ENCYCLOPEDIA OF TYPES OF ALGEBRAS 2010

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ABSTRACT. This is a cornucopia of types of algebras with some of their properties from the operadic point of view.

Introduction

The following is a list of some types of algebras together with their properties under an operadic and homological point of view. We keep the information to one page per type and we provide one reference as Ariadne's thread. More references are listed by the end of the paper. We work over a fixed field $\mathbb K$ though in many instances everything makes sense and holds over a commutative ground ring. The category of vector spaces over $\mathbb K$ is denoted by Vect. All tensor products are over $\mathbb K$ unless otherwise stated.

The items of a standard page (which is to be found at the end of this introduction) are as follows.

Sometimes a given type appears under different names in the literature. The choice made in Name is, most of the time, the most common one (up to a few exceptions). The other possibilities appear under the item Alternative.

The presentation given in Definit. is the most common one (with one exception). When others are used in the literature they are given in Alternative. The item oper. gives the generating operations. The item sym. gives their symmetry properties, if any. The item rel. gives the relation(s). They are supposed to hold for any value of the variables x, y, z, \ldots If, in the presentation, only binary operations appear, then the type is said to be binary. Analogously, there are ternary, k-ary, multi-ary types.

If, in the presentation, the relations involve only the composition of two operations at a time (hence 3 variables in the binary case, 5 variables in the ternary case), then the type is said to be *quadratic*.

For a given type of algebras \mathcal{P} the category of \mathcal{P} -algebras is denoted by \mathcal{P} -alg. For each type there is defined a notion of *free algebra*. By definition the free algebra of type \mathcal{P} over the vector space V is an algebra denoted by $\mathcal{P}(V)$ satisfying the following universal condition:

for any algebra A of type \mathcal{P} and any linear map $\phi: V \to A$ there is a unique \mathcal{P} -algebra morphism $\tilde{\phi}: \mathcal{P}(V) \to A$ which lifts ϕ . In other words the

forgetful functor

$$\mathcal{P}$$
-alg \longrightarrow Vect

admits a left adjoint

$$\mathcal{P}: \mathtt{Vect} \longrightarrow \mathcal{P} ext{-alg}.$$

In all the cases mentioned here the relations involved in the presentation of the given type are (or can be made) multilinear. Hence the functor $\mathcal{P}(V)$ is of the form (at least in characteristic zero),

$$\mathcal{P}(V) = \bigoplus_{n \ge 1} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n} ,$$

where $\mathcal{P}(n)$ is some \mathbb{S}_n -module. The \mathbb{S}_n -module $\mathcal{P}(n)$ is called the space of n-ary operations since for any algebra A there is a map

$$\mathcal{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \to A.$$

The functor $\mathcal{P}: \mathtt{Vect} \to \mathtt{Vect}$ inherits a monoid structure from the properties of the free algebra. Hence there exist transformations of functors $\iota: \mathrm{Id} \to \mathcal{P}$ and $\gamma: \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ such that γ is associative and unital. The monoid $(\mathcal{P}, \gamma, \iota)$ is called a *symmetric operad*.

The symmetric operad \mathcal{P} can also be described as a family of \mathbb{S}_n -modules $\mathcal{P}(n)$ together with maps

$$\gamma(i_1,\ldots,i_k):\mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_k)\longrightarrow\mathcal{P}(i_1+\cdots+i_k)$$

satisfying some compatibility with the action of the symmetric group and satisfying the associativity property.

If \mathbb{S}_n is acting freely on $\mathcal{P}(n)$, then $\mathcal{P}(n) = \mathcal{P}_n \otimes \mathbb{K}[\mathbb{S}_n]$ where \mathcal{P}_n is some vector space, and $\mathbb{K}[\mathbb{S}_n]$ is the regular representation. If, moreover, the maps $\gamma(i_1,\ldots,i_n)$ are induced by maps

$$\gamma_{i_1,\dots,i_k}: \mathcal{P}_k \otimes \mathcal{P}_{i_1} \otimes \dots \otimes \mathcal{P}_{i_k} \longrightarrow \mathcal{P}_{i_1+\dots+i_k},$$

then the operad \mathcal{P} comes a from a nonsymmetric operad (abbreviated ns operad), still denoted \mathcal{P} .

For more terminology and details about algebraic operads we refer to [LV11].

The generating series of the operad \mathcal{P} is defined as

$$f^{\mathcal{P}}(t) := \sum_{n>1} \frac{\dim \mathcal{P}(n)}{n!} t^n,$$

in the binary case. When dealing with a nonsymmetric operad it becomes

$$f^{\mathcal{P}}(t) := \sum_{n \ge 1} \dim \mathcal{P}_n \ t^n.$$

The Koszul duality theory of associative algebras has been extended to binary quadratic operads by Ginzburg and Kapranov, cf. [GK94], then to quadratic operads by Fresse, cf. [Fre06]. We give a conceptual treatment of this theory, together with applications in [LV11]. So, to any quadratic operad \mathcal{P} , there is associated a quadratic dual operad denoted \mathcal{P} !. It is

often a challenge to find a presentation of $\mathcal{P}^!$ out of a presentation of \mathcal{P} . One of the main results of the Koszul duality theory of operads is to show the existence of a natural differential map on the composite $\mathcal{P}^{!*} \circ \mathcal{P}$ given rise to the Koszul complex. If it is acyclic, then \mathcal{P} is said to be Koszul. One can show that, if \mathcal{P} is Koszul, then so is $\mathcal{P}^!$. In this case the generating series are inverse to each other for composition, up to sign, that is:

$$f^{\mathcal{P}!}(-f^{\mathcal{P}}(-t)) = t.$$

In the k-ary case we introduce the skew-generating series

$$g^{\mathcal{P}}(t) := \sum_{n \ge 1} (-1)^k \frac{\dim \mathcal{P}((k-1)n+1)}{n!} t^{((k-1)n+1)}.$$

If the operad \mathcal{P} is Koszul, then (cf. [Val07])

$$f^{\mathcal{P}^!}(-g^{\mathcal{P}}(-t)) = t.$$

The items Free alg., rep. $\mathcal{P}(n)$ or \mathcal{P}_n , dim $\mathcal{P}(n)$ or dim \mathcal{P}_n , and Gen. series speak for themselves.

Koszulity of an operad implies the existence of a small chain complex to compute the (co)homology of a \mathcal{P} -algebra. When possible, the information on it is given in the item Chain-cplx. Moreover it permits us to construct the notion of \mathcal{P} -algebra up to homotopy, whose associated operad, which is a differential graded operad, is denoted \mathcal{P}_{∞} .

The item Properties lists the main features of the operad. Set-theoretic means that there is a set operad \mathcal{P}_{Set} (monoid in the category of S-Sets) such that $\mathcal{P} = \mathbb{K}[\mathcal{P}_{Set}]$. Usually this property can be read on the presentation of the operad: no algebraic sums.

In the item Relationship we list some of the ways to obtain this operad under some natural constructions like tensor product (Hadamard product) or Manin products (white \circ or black \bullet), denoted \square and \blacksquare in the nonsymmetric framework, cf.[LV11]) for instance. We also list some of the most common functors to other types of algebras. Keep in mind that a functor $\mathcal{P} \to \mathcal{Q}$ induces a functor \mathcal{Q} -alg $\to \mathcal{P}$ -alg on the categories of algebras.

Though we describe only algebras without unit, for some types there is a possibility of introducing an element 1 which is either a unit or a partial unit for some of the operations, see the discuission in [Lod04]. We indicate it in the item Unit.

For binary operads the *opposite type* consists in defining new operations by $x \cdot y = yx$, etc. If the new type is isomorphic to the former one, then the operad is said to be *self-opposite*. When it is not the case, we mention whether the given type is called *right* or *left* in the item Comment.

In some cases the structure can be "integrated". For instance Lie algebras are integrated into Lie groups (Lie third problem). If so, we indicate it in the item Comment.

In the item Ref. we indicate a reference where information on the operad and/or on the (co)homology theory can be obtained. It is not necessarily the

first paper in which this type of algebras first appeared. For the three graces As, Com, Lie, the classical books Cartan-Eilenberg "Homological Algebra" and MacLane "Homology" are standard references.

Here is the list of the types included so far (with page number and letter K indicating that they are Koszul dual to each other):

sample	6	As	7	self-dual
\overline{Com}	8	Lie	9	K
Pois	10	none	11	self-dual
Leib	12	Zinb	13	K
Dend	14	Dias	15	K
PreLie	16	Perm	17	K
Dipt	18	$Dipt^!$	19	K
2as	20	2as!	21	K
Tridend	22	Trias	23	K
PostLie	24	ComTrias	25	K
CTD	26	$CTD^!$	27	K
Gerst	28	BV	29	
Mag	30	Nil_2	31	K
ComMag	32	ComMag!	33	K
Quadri	34	Quadri!	35	K
Dup	36	$Dup^!$	37	K
$As^{(2)}$	38	$As^{\langle 2 \rangle}$	39	
L- $dend$	40	Lie- adm	41	
PreLiePerm	42	Altern	43	
Param1rel	44	MagFine	45	
GenMag	46	NAP	47	
Moufang	48	Malcev	49	
Novikov	50	DoubleLie	51	
DiPreLie	52	Akivis	53	
Sabinin	54	$Jordan\ triples$	55	
$t - As^{(3)}$	56	$p -As^{(3)}$	57	
LTS	58	$Lie ext{-}Yamaguti$	59	
Interchange	60	HyperCom	61	
A_{∞}	62	C_{∞}	63	
L_{∞}	64	$Dend_{\infty}$	65	
\mathcal{P}_{∞}	66	Brace	67	
MB	68	2Pois	69	
Ξ^{\pm}	70	your own	71	

Notation

We use the notation \mathbb{S}_n for the symmetric group. Trees are very much in use in the description of operads. We use the following notation:

 $-PBT_n$ is the set of planar binary rooted trees with n-1 internal vertices (and hence n leaves). The number of elements of PBT_n is the Catalan number $c_n = \frac{1}{n+1} \binom{2n}{n}$. $-PT_n$ is the set of planar rooted trees with n leaves, whose vertices have

 $-PT_n$ is the set of planar rooted trees with n leaves, whose vertices have valency greater than 1+2 (one root, at least 2 inputs). So we have $PBT_n \subset PT_n$. The number of elements of PT_n is the super Catalan number, also called Schröder number, denoted C_n .

A planar binary rooted tree t is completely determined by its right part t^r and its left part t^l . More precisely t is the grafting of t^l and t^r :

$$t = t^l \vee t^r$$
.

Comments

Many thanks to Walter Moreira for setting up a software which computes the first dimensions of the operad from its presentation.

We remind the reader that we can replace the symmetric monoidal category Vect by many other symmetric monoidal categories. So there are notions of graded algebras, differential graded algebras, twisted algebras, and so forth. In the graded cases the Koszul sign rule is in order. Observe that there are also operads where the operations may have different degree (operad encoding Gerstenhaber algebras for instance).

We end this paper with a tableau of integer sequences appearing in this document.

This list of types of algebras is not as encyclopedic as the title suggests. We put only the types which are defined by a finite number of generating operations and whose relations are multilinear. Moreover we only put those which have been used some way or another. You will not find the "restricted types" (like divided power algebras), nor bialgebras.

Please report any error or comment to: gw.zinbiel@free.fr

We plan to update this encyclopedia every now and then.

1. Type of Algebras

Name Most common terminology

Notation my favorite notation for the operad (generic notation: \mathcal{P})

Def. oper. list of the generating operations

sym. their symmetry if any

rel. the relation(s)

Free alg. the free algebra as a functor in V

rep. $\mathcal{P}(n)$ the \mathbb{S}_n -representation $\mathcal{P}(n)$ and/or the space \mathcal{P}_n if nonsymmetric

 $\dim \mathcal{P}(n)$ the series (if close formula available), the list of the 7 first numbers

beginning at n=1

Gen. series close formula for $f^{\mathcal{P}}(t) = \sum_{n \geq 1} \frac{\dim \mathcal{P}(n)}{n!} t^n$ when available

Dual operad the Koszul dual operad

Chain-cplx Explicitation of the chain complex, if not too complicated

Properties among: nonsymmetric, binary, quadratic, set-theoretic, ternary, multi-ary,

cubic, Koszul.

