtain

$$\tanh \frac{\phi}{4} = \left(\tanh \frac{\phi_0}{4}\right) e^{-x}$$

where we chose the negative sign because $\phi \to 0$ as $x \to \infty$. This equation can be solved for $e^{\phi/2}$:

$$e^{\phi/2} = \frac{1 + \left(\tanh \frac{\phi_0}{4}\right) e^{-x}}{1 - \left(\tanh \frac{\phi_0}{4}\right) e^{-x}}$$

Figure 11.19 shows ϕ plotted against x for several values of ϕ_0.

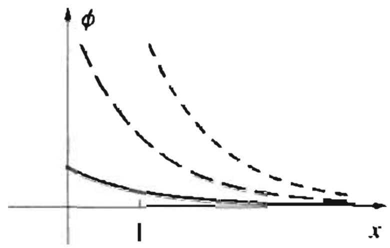

Figure 11.19
The potential calculated from the Poisson-Boltzmann equation (Example 4) plotted against x for $\phi_0 = 0.20$ (solid), 1.00 (long dashed), and 3.00 (short dashed).

Equations like the one in Example 4 occur often enough that it's worth seeing another way to solve them. Suppose we have the equation

$$\frac{d^2y}{dx^2} = f(y) \tag{8}$$

Multiply by dy/dx, use the relation

$$\frac{1}{2}\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$$

and then integrate both sides of Equation 8 to obtain

$$\frac{1}{2}\int \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 dx = \int f(y)\frac{dy}{dx}dx$$

or

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = \int f(y)dy + c \tag{9}$$

Of course, this is equivalent to what we did in Example 4, but in a slightly different procedure.

11.5 Problems

1. Solve the equation $x^2y''(x) - 2xy'(x) + 2y(x) = 0$.
2. Solve the equation $x^2y''(x) + 4xy'(x) - 4y(x) = 0$.
3. Suppose the roots of Equation 6 are repeated. In that case, we use reduction of order to find a second solution. Use this method to solve the equation $x^2y''(x) - 3xy'(x) + 4y(x) = 0$.
4. Show that if the two roots of Equation 6 are equal to m, then the two linearly independent solutions are x^m and $x^m \ln x$.
5. Suppose the roots of Equation 6 are complex conjugates of each other, as for the equation $x^2y''(x) + xy'(x) + y(x) = 0$. Show that the general solution to this equation is $y(x) = c_1x^i + c_2x^{-i}$. Now use Euler's identity to show that $x^{\pm i} = \cos(\ln x) \pm i\sin(\ln x)$, $x > 0$, and that the general solution to the above differential equation is $c_1\cos(\ln x) + c_2\sin(\ln x)$.
6. Use the method of the previous problem to solve $x^2y''(x) + 3xy'(x) + 5y(x) = 0$.

11.6 Systems of Linear Differential Equations

Systems of n simultaneous first-order linear differential equations such as

$$\begin{aligned} y_1' &= a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots + a_{1n}(x)y_n \\ y_2' &= a_{21}(x)y_1 + a_{22}(x)y_2 + \cdots + a_{2n}(x)y_n \\ &\vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\quad \ddots \qquad \vdots \\ y_n' &= a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots + a_{nn}(x)y_n \end{aligned} \tag{1}$$

where $\alpha \le x \le \beta$, occur quite frequently in physical applictions. We can write Equations 1 compactly in matrix notation:

$$\mathbf{y}' = \mathsf{A}(x)\mathbf{y} \qquad \alpha < x < \beta \tag{2}$$

where $\mathsf{A}(x)$ is an $n \times n$ matrix. We encountered and solved several homogeneous systems of differential equations in Section 10.3, where we discussed eigenvalue problems and the methods that we shall use in this section will be fairly similar. This section contains little new material, but will summarize the results for systems of linear equations in one place.

Because Equation 2 is linear, if $\mathbf{y}_1(x)$ and $\mathbf{y}_2(x)$ are solutions, then so is (Problem 1)

$$\mathbf{y}(x) = c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) \tag{3}$$

This result can be readily extended to any finite number of solutions. If A is an $n \times n$ matrix, then there are n linearly independent solutions to Equation 1. If $\mathbf{y}_1(x), \mathbf{y}_2(x), \ldots, \mathbf{y}_n(x)$ are n linearly independent solutions, then the general solution is given by

$$\mathbf{y}(x) = \sum_{j=1}^{n} c_j\mathbf{y}_j(x) \tag{4}$$

The n vectors are linearly independent if

$$c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) + \cdots + c_n\mathbf{y}_n(x) = \mathbf{0} \tag{5}$$

which implies that all the c_js are equal to zero. Otherwise, they are linearly dependent.

Equation 5 provides us with a convenient test for linear independence. Let the jth component of $\mathbf{y}_i(x)$ be denoted by $y_{ij}(x)$. Then Equation 5 can be expressed in matrix form as

$$\begin{pmatrix} c_1y_{11} + c_2y_{21} + \cdots + c_ny_{n1} \\ c_1y_{12} + c_2y_{22} + \cdots + c_ny_{n2} \\ \vdots \qquad\quad \vdots \qquad\quad \ddots \qquad \vdots \\ c_1y_{1n} + c_2y_{2n} + \cdots + c_ny_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{6}$$

If we define a matrix Y with elements y_{ij} and a column vector c by $(c_1, c_2, \ldots, c_n)^{\mathsf{T}}$, then Equation 6 can be written in matrix form as

$$\mathsf{Y}\mathbf{c} = \mathbf{0} \tag{7}$$

We learned in Chapter 9 that if $|\mathsf{Y}| \neq 0$, then Y^{-1} exists and the only solution to Equation 7 is $\mathbf{c} = \mathbf{0}$. Furthermore, if $|\mathsf{Y}| = 0$, then there is a nontrivial solution to Equation 7. Thus, we have the important result that the solutions $\mathbf{y}_1(x), \mathbf{y}_2(x), \ldots, \mathbf{y}_n(x)$ are linearly independent if and only if $|\mathsf{Y}| = 0$. Note that $|\mathsf{Y}|$ need be nonzero for any value of x in the interval (α, β) to obtain the result $\mathbf{c} = \mathbf{0}$, and so the condition $|\mathsf{Y}| \neq 0$ for any value of x implies that $|\mathsf{Y}| \neq 0$ for all values of x in (α, β).

Example 1:
Show that the following three vector functions are linearly independent on the entire real axis:

$$\mathbf{y}_1 = \begin{pmatrix} e^x \\ e^x \\ e^x \end{pmatrix}; \qquad \mathbf{y}_2 = \begin{pmatrix} e^x \\ -e^x \\ 2e^x \end{pmatrix}; \qquad \mathbf{y}_3 = \begin{pmatrix} e^{-x} \\ e^{-x} \\ 3e^{-x} \end{pmatrix}$$

SOLUTION: We need only to show that $|\mathsf{Y}| \neq 0$ for any value of x:

$$|\mathsf{Y}| = \begin{vmatrix} e^x & e^x & e^{-x} \\ e^x & -e^x & e^{-x} \\ e^x & 2e^x & 3e^{-x} \end{vmatrix} = -4e^x \neq 0$$

Up to this point, our statements apply even if the coefficient matrix in Equation 1 or Equation 2 depends upon x. For the remainder of this section, we shall assume that all the coefficients in Equation 1 are constants. Equation 2 becomes

$$\mathbf{y}' = \mathsf{A}\mathbf{y} \tag{8}$$

Being the solution to a set of first-order linear differential equations with constant coefficients, it is reasonable to expect $\mathbf{y}(x)$ to be of the form

$$\mathbf{y}(x) = \mathbf{v}e^{\lambda x} \tag{9}$$

where λ and $\mathbf{v}$ are to be determined. Substituting Equation 9 into Equation 8 gives the eigenvalue problem (after cancelling $e^{\lambda x}$ from both sides):

$$\mathsf{A}\mathbf{v} = \lambda\mathbf{v} \tag{10}$$

The eigenvalues λ are given by $|\mathsf{A} - \lambda\,\mathsf{I}| = 0$ and eigenvectors $\mathbf{v}$ are solutions to Equation 10.

Example 2:
Solve the equations

$$y_1' = y_1 + 2y_2$$
$$y_2' = 2y_1 + y_2$$

SOLUTION: The coefficient matrix of this system is

$$\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are $\lambda_1 = 3$, $\lambda_2 = -1$, $\mathbf{v}_1 = (1, 1)^{\mathsf{T}}$, and $\mathbf{v}_2 = (1, -1)^{\mathsf{T}}$. The general solution is

$$\begin{aligned} \mathbf{y}(x) &= c_1 e^{3x}\mathbf{v}_1 + c_2 e^{-x}\mathbf{v}_2 \\ &= c_1 e^{3x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

or

$$y_1(x) = c_1 e^{3x} + c_2 e^{-x}$$

and

$$y_2(x) = c_1 e^{3x} - c_2 e^{-x}$$

The eigenvalues in Example 2 are real and distinct, which is the simplest situation.

Now let's consider the case of complex eigenvalues. If the elements of A are real, any complex eigenvalues will occur as complex conjugate pairs, with corresponding complex conjugate pairs of eigenvectors. To see that this is so, start with the eigenvalue equation

$$(\mathsf{A} - \lambda\,\mathsf{I})\mathbf{v} = 0$$

and take its complex conjugate,

$$(\mathsf{A} - \lambda^*\,\mathsf{I})\mathbf{v}^* = 0$$

Thus, we see that λ, $\mathbf{v}$ and λ^*, $\mathbf{v}^*$ are eigenvalue-eigenvector pairs if the elements of A are real. The general solution to Equation 8 would include the terms

$$c_1\mathbf{v}e^{\lambda x} + c_2\mathbf{v}^* e^{\lambda^* x}$$

or, if $\lambda = a + ib$, the two terms would be

$$c_1\mathbf{v}e^{ax}e^{ibx} + c_2\mathbf{v}^* e^{ax}e^{-ibx}$$

We usually want to express our solutions in terms of real functions. We can do this using Euler's formula to write this part of the solution in terms of sines and cosines, but it is much easier algebraically to realize that if $\mathbf{v}e^{\lambda x}$ is a solution to Equation 8, then so are its real and imaginary parts separately, giving us two linearly independent real-valued solutions (Problem 12). The advantage to this is that we need to consider only one of the pair of complex conjugates.

Consider the pair of equations

$$\begin{aligned} y_1' &= 3y_1 + 5y_2 \\ y_2' &= -5y_1 + 3y_2 \end{aligned} \tag{11}$$

with $\mathbf{y}(0) = (0, 1)^{\mathsf{T}}$. The eigenvalues are $\lambda = 3 \pm 5i$ and the corresponding eigenvectors are $\mathbf{v}_1 = (-i, 1)^{\mathsf{T}}$ and $\mathbf{v}_2 = (i, 1)^{\mathsf{T}}$. (Note that $\mathbf{v}_1 = \mathbf{v}_2^*$.)

Therefore,

$$\mathbf{y}_+(x) = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{3x} e^{i5x}$$

is a solution to Equation 9. We use $e^{i5x} = \cos 5x + i \sin 5x$ to write $\mathbf{y}(x)$ as

$$\mathbf{y}_+(x) = \begin{pmatrix} \sin 5x \\ \cos 5x \end{pmatrix} e^{3x} + i \begin{pmatrix} -\cos 5x \\ \sin 5x \end{pmatrix} e^{3x}$$

Now the real and imaginary parts of $\mathbf{y}_+(x)$ are linearly independent solutions, so the general solution to Equation 11 expressed in terms of real functions is

$$\mathbf{y}(x) = c_1 \begin{pmatrix} \sin 5x \\ \cos 5x \end{pmatrix} e^{3x} + c_2 \begin{pmatrix} -\cos 5x \\ \sin 5x \end{pmatrix} e^{3x} \tag{12}$$

The initial conditions $\mathbf{y}(0) = (0, 1)^{\mathsf{T}}$ gives us $c_1 = 1$ and $c_2 = 0$, so the particular solution to Equation 9 is

$$\mathbf{y}(x) = \begin{pmatrix} \sin 5x \\ \cos 5x \end{pmatrix} e^{3x} \tag{13}$$

or

$$y_1(x) = e^{3x} \sin 5x \qquad \text{and} \qquad y_2(x) = e^{3x} \cos 5x$$

in component form. Thus, $y_1(x) = e^{3x} \sin 5x$ and $y_2(x) = e^{3x} \cos 5x$ are two linearly independent solutions to Equations 11.

Example 3:
Solve the equations

$$\begin{aligned} y_1' &= 4y_1 - 2y_2 \\ y_2' &= 5y_1 - 2y_2 \end{aligned}$$

with the initial conditions $\mathbf{y}(0) = (1, 2)^{\mathsf{T}}$.

SOLUTION: The eigenvalues are $\lambda = 1 \pm i$ and the corresponding eigenvectors are $(3 \pm i, 5)^T$. Choosing $\lambda = 1 + i$ and $\mathbf{v} = (3 + i, 5)^T$, we have

$$\mathbf{y}_+(x) = \begin{pmatrix} 3+i \\ 5 \end{pmatrix} e^{(1+i)x}$$

$$= \begin{pmatrix} 3\cos x - \sin x \\ 5\cos x \end{pmatrix} e^x + i \begin{pmatrix} \cos x + 3\sin x \\ 5\sin x \end{pmatrix} e^x$$

The general solution is a linear combination of the real and imaginary parts of $\mathbf{y}_+(x)$, so

$$\mathbf{y}(x) = c_1 \begin{pmatrix} 3\cos x - \sin x \\ 5\cos x \end{pmatrix} e^x + c_2 \begin{pmatrix} \cos x + 3\sin x \\ 5\sin x \end{pmatrix} e^x$$

The initial conditions give us $3c_1 + c_2 = 1$ and $5c_1 = 2$, or $c_1 = 2/5$ and $c_2 = -1/5$, so the particular solution is

$$\mathbf{y}(x) = \begin{pmatrix} \cos x - \sin x \\ 2\cos x - \sin x \end{pmatrix} e^x$$

or

$$y_1(x) = e^x \cos x - e^x \sin x$$

and

$$y_2(x) = 2e^x \cos x - e^x \sin x$$

in component form.

Example 4:
In Section 3, we wrote the equation describing a harmonic oscillator as the second-order equation

$$m\frac{d^2x}{dt^2} + kx = 0$$

In Chapter 13, it will be convenient to write this equation as two coupled first-order equations. Express this equation as coupled equations in matrix form for the momentum and the position of the oscillator. Find the general solution.

SOLUTION: Let $p = m\,dx/dt$ and $x = x$ to get the coupled equations in matrix form:

$$\begin{pmatrix} \dot{p} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & -k \\ 1/m & 0 \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are $\lambda = \pm i\omega_0$ and $(\pm ik/\omega_0, 1)^T$, where $\omega_0^2 = k/m$. Therefore,

$$\mathbf{u}_+ = \begin{pmatrix} \dfrac{ik}{\omega_0} \\ 1 \end{pmatrix} e^{i\omega_0 t}$$

is a solution corresponding to $\lambda = i\omega_0$. Using Euler's formula, we obtain

$$\mathbf{u}_+ = \begin{pmatrix} -\dfrac{k}{\omega_0} \sin \omega_0 t \\ \cos \omega_0 t \end{pmatrix} + i \begin{pmatrix} \dfrac{k}{\omega_0} \cos \omega_0 t \\ \sin \omega_0 t \end{pmatrix}$$

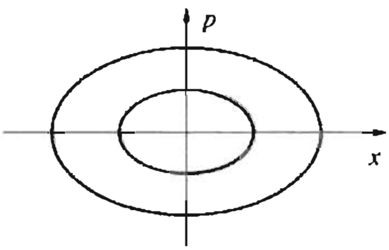

Figure 11.21
A parametric plot of the momentum $p(t)$ against the displacement $x(t)$ for a harmonic oscillator.

The real and imaginary parts of $\mathbf{u}_+$ constitute linearly independent solutions, so the general solution is given by

$$\begin{pmatrix} p \\ x \end{pmatrix} = c_1 \begin{pmatrix} -\dfrac{k}{\omega_0} \sin \omega_0 t \\ \cos \omega_0 t \end{pmatrix} + c_2 \begin{pmatrix} \dfrac{k}{\omega_0} \cos \omega_0 t \\ \sin \omega_0 t \end{pmatrix}$$

Notice that $\dot{p} = -kx$ and that $\dot{x} = p/m$. If we plot $p(t)$ against $x(t)$ parametrically, we obtain a family of ellipses, as shown in Figure 11.21. The curves (trajectories) in Figure 11.21 represent periodic motion.

11.6 Problems

1. Show that if $\mathbf{y}_1(x)$ and $\mathbf{y}_2(x)$ are solutions to Equation 2, then so is a linear combination of $\mathbf{y}_1(x)$ and $\mathbf{y}_2(x)$.

2. Test the following three vector functions for linear independence over the x axis:

$$\mathbf{y}_1 = \begin{pmatrix} e^x \\ -e^x \\ 2e^x \end{pmatrix} \qquad \mathbf{y}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{y}_3 = \begin{pmatrix} e^{-x} \\ e^{-x} \\ -e^{-x} \end{pmatrix}$$

3. Solve the system of equations

$$y_1' = 5y_1 + 4y_2$$
$$y_2' = -y_1$$

4. Solve the system of equations

$$y_1' = y_1 + y_2$$
$$y_2' = 4y_1 + y_2$$

5. Solve the system of equations

$$y_1' = 6y_1 + 8y_2$$
$$y_2' = -3y_1 - 4y_2$$

6. Solve the system of equations

$$y_1' = 3y_1 + 2y_2$$
$$y_2' = -y_1 + y_2$$

7. Solve the system of equations

$$y_1' = y_1 - 2y_2$$
$$y_2' = 2y_1 + y_2$$

8. Solve the system of equations

$$\mathbf{y}' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \mathbf{y}$$

9. Solve the system of equations

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 7 & -9 \\ 0 & 2 & -1 \end{pmatrix} \mathbf{y}$$

10. Solve the system of equations

$$\mathbf{y}' = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix} \mathbf{y}$$

11. Solve the system of equations

$$y_1' = 2y_1 + y_2 - y_3$$
$$y_2' = -4y_1 - 3y_2 - y_3$$
$$y_3' = 4y_1 + 4y_2 + 2y_3$$

12. Show that the real and imaginary parts of Equation 10 are separately solutions to Equations 9.

13. The rate equations for the chemical kinetic scheme $\mathrm{A} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} \mathrm{B} \xrightarrow{k_3} \mathrm{C}$ are

$$\frac{dA}{dt} = -k_1 A + k_2 B$$

$$\frac{dB}{dt} = k_1 A - (k_2 + k_3)B$$

$$\frac{dC}{dt} = k_3 B$$

Solve these equations for $k_1 = k_3 = 2$ and $k_2 = 1$ with the initial conditions $A(0) = A_0$, $B(0) = C(0) = 0$. Notice that the sum of these equations is $d(A + B + C)/dt = 0$. What does this mean physically?

14. Consider the two-mass, three-spring system shown in Figure 11.22.

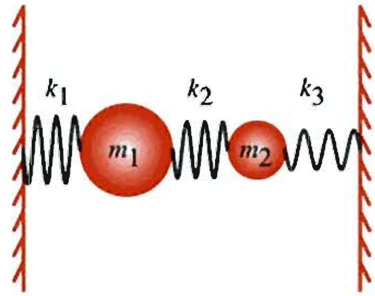

Figure 11.22
The two-mass, three-spring system referred to in Problem 14.

Assuming that the springs obey Hooke's law, show that the equations of motion are

$$m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2$$

$$m_2\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2$$

where x_1 and x_2 are the displacements of the masses from their equilibrium positions. Write these equations as four coupled first-order linear equations.

15. Write the third-order equation $y''' + a_2y'' + a_1y' + a_0y = 0$, where the a_js are constants, as three simultaneous first-order equations by letting $y = x_1$, $y' = x_1' = x_2$, $y'' = x_2' = x_3$, and $y''' = -a_2x_3 - a_1x_2 - a_0x_1$. Now show that the auxiliary equation associated with $y''' + a_2y'' + a_1y' + a_0y = 0$ is the same as the characteristic equation for the system of first-order equations. What does this tell you?

11.7 Two Invaluable Resources for Solutions to Differential Equations

The book *Ordinary Differential Equations and Their Solutions* by George M. Murphy (see the references at the end of the chapter) is to differential equations what a table of integrals is to integrals. The first half of the book reviews the methods that can be used to solve a great variety of differential equations, and the second half of the book lists over 200 pages of differential equations along with their solutions. A great feature of this list is that it is organized in such a way that any of the many differential equations included can be readily found. They are ordered as follows:

1. First Order and First Degree (751 equations)
2. First Order and Higher Degree (423 equations)
3. Second Order and Linear (596 equations)
4. Second Order and Nonlinear (196 equations)
5. Order Greater than Two and Linear (196 equations)
6. Order Greater than Two and Nonlinear (34 equations)

The tables include homogeneous as well as nonhomogeneous equations. Unfortunately, this 1960 book is out of print; nevertheless, it is such a great resource that it is worth checking to see if it is available in your library.

The second invaluable resource in the title of this short section is a Computer Algebra System such as Mathematica, Matlab, or Maple. Each of these programs can solve differential equations analytically (called symbolically). For example, the one-line command

```
DSolve [ y''[ x ] + 3 • y'[ x ] + 2 • y[ x ] == 12 • x • Exp[ 2 • x ], y[ x ], x ]
// Simplify
```

in Mathematica gives the solution

$$y(x) = e^{2x}\left(-\frac{7}{12} + x\right) + c_1 e^{-2x} + c_2 e^{-x}$$

to the nonhomogeneous differential equation in Example 2 of Section 4.

These programs can also yield analytic solutions to systems of equations. For example, the Mathematica command

```
DSolve [ {x'[ t ] == 4 • x[ t ] - y[ t ] + Exp [ -t ], y'[ t ] == 5 • x[ t ]
- 2 • y[ t ] + 2 • Exp[ -t ], x[ 0 ] == 0, y[ 0 ] == 0}, {x[ t ], y[ t ]}, t ]
```

gives the solution

$$x(t) = -\frac{3}{16}e^{-t} + \frac{3}{16}e^{3t} + \frac{1}{4}te^{-t}$$

$$y(t) = -\frac{3}{16}e^{-t} + \frac{3}{16}e^{3t} + \frac{5}{4}te^{-t}$$

to the two simultaneous differential equations

$$\dot{x} = 4x - y + e^{-t}$$

$$\dot{y} = 5x - 2y + 2e^{-t}$$

with $x(0) = 0$ and $y(0) = 0$.

The ability of these CAS to solve differential equations symbolically is somewhat limited, but you should be aware that these systems can also solve differential equations numerically. We're not going to discuss numerical methods for solving differential equations here, but they are discussed in most texts on differential equations. (See the references to Edwards and Penney and to Boyce and DiPrima at the end of the chapter.) For example, Mathematica is unable to provide a symbolic solution for

$$2y''(x) + y'(x) + 8y^3 = 0 \tag{1}$$

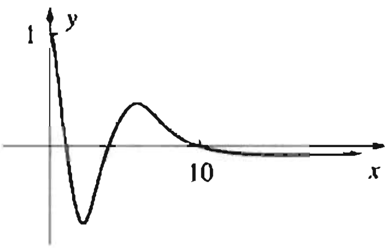

Figure 11.23
A plot of the numerical solution to Equation 1 with the initial conditions $y(0) = 1$ and $y'(0) = 0$.

but the command

```
NDSolve [ { 2 • y''[ x ] + y'[ x ] + 8 • y[ x ]^3 == 0, y[ 0 ] == 1,
y'[ 0 ] == 0}, y[ x ], {x, 0, 20 } ]
```

provides a numerical solution with the initial conditions $y(0) = 1$, $y'(0) = 0$ over the interval $0 \le x \le 20$, which is plotted in Figure 11.23.

Neither of these resources is a substitute for an understanding of how solutions are obtained, but they are great supplements.

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The Weierstrass Function: A Function That Is Everywhere Continuous, But Nowhere Differentiable.

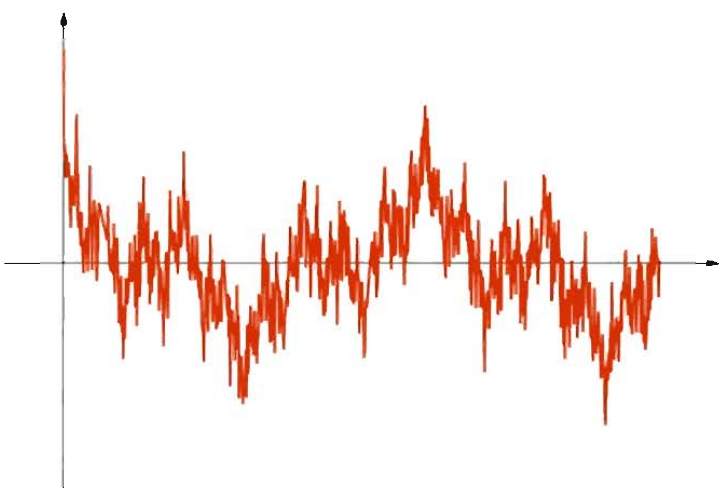
There are many functions that are continuous everywhere but not differentiable at certain points. One of the simplest examples is $f(x) = |x|$, which is continuous everywhere, but does not have a derivative at $x = 0$. In 1872, however, Weierstrass showed that the function

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$$

where $0 < b < 1$ and a is an odd positive integer, is continuous everywhere, but differentiable nowhere. We can use none other than the Weierstrass M test (Section 2.5) to show that $f(x)$ is uniformly convergent by taking $M_n = b^n$, in which case we have $|b^n \cos(a^n \pi x)| \leq b^n$. Because $f(x)$ is a uniformly convergent series of continuous functions, $f(x)$ itself must be continuous.

The series obtained from term-by-term differentiation of $f(x)$ diverges for $ab > 1$, which in itself does not prove that $f(x)$ is not differentiable, but does suggest caution. A careful analysis of $\{f(x+h) - f(x)\}/h$ shows that its magnitude diverges for $ab > 1$ for any value of x, and so $f(x)$ has a derivative nowhere.

What does such a function look like? We can't sum an infinite number of terms, but the figure below, which includes 1000 terms in the above summation for $a = 3$ and $b = 4/5$, gives you an idea.



CHAPTER 12

Series Solutions of Differential Equations

In the previous chapter, we learned how to solve any linear differential equation with constant coefficients and a few special types of differential equations whose coefficients are functions of x. Generally, however, there is no method that allows us to find an analytic solution to a differential equation of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_3(x)y(x) = 0$$

even if the a_j's are well-behaved functions of x, at least in a finite number of steps. There *is* a method, however, that does allow us to solve the above equation in terms of a power series in x. In its simplest application, we assume that $y(x)$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

then substitute it into the differential equation, and determine all the coefficients, $\{c_n\}$, in the power series. We can then use and manipulate the power series solution much as we would any simple analytic solution.

Surely, the coefficients a_j in the differential equation determine the efficacy of this series method. For example, the radius of convergence of the solution depends upon the radii of convergence of the series expansions of the coefficients $\{a_j(x)\}$. Furthermore, the behavior of the coefficients about the point $x = 0$ determine if such a power series exists. In most cases, the series has to be modified somewhat in order to serve as a solution, and we shall learn how to do this in this chapter.

You may think that resorting to series solutions is not a particularly convenient approach, but it turns out that many of the most important differential equations of the physical sciences can be solved only in terms of infinite series. These equations are very often second-order equations with non-constant coefficients and are named after the mathematicians who introduced them and applied them to significant problems. Thus, we have Bessel's equation with Bessel functions as its solutions, Legendre's equations with its Legendre polynomials and Legendre functions, and a host of others. Even though these "name" functions are formally defined only through power series, it is possible to deduce many of their properties and relations between them, as we shall do for Bessel functions in Section 6.

In time, you can be as comfortable with Bessel functions as you are with the trigonometric functions. In fact, if we were to *define* $\sin x$ and $\cos x$ formally as the odd and even power series solutions to the equation

$$y''(x) + y(x) = 0$$

we would still have our myriad of trigonometric identities.

12.1 The Power Series Method

We shall illustrate the method of solving differential equations by the power series method with a fairly simple example. Consider the equation

$$y''(x) + y(x) = 0 \tag{1}$$

We solved this equation several times in the previous chapter to obtain $y(x) = a\cos x + b\sin x$, but let's assume here that we are unable to solve it using the methods of the previous chapter. Now let's assume that $y(x)$ is a power series in x:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

Our task then is to determine the a_n in Equation 2 such that Equation 2 is a solution of Equation 1. Substitute Equation 2 into Equation 1 to obtain

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{3}$$

Notice that the first summation here vanishes if $n = 0$ or $n = 1$, so Equation 3 can be written as

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0 \tag{4}$$

The first summation in Equation 4 written out explicitly is

$$2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \cdots$$

Using this series as a guide, we can change the lower limit in the first summation from $n = 2$ to $n = 0$ by writing the summand as $(n+2)(n+1)a_{n+2}x^n$, so that Equation 4 now becomes

$$\sum_{n=0}^{\infty} [\,(n+2)(n+1)a_{n+2} + a_n\,]\,x^n = 0 \tag{5}$$

(Problem 1 provides practice in changing indices of summation if you need it.)

We now use the fact that if a power series vanishes identically over some interval, then each of the coefficients of the series must equal zero. Therefore, Equation 5 gives us

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \qquad n = 0, 1, 2, \ldots \tag{6}$$

Equation 6 is a *recursion formula* for a_{n+2} in terms of a_n. As you let $n = 0, 1, 2, \ldots$ in Equation 6, you get the even-subscripted a's in terms of a_0 and the odd-subscripted a's in terms of a_1. Thus, we get two separate sets of coefficients. Starting with $n = 0$, we have for even values of n,

$$a_2 = -\frac{a_0}{2 \cdot 1} \qquad a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_0}{4!}$$

$$a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6!}$$

The general result is

$$a_{2n} = \frac{(-1)^n}{(2n)!} a_0 \qquad n = 0, 1, 2, \ldots \tag{7}$$

Notice that we have determined only the even-subscripted coefficients (all in terms of a_0). To determine the odd-subscripted coefficients, we start with $n = 1$ and we use odd values of n in Equation 6:

$$a_3 = -\frac{a_1}{3 \cdot 2} \qquad a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{a_1}{5!}$$

$$a_7 = -\frac{a_5}{7 \cdot 6} = -\frac{a_1}{7!}$$

or

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1 \qquad n = 0, 1, 2, \ldots \tag{8}$$

If we substitute Equations 7 and 8 into Equation 2, we obtain

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{9}$$

Notice that the solution has two arbitrary constants as you would expect for the general solution for Equation 1. Furthermore, notice that the radius of convergence of each of the power series in Equation 9 is infinity. In fact, the two power series in Equation 9 are the Maclaurin series of $\cos x$ and $\sin x$, respectively, so Equation 9 is

$$y(x) = a_0 \cos x + a_1 \sin x \tag{10}$$

which we could have obtained easily since Equation 1 has constant coefficients. Nevertheless, we used the power series method to illustrate the procedure. Usually you will not be able to identify the resulting power series solution with any known function, but will have to deal with the power series as such.

Suppose now that we did not know that the two solutions in Equation 9 were the familiar sine and cosine functions from trigonometry. We might define two functions

$$s(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

The first task might be to evaluate these series numerically as a function of x and plot them. (This is certainly a lot easier nowadays than it was when most of the special functions were first studied.) We then might notice that $s'(x) = c(x)$ and that $c'(x) = -s(x)$. With a little inspiration, you might notice that

$$c(x) \pm i\, s(x) = e^{\pm ix} = \sum_{n=0}^{\infty} \frac{(\pm ix)^n}{n!}$$

and so on. If $s(x)$ and $c(x)$ occurred in a number of different problems, then they would eventually be given names accepted by the mathematical community and become part of the mathematical literature.

Example 1:
Solve the equation

$$y''(x) + 3xy'(x) + 3y(x) = 0$$

by the power series method.

SOLUTION: Substitute Equation 2 into the above equation to obtain

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 3\sum_{n=0}^{\infty} na_n x^n + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

Rewrite the first summation as $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$ and collect terms to get

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 3(n+1)a_n]x^n = 0$$

Setting the coefficients of x^n equal to zero gives the recursion formula:

$$a_{n+2} = -\frac{3}{n+2}a_n \qquad n = 0, 1, 2, \ldots$$

or

$$a_0 + a_1 x + 3a_0 x + \sum_{n=0}^{\infty} [\, (n+2)(n+1)a_{n+2} + 3a_{n+1} + a_{n+2}\,]\, x^{n+2} = 0$$

Thus, we see in this case that $a_0 = 0$, $a_1 = 0$, and

$$a_{n+2} = -\frac{3}{n^2 + 3n + 3} a_{n+1} \qquad n = 0, 1, 2, \ldots \tag{13}$$

But Equation 13 says that $a_2 = -a_1 = 0$, $a_3 = -a_2/7 = 0$, and so on. Every coefficient in Equation 2 is equal to zero, so there is no power series solution to Equation 11.

Equation 1 yields two linearly independent power series solutions, while Equation 11 yields none. The only difference between these two equations are the coefficients of $y''(x)$ and $y(x)$, so clearly there is some property of these coefficients that dictates when a power series solution can or can not be obtained. This is the subject of the next section.

12.1 Problems

1. Rewrite the following summations so that they begin with an $n = 0$ term:

   (a) $\sum_{n=1}^{\infty} n a_n x^{n-1}$ (b) $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ (c) $\sum_{n=2}^{\infty} (n-2) c_{n-2} x^n$

2. Determine a general expression for a_n in terms of a_0 from the following recursion formulas ($n \geq 0$ in all cases):

   (a) $a_{n+1} = -\dfrac{2a_n}{n+1}$ (b) $a_{n+1} = \dfrac{n+2}{2(n+1)} a_n$ (c) $a_{n+1} = -\dfrac{a_n}{(n+1)^2}$

3. Evaluate $11!!$ and $10!!$.

4. Show that $(2n+1)!! = \dfrac{(2n+1)!}{2^n n!}$ and that $(2n)!! = 2^n n!$.

5. Determine general expressions for a_{2n} in terms of a_0 and a_{2n+1} in terms of a_1 for the following recursion formulas:

   (a) $a_{n+2} = -\dfrac{a_n}{(n+1)(n+2)}$ (b) $a_{n+2} = \dfrac{n+1}{4(n+2)} a_n$

6. Solve the equation $y'(x) + y(x) = 0$ about $x = 0$ by the power series method.

7. If the differential equation is nonhomogeneous, we equate the coefficients of like powers of x on both sides of the equation. Solve the equation $y'(x) + y = 1 + x$ about $x = 0$ by the power series method.

8. If the differential equation is nonhomogeneous, we equate the coefficients of like powers of x on both sides of the equation. Solve the equation $xy'(x) - y = x^2 e^x$ about $x = 0$ by the power series method.

9. Solve the equation $(1 - x^2)y''(x) - 2xy'(x) + 2y(x) = 0$ about $x = 0$ using the power series method.

10. Solve the equation $y''(x) - 2xy'(x) + 4y(x) = 0$ about $x = 0$ using the power series method.

11. Starting with the two infinite series $s(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, show that $s^2(x) + c^2(x) = 1$. If you can't do it in general, show that it's true for sequential powers of x. Similarly, show that $s(2x) = 2s(x)c(x)$.

12. Starting with $s(x)$ and $c(x)$ of the previous problem, show that

$$\frac{s(x)}{c(x)} = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots.$$

13. Show that the radius of convergence of each of the series solutions in Example 1 is infinity.

14. Determine the radius of convergence of each of the following series:
(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ (b) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

15. Determine the radius of convergence of the following series:
(a) $\sum_{n=0}^{\infty}\frac{x^n}{2^n}$ (b) $\sum_{n=1}^{\infty}\frac{x^n}{n}$.

16. Show that the equation $x^2y''(x) + xy'(x) + (x^2 - \frac{1}{4})y(x) = 0$ has no power series solution about $x = 0$.

17. In all our examples, we have found a solution about $x = 0$. Suppose we wish to find a solution about some other point instead, say $x = 1$. We could substitute $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^2$ and then determine the a_n in the usual manner, but it is easier to let $z = x - 1$ and convert the differential equation so that z is the independent variable. Then find the power series solution about $z = 0$ and then substitute $x - 1$ for z. Transform the independent variable of the equation $xy''(x) + y'(x) + xy(x) = 0$ to $z = 2x - 1$.

18. The solutions to the equation $y''(x) + y(x) = 0$ are $\sin x$ and $\cos x$, an odd function and an even function, respectively. There must be some property of the differential equation that gives this result. Show that if $y_1(x)$ is a solution, then so is $y_1(-x)$, and $y_1(-x)$ is a constant multiple of $y_1(x)$. Show that this constant is ± 1, or that $y_1(x)$ must be either an even or an odd function x.

12.2 Ordinary Points and Singular Points of Differential Equations

Consider the linear differential equation

$$A(x)y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \qquad (1)$$

where $A(x)$, $P(x)$, and $Q(x)$ are polynomials containing no common factors. Suppose we want to solve Equation 1 in some interval containing the point x_0. If $A(x_0) \neq 0$, then the point x_0 is called an *ordinary point*. In this case, we can divide Equation 1 by $A(x)$ to obtain

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \qquad (2)$$

where $p(x) = P(x)/A(x)$ and $q(x) = Q(x)/A(x)$ are continuous in the neighborhood of x_0. The functions $p(x)$ and $q(x)$ in Equation 2 will usually be ratios of polynomials if $A(x)$, $P(x)$, and $Q(x)$ are polynomials, in which case they will have series expansions about the point x_0. We can actually relax the condition that $A(x)$, $P(x)$, and $Q(x)$ in Equation 1 be polynomials to the condition that $p(x)$ and $q(x)$ are analytic at x_0; that is, they have convergent series expansions about x_0. If $p(x)$ and $q(x)$ are analytic at x_0, then x_0 is an ordinary point of Equation 2.

We're now ready to state (but not prove) an important theorem concerning ordinary points.

If x_0 is an ordinary point of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{3}$$

then the general solution of Equation 3 consists of two linearly independent power series

$$y(x) = c_1 \sum_{n=0}^{\infty} a_n(x - x_0)^n + c_2 \sum_{n=0}^{\infty} b_n(x - x_0)^n$$

where c_1 and c_2 are arbitrary constants and the a's and b's are determined as in Section 1. Furthermore, the two power series are analytic at $x = x_0$ and the radius of convergence of each one is at least as large as the minimum of the radii of convergence of the series expansions of $p(x)$ and $q(x)$.

Let's consider the solution of

$$(1 - x^2)y''(x) - 6xy'(x) - 4y(x) = 0 \tag{4}$$

in the neighborhood of the point $x = 0$. This point is an ordinary point and

$$p(x) = -\frac{6x}{1 - x^2} \qquad \text{and} \qquad q(x) = -\frac{4}{1 - x^2}$$

We can obtain the series expansion of $p(x)$ and $q(x)$ by using the geometric series for $(1 - x^2)^{-1}$, in which case we have

$$p(x) = -6 \sum_{n=0}^{\infty} x^{2n+1} \qquad \text{and} \qquad q(x) = -4 \sum_{n=0}^{\infty} x^{2n}$$

The radius of convergence of each of these series is 1 (the same as that of the geometric series), so we expect that the two series solutions to Equation 4 converge for at least $|x| < 1$. It turns out that the power series solutions to Equation 4 are (Problem 1)

$$y(x) = a_0 \sum_{n=0}^{\infty} (n + 1)x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{2n + 3}{3} x^{2n+1} \tag{5}$$

which fortuitously can be expressed in closed form as (Problem 2)

$$y(x) = \frac{a_0}{(1-x^2)^2} + \frac{a_1(3x - x^3)}{3(1-x^2)^2} \tag{6}$$

The ratio test shows that the two series in Equation 5 converge for $|x| < 1$ (Problem 3).

Example 1:
Predict the radius of convergence of each of the power series solutions to

$$(1+4x^2)y''(x) - 8y(x) = 0$$

about the point $x = 0$.

SOLUTION: The point $x = 0$ is an ordinary point and

$$p(x) = 0 \qquad \text{and} \qquad q(x) = -\frac{8}{1+4x^2} = -8\sum_{n=0}^{\infty}(-1)^n(4x^2)^n$$

The radius of convergence of $p(x)$ is infinity, but that of $q(x)$ is $|4x^2| < 1$, or $|x| < 1/2$. Thus, we expect the two power series solutions to converge for at least $|x| < 1/2$. It turns out that the two solutions are given by (Problem 4)

$$y(x) = a_0(1+4x^2) + a_1\sum_{n=0}^{\infty}\frac{(-1)^{n+1}4^n x^{2n+1}}{4n^2-1}$$

The first term has an infinite radius of convergence (*at least* as large as $|x| < 1/2$) and that of the second term is 1/2.

Equation 1 and the equation in Example 1 of the previous section yielded two linearly independent power series solutions because $x = 0$ is an ordinary point of both equations. Furthermore, the radii of convergence of $p(x)$ and $q(x)$ in each case is infinity, and the solutions converge for all x. Equation 11 of the previous section,

$$4x^2y''(x) + (3x+1)y(x) = 0 \tag{7}$$

however, is a different story. In this case, the point $x = 0$ is not an ordinary point, and consequently $q(x)$ is not analytic about $x = 0$. Thus, we shouldn't be surprised that we found no power series solution.

A point that is not an ordinary point of a differential equation is called a *singular point*. For the case in which $A(x)$, $P(x)$, and $Q(x)$ in Equation 1 are polynomials with no common factors, a singular point is a point at which $A(x) = 0$. Notice that $x = 0$ is a singular point of Equation 7. All other points of Equation 7 are ordinary points. We expressed our statement about the radii of convergence of the two power series about an ordinary point in terms of the radii of convergence of $p(x) = P(x)/A(x)$ and $q(x) = Q(x)/A(x)$. We may express it in terms of singular points as well.

If $x = x_0$ is an ordinary point of

$$A(x)y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

then the general solution consists of two linearly independent power series

$$y(x) = c_1 \sum_{n=0}^{\infty} a_n(x - x_0)^n + c_2 \sum_{n=0}^{\infty} b_n(x - x_0)^n$$

The radius of convergence of each series is at least as great as the distance from x_0 to the nearest singular point (real or complex) of the differential equation.

For example, consider the equation

$$(1 - x^2)y''(x) - 2xy'(x) + y(x) = 0$$

The point $x = 0$ is an ordinary point, and the nearest singular points are $x = \pm 1$. Consequently, the radii of convergence of the two power series solutions are at least 1. The solutions are guaranteed to converge within the interval $|x| < 1$. This is the same as the interval of convergence of $p(x) = P(x)/A(x) = -2x/(1 - x^2)$ and $q(x) = Q(x)/A(x) = 1/(1 - x^2)$.

Consider now the equation

$$(1 + x^2)y''(x) - 2xy'(x) + y(x) = 0 \tag{8}$$

Once again, $x = 0$ is an ordinary point. The nearest singular points are $x = \pm i$. The distance from $x = 0$ to $x = \pm i$ is 1, and so the two power series solutions to Equation 8 converge (at least) for $|x| < 1$. Even though the singular points are complex, they still govern the convergence of the solutions, and also the convergence of

$$p(x) = -\frac{2x}{1 + x^2} \qquad \text{and} \qquad q(x) = \frac{1}{1 + x^2}$$

for that matter (see Section 2.7).

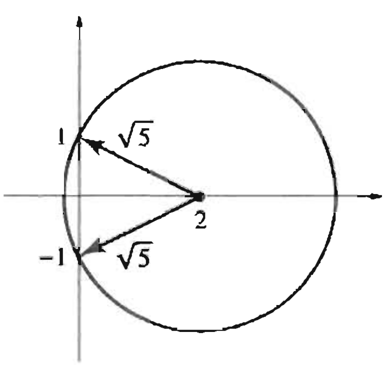


Figure 12.1
A pictorial aid to determining that the distance from the point $x = 2$ to the points $x = \pm i$ is $\sqrt{5}$.

Example 2:
Determine the radii of convergence of the power series solutions to Equation 8 about the ordinary point $x = 2$.

SOLUTION: The singular points occur at $x = \pm i$, and as Figure 12.1 shows, the distance from the point $x = 2$ to the points $x = \pm i$ is $\sqrt{5}$, so we expect the two power series in $(x - 2)$ to have radii of convergence equal to $\sqrt{5}$ (at least).

The solutions to differential equations often have dramatic behavior around singular points; for example, the solution may blow up or go through rapid oscillations. On the other hand, there may be two linearly independent solutions that are finite at a singular point. For example, the Euler-Cauchy equation

$$x^2y''(x) - 2xy'(x) + 2y(x) = 0$$

has the two linearly independent solutions $y_1(x) = x$ and $y_2(x) = x^2$, both of which are well behaved at the singular point $x = 0$, while the Euler-Cauchy equation

$$x^2y''(x) + 2xy'(x) - 12y(x) = 0$$

has the solutions $y_1(x) = x^3$ and $y_2(x) = x^{-4}$, one of which diverges at $x = 0$.

The functions $p(x) = P(x)/A(x)$ and $q(x) = Q(x)/A(x)$ diverge at a singular point $x = x_0$ because $A(x) = 0$ [provided $A(x)$, $P(x)$, and $Q(x)$ have no common factors involving x_0]. The nature of the solutions in the neighborhood of a singular point depends critically upon how strongly $p(x)$ and $q(x)$ diverge there. In particular, if $p(x)$ and $q(x)$ diverge at $x = x_0$, but

$$\lim_{x\to 0}(x - x_0)p(x) = \text{finite} \qquad \text{and} \qquad \lim_{x\to 0}(x - x_0)^2q(x) = \text{finite} \tag{9}$$

then we are able to find two linearly independent solutions. Equations 9 mean that $p(x)$ and $q(x)$ do not diverge more strongly than $1/(x - x_0)$ and $1/(x - x_0)^2$, respectively. The point $x = x_0$ in this case is called a *regular singular point*. If the singular point $x = x_0$ is not a regular singular point, it is called an *irregular singular point*. The study of the solutions to differential equations about irregular singular points is a fairly advanced topic and will not be discussed here. Fortunately, most of the equations of applied mathematics do not involve irregular singular points.

Example 3:
Classify the points $x = 1$, 0, and -1 for the equation

$$x^3(1 - x)y''(x) + (1 - x)y'(x) - 4xy(x) = 0$$

SOLUTION: The point $x = -1$ is an ordinary point. The point $x = 0$ is an irregular singular point because

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{x(1 - x)}{x^3(1 - x)} = \lim_{x\to 0} x^{-2} = \infty$$

Note that $\lim x^2q(x)$ is finite, but all it takes is for one of Equations 9 to fail.
The point $x = 1$ is a regular singular point because

$$\lim_{x\to 1}(x - 1)p(x) = \lim_{x\to 1} \frac{x - 1}{x^3} = 0$$

Starting with $n = 0$ and using even values of n, we find that

$$a_2 = -\frac{\alpha(\alpha+1)}{2\cdot 1}a_0$$

$$a_4 = -\frac{(\alpha-2)(\alpha+3)}{3\cdot 4}a_2 = \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}a_0$$

$$a_6 = -\frac{(\alpha-4)(\alpha+5)}{5\cdot 6}a_4 = \frac{\alpha(\alpha-2)(\alpha-4)(\alpha+1)(\alpha+3)(\alpha+5)}{6!}a_0$$

or generally

$$a_{2n} = (-1)^n\frac{\alpha(\alpha-2)\cdots(\alpha-2n+2)(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)}{(2n)!}a_0 \qquad n \geq 1 \quad (3)$$

Similarly, we find that

$$a_{2n+1} = (-1)^n\frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2n)}{(2n+1)!}a_1 \qquad n \geq 1 \quad (4)$$

The two linearly independent solutions of Equation 1 then are

$$y_1(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} \qquad \text{and} \qquad y_2(x) = \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1} \qquad (5)$$

For arbitrary values of α, both series of Equation 5 diverge at $x = \pm 1$. Yet in many applications, $x = \cos\theta$, where θ is the polar angle in spherical coordinates. Consequently, we often require a solution that is finite at the points $x = \pm 1$, which corresponds to $\theta = 0$ and $\theta = \pi/2$. It turns out that this is actually easy to accomplish. If we set α equal to zero or a positive integer, then one of the two series in Equation 5 will truncate, resulting in a polynomial. Thus, we will always have a polynomial solution to Equation 1 for integer values of α. This may be easiest to see if we write out the expressions for the first few coefficients in Equations 5 explicitly. Expressions for a_2, a_4, and a_6 appear above in Equation 3, and using Equation 4 for $n = 1$ and $n = 2$, we have

$$a_3 = -\frac{(\alpha-1)(\alpha+2)}{6}a_1; \quad a_5 = -\frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{120}a_1$$

So, if we let $\alpha = 0$, we get $a_2 = 0$, $a_4 = 0$, or $a_{2n} = 0$ for $n \geq 1$ and thus $y_1(x) = a_0$. If $\alpha = 1$, we get $a_3 = 0$, $a_5 = 0$, or $a_{2n+1} = 0$ for $n \geq 1$ and so $y_2(x) = a_1x$. If $\alpha = 2$, $a_2 = -3a_0$, and then $a_{2n} = 0$ for $n \geq 2$, and so $y_1(x) = a_0 + a_2x^2 = a_0(1 - 3x^2)$. If we continue in this manner, we generate a set of polynomial solutions to Equation 1. If we denote these solutions by $f_n(x)$, where n is the value of α, then we have (setting the arbitrary constants a_0 and a_1 equal to unity)

Example 2:
Show that the first few Legendre polynomials are orthogonal over the interval $(-1, 1)$.

SOLUTION: Using Equations 7,

$$\int_{-1}^{1} P_0(x)P_1(x)\,dx = \int_{-1}^{1} P_0(x)P_3(x)\,dx$$

$$= \int_{-1}^{1} P_1(x)P_2(x)\,dx = 0$$

because of the parity of the $P_n(x)$. In addition,

$$\int_{-1}^{1} P_0(x)P_2(x)\,dx = \frac{1}{2}\int_{0}^{1}(3x^2 - 1)\,dx = 0$$

$$\int_{-1}^{1} P_1(x)P_3(x)\,dx = \frac{1}{2}\int_{0}^{1}(5x^4 - 3x^2)\,dx = 0$$

You should keep in mind that when $\alpha = 0, 1, 2, \ldots$ in Equations 3 through 5, we obtain a polynomial solution from one of Equations 3 and 4 *plus* an infinite series that diverges at $x = \pm 1$ from the other. For example, if $\alpha = 0$, Equation 3 leads to $P_0(x) = 1$, while Equation 4 gives

$$y_2(x) = a_1\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \tag{10}$$

This infinite series is equal to $(1/2)\ln[(1+x)/(1-x)]$ (Problem 6). It is customary to denote this second solution by $Q_0(x)$, so we write

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \tag{11}$$

Thus, the complete solution to Equation 1 with $\alpha = 0$ is a linear combination of $P_0(x)$ and $Q_0(x)$:

$$y(x) = c_1P_0(x) + c_2Q_0(x) \tag{12}$$

For $\alpha = 1$, the complete solution is (Problem 8)

$$y(x) = c_1P_1(x) + c_2Q_1(x) \tag{13}$$

where

$$Q_1(x) = xQ_0(x) - 1 \tag{14}$$

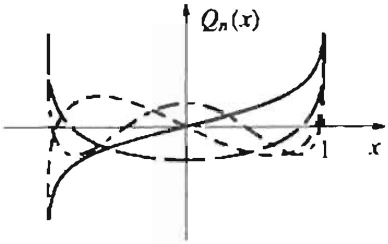


Figure 12.3
The first few second solutions, $Q_0(x)$ (solid), $Q_1(x)$ (long dash), $Q_2(x)$ (short dash), $Q_3(x)$ (dash-dot), to Legendre's equation plotted against x. Note that they all diverge at $x = \pm 1$.

Generally,

$$Q_n(x) = P_n(x)Q_0(x) + \text{a polynomial of degree } n - 1$$

The first few $Q_n(x)$ are plotted in Figure 12.3. As the functional forms indicate, they all diverge at $x = \pm 1$.

The general solution to Legendre's equation when α is zero or a positive integer is

$$y(x) = c_1 P_n(x) + c_2 Q_n(x) \qquad n = 0, 1, 2, \ldots \tag{15}$$

where $P_n(x)$ is the nth degree polynomial that is finite for all values of x and $Q_n(x)$ is a logarithmic function that diverges at $x = \pm 1$. Because we want $y(x)$ to be finite at $x = \pm 1$ in the vast majority of applications, we choose $c_2 = 0$ and work with the Legendre polynomials. This happens so frequently that you tend to forget that there are indeed non-polynomial solutions to Legendre's equation even when n is an integer.

12.3 Problems

1. Derive Equation 2.
2. Extend the polynomial solutions given in Equation 6 up to $f_5(x)$.
3. Show that the polynomials $f_4(x)$ and $f_5(x)$ that you generated in the previous problem satisfy Legendre's equation.
4. Use the polynomials $f_4(x)$ and $f_5(x)$ that you generated in Problem 2 to derive expressions for $P_4(x)$ and $P_5(x)$.
5. Use Equation 8 to generate the first four Legendre polynomials.
6. Show that the Maclaurin series of $\ln[(1 + x)/(1 - x)]$ is given by Equation 10.
7. Show that $\ln[(1 + x)/(1 - x)]$ is a solution to Legendre's equation for $\alpha = 0$.
8. Show that $Q_1(x) = x Q_0(x) - 1$ is a solution to Legendre's equation for $\alpha = 1$.
9. Show that the coefficient of x^n in $P_n(x)$ is $(2n)!/2^n(n!)^2$.
10. Show that Legendre's equation can be written as $[(1 - x^2)y'(x)]' + \alpha(\alpha + 1)y(x) = 0$.
11. In this problem, we'll show in general that the Legendre polynomials are orthogonal over the interval $(-1, 1)$. Start with Legendre's equation in the previous problem for $\alpha = n$ and $\alpha = m$. Multiply the first by $P_m(x)$ and the second by $P_n(x)$ and integrate both results from -1 to $+1$ by parts. Now equate the results and show the orthogonality.
12. Show that the first few Legendre polynomials satisfy the recursion formula
$$(n + 1)P_{n+1}(x) - (2n + 1)x P_n(x) + n P_{n-1}(x) = 0 \qquad n \geq 1$$
13. Use the formula $P_n(x) = \dfrac{(-1)^n}{2^n n!} \dfrac{d^n}{dx^n}(1 - x^2)^n$ to generate the first few Legendre polynomials. This formula is called *Rodrigues' formula*.
14. Argue that the solutions to Legendre's equation should be either even or odd functions of x (see Problem 18 of Section 1). Are they?

15. Often, in physical problems, x in Equation 1 is equal to $\cos\theta$, where θ is the polar angle in spherical coordinates $(0 \le \theta \le \pi/2)$. Express Equation 1 in terms of θ.

16. Show that $\dfrac{1}{(1-2xt+t^2)^{1/2}} = P_0(x) + tP_1(x) + t^2P_2(x) + \cdots$. The left side of this equation is called a *generating function* of the Legendre polynomials.

12.4 Solutions Near Regular Singular Points

We've already seen that a power series method may not work for solutions about regular singular points. In these cases we shall assume an expansion of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1}$$

If r happens to be an integer, then Equation 1 is just a power series, but often r will not be an integer. The general series given by Equation 1 is called a *Frobenius series*, and the use of such a series to find solutions about regular singular points is called the *method of Frobenius*. The theory of differential equations tells us that there is always at least one Frobenius series solution about a regular singular point. To determine the coefficients in the Frobenius series, we substitute Equation 1 into the differential equation and equate the coefficients of the various powers of x to zero, much as we did in the previous sections.

Let's start with the equation

$$2xy''(x) + 3y'(x) - y(x) = 0 \tag{2}$$

If we write Equation 2 in the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{3}$$

then we see that the point $x = 0$ is a *regular* singular point because $xp(x) = 3/2$ and $x^2q(x) = -x/2$ are analytic about $x = 0$. Therefore, we use the Frobenius method and substitute Equation 1 into Equation 2 to obtain (Problem 1)

$$a_0r(2r+1)x^{r-1} + \sum_{n=1}^{\infty} [(n+r)(2n+2r+1)a_n - a_{n-1}]\,x^{n+r-1} = 0 \tag{4}$$

The coefficient of the lowest power of x gives

$$a_0r(2r+1) = 0 \tag{5}$$

This equation for r, which is always obtained by setting the coefficient of the lowest power of x equal to zero, is called the *indicial equation*. Equation 5 tells us that $r = 0$ and $-1/2$. We reject the choice that $a_0 = 0$ because we have taken a_0 to be the coefficient of the lowest power of x in Equation 1. Therefore, we expect our two linearly independent solutions to be of the form

$$y_1(x) = \sum_{n=0}^{\infty} b_n x^n \qquad \text{and} \qquad y_2(x) = x^{-1/2} \sum_{n=0}^{\infty} c_n x^n$$

The first solution happens to be a power series, but the second is not.

If we set the remaining coefficients in Equation 4 (with $r = 0$) equal to zero, we obtain

$$b_n = \frac{b_{n-1}}{n(2n+1)} \qquad n \geq 1 \tag{6}$$

or (Problem 2)

$$b_n = \frac{2^n b_0}{(2n+1)!}$$

Similarly, for $r = -1/2$, we obtain (Problem 3)

$$c_n = \frac{2^n c_0}{(2n)!}$$

and so the general solution to Equation 2 is

$$y(x) = b_0 \sum_{n=0}^{\infty} \frac{2^n x^n}{(2n+1)!} + c_0 x^{-1/2} \sum_{n=0}^{\infty} \frac{2^n x^n}{(2n)!} \tag{7}$$

where b_0 and c_0 are arbitrary constants. [Do you recognize these series? (Problem 4)]

Example 1:
Solve the following equation about $x = 0$:

$$8x^2 y''(x) + 10xy'(x) - (1+x)y(x) = 0$$

SOLUTION: The point $x = 0$ is a *regular* singular point because

$$xp(x) = \frac{5}{4} \qquad \text{and} \qquad x^2 q(x) = -\frac{1+x}{8}$$

are both analytic about $x = 0$. If we substitute Equation 1 into the above differential equation, we obtain

$$\sum_{n=0}^{\infty} [2(n+r)(4n+4r+1) - 1]\, a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

The lowest power of x (which is x^r) comes from the $n = 0$ term of the first summation, so the indicial equation is

$$2r(4r+1) - 1 = 0$$

The two roots of this equation are $r = 1/4$ and $-1/2$. Thus, we expect two Frobenius series,

$$y_1(x) = x^{1/4}\sum_{n=0}^{\infty} b_n x^n \qquad \text{and} \qquad y_2(x) = x^{-1/2}\sum_{n=0}^{\infty} c_n x^n$$

The recursion formula for the coefficients in terms of r is

$$a_{n+1} = \frac{a_n}{2(n+1+r)(4n+4r+5)-1} \qquad n \geq 0$$

If we let $r = 1/4$ and $-1/2$ in turn, we find that (Problem 6)

$$y_1(x) = x^{1/4} + \frac{x^{5/4}}{14} + \frac{x^{9/4}}{616} + \cdots$$

and

$$y_2(x) = x^{-1/2} + \frac{x^{1/2}}{2} + \frac{x^{3/2}}{40} + \cdots$$

So far everything has been fairly straightforward. Finding series solutions often involves quite a bit of algebra, but the procedure itself is straightforward. Let's look at the equation

$$x^2y''(x) + 3xy'(x) + (1-2x)y(x) = 0 \tag{8}$$

The indicial equation associated with this equation is $(r^2+1)^2 = 0$, and so has a repeated root of -1. Clearly, we are not going to obtain two linearly independent solutions by the method that we have been using.

The general theory of differential equations tells us that there is at least one Frobenius series solution about a regular singular point, so the Frobenius method is always going to give us one solution. Once we have one solution, we can always find the other by the method of reduction of order. Recall that the method of reduction of order says that if $y_1(x)$ is a solution to Equation 3, then we can find another solution by assuming that $y_2(x) = u(x)y_1(x)$, where $u(x)$ is determined by substituting $y_2(x)$ into Equation 3. The final result for $y_2(x)$ is (Problem 11.3.14)

$$y_2(x) = y_1(x)\int \frac{e^{-\int p\,dx}}{y_1^2(x)}dx \tag{9}$$

Let's go back to Equation 8.

$$x^2y''(x) + 3xy'(x) + (1-2x)y(x) = 0$$

which we saw above has the indicial equation $(r+1)^2 = 0$, and so has a repeated root of $r = -1$. If we substitute Equation 1 with $r = -1$ into Equation 7, we eventually obtain the simple recursion relation (Problem 12),

$$a_n = \frac{2a_{n-1}}{n^2} \qquad n \geq 1$$

or

$$a_n = \frac{2^n}{(n!)^2} a_0 \tag{10}$$

so the Frobenius series solution is

$$y_1(x) = x^{-1} \sum_{n=0}^{\infty} \frac{(2x)^n}{(n!)^2} = \frac{1}{x} + 2 + x + \frac{2x^2}{9} + \cdots \tag{11}$$

To use Equation 9, we first note that $\int p\,dx = 3 \int \frac{dx}{x} = 3 \ln x$, so Equation 9 becomes

$$\begin{aligned} y_2(x) &= y_1(x) \int \frac{dx}{x^3 y_1^2(x)} \\ &= y_1(x) \int \frac{dx}{x + 4x^2 + 6x^3 + O(x^4)} \\ &= y_1(x) \int \frac{dx}{x[1 + 4x + 6x^2 + O(x^3)]} \end{aligned}$$

Now expand $1/[1 + 4x + 6x^2 + O(x^3)]$ using the geometric series (Problem 13)

$$\begin{aligned} y_2(x) &= y_1(x) \int \left[\frac{1}{x} - 4 + 10x + O(x^2)\right] dx \\ &= y_1(x)[\ln x - 4x - 5x^2 + O(x^3)] \\ &= y_1(x) \ln x - 4 - 13x + O(x^2) \end{aligned} \tag{12}$$

Thus, we see that a second solution to Equation 8 has the form $y_1(x) \ln x$ + a power series in x. The $y_1(x) \ln x$ term will always occur when the indicial equation has two equal roots, as in this case.

The Frobenius method sometimes runs into difficulty if the two roots of the indicial equation differ by an integer. In this case, you may or may not obtain a second solution of the form $y_1(x) \ln x$. Rather than go through a number of examples involving lots of algebra, we'll state the general results for the three cases in which (1) the two roots of the indicial equation do not differ by zero or an integer, (2) the two roots are equal, and (3) the two roots differ by an integer in the form of a general theorem:

Suppose that $x = 0$ is a regular singular point of the equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

and that $xp(x)$ and $x^2q(x)$ have series expansions about $x = 0$ that converge for $|x| < R$, where R is the smaller of the radii of convergence of $xp(x)$ and $x^2q(x)$. There are two linearly independent solutions that are valid for at least $0 < |x| < R$, whose form depends upon the relative values of r_1 and r_2. Let r_1 and r_2 be the two roots of the indicial equation, with $r_1 \geq r_2$.

1. *If $r_1 - r_2 \neq 0$ nor an integer, then*

$$y_1(x) = |x|^{r_1} \sum_{n=0}^{\infty} a_n x^n \qquad a_0 \neq 0 \tag{13}$$

and

$$y_2(x) = |x|^{r_2} \sum_{n=0}^{\infty} b_n x^n \qquad b_0 \neq 0 \tag{14}$$

2. *If $r_1 = r_2$, then*

$$y_1(x) = |x|^{r_1} \sum_{n=0}^{\infty} a_n x^n \qquad a_0 \neq 0 \tag{15}$$

and

$$y_2(x) = y_1(x) \ln x + |x|^{r_1+1} \sum_{n=0}^{\infty} b_n x^n \tag{16}$$

3. *If $r_1 - r_2 =$ an integer, then*

$$y_1(x) = |x|^{r_1} \sum_{n=0}^{\infty} a_n x^n \qquad a_0 \neq 0 \tag{17}$$

and

$$y_2(x) = cy_1(x) \ln |x| + |x|^{r_2} \sum_{n=0}^{\infty} b_n x^n \qquad b_0 \neq 0 \tag{18}$$

In each case, the a's and the b's may be determined by substituting $y_1(x)$ or $y_2(x)$ into the differential equation and setting the coefficients of the various powers of x equal to zero. In case 2, there is no requirement that $b_0 \neq 0$ and, in fact, the entire series may be absent. In case 3, the constant c may equal zero and so the complete solution is the sum of two Frobenius series, as in case 1. Finally, all the power series in the three cases define functions that are analytic at $x = 0$.

We'll illustrate the above theorem with a few examples to see how the solutions conform to these results. Consider the equation

$$4x^2y''(x) + (3x + 1)y(x) = 0 \tag{19}$$

for $x > 0$. Here we have

$$p(x) = 0 \qquad \text{and} \qquad q(x) = \frac{3}{4x} + \frac{1}{4x^2}$$

for $x > 0$. In this case,

$$p(x) = \frac{2}{x} \qquad \text{and} \qquad q(x) = \frac{2}{x(1-x)} = \frac{2}{x}\sum_{n=0}^{\infty} x^n$$

Substituting Equation 1 into Equation 25 yields the indicial equation $r^2 + r = 0$, or $r_1 = 0$ and $r_2 = -1$. The solution corresponding to $r_1 = 0$ is (Problem 19)

$$y_1(x) = -2 + 2x \tag{26}$$

and the solution corresponding to $r_2 = -1$ is (Problem 19)

$$y_2(x) = y_1(x)\ln x + x^{-1}\left[1 + x - 5x^2 + 2\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}\right] \tag{27}$$

Note that $x^2 q(x)$ converges for $x < 1$, and although $y_1(x)$ converges for all values of x, $y_2(x)$ converges only for $0 < x < 1$.

12.4 Problems

1. Derive Equation 4.
2. Show that the recursion formula $b_n = b_{n-1}/n(2n+1)$, $n \geq 1$, leads to $b_n = 2^n/(2n+1)!$.
3. Show that the recursion formula for the second solution of Equation 2 is $c_n = c_{n-1}/n(2n-1)$, $n \geq 1$, and that $c_n = 2^n c_0/(2n)!$.
4. Identify the series in Equation 7.
5. Using the results of the previous problem, show that $(2x)^{-1/2}\sinh\sqrt{2x}$ and $x^{-1/2}\cosh\sqrt{2x}$ are solutions to Equation 2.
6. Verify the solutions given in Example 1.
7. Use the Frobenius method to determine the solution of
   $4x^2y''(x) - 2x(x+2)y'(x) + (x+3)y(x) = 0$ about the point $x = 0$.

   Determine the indicial equation and its two roots for the equations in Problems 8 through 11.

8. $3x^2y''(x) + xy'(x) - (1+x)y(x) = 0$
9. $4x^2y''(x) + (1-2x)y(x) = 0$
10. $xy''(x) + y(x) = 0$
11. $x^2y''(x) + xy'(x) + (x^2-4)y(x) = 0$
12. Derive Equation 10.
13. Derive Equation 12.
14. Derive the Frobenius solution to Equation 19.
15. Derive Equation 21.
16. Derive Equation 23.

17. Derive Equation 24 by substituting Equation 18 into Equation 22.

18. Derive the Frobenius series solution (Equation 26) to Equation 25.

19. Use Equation 9 to derive the expression for $y_2(x)$ for Equation 25.

20. Show that Equation 24 can be written as $c_1x(1 - 2x + 2x^2) + c_2xe^{-2x}$.

12.5 Bessel's Equation

In Section 3, we studied Legendre's differential equation, whose solutions that are finite at $x = \pm 1$ are Legendre polynomials. We stated there that Legendre's equation is one of the most important equations in the physical sciences because it occurs in many problems involving spherical coordinates, as we shall see in Chapter 16. Another equally important differential equation is Bessel's equation, which occurs in many problems involving cylindrical coordinates. Bessel's equation nicely illustrates the various cases for the relative values of the roots of the indicial equation that we discussed in the previous section, so we'll spend this entire section discussing *Bessel's equation*:

$$x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0 \tag{1}$$

where $\nu \geq 0$ is a parameter that will subsequently label the various solutions, called *Bessel functions*. Equation 1 is called *Bessel's equation of order* ν.

The variable x usually takes on the values $0 \leq x < \infty$ in physical problems, so we will always take x to be equal to or greater than zero. First note that $x = 0$ is a regular singular point and that all other values of x are ordinary points. The functions $p(x)$ and $q(x)$, when Equation 1 is written in the form $y''(x) + p(x)y'(x) + q(x)y(x) = 0$, are

$$p(x) = \frac{1}{x} \qquad \text{and} \qquad q(x) = 1 - \frac{\nu^2}{x^2}$$

Because $xp(x) = 1$ and $x^2q(x) = x^2 - \nu^2$ are analytic everywhere, we expect the solutions to Equation 1 (the Bessel functions) to converge for $0 < x < \infty$, and perhaps for $0 \leq x < \infty$.

If we substitute $y(x) = \sum a_nx^{n+r}$ into Equation 1, we obtain

$$\begin{aligned}(r^2 - \nu^2)a_0x^r + [(r+1)^2 - \nu^2]a_1x^{r+1} + \{[(r+2)^2 - \nu^2]a_2 + a_0\}x^{r+2}&\\ + \cdots + \{[(r+n)^2 - \nu^2]a_n + a_{n-2}\}x^{r+n} + \cdots = 0&\end{aligned} \tag{2}$$

The $n = 0$ term gives the indicial equation because $a_0 \neq 0$ (essentially by definition) and so $r = \pm\nu$; the $n = 1$ term reads $[(r+1)^2 - \nu^2]a_1 = [(\pm\nu + 1)^2 - \nu^2]a_1 = (1 \pm 2\nu)a_1 = 0$; and the $n \geq 2$ terms give the recursion formula for the other a's:

$$[(r+n)^2 - \nu^2]a_n + a_{n-2} = 0 \qquad n \geq 2 \tag{3}$$

Taking ν to be zero or positive, we see that $r_2 - r_1 = 2\nu$, so depending upon the value of ν, Bessel's equation provides examples of all three cases in the general theorem of the previous section.

Many applications occur in which $\nu = 0$ or a positive integer, with $\nu = 0$ being particularly important. Thus, let's look at the case $\nu = 0$ first. If $\nu = 0$, we have $r_1 = r_2 = 0$. According to Equations 15 and 16 of the previous section, the two solutions to Equation 1 are of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (a_0 \neq 0) \tag{4}$$

and

$$y_2(x) = y_1(x) \ln x + x \sum_{n=0}^{\infty} b_n x^n \tag{5}$$

We can determine the a_n by substituting $y_1(x)$ into the zero-order Bessel equation, in which case we obtain $a_1 = 0$ and (Problem 1)

$$a_{2n} = -\frac{a_{2n-2}}{(2n)^2} = \cdots = \frac{(-1)^n a_0}{2^{2n}(n!)^2} \qquad n \geq 1 \tag{6}$$

Thus, if we denote $y_1(x)$ by the standard notation $J_0(x)$, we have

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \tag{7}$$

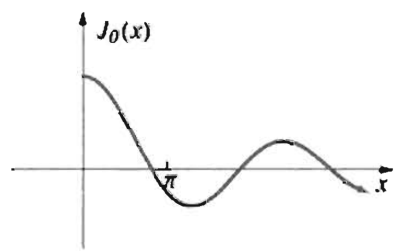


Figure 12.4
The zero-order Bessel function of the first kind, $J_0(x)$, plotted against x.

Equation 7 defines a *Bessel function of the first kind of order zero*. It is easy to show that $J_0(x)$ converges for $0 \leq x < \infty$ (Problem 2). The function $J_0(x)$ is oscillatory, as you can see in Figure 12.4.

We can obtain the second solution to Equation 1 (with $\nu = 0$) by substituting Equation 5 into Equation 1. This gives (Problem 3)

$$y_2(x) = J_0(x) \ln x + \left(\frac{x^2}{4} - \frac{3x^4}{128} + \cdots\right) \tag{8}$$

The general form of Equation 8 is (Problem 23)

$$y_2(x) = J_0(x) \ln x + \left[\frac{x^2}{2^2} - \frac{x^4}{2^2 \cdot 4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots\right] \tag{9}$$

If we define the quantity

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{j=1}^{n} \frac{1}{j} \tag{10}$$

where

$$\gamma = \lim_{n \to \infty} (H_n - \ln n) = 0.5772215 \cdots \tag{14}$$

is called *Euler's constant* (Section 3.4). This certainly unintuitive form of $Y_0(x)$ is chosen because of its convenient behavior as $x \to \infty$, as we shall see in the next section. Figure 12.5 shows $Y_0(x)$ plotted against x. Like $J_0(x)$, $Y_0(x)$ is an oscillatory function of x. Notice that $Y_0(x) \to -\infty$ as $x \to 0$ due to the $\ln x$ term in Equation 13. In summary, the general solution to the zero-order Bessel equation is

$$y(x) = c_1 J_0(x) + c_2 Y_0(x) \tag{15}$$

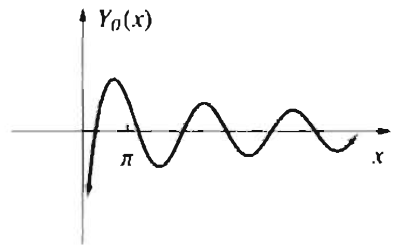


Figure 12.5
The zero-order Bessel function of the second kind, $Y_0(x)$, plotted against x.

Let's now look at the case where ν in Equation 1 is a positive integer, so that $r_1 - r_2 = 2n$, an integer. According to Equations 17 and 18 of the previous section, the two solutions to Bessel's equation of integer order are

$$y_1(x) = x^n \sum_{j=0}^{\infty} a_j x^j \qquad (a_0 \neq 0) \tag{16}$$

and

$$y_2(x) = c y_1(x) \ln x + x^{-n} \sum_{j=0}^{\infty} b_j x^j \qquad (b_0 \neq 0) \tag{17}$$

Substituting Equation 16 into Equation 1 (with $\nu = n$) gives (Problem 4)

$$a_{2j} = -\frac{a_{2j-2}}{(2j)(2j+2n)} = \frac{(-1)^j a_0}{2^{2j} j!(n+1)(n+2)\cdots(n+j)} \qquad j \geq 1$$

The factor $(n+1)\cdots(n+j)$ in the denominator of a_{2j} can be written as $\Gamma(n+j+1)/\Gamma(n+1)$, where $\Gamma(z)$ is a gamma function (Section 3.1), so a_{2j} can be written as

$$a_{2j} = \frac{(-1)^j \Gamma(n+1) a_0}{2^{2j} j! \Gamma(j+1+n)} \tag{18}$$

It is customary to choose a_0 to be $1/[2^n \Gamma(n+1)]$, and so Equation 18 becomes

$$a_{2j} = \frac{(-1)^j}{2^{2j+n} \Gamma(j+1) \Gamma(j+1+n)} \qquad j \geq 0$$

and one solution to Equation 1 is

$$J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j+1+n)} \left(\frac{x}{2}\right)^{2j+n} \tag{19}$$

Equation 19 defines a *Bessel function of the first kind of order n*. It's easy to show by the ratio test that $J_n(x)$ converges for $0 \leq x < \infty$ (Problem 5).

If we let $n = 1$ and $n = 2$ in Equation 19, we obtain the first-order Bessel functions of the first kind:

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \cdots \tag{20}$$

and

$$J_2(x) = \frac{x^2}{2^2 \cdot 2} - \frac{x^4}{2^4 \cdot 3!} + \frac{x^6}{2^7 \cdot 4!} + \cdots \tag{21}$$

The functions $J_0(x)$, $J_1(x)$, and $J_2(x)$ are oscillatory and are plotted in Figure 12.6. Because they are power series, $J_0(x)$, $J_1(x)$, $J_2(x)$, and all their derivatives are continuous functions of x.

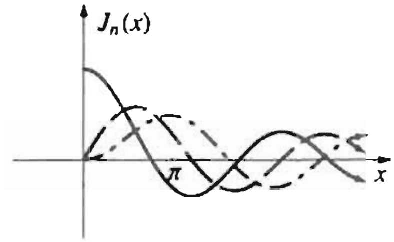


Figure 12.6
The Bessel functions $J_0(x)$ (solid), $J_1(x)$ (dashed), and $J_2(x)$ (dash-dot), plotted against x.

Example 2:
Show that $J_0'(x) = -J_1(x)$.

SOLUTION: We can differentiate a power series term by term within its radius of convergence. From Equation 7,

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n! n!} x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! n!}$$

and

$$J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-1} n!(n-1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)! n!}$$

By comparing the final result with Equation 19, we see that $J_0'(x) = -J_1(x)$.

Equation 19 does not give us a second linearly independent solution if we replace n by $-n$. In fact, Problem 6 has you show that

$$J_{-n}(x) = (-1)^n J_n(x) \qquad n = 1, 2, \ldots$$

Example 3:
Use Equation 19 to show that $J_{-1}(x) = -J_1(x)$.

SOLUTION: Equation 19 with $n = -1$ is

$$J_{-1}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j)} \left(\frac{x}{2}\right)^{2j-1}$$

The $j = 0$ term has $\Gamma(0)$ in the denominator, but $\Gamma(0) = \infty$. Therefore, the summation really starts with the $j = 1$ term.

into Equation 1 with $\nu = 1/2$, we obtain (Problem 7)

$$J_{1/2}(x) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1/2}}{(2n+1)!} \tag{24}$$

If we substitute Equation 23 into Equation 1 with $\nu = -1/2$, we obtain (Problem 8)

$$J_{-1/2}(x) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1/2}}{(2n)!} \tag{25}$$

These two series are linearly independent since one starts with $x^{1/2}$ and the other with $x^{-1/2}$. Thus, we can write the general solution to Equation 1 as

$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x) \tag{26}$$

where $J_{1/2}(x)$ and $J_{-1/2}(x)$ are given by Equations 24 and 25, respectively.

It turns out that $J_{1/2}(x)$ and $J_{-1/2}(x)$ can be expressed in terms of $\sin x$ and $\cos x$. From Equations 24 and 25 we see that

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x \qquad \text{and} \qquad J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x \tag{27}$$

Note that $J_{1/2}(x)$ converges for all values of x, but that $J_{-1/2}(x)$ diverges at $x = 0$. The two functions $J_{1/2}(x)$ and $J_{-1/2}(x)$ are plotted in Figure 12.8.

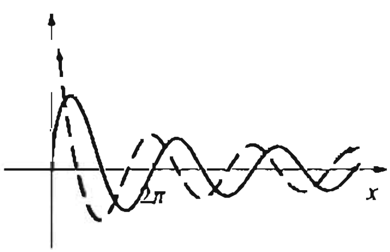


Figure 12.8
The Bessel functions $J_{1/2}(x)$ (solid) and $J_{-1/2}(x)$ (dashed) plotted against x.

Example 4:
Show that $J_{1/2}(x)$ and $J_{-1/2}(x)$ are linearly independent.

SOLUTION: We'll form the Wronskian determinant, so we need the derivatives of $J_{1/2}(x)$ and $J_{-1/2}(x)$. Using Equations 27, we have

$$J'_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\cos x - \frac{\sin x}{2x}\right)$$

$$J'_{-1/2}(x) = -\left(\frac{2}{\pi x}\right)^{1/2} \left(\sin x + \frac{\cos x}{2x}\right)$$

and so the Wronskian determinant is

$$W = \frac{2}{\pi} \begin{vmatrix} \dfrac{\sin x}{x^{1/2}} & \dfrac{\cos x}{x^{1/2}} \\ \dfrac{\cos x}{x^{1/2}} - \dfrac{\sin x}{2x^{3/2}} & -\dfrac{\sin x}{x^{1/2}} - \dfrac{\cos x}{2x^{3/2}} \end{vmatrix}$$

$$= -\frac{1}{x} \neq 0$$

equation is very convenient to know:

$$x^2y''(x) + (1 - 2\alpha)xy'(x) + (\beta^2\gamma^2x^{2\gamma} + \alpha^2 - \nu^2\gamma^2)y(x) = 0 \qquad (42)$$

The general solution to this equation is

$$y(x) = x^{\alpha}\,[\,c_1J_\nu(\beta x^\gamma) + c_2Y_\nu(\beta x^\gamma)\,] \qquad (43)$$

We will refer to this equation a number of times in later chapters.

Example 5:
Use Equation 42 to show that $J_\nu(ix)$ and $Y_\nu(ix)$ are solutions to the modified Bessel's equation, Equation 35.

SOLUTION: By comparing Equations 35 and 42, we see that $\alpha = 0$, $\gamma = 1$, $\nu = \nu$, and $\beta = i$. Therefore, according to Equation 43, the general solution to Equation 35 is

$$y(x) = c_1J_\nu(ix) + c_2Y_\nu(ix)$$

which we actually write as

$$y(x) = c_1I_\nu(x) + c_2K_\nu(x)$$

Example 6:
Consider a mass supported by a string swinging in a plane. Let the length of the string be played out at a constant rate b, so that the length of the string at any time is $l = a + bt$. If θ is the angle that the pendulum makes with the vertical, then Equation 8.2.12 shows that the transverse acceleration along the arc of motion is $l\ddot{\theta} + 2\dot{l}\dot{\theta}$, so that the equation of motion of this pendulum is

$$m(l\ddot{\theta} + 2\dot{l}\dot{\theta}) = -mg\sin\theta$$

or

$$(a + bt)\ddot{\theta} + 2b\dot{\theta} + g\theta = 0$$

for small θ. Find the general solution of this equation.

SOLUTION: First let $x = a + bt$, so that the equation of motion becomes

$$x^2\frac{d^2\theta}{dx^2} + 2x\frac{d\theta}{dx} + \frac{gx}{b^2} = 0$$

If we compare this equation to Equation 43, we see that $1 - 2\alpha = 2$, $\gamma = 1/2$, $\alpha^2 - \nu^2\gamma^2 = 0$, and $\beta^2\gamma^2 = g/b^2$. The general solution is

$$\theta(x) = x^{-1/2}\{c_1J_1[2(gx)^{1/2}/b] + c_2Y_1[(2(gx)^{1/2}/b]\}$$

We'll return to this problem in the next section, but note here that we will retain the $Y_1[(2(gx)^{1/2}/b]$ term because $x = a + bt$, and never equals zero if $a > 0$ and $b > 0$.

In the next, and last, section of this chapter, we shall study some more properties of Bessel functions.

12.5 Problems

1. Derive Equation 6.

2. Show that $J_0(x)$ given by Equation 7 converges for all values of x.

3. Show that the second solution to Equation 1 for $\nu = 0$ is given by Equation 8.

4. Derive Equation 18.

5. Show that $J_n(x)$ given by Equation 19 converges for all values of x.

6. Show that $J_{-n}(x) = (-1)^n J_n(x)$ when n is an integer. *Hint*: Recall that $\Gamma(z) = \infty$ when z is equal to zero or a negative integer.

7. Derive Equation 24 by substituting $y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n$ into Equation 1.

8. Derive Equation 25 by substituting $y_1(x) = x^{-1/2} \sum_{n=0}^{\infty} a_n x^n$ into Equation 1.

9. Show that $J_\nu(ix)$ is a solution to Equation 35.

10. Derive Equation 24 from Equation 30. *Hint*: Use the identity $\Gamma(n)\Gamma\left(n + \frac{1}{2}\right) = \dfrac{\pi^{1/2}\Gamma(2n)}{2^{2n-1}}$.

11. Derive Equation 25 from Equation 30. See the hint given in the previous problem.

12. Show that the substitution $x = iz$ transforms the modified Bessel equation, Equation 35, into Bessel's equation given by Equation 36.

13. Find the general solution of $y''(x) + 4x^2 y(x) = 0$.

14. Find the general solution of $x^{1/2} y''(x) + y(x) = 0$.

15. Find the general solution of $x^2 y''(x) + 5xy'(x) + x^2 y(x) = 0$.

16. The following equation occurs in a treatment of the stability of a flexible vertical rod, $y''(x) + a^2 xy(x) = 0$, where a depends upon the linear mass density, the Young's modulus, and the radius of the rod. Find the general solution to this equation.

17. Consider the differential equation $y''(x) + p(x)y'(x) + q(x)y(x) = 0$. Suppose that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions. Multiply the equation for $y_1(x)$ by $y_2(x)$ and the equation for $y_2(x)$ by $y_1(x)$.

12.6 Bessel Functions

Often in applications, the first few Bessel functions of the first kind are the most important. Recall that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \tag{1}$$

and

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \cdots \tag{2}$$

Generally, $J_n(x)$ is an even function of x when n is even and an odd function of x when n is odd. Also, $J_0(x) = 1$ when $x = 0$ and $J_n(x) = 0$ when $x = 0$ and $n \geq 1$. Extensive tables of the $J_n(x)$ are available for many values of n and there are built-in Bessel functions in many CAS. Figure 12.6 shows that $J_0(x)$ and $J_1(x)$ are oscillatory, each having an infinite number of zeros; or in other words, solutions to the equation $J_n(x) = 0$. These zeros are distinct and well tabulated, and often play key roles in physical problems, as we shall see in Chapter 16.

You can see directly from Equations 1 and 2 that

$$J_0'(x) = -J_1(x) \qquad \text{and} \qquad [x J_1(x)]' = x J_0(x) \tag{3}$$

We can use Equations 3 to derive a number of integrals involving $J_0(x)$ and $J_1(x)$.

Example 1:
Show that

$$\int x^3 J_0(x) dx = x(x^2 - 4) J_1(x) + 2x^2 J_0(x)$$

SOLUTION: Integrate by parts, letting "u" $= x^2$ and "dv" $= x J_0(x) dx$:

$$\int x^3 J_0(x) dx = x^3 J_1(x) - \int 2x^2 J_1(x) dx$$

For the remaining integral, let "u" $= x^2$ and "dv" $= J_1(x) dx$:

$$\begin{aligned}\int x^2 J_1(x) dx &= -x^2 J_0(x) + 2\int x J_0(x) dx \\ &= -x^2 J_0(x) + 2x J_1(x)\end{aligned}$$

Putting all this together gives

$$\begin{aligned}\int x^3 J_0(x) dx &= x^3 J_1(x) + 2x^2 J_0(x) - 4x J_1(x) \\ &= x(x^2 - 4) J_1(x) + 2x^2 J_0(x)\end{aligned}$$

Bessel functions have an interesting type of orthogonality condition. To see what it is, first write Equation 1 of the previous section with $\nu = 0$ in terms of z rather than x:

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + z^2 y(z) = 0$$

The solution to this equation is $J_0(z)$. Now let $z = \alpha x$ to obtain

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \alpha^2 xy = 0$$

whose solution is $J_0(\alpha x)$. Now let $u = J_0(\alpha_i x)$ and $v = J_0(\alpha_j x)$, which satisfy the equations

$$xu'' + u' + \alpha_i^2 xu = 0 \tag{4}$$

$$xv'' + v' + \alpha_j^2 xv = 0 \tag{5}$$

Multiply Equation 4 by v and Equation 5 by u and then subtract to get

$$x(u''v - uv'') + (u'v - uv') = (\alpha_j^2 - \alpha_i^2)xuv$$

The two terms on the left can be written as $[\, x(u'v - uv')\,]'$, and so we write

$$\frac{d}{dx}[\, x(u'v - uv')\,] = (\alpha_j^2 - \alpha_i^2)xuv \tag{6}$$

which upon integration between 0 and 1 gives

$$(\alpha_j^2 - \alpha_i^2)\int_0^1 xJ_0(\alpha_i x)J_0(\alpha_j x)dx = \alpha_i J_0'(\alpha_i)J_0(\alpha_j) - \alpha_j J_0(\alpha_i)J_0'(\alpha_j) \tag{7}$$

where $J_0'(\alpha_i)$ and $J_0'(\alpha_j)$ denote $J_0'(x)$ evaluated at $x = \alpha_i$ and $x = \alpha_j$, respectively. If α_i and α_j are two zeros of $J_0(x)$, then the right side of Equation 7 is equal to zero. Furthermore, $\alpha_i^2 \neq \alpha_j^2$ if $i \neq j$ because the roots of $J_0(x) = 0$ do not repeat, and so we have

$$\int_0^1 xJ_0(\alpha_i x)J_0(\alpha_j x)dx = 0 \qquad i \neq j \tag{8}$$

If $i = j$, we have

$$\int_0^1 xJ_0^2(ax) = \frac{1}{2}[\, J_0^2(a) + J_1^2(a)\,] \tag{9}$$

for any real constant (Problem 5).

Example 2:
Show that if α_i is *any* zero of $J_0(x)$, then

$$\int_0^{\alpha_i} J_1(x)dx = 1$$

SOLUTION: Simply use the first of Equations 3:

$$\int_0^{\alpha_i} J_1(x)dx = -\Big[J_0(x)\Big]_0^{\alpha_i} = 1$$

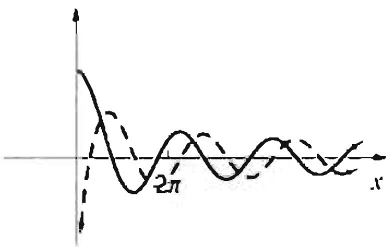


Figure 12.11
The zero-order Bessel functions $J_0(x)$ (solid) and $Y_0(x)$ (dashed) plotted against x.

The second solution to Bessel's equation when $\nu = 0$ is (Equation 13 of the previous section)

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right) J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}H_n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right] \tag{10}$$

Figure 12.11 shows $J_0(x)$ and $Y_0(x)$ plotted against x. The key point is that $J_0(x)$ is well behaved for all values of $x \geq 0$, whereas $Y_0(x)$ diverges as $x \to 0$. In certain scattering problems we need to know the behavior of $J_0(x)$ and $Y_0(x)$ for large values of x. Asymptotically,

$$J_0(x) \sim \left(\frac{2}{\pi x}\right)^{1/2}\left[\cos\left(x - \frac{\pi}{4}\right) + p(x)\right] \tag{11}$$

and

$$Y_0(x) \sim \left(\frac{2}{\pi x}\right)^{1/2}\left[\sin\left(x - \frac{\pi}{4}\right) + q(x)\right] \tag{12}$$

where $p(x)$ and $q(x) \to 0$ as $x \to \infty$. This convenient relation between these expressions is due to the "peculiar" definition of $Y_0(x)$ in Equation 10, as discussed in the previous section. The similarity of the asymptotic forms of $J_0(x)$ and $Y_0(x)$ to cosine and sine functions and the relation $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ has led to the definition of the following auxiliary Bessel functions:

$$\begin{aligned} H_0^{(1)}(x) &= J_0(x) + iY_0(x) \\ H_0^{(2)}(x) &= J_0(x) - iY_0(x) \end{aligned} \tag{13}$$

These auxiliary functions are called *Hankel functions*. Their primary use is in problems involving the scattering of electromagnetic radiation because they have the following asymptotic forms:

$$
\begin{aligned}
H_0^{(1)}(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} e^{i(x-\frac{\pi}{4})} \\
H_0^{(2)}(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} e^{-i(x-\frac{\pi}{4})}
\end{aligned} \tag{14}
$$

Just as there are numerous relations between the trigonometric functions, there are numerous relations between Bessel functions. For example, we showed above that $J_0'(x) = -J_1(x)$. By starting with the definition of $J_\nu(x)$,

$$
J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n+\nu)} \left(\frac{x}{2}\right)^{2n+\nu} \tag{15}
$$

it is straightforward to show that (Problem 6)

$$
\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x) \tag{16}
$$

and that (Problem 7)

$$
\frac{d}{dx}[x^{\nu}J_\nu(x)] = x^{\nu}J_{\nu-1}(x) \tag{17}
$$

Equation 17 gives $J_0'(x) = -J_1(x)$ when $\nu = 0$.

We can use Equations 16 and 17 to derive other relations between Bessel functions of various orders. For example, if we carry out the derivative in Equation 16 and then multiply by $x^{\nu+1}$, we obtain the recursion formula

$$
xJ_\nu'(x) = \nu J_\nu(x) - xJ_{\nu+1}(x) \tag{18}
$$

Similarly, if we carry out the derivative in Equation 17 and then divide by $x^{\nu-1}$, we get

$$
xJ_\nu'(x) = xJ_{\nu-1}(x) - \nu J_\nu(x) \tag{19}
$$

Equations 18 and 19 are recursion formulas involving derivatives. We can obtain a pure recursion formula (one without derivatives) by equating Equations 18 and 19:

$$
xJ_{\nu+1}(x) = 2\nu J_\nu(x) - xJ_{\nu-1}(x) \tag{20}
$$

Equation 20 gives $J_{\nu+1}(x)$ in terms of $J_\nu(x)$ and $J_{\nu-1}(x)$.

Example 3:
We found in the previous section that

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$$

Use the above formulas to find $J_{3/2}(x)$ and $J_{5/2}(x)$.

SOLUTION: Use Equation 16 to find $J_{3/2}(x)$:

$$J_{3/2}(x) = -x^{1/2}\left(\frac{2}{\pi}\right)^{1/2} \frac{d}{dx}\left(\frac{\sin x}{x}\right)$$

$$= \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{\sin x}{x} - \cos x\right)$$

Now use Equation 20:

$$J_{5/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right)$$

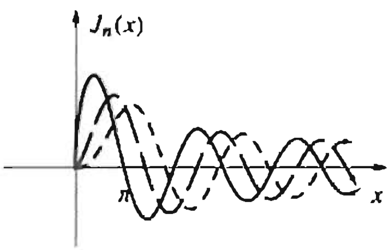


Figure 12.12
The Bessel functions, $J_{1/2}(x)$ (solid), $J_{3/2}(x)$ (long dashed), and $J_{5/2}(x)$ (short dashed), plotted against x.

Figure 12.12 shows $J_{1/2}(x)$, $J_{3/2}(x)$, and $J_{5/2}(x)$ plotted against x. Note that all three functions are equal to zero at $x = 0$ (Problem 8). The functions defined by $j_n(x) = (\pi/2x)^{1/2} J_{n+1/2}(x)$ are called *spherical Bessel functions*. The quantum-mechanical problem of a particle in a spherical cavity involves spherical Bessel functions (Section 16.8).

Although the $Y_\nu(x)$ do not play a role in most applied problems, we point out here that the $Y_\nu(x)$ satisfy the same recursion formulas as the $J_\nu(x)$ (Equations 16 through 20).

Using the result of Problem 18 of the previous section and Equation 18 for $J'_\nu(x)$ and $Y'_\nu(x)$, it is an easy matter to derive the relation (Problem 25)

$$J_\nu(x)Y_{\nu+1}(x) - J_{\nu+1}(x)Y_\nu(x) = -\frac{2}{\pi x} \tag{21}$$

Equation 21 is one of a family of such relations. These relations find frequent use in physical problems.

Example 4:
In Example 6 of the previous section, we showed that the general solution to the problem of a pendulum whose length increases at a constant rate is

$$\theta(x) = x^{-1/2}\{c_1 J_1[2(gx)^{1/2}/b] + c_2 Y_1[2(gx)^{1/2}/b]\}$$

where $x = a + bt$. Find the particular solution if $\theta = \theta_0$ and $d\theta/dt = 0$ at $t = 0$.

SOLUTION: The first initial condition gives

$$a^{1/2}\theta_0 = c_1 J_1(\lambda) + c_2 Y_1(\lambda) \tag{22}$$

where $\lambda = 2(ag)^{1/2}/b$. To implement the second initial condition, we write θ as

$$\theta(u) = \frac{2g^{1/2}}{b}\left[c_1 \frac{J_1(u)}{u} + c_2 \frac{Y_1(u)}{u} \right]$$

where $u = 2(gx)^{1/2}/b$ and then use Equation 16 with $\nu = 1$:

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{d\theta}{du}\frac{du}{dt} \\
&= \frac{2g^{1/2}}{b}\left[c_1 \frac{d}{du}\left(\frac{J_1}{u}\right)\cdot\frac{du}{dt} + c_2 \frac{d}{du}\left(\frac{Y_1}{u}\right)\cdot\frac{du}{dt} \right] \\
&= \frac{2g^{1/2}}{b}\left[-c_1 \frac{J_2}{u} - c_2 \frac{Y_2}{u} \right]\frac{g^{1/2}}{x} \\
&= -\frac{2g}{bxu}\left[c_1 J_2(u) + c_2 Y_2(u) \right]
\end{aligned}$$

Letting $t = 0$ and setting the result equal to zero gives

$$c_1 J_2(\lambda) + c_2 Y_2(\lambda) = 0 \tag{23}$$

Solving Equations 22 and 23 for c_1 and c_2 gives

$$c_1 = \frac{a^{1/2}\theta_0 Y_2(\lambda)}{J_1(\lambda)Y_2(\lambda) - J_2(\lambda)Y_1(\lambda)} = -\frac{\pi\lambda a^{1/2}\theta_0 Y_2(\lambda)}{2}$$

and

$$c_2 = \frac{a^{1/2}\theta_0 J_2(\lambda)}{J_2(\lambda)Y_1(\lambda) - J_1(\lambda)Y_2(\lambda)} = \frac{\pi\lambda a^{1/2}\theta_0 J_2(\lambda)}{2}$$

where we have used Equation 21 to rewrite the denominators. Finally, then, we have

$$\theta(t) = \frac{\pi\lambda\theta_0}{2}\left(\frac{a}{x}\right)^{1/2}\{ J_2(\lambda)Y_1[\,2(gx)^{1/2}/b\,] - Y_2(\lambda)J_1[\,2(gx)^{1/2}/b\,] \}$$

This result is plotted in Figure 12.13 for $\lambda = 600$ and t in units of $b/4g$ (Problem 28).

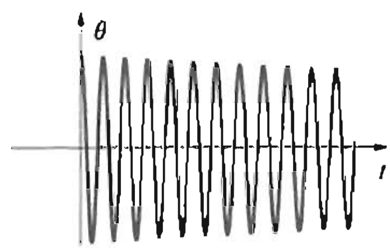


Figure 12.13
The solution to Example 4 (multiplied by $2/\pi\lambda^2\theta_0$) plotted against $4gt/b$ for $\lambda = 600$.

