Can the Cross Product be Generalized to Higher-Dimensional Space?

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Abstract This article uses the question in the title to motivate the **wedge product**. Many things that are traditionally expressed using the cross product in three-dimensional space have natural generalizations to any number of dimensions when expressed in terms of the wedge product. The wedge product can also be used to define the determinant of a linear transformation. Basic properties of the determinant are obvious from this definition.

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1 Introduction

In 3-dimensional space, the cross product of two vectors

$$\mathbf{a} = (a_1,\, a_2,\, a_3) \qquad\qquad \mathbf{b} = (b_1,\, b_2,\, b_3)$$

is defined by

$$\begin{aligned}\mathbf{a}\times\mathbf{b} \equiv (&a_2b_3 - a_3b_2,\\ &a_3b_1 - a_1b_3,\\ &a_1b_2 - a_2b_1).\end{aligned}$$

The fact that this has three components tempts us to call it a "vector," and for many purposes we can treat it as a vector. However, if we pay close enough attention to how the cross product is actually used in physics, we realize that treating the cross product as a vector is not necessary. If we discard this unnecessary part of the cross product idea, then it has a natural generalization to D-dimensional space. This natural replacement for the cross product is called the **wedge product** (also called the **exterior product**), the subject of this article.

The wedge product works in any number of dimensions. It also works for the product of any number of vectors, not just two vectors. It provides a natural way of defining the determinant of any linear transformation, and key properties of the determinant are more obvious from this definition than they are from the traditional one.

2 Key properties of the cross product

These properties of the cross product are important:

- Its inputs are two vectors, $\mathbf{a}$ and $\mathbf{b}$.
- It is antisymmetric in its two inputs: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- It is linear in both of its inputs. In particular, if we flip the sign of either one of its inputs, then the whole thing changes sign.
- Define the magnitude of $\mathbf{a} \times \mathbf{b}$, denoted $|\mathbf{a} \times \mathbf{b}|$, to be the square-root of the sum of the squares of the components. This quantity is equal to the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$.
- When the magnitude of $\mathbf{a} \times \mathbf{b}$ is nonzero, it has an orientation. If somebody gives us the components of $\mathbf{a} \times \mathbf{b}$ but doesn't tell us what $\mathbf{a}$ and $\mathbf{b}$ were, we can still determine which vectors lie within the plane defined by $\mathbf{a}$ and $\mathbf{b}$.

We can generalize the cross product to D-dimensional space in a way that preserves all of these properties, but the generalization is *not a vector*. To emphasize this, the generalization has a different name: the wedge product. The remaining sections define it and highlight some of its properties.

3 The wedge product of two vectors

In D-dimensional space, the wedge product of two vectors

$$\mathbf{a} = (a_1,\, a_2,\, ...,\, a_N) \qquad\qquad \mathbf{b} = (b_1,\, b_2,\, ...,\, b_N)$$

is an entity $\mathbf{a} \wedge \mathbf{b}$ with these components:

$$a_j b_k - a_k b_j \qquad\qquad \text{for all } j, k \in \{1,\, 2,\, ...,\, N\}.$$

We immediately see that this entity *cannot be a vector*, because the number of components is not equal to D. How many components does it have? Well, we might as well only count the components with $j < k$, because the components with $k > j$ are just the negatives of the ones with $j < k$. The number of components with $j < k$ is

$$\binom{D}{2} = \frac{D^2 - D}{2}.$$

This happens to be 3 when $D = 3$, but that's a distracting coincidence. The wedge product is *not a vector*. We can keep track of its components using two indices instead of one:

$$\boxed{(\mathbf{a} \wedge \mathbf{b})_{jk} \equiv a_j b_k - a_k b_j.} \tag{1}$$

If $\mathbf{a} \wedge \mathbf{b}$ isn't a vector, then what is it? We can answer that question the same way we answer any "what is it" question in math: by becoming familiar with its properties. That's what the remaining sections are about.

4 Properties of the wedge product of two vectors

The wedge product has all the properties that were listed in section 2. These properties are obvious from the definition:

- If we swap $\mathbf{a}$ and $\mathbf{b}$, then the components of the wedge product change sign: $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$.

- The wedge product is linear in both of its inputs. In particular, if we flip the sign of either one of its inputs, then the whole thing changes sign.