Alternative alternative terminology, and/or notation, and/or presentation.

Relationship some of the relationships with other operads, either under some

construction like symmetrizing, Hadamard product, Manin products, or

under the existence of functors

Unit whether one can assume the existence of a unit (or partial unit)

Comment whatever needs to be said which does not fit into the other items

Ref. a reference, usually dealing with the homology of the \mathcal{P} -algebras

(not necessarily containing all the results of this page)

Name Associative algebra

Notation As (as nonsymmetric operad) Ass (as symmetric operad)

Def. oper. xy, operadically μ sym.

rel. (xy)z = x(yz), operadically $\mu \circ_1 \mu = \mu \circ_2 \mu$ (associativity)

Free alg. $As(V) = \overline{T}(V) = \bigoplus_{n \geq 1} V^{\otimes n}$ tensor algebra (noncommutative polynomials) $(x_1 \dots x_p)(x_{p+1} \dots x_{p+q}) = x_1 \dots x_{p+q}$ (concatenation)

rep. $\mathcal{P}(n)$ $As(n) = \mathbb{K}[\mathbb{S}_n]$ (regular representation), $As_n = \mathbb{K}$

 $\dim \mathcal{P}(n)$ 1, 2, 6, 24, 120, 720, 5040, ..., n!, ...

Gen. series $f^{As}(t) = \frac{t}{1-t}$

 ${\tt Dual\ operad} \quad As! = As$

Chain-cplx non-unital Hochschild complex, $C_n^{As}(A) = A^{\otimes n}$ $b'(a_1,\ldots,a_n) = \sum_{i=1}^{i=n-1} (-1)^{i-1}(a_1,\ldots,a_ia_{i+1},\ldots,a_n)$ important variation: cyclic homology

Properties binary, quadratic, ns, set-theoretic, Koszul, self-dual.

Alternative Associative algebra often simply called *algebra*.

Can be presented with commutative operation $x \cdot y := xy + yx$ and anti-symmetric operation [x, y] = xy - yx satisfying

 $\begin{cases} [x \cdot y, z] = x \cdot [y, z] + [x, z] \cdot y, \\ (x \cdot y) \cdot z - x \cdot (y \cdot z) = [x, [y, z]] \end{cases}.$ (Livernet and Loday, unpublished)

 $\begin{array}{ll} \texttt{Relationship} & Ass\text{-alg} \to Lie\text{-alg}, \ [x,y] = xy - yx, \\ & Com\text{-alg} \to Ass\text{-alg (inclusion)}, \ \text{and many others} \end{array}$

Unit 1x = x = x1

Comment one the three graces

Name Commutative algebra

Notation Com

Def. oper. xy

sym. xy = yx (commutativity) rel. (xy)z = x(yz) (associativity)

Free alg. $Com(V) = \overline{S}(V)$ (polynomials)

if $V = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$, then $\mathbb{K}1 \oplus Com(V) = \mathbb{K}[x_1, \dots, x_n]$

rep. $\mathcal{P}(n)$ $Com(n) = \mathbb{K}$ (trivial representation)

 $\dim \mathcal{P}(n)$ 1, 1, 1, 1, 1, 1, 1, ..., 1, ...

Gen. series $f^{Com}(t) = \exp(t) - 1$

Dual operad Com! = Lie

Chain-cplx Harrison complex in char. 0, André-Quillen cplx in general

Properties binary, quadratic, set-theoretic, Koszul.

Alternative sometimes called associative and commutative algebra

other notation Comm

Relationship $Com\text{-alg} \rightarrow As\text{-alg}$,

Zinb-alg $\rightarrow Com$ -alg

Unit 1x = x = x1

Comment one of the three graces

Name Lie algebra

Notation Lie

Def. oper. [x, y] (bracket)

 $\mathtt{sym.} \qquad [x,y] = -[y,x] \text{ (anti-symmetry)}$

rel. [[x, y], z] = [x, [y, z]] + [[x, z], y] (Leibniz relation)

Free alg. Lie(V) = subspace of the tensor algebra T(V)

generated by V under the bracket

rep. $\mathcal{P}(n)$ $Lie(n) = \operatorname{Ind}_{C_n}^{\mathbb{S}_n}(\sqrt[n]{1})$

 $\dim \mathcal{P}(n)$ 1, 1, 2, 6, 24, 120, 720, ..., (n-1)!, ...

Gen. series $f^{Lie}(t) = -\log(1-t)$

Dual operad $Lie^! = Com$

Chain-cplx Chevalley-Eilenberg complex $C_n^{Lie}(\mathfrak{g}) = \Lambda^n \mathfrak{g}$

 $d(x_1 \wedge \ldots \wedge x_n) =$

 $\sum_{1 < i < j < n} (-1)^{j} (x_1 \wedge \ldots \wedge [x_i, x_j] \wedge \ldots \wedge \widehat{x_j} \wedge \ldots \wedge x_n)$

Properties binary, quadratic, Koszul.

Alternative The relation is more commonly written as the Jacobi identity:

[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0

Relationship As-alg $\rightarrow Lie$ -alg,

Lie-alg o Leib-alg,

PreLie-alg $\rightarrow Lie$ -alg

Unit no

Comment one of the three graces. Named after Sophus Lie.

Integration: Lie groups.

Name Poisson algebra

PoisNotation

 $xy, \{x, y\}$ Def. oper.

 $xy = yx, \{x, y\} = -\{y, x\}$

 $\begin{cases} \{\{x,y\},z\} = \{x,\{y,z\}\} + \{\{x,z\},y\}, \\ \{xy,z\} = x\{y,z\} + \{x,z\}y, \\ (xy)z - x(yz) = 0 \end{cases}$

 $Pois(V) \cong \overline{T}(V)$ (tensor module, iso as Schur functors) Free alg.

rep. $\mathcal{P}(n)$ $Pois(n) \cong \mathbb{K}[\mathbb{S}_n]$ (regular representation)

 $\dim \mathcal{P}(n)$ $1, 2!, 3!, 4!, 5!, 6!, 7!, \dots, n!, \dots$

 $f^{Pois}(t) = \frac{t}{1-t}$ Gen. series

Pois! = PoisDual operad

Isomorphic to the total complex of a certain bicomplex cons-Chain-cplx tructed from the action of the Eulerian idempotents

binary, quadratic, quasi-regular, Koszul, self-dual. Properties

Alternative Can be presented with one operation x * y satisfying the relation

> (x * y) * z = x * (y * z) + $\frac{1}{3}(+x*(z*y)-z*(x*y)-y*(x*z)+y*(z*x))$

Relationship Pois-alg $\rightleftharpoons Lie$ -alg, Pois-alg $\rightleftharpoons Com$ -alg,

1x = x = x1, [1, x] = 0 = [x, 1]Unit

Comment Named after Siméon Poisson.

[Fre06] B. Fresse, Théorie des opérades de Koszul et homologie des Ref. algèbres de Poisson, Ann.Math. Blaise Pascal 13 (2006), 237–312.

This page is inserted so that, in the following part, an operad and its dual appear on page 2n and 2n + 1 respectively. Since Pois is self-dual there is no point to write a page about its dual.

Let us take the opportunity to mention that if A is a \mathcal{P} -algebra and B is a $\mathcal{P}^!$ -algebra, then the tensor product $A \otimes B$ inherits naturally a structure of Lie algebra. If \mathcal{P} is nonsymmetric, then so is $\mathcal{P}^!$, and $A \otimes B$ is in fact an associative algebra.

In some cases (like the Leibniz case for instance), $A \otimes B$ is a pre-Lie algebra.

Name Leibniz algebra

Notation Leib

 $\mathtt{Def.}$ oper. [x,y]

sym.

rel. [[x, y], z] = [x, [y, z]] + [[x, z], y] (Leibniz relation)

Free alg. $Leib(V) \cong \overline{T}(V)$ (reduced tensor module, iso as Schur functors)

rep. $\mathcal{P}(n)$ $Leib(n) = \mathbb{K}[\mathbb{S}_n]$ (regular representation)

 $\dim \mathcal{P}(n)$ 1, 2!, 3!, 4!, 5!, 6!, 7!, ..., n!, ...

Gen. series $f^{Leib}(t) = \frac{t}{1-t}$

Dual operad Leib! = Zinb

 $\begin{array}{ll} \texttt{Chain-cplx} & C_n^{Leib}(\mathfrak{g}) = \mathfrak{g}^{\otimes n} \\ & d(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n} (-1)^j (x_1, \dots, [x_i, x_j], \dots, \widehat{x_j}, \dots, x_n) \end{array}$

Properties binary, quadratic, quasi-regular, Koszul.

Alternative Sometimes improperly called Loday algebra.

Relationship $Leib = Perm \circ Lie$ (Manin white product), see [Val08] Lie-alg $\rightarrow Leib$ -alg, Dias-alg $\rightarrow Leib$ -alg, Dend-alg $\rightarrow Leib$ -alg

Unit no

Comment Named after G.W. Leibniz. This is the *left* Leibniz algebra.

The opposite type is called right Leibniz algebra. Integration: "coquecigrues"! see for instance [Cov10]

Ref. [Lod93] J.-L. Loday, Une version non commutative des algèbres

de Lie: les algèbres de Leibniz.

Enseign. Math. (2) 39 (1993), no. 3-4, 269–293.

Name Zinbiel algebra

Notation Zinb

 $\texttt{Def. oper.} \qquad x \cdot y$

sym.

rel. $(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y)$ (Zinbiel relation)

Free alg. $Zinb(V) = \overline{T}(V), \quad \cdot = \text{halfshuffle}$

 $x_1 \dots x_p \cdot x_{p+1} \dots x_{p+q} = x_1 \operatorname{sh}_{p-1,q} (x_2 \dots x_p, x_{p+1} \dots x_{p+q})$

rep. $\mathcal{P}(n)$ $Zinb(n) \cong \mathbb{K}[\mathbb{S}_n]$ (regular representation)

 $\dim \mathcal{P}(n)$ 1, 2!, 3!, 4!, 5!, 6!, 7!, ..., n!, ...

Gen. series $f^{Zinb}(t) = \frac{t}{1-t}$

Dual operad Zinb! = Leib

Chain-cplx

Properties binary, quadratic, quasi-regular, Koszul.

Alternative Zinb = ComDend (commutative dendriform algebra)

Previously called dual Leibniz algebra.

Relationship $Zinb = PreLie \bullet Com$, see [Val08], Zinb = ComDend

Zinb-alg $\rightarrow Com$ -alg, $xy = x \cdot y + y \cdot x$, Zinb-alg $\rightarrow Dend$ -alg, $x \prec y = x \cdot y = y \succ x$

Unit $1 \cdot x = 0, \quad x \cdot 1 = x$

Comment symmetrization of the dot product gives, not only

a commutative alg., but in fact a divided power algebra. Named after G.W. Zinbiel. This is right Zinbiel algebra.

 ${\tt Ref.} \qquad \qquad [{\tt Lod95}] \ {\tt J.-L.} \ {\tt Loday}, \ {\it Cup\text{-}product for Leibniz cohomology and}$

dual Leibniz algebras. Math. Scand. 77 (1995), no. 2, 189-196.

Name Dendriform algebra

Notation Dend

 $\texttt{Def. oper.} \qquad x \prec y, \ x \succ y \ (\text{left and right operation})$

svm.