In Section 3.7, we encountered the idea of a *generating function*, and in particular, we saw that

$$f(x,t) = \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(t)\frac{t^n}{n!}$$

where the $B_n(x)$ are the Bernoulli polynomials. There is a generating function for the Bessel functions of integral order that has the form

$$G(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{24}$$

We can derive this generating function fairly easily from the recursion formula, Equation 20. Since many of the "name" functions of applied mathematics have generating functions that can be derived from their recursion relations, we shall derive Equation 24 from Equation 20. Perhaps it's easiest to write out Equation 24 for reference:

$$G(x,t) = \cdots + J_{-2}t^{-2} + J_{-1}t^{-1} + J_0 + J_1t + J_2t^2 + \cdots$$

Now multiply Equation 20 (with $\nu = n$) by t^n and then sum from $n = -\infty$ to $n = \infty$. The term on the left in Equation 20 gives

$$\begin{aligned}\sum_{n=-\infty}^{\infty} J_{n+1}t^n &= \cdots + J_{-1}t^{-2} + J_0t^{-1} + J_1 + J_2t + J_3t^2 + \cdots \\ &= \frac{G(x,t)}{t}\end{aligned}$$

The first term on the right gives

$$\begin{aligned}\sum_{n=-\infty}^{\infty} nJ_nt^n &= \cdots - 2J_{-2}t^{-2} - J_{-1}t^{-1} + 0 + J_1t + 2J_2t^2 + \cdots \\ &= t\frac{\partial G}{\partial t}\end{aligned}$$

The second term on the right gives

$$\begin{aligned}\sum_{n=-\infty}^{\infty} J_{n-1}t^n &= \cdots + J_{-2}t^{-1} + J_{-1} + J_0t + J_1t^2 + \cdots \\ &= tG(x,t)\end{aligned}$$

Putting all this together gives

$$\frac{\partial G}{\partial t} = \frac{x}{2}\left(1 + \frac{1}{t^2}\right)G$$

Integration gives

$$G(x,t) = A(x)\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right]$$

where $A(x)$ is the "constant" of the partial integration with respect to t. Equation 15 shows that all the $J_n(x) = 0$ at $x = 0$ except $J_0(x)$, which equals one at $x = 0$. Thus, $G(x = 0, t) = 1$, $A(x) = 1$, and we obtain Equation 24.

We can use Equation 24 to derive many relations involving Bessel functions. For example, using the fact that $J_{-n}(x) = (-1)^n J_n(x)$, Equation 24 can be written as

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = J_0(x) + J_1(x)\left(t - \frac{1}{t}\right) + J_2(x)\left(t^2 - \frac{1}{t^2}\right) + J_3(x)\left(t^3 - \frac{1}{t^3}\right) + \cdots$$

Letting $t = e^{i\theta}$ and using $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ and $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, we have

$$e^{ix\sin\theta} = J_0(x) + 2iJ_1(x)\sin\theta + 2J_2(x)\cos 2\theta + 2iJ_3(x)\sin 3\theta + \cdots \tag{25}$$

Equating real and imaginary parts gives

$$\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots \tag{26}$$

$$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \cdots \tag{27}$$

Example 5:
Use Equation 25 to show that

$$e^{ix\cos\theta} = J_0(x) + 2\sum_{n=1}^{\infty} i^n J_n(x)\cos nx$$

This formula is used in quantum-mechanical scattering theory.

SOLUTION: Let $\theta = \varphi + \frac{\pi}{2}$. Then $\sin(\varphi + \pi/2) = \cos\varphi$.

$$\begin{aligned}
\sin[n(\varphi + \pi/2)] &= \sin n\varphi\cos\frac{n\pi}{2} + \cos n\varphi\sin\frac{n\pi}{2} \\
&= (-1)^{\frac{n-1}{2}}\cos n\varphi \qquad \text{for } n \text{ odd} \\
\cos[n(\varphi + \pi/2)] &= \cos n\varphi\cos\frac{n\pi}{2} - \sin n\varphi\sin\frac{n\pi}{2} \\
&= (-1)^{\frac{n}{2}}\cos n\varphi \qquad \text{for } n \text{ even}
\end{aligned}$$

Finally, just as $J_{1/2}(x)$ and $J_{-1/2}(x)$ can be expressed in terms of $\sin x$ and $\cos x$,

$$I_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sinh x \qquad \text{and} \qquad I_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cosh x \quad (33)$$

There is a huge textbook literature on Bessel functions. Abramowitz and Stegun devote four entire chapters to Bessel functions (Bessel Functions of Integer Order; Bessel Functions of Fractional Order; Integrals of Bessel Functions; and Struve Functions and Related Functions). (Struve functions are closely related to Bessel functions.) The classic reference is Watson's 800-page "*A Treatise on the Theory of Bessel Function*."

12.6 Problems

1. Show that the general solution to $y''(x) + \frac{1}{x}y'(x) + \alpha^2 y(x) = 0$ is $y(x) = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$.

2. Show that $\int_0^1 x J_0(ax)dx = \frac{1}{a} J_1(a)$.

3. Show that $\int x \ln x \, J_0(x)dx = J_0(x) + x \ln x \, J_1(x)$.

4. Show that $\int J_0(x)J_1(x)dx = -\frac{1}{2}J_0^2(x)$.

5. Show that $\int_0^1 x J_0^2(ax)dx = \frac{1}{2}[J_0^2(a) + J_1^2(a)]$.

6. Starting with Equation 15, show that $[x^{-\nu}J_\nu(x)]' = -x^{-\nu}J_{\nu+1}(x)$.

7. Starting with Equation 15, show that $[x^{\nu}J_\nu(x)]' = x^{\nu}J_{\nu-1}(x)$.

8. Determine the behavior of $J_{1/2}(x) = (2/\pi x)^{1/2} \sin x$, $J_{3/2}(x)$, and $J_{5/2}(x)$ given in Example 3 for small values of x.

9. Assuming that we can integrate Equation 25 term by term, show that

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}d\theta.$$

10. Show that $e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} e^{in\theta}J_n(x)$.

11. Assuming that we can multiply Equation 26 by $\cos n\theta$ and integrate term by term, show that $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta \, d\theta$ for n even.

12. Assuming that we can multiply Equation 27 by $\sin n\theta$ and integrate term by term, show that $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta$ for n odd.

CHAPTER 13

Qualitative Methods for Nonlinear Differential Equations

We studied primarily linear differential equations in the previous two chapters. We saw that there are systematic methods for solving many of these equations, especially if we can be satisfied with series solutions. There are no general systematic methods for the analytic solutions of nonlinear differential equations. A method that may prove useful for one equation may be useless for others. In a sense, each nonlinear differential equation is a challenge unto itself. Nevertheless, even though we may not be able to find any type of analytic solution, we often can determine certain key properties of solutions, such as their behavior for large values of the independent variable or whether the solution is periodic.

Many of the equations that we shall be discussing in this chapter are motivated from classical mechanics, and so consequently will be of the form

$$\ddot{x} + f(x, \dot{x}) = 0$$

Even though it is a fairly simple matter to solve differential equations numerically, it is still useful to extract properties of the solutions before attacking them computationally for several very good reasons: 1. You can gain physical insight into the nature of the solution; 2. You may glean restrictions on the values of certain parameters and reduce the number of computer runs; and 3. You will have some checks for the final numerical solutions.

Section 1 presents an overview of the approach that we shall use throughout the chapter. We'll show that we can learn a great deal about the solutions of the equation that governs a pendulum of arbitrary amplitude.

$$\ddot{\theta} + \alpha^2 \sin\theta = 0$$

without ever solving it. The key concept that we introduce in this section is the *phase plane*. In Sections 2 and 3, we introduce critical points in the phase plane and their classification and stability. Then, in Section 4, we apply these ideas to nonlinear oscillators. We'll learn about an oscillator with no external driving force that will oscillate with an amplitude of 2 after transient terms die out, for just about *any* initial conditions. Finally, in Section 5, we'll study some models of population dynamics that involve coupled first-order nonlinear differential equations. At the end of the chapter, we shall briefly discuss chaotic systems.

13.1 The Phase Plane

Let's start off with a simple equation,

$$\ddot{x} + \omega^2 x = 0 \tag{1}$$

which is the equation of a simple harmonic oscillator. We know that the solution to this equation, $x(t) = x_0 \cos \omega t + (v_0/\omega) \sin \omega t$, where x_0 and v_0 are the initial position and velocity, is periodic with frequency ω. It so happens that we can deduce this important property of the solution without ever solving Equation 1. Multiply Equation 1 by $\dot{x}$ and use the two relations

$$\frac{1}{2}\frac{d}{dt}\dot{x}^2 = \dot{x}\ddot{x} \qquad \text{and} \qquad \frac{1}{2}\frac{d}{dt}x^2 = x\dot{x}$$

to get

$$\frac{d}{dt}(\dot{x}^2 + \omega^2 x^2) = 0$$

or

$$\dot{x}^2 + \omega^2 x^2 = \text{constant} \tag{2}$$

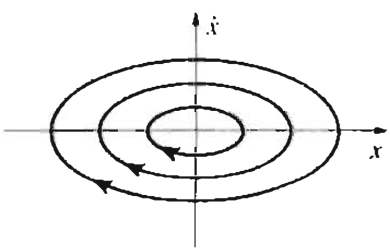


Figure 13.1
The phase portrait of a simple harmonic oscillator. The arrows indicate the forward direction of time.

Equation 2 simply expresses conservation of energy. Now plot Equation 2 in an $x, \dot{x}$ coordinate system, which shows a family of ellipses centered at the origin (Figure 13.1). The arrows on the ellipses indicate the forward direction of time. This $x, \dot{x}$ coordinate system is called the *phase plane*, and each ellipse represents a *trajectory* of the harmonic oscillations. The phase plane, including the family of trajectories, is called a *phase portrait*.

Figure 13.1 contains the essential physical properties of the solution of Equation 1. Each ellipse depicts oscillatory motion. If we write Equation 2 as

$$\frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2 = E \tag{3}$$

and use the fact that $\dot{x} = 0$ when x is at one of its extreme values, we see that x varies between $\pm(2E/k)^{1/2}$, or that the amplitude is $(2E/k)^{1/2}$. Of course, this is the same result that we would obtain from the explicit solution of Equation 1, which we happen to know in this case (Problem 1). Furthermore, you can show that the period of the oscillatory motion is $\tau = 2\pi/\omega$ (Problems 2 and 3).

Before leaving this simple example, we'll look at it from another point of view. If we let $\dot{x} = y$ in Equation 1, we can write it as the pair of first-order equations (see Example 11.6-4)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 x \end{aligned} \tag{4}$$

or

$$\dot{\mathbf{v}} = \mathsf{A}\mathbf{v} \tag{5}$$

in matrix form, where $\mathbf{v} = (x, y)^{\mathsf{T}}$ and

$$\mathsf{A} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \tag{6}$$

Note that $x = y = 0$ is a solution to Equation 4. Any point for which $\dot{x} = \dot{y} = 0$ is called a *critical point*. In this case, the critical point corresponds to the oscillator at rest in its equilibrium position. For this reason, critical points are sometimes called *equilibrium points*.

The eigenvalues and the corresponding eigenvectors of A in Equation 6 are $\pm i\omega$ and $(1, \pm i\omega)^{\mathsf{T}}$, so according to Section 11.6, the real-valued solution to Equation 5 is (Problem 4)

$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cos \omega t \\ -\omega \sin \omega t \end{pmatrix} + c_2 \begin{pmatrix} \sin \omega t \\ \omega \cos \omega t \end{pmatrix} \tag{7}$$

Equation 7 represents a family of ellipses centered at the origin, as in Figure 13.1 (Problem 5). The point (0, 0) is called a *center*. We can use Equations 4 to verify the directions of the arrows in Figure 13.1. Notice from Equation 4 that $\dot{x} > 0$ when $y > 0$, so x increases with time in the upper half of the phase plane, and that $\dot{x} < 0$ when $y < 0$, so x decreases with time in the lower half of the phase plane.

Now let's look at a damped harmonic oscillator and see what its phase plane behavior is like. The differential equation is

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = 0 \tag{8}$$

Let $\dot{x} = y$ and write Equation 8 as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\gamma y - \omega^2 x \end{aligned} \tag{9}$$

or

$$\dot{\mathbf{v}} = \mathsf{A}\mathbf{v} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \mathbf{v} \tag{10}$$

in matrix form. Note that the point (0, 0) is a critical point of Equation 9, corresponding to the final state of rest of the oscillator. The eigenvalues of the matrix in Equation 10 are

$$\lambda_{\pm} = -\frac{\gamma}{2} \pm \frac{1}{2}(\gamma^2 - 4\omega^2)^{1/2} \tag{11}$$

and so the nature of the solutions depends upon the relative values of γ and ω. If $\gamma^2 > 4\omega^2$, then both eigenvalues are real and negative, so $x(t)$ decays monotonically to zero. Recall from Chapter 11 that we called this behavior overdamped. As a concrete example, let $\gamma = 5$ and $\omega = 2$, so that the eigenvalues and corresponding eigenvectors of A are -4, -1 and $(-1, 4)^{\mathsf{T}}$, $(-1, 1)^{\mathsf{T}}$. Therefore, the solution to Equations 10 in this case is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} \tag{12}$$

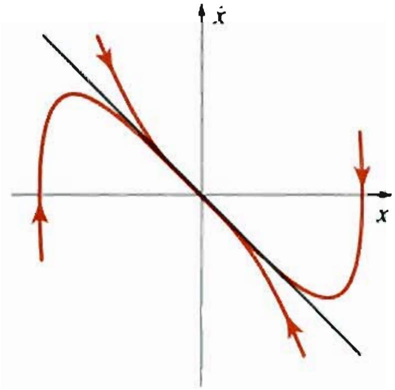


Figure 13.2
The phase portrait of an overdamped harmonic oscillator.

The phase portrait for this system is shown in Figure 13.2. Note that all the trajectories approach the origin tangentially to a single straight line as $t \to \infty$. It turns out that this straight line coincides with the eigenvector direction of $(-1, 1)^{\mathsf{T}}$, which is the line $y = \dot{x} = -x$ in the phase plane (Problem 8). The reason for this behavior is that as t increases, the first term in Equation 12 becomes negligible compared to the second term, and so the directions of the trajectories coincide with the direction of $(-1, 1)^{\mathsf{T}}$ as $t \to \infty$. The critical point (0, 0) in this case is called a *node*, and a *stable node*, in particular.

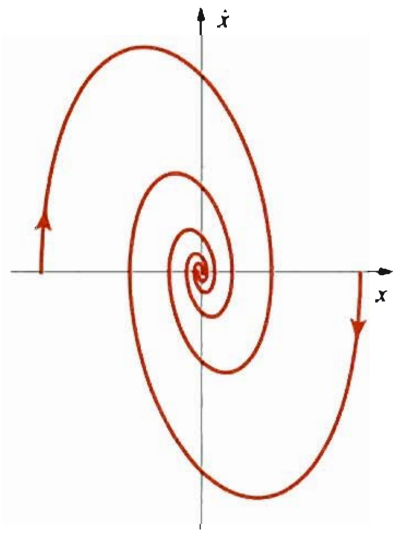


Figure 13.3
The phase portrait of an underdamped harmonic oscillator.

Example 1:

Investigate the phase portrait of the damped harmonic oscillator for the case where $\gamma^2 < 4\omega^2$.

SOLUTION: If $\gamma^2 < 4\omega^2$, then the eigenvalues given by Equation 11 are complex numbers of the form

$$\lambda_{\pm} = -\frac{\gamma}{2} \pm \frac{i}{2}(4\omega^2 - \gamma^2)^{1/2}$$

so the motion of the oscillator is underdamped and is described by a damped harmonic function. Figure 13.3 shows the phase portrait of this motion. Note that the trajectories spiral into the origin (the equilbrium position) as $t \to \infty$, as you should expect on physical grounds. The critical point in this case is called a *spiral point*.

So far we have discussed only cases where we could solve the differential equation analytically. Now let's consider a case that we cannot solve it analytically. In Section 3.5, we introduced the problem of a pendulum of arbitrary amplitude. Figure 13.4 shows a mass m suspended from a fixed point O by a light rod of length l, which swings in a fixed plane. If θ is the angle of the rod from vertical, then the equation of motion is given by (Problem 16)

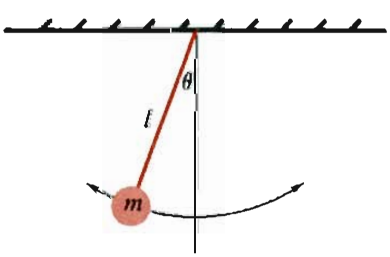


Figure 13.4
An illustration of a pendulum consisting of a mass m supported by a massless rigid rod of length l swinging in a fixed plane.

$$\ddot{\theta} + \omega^2 \sin\theta = 0 \tag{13}$$

where $\omega^2 = g/l$. Note that if the motion is restricted to small angles, then $\sin\theta \approx \theta$ and Equation 13 becomes that of a simple harmonic oscillator. Generally, however, Equation 13 is nonlinear and can be solved analytically only in terms of fairly advanced functions (elliptic functions).

We can learn a great deal, however, about the solutions to Equation 13 without solving it. First we'll write Equation 13 as two first-order equations by letting $\Omega = \dot{\theta}$, so that Equation 13 becomes

$$\begin{aligned} \dot{\theta} &= \Omega \\ \dot{\Omega} &= -\omega^2 \sin\theta \end{aligned} \tag{14}$$

The phase plane in this case will have coordinates Ω and θ. The critical points are given by the two equations

$$\begin{aligned} \dot{\theta} &= \Omega = 0 \\ \dot{\Omega} &= -\omega^2 \sin\theta = 0 \end{aligned} \tag{15}$$

or by $\Omega = 0$ and $\theta = \pm n\pi$, for $n = 0, 1, 2 \ldots$. The points with $n = 0, \pm 2, \pm 4, \ldots$ correspond to the pendulum hanging straight down at rest. We expect these positions to be stable in the sense that small displacements about those points will remain near those points. The points with $n = \pm 1, \pm 3, \ldots$ correspond to the pendulum balanced straight upwards at rest. We expect these positions to be unstable in the sense that small displacements from those points will result in the pendulum moving away from those points.

We can investigate the nature of the trajectories near these critical points by linearizing Equations 14 about these points. Let's take the point (0, 0) as typical of a stable critical point. If we linearize $\sin\theta$ about the point $\theta = 0$, Equations 14 become

$$\dot{\mathbf{v}} = \begin{pmatrix} \dot{\theta} \\ \dot{\Omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \Omega \end{pmatrix} \tag{16}$$

The eigenvalues of this system are $\lambda_\pm = \pm i\omega$, and so we see that the motion is oscillatory about the point (0, 0) and that the trajectories are ellipses centered about the origin, as in Figure 13.1. The point (0, 0) is a center.

Now let's look at the trajectories near an unstable critical point, such as $(0, \pi)$. If we linearize $\sin\theta$ about $\theta = \pi$ in Equation 14, we obtain (Problem 17)

$$\dot{\mathbf{v}} = \begin{pmatrix} \dot{\theta}_1 \\ \dot{\Omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \Omega \end{pmatrix} \tag{17}$$

where $\theta_1 = \theta - \pi$. The eigenvalues in this case are $\lambda_\pm = \pm\omega$ and the solutions are

$$\begin{pmatrix} \theta_1 \\ \Omega \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \omega \end{pmatrix} e^{\omega t} + c_2 \begin{pmatrix} 1 \\ -\omega \end{pmatrix} e^{-\omega t}$$

Figure 13.5 shows the trajectories about the point $(0, \pi)$ for various initial conditions with $\omega = 2$. Notice that the trajectories are hyperbolas with the eigenvector directions as asymptotes and that they are all repelled by the unstable critical point (Problem 20). Such a point is called a *saddle point*.

We have shown that the point (0, 0) is a center and that the point $(0, \pi)$ is a saddle point. It's easy to show generally that all the critical points $(0, \pm n\pi)$ with

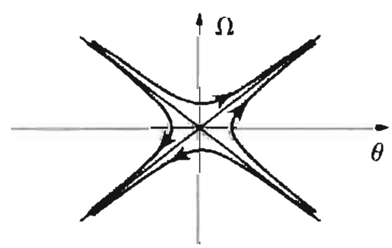


Figure 13.5
The trajectories of the pendulum described by Equation 13 about the point $\Omega = 0$, $\theta = \pi$ for various initial conditions with $\omega = 2$.

13.2 Critical Points in the Phase Plane

In the previous section we used mechanical systems as examples, and so one of the first-order differential equations was always of the form $\dot{x} = y$. That's why the equilibrium points all lay on the x axis. We'll be more general now and consider the pair of equations

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned} \tag{1}$$

with the initial conditions $x = x_0$ and $y = y_0$ at $t = t_0$. The first thing to note is that these equations do not contain time explicitly. Such equations are called *autonomous*. Physically, an autonomous system is one where its parameters are independent of time. Mathematically, the solutions to Equations 1 depend only upon the elapsed time $t - t_0$, where t_0 is some initial time (Problems 1 and 2). Autonomous systems of equations have the property that closed trajectories in the phase plane represent periodic solutions.

The solutions to Equations 1, $x(t)$ and $y(t)$, parametrically describe the trajectories in the phase plane. If the initial value problem associated with Equations 1 is unique, as it will be if $P(x, y)$ and $Q(x, y)$ are continuously differentiable, then the trajectories will never intersect each other (Problem 3).

The points in the phase plane where $P(x, y)$ and $Q(x, y)$ equal zero are called *critical points*, or equilibrium points. We'll denote critical points by (x_c, y_c), so that we have

$$P(x_c, y_c) = 0 \qquad \text{and} \qquad Q(x_c, y_c) = 0$$

Example 1:
Find the critical points of the system

$$\begin{aligned} \dot{x} &= 15x - 3x^2 - 4xy \\ \dot{y} &= 9y - 3y^2 - 2xy \end{aligned}$$

SOLUTION: We set each equation equal to zero:

$$\begin{aligned} 15x - 3x^2 - 4xy &= x(15 - 3x - 4y) = 0 \\ 9y - 3y^2 - 2xy &= y(9 - 3y - 2x) = 0 \end{aligned}$$

Certainly (0, 0) is a critical point. Let $x = 0$ and $y \neq 0$. The second equation gives $y = 3$, giving a critical point (0, 3). Now let $y = 0$ and $x \neq 0$. The first equation gives $x = 5$, for a critical point (5, 0). Finally, let $x \neq 0$ and $y \neq 0$ and solve the two equations

$$\begin{aligned} 15 - 3x - 4y &= 0 \\ 9 - 3y - 2x &= 0 \end{aligned}$$

simultaneously to give (9, −3) as the fourth critical point.

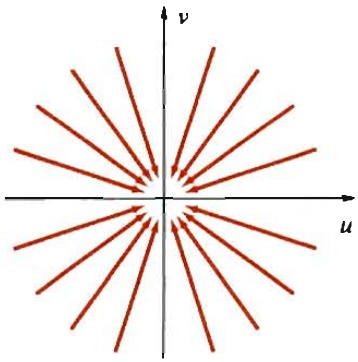


Figure 13.11
An illustration of a proper node with the trajectories approaching the critical point.

is illustrated by the system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The solution is given by $u(t) = u_0 e^{\alpha t}$ and $v(t) = v_0 e^{\alpha t}$. If $\alpha < 0$, the trajectories approach the origin and if $\alpha > 0$, the trajectories recede from the origin as t increases, as shown in Figure 13.11. This type of critical point is called a *proper node*.

The case in which c_2 and c_4 are not both equal to zero is illustrated by

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{13}$$

The eigenvalue $\lambda = 1$ is repeated in this case and we obtain only one independent eigenvector, $(0, 1)^{\mathrm{T}}$. However, Problem 12 helps you show that if we substitute Equations 11 into Equations 13, we obtain

$$u(t) = c_1 e^t \qquad \text{and} \qquad v(t) = (c_3 + c_1 t)e^t \tag{14}$$

We can solve for the trajectories explicitly in this case by solving the first of Equations 14 for t and substitute the result into the second of Equations 14. This gives us

$$v = \frac{c_3}{c_1} u + u \ln \frac{u}{c_1} \tag{15}$$

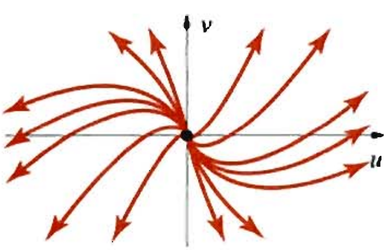


Figure 13.12
The trajectories given by Equation 15 for various values of c_1 and c_2. The critical point is an improper node.

The trajectories given by Equation 15 are shown in Figure 13.12. Note that all the trajectories recede from the critical point (the origin) tangentially to the vertical axis, and so the critical point in this case is an improper node.

Example 4:
Investigate the critical point of the system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{16}$$

SOLUTION: The critical point occurs at (0, 0) and the repeated eigenvalue is -1. To find a solution to Equations 16, substitute Equations 11 with $\lambda = -1$ into them to obtain

$$c_2 - c_1 - c_2 t = -c_1 - c_2 t$$

and

$$-c_3 + c_4 - c_4 t = c_1 - c_3 + (c_2 - c_4)t$$

Equating the coefficients of like powers of t gives $c_2 = 0$, $c_1 = c_4$, and c_3

Table 13.1

A summary of the type of critical point associated with the properties of the eigenvalues of the coefficient matrix of the pair of first-order differential equations.

eigenvalues	type of critical point	stability of the critical point
1. real, unequal, both negative	improper node	asymptotically stable
1. real, unequal, both positive	improper node	unstable
2. real, opposite signs	saddle	unstable
3. equal and positive	proper or improper node	unstable
3. equal and negative	proper or improper node	asymptotically stable
4. complex conjugates, real part > 0	spiral point	unstable
4. complex conjugates, real part < 0	spiral point	asymptotically stable
5. pure imaginary	center	stable

13.2 Problems

1. Let $\tau = t - t_0$ in Equations 1 and show that they take on the same form.

2. Show that the solution to $\dot{x} = y$ and $\dot{y} = -x$ for $x = x_0$ and $y = 0$ at $t = t_0$ is a function of $t - t_0$.

3. Argue that the uniqueness of the solutions to Equations 1 prevents trajectories in the phase plane from intersecting.

4. Find the critical points of

   (a) $\begin{aligned}\dot{x} &= 1 - xy\\ \dot{y} &= xy - y\end{aligned}$ (b) $\begin{aligned}\dot{x} &= 1 - y\\ \dot{y} &= x^2 - y^2\end{aligned}$ (c) $\begin{aligned}\dot{x} &= x - 3x^2 + xy\\ \dot{y} &= 4y - y^2 - 2xy\end{aligned}$ (d) $\begin{aligned}\dot{x} &= x - y\\ \dot{y} &= x^2 - 1\end{aligned}$

5. Modify the equations in Problem 4 so that the critical points occur at (0, 0).

6. Determine the coefficient matrices of the linearized equations of Problem 5.

7. Show that the general solution to Equations 7 is given by Equations 8.

8. Show that the eigenvector directions $(-1, 1)^{\mathrm{T}}$ and $(1, 2)^{\mathrm{T}}$ that pass through the origin correspond to the straight lines $v = -u$ and $v = 2u$, respectively, in the uv-plane.

9. Show that the straight line in the eigenvector direction $(-1 + \sqrt{2}, 1)^{\mathrm{T}}$ that passes through the origin is described by $v = u/(\sqrt{2} - 1)$.

10. Show that all the trajectories associated with Equations 8 (except for two) approach the origin tangentially to the eigenvector $(-1 + \sqrt{2}, 1)^{\mathrm{T}}$ as $t \to \infty$.

11. Show that Equation 10 is the general solution of Equation 9.

12. We'll derive Equations 14 in this problem. Substitute Equations 11 into Equations 13 to obtain $c_2 + (c_1 + c_2 t) = c_1 + c_2 t$ and $c_4 + (c_3 + c_4 t) = (c_1 + c_2 t) + (c_3 + c_4 t)$. Now equate similar powers of t on the two sides of each equation to show that $c_2 = 0$, $c_1 = c_4$, and c_3 is arbitrary. This gives us Equations 14.

13. Classify the singular point of the system $\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$.

SOLUTION: The eigenvalues of the system are $\pm i$, so the trajectories are circles centered at the critical point (a center). Any circle that starts at a distance equal to the radius of the circle from the center remains at that distance, so a center is necessarily a stable critical point.

Example 2:
Determine the nature of the stability of the critical point (0, 0) for the system

$$\dot{x} = x + 2y$$
$$\dot{y} = 2x + y$$

SOLUTION: The eigenvalues and corresponding eigenvectors are -1 and 3 and $(-1, 1)^T$ and $(1, 1)^T$, so the critical point is a saddle point and the trajectories are given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

There are many trajectories that start close to the origin and tend to infinity as t increases. (See Figure 13.16.) Because a saddle point occurs when the eigenvalues are real and of opposite sign, a saddle point is necessarily unstable.

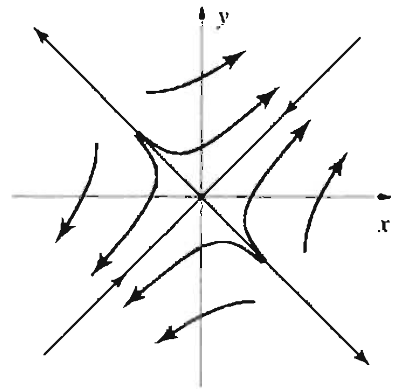


Figure 13.16
The trajectories around the saddle point associated with the system in Example 2, illustrating that a saddle point is unstable.

Example 3:
Determine the nature of the stability of the critical point (0, 0) for the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

SOLUTION: The eigenvalues and corresponding eigenvectors are -3 and -1 and $(-1, 1)^T$ and $(1, 1)^T$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

The two eigenvalues have the same sign, so the critical point is an improper node, and since they are both negative, the critical point is an asymptotically stable improper node. Furthermore, as $t \to \infty$, the second term in the equation becomes negligible compared to the first term and so the trajectories approach the node tangent to the eigenvector direction $(1, 1)^T$, or tangent to the straight line $y = x$ (see Figure 13.17).

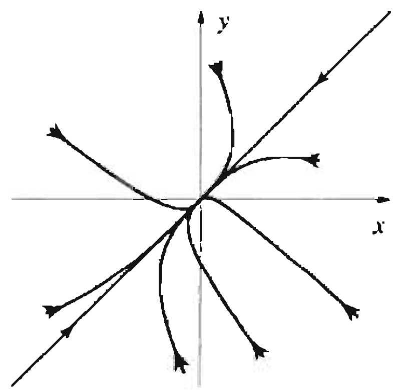


Figure 13.17
The trajectories described in Example 3, showing that the critical point (0, 0) is an asymptotically stable improper node. Note that all the trajectories approach the origin tangentially to the straight line $y = x$ (black) as t increases.

We can summarize all of our results regarding the type of critical point and the nature of its stability in terms of two quantities, $p = a + d$ and $q = ad - bc$, where

$$\mathsf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The eigenvalues of A are given by

$$\lambda_{\pm} = \frac{a+d}{2} \pm \frac{1}{2}[\,(a+d)^2 - 4(ad-bc)\,]^{1/2} = \frac{p}{2} \pm \frac{1}{2}[\,p^2 - 4q\,]^{1/2} \quad (1)$$

Let's look at the three cases, $p = 0$, $p > 0$, and $p < 0$, in turn.

1. If $p = 0$ and $q > 0$, then the eigenvalues are pure imaginary and the critical point is a center, which is necessarily stable.
2. If $p > 0$ and $q > 0$, then the eigenvalues are real, unequal, and positive if $\Delta = p^2 - 4q > 0$, and so the critical point is an unstable improper node. If $\Delta < 0$, on the other hand, then the eigenvalues are a complex conjugate pair with a positive real part, so the critical point is an unstable spiral point.
3. If $p < 0$ and $q > 0$, then the eigenvalues are real, unequal, and negative if $\Delta = p^2 - 4q > 0$, and so the critical point is a stable improper node. If $\Delta < 0$, on the other hand, then the eigenvalues are a complex conjugate pair with a negative real part, so the critical point is an asymptotically stable spiral point.
4. Finally, if $q < 0$, then the eigenvalues will be real and with opposite signs for any value of p, and so the critical point will be a saddle point, which is necessarily unstable.

We can summarize these results in a plot of q against p, as we show in Figure 13.18. Note that the critical points in the upper half plane are unstable if $p > 0$, stable if $p < 0$, and stable if $p = 0$. The critical points in the lower half plane are always saddle points. Also note that the curve $\Delta = 0$, which is the parabola described by $q = p^2/4$, separates various types of critical points in the figure. The eigenvalues given by Equation 1 are real and equal along the parabola. The nature of the critical point changes as we cross the parabola, but the stability of the critical point is not affected. This behavior illustrates the fact that small changes in the parameters of A can affect the type of node, but not its stability when the eigenvalues are real and equal. Another feature of Figure 13.18 to notice is that the positive vertical axis, where the eigenvalues are pure imaginary, separates a region describing an asymptotically stable spiral point from an unstable spiral point. The figure emphasizes just how sensitive the case of pure imaginary eigenvalues is. Even a small displacement from the vertical axis can lead to either a stable or an unstable crtical point. At this point, you should be able to relate Figure 13.18 to Table 13.1.

(b) *If the eigenvalues are pure imaginary ($\lambda_{\pm} = \pm i\nu$), then the critical point is either a center or a spiral point, and the spiral point may be asymptotically stable or unstable.*

(c) *In all other cases, the type of critical point and its stability is the same as that of the corresponding linear system.*

Note that the two cases where the results for the nonlinear system can differ from those of the linear system are the so-called sensitive cases that we discussed above and are illustrated in Figure 13.18 by the vertical positive axis and the parabola $\Delta = 0$. Thus, in a sense, the effect of the nonlinearity is equivalent to introducing a small uncertainty in the values of a, b, c, and d.

Example 4:
Find all the critical points and investigate the type and stability of each for the system

$$\dot{x} = x - y$$
$$\dot{y} = x^2 - y$$

SOLUTION: There are two critical points, (0, 0) and (1, 1). The linearized equations about (0, 0) are

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The eigenvalues are $\lambda_{\pm} = \pm 1$, and so the origin is a saddle point, which is necessarily unstable.

The linearized equations about (1, 1) are given by substituting $u = x - 1$ and $v = y - 1$ into the above differential equations and then linearizing them to obtain

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The eigenvalues are $\pm i$ and so the critical point (1, 1) is either a center or a spiral point, which may be either asymptotically stable or unstable.

Before we leave this section, there is one more topic to discuss. Consider the system

$$\begin{aligned} \dot{x} &= x + y - x(x^2 + y^2) \\ \dot{y} &= -x + y - y(x^2 + y^2) \end{aligned} \qquad (4)$$

The only critical point is at the origin. The presence of the $(x^2 + y^2)$ term in both equations suggests that we use polar coordinates in Equations 4. Letting $x = r\cos\theta$

13.4 Nonlinear Oscillators

In this section, we're going to apply the ideas that we have presented in the previous sections of this chapter to systems of equations that describe oscillating systems, whether they are mechanical or electrical systems. Let's start with a nonlinear oscillator described by

$$\ddot{x} + x - x^3 = 0 \tag{1}$$

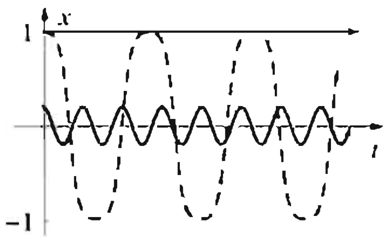


Figure 13.20
A numerical solution to the equation $\ddot{x} + x - x^3 = 0$ for the initial conditions (a) $x(0) = 0.20$ and $\dot{x}(0) = 0$ (solid color); (b) $x(0) = 0.99$ and $\dot{x}(0) = 0$ (dashed color); (c) $x(0) = 1$ and $\dot{x}(0) = 0$ (black). Note that curve c is simply a horizontal line.

Like most any differential equation, we can solve Equation 1 numerically using a CAS. For example, if we solve Equation 1 with the initial conditions $x(0) = 0.20$ and $\dot{x}(0) = 0$, we obtain Figure 13.20a. You might have expected what appears to be essentially harmonic behavior because the initial conditions restrict $x(t)$ to small oscillations, where Equation 1 will be almost linear. Figure 13.20b shows $x(t)$ for the initial conditions $x(0) = 0.99$ and $\dot{x}(0) = 0$. The behavior is still oscillatory, but does not look harmonic. Notice also that the frequency is about one-half of the frequency in Figure 13.20a, a good example where the frequency of a nonlinear system can depend upon the initial conditions. If we increase $x(0)$ to equal 1, we obtain Figure 13.20c, which is decidedly non-oscillatory. If we now increase $x(0)$ beyond 1, we get a deluge of error messages from our CAS. The same thing happens if we let $x(0) = 0.8$, $\dot{x}(0) = 0.8$ or $x(0) = 0$, $\dot{x}(0) = 1$, and so on. Thus, even though we could simply solve Equation 1 numerically, it will not give us much insight into why the solutions vary so much with the initial conditions. The methods that we have developed earlier, however, are easy to apply and will explain the numerical results that we mentioned above.

First, we'll write Equation 1 as two first-order equations and then find the critical points. Letting $\dot{x} = y$, Equation 1 becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^3 \end{aligned} \tag{2}$$

The critical points are given by $y = 0$ and $-x + x^3 = 0$, or by $(0, 0)$ and $(\pm 1, 0)$. The linearized equations about the origin are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues are $\lambda_{\pm} = \pm i$, so according to Poincaré's theorem, the critical point is *either* a center or a spiral point. We can distinguish between these two choices by realizing that there is no damping term in Equation 1, so we expect the critical point to be a center. (We'll verify this below.)

The linearized equations about the critical points $(\pm 1, 0)$ are (Problem 2)

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{3}$$

The eigenvalues are $\lambda_{\pm} = \pm\sqrt{2}$ and the corresponding eigenvectors are $(1, \sqrt{2})^{\mathsf{T}}$ and $(1, -\sqrt{2})^{\mathsf{T}}$, and the solution to Equations 3 is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{\sqrt{2}t} + c_2 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{-\sqrt{2}t}$$

The critical points at $(\pm 1, 0)$ are saddle points and the eigenvectors serve as asymptotic directions of the trajectories. The straight lines corresponding to the eigenvector directions are

$$v = \sqrt{2}u \qquad \text{and} \qquad v = -\sqrt{2}u$$

Therefore, the asymptotic directions of the trajectories around the critical point $(1, 0)$ are

$$y = \sqrt{2}(x-1) \qquad \text{and} \qquad y = -\sqrt{2}(x-1)$$

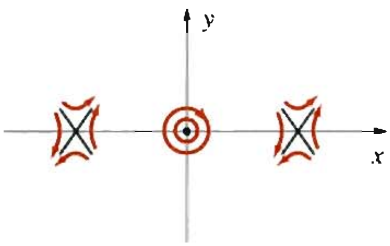


Figure 13.21
The phase portrait of Equations 2 about its three critical points at $(0, 0)$ and $(\pm 1, 0)$.

and those around the critical point $(-1, 0)$ are

$$y = \sqrt{2}(x+1) \qquad \text{and} \qquad y = -\sqrt{2}(x+1)$$

(Problem 3). Figure 13.21 shows the phase portrait of Equations 2 that we have deduced so far. We'll now fill in the rest of it. Divide the equation for $\dot{y}$ by the equation for $\dot{x}$ in Equations 2 to obtain

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{x^3 - x}{y}$$

An integration gives

$$y^2 = \frac{x^4}{2} - x^2 + c \tag{4}$$

where c is a constant. Looking at Figure 13.21, we see (guess?) that the largest closed trajectory passes through the points $(\pm 1, 0)$, so we set $x = \pm 1$ and $y = 0$ in Equation 4 to obtain $c = 1/2$. Thus, the largest closed trajectory (actually, it's the separatrix) is given by

$$y^2 + x^2 - \frac{1}{2}x^4 = \frac{1}{2} \tag{5}$$

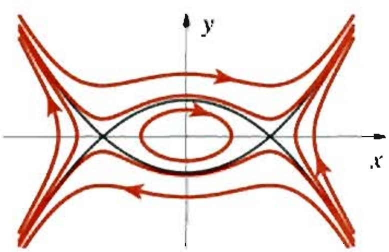


Figure 13.22
The family of closed trajectories given by Equation 6 for various values of C. The separatrix, which is shown in black, is given by Equation 5.

and the smaller trajectories are given by a family of curves described by

$$y^2 + x^2 - \frac{1}{2}x^4 = C \tag{6}$$

with $C < 1/2$. Figure 13.22 shows trajectories for various values of C, both greater than and less than 1/2. The closed curve that separates the region of closed trajectories from the region of asymptotically unstable trajectories is a *separatrix*. The separatrix is given by Equation 5.

We can now explain the behavior that we obtained numerically in Figure 13.20. Notice that when the initial values lie within the closed curve described by Equation 5, the motion is oscillatory. If the initial conditions are $x(0) = \pm 1$, $\dot{x}(0) = 0$, then $\dot{x} = \dot{y} = 0$ (the point is a critical point) and the solution is $x = \pm 1$, as shown in Figure 13.20c.

Example 1:
Include a damping term of the form $\dot{x}$ in Equation 1 and discuss its phase portrait.

SOLUTION: We start with

$$\ddot{x} + \dot{x} + x - x^3 = 0$$

which we write as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - y + x^3 \end{aligned}$$

The critical points are at (0, 0) and (± 1, 0). The eigenvalues associated with the equation linearized about the origin are $(-1 \pm i\sqrt{3})/2$, which says that the origin is an asymptotically stable spiral point.

The equations linearized about the critical points (± 1, 0) are the same:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are -2 and 1, and $(-1, 2)^{\mathsf{T}}$ and $(1, 1)^{\mathsf{T}}$, respectively, and so these two critical points are saddle points. Furthermore, the trajectories about the saddle point (+1, 0) are asymptotic to the two straight lines $y = x - 1$ and $y = -2(x - 1)$, and those about the critical point $(-1, 0)$ are asymptotic to $y = x + 1$ and $y = -2(x + 1)$. Figure 13.23 shows the phase portrait about the three critical points.

You can now fill in the other trajectories using numerical methods and judicious initial values to get the result shown in Figure 13.24.

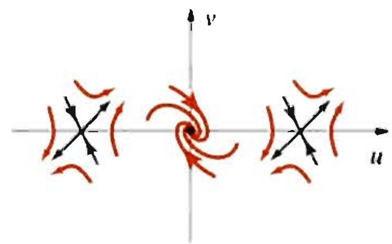


Figure 13.23
The phase portrait for the system described in Example 1 about its three critical points.

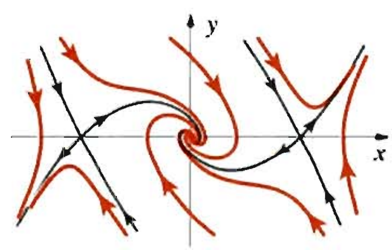


Figure 13.24
The complete phase portrait for the system described in Example 1.

In Section 1, we used the physical problem of an undamped pendulum of arbitrary amplitude to introduce the idea of a phase plane, critical points, and a phase portrait, which is shown in Figure 13.7. With the experience we have, it should be fairly straightforward to sketch the phase portrait of a damped pendulum of arbitrary amplitude, described by

$$\ddot{\theta} + \gamma\,\dot{\theta} + \omega^2 \sin\theta = 0 \tag{7}$$

We rewrite Equation 7 as

$$\begin{aligned}\dot{\theta} &= \Omega \\ \dot{\Omega} &= -\gamma\,\Omega - \omega^2 \sin\theta\end{aligned} \tag{8}$$

The critical points occur at $(n\pi, 0)$, with $n = 0, \pm 1, \pm 2, \ldots$, just as in the undamped case. Expanding $\sin\theta$ about $\theta = n\pi$, we obtain $\sin\theta = (-1)^n(\theta - n\pi) + \cdots$, and the linearized equations are

$$\begin{pmatrix}\dot{\theta}_n \\ \dot{\Omega}\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -(-1)^n\omega^2 & -\gamma\end{pmatrix}\begin{pmatrix}\theta_n \\ \Omega\end{pmatrix} \tag{9}$$

where $\theta_n = \theta - n\pi$. If n is odd, the eigenvalues are $\lambda_\pm = -\gamma/2 \pm \frac{1}{2}(\gamma^2 + 4\omega^2)^{1/2}$, so the critical points are saddle points. If n is even, the eigenvalues are $\lambda_\pm = -\gamma/2 \pm \frac{1}{2}(\gamma^2 - 4\omega^2)^{1/2}$, and the critical points are stable nodes if $\gamma^2 > 4\omega^2$ (overdamped) and asymptotically stable spiral points if $\gamma^2 < 4\omega^2$ (underdamped). Figure 13.25 shows the phase portrait for the underdamped case. Notice that every realized trajectory approaches a spiral point as $t \to \infty$.

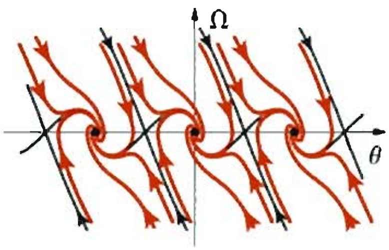


Figure 13.25
The phase portrait of an underdamped pendulum of arbitrary amplitude described by Equations 8.

Example 2:
Sketch the phase portrait for the overdamped case for the system described by Equation 8.

SOLUTION: In the overdamped case, the critical points at $(n\pi, 0)$ with $n = 0, 2, 4, \ldots$ are stable nodes like in Figure 13.2, rather than spiral points as in Figure 13.25. Therefore, the phase portrait looks like the one shown in Figure 13.26.

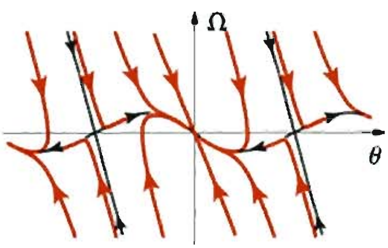


Figure 13.26
The phase portrait of an overdamped pendulum of arbitrary amplitude described by Equations 8.

We'll study one more equation in this section. The equation

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0 \qquad \epsilon > 0 \tag{10}$$

has some interesting properties. If $x < 1$, then the damping term is negative, which means that it causes $x(t)$ to increase with time. As x exceeds unity, the damping term becomes positive, meaning that it causes $x(t)$ to decrease. We might expect, then, that a balance will be reached, the solution will become stable and periodic, and that Equation 10 will admit a limit cycle. Furthermore, even with initial conditions such as $x(0) = 0.0010$ and $\dot{x}(0) = 0$, the system will approach its limit cycle and become periodic with its amplitude being independent of the value of $x(0)$. This behavior is shown in Figure 13.27.

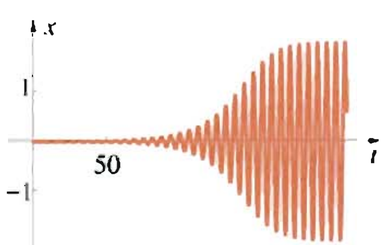


Figure 13.27
The displacement of a van der Pol oscillator plotted against time for $x(0) = 0.0010$ and $\dot{x}(0) = 0$.

Realize that the amplitude grows into its final value (determined by the limit cycle) without any input of energy. This type of behavior is called a *self-excited oscillation*, and can occur only in nonlinear systems. Equation 10 is called the *van der Pol equation*, and is a classic equation of nonlinear mechanics. It was originally derived to describe certain electrical circuits with feedback, but it (and its relatives)

13.4 Problems

1. Do you see why the point $x = 1$ in Equation 1 might be a special point? Plot the potential energy.
2. Verify Equations 3.
3. Show that the eigenvector directions of $(1, \pm\sqrt{2})^T$ passing through the points $(\pm 1, 0)$ are described by $y = \pm\sqrt{2}(x - 1)$ and $y = \pm\sqrt{2}(x + 1)$, respectively.
4. A spring governed by a potential energy of the form $V(x) = \frac{1}{2}kx^2 - \frac{1}{4}\beta x^4$ is said to be *hard* if $\beta < 0$ and *soft* if $\beta > 0$. For example, the spring associated with Equation 1 is soft, and is the more interesting case. Sketch the phase portrait for a nonlinear oscillation governed by a potential $V(x) = \frac{1}{2}x^2 - \frac{1}{40}x^4$ and see why we say that the soft case is more interesting physically. (Take $m = 1$.)
5. Sketch the phase portrait of the nonlinear oscillator in the previous problem, but let there be a damping term $+\dot{x}$.
6. Consider the nonlinear oscillator described by $\ddot{x} + x - \frac{1}{4}x^2 = 0$. If this equation is solved numerically with the initial conditions $x(0) = 4$, $\dot{x}(0) = 0$, the result is shown in Figure 13.29. Why do we obtain this result?

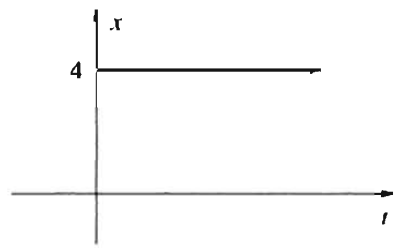


Figure 13.29
The numerical solution to $\ddot{x} + x - \frac{1}{4}x^2 = 0$ for the initial conditions $x(0) = 4$, $\dot{x}(0) = 0$.

7. Determine the phase portrait of the oscillator described in the previous problem.
8. Which of the curves in Problem 7 is the separatrix?
9. Show that it takes an infinite time for the system to reach the point $(4, 0)$ along the separatrix of the previous problem.
10. Show that the trajectories of mechanical systems cross the x axis at right angles except at critical points.
11. Determine and classify the critical points for the nonlinear oscillator described by $\ddot{x} + 9x - x^3 = 0$.
12. What is the reason for the critical points at $(\pm 3, 0)$ in the previous problem? *Hint*: Plot the potential energy.
13. The differential equation for an oscillator whose linear part of the force is repulsive rather than attractive is $\ddot{x} - x + \frac{1}{4}x^3 = 0$. Determine the nature of the critical points for this system. Sketch its phase portrait.
14. An oscillator moves in the potential $V(x) = \dfrac{1}{4} - \dfrac{1}{2}x^2 + \dfrac{1}{4}x^4$. Plot this potential. Determine the nature of the critical points and sketch the phase portrait.
15. An oscillator moves in the potential $V(x) = \dfrac{1}{4} - \dfrac{3x^2}{10} - \dfrac{x^3}{8} + \dfrac{3x^5}{40} + \dfrac{x^6}{10}$. Plot this potential. Use this plot to determine the nature of the critical points. Sketch the phase portrait. (See the previous problems.)
16. Plot the potential $V(x) = \dfrac{1}{4} - \dfrac{18x^2}{115} - \dfrac{7x^3}{23} - \dfrac{33x^4}{460} - \dfrac{21x^5}{115} + \dfrac{x^6}{10}$. (Note that there is no left-hand minimum here.) Sketch the phase portrait for this system and compare it to the phase portrait of the two previous problems.

17. This problem continues the three previous problems. Show that the critical points of $\ddot{x} + f(x) = 0$, where $f(x)$ is a polynomial, must either be centers or saddle points. Can you show that centers and saddle points must occur alternately along the x axis? *Hint*: Expand $f(x)$ about the critical points.

18. Show that each of the following equations has a limit cycle:

(a) $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x^3 = 0 \quad \epsilon > 0$

(b) $\ddot{x} + \epsilon(2x^2 - 1)\dot{x} + x + x^3 = 0 \quad \epsilon > 0$

19. In Example 3, we showed that the van der Pol equation has a limit cycle about the origin if $\epsilon > 0$. Show that the eigenvalues corresponding to the equation linearized about the origin are $\lambda_\pm = \epsilon/2 \pm (\epsilon^2 - 4)^{1/2}/2$. Show that the origin is an unstable spiral point if $0 < \epsilon < 2$. How do you reconcile this result with the existence of the limit cycle?

20. Another classic nonlinear oscillator equation is *Rayleigh's equation*, $\ddot{x} - \epsilon(\dot{x} - \frac{1}{3}\dot{x}^3) + x = 0$. Lord Rayleigh, certainly one the greatest physicists of the 19th century, first derived this equation to describe the vibrations of a violin string caused by moving the bow across it, but it has since been applied to the screech when chalk is dragged across a blackboard, the squeaking of an unoiled hinge, the waving of a flag in the wind, and a number of other systems. Use any CAS to show that Rayleigh's equation displays self-excited oscillations. Take $x(0) = 0.010$, $\dot{x}(0) = 0$, and $\epsilon = 0.0100$.

21. Use any CAS to show that Rayleigh's equation (see the previous problem) has a limit cycle of radius 2 for small values of ϵ.

13.5 Population Dynamics

The mathematical modelling of population growth is quite sophisticated and embraces a number of mathematical methods, but here we shall present only the fundamental ideas. Nevertheless, even at this simple level the basic equations are nonlinear and require the techniques that we have developed in this chapter.

We'll start off with the simplest system, describing a single population. We assume that the population grows at a rate that is proportional to the size of the population:

$$\dot{P} = kP \tag{1}$$

For $k > 0$, Equation 1 leads to an exponential growth, which might be valid during the initial stage of growth, but cannot persist indefinitely due to limited resources or overcrowding. We can include these effects by letting k be a function of P which decreases as P increases, so that the rate of growth decreases with increasing population. A simple functional dependence that reflects this effect is $k = a - bP$, in which case Equation 1 becomes

$$\dot{P} = aP - bP^2 \tag{2}$$

Both a and b are positive in Equation 2. You can see that the P^2 term leads to a decrease in $\dot{P}$ as P increases. Equation 2 is readily solved by separation of variables and its solution is (Problem 1)

$$P(t) = \frac{(a/b)P_0}{P_0 + (a/b - P_0)e^{-at}} \tag{3}$$

where $P_0 = P(0)$. Note that $P(t) \to a/b$ as $t \to \infty$, so that Equation 3 predicts a stable population. You can see this same result by setting $\dot{P}$ equal to zero in Equation 2. Equation 3 is plotted against t in Figure 13.30 for $a/b = 50$ and several values of P_0. Equation 1, or its solution, Equation 3, is called the *logistic equation*, and is one of the early equations of population dynamics.

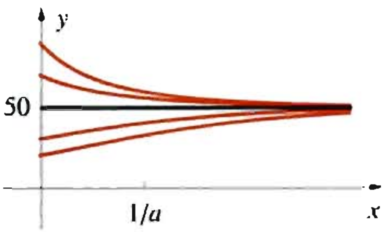


Figure 13.30
Equation 3 plotted against t for various values of P_0.

Example 1:
Modify Equation 1 to include the effect of the addition of individuals to the population due to immigration, for example, at a constant rate I.

SOLUTION: Equation 1 becomes

$$\dot{P} = aP - bP^2 + I$$

This equation can be solved by separation of variables and yields (Problem 2)

$$P(t) = \frac{a}{2b} + \frac{\kappa}{2b}\left(\frac{1 + Ae^{-\kappa t}}{1 - Ae^{-\kappa t}}\right) \tag{4}$$

where

$$\kappa = (a^2 + 4bI)^{1/2}$$

and

$$A = \frac{P_0 - a/2b - \kappa/2b}{P_0 - a/2b + \kappa/2b}$$

Note that Equation 4 says that $P(t) \to (a + \kappa)/2b$ as $t \to \infty$, independently of the value of P_0. Figure 13.31 shows $P(t)$ plotted against t for $a = 0.020$, $b = 0.00020$, and $I = 10$ for various values of P_0. The limiting value of $P(t)$ is $(a + \kappa)/2b = 279$.

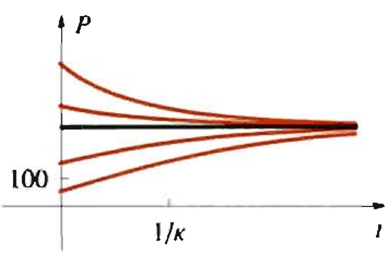


Figure 13.31
Equation 4 plotted against t for $a = 0.020$, $b = 0.00020$, and $I = 10$ for various values of P_0. The limiting value of P_0 is $[a + (a^2 + 4bI)^{1/2}]/2b = 279$.

A more interesting population study is that of two species that interact with each other. One species (the *prey*) lives on an abundant supply of food in its environment, while the other species (the *predator*) lives off the prey. Let $x(t)$ be the population of the prey and $y(t)$ be the population of the predator. Then $\dot{x}(t)$ increases proportionally to $x(t)$ (as in Equation 1) and decreases proportionally to its encounters with the predator. If we assume that the encounters are proportional to $x(t)y(t)$, then $\dot{x}(t)$ is given by

$$\dot{x}(t) = ax(t) - bx(t)y(t) \tag{5}$$

where a and b are positive constants. Similarly, if we assume that the only food source of the predators is the prey, then $\dot{y}(t)$ is given by

$$\dot{y}(t) = -cy(t) + dx(t)y(t) \tag{6}$$

where c and d are positive constants.

Equations 5 and 6 are classic equations of population dynamics and are called the *predator-prey equations*, or the *Lotka-Volterra equations*. They were introduced independently in the 1920s by the American biophysicist A.J. Lotka and the Italian mathematician Vito Volterra. There are extensive data of population records and the Lotka-Volterra equations have been applied to shark/fish populations, lynx/hare populations, bass/sunfish populations, ladybug/aphid populations, and many others. The constants a, b, c, and d are empirical constants that are used to fit the population data.

Equations 5 and 6 have two critical points, $(0, 0)$ and $(c/d, a/b)$. The origin is a saddle point and can be reached asymptotically only if $x(t) = 0$, in which case $y(t)$ decays exponentially to zero according to Equation 6. The other critical point is much more interesting. Equations 5 and 6 linearized about $(c/d, a/b)$ are (Problem 3)

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{bc}{d} \\ \dfrac{ad}{b} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{7}$$

The eigenvalues are $\lambda^2 = \pm i(ac)^{1/2}$, indicating that the critical point is a (stable) center for the linearized system. The trajectories are ellipses centered at the critical point (Problem 4).

Recall, however, that if the eigenvalues of the linearized system are $\pm i\mu$ (indicating a center), then the critical point of the parent nonlinear equations may be either a center or a spiral point. Equations 5 and 6 present no problem, however, because if we divide one by the other, we obtain

$$\frac{\dot{x}}{\dot{y}} = \frac{dx}{dy} = \frac{ax - bxy}{-cy + dxy} = \frac{x(a - by)}{y(dx - c)}$$

which can be readily integrated to get

$$a \ln y - by + c \ln x - dx = \text{constant} \tag{8}$$

It's not obvious (nor that easy to prove), but Equation 8 represents a family of closed curves in the phase plane. Equation 8 is shown in Figure 13.32 for $a = 1.0$, $b = 0.040$, $c = 4.0$, and $d = 0.020$. Note that the curves are centered at $x = c/d = 200$ and $y = a/b = 25$. Figure 13.33 shows that both $x(t)$ and $y(t)$ are periodic functions, but we have to solve Equations 5 and 6 numerically to obtain $x(t)$ and $y(t)$.

We can learn about the solutions to Equation 5 qualitatively by restricting ourselves to regions near the critical point, where we can use the linearized version,

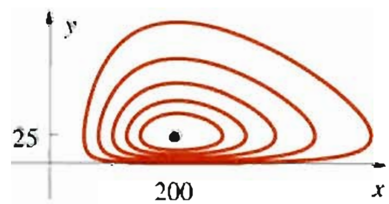


Figure 13.32
Equation 8 with $a = 1.0$, $b = 0.040$, $c = 4.0$, and $d = 0.020$ plotted in the phase plane.

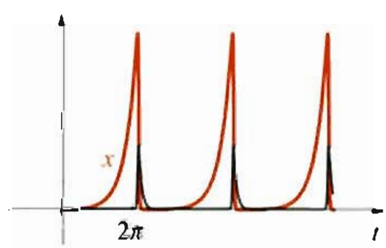


Figure 13.33
Plots of $x(t)$ (color) and $y(t)$ (black) from Equations 5 and 6 with $a = 1.0$, $b = 0.040$, $c = 4.0$, and $d = 0.020$ plotted against t.

Equation 7, to determine $x(t) = u(t) + c/d$ and $y(t) = v(t) + b/a$. The solutions to Equation 7 are (Problem 5)

$$x(t) = \frac{c}{d} + C\cos[(ac)^{1/2}t + \phi]$$
$$y(t) = \frac{a}{b} + \frac{a^{1/2}d}{bc^{1/2}}C\sin[(ac)^{1/2}t + \phi] \tag{9}$$

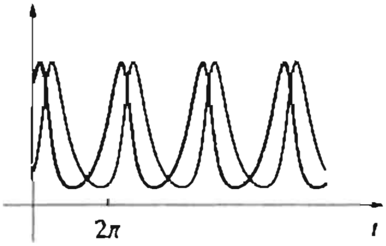


Figure 13.34
The numerical solution of Equations 5 and 6 for $x(t)$ (color) and $y(t)$ (black) for $a = b = c = d = 1$ and $x(0) = 2.0$ and $y(0) = 0.50$.

These equations say that $x(t)$ and $y(t)$ are periodic with a period $2\pi/(ac)^{1/2}$ that does not depend upon the initial conditions. Figure 13.34 shows $x(t)$ and $y(t)$ obtained numerically from Equations 5 and 6 with $a = b = c = d = 1$ and $x(0) = 2.0$ and $y(0) = 0.50$.

Note that the two curves have a period of about 2π. The predator curve lags behind that of the prey, as you might expect physically. As the population of the prey builds up, the predator has an ample food supply, and so its population grows at the expense of the prey. Then, as the density of the prey decreases, the density of the predator decreases, thus allowing the density of the prey to increase, and this cycle continues to play out. The following Example shows that phase difference between the two curves in Figure 13.34 is one quarter of the period, or $\pi/2(ac)^{1/2}$.

Example 2:
Use Equations 9 to show that the phase difference between $x(t)$ and $y(t)$ is $\pi/2(ac)^{1/2}$.

SOLUTION: Let's find the difference in times when $\cos[(ac)^{1/2}t + \phi] = 1$ and $\sin[(ac)^{1/2}t + \phi] = 1$. Let $t = t_c$ when the cosine equals 1 and $t = t_s$ when the sine equals 1. This occurs when $(ac)^{1/2}t + \phi = 2n\pi$ $(n = 0, 1, 2, \ldots)$ for the cosine and when $(ac)^{1/2}t + \phi = 2n\pi + \pi/2$ $(n = 0, 1, 2, \ldots)$ for the sine. The difference between t_c and t_s is $\pi/2(ac)^{1/2}$, or $[2\pi/(ac)^{1/2}]/4$.

Notice that neither Equation 5 nor 6 reduces to Equation 2 when $b = d = 0$, when the two populations do not interact. In the absence of any interaction, the population of the prey increases indefinitely, while that of the predator decreases indefinitely. We can, therefore, improve the simple Lotka-Volterra model by including negative quadratic terms in Equations 5 and 6. Let's consider the equations

$$\begin{aligned} \dot{x} &= 3x - xy - 2x^2 \\ \dot{y} &= -y + 2xy - y^2 \end{aligned} \tag{10}$$

These equations have critical points at (0, 0), (0, −1), (3/2, 0), and (1, 1). We can ignore the one at (0, −1) because $x(t)$ and $y(t)$ must be greater than zero. The critical point at the origin is a saddle point, which is approached only along the line $x = 0$. The critical point at (3/2, 0) is a saddle point with eigenvalues (−3, 2) and corresponding eigenvectors $(1, 0)^T$ and $(-3/10, 1)^T$. The asymptotes

We have considered only pairs of coupled nonlinear differential equations, but one of the requirements that a system of nonlinear first-order differential equations must fulfill to show chaotic behavior is that there be at least three coupled nonlinear equations. It may not be obvious at first sight that Equation 11 can be written as three first-order equations, but if we let $y = \dot{x}$ and $z = t$, then Equation 11 becomes

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x^3 - 0.050y + 7.50\cos z \qquad (12)\\ \dot{z} &= 1\end{aligned}$$

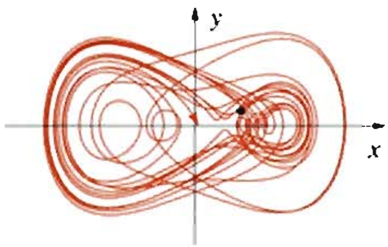


Figure 13.38
A parametric plot of $y(t)$ against $x(t)$ for Equations 12.

and these three equations lead to chaotic behavior. The parametric plot of $y(t)$ against $x(t)$ in Figure 13.38 suggests a chaotic behavior.

We don't mean to imply that all sets of three or more coupled nonlinear first-order equations lead to chaotic behavior. Both the forms of the equations and the numerical coefficients must be just so. Unfortunately, there are no simple general criteria that tell whether or not a set of equations will display chaotic behavior, but many systems have been studied both experimentally and computationally and there is an extensive literature on the subject. Problem 20 discusses the Lorenz equations, which are the classical equations of chaos theory.

13.5 Problems

1. Solve Equation 2 to obtain Equation 3.
2. Derive Equation 4.
3. Linearize Equations 5 and 6 about the critical point $(c/d, a/b)$.
4. Show that the solutions to Equations 7 are ellipses centered at $(c/d, a/b)$.
5. Derive Equations 9.

The following six problems require the use of a computer.

6. Divide $\dot{y}$ by $\dot{x}$ to construct a phase portrait for the predator-prey system

$$\begin{aligned}\dot{x} &= -2x - 0.0050xy\\ \dot{y} &= 4y - 0.015xy\end{aligned}$$

7. Compute $x(t)$ and $y(t)$ for the system in Problem 6 and compare the period to $2\pi/(ac)^{1/2}$.
8. Plot $x(t)$ and $y(t)$ parametrically for the system in Problem 6 and compare your result to the appropriate member of the family plotted in Problem 6.
9. Figure 13.39 shows the parametric plot of $y(t)$ against $x(t)$ for the system

$$\begin{aligned}\dot{x} &= 5x - 0.40xy\\ \dot{y} &= -10y + 0.20xy\end{aligned}$$

with $x(0) = 200$ and $y(0) = 10$. Interpret this result.

16. Analyze the competing species model

$$\dot{x} = 10x - 2x^2 - 5xy$$
$$\dot{y} = 14y - 4y^2 - 2xy$$

What is the ultimate fate of each species?

17. Analyze the competing species model

$$\dot{x} = 6x - 2x^2 - xy$$
$$\dot{y} = 8y - 3y^2 - 2xy$$

Show that the two species coexist in this case.

18. Equations similar to those in this section have been used to model chemical reactions involving (intermediate) species whose concentrations oscillate in time. Consider the kinetic scheme

$$\mathrm{A} + \mathrm{X} \xrightarrow{k_1} 2\,\mathrm{X}$$
$$\mathrm{X} + \mathrm{Y} \xrightarrow{k_2} 2\,\mathrm{Y}$$
$$\mathrm{Y} \xrightarrow{k_3} \mathrm{P}$$

Add these reactions to show that the overall reaction is $\mathrm{A} \longrightarrow \mathrm{P}$. (That's why X and Y are called "intermediates.") Show that the rate equations corresponding to this reaction scheme are

$$\dot{\mathrm{X}} = k_1\mathrm{XA} - k_2\mathrm{XY}$$
$$\dot{\mathrm{Y}} = k_2\mathrm{XY} - k_3\mathrm{Y}$$

where A, X, and Y represent concentrations. Take $\mathrm{A} = \mathrm{A}_0 = 1.00\ \mathrm{mol \cdot L^{-1}}$ (this can be readily achieved by adding A to the reaction system as the reaction proceeds), $k_1 = 1.00\ \mathrm{L \cdot mol^{-1} \cdot s^{-1}}$, $k_2 = 0.500\ \mathrm{L \cdot mol^{-1} \cdot s^{-1}}$, $k_3 = 0.100\ \mathrm{s^{-1}}$, and $\mathrm{X}(0) = \mathrm{Y}(0) = 1.00\ \mathrm{mol \cdot L^{-1}}$ and show that $\mathrm{X}(t)$ and $\mathrm{Y}(t)$ are periodic. *Hint*: Determine the phase portrait for this system.

19. Equations similar to those in this section have been used to model epidemics. Suppose that out of a total population of n individuals, x of them are considered to be susceptible, y of them are infected and can transmit the disease to a susceptible individual, and z of them are recovered and immune. Argue that the equations

$$\dot{x} = -\beta xy$$
$$\dot{y} = \beta xy - \gamma y$$
$$\dot{z} = \gamma y$$

can model the spread of the disease. Show that $x + y + z =$ constant. Take $\beta = 0.20$ and $\gamma = 0.040$, $x(0) = 1$ (in some units) and $y(0) = 0.030$ and solve the equations numerically for $x(t)$ and $y(t)$ and discuss the result.

20. The concept of chaotic systems came to the attention of physical scientists and engineers in 1963 when the MIT atmospheric scientist E.N. Lorenz proposed a simple model for thermally induced fluid convection in the atmosphere. His model consisted of three coupled nonlinear equations:

$$\dot{x} = \alpha(y - x)$$
$$\dot{y} = \beta x - y - xz$$
$$\dot{z} = xy - \gamma z$$

There are hundreds of papers discussing these equations, and many of them choose $\alpha = 10$, $\gamma = 8/3$, and allow β to vary.

Using these parameters, show that there are critical points at $(0, 0, 0)$, $([\frac{8}{3}(\beta - 1)]^{1/2}, [\frac{8}{3}(\beta - 1)]^{1/2}, \beta - 1)$, and $(-[\frac{8}{3}(\beta - 1)]^{1/2}, -[\frac{8}{3}(\beta - 1)]^{1/2}, \beta - 1)$. Use any CAS to show that the eigenvalues associated with $(0, 0, 0)$ are real and differ in sign (two negative and one positive) when $\beta > 1$, and the eigenvalues associated with the other two critical points are the same and that one is negative and the other two are of the form $a \pm ib$. Now show that $a < 0$ if $\beta < 24.74\ldots$ and $a > 0$ if $\beta > 24.74\ldots$. How would you interpret this result?

Now solve the Lorenz equations for $\beta = 15$ and $x(0) = 15$, $y(0) = 10$, and $z(0) = 15$ and then for $\beta = 15$ and $x(0) = 15$, $y(0) = 10.1$, and $z(0) = 15$, and show that the results are essentially identical. Now do the same thing with $\beta = 30$ and show that the two solutions differ markedly.

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14.1 Legendre Polynomials

We saw in Chapter 12 that the Legendre polynomials are solutions to the differential equation

$$(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0 \tag{1}$$

where n is an integer. The first few Legendre polynomials are

$$\begin{aligned} &P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2-1) \\ &P_3(x) = \frac{1}{2}(5x^3-3x) \qquad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned} \tag{2}$$

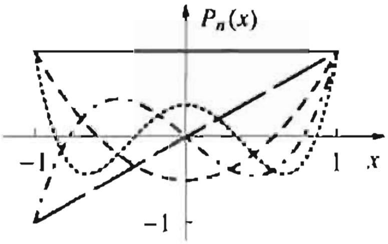

Figure 14.1
The Legendre polynomials. $P_0(x)$ (solid), $P_1(x)$ (long dashed), $P_2(x)$ (short dashed), $P_3(x)$ (dash-dot), and $P_4(x)$ (dotted).

and are plotted in Figure 14.1. Note that $P_n(x)$ has exactly $n-1$ distinct zeros in the open interval $(-1, 1)$. A general formula for $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n}\sum_{j=0}^{[n/2]}(-1)^j \frac{(2n-2j)!}{j!(n-j)!(n-2j)!}x^{n-2j} \tag{3}$$

where $[n/2] = n/2$ when n is even and $(n-1)/2$ when n is odd. Because x appears as x^{n-2j} in Equation 3, $P_n(x)$ is an even function if n is even and an odd function if n is odd (Problem 1).

It shouldn't be apparent at this point, but the Legendre polynomials form an orthogonal set of functions over the interval $[-1, 1]$. For example,

$$\int_{-1}^{1} P_1(x)P_3(x)dx = \frac{1}{2}\int_{-1}^{1} x(5x^3-3x)dx = 0$$

Generally,

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \qquad \text{if} \quad m \neq n \tag{4}$$

The orthogonality property of the Legendre polynomials follows from the differential equation that defines them. Let's go back to Equation 1 and write it in the form

$$-[(1-x^2)P_n'(x)]' = n(n+1)P_n(x) \tag{5}$$

with a similar equation for $P_m(x)$

$$-[(1-x^2)P_m'(x)]' = m(m+1)P_m(x) \tag{6}$$

Multiply Equation 5 by $P_m(x)$ and integrate from -1 to $+1$:

$$-\int_{-1}^{1}[(1-x^2)P_n'(x)]'P_m(x)\,dx = n(n+1)\int_{-1}^{1}P_n(x)P_m(x)dx$$

Integrate the left side by parts to obtain

$$\int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx = n(n+1)\int_{-1}^{1}P_n(x)P_m(x)\,dx \tag{7}$$

Now multiply Equation 6 by $P_n(x)$ and integrate by parts from -1 to $+1$ to get

$$\int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx = m(m+1)\int_{-1}^{1}P_n(x)P_m(x)\,dx \tag{8}$$

Subtract Equation 8 from Equation 7 to get

$$[n(n+1)-m(m+1)]\int_{-1}^{1}P_n(x)P_m(x)\,dx = 0$$

If $n \neq m$, then we get Equation 4, the orthogonality condition of the Legendre polynomials.

The Legendre polynomials also satisfy a number of recursion formulas. For example,

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \qquad n \geq 1 \tag{9}$$

Example 1:
Use Equation 9 to derive expressions for $P_2(x)$ and $P_3(x)$ from $P_0(x) = 1$ and $P_1(x) = x$.

SOLUTION: Let $n = 1$ in Equation 9:

$$2P_2(x) = 3xP_1(x) - P_0(x)$$

or

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

For $n = 2$:

$$3P_3(x) = 5xP_2(x) - 2P_1(x)$$

or

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Equation 9 is particularly well suited for numerical routines.

The function

$$G(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n \tag{10}$$

is called a *generating function* for the Legendre polynomials. If you know the function $G(x,t)$, then the coefficient of t^n in the Maclaurin expansion of $G(x,t)$

is $P_n(x)$. Thus, $G(x, t)$ is said to generate the Legendre polynomials. Problem 6 helps you derive an explicit expression for $G(x, t)$ from Equation 9. The result is

$$G(x, t) = \frac{1}{(1 - 2xt + t^2)^{1/2}} \tag{11}$$

Example 2:
Use Equation 11 to generate the first three Legendre polynomials.

SOLUTION: Use the expansion

$$(1 - z)^{-1/2} = 1 + \frac{z}{2} + \frac{3}{8}z^2 + \frac{5}{16}z^3 + \cdots$$

with $z = 2xt - t^2$.

$$\begin{aligned} G(x, t) &= 1 + xt - \frac{t^2}{2} + \frac{3}{8}[4x^2t^2 - 4xt^3 + O(t^4)] + \frac{5}{16}[8x^3t^3 + O(t^4)] \\ &= 1 + xt + \frac{3x^2 - 1}{2}t^2 + \frac{5x^3 - 3x}{2}t^3 + O(t^4) \end{aligned}$$

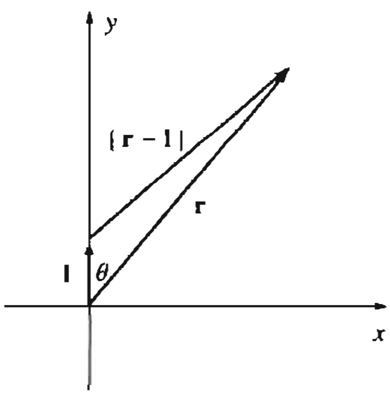

Figure 14.2
The geometry used to calculate the electrostatic potential at **r** due to a charge located at **l**.

Equation 11 may not look familiar, but you actually know the equation from electrostatics. Suppose an electric charge q is located at a point specified by a vector **l**. The electrostatic potential at a point specified by **r** (Figure 14.2) is given by

$$V = \frac{q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{l}|} \tag{12}$$

where ϵ_0 is the permittivity of free space. Using the law of cosines, $|\mathbf{r} - \mathbf{l}| = (r^2 + l^2 - 2rl\cos\theta)^{1/2}$, Equation 12 becomes

$$V = \frac{q}{4\pi\epsilon_0 r\left(1 - 2\frac{l}{r}\cos\theta + \frac{l^2}{r^2}\right)^{1/2}} \tag{13}$$

Comparing Equation 13 to Equation 11, with $x = \cos\theta$ and $t = l/r$, and using Equation 10 gives

$$V = \frac{q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{l^n P_n(\cos\theta)}{r^{n+1}} \tag{14}$$

Suppose now we have a set of charges q_j at positions specified by $\mathbf{l}_j$. The potential is given by

$$V = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\left[\sum_j q_j l_j^n P_n(\cos\theta_j)\right] \tag{15}$$

Equation 15 has an important physical interpretation. Let the summation over j in Equation 15 be denoted by

$$M_n = \sum_j q_j l_j^n P_n(\cos\theta_j) \tag{16}$$

Then V can be written as

$$V = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{M_n}{r^{n+1}} \tag{17}$$

Let's apply Equation 17 to the two-charge distribution shown in Figure 14.3. Using Equation 16, the first few M_n in Equation 17 are

$$M_0 = q - q = 0$$

$$M_1 = \frac{ql}{2}\cos\theta - \frac{ql}{2}\cos(\pi-\theta)$$

$$= ql\cos\theta = \mu\cos\theta$$

where μ is the magnitude of the dipole moment of the two-charge distribution, and

$$M_2 = \frac{ql^2}{2}P_2(\cos\theta) - \frac{ql^2}{2}P_2[\cos(\pi-\theta)] = 0$$

Equation 17 becomes

$$V = \frac{\mu\cos\theta}{4\pi\epsilon_0 r^2} + O\left(\frac{1}{r^4}\right) \tag{18}$$

Equation 18 is the electrostatic potential due to a dipole located at the origin of a coordinate system. If $r \gg l$, then the first term dominates all the others.

Figure 14.4 shows equipotential lines and the corresponding electric field for the dipole shown in Figure 14.3. Recall that the electric field is given by $\mathbf{E} = -\text{grad}\ V$, which in this case is

$$\mathbf{E} = \frac{\partial V}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\mathbf{e}_\theta$$

$$= -\frac{\mu\cos\theta}{4\pi\epsilon_0 r^3}\mathbf{e}_r - \frac{\mu\sin\theta}{4\pi\epsilon_0 r^3}\mathbf{e}_\theta$$

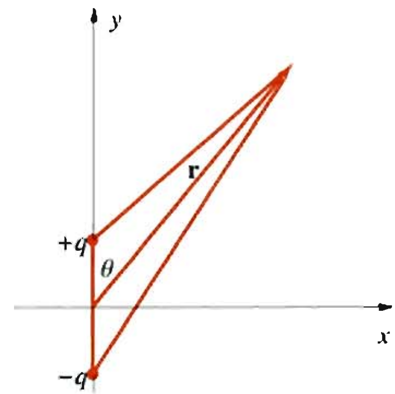

Figure 14.3
The two-charge distribution used to derive Equation 18.

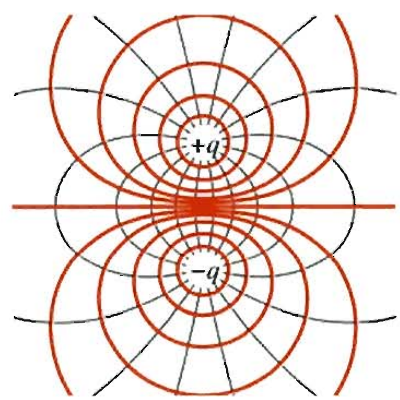

Figure 14.4
The equipotential lines (color) and the corresponding electric field (black) due to the dipole shown in Figure 14.3.

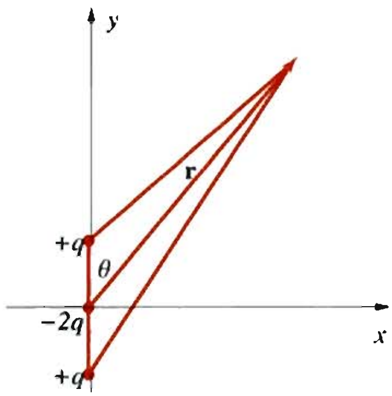

Figure 14.5
The linear quadrupole that is used to determine the electrostatic potential in Example 3.

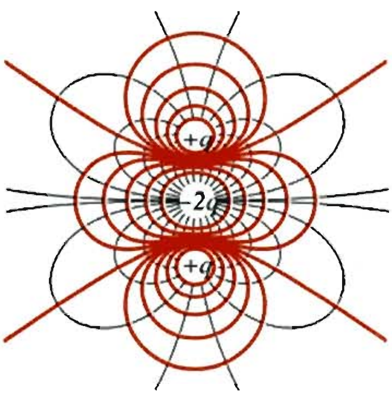

Figure 14.6
The equipotential lines (color) and the corresponding electric field (black) due to the linear quadrupole shown in Figure 14.5.

Example 3:
Apply Equation 17 to the point charge distribution shown in Figure 14.5.

SOLUTION: Using Equation 16, the first few M_n are $M_0 = 0$,

$$M_1 = \frac{ql}{2}\cos\theta + \frac{ql}{2}\cos(\pi - \theta) = 0$$

$$M_2 = \frac{ql^2}{4}P_2(\cos\theta) + \frac{ql^2}{4}P_2(\cos(\pi - \theta))$$

$$= \frac{ql^2}{2}P_2(\cos\theta) = \frac{ql^2}{2}\frac{3\cos^2\theta - 1}{2}$$

$$M_3 = \frac{ql^3}{8}P_3(\cos\theta) + \frac{ql^3}{8}P_3(\cos(\pi - \theta)) = 0$$

To calculate M_2 and M_3, we have used the fact that $P_2(-\cos\theta) = +P_2(\cos\theta)$ and that $P_3(-\cos\theta) = -P_3(\cos\theta)$.

Equation 17 becomes

$$V = \frac{Q}{4\pi\epsilon_0}\frac{P_2(\cos\theta)}{r^3} + O\left(\frac{1}{r^5}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{3\cos^2\theta - 1}{2r^3} + O\left(\frac{1}{r^5}\right)$$

where Q is the magnitude of the quadrupole moment of the charge distribution. If $r > l$, then the potential due to the quadrupole at the origin is given by the first term above. The equipotential lines and electric field (arrows) due to a quadrupole moment located at the origin are shown in Figure 14.6.

The quantities M_n in Equation 16 are called *multipole moments*, with M_1 being the magnitude of the dipole moment, μ, with M_2 being the magnitude of the quadrupole moment, Q, and so on, and Equation 17 is called a multipole expansion. Multipole expansions play a key role in the theory of the interactions between molecules.

Equation 11 is awkward to use to generate Legendre polynomials, but it is very useful for developing general properties of Legendre polynomials. For example, Problems 7 and 8 have you use Equation 11 to show generally that

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1} \tag{19}$$

Combining this result with Equation 4 gives

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm} \tag{20}$$

Example 4:
Show that the first few Legendre polynomials obey Equation 19.

SOLUTION:

$$\int_{-1}^{1} P_0^2(x)dx = 2 \qquad \int_{-1}^{1} P_1^2(x)dx = \int_{-1}^{1} x^2dx = \frac{2}{3}$$

$$\int_{-1}^{1} P_2^2(x)dx = \frac{1}{4}\int_{-1}^{1}(3x^2-1)^2dx = \frac{2}{5}$$

A useful property of Legendre polynomials, as well as other orthogonal polynomials that we shall encounter in this chapter, is that it is possible to expand a suitably behaved function $f(x)$ as an infinite series of Legendre polynomials

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{21}$$

We shall explain below what we mean by this equality, but first let's determine the a_n by multiplying both sides of Equation 21 by $P_m(x)$, integrating over x, and using Equation 20 to obtain

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)P_m(x)\,dx = \sum_{n=0}^{\infty} \frac{2a_n}{2n+1}\delta_{nm}$$
$$= \frac{2a_m}{2m+1}$$

or

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)\,dx \tag{22}$$

Equation 22, along with Equation 20, is reminiscent of an expansion of a vector in terms of a basis of orthogonal vectors. In fact, suppose that $f(x)$ is continuous on the interval $(-1, 1)$. It is easy to show that the set of all continuous functions on $(-1, 1)$ forms a vector space. It is an infinite dimensional vector space, and so requires an infinite number of basis vectors, as in Equation 21. The inner product in this vector space is defined by

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx$$

where $f(x)$ and $g(x)$ are any two vectors in the vector space. The a_m given by Equation 22 can be thought of as the components of $f(x)$ in the basis $\{P_m(x)\}$, $m = 0, 1, 2, \ldots$.

This inequality is called *Bessel's inequality*, which is valid for any value of N, including $N \to \infty$.

SOLUTION: Start with

$$\int_{-1}^{1} \left[f(x) - \sum_{n=0}^{N} a_n P_n(x) \right]^2 dx \geq 0$$

Then,

$$\int_{-1}^{1} f^2(x)dx - 2\sum_{n=0}^{N} a_n \int_{-1}^{1} f(x)P_n(x)\,dx$$
$$+ \sum_{n=0}^{N}\sum_{m=0}^{N} a_n a_m \int_{-1}^{1} P_n(x)P_m(x)\,dx \geq 0$$

or

$$\int_{-1}^{1} f^2(x)dx \geq \sum_{n=0}^{N} \frac{2}{2n+1} a_n^2 \tag{25}$$

You can use Bessel's inequality to show that $\lim_{n\to\infty} [2/(2n+1)]^{1/2} a_n = 0$ (Problem 18).

The Legendre series $\sum_{n=0}^{\infty} a_n P_n(x)$ of any square integrable function $f(x)$ converges to $f(x)$ in the mean. We say that the Legendre polynomials form a *complete set*; they span the vector space of square integrable functions. Because the Legendre polynomials form a complete set, Equation 25 implies that

$$\int_{-1}^{1} f^2(x)dx = \sum_{n=0}^{\infty} \frac{2}{2n+1} a_n^2 \tag{26}$$

Equation 26 is called *Parseval's equality*. You can think of it as an infinite vector space version of Pythagoras's theorem.

Equation 26 is a statement of the completeness of the Legendre polynomials. You might think that the Legendre polynomials are complete because there is an infinite number of them, but that is not so. If we snip out one of them, say $P_{78}(x)$, we still have an infinite number of them, but they are no longer complete. Although Equation 26 gives us a definition of completeness, it is difficult in practice to show that any set of functions is complete.

14.1 Problems

1. Use Equation 3 to generate the first few Legendre polynomials.

2. Show that $P_n(x)$ is orthogonal to every power of x less than n. In other words, show that $\int_{-1}^{1} x^s P_n(x)dx = 0$ if $s < n$. *Hint*: Use the fact that we can express x^s as a linear combination $x^s = \sum_{j=0}^{s} a_j P_j(x)$.

3. Use Equation 3 to show that $P_n(-x) = (-1)^n P_n(x)$.

4. Show explicitly that $P_1(x)$ is orthogonal to $P_2(x)$ and to $P_4(x)$.

5. Use Equation 9 to derive an expression for $P_4(x)$ from $P_2(x)$ and $P_3(x)$.

6. We'll see how to derive a generating function $G(x, t)$ from Equation 9 in this problem. Multiply Equation 9 by t^n and sum from $n = 0$ to ∞ to obtain $G'(t) - 2xtG' - xG + t(tG)' = 0$ or $(1 - 2xt + t^2)G' = (x - t)G$. Integrate this expression and use the condition $P_0 = 1$ to obtain Equation 11.

7. In this problem and the next problem, we're going to use the generating function (Equation 11) to show that the Legendre polynomials are orthogonal. Start with

$$G(x, t)G(x, u) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^n u^m$$
$$= (1 - 2xt + t^2)^{-1/2}(1 - 2xu + u^2)^{-1/2}$$

Now argue that the Legendre polynomials are orthogonal if $\int_{-1}^{1} G(x, t)G(x, u)dx =$ function of tu only.

8. Carry out the integration in the previous problem and show explicitly that the result is a function of tu only.

9. Let $x = 1$ in the generating function in Equation 11 to show that $P_n(1) = 1$.

10. Use the generating function in Equation 11 to show that $P_n(-1) = (-1)^n$.

11. The generating function in Equation 11 is a function of $2xt - t^2$, which we can write as $G(x, t) = F(2xt - t^2)$. First show that $\dfrac{\partial G}{\partial x} = 2tF'$ and $\dfrac{\partial G}{\partial t} = (2x - 2t)F'$, where F' means $dF/d(2xt - t^2)$. Now show that $(x - t)\dfrac{\partial G}{\partial x} - t\dfrac{\partial G}{\partial t} = 0$. Given that $G(x, t) = \sum_{n=0}^{\infty} P_n(x)t^n$, show that the Legendre polynomials satisfy the differential recursion formula $xP_n'(x) = nP_n(x) + P_{n-1}'(x)$.

12. We'll derive another differential recursion formula (see the previous problem) in this problem. First show that $\dfrac{\partial G}{\partial x} = t(1 - 2xt + t^2)^{-3/2} = \sum_{n=1}^{\infty} P_n'(x)t^n$ and that $\dfrac{\partial G}{\partial t} = \dfrac{x - t}{(1 - 2xt + t^2)^{3/2}} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$. Now multiply the first equation by $(1 - t^2)/t$, the second by $2t$, and subtract to show that

$$\sum_{n=1}^{\infty} P_n'(x)t^{n-1} - \sum_{n=1}^{\infty} P_n'(x)t^{n+1} - \sum_{n=1}^{\infty} 2nP_n(x)t^n = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Use this result to show that $(2n + 1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$.

13. We can derive Legendre's differential equation from the two differential recursion formulas in Problems 11 and 12. Use these two results to show that $xP_n'(x) = P_{n+1}'(x) - (n+1)P_n(x)$ or $xP_{n-1}'(x) = P_n'(x) - nP_{n-1}(x)$. Differentiate this with respect to x to obtain $xP_{n-1}''(x) = P_n''(x) - (n+1)P_{n-1}'(x)$. Now use this result and the result of Problem 11 to eliminate $P_{n-1}'(x)$ and $P_{n-1}''(x)$ to obtain $(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$.

14. Show that $\displaystyle\int_{-1}^{1} x^n P_n(x)dx = \frac{2^{n+1}(n!)^2}{(2n+1)(2n)!}$. *Hint*: Use the fact that $x^n = [2^n(n!)^2/(2n)!]P_n(x)+$ a lower degree polynomial.

15. The integral $\displaystyle\int_{-1}^{1} xP_n(x)P_m(x)dx$ occurs in atomic spectroscopy. Show that $I = \dfrac{2(n+1)}{(2n+1)(2n+3)}\delta_{m,n+1} + \dfrac{2n}{(2n+1)(2n-1)}\delta_{m,n-1}$.

16. Consider the sequence of functions $f_n(x) = x^n$ for $0 \le x \le 1$. Show that this sequence converges to $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$. Now show that it converges in the mean to $f(x) = 0$ for $0 \le x \le 1$.

17. Show that D_N^2 given by Equation 23 is a minimum with respect to α_n if they are equal to the a_n given by Equation 22.

18. Use Bessel's inequality to show that $\lim_{n\to\infty} [2/(2n+1)]^{1/2}a_n = 0$. (See Equation 25.)

19. Expand the function $f(x) = \sin \pi x$ in a series of Legendre polynomials. Plot the first few partial sums.

20. Expand $f(x) = 1 - x^2$, $-1 \le x \le 1$, in terms of Legendre polynomials. Verify Parseval's equality for this case.

21. Suppose we wish to expand a function defined on the interval (α, β) in terms of Legendre polynomials. Show that the transformation $u = (2x - \alpha - \beta)/(\beta - \alpha)$ maps the function onto the interval $(-1, 1)$.

22. Expand $f(x) = 1 - x^2/4$, $-2 \le x \le 2$, in terms of Legendre polynomials.

14.2 Orthogonal Polynomials

As we implied in the previous section, the Legendre polynomials are just one of a number of "name" polynomials that arise in applied mathematics. In this section we will present a general theory of orthogonal polynomials that encompasses any one of them as a special case.

Consider a set of functions $\phi_0(x), \phi_1(x), \ldots$. This set is said to be orthogonal over an interval $a \le x \le b$ with weight function $r(x) \ge 0$ if

$$\int_a^b r(x)\phi_i(x)\phi_j(x)dx = 0 \qquad i \ne j \tag{1}$$

Note the presence of the weight function $r(x)$, which is equal to one for the Legendre polynomials. If in addition to Equation 1, we have

$$\int_a^b r(x)\phi_i^2(x)dx = 1 \tag{2}$$

Table 14.1
Some commonly-used orthogonal polynomials.

name	symbol	interval	weight function
Legendre	$P_n(x)$	$-1 \le x \le 1$	1
Chebyshev(Tchebychef)	$T_n(x)$	$-1 \le x \le 1$	$(1-x^2)^{-1/2}$
Laguerre	$L_n(x)$	$0 \le x < \infty$	e^{-x}
Associated Laguerre	$L_n^{(\alpha)}(x)$	$0 \le x < \infty$	$x^{\alpha}e^{-x}$
Hermite	$H_n(x)$	$-\infty < x < \infty$	e^{-x^2}
Hermite	$He_n(x)$	$-\infty < x < \infty$	$e^{-x^2/2}$

The defining intervals and weight functions of some other commonly occurring sets of orthogonal polynomials are given in Table 14.1.

We devoted the entire first section of this chapter to the Legendre polynomials. The associated Laguerre polynomials reduce to the Laguerre polynomials (Example 1) when $\alpha = 0$ and also occur in the quantum-mechanical treatment of a hydrogen atom. The Hermite polynomials, $H_n(x)$, occur in the quantum-mechanical treatment of a harmonic oscillator and those designated by $He_n(x)$ in Table 14.1 are used in mathematical statistics. The first few Hermite polynomials, $H_n(x)$, are (Problem 1)

$$\begin{gathered} H_0(x) = 1 \qquad H_1(x) = 2x \qquad H_2(x) = 4x^2 - 2 \\ H_3(x) = 8x^3 - 12x \qquad H_4(x) = 16x^4 - 48x^2 + 12 \end{gathered} \tag{4}$$

Note that $H_n(-x) = (-1)^n H_n(x)$ and that the coefficient of x^n is 2^n (by convention). The Chebyshev polynomials are used in numerical analysis; the first few are given by

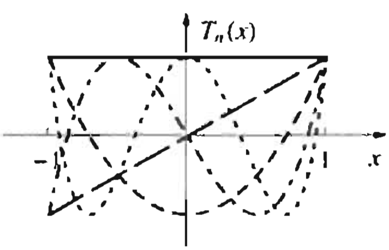

Figure 14.9
The Chebyshev polynomials, $T_0(x)$ (solid), $T_1(x)$ (long dashed), $T_2(x)$ (short dashed), $T_3(x)$ (dot-dash), and $T_4(x)$ (dotted), plotted against x.

$$\begin{gathered} T_0(x) = 1 \qquad T_1(x) = x \qquad T_2(x) = 2x^2 - 1 \\ T_3(x) = 4x^3 - 3x \qquad T_4(x) = 8x^4 - 8x^2 + 1 \end{gathered} \tag{5}$$

By convention, $T_n(1) = 1$. Note that $T_n(-x) = (-1)^n T_n(x)$. Figure 14.9 shows the first few Chebyshev polynomials plotted against x.

Example 2:
Show that $T_0(x)$ is orthogonal to $T_2(x)$ with respect to the weight function $(1-x^2)^{-1/2}$ over the interval $(-1, 1)$.

SOLUTION:

where we have used $T_1(x) = x$. Putting all this together, we have

$$G(x,t) = \frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n$$

Table 14.3 lists the generating functions associated with the recursion formulas in Table 14.2.

We can use a generating function to determine the value of

$$\int_a^b r(x)\phi_n^2(x)dx = h_n$$

just as we did in the previous section for the Legendre polynomials. Let's do this for the Hermite polynomials. We multiply the square of $G(x,t)$ by $r(x) = e^{-x^2}$ and integrate over x:

$$\int_{-\infty}^{\infty} e^{-x^2} e^{4xt - 2t^2} dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^n}{n!} \cdot \frac{t^m}{m!} \cdot \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx \qquad (8)$$

Using the orthogonality of the $H_n(x)$, the integral on the right becomes $\delta_{nm} h_m$, and we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^n t^m}{n!m!}\delta_{nm} h_m = \sum_{n=0}^{\infty} \frac{t^{2n}}{(n!)^2} h_n$$

Table 14.3
The generating functions of the orthogonal polynomials listed in Table 14.1.

	generating function	
$P_n(x)$	$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$	$\lvert t\rvert < 1$
$T_n(x)$	$\dfrac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n$	$\lvert t\rvert < 1$
$L_n^{(\alpha)}(x)$	$\dfrac{e^{-xt/(1-t)}}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n$	$\lvert t\rvert < 1$
$H_n(x)$	$e^{2xt - t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)t^n$	

14.2 Problems

1. Generate the first few polynomials that are orthogonal and normalized with respect to e^{-x^2} over the interval $(-\infty, \infty)$.
2. Show that $H_0(x)$ and $H_1(x)$ are orthogonal to $H_2(x)$ with respect to the weight function e^{-x^2} over the interval $(-\infty, \infty)$.
3. Derive an expression for $H_4(x)$ from $H_2(x)$ and $H_3(x)$ using the recursion formula in Table 14.2.
4. Use the recursion formula for the Laguerre polynomials ($\alpha = 0$) to verify the formula for $L_3(x)$ from the expressions in Example 1.
5. Show that the Laguerre polynomials that we derived in Example 1 are orthogonal.
6. Starting with the recursion formula for Hermite polynomials in Table 14.2, derive the generating function $G(x, t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)t^n$. Let $H_0(x) = 1$.
7. Use the recursion formula in Table 14.2 to derive the generating function of the Laguerre polynomials (with $\alpha = 0$).
8. Use the generating function in Table 14.3 to derive formulas for the first few Hermite polynomials.
9. Use the generating function for the Laguerre polynomials (with $\alpha = 0$) in Table 14.3 to show that $\int_0^{\infty} e^{-x} L_n(x)L_m(x)dx = \delta_{nm}$.
10. The integral $I = \int_{-\infty}^{\infty} e^{-x^2} H_n(x)xH_m(x)dx$ occurs in a discussion of the vibrational spectrum of a diatomic molecule when it is modelled as a harmonic oscillator. Show that this integral is equal to zero unless $m = n \pm 1$.
11. The average potential energy of a quantum-mechanical harmonic oscillator is directly related to the integral $I = \int_{-\infty}^{\infty} e^{-x^2} H_n(x)x^2H_n(x)dx$. Show that $I = (n + \frac{1}{2})\sqrt{\pi}\, 2^n n!$.
12. Derive Equation 10.
13. We shall derive Equation 7 in this problem. We choose a_n so that $\phi_{n+1}(x) - a_nx\phi_n(x)$ is at most of degree n (the x^{n+1} terms cancel). Then

$$\phi_{n+1}(x) - a_nx\phi_n(x) = \sum_{l=0}^{n} \alpha_l\phi_l(x) \tag{1}$$

Now multiply both sides by $r(x)\phi_i(x)$ ($i = 1, 2, \ldots, n$) and integrate over (a, b) and show that

$$\alpha_i = \frac{\int_a^b[\phi_{n+1}(x) - a_nx\phi_n(x)]r(x)\phi_i(x)dx}{\int_a^b r(x)\phi_i^2(x)dx} = -\frac{a_n\int_a^b r(x)x\phi_i(x)\phi_n(x)dx}{\int_a^b r(x)\phi_i^2(x)dx} \tag{2}$$

Now, argue that because $x\phi_i(x)$ is a polynomial of degree x^{i+1}, the numerator in equation 2 will equal zero unless $i = n - 1$ or n. Therefore, only α_{n-1} and α_n in equation 1 are nonzero, and so equation 1 reduces to Equation 7.
14. Derive Bessel's inequality, Equation 12.
15. Use a CAS to expand $f(x) = \sin 2\pi x$ in terms of Chebyshev polynomials.