The next two properties might not be so obvious:

- If we define the magnitude $|\mathbf{a} \wedge \mathbf{b}|$ to be the square-root of the sum of the squares of its components (counting only the components with $j < k$), then the magnitude is equal to the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$.

- When its magnitude is nonzero, the wedge product has an orientation. If somebody gives us the components of $\mathbf{a} \wedge \mathbf{b}$ but doesn't tell us what $\mathbf{a}$ and $\mathbf{b}$ were, we can still determine which vectors lie within the plane defined by $\mathbf{a}$ and $\mathbf{b}$.

These might not be obvious at first, but they are both easy to prove. The next section shows how.

5 Proof of the parallelogram-area property

First consider the parallelogram-area property:

$$\begin{aligned}
|\mathbf{a}\wedge\mathbf{b}|^2 \equiv \sum_{j<k}(a_jb_k-a_kb_j)^2 &= \frac{1}{2}\sum_{j,k}(a_jb_k-a_kb_j)^2 \\
&= \sum_{j,k} a_j^2b_k^2 - \sum_{j,k}(a_jb_j)(a_kb_k) \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \\
&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta,
\end{aligned}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This proves

$$|\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta,$$

which is the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$, as claimed. This works for any number of dimensions $D\geq 2$, not just $D=3$.

6 Proof of the orientation property

Now consider the orientation property. Suppose that somebody gives us the components $(\mathbf{a}\wedge\mathbf{b})_{jk}$ and asks us to determine whether another vector $\mathbf{c}$ lies within the plane defined by $\mathbf{a}$ and $\mathbf{b}$, but without telling us the individual components of $\mathbf{a}$ and $\mathbf{b}$. To solve this riddle, consider the quantities

$$T_{jkm} \equiv \frac{1}{2}\sum_\pi (-1)^\pi (\mathbf{a}\wedge\mathbf{b})_{\pi(j)\pi(k)} c_{\pi(m)},$$

where the sum is over all permutations of the D index-values, with a coefficient -1 for odd permutations and $+1$ for even permutations, so that swapping any two of T_{jkm}'s subscripts is equivalent to changing the overall sign. In words, T is **completely antisymmetric**. The quantities T_{jkm} are all zero if and only if the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar. To see this, use equation (1) to deduce

$$T_{jkm} = \sum_\pi (-1)^\pi a_{\pi(j)} b_{\pi(k)} c_{\pi(m)}.$$

This shows that T is completely antisymmetric in the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: it changes sign whenever two of them are swapped. If $\mathbf{c}$ can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$, then this antisymmetry implies $T = 0$. Therefore, if the three vectors are coplanar, then $T = 0$. What about the converse? We can always choose a basis in which all components of $\mathbf{a}$ are zero except possibly the first one, all components of $\mathbf{b}$ are zero except possibly the first two, and all components of $\mathbf{c}$ are zero except possibly the first three. In such a basis, the antisymmetry of T implies

$$T_{123} = a_1 b_2 c_3,$$

which is zero if *and only if* at least one of the three factors a_1, b_2, c_3 is zero. In other words, T is zero if and only if the three vetors are coplanar, so the riddle is solved.

This shows that the wedge product $\mathbf{a}\wedge\mathbf{b}$ encodes the orientation of the plane defined by the vectors $\mathbf{a}$ and $\mathbf{b}$.

7 The wedge product of multiple vectors, part 1

The definition of the wedge product of two vectors, equation (1), can also be written like this:

$$(\mathbf{a}\wedge\mathbf{b})_{jk} \equiv \sum_\pi (-1)^\pi a_{\pi(j)} b_{\pi(k)}.$$

We can generalize this to the wedge product of any number of vectors. For three vectors, it's the expression that appeared in the previous section:

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})_{jkm} \equiv \sum_\pi (-1)^\pi a_{\pi(j)} b_{\pi(k)} c_{\pi(m)}.$$

The pattern should be clear from these examples.

We can also define the wedge product of two wedge products: if A is the wedge product of K vectors and B is the wedge product of M vectors, then $A\wedge B$ is defined by antisymmetrizing over all $K+M$ indices and dividing by $K!\,M!$. (Notice that this reduces to equation (1) if $K=M=1$.) With this definition, we can easily see that $A\wedge B$ is equal to the wedge product of the $K+M$ original vectors that were used to construct A and B.

8 The wedge product of multiple vectors, part 2

The wedge product of K vectors in D-dimensional space has these properties:

- The wedge product is associative. Example:

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}.$$

- It changes sign if any two of the vectors are swapped.