 $\text{rel.} \quad \left\{ \begin{array}{rcl} (x \prec y) \prec z & = & x \prec (y \prec z) + x \prec (y \succ z), \\ (x \succ y) \prec z & = & x \succ (y \prec z), \\ (x \prec y) \succ z + (x \succ y) \succ z & = & x \succ (y \succ z). \end{array} \right.$

Free alg. $Dend(V) = \bigoplus_{n>1} \mathbb{K}[PBT_{n+1}] \otimes V^{\otimes n}$, for pb trees s and t:

 $s \prec t := s^l \lor (s^r * t)$, and $s \succ t := (s * t^l) \lor t^r$

where $x * y := x \prec y + x \succ y$.

rep. $\mathcal{P}(n)$ $Dend(n) = \mathbb{K}[PBT_{n+1}] \otimes \mathbb{K}[\mathbb{S}_n]$, so $Dend_n = \mathbb{K}[PBT_{n+1}]$

 $\dim \mathcal{P}(n) \qquad 1, 2 \times 2!, 5 \times 3!, 14 \times 4!, 42 \times 5!, 132 \times 6!, 429 \times 7!, \dots, c_n \times n!, \dots$

where $c_n = \frac{1}{n+1} {2n \choose n}$ is the Catalan number

Gen. series $f^{Dend}(t) = \frac{1-2t-\sqrt{1-4t}}{2t} = y, \quad y^2 - (1-2t)y + t = 0$

Dual operad Dend! = Dias

Chain-cplx Isomorphic to the total complex of a certain explicit bicomplex

Properties binary, quadratic, ns, Koszul.

Alternative Handy to introduce $x * y := x \prec y + x \succ y$ which is associative.

Relationship $Dend = PreLie \bullet As$, see [Val08]

 $Dend\text{-alg} \to As\text{-alg}, \quad x*y := x \prec y + y \succ x,$

Zinb-alg $\rightarrow Dend$ -alg, $x \prec y := x \cdot y =: y \succ x$ Dend-alg $\rightarrow PreLie$ -alg, $x \circ y := x \prec y - y \succ x$

Dend-alg $\rightarrow Brace$ -alg, see Ronco [Ron00]

Unit $1 \prec x = 0, \ x \prec 1 = x, \ 1 \succ x = x, \ x \succ 1 = 0.$

Comment dendro = tree in greek. There exist many variations.

Ref. [Lod01] J.-L. Loday, Dialgebras,

Springer Lecture Notes in Math. 1763 (2001), 7-66.

Name Diassociative algebra

Notation Dias

Def. oper. $x \dashv y, x \vdash y$ (left and right operation)

sym.

rel.
$$\begin{cases} (x \dashv y) \dashv z &= x \dashv (y \dashv z), \\ (x \dashv y) \dashv z &= x \dashv (y \vdash z), \\ (x \vdash y) \dashv z &= x \vdash (y \dashv z), \\ (x \dashv y) \vdash z &= x \vdash (y \vdash z), \\ (x \vdash y) \vdash z &= x \vdash (y \vdash z). \end{cases}$$

Free alg. $Dias(V) = \bigoplus_{n \geq 1} (\underbrace{V^{\otimes n} \oplus \cdots \oplus V^{\otimes n}}_{n \text{ copies}})$

noncommutative polynomials with one variable marked

rep. $\mathcal{P}(n)$ $Dias(n) = \mathbb{K}^n \otimes \mathbb{K}[\mathbb{S}_n], \quad Dias_n = \mathbb{K}^n$

 $\dim \mathcal{P}(n)$ 1, 2 × 2!, 3 × 3!, 4 × 4!, 5 × 5!, 6 × 6!, 7 × 7!, ..., n × n!, ...,

Gen. series $f^{Dias}(t) = \frac{t}{(1-t)^2}$

 ${\tt Dual\ operad} \quad {\it Dias}^! = {\it Dend}$

Chain-cplx see ref.

Properties binary, quadratic, ns, set-theoretic, Koszul.

Alternative Also called associative dialgebras, or for short, dialgebras.

Relationship $Dias = Perm \circ As = Perm \underset{\mathbf{H}}{\otimes} As$

 $\begin{array}{l} As\text{-alg} \rightarrow Dias\text{-alg}, \quad x\dashv y := \stackrel{\dots}{xy} =: x \vdash y \\ Dias\text{-alg} \rightarrow Leib\text{-alg}, \quad [x,y] := x\dashv y - x \vdash y \end{array}$

Unit Bar-unit: $x \dashv 1 = x = 1 \vdash x, \ 1 \dashv x = 0 = x \vdash 1$

Comment

Ref. [Lod01] J.-L. Loday, *Dialgebras*, Springer Lecture Notes in Math. 1763 (2001), 7-66.

Name Pre-Lie algebra

Notation PreLie

Def. oper. $x \circ y$

sym.

rel. $(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y)$

Free alg. $PreLie(V) = \{ rooted trees labeled by elements of V \}$

rep. $\mathcal{P}(n)$ $PreLie(n) = \mathbb{K}[\{\text{rooted trees, vertices labeled by } 1, \dots, n\}]$

 $\dim \mathcal{P}(n)$ 1, 2, 9, 64, 625, 1296, 117649, ..., n^{n-1} , ...

Gen. series $f^{PreLie}(t) = y$ which satisfies $y = t \exp(y)$

Dual operad PreLie! = Perm

Chain-cplx see ref.

Properties binary, quadratic, Koszul.

Alternative The relation is as(x, y, z) = as(x, z, y).

Relationship PreLie-alg $\rightarrow Lie$ -alg, $[x,y] = x \circ y - y \circ x$

Dend-alg o PreLie-alg, $x \circ y := x \prec y - y \succ x$

Brace-alg $\rightarrow PreLie$ -alg, forgetful functor

Unit $1 \circ x = x = x \circ 1$

Comment This is right pre-Lie algebra, also called right-symmetric algebra,

or Vinberg algebra. The opposite type is left symmetric.

First appeared in [Ger63, Vin63].

Ref. [CL01] F. Chapoton, M. Livernet, Pre-Lie algebras and the rooted

trees operad. Internat. Math. Res. Notices 2001, no. 8, 395–408.

Name Perm algebra

Notation Perm

Def. oper. xy

sym.

rel. (xy)z = x(yz) = x(zy)

Free alg. $Perm(V) = V \otimes S(V)$

rep. $\mathcal{P}(n)$ $Perm(n) = \mathbb{K}^n$

 $\dim \mathcal{P}(n)$ 1, 2, 3, 4, 5, 6, 7, ..., n, ...

Gen. series $f^{Perm}(t) = t \exp(t)$

 ${\tt Dual\ operad} \quad Perm! = PreLie$

Chain-cplx

Properties binary, quadratic, set-theoretic, Koszul.

Alternative

Relationship Perm = ComDias

 $Com ext{-alg} o Perm ext{-alg}, \ NAP ext{-alg} o Perm ext{-alg}, \ Perm ext{-alg} o Dias ext{-alg}$

Unit no, unless it is a commutative algebra

Comment

Ref. [Cha01] F. Chapoton, Un endofoncteur de la catégorie des

opérades. Dialgebras & related operads, 105–110,

Lect. Notes in Math., 1763, Springer, 2001.

Name Dipterous algebra

Notation Dipt

 $\text{Def. oper.} \qquad x*y, \ x \prec y$

sym.

rel. $\begin{cases} (x*y)*z = x*(y*z) \text{ (associativity)} \\ (x \prec y) \prec z = x \prec (y*z) \text{ (dipterous relation)} \end{cases}$

Free alg. $Dipt(V) = \bigoplus_{n \geq 1} (\mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n]) \otimes V^{\otimes n}$, for n = 1 the two copies of PT_1 are identified

 $\mathcal{D}(x)$ $\mathcal{D}(x)$ $\mathcal{D}(x)$ $\mathcal{D}(x)$ $\mathcal{D}(x)$ $\mathcal{D}(x)$ $\mathcal{D}(x)$ $\mathcal{D}(x)$ $\mathcal{D}(x)$

 $\begin{array}{ll} \operatorname{rep.} & \mathcal{P}(n) & & Dipt(n) = (\mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n]) \otimes \mathbb{K}[\mathbb{S}_n], n \geq 2, \\ & & Dipt_n = \mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n], n \geq 2 \end{array}$

Gen. series $f^{Dipt}(t) = rac{1-t-\sqrt{1-6t+t^2}}{2}$

Dual operad Dipt!

Chain-cplx Isomorphic to the total complex of a certain explicit bicomplex

Properties binary, quadratic, ns, Koszul.

Alternative "diptère" in French

Relationship Dend-alg $\to Dipt$ -alg, $x*y := x \prec y + y \succ x$, a variation: replace the dipterous relation by $(x \prec y)*z + (x*y) \prec z = x \prec (y*z) + x*(y \prec z)$ to get Hoch-algebras, see [Ler11]. Same properties.

Unit $1 \prec x = 0, x \prec 1 = x, 1 * x = x = x * 1.$

Comment dipterous = 2-fold in greek (the free algebra has 2 planar trees copies)

Ref. [LR03] J.-L. Loday, M. Ronco, Algèbres de Hopf colibres, C.R.Acad. Sci. Paris t. 337, Ser. I (2003), 153-158.

Name Dual dipterous algebra

Notation Dipt!

 $\texttt{Def. oper.} \qquad x\dashv y,\ x*y$

sym.

rel.
$$\begin{cases} (x*y)*z &= x*(y*z), \\ (x+y)+z &= x+(y*z), \\ (x*y)+z &= 0, \\ (x+y)*z &= 0, \\ 0 &= x*(y+z), \\ 0 &= x+(y+z). \end{cases}$$

Free alg. $Dipt^!(V) = \overline{T}(V) \oplus \overline{T}(V)$

rep. $\mathcal{P}(n)$ $Dipt^!(n) = \mathbb{K}^2 \otimes \mathbb{K}[\mathbb{S}_n], Dipt^!_n = \mathbb{K}^2, n \geq 2$

 $\dim \mathcal{P}(n)$ 1, 2 × 2!, 2 × 3!, 2 × 4!, 2 × 5!, 2 × 6!, 2 × 7!, ..., 2 × n!, ...,

Gen. series $f^{Dipt^!}(t) = \frac{t+t^2}{1-t}$

 ${\tt Dual\ operad} \quad Dipt^{!!} = Dipt$

Chain-cplx see ref.

Properties binary, quadratic, ns, Koszul.

Alternative

 $\text{Relationship} \quad As\text{-alg} \to Dipt^!\text{-alg}, \quad x\dashv y := xy =: x\vdash y$

Unit No

Comment

Ref. [LR03] J.-L. Loday, M. Ronco, Algèbres de Hopf colibres, C.R.Acad. Sci. Paris t. 337, Ser. I (2003), 153-158.

Name Two-associative algebra

Notation 2as

 ${\tt Def. oper.} \qquad x*y, \; x\cdot y,$

sym.

 $\operatorname{rel.} \quad \left\{ \begin{array}{lcl} (x*y)*z & = & x*(y*z), \\ (x\cdot y)\cdot z & = & x\cdot (y\cdot z). \end{array} \right.$

Free alg. $2as(V) = \bigoplus_{n \geq 1} (\mathbb{K}[T_n] \oplus \mathbb{K}[T_n]) \otimes V^{\otimes n}$, where $T_n = \text{planar trees}$

for n = 1 the two copies of T_1 are identified

rep. $\mathcal{P}(n)$ $2as(n) = (\mathbb{K}[T_n] \oplus \mathbb{K}[T_n]) \otimes \mathbb{K}[\mathbb{S}_n]$, so $Dipt_n = \mathbb{K}[T_n] \oplus \mathbb{K}[T_n]$, $n \geq 2$

 $\dim \mathcal{P}(n) \qquad 1, 2 \times 2!, 6 \times 3!, 22 \times 4!, 90 \times 5!, 394 \times 6!, 1806 \times 7!, \dots, 2\underline{C_n \times n}!, \dots$

where C_n is the Schröder number: $\sum_{n\geq 1} C_n t^n = \frac{1+t-\sqrt{1-6t+t^2}}{4}$

Gen. series $f^{2as}(t)=rac{1-t-\sqrt{1-6t+t^2}}{2}$

Dual operad 2as!