16. Use a CAS to expand $e^{-x^2/2}\cos 2x$ in terms of $e^{-x^2/2}H_n(x)$. Verify Parseval's equality for this case.

17. Use a CAS to expand $e^{-x}\cos 2x$ in terms of $e^{-x^2/2}H_n(x)$ and verify Bessel's inequality and Parseval's equality for this case.

18. Use a CAS to expand e^{-x^2} in terms of $e^{-x/2}L_n(x)$. Verify Bessel's inequality and Parseval's equality for this case.

19. Use a CAS to expand $e^{-|x|}$ in terms of $e^{-x^2/2}H_n(x)$. Verify Bessel's inequality and Parseval's equality for this case.

14.3 Sturm-Liouville Theory

In Section 1, we showed that the Legendre polynomials are orthogonal by starting with the differential equation that they satisfy. Also in Section 12.6, we showed that the Bessel functions, $J_n(x)$, obeyed a certain type of orthogonality condition,

$$\int_0^1 xJ_n(\alpha_i x)J_n(\alpha_j x)dx = 0 \qquad i \neq j \tag{1}$$

where α_i and α_j are (distinct) zeros of $J_n(x)$. We derived Equation 1 by starting with the differential equation for $J_n(x)$. In this section, we shall present a general theory that encompasses all the orthogonal polynomials that we have discussed up to now and also Equation 1, an orthogonality relation that does not involve polynomials. This theory is due to Sturm and Liouville and is called the *Sturm-Liouville theory*. Sturm-Liouville theory plays a central role in the mathematical formulation of quantum mechanics.

Consider the differential equation

$$[p(x)y'(x)]' + [q(x) + \lambda r(x)]y(x) = 0 \tag{2}$$

where λ is an unspecified parameter at this point. The functions $p(x)$, $p'(x)$, $q(x)$, and $r(x)$ are continuous on the interval $[a, b]$ and $p(x) \geq 0$ and $r(x) \geq 0$ everywhere in $[a, b]$. Equation 2 together with the boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \tag{3a}$$

and

$$\beta_1 y(b) + \beta_2 y'(b) = 0 \tag{3b}$$

constitute what we call a Sturm-Liouville problem. In Equations 3 we assume that the coefficients in Equations 3 are real and that at least one α and one $\beta \neq 0$.

Equations 2 and 3 describe a great variety of physical problems. Equation 2 may look restrictive, but even the general second-order equation $a(x)y''(x) + b(x)y'(x) + c(x)y(x) + \lambda d(x)y(x) = 0$ can be put into the form of Equation 1 if $a(x) \neq 0$ (Problem 1).

We tacitly assumed that the eigenvalues in Example 1 are real. We can use Equations 2 and 3 or Equations 3 and 5 to not only show that the eigenvalues of a Sturm-Liouville problem are real, but also that the eigenfunctions form an orthogonal set. Even though $p(x)$, $q(x)$, and $r(x)$ in Equation 2 are real, we'll allow for the possibility that λ and $y(x)$ are complex. Start with Equation 5 for $y_n(x)$ and $y_m(x)$ and take the complex conjugate of the equation for $y_m(x)$ to obtain

$$\mathcal{L}y_n(x) = \lambda_n r(x) y_n(x)$$

and

$$\mathcal{L}y_m^*(x) = \lambda_m^* r(x) y_m^*(x)$$

Multiply the first of these equations by $y_m^*(x)$ and the second by $y_n(x)$ and integrate both from a to b to obtain

$$\int_a^b y_m^*(x)\mathcal{L}y_n(x)dx = \lambda_n \int_a^b r(x)y_m^*(x)y_n(x)dx$$

$$\int_a^b y_n(x)\mathcal{L}y_m^*(x)dx = \lambda_m^* \int_a^b r(x)y_m^*(x)y_n(x)dx$$

Now subtract these two equations:

$$\int_a^b y_m^*(x)\mathcal{L}y_n(x)dx - \int_a^b y_n(x)\mathcal{L}y_m^*(x)dx = (\lambda_n - \lambda_m^*)\int_a^b r(x)y_m^*(x)y_n(x)dx \tag{6}$$

Using the definition of $\mathcal{L}$ given by Equation 4, the left side of Equation 6 is (Problem 2)

$$\int_a^b y_m^*(x)\mathcal{L}y_n(x)dx - \int_a^b y_n(x)\mathcal{L}y_m^*(x)dx$$
$$= \left[p(x)\{y_m^{*\prime}(x)y_n(x) - y_m^*(x)y_n'(x)\}\right]_a^b \tag{7}$$

Equation 7 is a key equation of Sturm-Liouville theory. It's easy to show that the boundary conditions in Equation 3 make the right side of Equation 7 equal to zero (Problem 3). If $p(a) \neq 0$ and $p(b) \neq 0$, we have what is called a *regular Sturm-Liouville problem*. In this case, $p(x) > 0$ for $a \leq x \leq b$. On the other hand, suppose that $p(a) = 0$. Then we really don't need the boundary condition 3a; all we require is that the solution and its derivative be finite at $x = a$. Similar situations occur if $p(b) = 0$, or if both $p(a) = 0$ and $p(b) = 0$. If $p(x) = 0$ at either (or both) boundary, then we have what is called a *singular Sturm-Liouville problem*. In this case, $p(x) \geq 0$ for $a \leq x \leq b$. We shall see below that some of the general properties of Sturm-Liouville systems depend upon whether it is a regular or a singular Sturm-Liouville problem.

Now if λ_m and λ_n are different, then $\lambda_m \neq \lambda_n$ and the integral in Equation 11 must equal zero. Thus, we see that

$$\int_a^b r(x)y_m^*(x)y_n(x)dx = 0 \tag{12}$$

The eigenfunctions corresponding to different eigenvalues of a Sturm-Liouville system are orthogonal.

We learned about a number of orthogonal polynomials in Section 2. All the polynomials discussed there arise from singular Sturm-Liouville problems. The differential equations that these polynomials satisfy are listed in Table 14.5. We can cast these equations into a Sturm-Liouville form by using the procedure outlined in Problem 1. We show there that the equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) + \lambda d(x)y(x) = 0 \tag{13}$$

can be written in the Sturm-Liouville form:

$$[p(x)y'(x)]' + [q(x) + \lambda r(x)]y(x) = 0 \tag{14}$$

where

$$p(x) = \exp \int \frac{b(x)}{a(x)}dx \tag{15}$$

and $q(x) = p(x)c(x)/a(x)$ and $r(x) = p(x)d(x)/a(x)$, so long as $a(x) \neq 0$. You can readily verify this result by substituting Equation 15 into Equation 14 and then comparing your result to Equation 13.

Table 14.7 lists the differential equations in Table 14.5 in Sturm-Liouville form. Note that they are all singular for one reason or another. The function $p(x) = 1 - x^2$ for Legendre's equation is equal to zero at $x = \pm 1$, so the boundary conditions at $x = \pm 1$ are simply that the solution (and its derivatives) is finite at these points. Recall from Chapter 12 that this is exactly the condition that leads to the solution being polynomials. The form of $r(x)$ in each differential equation in

Table 14.7

The Sturm-Liouville form of the defining differential equation for a few orthogonal functions.

	range of x	differential equation
$P_n(x)$	$-1 \leq x \leq 1$	$[(1 - x^2)y'(x)]' + \lambda(\lambda + 1)y(x) = 0$
$L_n^{(\alpha)}(x)$	$0 \leq x < \infty$	$[x^{\alpha+1}e^{-x}y'(x)]' + \lambda x^{\alpha}e^{-x}y(x) = 0$
$H_n(x)$	$-\infty < x < \infty$	$[e^{-x^2}y'(x)]' + 2\lambda e^{-x^2}y(x) = 0$
$T_n(x)$	$-1 \leq x \leq 1$	$[(1 - x^2)^{1/2}y'(x)]' + \lambda(1 - x^2)^{-1/2}y(x) = 0$
$J_0(x)$	$0 \leq x \leq 1$	$[xy'(x)]' + \lambda xy(x) = 0$

Table 14.7 is the weighting factor for the orthogonality property of the functions listed. Recall that they were given with no justification in Table 14.1.

Example 2:
Express Laguerre's differential equation (Table 14.5) in the form of a Sturm-Liouville equation and use Equation 12 to deduce the orthogonality condition.

SOLUTION: Laguerre's differential equation is

$$xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0$$

To convert to a Sturm-Liouville form, use the result of Problem 1, which says that

$$p(x) = \exp\left(\int \frac{b(x)}{a(x)}dx\right) = \exp\left(\int \frac{\alpha + 1 - x}{x}dx\right) = x^{\alpha+1}e^{-x}.$$

$q(x) = p(x)c(x)/a(x) = 0$ and $r(x) = p(x)d(x)/a(x) = x^{\alpha}e^{-x}$. Thus, the Sturm-Liouville form is

$$[x^{\alpha+1}e^{-x}y'(x)]' + nx^{\alpha}e^{-x}y(x) = 0$$

in agreement with Table 14.7. The orthogonality condition (Equation 12) is

$$\int_0^{\infty} x^{\alpha}e^{-x}L_n(x)L_m(x)dx = 0 \qquad m \neq n$$

in agreement with the previous section.

Let's go back to Example 1 in view of what we have learned about Sturm-Liouville problems. The eigenvalues are $\lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$ and the eigenfunctions are $\phi_n(x) = c_n \sin n\pi x$. Note that these eigenfunctions are orthogonal over (0, 1) because

$$\int_0^1 \sin m\pi x \ \sin n\pi x \ dx = 0 \qquad m \neq n$$

We can normalize them by writing

$$c_n^2 \int_0^1 \sin^2 n\pi x \ dx = c_n^2 \cdot \frac{1}{2} = 1$$

Thus, the orthonormal eigenfunctions are $\phi_n(x) = 2^{1/2} \sin n\pi x$, $n = 1, 2, \ldots$.

may or may not yield discrete eigenvalues. This is one of the primary differences between regular and singular Sturm-Liouville problems.

14.3 Problems

1. Show that the equation $a(x)y''(x) + b(x)y'(x) + c(x)y(x) + \lambda d(x)y(x) = 0$ can be put into a Sturm-Liouville form by dividing through by $a(x)$ and then multiplying by $p(x) = \exp\left(\int \frac{b(x)}{a(x)}dx\right)$.
2. Derive Equation 7.
3. Use the boundary conditions in Equations 3 to show that the right side of Equation 7 equals zero.
4. Show that if $p(a) = p(b)$, then the right side of Equation 7 will equal zero if $y(a) = y(b)$ and $y'(a) = y'(b)$ (periodic boundary conditions).
5. Show that the eigenfunctions in Example 3 are orthogonal.
6. Show explicitly that the eigenfunctions of Example 4 are orthogonal.
7. Determine the eigenvalues and eigenvectors of $y''(x) + \lambda^2 y(x) = 0$ with the boundary conditions $y(0) = 0$ and $y'(l) = 0$.
8. Determine the eigenvalues and eigenvectors of $y''(x) + \lambda^2 y(x) = 0$ with the boundary conditions $y'(0) = 0$ and $y(\pi) = 0$.
9. Determine the eigenvalues and eigenvectors of $y''(x) + \lambda^2 y(x) = 0$ with the boundary conditions $y'(0) = 0$ and $y'(1) = 0$.
10. Consider the Sturm-Liouville problem $y''(x) + \lambda y(x) = 0$ with $y(0) = 0$ and $\alpha y(l) - y'(l) = 0$. Determine the conditions for which there is a negative eigenvalue.
11. Write Hermite's differential equation in the form of a Sturm-Liouville equation. Identify $p(x)$, $q(x)$, and $r(x)$.
12. Write Chebyshev's equation in the form of a Sturm-Liouville equation. Identify $p(x)$, $q(x)$, and $r(x)$.
13. Use the result of Problem 11 to write the orthogonality condition for Hermite polynomials.
14. Use the result of Problem 12 to write the orthogonality condition for Chebyshev polynomials.
15. Classify the following Sturm-Liouville problems as regular, singular, or periodic:

    (a) $[xy'(x)]' + \lambda xy(x) = 0 \qquad [0, \infty)$

    (b) $[(1 - x^2)y'(x)]' + \lambda y(x) = 0 \qquad [-1, 1]$

    (c) $(1 - x^2)y''(x) - xy'(x) + \lambda y(x) = 0 \qquad [-1, 1]$

16. Determine the eigenvalues and eigenfunctions of $y''(x) + \lambda^2 y(x) = 0$ with the boundary conditions $y(-a) = y(a)$ and $y'(-a) = y'(a)$.
17. Show that the normalization constant of the eigenfunctions in Example 4 is $2\beta_n^{1/2}/(2\beta_n - \sin 2\beta_n)^{1/2}$.
18. Show that the eigenvalues and eigenfunctions of the equation $y''(x) + \lambda y(x) = 0$ with the boundary conditions $y(0) + y'(0) = 0$ and $y(1) = 0$ are $\lambda_0 = 0$ with $y_0(x) = x - 1$ and $\lambda_n = \beta_n^2$ with $y_n(x) = \beta_n \cos \beta_n x - \sin \beta_n x$ for $n \geq 1$, where the β_n are the positive roots of $\tan x = x$.

Example 3:
Expand $f(x) = x$ over the interval $(0, 1)$ in terms of the eigenfunctions in Example 4 of the previous section.

SOLUTION: The eigenfunctions are $y_n(x) = \sin \beta_n x$, where the β_n are the solutions of $\tan \beta_n = -\beta_n/5$. Thus,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \beta_n x$$

Multiply by $\sin \beta_m x$ and integrate to obtain

$$a_m \int_0^1 \sin^2 \beta_m x \, dx = \int_0^1 x \sin \beta_m x \, dx$$

$$a_m \left(\frac{2\beta_m - \sin 2\beta_m}{4\beta_m} \right) = \frac{\sin \beta_m - \beta_m \cos \beta_m}{\beta_m^2} = -\frac{6 \cos \beta_m}{5\beta_m}$$

where we have used $\tan \beta_m = -\beta_m/5$ in the last line. We have, then,

$$f(x) = -\frac{24}{5} \sum_{n=1}^{\infty} \frac{\cos \beta_m}{2\beta_m - \sin 2\beta_m} \sin \beta_m x$$

You can determine the β_n numerically, but they are also well-tabulated (see, for example, Table 4.19 of Abramowitz and Stegun). Figure 14.13 shows partial sums of the series for $f(x)$ consisting of up to 50 terms.

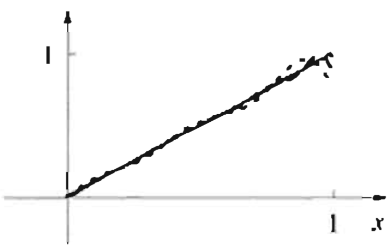

Figure 14.13
The partial sums of Example 3 consisting of 5 (dotted), 10 (short dashed), and 50 (solid) terms.

One important difference between a regular Sturm-Liouville problem and a singular Sturm-Liouville problem is that the eigenvalues of a singular problem may not be discrete. That is, there may be a continuous range of eigenvalues for which there are non-trivial solutions. When this occurs, we say that the problem has a *continuous spectrum* of eigenvalues. In some cases, the eigenvalues may be discrete over some interval and continuous over another. There is no general theory that you can use to determine the nature of the eigenvalues for any particular case. In those cases where the eigenvalues are discrete, you can show that the eigenvalues are real and that the eigenfunctions are orthogonal. Furthermore, there is a generalization of the theorem that we gave earlier in this section that guarantees the convergence of the series expansion of a function in terms of eigenfunctions. The Legendre polynomials are a good example of this case.

well-behaved given function, and where the boundary conditions are

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad \text{and} \qquad \beta_1 y(b) + \beta_2 y'(b) = 0 \tag{3}$$

We assume that both the solution $y(x)$ and the nonhomogeneous term $g(x)$ can be expanded in terms of the eigenfunctions of $\mathcal{L}$, which satisfy the eigenvalue problem

$$\mathcal{L}\phi_n(x) = \lambda_n r(x)\phi_n(x) \tag{4}$$

where the $\phi_n(x)$ satisfy the boundary conditions in Equation 3. For notational convenience only, we assume that the $\phi_n(x)$ are normalized and also write Equation 1 as

$$\mathcal{L}y(x) - \mu r(x)y(x) = r(x)f(x) \tag{5}$$

where $r(x)f(x) = g(x)$. We shall actually expand $f(x) = g(x)/r(x)$ instead of $g(x)$, and write

$$y(x) = \sum_{n=1}^{\infty} a_n\phi_n(x) \tag{6}$$

$$f(x) = \sum_{n=1}^{\infty} f_n\phi_n(x) \tag{7}$$

We know the f_n because they are given by

$$f_n = \int_a^b r(x)f(x)\phi_n(x)dx \tag{8}$$

and we wish to determine the a_n. Substitute Equations 6 and 7 into Equation 5 to obtain

$$r(x)\sum_{n=1}^{\infty} \lambda_n a_n\phi_n(x) - \mu r(x)\sum_{n=1}^{\infty} a_n\phi_n(x) = r(x)\sum_{n=1}^{\infty} f_n\phi_n(x) \tag{9}$$

Multiply both sides of Equation 9 by $\phi_l(x)$ and integrate over $[a, b]$ using the orthogonality of the $\phi_n(x)$ with respect to the weighting function $r(x)$ to find that

$$(\lambda_l - \mu)a_l = f_l \qquad l = 1, 2, \dots \tag{10}$$

If $\mu \neq \lambda_l$ for $l = 1, 2, \dots$, then a formal solution to Equation 5 is given by

$$y(x) = \sum_{n=1}^{\infty} a_n\phi_n(x) = \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \mu}\phi_n(x) \tag{11}$$

If $f(x)$ is continuous, then the series in Equation 11 converges point-wise to the solution of Equation 5.

If μ is not equal to any of the eigenvalues, then the solution to the boundary value problem given by Equations 1 and 3 will be unique for any continuous appropriate eigenvalue problem is (Equation 4)

$$\phi_n''(x) + \lambda_n \phi_n(x) = 0$$

The normalized solution that satisfies the boundary conditions is

$$\phi_n(x) = 2^{1/2} \sin n\pi x$$

and the eigenvalues are given by $\lambda_n = n^2\pi^2$. Equation 13 gives

$$G(x, z) = 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x \, \sin n\pi z}{n^2\pi^2 - 1}$$

Example 2:
Use the Green's function in Example 1 to solve the equation

$$y''(x) + y(x) = x$$

with the boundary conditions $y(0) = y(1) = 0$.

SOLUTION: According to Equation 15

$$\begin{aligned} y(x) &= \int_0^1 G(x, z)(-z)dz \\ &= 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x}{(n\pi)^2 - 1} \int_0^1 (-z) \sin n\pi z \, dz \\ &= 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x}{(n\pi)^2 - 1} \cdot \frac{(-1)^n}{n\pi} \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi x}{n[1 - (n\pi)^2]} \end{aligned}$$

This is perhaps a good time to point out that the definition of a Green's function varies from author to author, the difference being a factor of ± 1. Notice in Example 2 that x, the nonhomogeneous term of the differential equation, is represented in Equation 15 by $-z$. The difference can be traced back to the definition of $\mathcal{L}$ in Equation 2. Some authors define $\mathcal{L}$ as $[p(x)y'(x)]' + q(x)y(x)$, but then require that $p(x) < 0$ and $r(x) < 0$ for a regular Sturm-Liouville problem. There is no consensus on this issue, and if your work leads to working with Green's functions, then you'll simply refer to your favorite sourcebook on Green's functions. Whichever convention you use, your final answers will always be independent of that choice.

We can derive a differential equation for $G(x, z)$ by formally operating on the x dependence of both sides of Equation 15 with the operator $\mathcal{L} - \mu r(x)$. Using Equation 14 for the left side, we obtain

$$[\mathcal{L} - \mu r(x)]y(x) = g(x) = \int_a^b \{[\mathcal{L} - \mu r(x)]G(x, z)\}g(z)dz \tag{16}$$

Equation 16 says, however, that the term in curly brackets in the integrand is a Dirac delta function (Section 3.6) because

$$g(x) = \int_a^b \delta(x - z)g(z)dz$$

Thus, we have

$$\mathcal{L}G(x, z) - \mu r(x)G(x, z) = \delta(x - z) \tag{17}$$

as a differential equation for the Green's function, $G(x, z)$.

Let's see how to use Equation 17 by deriving the Green's function for the equation

$$y''(x) + y(x) = 0$$

with boundary conditions $y(0) = y(1) = 0$. (See Example 1.) We obtain the equation for the Green's function by letting $\mathcal{L} = -d^2/dx^2$, $\mu = 1$, and $r(x) = 1$, in which case Equation 17 becomes

$$G''(x, z) + G(x, z) = -\delta(x - z) \tag{18}$$

where the primes denote differentiation with respect to x. The boundary conditions on $G(x, z)$ in this case are $G(0, z) = 0$ and $G(1, z) = 0$. We work with the regions $0 \le x < z$ and $z < x \le 1$ separately. In each case, $G''(x, z) + G(x, z) = 0$, and the solutions are

$$G(x, z) = \begin{cases} c_1 \sin x + c_2 \cos x & 0 \le x < z \\ c_3 \sin x + c_4 \cos x & z < x \le 1 \end{cases}$$

We can apply the $G(0, z) = 0$ boundary condition to the first solution and $G(1, z) = 0$ to the second to obtain (Problem 20)

$$G(x, z) = \begin{cases} a \sin x & 0 \le x < z \\ b \sin(x - 1) & z < x \le 1 \end{cases} \tag{19}$$

We still have two constants, a and b, to determine. We do this by integrating Equation 18 from $z - \epsilon$ to $z + \epsilon$, where $\epsilon \to 0$.

$$\left[\frac{d^2G}{dx^2}\right]_{z-\epsilon}^{z+\epsilon} + \int_{z-\epsilon}^{z+\epsilon} G(x, z)dx = -\int_{z-\epsilon}^{z+\epsilon} \delta(x - z)dx = -1 \tag{20}$$

Equation 20 implies that one or both of the integrands on the left is discontinuous, for otherwise both integrals would equal zero as $\epsilon \to 0$. Since differentiation produces discontinuities whereas integration smooths them out, we shall assume that $G(x, z)$ is a continuous function of x and that dG/dx is discontinuous at $x = z$. Therefore,

$$\lim_{\epsilon \to 0} \int_{z-\epsilon}^{z+\epsilon} G(x, z)dx = 0 \tag{21}$$

and

$$\lim_{\epsilon \to 0} \int_{z-\epsilon}^{z+\epsilon} \frac{d^2G}{dx^2} = \lim_{\epsilon \to 0} \left[\frac{dG}{dx}\right]_{z-\epsilon}^{z+\epsilon} = -1 \tag{22}$$

Figure 14.15
An illustration that a Green's function is a continuous function of x and that its first derivative has a discontinuity of -1 at $x = z$.

Although $G(x, z)$ is continuous at $x = z$, its first derivative has a jump discontinuity there (Figure 14.15).

We can now use Equations 21 and 22 to determine a and b in Equation 20. The continuity of $G(x, z)$ at $x = z$ gives

$$a \sin z = b \sin(z - 1)$$

and Equation 22 gives

$$b \cos(z - 1) - a \cos z = -1$$

Solving these equations for a and b gives

$$a = -\frac{\sin(z - 1)}{\sin 1}$$

and

$$b = -\frac{\sin z}{\sin 1}$$

so finally we have

$$G(x, z) = \frac{1}{\sin 1}\begin{cases} -\sin x \sin(z - 1) & 0 \le x < z \\ -\sin z \sin(x - 1) & z < x \le 1 \end{cases} \tag{23}$$

Note that $G(x, z)$ is a symmetric function of x and z; that is, $G(x, z) = G(z, x)$. (If we allow for a complex vector space picture in which the eigenfunctions are complex, then the symmetry condition is $G(x, z) = G^*(z, x)$ (Problem 21).) Figure 14.16 shows $G(x, z)$ in Equation 23 plotted against x for $l = 1$. You can see that $G(x, z)$ is continuous at $x = z$, but that its slope is discontinuous there (Problem 22).

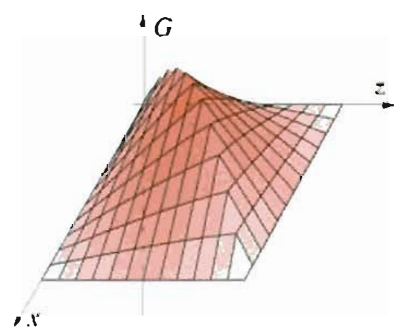

Figure 14.16
The Green's function given by Equation 23 plotted against x and z.

Because of the symmetry property of $G(x, z)$, we can also write Equation 23 as

$$G(x, z) = \frac{1}{\sin 1}\begin{cases} -\sin z \sin(x - 1) & 0 \le z < x \\ -\sin x \sin(z - 1) & x < z \le 1 \end{cases} \tag{24}$$

Transfinite Numbers

What would you say if someone told you that there is the same number of even numbers as there are integers? Or that there is the same number of rational numbers as there are integers? As you might expect, it depends upon what we mean when we say "the same number as." In the late 1800s, Georg Cantor investigated the properties of infinite sets, and he discovered that there is more than one type of infinity. He introduced the concept of a transfinite number to enumerate what is called the cardinality of an infinite set. He assigned the cardinality $\aleph_0$ (aleph naught) to the set of integers, and then he assigned cardinality $\aleph_0$ to the even integers by the deceptively simple process of matching the even numbers one-to-one with the integers by the arrangement

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & \cdots \end{array}$$

Similarly, you can show that the set of squares of integers, the set of odd numbers, or any infinite sequence has the same cardinality as the set of integers.

We said above that the number of rational numbers is the same as the number of integers. Cantor was able to show by an ingenious counting process that the set of all rational numbers can be matched up one-to-one with the integers, or that the set of all rational numbers has a cardinality of $\aleph_0$. We say that the set of all rational numbers is countable, or denumerable.

Even though the rational numbers are dense, in the sense that there is always at least one rational number between any two (the average of the two), there is still the set of irrational numbers (both algebraic numbers and transcendental numbers) in any line segment. Numbers that are solutions to polynomial equations with integer coefficients are called algebraic numbers; if not, they are called transcendental numbers (π and e are transcendental numbers). Cantor was able to show that the cardinality of the set of all algebraic numbers is $\aleph_0$, even though they are much more general than rational numbers. He furthermore showed that the cardinality of the set of all transcendental numbers is not denumerable, and he assigned it a cardinality of $\aleph_1$. The cardinality of the continuum of real numbers, which is also $\aleph_1$, is due to the transcendental numbers. In a sense, it is the transcendental numbers that fill up the continuum of real numbers. Cantor also showed that the set of the continuum of n-tuples in any finite dimension is still $\aleph_1$, and that there are sets with cardinalities greater than $\aleph_1$, thus developing a hierarchy of cardinalities. One of the great unsolved problems in mathematics is to determine if there is a cardinality between $\aleph_0$ and $\aleph_1$.

Joseph Fourier (1768–1830), who gave us Fourier series, was born on March 21, 1768, in Auxerre, France, where his father was a tailor. He was orphaned when he was 10 years old. Because of the talent that he displayed in his early school years, he received financial support to finish his education. In 1780, he entered the local École Royale Militaire, run by the Benedictine order, where he soon discovered mathematics. In 1787, he entered the Benedictine abbey of St. Benoit-sur-Loire with the intention of becoming a priest. He realized that his true calling was mathematics, and he left the abbey in 1789, returning to teach at his former school. In 1793, he became involved in the French Revolution, joining the local Revolutionary Committee. At first, Fourier was enamored with the goals of the revolution but became disillusioned by the Reign of Terror that followed. In July 1794, he was arrested as a result of a protest speech that he gave in Orléans, but he was freed when Napoleon came to power. In 1795, he began teaching at the École Polytechnique and succeeded Lagrange as Chair of Analysis and Mechanics in 1797. Fourier was a gifted orator and was an outstanding lecturer. In 1798, he, along with several other scientists, joined Napoleon's army in the invasion of Egypt as a scientific advisor. At the request of Napoleon, he served as Prefect in Grenoble. Among his duties and accomplishments in Grenoble were the draining of the swamps in the area and the construction of a highway between Grenoble and Turin. In 1822, he published his work, *Théorie analytique de la chaleur*, in which he introduced Fourier series. In his later years, he became an "insufferable bore" with his stories about the glories of working with Napoleon and the wonderful work he was going to do. His years in Egypt led him to believe that desert heat was the ideal condition for health, so he always wore heavy clothes and lived in very hot rooms. He died of heart disease on May 16, 1830, in Paris.

CHAPTER 15

Fourier Series

This chapter is devoted to a single topic, Fourier series, one of the most useful and important tools of applied mathematics. At the turn of the 19th century, the French mathematician and physicist Joseph Fourier analyzed the flow and the distribution of energy as heat in solid bodies. This work was summarized by Fourier in his book, *The Analytical Theory of Heat*, one of the most famous science books ever published. Newton had previously proposed that the rate of the temperature change of a body is proportional to the difference in the temperature of the body and that of its surroundings (Newton's law of cooling), but this observation applied only to temporal behavior. Fourier's work addressed the spatial distribution of the temperature within a solid body and gave us Fourier's law of heat conduction, which says that the flux of energy as heat throughout a body is proportional to the gradient of the temperature. Because Fourier was interested in both the spatial and the temporal distribution of the temperature throughout a solid body, he considered functions of temperature of the form $T = T(x, y, z, t)$, which leads to partial differential equations, as we shall see in the next chapter. Upon solving these equations, Fourier found it necessary to express temperature distributions as infinite series of sines and cosines of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$$

where the a_n and b_n depend upon $f(x)$. This type of series is now called a *Fourier series*. It might not be unexpected that a Fourier series would converge to a function $f(x)$ if $f(x)$ is continuous over some interval, but the amazing thing about Fourier series is that $f(x)$ does not even have to be continuous. We shall see in this chapter that a Fourier series will converge to $f(x)$ even if $f(x)$ is *discontinuous*, such as might occur initially across the boundary of a hot solid quenched in a cold liquid. At the time, it was incredible that a series of continuous functions could converge to a discontinuous function and Fourier's work was severely criticized. Nevertheless, Fourier's work not only survived almost two centuries of mathematical scrutiny, but has fostered several areas of modern mathematical research.

We introduce several variations of Fourier series in the first two sections of this chapter and then discuss the nature of the convergence of Fourier series in Section 3. In the last section, we show how to use Fourier series to solve linear nonhomogeneous ordinary differential equations. Then, in the next chapter, we shall show how Fourier series are used to solve partial differential equations.

15.1 Fourier Series as Eigenfunction Expansions

We saw in the previous chapter that the normalized eigenfunctions of the regular Sturm-Liouville problem

$$\phi_n''(x) + \lambda_n \phi_n(x) = 0 \tag{1}$$

with $\phi_n(0) = \phi_n(l) = 0$ are

$$\phi_n(x) = \left(\frac{2}{l}\right)^{1/2} \sin \frac{n\pi x}{l} \qquad n = 1, 2, \dots \tag{2}$$

These eigenfunctions form a complete orthonormal set over the interval $[0, l]$:

$$\frac{2}{l}\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \delta_{nm} \tag{3}$$

Similarly, the normalized eigenfunctions of the regular Sturm-Liouville problem

$$\phi_n''(x) + \lambda_n \phi_n(x) = 0 \tag{4}$$

with $\phi_n'(0) = \phi_n'(l) = 0$ are

$$\phi_n(x) = \left(\frac{2}{l}\right)^{1/2} \cos \frac{n\pi x}{l} \qquad n = 1, 2, \dots \tag{5}$$

and these also form a complete set over the interval $[0, l]$.

Now let's consider the periodic Sturm-Liouville problem

$$\phi_n''(x) + \lambda_n \phi_n(x) = 0 \tag{6}$$

with the periodic boundary conditions

$$\phi_n(-l) = \phi_n(l) \qquad \text{and} \qquad \phi_n'(-l) = \phi_n'(l) \tag{7}$$

The solution to Equation 6 is

$$\phi_n(x) = \alpha_n \sin \lambda_n^{1/2} x + \beta_n \cos \lambda_n^{1/2} x \tag{8}$$

Problem 9 has you show that the periodic boundary conditions given by Equations 7 give $\lambda_n = n^2\pi^2/l^2$, where $n = 0, 1, 2, \dots$. The boundary conditions in this case allow both α_n and β_n to be nonzero, so there are two sets of eigenfunctions, $\{\sin n\pi x/l\}$ and $\{\cos n\pi x/l\}$, corresponding to $\lambda_n = n^2\pi^2/l^2$. Furthermore, the zero eigenvalue, $\lambda_0 = 0$, is allowed in this case because the corresponding eigenfunction given by Equation 8 is the nontrivial function $\phi_0(x) =$ a constant.

Although Equations 6 and 7 do not constitute a regular Sturm-Liouville problem, we'll see that the powerful expansion theorem of Section 14.4 still applies in this case. Thus, the expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \tag{9}$$

converges for a generous class of functions, $f(x)$. By "converges," we mean either point-wise convergence if $f(x)$ is continuous, or convergence in the mean. We'll see below that the factor 2 in the a_0 term is included for convenience only.

We showed in Section 14.3 that the eigenvalues of a periodic Sturm-Liouville problem are real and that the eigenfunctions are orthogonal. In particular, we have (see Problems 1 through 8)

$$\begin{aligned} &\int_{-l}^{l} \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \int_{-l}^{l} \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\, dx = \delta_{nm} \\ &\int_{-l}^{l} \sin \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\, dx = 0 \end{aligned} \tag{10}$$

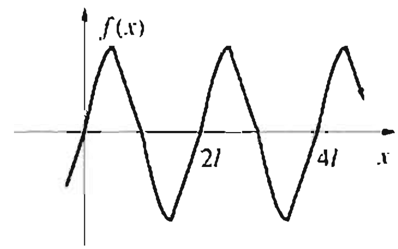

Figure 15.1
A periodic function with period $2l$.

where n and $m = 1, 2, \ldots$. It's easy to show that the special case $\phi_0(x) =$ constant is orthogonal to all the $\sin n\pi x/l$ and $\cos n\pi x/l$.

All the terms in Equation 9 are periodic functions of period $2l$ (Figure 15.1). (Recall that a function has period $2l$ if $f(x + 2l) = f(x)$ for all values of x.) Therefore the series in Equation 9 is particularly suited for the expansion of functions of period $2l$. In fact, if $f(x)$ is defined on the interval $[-l, l)$ and has period $2l$, then the series in Equation 9 is called the *Fourier series* of $f(x)$ if the coefficients are given by

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \qquad n = 0, 1, 2, \ldots \tag{11}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \qquad n = 1, 2, \ldots \tag{12}$$

These coefficients, called *Fourier coefficients*, are obtained using the orthogonality conditions in Equation 10. The inclusion of the "2" in the denominator of the a_0 term in Equation 9 allows us to use Equation 11 to calculate a_0. Otherwise, the integral for a_0 would have to be listed separately.

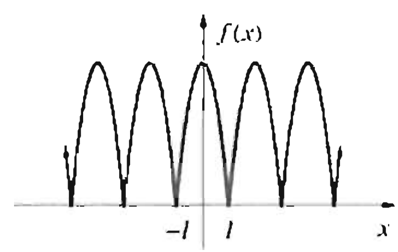

Figure 15.2
The periodic function defined by $f(x) = l^2 - x^2$ on the interval $[-l, l)$ and $f(x) = f(x + 2l)$ outside this interval.

Example 1:
Determine the Fourier series of $f(x) = l^2 - x^2$ for $(-l, l)$ and $f(x) = f(x + 2l)$ outside this interval (Figure 15.2).

SOLUTION:

$$a_n = \frac{1}{l}\int_{-l}^{l}(l^2 - x^2)\cos\frac{n\pi x}{l} = \frac{2}{l}\left[\frac{2x\cos(n\pi x/l)}{n^2\pi^2/l^2} + \frac{(n^2\pi^2x^2/l^2 - 2)\sin(n\pi x/l)}{n^3\pi^3/l^3}\right]_0^l$$

$$= \frac{4l^2}{\pi^2}\frac{(-1)^{n+1}}{n^2} \qquad n \neq 0$$

$$a_0 = \frac{4l^2}{3}$$

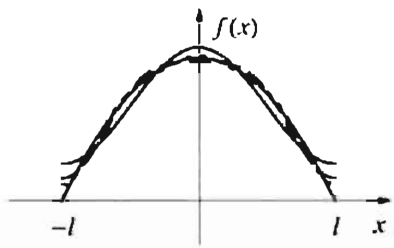

Figure 15.3
The function $f(x) = l^2 - x^2$ (solid) for $(-l, l)$ together with 1 (solid), 2 (dashed), and 3 (dotted) partial sums of the series in Example 1.

The b_n are equal to zero because $(l^2 - x^2)\sin(n\pi x/l)$ is an odd function of x. Thus,

$$f(x) = \frac{2l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

Figure 15.3 shows $f(x) = l^2 - x^2$ and the first few partial sums of the Fourier series representation of $f(x)$ over the interval $-l$ to l. Realize that the Fourier series represents not only $f(x)$ in the interval $-l$ to l, but its periodic extension as well (Figure 15.4).

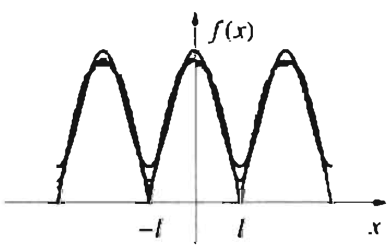

Figure 15.4
The periodic extension of the functions plotted in Figure 15.3.

Example 2:
Determine the Fourier series of $f(x) = x$ for $[-l, l)$ and $f(x + 2l) = f(x)$ outside this interval (Figure 15.5).

SOLUTION: The Fourier coefficients are

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l} = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l} = \frac{2}{l}\left[\frac{\sin(n\pi x/l)}{n^2\pi^2/l^2} - \frac{x\cos(n\pi x/l)}{n\pi/l}\right]_0^l$$

$$= \frac{(-1)^{n+1}2l}{n\pi} \qquad n = 1, 2, \ldots$$

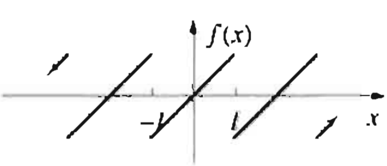

Figure 15.5
The periodic function defined by $f(x) = x$ on the interval $[-l, l)$ and by $f(x) = f(x + 2l)$ outside this interval.

The $a_n = 0$ because the integrand is an odd function of x. The Fourier series of $f(x)$ is

$$f(x) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}$$

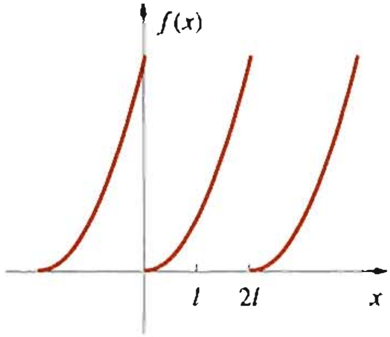

Figure 15.14
The function defined on the interval $[0, 2l)$ by $f(x) = x^2$ and $f(x + 2l) = f(x)$.

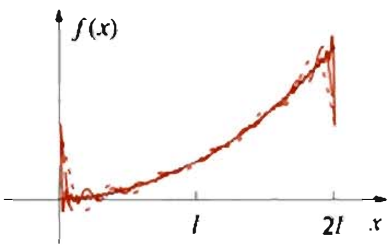

Figure 15.15
The function $f(x) = x^2$ over the interval $[0, 2l)$ (black) and the partial sums consisting of 5 terms (dotted color), 10 terms (long dashed color), and 50 terms (solid color) of the Fourier series representation of $f(x)$.

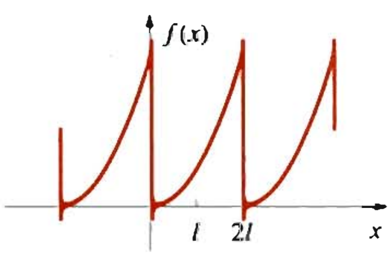

Figure 15.16
The Fourier series representation of $f(x)$ in Figure 15.15, showing that the Fourier series representation is a periodic function of period $2l$.

All our examples so far have been on a symmetric interval $-l$ to l. Suppose we want to expand $f(x) = x^2$ in $(0, 2l)$ with $f(x + 2l) = f(x)$ (see Figure 15.14). Problem 20 has you show that since $f(x)$ is periodic with period $2l$, the integrals for the Fourier coefficients (Equations 11 and 12) can be written as

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l+c}^{l+c} f(x)\cos\frac{n\pi x}{l}dx \tag{13}$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l+c}^{l+c} f(x)\sin\frac{n\pi x}{l}dx \tag{14}$$

where c is an arbitrary constant.

Therefore, the Fourier coefficents of the expansion of $f(x) = x^2$ in $(0, 2l)$ and $f(x) = f(x + 2l)$ are given by Equations 13 and 14 with $c = l$, or by

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{l}\int_0^{2l} x^2\cos\frac{n\pi x}{l}dx = \frac{4l^2}{n^2\pi^2} \qquad n \neq 0$$

$$a_0 = \frac{1}{l}\int_0^{2l} x^2 dx = \frac{8l^2}{3}$$

and

$$b_n = \frac{1}{l}\int_0^{2l} x^2\sin\frac{n\pi x}{l}dx = -\frac{4l^2}{\pi n} \qquad n = 1, 2, \ldots$$

Thus,

$$f(x) = \frac{4l^2}{3} + \frac{4l^2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{\pi n^2}\cos\frac{n\pi x}{l} - \frac{1}{n}\sin\frac{n\pi x}{l}\right)$$

Some partial sums of $f(x)$ are shown in Figure 15.15 for the interval $(0, 2l)$, and in Figure 15.16 for the periodic extension of $f(x)$. The $1/n$ terms in the Fourier series representation of $f(x)$ are due to the discontinuity of the periodic extension of $f(x)$ at $x = \pm nl$ with $n = 0, 1, 2, \ldots$.

Example 5:
Expand the function

$$f(x) = \begin{cases} 1 & -1/2 \leq x < 1/2 \\ 0 & 1/2 \leq x < 3/2 \end{cases}$$

with $f(x) = f(x + 2)$. (See Figure 15.17.)

SOLUTION: The period $2l = 2$, so we use Equations 13 and 14 with $l = 1$ and $c = 1/2$:

$$a_n = \int_{-1/2}^{3/2} f(x)\cos n\pi x\,dx = \int_{-1/2}^{1/2} \cos n\pi x\,dx = \frac{2}{n\pi}\sin\frac{n\pi}{2} \qquad n \neq 0$$

$$a_0 = 1$$

$$b_n = \int_{-1/2}^{1/2} \sin n\pi x\,dx = 0$$

Thus,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n}\cos n\pi x$$

$$= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos(2n-1)\pi x$$

The partial sums of $f(x)$ over the interval $(-1/2, 3/2)$ are plotted in Figure 15.18.

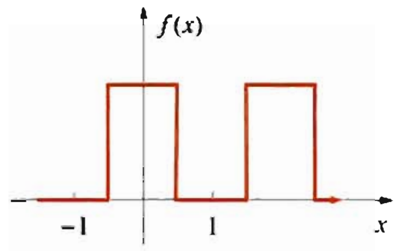

Figure 15.17
The function $f(x)$ from Example 5.

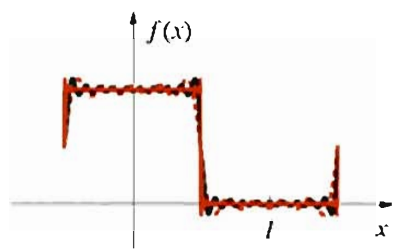

Figure 15.18
The partial sums of the Fourier series representation of $f(x)$ given in Example 5 consisting of 5 (dotted), 10 (solid black), and 100 (solid color) terms.

For simplicity only, most of the examples of Fourier series that we will discuss in this chapter will involve functions defined over a symmetric interval $(-l, l)$, but the above Example shows that it is not at all necessary that $f(x)$ be defined over a symmetric interval.

Another form of a Fourier series is the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{15}$$

Notice that the summation runs from $-\infty$ to ∞. We can determine the c_n in Equation 15 by realizing that $\{e^{in\pi x/l}\}$ is an orthogonal set over the interval $-l$ to l (Problem 24). Multiply Equation 15 by $e^{-ik\pi x/l}$ and integrate from $-l$ to l to obtain

$$\int_{-l}^{l} f(x)e^{-ik\pi x/l}dx = \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-k)\pi x/l}dx = \sum_{n=-\infty}^{\infty} 2lc_n\delta_{nk} = c_k 2l$$

so

$$c_k = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-ik\pi x/l}dx \tag{16}$$

Let's determine the complex Fourier series representation of $f(x) = x$ in $[-l, l]$ with $f(x + 2l) = f(x)$:

9. Show that the periodic boundary conditions given by Equation 7 give $\lambda_n = n^2\pi^2/l^2$ for $n = 0, 1, 2, \ldots$, and that both α_n and β_n in Equation 8 may be nonzero.

10. Show that the $a_n = 0$ in Equation 11 if $f(x)$ is an odd function and that the $b_n = 0$ in Equation 12 if $f(x)$ is an even function.

11. Find the Fourier series of $f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \end{cases}$.

12. Find the Fourier series of $f(x) = x^2$ for $-\pi \le x < \pi$.

13. Find the Fourier series of $f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ \sin x & 0 \le x < \pi \end{cases}$.

14. Find the Fourier series of $f(x) = \begin{cases} 0 & -\pi \le x < -\pi/2 \\ 1 & -\pi/2 \le x < \pi/2 \\ 0 & \pi/2 \le x < \pi \end{cases}$.

15. Find the Fourier series of $f(x) = \begin{cases} \dfrac{x}{l} & 0 \le x < l \\ \dfrac{2l - x}{l} & l \le x < 2l \end{cases}$.

16. Find the Fourier series of $f(x) = x^4$ for $0 \le x < 2\pi$.

17. Find the Fourier series of $f(x) = \begin{cases} 0 & -2 \le x < 0 \\ 2 & 0 \le x < 2 \end{cases}$.

18. Find the Fourier series of $f(x) = |\cos x|$ for $-\pi \le x < \pi$.

19. Find the Fourier series of $f(x) = \cos^2 x$ for $-\pi \le x < \pi$.

20. Verify Equations 13 and 14.

21. In this problem, we'll derive the Fourier series for a function defined in the general interval $[a, b)$. Show that if we define u by $x = \dfrac{b+a}{2} + \dfrac{b-a}{2\pi}u$, then the interval $a \le x < b$ becomes $-\pi \le u < \pi$, and that the function $f(x) = f[(b+a)/2 + (b-a)u/2\pi] = F(u)$ has a period 2π. Now show that $F(u) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nu + b_n \sin nu)$ becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi(2x - a - b)}{b - a} + b_n \sin\frac{n\pi(2x - a - b)}{b - a}\right], \text{ where}$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{n\pi(2x - a - b)}{b - a}\,dx \qquad n = 0, 1, 2 \ldots$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{n\pi(2x - a - b)}{b - a}\,dx \qquad n = 1, 2 \ldots$$

22. Show that the general formulas in the previous problem reduce to Equations 11 and 12.

23. Use the general formulas in Problem 21 to derive the Fourier series of

(a) $f(x) = \pi - x \quad 0 < x < \pi$ (b) $f(x) = \cos x \quad 0 \le x < \pi$

24. Show that $\{e^{in\pi x/l}\}$ is an orthogonal set over the interval $(-l, l)$.

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \tag{2}$$

show that $a_n = 0$ if $f(x)$ is an odd function of x and $b_n = 0$ if $f(x)$ is an even function. Thus, $f(x) = l^2 - x^2$ in Example 1 of Section 1 has a cosine series and $f(x) = x$ in Example 2 of Section 1 has a sine series.

Suppose now we have a function $f(x)$ defined only on the interval $(0, l)$. We may extend this function in two ways. One way is to extend $f(x)$ so that it is an even function of x over the interval $(-l, l)$. For example, if $f(x) = x$ in $(0, l)$, then the even extension, f_e, is defined by

$$f_e(x) = \begin{cases} -x & -l \le x < 0 \\ x & 0 \le x < l \end{cases}$$

This even extension is shown in Figure 15.20a. We can also extend $f(x)$ as an odd function of x, $f_o(x)$, by defining

$$f_o(x) = \begin{cases} x & -l \le x < 0 \\ x & 0 \le x < l \end{cases}$$

which is shown in Figure 15.20b.

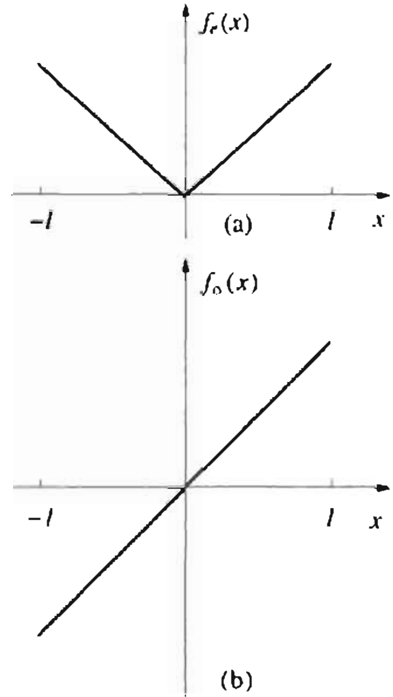

Figure 15.20
(a) The even extension of $f(x) = x$ in the interval $[0, l)$ plotted against x. (b) The odd extension of $f(x) = x$ in the interval $[0, l)$ plotted against x.

Generally, if a function $f(x)$ defined on the interval $(0, l)$ is neither even nor odd, then its even and odd extensions are given by

$$f_e(x) = \begin{cases} f(-x) & -l \le x < 0 \\ f(x) & 0 \le x < l \end{cases} \tag{3}$$

and

$$f_o(x) = \begin{cases} -f(-x) & -l \le x < 0 \\ f(x) & 0 \le x < l \end{cases} \tag{4}$$

For example, if $f(x) = x^2 - x$ on $[0, l]$, then

$$f_e(x) = \begin{cases} x^2 + x & -l \le x < 0 \\ x^2 - x & 0 \le x < l \end{cases}$$

and

$$f_o(x) = \begin{cases} -x^2 - x & -l \le x < 0 \\ x^2 - x & 0 \le x < l \end{cases}$$

These even and odd extensions are plotted against x in Figure 15.21.

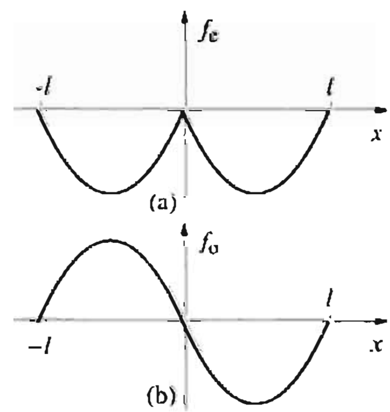

Figure 15.21
The (a) even and (b) odd extensions of $f(x)$ defined by $x^2 - x$ on $[0, l)$ plotted against x for $[-l, l)$.

of the sine series are given by

$$b_n = 2\int_0^1 (x^3 - x)\sin n\pi x\, dx = \frac{(-1)^n 12}{n^3\pi^3}$$

and the sine series is

$$f(x) = \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x \tag{9}$$

The coefficients of the cosine series are given by

$$a_n = 2\int_0^1 (x^3 - x)\cos n\pi x\, dx$$
$$= \begin{cases} \dfrac{2}{n^2\pi^2}[(-1)^n 2 + 1] + \dfrac{12}{n^4\pi^4}[1 - (-1)^n] & n \neq 0 \\ -\dfrac{1}{2} & n = 0 \end{cases}$$

and the cosine series is

$$f(x) = -\frac{1}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{1+(-1)^n 2}{n^2} + \frac{6}{\pi^2 n^4}[1 - (-1)^n]\right\}\cos n\pi x$$
$$= -\frac{1}{4} + \frac{2}{\pi^2}\left[\left(\frac{12}{\pi^2} - 1\right)\cos \pi x + \frac{3}{4}\cos 2\pi x\right.$$
$$\left. + \left(\frac{4}{27\pi^2} - \frac{1}{9}\right)\cos 3\pi x + \frac{3}{16}\cos 4\pi x + \cdots\right] \tag{10}$$

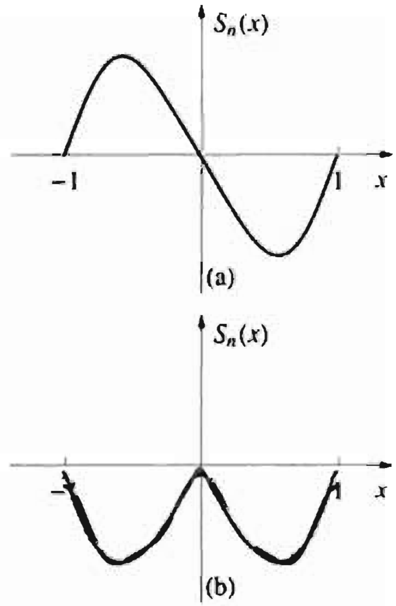

Figure 15.27
(a) The Fourier series of the odd extension of the function defined in Example 2 and the partial sums of Equation 9 containing just two terms. (b) The Fourier series of the even extension of the function defined in Example 2 and the partial sums of Equation 10 consisting of 4 terms (black), 8 terms (dotted), and 16 terms (solid color).

The continuity of $f_o(x)$ and its first derivatives leads to Fourier coefficients that decay as $1/n^3$, whereas the continuity of $f_e(x)$ but the discontinuity in $f_e'(x)$ leads to Fourier coefficients that decay as $1/n^2$. Some partial sums of Equations 9 and 10 are plotted in Figure 15.27. It takes almost twenty terms using the cosine series to achieve the same accuracy as just two terms of the sine series, as you may have expected.

Example 3:
Express the function

$$f(x) = \pi - x \qquad 0 \le x < \pi$$

as a Fourier sine series.

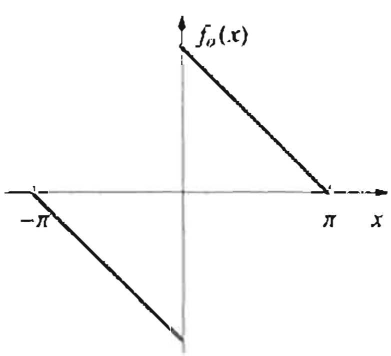

Figure 15.28
The odd extension of the function defined in Example 3 plotted against x.

SOLUTION: To express $f(x)$ as a Fourier sine series, we use the odd extension of $f(x)$ (see Figure 15.28):

8. Express the function $f(x) = x^3 - x$, $0 \le x < 2$, as both a Fourier sine series and cosine series.

9. Express the function $f(x) = x^2$, $0 \le x < l$, as a Fourier sine series.

10. Express the function $f(x) = \sin x$, $0 \le x < \pi$, as a Fourier cosine series.

11. Express the function $f(x) = \cos x$, $0 \le x < \pi$, as a Fourier sine series.

12. Express the function in Problem 10 as a Fourier sine series.

13. Express the function $f(x) = \begin{cases} x & 0 \le x < 1 \\ 2 - x & 1 \le x < 2 \end{cases}$ as a Fourier sine series.

14. Express the function in Problem 13 as a Fourier cosine series.

15. Express the function in Example 3 as a Fourier cosine series.

16. Consider the function $f(x) = \sin x$ defined on the interval $(0, \pi)$. Would you expect the Fourier series of its even or odd extension to converge more rapidly?

15.3 Convergence of Fourier Series

In the previous chapter, we learned about the convergence of eigenfunction expansions associated with Sturm-Liouville problems. Because the Fourier series that we have been discussing in this chapter can be viewed as eigenfunction expansions, it shouldn't be surprising that the convergence properties of Fourier series are similar to those in the previous chapter. Fourier series are so important in their own right, however, that we shall restate the convergence theorems here.

First, there's the property of convergence in the mean. Let $S_N(x)$ be the nth partial sum of the Fourier expansion of $f(x)$:

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \tag{1}$$

The mean square error in approximating $f(x)$ by $S_N(x)$ is

$$\begin{aligned} D_N^2 &= \int_{-l}^{l} [f(x) - S_N(x)]^2 dx \\ &= \int_{-l}^{l} f^2(x)dx - 2\int_{-l}^{l} f(x)S_N(x)dx + \int_{-l}^{l} S_N^2(x)dx \ge 0 \end{aligned} \tag{2}$$

But

$$\int_{-l}^{l} f(x)S_N(x)dx = \int_{-l}^{l} f(x)\left[\frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)\right]dx$$

Using the definition of the a_n and b_n (Equations 11 and 12 of Section 1), we have

$$\int_{-l}^{l} f(x)S_N(x)dx = \frac{la_0^2}{2} + l\sum_{n=1}^{N}(a_n^2 + b_n^2) \tag{3}$$

$$= \frac{2l}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots\right)$$

or

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$$

This was the first series involving π ever discovered. It converges much too slowly to be useful for calculating the value of π, however, but it was a wonderfully curious result when it was first discovered.

We frequently want to differentiate and integrate Fourier series term by term. This leads us to the following theorem:

Let $f(x)$ be piecewise continuous in $[-l, l)$ and periodic with period $2l$, then the Fourier series of $f(x)$ may be integrated term by term.

Interestingly, the above theorem holds even if the Fourier series for $f(x)$ does not converge pointwise to $f(x)$. The result of the term by term integration of the Fourier series is to introduce a factor of $1/n$ into the resulting series, hence enhancing its convergence. This is due to the fact that integration is a smoothing process.

Recall from Chapter 2 that if the functions $\{u_n(x)\}$ are continuous in $[a, b]$, then we can integrate the series $S(x) = \sum_{n=1}^{\infty} u_n(x)$ term by term if the series is uniformly convergent to $S(x)$. The fact that we can integrate a Fourier series term by term makes you wonder about the uniform convergence of Fourier series. We can state the following theorem:

If $f(x)$ is piecewise smooth in the interval $[a, b)$ and is periodic, then the Fourier series of $f(x)$ converges uniformly to $f(x)$ in every closed interval containing no discontinuity.

As the above theorem says, the partial sums of a Fourier series cannot approach $f(x)$ uniformly over any interval containing a point where $f(x)$ is discontinuous. This is nicely illustrated with the Fourier series for the step function

$$f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \end{cases}$$

whose Fourier series is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1} \tag{15}$$

Figure 15.31a shows the sum of the first ten terms of Equation 15 plotted against x. As more and more terms are taken, the small oscillations along each horizontal portion get smaller and smaller and eventually disappear, *except* for the two outer ones of each portion closest to the discontinuities. (See Figure 15.31b.) Even in the limit of an infinite number of terms, there persists a small "overshoot." This overshoot is called the *Gibbs phenomenon*, and is due to the fact that you cannot have uniform convergence at a point of discontinuity.

It so happens that the Gibbs phenomenon was not discovered by Gibbs, nor by a mathematician, nor by a theorist. It was actually discovered by the American experimental physicist Albert Michelson, of Michelson-Morley experiment fame. Michelson was a pioneer in interferometry and developed a harmonic analyzer, an optical-mechanical device that takes a periodic input and breaks it down into its individual Fourier components. He was able to calculate almost 100 terms of Fourier series, and noticed that there is always a small discrepancy around points of discontinuities. He brought this to the attention of J. Willard Gibbs (of thermodynamic and statistical mechanics fame), who analyzed the overshoot mathematically and showed that it approaches about 10% of the height of the discontinuity (Problem 17). Thus, the Gibbs phenomenon was discovered empirically by an experimentalist extraordinaire.

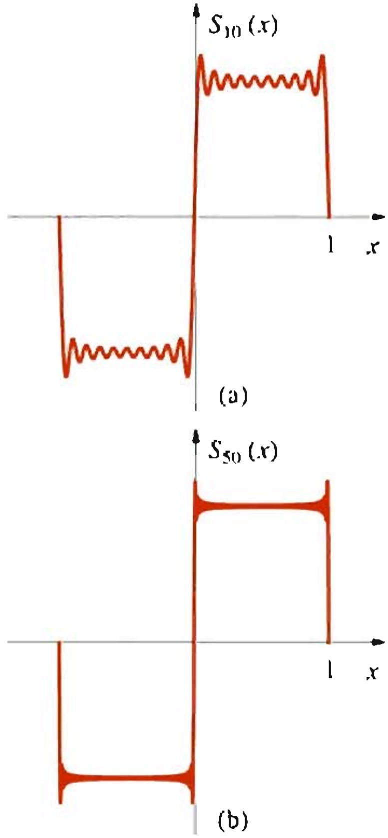

Figure 15.31
(a) The sum of the first 10 terms in Equation 15 plotted against x. (b) The sum of the first 50 terms in Equation 15 plotted against x.

Example 3:
Use the Fourier series in Example 2 of Section 1,

$$x = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l} \tag{16}$$

to derive a Fourier series for x^2 in the interval $[0, l)$.

SOLUTION: Integrate the above series term by term from 0 to x.

$$\begin{aligned}\frac{x^2}{2} &= \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^x \sin\frac{n\pi u}{l}\,du \\ &= \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left(1-\cos\frac{n\pi x}{l}\right) \\ &= \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} - \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}\end{aligned}$$

The first summation here is of the type that we studied in Section 3.7 and is equal to $\pi^2/12$. Thus, we have

$$x^2 = \frac{4l^2}{\pi^2}\left[\frac{\pi^2}{12} - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}\right]$$

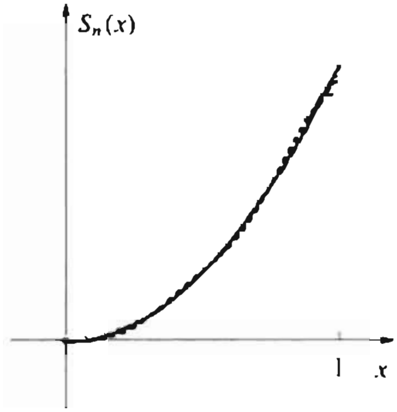

Figure 15.32
The function $f(x) = x^2$ (black) and the partial sums of the Fourier series in Example 3 consisting of 4 (dotted color), 8 (dashed color), and 16 (solid color) terms.

Figure 15.32 shows some partial sums of this series. Being a cosine series, it actually represents $f(x) = x^2$ in $[-l, l)$, as well as its periodic extension.

Let's differentiate the series for x in Equation 16 instead of integrating it. Differentiation gives

$$1 \overset{?}{=} 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos\frac{n\pi x}{l}$$

This series doesn't even converge! Notice that differentiation of $\cos n\pi x/l$ or $\sin n\pi x/l$ introduces a factor of n into the resulting series, so we can say that differentiation of a Fourier series is a threat to convergence. The relevant theorem regarding term by term differentiation of a Fourier series is

Let $f(x)$ be a continuous function in the interval $[-l, l]$ with $f(-l) = f(l)$, and let $f'(x)$ be piecewise smooth in $[-l, l]$. Then the Fourier series of $f'(x)$ can be obtained by differentiating the Fourier series of $f(x)$ term by term. Furthermore, the resulting series for $f'(x)$ converges to $f'(x)$ where it is continuous and to $[f'(x+) + f'(x-)]/2$ at the points where it is discontinuous.

Notice that the theorem governing term by term differentiation of a Fourier series is more demanding than for term by term integration. Recall from Chapter 2 that you can differentiate a series term by term only if the resulting series is uniformly convergent.

The reason that we couldn't differentiate the Fourier series for $f(x) = x$ in Equation 16 is that the function $f(x) = x$ represented by the Fourier series is *not* a continuous function when its periodic extension (which the Fourier series represents) is considered. It is continuous in its fundamental interval $[-l, l)$, but not as a periodic function.

We've observed several times that the rate of decay of Fourier coefficients depends upon whether or not the periodic extension of $f(x)$ is continuous. We found that if the periodic extension of $f(x)$ is discontinuous, then its Fourier coefficients decay as $1/n$, and that they decay at least as fast as $1/n^2$ if the periodic extension of $f(x)$ is continuous. We can formalize these observations in the following theorem:

If $f(x)$ and its first k derivatives satisfy the Dirichlet conditions on the interval $[-l, l]$ and if the periodic extensions of $f(x)$, $f'(x)$, ..., $f^{(k-1)}(x)$ are all continuous, then the Fourier coefficients of $f(x)$ decay at least as rapidly as $1/n^{k+1}$.

Example 4:
Determine the order of the Fourier coefficients of $f(x) = x^2$ on the interval $[-l, l]$.

SOLUTION: Figure 15.33 shows $f(x) = x^2$ in $[-l, l]$ and its periodic extension. The function $f(x)$ is continuous, but its first derivative is not. Therefore, $k = 1$ in the above theorem, and so we expect the coefficients in the Fourier series for $f(x)$ to decay as $1/n^2$ (at least).

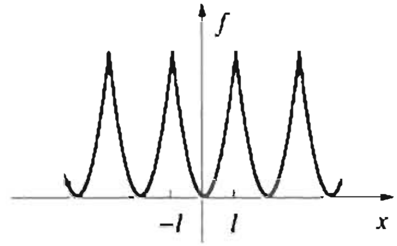

Figure 15.33
The function $f(x) = x^2$, $-l \le x < l$, and its periodic extension defined by $f(x + 2l) = f(x)$.

15.3 Problems

1. Derive Equation 4.

2. Use Parseval's equality and the Fourier series in Example 1 of Section 1 to show that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

3. Use Parseval's equality and the Fourier series in Example 3 of Section 1 to show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$.

   Hint: You need to use the result of Example 1 $\left(\sum_{n=1}^{\infty} n^{-2} = \pi^2/6\right)$.

4. Use Parseval's equality and the Fourier series in Example 5 of Section 1 to show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

5. Use Parseval's equality and the Fourier series in Example 4 of Section 1 to show that $\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}$ $= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2} = \frac{\pi^2 - 8}{16}$.

6. The Fourier sine series of $f(x) = x^3 - 4x$, $0 \le x < 2$, is $f(x) = \frac{96}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin \frac{n\pi x}{2}$. Use Parseval's equality to show that $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

7. The Fourier sine series of $f(x) = x(\pi - x)$, $0 \le x < \pi$, is $f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}$. Use Parseval's equality to show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$.

8. The Fourier series of $f(x) = x$ for $[-l, l)$ and $f(x + 2l) = f(x)$ outside that interval is (Example 2 of Section 1) $f(x) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l}$. Why does the series equal zero when $x = 0$?

9. Evaluate the series in Example 1 of Section 1 at $x = 0$ and $x = l$. Do these answers make sense?

15.4 Fourier Series and Ordinary Differential Equations

Fourier series are frequently used to solve linear nonhomogeneous differential equations with constant coefficients. For example, consider the equation

$$y''(x) - y(x) = f(x) \qquad 0 \leq x \leq l \tag{1}$$

with the boundary conditions $y(0) = y(l) = 0$. To solve Equation 1 by means of Fourier series, first express $f(x)$ as a Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \tag{2}$$

Now write $y(x)$ as a Fourier series.

$$y(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right) \tag{3}$$

where the A_n and B_n are to be determined. We substitute Equations 3 and 2 into Equation 1 to obtain

$$\begin{aligned} \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \left(1 - \frac{n^2\pi^2}{l^2}\right) \cos \frac{n\pi x}{l} + B_n \left(1 - \frac{n^2\pi^2}{l^2}\right) \sin \frac{n\pi x}{l} \\ = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \end{aligned} \tag{4}$$

Equation 4 gives us (Problem 1)

$$A_n = \frac{a_n l^2}{l^2 - n^2\pi^2} \qquad \text{and} \qquad B_n = \frac{b_n l^2}{l^2 - n^2\pi^2} \qquad n = 0, 1, \ldots \tag{5}$$

with $l \neq n\pi$. Realize that Equations 5 are valid only for Equation 1. (See Problem 7 for a slight generalization.)

We still have the boundary conditions to consider. The coefficients a_n and b_n are determined by $f(x)$, and particularly how we choose to extend $f(x)$ to the interval $[-l, 0)$. The $a_n = 0$ if we extend $f(x)$ such that it is an odd function over $[-l, l]$ and the $b_n = 0$ if we extend $f(x)$ such that it is an even function. The boundary conditions, $y(0) = y(l) = 0$, will be satisfied by $y(x)$ in Equation 3 if all the $A_n = 0$; this will be the case if all the $a_n = 0$, and so we choose the odd extension of $f(x)$ to develop its Fourier series in Equation 2. Thus,

$$y(x) = \sum_{N=1}^{\infty} \frac{b_n l^2}{l^2 - n^2\pi^2} \sin \frac{n\pi x}{l} \tag{6}$$

satisfies Equation 1 *and* the boundary conditions $y(0) = y(l) = 0$.

8. Use Fourier series to solve the equation $y''(x) + \beta^2 y(x) = x$ with boundary conditions $y(0) = y(1) = 0$. *Hint:* You need to use the result of Problem 7.

9. Solve the equation in Problem 8 by the method of undetermined coefficients.

10. The solution to Problem 9 is $y(x) = \dfrac{x}{\beta^2} - \dfrac{\sin \beta x}{\beta^2 \sin \beta}$. Expand $y(x)$ in a Fourier sine series and compare your result to the one obtained in Problem 8.

11. Derive Equation 8.

12. Show that Equation 12 is a solution to Equation 7, when the $x_n(t)$ satisfy Equation 11.

13. Show that Equation 14 is a particular solution of Equation 11.

14. Derive Equation 15.

15. Show that the third term in Equation 17 dominates if $m = 4$, $\gamma = 1/10$, $k = 98$, and $\omega = 1$. Derive the equation that corresponds to Equation 18.

16. Determine the steady periodic solution to $y''(t) + 2y'(t) + 3y(t) = f(t)$, where $f(t) = t$, $-\pi \leq t < \pi$, and $f(t + 2\pi) = f(t)$.

17. Consider the general linear second-order nonhomogeneous differential equation with constant coefficients $x''(t) + a_1 x'(t) + a_0 x(t) = f(t)$. Show that if $f(t) = Ae^{i\omega t}$, then the real part of the steady periodic response corresponds to the resulting $x^{sp}(t)$ if $f(t) = A\cos\omega t$ and the imaginary part corresponds to $x^{sp}(t)$ if $f(t) = B\sin\omega t$.

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CHAPTER 16

Partial Differential Equations

Many of the equations of the sciences and engineering describe how some physical quantity such as temperature or an electrical potential vary with position and time. This means that one or more spatial variables and time serve as independent variables. If we let $T = T(x, y, z, t)$ be the temperature, an equation that governs how $T(x, y, z, t)$ varies with x, y, z, and t is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$$

or

$$\alpha^2 \nabla^2 T = \frac{\partial T}{\partial t}$$

where α^2 is a constant called the thermal diffusivity. This equation, called the heat equation, is a *partial differential equation* because the dependent variable, T, occurs as partial derivatives. A study of partial differential equations is one of the most important areas of applied mathematics because so many physical processes (such as heat flow) are formulated as partial differential equations.

The field of partial differential equations is very large. We are going to discuss only a few special partial differential equations in this chapter, but ones that occur in a wide range of applied problems. We shall concentrate on Laplace's equation in Section 2, the wave equation in Sections 3 and 4, the heat equation (also called the diffusion equation) in Section 5, and the Schrödinger equation in Section 6. It turns out that these equations (and closely related versions of these equations) comprise a sizable fraction of the partial differential equations that many scientists and engineers encounter in practice. Furthermore, we shall see that they illustrate all the various types of linear partial differential equations according to one important classification of such equations that we shall discuss in Section 7. In this chapter, we shall use the method of separation of variables to solve these equations, and in the next chapter, we shall solve these equations by what are called integral transform methods.

16.1 Some Examples of Partial Differential Equations

We derived the heat equation in Section 7.4 using the divergence theorem. As we stated in the introduction, if we let $T = T(x, y, z, t)$ be the temperature at the point (x, y, z) at time t, then the heat equation is

$$\alpha^2 \nabla^2 T = \frac{\partial T}{\partial t} \tag{1}$$

where α^2 is the thermal diffusivity. Equation 1 is a second-order partial differential equation because the order of the highest derivative is two. Furthermore, it is a *linear* partial differential equation because T and its various partial derivatives appear only to the first power and there are no products of terms involving T or its derivatives. Many of the classic partial differential equations that occur in the physical sciences and engineering are linear and second-order.

Example 1:
Show that

$$T(x, y, z, t) = Ae^{-3\alpha^2 t} \sin x \sin y \sin z$$

satisfies the three-dimensional heat equation, Equation 1.

SOLUTION:

$$\nabla^2 T = -3Ae^{-3\alpha^2 t} \sin x \sin y \sin z = -3AT$$

$$\frac{\partial T}{\partial t} = -3\alpha^2 AT$$

so

$$\frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T$$

Let's consider the one-dimensional version of Equation 1:

$$\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{2}$$

Equation 2 might describe a long, thin bar of length l where its temperature varies only along its length x. Equation 2 is second-order with respect to the spatial coordinate and first order with respect to time. This suggests that some sort of general solution must be made to satisfy two boundary conditions and one initial condition. For example, we might specify the temperature at the ends of the bar. $x = 0$ and $x = l$,

$$T(0, t) = T_1 \qquad \text{and} \qquad T(l, t) = T_2$$

and the initial temperature profile of the bar

$$T(x, 0) = f(x)$$

It turns out that these conditions are sufficient to specify the solution $T(x, t)$ completely (and uniquely). We'll learn how to find such solutions in this and the next chapter.

Equation 1 (with different symbols) describes the diffusion of one substance through another. If $c(x, y, z, t)$ denotes the concentration of a given substance around the point (x, y, z), then $c(x, y, z, t)$ is given (approximately) by (Problem 1)

$$D\nabla^2 c = \frac{\partial c}{\partial t} \tag{3}$$

where D is the diffusion constant. Equation 3 is called the *diffusion equation*. We also derived the diffusion equation in Section 7.1.

Another important partial differential equation is the *wave equation*. For simplicity only, we shall derive the wave equation for the case of just one spatial variable. The physical system represented is that of a vibrating string. Consider a perfectly flexible homogeneous string stretched to a uniform tension τ between two points. Let $u(x, t)$ be the displacement of the string from its horizontal position (see Figure 16.1). The quantities τ_1 and τ_2 in Figure 16.1 are the tensions at the points P and Q on the string. Both τ_1 and τ_2 are tangential to the curve of the string. Assuming that there is only vertical motion of the string, the horizontal components of the tensions at all points along the string must be equal. Using the notation in Figure 16.1, we have that

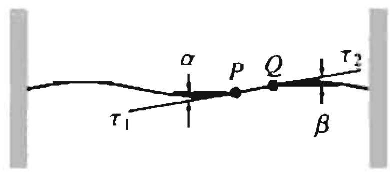

Figure 16.1
A vibrating string at an instant of time. The quantities shown in the figure are used in the derivation of the classical one-dimensional wave equation.

$$\tau_1 \cos\alpha = \tau_2 \cos\beta = \tau = \text{constant} \tag{4}$$

There is a net force in the vertical direction that causes the vertical motion of the string. Again, using the notation in Figure 16.1, we find that the net vertical force is

$$\text{net vertical force} = \tau_2 \sin\beta - \tau_1 \sin\alpha$$

By Newton's second law, this net force is equal to the mass $\rho\Delta x$ along the segment PQ times the acceleration of the string, $\partial^2 u/\partial t^2$. Thus, we write

$$\tau_2 \sin\beta - \tau_1 \sin\alpha = \rho\Delta x \frac{\partial^2 u}{\partial t^2} \tag{5}$$

Dividing Equation 5 by Equation 4 gives

$$\tan\beta - \tan\alpha = \frac{\rho\Delta x}{\tau}\frac{\partial^2 u}{\partial t^2} \tag{6}$$

But $\tan\beta$ and $\tan\alpha$ are the slopes of the curve of the string at $x + \Delta x$ and at x, respectively, and so

$$\tan\alpha = \frac{\partial u}{\partial x} = u_x(x) \qquad \text{and} \qquad \tan\beta = \frac{\partial u}{\partial x} = u_x(x + \Delta x) \tag{7}$$

where u_x denotes the partial derivative of u with respect to x. Substituting Equations 7 into Equation 6 gives

$$u_x(x + \Delta x) - u_x(x) = \frac{\rho\Delta x}{\tau}\frac{\partial^2 u}{\partial t^2} \tag{8}$$

Expanding $u_x(x + \Delta x)$ in a Taylor series about $\Delta x = 0$ gives

$$\begin{aligned} u_x(x + \Delta x) &= u_x + \frac{\partial u_x}{\partial x}\Delta x + O\,[(\Delta x)^2] \\ &= u_x + \frac{\partial^2 u}{\partial x^2}\Delta x + O\,[(\Delta x)^2] \end{aligned}$$

and so Equation 8 becomes

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{9}$$

in the limit $\Delta x \to 0$, where $v = (\tau/\rho)^{1/2}$ has units of speed.

Equation 9 is known as the classical one-dimensional wave equation. Its extension to more spatial variables is given by

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{10}$$

The two-dimensional equation describes the vibrations of a membrane and the three-dimensional equation describes the vibrational motion of an elastic solid.

Example 2:
Show that

$$u(x, t) = \sin x \sin vt$$

is a solution to the one-dimensional wave equation.

SOLUTION:

$$u_{xx} = -u \qquad \text{and} \qquad u_{tt} = -v^2 u$$

so $u(x, t) = \sin x \sin vt$ is a solution to Equation 9. Note that the solution to the heat equation in Example 1 decayed exponentially with time, while the above solution to the wave equation oscillates in time.

Being second-order in both x and t, it may not be surprising to learn that Equation 9 requires two boundary conditions and two initial conditions to yield a unique solution. For example, these conditions might be

$$u(0, t) = 0 \qquad u(l, t) = 0$$
$$u(x, 0) = f(x) \quad u_t(x, 0) = g(x)$$

The two boundary conditions say that the displacement is zero at the two ends of the string and the initial conditions give the initial displacement and the initial velocity of the string.

As Example 2 suggests, the time dependence of the wave equation is oscillatory. If we substitute

$$u(x, t) = X(x)e^{i\omega t}$$

into Equation 10, we obtain

$$\nabla^2 X(x) + \omega^2 X(x) = 0 \tag{11}$$

Equation 11 occurs in a number of physical problems and is called the *Helmholtz equation*.

Another partial differential equation that we shall discuss is *Laplace's equation*,

$$\nabla^2 \phi(x, y, z) = 0 \tag{12}$$

If T in Equation 1 is independent of time, then Equation 1 reduces to Laplace's equation, which in this case describes the equilibrium or the steady-state temperature distribution in a body.

Laplace's equation also governs the electrostatic potential in a charge-free region of space. If there is a charge density $\rho_c(x, y, z)$ within the region, then the electrostatic potential is given by *Poisson's equation*,

$$\nabla^2 \phi(x, y, z) = -\frac{\rho_c(x, y, z)}{\epsilon_0} \tag{13}$$

where ϵ_0 is the permittivity of free space. Poisson's equation is a nonhomogeneous version of Laplace's equation. We derived Poisson's equation in Section 7.4 from Gauss's law of electrostatics.

Example 3:
Show that

$$\phi(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{1/2}} \qquad (x, y, z) \neq (0, 0, 0)$$

satisfies Laplace's equation.

SOLUTION:

$$\phi_x = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\phi_{xx} = \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

with analogous equations for ϕ_{yy} and ϕ_{zz}. Thus,

$$\begin{aligned} \nabla^2\phi &= \phi_{xx} + \phi_{yy} + \phi_{zz} \\ &= \frac{(2x^2 - y^2 - z^2) + (2y^2 - x^2 - z^2) + (2z^2 - x^2 - y^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0 \end{aligned}$$

Example 4:
Show that if ϕ is spherically symmetric (in other words, depends only upon the radial coordinate r), then Laplace's equation becomes

$$\nabla^2\phi = \frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0$$

Solve this equation to show that

$$\phi(r) = \frac{c}{r} \qquad r \neq 0$$

where c is a constant.

SOLUTION: Referring to Table 8.3, we see that

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\phi}{\partial\phi^2}$$

If ϕ depends only upon r, then

$$\nabla\phi = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)$$

and $\nabla^2\phi = 0$ can be written as

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0$$

Integrating once gives $d\phi/dr = a/r^2$, and integrating once again gives

$$\phi(r) = -\frac{a}{r} + b$$

where a and b are constants. If we require that $\phi(r) \to 0$ as $r \to \infty$, then we find that $b = 0$. Replacing the constant $-a$ by another constant c gives $\phi(r) = c/r$ (Coulomb's law).

Before we conclude this section, there is one other partial differential equation that we must discuss. The central equation of non-relativistic quantum mechanics is the (time-independent) Schrödinger equation, which for a single particle of mass m in a potential $V(x, y, z)$ is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \tag{14}$$

where $\hbar = h/2\pi$ and h is the Planck constant. The dependent variable in Equation 14, $\psi(x, y, z)$, is the wave function of the particle. The wave function has the physical interpretation that $\psi^*(x, y, z)\psi(x, y, z)dxdydz$ is the probability that the particle will be observed to be in the volume element $dxdydz$ at the point (x, y, z). According to quantum mechanics, this probabilistic interpretation is the most complete description possible.

The Schrödinger equation cannot be derived from more fundamental principles, any more than Newton's classical equations of motion can. Each can be deduced from other basically equivalent approaches, but each one is essentially a postulate that has withstood intensive and long scrutiny.

If we define the so-called Hamiltonian operator by

$$\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x, y, z) \tag{15}$$

we can write Equation 14 as an eigenvalue problem

$$\mathcal{H}\,\psi_n(x, y, z) = E_n\psi_n(x, y, z) \tag{16}$$

The eigenvalues, $\{E_n\}$, have the physical interpretation that they are the allowed energies that the particle can have. Equation 16 is generally a partial differential equation, but it is just an ordinary differential equation in one dimension. Problem 14 works through the most basic non-trivial quantum-mechanical problem, a particle in a one-dimensional box.

16.1 Problems

1. Use the continuity equation (Equation 7.1.11) and Fick's law ($\mathbf{J} = -D$ grad c) to derive Equation 3.

2. Verify that $u(x, y, t) = \sin(x/2^{1/2}) \sin(y/2^{1/2}) \sin vt$ is a solution to the two-dimensional wave equation.

3. Verify that $u(x, t) = f(x \pm vt)$, where $f(z)$ is a differentiable function, is a solution to the one-dimensional wave equation.

4. Show that $c(x, t) = t^{-1/2} \exp(-x^2/4Dt)$ is a solution to the one-dimensional diffusion equation. What initial condition does this solution correspond to?

5. Verify that $c(r, t) = t^{-3/2} \exp(-r^2/4Dt)$ is a solution to the diffusion equation in spherical coordinates.

6. Starting with the continuity equation (Equation 7.1.11), show that $\dfrac{\partial c}{\partial t} = D\dfrac{\partial^2 c}{\partial x^2} + \beta\dfrac{\partial c}{\partial x}$ represents a system that is undergoing diffusion and moving with a constant drift velocity β. This equation could represent the diffusion of a large charged particle such as a protein in a uniform electric field.

7. Verify that $c(x, t) = t^{-1/2} \exp[-(x - x_0 + \beta t)^2/4Dt]$ is a solution to the equation in the previous problem. What initial condition does this solution correspond to?

8. Maxwell's equations for a nonconducting medium with no free charges can be written as $\operatorname{div}\mathbf{E} = 0$, $\operatorname{curl}\mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0$, $\operatorname{div}\mathbf{B} = 0$, and $\operatorname{curl}\mathbf{B} - \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} = 0$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric field and the magnetic induction, respectively. Show that if ζ represents any of the components of $\mathbf{E}$ and $\mathbf{B}$, then ζ satsifies $\nabla^2\zeta = \dfrac{1}{c^2}\dfrac{\partial^2\zeta}{\partial t^2}$. *Hint*: Use the vector identity curl curl $\mathbf{v}$ = grad div $\mathbf{v} - \nabla^2\mathbf{v}$.

9. Show that the substitution $T(x, t) = \phi(x, t)e^{-ht}$, where h is a constant, reduces the equation $\dfrac{\partial T}{\partial t} = \alpha^2\dfrac{\partial^2 T}{\partial x^2} - hT$ to the one-dimensional heat equation.

10. Show that the kinetic energy and the potential energy of a uniform vibrating string are given by $K = \dfrac{\rho}{2}\displaystyle\int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx$ and $V = \dfrac{\tau}{2}\displaystyle\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx$.

11. If all the points of a uniform string experience a frictional force proportional to the velocity, show that the wave equation becomes $\dfrac{1}{v^2}\dfrac{\partial^2 u}{\partial t^2} + \gamma\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0$, where γ is the proportional frictional coefficient.

12. Show that the substitution $u(x, t) = f(x) \sin vt$ reduces the one-dimensional wave equation to an ordinary differential equation in x. Does this substitution make sense physically?

13. Use the divergence theorem to derive the heat equation if a heat source is present within the region of interest. Let $F(x, y, z, t)$ be the rate of energy as heat generated per unit volume.

14. The time-independent Schrödinger equation for a free particle constrained to the one-dimensional region 0 to a is $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} = E\psi$. Because $\psi^*\psi\, dx$ represents the probability that the particle will be located within the interval $(x, x + dx)$, we require that $\psi(x) = 0$ for $x \leq 0$ and $x \geq a$. Show that the energy of the particle is *quantized*, and in particular, that $E_n = \dfrac{n^2h^2}{8ma^2}$, where $n = 1, 2, \ldots$ and $\hbar = h/2\pi$.

boundary, then the potential within the region enclosed by the boundary is uniquely determined.

The other commonly occurring boundary condition is the specification of the normal derivative of u on the boundary, or

$$\nabla^2 u = 0 \quad \text{in } R \qquad \text{with} \qquad \frac{\partial u}{\partial n} = f \quad \text{on } B \tag{4}$$

where $\partial u/\partial n$ is the normal derivative of u, taken in the outward direction. This type of boundary condition is called a *Neumann boundary condition* and Equation 4 constitutes what is called a *Neumann problem*. You might remember from electrostatics that if you specify the charge density (which is directly related to $\partial u/\partial n$) on a boundary, then the potential is determined to within an additive constant.

The proof of the uniqueness of the solutions to the two-dimensional Dirichlet problem and the two-dimensional Neumann problem (to within an additive constant) are given in Problems 24 through 27. The Dirichlet problem will always have a solution (existence theorem), but the Neumann problem may not. Problem 28 helps you show that the Neumann problem has a solution only if

$$\oint_B \frac{\partial u}{\partial n} ds = \oint_B f \, ds = 0 \tag{5}$$

where the integral is along the boundary. Equation 5 is called a *compatibility condition*. We can give a simple physical interpretation of this condition. Suppose that u represents the steady-state temperature in some region. Then $\partial u/\partial n = f$ represents the flux of energy as heat across the boundary. Equation 5 says that the net flux across the boundary must be zero in order to maintain a steady-state temperature.

Now that we've discussed some general properties, let's solve Laplace's equation for a few special cases. Consider the Dirichlet problem for a rectangle:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad \begin{array}{l} 0 < x < a \\ 0 < y < b \end{array} \tag{6}$$

$$\begin{array}{ll} T(0, y) = 0 & \qquad T(a, y) = 0 \\ T(x, 0) = 0 & \qquad T(x, b) = f(x) \end{array} \tag{7}$$

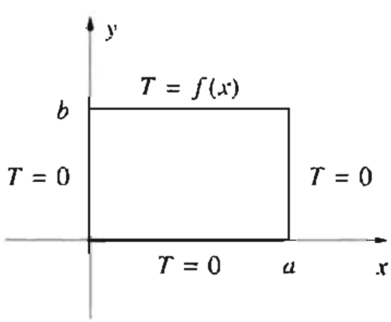

Figure 16.2
A summary of the boundary conditions for Equations 6 and 7.

This problem might correspond to a determination of the steady-state temperature distribution in a rectangular plate with the temperature prescribed on each of its edges (Figure 16.2).

Laplace's equation, as well as many other partial differential equations that arise in physics, can be solved by a method called *separation of variables*. The key step in the method of separation of variables is to assume that $T(x, y)$ in this case factors into a function of x only times a function of y only, or that

$$T(x, y) = X(x)Y(y) \tag{8}$$

If we substitute Equation 8 into Equation 6, we obtain

$$Y(y)\frac{d^2X}{dx^2} + X(x)\frac{dY^2}{dy^2} = 0$$

Now divide by $T(x, y) = X(x)Y(y)$ and obtain

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = -\frac{1}{Y(y)}\frac{dY^2}{dy^2} \tag{9}$$

The left side of Equation 9 is a function of x only and the right side is a function of y only. Because x and y are independent variables, each side of Equation 9 can be varied independently. The only way for the equality of the two sides of Equation 9 to be preserved under any variation of x and y is that each side be equal to a constant. If we let this constant be k, then we can write

$$X''(x) - kX(x) = 0 \qquad \text{and} \qquad Y''(y) + kY(y) = 0 \tag{10}$$

where k is called the *separation constant* and will be determined by the boundary conditions, Equations 7. The boundary conditions can be expressed in terms of $X(x)$ and $Y(y)$ by

$$X(0) = X(a) = 0, \quad Y(0) = 0, \quad T(x, b) = f(x) \tag{11}$$

Notice that the method of separation of variables has given us two *ordinary* differential equations, one for each independent variable.

We do not know at this point if the value of k is positive, zero, or negative. Let's first assume that $k = 0$. In this case, Equations 10 give $X(x) = \alpha_1 x + \beta_1$ and $Y(y) = \alpha_2 y + \beta_2$, and we find that the only way to satisfy $X(0) = X(a) = 0$ is for $\alpha_1 = \beta_1 = 0$, which gives us a trivial solution. Problem 1 asks you to show that the same situation occurs if k is positive. This leaves only the possibility that $k < 0$. To emphasize that we consider k to be negative, we write it as $-\lambda^2$ (with λ real), so that Equations 10 become

$$X''(x) + \lambda^2 X(x) = 0 \qquad \text{and} \qquad Y''(y) - \lambda^2 Y(y) = 0 \tag{12}$$

Solving the first of Equations 12 for $X(x)$ and applying the boundary conditions give $\lambda_n = n\pi/a$ and

$$X_n(x) = \sin\frac{n\pi x}{a} \qquad n = 1, 2, \ldots \tag{13}$$

Solving the second of Equations 12 for $Y(y)$ gives

$$Y_n(y) = c_1\cosh\frac{n\pi y}{a} + c_2\sinh\frac{n\pi y}{a} \tag{14}$$

The boundary condition $Y(0) = 0$ forces c_1 to be equal to zero. So far, then, we have

$$T_n(x, y) = \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} \qquad n = 1, 2, \ldots$$

Because Laplace's equation is linear, the complete solution to Equation 6 is given by a superposition of the $T_n(x, y)$.

$$T(x, y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} \tag{15}$$

Equation 15 satisfies Equation 6 and the three homogeneous boundary conditions. We can determine all the c_n in Equation 15 by using the fourth boundary condition. Letting $y = b$ in Equation 15, we have

$$T(x, b) = f(x) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a} \qquad 0 < x < a$$

Using the orthogonality of the $\{\sin n\pi x/a\}$ over the interval $(0, a)$ gives

$$c_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \tag{16}$$

If $f(x) = T_0 x(a - x)$, for example, then

$$c_n \sinh\frac{n\pi b}{a} = \frac{2T_0}{a}\int_0^a x(a - x)\sin\frac{n\pi x}{a}dx$$

$$= 4T_0 a^2\frac{1 - (-1)^n}{n^3\pi^3}$$

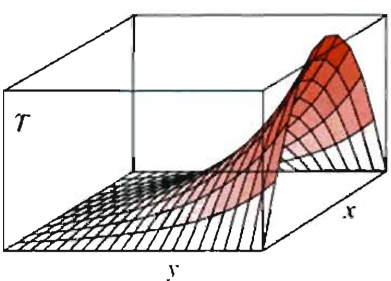

Figure 16.3
Equation 17 plotted against x and y.

and so the solution is

$$T(x, y) = \frac{8T_0 a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{[1 - (-1)^n]}{n^3}\frac{\sinh\frac{n\pi y}{a}}{\sinh\frac{n\pi b}{a}}\sin\frac{n\pi x}{a} \tag{17}$$

Figure 16.3 shows the steady-state temperature given by Equation 17 plotted against x and y.

Example 1:
Determine the steady-state temperature distribution in a rectangular plate that is insulated along the edges $x = 0$ and $x = a$, so that

$$T_x(0, y) = T_x(a, y) = 0$$

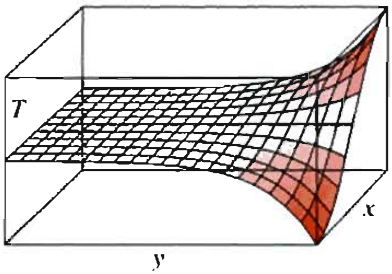

Figure 16.5
The steady-state temperature distribution for the system described in Example 1.

It's easy to verify that $T(x, y)$ satisfies Laplace's equation and all the boundary conditions. Figure 16.5 shows $T(x, y)$ plotted over the rectangular region.

Example 2:
The velocity profile $u(x, y)$ for the steady flow of a viscous fluid through a rectangular conduit is governed by the equation

$$\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\gamma$$

where η is the coefficient of viscosity and γ is a constant pressure head. Solve this equation with the "non-slip" boundary conditions

$$u(-a, y)=u(a, y)=u(x, b)=u(x, -b)=0$$

The geometry is shown in Figure 16.6.

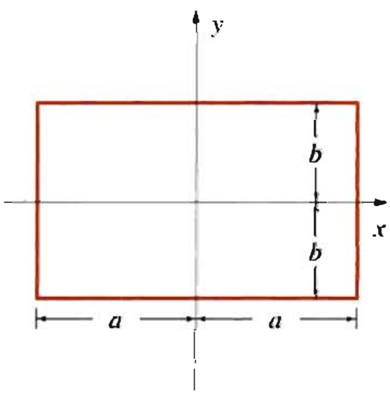

Figure 16.6
The coordinates for a rectangular cross-section describing unidirectional flow in the rectangular conduit described in Example 2.

SOLUTION: As with ordinary differential equations, the solution to the above equation is given by

$$u(x, y)=u_{\mathrm{c}}(x, y)+u_{\mathrm{p}}(x, y)$$

where $u_{\mathrm{c}}(x, y)$ is the complementary solution (that is, the solution to the homogeneous equation), and $u_{\mathrm{p}}(x, y)$ is any particular solution to the homogeneous equation. Certainly

$$u_{\mathrm{p}}(x, y)=\frac{\gamma}{2\eta}y^2+c_1y+c_2$$

is a particular solution.

The boundary conditions that we shall use are the so-called non-slip boundary conditions, which say that the velocity of the fluid at a boundary is zero. Applying the non-slip boundary conditions to $u_{\mathrm{p}}(x, y)$ at $y=\pm b$ in Figure 16.6 gives us

$$u_{\mathrm{p}}(x, y)=\frac{\gamma}{2\eta}(y^2-b^2)$$

We use separation of variables to find the complementary solution. Letting $u(x, y)=X(x)Y(y)$, we find that

$$X''(x)-\lambda^2X(x)=0 \qquad \text{and} \qquad Y(y)+\lambda^2Y(y)=0$$

where λ^2 is the separation constant. The solution to the equation for $Y(y)$ is

$$Y(y)=c_1\cos\lambda y+c_2\sin\lambda y$$

We've taken $-a \le x \le a$ and $-b \le y \le b$. Physically, we expect the velocity profile to be symmetric in x and y, so we set $c_2 = 0$ in $Y(y)$. The no-slip boundary condition, $u(x, -b) = u(x, b) = 0$, gives us $c_1 \cos \lambda b = 0$, or $\lambda_n = (2n-1)\pi/2b$, with $n = 1, 2, \ldots$. Thus, we have

$$Y_n(y) = \cos \frac{(2n-1)\pi y}{2b} \qquad n = 1, 2, \ldots$$

The solution for $X(x)$ is given by $c_3 \cosh \lambda x + c_4 \sinh \lambda x$. We'll set $c_4 = 0$ so that $X(x)$ is an even function of x, and so we have

$$X_n(x) = \cosh \frac{(2n-1)\pi x}{2b} \qquad n = 1, 2, \ldots$$

The complementary solution is a superposition of $X_n(x)Y_n(y)$.

$$u_c(x, y) = \sum_{n=1}^{\infty} c_n \cosh \frac{(2n-1)\pi x}{2b} \cos \frac{(2n-1)\pi y}{2b}$$

and the total solution is given by

$$u(x, y) = \frac{\gamma}{2\eta}(y^2 - b^2) + \sum_{n=1}^{\infty} c_n \cosh \frac{(2n-1)\pi x}{2b} \cos \frac{(2n-1)\pi y}{2b}$$

We can determine the c_n by requiring that $u(x, y) = 0$ at $x = \pm a$, or from

$$u(\pm a, y) = \frac{\gamma}{2\eta}(y^2 - b^2) + \sum_{n=1}^{\infty} c_n \cosh \frac{(2n-1)\pi a}{2b} \cos \frac{(2n-1)\pi y}{2b} = 0$$

so that

$$c_n \cosh \frac{(2n-1)\pi a}{2b} = \frac{1}{b}\int_{-b}^{b} \frac{\gamma}{2\eta}(b^2 - y^2) \cos \frac{(2n-1)\pi y}{2b}\, dy$$

$$= \frac{16\gamma b^2}{\eta \pi^3} \cdot \frac{(-1)^{n+1}}{(2n-1)^3}$$

The final solution is

$$u(x, y) = \frac{\gamma}{2\eta}(y^2 - b^2) + \frac{16\gamma b^2}{\eta\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} \frac{\cosh \frac{(2n-1)\pi x}{2b}}{\cosh \frac{(2n-1)\pi a}{2b}} \cos \frac{(2n-1)\pi y}{2b}$$

The velocity contours are shown in Figure 16.7.

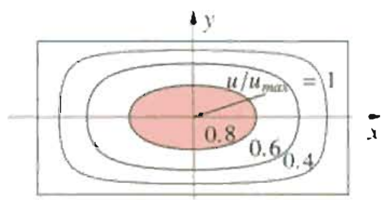

Figure 16.7
Velocity contours for the steady unidirectional flow through a rectangular conduit for $a = 2b$.

To keep things fairly simple, we chose three of the boundary conditions to be equal to zero in all the above examples. Problems 6 through 10 show how to handle the more general case where have

$$c_n I_0\left(\frac{n\pi a}{l}\right) = \frac{2}{l}\int_0^l T(a,z)\sin\frac{n\pi z}{l}dz$$

$$= \frac{2T_0}{l}\int_0^l z(l-z)\sin\frac{n\pi z}{l}dz$$

$$= \frac{4T_0 l^2}{n^3\pi^3}[1-(-1)^n] \qquad n = 1, 2, \ldots$$

The final solution is

$$T(r,z) = \frac{4T_0 l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\frac{I_0(n\pi r/a)}{I_0(n\pi l/a)}\sin\frac{n\pi z}{l}$$

This result is plotted in Figure 16.8.