- It is linear in each vector. In particular, if we flip the sign of any one vector, then the whole thing changes sign.

- It is nonzero if and only if all of the vectors are linearly independent. In other words, it is zero if and only if one of the vectors can be written as a linear combination of the others.

- If we define its magnitude to be the square-root of the sum of the squares of its D-choose-K components, then the magnitude is the volume of the (hyper)parallelepipid defined by the K vectors.

Geometrically, the wedge product of K vectors represents an oriented element of K-volume. If $K = N$, then it has only one component. The significance of this case is highlighted in the next section. If $K > N$, then it is identically zero because that many vectors cannot all be linearly independent.

These properties obviously still hold when $K = 1$, which is a single ordinary vector, so the pattern is complete.

9 The determinant of a linear transformation

Let V be an D-dimensional vector space over the real or complex numbers. The wedge product of D vectors has only one component, so if $\mathbf{v}_1, ..., \mathbf{v}_D$ is a set of D linearly independent vectors in V, and if $\mathbf{u}_1, ..., \mathbf{u}_D$ is another set of D linearly independent vectors V, then

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_D \propto \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_D.$$

Such a product is called a **top-dimensional form**, because it has the largest number of vector factors that any nonzero wedge product can have in a D-dimensional vector space.[1] In words: all top-dimensional forms are proportional to each other.

We can use this fact to define the determinant of a linear transformation. Let V be an D-dimensional vector space over the real or complex numbers, let $\mathbf{v}_1, ..., \mathbf{v}_D$ be any set of D linearly independent vectors in V, and let $L : V \to V$ be any linear transformation of V. Then

$$(L\mathbf{v}_1) \wedge (L\mathbf{v}_2) \wedge \cdots \wedge (L\mathbf{v}_D) = (\det L)\, \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_D$$

for some real or complex number $\det L$ that is uniquely determined by this equation. The number $\det L$ is called the **determinant** of L. From this definition, we can easily deduce some key properties of the determinant:

- If L and M are two linear transformations, then $\det(LM) = (\det L)(\det M)$.
- If L is has D linearly independent eigenvectors,[2] then $\det L$ is equal to the product of the D eigenvalues of L. To prove this, take the factors $\mathbf{v}_n$ to be the eigenvectors of L.
- $\det L \neq 0$ if and only if L is invertible.
- If L is invertible, then $\det(L^{-1}) = 1/\det L$.

[1] The wedge product of any number of vectors is zero if the vectors are not all linearly independent of each other.

[2] Recall that a vector $\mathbf{v}$ is called an **eigenvector** of L with **eigenvalue** λ if $L\mathbf{v} = \lambda\mathbf{v}$.

10 Pseudovectors

If $\mathbf{a}$ and $\mathbf{b}$ are two vectors in 3-dimensional space, then their cross product $\mathbf{c} \equiv \mathbf{a}\times\mathbf{b}$ has the same number of components (three) as a vector. Even more, it transforms like a vector under ordinary rotations: $\mathbf{a}\times\mathbf{b} \to (R\mathbf{a})\times(R\mathbf{b})$ is the same as $\mathbf{c} \to R\mathbf{c}$, for any rotation R with unit determinant.[3] As a consequence, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are three vectors, then the quantity $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$ is invariant under ordinary rotations $\mathbf{v}_n \to R\mathbf{v}_n$. The quantity $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$ can be interpreted as the volume of the parallelepiped with edges $\mathbf{v}_n$.

This has a natural generalization to D-dimensional space. If $\mathbf{v}_1, ..., \mathbf{v}_{D-1}$ are $D-1$ vectors, then their wedge product

$$\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{D-1} \tag{2}$$

has the same number of components (namely D) as a vector, and those components can be arranged so that it transforms like a vector under ordinary rotations. The proof is simple: given another vector $\mathbf{v}_D$, the wedge product of (2) with $\mathbf{v}_D$ is invariant under any transformation $\mathbf{v}_n \to R\mathbf{v}_n$ with $\det R = 1$. This is a consequence of the definition of the determinant given in the previous section. Therefore, we can arrange the components of the quantity (2) so that its “dot product” with $\mathbf{v}_D$ is invariant under rotations. When the D components of (2) are arranged this way, it’s called a **pseudovector** (or **axial vector**).

[3] The unit-determinant condition excludes reflections.