Chain-cplx Isomorphic to the total complex of a certain explicit bicomplex

Properties binary, quadratic, ns, set-theoretic, Koszul.

Alternative

Relationship 2as-alg $\rightarrow Dup$ -alg, 2as-alg $\rightarrow B_{\infty}$ -alg

Unit $1 \cdot x = x = x \cdot 1, \quad 1 * x = x = x * 1.$

Comment

Ref. [LR06] J.-L. Loday, M. Ronco, On the structure of cofree Hopf

algebras, J. reine angew. Math. 592 (2006), 123–155.

Name Dual 2-associative algebra

Notation 2as!

 $\texttt{Def. oper.} \qquad x \cdot y, \ x * y$

sym.

rel. $\begin{cases} (x*y)*z &= x*(y*z), \\ (x\cdot y)\cdot z &= x\cdot (y\cdot z), \\ (x\cdot y)*z &= 0, \\ (x*y)\cdot z &= 0, \\ 0 &= x*(y\cdot z) \\ 0 &= x\cdot (y*z). \end{cases}$

Free alg. $2as^!(V) = V \oplus \oplus_{n \geq 2} V^{\otimes n} \oplus V^{\otimes n}$

rep. $\mathcal{P}(n)$ $Dipt^!(n) = \mathbb{K}[\mathbb{S}_n] \oplus \mathbb{K}[\mathbb{S}_n], \quad Dipt^!_n = \mathbb{K} \oplus \mathbb{K}, n \geq 2.$

 $\dim \mathcal{P}(n)$ 1,2 × 2!,2 × 3!,2 × 4!,2 × 5!,2 × 6!,2 × 7!,...,2 × n!,...

Gen. series $f^{2as^!}(t) = \frac{t+t^2}{1-t}$

Dual operad (2as!)! = 2as

Chain-cplx see ref.

Properties binary, quadratic, Koszul.

Alternative

Relationship

Unit No

Comment

Ref. [LR06] J.-L. Loday, M. Ronco, On the structure of cofree Hopf algebras, J. reine angew. Math. 592 (2006), 123–155.

Name Tridendriform algebra

Notation Tridend

 $\text{Def. oper.} \qquad x \prec y, \ x \succ y, \ x \cdot y$

rel. $\begin{cases} (x \prec y) \prec z &= x \prec (y * z) , \\ (x \succ y) \prec z &= x \succ (y \prec z) , \\ (x * y) \succ z &= x \succ (y \succ z) , \\ (x \succ y) \cdot z &= x \succ (y \succ z) , \\ (x \prec y) \cdot z &= x \cdot (y \succ z) , \\ (x \cdot y) \prec z &= x \cdot (y \prec z) , \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z) . \end{cases}$

Free alg. Planar rooted trees with variables in between the leaves

rep. $\mathcal{P}(n)$ $Tridend(n) = \mathbb{K}[PT_n] \otimes \mathbb{K}[\mathbb{S}_n], \ Tridend_n = \mathbb{K}[PT_n]$

Gen. series $f^{Tridend}(t) = \frac{-1+3t+\sqrt{1-6t+t^2}}{4t}$

 ${\tt Dual\ operad} \quad Triend^! = Trias$

Chain-cplx Isomorphic to the total complex of a certain explicit tricomplex

 ${\tt Properties} \qquad {\tt binary, \, quadratic, \, ns, \, Koszul.}$

Alternative sometimes called dendriform trialgebras

Relationship Tridend-alg $\rightarrow As$ -alg, $x*y := x \prec y + y \succ x + x \cdot y$, ComTridend-alg $\rightarrow Tridend$ -alg, $x \prec y := x \cdot y =: y \succ x$

Unit $1 \prec x = 0, \ x \prec 1 = x, \ 1 \succ x = x, \ x \succ 1 = 0, \ 1 \cdot x = 0 = x \cdot 1.$

Comment There exist several variations (see [Cha02] for instance).

Ref. [LR04] J.-L. Loday and M. Ronco, *Trialgebras and families of polytopes*, Contemporary Mathematics (AMS) 346 (2004), 369–398.

Name Triassociative algebra

Notation Trias

 $x \dashv y, x \vdash y, x \perp y$ (no symmetry) per. $x \dashv y, \ x \vdash y, \ x \perp y \quad \text{(no symme)}$ $\begin{cases} (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (x \dashv y) \dashv z = x \vdash (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \vdash z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (x \vdash y) \dashv z = x \dashv (y \perp z), \\ (x \dashv y) \dashv z = x \dashv (y \perp z), \\ (x \perp y) \dashv z = x \perp (y \dashv z), \\ (x \dashv y) \perp z = x \perp (y \perp z), \\ (x \vdash y) \perp z = x \vdash (y \perp z), \\ (x \perp y) \vdash z = x \vdash (y \perp z), \\ (x \perp y) \perp z = x \perp (y \perp z). \end{cases}$ one relation for each cell of the

one relation for each cell of the pentagon.

Free alg. noncommutative polynomials with several variables marked

 $Trias(n) = \mathbb{K}^{2^{n}-1} \otimes \mathbb{K}[\mathbb{S}_n], \quad Trias_n = \mathbb{K}^{2^{n}-1}$ rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ $1, 3 \times 2!, 7 \times 3!, 15 \times 4!, 31 \times 5!, 63 \times 6!, 127 \times 7!, \dots, (2^{n} - 1) \times n!, \dots$

 $f^{Trias}(t) = \frac{t}{(1-t)(1-2t)}$ Gen. series

Trias! = TridendDual operad

binary, quadratic, ns, set-theoretic, Koszul. Properties

Alternative Also called associative trialgebra, or for short, trialgebra.

As-alg $\rightarrow Trias$ -alg, $x \dashv y = x \vdash y = x \perp y = xy$ Relationship

Bar-unit: $x \dashv 1 = x = 1 \vdash x, \ 1 \dashv x = 0 = x \vdash 1, \ 1 \perp x = 0 = x \perp 1$ Unit

Comment Relations easy to understand in terms of planar trees

Ref. [LR04] J.-L. Loday and M. Ronco, Trialgebras and families of polytopes, Contemporary Mathematics (AMS) 346 (2004), 369–398.

Name PostLie algebra

Notation $x \circ y$, [x,y]

Def. oper. $x \circ y$, [x, y] sym. [x, y] = -[y, y]

 $\begin{aligned} & \text{sym.} & & [x,y] = -[y,x] \\ & \text{rel.} & & [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \end{aligned}$

 $(x \circ y) \circ z - x \circ (y \circ z) - (x \circ z) \circ y + x \circ (z \circ y) = x \circ [y, z]$

 $[x,y] \circ z = [x \circ z, y] + [x,y \circ z]$

Free alg. $PostLie(V) \cong Lie(??(V))$

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ 1, 3, 20, 210, 3024, ..., a(n), ...

Gen. series $f(t) = -\log\left(\frac{1+\sqrt{1-4t}}{2}\right)$

Chain-cplx

Properties binary, quadratic, Koszul.

Alternative

Relationship PostLie-alg \rightarrow ??-alg, $xy = x \circ y$???

 $PostLie\text{-alg} \rightarrow Lie\text{-alg}, \ \{x,y\} = x \circ y - y \circ x + [x,y]$

PreLie-alg $\rightarrow PostLie$ -alg, [x,y] = 0

Unit

Comment

 ${\tt Ref.} \qquad \qquad [{\tt Val07}] \ {\tt Vallette} \ {\tt B.}, \ {\tt Homology} \ {\tt of} \ {\tt generalized} \ {\tt partition} \ {\tt posets},$

J. Pure Appl. Algebra 208 (2007), no. 2, 699–725.

Name Commutative triassociative algebra

Notation ComTrias

 $\begin{array}{llll} \text{Def. oper.} & x\dashv y, x\perp y \\ & \text{sym.} & x\perp y=y\perp x \\ & & \left((x\dashv y)\dashv z\right)=x\dashv (y\dashv z), \\ & (x\dashv y)\dashv z=x\dashv (z\dashv y), \\ & (x\dashv y)\dashv z=x\dashv (y\perp z), \\ & (x\perp y)\dashv z=x\perp (y\dashv z), \\ & (x\perp y)\perp z=x\perp (y\perp z). \end{array}$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t)

 ${\tt Dual\ operad} \quad ComTrias^! = PostLie$

Chain-cplx

Properties binary, quadratic, Koszul.

Alternative Triassociative with the following symmetry:

 $x \dashv y = y \vdash x \text{ and } x \perp y = y \perp x$

Relationship ComTrias-alg $\rightarrow Perm$ -alg

Unit $x \dashv 1 = x, 1 \dashv x = 0, 1 \perp x = 0$

Comment

Name Commutative tridendriform algebra

 ${\tt Notation} \qquad CTD$

 $\begin{array}{lll} \text{Def. oper.} & x \prec y, \ x \cdot y \\ & \text{sym.} & x \cdot y = y \cdot x \\ & \\ \text{rel.} & \begin{cases} (x \prec y) \prec z &= \ x \prec (y \prec z) + x \prec (z \prec y), \\ (x \cdot y) \prec z &= \ x \cdot (y \prec z) \ , \\ (x \prec z) \cdot y &= \ x \cdot (y \prec z) \ , \\ (x \cdot y) \cdot z &= \ x \cdot (y \cdot z) \ . \end{cases}$

Free alg. CTD(V) =quasi-shuffle algebra on V = QSym(V)

rep. $\mathcal{P}(n)$ CTD(n) =

 $\dim \mathcal{P}(n)$ 1, 3, 13, 75, 541, 4683, ...

Gen. series $f^{CTD}(t) = \frac{\exp(t) - 1}{2 - \exp(t)}$

Dual operad CTD! =

Chain-cplx

Properties binary, quadratic, Koszul.

Alternative Handy to introduce $x*y := x \prec y + y \prec x + x \cdot y$ (assoc. and comm.) equivalently: tridendriform with symmetry:

 $x \prec y = y \succ x, \ x \cdot y = y \cdot x$

Relationship CTD-alg o Tridend-alg

Unit $1 \prec x = 0, x \prec 1 = x, 1 \cdot x = 0 = x \cdot 1.$

Comment

Ref. [Lod07] J.-L. Loday, On the algebra of quasi-shuffles, Manuscripta mathematica 123 (1), (2007), 79–93. Name Dual CTD algebra

Notation $CTD^!$

 $\begin{array}{ll} \text{Def. oper.} & x\dashv y, \ [x,y] \\ & \text{sym.} & [x,y] = -[y,x] \\ & \text{rel.} \end{array}$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ 1,3,...

 ${\tt Gen. \ series} \quad f^{CTD^!}(t) =$

 ${\tt Dual\ operad} \quad (CTD^!)^! = CTD$

Chain-cplx

Properties binary, quadratic, Koszul.

Alternative

Relationship Trias-alg $\to CTD^!$ -alg

Unit

Comment

Name Gerstenhaber algebra

underlying objects: graded vector spaces

Notation Gerst

Def. oper. m = binary operation of degree 0, c = binary operation of degree 1

sym. m symmetric, c antisymmetric

rel. $c \circ_1 c + (c \circ_1 c)^{(123)} + (c \circ_1 c)^{(321)} = 0,$

 $c \circ_1 m - m \circ_2 c - (m \circ_1 c)^{(23)} = 0,$

 $m \circ_1 m - m \circ_2 m = 0$.

Free alg.

 $\mathtt{rep.}\ \mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t) =

Dual operad

Chain-cplx

Properties binary, quadratic, Koszul.

Alternative

Relationship

Unit

Comment To get the relations in terms of elements, do not forget to apply

the Koszul sign rule. Observe that the last relation is associativity.