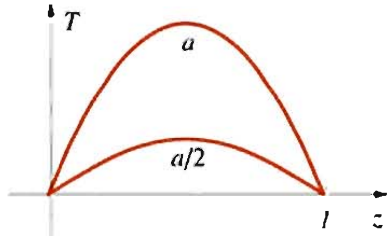

Figure 16.8
The steady-state temperature distribution $T(r,z)$ given in Example 3 for $r = a$ and $r = a/2$.

A standard application of Laplace's equation in spherical coordinates is the problem of a conducting sphere in a uniform electric field (Figure 16.9). The sphere is centered at the origin of a spherical coordinate system and we take the z axis to be in the direction of the external field. One of the boundary conditions is that the potential is constant over the surface of the sphere, so it is natural to use spherical coordinates. Laplace's equation in spherical coordinates is (Table 8.3)

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \quad (18)$$

Since we have taken the z axis of our spherical coordinate system to lie along the direction of the electric field in Figure 16.9, u will be independent of ϕ, and Equation 18 becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0 \qquad (19)$$

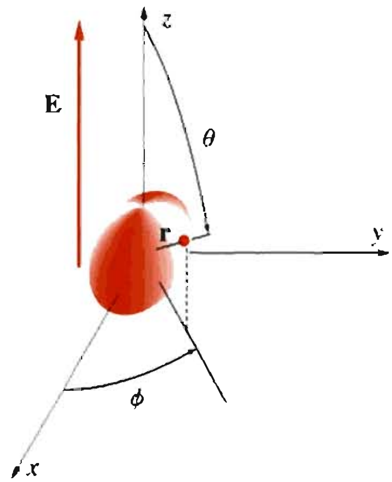

Figure 16.9
The geometry associated with a conducting sphere in a uniform electric field.

Two boundary conditions are

$$u(a,\theta) = 0 \qquad \text{and} \qquad u(r,\theta) \to -E_0 z = -E_0 r\cos\theta \quad \text{as} \quad r\to\infty \quad (20)$$

The first of these says that the surface of the conducting sphere is an equipotential surface (we take the potential to be zero) and the second simply gives the form of $u(r,\theta)$ far away from the disturbing influence of the sphere centered at the origin of the coordinate system. (Recall that $E = -\partial u/\partial z$.)

We seek a solution of the form $u(r,\theta) = R(r)\Theta(\theta)$. This substitution leads to

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - \lambda R(r) = 0 \qquad (21)$$

and

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + \lambda\sin\theta\,\Theta(\theta) = 0 \tag{22}$$

where λ is the separation constant. The equation in $R(r)$ is

$$r^2R''(r) + 2rR'(r) - \lambda R(r) = 0 \tag{23}$$

This is an Euler equation (Section 11.5). We substitute $R(r) = r^m$ and find that $m(m-1) + 2m - \lambda = 0$, so that

$$R(r) = c_1r^{m_1} + c_2r^{m_2} \tag{24}$$

where m_1 and m_2 are the roots of $m^2 + m - \lambda = 0$, where λ is yet to be determined.

The equation for $\Theta(\theta)$ may not look at all recognizable, but if we let $x = \cos\theta$ and $\Theta(\theta) = P(x)$, then Equation 22 becomes (Problem 20)

$$(1-x^2)P''(x) - 2xP'(x) + \lambda P(x) = 0 \tag{25}$$

This equation is Legendre's equation, which is the sole topic of Section 12.3. Legendre's equation often arises as the separated equation in θ in problems involving spherical coordinates. We learned in Section 12.3 that Equation 25 has a finite solution at $x = 1$ (at $\theta = 0$) only if $\lambda = n(n+1)$ where $n = 0, 1, 2, \ldots$ and that the solutions in this case are the Legendre polynomials. Thus, we write Equation 25 as

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \qquad n = 0, 1, 2, \ldots \tag{26}$$

Therefore, the solution to Equation 22 becomes

$$\Theta_n(\theta) = P_n(\cos\theta) \tag{27}$$

where

$$\Theta_0(\theta) = 1 \qquad \Theta_1(\theta) = \cos\theta \qquad \Theta_2(\theta) = \frac{1}{2}(3\cos^2\theta - 1) \qquad \cdots$$

Given that $\lambda = n(n+1)$, the two roots of the equation $m^2 + m - \lambda = 0$ are $m_1 = n$ and $m_2 = -(n+1)$, and so Equation 24 becomes

$$R_n(r) = c_1r^n + \frac{c_2}{r^{n+1}} \qquad n = 0, 1, 2, \ldots \tag{28}$$

The general solution to Equation 19 is a superposition of the $R_n(r)\Theta_n(\theta)$

$$u(r,\theta) = \sum_{n=0}^{\infty} a_nr^nP_n(\cos\theta) + \sum_{n=0}^{\infty} b_nr^{-(n+1)}P_n(\cos\theta) \tag{29}$$

13. Substitute a_n and b_n of Problem 11 into $u(r, \theta)$ and interchange the order of summation and integration to get *Poisson's integral formula for a disk*:

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{f(x)dx}{a^2 - 2ar\cos(x - \theta) + r^2} \tag{25}$$

14. Let $r = 0$ in the solution to the previous problem and show that

$$u(0, \theta) = \frac{1}{2\pi} \int_{\pi}^{\pi} u(a, \theta) d\theta \tag{26}$$

Interpret this result physically. This property of the solutions to Laplace's equation is called the *mean value property*.

15. Use Poisson's integral formula for a disk (Problem 13) to show that if the electrostatic potential on a circle is a constant, then the potential within the circle will be a constant.

16. Use Poisson's integral formula for a disk (Problem 13) to show that the maximum value of the electrostatic potential within a circular region must occur on the boundary. This result is known as the *maximum principle*. *Hint*: Show that if $f(\theta) \leq M$ (a constant), then $u(r, \theta) \leq M$.

17. Verify the general solution of $r^2 R''(r) + rR'(r) - \frac{n^2\pi^2}{l^2} r^2 R(r) = 0$ given in Example 3.

18. Determine the potential within a circular annulus of radii equal to 1 and 2 with $u(1, \theta) = \sin^2\theta$ and $u(2, \theta) = 0$ (Figure 16.11).

19. The curved surface of a right-circular cylinder of radius a (Figure 16.12) is maintained at zero temperature, and its two plane ends at $z = \pm l$ are fixed at $T(r, \theta, l) = T_0(1 - r^2/a^2)$ at $z = l$ and at zero at $z = 0$. Determine the steady-state temperature in the cylinder. *Hint*: You need the integral $\int_0^1 (1 - x^2) x J_0(\alpha_n x)\, dx = 2J_2(\alpha_n)/\alpha_n^2$.

20. Let $x = \cos\theta$ in Equation 22 to derive Legendre's equation, Equation 25.

21. Determine the electric potential inside a hollow sphere of radius a given that the potential on its surface is $u_0 \cos 2\theta$.

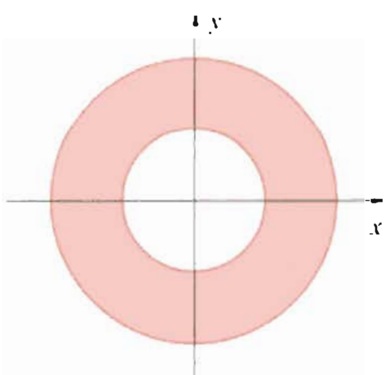

Figure 16.11
A circular annulus of inner radius 1 and outer radius 2. (See Problem 18.)

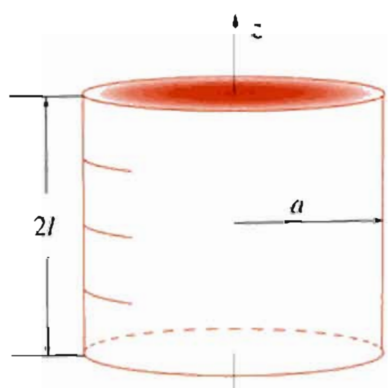

Figure 16.12
The right cylinder described in Problem 19.

SOLUTION: We can see this mathematically by considering only the first harmonic in Equation 6.

$$u_1(x,t) = A_1 \cos \omega_1 t \sin \frac{\pi x}{l}$$

where, for convenience, we have set $\phi = 0$. Using the trigonometric identity

$$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$$

and Equation 7 for ω_1, $u_1(x,t)$ becomes

$$\begin{aligned} u_1(x,t) &= \frac{A_1}{2}\sin\left(\frac{\pi x}{l} + \frac{\pi vt}{l}\right) + \frac{A_1}{2}\sin\left(\frac{\pi x}{l} - \frac{\pi vt}{l}\right) \\ &\frac{A_1}{2}\sin\left[\frac{\pi}{l}(x+vt)\right] + \frac{A_1}{2}\sin\left[\frac{\pi}{l}(x-vt)\right] \end{aligned}$$

The wavelength of the first harmonic is $\lambda = 2l$ (see Figure 16.14a), and so $u_1(x,t)$ can be written as

$$u_1(x,t) = \frac{A_1}{2}\sin\left[\frac{2\pi}{\lambda}(x+vt)\right] + \frac{A_1}{2}\sin\left[\frac{2\pi}{\lambda}(x-vt)\right]$$

Each of the terms in $u_1(x,t)$ represents a *traveling wave*. If we look at some position x and let t vary, the first term in $u_1(x,t)$ would appear to be a wave of wavelength λ and frequency $\nu = v/\lambda$, traveling to the left. The second term in $u_1(x,t)$ would be a similar wave traveling to the right. Thus, we see that a standing wave is the superposition of two similar traveling waves, traveling opposite directions.

It is instructive to consider a simple case in which $u(x,t)$ consists of only the first two harmonics and is of the form (Equation 6)

$$u(x,t) = \cos\omega_1 t \sin\frac{\pi x}{l} + \frac{1}{2}\cos\left(\omega_2 t + \frac{\pi}{2}\right)\sin\frac{2\pi x}{l} \tag{8}$$

Equation 8 is illustrated in Figure 16.15. The left side of Figure 16.15 shows the time dependence of each mode separately. Notice that $u_2(x,t)$ has gone through one complete oscillation in the time depicted while $u_1(x,t)$ has gone through only one-half cycle, nicely illustrating that $\omega_2 = 2\omega_1$. The right side of Figure 16.15 shows the sum of the two harmonics, or the actual motion of the string, as a function of time. It is interesting to see how a superposition of the standing waves in the left side of the figure yields the traveling wave in the right side.

We're not finished solving Equation 1. Equation 5 (or equivalently, Equation 6) still has two infinite sets of constants to be determined. Note that Equation 5 is of the form of a Fourier sine series, where the Fourier coefficients are $a_n \cos\omega_n t + b_n \sin\omega_n t$. We can evaluate all the a_n and b_n from the two initial conditions.

where $\omega_n = n\pi v/l$. The initial condition $u_t(x, 0) = 0$ implies that the $b_n = 0$. The a_n are given by the initial condition $u(x, 0) = u_0 \sin 2\pi x/l$,

$$a_n = \frac{2u_0}{l}\int_0^l \sin\frac{2\pi x}{l}\sin\frac{n\pi x}{l}dx = u_0\,\delta_{n2}$$

where δ_{n2} is a Kronecker delta. The complete solution is

$$u(x, t) = u_0 \sin\frac{2\pi x}{l}\cos\frac{2\pi vt}{l} \tag{13}$$

It's easy to see that this solution satisfies the wave equation along with the given boundary conditions and initial conditions (Problem 4).

Example 2:
Express Equation 13 as the sum of two traveling waves.

SOLUTION: We simply use the trigonometric relation

$$\sin\alpha\ \cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$$

to write

$$u(x, t) = \frac{1}{2}\sin\left[\frac{2\pi}{l}(x+vt)\right] + \frac{1}{2}\sin\left[\frac{2\pi}{l}(x-vt)\right]$$

The first term on the right is a sine wave traveling from right to left and the second term is a sine wave travelling from left to right.

Example 3:
Suppose a string is initially displaced a small distance b at the middle of the string and then released (in other words, plucked in the middle). Determine the subsequent motion of the string.

SOLUTION: The initial conditions translate into

$$u(x, 0) = \begin{cases} \dfrac{2bx}{l} & 0 \le x \le \dfrac{l}{2} \\ \dfrac{2b}{l}(l-x) & \dfrac{l}{2} \le x \le l \end{cases}$$

and $u_t(x, 0) = 0$. Equation 12 gives $b_n = 0$ and Equation 10 gives (Problem 5)

$$a_n = \frac{2}{l}\left[\int_0^{l/2}\frac{2bx}{l}\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}\frac{2b}{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

the string is $\phi(x)$ and that its initial velocity is $\psi(x)$, so that

$$u(x, 0) = \phi(x) \qquad \text{and} \qquad u_t(x, 0) = \psi(x) \tag{23}$$

Then,

$$\phi(x) = f(x) + g(x) \tag{24}$$

and

$$\psi(x) = -vf'(x) + vg'(x) \tag{25}$$

where the primes here mean differentiating with respect to the arguments of f and g. Integrating Equation 25 from 0 to x gives

$$f(x) - g(x) = -\frac{1}{v}\int_0^x \psi(u)du + c$$

where $c = f(0) - g(0)$. Adding and subtracting this result from Equation 24 gives

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2v}\int_0^x \psi(u)du + \frac{c}{2} \tag{26}$$

and

$$g(x) = \frac{1}{2}\phi(x) + \frac{1}{2v}\int_0^x \psi(u)du - \frac{c}{2} \tag{27}$$

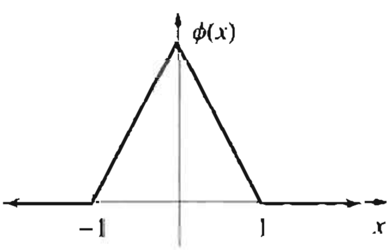

Figure 16.18
A plot of the initial disturbance $\phi(x) = 1 - |x|$ when $-1 \leq x \leq 1$ and $\phi(x) = 0$ otherwise.

Therefore, the solution to Equation 14 is (Problem 17)

$$u(x, t) = \frac{\phi(x - vt) + \phi(x + vt)}{2} + \frac{1}{2v}\int_{x-vt}^{x+vt} \psi(u)du \tag{28}$$

Equation 28 is known as *d'Alembert's solution* to the one-dimensional wave equation. It gives us the displacement $u(x, t)$ in terms of the given initial displacement $u(x, 0) = \phi(x)$ and the initial velocity $u_t(x, 0) = \psi(x)$.

Suppose that $\phi(x) = u(x, 0)$ is given by

$$\phi(x) = \begin{cases} 1 - |x| & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

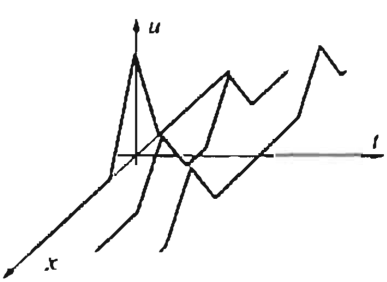

Figure 16.19
An initial triangular waveform breaking into two and moving in opposite directions.

and that $\psi(x) = u_t(x, 0) = 0$. Figure 16.18 shows that $\phi(x)$ is a triangular waveform centered at $x = 0$. Now, as time increases, the displacement is given by

$$u(x, t) = \frac{1}{2}\phi(x - vt) + \frac{1}{2}\phi(x + vt) \tag{30}$$

This result suggests that as time increases, the original triangular waveform splits into two, with one half moving to the right and the other half moving to the left, as in Figure 16.19.

Example 5:
Consider an infinitely long string that is released from rest with a displacement

$$\phi(x) = e^{-x^2}$$

Determine the subsequent motion of the string.

SOLUTION: Using Equation 28 with $\psi(x) = 0$,

$$u(x,t) = \frac{1}{2}[e^{-(x-vt)^2} + e^{-(x+vt)^2}]$$

The displacement of the string at $vt = 0$, 1.5, 2, and 8 is shown in Figure 16.20. Note that one half of the initial disturbance propagates to the right and the other half propagates to the left.

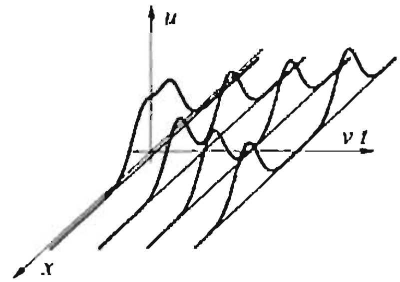

Figure 16.20
The displacement in Example 5 at $vt = 0.25$, 0.50, 0.75, and 1.0.

When d'Alembert discovered his general solution to the one-dimensional wave equation, he initially thought that he had discovered a method that is applicable to other partial differential equations, but it was not to be.

16.3 Problems

1. Verify Equation 4.
2. Show that $F \cos \omega t + G \sin \omega t$ can be written as $A \cos(\omega t + \phi)$.
3. Show that the number of nodes of $\sin n\pi x/l$ between 0 and l is $n - 1$.
4. Show that Equation 13 satisfies the wave equation and the boundary conditions $u(0,t) = u(l,t) = 0$ and the initial conditions $u(x,0) = u_0 \sin 2\pi x/l$ and $u_t(x,0) = 0$.
5. Verify that $b_n = 0$ and that $a_n = \dfrac{8b}{n^2\pi^2} \sin \dfrac{n\pi}{2}$ in Example 3.
6. Solve the one-dimensional wave equation subject to the conditions $u(0,t) = u(l,t) = 0$, $u(x,0) = \sin(3\pi x/l)$, and $u_t(x,0) = 0$.
7. If $u(x,0) = u_0 \sin^3 \pi x/l$ in the previous problem, can you predict which normal modes will be excited?
8. Solve the one-dimensional wave equation subject to the conditions $u(0,t) = u(l,t) = 0$, $u(x,0) = 0$, and $u_t(x,0) = u_{t0}$.
9. Solve the one-dimensional wave equation subject to the conditions $u(0,t) = u(l,t) = 0$, $u(x,0) = u_0 x(l-x)$, and $u_t(x,0) = u_{t0}$.
10. Show that $u(x,t) = f(x \pm vt)$, where $f(z)$ is a differentiable function, is a (general) solution to the one-dimensional wave equation.

11. If a vibrating string has a resistance to motion that is proportional to the velocity of the string, then the wave equation takes the form (see Problem 11 of Section 1) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t}$, where γ is a constant that is a measure of the resistance. Solve this equation subject to the conditions $u(0, t) = u(l, t) = 0$, $u(x, 0) = u_0 \sin \pi x / l$, and $u_t(x, 0) = 0$. Assume that $\gamma < 1/lv$.

12. Problem 15 of Section 1 shows that if the linear mass density of a string is not uniform, then the wave equation takes the form $\tau \frac{\partial^2 u}{\partial x^2} = \rho(x)\frac{\partial^2 u}{\partial t^2}$. Show that separation of variables leads to the Sturm-Liouville equation $\tau X''(x) + \lambda \rho(x) X(x) = 0$.

13. The kinetic energy and potential energy of a vibrating string are (see Problem 10 of Section 1) $K = \frac{\rho}{2}\int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx$ and $V = \frac{\tau}{2}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx$. Show that the total energy of the string in Problem 4 is a constant.

14. Using the equations in the previous problem, show that the total energy of each normal mode in Equation 6 is a constant.

15. Use a CAS to show that $u_1(x, t) = e^{-16(x-t)^2}$ and $u_2(x, t) = 1.5e^{-(x-t)^2}$ are two Gaussian waveforms traveling in different directions.

16. Show that Equation 22 is a solution to Equation 14.

17. Derive Equation 28 from Equations 26 and 27.

18. Use d'Alembert's formula to find the solution to the wave equation that satisfies the initial conditions $u(x, 0) = \sin \pi x / l$ and $u_t(x, 0) = 0$ for $-\infty < x < \infty$.

19. Repeat the previous problem with the initial conditions $u(x, 0) = \sin(\pi x / l)\cos(\pi x / l)$ and $u_t(x, 0) = \cos(\pi x / l)$.

20. Consider an infinitely long string that is released from rest with a displacement $\phi(x) = 1/(1 - 4x^2)$. Determine the subsequent motion of the string.

21. Show that if ∇^2 depends upon only the radial coordinate r in spherical coordinates, then $\nabla^2 u = k^2 u$ can be written as $\frac{d^2(ru)}{dr^2} = k^2(ru)$. Now show that if ∇^2 depends upon only r, then the wave equation becomes $\nabla^2 u = \frac{\partial^2(ru)}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2(ru)}{\partial t^2}$. Show that the general solution to this equation is $u(r, t) = \frac{1}{r} f(r + vt) + \frac{1}{r} g(r - vt)$. Interpret this result physically.

22. Consider the wave equation in the previous problem. First show that $ru(r, t) = rR(r)e^{\pm i\omega t}$ reduces the partial differential equation to an ordinary differential equation. Now show that the solution to that equation is of the form $rR(r) = e^{\pm i\omega r/v}$. Finally, show that $u(r, t)$ is of the general form given in the previous problem.

$$\frac{1}{R(r)}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)=\frac{1}{v^2T(t)}\frac{d^2T}{dt^2}=k \tag{23}$$

where k is a separation constant. Problem 13 has you show that k must be negative, so we'll write it as $k=-\lambda^2$, which assures that k is negative so long as λ is real. The solution to the equation for $T(t)$ is

$$T(t)=c_1\cos\lambda vt+c_2\sin\lambda vt \tag{24}$$

We could have anticipated that the separation constant k must be negative because we expect physically that $T(t)$ should be oscillatory.

The equation for $R(r)$ is

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\lambda^2R=0 \tag{25}$$

Do you recognize this equation? This is Bessel's equation of order zero. Referring to Equation 12.5.42, we find that the solution to Equation 25 is (Problem 14)

$$R(r)=a_1J_0(\lambda r)+a_2Y_0(\lambda r) \tag{26}$$

where $J_0(\lambda r)$ and $Y_0(\lambda r)$ are zero-order Bessel functions of the first and second kind, respectively, and where λ is yet to be determined.

A solution to Equation 20 is the product of Equations 24 and 26:

$$u_\lambda(r,t)=[a_1J_0(\lambda r)+a_2Y_0(\lambda r)](c_1\cos\lambda vt+c_2\sin\lambda vt) \tag{27}$$

Recall from Section 12.5 that $J_0(z)$ is finite for all values of z, but that $Y_0(z)$ diverges as $z\to 0$ (Problem 15). In order to assure that the displacement of the membrane remains finite as $r\to 0$, we must set $a_2=0$ in Equation 27. Therefore, we have

$$u_\lambda(r,t)=J_0(\lambda r)\,(b_1\cos\lambda vt+b_2\sin\lambda vt) \tag{28}$$

where $b_1=a_1c_1$ and $b_2=a_1c_2$.

We can now determine λ by applying the boundary condition, $u(a,t)=0$. This boundary condition implies that

$$J_0(\lambda a)=0 \tag{29}$$

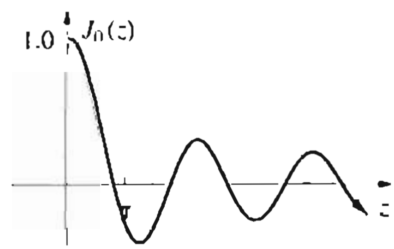

Figure 16.23
The zeroth order Bessel function of the first kind, $J_0(z)$, plotted against z.

Recall that $J_0(z)$ is an oscillatory function (see Figure 16.23) and has an infinite number of discrete zeros (that is, values of z where $J_0(z)=0$). Let these zeros be denoted by $\alpha_1, \alpha_2, \alpha_3, \ldots$. Their numerical values are $\alpha_1=2.4048$, $\alpha_2=5.5201$, $\alpha_3=8.6537$, $\alpha_4=11.7913$, and so on. Equation 29 then says that

$$\lambda_n=\frac{\alpha_n}{a} \qquad n=1,2,3,\ldots \tag{30}$$

8. Express $f(x, y) = x^2y^2$ for $-a \le x < a$ and $-b \le y < b$ as a double Fourier cosine series.

9. Express $f(x, y) = \cos\dfrac{2\pi x}{a}\cos^3\dfrac{2\pi y}{b}$ for $-a \le x < a$ and $-b \le y < b$ as a double Fourier cosine series.

10. Solve the two-dimensional wave equation for the case where a rectangular membrane is given the initial shape $u(x, y) = u_0 \sin\dfrac{2\pi x}{a}\sin\dfrac{3\pi y}{b}$ and is released from rest.

11. Solve Equation 1 with the conditions $u(0, y, t) = u(a, y, t) = 0$; $u_y(x, 0, t) = u_y(x, b, t) = 0$; $u(x, y, 0) = f(x, y)$; and $u_t(x, y, 0) = 0$.

12. Find the solution to the previous problem if $f(x, y) = (\sin^3 \pi x/a)(\cos^2 2\pi y/b)$.

13. Why must k be negative in Equation 23?

14. Use Equation 12.5.42 to verify Equation 26.

15. Use the definition of $Y_0(z)$ in Section 12.5 to show that $Y_0(z) \to -\infty$ as $z \to 0$.

16. Use any CAS to determine numerically the values of the first few zeros of $J_0(x)$.

17. Show that the $n = 3$ mode associated with Equation 32 has a nodal circle at $r = 0.2779\,a$ and one at $r = 0.6379\,a$ and that it vibrates with a frequency $3.598\,\omega_1$.

18. Show that the initial condition $u_t(a, 0) = 0$ requires that all the b_{2n} in Equation 32 must vanish.

19. Discuss how the solution to the vibrations of an annulus would differ from that of a circular membrane.

16.5 The Heat Equation

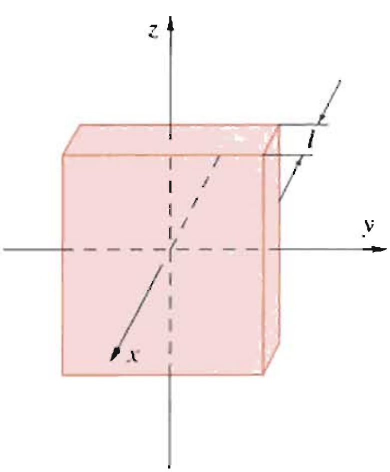

Figure 16.27
A slab of uniform material of thickness l.

Consider a slab of uniform material of thickness l in the x direction as shown in Figure 16.27. Let the y and z dimensions of the slab be much greater than l, so that we can ignore any edge effects at the y and z boundaries. The temperature distribution in the slab will vary only in the x direction, so the governing equation is the one-dimensional heat equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t} \qquad \begin{array}{c} 0 \le x \le l \\ 0 < t \end{array} \tag{1}$$

To find a unique solution, we must have two boundary conditions and one initial condition (unlike the wave equation, which requires two initial conditions). Let's assume that the two faces of the slab are maintained at $T = 0$ and that the initial temperature profile in the slab is given by $f(x)$. In terms of equations, we have

$$T(0, t) = 0 \qquad T(l, t) = 0 \qquad T(x, 0) = f(x) \tag{2}$$

Equations 1 and 2 could also describe a thin homogeneous rod or wire of length l that is insulated along its lateral surface, as shown in Figure 16.28, so that its temperature varies only in the x direction. Figure 16.29 summarizes Equations 1 and 2.

To solve Equation 1 by separation of variables, substitute $T(x, t) = X(x)\Theta(t)$ into Equation 1 and divide by $T(x, t) = X(x)\Theta(t)$ to get

$$\frac{X''(x)}{X(x)} = \frac{\Theta'(t)}{\alpha^2\Theta(t)} = -\lambda^2 \tag{3}$$

or

$$X''(x) + \lambda^2 X(x) = 0 \tag{4}$$

and

$$\Theta'(t) + \alpha^2\lambda^2\Theta(t) = 0 \tag{5}$$

We wrote the separation constant as $-\lambda^2$ because the boundary conditions require that it be negative (Problem 1). The two initial conditions are equivalent to $X(0) = X(l) = 0$. The solution to Equation 4 with these boundary conditions is

$$X(x) = \sin\frac{n\pi x}{l} \qquad n = 1, 2, \ldots \tag{6}$$

The solution to Equation 5 gives

$$\Theta(t) = e^{-\alpha^2\lambda^2 t} = e^{-\alpha^2 n^2\pi^2 t/l^2} \tag{7}$$

and so

$$T_n(x, t) = c_n \sin\frac{n\pi x}{l} e^{-\alpha^2 n^2\pi^2 t/l^2} \qquad n = 1, 2, \ldots \tag{8}$$

Equation 8 with $n = 1, 2, \ldots$ satisfies Equation 1 and the boundary conditions. The general solution is a superposition of the solutions in Equation 8:

$$T(x, t) = \sum_{n=1}^{\infty} T_n(x, t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} e^{-\alpha^2 n^2\pi^2 t/l^2} \tag{9}$$

We have yet to impose the initial condition. If we set $t = 0$ in Equation 9, then we obtain

$$T(x, 0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \tag{10}$$

Equation 10 is a Fourier sine series and the c_n are given by

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \tag{11}$$

Figure 16.28
A thin homogeneous wire of length l that is insulated along its lateral surface so that the temperature varies only in the x direction.

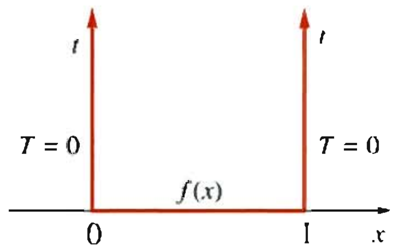

Figure 16.29
A summary of the boundary conditions and initial conditions for Equations 1 and 2.

Example 1:
Solve Equation 1 subject to the boundary conditions $T(0, t) = T(l, t) = 0$ and the initial condition $T(x, 0) = T_0 =$ constant for $0 < x < l$.

SOLUTION: The solution is given by Equation 9 with

$$\begin{aligned} c_n &= \frac{2T_0}{l}\int_0^l \sin\frac{n\pi x}{l}\,dx = \frac{2u_0}{n\pi}[1-(-1)^n] \qquad n = 1, 2, \ldots \\ &= \frac{4T_0}{n\pi} \qquad n = 1, 3, 5, \ldots \end{aligned}$$

and zero otherwise. Thus,

$$T(x, t) = \frac{4T_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin\dfrac{(2n-1)\pi x}{l}}{2n-1}e^{-\alpha^2(2n-1)^2\pi^2 t/l^2}$$

Figure 16.30 shows $T(x, t)$ plotted against x for several values of t (actually $\alpha^2 t/l^2$).

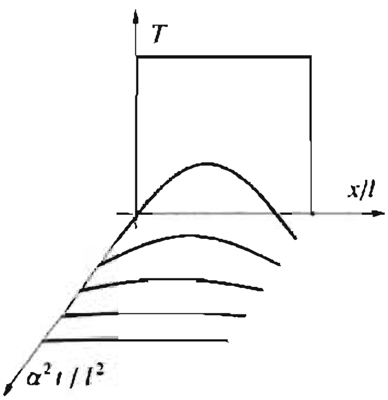

Figure 16.30
The solution to the equations in Example 1 plotted against x for several values of $\alpha^2 t/l^2$.

Figure 16.30 illustrates a difference between the wave equation and the heat equation (or the diffusion equation). Notice that the box-like initial temperature profile smooths out as time increases. This is typical of solutions to the heat equation. Compare this to Figure 16.19, which shows an initial triangular waveform splitting into two triangular waveforms moving in opposite directions. The underlying reason for this behavior is that the wave equation is time-reversible, in the sense that changing t to $-t$ leaves the equation unaltered. If you run time backward, the solutions simply retrace themselves, as you can see by changing t to $-t$ in the wave equation. Newton's equations of motion are also time-reversible, and thus we expect mechanical systems in general to be time-reversible. This is not so for the heat equation; it is not time-reversible. Unlike mechanical systems, the heat equation admits a certain directionality in time, which leads to the smoothing process illustrated in Figure 16.30. A molecular (statistical mechanical) derivation of the heat equation would show that it rests upon several statistical assumptions, which lead to its unidirectionality.

You may have wondered how the initial temperature throughout the material could be T_0 and yet the boundary conditions say that $T(0, t) = T(l, t) = 0$. In practice there really isn't a sharp discontinuity at the boundary; the temperature might fall from T_0 to 0 over a very short distance. What we have formulated in Example 1 (and most other problems that we consider) is an idealization of the actual situation. Unless we are *specifically* interested in a small boundary layer at the edges of the material, the discrepancy is of no physical importance. Part of the challenge in representing a physical system by a mathematical formalism is to model the system such that the important physical features are faithfully represented and yet keep the mathematical description as simple as possible.

So far, the only boundary conditions that we have considered are $T(0, t) = T(l, t) = 0$. In other words, the temperature is prescribed on the boundary. This type of boundary condition, prescribing a value of T (not necessarily zero) is called a *Dirichlet boundary condition* or a *boundary condition of the first kind*. Another common boundary condition is to prescribe the flux of energy as heat across a boundary. Mathematically, this amounts to prescribing the normal gradient of T at the boundary. For example, if the surface is insulated, then the flux across the surface is zero. This type of boundary condition is called a *Neumann boundary condition* or a *boundary condition of the second kind*. There is also a *boundary condition of the third kind*. Let the surface be in contact with the surrounding medium, which we model as an infinite heat bath at temperature T_0. According to Newton's law of cooling, the flux of energy as heat across the surface is given by

$$\text{flux} = a[T(0, t) - T_0]$$

The constant a is called the *coeffcient of surface heat transfer*. This flux must be equal to the flux from the body into the surface of the material due to conduction, or

$$\text{flux} = -\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0}$$

where λ is the thermal conductivity. Equating these two fluxes gives the boundary condition of the third kind:

$$\lambda \frac{\partial T}{\partial x} + a(T - T_0) = 0 \qquad \text{(at a boundary)} \tag{12}$$

Equation 12 is also called a *Robin boundary condition* or a *radiation boundary condition*. In practice, it is often difficult to prescribe a fixed surface temperature (as in a Dirichlet boundary condition) and a radiation boundary condition may be more appropriate. Usually, however, a Dirichlet boundary condition is simpler to implement.

We'll now solve the one-dimensional heat equation for boundary conditions of the second and third kind.

Example 2:

A thin wire of length l is totally insulated and its initial temperature profile is given by $T(x, 0) = T_0 \sin^2 \pi x/l$. Determine the subsequent temperature profiles. (Figure 16.31 summarizes these conditions.)

SOLUTION: The mathematical formulation of this problem is to solve

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial T}{\partial t}$$

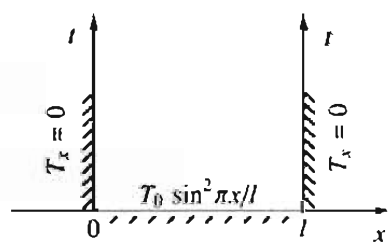

Figure 16.31
A summary of the boundary conditions and the initial condition for Example 2.

with the boundary conditions $T_x(0, t) = T_x(l, t) = 0$ and the initial condition $T(x, 0) = T_0 \sin^2 \pi/l$. Separation of variables gives the two equations

$$X''(x) + \lambda^2 X(x) = 0$$

and

$$\Theta'(t) + \lambda^2 \alpha^2 \Theta(t) = 0$$

with $X'(0) = X'(l) = 0$. The separation constant has been taken to $-\lambda^2$ because it is not possible to satisfy the boundary conditions if it is positive.

We now determine the allowed values of the separation constant λ. Note that $\lambda = 0$ yields $X(x) = ax + b$. The boundary conditions require that $a = 0$, but b is arbitrary. So far, then, we have $X_0(x) = b$ and $\lambda = 0$. For $\lambda \neq 0$, we have $X_n(x) = \cos n\pi x/l$ where $n = 1, 2, \ldots$, and so we have

$$X_n(x) = \begin{cases} 1 & n = 0 \\ \cos \dfrac{n\pi x}{l} & n = 1, 2, \ldots \end{cases}$$

where we set the value of b equal to 1 arbitrarily. The equation for $\Theta_n(t)$ gives

$$\Theta_n(t) = \begin{cases} \text{constant} & n = 0 \\ e^{-\alpha^2\pi^2 n^2 t/l^2} & n = 1, 2, \ldots \end{cases}$$

The superposition of $X_n(x)\Theta_n(t)$ gives

$$T(x, t) = \text{constant} + \sum_{n=1}^{\infty} a_n e^{-\alpha^2\pi^2 n^2 t/l^2} \cos \frac{n\pi x}{l}$$

This result would be of the form of a Fourier cosine series if we let constant $= a_0/2$, and so we write

$$T(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\alpha^2\pi^2 n^2 t/l^2} \cos \frac{n\pi x}{l}$$

We now use the initial condition to write

$$T(x, 0) = T_0 \sin^2 \frac{\pi x}{l} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

and so the a_n are given by

$$\begin{aligned} a_n &= \frac{2T_0}{l} \int_0^l \sin^2 \frac{\pi x}{l} \cos \frac{n\pi x}{l} dx \\ &= \frac{2T_0}{l} \int_0^l \left(\frac{1 - \cos \dfrac{2\pi x}{l}}{2} \right) \cos \frac{n\pi x}{l} dx \end{aligned}$$

regular singular points at $x = \pm 1$ (Problem 5), so we expect that there is at least one Frobenius series solution about $x = \pm 1$. We won't go through that process here, but it turns out that β must equal $l(l+1)$ with $l = 0, 1, 2, \ldots$ (as in the case of Legendre's equation) for Equation 15 to have solutions that are finite at $x = \pm 1$ ($\theta = 0$ and π). Furthermore, it turns out that m must be $\leq l$. Thus, we see that a rigid rotator is restricted to have only energies that satisfy $\beta = l(l+1)$, where $l = 0, 1, 2, \ldots$. Using Equation 8, we see that the energy of a rigid rotator is given by

$$E = \frac{\hbar^2}{2I} l(l+1) \qquad l = 0, 1, 2, \ldots \tag{16}$$

These allowed energies account for the experimentally observed microwave spectra of diatomic molecules.

The solutions to Equation 15 that are well behaved at $x = \pm 1$ are called *associated Legendre functions*, which are customarily denoted by $P_l^{|m|}(x)$. The superscript is written as $|m|$ because m appears only as m^2 in Equation 15. The associated Legendre functions can be defined in terms of Legendre polynomials by the relation

$$P_l^{|m|}(x) = (1 - x^2)^{|m|/2} \frac{d^m}{dx^m} P_l(x) \tag{17}$$

Note that $P_l^{|m|}(x) = 0$ if $m > l$ because $P_l(x)$ is an lth degree polynomial. Note also that the $P_l^{|m|}(x)$ are polynomials only if m is even. The first few associated Legendre functions are given in Table 16.1.

Just as the Legendre polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} P_l(x) P_n(x) dx = \int_0^{\pi} d\theta \sin\theta \, P_l(\cos\theta) P_n(\cos\theta) = \frac{\delta_{ln}}{2l+1}$$

Table 16.1
The first few associated Legendre functions, $P_l^{|m|}(x)$.

$P_0^0(x) = 1$
$P_1^0(x) = x = \cos\theta$
$P_1^1(x) = (1 - x^2)^{1/2} = \sin\theta$
$P_2^0 = \frac{1}{2}(3x^2 - 1) = \frac{1}{2}(3\cos^2\theta - 1)$
$P_2^1(x) = 3x(1 - x^2)^{1/2} = 3\cos\theta\sin\theta$
$P_2^2(x) = 3(1 - x^2) = 3\sin^2\theta$

Remember now that we're in the process of solving the Schrödinger equation for a rigid rotator, Equation 7. Putting everything together, the rigid-rotator wave functions are $P_l^{|m|}(\cos\theta)\,\Theta_m(\phi)$. By referring to Equations 14 and 19, we see that the functions

$$Y_l^m(\theta,\phi) = \left[\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2} P_l^{|m|}(\cos\theta)e^{im\phi} \tag{20}$$

are normalized solutions to Equation 7.

The $Y_l^m(\theta,\phi)$ form an orthonormal set.

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; Y_l^m(\theta,\phi)^* Y_n^k(\theta,\phi) = \delta_{nl}\delta_{mk} \tag{21}$$

Note that the $Y_l^m(\theta,\phi)$ are orthonormal with respect to $\sin\theta\, d\theta d\phi$ and not just $d\theta d\phi$. The factor $\sin\theta\, d\theta d\phi$ has a simple physical interpretation. The differential volume element in spherical coordinates is $r^2 \sin\theta\, drd\theta d\phi$. If r is a constant, as it is in the case of a rigid rotator, and set equal to unity for convenience, then the spherical coordinate volume element becomes a spherical surface element, $dA = \sin\, d\theta d\phi$. If this surface element is integrated over θ and ϕ, we obtain 4π, the surface area of a sphere of unit radius. According to Equation 21, the $Y_l^m(\theta,\phi)$ are orthonormal over a spherical surface and so are called *spherical harmonics*. Spherical harmonics occur in a great variety of physical problems involving spherical coordinates. The first few spherical harmonics are given in Table 16.2.

Example 3:
Show that $Y_1^1(\theta,\phi)$ is normalized and orthogonal to $Y_1^0(\theta,\phi)$.

SOLUTION:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta Y_1^1(\theta,\phi)^* Y_1^1(\theta,\phi) = \frac{3}{8\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3\theta$$

$$= \frac{3}{4}\int_{-1}^{1} dx(1-x^2) = 1$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta Y_1^1(\theta,\phi)^* Y_1^0(\theta,\phi) =$$

$$\frac{3}{4\pi\sqrt{2}}\int_0^{2\pi} d\phi\, e^{i\phi} \int_0^{\pi} d\theta \sin^2\theta\cos\theta = 0$$

In summary, the Schrödinger equation for a rigid rotator is

$$\mathcal{H}Y_l^m(\theta,\phi) = \frac{\hbar^2 l(l+1)}{2I}Y_l^m(\theta,\phi) \tag{22}$$

with $\mathcal{H}$ given by

$$\mathcal{H} = -\frac{\hbar^2}{2I}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

C. THE ELECTRON IN A HYDROGEN ATOM

The spherical harmonics occur whenever the Schrödinger equation is solved for a potential of the form $V = V(r)$. It's straightforward to show that if $V = V(r)$, then (Problem 13)

$$\psi(r,\theta,\phi) = R(r)Y_l^m(\theta,\phi) \tag{23}$$

where $R(r)$ is the solution to the ordinary differential equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{\hbar^2 l(l+1)}{2mr^2} + V(r) - E\right]R(r) = 0 \tag{24}$$

The final problem that we shall discuss in this section is the Schrödinger equation for a hydrogen atom, one of the great triumphs of quantum mechanics. As our model, we shall picture the hydrogen atom as a proton fixed at the origin and an electron of reduced mass μ interacting with the proton through a coulombic potential,

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} \tag{25}$$

In Equation 25, e is the charge on the proton, ϵ_0 is the permittivity of free space, and r is the distance between the electron and the proton. The model suggests that we use a spherical coordinate system with the proton at the origin. Therefore, the Schrödinger equation for a hydrogen atom can be written as

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r,\theta,\phi) - \frac{e^2}{4\pi\varepsilon_0 r}\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

or

$$\begin{aligned}-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]& \\ -\frac{e^2}{4\pi\varepsilon_0 r}\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)&\end{aligned} \tag{26}$$

Note that this equation is of the form $\mathcal{H}\psi = E\psi$, regardless of its apparent

The orthogonality condition is given by

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr\, r^2 \left(\frac{\rho(2-\rho)e^{-\rho}\cos\theta}{32\pi a_0^3} \right)$$

$$= \frac{2\pi}{32\pi} \int_0^{\pi} d\theta \sin\theta \cos\theta \int_0^{\infty} d\rho\, \rho^3(2-\rho)e^{-\rho} = 0$$

because of the integral over θ.

16.6 Problems

1. Derive Equation 3.
2. Show that the normalization constant of $\psi^*(x,y,z)\psi(x,y,z)$ in Equation 3 is $A_xA_yA_z = (8/abc)^{1/2}$.
3. Determine the allowed energies of a particle in a two-dimensional rectangular box of sides a and b.
4. Show that the allowed energies of a particle constrained to the circular potential-free region are given by $E_{nm} = \dfrac{\hbar^2}{2ma^2}\beta_{nm}^2$, where β_{nm} is the mth zero of the first-order Bessel function $J_n(x)$.
5. Show that Equation 15 with $m \neq 0$ has regular singular points at $x = \pm 1$.
6. Show that the first few associated Legendre functions in Table 16.1 satisfy Equation 18.
7. Show that the first few associated Legendre functions in Table 16.1 satisfy the recursion formula
$$(2l+1)xP_l^{|m|}(x) - (l-|m|+1)P_{l+1}^{|m|}(x) - (l+|m|)P_{l-1}^{|m|}(x) = 0.$$
8. Use a CAS to show that the first few Legendre functions satisfy the recursion formula
$$P_n^{|m+2|}(x) - \frac{2(m+1)x}{(1-x^2)^{1/2}}P_n^{|m+1|}(x) + (n-m)(n+m+1)P_n^{|m|}(x) = 0.$$
Be sure that your CAS defines $P_n^{|m|}(x)$ exactly as we have done. A few authors include a factor of $(-1)^m$.
9. Show that the spherical harmonics satisfy the equation
$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^m}{\partial\theta}\right) + \frac{\partial^2 Y_l^m}{\partial\phi^2} + l(l+1)\sin^2\theta\, Y_l^m = 0.$$
10. Show explicitly that the first few spherical harmonics in Table 16.2 satisfy the equation in Problem 9.
11. Show that the first few spherical harmonics in Table 16.2 are orthogonal to each other.
12. Using Table 16.2, show that $|Y_1^1(\theta,\phi)|^2 + |Y_1^0(\theta,\phi)|^2 + |Y_1^{-1}(\theta,\phi)|^2 = 3/4\pi$. This is a special case of a general theorem, $\sum_{m=-l}^{l} |Y_l^m(\theta,\phi)|^2 = \text{constant}$, known as *Unsöld's theorem*.
13. Show that if $V = V(r)$ in the Schrödinger equation, then $\psi(r,\theta,\phi) = R(r)Y_l^m(\theta,\phi)$, where $R(r)$ satisfies Equation 24.

14. The equation $\nabla^2 u + k^2 u = 0$ is called the *Helmholtz equation*. Show that $u(r) = f(r)Y_l^m(\theta, \phi)$ is a solution to the Helmholtz equation in spherical coordinates.

15. The *Laguerre polynomials* can be defined by the Rodriguez formula $L_n(x) = e^x \dfrac{d^n}{dx^n}(e^{-x}x^n)$. Use this formula to generate the first few Laguerre polynomials.

16. The associated Laguerre polynomials can be defined in terms of the Laguerre polynomials (see the previous problem) by $L_n^\alpha(x) = \dfrac{d^\alpha}{dx^\alpha} L_n(x)$. Use this formula to generate the first few entries in Table 16.3.

17. Show explicitly that $\psi_{210}(r, \theta, \phi)$ satisfies Equation 27 with E given by Equation 28.

18. In this problem, we shall determine the allowed energies of a particle constrained to lie within a potential-free sphere. Using Equation 24 as a start, show that the radial part of the Schrödinger equation for this system is $R''(r) + \dfrac{2}{r}R'(r) + \left[\dfrac{2mE}{\hbar^2} - \dfrac{l(l+1)}{r^2}\right]R(r) = 0$ with the boundary condition $R(a) = 0$. Show that the solution to this equation is $R_{l\beta}(r) = c_1 r^{-1/2} J_{l+\frac{1}{2}}(\beta r) + c_2 r^{-1/2} Y_{l+\frac{1}{2}}(\beta r)$ where $\beta = (2mE)^{1/2}/\hbar$. Why must $c_2 = 0$? Now show that βa must be a zero of $r^{-1/2} J_{l+\frac{1}{2}}(r)$. If β_{lm} is the mth zero of $r^{-1/2} J_{l+\frac{1}{2}}(r)$, show that the allowed energies are given by $E_{ln} = \beta_{ln}^2 \dfrac{\hbar^2}{2ma^2}$.

16.7 The Classification of Partial Differential Equations

In Section 2, we learned that the wave equation has a general solution

$$u(x, t) = f(x + vt) + g(x - vt) \tag{1}$$

where f and g are suitably well-behaved functions. You might have wondered if the other partial differential equations that we have discussed have similar general solutions. Recall that we derived Equation 1 by transforming the wave equation into the form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{2}$$

under the linear transformation

$$\xi = x + vt \qquad \text{and} \qquad \eta = x - vt \tag{3}$$

Consider the second-order equation in two independent variables,

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = 0 \tag{4}$$

where a, b, and c are constants (for simplicity, that's all we'll discuss in this section), and the linear transformation

$$\xi = x + \alpha t \qquad \text{and} \qquad \eta = x + \beta t \tag{5}$$

$a = c = 1$ and $b = 0$, so

$$b^2 - 4ac = -4 < 0$$

Thus, Laplace's equation is an elliptic partial differential equation.

We see from this Example that $\alpha = -v$ and $\beta = v$ for the wave equation, so that the linear transformation in Equation 3 is $\xi = x - vt$ and $\eta = x + vt$, as we found in Section 2.

We didn't include the heat equation in Example 1 because it is not of the form of Equation 4. A more general second-order partial differential equation is

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + hu = 0 \tag{10}$$

The above classification scheme in Equation 9 applies to Equation 10 also; the classification scheme depends only upon the coefficients of the second derivatives in Equation 10. Therefore, if we consider the heat equation,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$$

we see that $a = 1$ and $b = c = 0$, so that $b^2 = 4ac = 0$ and the heat equation is parabolic.

Example 2:
Classify the Helmholtz equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

SOLUTION: The classification scheme depends only upon the coefficients of the second derivatives, so the Helmholtz equation is elliptic, like Laplace's equation.

We can derive a general solution for Laplace's equation, much like we did for the wave equation. According to Example 1, $a = c = 1$ and $b = 0$ in Equations 7 and 8, so $\alpha = i$ and $\beta = -i$. If we substitute these values into Equation 6, we obtain

$$4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{11}$$

where

$$\xi = x + iy \qquad \text{and} \qquad \eta = x - iy \tag{12}$$

Integration of Equation 11 gives (Problem 11)

$$u(\xi, \eta) = f(\xi) + g(\eta)$$

where f and g are twice-differentiable functions. Using Equations 12, we have the following as the general solution to Laplace's equation:

$$u(x, y) = f(x + iy) + g(x - iy) \tag{13}$$

Example 3:
In Example 1 of Section 2, we found that a certain solution to Laplace's equation is

$$u_n(x, y) = \cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

where $n = 1, 2, \ldots$. Show that this solution is of the form of Equation 13.

SOLUTION: First use the relation

$$\sinh z = -i\sin iz$$

to write

$$\sinh\frac{n\pi y}{a} = -i\sin\frac{in\pi y}{a}$$

Now use the relation

$$\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta)$$

to write

$$\begin{aligned} u_n(x, y) &= -\frac{i}{2}\left[\sin\frac{n\pi}{a}(x+iy) - \sin\frac{n\pi}{a}(x-iy)\right] \\ &= f(x+iy) + g(x-iy) \end{aligned}$$

Usually the general solutions to partial differential equations are not that useful due to the difficulty of determining the unknown functions in terms of the boundary conditions. It does turn out, however, that the different types of partial differential equations require different types of boundary conditions in order to be well posed. By a well-posed equation, we mean an equation whose solution that obeys the boundary conditions is unique and *stable*. By stable, we mean that the solution

5. Set $b' = 0$ in Problem 4 to eliminate the cross term in $a'x'^2 + b'x'y' + c'y'^2 + d'x' + e'y' + f = 0$ and argue that the resulting equation is that of an ellipse if $a'c' > 0$, that of a hyperbola if $a'c' < 0$, and that of a parabola if $a'c' = 0$.

6. Show that $ax^2 + bxy + cy^2 = d$ is (a) the equation of a hyperbola if $b^2 - 4ac > 0$; (b) the equation of an ellipse if $b^2 - 4ac < 0$; and (c) the equation of a parabola if $b^2 = 4ac$.

7. Determine whether the following partial differential equations are hyperbolic, elliptic, or parabolic:

   (a) $u_{xx} - u_{yy} = 0$ (b) $u_{xx} - 2u_{xy} + 2u_{yy} = 0$

   (c) $u_{xx} - 3u_{xy} = 0$ (d) $u_{xy} - u_{yy} = 0$

8. Determine whether the following partial differential equations are hyperbolic, elliptic, or parabolic:

   (a) $u_{xx} - 2u_{xy} + u_x + u = 0$ (b) $u_{xy} - u_y - u_x + 3u = 0$

   (c) $u_{xx} + u_{yy} - u_y + 4u = 0$ (d) $2u_{xy} - u_{xx} - u_{yy} + u_y - u_x = 0$

9. Show that if ∇^2 depends upon only the radial coordinate r in spherical coordinates, then $\nabla^2 u = k^2 u$ can be written as $\dfrac{d^2(ru)}{dr^2} = k^2(ru)$. Now show that if ∇^2 depends upon only r, then the wave equation becomes $\nabla^2 u = \dfrac{\partial^2(ru)}{\partial r^2} = \dfrac{1}{v^2}\dfrac{\partial^2(ru)}{\partial t^2}$. Show that the general solution to this equation is $u(r,t) = \dfrac{1}{r}f(r+vt) + \dfrac{1}{r}g(r-vt)$. Interpret this result physically.

10. Consider the wave equation in the previous problem. First show that $ru(r,t) = rR(r)e^{\pm i\omega t}$ reduces the partial differential equation to an ordinary differential equation. Now show that the solution to that equation is of the form $rR(r) = e^{\pm i\omega r/v}$. Finally, show that $u(r,t)$ is of the general form given in the previous problem.

11. Show that the solution to Equation 11 is $u(\xi,\eta) = f(\xi) + g(\eta)$ where ξ and η are given in Equation 12.

12. Equation 15 of Section 2 shows that $u_n(x,y) = \sin\dfrac{n\pi x}{a}\sinh\dfrac{n\pi y}{a}$ is a solution to Laplace's equation. Show that this solution is of the form of Equation 13.

13. Example 2 of Section 2 shows that $u_n(x,y) = \cosh\dfrac{(2n-1)\pi x}{a}\cos\dfrac{(2n-1)\pi y}{a}$ is a solution to Laplace's equation. Show that this solution is of the form of Equation 13.

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CHAPTER 17

Integral Transforms

We spent the entire previous chapter learning how to solve partial differential equations. We learned that three of the most commonly used partial differential equations (the heat equation, the wave equation, and Laplace's equation) are linear partial differential equations with constant coefficients, and can be readily solved by the method of separation of variables. There is another method for solving these equations, which is often more convenient, using integral transforms, which is the subject of this chapter. An integral transform is a relation of the form

$$\hat{F}(p) = \int_a^b K(p, x) f(x) dx$$

Given a function $K(p, x)$, called the kernel, this relation transforms $f(x)$ into $\hat{F}(p)$. There are a number of kernels that find use in applied mathematics, but the two most commonly used ones are a Laplace transform,

$$\hat{F}(s) = \int_0^\infty e^{-sx} f(x) dx$$

and a Fourier transform,

$$\hat{F}(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^\infty e^{-ikx} f(x) dx$$

We'll learn how to use these transforms to solve partial differential equations, and we'll see that the type of boundary conditions determines which type of transform to use.

A characteristic property of these integral transforms is that they can be used to reduce the number of independent variables in a partial differential equation by one. Thus, the one-dimensional heat equation or wave equation can be transformed into an ordinary differential equation in the transform function $\hat{F}(p)$. If you apply the transform method to an ordinary differential equation (only one independent variable), then you get just an algebraic equation for the transform function. In either case, it's usually much easier to solve the resultant equation for the transform function than it is to solve the original equation. The final step in the method is to undo the transform and deduce the function $f(x)$ from $\hat{F}(p)$. This process is called inversion.

We'll learn some of the properties of Laplace transforms in Section 1, and then in Section 2, we'll learn a number of methods to invert them. Then in the next two sections, we'll learn how to use Laplace tranforms to solve ordinary differential equations and partial differential equations. We'll learn about Fourier transforms in Section 5 and then use them to solve partial differential equations in Section 6.

17.1 The Laplace Transform

The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = \hat{F}(s) = \int_0^\infty e^{-st} f(t)dt \tag{1}$$

where s is a parameter that may be complex, although we'll consider it to be real in most of this chapter. For the integral in Equation 1 to converge, s must be greater than zero if s is real, or the real part of s must be greater than zero if s is complex. Note that the Laplace transform converts a function of t to a function of s. For example,

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st} dt = \frac{2}{s^2}$$

and

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a} \tag{2}$$

The evaluation of $\mathcal{L}\{f(t)\}$ consists of no more than performing the integration in Equation 1.

Example 1:
Determine $\mathcal{L}\{t^n\}$, where n is an integer.

SOLUTION:

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}} \tag{3}$$

where we have used the fact that $\int_0^\infty x^n e^{-x} dx = n!$.

Example 2:
Determine the Laplace transform of

$$f(t) = \begin{cases} 2 & 0 \le t < 1 \\ t & 1 \le t \end{cases}$$

SOLUTION:

$$\mathcal{L}\{f(t)\} = \hat{F}(s) = \int_0^1 2e^{-st} f(t)dt + \int_1^\infty te^{-st}dt$$

$$= \frac{2}{s}(1 - e^{-s}) + \frac{1+s}{s^2}e^{-s}$$

Example 2 shows that $f(t)$ does not have to be continuous in order for its Laplace transform to exist. In fact, if $f(t)$ is only piecewise continuous over every finite interval for $t \ge 0$ and of *exponential order* as $t \to \infty$, then $\mathcal{L}\{f(t)\}$ exists. By exponential order as $t \to \infty$, we mean that there exists a constant α such that

$$\lim_{t\to\infty} e^{-\alpha t} f(t)$$

is finite. For example, t^n (n an integer) is of exponential order because $t^n e^{-bt} \to 0$ as $t \to \infty$ for any $b > 0$. An example of a function that is not of exponential order is e^{t^2}, which diverges much faster than e^{bt} for any value of b (Problem 5).

The above conditions for $\mathcal{L}\{f(t)\}$ to exist are sufficient, but not necessary. A classic example of a function that is not continuous for $t \ge 0$ but whose Laplace transform exists is $f(t) = t^{-1/2}$. In this case,

$$\mathcal{L}\{f(t)\} = \hat{F}(s) = \int_0^\infty t^{-1/2}e^{-st}dt = \left(\frac{\pi}{s}\right)^{1/2}$$

Table 17.1 lists a few Laplace transforms for easy reference. There are many extensive tables of Laplace transforms. The *CRC Mathematical Tables* and Abramowitz and Stegun list over a hundred. The most extensive readily available table is Volume 1 of *Tables of Integral Transforms* by Erdélyi et al., which lists over a hundred pages of Laplace transforms. Furthermore, most CAS have built-in Laplace transform routines. For example,

```
LaplaceTransform [ f ( t ), t, s ]
```

in Mathematica gives the Laplace transform of $f(t)$ in terms of s.

Even with extensive tables and computer programs, it's still important to be aware of some general properties of Laplace transforms. Two of these are called

Table 17.1
A short table of Laplace transforms.

$f(t)$	$\mathcal{L}\{f(t)\} = \hat{F}(s)$		$f(t)$	$\mathcal{L}\{f(t)\} = \hat{F}(s)$	
t^n	$\dfrac{n!}{s^{n+1}}$	$n = 0, 1, 2, \ldots$	t^k	$\dfrac{\Gamma(k+1)}{s^{k+1}}$	$k > -1$
e^{at}	$\dfrac{1}{s-a}$	$s > a$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	$s > a$
$\sin at$	$\dfrac{a}{s^2+a^2}$	$s > 0$	$\cos at$	$\dfrac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\dfrac{a}{s^2-a^2}$	$s > a$	$\cosh at$	$\dfrac{s}{s^2-a^2}$	$s > a$
$t \sin at$	$\dfrac{2as}{(s^2+a^2)^2}$		$t \cos at$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	

translation properties. The first says that if $\mathcal{L}\{f(t)\} = \hat{F}(s)$, then

$$\mathcal{L}\{e^{at} f(t)\} = \hat{F}(s-a) \qquad s > a \tag{4}$$

Note in Table 17.1, for example, that $\mathcal{L}\{t^n\} = n!/s^{n+1}$ whereas $\mathcal{L}\{t^n e^{at}\} = n!/(s-a)^{n+1}$, in agreement with Equation 4.

Example 3:
Show that

$$\mathcal{L}\{e^{-2t} \cos \omega t\} = \frac{s+2}{s^2+4s+4+\omega^2}$$

SOLUTION: First use

$$\mathcal{L}\{\cos \omega t\} = \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{s}{s^2+\omega^2}$$

and then replace s by $s+2$ to obtain

$$\mathcal{L}\{e^{-2t} \cos \omega t\} = \frac{s+2}{s^2+4s+4+\omega^2}$$

Figure 17.1
A function $f(t)$ and $f(t-a)$ plotted against t. Note that $f(t-a)$ is simply $f(t)$ shifted to the right by a units.

There is a second translation property of Laplace transforms. Let's consider the function defined by

$$g(t) = \begin{cases} 0 & 0 \le t < a \\ f(t-a) & a \le t \end{cases} \tag{5}$$

Note that $g(t)$ is simply $f(t)$ shifted a units to the right (Figure 17.1). It's easy to

17.1 Problems

1. Show that $\mathcal{L}\{\sin \omega t\} = \dfrac{\omega}{s^2+\omega^2}$ and $\mathcal{L}\{\cos \omega t\} = \dfrac{s}{s^2+\omega^2}$.

2. Show that $\mathcal{L}\{t^n e^{-at}\} = \dfrac{n!}{(s+a)^{n+1}}$ where $s + a > 0$ and n is an integer.

3. Show that $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$ and $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$.

4. Show that the following functions are of exponential order as $t \to \infty$:

(a) $t^n \cos t$ (b) $t^n e^{at}$ (c) $\cosh at$

5. Show that e^{t^2} is not of exponential order as $t \to \infty$.

6. Show that $\lim\limits_{s\to\infty} \hat{F}(s) = 0$ if $f(t)$ is piecewise continuous over every finite interval for $t \geq 0$ and is of exponential order. The significance of this result is that there are no Laplace transforms that are polynomials in s or trigonometric functions of s.

7. Derive Equation 10.

8. Suppose that $f(t)$ is not continuous, as required by Equation 10. Show that Equation 10 is modified to read $\mathcal{L}\{f'(t)\} = s\hat{F}(s) - f(0) - e^{-st_1}[f(t_{1+}) - f(t_{1-})]$ if $f(t)$ has a (finite) discontinuity at $t = t_1$.

9. Show that $(1 - e^{-x})^2/(1 - e^{-2x}) = \tanh(x/2)$.

10. If $f(t)$ is piecewise continuous and of exponential order as $t \to \infty$, then it is valid to differentiate Equation 1 with respect to s to obtain $\hat{F}^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$. Use this result to show that $\mathcal{L}\{t \sin \omega t\} = \dfrac{2\omega s}{(s^2+\omega^2)^2}$.

11. Use the result of Problem 10 to show that $\mathcal{L}\{t \cos \omega t\} = (s^2 - \omega^2)/(s^2 + \omega^2)^2$.

12. Use the result of Problem 10 to evaluate $I(\alpha) = \displaystyle\int_0^\infty t^2 e^{-\alpha t} \sin t \, dt$.

13. Show that if $f(t + 2\pi/\omega) = f(t)$ and if $f(t) = \begin{cases} \sin \omega t & 0 \leq t \leq \dfrac{\pi}{\omega} \\ 0 & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$

then $\hat{F}(s) = \mathcal{L}\{f(t)\} = \dfrac{\omega}{s^2+\omega^2}(1 - e^{-s\pi/\omega})^{-1}$.

14. Show that the Laplace transform of $\delta(t)$ is equal to 1. Why doesn't $\hat{F}(s) \to 0$ as $s \to \infty$ in this case?

15. Show that $\mathcal{L}\{e^{at} f(t)\} = \hat{F}(s - a)$. Use this result to show that $\mathcal{L}\{e^{at} \cos \omega t\} = \dfrac{s-a}{s^2 - 2as + a^2 + \omega^2}$.

16. Show that if $f(t + 2) = f(t)$ and if $f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2 - t & 1 < t < 2 \end{cases}$, then $\mathcal{L}\{f'(t)\} = \dfrac{\tanh(s/2)}{s^2}$.

17. Show that $\mathcal{L}\{S_k(t)\} = \dfrac{e^{-ks}}{s}$ if $S_k(t) = \begin{cases} 0 & 0 \leq t < k \\ 1 & k \leq t \end{cases}$.

18. Determine $\mathcal{L}\{t^{-1/2} e^{-a^2/t}\}$. *Hint*: Use the integral that you evaluated in Problem 1.9.14.

19. Show that $\mathcal{L}\{t^{-3/2} e^{-a^2/t}\} = \left(\dfrac{\pi}{a^2}\right)^{1/2} e^{-2as^{1/2}}$. *Hint*: Use the result of the previous problem.

20. Recall the definition of the complementary error function given in Section 3.3, namely $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du$. It turns out that $\mathcal{L}\{\operatorname{erfc}(a/\sqrt{t})\}$ for $a > 0$ occurs in a number of physical problems. We will evaluate $\mathcal{L}\{\operatorname{erfc}(a/\sqrt{t})\}$ in this problem. First let $u = a/\sqrt{y}$ in $\operatorname{erfc}(a/\sqrt{t})$ and show that $\operatorname{erfc}(a/\sqrt{t}) = \frac{a}{\sqrt{\pi}}\int_0^t e^{-a^2/y}\frac{dy}{y^{3/2}}$, $a > 0$. Now write $\hat{F}(s) = \mathcal{L}\{\operatorname{erfc}(a/\sqrt{t})\} = \int_0^\infty e^{-st}\operatorname{erfc}(a/\sqrt{t})dt$ as a double integral and then interchange the order of integration to show that $\hat{F}(s) = \frac{a}{\sqrt{\pi}s}\int_0^\infty e^{-sy-a^2/y}\frac{dy}{y^{3/2}}$. Now let $y = x^2$ and show that

$$\begin{aligned}\hat{F}(s) &= \frac{2a}{\sqrt{\pi}s}\int_0^\infty e^{-sx^2-a^2/x^2}\frac{dx}{x^2}\\ &= -\frac{1}{\sqrt{\pi}s}\frac{d}{da}\int_0^\infty e^{-sx^2-a^2/x^2}dx\end{aligned}$$

Finally, use the fact that (see Problem 1.9.14) $\int_0^\infty e^{-sx^2-a^2/x^2}dx = \frac{1}{2}\left(\frac{\pi}{s}\right)^{1/2}e^{-2as^{1/2}}$ to show that $\hat{F}(s) = \mathcal{L}\{\operatorname{erfc}(a/\sqrt{t})\} = \frac{1}{s}e^{-2as^{1/2}}$.

21. Show that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$, where $J_0(t)$ is a zero-order Bessel function of the first kind. *Hint*: Use the series definition of $J_0(t)$ and also use the result of Problem 2.7.13. (See also Problem 23.)

22. Show that if $\hat{F}(s) = \mathcal{L}\{f(t)\}$, then $\mathcal{L}\{f(at)\} = \frac{1}{a}\hat{F}\left(\frac{s}{a}\right)$. Use this result to show that $\mathcal{L}\{J_0(at)\} = \frac{1}{(s^2+a^2)^{1/2}}$.

23. Here is another way to evaluate $\mathcal{L}\{J_0(t)\}$. Start with (Section 12.6) $J_0(t) = \frac{1}{\pi}\int_0^\pi \cos(t\sin\theta)d\theta$. Now let $\phi = \theta - \pi/2$ and show that $J_0(t) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(t\cos\theta)d\theta = \frac{2}{\pi}\int_0^{\pi/2}\cos(t\cos\theta)d\theta$. Now take the Laplace transform of $J_0(t)$ and interchange the orders of integration to get $\int_0^\infty e^{-st}J_0(t)dt = \frac{2s}{\pi}\int_0^{\pi/2}\frac{d\theta}{s^2+\cos^2\theta}$. Now use the fact that $\int_0^{\pi/2}\frac{d\theta}{s^2+\cos^2\theta} = \frac{\pi}{2s(s^2+1)^{1/2}}$ to show that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$.

24. Determine the Laplace transform of $H(t) - H(t-1/2)$.

25. Show that if $f(t)$ is piecewise continuous and of exponential order and, in addition, if $\lim_{t\to 0} f(t)/t$ exists, then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \hat{F}(s')ds'$.

26. Show that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\hat{F}(s)$ provided these limits exist. *Hint*: Start with $\mathcal{L}\{f'(t)\}$.

27. Show that $\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$ provided these limits exist. *Hint*: Start with $\mathcal{L}\{f'(t)\}$.

or $\alpha = \beta = 1/2$, $\gamma = -1$, and $\delta = 0$. Therefore, we have

$$\frac{2a^2 s}{s^4 - a^4} = \frac{1}{2(s+a)} + \frac{1}{2(s-a)} - \frac{s}{s^2 + a^2}$$

We can readily use the result of Example 2 to invert $\hat{F}(s)$. Referring to Table 17.1, we see that

$$\begin{aligned} f(t) &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} \\ &= \frac{e^{-at}}{2} + \frac{e^{at}}{2} - \cos at \\ &= \cosh at - \cos at \end{aligned}$$

If we try to use the result of Example 1 to invert $\hat{F}(s)$ in that case, we don't seem to be able to use Table 17.1. We can use Equation 4 of the previous section, however, which says that

$$\mathcal{L}\{e^{at} f(t)\} = \hat{F}(s-a) \qquad s > a \tag{2}$$

Let's use this result to find the inverse of

$$\hat{F}(s) = \frac{1}{s^2 + 4s + 13}$$

Add and subtract 4 in the denominator to write it as $(s+2)^2 + 9$, so that

$$\hat{F}(s) = \frac{1}{(s+2)^2 + 9}$$

Table 17.1 shows that $\mathcal{L}\{\sin at\} = a/(s^2 + a^2)$, so that we use Equation 2 to write

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4s + 13}\right\} = \frac{1}{3}e^{-2t}\sin 3t$$

Example 3:
Find the inverse of

$$\hat{F}(s) = \frac{s+1}{s^2 - 4s + 5}$$

SOLUTION: We first write the denominator as $(s-2)^2 + 1$ to get

$$\hat{F}(s) = \frac{s+1}{(s-2)^2 + 1}$$

The form of the denominator suggests that we are going to have a result such as $\hat{F}(s-2)$, and so we write $\hat{F}(s)$ as

$$\hat{F}(s) = \frac{(s-2)+3}{(s-2)^2+1} = \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$$

Referring to Table 17.1 and using Equation 2, we have

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s+5}\right\} = e^{2t}\cos t + 3e^{2t}\sin t$$

We can also use the second translation property to invert Laplace transforms that contain factors of e^{-as}. Recall that this property says that if (see Equations 5 through 9 of the previous section)

$$g(t) = \begin{cases} 0 & 0 \le t < a \\ f(t-a) & a \le t \end{cases} \tag{3}$$

then

$$\mathcal{L}\{g(t)\} = \int_a^\infty dt\, e^{-st} f(t-a) = \int_0^\infty dt\, e^{-st} H(t-a) f(t-a) = e^{-as}\hat{F}(s) \tag{4}$$

Let's use this result to determine $\mathcal{L}^{-1}\{e^{-2s}/s^2\}$. Equation 4 tells us that $\hat{F}(s) = 1/s^2$, so that (see Table 17.1) $f(t) = t$, and

$$g(t) = \mathcal{L}^{-1}\{e^{-2s}/s^2\} = \begin{cases} 0 & 0 \le t < 2 \\ t-2 & 2 \le t \end{cases}$$

or $g(t) = (t-2)H(t-2)$, where $H(x)$ is a unit step function. It's easy to show that $\mathcal{L}\{g(t)\} = e^{-as}/s^2$.

Example 4:
Determine

$$\mathcal{L}^{-1}\left\{\frac{1-e^{-3s}}{s^2}\right\}$$

SOLUTION:

$$\mathcal{L}^{-1}\left\{\frac{1-e^{-3s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}$$

The first term gives $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$. Using Equation 4 we see that the second

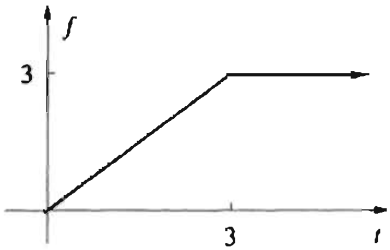

Figure 17.4
The solution to Example 4, $f(t) = t - (t-3)H(t-3)$, plotted against t.

term gives

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} = (t-3)H(t-3)$$

Therefore,

$$f(t) = t - (t-3)H(t-3) \qquad t \geq 0$$

which is plotted in Figure 17.4.

Let's invert

$$\hat{F}(s) = \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})} \tag{5}$$

The first term gives t, but what about the second? Let's expand $1/(1-e^{-s})$ as a geometric series to obtain

$$\begin{aligned}\hat{F}(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s}(1 + e^{-s} + e^{-2s} + \cdots) \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \cdots\end{aligned}$$

Inverting term by term using Equation 4 gives

$$\begin{aligned} f(t) &= t - H(t-1) - H(t-2) - H(t-3) + \cdots \\ &= \begin{cases} t & 0 < t < 1 \\ t-1 & 1 < t < 2 \\ t-2 & 2 < t < 3 \\ \text{and so on} \end{cases} \end{aligned} \tag{6}$$

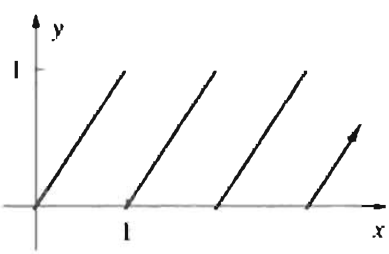

Figure 17.5
The function $f(t) = t - H(t-1) - H(t-2) - H(t-3) + \cdots$ plotted against t.

This result is plotted in Figure 17.5, which shows that $f(t)$ is a periodic sawtooth function of period 1.

It may not be surprising that the inverse of $\hat{F}(s)$ in Equation 5 is a periodic function if you remember Equation 18 of the previous section. Recall that if $f(t) = f(t+\tau)$, then its Laplace transform is given by

$$\hat{F}(s) = \frac{\int_0^{\tau} e^{-st} f(t)dt}{1 - e^{-s\tau}} \tag{7}$$

and so the denominator of Equation 5 suggests that $f(t)$ *might* be periodic. (See Problem 26, however.)

each term and use Equation 2 to obtain

$$s^2\hat{Y}(s) - sy_0 - y_1 + 3s\hat{Y}(s) - 3y_0 + 2\hat{Y}(s) = 0$$

Solving for $\hat{Y}(s)$ gives

$$\begin{aligned}\hat{Y}(s) &= \frac{sy_0 + 3y_0 + y_1}{s^2 + 3s + 2} \\ &= \frac{sy_0 + 3y_0 + y_1}{(s+1)(s+2)}\end{aligned}$$

We use partial fractions to write

$$\frac{sy_0 + 3y_0 + y_1}{(s+1)(s+2)} = \frac{2y_0 + y_1}{s+1} - \frac{y_0 + y_1}{s+2}$$

Referring to Table 17.1, we see that

$$y(t) = \mathcal{L}^{-1}\{\hat{Y}(s)\} = (2y_0 + y_1)e^{-t} - (y_0 + y_1)e^{-2t}$$

This is the general solution to Equation 3, which could easily have been obtained from Section 11.3 (Problem 1). Note that we obtain the solution with the initial conditions built in. This is because of the nature of Equation 2, which includes the initial conditions. Note also that the method works so smoothly because Equation 3 has constant coefficients. It's not necessary that this be so, but applications to differential equations with non-constant coefficients most often rely on fortuitous cancellations of terms or special results. The real power and convenience of the Laplace transform method is for nonhomogeneous differential equations (usually with constant coefficients).

Example 1:
Solve

$$y''(t) + 3y'(t) + 2y(t) = e^t$$

with $y(0) = 1$ and $y'(0) = 0$.

SOLUTION: Taking the Laplace transform of both sides yields

$$s^2\hat{Y}(s) - s + 3s\hat{Y}(s) - 3 + 2\hat{Y}(s) = \frac{1}{s-1}$$

Solve for $\hat{Y}(s)$:

$$\hat{Y}(s) = \frac{1}{(s-1)(s^2+3s+2)} + \frac{3+s}{s^2+3s+2}$$

where we used the result of Problem 10 of the previous section. (See also Problem 22.) Note that in this case the displacement increases with time due to the resonance condition $\omega = \omega_0$.

The next Example shows how we can solve a nonhomogeneous equation in terms of a convolution integral.

Example 3:
Solve

$$y''(t) + \omega_0^2 y(t) = g(t)$$

with $y(0) = y_0$ and $y'(0) = v_0$.

SOLUTION: Take the Laplace transform of both sides and solve for $\hat{Y}(s)$:

$$\hat{Y}(s) = \frac{y_0 s + v_0}{s^2 + \omega_0^2} + \frac{\hat{G}(s)}{s^2 + \omega_0^2}$$

The inverse of $\hat{Y}(s)$ is

$$y(t) = y_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t + \frac{1}{\omega_0} \int_0^t g(t-u) \sin \omega_0 u \, du$$

This is the general solution to the above differential equation, which you could obtain using variation of parameters.

One final set of problems that we shall consider before we finish this section involves the bending of a beam or bar under an applied load. These systems are different from the others that we have discussed because they are described by a fourth-order differential equation, the independent variable is a distance, and the boundary is not necessarily specified at $x = 0$. Consider a uniform beam that is slightly bent due to some applied load, as shown in Figure 17.8. The theory of elasticity tells us that if $y(x)$ denotes the deflection of the beam from the horizontal, then $y(x)$ satisfies

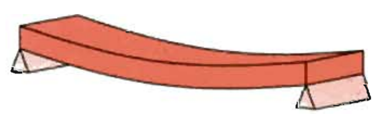

Figure 17.8
A uniform beam supported at its ends and under an applied load.

$$\gamma \frac{d^4 y}{dx^4} = w(x) \tag{4}$$

where γ is a constant that depends upon the material of the beam and its shape and $w(x)$ represents the distribution of the load along the beam. The typical boundary

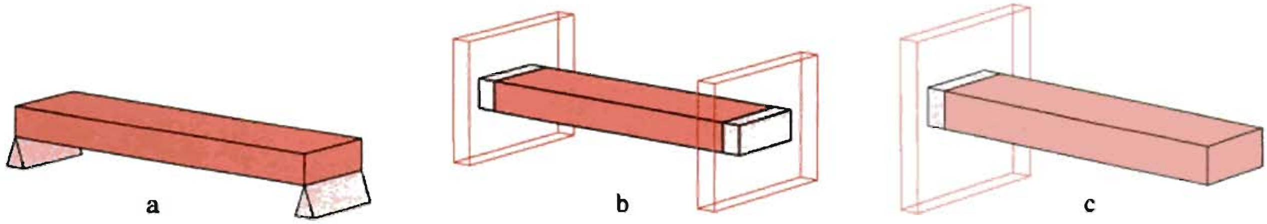

Figure 17.9
An illustration of the typical boundary conditions associated with the calculation of the deflection of a uniform beam. (a) The ends are supported; (b) the ends are clamped, as imbedded in a support; and (c) one end is clamped and the other end is free.

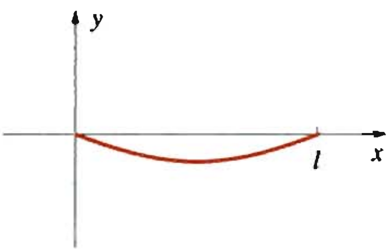

Figure 17.10
The deflection of a uniform beam supported at its two ends under a uniform load $-w_0$.

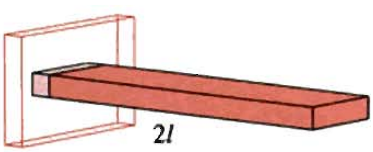

Figure 17.11
A beam of length $2l$ that is clamped at its left end and free at its right end.

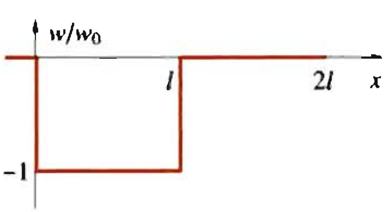

Figure 17.12
A plot of the load $w(x) = -w_0[H(x) - H(x - l)]$ that occurs in Example 4.

conditions are (Figure 17.9):

1. Simply supported, as resting on a support:
   $y = 0$ and $y'' = 0$ at the point.
2. Clamped end, as imbedded in a support:
   $y = 0$ and $y' = 0$ at the point.
3. Free end:
   $y'' = 0$ and $y''' = 0$ at the point.

For example, let's determine the shape of a beam of length 1 that is supported at both ends and is under a uniform load $-w_0$ acting downward in Figure 17.8. Equation 4 is very simple in this case and yields

$$yy(x) = -\frac{w_0 x^4}{24} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4$$

There are four boundary conditions to determine the four constants of integration:

$$y(0) = y(1) = 0 \qquad \text{and} \qquad y''(0) = y''(1) = 0$$

The resulting curve is

$$y(x) = -\frac{w_0}{24y}(-x^4 + 2x^3 - x)$$

Figure 17.10 shows this deflection. This example is pretty simple and doesn't require the use of Laplace transforms.

The following Example is a little more demanding.

Example 4:
Consider the beam of length $2l$ in Figure 17.11. The beam is clamped at its left end and its right end is free. Determine the resulting deflection of the beam under the load (see Figure 17.12).

$$w(x) = -w_0[H(x) - H(x - l)]$$

SOLUTION: The boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y''(2l) = 0$, and $y'''(2l) = 0$. Take the Laplace transform of Equation 4 to obtain

$$s^4\bar{Y}(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{-w_0}{\gamma}\frac{1-e^{-sl}}{s}$$

Now $y(0) = y'(0) = 0$, but we are not given $y''(0)$ or $y'''(0)$. Denote these by α and β, respectively.

Solving for $\bar{Y}(s)$ gives

$$\bar{Y}(s) = \frac{\alpha}{s^3} + \frac{\beta}{s^4} - \frac{w_0}{\gamma}\frac{1-e^{-sl}}{s^5}$$

so that

$$y(x) = \frac{\alpha x^2}{2} + \frac{\beta x^3}{6} - \frac{w_0}{24\gamma}[x^4 - (x-l)^4 H(x-l)]$$

We can now determine α and β by using $y''(2l) = y'''(2l) = 0$. For $l < x \le 2l$,

$$y(x) = \frac{\alpha x^2}{2} + \frac{\beta x^3}{6} - \frac{w_0}{24\gamma}[x^4 - (x-l)^4]$$

This gives $y''(2l) = \alpha + 2\beta l - 3w_0l^2/2\gamma$ and $y'''(2l) = \beta - w_0l/\gamma$, so that

$$\alpha = -\frac{w_0l^2}{2\gamma} \quad \text{and} \quad \beta = \frac{w_0l}{\gamma}$$

Finally, then, the deflection of the beam is given by

$$y(x) = -\frac{w_0l^2x^2}{4\gamma} + \frac{w_0lx^3}{6\gamma} - \frac{w_0}{24\gamma}[x^4 - (x-l)^4H(x-l)]$$

This result is plotted in Figure 17.13. A beam with one end clamped and the other free is called a *cantilever*.

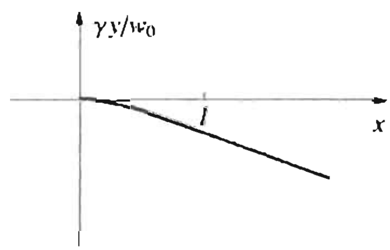

Figure 17.13
The deflection of the beam described in Example 4.

This Example illustrates that it is not necessary for all the boundary conditions to be specified at $x = 0$ (or $t = 0$). Problems 16 through 20 involve other calculations of the deflection of beams.

17.3 Problems

1. Solve Equation 3 using the method given in Section 11.3.

Use Laplace transforms to solve the following differential equations:

2. $y''(t) + 4y'(t) + 4y(t) = 9e^t$ with $y(0) = y'(0) = 0$.

3. $y''(t) - 4y(t) = 5\sin 2t$ with $y(0) = 0$ and $y'(0) = 1$.

17.4 Laplace Transforms and Partial Differential Equations

Laplace transforms can be used to solve partial differential equations. As usual, the partial differential equations must be linear with constant coefficients, but as we saw in the previous chapter, this still provides us with a large number of physical applications.

As our first example, consider the one-dimensional heat equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \qquad \begin{matrix} 0 \le x \\ 0 < t \end{matrix} \tag{1}$$

over the semi-infinite region $x \geq 0$. This equation describes the conduction of heat in a long rod whose lateral surface is insulated. Let the rod be initially at zero temperature and let the left end face of the rod ($x = 0$) be kept at a fixed temperature u_0. These two conditions translate into

$$u(x, 0) = 0 \qquad x > 0 \qquad u(0, t) = u_0 \tag{2}$$

We also impose the physical condition that $u(x, t) = 0$ as $x \to \infty$.

Let's take the Laplace transform of Equation 1 with respect to the time variable. Denoting $\mathcal{L}\{u(x, t)\}$ by $\hat{U}(x, s)$, the left side becomes

$$\int_0^\infty e^{-st}\frac{\partial^2 u}{\partial x^2}dx = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st}u(x, t)dt = \frac{\partial^2 \hat{U}(x, s)}{\partial x^2} \tag{3}$$

provided that it is legitimate to interchange the orders of integration and differentiation. We know the conditions for doing this from Section 1.8, but since we don't know $u(x, t)$ yet, it's difficult to access it. Nevertheless, we would expect any function that describes the physical situation of the temperature in a rod to be suitably well behaved. In all cases involving integral transforms to solve partial differential equations, we should verify that our final result satisfies both the partial differential equation *and* its associated boundary conditions and initial conditions.

Example 1:
Show explicitly that

$$\mathcal{L}\left\{\frac{\partial}{\partial x}\cos xt\right\} = \frac{\partial}{\partial x}\mathcal{L}\{\cos xt\}$$

where $\mathcal{L}$ is taken with respect to t.

SOLUTION:

$$\mathcal{L}\left\{\frac{\partial}{\partial x}\cos xt\right\} = \mathcal{L}\{-t\sin xt\} = -\frac{2sx}{(s^2+x^2)^2}$$

$$\frac{\partial}{\partial x}\mathcal{L}\{\cos xt\} = \frac{\partial}{\partial x}\frac{s}{s^2+x^2} = -\frac{2sx}{(s^2+x^2)^2}$$

The Laplace transform of the right side of Equation 1 gives

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\hat{U}(s, x) - u(x, 0) = s\hat{U}(s, x) \tag{4}$$

Finally, the Laplace transform of the boundary condition $u(0, t) = u_0$ gives

$$\mathcal{L}\{u(0, t)\} = \hat{U}(0, s) = \frac{u_0}{s} \tag{5}$$

Equations 3 and 4 give us

$$\frac{d^2\hat{U}(x, s)}{dx^2} - \frac{s}{\alpha^2}\hat{U}(x, s) = 0 \tag{6}$$

We have written the derivative of $\hat{U}(x, s)$ as an ordinary derivative because we consider s to be a fixed constant at this point.

Notice that the time variable has been eliminated (transformed out) in Equation 1, leaving an equation in just one independent variable. Generally, applying a transform to a differential equation will reduce the number of independent variables by one. In the case of an ordinary differential equation, the result is an algebraic equation.

The solution to Equation 6 is

$$\hat{U}(x, s) = c_1(s)e^{xs^{1/2}/\alpha} + c_2(s)e^{-xs^{1/2}/\alpha} \tag{7}$$

where we recognize that the two "constants of integration" may depend upon s. The condition that $u(x, t) \to 0$ as $x \to \infty$ forces us to choose $c_1(s) = 0$. Setting $x = 0$ and using Equation 5 yields

$$\hat{U}(0, s) = c_2(s) = \frac{u_0}{s}$$

Therefore, we have

$$\hat{U}(s, x) = \frac{u_0}{s}e^{-xs^{1/2}/\alpha} \tag{8}$$

as the Laplace transform of our solution. The inverse of $e^{-xs^{1/2}/\alpha}$ is $xe^{-x^2/4\alpha^2 t}/2(\pi\alpha^2 t^3)^{1/2}$ (see Problem 2-22), and so (Problem 2-20)

$$u(x, t) = \frac{u_0 x}{(4\pi\alpha^2)^{1/2}}\int_0^t y^{-3/2}e^{-x^2/4\alpha^2 y}dy$$

We can express this result in terms of a complementary error function by letting $z^2 = x^2/4\alpha^2 y$:

$$u(x, t) = \frac{2u_0}{\sqrt{\pi}}\int_{x/(4\alpha^2 t)^{1/2}}^{\infty} e^{-z^2}dz$$

$$= \frac{2u_0}{\sqrt{\pi}} \left\{ \int_0^{\infty} e^{-z^2} dz - \int_0^{x/(4\alpha^2 t)^{1/2}} e^{-z^2} dz \right\} \qquad (9)$$

$$= u_0\{1 - \mathrm{erf}[(x/(4\alpha^2 t)^{1/2}]\}$$

$$= u_0 \,\mathrm{erfc}[(x/(4\alpha^2 t)^{1/2}] \qquad (10)$$

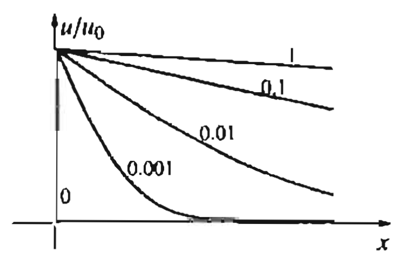

Figure 17.14
The plot of $u(x, t) = u_0 \,\mathrm{erfc}[x/(4\alpha^2 t)^{1/2}]$ against x for various values of t.

Equation 10 is plotted against x for various values of t in Figure 17.14. Note that the temperature approaches u_0 for any finite value of x but that $u(x, t) \to 0$ as $x \to \infty$.

Let's now verify that Equation 10 is indeed a solution to Equation 1. Using Leibnitz's rule (Problem 2) with Equation 9, we see that

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_0 x}{(4\pi\alpha^6 t^3)^{1/2}} e^{-x^2/4\alpha^2 t}$$

and that

$$\frac{\partial u}{\partial t} = \frac{u_0 x}{(4\pi\alpha^2 t^3)^{1/2}} e^{-x^2/4\alpha^2 t}$$

so that Equation 1 is satisfied. Furthermore, $u(x, 0) = \mathrm{erfc}(\infty) = 0$, $u(0, t) = u_0 \,\mathrm{erfc}(0) = u_0$, and $u(x, t) \to 0$ as $x \to 0$. Thus, we see that Equation 10 satisfies Equation 1 and its associated boundary conditions and initial conditions. Having done this once, the verification of the subsequent solutions will be left as exercises.

Before we do some other examples, let's consider taking the Laplace transform with respect to the x variable in Equation 1. Note that if we were to do that, we would need to have both $u(0, t)$ and $u_x(0, t)$ specified. So even though $0 \le x < \infty$ in this case, we cannot transform with respect to x.

There are really no new principles involved here, and the best way to illustrate the application of Laplace tranforms to solving partial differential equations is by example. Consequently, the rest of this section will consist of various Examples. The next Example specifies the flux of heat rather than the temperature at $x = 0$.

Example 2:
Solve the heat equation for a semi-infinite rod whose lateral surface is insulated, whose initial temperature is zero, and with a constant heat flux maintained at its left-end face.

SOLUTION: The problem to be solved is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \qquad x > 0,\ t > 0$$

with $u(x, 0) = 0$, $-\kappa u_x(0, t) = \beta_0$, and $u(x, t) = 0$ as $x \to \infty$. Take the

Laplace transform with respect to time to obtain

$$\frac{d^2\hat{U}(x,s)}{dx^2} - \frac{s}{\alpha^2}\hat{U}(x,s) = 0$$

with $-\kappa \hat{U}_x(0,s) = \beta_0/s$ and $\hat{U}(x,s) = 0$ as $x \to \infty$. Solving for $\hat{U}(x,s)$ gives

$$\hat{U}(x,s) = c(s)e^{-xs^{1/2}/\alpha}$$

Setting $-\kappa \hat{U}_x(0,s) = \beta_0/s$ gives $c(s) = \beta_0\alpha/\kappa s^{3/2}$, and

$$\hat{U}(x,s) = \frac{\beta_0\alpha}{\kappa s^{3/2}}e^{-xs^{1/2}/\alpha}$$

Now, according to Problem 2-24,

$$\mathcal{L}^{-1}\left\{\frac{e^{-xs^{1/2}/\alpha}}{s^{1/2}}\right\} = (\pi t)^{-1/2}e^{-x^2/4\alpha^2 t}$$

and so (Problem 2-20)

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{e^{-xs^{1/2}/\alpha}}{s^{1/2}}\right\} = \int_0^t (\pi y)^{-1/2}e^{-x^2/4\alpha^2 y}dy$$

and

$$\begin{aligned} u(x,t) &= \frac{\beta_0\alpha}{\kappa\pi^{1/2}}\int_0^t y^{-1/2}e^{-x^2/4\alpha^2 y}dy \\ &= \frac{\beta_0 x}{\kappa\pi^{1/2}}\int_{x/(4\alpha^2 t)^{1/2}}^{\infty} e^{-z^2}\frac{dz}{z^2} \end{aligned}$$

where we let $z = x/(4\alpha^2 y)^{1/2}$. We can express this result in terms of the error function by integrating by parts, letting "u" $= e^{-z^2}$ and "dv" $= dz/z^2$, to get (see also Problem 3.3.16)

$$\begin{aligned} u(x,t) &= \frac{\beta_0}{\kappa\pi^{1/2}}\left[(4\alpha^2 t)^{1/2}e^{-x^2/4\alpha^2 t} - 2x\int_{x/(4\alpha^2 t)^{1/2}}^{\infty} e^{-z^2}dz\right] \\ &= \frac{\beta_0}{\kappa}\left\{\left(\frac{4\alpha^2 t}{\pi}\right)^{1/2} e^{-x^2/4\alpha^2 t} - x\,\mathrm{erfc}\,[x/(4\alpha^2 t)^{1/2}]\right\} \end{aligned}$$

This solution is plotted in Figure 17.15. Problem 3 asks you to show that this result satisfies the above partial differential equation and all its conditions.

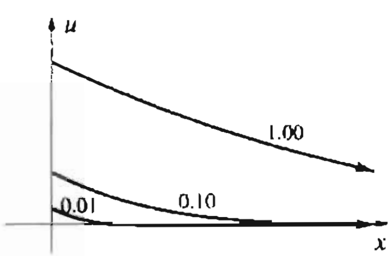

Figure 17.15
The solution given in Example 2 plotted against x for various values of $4\alpha^2 t$.

SOLUTION: Using k as the transform variable, we have

$$\hat{F}(k) = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-x^2/2\sigma^2}$$
$$= \frac{1}{\pi\sigma}\int_{0}^{\infty} dx\, e^{-x^2/2\sigma^2}\cos kx = \frac{1}{(2\pi)^{1/2}} e^{-\sigma^2 k^2/2}$$

The inverse of $\hat{F}(k)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-\sigma^2 k^2/2}\cos kx$$
$$= \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-x^2/2\sigma^2}$$

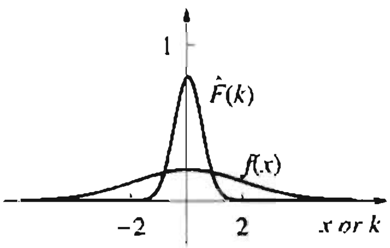

Figure 17.19
The Fourier transform pair (a) $f(x) = (2\pi\sigma^2)^{-1/2}\exp(-x^2/2\sigma^2)$ and (b) $\hat{F}(k) = (\sigma^2/2\pi)^{1/2}\exp(-\sigma^2 k^2/2)$.

Problems 1 through 6 involve finding the Fourier transform of some elementary functions.

Example 1 tells us that the Fourier transform of a Gaussian distribution is a Gaussian distribution. In addition, the widths of the two normalized distributions are reciprocals; $f(x)$ gets narrower and $\hat{F}(k)$ gets wider as σ decreases (Figure 17.19). This reciprocal relationship is an illustration of the uncertainty

Table 17.2
Some Fourier transform pairs. In all cases, $a > 0$.

$f(x)$	$\hat{F}(k)$	$f(x)$	$\hat{F}(k)$
1	$(2\pi)^{1/2}\delta(k)$	$^{*}\mathrm{sgn}(x)$	$-\left(\frac{2}{\pi}\right)^{1/2}\frac{i}{k}$
e^{ik_0x}	$(2\pi)^{1/2}\delta(k-k_0)$	e^{-ik_0x}	$(2\pi)^{1/2}\delta(k+k_0)$
$e^{-a\lvert x\rvert}$	$\left(\frac{2}{\pi}\right)^{1/2}\frac{a}{k^2+a^2}$	$xe^{-a\lvert x\rvert}$	$-\left(\frac{2}{\pi}\right)^{1/2}\frac{2aik}{(k^2+a^2)^2}$
$e^{-a^2x^2}$	$\frac{1}{(2a^2)^{1/2}}e^{-k^2/4a^2}$	$1/x$	$-i\left(\frac{\pi}{2}\right)^{1/2}\mathrm{sgn}\,(k)$
$\frac{1}{x^2+a^2}$	$\left(\frac{\pi}{2a^2}\right)^{1/2}e^{-a\lvert k\rvert}$	$^{**}H(x)$	$\left(\frac{\pi}{2}\right)^{1/2}\delta(k) - \frac{i}{(2\pi)^{1/2}k}$

* signum $x = -1$ for $x < 0$ and 1 for $x > 0$
** unit step function $= 0$ for $x < 0$ and 1 for $x > 0$
Note that $H(x) = \frac{1}{2}[1 + \mathrm{sgn}(x)]$.

Example 3:
Find the Fourier transform of $f(x) = \delta(x - x_0)$.

SOLUTION: Using Equation 9 formally, we have

$$\hat{F}(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \delta(x - x_0) e^{-ikx} dx = \frac{1}{(2\pi)^{1/2}} e^{-ikx_0}$$

We say "formally" because $\delta(x - x_0)$ is not piecewise smooth, but nevertheless the result is still valid (and useful).

If we invert $\hat{F}(k)$ using Equation 9, we have

$$f(x) = \delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)} dk \tag{11}$$

which is one of the most useful definitions of a delta function.

There are some general properties of Fourier transforms that are useful in applications. Two of these are the so-called *shifting properties*:

$$e^{-iak} \hat{F}(k) = \mathcal{F}\{f(x - a)\} \tag{12}$$

and

$$\mathcal{F}\{e^{iax} f(x)\} = \hat{F}(k - a) \tag{13}$$

The proofs of these are fairly straightforward (Problems 8 and 9).

Example 4:
Find the inverse of

$$\hat{F}(k) = e^{-ikx_0} e^{-k^2/4a^2}$$

SOLUTION: Table 17.2 shows that

$$\mathcal{F}^{-1}\{e^{-k^2/4a^2}\} = (2a^2)^{1/2} e^{-a^2x^2}$$

Using this result and Equation 12 gives

$$\mathcal{F}^{-1}\{e^{-ikx_0} e^{-k^2/4a^2}\} = (2a^2)^{1/2} e^{-a^2(x-x_0)^2}$$

An important property for the application of Fourier transforms to the solution of boundary value problems is the *derivative property*. It's easy to see using integration by parts that

$$\mathcal{F}\{f^{(n)}(x)\} = (ik)^n \hat{F}(k) \tag{14}$$

if $f(x)$ and all its derivatives vanish at infinity. Note that this formula differs from the corresponding relation for Laplace transforms in that no initial conditions are required. We'll see in the next section that Fourier transforms are particularly appropriate if the independent variable being eliminated ranges from $-\infty$ to $+\infty$.