Name Batalin-Vilkovisky algebra

underlying objects: graded vector spaces

Notation BV-alg

Def. oper. Δ unary degree 1, m binary degree 0, c binary degree 1

sym. m symmetric, c antisymmetric

rel. $m \circ_1 m - m \circ_2 m = 0$,

 $\Delta^2 = 0,$

 $c = \Delta \circ_1 m + m \circ_1 \Delta + m \circ_2 \Delta,$ $c \circ_1 c + (c \circ_1 c)^{(123)} + (c \circ_1 c)^{(321)} = 0,$

 $c \circ_1 m - m \circ_2 c - (m \circ_1 c)^{(23)} = 0,$

 $\Delta \circ_1 c + c \circ_1 \Delta + c \circ_2 \Delta = 0 .$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t) =

Dual operad

Chain-cplx

Properties unary and binary, inhomogeneous quadratic, Koszul.

Alternative Generated by Δ and m only.

Relationship $BV\operatorname{-alg} o Gerst\operatorname{-alg}$

Unit

Comment Defined by J.-L. Koszul. To get the relations in terms of elements,

do not forget to apply the Koszul sign rule.

Ref. [GCTV09]

Name Magmatic algebra

Notation Mag

Def. oper. xy

sym.

rel.

Free alg. any parenthesizing of words

 $Mag(n) = \mathbb{K}[PBT_n] \otimes \mathbb{K}[\mathbb{S}_n], \quad Mag_n = \mathbb{K}[PBT_n]$ rep. $\mathcal{P}(n)$

(planar binary trees with n leaves)

 $1, 1 \times 2!, 2 \times 3!, 5 \times 4!, 14 \times 5!, 42 \times 6!, 132 \times 7!, \dots, c_{n-1} \times n!, \dots$ where $c_n = \frac{1}{n+1} \binom{2n}{n}$ (Catalan number) $\dim \mathcal{P}(n)$

 $f^{Mag}(t) = (1/2)(1 - \sqrt{1 - 4t})$ Gen. series

 $Mag! = Nil_2$ Dual operad

Chain-cplx

Properties binary, quadratic, ns, set-theoretic, Koszul.

Alternative sometimes (improperly) called nonassociative algebra.

Relationship many "inclusions" (all types of alg. with only one gen. op.)

1x = x = x1Unit

Comment

Name 2-Nilpotent algebra

Notation Nil_2

Def. oper. xy

sym.

rel. (xy)z = 0 = x(yz)

Free alg. $Nil_2(V) = V \oplus V^{\otimes 2}$

 $\text{rep. } \mathcal{P}(n) \qquad Nil_2(2) = \mathbb{K}[\mathbb{S}_2], \; Nil_2(n) = 0 \text{ for } n \geq 3$

 $\dim \mathcal{P}(n)$ 1, 2, 0, 0, 0, 0, . . .

Gen. series $f^{\mathcal{P}}(t) = t + t^2$

 ${\tt Dual\ operad} \quad Nil_2{}^! = Mag$

Chain-cplx

Properties binary, quadratic, ns, Koszul.

Alternative

Relationship

Unit no

Comment

Name Commutative magmatic algebra

Notation ComMag

Def. oper. $x \cdot y$

 $\mathtt{sym.} \qquad x \cdot y = y \cdot x$

rel. none

Free alg. any parenthesizing of commutative words

rep. $\mathcal{P}(n)$ $ComMag(n) = \mathbb{K}[shBT(n)]$

 $shBT(n) = \{shuffle \text{ binary trees with } n \text{ leaves} \}$

 $\dim \mathcal{P}(n) \qquad \dim ComMag(n) = (2n-3)!! = 1 \times 3 \times \cdots \times (2n-3)$

Gen. series $f^{ComMag}(t) =$

Dual operad $ComMag^!$ -alg: [x, y] antisymmetric, [[x, y], z] = 0.

Chain-cplx

Properties binary, quadratic, set-theoretic, Koszul.

Alternative

Relationship $ComMag \rightarrow PreLie, x \cdot y := \{x, y\} + \{y, x\}, \text{ cf. [BL11]}$

Unit 1x = x = x1

Comment

Name Anti-symmetric nilpotent algebra

Notation $ComMag^!$

 ${\tt Def. oper.} \qquad x \cdot y$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series

Dual operad ComMag.

Chain-cplx

Properties binary, quadratic, set-theoretic, Koszul.

Alternative

Relationship

Unit

Comment

Name Quadri-algebra

Notation Quadri

Def. oper. $x \nwarrow y, \ x \nearrow y, \ x \searrow y, \ x \swarrow y$ called NW, NE, SE, SW oper. rel.

$$(x \nwarrow y) \nwarrow z = x \nwarrow (y \star z) \qquad (x \nearrow y) \nwarrow z = x \nearrow (y \prec z) \qquad (x \land y) \nearrow z = x \nearrow (y \succ z)$$

$$(x \swarrow y) \nwarrow z = x \swarrow (y \land z) \qquad (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z) \qquad (x \lor y) \nearrow z = x \searrow (y \nearrow z)$$

$$(x \prec y) \swarrow z = x \swarrow (y \lor z) \qquad (x \succ y) \swarrow z = x \searrow (y \swarrow z) \qquad (x \star y) \searrow z = x \searrow (y \searrow z)$$

where

$$\begin{array}{ll} x \succ y := x \nearrow y + x \searrow y, & x \prec y := x \nwarrow y + x \swarrow y \\ x \lor y := x \searrow y + x \swarrow y, & x \land y := x \nearrow y + x \nwarrow y \\ x \star y := x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y \\ &= x \succ y + x \prec y = x \lor y + x \land y \end{array}$$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n) \qquad 1, 4 \times 2!, 23 \times 3!, 156 \times 4!, 1162 \times 5!, 9192 \times 6!, \dots$ $\dim \mathcal{P}_n = \frac{1}{n} \sum_{j=n}^{2n-1} {3n \choose n+1+j} {j-1 \choose j-n}$

Gen. series f(t)

Dual operad Quadri!

Properties binary, quadratic, ns, Koszul.

Alternative $Quadri = Dend \blacksquare Dend = PreLie \bullet Dend$

Relationship Related to dendriform in several ways

Unit partial unit (like in dendriform)

Comment There exist several variations like $\mathcal{P} \blacksquare \mathcal{P}$ (cf. Ph. Leroux [Ler04])

Ref. [AL04] M. Aguiar, J.-L. Loday, *Quadri-algebras*, J. Pure Applied Algebra 191 (2004), 205–221.

Name Dual quadri-algebra

Notation Quad!

Def. oper. $x \nwarrow y, x \nearrow y, x \searrow y, x \swarrow y$

sym.

rel. A FAIRE

Free alg.

 $\operatorname{rep.} \mathcal{P}(n)$

Gen. series f(t)

Dual operad

Chain-cplx

Properties binary, quadratic, Koszul.

Alternative

Relationship

Unit

Comment

Ref. [Val08] B.Vallette, Manin products, Koszul duality,

Loday algebras and Deligne conjecture.

J. Reine Angew. Math. 620 (2008), 105–164.

Name Duplicial algebra

Notation Dup

 $\texttt{Def. oper.} \qquad x \prec y, \ x \succ y$

sym

rel. $\begin{cases} (x \prec y) \prec z &= x \prec (y \prec z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x \succ y) \succ z &= x \succ (y \succ z). \end{cases}$

Free alg. $Dup(V) = \bigoplus_{n \geq 1} \mathbb{K}[PBT_{n+1}] \otimes V^{\otimes n}$, for p.b. trees s and t: $x \succ y$ over operation is grafting of x on the leftmost leaf of y $x \prec y$ under operation is grafting of y on the rightmost leaf of x

 $\texttt{rep. } \mathcal{P}(n) \qquad Dup(n) = \mathbb{K}[PBT_{n+1}] \otimes \mathbb{K}[\mathbb{S}_n], \, \text{so} \, \, Dup_n = \mathbb{K}[PBT_{n+1}]$

Gen. series $f^{Dup}(t) = \frac{1-2t-\sqrt{1-4t}}{2t} = y, \quad y^2 - (1-2t)y + t = 0$

Dual operad Dup!

Chain-cplx Isomorphic to the total complex of a certain explicit bicomplex

 ${\tt Properties} \qquad {\tt binary, \, quadratic, \, ns, \, set\text{-}theoretic, \, Koszul.}$

Alternative

Relationship $2as\text{-alg} \to Dup\text{-alg} \to As^2\text{-alg}$

Unit

Comment The associator of $xy := x \succ y - x \prec y$ is

 $as(x,y,z) = x \prec (y \succ z) - (x \prec y) \succ z.$

Appeared first in [BF03]

Ref. [Lod08] J.-L. Loday, Generalized bialgebras and triples of operads,

Astérisque (2008), no 320, x+116 p.

Name Dual duplicial algebra

Dup!Notation

Def. oper. $x \prec y, \ x \succ y$

 $\text{rel.} \quad \left\{ \begin{array}{ll} (x \prec y) \prec z &=& x \prec (y \prec z), \\ (x \prec y) \succ z &=& 0, \\ (x \succ y) \prec z &=& x \succ (y \prec z), \\ 0 &=& x \prec (y \succ z), \\ (x \succ y) \succ z &=& x \succ (y \succ z). \end{array} \right.$

 $Dup!(V) = \bigoplus_{n \ge 1} \ n \ V^{\otimes n}$ Free alg. noncommutative polynomials with one variable marked

rep. $\mathcal{P}(n)$ $Dup!(n) = \mathbb{K}^n \otimes \mathbb{K}[\mathbb{S}_n], \quad Dup!_n = \mathbb{K}^n$

 $\dim \mathcal{P}(n)$ $1, 2 \times 2!, 3 \times 3!, 4 \times 4!, 5 \times 5!, 6 \times 6!, 7 \times 7!, \dots, n \times n!, \dots,$

Gen. series $f^{Dup!}(t) = \frac{t}{(1-t)^2}$

Dual operad Dup!! = Dup

see ref. Chain-cplx

Properties binary, quadratic, Koszul.

Alternative

Relationship

Unit

Comment

Name $As^{(2)}$ -algebra

Notation $As^{(2)}$

 $\texttt{Def. oper.} \qquad x*y, x\cdot y$

sym.

rel. $(x \circ_1 y) \circ_2 z = x \circ_1 (y \circ_2 z)$ any values (4 relations)

Free alg.

rep. $\mathcal{P}(n)$ $As_n^{(2)} = \mathbb{K}[\{0,1\}^{n-1}]$

 $\dim \mathcal{P}(n) \qquad 2^{n-1}$

Gen. series $f^{As^{(2)}}(t)=rac{t}{1-2t}$

 ${\tt Dual\ operad} \quad As^{(2)!} = As^{(2)}$

Chain-cplx

Properties binary, quadratic, set-theoretic, ns.

Alternative

Relationship $2as\text{-alg} \to Dup\text{-alg} \to As^{(2)}\text{-alg}$

Unit

Comment Variations: $As^{(k)}$

Name $As^{\langle 2 \rangle}$ -algebra

Notation $As^{\langle 2 \rangle}$, compatible products algebra

 $\texttt{Def. oper.} \qquad x*y, x\cdot y$

sym.

rel. (x*y)*z = x*(y*z) $(x*y)\cdot z + (x\cdot y)*z = x*(y\cdot z) + x\cdot (y*z)$ $(x\cdot y)\cdot z = x\cdot (y\cdot z)$

Free alg. similar to dendriform

rep. \mathcal{P}_n $As_n^{\langle 2 \rangle} = \mathbb{K}[PBT_{n+1}]$

 $\dim \mathcal{P}_n \qquad c_n = \frac{1}{n+1} \binom{2n}{n}$

Gen. series $f^{As^{\langle 2 \rangle}}(t) = rac{1-2t-\sqrt{1-4t}}{2t}$

Dual operad An $(As^{\langle 2 \rangle})!$ -algebra has 2 generating binary operations and 5 relations:

 $(x \circ_i y) \circ_j z = x \circ_i (y \circ_j z)$ and $(x * y) \cdot z + (x \cdot y) * z = x * (y \cdot z) + x \cdot (y * z)$

Chain-cplx

Properties binary, quadratic, ns, Koszul.