There is also a formula involving the derivative of a Fourier transform. If $f(x)$ is piecewise smooth and absolutely integrable, then

$$\hat{F}^{(n)}(k) = (-i)^n \mathcal{F}\{x^n f(x)\} \tag{15}$$

This formula follows easily from Equation 8.

Example 5:
Use Equation 15 to verify the entry for $xe^{-a|x|}$ in Table 17.2.

SOLUTION:

$$\mathcal{F}\{e^{-a|x|}\} = \left(\frac{2}{\pi}\right)^{1/2} \frac{a}{k^2 + a^2}$$

$$\mathcal{F}\{xe^{-a|x|}\} = i\left(\frac{2}{\pi}\right)^{1/2} \left(-\frac{2ak}{(k^2 + a^2)^2}\right) = -\left(\frac{2}{\pi}\right)^{1/2} \frac{2aik}{(k^2 + a^2)^2}$$

There is a convolution theorem involving Fourier transforms. We define the convolution of $f(x)$ and $g(x)$ by

$$f * g = \int_{-\infty}^{\infty} f(u)g(x-u)du = \int_{-\infty}^{\infty} f(x-u)g(u)du \tag{16}$$

The Fourier transform of $f * g$ is

$$\begin{aligned}\mathcal{F}\{f * g\} &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} du\, f(x-u)g(u) \\ &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} du\, g(u) \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x-u)\end{aligned}$$

Now let $z = x - u$ to write

$$\mathcal{F}\{f * g\} = \int_{-\infty}^{\infty} du\, g(u)\hat{F}(k)e^{-iku} = (2\pi)^{1/2}\hat{F}(k)\hat{G}(k) \tag{17}$$

or

$$\mathcal{F}^{-1}\{\hat{F}(k)\hat{G}(k)\} = \frac{1}{(2\pi)^{1/2}} f * g \tag{18}$$

You may have noticed that the convolution integral in Equation 16 differs from the one that we defined in Section 2 in that the limits in Equation 16 are $-\infty$ to ∞ instead of 0 to t as in Equation 17.2.12. If $f(x)$ and $g(x)$ are equal to zero when $x < 0$, however, the two integrals are the same.

We frequently use Fourier transforms in three dimensions:

$$\hat{F}(k_x, k_y, k_z) = \frac{1}{(2\pi)^{3/2}} \iiint\limits_{-\infty}^{\infty} f(x, y, z) e^{-i\mathbf{k}\cdot\mathbf{r}} dx dy dz \tag{19}$$

where $\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z$. The inverse transform, in a more common notation, is

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \iiint \hat{F}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \tag{20}$$

where $f(\mathbf{r})$ denotes $f(x, y, z)$, $\hat{F}(\mathbf{k})$ denotes $\hat{F}(k_x, k_y, k_z)$, and $d\mathbf{k} = dk_x dk_y dk_z$. Suppose now that $f(\mathbf{r})$ depends only upon $|\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2}$, as is commonly the case. To evaluate $\hat{F}(\mathbf{k})$ when $f(\mathbf{r}) = f(|\mathbf{r}|)$, introduce spherical coordinates into Equation 19 and write

$$\hat{F}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\phi\, f(r) e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{21}$$

Choose the z axis (the polar axis) of our spherical coordinate system to point along $\mathbf{k}$, so that $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$. We can then integrate over θ and ϕ in Equation 21 to get (Problem 15)

$$\hat{F}(k) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(r) \frac{r \sin kr}{k} dr \tag{22}$$

The inverse formula is

$$f(r) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \hat{F}(k) \frac{k \sin kr}{r} dk \tag{23}$$

Notice that $\hat{F}(\mathbf{k}) = \hat{F}(|\mathbf{k}|) = \hat{F}(k)$ when $f(\mathbf{r}) = f(|\mathbf{r}|) = f(r)$.

Example 6:

Determine the Fourier transform of

$$f(r) = \frac{Z^3}{\pi} e^{-2Zr}$$

given that r is the radial coordinate of a spherical coordinate system.

SOLUTION: We use Equation 22:

$$\hat{F}(k) = \left(\frac{2}{\pi}\right)^{1/2} \frac{Z^3}{\pi k} \int_0^\infty r e^{-2Zr} \sin kr \, dr$$
$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{4Z^4/\pi}{(k^2 + 4Z^2)^2}$$

As a final topic in this section, we shall derive Parseval's theorem for Fourier transforms (see Section 15.3 for Parseval's theorem for Fourier series). Start with

$$\int_{-\infty}^{\infty} f^*(t) f(t) dt = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

where we have allowed $f(t)$ to be complex for generality. Use Equation 9 to write

$$\int_{-\infty}^{\infty} dt \, |f(t)|^2 = \int_{-\infty}^{\infty} dt \, (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega \, \hat{F}^*(\omega) e^{-i\omega t} (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega' \hat{F}(\omega') e^{i\omega' t}$$
$$= \int_{-\infty}^{\infty} d\omega \hat{F}^*(\omega) \int_{-\infty}^{\infty} d\omega' \hat{F}(\omega') \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{i(\omega' - \omega)t}$$
$$= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \hat{F}^*(\omega) \hat{F}(\omega') \delta(\omega' - \omega)$$
$$= \int_{-\infty}^{\infty} d\omega \hat{F}^*(\omega) \hat{F}(\omega) = \int_{-\infty}^{\infty} d\omega \, |\hat{F}(\omega)|^2 \tag{24}$$

We used Equation 11 in going from the second line to the third line.

Equation 24 is known as *Parseval's theorem*. We could have derived Equation 24 by starting with the convolution theorem (Problem 20), but the above derivation illustrates the manipulative utility of the delta function. The following Example is a nice application of Parseval's theorem.

Example 7:
Let the electric field in a radiated wave be described by

$$E(t) = \begin{cases} 0 & t < 0 \\ e^{-t/\tau} \sin \omega_0 t & t > 0 \end{cases}$$

Use Parseval's theorem to find the power radiated in the frequency interval $(\omega, \omega + d\omega)$.

SOLUTION: The Fourier transform of $E(t)$ is

$$\hat{E}(\omega) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{-t/\tau} e^{-i\omega t} \sin \omega_0 t \, dt$$

We'll evaluate this integral by writing $\sin \omega_0 t$ as $(e^{i\omega_0 t} - e^{-i\omega_0 t})/2i$.

$$\hat{E}(\omega) = \frac{1}{2(2\pi)^{1/2}} \left(\frac{1}{\omega + \omega_0 - \frac{i}{\tau}} - \frac{1}{\omega - \omega_0 - \frac{i}{\tau}} \right)$$

The total power radiated is given by

$$\int_{-\infty}^{\infty} |E(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega$$

Using the fact that the second term in $|\hat{E}(\omega)|$ will dominate for $\omega > 0$, we have

$$|\hat{E}(\omega)|^2 d\omega \approx \frac{1}{8\pi} \frac{d\omega}{(\omega - \omega_0)^2 + \frac{1}{\tau^2}}$$

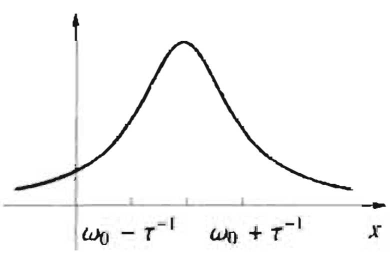

Figure 17.21
The frequency spectrum associated with the radiated wave described in Example 7 for $\omega_0 = 20$ and $\tau = 1$.

This frequency spectrum is shown in Figure 17.21.

When $\omega = \omega_0 \pm 1/\tau$, the height of the spectrum falls off by a factor of 2, as shown in the figure. Therefore, the width at half power is given by $2/\tau$. This result is reminiscent of the uncertainty principle. In this case, we have that the width of a frequency spectrum varies inversely as the duration of the signal. The finite wave train discussed in Example 2 is another example of this version of the uncertainty principle.

17.5 Problems

1. Find the Fourier transform of $f(t) = 1/(t^2 + a^2)$.

2. Find the Fourier transform of $f(t) = \begin{cases} 1 & -a \le t \le a \\ 0 & \text{otherwise} \end{cases}$. Discuss the reciprocal relation between the width of $f(t)$ and $\hat{F}(\omega)$.

3. Find the Fourier transform of $te^{-a|t|}$, $a > 0$.

4. Find the Fourier transform of $1/x$. *Hint*: Be sure to use the fact that $1/x$ is an odd function and proceed formally.

5. Use the result of the previous problem to show that $\mathcal{F}\{\text{sgn}(x)\} = -i \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{k}$.

6. Use the result of the previous problem to show that $\mathcal{F}\{H(x)\} = \left(\frac{\pi}{2}\right)^{1/2} \left[\delta(k) - \frac{i}{\pi k}\right]$, where $H(x)$ is the unit step function. *Hint*: Express $H(x)$ in terms of $\text{sgn}(x)$.

the Fourier transform of Equation 2 yields

$$\frac{d\hat{C}(k,t)}{dt} = -Dk^2\hat{C}(k,t) \tag{3}$$

with the initial condition

$$\hat{C}(k,0) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ikx}c(x,0)dx = (2\pi)^{-1/2}c_0e^{-ikx_0} \tag{4}$$

The solution to Equation 3 is

$$\hat{C}(k,t) = \hat{C}(k,0)e^{-k^2Dt} = \frac{c_0}{(2\pi)^{1/2}}e^{-ikx_0}e^{-k^2Dt} \tag{5}$$

We can determine $c(x,t)$ from the formula

$$\begin{aligned} c(x,t) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ikx}\hat{C}(k,t)dk \\ &= \frac{c_0}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}e^{-k^2Dt}dk \\ &= \frac{c_0}{(4\pi Dt)^{1/2}}e^{-(x-x_0)^2/4Dt} \end{aligned} \tag{6}$$

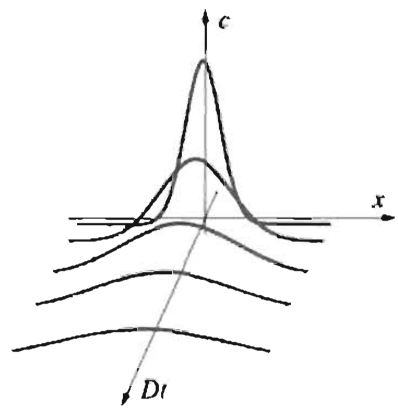

Figure 17.22
The fundamental solution of the one-dimensional diffusion equation plotted against x for increasing values of Dt.

Figure 17.22 shows plots of $c(x,t)$ for various values of t. Notice how the diffusing species spreads out as time increases.

It's also easy to solve Equation 2 for a general initial condition $c(x,0) = f(x)$.

Example 1:
Solve Equation 2 with the initial condition $c(x,0) = f(x)$.

SOLUTION: Let $\hat{C}(k,0) = \hat{F}(k)$. Then Equation 5 changes to

$$\hat{C}(k,t) = (2\pi)^{-1/2}\hat{F}(k)e^{-k^2Dt}$$

We use the Fourier transform convolution theorem to write

$$c(x,t) = \frac{1}{(4\pi Dt)^{1/2}}\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/4Dt}du \tag{7}$$

Note that Equation 7 is the convolution of $c(x,0)$ with Equation 6. We can give a nice physical interpretation of Equation 7. Picture $f(u)$ as consisting of a sum of delta functions spread along the x axis. Then Equation 7 represents the sum of the results of each delta function at the point x. Because of this, Equation 6 is called the *fundamental solution* of the one-dimensional diffusion equation over an infinite interval. It is also the Green's function for this system.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\cos kvt \int_{-\infty}^{\infty} dx'\, e^{-ikx'} f(x')$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} dx'\, f(x') \int_{-\infty}^{\infty} dk\, [e^{ik(x-x'+vt)} + e^{ik(x-x'-vt)}]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx'\, f(x')[\delta(x-x'+vt) + \delta(x-x'-vt)]$$

$$= \frac{1}{2}[f(x-vt) + f(x+vt)]$$

We used Equation 11 of the previous section in going from the third line to the fourth line. Thus we obtain the D'Alembert solution of the wave equation. Problem 13 has you derive the more general result that you get when $u_t(x, 0) \neq 0$.

Recall that a Dirichlet problem consists of solving Laplace's equation with the potential function specified on the boundary. The next Example illustrates the use of Fourier transforms to solve a Dirichlet problem.

Example 4:
Determine the potential function $u(x, y)$ that satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \begin{array}{c} -\infty < x < \infty \\ 0 < y \end{array}$$

with $u(x, 0) = f(x)$ for $-\infty < x < \infty$ and $u(r) \to 0$ as $r \to \infty$, where $r = (x^2 + y^2)^{1/2}$. Give a physical interpretation of this problem.

SOLUTION: This problem might represent the steady temperature distribution in a large thin rectangular sheet, with its lateral surfaces insulated and the temperature specified by $f(x)$ along one edge.

We take the Fourier transform with respect to x to obtain

$$\frac{d^2\hat{U}}{dy^2} - k^2\hat{U} = 0 \qquad 0 \le y < \infty$$

with $\hat{U}(k, 0) = \hat{F}(k)$ and $\hat{U}(k, y) \to 0$ as $y \to \infty$. The solution that satisfies these conditions is

$$\hat{U}(k, y) = \hat{F}(k)e^{-|k|y} \qquad 0 \le y < \infty$$

The inverse of $e^{-|k|y}$ is

$$\mathcal{F}^{-1}\{e^{-|k|y}\} = \left(\frac{2}{\pi}\right)^{1/2} \frac{y}{x^2 + y^2} \qquad 0 \le y < \infty$$

and so

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(z)\, dz}{(x - z)^2 + y^2}$$

This equation is well known in electrostatics and is called *Poisson's integral formula* for the half-plane.

As a final Example in this section, we shall determine the fundamental solution of the two-dimensional isotropic diffusion equation.

Example 5:
We wish to solve

$$D\nabla^2 c = \frac{\partial c}{\partial t} \qquad \begin{matrix} 0 \le r < \infty \\ 0 \le t \end{matrix} \tag{14}$$

with $c(x, y, 0) = c_0\delta(x - x_0)\delta(y - y_0)$ and $c(r, t) \to 0$ as $r \to \infty$.

SOLUTION: First note that

$$\hat{C}(k_x, k_y, t) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{r}} c(x, y, t)\, dx\, dy$$

$$= \frac{1}{2\pi} \int_0^{\infty} dr\, r c(r, t) \int_0^{2\pi} e^{-ikr\cos\theta}\, d\theta$$

The integral over θ is $2\pi J_0(kr)$ (see Problem 12.6.9), and so

$$\hat{C}(k, t) = \int_0^{\infty} dr\, r J_0(kr) c(r, t) \tag{15}$$

The inverse of $\hat{C}(k, t)$ is (Problem 15)

$$c(r, t) = \int_0^{\infty} dk\, k J_0(kr) \hat{C}(k, t) \tag{16}$$

Taking the Fourier transform of Equation 14 gives

$$\frac{d\hat{C}}{dt} = -k^2 D\hat{C}$$

with $\hat{C}(r, 0) = \frac{c_0}{2\pi} e^{-i\mathbf{k}\cdot\mathbf{r}_0}$. The solution to this equation is

$$\hat{C}(k, t) = \frac{c_0}{2\pi} e^{-i\mathbf{k}\cdot\mathbf{r}_0} e^{-k^2 Dt}$$

17.7 The Inversion Formula for Laplace Transforms

Up to this point we haven't presented an explicit formula for the inversion of Laplace transforms, although we were able to invert many of them using tables and special techniques such as partial fractions. When we derived the Fourier integral theorem, on the other hand, we were led directly to a Fourier transform pair, which gave us explicit integral formulas to go from $f(x)$ and $\hat{F}(k)$ and back. In this brief section, we shall derive an integral formula for the inverse of a Laplace transform. We'll see that it involves integration in the complex plane, a topic that we have not considered before.

Recall the Fourier integral theorem, which says that if $f(t)$ is everywhere piecewise smooth and absolutely integrable, then

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dz\, e^{i\omega(t-z)}u(z) \tag{1}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \int_{-\infty}^{\infty} dz\, e^{-i\omega z}u(z) \tag{2}$$

Equation 2 leads directly to the Fourier transform pair

$$\hat{U}(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-i\omega t}u(t)\, dt \tag{3}$$

and

$$u(t) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{i\omega t}\hat{U}(\omega)\, d\omega \tag{4}$$

Now let's consider the function

$$g(t) = e^{-ct}f(t)H(t) \tag{5}$$

where c is a positive real constant, $f(t)$ is piecewise smooth for $t \geq 0$, and $H(t)$ is the unit step function. We assume that $g(t)$ is absolutely integrable, and so $f(t)$ must be such that

$$\int_0^{\infty} e^{-ct}|f(t)|\, dt = \text{finite} \tag{6}$$

The function $g(t)$ satisfies the conditions of the Fourier integral theorem, so we can write (Equation 2)

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{ipt} \int_{-\infty}^{\infty} dx\, e^{-ipx}g(x) \tag{7}$$

or, using Equation 5 in both sides of Equation 7,

CHAPTER 18

Functions of a Complex Variable: Theory

We discussed complex numbers and functions of a complex variable in Chapter 4, and we have seen several examples where functions defined in terms of the complex variable $z = x + iy$ somehow underlie the properties of functions $f(x)$ of a real variable. For example, the power series

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \qquad |x| < 1$$

has an interval of convergence $-1 < x < 1$ even though the left side appears to be a perfectly well-behaved function of x for all values of x. As we pointed out in Chapter 4, the reason that x is restricted to the interval $-1 < x < 1$ is that $f(x)$, considered as a function of a complex variable

$$f(z) = \frac{1}{1+z^2}$$

has singularities at $z = \pm i$, and so z is restricted to lie within a unit circle centered at the origin. Thus, x, the real part of z, is restricted to the interval $-1 < x < 1$.

Another hint that functions of a complex variable underlie functions of a real variable occurred in Section 12.2 where we saw that the radius of convergence of a power series solution to a differential equation about the point x_0 is at least as great as the distance from x_0 to the nearest singular point (*real* or *complex*) of the differential equation.

It turns out that the analysis of functions of a complex variable is one of the richest areas of applied mathematics. In this chapter, we shall learn about the general properties of functions of a complex variable. Some of these properties are really remarkable. We'll see that if $f(z)$ has a first derivative in some region of the complex plane, then all its derivatives exist there. Furthermore, we'll see that if $f(z)$ is differentiable on a simple closed curve in the complex plane, then the values of $f(z)$ throughout the region enclosed by the curve are determined by the values of $f(z)$ on the curve.

In Section 1, we'll define limits and continuity of functions of a complex variable, which will lead to the definition of the derivative of $f(z)$ in Section 2. Then we'll learn two important (and easy to use) integral theorems in Sections 3 and 4. In Section 5, we'll study Taylor series of functions of a complex variable and then Laurent series, which are series of the form

$$F(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

Notice that a Laurent series is a series in both ascending and descending powers of $z - a$. Finally, in Section 6, we'll learn what is called the residue theorem, which leads to a powerful method for evaluating not only integrals of functions of a complex variable, but for evaluating integrals of functions of real variables as well.

18.1 Functions, Limits, and Continuity

Let $z = x + iy$, where x and y are real, represent a complex variable. If we have a prescription for assigning one or more values of another complex variable w to each value of z, then we say that w is a function of z and write $w = f(z)$. Because w is complex and depends upon x and y through z, we often write $w = f(z)$ as

$$w = f(z) = u(x, y) + i\, v(x, y) \tag{1}$$

where $u(x, y)$ and $v(x, y)$ are real functions of x and y. If only one value of w corresponds to each value of z, then $w = f(z)$ is called a single-valued function of z. If more than one value of $w = f(z)$ corresponds to each value of z, then $w = f(z)$ is called a multiple-valued or multivalued function of z. A multivalued function can be considered to be a collection of single-valued functions. For example, consider the two-valued function $f(z) = z^{1/2}$. We can see the nature of the two values of $f(z)$ by expressing it in polar form,

$$w = f(z) = z^{1/2} = r^{1/2}e^{i\theta/2}$$

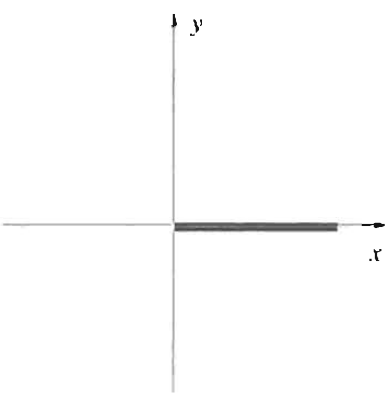

Figure 18.1
An illustration of a branch cut along the positive x axis for the function $f(z) = z^{1/2}$.

where $r^{1/2} > 0$. Now, when $\theta = 0$, $w = r^{1/2}$. As θ increases from $\theta = 0$ in a counterclockwise direction, w varies as $r^{1/2}e^{i\theta/2}$. After a complete circuit, $\theta = 2\pi$ and $w = r^{1/2}e^{i\pi} = -r^{1/2}$.

We can make $f(z) = z^{1/2}$ single-valued by introducing a branch cut into the z plane as we did in Figure 4.26, and as shown again in Figure 18.1. The multivalued function $f(z) = z^{1/2}$ is separated into two single-valued functions by the *branch cut* along the positive x axis. When θ crosses the branch cut, we go from one (single-valued) *branch* of $f(z)$ to another. For one branch, $0 \le \theta < 2\pi$, and for the other, $2\pi \le \theta < 4\pi$. Note that $\theta < 2\pi$ on the first branch and that $\theta < 4\pi$ on the second branch. Crossing the branch cut as θ goes through 2π is like going from $w = +|z^{1/2}| = r^{1/2}$ to $w = -|z^{1/2}| = -r^{1/2}$. As θ goes through 4π, we simply go from the second branch back to the first branch.

We saw in Section 4.5 that $\ln z$ is a multivalued function (an infinity of single-valued functions in this case). The following Example reviews the multivalued nature of $\ln z$.

Example 1:
Determine the branches of the logarithmic function, $w = f(z) = \ln z$.

SOLUTION: Express z in polar form, $z = re^{i\theta}$. Then $\ln z$ can be written as

$$\ln z = \ln r + i\theta$$

But $e^{i\theta} = e^{i\theta_0 + 2\pi n i}$, where θ_0, which lies in the interval $[0, 2\pi)$, is called the principal argument of z, Arg z, and $n = 0, \pm 1, \pm 2, \ldots$. Thus,

$$\ln z = \ln r + i(\theta_0 + 2\pi n) \qquad n = 0, \pm 1, \pm 2, \ldots$$

and we see that $\ln z$ has an infinite number of branches. If we restrict θ to the interval $[0, 2\pi)$, then we write

$$\text{Ln}\, z = \ln r + i\theta_0 = \ln r + i \,\text{Arg}\, z \qquad 0 \leq \theta_0 < 2\pi$$

where Arg $z = \theta_0$ denotes the *principal argument* of z and Ln z is the *principal branch* of $\ln z$. Thus, we can also write

$$\ln z = \text{Ln}\, z + 2\pi n i \qquad n = 0, \pm 1, \pm 2, \ldots$$

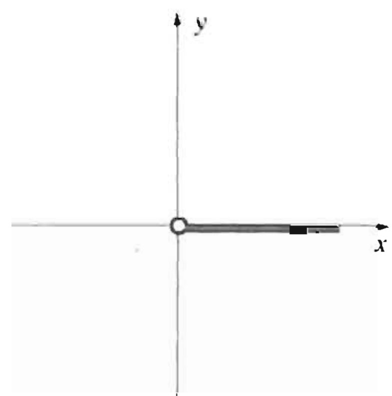

Figure 18.2
A branch cut along the positive x axis for $f(z) = \ln z$ and a branch point at the origin.

Figure 18.2 shows a branch cut along the positive x axis for $f(z) = \ln z$. As in the case of $f(z) = z^{1/2}$, the angle θ is equal to zero on the upper part of the branch cut, and θ increases in a counterclockwise direction until it reaches the lower part of the branch cut, where it approaches but never equals 2π. Each time θ crosses the branch cut, we move from one branch of $\ln z$ to another branch. Unlike for $f(z) = z^{1/2}$, however, there is an infinite number of branches for $f(z) = \ln z$ because we go from one branch (with n increasing by one unit) to another as θ increase. The origin in Figure 18.2 is also excluded because z can never equal zero. The origin in this case is called a *branch point*. With the idea of branches of functions at our disposal, we shall always consider a function to be single-valued.

We can define the limit of a function of a complex variable much as we defined the limit of a function of a real variable in Chapter 1. First, we must define a δ *neighborhood* in the complex plane. The neighborhood of a point z_0 is the set of all points such that $|z - z_0| < \delta$, where δ is a positive constant. A *deleted δ neighborhood* of z is the set of all points such that $0 < |z - z_0| < \delta$. In other words, it's a δ neighborhood of z_0, excluding the point z_0 itself. We now say that the limit of a function $f(z)$ defined in a δ neighborhood of z_0 is equal to l as z approaches z_0 if $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$. We write this as

$$\lim_{z \to z_0} f(z) = l \tag{2}$$

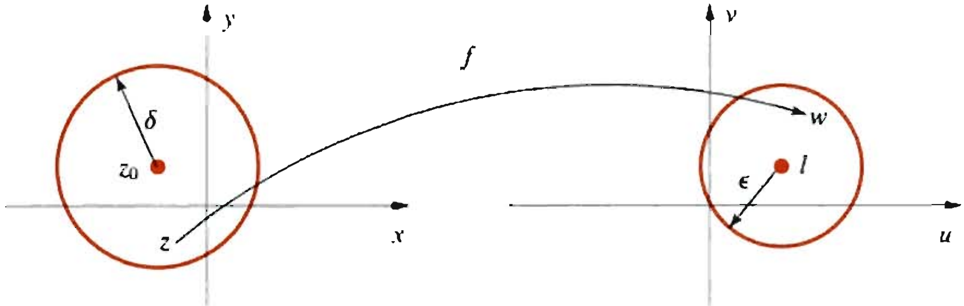

Figure 18.3
An illustration of the limit of a function $f(z)$ in the complex plane. (a) The point z lies within a δ neighborhood of z_0. (b) The point $w = f(z)$ lies within an ϵ neighborhood of l.

Figure 18.3 illustrates the idea of a limit in the complex plane. As the figure implies, the limit in Equation 2 must be independent of the direction in which z approaches z_0. Consider the limit of $f(z) = z^*/z$ as $z \to 0$. We can let $z \to 0$ along the x axis by setting y equal to zero and then letting $x \to 0$, in which case we get

$$\lim_{z\to 0} \frac{z^*}{z} = \lim_{x\to 0} \frac{x}{x} = 1$$

We can also take the limit along the y axis by setting x equal to zero first:

$$\lim_{z\to 0} \frac{z^*}{z} = \lim_{y\to 0} \frac{-iy}{iy} = -1$$

We obtain different values for the limit depending upon how we calculate it, so we say lim z^*/z as $z \to 0$ does not exist. (See also Problem 19.)

If we write $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, then $\lim\limits_{z\to z_0} f(z)$ exists if and only if $u(x, y)$ and $v(x, y)$ have limits as $(x, y) \to (x_0, y_0)$. In this case,

$$\lim_{z\to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x, y) + i \lim_{(x,y)\to(x_0,y_0)} v(x, y) \tag{3}$$

Once again, the two limits in Equation 3 must be independent of the direction in which (x, y) approaches (x_0, y_0).

Example 2:
Does the following limit exist?

$$\lim_{z\to 0} \left(\frac{1}{1+e^{1/x}} + iy^3 \right)$$

SOLUTION: Certainly the limit of $v(x, y) = y^3$ exists as $(x, y) \to (0, 0)$. What about the limit of $u(x, y) = 1/(1 + e^{1/x})$? Well,

$$\lim_{x \to 0+} e^{1/x} = \infty \qquad \text{and} \qquad \lim_{x \to 0-} e^{1/x} = 0$$

so $\lim_{x \to 0} u(x, y)$ depends upon the direction in which $x \to 0$. Therefore, the limit does not exist.

It's easy to show that if $f(z) \to a$ and $g(z) \to b$ as $z \to z_0$, then $f(z) \pm g(z) \to a \pm b$, $f(z)g(z) \to ab$, and $f(z)/g(z) \to a/b$ as $z \to z_0$, provided that $b \neq 0$.

Now that we have defined a limit in the complex plane, we can define continuity. Let $f(z)$ be defined in some δ neighborhood of z_0. Then $f(z)$ is continuous at z_0 provided that $f(z) \to f(z_0)$ as $z \to z_0$. This statement is equivalent to

1. $\lim_{z \to z_0} f(z) = l$ must exist.
2. $f(z_0)$ must be defined at z_0.
3. $f(z_0) = l$.

A function is continuous in a region R if it is continuous at each point of R.

As with limits, if $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, then $f(z)$ is continuous at z_0 if and only if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0). Furthermore, if $f(z)$ and $g(z)$ are continuous at z_0, then so is $f(z) \pm g(z)$, $f(z)g(z)$, and $f(z)/g(z)$, provided that $g(z_0) \neq 0$.

Example 3:
Show that e^{iz} is continuous everywhere.

SOLUTION: Write e^{iz} as

$$\begin{aligned} e^{iz} &= e^{i(x+iy)} = e^{-y}e^{ix} \\ &= e^{-y}(\cos x + i \sin x) \end{aligned}$$

Both $e^{-y} \cos x$ and $e^{-y} \sin x$ are continuous everywhere, and so is e^{iz}.

Example 4:
Determine the points where $f(z) = \cot z$ is discontinuous.

SOLUTION: First write

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

Example 3 shows that e^{iz} (and e^{-iz}) are continuous everywhere. Consequently $\cos z = (e^{iz} + e^{-iz})/2$ and $\sin z = (e^{iz} - e^{-iz})/2i$ are continuous everywhere. Therefore, $\cot z$ is continuous everywhere *except* where $\sin z = 0$, or at the points $0, \pm\pi, \pm 2\pi, \ldots$ (Problem 14).

18.1 Problems

The first nine problems constitute a review of some of the topics that we discussed in Chapter 4.

1. Show that $\left|\dfrac{z-i}{z+i}\right| < 1$ when $\operatorname{Im} z > 0$.
2. Express e^{iaz} in the form $u + iv$. Assume that a is real.
3. Describe the upper half of the disk $(x - x_0)^2 + (y - y_0)^2 \le R^2$ in terms of z.
4. Determine all the possible arguments of z_1, z_2, z_1z_2, and z_1/z_2 if $z_1 = 1 + i$ and $z_2 = 1 - i$.
5. Show that $\arg z_1z_2 = \arg z_1 + \arg z_2$, but that $\operatorname{Arg} z_1z_2 \neq \operatorname{Arg} z_1 + \operatorname{Arg} z_2$.
6. Show that $\sin iz = i \sinh z$.
7. Show that $\sin z = \sin x \cosh y + i \cos x \sinh y$.
8. Show that $\sin^2 z + \cos^2 z = 1$.
9. Determine the values of $\ln(1 + i)$.
10. Let $w = f(z) = z^3$. Determine the value of $f(2 + i)$.
11. Show that $f(z) = z^{1/2}$ is multivalued by determining $u(x, y)$ and $v(x, y)$ explicitly.
12. Determine $\displaystyle\lim_{z \to 1+2i} \left(x^2 + y^3 + i \sin \frac{\pi y}{2}\right)$.
13. Determine $\displaystyle\lim_{z \to \infty} \frac{z}{1 + iz}$. *Hint*: Let $z = 1/\eta$ and then let $\eta \to 0$.
14. Show that the zeros of $\sin z$ are all real and equal to $n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$.
15. Express $f(z) = z^2 + (z^*)^2$ in the form $w = u + iv$.
16. Is $\operatorname{Arg} z$ single-valued? Is $\arg z$?
17. Determine $\displaystyle\lim_{z \to i} \left(xe^{xy} + i\frac{e^{xy}}{x + 1}\right)$.
18. Where is the function $f(z) = \begin{cases} \dfrac{\operatorname{Re}(z)}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$ continuous?
19. Show that $\displaystyle\lim_{z \to 0} \frac{z^*}{z}$ does not exist by taking the limit along the ray $y = mx$, where m is a constant.
20. Show that $\displaystyle\lim_{z \to i\pi/2} z^4 \cosh \frac{z}{3} = \frac{\sqrt{3}\,\pi^4}{32}$.

18.2 Differentiation: The Cauchy-Riemann Equations

The definition of the derivative of a function of a complex variable is similar to the one we use for functions of a real variable. We define the derivative of $f(z)$ at some point z_0 by

$$f'(z_0) = \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right] \tag{1}$$

if the limit exists. Clearly, $f(z)$ must be defined in a neighborhood of $z = z_0$. For the limit in Equation 1 to exist, it must be independent of the manner in which z approaches z_0.

For example, if $f(z) = 1/z$, then

$$f'(z_0) = \lim_{z\to z_0}\frac{1/z - 1/z_0}{z-z_0} = \lim_{z\to z_0}\frac{z_0 - z}{zz_0(z-z_0)} = -\frac{1}{z_0^2}$$

Therefore, $f'(z)$ exists for all values of z except for $z = 0$.

Example 1:
For what values of $z = z_0$ does $f'(z)$ exist if $f(z) = 2x + 3iy$?

SOLUTION:

$$\begin{aligned} f'(z) &= \lim_{z\to z_0}\left[\frac{2(x-x_0)+3i(y-y_0)}{(x-x_0)+i(y-y_0)}\right] \\ &= \lim_{z\to z_0}\left[\frac{2(x-x_0)^2+3(y-y_0)^2+i(x-x_0)(y-y_0)}{(x-x_0)^2+(y-y_0)^2}\right] \end{aligned}$$

Let's look at the limit of the real part of the expression in brackets.

$$\lim_{(x,y)\to(x_0,y_0)}\left[\frac{2(x-x_0)^2+3(y-y_0)^2}{(x-x_0)^2+(y-y_0)^2}\right]$$

If we let $x \to x_0$ first, then the limit is equal to 3. If we let $y \to y_0$ first, then the limit is equal to 2. Therefore, the limit does not exist and $f(z) = 2x + 3iy$ is differentiable nowhere.

It turns out that $u(x, y)$ and $v(x, y)$ must satisfy certain relations for $f(z) = u(x, y) + iv(x, y)$ to be differentiable. These relations are known as the *Cauchy-Riemann equations*, and are easy to derive. Let's evaluate $f'(z_0)$ by letting $y \to y_0$ first and then letting $x \to x_0$. In this case

Points at which $f(z)$ is analytic are called *regular points*. Points at which $f(z)$ is not analytic are called *singular points*. Singular points are often, but not necessarily, points at which $f(z)$ becomes unbounded. For example, the point $z = 2$ is a singular point of $f(z) = 1/(z-2)$. Let z_0 be a singular point of $f(z)$. If there is a deleted δ neighborhood about z_0 containing no singularities, then z_0 is said to be an *isolated singular point*. For example, the point $z = 2$ is an isolated singular point of $f(z) = 1/(z-2)$.

The function $1/\sin(1/z)$ is a classic example of a function with a singular point that is not isolated. Note that $\sin(1/z) = 0$ when $(1/z) = n\pi$, with $n = 0, \pm1, \pm2, \ldots$ or when $z = 1/n\pi$. The point $z = 0$ is a singular point, but it is not isolated because no matter how small you take δ to be, there is an infinite number of singular points within any deleted δ neighborhood of $z = 0$. We shall deal only with isolated singular points.

If $\lim\limits_{z\to z_0}(z-z_0)^n f(z) = a \neq 0$ for some positive integer n, then z_0 is called a *pole of order n*. For example, the point $z = 1$ is a pole of order 2 for $f(z) = 1/(z-1)^2$. A pole of order 1, as is the point $z = 2$ for $f(z) = 1/(z-2)$, is called a *simple pole*.

Example 4:
Determine all the poles of

$$f(z) = \frac{2z+1}{z^4 - 2z^3 + 2z^2 - 2z + 1}$$

SOLUTION: Any CAS will give the four zeros of the denominator to be $z = i, -i, 1, 1$, and so we write $f(z)$ as

$$f(z) = \frac{2z+1}{(z+i)(z-i)(z-1)^2}$$

Thus, $f(z)$ has three poles: simple poles at i and $-i$ and a pole of order 2 at 1.

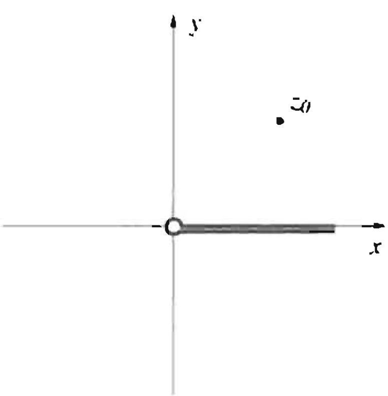

Figure 18.4
An illustration that $z_0 \neq 0$ is not a branch point of $\ln z$, but $z_0 = 0$ is.

Branch points are also considered to be singularities. We have seen several times that $\ln z$ has a branch point at $z = 0$. Branch points are associated with multivalued functions and are examples of non-isolated singularities. A branch point z_0 has the property that there is some small enough circle around z_0 such that $f(z)$ varies continuously with z but does not return to its original values when z goes around the circle. We can use this definition to show that $z = 0$ is the only branch point of $\ln z$. Figure 18.4 shows a point $z_0 \neq 0$ with a circle around it in the complex plane. As z travels around the circle, the path does not cross the branch cut. Therefore, both $|z|$ and θ return to their original values and $\ln z$ undergoes no change. For $z_0 = 0$, however, the path must cross the branch cut and so the imaginary part of $\ln z$ changes by 2π as z travels around the circle. Therefore, $z_0 = 0$ is a branch point, and the only branch point.

$$\oint_C f(z)\,dz = i\int_0^{2\pi} e^{i(n+1)\theta}d\theta$$

$$= \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

This turns out to be a very well used result later on and should be remembered. Note that we indicated the closed contour by writing $\oint_C f(z)\,dz$. We shall always evaluate contour integrals around closed paths in a counterclockwise direction unless we state otherwise.

Another useful method to evaluate contour integrals is to express $\int_C f(z)\,dz$ in terms of real line integrals by writing $f(z) = u(x, y) + iv(x, y)$ and $dz = dx + i\,dy$:

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \tag{6}$$

Example 3:

Evaluate $\int_C f(z)\,dz$ from $z = (0, 0)$ to $z = (2, 4)$, where $f(z) = x^2 - iy^2$ and C is the parabola $y = x^2$.

SOLUTION: Use Equation 6 to write

$$\int_C f(z)\,dz = \int_C (x^2dx + y^2dy) + i\int_C (-y^2dx + x^2dy)$$

$$= \int_0^2 (x^2dx + 2x^5dx) + i\int_0^2 (-x^4dx + 2x^3dx)$$

$$= 24 + \frac{8i}{5}$$

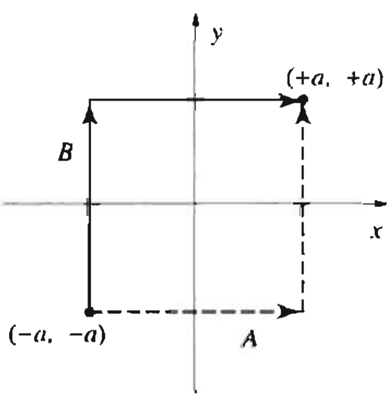

Figure 18.8
Two different paths along which to integrate $f(z) = \cos z$.

If we evaluate the integral in Example 3 along the straight line connecting the points (0, 0) and (2, 4) ($y = 2x$), then the value of the integral comes out to be $24 - 16i/3$ (Problem 13), so the value of the integral depends upon the path from the point (0, 0) to the point (2, 4). This isn't always the case, however. Let's look at the integral of $f(z) = \cos z$ along the two paths shown in Figure 18.8. We first write

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

$$= u(x, y) + iv(x, y)$$

We can use general formulas for $\int f(z)\,dz$ integrals so long as the antiderivative of $f(z)$ is analytic in the region containing the integration contour. This theorem also explains why the integral of $f(z) = 1/z$ depends upon the two paths in Figure 18.8. The antiderivative, $F(z) = \ln z$, is *not* single-valued and consequently not analytic, having a branch point at $z = 0$.

We can integrate $f(z) = 1/z$ if we keep track of which branch of $\ln z$ we are on at various values of z (or θ if the contour is a circle or an arc of a circle). Consider

$$I = \oint_C \frac{dz}{z}$$

where C is the unit circle. Let's integrate from $\theta = \theta_0$ $(0 \le \theta_0 < 2\pi)$ to $\theta = \theta_0 + 2\pi$, and write

$$I = \Big[\ln z\Big]_{\theta_0}^{\theta_0+2\pi}$$

Recall that we have defined $\ln z$ with a branch cut along the positive x axis (Figure 18.4), so that

$$\ln z = \ln r + (\theta + 2\pi n)i \qquad n = 0, 1, 2, \ldots$$

The value of $n = 0$ gives us the principal branch of $\ln z$, with $0 \le \theta < 2\pi$, but not equal to 2π. If $\theta = 2\pi$, we move to the second branch of $\ln z$, with $n = 1$, and $2\pi \le \theta < 4\pi$. Therefore,

$$I = \Big[\ln z\Big]_{\theta_0}^{\theta_0+2\pi} = \ln r + (\theta_0 + 2\pi)i - \ln r - \theta_0 i = 2\pi i$$

Note that this result is in agreement with the result that we obtained on page 886 for the path shown in Figure 18.8.

There is one other consequence of the Cauchy-Goursat theorem that we shall discuss before concluding this section. Consider the situation in Figure 18.15, where C_0 and C_1 are two simple closed curves, the point z_0 may or may not be a singularity of $f(z)$, and where there are no singularities of $f(z)$ in the region bounded by and including C_0 and C_1. Then, a direct consequence of the Cauchy-Goursat theorem is that

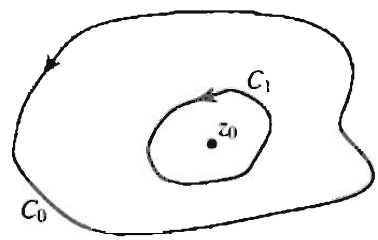

Figure 18.15
Two simple closed curves, C_0 and C_1, in the complex plane surrounding a point z_0.

$$\oint_{C_0} f(z)\,dz = \oint_{C_1} f(z)\,dz \tag{12}$$

where both integrals are taken in a counterclockwise direction, as indicated in Figure 18.15.

We're going to prove this result below not only because it is fairly straightforward, but also because the proof involves a procedure that we will use a number of times later on. Before doing that, however, we want to emphasize the utility of Equation 12. Suppose we are asked to evaluate

$$I = \oint_C \frac{dz}{z}$$

Now, given any ϵ, there is a δ such that

$$|f(z) - f(a)| < \epsilon \qquad \text{when} \quad |z - a| < \delta$$

Therefore, the integral on the right side of Equation 3 satisfies

$$\left|\oint_{C_\epsilon} \frac{[f(z) - f(a)]}{z - a}\,dz\right| \le \epsilon \left|\oint_{C_\epsilon} \frac{dz}{z - a}\right| \le 2\pi\epsilon$$

and Equation 3 becomes

$$\left|\oint_C \frac{f(z)\,dz}{z - a}\,dz - 2\pi i f(a)\right| \le 2\pi\epsilon \tag{4}$$

Because ϵ can be made arbitrarily small, Equation 4 leads to

$$\oint_C \frac{f(z)}{z - a}\,dz = 2\pi i f(a)$$

Equation 1 is an invaluable tool for evaluating contour integrals.

Example 1:
Evaluate $\oint_C \frac{e^z}{z-2}\,dz$ where C is the rectangle shown in Figure 18.23.

SOLUTION: Using Equation 1 with $f(z) = e^z$ and $a = 2$, we have

$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i e^2$$

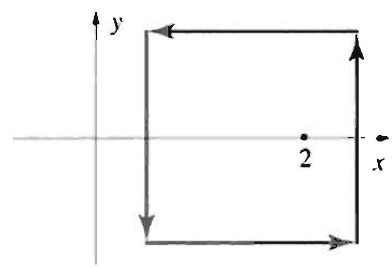

Figure 18.23
A rectangle surrounding the point $z = 2$.

Example 2:
Evaluate

$$I = \oint_C \frac{\cos \pi z}{z^2 + (1 - i)z - i}\,dz$$

for the two different contours shown in Figure 18.24.

SOLUTION: The denominator of the integrand factors into $(z - i)(z + 1)$, so there are simple poles at $z = i$ and $z = -1$. Contour 1 surrounds both poles, and we can deform the contour into two small circles surrounding each pole. For the pole at $z = i$, we have $f(z) = \cos \pi z/(z + 1)$ and $a = i$ in Equation 1, so we have

$$I(\text{at } z = i) = 2\pi i f(i) = \frac{2\pi i \cos i\pi}{1 + i} = \frac{2\pi i \cosh \pi}{1 + i}$$

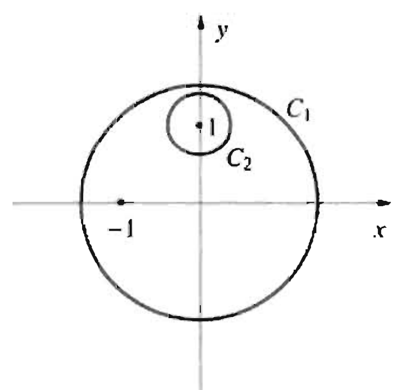

Figure 18.24
The two contours to be used to evaluate the integral in Example 2.

For the pole at $z=-1$, we have $f(z)=\cos\pi z/(z-i)$, so

$$I(\text{at } z=-1)=2\pi i f(-1)=\frac{2\pi i\cos(-\pi)}{-1-i}=\frac{2\pi i}{1+i}$$

Therefore,

$$\oint_{C_1}\frac{\cos\pi z}{z^2+(1-i)z-i}dz=\frac{2\pi i\cosh\pi}{1+i}+\frac{2\pi i}{1+i}=\frac{2\pi i}{1+i}(\cosh\pi+1)$$

Contour 2 surrounds only the pole at $z=i$, so $f(z)=\cos\pi z/(z+1)$ at $a=i$:

$$I=2\pi i f(i)=\frac{2\pi i\cosh\pi}{1+i}$$

Both of the above Examples have simple poles. What if we want to evaluate something like

$$I=\oint_C\frac{f(z)}{(z-a)^3}dz$$

where $f(z)$ satisfies the conditions of the Cauchy integral formula and C encloses the point $z=a$? We can't use Equation 1 because $f(z)/(z-a)^2$ may not be analytic at $z=a$. It so happens that we can use Equation 1 to derive a result that does allow us to evaluate integrals like the one above. Let's form the derivative of $f(z)$ at $z=a$ using $f(a)$ from the right side of Equation 1. First write

$$\begin{aligned}\frac{f(a+h)-f(a)}{h}&=\frac{1}{2\pi ih}\oint_C\left(\frac{1}{z-a-h}-\frac{1}{z-a}\right)f(z)\,dz\\&=\frac{1}{2\pi i}\oint_C\frac{f(z)\,dz}{(z-a-h)(z-a)}\end{aligned}$$

Using the identity

$$\frac{1}{x-h}=\frac{1}{x}\left(1+\frac{h}{x-h}\right)$$

we can let $x=z-a$ to write

$$\begin{aligned}\frac{f(a+h)-f(a)}{h}&=\frac{1}{2\pi i}\oint_C\frac{f(z)\,dz}{(z-a)^2}+\frac{h}{2\pi i}\oint_C\frac{f(z)\,dz}{(z-a)^2(z-a-h)}\\&=\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-a)^2}\,dz+\Delta I\end{aligned}\tag{5}$$

The last term here, ΔI, is proportional to h, so we may hope that this term vanishes as $h\to0$. If that's so, then Equation 5 becomes

$$f'(a)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-a)^2}\,dz\tag{6}$$

$$I_1 = 2\pi i \left[\frac{d}{dz}\left(\frac{\cos z}{(z-i)^3}\right)\right]_{z=0} = -6\pi i$$

For C_2, take $f(z) = \cos z/z^2$, $a = i$, and $n = 2$:

$$I_2 = \pi i \left[\frac{d^2}{dz^2}\frac{\cos z}{z^2}\right]_{z=i} = \pi i\,[7\cos i + 4i \sin i\,] = \pi i(7\cosh 1 + 4\sinh 1)$$

The total integral is

$$I = I_1 + I_2 = -6\pi i + \pi i(7\cosh 1 + 4\sinh 1)$$

There are a number of important consequences of the Cauchy integral formula. Many of these are what you might call theoretical, and are discussed in texts on complex variables, but some are directly applicable to physical problems. Two that we shall use in Section 19.5 are Poisson's integral formula for a half plane and Poisson's integral formula for a circle. We derived these formulas in Sections 17.6 and 16.2, but they can be derived directly from the Cauchy integral formula (Problems 15 through 20).

18.4 Problems

1. Show that $f'(0) = 0$ but that $f''(0)$ does not exist for $f(x) = x^3 \sin(1/x)$.

2. Evaluate $I = \oint_C \frac{e^z - 1}{z - a}dz$ where C is given by $|z| = 1$ for all values of a.

3. Evaluate $I = \oint_C \frac{\tan z}{4z - \pi}dz$ where C is given by (a) $|2z| = 1$ and (b) $|z| = 1$.

4. Evaluate $I = \oint_C \frac{z^2}{z^2 - 4}dz$ where C is given by $|z| = 1$.

5. Evaluate $I = \oint_C \frac{e^z}{z - i\pi}dz$ where C is given by (a) $|z - 1| = 4$ and (b) $|z| = 3$.

6. Evaluate $I = \oint_C \frac{\sin z}{z^2 - \pi^2/4}dz$ where C is given by (a) $|z - 1| = 2$ and (b) $|z - 1| = 3$.

7. Evaluate $I = \oint_C \frac{e^{iz}}{(z^2 + 1)}dz$ where C is given by $|z| = 2$ by (a) using partial fractions and (b) deformation of contour.

8. Evaluate $I = \oint_C \frac{\sin^4 z}{(z - \pi/3)^3}dz$ where C is given by $|z| = 2$.

9. Evaluate $I = \oint_C \frac{\sin z}{z^{n+1}}dz$ where $n \geq 0$ and C is given by $|z| = 2$.

Show that this will be so if $\zeta_2 = \zeta_1^*$, and then show that this leads to

$$f(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi, 0)}{(\xi - x)^2 + y^2} d\xi$$

Now let $f(z) = f(x, y) = u(x, y) + iv(x, y)$ and equate real parts to obtain

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u(\xi, 0) d\xi}{(\xi - x)^2 + y^2}$$

which is Poisson's integral formula for a half plane.

16. Use the result of the previous problem to determine the electrostatic potential in the upper half plane subject to the Dirichlet boundary conditions $u = u_1$ for $x < x_0$ and $u = u_2$ for $x > x_0$.

17. Plot your result in the previous problem for $x_0 = 0$, $u_1 = 100$, $u_2 = 0$, and $y = 1$. Realize that the arctangents in the solution must lie between 0 and π because of the geometry of the problem. *Hint*: The formula $\tan^{-1}(-x) = -\tan^{-1} x$ is not valid in this case because $0 \le \tan^{-1}(y/x) \le \pi$.

18. Use the result of Problem 15 to solve the Dirichlet problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \begin{matrix} -\infty < x < \infty \\ 0 < y \end{matrix} \qquad \text{if} \lim_{y \to 0+} u(x, y) = \begin{cases} u_0 & x < -1 \\ u_1 & -1 < x < 1 \\ u_2 & x > 1 \end{cases}.$$

19. Plot your result in the previous problem for $u_0 = 1$, $u_1 = 10$, $u_2 = 5$, and $y = 1$. Realize that the arctangents in the solution must lie between 0 and π because of the geometry of the problem. *Hint*: The formula $\tan^{-1}(-x) = -\tan^{-1} x$ is not valid in this case because $0 \le \tan^{-1}(y/x) \le \pi$.

20. Derive Poisson's integral formula for a circle from Equation 1. *Hint*: Use the same trick that we used to derive Poisson's integral for a half plane in Problem 15. Take the point outside the circle to be the point R^2/ζ^*, which is the reflection of the point ζ through the origin (*inverse* of ζ).

21. In this problem, we derive *Cauchy's inequality*. Show that if $f(z)$ is analytic inside and on a circle C of radius R centered at $z = a$, then $|f^{(n)}(a)| \le \dfrac{n!M}{R^n}$ $\quad n = 0, 1, 2, \ldots$, where $|f(z)| \le M$.

22. Show that if $f(z)$ is analytic inside and on a circle C with center at $z = a$, then $f(a) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f(a + re^{i\theta}) d\theta$. Interpret this result.

23. Recall that an entire function is a function that is analytic throughout the (finite) complex plane. For example, e^z, $\cos z$, $\sin z$, and any polynomial in z are entire functions. Prove *Liouville's theorem*, which says that any bounded entire function must be a constant. *Hint*: Start with Cauchy's inequality with $n = 1$, Problem 21.

24. Evaluate $\displaystyle\oint_C \frac{dz}{2 + z^*}$ where C is given by (a) $|z| = 1$, and (b) $|z| = 3$.

25. We'll prove that ΔI in Equation 5 vanishes as $h \to 0$ in this problem. First deform C to C_ρ, a circle of radius ρ centered at a, and lying entirely within C. The point $a + h$ will lie within C_ρ if we choose $\rho > 2|h|$, and since we anticipate that $h \to 0$, the circle C_ρ will lie entirely with C. Now use the triangle inequality, $|z_1 - z_2| \ge |z_1| - |z_2|$, to show that the magnitude of the denominator of ΔI satisfies $|(z - a)^2(z - a - h)| \ge \rho^2(\rho - |h|) \ge \rho^3/2$. Now let $M \ge |f(z)|$ and show that $|\Delta I| \le \dfrac{|h|M}{\pi\rho^3} \cdot 2\pi\rho = \dfrac{2M|h|}{\rho^2}$, which goes to zero as $h \to 0$.

18.5 Taylor Series and Laurent Series

We spent all of Chapter 2 discussing infinite series of real quantities. Many of the results of Chapter 2 carry over to infinite series in the complex plane with little change. For example, if

$$S_n(z) = \sum_{j=0}^{n} f_j(z)$$

then the series converges to $S(z)$ if there exists a value of N such that

$$|S(z) - S_n(z)| < \epsilon \qquad \text{when} \quad n > N(\epsilon, z)$$

where we emphasize that N depends upon the value of ϵ and possibly z. If N does not depend upon the value of z, then the series converges *uniformly* to $S(z)$. We can use the ratio test for convergence. The *ratio test* says that

> *The series* $\sum_{n=0}^{\infty} f_n(z)$ *converges if* $\lim_{n\to\infty} \left|\frac{f_{n+1}}{f_n}\right| = L < 1$ *and diverges if* $L > 1$. *The test is inconclusive if* $L = 1$.

Example 1:
Use the ratio test to determine for what values of z the geometric series $S_N = \sum_{n=0}^{\infty} z^n$ converges.

SOLUTION:

$$\lim_{n\to\infty} \left|\frac{z^{n+1}}{z^n}\right| = |z|$$

and so $S_N(z)$ converges (absolutely) for $|z| < 1$ and diverges if $|z| > 1$. You can see that it diverges for $|z| = 1$ because the nth term does not go to zero as $n \to \infty$.

Recall that the geometric series can be summed into closed form by multiplying S_n by z and then subtracting S_n to get

$$\begin{aligned} zS_n - S_n &= z + z^2 + \cdots + z^{n+1} - (1 + z + z^2 + \cdots + z^n) \\ &= z^{n+1} - 1 \end{aligned}$$

Solving for S_n gives

$$S_n(z) = \frac{1 - z^{n+1}}{1 - z} \tag{1}$$

For $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$ and we obtain

$$S(z) = \frac{1}{1-z} = \sum_{j=0}^{\infty} z^j \qquad |z| < 1 \tag{2}$$

Equation 2 is a power series about the point $z = 0$. Power series play a central role in complex variable theory. We have the following theorem for power series in a complex variable:

Consider the power series about the point $z = a$:

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \tag{3}$$

where the series converges for $|z - a| < R$ and diverges for $|z - a| > R$. The circle $|z - a| = R$ (on which the series may or may not converge) is called the circle of convergence. *The radius of this circle is given by*

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| \tag{4}$$

If $R = 0$, then the series converges only at the point $z = a$. If $R = \infty$, then the series converges in the entire complex plane.

Example 2:
Derive Equation 4.

SOLUTION: Simply apply the ratio test to Equation 4:

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(z-a)^{n+1}}{c_n(z-a)^n} \right| = |z-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$
$$= \frac{|z-a|}{R}$$

The series converges if $|z - a| < R$ and diverges if $|z - a| > R$.

We learned in Chapter 2 that power series converge uniformly and that uniformly convergent series are special in the sense that they can be manipulated much like polynomials. This is also true for power series of complex variables.

Let $|z - a| = R$ be the circle of convergence of the power series in Equation 3. Then this series converges uniformly *and* absolutely *within and on any circle interior to the circle of convergence. Not only do power series converge uniformly and absolutely, but also the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ describes an* analytic function *within the circle of convergence.*

The above properties of power series explain why power series play such an important role in complex variable theory.

The Weierstrass M-test for series of a real variable carries over essentially unchanged for a series in a complex variable. Consider the series $\sum_{n=0}^{\infty} f_n(z)$. The Weierstrass M-test says that if there is a set of positive constants $\{M_n\}$ such that $|f_n(z)| \leq M_n$ for all n and for all z in some region R, and if the series $\sum_{n=0}^{\infty} M_n$ converges, then the series $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly.

Example 3:

Show that the series $\sum_{n=1}^{\infty} 1/(n^2 + z)$ converges uniformly for $\mathrm{Re}\, z \geq 0$.

SOLUTION:

$$\left|\frac{1}{n^2 + z}\right| = \left[\frac{1}{(n^2 + x)^2 + y^2}\right]^{1/2} \leq \frac{1}{n^2}$$

for $\mathrm{Re}\, z = x \geq 0$. The series $1/n^2$ converges, and so the series $\sum_{n=1}^{\infty} 1/(n^2 + z)$ converges uniformly for $\mathrm{Re}\, z \geq 0$.

We'll now prove that every analytic function can be represented by a power series. Start with the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{\zeta - z} \tag{5}$$

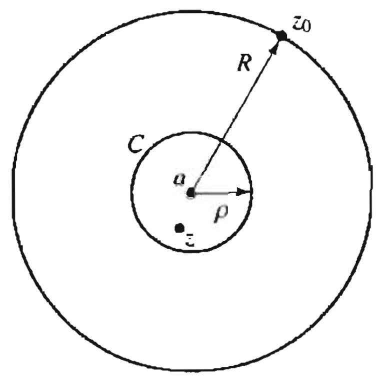

Figure 18.27
The geometry used to derive Equation 9 from Equation 5.

and the geometry shown in Figure 18.27. The quantity R in Figure 18.27 is the radius of the largest circle about the point $z = a$ within which $f(z)$ is analytic. In other words, R is the distance from a to z_0, the singularity of $f(z)$ nearest a. The circle C in Equation 5 is a circle centered at a whose radius ρ is greater than $|z - a|$ (but less than R), so that z lies within C. Write $1/(\zeta - z)$ as

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\left[\frac{1}{1 - (z - a)/(\zeta - a)}\right]$$

where $(z - a)/(\zeta - a) < 1$ because ζ lies on C and z is within C. Use the algebraic identity

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1 - x} \qquad (x \neq 1) \tag{6}$$

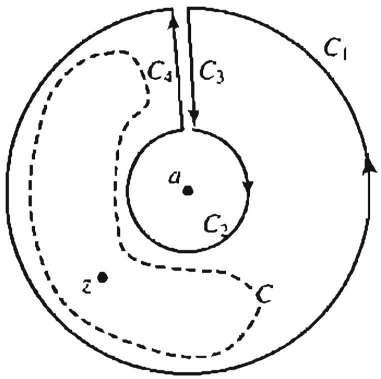

Figure 18.29
The geometry used to derive Equations 12 through 14. The point z lies in the region between C_1 and C_2.

where C is any simple, closed, piecewise smooth curve lying in the annular region between C_1 and C_2. (See Figure 18.29.)

Referring to Figure 18.29, we see that $f(z)$ is analytic within the region bounded by the closed path $C_1 + C_2 + C_3 + C_4$, and so we can use the Cauchy integral formula to write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_1+C_2+C_3+C_4} \frac{f(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}d\zeta = I_{C_1} - I_{C_2} \end{aligned} \tag{15}$$

Both C_1 and C_2 are taken in the counterclockwise direction, and we have recognized that the contributions along C_3 and C_4 cancel. In the integral I_{C_1} on the right side of Equation 15, we take

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\frac{1}{1 - \dfrac{z-a}{\zeta - a}} = \frac{1}{\zeta - a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta - a}\right)^n \tag{16}$$

with $(z-a)/(\zeta - a) < 1$ on C_1, because z lies in the region between C_1 and C_2 and ζ lies on C_1 in Figure 18.29. Substituting this expansion into I_{C_1} leads to (Problem 16)

$$I_{C_1} = \sum_{n=0}^{\infty}\left\{\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{(\zeta - a)^{n+1}}d\zeta\right\}(z-a)^n = \sum_{n=0}^{\infty} a_n(z-a)^n \tag{17}$$

where the a_n are the same as in Equation 13.

For the integral I_{C_2} on the right side of Equation 15, we take

$$\frac{1}{\zeta - z} = -\frac{1}{z-a}\left(\frac{1}{1 - \dfrac{\zeta - a}{z-a}}\right) = -\frac{1}{z-a}\sum_{n=0}^{\infty}\left(\frac{\zeta - a}{z-a}\right)^n \tag{18}$$

with $(\zeta - a)/(z-a) < 1$ on C_2 because ζ lies in C_2 in Figure 18.29. Substituting this expansion into I_{C_2} leads to (Problem 27)

$$\begin{aligned} I_{C_2} &= -\sum_{n=0}^{\infty}\left\{\frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{(\zeta - a)^{-n}}d\zeta\right\}(z-a)^{-n-1} \\ &= -\sum_{n=1}^{\infty}\left\{\frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{(\zeta - a)^{-n+1}}d\zeta\right\}(z-a)^{-n} \\ &= -\sum_{n=1}^{\infty} b_n(z-a)^{-n} \end{aligned} \tag{19}$$

where the b_n are the same as in Equation 14. Substituting Equations 18 and 19 back into Equation 15 gives Equation 12 with the coefficients given by Equations 13 and 14. Because $(z-a)/(\zeta-a) < 1$ on C_1, we see that the first series in Equation 12 converges for $|z| < C_1$. Similarly, because $(\zeta - a)/(z-a) < 1$ on C_2, we see that the second series in Equation 12 converges for $|z| > C_2$. The common region of convergence of Equation 12, then, is $C_1 < |z| < C_2$.

Note that the Laurent series assumes that $f(z)$ is analytic only in the annulus determined by C_1 and C_2. If $f(z)$ happens to be analytic within C_1, then

$$\begin{aligned} b_n &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta-a)^{1-n}}d\zeta \\ &= \frac{1}{2\pi i}\oint_{C_1} (\zeta-a)^{n-1}f(\zeta)d\zeta = 0 \qquad \text{for} \quad n \geq 1 \end{aligned}$$

according to Cauchy's integral formula because $(\zeta-a)^{n-1}f(\zeta)$ is analytic within C_1. Thus, all the coefficients of the negative powers of $z-a$ in Equation 12 vanish and we have a Taylor series. We can consider a Taylor series to be a special case of a Laurent series.

Let's consider the Laurent series for

$$f(z) = \frac{1}{(1-z)(2-z)} \tag{20}$$

There are singularities at $z = 1$ and $z = 2$. Therefore, we have three regions ($|z| < 1$, $1 < |z| < 2$, and $2 < |z|$) to consider. For $|z| < 1$, we can just expand both $1/(1-z)$ and $1/(2-z)$ in geometric series and obtain

$$\begin{aligned} f(z) &= \frac{1}{2}(1+z+z^2+\cdots)\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right) \\ &= \frac{1}{2}+\frac{3z}{4}+\frac{7z^2}{8}+\cdots \qquad |z| < 1 \end{aligned} \tag{21}$$

This is just a Taylor series because there are no singularities within $|z| = 1$.

Now let's look at the annular region, $1 < |z| < 2$. We expand the first factor in $f(z)$ as

$$\frac{1}{1-z} = -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

which converges for $1 < |z|$. We expand the second factor in $f(z)$ as

$$\frac{1}{2-z} = \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

which converges for $|z| < 2$. The product of these two expansions is (Problem 17)

The series for e^{z-1} converges for all values of $|z-1|$, so the Laurent series converges for $0 < |z-1| < \infty$.

We'll see in the next section that it turns out in practice that the most important term in a Laurent expansion is the term $b_1/(z-a)$.

Example 6:
Find the coefficient of the first few terms of ascending powers of z of

$$f(z) = \frac{1}{e^z - 1}$$

about $z = 0$, which is valid in the annulus $0 < |z| < 2\pi$ (Problem 19).

SOLUTION:

$$\begin{aligned}
\frac{1}{e^z - 1} &= \frac{1}{z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots} \\
&= \frac{1}{z}\frac{1}{1 + \frac{z}{2} + \frac{z^2}{6} + O(z^3)} \\
&= \frac{1}{z}\left\{1 - \frac{z}{2} + \frac{z^2}{6} + O(z^3) + \left[\frac{z}{2} + \frac{z^2}{6} + O(z^3)\right]^2 + O(z^3)\right\} \\
&= \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + O(z^3)
\end{aligned}$$

18.5 Problems

1. The series $\sum_{n=0}^{\infty} e^{nz}$ converges for which values of z?

2. Sum the series $\sum_{n=0}^{\infty} \left(\frac{1+i}{4}\right)^n$.

3. Sum the series $\sum_{n=1}^{\infty} \left(\frac{1-i}{3}\right)^n$.

4. Sum the series $\sum_{n=0}^{\infty} nz^n$.

5. For which values of z does $\sum_{n=0}^{\infty} e^{n/z}$ converge? Sum the series.

6. Determine the radii of convergence of the following power series:

   (a) $\sum_{n=0}^{\infty} \frac{z^{2n}}{a^n}$ (b) $\sum_{n=1}^{\infty} \frac{(z-i)^n}{n^2}$ (c) $\sum_{n=0}^{\infty} (n+2^n)z^n$ (d) $\sum_{n=1}^{\infty} \frac{n!z^n}{n^n}$

7. For which values of z does the series $\sum_{n=0}^{\infty} \frac{1}{n^2+z}$ converge uniformly?

8. Obtain a Taylor series expansion of e^z about the point $z = 2$.

9. Obtain a Taylor series expansion of $\cosh z$ about the point $z = 0$.

10. Obtain a Taylor series expansion of $\cosh z^2$ about the point $z = 0$. (See the previous problem.)

11. Obtain a Taylor series expansion of $\mathrm{Ln}\,(1+z)$ about the point $z = 0$.

12. Obtain a Taylor series expansion of $\mathrm{Ln}\,z$ about the point $z = 1$.

13. Obtain a Taylor series expansion of $I(z) = \int_0^z e^{-zw}dw$ about the point $z = 0$.

14. The answer to the previous problem is $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(n+1)!}$. Can you express this in closed form? Evaluate the integral in the previous problem and compare your answer.

15. Show that the remainder in Equation 9 goes to zero as $n \to \infty$.

16. Verify Equation 17.

17. Show that the product of $-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$ and $\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$ is $\cdots-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{z}-1-\frac{z}{2}-\frac{z^2}{4}-\cdots$. (Be sure to keep track of the various powers of z.)

18. Verify Equation 23.

19. Why is the Laurent series of $1/(e^z-1)$ given in Example 6 valid in the annulus $0 < |z| < 2\pi$.

20. How many Laurent series about $z = 0$ are there for $f(z) = 1/(1-z)$? For which values of $|z|$ are these series valid? Find these series.

21. Find the Laurent series of $f(z) = 1/(1+z^2)$ for $|z| > 1$.

22. Find the Laurent series of $f(z) = e^z/(1+z)$ about the point $z = -1$.

23. Find all the terms involving negative powers of $z-1$ and the first few terms with positive powers for the Laurent series of $f(z) = e^z/(z^2-1)$ about the point $z = 1$.

24. Find the Laurent series for $f(z) = z^3\sin(1/z)$ about the point $z = 0$.

25. Find the first few terms with positive and negative powers of z (if any) for all the Laurent series of $f(z) = 1/(z+1)(z+2)$ about the point $z = 0$.

26. When we derived the Laurent series for $f(z)$ in the text, we substituted the series given by Equation 16 into I_{C_1} in Equation 15 and integrated term by term. This is a valid procedure if the series in Equation 16 is uniformly convergent. Show that the series is uniformly convergent in ζ.

27. This problem is related to the previous problem. Show that the series in Equation 18 is uniformly convergent in ζ.

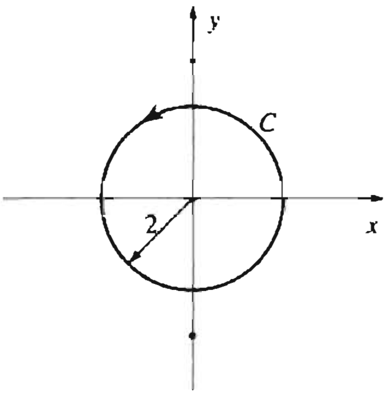

Figure 18.30
The contour used in Example 1 and the singular points of the integrand.

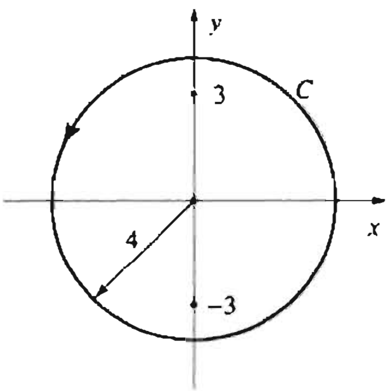

Figure 18.31
The contour described by $|z| = 4$ and the singular points of the integrand of the integral in Example 1.

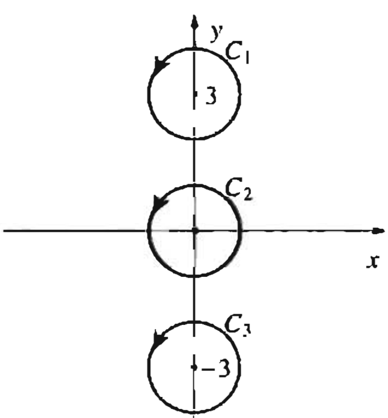

Figure 18.32
The deformation of the contour in Figure 18.31.

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \frac{1}{24z^4} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \frac{1}{24z^2} + \cdots$$

so we see that the residue of $f(z)$ at $z = 0$ is $1/6$. Therefore, Equation 2 tells us that

$$\oint_C z^2 e^{1/z} dz = \frac{2\pi i}{6} = \frac{\pi i}{3}$$

Example 1:
Use the residue theorem to evaluate

$$I = \oint_C \frac{\cos z}{z^3(z^2+9)} dz$$

where C is the circle given by $|z| = 2$ (Figure 18.30).

SOLUTION: The singularities of the integrand occur at $z = 0$ and $z = \pm 3i$, so C encloses only the singularity at $z = 0$. The Laurent series for $f(z) = \cos z/z^3(z^2+9)$ in the annulus $0 < |z| < 3$ is

$$\frac{\cos z}{z^3(z^2+9)} = \frac{1}{9z^3}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\left(1 - \frac{z^2}{9} + \frac{z^4}{81} - \cdots\right)$$

$$= \frac{1}{9z^3} - \frac{11}{162z} + O(z)$$

The residue of $f(z)$ at $z = 0$ is $-11/162$, and the residue theorem tells us that

$$\oint_C \frac{\cos z}{z^3(z^2+9)} dz = -\frac{11\pi i}{81}$$

What if the contour in Example 1 enclosed all three singular points of the integrand? Let's suppose that C is the circle given by $|z| = 4$, as shown in Figure 18.31. Because there are only three isolated singularities, we can deform C as shown in Figure 18.32 so that

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz$$

and then apply the residue theorem to each integral separately. This leads to the general result that if $f(z)$ has a number of isolated singularities within C, then

$$\oint_C f(z)\,dz = 2\pi i \text{ [sum of the residues of } f(z) \text{ inside } C \text{]} \tag{4}$$

The two most important types of singularities for our purposes are best discussed in terms of Laurent series. Consider the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{N} \frac{b_n}{(z-a)^n} \tag{5}$$

where the expansion in negative powers terminates at the Nth term (assuming that $b_N \neq 0$). In this case, the singularity of $f(z)$ at $z = a$ is said to be a *pole of order* N. For example,

$$f(z) = \frac{e^z}{z^2} = \frac{1 + z + \dfrac{z^2}{2} + \cdots}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \cdots$$

has a pole of order 2 at $z = 0$. If $N = 1$, the pole is called a *simple pole*. The function $f(z) = e^z/z$ has a simple pole at the origin. If $N = \infty$ (in other words, if the negative powers of the Laurent series continue indefinitely), the singularity is called an *essential singularity*. The function $e^{1/z}$ has an essential singularity at $z = 0$.

We can determine the order of a pole of $f(z)$ fairly easily by using the fact that a is a pole of order N if and only if

$$f(z) = \frac{g(z)}{(z-a)^N} \qquad g(a) \neq 0 \tag{6}$$

where $g(z)$ is analytic at $z = a$. To prove this, start with Equation 6 and expand $g(z)$ in a Taylor series about $z - a$ and get

$$g(z) = \frac{g(a)}{(z-a)^N} + \frac{g'(a)}{(z-a)^{N-1}} + \frac{g''(a)/2!}{(z-a)^{N-2}} + \cdots$$

which is essentially the definition of a pole of order N, according to Equation 6.

Example 3:
Classify the singular points of

$$f(z) = \frac{1}{(z-1)^3(z+2i)}$$

SOLUTION: There are singular points at $z = 1$ and at $z = -2i$. Use Equation 6 to write $f(z)$ as

$$f(z) = \frac{g(z)}{(z-1)^3}$$

where $g(z) = 1/(z+2i)$. Because $g(z)$ is analytic at $z = 1$, the pole at $z = 1$

is a pole of order 3. Similarly, for the pole at $z = -2i$, we write

$$f(z) = \frac{g(z)}{z + 2i}$$

where now $g(z) = 1/(z-1)^3$. Because $g(z)$ is analytic at $z = -2i$, the pole at $z = -2i$ is a simple pole.

Example 4:
Classify the singular point at $z = 0$ of

$$f(z) = \frac{e^z}{1 - \cosh z}$$

SOLUTION: It is not convenient to use Equation 6 here. Using the Taylor series of $\cosh z$,

$$\cosh z = 1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)$$

we see that the denominator can be written as

$$-\frac{z^2}{2} + O(z^4) = -\frac{z^2}{2}[1 + O(z^2)]$$

Therefore,

$$f(z) = \frac{1 + z + O(z^2)}{-\frac{z^2}{2}[1 + O(z^2)]} = -\frac{2}{z^2} - \frac{2}{z} + \cdots$$

and so we see that the point $z = 0$ is a pole of order 2.

With our discussion of singular points behind us, we're ready to show how to find the residue of $f(z)$ at a pole of order N. Suppose $f(z)$ has a simple pole at $z = a$. Then, according to Equation 5,

$$f(z) = \frac{b_1}{z - a} + a_0 + a_1(z - a) + \cdots$$

If we multiply both sides by $z - a$ and then let $z \to a$, we have

$$b_1 = \lim_{z \to a} (z - a) f(z) \qquad \text{(simple pole)} \tag{7}$$

Suppose now that $f(z)$ has a pole of order N, as in Equation 5. If we multiply by $(z - a)^N$, we have

$$(z - a)^N f(z) = b_N + b_{N-1}(z - a) + \cdots + b_1(z - a)^{N-1} + a_0(z - a)^N + \cdots \tag{8}$$

This equation represents the Taylor series expansion of $(z-a)^N f(z)$. If we let $G(z) = (z-a)^N f(z)$, then Equation 8 can be written as

$$G(z) = G(a) + G'(a)(z-a) + \cdots + \frac{1}{(N-1)!}\left(\frac{d^{N-1}G}{dz^{N-1}}\right)_{z=a}(z-a)^{N-1} + \cdots \tag{9}$$

By comparing Equation 9 with Equation 8, we see that

$$b_1 = \frac{1}{(N-1)!}\lim_{z\to a}\left[\frac{d^{N-1}(z-a)^N f(z)}{dz^{N-1}}\right] \tag{10}$$

Note that Equation 10 reduces to Equation 7 when $N = 1$.

The following two Examples illustrate the use of Equations 7 and 10.

Example 5:
Find all the residues of $f(z)$ given in Example 3.

SOLUTION: The point $z = 1$ is a pole of order 3. In this case,

$$\text{Res}\,(z=1) = \frac{1}{2!}\lim_{z\to 1}\left[\frac{d^2}{dz^2}\frac{1}{(z+2i)}\right] = \frac{1}{(1+2i)^3} = -\frac{11}{125} + \frac{2i}{125}$$

The point $z = -2i$ is a simple pole, so

$$\text{Res}\,(z=-2i) = \lim_{z\to -2i}\frac{1}{(z-1)^3} = -\frac{1}{(1+2i)^3} = \frac{11}{125} - \frac{2i}{125}$$

Example 6:
Find the residue at $z = 0$ of $f(z)$ given in Example 4.

SOLUTION: The point $z = 0$ is a pole of order 2. Using Equation 10,

$$\text{Res}\,(z=0) = \lim_{z\to 0}\left[\frac{d}{dz}\frac{z^2 e^z}{(1-\cosh z)}\right]_{z=0} = -2$$

In this case, it is just as easy to determine the residue from the final result in Example 4:

$$f(z) = \frac{1 + z + O(z^2)}{-\frac{z^2}{2}[1 + O(z^2)]} = -\frac{2}{z^2} - \frac{2}{z} + O(1)$$

28. Evaluate $\oint_C \frac{dz}{\sinh z}$, where C is the ellipse described by $4x^2 + y^2 = 16$. (See Problem 18.)

29. Evaluate $\oint_C \frac{z^3 dz}{(z+2)^2(z^2-4)} dz$, where C is the square with vertices $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$. (See Problem 19.)

30. Use any CAS to determine the residues in Problems 1 through 10.

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CHAPTER 19

Functions of a Complex Variable: Applications

This chapter continues our study of functions of a complex variable. Complex variable theory, or analytic function theory as it is sometimes called, has numerous applications in the physical sciences and engineering. We saw in Section 17.7 that the inversion formula for Laplace transforms involves integration in the complex plane, and we shall learn how to use the residue theorem to invert Laplace transforms analytically in Section 1. The inversion formula for Laplace transforms is expressed in terms of a contour integral, so it's not surprising that we can use the residue theorem to evaluate the integrals. What might be surprising, however, is that we can use the residue theorem to evaluate *real* definite integrals such as

$$I = \int_0^\infty \frac{dx}{1+x^4}$$

and

$$I = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

In fact, we shall see in Section 2 that the residue theorem can be used to evaluate a great variety of real definite integrals.

Not only can the residue theorem be used to evaluate real definite integrals, but it can also be used to derive closed expressions for summations such as

$$S = \sum_{n=0}^{\infty} \frac{1}{n^2+a^2}$$

and

$$S = \sum_{n=-\infty}^{\infty} \frac{(-1)^n n^2}{(n+a)^4}$$

We'll develop a general method for evaluating such summations in Section 3. Complex variable theory can also be used to determine the number of roots there are to an equation in a given region of the complex plane. For example, in Section 4, we'll learn how to determine that there are two solutions to the equation $e^z = 3z^2$ that lie within the unit circle. We'll do this using a theorem that says that if $f(z)$

is analytic inside and on a closed curve C except for possibly a finite number of poles and if $f(z) \neq 0$ on C, then

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz = Z - P$$

where Z is the total number of zeros and P is the total number of poles within C. A practical application of this theorem and its related material in Section 4 is the determination of the stability of mechanical and electrical systems. Basically, this theory rests upon a determination of the number of zeros of a polynomial equation such as $Q(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ that lie in the right half of the complex plane. We shall develop a rather simple and convenient graphical procedure called the Nyquist stability criterion for doing this.

The last two sections deal with solving Laplace's equation in two dimensions by means of a procedure called conformal mapping. We briefly discussed the transformation or mapping of curves from the z plane to the w plane described by analytic functions $w = f(z)$ in Chapter 4. We've learned that the two functions $u(x, y)$ and $v(x, y)$ in the relation $w = f(z) = u(x, y) + iv(x, y)$ are harmonic; in other words, they each satisfy Laplace's equation in two dimensions. We can utilize this property of $u(x, y)$ and $v(x, y)$ in the following way. Suppose that we want to solve Laplace's equation in a somewhat complicated region, such as the region between two parallel non-coaxial cylinders. If we can find a mapping $w = f(z)$ that transforms the region in question (in the z-plane) into the region between two parallel coaxial cylinders (in the w-plane), then we can solve Laplace's equation in the w-plane, where the geometry is simple, and then transform the resulting solution back into the z-plane to produce the solution to the original problem. The procedure is reminiscent of integral transform methods, where we transform a given problem into a simpler problem in transform space, solve it and then find the inverse transformation to obtain the solution to the original problem.

19.1 The Inversion Formula for Laplace Transforms

In the final section of Chapter 17, we presented a formula for the inverse of a Laplace transform. We showed that if

$$\hat{F}(s) = \int_0^{\infty} e^{-st} f(t)\, dt \tag{1}$$

then

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\hat{F}(s)\, ds \tag{2}$$

where c lies to the right of all the singular points of $\hat{F}(s)$. At the time, we didn't know how to handle the integral in Equation 2, but now we recognize it as an integral in the complex plane.

A standard way to evaluate the integral in Equation 2 is by using a closed contour C like the one shown in Figure 19.1. The contour shown in Figure 19.1 is called the *Bromwich contour*. Using this contour, Equation 2 becomes

$$f(t) = \frac{1}{2\pi i}\oint_C e^{st}\hat{F}(s)\,ds - \frac{1}{2\pi i}\oint_\Gamma e^{st}\hat{F}(s)\,ds \tag{3}$$

If we can show that the integral along Γ vanishes as R, the radius of the arc, goes to infinity, then

$$f(t) = \frac{1}{2\pi i}\oint_C e^{st}\hat{F}(s)\,ds \tag{4}$$

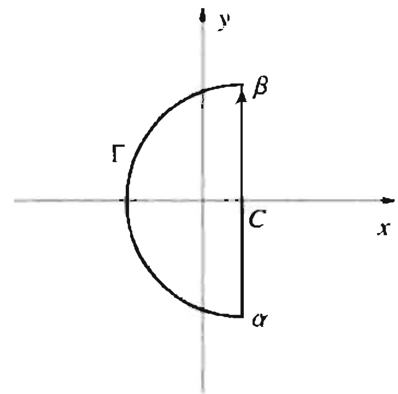


Figure 19.1
A Bromwich contour for inverting Laplace transforms. The closed contour C consists of the vertical line (α, β) and the circular arc Γ.

and assuming that the singularities of $\hat{F}(s)$ are isolated, we can determine $f(t)$ in terms of the residues lying within the closed curve C.

Now let's show that the integral along Γ vanishes as $R \to \infty$. We first let $s = Re^{i\theta}$ along Γ. Now, assuming that the real parts of all the singular points of $\hat{F}(s)$ are less than some finite number c, then the integral over Γ (in a counterclockwise direction) approaches

$$I_\Gamma = \frac{1}{2\pi i}\int_{3\pi/2}^{\pi/2} e^{Rte^{i\theta}}\hat{F}(Re^{i\theta})iRe^{i\theta}d\theta$$

as $R \to \infty$. But

$$|I_\Gamma| \le \frac{R}{2\pi}\int_{3\pi/2}^{\pi/2} e^{Rt\cos\theta}|\hat{F}(Re^{i\theta})|\,d\theta$$

Now assume that the magnitude of $\hat{F}(s)$ along Γ is $\le M/R^k$, where $k > 1$ and M is a constant. Under this condition

$$|I_\Gamma| \le \frac{M}{2\pi R^{k-1}}\int_{3\pi/2}^{\pi/2} e^{Rt\cos\theta}\,d\theta \tag{5}$$

Now, $\cos\theta \le 0$ when $\pi/2 \le \theta \le 3\pi/2$, and if you plot the integrand of Equation 5 for increasing values of Rt, you'll see that it goes to zero everywhere, except at the end points ($\theta = \pi/2$ and $3\pi/2$), which contribute less and less to the integral as Rt increases. Thus, we see that $|I_\Gamma| \to 0$ as $R \to \infty$, and that $f(t)$ is given by Equation 4. (Problem 22 takes you through a more rigorous demonstration that this is so.)

Now let's use Equation 4 to find the inverse of $\hat{F}(s) = a/(s^2 + a^2)$. Using Equation 4,

$$f(t) = \frac{1}{2\pi i}\oint_C \frac{ae^{st}}{s^2 + a^2}ds$$

The singular points of the integrand occur at $\pm ia$ and the residues there are

$$\text{Res}\,(s = ia) = \lim_{s \to ia} \frac{ae^{st}}{s + ia} = \frac{e^{iat}}{2i}$$

and

$$\text{Res}\,(s = -ia) = \lim_{s \to -ia} \frac{ae^{st}}{s - ia} = \frac{e^{-iat}}{-2i}$$

Thus, Equation 4 gives

$$f(t) = \frac{1}{2\pi i}\left[2\pi i\left(\frac{e^{iat}}{2i} - \frac{e^{-iat}}{2i}\right)\right] = \sin at$$

This is one of the entries in Table 17.1.

Example 1:
Use Equation 4 to find the inverse of

$$\hat{F}(s) = \frac{s}{s^2 + a^2}$$

SOLUTION: There are simple poles at $s = \pm ia$. The residues at those poles are

$$\text{Res}\,(s = ia) = \lim_{s \to ia} \frac{se^{st}}{s + ia} = \frac{iae^{iat}}{2ia} = \frac{e^{iat}}{2}$$

and $\text{Res}\,(s = -ia) = e^{-iat}/2$. Therefore, Equation 4 gives

$$f(t) = \frac{1}{2\pi i}\left[2\pi i\left(\frac{e^{iat}}{2} + \frac{e^{-iat}}{2}\right)\right] = \cos at$$

The next Example is a little more involved.

Example 2:
Use Equation 4 to find the inverse of

$$\hat{F}(s) = \frac{s^2}{(s^2 + 4)^2}$$

SOLUTION: The singular points of $\hat{F}(s)$ are poles of order 2 at $s = \pm 2i$. The residue of $e^{st}\hat{F}(s)$ at those poles are

$$\text{Res}\,(s = 2i) = \lim_{s \to 2i} \frac{d}{ds}\frac{s^2 e^{st}}{(s + 2i)^2} = \frac{e^{2it}}{4}\left(t - \frac{i}{2}\right)$$

and

$$\operatorname{Res}(s=-2i) = \frac{e^{2it}}{4}\left(t + \frac{i}{2}\right)$$

Therefore,

$$f(t) = \frac{\sin 2t}{4} + \frac{t\cos 2t}{2}$$

You can verify this result by showing that $\mathcal{L}\{f(t)\} = s^2/((s^2+4)^2$ (Problem 1).

Example 3:
Find the inverse of

$$\hat{F}(s) = \frac{1}{s^4+4}$$

SOLUTION: There are four singular points, at $s = 1+i$, $-1+i$, $-1-i$, and $i-i$ (Problem 2). The denominator of $\hat{F}(s)$ factors into $(s-1-i)(s+1-i)(s+1+i)(s-1+i)$. The residues at these points are

$$\operatorname{Res}(s=1+i) = \lim_{s\to 1+i}\left[\frac{e^{st}}{(s+1-i)(s+1+i)(s-1+i)}\right]$$

$$= \frac{e^t e^{it}}{8i(1+i)}$$

$$\operatorname{Res}(s=-1+i) = \frac{e^{-t}e^{it}}{8i(1-i)}$$

$$\operatorname{Res}(s=-1-i) = -\frac{e^{-t}e^{-it}}{8i(1+i)}$$

$$\operatorname{Res}(s=1-i) = -\frac{e^{t}e^{-it}}{8i(1-i)}$$

The sum of the residues is

$$\text{sum} = \frac{e^t e^{it}(1-i) + e^{-t}e^{it}(1+i) - e^{-t}e^{-it}(1-i) - e^t e^{-it}(1+i)}{16i}$$

$$= \frac{e^{it}(e^t + e^{-t}) - e^{-it}(e^t + e^{-t}) - ie^{it}(e^t - e^{-t}) - ie^{-it}(e^t - e^{-t})}{16i}$$

$$= \frac{(e^{it} - e^{-it})(e^t + e^{-t}) - i(e^{it} + e^{-it})(e^t - e^{-t})}{16i}$$

$$= \frac{\sin t\ \cosh t}{4} - \frac{\cos t\ \sinh t}{4}$$

19.2 Evaluation of Real, Definite Integrals

The inversion formula for Laplace transforms is a contour integral, so the application of the residue theorem to the determination of the inverses of Laplace transforms is obvious. Less obvious, however, is that we can use the residue theorem to evaluate a wide variety of integrals of a real variable. For starters, let's consider the integral

$$I = \int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} \tag{1}$$

Integrals of trigonometric functions like Equation 1 can be converted to contour integrals by the substitutions

$$\begin{aligned} z &= e^{i\theta} \quad , \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2} = \frac{z^2 + 1}{2z} \\ d\theta &= \frac{dz}{iz} \quad , \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i} = \frac{z^2 - 1}{2iz} \end{aligned} \tag{2}$$

and then integrating around the unit circle $|z| = 1$. Using Equations 2, Equation 1 becomes

$$I = \oint \frac{2i\, dz}{3z^2 - 10z + 3} = \oint \frac{2i\, dz}{(3z - 1)(z - 3)} \tag{3}$$

The integrand has simple poles at $z = 1/3$ and $z = 3$. Only the pole at $z = 1/3$ lies within the contour integral ($|z| = 1$), and so

$$\begin{aligned} I &= 2\pi i \text{ Res } (z = 1/3) \\ &= 2\pi i \lim_{z \to 1/3} \left[\frac{2i(z - 1/3)}{(3z - 1)(z - 3)} \right] = \frac{\pi}{2} \end{aligned}$$

Example 1:
Evaluate

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta} d\theta \tag{4}$$

SOLUTION: Use Equations 2 to write

$$I = -\frac{i}{4} \oint \frac{(z^2 - 1)^2}{z^2(2z^2 - 5z + 2)} dz = -\frac{i}{4} \oint \frac{(z^2 - 1)^2}{z^2(2z - 1)(z - 2)} dz$$

The integrand has a pole of order 2 at $z = 0$ and simple poles at $z = 1/2$ and $z = 2$. The residues at the two poles lying within $|z| = 1$ are

$$\text{Res } (z = 0) = \lim_{z \to 0} \left[\frac{d}{dz} \frac{(z^2 - 1)^2}{(2z - 1)(z - 2)} \right] = \frac{5}{4}$$

$$\text{Res } (z = 1/2) = \lim_{z \to 1/2} \left[\frac{(z - 1/2)(z^2 - 1)^2}{z^2(2z - 1)(z - 2)} \right] = -\frac{3}{4}$$

residue at $z = 1 + i$, the pole that lies in the upper half plane, is

$$\operatorname{Res}(z = 1 + i) = \lim_{z \to 1+i} \frac{d}{dz} \frac{1}{(z - 1 + i)^2} = -\frac{i}{4}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 2)^2} = 2\pi i \left(-\frac{i}{4}\right) = \frac{\pi}{2}$$

Another general type of real integral that can be evaluated by contour integration is of the form

$$\int_{-\infty}^{\infty} F(x) \cos mx \, dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} F(x) \sin mx \, dx$$

where $F(x)$ is a rational function of x. In this case, we replace $F(x)$ by $F(z)$, replace $\cos mx$ by $(e^{imz} + e^{-imz})/2$ and $\sin mx$ by $(e^{imz} - e^{-imz})/2i$ and use the contour shown in Figure 19.5. As usual, $F(z)$ must be such that the integral along Γ in Figure 19.5 vanishes as $R \to \infty$, which means that $|F(z)| \leq M/R^k$ for large values of R, where $k > 1$ and M is a constant.

Example 4:
Evaluate

$$\int_0^{\infty} \frac{\cos \omega x}{1 + x^2} dx$$

where $\omega > 0$.

SOLUTION: Start with

$$\oint_C \frac{e^{i\omega z}}{1 + z^2} dz$$

where C is the contour shown in Figure 19.5. We must be sure to show that the integral along Γ vanishes at $R \to \infty$. The magnitude of the integrand along Γ is equal to

$$\left[\frac{e^{i\omega(R\cos\theta + i\sin\theta)}}{1 + R^2 e^{2i\theta}} \cdot \frac{e^{-i\omega(R\cos\theta - i\sin\theta)}}{1 + R^2 e^{-2i\theta}}\right]^{1/2} = \frac{e^{-\omega R \sin\theta}}{(1 + 2R^2 \cos 2\theta + R^4)^{1/2}}$$

This quantity vanishes as $R \to \infty$ because $\omega > 0$ and $\sin\theta > 0$ all along Γ. The integrand has two simple poles, at $z = \pm i$. Only the pole at $z = i$ lies within C, and the residue there is

$$\operatorname{Res}(z = i) = \lim_{z \to i} \left[\frac{e^{i\omega z}}{z + i}\right] = \frac{e^{-\omega}}{2i}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{\cos \omega x}{1+x^2} dx = \pi e^{-\omega}$$

or, using the fact that the integrand is an even function of x,

$$\int_0^{\infty} \frac{\cos \omega x}{1+x^2} dx = \frac{\pi}{2} e^{-\omega} \qquad \omega > 0$$

Example 5:
Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+2} dx$$

SOLUTION: Start with

$$\int_C \frac{z e^{i\pi z}}{z^2+2z+2} dz$$

where C is the contour shown in Figure 19.5. The argument that the integral along Γ vanishes as $R \to \infty$ is similar to that outlined in the previous Example. There are simple poles at $z = -1 \pm i$, and only the point $-1+i$ lies within the contour. The residue of the above integrand at $z = -1+i$ is

$$\operatorname{Res}(z = -1+i) = \lim_{z \to -1+i} \frac{z e^{i\pi z}}{z+1+i} = \frac{1-i}{2i} e^{-\pi}$$

and so

$$\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2+2x+2} dx = 2\pi i \left[\frac{(1-i)e^{-\pi}}{2i} \right] = \pi e^{-\pi} - i\pi e^{-\pi}$$

By using Euler's relation on the left, we get the two results

$$\int_{-\infty}^{\infty} \frac{x \cos \pi x \, dx}{x^2+2x+2} = \pi e^{-\pi} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{x \sin \pi x \, dx}{x^2+2x+2} = -\pi e^{-\pi}$$

We'll finish the section with the integral of a function with a branch cut. Let's evaluate

$$I = \int_0^{\infty} \frac{x^{p-1}}{1+x} dx$$

where $0 < p < 1$. Start with

$$\oint_C \frac{z^{p-1}}{1+z}dz \tag{10}$$

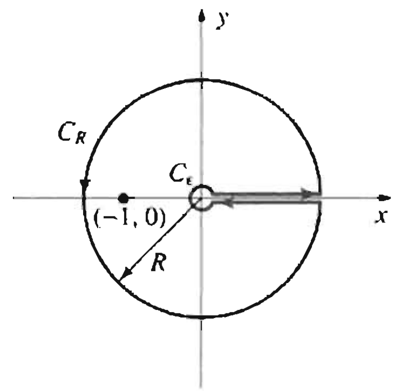


Figure 19.7
The contour used to evaluate the integral in Equation 10.

where C is the contour shown in Figure 19.7. The integral along Γ vanishes as $R \to \infty$ because $0 < p < 1$. Similarly, the integral around C_ϵ vanishes as $\epsilon \to 0$ (Problem 13). Using $z = x$ along the upper part of the branch cut and $z = xe^{2\pi i}$ along the lower part, we can write

$$\int_0^\infty \frac{x^{p-1}dx}{1+x} + \int_\infty^0 \frac{e^{2\pi i(p-1)}x^{p-1}dx}{1+x} = 2\pi i \text{ (sum of the residues)}$$

The only singular point of the integrand within the contour shown in Figure 19.7 is at $z = -1$, and the residue there is $(-1)^{p-1} = e^{i\pi(p-1)}$. Therefore,

$$[1 - e^{2\pi i(p-1)}]\int_0^\infty \frac{x^{p-1}}{1+x}dx = 2\pi i e^{(p-1)\pi i}$$

or

$$\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{2\pi i e^{(p-1)\pi i}}{1-e^{2\pi i(p-1)}} = \frac{2\pi i e^{-i\pi}}{e^{-p\pi i} - e^{p\pi i}}$$

$$= \frac{\pi}{\sin p\pi} \qquad 0 < p < 1$$

Example 6:
Evaluate

$$I = \int_0^\infty \frac{x^{1/3}dx}{(1+x)^2}$$

SOLUTION: Use the contour in Figure 19.7 to evaluate

$$\oint_C \frac{z^{1/3}dz}{(1+z)^2}$$

As before, the integrals along C_R and C_ϵ vanish in the limits. Use $z = x$ along the upper part of the branch cut and $z = xe^{2\pi i}$ along the lower part to write

$$\int_0^\infty \frac{x^{1/3}dx}{(1+x)^2} + \int_\infty^0 \frac{e^{2\pi i/3}x^{1/3}dx}{(1+x)^2} = 2\pi i \ \text{Res}\,(z=-1)$$

The singularity at $z = -1$ is a pole of order 2, and the residue there is given by

$$\text{Res}\,(z=-1) = \lim_{z\to -1}\frac{dz^{1/3}}{dz} = -\frac{e^{i\pi/3}}{3}$$

Therefore,

$$\int_0^\infty \frac{x^{1/3}dx}{(1+x)^2} = -\frac{2\pi i e^{i\pi/3}}{3(1-e^{2\pi i/3})} = \frac{\pi}{3\sin(\pi/3)} = \frac{2\pi}{3\sqrt{3}}$$

It simply takes practice to become facile with contour integration. The problems not only reflect the methods that we have used in this section, but illustrate other methods as well.

19.2 Problems

1. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{3+\cos\theta} = \frac{\pi}{\sqrt{2}}$.

2. Show that $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{3+\cos\theta}\,d\theta = 0$.

3. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{(a^2-b^2)^{1/2}}$.

4. Show that $\displaystyle\int_0^{\pi} \frac{d\theta}{\cos^2\theta + 4\sin^2\theta} = \frac{\pi}{2}$.

5. Show that $\displaystyle\int_0^{\pi} \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{\pi}{ab}$.

6. Show that $\displaystyle\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \int_\pi^{2\pi} \frac{d\theta}{a+b\cos\theta}$.

7. Calculate all the residues of $f(z) = 1/(1+z^4)$.

8. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$ using the contour shown in Figure 19.5. Use the result of the previous problem.

9. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\,dx = \frac{\pi}{2}$.

10. Show that $\displaystyle\int_0^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}}$.

11. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2-2x+2)^2}\,dx = \frac{\pi}{2}$.

12. Show that the integral along Γ in Figure 19.5 vanishes as $R \to \infty$ if $F(x)$ is a ratio of two polynomials with the degree of the denominator at least two greater than the degree of the numerator.

13. Show that the integrals on C_R and C_ϵ in Equation 10 vanish as $R \to \infty$ and $\epsilon \to 0$, respectively.

14. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin\omega x}{x^2+a^2}\,dx = \pi e^{-\omega a}$ where $\omega > 0$.

15. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos\pi x\,dx}{x^2-2x+2}$ and $\displaystyle\int_{-\infty}^{\infty} \frac{\sin\pi x\,dx}{x^2-2x+2}$.

16. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2\omega x}{1+x^2}\,dx = \frac{\pi}{2}(1-e^{-2\omega})$ for $\omega \geq 0$.

17. Can you show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2\omega x}{a^2+x^2}\,dx = \frac{\pi}{2a}(1-e^{-2a\omega})$ without re-doing the previous problem?

18. An integral that has occurred a number of times in the text is $I = \displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$. We will evaluate this integral

determine how many solutions to the equation

$$e^z = 3z^2$$

there are within the unit circle $|z| = 1$ and we'll learn how to determine if a polynomial such as

$$f(s) = s^2 + 2s + 3 + 1$$

has any zeros whose real part is positive. This second problem occurs in the assessment of the stability of mechanical and electrical systems, particularly in regard to feedback and control systems.

The method of the summation of series rests upon the fact that $\cot \pi z$ has simple poles at $z = 0, \pm 1, \pm 2, \ldots$ (Problem 1). Consider the contour, C_N, shown in Figure 19.12. First we'll show that $|\cot \pi z|$ is bounded on the contour. Along the right (vertical) side, $x = N + 1/2$ and so $z = (N + 1/2) + iy$, where $-(N + 1/2) \le y \le N + 1/2$. Therefore,

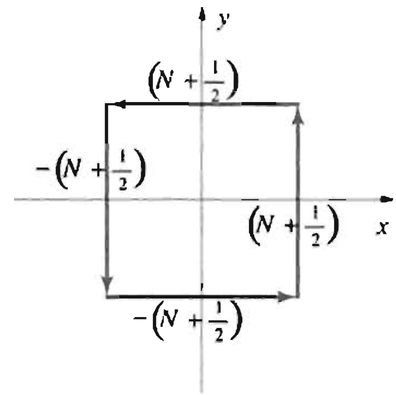


Figure 19.12
The contour used to evaluate the integral in Equation 1. Each side of the square is at a distance $N + \frac{1}{2}$ (where N is an integer) from the origin.

$$\cot \pi z = \frac{\cos[\pi(N + \frac{1}{2}) + i\pi y]}{\sin[\pi(N + \frac{1}{2}) + i\pi y]}$$

$$= \frac{\cos[(N + \frac{1}{2})\pi]\cosh \pi y - i \sin[(N + \frac{1}{2})\pi]\sinh \pi y}{\sin[(N + \frac{1}{2})\pi]\cosh \pi y + i \cos[(N + \frac{1}{2})\pi]\sinh \pi y}$$

Using the fact that $\cos(N + 1/2)\pi = 0$ and $\sin(N + 1/2)\pi = \pm 1$, we can see that

$$|\cot \pi z| = \frac{|\sinh \pi y|}{|\cosh \pi y|} = |\tanh \pi y|$$

But $-1 \le \tanh \pi y \le 1$ (see Figure 19.13), so $|\cot \pi z| \le 1$ on the right vertical part of the contour C_N.

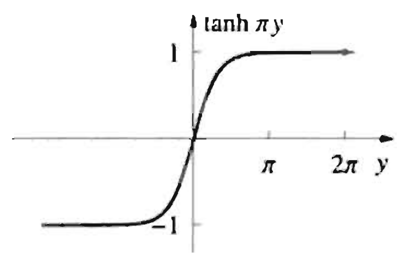


Figure 19.13
A pictorial representation of the inequalities $-1 \le \tanh \pi y \le 1$.

A similar argument shows that this is true of the left vertical part of the contour, also. Along the top side of C_N, $y = N + 1/2$ and $-(N + 1/2) \le x \le (N + 1/2)$, and so

$$|\cot \pi z| = \frac{|\cos \pi x \cosh \pi y - i \sin \pi x \sinh \pi y|}{|\sin \pi x \cosh \pi y + i \cos \pi x \sinh \pi y|}$$

Anticipating that we are going to let the value of N approach infinity, we recognize that $\cosh \pi y$ and $\sinh \pi y$ are essentially equal for large values of $y = N + 1/2$ (see Figure 19.14, which shows that $\cosh \pi y$ and $\sinh \pi y$ are almost equal even for $N = 1$). Therefore,

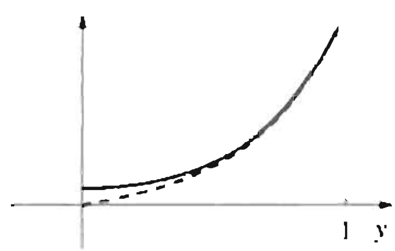


Figure 19.14
An illustration of the fact that $\cosh \pi y$ and $\sinh \pi y$ are essentially equal for $y \gtrsim 1$.

$$|\cot \pi z| \longrightarrow \frac{|\cos \pi x - i \sin \pi x|}{|\sin \pi x + i \cos \pi x|} = 1$$

along the top side of C_N with a similar result along the bottom side of C_N.

4. Show that $\sum_{n=-\infty}^{\infty} \frac{1}{(n^2-a^2)^2} = \frac{\pi^2}{2a^2 \sin^2 \pi a} + \frac{\pi \cot \pi a}{2a^3}$, where a is real and not equal to $0, \pm 1, \pm 2, \ldots$.

5. Show that your answer to the previous problem reduces to $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ as $a \to 0$.

6. Show that $\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+a^2)^2} = \frac{\pi \coth \pi a}{2a^3} + \frac{\pi^2 \operatorname{cosech}^2 \pi a}{2a^2}$ where $a > 0$.

7. Show that $\sum_{n=1}^{\infty} \frac{1}{(n^2+a^2)^2} = \frac{\pi \coth \pi a}{4a^3} + \frac{\pi^2 \operatorname{cosech}^2 \pi a}{4a^2} - \frac{1}{2a^4}$ where $a > 0$.

8. Starting with the defining equation for the Bernoulli numbers, $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$, show that

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} B_{2n} x^{2n-1} = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + \cdots$$

9. Evaluate $S = \sum_{n=1}^{\infty} \frac{1}{n^6}$.

10. Show that $\sum_{n=-\infty}^{\infty} \frac{n^2}{n^4 - a^4} = \frac{\pi \coth \pi a}{2a} - \frac{\pi \cot \pi a}{2a}$, where a is real and not equal to $0, \pm 1, \pm 2, \ldots$.

11. Let $a \to 0$ in the previous problem to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

12. Derive Equation 7.

13. Derive Equation 8.

14. Show that $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$, where a is real and not equal to $0, \pm 1, \pm 2, \ldots$.

15. Use the defining equation of the Bernoulli numbers, $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$, to show that

$$\operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \cdots.$$

16. Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$.

17. Let $a \to 0$ in the result of Example 3 to show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

18. Use the result of Problem 3 to show that $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

Example 2:
Re-do Example 1 with C given by $|z| = 3$.

SOLUTION: In this case, all the poles of $f'(z)/f(z)$ lie within C, and the expression for $f'(z)/f(z)$ in Example 1 shows that the sum of the residues is -3, in agreement with Equation 4.

Let's sketch the proof of Equation 4. Suppose that $f(z)$ has a zero of multiplicity n at $z = a_i$. Then the Taylor expansion of $f(z)$ has the form

$$f(z) = c_n(z - a_i)^n + c_{n+1}(z - a_i)^{n+1} + \cdots$$

Therefore,

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{nc_n(z - a_i)^{n-1} + (n+1)c_{n+1}(z - a_i)^n + \cdots}{c_n(z - a_i)^n + c_{n+1}(z - a_i)^{n+1} + \cdots} \\ &= \frac{n}{z - a_i} + 1 + O(z - a_i)\end{aligned}$$

and

$$\text{Res}\left\{\frac{f'(z)}{f(z)} \text{ at } z = a_i\right\} = n_i$$

Using the same approach, if $f(z)$ has a pole of order m_j at $z = b_j$, then (Problem 4)

$$\text{Res}\left\{\frac{f'(z)}{f(z)} \text{ at } z = b_j\right\} = -m_j$$

Equation 4 follows by evaluating the integral by the method of residues using the above results.

We evaluated the integral in Equation 4 explicitly for the case $f(z) = z/(2z^2 + 1)$, but there is another, geometric, way to evaluate it that is often more convenient. First, we recognize that $f'(z)/f(z) = d \ln f(z)/dz$, and write

$$\begin{aligned}\oint_C \frac{f'(z)}{f(z)} dz &= \oint_C \frac{d \ln f(z)}{dz} dz \\ &= \oint_C d \ln f(z) = \Delta_C \ln f(z)\end{aligned}$$

where $\Delta_C \ln f(z)$ is the change in $\ln f(z)$ as z makes one circuit around C in a counterclockwise (positive) direction. Now write $\ln f(z)$ as

$$\ln f(z) = \ln |f(z)| + i \arg f(z)$$

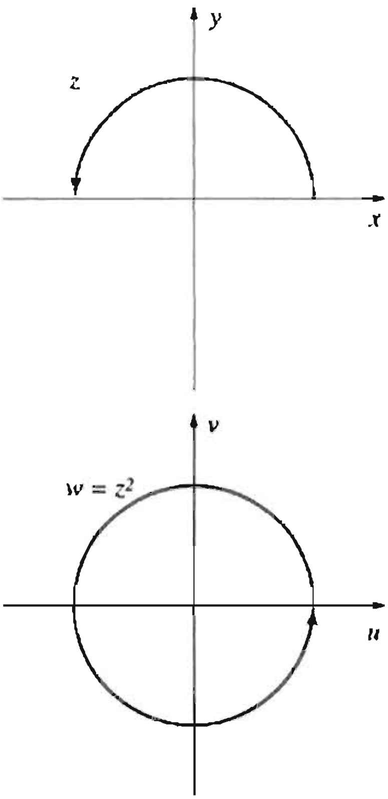


Figure 19.15
An illustration of the fact that $w = f(z) = z^2$ makes a complete circuit in the w-plane when z makes one half a circuit in the z-plane.

so that

$$\Delta_C \ln f(z) = \Delta_C \ln|f(z)| + i\Delta_C \arg f(z)$$

There is no change in $|f(z)|$ as z makes one circuit around C, but $\arg f(z)$ may very well change (as we'll soon see). Therefore, we may write Equation 4 as

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = Z - P = \frac{1}{2\pi}\Delta_C \arg f(z) \tag{5}$$

This result is known as the *argument principle*, or the *principle of the argument*.

As we implied above, Equation 5 has a nice geometric interpretation. Let C_w be the image of C in the w-plane under the mapping $w = f(z)$. In other words, as z traces out a path C in the z-plane, $w = f(z)$ traces out a path C_w (the image curve) in the w-plane. The image curve will also be a closed contour, but it need not be one-to-one with C, as in the case of $w = z^2$, where w makes two circuits around the origin in the w-plane as z makes one circuit (Figure 19.15).

Let's use Equation 5 to see how many zeros of

$$f(z) = z^2 + 2z + 2 \tag{6}$$

lie within the circle $|z| = 1$. (The zeros of $f(z)$ are clearly $z = -1 \pm i$, but we'll consider more challenging cases soon.) Let $z = x + iy$ and write

$$\begin{aligned} w = f(z) &= u(x, u) + i\,v(x, y) \\ &= x^2 - y^2 + 2x + 2 + i\,(2xy + 2y) \end{aligned}$$

We wish to plot w as z makes one circuit around $|z| = 1$. We can do this by writing x and y as $x = \cos\theta$ and $y = \sin\theta$, $0 \le \theta < 2\pi$, in which case we have

$$\begin{aligned} u(\theta) &= \cos^2\theta - \sin^2\theta + 2\cos\theta + 2 \\ v(\theta) &= 2\cos\theta\sin\theta + 2\sin\theta \end{aligned} \tag{7}$$

We now plot v parametrically against u. (This can be done easily using any computer algebra system.) The result is shown in Figure 19.16a. Note carefully that the change in $\arg w$ is zero because the curve C_w does not enclose the origin as you can see in Figure 19.16a, where a few of the values of $\arg w$ are shown. Thus, there are no zeros of $f(z)$ lying within the unit circle.

Now let C be the circle $|z| = 3$. In this case, $x(\theta) = 3\cos\theta$ and $y(\theta) = 3\sin\theta$, and the curve C_w is shown in Figure 19.16b. Note that C_w encircles the origin two times, so $\Delta_C \arg w = 2$ in this case, having picked up the two zeros of $f(z)$ at $z = -1 \pm i$.

Thus, we can express Equation 5 as

$$Z - P = N \tag{8}$$

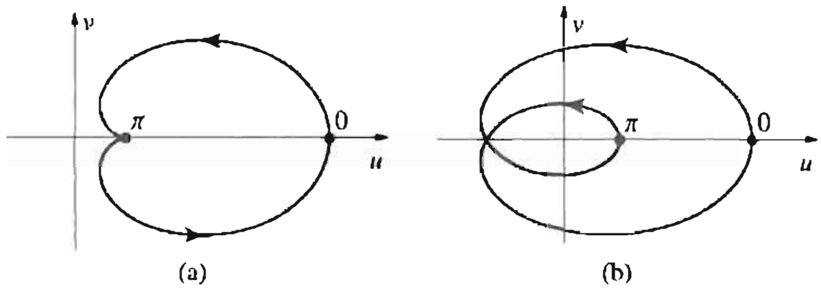


Figure 19.16
(a) A plot of $v(x, y)$ against $u(x, y)$ in the w-plane for $w = f(z) = z^2 + 2z + 2$ for one circuit of z around a unit circle in the z-plane. The numbers 0 and π represent the values of θ in the parametric representation of $v(\theta)$ and $u(\theta)$ in Equations 7. (b) Similar to (a), but in this case z traverses the circle $|z| = 3$ once.

where N is the number of times that C_w, the image curve of C, encircles the origin at $w = 0$. The quantity N is sometimes called the *winding number* of $w = f(z)$.

Example 3:
How many solutions does the equation

$$e^z = 3z^2$$

have within the unit circle?

SOLUTION: We wish to determine how many zeros the function $f(z) = e^z - 3z^2$ has within the unit circle. There are no poles, so Equation 5 will give us the number of zeros. Write

$$\begin{aligned} w = f(z) &= e^z - 3z^2 \\ &= e^x \cos y - 3(x^2 - y^2) + i\,(e^x \sin y - 6xy) \\ &= u(x, y) + i\,v(xy) \end{aligned}$$

Let $x = \cos\theta$ and $y = \sin\theta$, $0 \le \theta < 2\pi$, and write

$$u(\theta) = e^{\cos\theta} \cos(\sin\theta) - 3(\cos^2\theta - \sin^2\theta)$$

$$v(\theta) = e^{\cos\theta} \sin(\sin\theta) - 6\cos\theta\sin\theta$$

A parametric plot of $v(\theta)$ against $u(\theta)$ is shown in Figure 19.17. We see that C_w encircles the origin two times, and so there are two zeros of $f(z) = e^z - 3z^2$, or two solutions to the equation $e^z = 3z^2$, that lie within the unit circle $|z| = 1$.

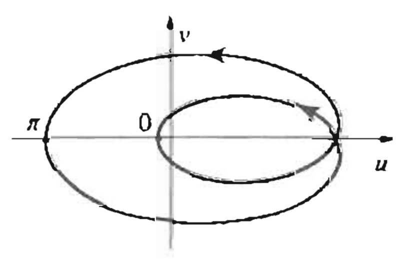


Figure 19.17
A plot of $v(\theta)$ against $u(\theta)$ in the w-plane as z traverses the unit circle once for the function given in Example 3.

There is a companion theorem to the argument principle, called *Rouche's theorem*, that is a great aid to the location of the zeros of a function.

Rouche's theorem: *Let $f(z)$ and $g(z)$ be analytic inside and on a closed curve C, and suppose that $|f(z)| > |g(z)|$ all along C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros within C.*

The proof of Rouche's theorem is developed in Problem 9. The following Example illustrates the utility of Rouche's theorem.

Example 4:
How many roots to the equation

$$z^8 - 4z^5 + z^2 - 1 = 0$$

are there within the unit circle?

SOLUTION: Let

$$f(z) + g(z) = z^8 - 4z^5 + z^2 - 1$$

We want $|f(z)| > |g(z)|$ on the unit circle, so choose $f(z) = -4z^5$ and $g(z) = z^8 + z^2 - 1$. This choice assures us that $|f(z)| > |g(z)|$ on the unit circle because

$$|g(z)| = |z^8 + z^2 - 1| \le |z^8| + |z^2| + |1| = 3 < |f(z)| = 4$$

on the unit circle. According to Rouche's theorem, $f(z) + g(z) = z^8 - 4z^5 + z^2 - 1$ has the same number of zeros within $|z| = 1$ as $f(z) = -4z^5$. But $f(z)$ has a zero of multiplicity 5 at $z = 0$, so $z^8 - 4z^5 + z^2 - 1 = 0$ has five roots within $|z| = 1$.

Example 5:
Show that all the roots of

$$z^7 - 2z^4 + 5z - 9 = 0$$

lie between the circles $|z| = 1$ and $|z| = 2$.

SOLUTION: Consider the circle $|z| = 1$, and choose $f(z) = -9$ and $g(z) = z^7 - 2z^4 + 5z$. On the circle $|z| = 1$, we have

$$|g(z)| = |z^7 - 2z^4 + 5z| \le |z^7| + |-2z^4| + |5z| = 8 < |f(z)| = 9$$

According to Rouche's theorem, $f(z) + g(z)$ has the same number of zeros inside $|z| = 1$ as $f(z) = -9$, which is zero.

Now consider the circle $|z| = 2$, and choose $f(z) = z^7$ and $g(z) = -2z^4 + 5z - 9$. On the circle $|z| = 2$, we have

$$|g(z)| \le |2z^4| + |5z| + |9| = 51 < |f(z)| = |z^7| = 128$$

In this case, $f(z)$ has 7 zeros within $|z| = 2$, so all 7 roots of $z^7 - 2z^4 + 5z - 9 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

extremes ($w \to \pm\infty$). Notice that $u(w)$ and $v(w)$ are both negative as $w \to +\infty$ and that $u(w) < 0$ and $v(w) > 0$ as $w \to -\infty$, as indicated in the figure.

Now let's investigate the behavior of arg w as we travel along the curve from $w = \infty$ to $w = -\infty$ in Figure 19.19, which corresponds to the contribution from the integral along the vertical axis in Figure 19.18. The arg w is shown as the angle ϕ in Figure 19.20. If we start at arg $w = 0$ and follow the curve to $w = \infty$, arg w increases from 0 to $+3\pi/2$, and if we start at arg w and follow the curve to $w = -\infty$, arg w goes from 0 to $-3\pi/2$ (Problem 21). Therefore, if we start at $w = +\infty$, where arg $w = 3\pi/2$ and move along the curve toward $w = -\infty$, where arg $w = -3\pi/2$, we see that the change of arg w is equal to $-3\pi/2 - 3\pi/2 = -3\pi$. This result, together with 3π from the semicircular arc, gives a net change of 0. Thus, we conclude that there are no zeros of $Q(s) = s^3 + 6s^2 + 10s + 6$ in the right half plane.

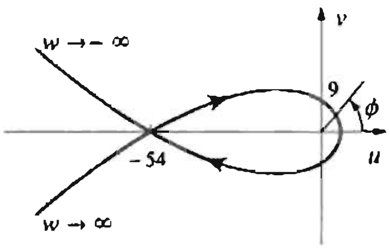


Figure 19.20
An enlargement of the curve in Figure 19.19, showing $\phi = \arg w$ and how it varies as you move along the curve toward its two extremes. The arrows point in the direction of decreasing w along the path.

Example 6:
Determine if

$$Q(s) = s^3 + 2s^2 + s + 4$$

has any zeros that lie in the right half plane.

SOLUTION: According to Equation 10, the contribution to arg w from the semicircular arc is 3π. The contribution to arg w from the imaginary axis is found from

$$Q(iw) = -iw^3 - 2w^2 + iw + 4$$
$$= 4 - 2w^2 + i(w - w^3)$$

which gives

$$u(w) = 2(2 - w^2) \qquad \text{and} \qquad v(w) = w - w^3$$

Figure 19.21 shows the image curve in the w-plane. As $w \to \pm\infty$, $dv/du \to 3w/4 \to \pm\infty$, which says that the curve in Figure 19.21 is vertical at its two extremes ($w = \pm\infty$). If we start with arg $w = 0$ and follow the curve toward $w = +\infty$, then we see that arg $w \to -\pi/2$. Notice that because the curve in Figure 19.21 does not enclose the origin, arg w starts at 0, builds up to a small positive value, and then decreases to zero and continues to decrease toward $-\pi/2$ (Problem 22). Similarly, if we start at arg $w = 0$ and follow the curve toward $w = -\infty$, then arg w goes through small negative values, passes through zero, and then increases toward $\pi/2$ (Problem 22). Thus, if we start at $w = \infty$, where arg $w = -\pi/2$, and move along the curve toward $w = -\infty$, where arg $w = \pi/2$, we see that the change of arg w is equal to $+\pi$. This result together with the 3π from the semicircular arc gives a total of 4π. Thus, there are two zeros of $Q(s) = s^3 + 2s^2 + s + 4$ in the right half plane.

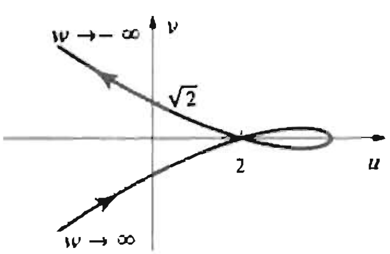


Figure 19.21
The image curve for Example 6. Note that this curve does not enclose the origin.

14. Use Rouche's theorem to determine how many zeros the function $z^4 - 5z + 1$ has in the annulus $1 < |z| < 2$. See the hint to the previous problem.

15. Use Rouche's theorem to prove that there are n roots of the equation $e^z = az^n$ within the circle $|z| = 1$ if $a > e$. Compare this result to the result of Example 3.

16. A classic application of Rouche's theorem involves the proof of the *fundamental theorem of algebra*, that every polynomial of nth degree with complex coefficients, $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1z + a_0 = 0$, has exactly n complex roots. Use Rouche's theorem to prove this result. *Hint*: Take $f(z) = z^n$ and $g(z) = a_{n-1}z^{z-1} + \cdots + a_1z + a_0$.

17. Determine if $Q(s) = s^5 + 9s^4 + 24s^3 + 12s^2 - 60s - 60$ has any zeros that lie in the right half plane.

18. Determine if $Q(s) = s^5 + 3s^4 + 5s^3 + 5s^2 + 3s + 1$ has any zeros in the right half plane.

19. Determine if $Q(s) = s^4 + s^3 + s^2 + 10s + 10$ has any zeros that lie in the right half plane. *Caution*: Be sure to investigate the image curve as $w \to \pm\infty$.

20. Determine if $Q(s) = s^6 + 2s^5 + 3s^3 + s^2 + 6s + 2$ has any zeros that lie in the right half plane. *Caution*: Be sure to investigate the image curve as $w \to \pm\infty$.

21. Use the relation $\arg w = \tan^{-1}[v(w)/u(w)]$ to plot $\arg w$ against w for the curve in Figures 19.19 and 19.20. Be sure to use the appropriate branch of the arctangent.

22. Use the relation $\arg w = \tan^{-1}[v(w)/u(w)]$ to plot $\arg w$ against w for the curve in Figure 19.21. Be sure to use the appropriate branch of the arctangent.

19.5 Conformal Mapping

We know that if $f(z) = u(x, y) + i\,v(x, y)$ is analytic in some region R, then $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

in R. If the second derivatives of $u(x, y)$ and $v(x, y)$ exist and are continuous in R, then it follows from the Cauchy-Riemann equations that both $u(x, y)$ and $v(x, y)$ satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

in R. It may not be surprising, then, that the theory of analytic functions can be applied to two-dimensional boundary value problems involving Laplace's equation. By two-dimensional, we actually mean a three-dimensional system that is symmetric in one of its dimensions so that its cross section is invariant in that dimension. Before we can discuss this type of application, we must review the idea of a mapping from the z-plane to the w-plane from Chapter 4.

Let's start off with some simple mappings and then build them into more complicated mappings. Consider some region in the z-plane and see how it is

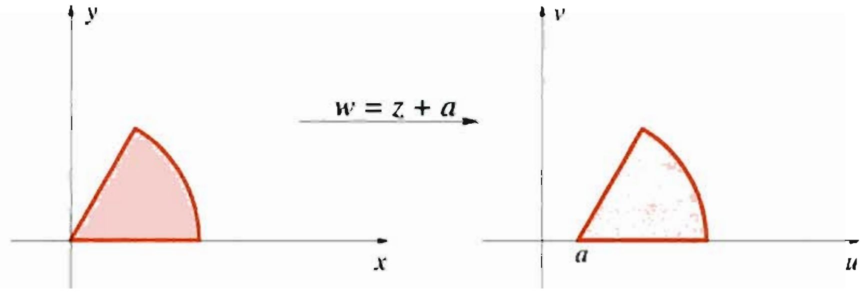


Figure 19.22
The mapping $w = f(z) = z + a$ translates a region in the z-plane a units to the right in the w-plane.

transformed into the w-plane under the mapping $w = f(z)$. For example, the mapping

$$w = f(z) = z + a \tag{1}$$

simply translates the region a units to the right because each point z in R is increased by a units, as shown in Figure 19.22. The point in the w-plane corresponding to z in the z-plane under the mapping $w = f(z)$ is called the *image* of z.

The mapping

$$w = f(z) = bz \tag{2}$$

does two things. If we write this mapping in polar form by writing $w = \rho e^{i\phi}$, $b = |b|e^{i\beta}$, and $z = re^{i\theta}$, then

$$\rho e^{i\phi} = |b|e^{i\beta}re^{i\theta} = |b|re^{i(\theta+\beta)}$$

We see that the magnitude of z is scaled by a factor of $|b|$ and z is also rotated through an angle β. This mapping is called a *scaling-rotation mapping*. Figure 19.23 shows the mapping of our test region in Figure 19.22 by $w = (1 + \sqrt{3}\,i)z = 2e^{i\pi/3}z$.

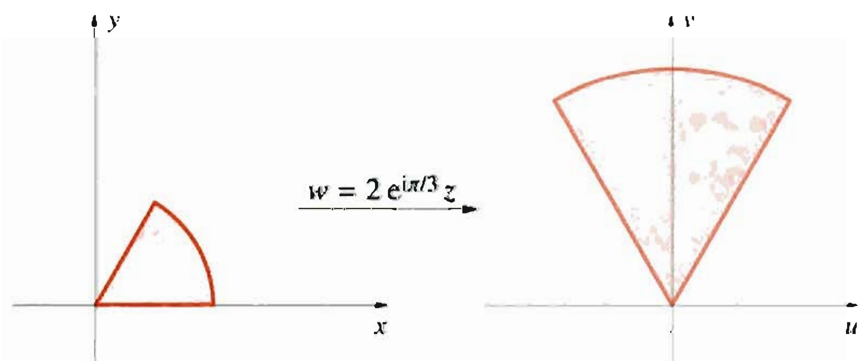


Figure 19.23
The mapping of the test region in Figure 19.22 by the scaling-rotation mapping $w = (1 + \sqrt{3}\,i)z = 2e^{i\pi/3}z$.

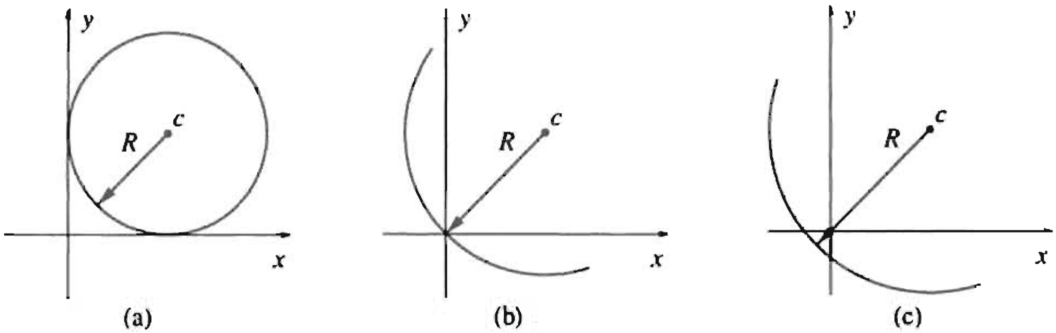


Figure 19.25
The three cases considered in Example 1. (a) If $R < |c|$, then the point $z = 0$ lies outside the disk $|z - c| < R$; (b) if $R = |c|$, then the point $z = 0$ lies on the circle $|z - c| = R$; and (c) if $R > |c|$, then the point $z = 0$ lies inside the disk $|z - c| < R$.

Now let $z = 1/w$ to get

$$(R^2 - |c|^2)ww^* + (cw + c^*w^*) > 1 \tag{5}$$

The three cases to be discussed depend upon the value of $R^2 - |c|^2$, as shown in Figure 19.25.

(a) When $R < |c|$, divide Inequality 5 by $|c|^2 - R^2$ to get

$$-ww^* + \frac{cw + c^*w^*}{|c|^2 - R^2} > \frac{1}{|c|^2 - R^2} > 0$$

which can be transformed into

$$ww^* - \frac{cw + c^*w^*}{|c|^2 - R^2} < \frac{1}{|c|^2 - R^2} \tag{6}$$

by multiplying both sides by -1. Now add $|c|^2/(|c|^2 - R^2)^2$ to both sides of Inequality 6 and take the square root to obtain

$$\left| w - \frac{c^*}{|c|^2 - R^2} \right| < \frac{(2|c|^2 - R^2)^{1/2}}{|c|^2 - R^2} \tag{7}$$

Thus, the interior of a disk in the z-plane is mapped into the interior of a disk in the w-plane if $R < |c|$. (See Figure 19.26a.)

(b) When $R = |c|$, the point $z = 0$ lies on the circle $|z - c| = R$. In that case, Inequality 5 becomes simply

$$cw + c^*w^* > 1$$

If we let $c = a + ib$ and $w = u + iv$, this inequality becomes

$$au - bv \geq \frac{1}{2}$$

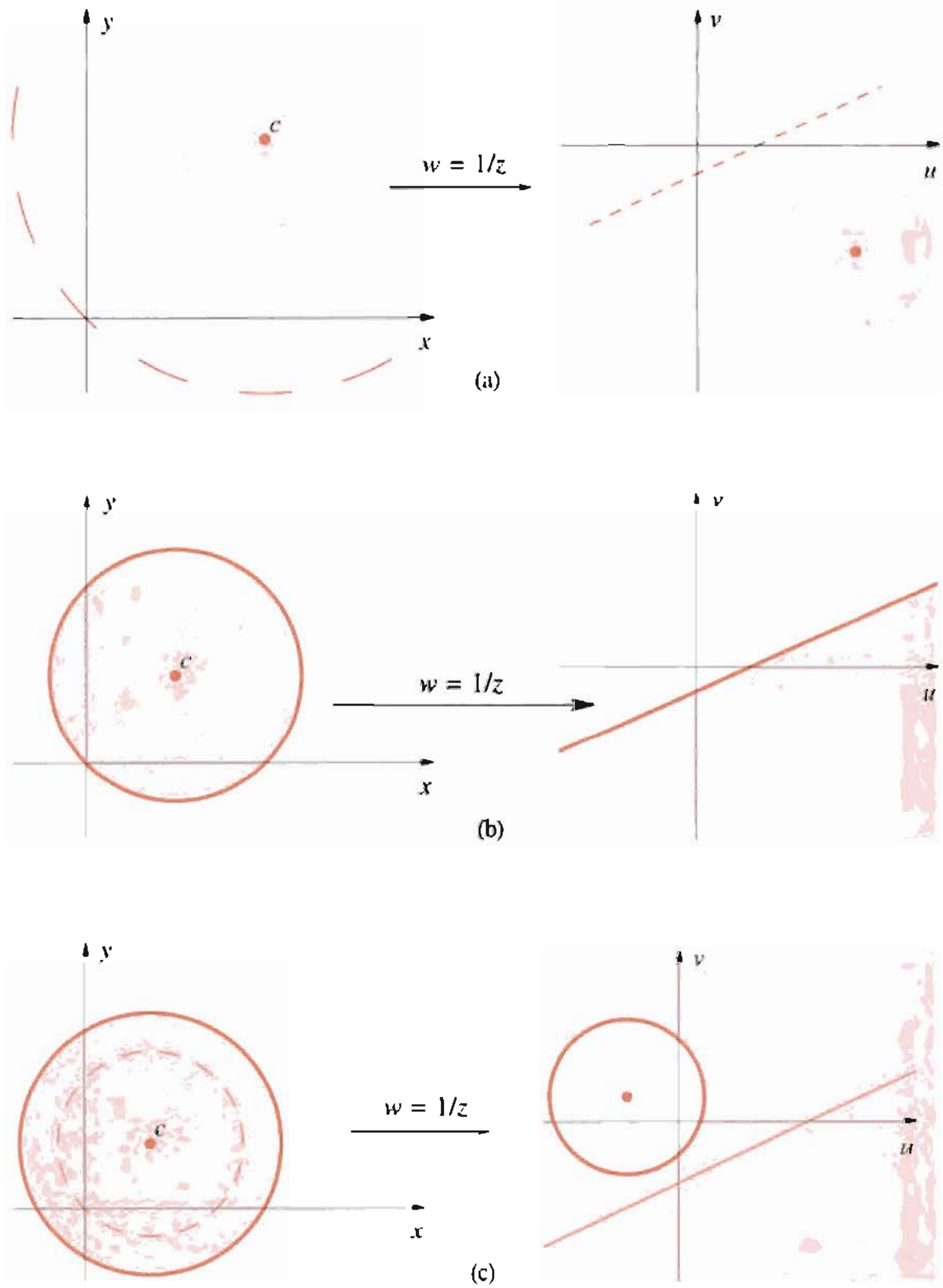


Figure 19.26
An illustration of the mapping discussed in Example 1. (a) $R < |c|$ and the interior of a disk in the z-plane is mapped into the interior of a disk in the w-plane. (b) $R = |c|$ and the disk in the z-plane is mapped into the region below a straight line in the w-plane. (c) $R > |c|$ and the interior of a disk in the z-plane is mapped into the exterior of a disk in the w-plane.

$z_1(t)$ and $z_2(t)$ are mapped into the curves

$$w_1(t) = f(z_1(t)) \qquad \text{and} \qquad w_2(t) = f(z_2(t))$$

The slopes of $w_1(t)$ and $w_2(t)$ at t_0 are given by

$$\left(\frac{dw_1}{dt}\right)_{t=t_0} = \left(\frac{df}{dz_1}\right)_{t=t_0}\left(\frac{dz_1}{dt}\right)_{t=t_0} = f'(t_0)\left(\frac{dz_1}{dt}\right)_{t=t_0} \tag{9}$$

and

$$\left(\frac{dw_2}{dt}\right)_{t=t_0} = \left(\frac{df}{dz_2}\right)_{t=t_0}\left(\frac{dz_2}{dt}\right)_{t=t_0} = f'(t_0)\left(\frac{dz_2}{dt}\right)_{t=t_0} \tag{10}$$

Expressing $(dw_1/dt)_{t=t_0}$, $f'(t_0)$, and $(dz_1/dt)_{t=t_0}$ in Equation 9 in polar form gives

$$\left(\frac{dw_1}{dt}\right)_{t=t_0} = \left|\frac{dw_1}{dt}\right|_{t=t_0} e^{i\phi_1(t_0)} = |\, f'(t_0)\,|\, e^{i\beta(t_0)} \left|\frac{dz_1}{dt}\right|_{t=t_0} e^{i\theta_1(t_0)} \tag{11}$$

with a similar result for $(dw_2/dt)_{t=t_0}$. Provided that $f'(t_0) \neq 0$, we can equate the arguments on the two sides of Equation 11 to write

$$\phi_1(t_0) = \beta(t_0) + \theta_1(t_0) \tag{12}$$

The same procedure for Equation 10 yields

$$\phi_2(t_0) = \beta(t_0) + \theta_2(t_0) \tag{13}$$

Subtracting Equations 12 and 13 yields

$$\phi_2(t_0) - \phi_1(t_0) = \theta_2(t_0) - \theta_1(t_0) \tag{14}$$

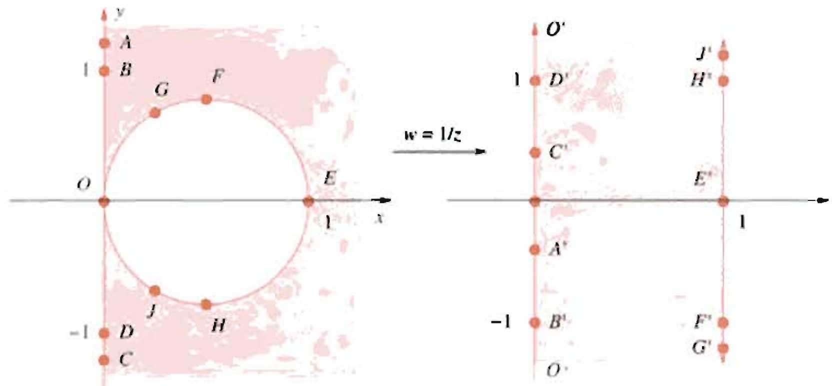
Thus, we see that the angle between two smooth curves at their point of intersection, t_0, in the z-plane is carried over to the w-plane by a mapping $w = f(z)$ that is analytic, provided that $f'(t_0) \neq 0$. A mapping with this property is said to be *conformal*, or *angle preserving*.

Example 2:
Where is the mapping defined by

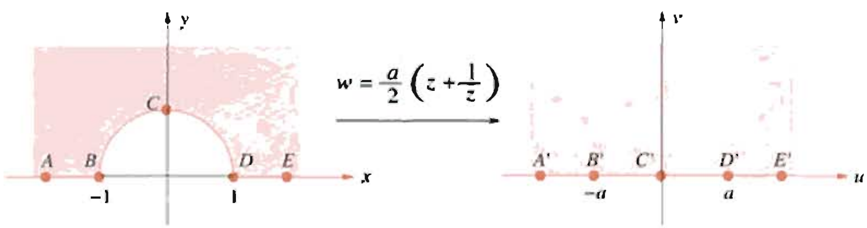
$$w = \frac{3z + 1}{2z - 1}$$

conformal?

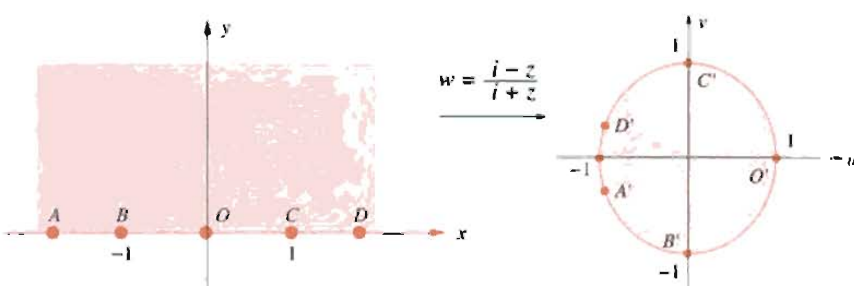
SOLUTION: The function $w = f(z)$ is analytic everywhere except at $z = 1/2$. Furthermore, $dw/dz = -5/(2z - 1)^2$ never equals zero, so the mapping is conformal everywhere except at $z = 1/2$.



7. The region exterior to the unit disk $\left|z-\frac{1}{2}\right| \le \frac{1}{2}$ in the positive x half plane into the infinite vertical strip $0 \le u \le 1$ by $w = 1/z$.



8. The upper half plane with the half disk $|z| \le 1$, $0 \le \theta \le \pi$ removed onto the upper half plane by $w = \frac{a}{2}\left(z + \frac{1}{z}\right)$.



9. The upper half plane into the unit disk centered at $u = 0$, $v = 0$ by $w = \dfrac{i - z}{i + z}$.

Example 4:
Verify the mapping shown in Entry 14 in Table 19.1.

SOLUTION: Let $z = re^{i\theta}$, so that $w = \ln r + i\theta$, or

$$u = \ln r \qquad \text{and} \qquad v = \theta$$

The two arcs are mapped into vertical lines $u = \ln r_1$ and $u = \ln r_2$ and the angles θ_1 and θ_2 are mapped into $v = \theta_1$ and $v = \theta_2$, yielding the rectangle shown in Entry 14. Note that if $r_1 = 0$ and $r_2 = \infty$, then the entire angular arc $\theta_1 \leq \theta \leq \theta_2$ is mapped into the infinite strip $\theta_1 \leq v \leq \theta_2$.

The most important property of conformal transformations and the one that we shall exploit in the next section is the following:

Consider some region R_z in the z-plane that is mapped into a region R_w in the w-plane by a conformal transformation $w = f(z) = u(x, y) + v(x, y)$. Let this mapping be one-to-one, so that the inverse of $f(z)$, $z = f^{-1}(w)$, exists.

If $\phi(x, y)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad in \quad D_z$$

then $\Phi(u, v)$ satisfies

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0 \qquad in \quad D_w$$

where $\phi(x, y) = \Phi(u, v) = \Phi[\,u(x, y), v(x, y)\,]$. Furthermore, if $\phi(x, y) = g(x, y)$ along some boundary curve C_z in the z-plane, then $\Phi(u, v) = g(x(u, v), y(u, v))$ along the image curve C_w in the w-plane.

Problem 20 takes you through the straightforward, but lengthy, proof of this result.

Before going on to the next section, we'll present a short problem to illustrate the above result. Consider Figure 19.28, which shows the cross-section of a long cylindrical conducting sheet touching, but insulated from, a planar conducting sheet perpendicular to the page. Let the planar sheet be held at a potential V_0 and the cylindrical sheet be held at a potential V_1. We wish to solve Laplace's equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

for the potential in the shaded region in Figure 19.28.

None of the methods that we have learned to solve partial differential equations will work here. However, if you scan through Table 19.1, you'll see that Entry 7

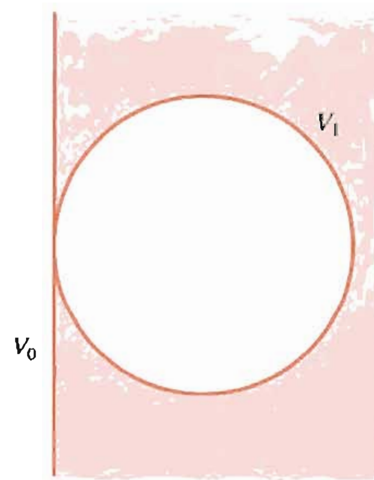


Figure 19.28
The cross-section of a long cylindrical conducting sheet touching, but insulated from, a planar conducting sheet perpendicular to the page.

12. Verify the mapping shown in Entry 7 in Table 19.1.

13. Verify the mapping shown in Entry 8 in Table 19.1.

14. Verify the mapping shown in Entry 9 in Table 19.1.

15. Verify the mapping shown in Entry 10 in Table 19.1.

16. Verify the mapping shown in Entry 13 in Table 19.1.

17. Consider the two parametrized curves, C_1: $z_1 = t + it^2$ and C_2: $z_2 = t + 2it$ $(t \geq 0)$, in the z-plane. Show explicitly that the angle between these two curves at their point of intersection at $t = 2$ is preserved under the mapping $w = z^2$.

18. Consider the two parametrized curves, C_1: $z_1 = t$ and C_2: $z_2 = it$ $(0 \leq t \leq 1)$, in the z-plane. Show that the angle between these two curves at their point of intersection at $z = 0$ is $\pi/2$ radians. Now consider the mapping $w = z^2$. Show that the intersection angle between the image curves in the w-plane is π radians. Why isn't the intersection angle preserved?

19. In the next problem, you need to use the relation

$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$. Derive this result.

20. In this problem we'll show that Laplace's equation remains invariant under a conformal transformation. Let $w = f(z) = u(x, y) + i\,v(x, y)$. Because $f(z)$ is invertible, we can write $u = u(x, y)$, $v = v(x, y)$ and $x = x(u, v)$, $y = y(u, v)$. Now let $\phi(x, y)$ satisfy Laplace's equation in some region R_z, and let $\Phi(u, v) = \phi(x(u, v), y(u, v))$ be the function that results from $\phi(x, y)$ under the conformal transformation. We wish to show that $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ implies that $\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2} = 0$, assuming that $w = f(z)$ is conformal. Apply the chain rule to $\Phi(u, v) = \phi(x(u, v), y(u, v))$ to show that $\frac{\partial\phi}{\partial x} = \frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\Phi}{\partial v}\frac{\partial v}{\partial x}$ and $\frac{\partial\phi}{\partial y} = \frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\Phi}{\partial v}\frac{\partial v}{\partial y}$. Now apply the chain rule a second time to get

$$\begin{aligned}
\frac{\partial^2\phi}{\partial x^2} &= \frac{\partial\Phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial\Phi}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial u}\right) + \frac{\partial v}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial v}\right) \\
&= \frac{\partial\Phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial\Phi}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\left(\frac{\partial^2\Phi}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2\Phi}{\partial v\partial u}\frac{\partial v}{\partial x}\right) + \frac{\partial v}{\partial x}\left(\frac{\partial^2\Phi}{\partial u\partial v}\frac{\partial u}{\partial x} + \frac{\partial^2\Phi}{\partial v^2}\frac{\partial v}{\partial x}\right)
\end{aligned}$$

with a similar equation for $\partial^2\phi/\partial y^2$. Now show that

$$\begin{aligned}
\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= \frac{\partial\Phi}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial\Phi}{\partial v}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\
&\quad + \frac{\partial^2\Phi}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{\partial^2\Phi}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \\
&\quad + 2\frac{\partial^2\Phi}{\partial u\partial v}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]
\end{aligned}$$

Now show that this result reduces to $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = |f'(z)|^2\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right)$.

We could also have solved the problem in Example 3 using Poisson's integral formula for the unit circle (Problem 6).

Let's do one last Example.

Example 4:
Suppose we have two parallel conducting cylindrical sheets as shown in Figure 19.35. Calculate the electrostatic potential in the space surrounding these sheets if the larger one is held at a potential V_1 and the smaller at a potential V_2.

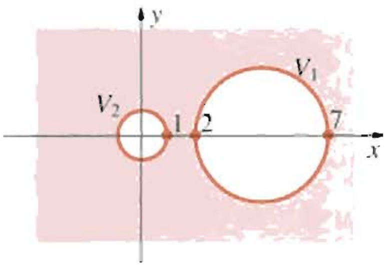


Figure 19.35
The geometry referred to in Example 4.

SOLUTION: Entry 11 of Table 19.1 maps the region surrounding the cylinders into the annular region between two concentric cylinders. According to Entry 11, the mapping is given by

$$w = \frac{z-a}{az-1}$$

where

$$a = \frac{1 + 14 + \sqrt{3 \cdot 48}}{7+2} = 3$$

The outer radius of the annular region is 1 and the inner radius is

$$R = \frac{14 - 1 - 12}{7-2} = \frac{1}{5}$$

Furthermore, the smaller circle in the z-plane is mapped onto the outer circle in the w-plane and the larger circle is mapped onto the inner circle, so the potential is V_2 on the large circle and V_1 on the smaller circle.

Letting ρ be the radial coordinate in the w-plane, the electrostatic potential in the annular region is given by (Problem 2)

$$\Phi(\rho) = A \ln \rho + B$$

The potential is V_2 at $\rho = 1$ and V_1 at $\rho = 1/5$, so

$$\Phi(\rho) = \frac{V_2 - V_1}{\ln 5} \ln \rho + V_2$$

But $\rho = |w|$, so

$$\Phi(u, v) = \frac{V_2 - V_1}{\ln 5} \ln |w| + V_2$$

From $w = (z-3)/(3z-1)$, we find that

$$|w| = \left| \frac{(x-3) + iy}{3x - 1 + 3iy} \right| = \left[\frac{(x-3)^2 + y^2}{(3x-1)^2 + 9y^2} \right]^{1/2}$$

and so

$$\phi(x, y) = \frac{V_2 - V_1}{2 \ln 5} \ln \frac{(x-3)^2 + y^2}{(3x-1)^2 + 9y^2} + V_2$$

This solution is plotted in Figure 19.36 for $V_1 = 0$ and $V_2 = 100$.

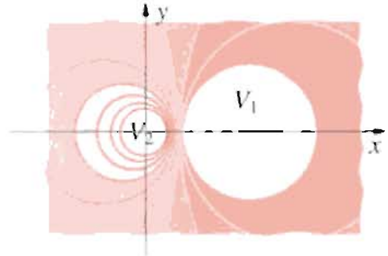


Figure 19.36
The solution given in Example 4 for $V_1 = 0$ and $V_2 = 100$.

19.6 Problems

The first two problems present two simple Dirichlet boundary value problems whose results will be used a number of times.

1. Solve Laplace's equation in the strip $-\infty < u < \infty$, $0 \le v \le a$, assuming that $\Phi(u, 0) = \Phi_0$ and $\Phi(u, a) = \Phi_1$.
2. Solve Laplace's equation in the annulus of radii R_1 and R_2, with $R_2 > R_1$, assuming that $\Phi(R_1, \theta) = \Phi_1$ and $\Phi(R_2, \theta) = \Phi_2$.
3. Verify the mapping that is used in Example 1.
4. Right after Example 1, we solved Example 1 by using $w = z^2$ to map the first quadrant into the upper half plane and then using Poisson's integral formula for the upper half plane. We can by-pass the use of Poisson's integral formula by using the inverse transformation in Entry 2 of Table 19.1. Use the sequential mappings $w_1 = z^2$ and $w_2 = \text{Ln}\, w_1$ to solve Example 1.
5. In the text, we used Poisson's integral formula for the upper half plane to derive Equation 3, the temperature distribution in the upper half w-plane when $\Phi(u, 0) = \begin{cases} T_0 & u < 0 \\ T_1 & u > 0 \end{cases}$. Derive Equation 3 by solving Laplace's equation in polar coordinates.
6. Use Poisson's integral formula for the unit circle to derive Equation 7.
7. Determine the steady-state temperature distribution in the wedge shown in Figure 19.37.
8. Determine the steady-state temperature distribution in the annular segment shown in Figure 19.38. The arcs are insulated.

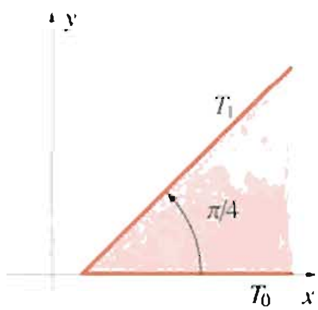


Figure 19.37
The wedge referred to in Problem 7.

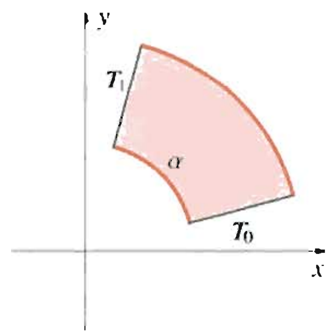


Figure 19.38
The region referred to in Problem 8.

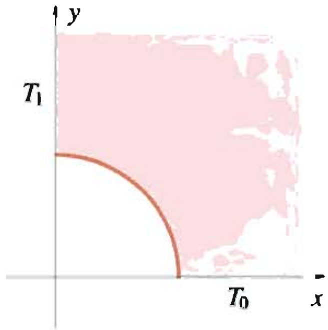


Figure 19.39
The region referred to in Problem 9.

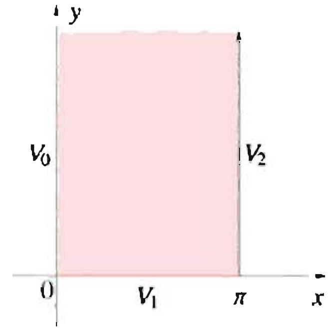


Figure 19.40
The region referred to in Problem 10.

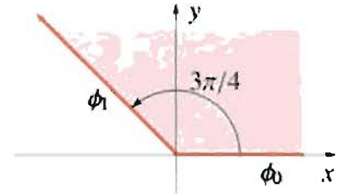


Figure 19.41
The region referred to in Problem 11.

9. Determine the steady-state temperature distribution in the region shown in Figure 19.39. The arc is insulated.

10. How would you modify Example 2 if the region in the z-plane were like that shown in Figure 19.40 rather than the one shown in Entry 4 in Table 19.1.

11. Determine the electrostatic potential in the region shown in Figure 19.41.

12. Verify the mapping used in Example 2.

13. Generalize Equation 6 to the case $\Phi(\zeta, 0) = \begin{cases} V_0 & \zeta < -a \\ V_1 & a < \zeta < b \\ V_2 & \zeta > b \end{cases}$.

14. Show that Equation 6 can be written as

$$\Phi(u, v) = \frac{V_0 - V_1}{\pi} \arg(w_2 + a) + \frac{V_1 - V_2}{\pi} \arg(w_2 - a) + V_2.$$

15. Show that the resulting potential $\phi(x, y)$ in Example 4 satisfies the two boundary conditions.

16. Determine the electrostatic potential in the region surrounding two parallel cylindrical sheets of unit radius whose centers are 4 units apart. Let one sheet be held at a potential ϕ_0 and the other at $-\phi_0$. Show that your result satisfies the boundary conditions. Use a CAS to plot your result.

17. Consider two parallel cylindrical sheets whose cross-sections are shown in Figure 19.42. Let the radius of the inner sheet be 1/2 and that of the outer sheet be 1, and let the surface of the inner sheet be held at a potential V_0 and that of the outer sheet be V_1. Calculate the potential between the cylinders. Show that your result satisfies the boundary conditions. Use a CAS to plot your result.

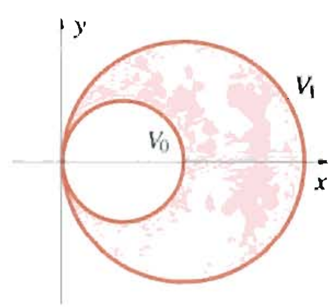


Figure 19.42
The geometry associated with Problem 17.

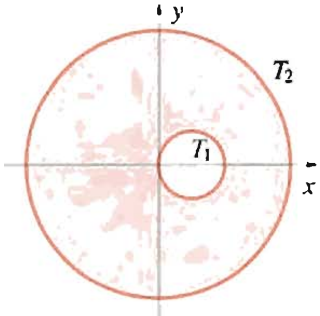


Figure 19.43
The region referred to in Problem 18.

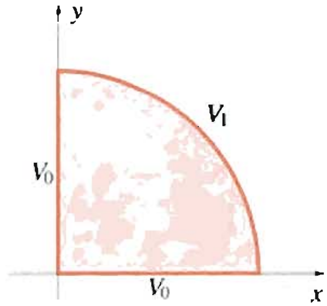


Figure 19.44
The region referred to in Problem 19.

18. Determine the temperature distribution in the region shown in Figure 19.43. Take the inner cylinder to be centered at $x = 1/4$ and its radius to be $1/4$ and the radius of the outer cylinder to be 1. Show that your result satisfies the boundary conditions. Use a CAS to plot your result.

19. Find the electrostatic potential within the region shown in Figure 19.44. Take the radius of the arc to be 1.

19.7 Conformal Mapping and Fluid Flow

One of the nicest applications of conformal mapping to physical problems involves (two-dimensional) fluid flow. The partial differential equations that describe the flow of fluids are derived from mass balance and momentum balance considerations and are fairly complicated in general. However, if we can ignore viscosity and if the fluid is incompressible, then the velocity of the fluid under steady-state or stationary conditions is derivable from a *velocity potential*, $\phi(x, y)$, such that

$$v_x = \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad v_y = \frac{\partial \phi}{\partial y} \tag{1}$$

where v_x and v_y are the x- and y-components of the velocity of a small element of the fluid. The mathematical condition that the fluid be incompressible is that $\operatorname{div} \mathbf{v} = 0$ (Problem 1), which in two dimensions is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{2}$$

If we combine Equations 1 and 2, we see that the velocity potential satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{3}$$

It turns out to be convenient to consider $\phi(x, y)$ to be the real part of a *complex potential*

$$\Omega(z) = \Omega(x, y) = \phi(x, y) + i\,\psi(x, y) \tag{4}$$

If $\Omega(x, y)$ is an analytic function of $z = x + iy$, then not only does $\phi(x, y)$ satisfy Laplace's equation, but so does $\psi(x, y)$:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{5}$$

Now we know from Section 18.2 that the two families of curves

$$\phi(x, y) = c_1 \qquad \text{and} \qquad \psi(x, y) = c_2$$

are orthogonal. (See Problem 2, also.) Therefore, because of Equation 1, the velocity vector $\mathbf{v} = \mathbf{i}\,v_x + \mathbf{j}\,v_y = \text{grad}\,\phi$, and thus is normal to the $\phi(x, y) = c_1$ curves, and consequently, tangent to the $\psi(x, y) = c_2$ curves. Therefore, the fluid flow follows along the $\psi(x, y) = c_2$ curves, which are called *streamlines*. The function $\psi(x, y)$ itself is called the *stream function*.

Example 1:
Interpret the flow described by the complex potential

$$\Omega(z) = v_0 z$$

SOLUTION: The velocity potential and the stream function are

$$\phi(x, y) = v_0 x \qquad \text{and} \qquad \psi(x, y) = v_0 y$$

The velocity is simply $v_x = v_0$, which represents uniform motion in the x direction. The streamlines are $v_0 y =$ constant, which are horizontal lines. Thus, the complex potential $\Omega(z) = v_0 z$ represents uniform flow in the x direction. The complex potential $\Omega(z) = v_0 e^{-i\alpha} z$ represents flow at an angle α with respect to the x axis (Problem 3).

Example 2:
Interpret the flow described by

$$\Omega(z) = v_0 z^2$$

SOLUTION: The streamlines in this case are given by the family of rectangular hyperbolas

$$\psi(x, y) = 2xy = c$$

becomes the complex potential,

$$\Omega(z) = \frac{v_0 a}{2}\left(z + \frac{1}{z}\right)$$

in the z-plane. We want $\Omega(z) \to v_0 z$ as $r \to \infty$, so we choose $a = 2$. The streamlines are given by

$$\psi(r, \theta) = v_0\left(r - \frac{1}{r}\right)\sin\theta = \text{constant}$$

and are shown in Figure 19.46. Even though we mapped from one upper half plane to another, the problem is symmetric about the x axis and so we show streamlines both above and below the x axis in Figure 19.46.

Problem 6 has you show that the speed v of a fluid element at any point in the fluid is given by

$$v = \left|\frac{d\Omega}{dz}\right| \tag{6}$$

which for the complex potential in Example 4 gives

$$v = v_0\left|1 - \frac{1}{z^2}\right|$$

Thus, we see that $v = 0$ at the points $z = \pm 1$. Points at which $v = 0$ are called *stagnation points*. Problem 11 asks you to show that the stagnation points of the flow depicted in Figure 19.46 occur at the front and back ends of the cylinder. The corner of the right-angle region shown in Figure 19.45a is also a stagnation point.

The map

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{7}$$

has played a venerable role in aerodynamics. By letting $z = ae^{i\theta}$, we see that

$$w = \frac{1}{2}\left(a + \frac{1}{a}\right)\cos\theta + \frac{i}{2}\left(a - \frac{1}{a}\right)\sin\theta$$

Using $\cos^2\theta + \sin^2\theta = 1$, we obtain

$$\frac{u^2}{\left[\frac{1}{2}\left(a + \frac{1}{a}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(a - \frac{1}{a}\right)\right]^2} = 1$$

Thus we see that Equation 7 maps circles $|z| = a$ onto ellipses in the w-plane.

Now, if the circle in the z-plane is not centered at the origin, passes through the point $z = 1$, and contains the point $z = -1$ within it, you get quite a different result.

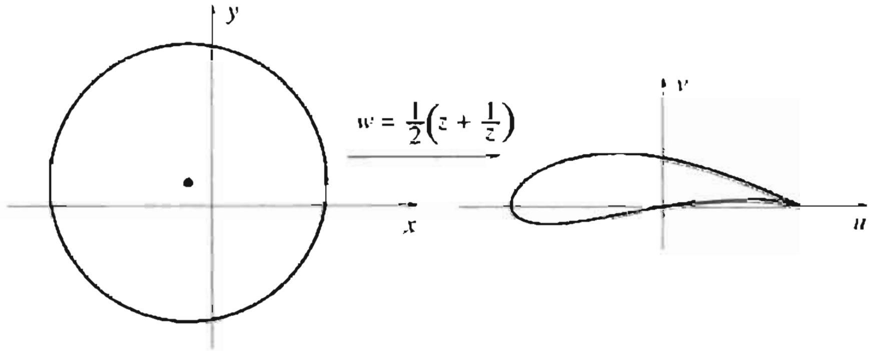


Figure 19.50
The mapping of the circle described by $\left|z + \frac{1}{5} - \frac{i}{5}\right| = \frac{(37)^{1/2}}{5}$ into the w-plane by the mapping in Equation 7.

In this case, the figure in the w-plane begins to resemble an airfoil (Figure 19.50). The resultant figures in the w-plane are called *Joukowski profiles* and Equation 7 is called a *Joukowski map* after Nikolai Joukowski, who is known as the father of Russian aviation. By starting with figures in the z-plane that are nearly circles, it's possible to produce figures in the w-plane that describe a great variety of airfoils.

Before we leave this section, we should point out that the concept of a complex potential is not limited to a treatment of fluid flow. In electrostatics, $\phi(x, y)$ is the electrostatic potential from which we calculate the electric field intensity according to

$$\mathbf{E} = -\mathbf{i}\,\frac{\partial \phi}{\partial x} - \mathbf{j}\,\frac{\partial \phi}{\partial y}$$

The curves $\phi(x, y) = c_1$ and $\psi(x, y) = c_2$ are called the equipotential curves and the flux lines, respectively. For steady heat flow, $\phi(x, y)$ corresponds to the temperature and $\psi(x, y)$ corresponds to heat flow lines.

19.7 Problems

1. Show that Equation 2 governs an incompressible fluid. *Hint*: Start with the continuity equation.
2. Show that if $f(z) = u(x, y) + i\,v(x, y)$ is an analytic function, then the two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal.
3. Show that the complex potential $\Omega(z) = v_0 e^{-i\alpha} z$ represents uniform flow that makes an angle α with the x axis.
4. Derive a complex potential that corresponds to uniform flow in the x direction.
5. Show that $d\Omega/dz = v_x - i\,v_y$. Verify this result for uniform flow making an angle α with the x axis.
6. Show that $|v(z)| = |d\Omega/dz|$.
7. Deduce the type of fluid flow governed by the complex potential $\Omega(z) = ik \ln z$.
8. Interpret the type of fluid flow associated with the complex potential $\Omega(z) = k \ln z$. (See the next problem also.)

9. Show that the electrostatic potential due to a line of uniform continuous charge density λ is given by $(\lambda/2\pi\epsilon_0)\ln(a/r)$, where r is the distance from the wire and a is an arbitrary constant.

10. Re-do Example 4 for a cylinder of radius b.

11. Show that the stagnation points of the flow depicted in Figure 19.46 occur at the front and back ends of the cylinder.

12. Find the streamlines for the flow of an incompressible, non-viscous fluid within the corner shown in Figure 19.45a.

13. Find the streamlines for the flow of an incompressible, non-viscous fluid over a wedge, as shown in Figure 19.51.

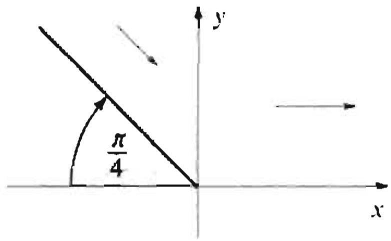


Figure 19.51
The wedge referred to in Problem 13.

14. Use a CAS to verify the mapping in Figure 19.50.

15. Let's look at the circle $\left|z+\frac{1}{2}\right|=\frac{3}{2}$ and its resulting mapping in the w-plane under $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$. The outer angle to the circle in the z-plane at $z=1$ is π, but the angle at $w=1$, the image point of $z=1$, is 2π in the w-plane. Why isn't the angle preserved at this point?

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Johann Bernoulli (1667–1748), who, along with his brother Jacob, was a great promulgator of the Leibnitz (differential) formulation of calculus, was born on July 27, 1667, in Basel, Switzerland. Johann also had little interest or talent for the family business, and so his father reluctantly allowed him to study medicine at the University of Basel. While at Basel, he studied mathematics under his brother Jacob, becoming Jacob's equal in just two years. In 1691, he traveled to Paris, where he met the Marquis de l'Hôpital, who hired Bernoulli to teach him the new calculus of Leibnitz. In 1696, l'Hôpital published the first calculus book, which was based mainly on Bernoulli's lessons. Bernoulli later claimed that l'Hôpital's rule was actually his work. Bernoulli was largely responsible for the spread of Leibnitz's formulation of the calculus. Upon his return to Basel, he continued with his medical studies, as well as with mathematics. At this time, he married Dorothea Falkner. Three of their sons became mathematicians, which was to become a family tradition. During this time, Johann and Jacob had a serious falling-out over mathematics. Both brothers had difficult personalities, but it seems that the greater portion of blame lies with Johann. Johann has been described as intolerant, mean, and nasty to anyone who disagreed with him, including his own son, Daniel. In 1695, he accepted the Chair of Mathematics at the University of Groningen in Holland over the objections of his wife and father-in-law. During his ten years at Groningen, he was involved in a number of disputes, including religious ones. In 1705, the family returned to Basel. While en route, his brother Jacob died, and Johann succeeded his brother at Basel. In 1718, he laid the foundation for the calculus of variations when he revisited the brachistochrone problem, the cause of the bitter hostility between the brothers. Bernoulli died on January 1, 1748, in Basel. Johann Bernoulli was known as the "Archimedes of his age," which is inscribed on his tombstone.

CHAPTER 20

Calculus of Variations

A standard problem in calculus is to find extrema of functions, whether they are functions of a single variable or functions of many variables. A standard problem in the calculus of variations is the following: What curve lying in a plane connecting two given points has the shortest arc length? In other words, what is the shortest distance between two points in a plane? Of course, the answer is a straight line, but the calculus of variations provides us with a systematic procedure for proving it. We can formulate this problem by specifying the two points as (x_1, y_1) and (x_2, y_2) and then expressing the arc length in terms of an integral involving $y(x)$ as

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} (dx^2 + dy^2)^{1/2} = \int_{x_1}^{x_2} (1 + y')^{1/2} dx$$

The value of I depends upon the path that we take from (x_1, y_1) to (x_2, y_2), and we want to minimize I with respect to all possible functions $y(x)$, or at least with respect to all functions within a certain class, such as all smooth functions. When we learn how to do this in Section 1, we'll see that $y'(x) =$ constant does indeed minimize I, above.

The determination of the shortest distance between two points in a plane is a fairly simple problem, but what about the shortest distance between two points on some other surface, such as a sphere or a right circular cone? Such curves are called *geodesics*, and geodesics play an important role in the theory of relativity. As we shall see, the calculus of variations provides us with a method to determine geodesics on most any surface.

Another standard problem of the calculus of variations, the one that actually initiated the development of the field, is the so-called brachistochrone problem, proposed by Johann Bernoulli in 1696. Consider two points a and b in a vertical plane with the x axis being horizontal and the y axis being directed downward. (See Figure 20.2.) If a particle starts from rest at point a, then along which path will the particle reach point b in the shortest time? We'll see that in this case, we need to minimize the functional

$$I = \int_a^b \frac{(1 + y'^2)^{1/2}}{y^{1/2}} dx$$

with respect to $y(x)$. The name *brachistochrone* derives from the Greek for shortest (brachistos) time (chronos).

Not only does the calculus of variations allow us to solve problems like those above, but many others, such as the determination of the closed curve of a given length that encloses the largest area and the shape of a uniform flexible cable of given length suspended at its ends by two given points. As important as these problems might be mathematically, however, it turns out that many of the laws of physics can be expressed in variational form. For example, the laws of classical mechanics can be expressed succinctly by Hamilton's principle, which says that a (conservative) mechanical system will evolve along a trajectory such that the integral

$$I = \int_{t_1}^{t_2} (K - V)\, dt$$

where K is the kinetic energy and V is the potential energy, is an extremum with respect to all trajectories. The laws of optics can be formulated by Fermat's principle, which says that the time it takes light to travel from one fixed point to another is an extremum with respect to time. Furthermore, the Schrödinger equation can be formulated as a variational solution to the problem of minimizing the energy of a quantum-mechanical system with respect to all possible wave functions. This has produced a huge industry involving the calculation of atomic and molecular properties from first principles. We shall see examples of all the calculations that we have discussed here throughout this chapter.

20.1 The Euler Equation

All the integrals that we mentioned in the introduction are special forms of the integral

$$I = \int_a^b F(y, y', x)\, dx \tag{1}$$

The value of I depends upon $y(x)$ in the sense that different functions $y(x)$ will yield different values of I. Equation 1 represents a mapping of some given class of functions into a set of numbers. We say that I is a *functional* of $y(x)$ and write $I = I[y(x)]$. The problem that we wish to solve is the determination of the particular function $y(x)$ that makes I an extremum. The simplest (and common) case assumes that $y(x)$ is prescribed at the end points a and b. Therefore, we use a *trial function* of the form

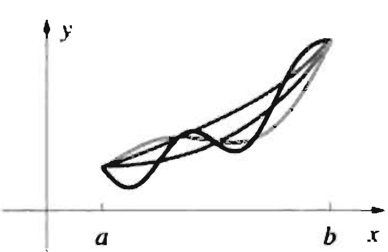


Figure 20.1
Examples of trial functions that can be used to minimize the functional given by Equation 1.

$$Y(x, \epsilon) = y(x) + \epsilon\eta(x) \tag{2}$$

where $\eta(x)$ is chosen such that $\eta(a) = \eta(b) = 0$, so that $Y(x)$ and $y(x)$ coincide at the end points (Figure 20.1). As the notation suggests, we plan to let $\epsilon \to 0$

eventually. We will always assume that the partial derivatives of $F(y, y', x)$ exist and are differentiable and that $y(x)$ is twice differentiable.

If we substitute Equation 2 into Equation 1, we obtain

$$I(\epsilon) = \int_a^b F[Y(x, \epsilon), Y'(x, \epsilon, x)]\, dx \tag{3}$$

Because $y(x)$ is the function that extremizes the value of I, $I(\epsilon)$ takes on its extreme value when $\epsilon = 0$, and so we can write

$$\frac{dI}{d\epsilon} = 0 \qquad \text{when} \quad \epsilon = 0 \tag{4}$$

If we differentiate $I(\epsilon)$ in Equation 3 with respect to ϵ, we obtain

$$\begin{aligned}\frac{dI}{d\epsilon} &= \int_a^b \left(\frac{\partial F}{\partial Y}\frac{\partial Y}{\partial \epsilon} + \frac{\partial F}{\partial Y'}\frac{\partial Y'}{\partial \epsilon} \right) dx \\ &= \int_a^b \left[\frac{\partial F}{\partial Y}\eta(x) + \frac{\partial F}{\partial Y'}\eta'(x) \right] dx\end{aligned}$$

Assuming that $\partial F/\partial Y$ and $\partial F/\partial Y'$ are continuous functions of ϵ, $\partial F/\partial Y \to \partial F/\partial y$ and $\partial F/\partial Y' \to \partial F/\partial y'$ as $\epsilon \to 0$, and so we have

$$\int_a^b \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\frac{d\eta}{dx} \right] dx = 0 \tag{5}$$

Equation 5 is a *necessary* condition for I to be an extreme with respect to all possible trial functions $Y(x)$ in Equation 2. It certainly isn't a sufficient condition, which depends upon properties of the second derivative of I with respect to ϵ. It turns out that an examination of this second derivative is fairly involved, so we'll simply accept the fact that Equation 5 gives us an *extremum*, or that I is *stationary*, and appeal to physical arguments that it gives a maximum or a minimum.

We can cast Equation 5 into a more convenient form by integrating the second term by parts to obtain

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right] \eta(x)dx + \left[\eta(x)\frac{\partial F}{\partial y'} \right]_a^b = 0 \tag{6}$$

Assuming that $\eta(a) = \eta(b) = 0$ (*fixed end points*), Equation 6 becomes

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right] \eta(x)\, dx = 0 \tag{7}$$

Because Equation 7 must hold for arbitrary $\eta(x)$ (other than $\eta(a) = \eta(b) = 0$), we see that the condition that I be an extremum is that

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 \tag{8}$$

Equation 8, called the *Euler equation*, is a central equation of the calculus of variations. Realize that it is an ordinary, nonlinear, second-order differential equation in $y(x)$ because $F(y, y', x)$ is a given function of y and y'.

Example 1:

Use Equation 8 to show that the shortest curve connecting two points in a plane is a straight line.

SOLUTION: The integral in question is

$$I = \int_a^b ds = \int_a^b (dx^2 + dy^2)^{1/2}$$
$$= \int_a^b (1 + y'^2)^{1/2} dx$$

where $y' = dy/dx$. The integrand is independent of y, so Equation 8 becomes simply

$$\frac{d}{dx}\left[\frac{\partial(1 + y'^2)^{1/2}}{\partial y'}\right] = \frac{d}{dx}\frac{y'}{(1 + y'^2)^{1/2}} = 0$$

or $y'(x) =$ constant. Thus, we see that the curve is a straight line.

The result of Example 1 comes as no great surprise. You probably also know that the shortest distance between two points on the surface of a sphere lies on a great circle. We'll prove this result in Example 2, but before doing so we'll simplify Equation 8 for the special case that F does not depend explicitly on the independent variable x. We start with the relation

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{d}{dx}\frac{\partial F}{\partial y'} + y''\frac{\partial F}{\partial y'}$$

and then use Equation 8 for $d(\partial F/\partial y')/dx$ on the right side of this equation to write

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'}$$
$$= \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx} \tag{9}$$

We can express the right side of Equation 9 in a compact form by first recalling that the total derivative of $F(y, y', x)$ with respect to x is given by

$$\frac{dF}{dx} = \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx} + \frac{\partial F}{\partial x}$$

20.1 Problems

1. We'll evaluate the integral for ϕ in Example 2 in this problem. First factor $c_1^2 \sin^4\theta$ from the denominator and use the identity $\csc^2\theta = 1 + \cot^2\theta$ to get $\phi = \displaystyle\int \frac{\csc^2\theta\, d\theta}{[(a/c_1)^2 - 1 - \cot^2\theta]^{1/2}}$. Now notice that this integrand is of the form $d \sin^{-1} u = du/(1-u^2)^{1/2}$ if you let $u = \cot\theta/[(a/c_1)^2 - 1]^{1/2}$. Show that this substitution leads to ϕ given in Example 2.

2. Determine the Euler equation for $I = \displaystyle\int_a^b (y^2 - y'^2)dx$.

3. Determine the Euler equation for $I = \displaystyle\int_a^b (py'^2 - qy^2)dx$. Does the result look familiar?

4. Find the general solution of the Euler equation associated with $I = \displaystyle\int_a^b x(1 - y'^2)^{1/2}dx$.

5. Determine the equation of the curve that passes through the points (0, 0) and (1, 1) and for which $I = \displaystyle\int_0^1 \frac{(1+y'^2)^{1/2}}{y}dx$ is an extremum.

6. Determine the Euler equation for $I = \displaystyle\int_a^b (xy'^2 - yy' + y)dx$.

7. Re-do Example 1 in plane polar coordinates. Take θ to be the independent variable.

8. Determine the geodesic on the surface of a right circular cylinder of radius a. *Hint*: Take $ds^2 = a^2 d\theta^2 + dz^2$.

9. Determine the geodesics on the surface of a right circular cone. *Hint*: Use $ds^2 = dr^2 + r^2\sin^2\alpha\, d\phi^2$ because the surface of a cone is described by $\sin\theta = \sin\alpha =$ constant.

10. Set up the equations to determine c, θ_{initial}, and θ_{final} for the solution to the brachistochrone problem given by Equations 14.

11. Show that the time of descent of the mass in the brachistochrone problem is given by $T = \left(\dfrac{c}{2g}\right)^{1/2}\theta_{\text{final}}$. *Hint*: Use Equation 13.

12. Evaluate the integral that leads to Equation 15 by making the substitution $y = c_1\cosh u$.

13. Show that $c_2 = 0$ in Equation 15, thus yielding Equation 16.

14. Show that the roots of $1 - c\cosh(1/2c) = 0$ are $c = 0.2351$ and $c = 0.8483$.

15. We'll explore the soap film problem numerically in this problem. Use a CAS to show that the area given by Equation 18 is equal to $2\pi a^2$ when $b/a = 0.527696$ and c is correspondingly equal to $0.825519a$. Show, however, that there are still two roots to Equation 17 for this value of b/a. Show that the area corresponding to this other root is greater than $2\pi a^2$. Now show that the value of $b/a = 0.662744$ corresponds to the point where there is no real solution to Equation 17. This calculation illustrates that we obtain only a local minimum for $0.527696 < b/a < 0.662744$.

16. Derive Equation 16 using Equation 8 instead of Equation 10.

If the force $\mathbf{F}$ is conservative, then it can be written as the gradient of a potential function $V(x, y, z)$:

$$\mathbf{F} = -\text{grad } V = -\mathbf{i}\frac{\partial V}{\partial x} - \mathbf{j}\frac{\partial V}{\partial y} - \mathbf{k}\frac{\partial V}{\partial z}$$

and we can write $\mathbf{F} \cdot \delta\mathbf{r}$ in Equation 3 as

$$-\delta V = -\frac{\partial V}{\partial x}\delta x - \frac{\partial V}{\partial y}\delta y - \frac{\partial V}{\partial z}\delta z$$

which essentially defines δV. Therefore, Equation 3 can be written as

$$\int_{t_1}^{t_2} (\delta K - \delta V)\, dt = 0$$

or

$$\delta \int_{t_1}^{t_2} (K - V)\, dt = 0 \tag{4}$$

Equation 4 is Hamilton's principle for a conservative system. We derived it for a single particle, but the derivation can be extended to a system of particles by summing and even to continuous systems by integrating. The quantity $K - V$ in Equation 4 is called the *Lagrangian*, L, and so we can write Hamilton's principle as

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \tag{5}$$

We said above that the advantage of Hamilton's principle is that it can be applied equally conveniently to any coordinate system. The Lagrangian can be expressed in terms of any set of coordinates, whether they are distances, angles, or whatever. For example, we might use the three spherical coordinates r, θ, and ϕ instead of x, y, and z. These new coordinates are called *generalized coordinates* and are customarily denoted by q_j. If it takes $3N$ coordinates to specify a system (as it would for N point masses), the generalized coordinates are $q_1, q_2, \ldots, q_{3N}$. The time derivatives of the generalized coordinates, $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{3N}$, are called *generalized velocities*. Hamilton's principle, Equation 5, can be written as

$$\delta \int_{t_1}^{t_2} L(q_1, q_2, \ldots, q_{3N}, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{3N}, t)\, dt = 0 \tag{6}$$

Equation 6 constitutes a variational problem involving $3N$ dependent variables, rather than just one as we treated in the previous section. For simplicity, let's assume that we have just two dependent variables, and derive the Euler equation that extremizes

$$I = \delta \int_{t_1}^{t_2} L(q_1, q_2, \dot{q}_1, \dot{q}_2, t)\, dt \tag{7}$$

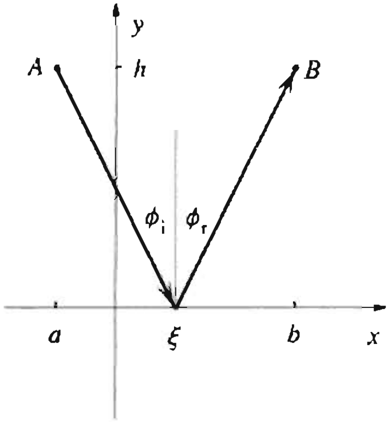


Figure 20.10
The geometry associated with Example 3.

Example 3:
Use the geometry in Figure 20.10 to derive the law of refraction from Fermat's principle that the angle of incidence is equal to the angle of reflection. Assume that n is a constant.

SOLUTION: Consider the geometry in Figure 20.10. We know from Equation 13 that the light will travel in a straight line because n is a constant, so we want to fix P to be on the x axis and find the value of $P(\xi, 0)$ that minimizes the distance APB. The total distance is

$$d = [(\xi - a)^2 + h^2]^{1/2} + [(\xi - b)^2 + h^2]^{1/2}$$

Minimize this expression with respect to ξ to obtain

$$\frac{\xi - a}{[(\xi - a)^2 + h^2]^{1/2}} - \frac{\xi - b}{[(\xi - b)^2 + h^2]^{1/2}} = 0$$

or $(\xi - a)^2 = (\xi - b)^2$, or

$$\xi = \frac{a + b}{2}$$

which is midway between a and b. Thus, we see that the angle of incidence is equal to the angle of reflection.

20.2 Problems

1. Derive Equation 8.
2. Find an extremum of $I = \int_{t_1}^{t_2} (\dot{x}^2 + \dot{y}^2)^{1/2} dt$, where $x(t_j) = x_j$ and $y(t_j) = y_j$ for $j = 1$ and 2 are given. Describe the result.
3. Extend Problem 2 to three dimensions.
4. Derive and solve Lagrange's equation of motion for a mass falling vertically under the influence of gravity.
5. Derive Lagrange's equation of motion for a particle in a gravitational field constrained to lie on a circle of radius a in a fixed vertical plane.
6. Derive Lagrange's equation of motion for a simple pendulum of length l in terms of θ, the angle that the pendulum makes with the vertical.
7. Derive Lagrange's equations of motion of the double pendulum shown in Figure 20.11.
8. What form do Lagrange's equations of motion in the previous problem take on for small oscillations?
9. Derive Lagrange's equation of motion for a mass hanging from a spring whose force constant is k, given that the point of suspension moves vertically according to $x = a \sin \omega t$. Neglect gravity.

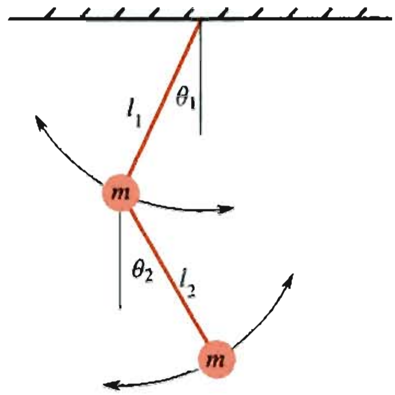


Figure 20.11
The double pendulum referred to in Problem 7.

10. Derive Lagrange's equations of motion for a particle moving in two dimensions under a central potential $V(r)$. Which of these equations illustrates the law of conservation of momentum? Is angular momentum conserved if the potential depends upon θ as well?

11. Derive Equation 12.

12. Calculate the path that a ray of light will travel if the index of refraction varies as a/y, where a is a constant.

13. Calculate the path that a ray of light will travel if the index of refraction varies as ay, where a is a constant.

14. Show that a geodesic on a surface described by orthogonal curvilinear coordinates α and β is given by an extremum of $I = \int_a^b \left[h_\alpha^2 + h_\beta^2\left(\frac{d\beta}{d\alpha}\right)^2\right]^{1/2} d\alpha$, where $h_\alpha^2 = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2$ and $h_\beta^2 = \left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 + \left(\frac{\partial z}{\partial \beta}\right)^2$.

15. Use the result of the previous problem to derive an expression for the integral to minimize for the geodesics on the surface of a right cylinder.

20.3 Variational Problems with Constraints

Just as we minimized functions of several variables with constraints in Section 6.9, we can find extrema of functionals subject to certain constraints. For example, suppose we wish to find an extreme value of

$$I = \int_a^b F(y, y', x)\, dx \tag{1}$$

subject to the constraint condition

$$J = \int_a^b G(y, y', x)\, dx \tag{2}$$

Obviously, $2l \geq 2a$.

Because $F + \lambda G$ does not depend explicitly on x, we can use the first integral given by Equation 6 to write

$$\rho g y - \lambda = (1 + y'^2)^{1/2} c_1$$

or

$$y'^2 = \left(\frac{\rho g y - \lambda}{c_1}\right)^2 - 1$$

Solving for x, as we did for the soap film problem in Section 1, gives

$$x = c_1 \int \frac{dy}{[(\rho g y - \lambda)^2 - c_1^2]^{1/2}} \tag{7}$$

Following Section 1, let $\rho g y - \lambda = c_1 \cosh z$ and get $x = c_1 z/\rho g + c_2$, or (Problem 3)

$$y = \frac{\lambda}{\rho g} + \frac{c_1}{\rho g} \cosh \frac{\rho g (x - c_2)}{c_1} \tag{8}$$

There are three constants (c_1, c_2, and λ) in Equation 7 and three conditions ($y(\pm a) = 0$ and $J = 2l$) to specify them. Problem 4 has you show that $y(x)$ can be written as

$$y(x) = \frac{\alpha}{\rho g}\left(\cosh \frac{\rho g x}{\alpha} - \cosh \frac{\rho g a}{\alpha}\right) \tag{9}$$

where α is given by $\alpha \sinh(\rho g a/\alpha) = \rho g l$. Thus, we see that the hanging cable is described by a catenary (Figure 20.12).

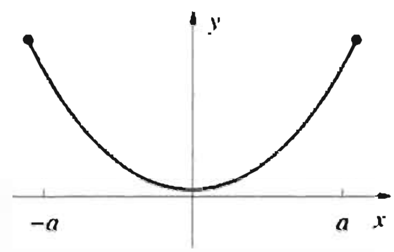


Figure 20.12
The solution to Example 1 with $\rho g = a = 1$ and $l = 1.25$. (α turns out to be 0.8455.)

Example 2:
Determine the curve of length l which passes through the points (0, 0) and (1, 0) and for which the area between the curve and the x axis is a maximum.

SOLUTION: The area is given by

$$I = \int_0^1 y\, dx$$

and the (fixed) length is given by

$$J = \int_0^1 (1 + y'^2)^{1/2} dx = l$$

where $l > 1$. In this case,

$$F + \lambda G = y + \lambda(1 + y'^2)^{1/2}$$

does not depend explicitly on x, so we can use the first integral given by Equation 6, which gives

$$\frac{\lambda y'^2}{(1+y'^2)^{1/2}} - y - \lambda(1+y'^2)^{1/2} = c$$

or

$$\lambda = (c_1 - y)(1+y'^2)^{1/2}$$

where $c_1 = -c$. Solving for dy/dx and then integrating gives

$$x = \pm \int \frac{(c_1 - y)dy}{[\lambda^2 - (c_1 - y)^2]^{1/2}}$$

Let $z = c_1 - y$, integrate, and then rearrange to get

$$(x - c_2)^2 + (y - c_1)^2 = \lambda^2$$

Thus, our result is a circle of radius λ centered at (c_2, c_1). We can determine c_1, c_2, and λ using the fact that the circle passes through the points $(0, 0)$ and $(1, 0)$ and using the constraint that the arc length is l. The result is $c_1 = 0$, $c_2 = 1/2$, and $\lambda = 1/2$, which yields a maximum because the straight line $y = 0$ produces a minimum.

Another type of problem that can be solved using the methods of this section is called an *isoperimetric problem*: Of all the closed plane curves with perimeter l, which one encloses the largest area?

Example 3:
Determine the curve of length l that encloses the largest area.

SOLUTION: We use the equation (Problem 7.2.18)

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$
$$= \frac{1}{2}\oint_C (xy' - y)\,dx$$

with the constraint

$$\int_C (1+y'^2)^{1/2} dx = l$$

We use Equation 4 with

$$F + \lambda G = \frac{1}{2}(xy' - y) + \lambda(1+y'^2)^{1/2}$$

to obtain

$$\frac{d}{dx}\left[\frac{x}{2} + \frac{\lambda y'}{(1+y'^2)^{1/2}}\right] + \frac{1}{2} = 0$$

or

$$\frac{d}{dx}\left[\frac{\lambda y'}{(1+y'^2)^{1/2}}\right] + 1 = 0$$

Integrate once to obtain

$$\frac{\lambda y'}{(1+y'^2)^{1/2}} + x = c_1$$

Solve for y' to get

$$\frac{dy}{dx} = \pm\frac{x - c_1}{[\lambda^2 - (x-c_1)^2]^{1/2}} \tag{10}$$

Let $z = x - c_1$ and integrate again (Problem 11)

$$y(z) = \pm(\lambda^2 - z^2)^{1/2} + c_2$$

or

$$y(x) = \pm[\lambda^2 - (x-c_1)^2]^{1/2} + c_2 \tag{11}$$

This result can be written as

$$(x-c_1)^2 + (y-c_2)^2 = \lambda^2$$

which is the equation of a circle centered at (c_1, c_2) and of radius $\lambda = l/2\pi$. In other words, a circle is the curve of a given length that encloses the largest area.

20.3 Problems

1. Derive Equation 4.
2. Derive Equation 6.
3. Derive Equation 8 from Equation 7.
4. Show that Equation 8 can be written as Equation 9.
5. Show that a function which extremizes I with the constraint J = constant also extremizes J with the constraint I = constant.
6. Find the equation of the shortest curve that passes through the points $(a, 0)$ and $(b, 0)$ and encloses a given area A between the curve and the x axis. *Hint*: Use the result of the previous problem.
7. Suppose we wish to find an extremum of the integral $\int_a^b F(u, v, u_x, v_x, x)dx$ subject to the

Example 2:
Use the trial function given by Equation 12 to calculate the smallest eigenvalue of the problem in Example 1.

SOLUTION: We must first calculate the elements of the secular determinant. The integrals are conveniently evaluated in terms of the beta function, in which case we have

$$L_{11} = 2B(2,2) = \frac{2\Gamma(2)\Gamma(2)}{\Gamma(4)} = \frac{1}{3}$$

$$L_{12} = L_{21} = 2B(3,3) = \frac{2\Gamma(3)\Gamma(3)}{\Gamma(6)} = \frac{1}{15}$$

$$\begin{aligned} L_{22} &= -2B(3,3) + 12B(4,3) - 12B(5,3) \\ &= -\frac{2\Gamma(3)\Gamma(3)}{\Gamma(6)} + \frac{12\Gamma(4)\Gamma(3)}{\Gamma(7)} - \frac{12\Gamma(5)\Gamma(3)}{\Gamma(8)} = \frac{2}{105}. \end{aligned}$$

$$S_{11} = B(4,3) = \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} = \frac{1}{60}$$

$$S_{12} = B(5,4) = \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)} = \frac{1}{280}$$

$$S_{22} = B(6,5) = \frac{\Gamma(6)\Gamma(5)}{\Gamma(11)} = \frac{1}{1260}$$

The secular determinant (Equation 23) becomes

$$\begin{vmatrix} \frac{1}{3} - \frac{\lambda_\phi}{60} & \frac{1}{15} - \frac{\lambda_\phi}{280} \\ \frac{1}{15} - \frac{\lambda_\phi}{280} & \frac{2}{105} - \frac{\lambda_\phi}{1260} \end{vmatrix} = 0$$

which gives

$$\lambda_\phi^2 - 224\lambda_\phi + 4032 = 0$$

The two roots to this equation are 19.7395 . . . and 204.26 . . . . We take the lower root and obtain $\lambda = 19.7395\ldots$ compared with $\lambda = 20$ from using just one term.

So far in this section we have discussed how to find approximate eigenvalues associated with Sturm-Liouville problems using the Rayleigh-Ritz method. It turns out that there is a very close connection between Sturm-Liouville problems and the calculus of variations. Consider the functional

$$I = \int_a^b [p(x)y'^2(x) - q(x)y^2(x)]\,dx \tag{24}$$

where p and q are given functions of x. The extremum of I subject to the constraint

$$J = \int_a^b r(x)y^2(x)\,dx = \text{constant} \tag{25}$$

where $r(x)$ is a given function of x, is given by (Problem 6)

$$\int_a^b \left\{ \frac{d}{dx}[p(x)y'(x)] + [q(x) + \lambda r(x)]\,y(x) \right\} \eta(x)\,dx - \Big[p(x)y'(x)\eta(x)\Big]_a^b = 0 \tag{26}$$

Note that this result is equivalent to the Sturm-Liouville differential equation,

$$\frac{d}{dx}[p(x)y'(x)] + [q(x) + \lambda r(x)]\,y(x) = 0$$

along with the boundary conditions that $p(x)y'(x)$ vanish at an end where $y(x)$ is not prescribed. In particular, you can see that the homogeneous boundary conditions

$$y(a) = 0 \quad \text{or} \quad y'(a) = 0 \qquad \text{and} \qquad y(b) = 0 \quad \text{or} \quad y'(b) = 0$$

easily satisfy these conditions. Thus, determining an extremum of Equation 24 with the constraint given by Equation 25 is equivalent to solving a Sturm-Liouville problem.

We can carry this development one step further. It is straightforward to show that finding an extremum of Equation 24 subject to Equation 25 is equivalent to determining an extremum of the ratio

$$\lambda = \frac{\displaystyle\int_a^b [p(x)y'^2(x) - q(x)y^2(x)]\,dx}{\displaystyle\int_a^b r(x)y^2(x)\,dx} \tag{27}$$

The condition that λ be an extremum with respect to $y(x)$ leads directly to Equation 26 (Problem 7). Of course, Equations 26 and 5 lead to the same result, and, in fact, are equivalent (Problem 8).

Example 3:

Use Equation 27 along with the trial function $y(x) = c_1x(1 - x)$ to calculate λ in Equation 1. Compare your result to $\lambda = 10$, the one obtained earlier in this section. The exact value is $\pi^2 = 9.8696\ldots$.

SOLUTION: Comparing Equation 1 with Equation 5, we see that $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. Therefore,

11. Following Problems 9 and 10, use a trial function of the form $\phi(x) = c_1x(1-x) + c_2x^2(1-x)$ to estimate the first zero of $J_1(z)$.

12. Use a trial function of the form $\phi(x) = (1-x)(c_1x + c_2x^2)$ to estimate the smallest eigenvalue of $u'' + \lambda u = 0$ with $u(0) = u(1)$. Compare your result to the one that you obtained in Problem 5.

13. Describe what happens if you use a trial function of the form $\phi(x) = c_1 \sin \pi x + c_2 \sin 2\pi x$ in Problem 12.

14. The ground state electronic energy E_0 of a hydrogen atom is given by $E_0 = \dfrac{\int \psi_0^* \mathcal{H} \psi_0 \, dv}{\int \psi_0^* \psi_0 \, dv}$, where ψ_0 is the ground state wave function, dv is the volume element, and $\mathcal{H}$ is the Hamiltonian operator $\mathcal{H} = -\dfrac{h^2}{8\pi^2 \mu r^2} \dfrac{d}{dr}\left(r^2 \dfrac{d}{dr}\right) - \dfrac{e^2}{4\pi \varepsilon_0 r}$, where h is the Planck constant, μ is the reduced mass of the electron, e is the protonic charge, and ε_0 is the permittivity of free space. Using a Gaussian trial function $\phi(x) = e^{-\alpha r^2}$, show that $E(\alpha) = \dfrac{3h^2\alpha}{2\mu} - \dfrac{e^2\alpha^{1/2}}{2^{1/2}\varepsilon_0 \pi^{3/2}}$ and that $E_{\min} = -\dfrac{4}{3\pi} \dfrac{\mu e^4}{4\varepsilon_0^2 h^2}$. Compare your result to the exact energy, $E_0 = (-1/2)(\mu e^4/4\varepsilon_0^2 h^2)$.

15. Re-do the previous problem using a trial function of the form $\phi(r) = e^{-\alpha r}$. Compare your result to the exact energy, $E_0 = (-1/2)(\mu e^4/4\varepsilon_0^2 h^2)$. Why do you think that the agreement is so good?

16. Show that $\phi(r) = e^{-r/a_0}$ with $a_0 = h^2\varepsilon_0/\pi \mu e^2$ is an eigenfunction of the Hamiltonian operator given in Problem 14. Now suppose that one were to use a trial function of the form $\phi(x) = c_1 e^{-\alpha r} + c_2 e^{\beta r^2}$ to carry out a variational calculation for the ground state of the hydrogen atom. Can you guess without doing any calculations what c_1, c_2, α, and E_0 will come out to be? What about a trial function of the form $\phi(x) = \sum_{k=1}^{5} c_k e^{-\alpha_k r - \beta_k r^2}$?

17. The ground state energy of a quantum-mechanical harmonic oscillator is given by $E_0 = \dfrac{\int_{-\infty}^{\infty} \psi_0^* \mathcal{H} \psi_0 \, dx}{\int_{-\infty}^{\infty} \psi_0^* \psi_0 \, dx}$, where ψ_0 is the ground state wave function and $\mathcal{H}$ is the Hamiltonian operator $\mathcal{H} = -\dfrac{h^2}{8\pi^2 \mu} \dfrac{d^2}{dx^2} + \dfrac{k}{2}x^2$, where h is the Planck constant, μ is the reduced mass, and k is the force constant. Use a trial function of the form $\phi(x) = \cos \lambda x$ with $-\pi/2\lambda \le x \le \pi/2\lambda$, where λ is a variational parameter, to calculate an upper bound to E_0. Compare your result to the exact value, $E_0 = \dfrac{h}{4\pi}\left(\dfrac{k}{\mu}\right)^{1/2}$.

18. Re-do the previous problem using a trial function of the form $\phi(x) = 1/(1 + \beta x^2)$, where β is a variational parameter.

19. Re-do Problem 17 using a trial function of the form $\phi(x) = e^{-\beta x^2}$, where β is a variational parameter. Why do you think the agreement is so good?

20. Show that $\psi_0(x) = e^{-\alpha x^2/2}$ with $\alpha = (4\pi^2 k\mu/h^2)^{1/2}$ is an eigenfunction of the quantum-mechanical harmonic oscillator Hamiltonian operator given in Problem 17.

21. Show that the normalized trial function corresponding to Problem 5 is $4.404x(1-x) + 4.990x^2(1-x)^2$.

22. Show that the normalized trial function in Example 1 is $(60)^{1/2}x(1-x)$ and the normalized exact solution is $3.6851x^{1/2}J_{1/3}(\frac{2}{3}\lambda_1^{1/2}x^{3/2})$.

Given that $\eta(x, y)$ is sufficiently arbitrary, we conclude that the term in brackets in Equation 5 must equal zero and write the *Euler equation in two-dimensions* as follows:

$$\frac{\partial F}{\partial u} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) \tag{6}$$

The extension of Equation 6 to higher dimensions is straightforward (Problem 2).