Alternative

Relationship $(As^{\langle 2 \rangle})! - \mathsf{alg} \to As^{\langle 2 \rangle} - \mathsf{alg} \to As^{\langle 2 \rangle} - \mathsf{alg}$

Unit

Comment equivalently $\lambda \ x * y + \mu \ x \cdot y$ is associative for any λ, μ

Variations: $As^{\langle k \rangle}$, Hoch-alg.

Ref. for instance [Gon05], [OS06], [Dot09].

Name L-dendriform algebra

Notation L-dend

 $\texttt{Def. oper.} \qquad x \rhd y \text{ and } x \vartriangleleft y$

sym.

 $\begin{array}{ll} \texttt{rel.} & x \rhd (y \rhd z) - (x \bullet y) \rhd z = y \rhd (x \rhd z) - (y \bullet x) \rhd z \\ & x \rhd (y \lhd z) - (x \rhd y) \lhd z = y \lhd (x \bullet z) - (y \lhd x) \lhd z \end{array}$

where $x \bullet y := x \triangleright y + x \triangleleft y$.

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t) =

Dual operad

Chain-cplx

Properties L-dend = $PreLie \bullet PreLie$?

Alternative

 $\text{Relationship} \quad Dend\text{-alg} \to L\text{-}dend\text{-alg via } x\rhd y:=x\succ y \text{ and } x\vartriangleleft y:=x\vartriangleleft y$

L-dend-alg $\rightarrow PreLie$ -alg, $(A, \triangleright, \triangleleft) \mapsto (A, \bullet)$

Unit

Comment Various variations like L-quad-alg [Bai10]

Ref. Bai C.M., Liu L.G., Ni X., L-dendriform algebras, preprint (2010).

Name Lie-admissible algebra

Notation Lie-adm

Def. oper. xy

sym.

rel. [x,y]=xy-yx is a Lie bracket, that is $\sum_{\sigma} \mathrm{sgn}(\sigma) \sigma \big((xy)z-x(yz) \big) = 0$

Free alg.

 $\texttt{rep.} \ \mathcal{P}(n) \qquad \textit{Lie-adm}(n) = ?$

 $\dim \mathcal{P}(n) \qquad 1, 2, 11, \dots$

 $\mbox{Gen. series} \quad f^{Lie\text{-}adm}(t) = ? \mbox{ its dimensions are } 1,2,1,??$

Dual operad Lie-adm!

Chain-cplx

Properties binary, quadratic, Koszul??.

Alternative

Relationship As-alg o PreLie-alg o Lie-adm-alg

Unit

Comment

Name PreLiePerm algebra

 $Notation \qquad PreLiePerm$

 $\text{Def. oper.} \qquad x \prec y, \ x \succ y, \ x \ast y$

 $\text{rel.} \quad \left\{ \begin{array}{ll} (x \prec y) \prec z &=& x \prec (y * z), \\ (x \prec y) \prec z &=& x \prec (z * y), \\ (x * y) \succ z &=& x \succ (y \succ z), \\ (x \succ y) \prec z &=& x \succ (z \succ y), \\ (x \succ y) \prec z &=& x \succ (y \prec z), \end{array} \right.$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ 1, 6,

Gen. series

 $\texttt{Dual operad} \quad PermPreLie = Perm \circ PreLie = Perm \otimes PreLie$

Chain-cplx

Properties binary, quadratic, set-theoretic.

Alternative

Relationship $PreLiePerm = PreLie \bullet Perm$, see [Val08] Zinb-alg $\rightarrow PreLiePerm$ -alg $\rightarrow Dend$ -alg

Unit no

Comment

 ${\tt Ref.} \qquad \qquad [{\tt Val08}] \ {\tt B.Vallette}, \ {\it Manin products}, \ {\it Koszul duality},$

Loday algebras and Deligne conjecture.

J. Reine Angew. Math. 620 (2008), 105–164.

Name Alternative algebra

Notation Altern

 ${\tt Def. oper.} \qquad xy$

sym.

rel. (xy)z - x(yz) = -(yx)z + y(xz),(xy)z - x(yz) = -(xz)y + x(zy),

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ 1, 2, 7, 32, 175, ??

Gen. series f(t)

Dual operad (xy)z = x(yz),

xyz + yxz + zxy + xzy + yzx + zyx = 0

 $\dim Altern!(n) = 1, 2, 5, 12, 15, \dots$

Chain-cplx

Properties binary, quadratic, nonKoszul [DZ09].

Alternative Equivalent presentation: the associator as(x, y, z)

is skew-symmetric: $\sigma \cdot as(x, y, z) = \operatorname{sgn}(\sigma)as(x, y, z)$

Relationship As-alg o Altern-alg

Unit 1x = x = x1

Comment The octonions are an example of alternative algebra

Integration: Moufang loops

 $\dim \mathcal{P}(n)$ computed by W. Moreira

Ref. [She04] Shestakov, I. P., Moufang loops and alternative

algebras.

Proc. Amer. Math. Soc. 132 (2004), no. 2, 313–316.

Name Parametrized-one-relation algebra

Notation Param1rel

 $\operatorname{Def.}$ oper. xy

sym. none

rel. $(xy)z = \sum_{\sigma \in \mathbb{S}_3} a_{\sigma} \sigma \cdot x(yz)$ where $a_{\sigma} \in \mathbb{K}$

Free alg.

 $\operatorname{rep.} \mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t) =

Dual operad $x(yz) = \sum_{\sigma \in \mathbb{S}_3} \mathrm{sgn}(\sigma) a_\sigma \sigma^{-1} \cdot (xy) z$

 ${\tt Chain-cplx}$

Properties

Alternative

Relationship Many classical examples are particular case: As, Leib, Zinb, Pois

Unit

Comment Problem: for which families of parameters $\{a_{\sigma}\}_{{\sigma}\in\mathbb{S}_3}$

is the operad a Koszul operad?

Name Magmatic-Fine algebra

MagFineNotation

 $(x_1,\ldots,x_n)_i^n$ for $1 \le i \le n-2, n \ge 3$ Def. oper. sym.

rel.

Free alg. Described in terms of some coloured planar rooted trees

rep. $\mathcal{P}(n)$ a sum of regular representations indexed as said above

 $\dim \mathcal{P}(n)$ $1, 0, 1 \times 3!, 2 \times 4!, 6 \times 5!, 18 \times 6!, 57 \times 7!, \dots, F_{n-1} \times n!, \dots$ where $F_n = \text{Fine number}$

 $f^{MagFine}(t) = \frac{1+2t-\sqrt{1-4t}}{2(2+t)}$ Gen. series

MagFine! same generating operations, any composition Dual operad

is trivial dim $MagFine_n^! = n - 2$ $f^{MagFine!}(t) = t + \frac{t^3}{(1-t)^2}$

Chain-cplx

multi-ary, quadratic, ns, Koszul. Properties

Alternative

Relationship

Unit

Comment

[HLR08] Holtkamp, R., Loday, J.-L., Ronco, M., Ref.

Coassociative magmatic bialgebras and the Fine numbers,

J.Alg.Comb. 28 (2008), 97–114.

Name Generic magmatic algebra

GenMagNotation

Def. oper. a_n generating operations of arity $n, a_1 = 1$

sym.

rel.

Free alg.

rep. $\mathcal{P}(n)$ a sum of regular representations

 $\dim \mathcal{P}(n)$

 $f^{GenMag}(t) = \sum_{n} b_n t^n, b_n = \text{polynomial in } a_1, \dots, a_n$ Gen. series

same generating operations, any composition is trivial Dual operad

 $\dim GenMag_n^! = a_n, \ f^{GenMag_n^!}(t) = \sum_n a_n t^n$

Chain-cplx

multi-ary, quadratic, ns, Koszul. Properties

Alternative

Relationship For MagFine! $a_n = n - 2$.

Unit

give a nice proof of the inversion formula for a generic power Comment

series (computation of the polynomial b_n)

[Lod05] Loday, J.-L., Inversion of integral series Ref.

enumerating planar trees. Séminaire lotharingien Comb. 53

(2005), exposé B53d, 16pp.

Name Nonassociative permutative algebra

Notation NAP

Def. oper. xy

sym.

rel. (xy)z = (xz)y

Free alg. NAP(V) can be described in terms of rooted trees

rep. $\mathcal{P}(n)$ NAP(n) = PreLie(n) as \mathbb{S}_n -modules

 $\dim \mathcal{P}(n)$ 1, 2, 9, 64, 625, 1296, 117649, ..., n^{n-1} , ...

Gen. series $f^{NAP}(t) = y$ which satisfies $y = t \exp(y)$

Dual operad NAP!

 ${\tt Chain-cplx}$

Properties binary, quadratic, set-theoretic, Koszul.

Alternative

Relationship $Perm\text{-alg} \to NAP\text{-alg}$

Unit no

Comment This is right NAP algebra

Ref. [Liv06] Livernet, M., A rigidity theorem for pre-Lie algebras,

J. Pure Appl. Algebra 207 (2006), no. 1, 1–18.

Name Moufang algebra

Notation Moufang

 $\operatorname{Def.}$ oper. xy

sym.

rel. x(yz) + z(yx) = (xy)z + (zy)x

((xy)z)t + ((zy)x)t = x(y(zt)) + z(y(xt)),t(x(yz) + z(yx)) = ((tx)y)z + ((tz)y)x,

(xy)(tz) + (zy)(tx) = (x(yt))z + (z(yt))x

Free alg.

 $\mathtt{rep.}\ \mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ 1, 2, 7, 40, ??

Gen. series f(t) =

Dual operad

Chain-cplx

Properties binary

Alternative Relation sometimes written in terms of the Jacobiator

Relationship Altern-alg $\xrightarrow{-} Moufang$ -alg $\to NCJordan$ -alg

Unit

Comment Integration: Moufang loop.

From this presentation there is an obvious definition of

"nonantisymmetric Malcev algebra".

Ref. [PIS04] Shestakov, I., Pérez-Izquierdo, J.M., An envelope

for Malcev algebras. J. Alg. 272 (2004), 379-393.

Name Malcev algebra

Notation Malcev

 $\operatorname{Def.}$ oper. xy

 $\mathtt{sym.} \qquad xy = -yx$

rel. ((xy)z)t + (x(yz))t + x((yz)t) + x(y(zt)) = (xy)(zt)

Free alg.

 $\mathtt{rep.}\ \mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ 1, 1, 3, 9, ??

Gen. series f(t)

Dual operad

Chain-cplx

Properties cubic.

Alternative

 $\texttt{Relationship} \quad Altern\texttt{-alg} \xrightarrow{-} Malcev\texttt{-alg}, \quad Lie\texttt{-alg} \to Malcev\texttt{-alg}$

Unit

Comment

Ref. [PIS04] Shestakov, I., Pérez-Izquierdo, J.M., An envelope

for Malcev algebras. J. Alg. 272 (2004), 379–393.

Name Novikov algebra

Notation Novikov

 ${\tt Def. oper.} \qquad xy$

sym.

rel. (xy)z - x(yz) = (xz)y - x(zy)x(yz) = y(xz)

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ 1, 2,

 ${\tt Gen. \ series} \quad f^{Novikov}(t) =$

Dual operad

Chain-cplx

Properties binary, quadratic.

Alternative Novikov is pre-Lie + x(yz) = y(xz)

 $\texttt{Relationship} \quad Novikov\text{-alg} \rightarrow PreLie\text{-alg}$

Unit

Comment

Name Double Lie algebra

 $Notation \qquad Double Lie$

 ${\tt Def. oper.} \qquad [x,y], \{x,y\}$

 $\mathrm{sym}. \qquad [x,y] = -[y,x], \{x,y\} = \{y,x\}$

rel. Any linear combination is a Lie bracket

Free alg.