Example 1:
Show that the functional

$$I = \iint\limits_R \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] dxdydz$$

leads to Laplace's equation.

SOLUTION:

$$\frac{\partial F}{\partial u} = 0, \qquad \frac{\partial F}{\partial u_x} = \frac{\partial u}{\partial x}, \qquad \frac{\partial F}{\partial u_y} = \frac{\partial u}{\partial y}, \qquad \frac{\partial F}{\partial u_z} = \frac{\partial u}{\partial z}$$

and so Equation 6 gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

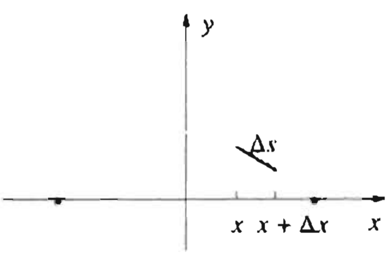


Figure 20.16
A vibrating string at an instant of time.

We can derive the wave equation using the calculus of variations, and using Hamilton's principle in particular. Consider a flexible string fixed at its two endpoints. For simplicity, assume that the tension, τ, in the string and the linear mass density, ρ, of the string are constants. In the course of a vibration, an element of length dx becomes an element of length ds (see Figure 20.16). For small displacements,

$$ds = \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]^{1/2} dx \approx dx + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx + \cdots$$

where u is the displacement and where we have used the binomial theorem. The extension of the element is approximately

$$dl = ds - dx \approx \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

and the work done against the tension τ, which is the potential energy, is

$$dV \approx \frac{\tau}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

The potential energy of the string is given by (small displacement)

$$V = \frac{\tau}{2}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx$$

The kinetic energy of the string is given by

$$K = \frac{\rho}{2}\int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx$$

and so the Lagrangian of the string is

$$L = \frac{\rho}{2}\int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx - \frac{\tau}{2}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{7}$$

According to Hamilton's principle, the integral

$$I = \int_{t_1}^{t_2}\int_0^l \frac{1}{2}\left[\rho\left(\frac{\partial u}{\partial t}\right)^2 - \tau\left(\frac{\partial u}{\partial x}\right)^2\right] dx\,dt \tag{8}$$

must be stationary. If we apply Equation 6 to this equation (with $y \to t$ and $u_y \to u_t$), we obtain

$$\frac{\partial}{\partial x}\left(-\tau\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial t}\left(\rho\frac{\partial u}{\partial t}\right) = 0$$

or

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{9}$$

where $v = (\tau/\rho)^{1/2}$. Thus, we see that the wave equation can be thought of as a Euler-Lagrange equation.

We can use the Rayleigh-Ritz variational principle to calculate upper bounds to the eigenvalues of multidimensional problems. For example, let's calculate an upper bound to the lowest vibrational frequency of the square membrane shown in Figure 20.17. The wave equation for this system is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{10}$$

Realizing that the time dependence of $u(x, y, t)$ is harmonic, substitute $u(x, y, t) = \phi(x, y)e^{i\omega t}$ into Equation 10 to get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\omega^2}{v^2}\phi = 0 \tag{11}$$

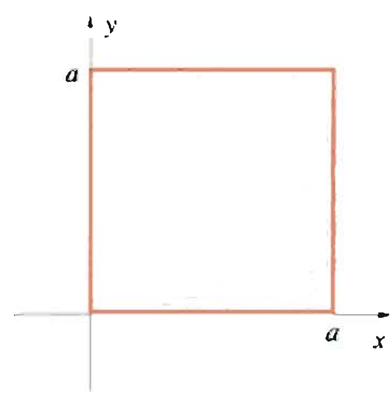


Figure 20.17
A square membrane clamped along its perimeter.

effect, which consists of a train of pulses that occur at random times. The shot effect describes the noise in electronic devices, or the noise generated by the dumping of shot onto a metallic surface.

21.1 Discrete Random Variables

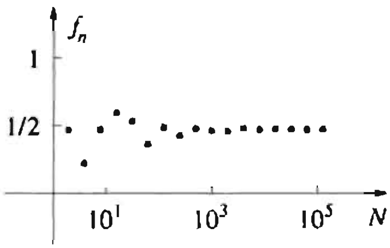


Figure 21.1
A computer simulation of the tossing of a fair coin. The graph shows the ratio of the number of heads to the number of tosses as a function of N, the number of tosses.

There are a number of deep and philosophical questions regarding the definition of probability, but here we address only the most pragmatic aspects. We all would agree that when a fair coin is tossed N times, the ratio of the number of heads H to the number of tosses is around 1/2, and approaches 1/2 more closely as N increases. Figure 21.1 shows a computer generated simulation of the tossing of a fair coin. Other examples of this sort of limiting behavior are the rolling of dice, the drawing of cards, and so on. We can use this type of observation to define probability. In general terms, suppose that an experiment is performed N times and that a particular outcome occurs S times. If the ratio S/N approaches a limit as $N \to \infty$, we say that this limit is the *probability* of that particular outcome.

Suppose now that an experiment has n possible outcomes $E_1, E_2, \ldots, E_n$, where one and only one outcome can occur in a given experiment. The set of all possible outcomes, $\{E_j,\ j = 1, 2, \ldots, n\}$ in this case, is called the *sample space* of the experiment. As described above, the probability of any particular outcome is

$$p(E_j) = \lim_{N\to\infty} \frac{N_j}{N} \qquad j = 1, 2, \ldots, n \tag{1}$$

where N_j is the number of times the event j occurs in N trials. We can see from Equation 1 that if some event, say the αth, is certain to occur, then $N_\alpha = N$ and $p(E_\alpha) = 1$. On the other hand, if it is certain not to occur, then $N_\alpha = 0$ and $p(E_\alpha) = 0$. Because these two cases represent the possible extremes, we have that $0 \le p(E) \le 1$ as a condition that a probability must satisfy. Furthermore, because $N_1 + N_2 + \cdots + N_n = N$, we have

$$\sum_{j=1}^{n} p(E_j) = 1 \tag{2}$$

This result says that it is certain that one of the n events will occur when the experiment is performed (in other words, that something will happen). This condition is called *normalization*.

Now consider the more complicated case of an experiment in which two events A and B can or cannot occur. We can set up Table 21.1. Since the four cases given in Table 21.1 account for all the possibilities, we have

$$N = N_1 + N_2 + N_3 + N_4 \tag{3}$$

Table 21.1
An experiment in which A and B are the two possible outcomes of interest.

Result	Number of occurrences
1. A and B	N_1
2. A, but not B	N_2
3. B, but not A	N_3
4. Neither A nor B	N_4

If we let N be suitably large, the probability that A occurs is

$$p(A) = \frac{N_1 + N_2}{N} \tag{4}$$

and that B occurs is

$$p(B) = \frac{N_1 + N_3}{N} \tag{5}$$

The probability that *both* A and B occur, $p(A, B)$, is given by

$$p(A, B) = \frac{N_1}{N} \tag{6}$$

This is known as the *joint probability* of A and B. The probability that *either* A or B occurs is given by

$$p(A + B) = \frac{N_1 + N_2 + N_3}{N} \tag{7}$$

We can also define *conditional probabilities*. The probability that A occurs, *given that B has occurred*, is

$$p(A|B) = \frac{N_1}{N_1 + N_3} \tag{8}$$

Similarly the probability that B occurs, given that A has occurred, is

$$p(B|A) = \frac{N_1}{N_1 + N_2} \tag{9}$$

We can use the formulas given above to deduce the following relations:

$$\begin{aligned} p(A + B) &= \frac{N_1}{N} + \frac{N_2}{N} + \frac{N_3}{N} \\ &= p(A) + p(B) - p(A, B) \end{aligned} \tag{10}$$

$$p(B|A)p(A) = \frac{N_1}{N_1 + N_2} \cdot \frac{N_1 + N_2}{N}$$

$$= \frac{N_1}{N} = p(A, B) \tag{11}$$

$$p(A|B)p(B) = \frac{N_1}{N_1 + N_3} \cdot \frac{N_1 + N_3}{N}$$

$$= \frac{N_1}{N} = p(A, B) \tag{12}$$

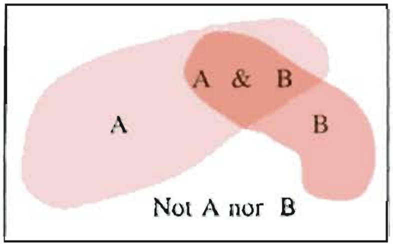


Figure 21.2
A Venn diagram, which illustrates the probability space of the outcomes of A and B.

Equations 10 through 12 can be seen pictorially in Figure 21.2, where the rectangular area E represents the space of all possible outcomes of an experiment and the areas A and B represent the space of outcomes of A and B, respectively. If E is a unit area, then the area A is equal to $p(A)$ and the area B is equal to $p(B)$. The region of overlap between the A and B areas represents the simultaneous occurrence of A and B, or $p(A, B)$. Equation 10 then follows immediately since $p(A + B)$ is the total shaded area in Figure 21.2. Equation 11 follows by noting that $p(B|A) = p(A, B)/p(A)$.

We can also use Equations 10 and 12 and Figure 21.2 to formulate some definitions. If the occurrence of A precludes the occurrence of B and vice versa, there is no overlap region in Figure 21.2 and $p(A, B) = 0$. Therefore, $p(A + B) = p(A) + p(B)$ and $p(B|A) = p(A|B) = 0$. When this is so, we say that the two events A and B are *mutually exclusive*. If the occurrence of A has no effect on the occurrence of B, we have $p(B|A) = p(B)$ and say that the two events are *independent*. An important consequence of the independence of two events is that their joint probability factors, so that $p(A, B) = p(A)p(B)$.

Many events are descriptive in nature, such as the event that the next card drawn from a deck of cards will be red. In many other cases, events have a natural numerical value. Suppose that all the possible events $E_1, E_2, \ldots, E_n$ from an experiment have a numerical value. Then we can represent the outcomes by a random variable, say X. A *random variable* is a rule or a formula that assigns numerical values to each of the simple events in an experiment. We denote a random variable by a capital letter and a particular observation of the random variable by a lowercase letter. The probability that we observe the value x from an experiment is written as Prob $\{X = x\}$ or simply $p(x)$. It is often convenient to interpret a set of probabilities $\{p(x_j)\}$ as a unit mass distributed along the x axis such that m_j is the quantity of mass located at the point x_j. Because $\sum_j p(x_j) = 1$, we must have $\sum_j m_j = 1$. The probability distribution, then, can be pictured as the distribution of a unit mass along the x axis (Figure 21.3).

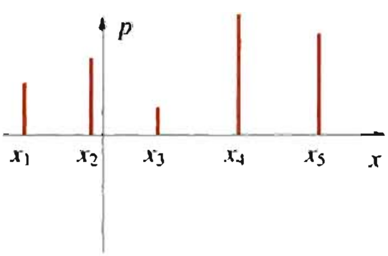


Figure 21.3
The discrete probability frequency function, or probability density. $p(x_j) = \text{Prob}\,\{X = x_j\}$.

We call the variable X a random variable if it takes on the values $x_1, x_2, \ldots, x_n$ with respective probabilities $p(x_1), p(x_2), \ldots, p(x_n)$. The set $\{x_j\}$ over which X varies is called the *sample space*. The *expected value* of X or the *mean value* of X

is given by

$$\mu = E[X] = \sum_{j=1}^{n} x_j p(x_j) \tag{13}$$

By referring to our analogy of mass distribution, Equation 13 says that the expectation value is the center of mass. More generally, if $f(X)$ is a function of X, we define the expectation value of $f(X)$ by

$$E[f(X)] = \sum_{j=1}^{n} f(x_j)p(x_j) \tag{14}$$

If $f(X) = X^n$, then $E[X^n]$ is called the nth *moment* of the probability distribution $\{p(x_j)\}$ and is often denoted by m_n. Note that the second moment corresponds to the moment of inertia of a mass distribution.

You should be aware of the fact that $E[X^2]$ does not necessarily equal $E[X]^2$ as the following Example shows.

Example 1:
Given the following probability data, calculate $E[X]$, $E[X^2]$, and $E[X]^2$.

x	1	2	3	4	5
$p(x)$	0.10	0.15	0.05	0.50	0.20

SOLUTION:

$$E[X] = (1)(0.10) + (2)(0.15) + (3)(0.05) + (4)(0.50) + (5)(0.20)$$
$$= 3.55$$
$$E[X]^2 = 12.60$$
$$E[X^2] = (1)(0.10) + (4)(0.15) + (9)(0.05) + (16)(0.50) + (25)(0.20)$$
$$= 14.15$$

Note that $E[X^2] > E[X]^2$. We shall prove below that this is a general result.

Example 2:
An important discrete probability distribution is the *Poisson distribution*, where $x_j = j$ and

$$p(j) = \frac{a^j}{j!}e^{-a} \qquad j = 0, 1, 2, \ldots$$

The quantity $\text{Cov}[X, Y] = E[(X - \mu_x)(Y - \mu_y)]$ is called the *covariance* of X and Y. Problem 7 has you show that $E[(X - \mu_x)(Y - \mu_y)] = 0$ if X and Y are independent. In such a case, X and Y are said to be uncorrelated. If $\text{Cov}[X, Y] \neq 0$, then the magnitude of $E[(X - \mu_x)(Y - \mu_y)]$ is a measure of the lack of independence (degree of correlation) of X and Y. However, $\text{Cov}[X, Y] = 0$ does not imply independence.

We shall now discuss two well-known and important discrete probability distributions. First let us consider n successive tosses of a coin and ask what is the probability that heads comes up exactly m times. The results of any n successive tosses can be represented by a sequence of h's and t's:

$$hhhthttt \cdots tth$$

There are n positions in any sequence and there are two choices (h or t) for each position. Thus there are 2^n possible sequences, and since they are all equally likely, the probability of any particular one will be 2^{-n}. The number of such sequences with heads exactly m times is equal to the number of ways we can arrange m heads among n positions, or

$$\frac{n!}{m!(n-m)!}$$

The probability of exactly m heads, $p(m)$, then, is

$$p(m) = \frac{n!}{m!(n-m)!}\left(\frac{1}{2}\right)^n$$

More generally, if a "successful" outcome occurs with probability p and an "unsuccessful" outcome occurs with probability q, where $p + q = 1$, then the probability of m "successful" outcomes is given by

$$p(m) = \frac{n!}{m!(n-m)!}p^m q^{n-m} = \frac{n!}{m!(n-m)!}p^m(1-p)^{n-m} \tag{23}$$

This distribution is known as the *binomial distribution* and is applicable to the case of repeated independent trials such as the drawing of cards from a deck, replacing the drawn card after each draw. Equation 23 is called the binomial distribution because the binomial theorem of algebra says that

$$(x+y)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}x^m y^{n-m} \tag{24}$$

Figure 21.5 shows the binomial distribution plotted against m for several values of n. Note that the distribution becomes more and more bell-shaped as n increases.

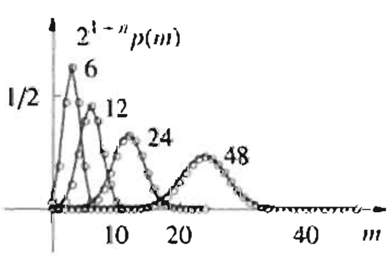


Figure 21.5
The binomial distribution plotted against m for several values of n (6, 12, 24, and 48). Note that the distribution becomes increasingly bell-shaped as n increases.

Example 4:
Show that the binomial distribution is normalized and that $E[M] = np$ and $\text{Var}[M] = \sigma^2 = npq$.

SOLUTION: Equation 24 with $x = p$ and $y = q = 1 - p$ shows that Equation 23 is normalized. The mean is given by

$$E[M] = \sum_{m=0}^{n} mp(m) = \sum_{m=0}^{n} \frac{mn!}{m!(n-m)!} p^m (1-p)^{n-m}$$

We can evaluate this sum by differentiating both sides of Equation 24 with respect to x and then multiplying the result by x to obtain

$$nx(x+y)^{n-1} = \sum_{m=0}^{n} \frac{mn!}{m!(n-m)!} x^m y^{n-m}$$

Letting $x = p$ and $y = 1 - p$ gives

$$E[M] = np$$

Another such differentiation gives

$$E[M(M-1)] = n(n-1)x^2(x+y)^{n-2}$$

Letting $x = p$ and $y = 1 - p$ gives $E[M(M-1)] = n(n-1)p^2$, or

$$\begin{aligned}\sigma^2 &= E[M^2] - E[M]^2 = E[M(M-1)] + E[M] - E[M]^2 \\ &= n^2p^2 - np^2 + np - n^2p^2 = np - np^2 = np(1-p)\end{aligned}$$

The binomial distribution is used to describe a one-dimensional random walk (Problem 12).

Although the binomial distribution is important and useful for its own sake, it also serves as the basis for two other perhaps better known distributions, namely the Gaussian distribution, which we discuss in the next section, and the Poisson distribution, which we derive now. Consider the following problem: We randomly distribute n points over some interval $(0, t)$, and then ask what is the probability that exactly m of these points will lie in some subinterval Δt (Figure 21.6). We can think of this problem as consisting of n repeated, independent trials of placing a single particle in the interval $(0, t)$ with "success" being the probability that the particle will lie in the subinterval Δt, which is equal to $p = \Delta t/t$. Thus, the probability that m of the n particles will lie in Δt is given by

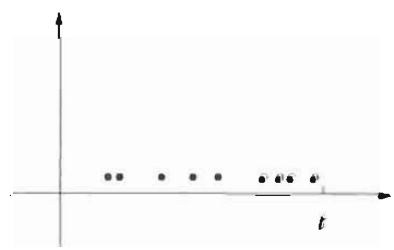


Figure 21.6
n points distributed randomly over the interval $(0, t)$.

$$\begin{aligned}p(m) &= \frac{n!}{m!(n-m)!} p^m (1-p)^{n-m} \\ &= \frac{n!}{m!(n-m)!} \left(\frac{\Delta t}{t}\right)^m \left(1 - \frac{\Delta t}{t}\right)^{n-m}\end{aligned} \tag{25}$$

The average value of m is $\mu = np = n\Delta t/t$, which we denote by $\lambda\Delta t$. We are often interested in the case where n is large and Δt is small (with μ fixed at $\lambda\Delta t$), in which case Equation 25 can be cast in a much more convenient form (Problem 14):

$$p(m) = \frac{(\lambda \Delta t)^m}{m!} e^{-\lambda \Delta t} \tag{26}$$

Equation 26 is the probability that there will be m points in an interval Δt if n points are randomly distributed over the interval $(0, t)$, n is large, Δt is small, and $\lambda = n/t$ is the number of particles per unit of t. Equation 26 is a *Poisson distribution*, and we showed in Example 2 that $\mu = \sigma^2 = \lambda \Delta t$.

Equation 26 applies to radioactive decay, among many other physical phenomena. In this case, λ is the mean rate of decay and Equation 26 tells us the chance that we will observe m decays in a time interval Δt. Notice that the probability of observing no decay in an interval Δt is given by $p(0) = e^{-\lambda \Delta t}$.

Example 5:
A radioactive sample is observed to emit alpha particles at a rate of 1.5 per minute. Determine the average or expected number of alpha particles that you would observe in two minutes. Calculate the probability that you would observe 0, 1, 2, 3, 4, and equal to or greater than 5 counts.

SOLUTION: According to Equation 26, the expected number of counts in a two-minute interval is $\lambda \Delta t = 3.0$ counts. The probability of observing m counts is given by Equation 26, or

m	0	1	2	3	4
Prob $\{M = m\}$	0.0498	0.149	0.224	0.224	0.168

The probability that $m \geq 5$ is given by

$$\text{Prob}\{M \geq 5\} = 1 - \sum_{m=0}^{4} \text{Prob}\{M = m\} = 0.185$$

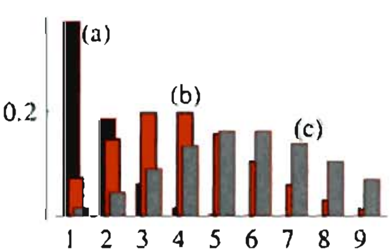


Figure 21.7
The Poisson distribution plotted against m for (a) $a = 1.0$ (black), (b) $a = 4.0$ (color), and (c) $a = 6.0$ (gray).

The Poisson distribution has an amazingly broad range of physical applications, including radioactive decay, aerial search, the arrival of electrons striking a cathode, the transmission of a nervous impulse across a synapse, the distribution of galaxies, and so forth. We'll write Equation 26 in the general case as

$$p(m) = \frac{a^m}{m!} e^{-a} \tag{27}$$

where a is equal to $E[M]$. Figure 21.7 shows $p(m)$ plotted against m for several values of a. For example, suppose it is known that there are 300 errors in a book containing 500 pages. What is the probability that a page contains no errors? Three or more errors? Here the "time interval" is the area of a page, and we wish to calculate the frequency of errors on a page given that on the average there are

and

$$p(+ \mid \overline{D}) + p(- \mid \overline{D}) = \alpha + p(- \mid \overline{D}) = 1$$

What does a positive test result indicate? What about a negative result? These questions are answered using Bayes's formula.

Let's consider a rare disease where $p(D) = 1.00 \times 10^{-4}$ (1 in 10,000) and where the test has $\alpha = 0.0100$ and $\beta = 0.0500$. Now suppose that you test positive for the disease. The probability that you have the disease given that you tested positive is given by Equation 29:

$$\begin{aligned} P(D \mid +) &= \frac{p(D)p(+ \mid D)}{p(D)p(+ \mid D) + p(\overline{D})p(+ \mid \overline{D})} \\ &= \frac{p(D)(1-\beta)}{p(D)(1-\beta) + p(\overline{D})\alpha} \\ &= \frac{(1.00 \times 10^{-4})(0.950)}{(1.00 \times 10^{-4})(0.950) + (0.9999)(0.0100)} \\ &= 0.00941 \end{aligned}$$

or about 1 in 100. This result may not seem too informative. However, the chance of having the disease has increased 100-fold.

What about a negative test result? In other words, what is the value of $p(D \mid -)$. In this case,

$$\begin{aligned} p(D \mid -) &= \frac{p(D)p(- \mid D)}{p(D)p(- \mid D) + p(\overline{D})p(- \mid \overline{D})} \\ &= \frac{p(D)\beta}{p(D)\beta + p(\overline{D})(1-\alpha)} \\ &= \frac{(1.00 \times 10^{-4})(0.0500)}{(1.00 \times 10^{-4})(0.0500) + (0.9999)(0.9900)} \\ &= 5.05 \times 10^{-6} \end{aligned}$$

Thus, there is a very small chance that you'll have the disease given that you test negative.

Let's calculate the chance that you don't have the disease and yet test positive. In other words, let's calculate $p(\overline{D} \mid +)$. This is given by

$$\begin{aligned} p(\overline{D} \mid +) &= 1 - p(D \mid +) \\ &= 0.9906 \end{aligned}$$

Thus, there is a 99% chance that you don't have the disease given that you test positive. Finally, the probability that you don't have the disease given that you test

21.1 Problems

Probability problems can be somewhat tricky to solve and sometimes the results are fairly counterintuitive. The first two problems are examples whose results many people find surprising (or even unconvincing).

1. If heads comes up ten times in a row, many people feel strongly that the next toss is more likely to be tails than heads. How do you explain to them that they are dead wrong?

2. Consider a group of n persons. Calculate the probability that at least two of them have a birthday on the same day of the year for $n = 50$ (exclude leap year). What is the smallest value for which the probability is greater than 1/2? *Hint*: First calculate the total number of outcomes and then calculate the number of ways that no two persons have the same birthday. Now calculate their ratio to get the probability that no two have the same birthday and then subtract this result from 1 to get the probability that at least two do have the same birthday.

3. Let $p(A) = 0.150$ and $p(B) = 0.420$ be independent events. Calculate the probability that either A or B occurs and the probability that both occur. What if A and B are mutually exclusive?

4. Show that $E[X - \mu] = 0$.

5. Show that $\text{Var}\,[X] = \sigma^2 = E[X^2] - E[X]^2$.

6. Show that $E[aX + bY] = aE[X] + bE[Y]$.

7. Show that $E[(X - \mu_x)(Y - \mu_y)] = 0$ if X and Y are independent.

8. Show that $\text{Var}\,[X + Y] = \text{Var}\,[X] + \text{Var}\,[Y]$ if X and Y are independent.

9. Show that $E[\{(X - \mu_x) + (Y - \mu_y)\}^3] = E[(X - \mu_x)^3] + E[(Y - \mu_y)^3]$ if X and Y are independent. In other words, the third central moment of the sum of independent random variables is additive.

10. Show that $E[\{(X - \mu_x) + (Y - \mu_y)\}^4] \neq E[(X - \mu_x)^4] + E[(Y - \mu_y)^4]$ if X and Y are independent. In other words, the fourth central moment of the sum of independent random variables is *not* additive.

11. Show that $\text{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \text{Cov}\,[X_i, X_j]$.

12. We shall investigate a one-dimensional random walk in this problem. Let a particle start at the origin and take one step to the right if a toss of a coin yields heads and one step to the left if the toss yields tails. Show that the probability of being at the point m after n tosses of the coin is

$$P(n, m) = \begin{cases} \dfrac{1}{2^n} \dfrac{n!}{\left(\dfrac{n+m}{2}\right)!\left(\dfrac{n-m}{2}\right)!} & n \text{ and } m \text{ both even or both odd} \\ 0 & \text{otherwise} \end{cases}$$

Show that the average position after n tosses is zero. Plot $P(n, m)$ for all values of m for $n = 3$ and 4.

13. We show in this problem that $P(n, m)$ in the previous problem becomes a bell-shaped curve as n and m become large. Take the logarithm of $P(n, m)$ and use Stirling's approximation to obtain

$$\ln P = n \ln n - \left(\frac{n+m}{2}\right) \ln \left(\frac{n+m}{2}\right) - \left(\frac{n-m}{2}\right) \ln \left(\frac{n-m}{2}\right) - n \ln 2$$

The cumulative distribution function $P(x)$ is given by

$$P(x) = \text{Prob}\{X \leq x\} = \int_{-\infty}^{x} p(x)\,dx \tag{4}$$

If $P(x)$ is continuous. Equation 4 gives

$$p(x) = \frac{dP}{dx} \tag{5}$$

The expected value of a function of a continuous random variable is

$$E[f(X)] = \int_{-\infty}^{\infty} f(x)p(x)\,dx \tag{6}$$

The central moments are simply $E[(X-\mu)^n]$.

Example 1:
A uniform probability density is given by

$$p(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

First show that $p(x)$ is normalized and then calculate its first two central moments.

SOLUTION:

$$\int_a^b p(x)\,dx = \frac{b-a}{b-a} = 1$$

$$\mu = E[X] = \int_a^b xp(x)\,dx = \frac{1}{2}\frac{b^2-a^2}{b-a} = \frac{b-a}{2}$$

$$E[X^2] = \int_a^b x^2 p(x)\,dx = \frac{1}{3}\frac{b^3-a^3}{b-a} = \frac{b^2+ab+a^2}{3}$$

$$\sigma^2 = \text{Var}[X] = E[X^2] - E[X]^2 = \frac{(b-a)^2}{12}$$

We can extend these ideas to more than one random variable. Let X and Y be two continuous random variables. The joint probability density is

$$p(x,y)\,dxdy = \text{Prob}\{x \leq X \leq x+dx \text{ and } y \leq Y \leq y+dy\} \tag{7}$$

and the associated cumulative distribution function is

$$P(x,y) = \text{Prob}\{X \leq x \text{ and } Y \leq y\} = \int_{-\infty}^{y} dy \int_{-\infty}^{x} dx\; p(x,y) \tag{8}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\mathbf{t}^{\mathrm{T}}\cdot\mathbf{x}} e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathsf{A}^{-1}\mathbf{x}} dx_1 dx_2 \ldots dx_n = \frac{(2\pi)^{n/2}}{\sqrt{|\mathsf{A}|}} e^{-\frac{1}{2}i\mathbf{t}^{\mathrm{T}}\mathsf{A}\mathbf{t}} \tag{25}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_n)$.

Another useful property of an n-dimensionsal Gaussian distribution, which is proven in Problem 10, is the following: Let $X_1, X_2, \ldots, X_n$ be n independent normally distributed random variables with means equal to zero and variances σ_j^2. Let $Y_1, Y_2, \ldots, Y_n$ be linear combinations of the X_j; i.e.,

$$\mathbf{Y} = \mathsf{M}\mathbf{X} \tag{26}$$

where M is nonsingular. Then the Y_j have an m-dimensional joint normal distribution given by

$$p(y_1, y_2, \ldots, y_n)dy_1dy_2 \ldots dy_n = \frac{1}{(2\pi)^{m/2}\sqrt{|\mathsf{D}|}} e^{-\frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathsf{D}^{-1}\mathbf{y}} dy_1dy_2 \ldots dy_n \tag{27}$$

where

$$\mathsf{D} = \mathsf{MAM}^{\mathrm{T}} \tag{28}$$

21.2 Problems

1. Verify Equations 11 and 12 if X and Y are independent random variables.
2. Show that $E[XY]^2 \le E[X^2]\,E[Y^2]$. *Hint*: Follow the proof of the Schwartz inequality in Section 9.6.
3. Show that the correlation coefficient ρ_{xy} satisfies $-1 \le \rho_{xy} \le 1$. *Hint*: Use the inequality in the previous problem with X replaced by $X - \mu_x$ and Y by $Y - \mu_y$.
4. Show that $p(x)$ given by Equation 17 is normalized.
5. Equation 24 shows that the Fourier transform $\hat{P}(s)$ [without the factor of $(2\pi)^{1/2}$] of a Gaussian distribution is $e^{-\sigma^2 s^2/2}$. Show that $\hat{P}(0) = 1$ and that $\left(\dfrac{d\hat{P}(s)}{ds}\right)_{s=0} = 0$ and $\left(\dfrac{d^2\hat{P}(s)}{ds^2}\right)_{s=0} = -\sigma^2$. Does this suggest anything to you? See the next problem.
6. Show that the derivatives of the Fourier transform of any continuous probability density $\hat{P}(s) = \int_{-\infty}^{\infty} e^{isx}p(x)dx$ are related to the moments of the probability density by $\left(\dfrac{d^n\hat{P}(s)}{ds^n}\right)_{s=0} = i^n E[X^n]$.
7. Use Equation 22 to write out the Gaussian distribution for two random variables, X and Y.
8. Show that your result for $p(x, y)$ in the previous problem reduces to $p(x)p(y)$ when $\rho_{xy} = 0$.
9. In this problem, we show that

$$\int_{-\infty}^{\infty} \cdots \int e^{i\mathbf{t}^{\mathrm{T}}\mathbf{x} - \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathsf{A}^{-1}\mathbf{x}} dx_1 dx_2 \ldots dx_n = \frac{(2\pi)^{n/2}}{(\det \mathsf{A})^{1/2}} e^{-\frac{1}{2}\mathbf{t}^{\mathrm{T}}\mathsf{A}\mathbf{t}} \tag{1}$$

Let S be the normalized modal matrix of A^{-1}, so that $S^TA^{-1}S = D$ and $x^TA^{-1}x = x'^TDx'$, where $x = Sx'$. (The quadratic form is now diagonalized.) Now let $u = S^{-1}t = S^Tt$ so that $t^Tx = (Su)^Tx = u^TS^Tx = u^Tx'$. Thus, we have

$$\int_{-\infty}^{\infty}\cdots\int e^{iu^Tx'-\frac{1}{2}x'^TDx'}dx_1'dx_2'\dots dx_n' = \prod_{j=1}^{n}\int_{-\infty}^{\infty} e^{iu_jx_j'}e^{-\frac{1}{2}\lambda_jx_j'^2}dx_j'$$

where the λ_j are the eigenvalues of A^{-1}. Now use

$$\int_{-\infty}^{\infty} e^{iu_jx_j'}e^{-\frac{1}{2}\lambda_jx_j'^2}dx_j' = \left(\frac{2\pi}{\lambda_j}\right)^{1/2} e^{-u_j^2/2\lambda_j}$$

and $u^TD^{-1}u = t^TS\,D^{-1}S^Tt = t^TAt$ to verify equation 1.

10. We shall derive Equation 27 in this problem. Substitute $x = M^{-1}y$ into the exponent of Equation 22 to obtain $x^TA^{-1}x = y^T(M^{-1})^TA^{-1}M^{-1}y$. Now use the relation $(SR)^{-1} = R^{-1}S^{-1}$ to show that $x^TA^{-1}x = y^T(MAM^T)^{-1}y = y^TD^{-1}y$, which defines D. Lastly, show that $\det D = \det A(\det M)^2$.

The distributions of the speeds of molecules in the gas phase are governed by Gaussian distributions. The following problems involve the kinetic theory of gases.

11. The distribution of the components of the velocity of the molecules of a gas is given by $f(u_x)du_x = \left(\frac{m}{2\pi k_BT}\right)^{1/2} e^{-mu_x^2/2k_BT}du_x$, where m is the mass of the molecule, k_B is the Boltzmann constant, and T is the kelvin temperature. Determine $E[U_x]$, $E[U_x^2]$, and $E[\frac{1}{2}mU_x^2]$.

12. Using the distribution in the previous problem, derive an expression for the probability that $-u_{x0} \le U_x \le u_{x0}$. Express your result in terms of the error function of $w_0 = (m/2k_BT)^{1/2}u_{x0}$. Calculate the probability that $(-2k_BT/m) \le U_x \le (2k_BT/m)^{1/2}$.

13. Use the result of the previous problem to show that Prob $\{|U_x| \ge u_{x0}\} = 1 - \text{erf}\,(w_0)$. Calculate Prob $\{|U_x| \ge (k_BT/m)^{1/2}\}$ and Prob $\{|U_x| \ge (2k_BT/m)^{1/2}\}$.

14. Use the result of Problem 12 to plot the probability that $-u_{x0} \le U_x \le u_{x0}$ against $u_{x0}/(2k_BT/m)^{1/2}$.

15. We are often more interested in the distribution of molecular speeds rather than just components of velocity. To determine the distribution of molecular speeds, first write

$$f(u_x)f(u_y)f(u_z)du_xdu_ydu_z = \left(\frac{m}{2\pi k_BT}\right)^{3/2} e^{-m(u_x^2+u_y^2+u_z^2)/2k_BT}du_xdu_ydu_z$$

Now, using the fact that the speed u is given by $u^2 = u_x^2 + u_y^2 + u_z^2$, convert the above equation to $F(u)du = 4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2} u^2e^{-mu^2/2k_BT}du$. Now determine $E[U]$, $E[U^2]$, and $E[\frac{1}{2}mU^2]$.

16. Another distribution that is frequently used in the kinetic theory of gases is the distribution of energy, or kinetic energy in particular. Let $\epsilon = mu^2/2$ and use the result of the previous problem to show that $F(\epsilon)d\epsilon = \frac{2\pi}{(\pi k_BT)^{3/2}}\epsilon^{1/2}e^{-\epsilon/k_BT}d\epsilon$. Now calculate $E[\epsilon]$ and compare your result to $E[\frac{1}{2}mU^2]$ that you obtained in the previous problem.

17. In this problem, we shall prove an interesting theorem called *Chebyshev's inequality*, which says that Prob $[|X - c| \geq \epsilon] \leq \frac{1}{\epsilon^2} E[(X - c)^2]$, where c and ϵ are any real numbers. If $p(x)$ is the density function of X, then
Prob $[|X - c| \geq \epsilon] = \int_{|x-c|\geq\epsilon} p(x)\,dx$. Now argue that

$$\text{Prob}\,[|X - c| \geq \epsilon] = \int\limits_{|x-c|\geq\epsilon} p(x)\,dx \leq \int\limits_{|x-c|\geq\epsilon} \frac{(x-c)^2}{\epsilon^2} p(x)\,dx \leq \frac{1}{\epsilon^2} E[(X-c)^2]$$

18. Use Chebyshev's inequality (previous problem) to show that Prob $\left[\left|\frac{X-\mu}{\sigma_x}\right| \geq a\right] \leq \frac{1}{a^2}$. Interpret this result.

21.3 Characteristic Functions

Given a probability density function $p(x)$, we can uniquely define a new function $\phi(s)$ by

$$\phi(s) = E[e^{isx}] = \int_{-\infty}^{\infty} e^{isx} p(x)\,dx \tag{1}$$

The function $\phi(s)$, called the *characteristic function* of $p(x)$, plays a central role in probability theory and its applications. Notice that the characteristic function is essentially the Fourier transform of $p(x)$ (without the prefactor of $1/(2\pi)^{1/2}$). It is also equal to the expectation value of e^{isx}, as Equation 1 indicates.

We shall see that the characteristic function has a number of important and useful properties. The first that we demonstrate is the following. If we repeatedly differentiate $\phi(s)$ with respect to s and then set s equal to zero, we obtain

$$\phi^{(n)}(0) = i^n E[X^n] \qquad n = 0, 1, 2, \ldots \tag{2}$$

We see then that if the characteristic function is known, then it is a simple matter to compute the nth moment of $p(x)$.

If all the moments are known, then the Maclaurin expansion of $\phi(s)$, and hence $\phi(s)$ itself, is known since

$$\phi(s) = \phi(0) + \phi'(0)s + \phi''(0)\frac{s^2}{2!} + \cdots \tag{3}$$

Now if $\phi(s)$ is known, then $p(x)$ is known uniquely through the inversion theorem of Fourier transforms, namely (Problem 6)

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx}\phi(s)\,ds \tag{4}$$

Thus, a knowledge of all the moments of a probability density is equivalent to knowing the probability density itself.

The first few derivatives of $\phi(s)$ yield

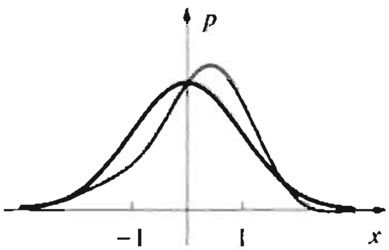
$$\phi'(0) = i\mu \qquad \phi''(0) = -\mu^2 - \sigma^2 = -E[X^2] \qquad \phi'''(0) = i^3(\mu^3 + 3\mu\sigma^2)$$

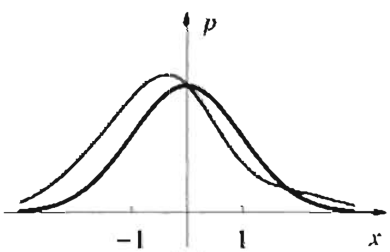
and the first few derivatives of $\Phi(s)$ yield

$$\Phi'(0) = 0 \qquad \Phi''(0) = -\sigma^2 \qquad \Phi'''(0) = 0$$

Although one does not often know all the moments without knowing $p(x)$, it often occurs that one knows the first few moments. The practical question then arises as to just how much we can say about $p(x)$ if we know its first few moments. Obviously, $p(x)$ cannot be specified completely, but nevertheless, the first few moments do describe the main features of $p(x)$. For example, all distributions have a unit area under the curve. The mean is a measure of where the curve is "located" and the variance tells the spread of the curve about the mean. More precisely, it is the magnitude of the square root of the variance *relative* to the mean, σ/μ (that is, a measure of the spread). This ratio is called the *coefficient of variation* and is usually denoted by γ. The reason that σ must be compared to μ can be seen from the following example. In statistical mechanics, one finds that fluctuations of the order of 10^{10} molecules might exist in an open macroscopic system. This might seem like an enormous number until it is realized that there are 10^{20} or so molecules in the system. Thus, we see that 10^{10} represents a percentage fluctuation of $10^{-8}\%$! Putting it another way, picture a Gaussian curve centered at $x = 10^{20}$ with a spread of a few multiples, say n, of 10^{10}. Most of the curve appears between $10^{20} + (n \times 10^{10})$ and $10^{20} - (n \times 10^{10})$, which for all practical purposes is a delta function. Although the standard deviation $\sigma = 10^{10}$, the coefficient of variation $\gamma = 10^{-10}$ and is a more realistic indication of the spread.

The relative third central moment, μ_3/σ^3, called the *coefficient of skewness*, is a measure of the asymmetry of the curve about the mean. If this moment is zero, the curve is symmetric about the mean. If it is positive, the curve is skewed to the high side of the mean, and vice versa. The relative fourth central moment, $\mu_4/\sigma^4 - 3$, called the *kurtosis*, is a measure of the peakedness of the curve about the mean relative to a Gaussian distribution (Figure 21.11). The physical interpretation of the higher central moments becomes progressively more obscure. Thus, we see that the first few moments characterize the general features of the probability distribution. In Problem 16, we show how a knowledge of the first few moments can be used to approximate their associated probability distribution.





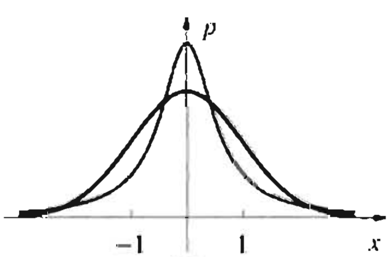


Figure 21.11
An illustration of (a) a distribution skewed to the high side of the mean, (b) a distribution skewed to the low side, and (c) a distribution with a high degree of peakedness.

Another approach that has found wide application to physical problems is the use of the first few moments to determine rigorous upper and lower bounds on the cumulative distribution function $P(x)$. This approach is called the *method of moments* and has been developed and applied to a wide range of physical problems. This is a particularly beautiful theory, but it is beyond the scope of this chapter. (See the reference to Shohat and Tamarkin at the end of the chapter.)

A particularly important property of the characteristic function is illustrated and so we can drop the $O(As^3)$ term and write

$$\phi_Z(s) \to e^{iAs}e^{-As^2/2} \tag{13}$$

But if we compare this result to either Example 2 or Example 3, we see that $\phi_Z(s)$ is the characteristic function of a normal distribution with mean A and variance A (Figure 21.12). (Problem 10 offers a more rigorous derivation of Equation 13.)

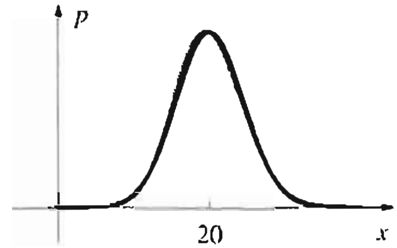


Figure 21.12
The envelope of a Poisson distribution with $a = 20$ compared to a normal distribution with $\mu = 20$ and $\sigma^2 = 20$.

We have shown that if Z is the sum of n independent Poisson distributed random variables, then Z becomes normally distributed as n becomes large. Problem 13 offers a simple proof that the sum of n independent binomially distributed random variables becomes normally distributed as n becomes large. Thus, it might appear that the behavior is more general than it may seem. In fact, there is a famous theorem in probability theory that says that the sum of n random variables becomes Gaussian as n becomes large regardless of the probability distribution of the individual X_i so long as they are independent, identically distributed, and that their mean and variance exist. This theorem is called the *central limit theorem* and states that

If $X_1, X_2, \ldots, X_n$ are independent, identically distributed random variables with mean μ and variance σ^2, then as $n \to \infty$,

$$Y = \frac{X_1 + X_2 + \cdots + X_n}{n} \tag{14}$$

is normally distributed with mean μ and variance σ^2/n.

Note that Y can be thought of as an average of n measurements of X. This theorem is the basis for many statistical methods, as we shall see in the next chapter.

The proof of this theorem provides a good example of the power of using characteristic functions. The characteristic function of Y is

$$\begin{aligned}\phi_Y(s) &= \int_{\infty}^{\infty} e^{isy} q(y)dy \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{is}{n}(x_1 + \cdots + x_n)p(x_1)\cdots p(x_n)\,dx_1\cdots dx_n \\ &= \left[\int_{-\infty}^{\infty} e^{isx/n}p(x)dx\right]^n = \left[\phi\left(\frac{s}{n}\right)\right]^n \end{aligned} \tag{15}$$

where $\phi(s)$ is the characteristic function of $p(x)$, namely,

$$\phi(s) = \int_{-\infty}^{\infty} e^{isx}\,p(x)\,dx = 1 + is\mu - \frac{s^2\sigma^2}{2} + O(s^3)$$

Using this expansion in Equation 15 gives

$$\phi_Y(s) = \left[1 + \frac{is\mu}{n} - \frac{s^2\sigma^2}{2n^2} + O\left(\frac{s^3}{n^3}\right)\right]^n = \left[1 + \frac{is\mu - s^2\sigma^2/2n + O(s^3/n^2)}{n}\right]^n$$

Using the definition

$$e^{w} = \lim_{n\to\infty}\left(1+\frac{w}{n}\right)^n$$

this expression becomes

$$\phi_Y(s) \to e^{is\mu}e^{-s^2\sigma^2/2n} \tag{16}$$

for large n. Equation 16 is the characteristic function of a Gaussian distribution with mean μ and variance σ^2/n. Note that it is independent of the distribution of x. This result says that the average of repeated measurements will be approximately normal, and hence explains why the expression "normal curve of error" occurs so frequently in the analysis of experimental data and why the central limit theorem plays a fundamental role in mathematical statistics. In addition, because many physical phenomena (such as Brownian motion) are the result of a large number of repeated small effects, the Gaussian distribution occurs often in physical applications.

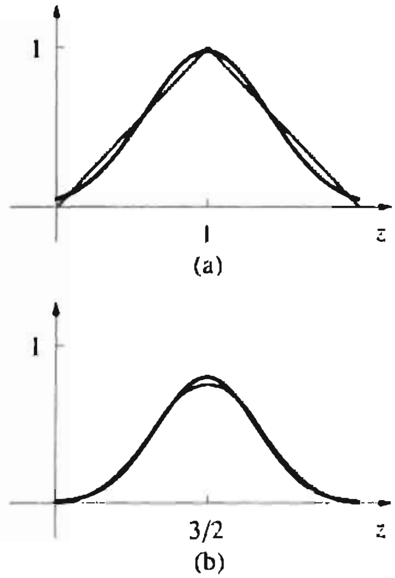


Figure 21.13
The exact density function (color) and its Gaussian approximation (black) for the sum of (a) two independent uniformly distributed random variables and (b) three independent uniformly distributed random variables.

Figure 21.13a and b show the exact density function and its Gaussian approximation for the sum of two and three independent uniformly distributed random variables, whose density function is

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Notice that the Gaussian distribution in this case is a fairly good approximation even for $n = 2$ and $n = 3$. Problems 14 and 15 have you obtain the results in Figure 21.13.

There are several variations of the characteristic function that are used. For example, if the range of X is not $(-\infty, \infty)$ but $(0, \infty)$, then the Laplace transform is often more convenient and we write

$$\varphi(s) = \int_0^\infty e^{-sx} p(x)\,dx \tag{17}$$

This type of characteristic function has properties similar to those of the Fourier transform characteristic function (Problem 7).

21.3 Problems

1. Determine the probability density function of $Y = aX + b$ $(a > 0)$ if X is uniformly distributed in $[0, 1]$.
2. Determine the probability density function of $Y = aX + b$ $(a > 0)$ if X is normally distributed with mean μ and variance σ^2.
3. Use Equation 9 to determine the probability density function of $Z = X + Y$ if X and Y are independent normally distributed random variables with mean 0 and variance 1.

14. Let X_j be independent random variables uniformly distributed in the interval $(0, T)$. Show that $E[X_j] = T/2$ and $\mathrm{Var}\,[X_j] = T^2/12$. Show that the density function for $X = X_1 + X_2$ is $p_X(x) = \begin{cases} x & 0 < x < T \\ 2T - x & T < x < 2T \end{cases}$. Compare this exact density to a Gaussian approximation for $p_X(x)$.

15. Extend the calculation in the previous problem to the sum of three independent random variables.

16. Many distributions, although not exactly normal, do approximate a normal distribution in some sense. There is a systematic expansion of an arbitrary distribution about a normal distribution. Such an expansion has found a number of physical applications and is called a *Gram-Charlier series*. Let $f(x)$ be some probability density that looks somewhat Gaussian and let $\varphi(x)$ be a normal distribution. Without loss of generality, let the variable be taken to be a standardized random variable $X = (X - \mu)/\sigma$; in other words, one with zero mean and unit variance, so that $\varphi(x) = (2\pi)^{-1/2}e^{-x^2/2}$. Then we can write

$$f(x) = \varphi(x) + c_1\varphi'(x) + \frac{c_2}{2!}\varphi''(x) + \frac{c_3}{3!}\varphi'''(x) + \cdots \tag{1}$$

Now show that the nth derivative of $e^{-x^2/2}$ is equal to the Hermite polynomial $He_n(x)$ in Table 14.1. Therefore, equation 1 becomes

$$f(x) = e^{-x^2/2}[a_0He_0(x) + a_1He_1(x) + a_2He_2(x) + \cdots] \tag{2}$$

which is simply an expansion of $f(x)$ in a set of orthogonal polynomials, since the $He_n(x)$ satisfy the orthogonality relation (Table 14.4):

$$\int_{-\infty}^{\infty} dx\, e^{-x^2/2}He_m(x)He_n(x) = (2\pi)^{1/2}n!\delta_{mn} \tag{3}$$

Show that the a_j in equation 2 are given by $a_j = \dfrac{1}{(2\pi)^{1/2}j!}\displaystyle\int_{-\infty}^{\infty} He_j(x)f(x)dx$. Now show that $a_0 = (2\pi)^{-1/2}$, $a_1 = 0$, $a_2 = 0$, $a_3 = E[X^3]/3!(2\pi)^{1/2}$, and $a_4 = (E[X^4] - 3)/4!(2\pi)^{1/2}$. Equation 2 becomes

$$f(x) = \frac{1}{(2\pi)^{1/2}}e^{-x^2/2} + e^{-x^2/2}\sum_{n=3}^{\infty} a_nHe_n(x) \tag{4}$$

which is the form usually presented as the Gram-Charlier series. The leading term is a normal distribution and the remaining terms represent deviations of $f(x)$ from normal behavior. It should be pointed out that the expansion in equation 4 is valid for only a fairly small class of functions, but often physical grounds can be argued for its use.

21.4 Stochastic Processes—General

In the first part of this chapter, we introduced the idea of a random variable and its associated probability functions. In a sense, we could say that we discussed static probability functions since the concept of time did not enter the discussion. In the remainder of this chapter, we discuss random variables and probability distributions that depend upon time, and hence, we discuss the dynamics of probability functions, or *stochastic processes*.

Let's start by considering a counting process. A random process is said to be a *counting process* if the random variable $X(t)$ denotes the total number of events (the detection of a radioactive decay particle, the arrival of customers in a store) that have occurred in the time interval $(0, t)$. Clearly, $X(0) = 0$, $X(t_2) \geq X(t_1)$ if $t_2 \geq t_1$, and $X(t_2) - X(t_1)$ is equal to the number of events that occur in the interval (t_1, t_2) with $t_2 \geq t_1$. The figure in the introduction to this chapter (reproduced as Figure 21.14) is a computer-generated illustration of a counting process. Note that the figure is a staircase shape with unit steps occurring at the random time points t_j.

A counting process is said to be a *Poisson process* if $X(0) = 0$, if the number of events that occur in non-overlapping time intervals are independent, and if the number of events that occur in a time interval $t_2 - t_1$ satisfies a Poisson distribution, so that

$$\text{Prob}\{X(t_2) - X(t_1)\} = e^{-\lambda(t_2-t_1)}\frac{[\lambda(t_2 - t_1)]^n}{n!} \qquad n = 0, 1, 2, \ldots$$

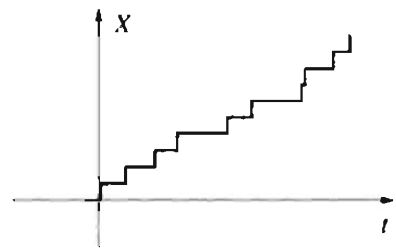


Figure 21.14
A computer-generated illustration of a counting process.

Figure 21.14 is actually that of a Poisson counting process.

Figure 21.14 is the outcome of one particular computer run. If we run the experiment a number of times, we will generate a number of staircase forms with different values of the t_j. We can label the various outcomes by some index, say n, so that each curve is given by $x(t; n)$. The mathematical abstraction is to consider a set of functions $x(t; n)$, $n = 1, 2, \ldots, N$, where $N \to \infty$. Such a set of functions is called an *ensemble* and any one member is called a *realization* of the process. The family of random variables $X(t, n)$ is called a *random process* or a *stochastic process*. A (discrete) stochastic process can be thought of as a function of two variables, t and n. Roughly speaking, the random variable $X(t)$ does not depend upon time in a completely definite (deterministic) way, but only in some probabilistic sense.

A random process $X(t)$ can be described by a set of probability distributions. Consider an ensemble of curves described by $x(t; n)$. At any given time, find the fraction of the total number of curves where $X(t)$ has the value x_1. This fraction, $w_1(x_1, t)$, is called the *first probability distribution*. Now define the *second probability distribution*, $w_2(x_1, t_1 : x_2, t_2)$, as the joint probability of X having a value x_1 at time t_1 *and* a value x_2 at time t_2. This process can be continued on through the third, fourth, and all subsequent probability distributions. This set of functions completely characterizes the random process in a statistical sense. If we know the functions w_j for all j, we know all that can be known about the random process.

In general, the complete determination of the set of probability distributions given above is not feasible. For example, the determination of just $w_1(x_1, t)$ would require that we determine the number of observations that X equal x for all values of t. Thus, we would have to observe the time evolution of a large number of realizations. Fortunately, however, there are several reasonable assumptions that can be made that greatly simplify matters. The first and least restrictive assumption we discuss is to assume that the random process is stationary.

By *stationary*, we mean that the form of the probability distribution functions $w_n(x_1, t_1; x_2, t_2; \ldots; x_n, t_n)$ do not depend upon a shift of the origin of time. In a sense, we assume that the underlying probabilistic mechanism of the process does not change with time. More precisely, we say that the random process is stationary when the probability distributions of $\{X(t, n)\}$ and $\{X(t+\tau, n)\}$ are the same for any value of τ. Thus, $w_n(x_1, t_1; \ldots; x_n, t_n) = w_n(x_1, t_1+\tau; \ldots; x_n, t_n+\tau)$.

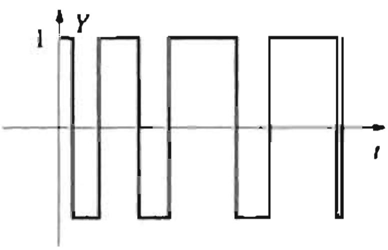


Figure 21.15
A semirandom telegraph signal. Notice that $Y(0) = 1$.

Example 1:
Consider the random process

$$Y(t) = (-1)^{X(t)}$$

where $X(t)$ is a Poisson process with mean λt, so that λ is the mean rate at which Poisson events occur. Note that $Y(t)$ starts at $Y(0) = 1$ and then switches between 1 and -1 at random Poisson times $T_1, T_2, \ldots$, as shown in Figure 21.15. This process is known as a *semirandom telegraph signal* because the initial value is $Y(0) = 1$, instead of being random. Determine $E[Y(t)]$ for this process.

SOLUTION: The equation $Y(t) = (-1)^{X(t)}$ says that

$$Y(t) = \begin{cases} 1 & \text{if } X(t) \text{ is even} \\ -1 & \text{if } X(t) \text{ is odd} \end{cases}$$

But $X(t)$ is a Poisson process, so

$$\begin{aligned} \text{Prob}\{X(t) \text{ is even}\} &= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right] \\ &= e^{-\lambda t}\cosh \lambda t \end{aligned}$$

and

$$\begin{aligned} \text{Prob}\{X(t) \text{ is odd}\} &= e^{-\lambda t}\left[\frac{\lambda t}{1!} + \frac{(\lambda t)^3}{3!} + \cdots\right] \\ &= e^{-\lambda t}\sinh \lambda t \end{aligned}$$

The expectation value of $Y(t)$ is given by

$$\begin{aligned} E[Y(t)] &= (1)\text{Prob}\{X(t) \text{ is even}\} + (-1)\text{Prob}\{X(t) \text{ is odd}\} \\ &= e^{-\lambda t}(\cosh \lambda t - \sinh \lambda t) = e^{-2\lambda t} \end{aligned}$$

Thus, we see that this process is not stationary because $E[Y(t)]$ depends upon t.

The process described in Example 1 is a semirandom telegraph signal because $Y(0) = 1$. If $Y(0)$ takes on the values ± 1 with equal probability, then the process is called a *random telegraph signal*.

Example 2:
Show that a random telegraph signal has a zero mean.

SOLUTION: Let $Y(0) = y_0$, where $\text{Prob}\{y_0 = 1\} = 1/2$ and $\text{Prob}\{y_0 = -1\} = 1/2$. Then

$$Y(t) = \begin{cases} 1 & \text{if } X(t) \text{ is even and } y_0 = 1 \\ & \text{if } X(t) \text{ is odd and } y_0 = -1 \\ -1 & \text{if } X(t) \text{ is odd and } y_0 = 1 \\ & \text{if } X(t) \text{ is even and } y_0 = -1 \end{cases}$$

These four events are mutually exclusive, so

$$\begin{aligned} E[Y(t)] = &(1)\,\text{Prob}\,[X(t) \text{ is even}]\,\text{Prob}\,[y_0 = 1] \\ &+ (1)\,\text{Prob}\,[X(t) \text{ is odd}]\,\text{Prob}\,[y_0 = -1] \\ &+ (-1)\,\text{Prob}\,[X(t) \text{ is odd}]\,\text{Prob}\,[y_0 = 1] \\ &+ (-1)\,\text{Prob}\,[X(t) \text{ is even}]\,\text{Prob}\,[y_0 = -1] \\ = &(1)\frac{e^{-\lambda t}}{2} + (-1)\frac{e^{-\lambda t}}{2} = 0 \end{aligned}$$

For a stationary random process, we may, in principle at least, determine the various probability distributions from the experimental observation of $x(t)$ for one system over a long period of time. This long time record can be cut up into pieces of length T (where T is much longer than any "periodicities" occurring in the process), and these pieces may be treated as observations of different systems in an ensemble. The underlying assumption here is the so-called *ergodic hypothesis*, which states that for a stationary random process, a large number of observations made on a single system at N arbitrary instants of time have the same statistical properties as observing N arbitrarily chosen systems at the same time from an ensemble of similar systems. The subject of ergodicity is extremely involved, but in almost all physical applications of stochastic processes, we assume that a stationary process is ergodic.

The ergodic hypothesis and the assumption of stationarity have an important consequence. In dealing with general random processes, there are two types of mean values that we encounter. One is obtained by observations made on many systems at some fixed time t. If we denote this *ensemble average* by $\langle f(x)\rangle_{\text{ensemble}}$, then

$$\langle f(x)\rangle_{\text{ensemble}} = \sum_x f(x) w_1(x) \tag{1}$$

and consequently, has generated a great deal of research, much of it arcane and abstruse (a check of the University of California library holdings show 188 titles involving the key words "ergodic theory").

The probability distributions of stationary random processes are indeed simpler than those of the general case, but it is necessary, nevertheless, to make further restrictions as well. This leads us to another important classification of random processes. A random process is said to be purely random when values of x at different times are completely uncorrelated. The probability distributions in the case become

$$\begin{aligned} w_2(x_1, t_1; x_2, t_2) &= w_1(x_1, t_1)w_1(x_2, t_2) \\ w_3(x_1, t_1; x_2, t_2; x_3, t_3) &= w_1(x_1, t_1)w_1(x_2, t_2)w_1(x_3, t_3) \end{aligned} \tag{4}$$

and so on. The random process is completely specified by w_1. Purely random processes do not occur often in physical applications since in most real situations $x_1(t_1)$ and $x_2(t_2)$ will be correlated at least for small values of $t_2 - t_1$.

The next more complicated case, and one that turns out to be a reasonable abstraction for a large number of physical processes is to assume that the process is a *Markov process*. In order to define a Markov process, first divide the time axis into small intervals of length δ, and let $t_j = j\delta$. Now introduce the conditional probability $p_n(x_n, t_n|x_0, t_0; x_1, t_1; \cdots; x_{n-1}, t_{n-1})$ $(t_0 < t_1 < \cdots < t_n)$ that $X = x_n$ at time t_n given that $X(t_0) = x_0$, $X(t_1) = x_1, \ldots, X(t_{n-1}) = x_{n-1}$. We define a Markov process by the requirement that

$$p_n(x_n, t_n|x_0, t_0; x_1, t_1; \cdots; x_{n-1}, t_{n-1}) = p_2(x_n, t_n|x_{n-1}, t_{n-1}) \tag{5}$$

In other words, the probability that the system "is in the state x_n" at time t_n depends only on its state directly preceding t_n and not on the entire previous history of the process. Thus, all the conditional probabilities $p_n(x_n, t_n|x_1, t_1; \ldots; x_{n-1}, t_{n-1})$ reduce to $p_2(x_n, t_n|x_{n-1}, t_{n-1})$. Furthermore, because of the general relation

$$w_2(x_1, t_1; x_2, t_2) = p_2(x_2, t_2|x_1, t_1)w_1(x_1, t_1) \tag{6}$$

a Markov process is completely specified by the first two distributions $w_1(x_1, t_1)$ and $w_2(x_1, t_1; x_2, t_2)$. Since we shall consider only stationary processes, Equation 6 reads

$$w_2(x_1, x_2; t_2 - t_1) = w_1(x_1)p_2(x_2, t_2 - t_1) \tag{7}$$

Markov processes allow us to introduce *transition probabilities*. To see what we mean by this, we'll consider a *Markov chain*, which is a Markov process in which time as well as X takes on discrete values. Thus, a Markov chain evolves in discrete steps, $n = 0, 1, 2, \ldots$. If $X(n) = i$, then the Markov chain is said to be in the state i after n steps. As a concrete example, consider the three-state system shown in Figure 21.17. We define a probability transition matrix P having p_{ij} as its matrix elements where p_{ij} is the probability that the system undergoes a transition

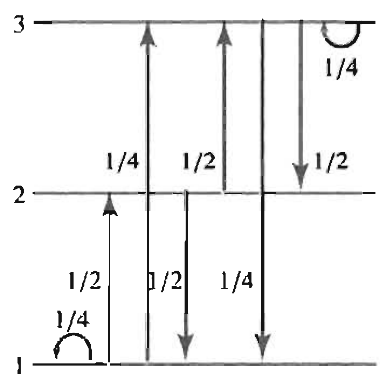


Figure 21.17
A three-state Markov chain with a transition matrix given by Equation 8.

probability distribution, $w_1(x, t)dx$, is the fraction of the members of the ensemble for which $X(t)$ lies between x and $x + dx$ at the time t. We now define the second probability distribution, $w_2(x_1, t_1; x_2, t_2)dx_1dx_2$ as the joint probability of finding X between x_1 and $x_1 + dx_1$ at time t_1 *and* between x_2 and $x_2 + dx_2$ at time t_2. As in the discrete case, this process can be continued on through the third, fourth, and all subsequent probability distributions. This set of functions completely characterizes the random process in a statistical sense.

The probability distributions for a stationary random process become $w_1(x)$, $w_2(x_1, x_2; t_2 - t_1)$, $w_3(x_1, x_2, x_3; t_2 - t_1, t_3 - t_1)$, and so on. Equations 1 and 2 become

$$\langle f(x)\rangle_{\text{ensemble}} = \int_{-\infty}^{\infty} f(x)w_1(x)\,dx \tag{11}$$

and

$$\langle f(x)\rangle_{\text{time}} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x(t))\,dt \tag{12}$$

which are equal according to the ergodic hypothesis.

There are two functions associated with stationary random processes that are central to the theory of stochastic processes. These two functions are the *autocorrelation function* and the *spectral density*. We shall see that stationary random processes can be well characterized by either of these two functions. The correlation function of a continuous stationary random process is defined by

$$R_x(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)\,dt \tag{13}$$

According to the ergodic hypothesis, we could also write

$$R_x(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2w_2(x_1, x_2; \tau)\,dx_1dx_2 \tag{14}$$

Note that if x_1 and x_2 are uncorrelated, then $w_2(x_1, x_2; \tau) = w_1(x_1)w_1(x_2)$ and $R_x = \langle x\rangle^2_{\text{ensemble}}$.

On physical grounds, x_1 and x_2 usually will be more correlated for small τ than for large values of τ, and hence $R(\tau)$ should be a bounded, continuous function of τ. In fact, we can prove that $R(0) \geq |R(\tau)|$ (Problem 11). In addition, $R_x(\tau)$ is an even function of τ (Problem 12).

Example 3:
Determine the autocorrelation function of the random telegraph signal described in Example 2.

SOLUTION: We need to evaluate $E[Y(t)Y(t+\tau)]$ first. When $\tau = 0$, $Y^2(t) = 1$, so the value of $Y(t)Y(t+\tau)$ depends only upon the number of events in the interval $(t, t+\tau)$. The product $Y(t)Y(t+\tau) = 1$ if there is an even number of events and $Y(t)Y(t+\tau) = -1$ if there is an odd number of events. Therefore, following Example 1,

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] = (1)e^{-\lambda\tau}\cosh\lambda\tau + (-1)e^{-\lambda\tau}\sinh\lambda\tau \\ &= e^{-2\lambda\tau} \end{aligned} \tag{15}$$

which we can write as $e^{-2\lambda|\tau|}$ because $R_Y(\tau)$ is an even function of τ (Problem 12). Exponentially decaying autocorrelation functions occur frequently in physical problems.

The other of the two central functions in the theory of stochastic processes is the spectral density of $x(t)$. Let

$$x_T(t) = \begin{cases} x(t) & -T \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

Let the Fourier transform of $x_T(t)$ be $\hat{A}(\omega)$,

$$\hat{A}(\omega) = \int_{-\infty}^{\infty} x_T(t)e^{-i\omega t}dt = \int_{-T}^{T} x(t)e^{-i\omega t}dt \tag{17}$$

so that by Fourier inversion

$$x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{A}(\omega)e^{i\omega t}d\omega \tag{18}$$

The *spectral density* $S(\omega)$ of $x(t)$ is defined by

$$S(\omega) = \lim_{T\to\infty}\frac{1}{2T}|\hat{A}(\omega)|^2 \tag{19}$$

Problem 13 has you show that $S(\omega)$ is an even function of ω if $x(t)$ is real.

Example 4:
Show that

$$\langle x^2\rangle_{\text{time}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega$$

SOLUTION: Using Parseval's theorem (Section 17.5), we can write

$$\int_{-\infty}^{\infty} x_T^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{A}(\omega)|^2 d\omega$$

The value of these sums can be easily determined using a CAS, and we obtain Prob $\{X(t) > 75\} = 0.000372$. The probability that X exceeds 50 calls is 0.462, which is about what you might expect.

We can also derive the probability density for arrival times in Figures 21.21 and 21.22. Let t_1 be the time at which the first event occurs. Then, the events $[T_1 > t]$ and $[X(t) = 0]$ are equivalent, and so

$$\text{Prob}\,\{T_1 > t\} = \text{Prob}\,\{X(t) = 0\} = e^{-\lambda t} \tag{4a}$$

Thus, the cumulative distribution function of T_1 is given by

$$\text{Prob}\,\{T_1 \le t\} = 1 - e^{-\lambda t} \tag{5a}$$

and the density function is the derivative of Equation 5a,

$$p_{T_1}(t) = \lambda e^{-\lambda t} \qquad t > 0 \tag{6a}$$

Equation 5a says that the time to the first occurrence in a Poisson process is governed by an exponential probability density. The expectation value of T_1 is

$$E[T_1] = \int_0^\infty \lambda t e^{-\lambda t} dt = \frac{1}{\lambda} \tag{7a}$$

What is the probability law for the occurrence of the mth event in a Poisson process? The events $[T_m > t]$ and $[X(t) < m - 1]$ are equivalent, and so we have

$$\text{Prob}\,\{T_m > t\} = \text{Prob}\,\{X(t) < m - 1\} = \sum_{k=0}^{m-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

or

$$\text{Prob}\,\{T_m \le t\} = 1 - \sum_{k=0}^{m-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \tag{8a}$$

Therefore, the density function of T_m is the derivative of Equation 8a and gives (Problem 3)

$$p_{T_m}(t) = \frac{\lambda^m t^{m-1}}{(m-1)!} e^{-\lambda t} \qquad t > 0 \tag{9a}$$

Equation 9a is called the *Erlang density function*, after the Danish mathematician A.K. Erlang, who so successfully applied stochastic processes to the analysis of telephone traffic. The expectation value of T_m is given by

$$E[T_m] = \int_0^\infty \frac{\lambda^m t^m}{(m-1)!} e^{-\lambda t} dt = \frac{m}{\lambda}$$

which is the expected time to the mth call.

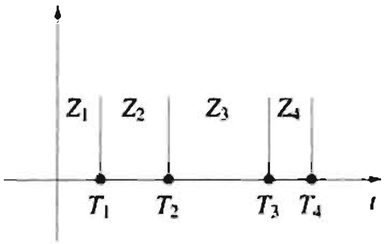


Figure 21.22
The arrival times $T_1, T_2, \ldots$ and the interarrival times $Z_1, Z_2, \ldots$ of a Poisson process.

There is one other property of Poisson processes that we should discuss. Let's consider the time intervals between successive arrival times. Let $Z_n = T_n - T_{n-1}$ be the nth interarrival time. (See Figure 21.22.) The events $[Z_1 > t]$ and $[X(t) = 0]$ are equivalent, so

$$\text{Prob}\{Z_1 > t\} = \text{Prob}\{X(t) = 0\} = e^{-\lambda t}$$

and

$$\text{Prob}\{Z_1 \le t\} = 1 - e^{-\lambda t} \tag{10a}$$

and the probability density of $Z_1(t)$ is

$$p_{Z_1}(t) = \lambda e^{-\lambda t} \tag{11a}$$

Now

$$\text{Prob}\{Z_2 > t\} = \int_0^\infty \text{Prob}\{Z_2 > t \mid Z_1 = \tau\} p_{Z_1}(\tau)d\tau \tag{12a}$$

If you look at Figure 21.22, you can see that $\text{Prob}\{Z_2 > t \mid Z_1 = \tau\} = \text{Prob}\{X(t+\tau) - X(\tau) = 0\}$. But $\text{Prob}\{X(t+\tau) - X(\tau) = 0\}$ is the probability that no event occurs in the interval $(\tau, t+\tau)$, which according to Equation 4a is

$$\text{Prob}\{X(t+\tau) - X(\tau) = 0\} = \text{Prob}\{T_1 > t\} = e^{-\lambda t}$$

Substituting these results into Equation 12a gives

$$\text{Prob}\{Z_2 > t\} = e^{-\lambda t}\int_0^\infty p_{Z_1}(\tau)\,d\tau = e^{-\lambda t} \tag{13a}$$

or

$$\text{Prob}\{Z_2 \le t\} = 1 - e^{-\lambda t} \tag{14a}$$

Thus both Z_1 and Z_2 are exponentially distributed random variables. Repeating the same argument shows that the interarrival times of a Poisson process are independent, identically distributed random variables.

Example 2:
Use the fact that

$$T_m = Z_1 + Z_2 + \cdots + Z_m$$

and that the Z_j are independent, exponentially distributed random variables to derive the Erlang density function, Equation 9a.

SOLUTION: Because $t \ge 0$, we can use a Laplace transform as the characteristic function of Z_j and write

The arrival times are independent and occur at random, so the probability of an arrival occurring between t_j and $t_j + dt_j$ is simply the uniform distribution dt_j/T. Thus,

$$E[I_k(t)] = \int_0^T \frac{dt_1}{T} \int_0^T \frac{dt_2}{T} \cdots \int_0^T \frac{dt_k}{T} \sum_{j=1}^{k} f(t - t_j)$$

$$= \frac{1}{T} \sum_{j=1}^{k} \int_0^T f(t - t_j)\, dt_j = \frac{k}{T} \int_0^T dz\, f(t - z)$$

$$= \frac{k}{T} \int_{-\infty}^{\infty} dz\, f(t - z) = \frac{k}{T} \int_{-\infty}^{\infty} f(u)\, du \tag{3b}$$

This last line follows because $f(u)$ is nonzero only for some small time interval, and so the integral from 0 to T can be written as an integral over all values of $t - t_j$.

Now, the probability that exactly k events occur in the time interval $(0, T)$ is given by a Poisson probability

$$p_k(t) = \frac{(\lambda T)^k}{k!} e^{-\lambda T} \tag{4b}$$

and so

$$E[I(t)] = \sum_{k=0}^{\infty} E[I_k(t)] p_k(t) \tag{5b}$$

Substitute Equation 3b into 5b and use the fact that $\langle k \rangle = \lambda T$ to obtain

$$E[I(t)] = \langle I(t) \rangle = \lambda \int_{-\infty}^{\infty} f(u)\, du = \lambda q \tag{6b}$$

This result is known as *Campbell's theorem*. Note that $\langle I(t) \rangle$ is independent of t as it should be for a stationary process.

We can use the same approach to derive the characteristic function of $I(t)$, which is given by

$$\varphi(s) = E[e^{isI(t)}] = \sum_{k=0}^{\infty} p_k(t) \langle e^{isI_k(t)} \rangle \tag{7b}$$

But

$$\langle e^{isI_k(t)} \rangle = \langle e^{is \sum_{j=1}^{k} f(t - t_j)} \rangle$$

$$= \prod_{j=1}^{k} \langle e^{isf(t - t_j)} \rangle = \langle e^{isf(t - t_j)} \rangle^k \tag{8b}$$

where t_j is any arrival time. Now

$$\langle e^{isf(t-t_j)}\rangle = \frac{1}{T}\int_0^T e^{isf(t-t_j)}dt_j$$

$$= 1 + \frac{1}{T}\int_0^T (e^{isf(t-t_j)} - 1)dt_j$$

$$= 1 + \frac{1}{T}\int_{-\infty}^{\infty} (e^{isf(t-t_j)} - 1)dt_j$$

$$= 1 + \frac{1}{T}\int_{-\infty}^{\infty} (e^{isf(u)} - 1)du$$

Let this last line be denoted by $1+\alpha$, so that Equations 7b and 8b become (Problem 5)

$$\varphi(s) = \sum_{k=0}^{\infty} p_k(T)(1+\alpha)^k = e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k(1+\alpha)^k}{k!}$$

$$= e^{-\lambda T}e^{\lambda T(1+\alpha)}$$

$$= \exp\left\{\lambda\int_{-\infty}^{\infty}[e^{isf(u)} - 1]du\right\} \tag{9b}$$

Example 3:
Use Equation 9b to derive the first few moments of $I(t)$.

SOLUTION: First note that $\varphi(0) = 1$, as it should. Differentiating once with respect to s gives

$$\frac{d\varphi}{ds} = i\varphi(s)\lambda\int_{-\infty}^{\infty} f(u)du$$

which gives Equation 6b when $s = 0$. Another differentiation gives

$$\langle I^2(t)\rangle = \lambda\int_{-\infty}^{\infty} f^2(u)du + \lambda^2\left[\int_{-\infty}^{\infty} f(u)du\right]^2$$

or

$$\sigma_I^2 = \langle[I(t) - \langle I\rangle]^2\rangle = \lambda\int_{-\infty}^{\infty} f^2(u)du \tag{10b}$$

If we substitute Equation 9b into

$$p(I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isI}\varphi(s)ds$$

then we find that the probability density function for $I(t)$ is given by

$$p(I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-iIs + \lambda\int_{-\infty}^{\infty}[e^{isf(u)} - 1]du\right\} ds \tag{11b}$$

Equation 11b may look pretty hopeless at first sight, but we should suspect that $p(I)$ should approach a normal distribution because of the central limit theorem. Realize that $I(t)$ is the sum of a large number of random variables. Problem 6, which is not difficult, has you show that $p(I)dI$ becomes

$$p(I)dI = \frac{1}{(2\pi\sigma_I^2)^{1/2}} e^{-(I-\langle I\rangle)^2/2\sigma_I^2} dI \tag{12b}$$

where $\langle I\rangle$ is given by Equation 6b and σ_I^2 in Example 1. Thus, we see that $p(I)$ is indeed a normal distribution.

We can derive an expression for the correlation function using the same method that we used to calculate $\langle I\rangle$ and $\varphi(s)$. The final result is (Problem 7)

$$R(\tau) = \lambda\int_{-\infty}^{\infty} f(u)f(u+\tau)du + \lambda^2\left[\int_{-\infty}^{\infty} f(u)du\right]^2 \tag{13b}$$

Note that R depends only upon τ, as we expect for a stationary process. Also note that the second term on the right side of Equation 13b is equal to $\langle I\rangle^2$. This is due to the fact that $R(\tau) \to \langle I\rangle^2$ as $\tau \to \infty$. If we let $R_0(\tau)$ be the correlation function of $I(t) - \langle I\rangle$, then $R_0(\tau) \to 0$ as $\tau \to \infty$, and we have

$$R_0(\tau) = \lambda\int_{-\infty}^{\infty} f(u)f(u+\tau)du \tag{14b}$$

Note that if we let $\tau = 0$ in Equation 14b, we get Equation 10b for σ_I^2.

Example 4:
Determine $R_0(\tau)$ if

$$f(u) = \begin{cases} \dfrac{q}{\tau_c} e^{-u/\tau_c} & u \geq 0 \\ 0 & u < 0 \end{cases}$$

SOLUTION: Using Equation 14b gives

$$\begin{aligned} R_0(\tau) &= \frac{\lambda q^2}{\tau_c^2}\int_0^{\infty} e^{-u/\tau_c} e^{-(u+\tau)/\tau_c} du \\ &= \frac{\lambda q^2}{2\tau_c} e^{-|\tau|/\tau_c} \end{aligned}$$

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CHAPTER 22

Mathematical Statistics

Suppose we have a large group of objects or individuals and we wish to know the average mass of the objects or the average age of the individuals or some other parameter of the *population* (the entire group). We could, of course, measure and record the appropriate property for every member of the population, but this is usually impractical for reasons of time, expense, effort, or whatever. Instead, we examine a small portion of the population (called a *sample*) and then infer the properties of the entire population from this sample. This process is called *statistical inference* and is one of the primary undertakings of statistics. For example, suppose that we know on theoretical grounds or through extensive experience that the properties of a certain population can be described by a normal distribution, but that we do not know either the mean or the variance. We could select a sample and then calculate its mean and variance, but how do these values reflect the mean and variance of the entire population? This problem is called *estimation of parameters*, and in Section 1 we shall learn how to choose functions of the sample data that give good estimates of the population parameters. Of course, these will not be exact, but in Section 3 we shall also learn how to assign ranges for these statistical estimates such that we can say that there is a certain probability (say 99%) that these estimates will lie within a certain distance from the true population value. In other words, we shall determine *confidence intervals* for these estimates.

In the final section we shall discuss *regression analysis*, where we use curve-fitting procedures in order to estimate one variable in terms of another. The standard curve-fitting procedure is a least-squares procedure, originally due to Gauss but developed into a powerful predictive tool by statisticians. We'll not only determine the optimum straight line through a set of data, but we'll also derive confidence intervals for this line. We conclude the chapter with a brief discussion of the theory of errors of measurement.

22.1 Estimation of Parameters

Suppose we have a population that is described by a probability distribution $f(x;\theta)$, where θ is a parameter. For example, θ might be the probability p in a binomial distribution or the mean λ in a Poisson distribution. We choose a random sample $x_1, x_2, \ldots, x_n$ of the population and use these data to estimate the parameter θ. More precisely, we use a function of these data:

$$\hat{\theta} = \hat{\theta}(x_1, x_2, \ldots, x_n) \tag{1}$$

Such a function is called a *sample statistic*, or simply a *statistic*. Each population parameter will have an associated statistic that we use to estimate the parameter. Because these estimates are single values, they are called *point estimates*.

It is certainly natural to use the sample mean (denoted by $\overline{x}$),

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} \tag{2}$$

as an estimate of the population mean μ. If we denote estimates by a carat, then we write

$$\hat{\mu} = \frac{x_1 + x_2 + \cdots + x_n}{n} \tag{3}$$

Similarly, we might use the *sample variance*,

$$\hat{s}^2 = \frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n} \tag{4}$$

as an estimate for the population variance σ^2. It turns out that we obtain better estimates of σ^2 by having $n - 1$ rather than n in the denominator of Equation 4. To see why this is so, we have to consider the nature of estimates a little more fully.

When we sample from a population, we obtain n data points, $x_1, x_2, \ldots, x_n$. If the sample data are taken randomly, then each observation, x_j, may be considered to be the value of a corresponding random variable X_j. As we learned in the previous chapter, it is convenient to work with independent random variables because their joint probability distribution factors into a product of individual probability distributions. For the n random variables $\{X_j;\ j = 1, 2, \ldots, n\}$ to be independent, we must not disturb the population in any way when we sample it. If the population is very large, the sampling has a negligible effect on the population, but this is not the case if the population is "small enough." Consequently, we shall always assume that we sample with replacement. There are statistical methods for dealing with small populations or for sampling without replacement, but we shall not consider them here.

We may regard $\hat{\theta} = \hat{\theta}(x_1, x_2, \ldots, x_n)$ as a single observation of the random variable

$$\hat{\Theta} = \hat{\Theta}(X_1, X_2, \ldots, X_n) \tag{5}$$

This random variable is called an *estimator* of the population parameter θ. For example, the sample mean

$$\hat{M} = \frac{X_1 + X_2 + \cdots + X_n}{n} \tag{6}$$

is a random variable that is an estimator of the population mean. The expectation value of $\hat{M}$ is

$$E[\hat{M}] = \frac{1}{n} E[X_1 + X_2 + \cdots + X_n]$$
$$= \frac{E[X_1] + E[X_2] + \cdots + E[X_n]}{n} \tag{7}$$

Because $E[X_j]$ is the expectation value for the population, $E[X_j] = \mu$ and

$$E[\hat{M}] = \frac{n\mu}{n} = \mu \tag{8}$$

Thus, the expectation value of $\hat{M}$ is μ, the population mean. Generally, if

$$E[\hat{\Theta}] = \theta \tag{9}$$

the estimator $\hat{\Theta}$ is said to be *unbiased*. Equation 8 says that the sample mean is an unbiased estimator of μ.

The following Example investigates whether or not the random variable corresponding to Equation 4 is an unbiased estimator of the population variance.

Example 1:
The random variable expression corresponding to Equation 4 is

$$\hat{S}^2 = \frac{(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2}{n} = \frac{1}{n} \sum_{j=1}^{n} (X_j - \overline{X})^2$$

where

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

Show that $\hat{S}^2$ is *not* an unbiased estimator of the population variance σ^2.

SOLUTION: According to Equation 9, we want to show that $E[\hat{S}^2] \neq \sigma^2$.

$$E[\hat{S}^2] = \frac{1}{n} E\left[\sum_{j=1}^{n} (X_j - \overline{X})^2\right]$$
$$= \frac{1}{n} E\left[\sum_{j=1}^{n} \{(X_j - \mu) - (\overline{X} - \mu)\}^2\right]$$
$$= \frac{1}{n} E\left[\sum_{j=1}^{n} (X_j - \mu)^2 - 2(\overline{X} - \mu) \sum_{j=1}^{n} (X_j - \mu) + n(\overline{X} - \mu)^2\right] \tag{10}$$

Let's look at each of these terms in turn. The first term is just

$$\frac{1}{n}\sum_{j=1}^{n} E[(X_j-\mu)^2] = \frac{n\sigma^2}{n} = \sigma^2 \tag{11}$$

Use the relation $\sum X_j = n\overline{X}$ in the second term to write

$$\sum_{j=1}^{n}(X_j-\mu) = n\overline{X} - n\mu = n(\overline{X}-\mu)$$

so that the second term becomes $-2E[(\overline{X}-\mu)^2]$. The third term contributes $E[(\overline{X}-\mu)^2]$, so that

$$E[\hat{S}^2] = \sigma^2 - E[(\overline{X}-\mu)^2]$$

Now let's consider the term $E[(\overline{X}-\mu)^2]$. The X_j are independent random variables, so

$$\begin{aligned}
E[(\overline{X}-\mu)^2] &= E\left[\left(\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right)^2\right]\\
&= \frac{1}{n^2}E[(X_1-\mu+X_2-\mu+\cdots+X_n-\mu)^2]\\
&= \frac{1}{n^2}\sum_{j=1}^{n}E[(X_j-\mu)^2] + \frac{2}{n^2}\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}E[(X_i-\mu)(X_j-\mu)]
\end{aligned}$$

The second term vanishes because the X_j are independent random variables, so Equation 11 tells us that

$$E[(\overline{X}-\mu)^2] = \frac{\sigma^2}{n} \tag{12}$$

Therefore, Equation 10 becomes

$$E[\hat{S}^2] = \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$$

Thus, we see that $E[\hat{S}^2] \neq \sigma^2$, and so $\hat{S}^2$ is not an unbiased estimator of σ^2.

Note, however, that

$$\hat{S}_X^2 = \frac{(X_1-\overline{X})^2+(X_2-\overline{X})^2+\cdots+(X_n-\overline{X})^2}{n-1}$$

is an unbiased estimator. The observation of $\hat{S}_X^2$ is

$$s_x^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j-\overline{x})^2 \tag{13}$$

22.1 Problems

1. Consider a population consisting of the four numbers 1, 5, 8, and 10, each of equal probability. Now consider all the possible samples of size two that can be taken (with replacement) from this population. Verify Equation 7 by showing that the mean of all the possible sample means is equal to the mean of the population. Also, verify Equation 12 by showing that the variance of the sample means is equal to the variance of the population divided by the sample size n.

2. Show that Equations 14 are equivalent to the statement $\lim_{n\to\infty} \text{Prob}\{|\Theta_n - \theta| < \epsilon\} = 1$. *Hint*: Use Chebyshev's inequality.

3. Determine the maximum likelihood estimate of λ in the exponential distribution $p(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

4. Determine the maximum likelihood estimate of p in a binomial distribution.

5. The distribution $xe^{-x^2/2\alpha}dx/\alpha$, $x \geq 0$, $\alpha > 0$, is called a *Rayleigh distribution* (Figure 22.1). First show that a Rayleigh distribution is normalized. Then show that the maximum likelihood estimator of α is $\alpha = (1/2n)\sum_j x_j^2$.

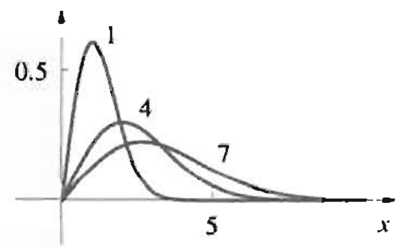

Figure 22.1
The Rayleigh distribution $xe^{-x^2/2\alpha}/\alpha$, $x \geq 0$, $\alpha > 0$, for several values of α.

6. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two independently determined estimates of some population parameter θ. Show that any linear combination of $\hat{\theta}_1$ and $\hat{\theta}_2$, $\hat{\theta} = c_1\hat{\theta}_1 + c_2\hat{\theta}_2$ with $c_1 + c_2 = 1$, is an unbiased estimate if $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased.

7. Let $\hat{\theta}$ be the maximum likelihood estimate of θ and let $g(\theta)$ be a monotonically increasing function of θ. Show that $g(\hat{\theta})$ is the maximum likelihood estimate of $g(\theta)$.

8. The fourth central moment of a normal distribution is $3\sigma^4$. Use the result of the previous problem to find the maximum likelihood estimate of $3\sigma^4$.

9. Show that $\partial^2 \ln L/\partial p^2 < 0$ at $p = \hat{p}$ for the likelihood function is given in Example 3.

10. Show that $\partial^2 \ln L/\partial\alpha^2 < 0$ at $\alpha = \hat{\alpha}$ for a Rayleigh distribution. (See Problem 5.)

11. Find the maximum likelihood estimate of the mean of a normal distribution if σ^2 is known. Show that $\partial^2 \ln L/\partial\mu^2 < 0$ at $\mu = \hat{\mu}$.

12. Find the maximum likelihood estimate of the variance of a normal distribution if μ is known. Show that $\partial^2 \ln L/\partial\sigma^2\partial\sigma^2 < 0$ at $\sigma^2 = \hat{\sigma}^2$.

13. Verify the matrix elements given in Example 4.

14. Verify Equations 26.

15. Show that $L(\mu, \sigma^2)$ for a normal distribution is a maximum at its maximum likelihood estimates.

22.2 Three Key Distributions Used in Statistical Tests

In the previous section, we learned how to determine point estimates of population parameters. In the next section, we'll learn how to assess a point estimate, in the sense of providing an interval where there is a given probability (say 0.95 or 0.99) that the population parameter will lie within that interval. Such intervals are called confidence intervals, but before we can discuss confidence intervals, we must discuss three distributions that are used extensively in the determination of confidence intervals. These three distributions are the normal distribution, the chi-square distribution, and the t-distribution.

A. THE NORMAL DISTRIBUTION

The normal distribution occurs in a wide variety of statistical applications, and we'll use it numerically in the next section when we calculate confidence intervals of the mean of a normal distribution when the variance is known. We have discussed the normal, or Gaussian, distribution throughout Chapter 21, but we shall present its relevant properties here for completeness. The normal distribution with mean μ and variance σ^2 is given by

$$p(x)dx = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(x-\mu)^2/2\sigma^2} dx \qquad -\infty < x < \infty \tag{1}$$

It's easy to show that $p(x)$ is normalized and that $E[X] = \mu$ and $E[(X-\mu)^2] = \sigma^2$. The effective width of a normal distribution is governed by the value of σ. Small values of σ produce narrow distributions, and in fact, $p(x)$ becomes a Dirac delta function, $\delta(x-\mu)$, as $\sigma \to 0$. Figure 22.2 shows the area under a normal curve that lies within $\mu \pm \sigma$, $\mu \pm 2\sigma$, and $\mu \pm 3\sigma$. The percentage areas corresponding to these intervals are 68.26%, 95.44%, and 99.74%, respectively (Problem 1).

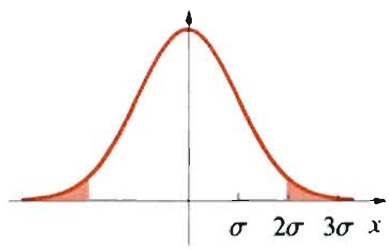

Figure 22.2
The area of a normal distribution between $\mu - n\sigma$ and $\mu + n\sigma$ is 0.6826, 0.9544, and 0.9974 for $n = 1$, 2, and 3, respectively.

In theoretical discussions, it is convenient to express Equation 1 in terms of a standardized random variable, with zero mean and unit variance. If we let

$$Z = \frac{X-\mu}{\sigma} \tag{2}$$

then Equation 1 becomes

$$p(z)dz = \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} dz \tag{3}$$

Equation 3 represents what is called a *standardized normal distribution*. The variable Z in Equation 2 is called the *standardized normal variable*.

A more general version of Equations 2 and 3 says that if the random variable X is normally distributed with mean μ_x and variance σ_x^2, then the random variable

$$Y = c_1 X + c_2 \qquad c_1 \neq 0 \tag{4}$$

is normally distributed with mean $\mu_y = c_1\mu_x + c_2$ and variance $\sigma_y^2 = c_1^2\sigma_x^2$ (Problem 2).

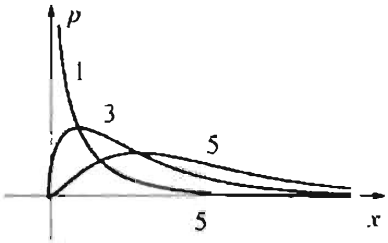

Figure 22.4
The chi-square distribution for $n = 1, 3,$ and 5 degrees of freedom.

and the integral of $p(x)$ from 0 to x gives the (cumulative) distribution function of χ^2:

$$P(x) = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^x u^{(n-2)/2} e^{-u/2} du \qquad x \geq 0 \tag{14}$$

The positive integer n in Equations 13 and 14 is called the *number of degrees of freedom* of the χ^2 distribution.

Figure 22.4 shows $p(x)$ plotted against x for several values of n. Note that $p(x)$ decreases monotonically for $n = 1$ and 2, and has a maximum value at $x = n - 2$ for $n > 2$ (Problem 14).

Example 2:
Determine the mean and variance of the chi-square distribution.

SOLUTION: We'll first show that $p(x)$ given by Equation 13 is normalized.

$$\int_0^\infty p(x)\, dx = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty x^{(n-2)/2} e^{-x/2} dx$$

$$= \frac{1}{\Gamma(n/2)} \int_0^\infty u^{(n-2)/2} e^{-u} du = 1$$

Then

$$E[X] = \mu = \int_0^\infty x p(x)\, dx = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty x^{n/2} e^{-x/2} dx$$

$$= \frac{2}{\Gamma(n/2)} \int_0^\infty u^{n/2} e^{-u} du = \frac{2\Gamma((n+2)/2)}{\Gamma(n/2)} = n$$

and

$$E[X^2] = \frac{4}{\Gamma(n/2)} \int_0^\infty u^{(n+2)/2} e^{-u} du = \frac{4\Gamma((n+4)/2)}{\Gamma(n/2)} = n^2 + 2n$$

$$\sigma^2 = E[X^2] - E[X]^2 = 2n$$

We might expect from the definition of χ^2 (Equation 11) and the central limit theorem that the χ^2 distribution becomes asymptotically normal for large n. In fact, we have

$$P(x) \sim F\left(\frac{x - n}{\sqrt{2n}}\right) \tag{15}$$

Table 22.4
Some values of a that satisfy $F(a) = (1+\eta)/2$ for several values of the confidence level η for a t-distribution.

$n\backslash\eta$	0.90	0.95	0.99	0.999
5	2.02	2.57	4.03	6.87
6	1.94	2.45	3.71	5.96
8	1.86	2.31	3.36	5.04
9	1.83	2.26	3.25	4.78
10	1.81	2.23	3.17	4.59
14	1.76	2.15	2.98	4.14
15	1.75	2.13	2.95	4.07
19	1.73	2.09	2.86	3.88
20	1.73	2.09	2.85	3.85
25	1.71	2.06	2.79	3.73
50	1.68	2.01	2.68	3.50
100	1.66	1.98	2.63	3.39
∞	1.65	1.96	2.58	3.30

C. CONFIDENCE INTERVALS FOR THE VARIANCE OF A NORMAL DISTRIBUTION

We learned in Section 2 that if $X_1, X_2, \ldots, X_n$ are independent normally distributed random variables with zero mean and unit variance, then

$$\chi^2 = \sum_{j=1}^{n} X_j^2 \tag{3}$$

is governed by a chi-square distribution with n degrees of freedom. It turns out that if $X_1, X_2, \ldots, X_n$ are independent normally distributed random variables with mean μ and variance σ^2, then the random variable

$$Y = \frac{1}{\sigma^2}\sum_{j=1}^{n}(X_j - \overline{X})^2 \tag{4}$$

is governed by a chi-square distribution with $n-1$ degrees of freedom. The reason that the distribution has $n-1$ degrees of freedom instead of n, as for Equation 3, is that the n terms in Equation 4 are *not* independent because $\overline{X}$ is given by

$$\overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

The proof that Y is governed by a chi-square distribution with $n-1$ degrees of freedom is a little lengthy and will not be given here. Nevertheless, the fact

Table 22.5

A step-wise procedure for determining the confidence intervals for the variance of a normal distribution.

Step 1: Calculate the sample mean $\bar{x}$ and $\sum_{j=1}^{n}(x_j - \bar{x})^2$.

Step 2: Choose a confidence level, η.

Step 3: Use a table of the chi-square distribution with $n - 1$ degrees of freedom (see also Table 22.6) to determine values of a_1 and a_2 such that

$$\text{Prob}\,\{a_1 \le x \le a_2\} = \int_{a_1}^{a_2} p(x)\,dx = \eta$$

In particular, we choose a_1 and a_2 such that the area to the left of a_1 is equal to the area to the right of a_2 in Figure 22.10. In terms of the cumulative distribution function $P(x)$, a_1 and a_2 are chosen such that

$$P(a_1) = \frac{1-\eta}{2} \qquad \text{and} \qquad P(a_2) = \frac{1+\eta}{2}$$

Step 4: Calculate $c_1 = (n-1)s^2/a_2$ and $c_2 = (n-1)s^2/a_1$. The confidence interval for the variance is

$$\text{Conf}\,\{c_1 \le \sigma^2 \le c_2\}$$

(The theoretical justification for this procedure is given in the references at the end of the chapter.)

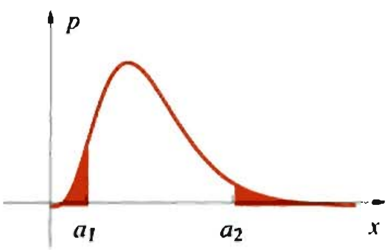

Figure 22.10
The two values a_1 and a_2 in the test given in Table 22.5 are chosen such that the area to the left of a_1 is equal to the area to the right of a_2.

that Equation 3 yields a statistic for the variance makes it plausible that the determination of confidence intervals for the variance involves the chi-square distribution. The step-wise procedure for doing so is described in Table 22.5.

Example 4:

Determine a 95% confidence interval for the variance of a normal distribution for the following data.

41.60	37.74	36.82	42.36	33.26	39.26
42.31	46.35	41.88	39.78	37.82	42.80
40.70	38.99	41.33	40.69	32.09	40.58
37.58	34.66	43.18			

SOLUTION:

Step 1: $\bar{x} = 39.61$ and $s^2 = 11.96$

Step 2: $\eta = 0.95$

Table 22.6
A few values of a_1 and a_2 that satisfy $P(a_1) = (1 - \eta)/2$ and $P(a_2) = (1 + \eta)/2$ for several values of the confidence level η for a chi-square distribution.

$n \setminus \eta$	0.90	0.95	0.99
9	3.33, 16.9	2.70, 19.0	1.73, 23.6
10	3.94, 18.3	3.25, 20.5	2.16, 25.2
14	6.57, 23.7	5.63, 26.1	4.07, 31.3
15	7.26, 25.0	6.26, 27.5	4.60, 32.8
19	10.1, 30.1	8.91, 32.9	6.84, 38.6
20	10.9, 34.1	9.59, 34.2	7.43, 40.0
30	18.5, 43.8	16.8, 47.0	13.8, 53.7
50	34.8, 67.5	32.4, 71.4	28.0, 79.5
100	77.9, 124.3	74.2, 129.6	67.3, 140.2

Step 3: These are 21 data points, so using Table 22.6, we find that $a_1 = 9.56$ and $a_2 = 34.2$ for $n = 20$.

Step 4: $c_1 = (20)(11.96)/34.2 = 7.00$ and $c_2 = (20)(11.96)/9.56 = 24.9$.

Therefore,

$$\text{Conf}\,\{7.00 \le \sigma^2 \le 24.9\}$$

Problem 15 has you show that the 99% confidence interval is $[\,5.98,\ 32.2\,]$.

In all the cases that we have discussed in this section, we have assumed that the distribution is normal. What if the distribution is not normal, or perhaps even unknown? We saw above that both the t-distribution and the chi-square distribution become asymptotically normal as the number of degrees of freedom become large. This result is a consequence of the central limit theorem, which says that if $X_1, X_2, \ldots, X_n$ are independent, identically distributed random variables, then the distribution function of the random variable,

$$Y = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

is asymptotically normal with mean μ and variance $n\sigma^2$ as $n \to \infty$. Often a value $n \ge 30$ is sufficient to approximate the sample distribution by a normal distribution. The central limit theorem is the primary reason that the normal distribution is of such importance in statistics.

D. CONFIDENCE INTERVALS OF THE PROBABILITY OF SUCCESS IN REPEATED TRIALS

An experiment that has exactly two outcomes, often referred to as success and failure, is called a *trial*. If we define a random variable Y that is equal to 1 for a success and equal to 0 for a failure, then $X = Y_1 + Y_2 + \cdots + Y_n$ is equal to the number of successes in n (independent) trials. We know from the previous chapter that the probability density of X is the binomial distribution

$$f(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x} \qquad x = 0, 1, \ldots, n \tag{5}$$

where $p + q = 1$. Recall that the mean and variance of the binomial distribution are np and npq, respectively.

Suppose now we have data such as

0	0	1	0	1	0	0	0	0	1
0	0	0	0	1	0	1	1	0	0
1	1	0	0	1	0	0	0	1	0

for 30 repeated trials. The number of successes is 10, which leads to $10/30 = 0.33$ as an estimate of the probability of a success. We would like to assign a confidence interval to this estimate of the parameter p in Equation 5. This turns out to be fairly easy if n is sufficiently large, because the binomial distribution is well approximated by a normal distribution for large values of n. In particular, the distribution function of the random variable

$$Z = \frac{X - \mu}{\sigma} = \frac{X - np}{(npq)^{1/2}} \tag{6}$$

is approximately normal with zero mean and unit variance for large values of n. We now choose a confidence limit η and use tables of the normal distribution to find the value of a that satisfies the relation

$$\text{Prob}\{-a \le Z \le a\} = \frac{1}{(2\pi)^{1/2}} \int_{-a}^{a} e^{-u^2/2} du = \eta \tag{7}$$

Using the expression for Z in Equation 6, the inequality in Equation 7 becomes

$$-a \le \frac{X - np}{(npq)^{1/2}} \le a \tag{8}$$

If we let $\hat{p} = x/n$ (the ratio of the number of successes to the number of trials) be our estimate of p, then we have at the two extremes of Inequality 8 that

$$\hat{p} - p = \pm a \left[\frac{p(1-p)}{n}\right]^{1/2} \tag{9}$$

Squaring both sides yields a quadratic equation in p, whose two solutions are (Problem 18)

$$p_{\pm} = \frac{\hat{p} + \dfrac{a^2}{2n} \pm a\left[\dfrac{\hat{p}(1-\hat{p})}{n} + \dfrac{a^2}{4n^2}\right]^{1/2}}{1 + \dfrac{a^2}{n}} \tag{10}$$

The corresponding confidence interval is

$$\text{Conf}\,\{p_- \leq p \leq p_+\} \tag{11}$$

For sufficiently large values of n, we can neglect the terms in a^2/n and a^2/n^2 in Equation 10 and write the confidence interval as

$$\text{Conf}\,\left\{\hat{p} - a\left[\frac{\hat{p}(1-\hat{p})}{n}\right]^{1/2} \leq p \leq \hat{p} + a\left[\frac{\hat{p}(1-\hat{p})}{n}\right]^{1/2}\right\} \tag{12}$$

Example 5:
Suppose an urn contains red balls and white balls. A random drawing (with replacement) of 50 balls showed that 22 were red and 28 were white. Use both Equations 10 and 11 to calculate confidence intervals for the actual fraction of red balls in the urn.

SOLUTION: A 95% confidence level means that the area in the two extreme wings of the normal distribution are each 0.025, and so we find that $a = 1.96$ for

$$F(a) = \frac{1+\eta}{2} = 0.975$$

Equation 12 with $\hat{p} = 22/50$ and $n = 50$ gives

$$\text{Conf}\,\{0.30 \leq p \leq 0.58\}$$

Equation 10 gives

$$\text{Conf}\,\{0.31 \leq p \leq 0.58\}$$

Example 6:
How many balls should we draw from the urn in Example 5 if we want a confidence level of 95% that $\hat{p}$ and p differ by no more than 0.050?

SOLUTION: We start with Equation 12:

$$\hat{p} - p = \pm a\left[\frac{\hat{p}(1-\hat{p})}{n}\right]^{1/2}$$

Table 22.7
The step-wise procedure for the chi-square goodness-of-fit test.

Step 1: Divide the x axis into n intervals (not necessarily of equal length). $I_1, I_2, \ldots, I_n$, such that the number of observed sample values o_j in each interval is at least 5.

Step 2: Using the hypothesized population distribution $F(x)$, calculate the expected number of sample values $e_j = np_j$ in each interval, where n is the total number of sample values and p_j is the probability that x lies in the interval I_j.

Step 3: Calculate the quantity

$$\chi^2 = \sum_{j=1}^{n} \frac{(o_j - e_j)^2}{e_j} \tag{1}$$

Step 4: Choose a significance level α (say 0.010 or 0.050) and then use a table of the chi-square distribution (see Table 22.8) with $n - 1$ degrees of freedom to determine c_α such that

$$\text{Prob}\,\{\chi^2 > c_\alpha\} = \alpha \tag{2}$$

Step 5: If $\chi^2 > c_\alpha$, we reject the assumption that $F(x)$ is the population distribution. If $\chi^2 \leq c_\alpha$, we accept the assumption, or at least we don't reject it.

This step-wise process is called the *chi-square test for the goodness of fit*.

chi-square distribution that allows us to do this. We'll present this method as a step-wise process (Table 22.7), as we did for the various tests in the preceding section.

You can see from Equation 1 that χ^2 will be small if the sample distribution and the population distribution are similar. Tables of the chi-square distribution give values of the cumulative distribution function, Prob $\{\chi^2 < x\}$, so Equation 2 can be expressed as

$$\text{Prob}\,\{\chi^2 \leq c_\alpha\} = 1 - \alpha \tag{3}$$

An extension of the above procedure says that if the population distribution has r unknown parameters, then we use the corresponding maximum likelihood estimates and then use the chi-square distribution with $n - r - 1$ degrees of freedom.

The chi-square test is best illustrated by Examples.

Example 1:

A coin is tossed 50 times and heads comes up 31 times. Is it a fair coin?

Table 22.8

A short table of the chi-square distribution for n degrees of freedom. The table lists the values of c_α for the corresponding values of n and α.

	α					
n	0.10	0.050	0.025	0.010	0.0050	0.0010
1	2.71	3.84	5.02	6.63	7.88	10.83
2	4.61	5.99	7.38	9.21	10.60	13.82
3	6.25	7.81	9.35	11.34	12.84	16.27
4	7.78	9.49	11.14	13.28	14.86	18.47
5	9.24	11.07	12.83	15.09	16.75	20.52
6	10.64	12.59	14.45	16.81	18.55	22.46
7	12.02	14.07	16.01	18.48	20.28	24.32
8	13.36	15.51	17.53	20.09	21.96	26.13
9	14.68	16.92	19.02	21.67	23.59	27.88
10	15.99	18.31	20.48	23.21	25.19	29.59

SOLUTION: We partition the results into two intervals, heads and tails. In Equation 1, $o_1 = 31$ and $o_2 = 19$. The expected values, based on the assumption that the coin is fair, are $e_1 = e_2 = 25$. The value of χ^2 is

$$\chi^2 = \frac{(31-25)^2}{25} + \frac{(19-25)^2}{25} = 2.88$$

Choosing a value of $\alpha = 0.050$ and using one degree of freedom, we find from Table 22.8 that $c_\alpha = 3.84$. Because $\chi^2 < 3.84$, we accept the assumption that the coin is fair, or at least we do not reject it.

Example 2:

Suppose now we toss the coin in Example 1 50 more times and get 62 heads and 38 tails (same ratio of heads to tails as in Example 1). What do you conclude now about the coin?

SOLUTION: In this case

$$\chi^2 = \frac{(62-50)^2}{50} + \frac{(38-50)^2}{50} = 5.76$$

which is greater than 3.84. Thus, we reject the assumption that the coin is fair.

Why do you suppose we concluded that the coin was fair in Example 1 and unfair in Example 2, based on the same ratio of heads to tails?

Example 3:
Someone gives you an algorithm to generate random digits 0, 1, 2, . . . , 9 and it gives the following result when you use it to generate 200 random digits.

	0	1	2	3	4	5	6	7	8	9
observed frequency	27	9	14	19	12	18	24	20	30	27

What do you think about this algorithm?

SOLUTION: The expected number of occurences in each case is 20, and so

$$\chi^2 = \frac{(27-20)^2 + (9-20)^2 + (14-20)^2 + \cdots + (12-20)^2}{20} = 22$$

Choosing a value of $\alpha = 0.010$, we find from Table 22.8 with 9 degrees of freedom that $c_\alpha = 21.67$. Because $\chi^2 > 21.67$, we should be suspicious of this algorithm.

Let's do an Example with a continuous distribution.

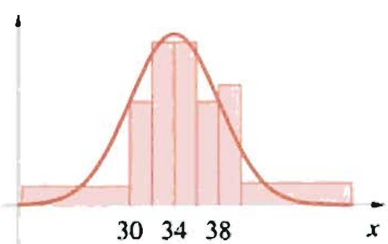

Figure 22.12
A bar graph of the data used in Example 4.

Example 4:
We suspect that a population is described by a normal distribution with $\mu = 35$ and $\sigma = 4.0$. The population is sampled 100 times with the following results. (See Figure 22.12 for a bar graph of the data.)

x interval	≤ 30	$30 < x \leq 32$	$32 < x \leq 34$	$34 < x \leq 36$
frequency	11	12	19	19
x interval	$36 < x \leq 38$	$38 < x \leq 40$	$x > 40$	
frequency	12	14	13	

Use these data to assess the assumption that the population is normally distributed with $\mu = 35$ and $\sigma = 4.0$.

SOLUTION: Using tables of the normal distribution, we find that the corresponding expected values for the intervals in the above table are 10.56, 12.10, 17.47, 19.74, 17.47, 12.10, and 10.56. The corresponding value of

8. A pair of dice are rolled 180 times and a total of seven comes up 40 times and a total of ten comes up 20 times. Test the assumption that the dice are fair at a 5.0% significance level.

9. Sixty families with four children were sampled and the number of families with x daughters turned out to be the following:

x	0	1	2	3	4
frequency	5	12	24	12	7

Test the assumption that these data satisfy a binomial distribution with $p = 1/2$.

10. Test the assumption that the following data obey a binomial distribution with $n = 4$ and $p = 1/2$:

x	0	1	2	3	4
frequency	19	34	34	11	2

11. Test the assumption that the data in the previous problem obey a binomial distribution with $n = 4$? The mean of the data is 1.43.

12. The grades on 150 exams are distributed according to the following:

interval	< 50	50–60	60–70	70–80	80–90	90–100
number	34	33	41	27	9	6

Do you think that these grades conform to a normal distribution with $\mu = 62$ and $\sigma = 15$?

13. Test the assumption that the following data obey a normal distribution with zero mean and unit variance:

interval	< -1.5	$(-1.5, -1.0)$	$(-1.0, -0.50)$	$(-0.50, 0)$
frequency	6	9	24	22
interval	$(0, 0.50)$	$(0.50, 1.0)$	$(1.0, 1.5)$	(> 1.5)
frequency	18	9	6	6

14. Test the assumption that the following data obey a normal distribution with zero mean and unit variance:

interval	< -2	$(-2, -1)$	$(-1, 0)$	$(0, 1)$	$(1, 2)$	(> 2)
frequency	14	10	25	24	13	14

15. Test the assumption that the following data obey a normal distribution with zero mean and unit variance:

interval	< -2	$(-2, -1)$	$(-1, 0)$	$(0, 1)$	$(1, 2)$	> 2
frequency	18	37	83	98	46	18

16. Test the assumption that the data in the previous problem conform to a normal distribution? Take $\overline{x} = 0.0863$ and $s^2 = 1.75$.

22.5 Regression and Correlation

A common and practical experiment in all sciences involves the repeated measurement of two different physical quantities for the purpose of determining a numerical relationship between them. We often seek a linear relationship, even transforming variables to obtain a linear relationship. For example, Figure 22.13a shows the vapor pressure P of water plotted against the celsius temperature, which is hardly linear. Thermodynamics tells us, however, that we should obtain a linear (or at least almost linear) plot if we plot ln P against the reciprocal of the kelvin temperature, as shown in Figure 22.13b.

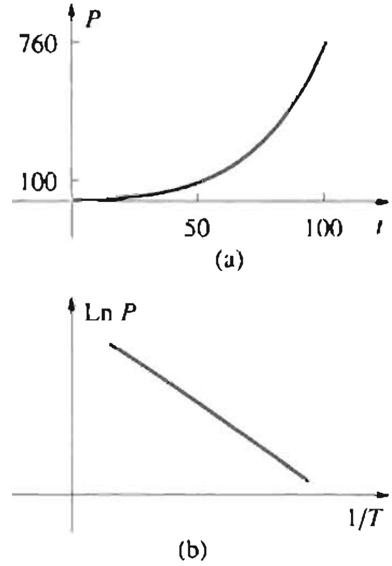

Figure 22.13
(a) The vapor pressure of water plotted against the celsius temperature. (b) The logarithm of the vapor pressure of water plotted against the reciprocal of the kelvin temperature.

This section involves regression and correlation. In regression analysis, we assume that one of the two variables, call it x, can be measured without appreciable error, and so we regard it as an ordinary variable. The other variable, Y, which is a function of x, is subject to some imprecision or uncertainty and is regarded as a random variable. We call x the independent variable and Y the dependent variable. In regression analysis, the mean value of Y is assumed to be a function of x, $\mu = \mu(x)$. The curve $y = \mu(x)$ is called the *regression curve of Y on x*. In the linear case, $\mu(x) = a + bx$, which is called the *regression line of Y on x* and the slope b is called the *regression coefficient*. In correlation analysis, both variables are regarded as random variables, and we wish to determine if they are, in fact, related.

Table 22.9 lists some typical data for the resistivity ρ of nichrome as a function of the celsius temperature t. These data are plotted in Figure 22.14. Both Figure 22.14 and theory suggest that the relation between ρ and t is linear. We want to draw a straight line through the data in Figure 22.14 in the most objective

Table 22.9
Typical data for the resistivity ρ of nichrome as a function of the celsius temperature. The resistivity is expressed in units of meter · ohm × 10^{-7}.

t /°C	20	25	30	35	40	45	50
ρ	9.137	8.913	8.665	8.528	8.242	8.203	7.972

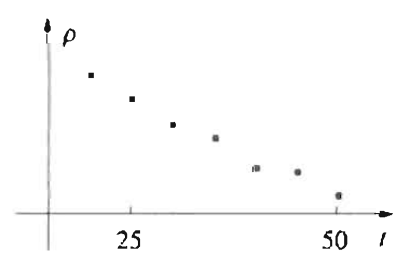

Figure 22.14
The resistivity ρ of nichrome as a function of the celsius temperature. The resistivity is expressed in units of meters · ohms × 10^{-7}.

and so

$$b = \frac{-4.449}{116.7} = -0.03812$$

and

$$a = 8.523 + (0.03812)(35) = 9.858$$

Therefore, the regression line is $\rho = 9.858 - 0.03812\, t$ (Figure 22.16).

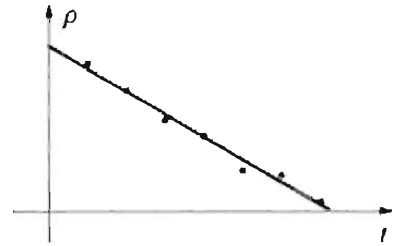

Figure 22.16
The data in Table 22.9 along with the regression line $\rho = 9.858 - 0.03812\, t$.

Example 1:
Table 22.10 lists some data for the molar heat capacity of carbon monoxide as a function of the kelvin temperature. Determine the regression line for these data.

SOLUTION: The necessary quantities from these data are

$$\overline{T} = 800 \qquad \overline{C}_P = 32.34$$
$$s_T^2 = 18750 \qquad s_{TC} = 133.78$$

and so $b = s_{TC}/s_T^2 = 0.007135$ and $a = 26.64$. The least-squares straight line is

$$C_P = 26.64 + 0.007135\, T$$

These data and results are plotted in Figure 22.17.

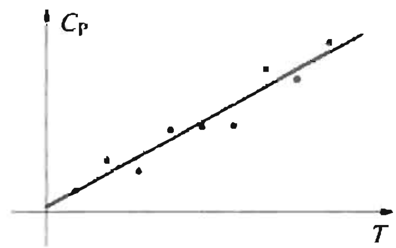

Figure 22.17
The data in Table 22.10 along with the regression line $C_P = 26.64 + 0.007135\, T$.

Table 22.10
The molar heat capacity (joules per mole per kelvin) of carbon monoxide as a function of the kelvin temperature.

T	600	650	700	750	800	850	900	950	1000
C_P	30.93	31.54	31.32	32.18	32.25	32.27	33.41	33.21	33.97

At one time the calculations of a and b were fairly tedious and older statistics books offer various short cuts for handling lots of data, but nowadays even the most modest CAS calculates not only linear regression lines, but regression curves for polynomials and sums of essentially arbitrary functions (Problem 6).

We said at the beginning of this section that we consider the Y_j to be random variables. We'll now assume further that each one is normally distributed with

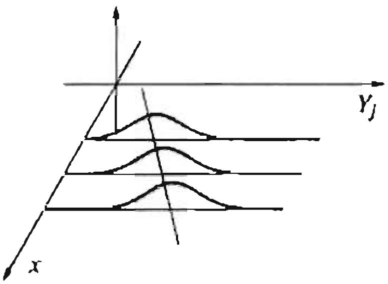

Figure 22.18
An illustration of the assumption that each random variable Y_j is normally distributed with mean $\mu = \alpha + \beta x$ and variance σ^2, which is independent of x.

mean

$$\mu(x) = \alpha + \beta x \tag{7}$$

and variance σ^2, which is independent of x (Figure 22.18). Notice that we use the notation $\alpha + \beta x$ in Equation 7. Equation 7 is the *regression line of the population* and β is the *regression coefficient of the population*. The values of a and b depend upon the particular sample, but α and β are population parameters. We'll now show that a and b given by Equations 3 through 6 are the maximum likelihood estimators of α and β, respectively.

Because we are assuming that the Y_j are independent and normally distributed, the likelihood function is given by

$$L(\alpha, \beta, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j - \alpha - \beta x_j)^2\right] \tag{8}$$

(Notice that maximizing $L(\alpha, \beta, \sigma^2)$ is equivalent to minimizing $S(a, b)$ in Equation 2.) Taking the logarithm of $L(\alpha, \beta, \sigma^2)$ and then maximizing $\ln L(\alpha, \beta, \sigma^2)$ with respect to α and β gives (Problem 7)

$$\hat{\alpha} = a = \overline{y} - b\overline{x} \tag{9}$$

$$\hat{\beta} = b = \frac{\sum_{j=1}^{n} x_j y_j - n\,\overline{x}\,\overline{y}}{\sum_{j=1}^{n} x_j^2 - n\overline{x}^2} \tag{10}$$

as the maximum likelihood estimators of the population regression parameters α and β.

We learned in Section 1 that maximum likelihood estimators are normally distributed for large values of n. In that case, we can determine confidence intervals for the population regression parameters α and β. As we did in Section 3 for other types of confidence intervals, we give a step-by-step procedure for determining confidence intervals for α and β in Table 22.11. (See the references at the end of the chapter for the theoretical justification of this procedure.)

Example 2:
Use the data in Table 22.10 to determine 95% confidence intervals for α and β.

SOLUTION: Example 1 gives us $\overline{T} = 800$, $\overline{C}_P = 32.34$, $s_T^2 = 18750$, $s_{TC} = 133.78$, $n = 9$, $a = 26.64$, and $b = 0.007135$. Furthermore, the data in Table 22.10 yield $s_C^2 = 1.0299$. We now follow the step-by-step procedure in Table 22.11.

Table 22.11
A step-wise procedure for determining the confidence intervals for the population regression parameters α and β.

Step 1: Calculate the two quantities

$$\sigma_\alpha^2 = \frac{[(n-1)s_x^2 + n\overline{x}^2](s_y^2 - b^2 s_x^2)}{n(n-2)s_x^2}$$

and

$$\sigma_\beta^2 = \frac{s_y^2 - b^2 s_x^2}{(n-2)s_x^2}$$

Step 2: Choose a confidence interval, η.

Step 3: Use a table of the standardized normal distribution, $p(z)$, (see also Table 22.2) to determine a value of γ such that

$$\text{Prob}\,\{-\gamma \le z \le \gamma\} = \int_{-\gamma}^{\gamma} f(z)\,dz = \eta$$

or, in terms of the cumulative distribution function, choose γ such that

$$F(\gamma) = \frac{1+\eta}{2}$$

Step 4: Calculate $\gamma\sigma_a$ and $\gamma\sigma_b$. The confidence intervals of α and β are given by

$$\text{Conf}\,\{a - \gamma\sigma_a < \alpha < a + \gamma\sigma_a\}$$

and

$$\text{Conf}\,\{b - \gamma\sigma_b < \beta < b + \gamma\sigma_b\}$$

The justification for this procedure is given in the references at the end of the chapter.

Step 1:

$$\sigma_a^2 = \frac{[(n-1)s_T^2 + n\overline{T}^2][s_C^2 - b^2 s_T^2]}{n(n-2)s_T^2}$$

$$= \frac{[(8)(18750) + (10)(800)^2][1.0299 - (0.007135)^2(18750)]}{(9)(7)(18750)}$$

$$= 0.4179$$

$$\sigma_b^2 = \frac{s_C^2 - b^2 s_T^2}{(n-2)s_T^2} = \frac{1.0299 - (0.007135)^2(18750)}{(7)(18750)}$$

$$= 5.743 \times 10^{-7}$$

or

$$\sigma_a = 0.6464 \qquad \text{and} \qquad \sigma_b = 0.000758$$

Step 2: $\eta = 0.95$

Step 3: Using Table 22.2, we find that $\gamma = 1.960$.

Step 4:

$$\text{Conf}\,\{25.37 < \alpha < 27.91\}$$

and

$$\text{Conf}\,\{0.00565 < \beta < 0.00862\}$$

Up to this point, we have assumed that the independent variables (the x_j) are determined with little or no uncertainty, so that we may regard them as ordinary variables. From here on, we'll treat both X and Y as random variables. We define a *correlation coefficient of the sample* by

$$r = \frac{s_{xy}}{s_x s_y} \qquad s_x > 0,\ s_y > 0 \tag{11}$$

Since s_{xy} may be either positive or negative and $s_x > 0$ and $s_y > 0$, r may be either positive or negative. Problem 4 has you show that

$$s_y^2 \geq \frac{s_{xy}^2}{s_x^2} \tag{12}$$

and so we see that $r^2 \leq 1$, or that $-1 \leq r \leq 1$. Furthermore, if $r^2 = 1$, then $\sigma_\beta^2 = 0$ and all the sample pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ will lie on a straight line. The converse is also true; that is, if all the sample pairs lie on a straight line, then $r^2 = 1$ (Problem 11). Suppose, on the other hand, that there is no relationship between the x_j and the y_j. Then the terms in the sum

$$\sum_{j=1}^{n}(x_j - \overline{x})(y_j - \overline{y})$$

will be equally likely to be positive and negative, and so s_{xy}, and consequently r, will be zero. In this case, the x_j and the y_j are said to be *uncorrelated*. This all suggests that the value of r is a measure of the *linear correlation* between the x_js and the y_js.

Example 3:
Determine the value of r^2 for the data given in Table 22.10.

Table 22.12
A step-wise procedure for determining confidence intervals for the population correlation coefficient.

Step 1: Calculate the quantity

$$z = \frac{1}{2} \ln \frac{1+r}{1-r}$$

Step 2: Choose a confidence level, η (for example, 95%, 99%, ...)

Step 3: Using tables of the standardized normal distribution (see also Table 22.2), determine a value of ξ such that

$$F(\xi) = \frac{1+\eta}{2}$$

Step 4: Calculate

$$\rho_1 = \tanh\left(z - \frac{\xi}{\sqrt{n-3}}\right) \quad \text{and} \quad \rho_2 = \tanh\left(z + \frac{\xi}{\sqrt{n-3}}\right)$$

The confidence interval for ρ is given by

$$\text{Conf}\,\{\rho_1 \le \rho \le \rho_2\}$$

Example 4:
Determine 95% confidence intervals for the population correlation coefficent ρ using the data in Table 22.10.

SOLUTION: According to Example 3, $r = (0.927)^{1/2} = 0.963$.

Step 1: $z = \frac{1}{2} \ln \frac{1+r}{1-r} = \frac{1}{2} \ln \frac{1.963}{0.037} = 1.98$

Step 2: $\eta = 0.950$

Step 3: Using Table 22.2, we find that $\xi = 1.960$

Step 4: $\rho_1 = \tanh\left(1.98 - \frac{1.960}{\sqrt{6}}\right) = \tanh(1.18) = 0.827$

Step 5: $\rho_2 = \tanh\left(1.98 + \frac{1.960}{\sqrt{6}}\right) = \tanh(2.78) = 0.992$

Therefore,

$$\text{Conf}\,\{0.827 \le \rho \le 0.992\}$$

Before we finish this section, we should discuss the important topic of error propagation in measurements. Let $f(x, y)$ be some quantity whose value we can determine by measuring x and y separately. For example, $f(x, y)$ might be an area of a rectangle, which we determine by measuring its width, x, and length, y, to give $A = xy$. Suppose now we measure x and y, so that we have the pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. We can calculate the means and sample variances of x and y according to

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j \qquad \text{and} \qquad \bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j \tag{15}$$

$$s_x^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2 \qquad \text{and} \qquad s_y^2 = \frac{1}{n-1}\sum_{j=1}^{n}(y_j - \bar{y})^2 \tag{16}$$

We can also calculate n values of $f(x, y)$ from the n pairs (x_j, y_j) according to $f_j = f(x_j, y_j)$, from which we can calculate

$$\bar{f} = \frac{1}{n}\sum_{j=1}^{n} f(x_j, y_j) \qquad \text{and} \qquad s_f^2 = \frac{1}{n-1}\sum_{j=1}^{n}(f_j - \bar{f})^2 \tag{17}$$

Just as s_x and s_y are measures of the imprecision of the values of the x_js and y_js, we consider s_f to be a measure of the imprecision of the $f(x_j, y_j)$.

Assuming, as usual, that the imprecisions are small, we can expand $f_j = f(x_j, y_j)$ about $x_j = \bar{x}$ and $y_j = \bar{y}$ to obtain

$$f_j = f(x_j, y_j) = f(\bar{x}, \bar{y}) + \left(\frac{\partial f}{\partial x}\right)_{\bar{x},\bar{y}}(x_j - \bar{x}) + \left(\frac{\partial f}{\partial y}\right)_{\bar{x},\bar{y}}(y_j - \bar{y}) + \cdots \tag{18}$$

If we divide both sides of Equation 18 by n and then sum over j, we obtain

$$\bar{f} = f(\bar{x}, \bar{y}) \tag{19}$$

Thus, if $f(x, y) = xy$ represents the area of the rectangle, then $\bar{A} = \bar{x}\,\bar{y}$.

We can also determine s_f^2 in terms of the imprecisions of the (x_j, y_j) data pairs. Substitute Equations 18 and 19 into s_f^2 (Equation 17) to get (Problem 21)

$$s_f^2 = \left(\frac{\partial f}{\partial x}\right)_{\bar{x},\bar{y}}^2 s_x^2 + \left(\frac{\partial f}{\partial y}\right)_{\bar{x},\bar{y}}^2 s_y^2 + 2\left(\frac{\partial f}{\partial x}\right)_{\bar{x},\bar{y}}\left(\frac{\partial f}{\partial y}\right)_{\bar{x},\bar{y}} s_{xy}^2 \tag{20}$$

where

$$s_{xy} = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})(y_j - \bar{y}) \tag{21}$$

If the values of the x_j and the y_j are independent, then $s_{xy} = 0$ and Equation 20

5. Use the result of the previous problem to show that all the points of a sample lie on the corresponding regression line if and only if $s_{xy}^2 = s_x^2 s_y^2$.

6. Derive formulas for a, b, and c in the regression curve $y = a + bx + cx^2$ for a parabolic fit.

7. Show that the maximum likelihood estimates of α and β in Equation 8 are given by Equations 9 and 10.

8. Use the following data to determine the least-squares regression line:

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	
y	−0.2765	0.0605	1.132	1.854	2.300	2.925	4.422	3.248	
x	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00
y	4.120	4.453	5.631	5.125	5.412	6.684	7.307	7.726	8.400

9. Use the data in the previous problem to determine confidence intervals for α and β, assuming that α and β are normally distributed.

10. Use the data in the previous problem to determine the correlation coefficient r.

11. Show that the correlation coefficient $r = \pm 1$ if the sample pairs lie on a straight line.

12. Show that if the two random variables X and Y are jointly normally distributed, then they are independent if they are uncorrelated.

13. Let η' be the viscosity of a polymer solution and η be the viscosity of the solvent. Then $(\eta' - \eta)/\eta = \eta_{sp}$ is called the intrinsic viscosity of the solution. Theory says that η_{sp}/c should vary linearly with c, where c is the concentration of the polymer in the solution. Table 22.13 gives some typical data. Use these data to determine the least-squares regression line.

Table 22.13

The intrinsic viscosity of cellulose nitrate in alcohol. The units of concentration are grams per liter.

c	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
η_{sp}/c	9.04	10.2	11.7	13.6	14.8	16.4	17.7	19.9	20.5	22.3	23.9

14. Use the data in the previous problem to determine a 99% confidence interval for α and β, assuming that α and β are normally distributed.

15. Use the data in the previous two problems to determine a 99% confidence interval for the population correlation coefficient ρ.

16. In the photoelectric effect, a metallic surface is irradiated with electromagnetic radiation and electrons are ejected from the surface. The energies of the ejected electrons are determined by measuring the potential at which the photoelectric current drops to zero. According to the theory of the photoelectric effect, the stopping potential ϕ_s is linearly related to the frequency of the radiation by $\phi_s = a\nu + b$. Use the data in Table 22.14 to determine the least-squares values of a and b and the value of the correlation coefficient r.

Table 22.14

The stopping potential for the photoelectric effect on sodium metal. The frequency ν is given as 10^{-15} Hz and the stopping potential ϕ_s is given in volts.

ν	0.46	0.48	0.50	0.52	0.54	0.56	0.58	0.60
ϕ_s	0.0834	0.315	0.182	0.340	0.786	0.352	0.731	0.430

17. Use the data in the previous problem to determine a 90% confidence interval for α and β, assuming that α and β are normally distributed.

18. Use the data in the previous two problems to determine a 90% confidence interval for the population correlation coefficient ρ.

19. Table 22.15 gives the vapor pressure of water P (in torr) and the corresponding temperature t (in °C). Thermodynamics tells us that a plot of $\ln P$ against $1/T$, where T is the kelvin temperature $T = t + 273.15$, should approximate a straight line. Determine the least-squares fit to these data.

Table 22.15

The vapor pressure of water, P, in torr and the corresponding celsius temperature, t.

t	0	5	10	15	20	25	30
P	4.6	6.5	9.2	12.8	17.4	23.8	31.6
t	35	40	45	50	55	60	65
P	42.2	55.3	71.9	92.5	118.0	149.4	187.5
t	70	75	80	85	90	95	100
P	233.7	289.1	355.1	433.6	525.8	633.9	760

20. Use the data in the previous problem to determine a 99.5% confidence interval for α and β, assuming that α and β are normally distributed.

21. Substitute Equations 18 and 19 into Equation 17 to obtain Equation 20.

22. Show that s_{xy} defined by Equation 21 satisfies the inequality $|s_{xy}| \leq s_x s_y$. *Hint*: Start with

$A(t) = \sum_{j=1}^{n}[(x_j - \overline{x}) + t(y_j - \overline{y})]^2 \geq 0$ and use the fact that $A_{\text{min}} = s_x^2 - s_{xy}^2/s_y^2 \geq 0$.

23. Suppose we have two independent measurements on a quantity y and that the results are reported as $y_1 \pm \sigma_1$ and $y_2 \pm \sigma_2$. How should we combine these results to find the average? Surely $\overline{y} = \frac{1}{2}(y_1 + y_2)$ is unsuitable; the value with the smaller value of σ should somehow be weighted more. Use the maximum likelihood method to show that $\overline{y} = \dfrac{w_1 y_1 + w_2 y_2}{w_1 + w_2}$ where $w_j = 1/\sigma_j^2$.

Chapter 1

Section 1.1

1. (a) $-4 \le x \le 4$, (b) $-\infty < x < \infty$, (c) $x > 0$, (d) $x \neq 1$; **9.** An ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b^2 = a^2 - c^2$; **10.** $\dfrac{(x-2)^2}{a^2} + \dfrac{(y+1)^2}{b^2} = 1$; **11.** A parabola; **15.** (a) Periodic, $\pi/2$, (b) Periodic, π, (c) Not periodic; **16.** (a) Odd, (b) Neither, (c) Even, (d) Neither; **19.** $(x^2+1)^2 + 1 = x^4 + 2x^2 + 2$; **20.** $y^2/(1+y^2)$; **22.** (a) $1 < x < 2$, (b) $-3 < x < 1/5$; **23.** A staircase; **24.** A sequence of square waves.

Section 1.2

4. (a) 3, (b) 0, (c) 1, (d) 2/3; **5.** (a) 2, (b) 1/2, (c) -6, (d) 1/2; **6.** (a) $2x$, (b) $-1/x^2$, (c) $\cos x$, (d) $-\sin x$; **7.** (a) 0, (b) 1/2; **8.** $1/2\sqrt{2}$; **9.** $\lim\limits_{x\to 0+} = 1$ and $\lim\limits_{x\to 0-} = -1$; **10.** -1; **11.** 4; **14.** 10^{-3}; **15.** 2; **16.** 1; **17.** 2/3; **18.** ∞; **19.** $-\infty$; **20.** -1.

Section 1.3

2. No, but the point $x = 1$ is a removable discontinuity; **3.** Use the intermediate value theorem and the fact that $f(1) = 5$ and $f(-1) = -3$; **4.** Let $f(x) = \cos x - x$ and apply the intermediate value theorem; **5.** (a) -1, (b) 1, (c) does not exist; **6.** $x = n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$; **9.** We say that $g(x)$ diverges more strongly than $f(x)$ at the point $x = 2$. Furthermore, the one-sided limits of $f(x)$ differ in sign, but those of $g(x)$ do not; **10.** Yes; 3/2; **11.** Let $f(x) = x3^x - 2$ and use the intermediate value theorem and the fact that $f(0) = -2$ and $f(1) = 1$; **12.** $\alpha = \beta = -1/2$.

Section 1.4

1. (a) $(1 - 4x - 2x^2)e^{-x^2}$, (b) $\dfrac{\cos x}{x} - \dfrac{\sin x}{x^2}$, (c) $2x \tan 2x + 2x^2 \sec^2 x$; **2.** (a) $-\dfrac{3}{2}(x-1)^{-5/2}$, (b) $\dfrac{1}{2}\dfrac{2x-3}{(x^2-3x+1)^{1/2}}$, (c) $a^x \ln a$; **3.** (a) $\dfrac{-e^{-x}}{1+e^{-2x}}$, (b) $\dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$, (c) $x^{\sin x}\left(\ln x \cos x + \dfrac{\sin x}{x}\right)$; **5.** No; **8.** $x = 1$ is a maximum and $x = 2$ and -2 are minima. There are inflection points at $x = \frac{1}{3}(1 \pm \sqrt{13})$; **9.** $x = 0$; **10.** Let $\xi = x - \frac{3}{2}$ and note that $F(\xi) = \left(\xi + \frac{1}{2}\right)^2 \left(\xi - \frac{1}{2}\right)^2$ is symmetric about $\xi = 0$; **13.** $x^x(1 + \ln x)$; **16.** $h_{\max} = 25/2$; **18.** $(2^{-1/3}, \pm 2^{1/6})$; **19.** 1; **20.** $3r^2\sqrt{3}/4$; **26.** $18x^3e^{-x}\cos(3x^2+2)\sin^2(3x^2+2) + 2xe^{-x}\sin^3(3x^2+2) - x^2e^{-x}\sin^3(3x^2+2)$; **27.** $\dfrac{18\cos 5 \sin^2 5}{e} + \dfrac{\sin^3 5}{e} = 1.40284$; **29.** 34.4082.

Section 1.5

1. Use $\Delta x = -5$ and obtain 4.933; **2.** $\Delta V = 3.84$ cm^2, or 0.75%; **3.** $\epsilon(x) = (3x+1)\Delta x + (\Delta x)^2 \to 0$ as $\Delta x \to 0$; **4.** $\epsilon(x) = \dfrac{(x+a)\Delta x}{(x+a)(x+a+\Delta x)} \to 0$ as $\Delta x \to 0$; **5.** $\Delta y = y(10.10) - y(10.00) = 1.91$, $dy = f'(x)\Delta y = (2x-1)\Delta x = (19.0)(0.10) = 1.90$; **6.** $\Delta y = (4.35)^{1/2} - (4.00)^{1/2} = 0.086$.

$dy = f'(x)\Delta x = \dfrac{\Delta x}{2x^{1/2}} = \dfrac{0.35}{2(4.00)^{1/2}} = 0.0875$; **7.** $d\cos\theta = (-\sin\theta)d\theta =$ $(-0.4226)(0.20°) = -0.085$; **8.** (a) $dx/8x^{1/2}$, (b) $-dx/x^5$, (c) $2\tan x\, dx/\cos^2 x$, (d) $-dx/4x^{3/2}$; **9.** (a) $2x\, dx/3(x^2-2)^{2/3}$, (b) $\cos x^{1/2}dx/2x^{1/2}$, (c) $-e^{\cos x}\sin x\, dx$; **10.** Yes; **11.** (a) $-2\tan y/x$, (b) $(3x^2-6y)/(6x-2y)$, (c) $-b^2x/a^2y$.

Section 1.6

1. 0.7071; **2.** 2.71828; **4.** (a) 0, (b) 0, (c) 0, (d) 0; **5.** (a) 1, (b) 1/2, (c) −1, (d) 1; **6.** (a) 0, (b) 1, (c) 1, (d) 1; **9.** 1; **10.** 1; **11.** The next to the last displayed limit is *not* an indeterminate form.

Section 1.7

1. Integrate the formula for the derivative of a product; **2.** (a) $-(1+x)e^{-x}$, (b) $\sin x - x\cos x$, (c) $(\ln x)^2/2$, (d) $x\ln x - x$; **3.** (a) $-\cos^{-1}(x/a)$, (b) $\ln\tan(3\pi/8) = 0.8814$, (c) $\pi/8 + 1/4$, (d) $a^2\pi/4$; **4.** (a) $\ln|x + \sqrt{x^2-a^2}|$, (b) $\frac{1}{2}(\sqrt{2} + \sinh^{-1} 1)$, (c) $\sinh^{-1} 1$, (d) $\tanh^{-1}(1/2) = 0.5493$; **5.** 4/3; **6.** $\pi a^2/2$; **7.** $\sqrt{17} + \frac{1}{4}\ln(4+\sqrt{17}) = 4.6468$; **8.** $\pi r^2 h/3$; **14.** $S_3 = n^2(n+1)^2/4$; **16.** $\sum_{j=1}^{n}(2j-1) = 2\sum_{j=1}^{n} - \sum_{j=1}^{n} 1 = n(n+1) - n = n^2$; **21.** 1/3.

Section 1.8

1. $\pi/2$; **2.** $(2\pi\sigma^3/3)[1-(\lambda^3-1)(e^{\epsilon/k_BT}-1)]$; **3.** π; **4.** $\pi/2$; **5.** 2; **6.** The integral diverges; **20.** It does not converge; **23.** 0.4036; **24.** 0.3857; **25.** ln 2; **26.** $(\pi^{1/2}/2a)e^{-b^2/4a^2}$; **27.** $(\pi^{1/2}/2a)e^{-2ab}$.

Section 1.9

3. Differentiate $\int_0^\infty e^{-\alpha t}dt = 1/\alpha$ with respect to α n times; **5.** Differentiate $\int_0^\infty e^{-\alpha t^2}dt = (\pi/4\alpha)^{1/2}$ with respect to α n times; **9.** Yes. The second integral is uniformly convergent; **13.** $\cot^{-1} a$.

Chapter 2

Section 2.1

9. (a) Increasing, (b) Decreasing if $n > 2$, (c) Decreasing, (d) Decreasing; **12.** (a) $s_n \to 1$, (b) $s_n \to 1$, (c) $s_n \to e$, (d) e^{-2}; **13.** 0.

Section 2.2

1. 3/2; **2.** 1/3; **4.** 3/11; **6.** 1; **7.** $\frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$; **8.** 3/2; **9.** 1/4; **10.** $\frac{2^n-1}{2^n} \to 1$ as $n \to \infty$; **11.** 1/24; **12.** (a) $|x| < 1/2$, (b) $0 < x < 2$, (c) $-1 < x < 2$, (d) $x < 0$; **14.** No.

Section 2.3

1. No; **2.** Yes; **4.** About 25; **8.** No. S as written is an indeterminate form, $\infty - \infty$; **10.** (a) Converges, (b) Converges, (c) Converges, (d) Diverges; **11.** Yes; **12.** No, yes; **13.** Yes. Use the fact that $1 + 2 + \cdots + n = n(n+1)/2$.

Section 2.4

1.

N	u_{N+1}	S_N	R_N	$\|R_N\|$
1	0.0625	1.0000	−0.0529	0.0529
2	0.0123	0.9375	0.0095	0.0095
3	0.0039	0.9498	−0.0028	0.0028
4	0.0016	0.9459	0.0011	0.0011
5	0.0008	0.9475	−0.0005	0.0005

2.

N	v_{N+1}	S_N	R_N	$\lvert R_N\rvert$
1	0.2500	1.0000	−0.1775	0.1775
2	0.1111	0.7500	0.0725	0.0725
3	0.0625	0.8611	−0.0386	0.0386
4	0.0400	0.7986	0.0239	0.0239
5	0.0278	0.8386	−0.0161	0.0161

3. About 9; **4.** About 14; **5.** About 21; **9.** Diverges; **10.** Converges; **12.** (a) Conditionally convergent, (b) Conditionally convergent, (c) Absolutely convergent, (d) Absolutely convergent, (e) Conditionally convergent, (f) Diverges; **13.** Conditionally convergent; **14.** Diverges (Raabe's test); **15.** Converges absolutely for $|x| < 1$; **16.** Converges;

19.

N	v_{N+1}	S_N	R_N	$\lvert R_N\rvert$
1	0.5000	1.0000	−0.4167	0.4617
2	0.3333	0.5000	0.0833	0.0833
3	0.2500	0.8333	−0.2500	0.2500
4	0	0.5833	0	0
5	0			

20. Converges (Raabe's test).

Section 2.5

1. (a) All values of x, (b) $0 < x < 4$ and conditionally at $x = 0$, (c) $-1 \le x \le 1$, (d) $-2 < x < 0$ and conditionally at $x = -2$; **2.** (a) $x = 0$ only, (b) $-1 < x < 3$; **3.** All values of x; **6.** (a) Take $M = 1/n^2$, (b) Take $M = 1/n^2$; **10.** No, because the terms of the series are continuous and $S(x)$ is discontinuous; **12.** $1 + 2x + 3x^2 + \cdots$; **13.** $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$; **14.** Yes. Use the M test with $M = 1/n^2$; **15.** $S_n(x)$ does not converge uniformly to $S(x)$.

Section 2.6

2. (a) $-1 \le x \le 1$, (b) $-\infty < x < \infty$; **3.** (a) $-\infty < x < \infty$, (b) $1 \le x \le 3$; **4.** (a) $-1 \le x < 1$, (b) $-2 \le x < 0$; **5.** $x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$; **6.** 0.7357; **7.** Differentiate the geometric series twice; **9.** $S(x) = \sin x$ and $C(x) = \cos x$; **10.** Let $x = \sqrt{3}/3$ and $\tan^{-1}\sqrt{3}/3 = \pi/6$; use 11 terms, 3.141593.

Section 2.7

2. Each number is the sum of the two above it; **3.** Use the binomial expansion with $x = 1$; **6.** Use Equation 3; **8.** Choose $n > 10$, 2.718282; **9.** Multiply the expansion of e^{-x^2} by x: $x\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!}$; **10.** Use Equation 7 with $\alpha = 1/2$, $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$; **14.** Use Equation 7 with $\alpha = -1/2$ and "x" $= -x^2$; **19.** $1 - \frac{x^2t^2}{2} + \left(\frac{x^2}{6} + \frac{x^4}{24}\right)t^4 + O(t^6)$; **20.** $-x + \frac{4x^3}{3} - \frac{17}{15}x^5 + O(x^7)$; **21.** $(x-1) + \frac{5}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + \frac{1}{4}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{60}(x-1)^6 + O((x-1)^7)$; **22.** $-\pi^3 - 3\pi^2(x-\pi) - \left(3\pi + \frac{\pi^3}{2}\right)(x-\pi)^2 - \left(1 + \frac{3\pi^2}{2}\right)(x-\pi)^3 - \left(\frac{3\pi}{2} + \frac{5\pi^3}{24}\right)(x-\pi)^4 - \left(\frac{1}{2} + \frac{5\pi^2}{8}\right)(x-\pi)^5 + O((x-\pi)^6)$.

Section 2.8

1. Yes; **2.** 0.42872; **3.** No. It diverges as $\ln\epsilon$ as $\epsilon \to 0$; **4.** It tells you that the integral diverges; **5.** $(1+x^4)^{1/2} = 1 + \frac{x^4}{2} - \frac{x^8}{8} + \cdots$ Integration of three terms gives 0.5031; **6.** 0.503098; **7.** $\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{3}{2\cdot 4}\right)^2 k^4 + \left(\frac{3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right]$; **9.** $\sin\sqrt{x} = x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \cdots$. 0.60234; **10.** 0.602337; **11.** 1; It's a telescoping series; **12.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{4n+2}$; **13.** 10!/21!; **14.** Mathematica beeps that $f'(0)$ is an indeterminate expression. $f'(0) = -1/6$; **19.** $1 - \frac{3}{4}\kappa R + \frac{3}{5}(\kappa R)^2 - \frac{1}{2}(\kappa R)^3 + \frac{3}{7}(\kappa R)^4 - \frac{3}{8}(\kappa R)^5 + O((\kappa R)^6)$; **21.** $1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147456} - \frac{x^{10}}{14745600} + O(x^{12})$; **23.** $1/A(t) = 1/A_0 + kt$.

Section 2.9

1. Integrate by parts, letting "u" $= 1/u$ and "dv" $= ue^{-u^2}du$; **4.** $-f(x)\sin x + g(x)\cos x$; **7.** Write Equation 14 as $1 + \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)!}{2^n x^{2n} 2^{n-1}(n-1)!}$; **9.** $F(10) = 0.099010$ using $1/s - 1/s^3 + 1/s^5$; **10.** $F(s) = \frac{1}{s} - \frac{1}{3s^3} + \cdots$ To two terms, $F(s) = 0.99666$. The "exact" result is $\tan^{-1}(0.10) = 0.996687$.

Chapter 3

Section 3.1

1. $\frac{1}{a^{5/2}}\Gamma\left(\frac{5}{2}\right) = \frac{1}{a^{5/2}}\frac{3\sqrt{\pi}}{4}$; **2.** 3/8; **3.** $\frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$; **4.** $\sqrt{\pi}/8$; **5.** $(-1)^n\Gamma(n+1)$; **6.** $(2\pi)^{1/2}$; **13.** 8.8553; **15.** (a) $10!! = 3840$, (b) $7!! = 105$; **17.** $\frac{1}{n}\Gamma\left(\frac{m+1}{n}\right)$.

Section 3.2

1. Use Equation 2; **5.** $5\pi/8$; **6.** $B(4/3, 4/3) = \Gamma(1/3)^2/10\,\Gamma(2/3) = 0.52999$; **7.** $\frac{\Gamma(1/3)\Gamma(5/3)}{\Gamma(2)} = 2.4184$; **8.** 4; **9.** $2\pi/3\sqrt{3}$; **10.** $\Gamma(1/6)\sqrt{\pi}/6a^2\Gamma(2/3)$; **11.** $2^{1/3}B(2/3, 2/3)/2 = 1.29355$; **12.** $3\pi/8$.

Section 3.3

2. $\sqrt{\pi}\,\mathrm{erf}\,(2\sqrt{2})/2 = 0.88617$; **6.** $\mathrm{erf}\,(\sqrt{2}) = 0.955$; **9.** $\frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} + \cdots\right)$; **10.** Six terms give 0.8426; **15.** Let $a = 1/4$, $2b = a$, and $c = 0$ in Problem 11.

Section 3.4

1. Integrate $E_n(x)$ by parts, letting z^{-n} be "u" and $e^{-xz}dz$ be "dv"; **4.** $\frac{e^{-x}}{x+n}\left[1 + \frac{n}{(x+n)^2} + \cdots\right]$; **5.** Expand $\sin u/u$ and integrate term by term.

Section 3.5

6. $4\sqrt{2}\,E(1/\sqrt{2}) = 7.640$; **7.** $16E(\sqrt{3}/2) = 19.38$; **9.** First let $\cos\theta = 2\cos^2(\theta/2) - 1$ and then let $\theta/2 = \pi/2 - \phi$. $K(1/\sqrt{2}) - F(1/\sqrt{2}, \pi/4)$; **10.** $1.8541 - 0.8260 = 1.0281$; **16.** $K(3/4)/4 = 0.4777$; **17.** $K(1/\sqrt{5})/\sqrt{5} = 0.7422$.

parametric equations of the tangent line are $x = u$, $y = 1 - 4u$, $z = -2 + 6u$ and the equation of the normal plane is $x - 4y + 6z + 16 = 0$.

Chapter 6

Section 6.1

1. $\sqrt{29}$; **2.** $-1 \le (x - y)/(x + y) \le 1$, or $x \ge 0$, $y \ge 0$, $(x, y) \ne (0, 0)$; **3.** No; **4.** Yes, yes; **6.** It bisects the second and fourth xy-quadrants and the z axis lies in the plane; **7.** $(x-1)^2/a^2 + (y+1)^2/b^2 + z^2/c^2 = 1$; **11.** Complete the square to get $(x-1)^2 + y^2 + (z+2)^2 = 13$. It is a sphere of radius $\sqrt{13}$ centered at $(1, 0, -2)$; **12.** It is an elliptic cylinder whose axis is the z axis; **13.** Complete the square and divide by 4 to get $(x+1)^2/4 + y^2 = 2(z-2)$, the equation of an elliptic paraboloid; **16.** $\sqrt{7}$; **17.** Because $z = \pm c$ when $x = y = 0$.

Section 6.2

7. (a) 4, (b) $(a^2 + b^2)/ab$; **8.** (a) Does not exist, (b) 0; **9.** (a) limit $= \pi/2$ as $x \to 0+$ and $-\pi/2$ as $x \to 0-$. Does not exist. (b) Does not exist; **10.** (a) 0, (b) Does not exist; **11.** (a) You get two different results (0 and 1). Therefore, the limit does not exist. (b) You get $-5/3$ in each case. Therefore, it *may* exist, and, in fact, does exist; **12.** Both sequential limits give 1. But the limit is equal to $(1 - m^2)^2/(1 + m^2)^2$. The two sequential limits correspond to $m = 0$ and ∞; **13.** No. It is not defined at $(0, 0)$; **14.** Yes. The limit of $f(x, y)$ is zero. (See Problem 4.); **15.** No. The limit of $f(x, y) = 1$ if x and y approach zero along the curve $x = t^2$ and $y = t$; **16.** (a), (b).

Section 6.3

1. (a) $f_x = e^y$, $f_y = 1 + xe^y$, $f_{xx} = 0$, $f_{yy} = xe^y$, $f_{xy} = f_{yx} = e^y$, (b) $f_x = y\cos x + 2x$, $f_y = \sin x$, $f_{xx} = 2 - y\sin x$, $f_{yy} = 0$, $f_{xy} = f_{yx} = \cos x$; **2.** (a) $f_x = -y/(x^2 + y^2)$, $f_y = x/(x^2 + y^2)$, $f_{xx} = 2xy/(x^2 + y^2)^2$, $f_{yy} = -2xy/(x^2 + y^2)^2$, $f_{xy} = f_{yx} = (y^2 - x^2)/(x^2 + y^2)^2$, (b) $f_x = -2xe^{-(x^2+y^2)}$, $f_y = -2ye^{-(x^2+y^2)}$, $f_{xx} = (4x^2 - 2)e^{-(x^2+y^2)}$, $f_{yy} = (4y^2 - 1)e^{-(x^2+y^2)}$, $f_{xy} = f_{yx} = 4xye^{-(x^2+y^2)}$; **11.** $x + 7y - z = 4$; **12.** $3x - y - 4z = 0$; **13.** $4x + 2y - z = 3$; **14.** Let $y = mx$ and see that the limit depends upon m; **15.** To see that $f(x, y)$ is discontinuous at $(0, 0)$, let $y = mx$; **16.** $\kappa_T = 1/P$, $\alpha = 1/T$; **17.** $U = 3Nk_BT/2$; **18.** $(\partial U/\partial V)_T = 0$ for an ideal gas and a/V^2 for a van der Waals gas.

Section 6.4

2. $-6t\sin 3t^2$; **3.** $4t^3(1 - t^4)e^{-t^4}$; **4.** $(\sin^2 t - \cos^2 t)\,e^{-\cos t \sin t} = -\cos 2t\; e^{-(\sin 2t)/2}$; **5.** $2t + 2t^4e^{t^2} + 3t^2e^{t^2}$; **6.** $xe^{-2x}\sin(xe^{-x}) - e^{-x}\cos(xe^{-x}) - e^{-2x}\sin(xe^{-x})$; **7.** $2r(1 - \sin\theta\cos\theta)$; **8.** 1; **9.** $(te^s + \cos s)e^{te^s + \sin s}$; **10.** $2(e - 1)$; **16.** $U - TS + PV = G$, the Gibbs energy; **17.** $A = U - TS$, the Helmholtz energy; **18.** $V = \sum_j n_j\overline{V}_j$ where $\overline{V}_j = (\partial V/\partial n_j)_{T,V}$.

Section 6.5

1. $\epsilon_1 = e^y\,\Delta x + \cos y\,\Delta y + \cdots$, $\epsilon_2 = 2xe^y\,\Delta x + (x^2e^y)/2\,\Delta y - \dfrac{1}{2}x\sin y\,\Delta y + \cdots$ Not unique; **3.** (a) $df = 2x\sin y\,dx + x^2\cos y\,dy$, (b) $dg = 3u^2e^v\,du + (u^3 + 1 + v)e^v\,dv$; **4.** (a) $df = (2xy + y^2 + z^2)\,dx + (x^2 + 2xy)\,dy + 2xz\,dz$, (b) $df = \dfrac{yz}{1+z^2}dx + \dfrac{xz}{1+z^2}dy + \dfrac{xy(1-z^2)}{(1+z^2)^2}dz$; **5.** $dP = \dfrac{R}{V-b}dT + \left(\dfrac{2a}{V^3} - \dfrac{RT}{(V-b)^2}\right)dV$; **7.** $\partial z/\partial x = -2/3$ and $\partial z/\partial y = -1/3$ at $(1, 1, 1)$; **8.** (a) $dz = -\dfrac{2z}{x}dx - \dfrac{z}{y}dy$, (b) $dz = \dfrac{-6z^2\,dx - 16yz\,dz}{5z^4 + 12xz + 8y^2}$; **9.** 54.5, 1.625%; **10.** 2.31×10^{-3} N $\cdot$ m^{-1}; **11.** Yes. It is the differential of πr^2h. No; **12.** No. Yes; **13.** Yes. It is the differential of $x^2y + xy^2 +$ constant; **14.** $f(x, y) = x^2\sin y + e^y +$ constant.

Section 6.6

1. $(2xy + 3y^3)\,\mathbf{i} + (x^2 + 9xy^2)\,\mathbf{j}$; **2.** $(y\cos xy + ye^z)\,\mathbf{i} + (x\cos xy + xe^z)\,\mathbf{j} + xye^z\,\mathbf{k}$; **3.** 1; **4.** $-1/\sqrt{2}$; **5.** $-1/\sqrt{3}$; **6.** 1; **7.** $\sqrt{12}$; **8.** The plane $8x + 3y + 2z = 0$; **9.** The curves are orthogonal; **10.** $2x\,\mathbf{i} + z\,\mathbf{j} + y\,\mathbf{k}$; **11.** $\mathbf{u} = -(6\,\mathbf{i} - 8\,\mathbf{j} + 10\,\mathbf{k})/\sqrt{200}$; **12.** $-100e^{-3/4}(\mathbf{i} + \mathbf{j})$; the rate of descent is 66.8; **16.** $(2\,\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{6}$; **17.** $4x - 2y + 2z = 8$; **18.** $\mathbf{u} = -(\mathbf{i} - \mathbf{j} + \mathbf{k})/\sqrt{3}$; **19.** $70.5°$; **20.** $y - 4x + 2 = 0$; **21.** $x + y = 1$.

Section 6.7

1. (a) 0, (b) $2 + 2x + 3x^2$, (c) $y^2 + 4xy$, (d) $2xy\cos xy$; **2.** (a) Linear, (b) Linear, (c) Nonlinear, (d) Nonlinear, nonlinear if c_1 and c_2 are complex; **3.** (a) d^4/dx^4, (b) $d^2dx^2 + 2xd/dx - (1 + x^2)$; **4.** $-(a^2 + b^2 + c^2)\cos ax \cos by \cos cz$; **5.** (a) Commute, (b) Do not commute, (c) Commute; **6.** (a) $-2\cos x$, (b) $6k^3x - 18hk^2y$; **7.** $\hat{P}$ and $\hat{Q}$ must commute; **10.** $e + e(x - 1) + e(y - 1) + \frac{e}{2}(x - 1)^2 + \frac{e}{2}(y - 1)^2 + 2e(x - 1)(y - 1) + \cdots$; **11.** $xy + O[(xy)^3]$; **12.** $\sin 1 + \cos 1(x - 1) + \cos 1(y - 1) - \frac{\sin 1}{2}(x - 1)^2 - \frac{\sin 1}{2}(y - 1)^2 + (\cos 1 - \sin 1)(x - 1)(y - 1) + \cdots$;
13. $x - \frac{x^3}{6} - \frac{xy^2}{2} + \cdots$

Section 6.8

1. It has a maximum at $x = 0$. $f(x)$ is not differentiable at $x = 0$; **2.** $f'(0) = f''(0) = f'''(0) = 0$, $f^{(4)}(0) = -24$, $f(x)$ has a local (actually global) maximum at $x = 0$; **3.** There is a minimum at $x = 1/e$. There is a maximum at $x = 1$ in $(0, 1]$, but there is no maximum in the open interval $(0, 1)$; **4.** A maximum at $x = -1$ and a minimum at $x = 1$; **5.** There is an inflection point at $x = 0$; **7.** (a) $(0, 0)$, $(0, 1/3)$, $(1/6, 1/6)$, (b) $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$; **8.** (a) $(0, 0)$, $(5/2, -5/4)$, (b) $(-1/2, 11/8)$, $(1, 1)$, $(-2, 4)$; **9.** (a) $(0, 0)$, $D = 0$, no conclusion; $(0, 1/3)$, $D < 0$, saddle point; $(1/6, 1/6)$, $f_{xx} > 0$, $D > 0$, a minimum, (b) $(0, 0)$, $f_{xx} < 0$, $D > 0$, a maximum; $(\sqrt{2}, -\sqrt{2})$, $f_{xx} > 0$, $D > 0$, a minimum; $(-\sqrt{2}, \sqrt{2})$, $f_{xx} > 0$, $D > 0$, a minimum; **10.** (a) $(0, 0)$, $f_{xx} < 0$, $D > 0$, a maximum; $(5/2, -5/4)$, $D < 0$, a saddle point, (b) $(1, 1)$, $D < 0$, a saddle point; $(-2, 4)$, $D < 0$, a saddle point; $(-1/2, 11/8)$, $f_{xx} < 0$, $D > 0$, a maximum; **11.** (a) A saddle point at $(0, 0)$, (b) A minimum at $(-4, 2)$; **12.** (a) A saddle point at $(0, -2)$ and a minimum at $(-5, 3)$, (b) A minimum at $(-2, 1)$; **15.** $(50)^3$; **16.** A cube whose volume is $(A/6)^{3/2}$. Note that $V = l^3$ if $A = 6l^2$.

Section 6.9

1. $\sqrt{2}$; **2.** $2ab$; **3.** $8abc/3^{3/2}$; **4.** A square; **5.** $(a^2 + b^2 + c^2)^{1/2}R$; **6.** A cube; **7.** $A^{3/2}/3(6\pi)^{1/2}$; **8.** $\dfrac{2^{1/2}A^{3/2}}{3^{7/4}\pi^{1/2}}$; **9.** $3(2/7)^{1/2}$; **10.** $1/(a^2 + b^2 + c^2)^{1/2}$; **11.** $6(2/7)^{1/2}$; **12.** 2; **13.** $5/\sqrt{3}$; **14.** $(21/13, 2, 63/26)$; **15.** 2; **16.** $(3/2)^{1/2}$; **17.** Closest, $(-1/\sqrt{26}, -3/\sqrt{26}, -4/\sqrt{26})$; and farthest, $(1/\sqrt{26}, 3/\sqrt{26}, 4/\sqrt{26})$; **18.** $(a/n)^n$.

Section 6.10

1. $a^2b^2/24$; **2.** $\frac{4}{3}a^{1/2}(a - 1)$; **3.** $M = 1/24$, $x_{cm} = y_{cm} = 2/5$; **4.** $M = 64/189$, $x_{cm} = 36/55$, $y_{cm} = -1/5$; **5.** $M = 3\pi a^4/16$, $x_{cm} = 16a/15\pi$, $y_{cm} = 64a/45\pi$; **6.** 4/3; **7.** 3/2; **8.** 85/16; **9.** $x_{cm} = a/4$, $y_{cm} = b/4$, $z_{cm} = c/4$; **10.** $a^2/10 + b^2/10$; **11.** $\pi a^6/24$; **12.** $\pi a^6/24$; **13.** (a) $\pi/2$, (b) 1/2; **20.** $(e - 2)/2$; **21.** $\frac{4}{3}[\sqrt{2} + \ln(1 + \sqrt{2})]$; **22.** $\pi/2$; **23.** $\pi a^6/24$.

Chapter 7

Section 7.1

1. $(-4\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k})/\sqrt{21}$; **2.** $(-\mathbf{i} - \mathbf{j} + 2\,\mathbf{k})/\sqrt{6}$; **3.** $\operatorname{div}\mathbf{A} = y^2 + 2xz - x^2$, $\operatorname{curl}\mathbf{A} = -2xy\,\mathbf{i} + 2xz\,\mathbf{j} + (2yz - 2xy)\,\mathbf{k}$; **4.** $\operatorname{div}\mathbf{A} = 3$, $\operatorname{curl}\mathbf{A} = x(\sin xy - \sin xz)\,\mathbf{i} + y(\sin yz - \sin xy)\,\mathbf{j} + z(\sin xz - \sin yz)\,\mathbf{k}$; **10.** (a) Incompressible, (b) Not incompressible, (c) $\mathbf{v} = 2x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ represents irrotational flow; **14.** (b) $y^3 + x^2y$; **19.** $(0, 0, 0)$;

clockwise direction looking down the North Pole. Because ϕ increases in a counterclockwise direction, $\beta = 360° - \phi$; **2.** Use Equation 2 with $dr = d\theta = 0$, $r = a$, $\theta = \pi/2$, and $0 \le \phi \le 2\pi$; **4.** $4\pi a^5/15$; **5.** $\iiint x^2\,dV = \iiint y^2\,dV = \iiint z^2\,dV$ by spherical symmetry; **6.** $\iiint r^2\,dV = 3\iiint x^2\,dV = 4\pi a^5/5$; **7.** $\pi/3$; **8.** $8\pi/15$; **12.** $\cos\phi\,\mathbf{e}_r - \dfrac{\sin\phi}{\sin\theta}\,\mathbf{e}_\phi$; **21.** $4\pi(a^2+b^2+c^2)^n R^{2n+3}/(2n+1)(2n+3)$.

Section 8.5

1. The coordinate surfaces are right cylinders of radius r whose axes are the z axis (r-surfaces); planes containing the z axis (θ-surfaces), and planes perpendicular to the z axis (z-surfaces). The coordinate curves are straight lines through the origin and parallel to the xy-plane (r-curve), circles centered on the z axis (θ-curve), and straight lines perpendicular to the xy-plane (z-curves); **2.** The coordinate surfaces are spherical surfaces of radius r centered at the origin (r-surfaces); right circular cones of angle θ whose axes are the z axis (θ-surfaces); and planes containing the z axis (ϕ-surfaces). The coordinate curves are straight lines through the origin (r-curves), great circles lying in planes containing the z axis (θ-curves) and circles parallel to the xy-plane (ϕ-curves); **5.** $r^2\cos^2\theta\,\sin\theta\,\mathbf{e}_r - r^2\cos\theta\,\sin^2\theta\,\mathbf{e}_\theta + r\cos\theta\,\mathbf{e}_z$; **6.** Use the fact that $\mathbf{j}\times\mathbf{k} = \mathbf{i}$; **8.** r; **9.** $r\sin\theta$; **15.** 4π; **16.** $\pi R(R+S)$.

Section 8.6

2. $2\pi\,[\,b^2 + (cb/e)\sin^{-1}e]$, where $e = (c^2-b^2)^{1/2}/c$; **9.** $2\pi[\,b^2 + (cb/e)\sin^{-1}e\,]$, where $e = 1/\lambda_0$; **13.** $h_\eta = a\,(\cosh^2\eta - \sin^2\theta)^{1/2} = h_\theta$, $h_\phi = a\cosh\eta\sin\theta$; **14.** $4\pi b^2c/3$; **15.** $2\pi b^2 + \dfrac{\pi b^2}{e}\ln\dfrac{1+e}{1-e}$, where $e = (b^2-c^2)^{1/2}/b$.

Chapter 9

Section 9.1

1. 5; **2.** -5; **3.** 0, because two columns are equal; 0, because column $1 = 2\times$ column 3; **4.** 5; **6.** -1; **7.** $x^4 - 3x^2 = 0$; $0, 0, \pm\sqrt{3}$; **8.** $x^4 - 4x^2 = 0$; $0, 0, \pm 2$; **10.** (a) -1, (b) -1, (c) $+1$; **15.** $(9/5, 1/5)$; **16.** $(1, 3, -4)$; **17.** The determinant of the factors of x and y is equal to zero. The two equations are inconsistent; **20.** -19; **21.** -75.

Section 9.2

1. $(13/12, 7/12, -7/12)$; **2.** $(11, -4, 3)$; **3.** There is no solution; **4.** There is no solution; **5.** There is no solution; **6.** $(1, 2, 3, 4)$; **7.** $x_1 = 2x_2 + 3x_4 + 16/5$, $x_3 = -3x_4 - 2/5$; **8.** $x_1 = x_4$, $x_2 = 0$, $x_3 = 0$, $x_4 =$ arbitrary; **9.** $x_1 = -3x_3 + 3x_5$, $x_2 = 3 - x_3 + 3x_5$, $x_4 = 1 - x_5$; **10.** $1 \pm i$; **11.** None; **12.** $\lambda \ne 1$; **13.** $4 \pm \sqrt{3}$.

Section 9.3

1. $\begin{pmatrix} 5 & -3 & -2 \\ -11 & 4 & -6 \\ 3 & -1 & -1 \end{pmatrix}$, $\begin{pmatrix} -7 & 6 & 1 \\ 19 & -2 & 12 \\ 6 & 5 & 5 \end{pmatrix}$; **4.** $(\mathsf{A}+\mathsf{B})^2 = \mathsf{A}^2 + 2\mathsf{AB} + \mathsf{B}^2$ only if A and B commute, $(\mathsf{AB})^2 = (\mathsf{ABAB}) = \mathsf{A}^2\mathsf{B}^2$ only if A and B commute; **7.** $\det \mathsf{C}_3 = 1$, $\det\sigma_v = -1$, $\det\sigma'_v = -1$, $\det\sigma''_v = -1$; **8.** They all are; **9.** (a) $\mathsf{A}_{\text{cof}} = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 3 & 0 \\ -2 & 3 & 1 \end{pmatrix}$, $\operatorname{adj}\mathsf{A} = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 3 & 3 \\ 1 & 0 & 1 \end{pmatrix}$, (b) $\mathsf{A}_{\text{cof}} = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ -4 & 2 & 5 \end{pmatrix}$, $\operatorname{adj}\mathsf{A} = \begin{pmatrix} 2 & -1 & -4 \\ -1 & 1 & 2 \\ -2 & 1 & 5 \end{pmatrix}$; **12.** (a) $\mathsf{B} = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}$, $\mathsf{BA} \ne 0$, (b) No. The resulting equations are inconsistent;

the diagonal; **12.** $\begin{pmatrix} 2e^{-2}-e^{-1} & -2e^{-2}+2e^{-1} \\ e^{-2}-e^{-1} & -e^{-2}+2e^{-1} \end{pmatrix}$; **13.** $\begin{pmatrix} -1 & 1/\pi \\ 0 & 0 \end{pmatrix}$; **14.** A and B must commute; **15.** $\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$; **16.** $e^{At} = \frac{1}{2}\begin{pmatrix} e^{3t}+e^{-t} & e^{3t}-e^{-t} \\ e^{3t}-e^{-t} & e^{3t}+e^{-t} \end{pmatrix}$; **19.** $S = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = S^{\mathsf{T}} = S^{-1}$.

Section 10.6

1. (a) $A = \begin{pmatrix} 3 & -\frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, (b) $A = \begin{pmatrix} 1 & -3/2 \\ -3/2 & 6 \end{pmatrix}$, (c) $A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$; **3.** $x'^2 + 6y'^2$, $x = \frac{1}{\sqrt{5}}(x' + 2y')$, $y = \frac{1}{\sqrt{5}}(-2x' + y')$; **4.** $x'^2 - 2y'^2 + 4z'^2 = 1$, a hyperboloid of one sheet; **5.** $x'^2 + 3y'^2 + 8z'^2 = 8$; an ellipsoid; **6.** $x'^2 + y'^2 + 10z'^2 = 1$, an ellipsoid; **7.** $13\left(y' + \frac{11}{\sqrt{17}}\right)^2 - 4\left(x' - \frac{7}{\sqrt{17}}\right)^2 = 81$; **8.** $-x'^2 - y'^2 + 2z'^2 = 1$, a hyperboloid of two sheets; **10.** Use $\int_{-\infty}^{\infty} dx\, x^{2n} e^{-ax^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n a^n}\left(\frac{\pi}{a}\right)^{1/2}$; **11.** $\langle (x-\mu)^2 \rangle$ is the spread of the curve about the vertical line $x = \mu$; **17.** $\lambda = 2$ and 8, $I_{xx} = 8$ and $I_{yy} = 2$ if $\theta = 90°$ and $I_{xx} = 2$ and $I_{yy} = 8$ if $\theta = 0°$; **19.** Use $x_i = \sum_j s_{ij} x'_j$.

Chapter 11

Section 11.1

2. (a) $y(x) = c/(x-1)$, (b) $x^2 y = ce^{y/a}$, (c) $s(t) = \frac{15}{16}(1 - e^{-16t})$; **3.** $y = c_1 x$ or $y = c_2/x$; **4.** $T(t) = 20 + 180e^{-0.1352\,t}$ (t in minutes), $T(t) = 25°$ when $t = 26.5$ minutes, or 20.5 minutes after $T = 100°$C; **5.** $x(t) = \frac{Ae^{at}}{1 + \frac{Ab}{a}e^{at}}$, where A is a constant, $x(t) \to a/b$ as $t \to \infty$; **6.** (a) $x^2 y + \frac{y^3}{3} = c$, (b) not exact, (c) $\frac{x^2 y^2}{2} + 4x = c$, (d) $xy - \cos x - 2\sin y = c$; **7.** (a) $v = xy$, (b) $v = y/x$, (c) $v = x + y$, (d) $v = y/x$; **8.** $v(t) = \frac{mg}{\alpha}(1 - e^{-\alpha t/m})$; **9.** $x - x^2 y = cy$; **10.** $x^2 + y^2 = c$; **11.** $x^2 - y^2 = c_1$; **12.** $c_1(1 - \sin\theta)$; **13.** $y^2 = c + x^2$; **15.** $20h^{3/2} - 6h^{5/2} = 14 - (1.26 \times 10^{-3})\,t$, where h is in meters; 3.06 hours; **16.** $x(t) = 10^5 e^{-0.0050t}$, where x is in grams and t is in minutes; 7.4×10^4 grams; **17.** $x(t) = 200(25 + 475\,e^{-0.0050t})$, where x is in grams and t is in minutes; 7.54×10^5 grams; **19.** (a) $\ln y - x^3/3y^3 = c$, (b) $\ln y + x^2/2y^2 = c$; **21.** $y = 2x + c + 3(y-2)\ln(1 + x - y)$; **22.** $\alpha = 0$, $\beta = -3$.

Section 11.2

1. (a) $y(x) = \frac{1}{3} + ce^{-x^3}$, (b) $y(x) = \frac{x^3}{5} + \frac{2x}{3} + \frac{c}{x^2}$; **2.** (a) $x(y) = \frac{2}{5}y^2 + \frac{c}{y^3}$, (b) $s(t) = (c + t^2)te^{3t}/2$; **3.** (a) $y(x) = x$, (b) $y(x) = \cos x\,(\sin x - 1)$; **4.** $y(x) = \begin{cases} \frac{1}{2}(1 + 3e^{-x^2}) & 0 \le x < 2 \\ \frac{1}{2}(e^4 + 3)e^{-x^2} & x \ge 2 \end{cases}$; **5.** (a) $x(y) = ce^y - y^2 - 2y - 2$, (b) $i(t) = \frac{3}{2}\sin t + \frac{1}{2}\cos t + ce^{-3t}$; **6.** $i(t) = \begin{cases} \frac{E_0}{R}(1 - e^{-Rt/L}) & 0 \le t < 1 \\ \frac{E_0}{R}(e^{R/L} - 1)e^{-Rt/L} & t > 1 \end{cases}$; **7.** $i(t) = \frac{E_0 L}{R^2 + \omega^2 L^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t + \omega e^{-Rt/L}\right)$; **8.** $A(t) = A_0 e^{-k_1 t}$, $B(t) = \frac{k_1 A_0}{k_2 - k_1}(e^{-k_1 t} - e^{-k_2 t})$; **12.** (a) $y(x) = 1/(cx - \frac{3}{2}x^3)$, (b) $x(y) = 1/(cy^4 + 2y^2)^{1/2}$; **13.** $y(x) = e^x/(1 + x^2)^{1/2}$; **14.** (a) $x^3 - 2y^3 = cx$, (b) $2x + 3(x + y) + 2\ln(2 - x - y) = c$; **15.** (a) $u(t) = 2 + ce^{-t^2}$, (b) $1/y^3 = -1 - 2x + ce^x$; **16.** $A(t) = \frac{k_2(A_0 + B_0)}{k_1 + k_2} + \frac{k_1 A_0 - k_2 B_0}{k_1 + k_2}e^{-(k_1+k_2)t}$, $B(t) = A_0 + B_0 - A(t)$;

17. $A(t)=A_0e^{-k_1t}$, $B(t)=\dfrac{k_1A_0}{k_2-k_1}(e^{-k_1t}-e^{-k_2t})$; **18.** $y(x)=e^{x^2}\left[c+\dfrac{\pi^{1/2}}{2}\text{erf}\,(x)\right]$;
19. $y(x)=x+1/x$; **20.** 3980 grams of salt; 179 minutes;
21. $xy(x)=2+\displaystyle\int_1^x \frac{\sin u}{u}du \quad x\geq 1$.

SECTION 11.3

1. Yes, yes; **2.** No. $e^x=\cosh x+\sinh x$; **3.** Yes; **4.** (a) $y(x)=c_1e^{2x}+c_2e^{-x}$, (b) $y(x)=(c_1+c_2x)e^{3x}$, (c) $y(x)=e^{-2x}(c_1e^{\sqrt{3}x}+c_2e^{-\sqrt{3}x})$; **5.** (a) $y(x)=c_1e^{2x}+c_2e^{-2x}$, (b) $y(x)=e^{-x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)$, (c) $y(x)=c_1\cos 3x+c_2\sin 3x$; **6.** (a) $y(x)=c_1+c_2e^{-6x}$, (b) $y(x)=c_1e^x+c_2e^{3x}$, (c) $y(x)=c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x$; **7.** (a) $y(x)=e^{-x}\left(\dfrac{2}{3}+\dfrac{e^{3x}}{3}\right)$, (b) $y(x)=e^{3x}(1-3x)$, (c) $y(x)=e^{-2x}\left[\left(\dfrac{1}{2}+\dfrac{1}{\sqrt{3}}\right)e^{\sqrt{3}x}+\left(\dfrac{1}{2}-\dfrac{1}{\sqrt{3}}\right)e^{-\sqrt{3}x}\right]$;
8. (a) $y(x)=(e^{-x}/3)(e^{3x}-1)$, (b) $y(x)=xe^{3x}$, (c) $y(x)=\dfrac{e^{-2x}}{2\sqrt{3}}(e^{\sqrt{3}x}-e^{-\sqrt{3}x})$;
9. (a) $y(x)=2e^{2x}$, (b) $y(x)=e^{2x}(-3+2e^x)$, (c) $y(x)=2e^{2x}$; **10.** $y(x)=c_1e^{-x}+c_2e^x+c_3e^{2x}$; **11.** $y(x)=e^{2x}(1+c_1x+c_2x^2)$; **12.** $y_2(x)=x^2(c_1/x^4)=c_1/x^2$; **13.** $y_2(x)=cx\ln x$.

SECTION 11.4

1. Because e^x is part of the solution to the homogeneous equation; **2.** $y(x)=c_1e^{-x}+c_2e^{-2x}+x-1$; **3.** $y(x)=c_1+c_2\sin x+x$ (x occurs in all three solutions to the inhomogeneous equation); **4.** (a) $y_p(x)=xe^x$, (b) $y_p(x)=-2-x^2$; **5.** (a) $y_p(x)=x+\frac{1}{2}e^{-x}$, (b) $y_p(x)=-11x-2x^3$; **6.** (a) $y_p(x)=\frac{1}{2}e^x$, (b) $y_p(x)=\frac{45}{8}+\frac{21}{4}x+\frac{9}{4}x^2+\frac{1}{2}x^3$;
7. (a) $y_p(x)=-\frac{1}{2}x\cos x$, (b) $y_p(x)=\frac{1}{2}xe^x$; **8.** $y(x)=c_1e^{2x}+c_2e^{-3x}-\frac{1}{36}-\frac{x}{6}$;
9. $y(x)=c_1\cos 2x+c_2\sin 2x-\frac{6}{5}\cos 3x-4\sin 3x$; **11.** Use the formula in Problem 11.3.18;
20. It is the average rate at which energy is dissipated; **26.** $y(x)=c_1+c_2e^{2x}-\frac{1}{2}e^x\sin x$;
27. $y(x)=c_1\cos x+c_2\sin x-x\cos x+\sin x\cdot\ln\sin x$; **28.** $y(x)=c_1e^{-x}+c_2xe^{-x}-xe^{-x}+e^{-x}x\ln x$; **29.** $y(x)=c_1e^{-x}+c_2xe^{-x}-\frac{3}{4}x^2e^{-x}+\frac{1}{2}x^2e^{-x}\ln x$.

SECTION 11.5

1. $y(x)=c_1x+c_2x^2$; **2.** $y(x)=c_1x+c_2/x^4$; **3.** $y(x)=c_1x^2+c_2x^2\ln x$;
6. $y(x)=x^{-1}[c_1\cos(2\ln x)+c_2\sin(2\ln x)]$; **7.** $y(x)=c_1x+c_2x\ln x$; **8.** $y(x)=x\,[\,c_1\cos(\sqrt{3}\,\ln x)+c_2\sin(\sqrt{3}\,\ln x)]$; **9.** $y(x)=c_1x+c_2x^2+c_3x^3$; **10.** $y(x)=c_1x+c_2x\cos(\ln x)+c_3x\sin(\ln x)$; **11.** $y(x)=c_1/x+c_2x^3-x^2/3$; **12.** $y(x)=c_1x+\dfrac{c_2}{x^2}-\dfrac{2}{3}x+2x\ln x$; **13.** $y(x)=c_1+c_2\ln x+x^2/4$; **14.** $y(x)=\sin^{-1}(\alpha e^x)+\beta$;
15. $y(x)=$ constant and $y(x)=Ae^{\alpha x}$; **17.** $x(y)=y\ln y+c_1y+c_2$, or $y=$ constant.

SECTION 11.6

2. They are linearly independent; **3.** $y_1(x)=c_1e^x+4c_2e^{4x}$, $y_2(x)=-c_1e^x-c_2e^{4x}$;
4. $y_1(x)=c_1e^{-x}+c_2e^{3x}$, $y_2(x)=-2c_1e^{-x}+2c_2e^{3x}$; **5.** $y_1(x)=-4c_1-2c_2e^{2x}$, $y_2(x)=3c_1+c_2e^{2x}$; **6.** $y_1(x)=c_1e^{2x}(\sin x-\cos x)+c_2e^{2x}(\cos x+\sin x)$, $y_2(x)=c_1e^{2x}\cos x-c_2e^{2x}\sin x$; **7.** $y_1(x)=e^x(c_2\cos 2x-c_1\sin 2x)$, $y_2(x)=e^x(c_1\cos 2x+c_2\sin 2x)$; **8.** $y_1(x)=2c_1e^x+c_2e^{6x}$, $y_2(x)=-3c_1e^x+c_2e^{6x}$;
9. $\mathbf{y}(x)=c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^x+c_2\begin{pmatrix}1\\3\\2\end{pmatrix}e^{2x}+c_3\begin{pmatrix}1\\6\\3\end{pmatrix}e^{3x}$;
10. $\mathbf{y}(x)=c_1\begin{pmatrix}2\\2\\-1\end{pmatrix}e^{3x}+c_2\begin{pmatrix}-1\\2\\2\end{pmatrix}e^{6x}+c_3\begin{pmatrix}2\\-1\\2\end{pmatrix}e^{9x}$; **11.** $y_1(x)=(\alpha+\beta)\cos 2x+(\beta-\alpha)\sin 2x+c_3e^x$, $y_2(x)=-2\alpha\cos 2x-2\beta\sin 2x-c_3e^x$, $y_3(x)=$

$-2\alpha\cos 2x - 2\beta\sin 2x$; **13.** $y(t) = \frac{A_0}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix}e^{-4t} + \frac{2A_0}{3}\begin{pmatrix}1\\1\\-2\end{pmatrix}e^{-t} + A_0\begin{pmatrix}0\\0\\1\end{pmatrix}$;
14. Let $x_1 = u_1$, $\dot{x}_1 = u_2$, $x_2 = u_3$, $\dot{x}_2 = u_4$; $\dot{u}_1 = u_2$, $m_1\dot{u}_2 = -(k_1 + k_2)u_1 + k_2u_3$, $\dot{u}_3 = u_4$, $m_2\dot{u}_4 = k_2u_1 - (k_2 + k_3)u_3$.

Chapter 12

Section 12.1

1. (a) $\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$, (b) $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$, (c) $\sum_{n=0}^{\infty}nc_nx^{n+2}$; **2.** (a) $(-1)^n\frac{2^n}{n!}a_0$, (b) $\frac{n+1}{2^n}$, (c) $\frac{(-1)^n}{(n!)^2}$; **3.** 10 395 and 3840; **5.** (a) $a_{2n} = \frac{(-1)^n}{(2n)!}a_0$, $a_{2n+1} = \frac{(-1)^n}{(2n+1)!}a_1$, (b) $a_{2n} = \frac{(2n-1)!!}{2^{3n}n!}$, $a_{2n+1} = \frac{n!}{2^n(2n+1)!!}a_1$;
6. $y(x) = \sum_{n=0}^{\infty}\frac{(-1)^nx^n}{n!} = e^{-x}$; **7.** $y(x) = x + c\sum_{n=0}^{\infty}\frac{(-1)^nx^n}{n!} = x + ce^{-x}$;
8. $y(x) = cx + \sum_{n=2}^{\infty}\frac{x^n}{(n-1)!} = cx + x(e^x - 1) = c_1x + xe^x$; **9.** $y(x) = a_1x + a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} + \cdots\right) = a_1x + a_0\left(1 - x\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}\right) = a_1x + a_0\left(1 - \frac{x}{2}\ln\frac{1+x}{1-x}\right)$;
10. $y(x) = a_0(1 - 2x^2) + a_1\left(x - \frac{1}{3}x^3 - \sum_{n=2}^{\infty}\frac{2^n(2n-3)!!x^{2n+1}}{(2n+1)!}\right)$; **14.** (a) $R < 1$, (b) $R = \infty$; **15.** (a) $R < 2$, (b) $R < 1$; **16.** All the a_n's equal to zero;
17. $2(z+1)\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + \frac{z+1}{2}y = 0$.

Section 12.2

5. At least 2; **6.** At least $(20)^{1/2}$; **7.** At least $\sqrt{5}/2$; **8.** At least 1; **9.** $x = 0$ is a regular singular point; **10.** $x = 0$ and $x = 1$ are regular singular points; **11.** $x = -1$ is an irregular singular point; **12.** $x = 0$ is an irregular singular point and $x = \pm i$ are regular singular points; **13.** An irregular singular point; **14.** A regular singular point; **15.** A regular singular point; **16.** All points except $x \neq 0$; **17.** All x; **18.** All x; **19.** All points except $x = 0$; **20.** All points except $x = \pm 1$.

Section 12.3

2. $f_4(x) = 3 - 30x^2 + 35x^4$, $f_5(x) = 15x - 70x^3 + 63x^5$; **4.** $P_4(x) = \frac{1}{8}(3 - 30x^2 + 35x^4)$, $P_5(x) = \frac{1}{8}(15x - 70x^3 + 63x^5)$; **9.** Use Equation 8 with $j = 0$;
15. $\frac{d^2\Theta_n(\theta)}{d\theta^2} + \cot\theta\frac{d\Theta_n(\theta)}{d\theta} + n(n+1)\Theta_n(\theta) = 0$.

Section 12.4

4. $\cosh(2x)^{1/2}/x^{1/2}$ and $\sinh(2x)^{1/2}/x^{1/2}$; **7.** $y(x) = a_0x^{1/2} + a_1x^{1/2}\sum_{n=0}^{\infty}\frac{x^n}{2^nn!}$; **8.** $r = 1$ and $-1/3$; **9.** $r = 1/2$ and $1/2$; **10.** $r = 0$ and 1; **11.** $r = \pm 2$.

Section 12.5

13. $y(x) = c_1x^{1/2}J_{1/4}(x^2) + c_2x^{1/2}Y_{1/4}(x^2)$; **14.** $y(x) = c_1x^{1/2}J_{2/3}(\frac{4}{3}x^{3/4}) + c_2x^{1/2}Y_{2/3}(\frac{4}{3}x^{3/4})$; **15.** $y(x) = c_1x^{-2}J_2(x) + c_2x^{-2}Y_2(x)$; **16.** $y(x) = c_1x^{1/2}J_{1/3}(\frac{2}{3}ax^{3/2}) + c_2x^{1/2}Y_{1/3}(\frac{2}{3}ax^{3/2})$; **22.** $(r+1)(r+2)\cdots(r+n)\left(\frac{1}{r+1} + \frac{1}{r+2} + \cdots + \frac{1}{r+n}\right)$.

SECTION 12.6

1. Multiply by x^2 and use Equation 12.5.42; **8.** $J_{1/2} = \left(\frac{2}{\pi}\right)^{1/2}\left[x^{1/2} - \frac{x^{5/2}}{6} + O(x^{9/2})\right]$, $J_{3/2} = \left(\frac{2}{\pi}\right)^{1/2}\left[\frac{x^{3/2}}{3} - \frac{x^{7/2}}{30} + O(x^{11/2})\right]$, $J_{5/2} = \left(\frac{2}{\pi}\right)^{1/2}\left[\frac{x^{5/2}}{15} - \frac{x^{9/2}}{210} + O(x^{13/2})\right]$; **10.** Let $t = e^{i\theta}$ in Equation 22; **11.** Use the fact that $\int_0^\pi \cos n\theta \cos m\theta = \delta_{mn}\pi$; **12.** Use the fact that $\int_0^\pi \sin n\theta \sin m\theta = \delta_{mn}\pi$; **16.** Let $a = 0$ and $b = 1$.

Chapter 13

SECTION 13.1

6. (a) $\dot{x} = y$, $\dot{y} = -\gamma y - kx + \beta x^3$, (b) $\dot{x} = y$, $\dot{y} = \epsilon y(1 - \frac{1}{3}y^2) - x$, (c) $\dot{x} = y$, $\dot{y} = \epsilon(1 - x^2)y - x$, (d) $\dot{x} = y$, $\dot{y} = -\gamma y - \omega^2 \sin x$; **7.** The trajectories move with a constant speed in the phase plane; **10.** (a) (0, 0) and (4, 0), (b) (0, 0) and (3, 2), (c) (0, 0) and $(2, \frac{1}{2})$, (d) All the points on the line $x = 2y$; **11.** (a) (0, 0), (0, 3), (5, 0), and (9, −3), (b) (0, 0), (0, 7), (3, 0), and (18, −20); **12.** $\mathbf{v}(t) = e^{-t}\begin{pmatrix}\alpha+\beta\\ \alpha\end{pmatrix}\cos 2t + e^{-t}\begin{pmatrix}\alpha-\beta\\ \beta\end{pmatrix}\sin 2t$; **13.** $\mathbf{v}(t) = \begin{pmatrix}\alpha-2\beta\\ \beta\end{pmatrix}\cos t + \begin{pmatrix}-2\alpha-\beta\\ \alpha\end{pmatrix}\sin t$; **14.** (a) Circles about the origin, (b) Spiral into the origin; **20.** As $t \to \infty$, the eigenvector $(1, \omega)^T$ dominates, as $t \to -\infty$, the eigenvector $(1, -\omega)^T$ dominates; **21.** When $\Omega > 0$, θ increases with time; when $\Omega < 0$, θ decreases with time.

SECTION 13.2

3. A trajectory could go off in two different directions at an intersection; **4.** (a) (1, 1), (b) (1, 1), (−1, 1), (c) (0, 0), (1, 2), (0, 4), (1/3, 0), (d) (−1, −1), (1, 1); **5.** (a) (1, 1), $\dot{u} = -u - v - uv$, $\dot{v} = u + uv$, (b) (1, 1), $\dot{u} = -v$, $\dot{v} = 2u - 2v + u^2 - v^2$; (−1, 1), $\dot{u} = -v$, $\dot{v} = -2u - 2v + u^2 - v^2$, (c) (0, 0), $\dot{u} = u - 3u^2 + uv$, $\dot{v} = 4v - v^2 - 2uv$; (1, 2), $\dot{u} = -3u + v - 3u^2 + uv$, $\dot{v} = -4u - 2v - 2uv - v^2$; (0, 4), $\dot{u} = 5u - 3u^2 + uv$, $\dot{v} = -4v - v^2 - 2uv$; (1/3, 0), $\dot{u} = -u + \frac{1}{3}v - 3u^2 + uv$, $\dot{v} = \frac{10}{3}v - v^2 - 2uv$, (d) (1, 1), $\dot{u} = u - v$, $\dot{v} = 2u + u^2$; (−1, −1), $\dot{u} = u - v$, $\dot{v} = -2u + u^2$; **6.** (a) (1, 1), $\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$, (b) (1, 1), $\begin{pmatrix}0 & -1\\ 2 & -2\end{pmatrix}$; (−1, 1), $\begin{pmatrix}0 & -1\\ -2 & -2\end{pmatrix}$, (c) (0, 0), $\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}$; (1, 2), $\begin{pmatrix}-3 & 1\\ -4 & -2\end{pmatrix}$; (0, 4), $\begin{pmatrix}5 & 0\\ 0 & -4\end{pmatrix}$; (1/3, 0), $\begin{pmatrix}-1 & 1/3\\ 0 & 10/3\end{pmatrix}$, (d) (1, 1), $\begin{pmatrix}1 & -1\\ 2 & 0\end{pmatrix}$; (−1, −1), $\begin{pmatrix}1 & -1\\ -2 & 0\end{pmatrix}$; **13.** A (stable) proper node; **14.** An (asymptotically stable) improper node; **17.** (a) Saddle point, (b) Node, (c) Spiral point.

SECTION 13.3

1. A stable proper node; **2.** An asymptotically stable improper node; **3.** A saddle point (unstable); **4.** An asymptotically stable node; **5.** A saddle point (unstable); **6.** An asymptotically stable spiral point; **7.** (−2, −1), a center (stable); **8.** (1, 1), an asymptotically stable spiral point; **9.** (2, −1), a saddle point (unstable); **10.** (−1, −1), an asymptotically stable node; **11.** $\lambda_\pm = a \pm i$, a center if $a = 0$, a stable spiral point if $a < 0$, and an unstable spiral point if $a > 0$; **12.** $\lambda_\pm = -1 \pm ia^{1/2}$; An asymptotically stable node if $a = 0$, an asymptotically stable spiral point if $a > 0$, an asymptotically stable node if $-1 < a < 0$, and a saddle point if $a < -1$; **13.** (0, 0), a center (stable); (−10, 10), a saddle point (unstable); **14.** (0, 0), a saddle point (unstable); (3, 2), a center (stable); **15.** (2, 1), a saddle point (unstable); (−2, −1), an asymptotically stable spiral point; **16.** (0, 0), an unstable node; (−1, 1), a saddle point (unstable); **17.** (1, 1), an asymptotically stable spiral point.

Section 13.4

1. $V(x)$ goes through a maximum at $x = 1$; **6.** The point (4, 0) is a critical point; **8.** The one with $\frac{y^2}{2} + \frac{x^2}{2} - \frac{x^3}{12} = \frac{8}{3}$; **10.** $\dot{x} = y = 0$ along the x axis. Therefore, $\frac{dy}{dx} = \frac{\dot{x}}{\dot{y}} = \frac{f(x, y)}{y} \to \infty$ unless $f(x, y) = 0$, which it doesn't unless (x, y) is a critical point; **11.** (0, 0) is a center; both $(\pm 3, 0)$ are saddle points; **12.** The potential energy has a maximum at $x = \pm 3$; **13.** (0, 0) is a saddle point and $(\pm 2, 0)$ are centers; **14.** (0, 0) is a saddle point and $(\pm 1, 0)$ are centers; **15.** (0, 0) is a saddle point and $(\pm 1, 0)$ are centers; **16.** (0, 0) is a saddle point, (2.1338, 0) is a center, and (−0.332755, 0) is a center; **18.** (a) $F(x) = \epsilon(\frac{1}{3}x^3 - x)$ and $G(x) = \frac{1}{4}x^4$. There is a limit cycle. (b) $F(x) = \epsilon(\frac{1}{3}x^3 - x)$ and $G(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4$. There is a limit cycle.

Section 13.5

3. $\dot{u} = -bcv/d$ and $\dot{v} = adu/b$; **9.** Each species goes almost to extinction; **11.** (50, 12.5) is a critical point. The number of each species oscillates slightly about its critical value; **13.** (0, 0) and (4, 0) are saddle points and (2, 2) is an asymptotically stable spiral point; **14.** The (critical) point (190, 25) is an asymptotically stable spiral point and (0, 5/4) is a saddle point; **15.** y survives and x becomes extinct; **16.** y survives and x becomes extinct; **17.** Both species survive. They coexist; **18.** The phase portrait is given by $0.500(X + Y) - 0.100 \ln X - \ln Y = 1.00$; **20.** When $a < 0$, the two critical points are asymptotically stable, but when $a > 0$, they are asymptotically unstable. The value $\beta = 24.74$ is the borderline between a normal system and a chaotic system.

Chapter 14

Section 14.1

5. $P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$; **7.** If the Legendre polynomials are orthogonal, then only the $n = m$ terms in the integral of the product of $G(x, t)$ and $G(x, u)$ are nonzero; **8.** $I = (tu)^{-1/2} \ln\{[1 + (tu)^{1/2}]/[1 - (tu)^{1/2}]\}$; **9.** $G(1, t) = (1-t)^{-1} = 1 + t + t^2 + \cdots$; **10.** $G(-1, t) = (1+t)^{-1} = 1 - t + t^2 - t^3 + \cdots$; **14.** Use the result of Problem 2 and Equation 20; **15.** Use Equations 9 and 20; **18.** The sequence of partial sums in Equation 25 is monotonically increasing and bounded by the left side. Thus, the series converges, and so $[2/(2n+1)]^{1/2}a_n$ must go to zero as n increases; **19.** $\sin \pi x = \frac{3}{\pi} P_1(x) + \frac{7(\pi^2 - 15)}{\pi^3} P_3(x) + \frac{11(945 - 105\pi^2 + \pi^4)}{\pi^5} P_5(x) + \cdots$; **20.** $1 - x^2 = \frac{2}{3} - \frac{2}{3}P_2(x)$; **21.** $u = 1$ when $x = \beta$ and $u = -1$ when $x = \alpha$; **22.** $f(x) = \frac{2}{3} - \frac{2}{3}P_2(x/2)$.

Section 14.2

1. $\phi_0(x) = \pi^{-1/4}$, $\phi_1(x) = (4/\pi)^{1/4}x$, $\phi_2(x) = (4\pi)^{-1/4}(2x^2 - 1)$; **3.** $16x^4 - 48x^2 + 12$; **10.** Use the recursion formula in Table 14.2; **11.** Use the recursion formula in Table 14.2 successively to get $x^2 H_n(x)$ in terms of $H_{n+2}(x)$, $H_n(x)$, and $H_{n-2}(x)$.

Section 14.3

7. $\lambda_n = (2n-1)\pi/2l$, $y_n(x) = \sin(2n-1)\pi x/2l$, $n = 1, 2, \ldots$; **8.** $\lambda_n = (2n-1)/2$, $y_n(x) = \cos(2n-1)x/2$, $n = 1, 2, \ldots$; **9.** $\lambda_n = n\pi$, $y_n(x) = \cos n\pi x$, $n = 1, 2, \ldots$; **10.** If $\tanh x = x/\alpha l$ has a solution, or if $\alpha l > 1$; **11.** $p(x) = r(x) = e^{-x^2}$ and $q(x) = 0$; **12.** $p(x) = (1-x^2)^{1/2}$, $q(x) = 0$, and $r(x) = (1-x^2)^{-1/2}$; **13.** Use Equation 12 with $r(x) = e^{-x^2}$; **14.** Use Equation 12 with $r(x) = (1-x^2)^{-1/2}$; **15.** (a) $p(x) = x = 0$ when $x = 0$. Singular. (b) $p(x) = (1-x^2) = 0$ when $x = \pm 1$. Singular. (c) $p(x) = (1-x^2)^{1/2} = 0$ when $x = \pm 1$. Singular; **16.** $\lambda = 0$: $y_0(x) =$ constant, $\lambda \neq 0$: $\lambda = n\pi/a$, $n = 1, 2, \ldots$, $y_n(x) = \cos n\pi x/a$ and $\sin n\pi x/a$.

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