 $\mathtt{rep.}\ \mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t) =

Dual operad

Chain-cplx

Properties binary, quadratic.

Alternative

Relationship

Unit

Comment

Ref. [DK07] Dotsenko V., Khoroshkin A., Character formulas for the operad of two compatible brackets and for the bihamiltonian operad,

Functional Analysis and Its Applications, 41 (2007), no.1, 1-17.

Name DipreLie algebra

 ${\tt Notation} \qquad \qquad DipreLie$

 ${\tt Def. oper.} \qquad x \circ y, \ x \bullet y$

sym.

rel. $(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y)$ $(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y)$

 $(x \circ y) \bullet z - x \circ (y \bullet z) = (x \bullet z) \circ y - x \bullet (z \circ y)$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t) =

Dual operad

Chain-cplx

Properties binary, quadratic.

Alternative

Relationship relationship with the Jacobian conjecture (T. Maszczsyk)

Unit

Comment

Ref. Maszczsyk T., unpublished.

Name Akivis algebra

Notation Akivis

 $\begin{array}{ll} \text{Def. oper.} & [x,y], (x,y,z) \\ & \text{sym.} & [x,y] = -[y,x] \end{array}$

rel. $\begin{aligned} &[[x,y],z]+[[y,z],x]+[[z,x],y]=\\ &(x,y,z)+(y,z,x)+(z,x,y)-(x,z,y)-(y,x,z)-(z,y,x)\\ &(\text{Akivis relation}) \end{aligned}$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n) \qquad 1, 1, 8, \dots,$

Gen. series f(t) =

Dual operad

Chain-cplx

Properties binary and ternary, quadratic.

Alternative relation also called "nonassociative Jacobi identity"

Relationship Akivis-alg $\rightarrow Sabinin$ -alg,

Mag-alg $\rightarrow Akivis$ -alg, [x,y] = xy - yx,

(x, y, z) = (xy)z - x(yz)

Unit

Comment

Ref. [BHP05]

Name Sabinin algebra

Notation Sabinin

Def. oper. $\langle x_1, \dots, x_m; y, z \rangle, m \geq 0$ $\Phi(x_1, \dots, x_m; y_1, \dots, y_n), \quad m \geq 1, n \geq 2,$

 $sym. \qquad \langle x_1, \dots, x_m; y, z \rangle = -\langle x_1, \dots, x_m; z, y \rangle$

 $\Phi(x_1,\ldots,x_m;y_1,\ldots,y_n)=\Phi(\omega(x_1,\ldots,x_m);\theta(y_1,\ldots,y_n)),\omega\in\mathbb{S}_m,\theta\in\mathbb{S}_n$

rel. $\langle x_1, \dots, x_r, u, v, x_{r+1}, \dots, x_m; y, z \rangle - \langle x_1, \dots, x_r, v, u, x_{r+1}, \dots, x_m; y, z \rangle + \sum_{k=0}^r \sum_{\sigma} \langle x_{\sigma(1)}, \dots, x_{\sigma(k)}; \langle x_{\sigma(k+1)}, \dots, x_{\sigma(r)}; u, v \rangle, x_{r+1}, \dots, x_m; y, z \rangle$

where σ is a (k, r - k)-shuffle

 $K_{u,v,w}[\langle x_1,\ldots,x_r;y,z\rangle + \sum_{k=0}^r \sum_{\sigma} \langle x_{\sigma(1)},\ldots,x_{\sigma(k)};\langle x_{\sigma(k+1)},\ldots,x_{\sigma(r)};v,w\rangle,u\rangle] = 0$ where $K_{u,v,w}$ is the sum over all cyclic permutations

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ 1, 1, 8, 78, 1104, ...

Gen. series $f^{Sab}(t) = \log(1 + (1/2)(1 - \sqrt{1 - 4t}))$

Dual operad

Chain-cplx

Properties quadratic, ns.

Alternative There exists a more compact form of the relations which uses the tensor algebra over the Sabinin algebra

 $\begin{array}{ll} \text{Relationship} & Mag\text{-alg} \longrightarrow Sabinin\text{-alg}, \langle y,z \rangle = yz - zy, \langle x;y,z \rangle =?? \\ & Akivis\text{-alg} \longrightarrow Sabinin\text{-alg}, \langle y,z \rangle = -[y,z], \end{array}$

 $\langle x; y, z \rangle = (x, z, y) - (x, y, z), \langle x_1, \dots, x_m; y, z \rangle = 0, m \ge 2$

Unit

Comment Integration: local analytic loop

Ref. [PI07] D. Pérez-Izquierdo, Algebras, hyperalgebras, nonassociative bialgebras and loops. Adv. in Maths 208 (2007), 834–876.

Name Jordan triples

 ${\tt Notation} \hspace{1.5cm} JT$

 ${\tt Def. oper.} \qquad (xyz) \text{ or } (x,y,z)$

 $\mathtt{sym.} \qquad (xyz) = (zyx)$

rel. (xy(ztu)) = ((xyz)tu) - (z(txy)u) + (zt(xyu))

Free alg.

rep. $\mathcal{P}(2n-1)$

 $\dim \mathcal{P}(2n-1)$ 1, 3, 50, ??

Gen. series f(t)

Dual operad

Chain-cplx

Properties ternary, quadratic, Koszul?.

Alternative

Relationship

Unit

Comment Remark that the quadratic relation, as written here, has a

Leibniz flavor. Dimension computed by Walter Moreira

Name Totally associative ternary algebra

Notation $t\text{-}As^{\langle 3 \rangle}$

 $\texttt{Def. oper.} \qquad (xyz)$

sym.

rel. ((xyz)uv) = (x(yzu)v) = (xy(zuv))

Free alg.

rep. $\mathcal{P}(2n-1)$

 $\dim \mathcal{P}(2n-1)$ 1, 3!, 5!, ..., (2n+1)!, ...

Gen. series $f^{t-As^3}(t)=rac{t}{1-t^2}$

Dual operad $t ext{-}As^{\langle 3 \rangle} \; ! = p ext{-}As^{\langle 3 \rangle}$

Chain-cplx

Properties ternary, quadratic, ns, set-theoretic, Koszul?.

Alternative

Relationship $As ext{-alg} o t ext{-}As^{\langle 3 \rangle}$

Unit

Comment

Ref. [Gne97] Gnedbaye, A.V., $Op\'{e}rades\ des\ alg\`{e}bres\ (k+1)$ -aires.

Operads: Proceedings of Renaissance Conferences, 83–113,

Contemp. Math., 202, Amer. Math. Soc., Providence, RI, 1997.

Name Partially associative ternary algebra

Notation $p\text{-}As^{\langle 3 \rangle}$

Def. oper. (xyz)

sym.

rel. ((xyz)uv) + (x(yzu)v) + (xy(zuv)) = 0

Free alg.

rep. $\mathcal{P}(2n-1)$

 $\dim \mathcal{P}(2n-1)$

Gen. series f(t)

Dual operad $p\text{-}As^{\langle 3 \rangle} \; ! = t\text{-}As^{\langle 3 \rangle}$

Chain-cplx

Properties ternary, quadratic, ns, Koszul?.

Alternative

Relationship

Unit

Comment

Ref. [Gne97] Gnedbaye, A.V., $Op\'{e}rades\ des\ alg\`{e}bres\ (k+1)$ -aires.

Operads: Proceedings of Renaissance Conferences, 83–113,

Contemp. Math., 202, Amer. Math. Soc., Providence, RI, 1997.

Name Lie triple systems

Notation LTS

Def. oper. [xyz]

sym. [xyz] = -[yxz]

[xyz] + [yzx] + [zxy] = 0

rel. [xy[ztu]] = [[xyz]tu] - [z[txy]u] + [zt[xyu]]

Free alg.

 $\mathtt{rep.}\ \mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t)

Dual operad $LT^!$

Chain-cplx

Properties ternary, quadratic, Koszul?.

Alternative the relation admits many different versions due to the symmetry

Relationship Lie-alg $\rightarrow LTS$ -alg, [xyz] = [[xy]z]

Comment Appreciate the Leibniz presentation

Integration: symmetric spaces

Ref. [Loo69] Loos O., Symmetric spaces. I. General theory.

W. A. Benjamin, Inc., New York-Amsterdam (1969) viii+198 pp.

Name Lie-Yamaguti algebra

Lie-YamagutiNotation

 $x \cdot y$, [x, y, z]Def. oper.

 $x \cdot y = -y \cdot x$, [x, y, z] = -[y, x, z]

 $\begin{aligned} &[x,y,z] + [y,z,x] + [z,x,y] + (x \cdot y) \cdot z + (y \cdot z) \cdot x + (z \cdot x) \cdot y = 0 \\ &\sum_{cyclic} [x \cdot y,z,t] = 0 \\ &[x,y,u \cdot v] = u \cdot [x,y,v] + [x,y,u] \cdot v \\ &[x,y,[z,t,u]] = [[x,y,z],t,u] - [z,[t,x,y],u] + [z,t,[x,y,u]] \end{aligned}$

rel.

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t)

LT!Dual operad

Chain-cplx

binary and ternary, quadratic, Koszul?. Properties

Alternative Generalized Lie triple systems

Relationship LTS-alg $\rightarrow LY$ -alg, $x \cdot y = 0, [x, y, z] = [xyz]$

Unit

Comment

Ref. [KW01] R. Kinyon, M. Weinstein, A., Leibniz algebras,

Courant algebroids, and multiplications on reductive homogeneous spaces. Amer. J. Math. 123 (2001), no. 3, 525–550.

Name Interchange algebra

 ${\tt Notation} \qquad \qquad Interchange$

Def. oper. $x \cdot y$, x * y

sym.

rel. $(x \cdot y) * (z \cdot t) = (x * z) \cdot (y * t)$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t) =

Dual operad

Chain-cplx

Properties binary, cubic, set-theoretic.

Alternative

Relationship Strongly related with the notions of 2-category and 2-group

Unit if a unit for both, then $* = \cdot$ and they are commutative

Comment Many variations depending on the hypotheses on * and \cdot

Most common \cdot and * are associative.

Name Hypercommutative algebra

underlying objects: graded vector spaces

HyperComNotation

 (x_1,\ldots,x_n) n-ary operation of degree 2(n-2) for $n\geq 2$ Def. oper.

totally symmetric sym.

rel.

 $\begin{array}{l} \sum_{S_1 \sqcup S_2 = \{1, \ldots, n\}} ((a,b,x_{S_1}),c,x_{S_2}) = \\ \sum_{S_1 \sqcup S_2 = \{1, \ldots, n\}} (-1)^{|c||x_{S_1}|} (a,(b,x_{S_1},c),x_{S_2}) \ , \\ \text{for any } n \geq 0. \end{array}$

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t) =

Dual operad Gravity algebra, see ref below

Chain-cplx

Properties

Alternative

Relationship

Unit

Comment

[Get95] Ref.

Name Associative algebra up to homotopy

operad with underlying space in dgVect

Notation A_{∞}

Def. oper. m_n for $n \ge 2$ (operation of arity n and degree n-2)

sym.

rel. $\partial(m_n) = \sum_{\substack{n=p+q+r \ k=p+1+r \ k>1}} (-1)^{p+qr} m_k \circ (\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}).$

Free alg.

 \mathcal{P}_n $(A_{\infty})_n = \mathbb{K}[PT_n]$ isomorphic to $C_{\bullet}(\mathcal{K}^{n-2})$ as chain complex

where K^n is the Stasheff polytope of dimension n

 $\dim \mathcal{P}_n$

Gen. series f(t)

Dual operad

Chain-cplx

Properties multi-ary, quadratic, ns, minimal model for As

Alternative Cobar construction on As^{i} : $A_{\infty} = As_{\infty} := \Omega As^{i}$

Relationship Many, see the literature.

Unit Good question!

Comment There are two levels of morphisms between A_{∞} -algebras:

the morphisms and the ∞ -morphisms, see [LV11] for instance.

Ref. [Sta63] J. Stasheff, Homotopy associativity of H-spaces. I, II.

TAMS 108 (1963), 275-292; ibid. 108 (1963), 293312.

Name Commutative algebra up to homotopy

operad with underlying space in dgVect

Notation C_{∞}

Def. oper. m_n for $n \ge 2$ (operation of arity n and degree n-2) which vanishes on the sum of (p, n-p)-shuffles, $1 \le p \le n-1$.

sym.

rel. $\partial(m_n) = \sum_{\substack{k=p+q+r \ k>1,q>1}} \sum_{\substack{k=p+1+r \ k>1,q>1}} p+qr m_k \circ (\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}).$

Free alg.

 $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t)

Dual operad

Chain-cplx

Properties multi-ary, quadratic, minimal model for Com

Alternative Cobar construction on Com^{\dagger} : $Com_{\infty} = Com_{\infty} := \Omega Com^{\dagger}$

Relationship

Unit

Comment

Ref. [Kad85] T. Kadeishvili, The category of differential coalgebras

and the category of $A(\infty)$ -algebras.

Proc. Tbilisi Math.Inst. 77 (1985), 50-70.

Name Lie algebra up to homotopy

operad with underlying space in dgVect

Notation L_{∞}

Def. oper. ℓ_n , n-ary operation of degree n-2, for all $n \geq 2$

sym.

rel. $\sum_{\substack{p+q=n+1 \ p,q>1}} \sum_{\sigma \in Sh_{p,q}^{-1}} \mathrm{sgn}(\sigma) (-1)^{(p-1)q} (\ell_p \circ_1 \ell_q)^{\sigma} = \partial_A(\ell_n)$,

Free alg.

 $\mathtt{rep.}\ \mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t)

Dual operad

 ${\tt Chain-cplx}$

Properties multi-ary, quadratic, minimal model for Lie

Alternative Cobar construction on $Lie^{\mathrm{i}}\colon Lie_{\infty}=Lie_{\infty}:=\Omega Lie^{\mathrm{i}}$

Relationship

Unit

Comment

Name Dendriform algebra up to homotopy

operad with underlying space in dgVect

Notation $Dend_{\infty}$

Def. oper. $m_{n,i}$ is an n-ary operation, $1 \le i \le n$, for all $n \ge 2$

sym. none

rel. $\partial(m_{n,i}) = \sum (-1)^{p+qr} m_{p+1+r,\ell}(\underbrace{\mathrm{id},\cdots,\mathrm{id}}_p,m_{q,j},\underbrace{\mathrm{id},\cdots,\mathrm{id}}_r)$

sum extended to all the quintuples p,q,r,ℓ,j satisfying: $p\geq 0, q\geq 2, r\geq 0, p+q+r=n, 1\leq \ell\leq p+1+q, 1\leq j\leq q$

and either one of the following:

 $i = q + \ell$, when $1 \le p + 1 \le \ell - 1$, $i = \ell - 1 + j$, when $p + 1 = \ell$,

 $i = \ell$, when $\ell + 1 \le p + 1$.

Free alg.

rep. \mathcal{P}_n

 $\dim \mathcal{P}_n$

Gen. series f(t) =

Dual operad

Chain-cplx

Properties multi-ary, quadratic, ns, minimal model for Dend

Alternative

Relationship

Unit

Comment

Ref. See for instance [LV11]

Name \mathcal{P} -algebra up to homotopy

operad with underlying space in dgVect

Notation \mathcal{P}_{∞}

the operad \mathcal{P} is supposed to be quadratic and Koszul

Def. oper.

sym.

rel.

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$

Gen. series f(t)

Dual operad

Chain-cplx

Properties quadratic, ns if $\mathcal{P}(n) = \mathcal{P}_n \otimes \mathbb{K}[\mathbb{S}_n]$.

Alternative Cobar construction on $\mathcal{P}^{i}\colon \mathcal{P}_{\infty}:=\Omega \ \mathcal{P}^{i}$

Relationship

Unit

Comment

Ref. See for instance [LV11]

Name Brace algebra

Notation Brace

Def. oper. $\{x_0; x_1, \ldots, x_n\}$ for $n \ge 0$

 $\operatorname{sym}. \qquad \{x;\emptyset\} = x$

rel. $\{\{x; y_1, \dots, y_n\}; z_1, \dots, z_m\} = \sum_{m=1}^{\infty} \{x_m, \dots, x_m\}$

 $\sum \{x; \ldots, \{y_1; \ldots\}, \ldots, \{y_n; \ldots, \}, \ldots \}.$ the dots are filled with the variables z_i 's (in order).

Free alg.

rep. $\mathcal{P}(n)$ $Brace(n) = \mathbb{K}[PBT_{n+1}] \otimes \mathbb{K}[\mathbb{S}_n]$

 $\dim \mathcal{P}(n) \qquad 1, 1 \times 2!, 2 \times 3!, 5 \times 4!, 14 \times 5!, 42 \times 6!, 132 \times 7!, \dots, c_{n-1} \times n!, \dots$

Gen. series f(t)

Dual operad

Chain-cplx

Properties multi-ary, quadratic, quasi-regular.

Alternative

Relationship Brace-alg $\rightarrow MB$ -alg

Brace-alg $\rightarrow PreLie$ -alg, ($\{-; -\}$ is a pre-Lie product) If A is a brace algebra, then $T^c(A)$ is a cofree Hopf algebra

Unit

Comment There exists a notion of brace algebra with differentials useful

in algebraic topology

Ref. [Ron02] Ronco, M. Eulerian idempotents and Milnor-Moore

theorem for certain non-cocommutative Hopf algebras.

J. Algebra 254 (2002), no. 1, 152–172.

Name Multi-brace algebra

Notation MB

Def. oper. $(x_1,\ldots,x_p;y_1,\ldots,y_q)$ for $p\geq 1, q\geq 1$

sym.

rel. \mathcal{R}_{ijk} , see ref

Free alg.

rep. $\mathcal{P}(n)$ $MB(n) = (\mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n]) \otimes \mathbb{K}[\mathbb{S}_n], n \geq 2$

 $\dim \mathcal{P}(n) \qquad 1, 1 \times 2!, 6 \times 3!, 22 \times 4!, 90 \times 5!, ??? \times 6!, ??? \times 7!, \dots, 2C_n \times n!, \dots$

 $C_n =$ Schröder number (super Catalan)

Gen. series f(t)

Dual operad

Chain-cplx

Properties multi-ary, quadratic, quasi-regular.

Alternative Used to be denoted by B_{∞} or \mathbf{B}_{∞}

confusing notation with respect to algebras up to homotopy

Relationship Brace-alg $\rightarrow MB$ -alg

Brace-alg $\to PreLie$ -alg, ($\{-; -\}$ is a pre-Lie product) If A is a brace algebra, then $T^c(A)$ is a cofree Hopf algebra

(and vice-versa)

Unit

Comment There exists a notion of MB-infinity algebra with differentials

useful in algebraic topology (and called B_{∞} -algebra)

Ref. [LR06] Loday, J.-L., and Ronco, M. On the structure of cofree

Hopf algebras J. reine angew. Math. 592 (2006) 123–155.

Name Double Poisson algebra

Notation 2Pois

Def. oper.

sym.

rel.

Free alg.

rep. $\mathcal{P}(n)$

 $\dim \mathcal{P}(n)$ 1,...,??,...

Gen. series f(t)

Dual operad

Chain-cplx

Properties

Alternative

Relationship

Unit

Comment

Ref. M. Van den Bergh,

Name \mathcal{X}^{\pm} -algebra

Notation \mathcal{X}^{\pm} -alg

Def. oper. $x \nwarrow y, x \nearrow y, x \searrow y, x \swarrow y$.

$$(\nwarrow) \nwarrow = \nwarrow (\nwarrow) + \nwarrow (\swarrow), \quad (\swarrow) \nwarrow = \swarrow (\nwarrow), \quad (\nwarrow) \swarrow + (\swarrow) \swarrow = \swarrow (\swarrow), \\ (\nwarrow) \nwarrow = \nwarrow (\searrow) + \nwarrow (\nearrow), \quad (\swarrow) \nwarrow = \swarrow (\nearrow), \quad (\nwarrow) \swarrow + (\swarrow) \swarrow = \swarrow (\searrow), \\ (\nearrow) \nwarrow = \nearrow (\nwarrow) + \nearrow (\swarrow), \quad (\searrow) \nwarrow = \searrow (\nwarrow), \quad (\nearrow) \swarrow + (\searrow) \swarrow = \searrow (\swarrow), \\ (\nwarrow) \nearrow = \nearrow (\nearrow) + \nearrow (\searrow), \quad (\swarrow) \nearrow = \searrow (\nearrow), \quad (\nwarrow) \searrow + (\swarrow) \searrow = \searrow (\searrow), \\ (\nearrow) \nearrow = \nearrow (\nearrow) + \nearrow (\searrow), \quad (\searrow) \nearrow = \searrow (\nearrow), \quad (\nearrow) \searrow + (\searrow) \searrow = \searrow (\searrow).$$

$$(\nearrow) \searrow - (\nwarrow) \searrow = + \nwarrow (\swarrow) - \nwarrow (\searrow), \quad (16+) \\ (\nearrow) \searrow - (\nwarrow) \searrow = - \nwarrow (\swarrow) + \nwarrow (\searrow). \quad (16-)$$

Free alg.

rep. \mathcal{P}_n

 $\dim \mathcal{P}_n$

Gen. series f(t) =

Dual operad Both sef-dual.

Chain-cplx

Properties binary, quadratic, ns.

Alternative

Relationship A quotient of *Dend* ■ *Dias* (Manin black product in ns operads) Fit into the "operadic butterfly" diagram, see the reference.

Unit

Comment

Ref. [Lod06] Loday J.-L., Completing the operadic butterfly, Georgian Math Journal 13 (2006), no 4. 741–749.

```
Name
                put your own type of algebras
Notation
Def. oper.
       sym.
       rel.
Free alg.
rep. \mathcal{P}(n)
\dim \mathcal{P}(n)
                f(t) =
Gen. series
Dual operad
Chain-cplx
Properties
Alternative
{\tt Relationship}
Unit
```

Comment

Integer sequences which appear in this paper, up to some shift and up to multiplication by n! or (n-1)!.

1	1	1	1	1		1		Com, As
1	2	0	0	0		0		Nil
1	2	2	2	2		2		Dual2as
1	2	3	4	5		n		Dias, Perm
1	2	5	12	15	• • •	??	• • •	Altern!
1	2	5	14	42		c_{n-1}		Mag, Dend, brace, Dup
1	2	6	18	57		f_{n+2}		MagFine
1	2	6	22	90		$2C_n$		Dipt, 2as, brace
1	2	6	24	120	• • •	n!	• • •	As, Lie, Leib, Zinb
1	2	7			• • •	??		Lie- adm
1	2	7	32	175	• • •	??		Altern
1	2	7	40		• • •	??		Moufang
1	2	9	64	625		n^{n-1}		PreLie, NAP
1	2	10	26	76		??		Parastat
1	3	7	15	31		$(2^n - 1)$		Trias
1	3	9			• • •	??	• • •	Malcev
1	3	11	45	197		C_n		TriDend
1	3	13	75	541	• • •	??		CTD
1	3	16	125	6^{5}	• • •	$(n+1)^{n-1}$	• • •	Park
1	3	20	210	3024	• • •	a(n)	• • •	PostLie
1	3	50			• • •	??	• • •	Jordan triples
1	4	9	16	25		$n^2n!$		$Quadri^!$
1	4	23	156	1162	• • •	??		Quadri
1	4	23	181		• • •	??		see $PreLie$
1	4	27	256		• • •	n^n	• • •	
1	8				• • •	??		Akivis
1	8	78	1104			??		Sabinin

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