Clifford Algebra, Lorentz Transformations, and Spin Groups

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Abstract Article 03910 introduced **Clifford algebra**, which is sometimes also called **geometric algebra**. This article uses Clifford algebra to construct the **spin group**, which is a double cover of the part of the Lorentz group that is generated by pairs of reflections. This is a prerequisite for the idea of a **spinor field**, which is an important ingredient in our current understanding of nature. The construction and basic topological properties of the spin group are explained for arbitrary signatures (p, q) with $p + q \geq 2$, including euclidean signatures (either p or q is 0), lorentzian signatures (either p or q is 1), and other signatures (p and q are both ≥ 2).

Contents

1 Introduction

In quantum field theory, observables are typically expressed in terms of (quantum) fields. This is useful partly because fields simplify the task of describing how observables at different times are related to each other.

The fields themselves are not necessarily observables, so a given transformation might not affect any observables even if it does affect the fields. This can have interesting topological implications. Consider a one-parameter family of transformations $T(r)$, one for each $0 \leq r \leq 1$. If the first and last transformations ($T(0)$ and $T(1)$) both leave all observables unaffected, then the function $T(r)$ defines a closed loop in the space of possible effects on the observables. If the fields are not observables, then we can have a situation in which $T(0)$ doesn't affect the fields but $T(1)$ does, even though neither one affects observables. Then the function $T(r)$ describes a path that is not closed in the space of possible effects on the fields, even though it is closed in the space of possible effects on the observables.

In models with spinor fields, the situation described in the previous paragraph does occur. The *spin group*,[1] a group of transformations of the spinor fields, is a *double cover*[2] of the Lorentz group that acts on observables.[3] The Lorentz group has closed paths that come from non-closed paths in the spin group.

Using Clifford algebra[4] to make the math easy, this article explains how to construct the spin group. The fact that every Lorentz transformation is a composition of reflections is then used to build some geometric intuition about the topology of the spin group in various dimensions and signatures, including non-lorentzian signatures.

[1]Section 12

[2]Section 2

[3]Spinor fields are also fermion fields, which means that field operators separated by a spacelike interval anticommute with each other. The two-to-one relationship between the spin group and the Lorentz group is possible because a product of an even number of fermion field operators can be an observable, but a product of an odd number fermion field operators cannot.

[4]Article 03910

2 Group theory and topology: some basic definitions

A **homomorphism** from one group Σ to another group G is a map that respects the group structure.[5,6] The group Σ is called a **double cover** of G if a homomorphism $\rho : \Sigma \to G$ exists that assigns two different elements of Σ to each element of G. In this article, G and Σ will both be Lie groups. Roughly, a **Lie group** is both a group and a smooth manifold whose group structure and smooth structure are compatible with each other in a natural way.[7] In a context where all of the groups are Lie groups, all of the homomorphisms are usually understood to be **Lie group homomorphisms** – maps that respect both the smooth structure and the group structure. In this article, the name *Lie group homomorphism* will be abbreviated *homomorphism*.

A manifold is called **connected** if every point can be reached from every other point by a continuous path. A Lie group may or may not be connected. Two continuous closed paths are called **homotopic** to each other if one can be continuously morphed into the other without breaking it during the process. A manifold is called **simply connected** if all continuous closed paths are homotopic to each other.[8,9] Examples: the two-dimensional sphere (the surface of a three-dimensional ball) is simply connected, but a circle (the one-dimensional boundary of a two-dimensional disk) is merely connected, not simply connected.

[5] Article 29682

[6] In sections 18-25, the symbols G and Σ will be used as abbreviations for the groups $SO^+(p,q)$ and $\text{Spin}^+(p,q)$. These groups will be defined in sections 4 and 12, respectively.

[7] Chapter 7 in Lee (2013) introduces Lie groups. Appendix A in Harlow and Ooguri (2018) gives a more concise introduction.

[8] Hatcher (2001), proposition 1.6 and the text above it

[9] According to this definition, *simply connected* implies *connected*.

3 Scalar product and signature

In this article, the only aspect of spacetime that matters is the scalar product between vectors at a single point in spacetime, as defined by the metric tensor. If spacetime is a d-dimensional manifold, then the space of vectors at a given point is a d-dimensional vector space over the real numbers $\mathbb{R}$. Given two vectors $\mathbf{u} = (u_1, ..., u_d)$ and $\mathbf{v} = (u_1, ..., u_d)$ in this d-dimensional vector space,[10] their scalar product is

$$g(\mathbf{u}, \mathbf{v}) = (u_1 v_1 + \cdots + u_p v_p) - (u_{p+1} v_{p+1} + \cdots + u_{p+q} v_{p+q}), \tag{1}$$

where (p, q) is a pair of nonnegative integers with $p + q = d$. Two vectors $\mathbf{u}, \mathbf{v}$ are called **orthogonal** to each other if $g(\mathbf{u}, \mathbf{v}) = 0$. In this context, referring to $g(\mathbf{v}, \mathbf{v})$ as the **norm** of $\mathbf{v}$ is convenient.[11] The norm may be positive, negative, or zero. A nonzero vector $\mathbf{v}$ with $g(\mathbf{v}, \mathbf{v}) = 0$ will be called **self-orthogonal**.[12] Such vectors exist if p and q are both nonzero.

The pair (p, q) is called the **signature**. The signature is called **euclidean** if either p or q is equal to 0. The signature is called **lorentzian** if either p or q is equal to 1, which is normally implied when the word *spacetime* is used. This article considers all signatures, not just euclidean and lorentzian, even though those two cases have the most immediate relevance to physics.

[10] This article uses lowercase boldface letters to denote vectors.

[11] Porteus (1995) (chapter 4, page 22) calls this the **quadratic norm**.

[12] When the signature is lorentzian, a self-orthogonal vector is often called **lightlike**. A vector $\mathbf{v}$ with $g(\mathbf{v}, \mathbf{v}) \neq 0$ may be called either **timelike** and **spacelike**, depending on the sign of $g(\mathbf{v}, \mathbf{v})$ and on whether the signature is $(p, 1)$ or $(1, p)$ (article 48968). This article uses the language *positive norm* and *negative norm* instead.

4 Groups of isometries

An **isometry** is a linear transformation of the d-dimensional vector space that leaves the scalar product invariant. In symbols: a linear transformation $\mathbf{v} \to \rho(\mathbf{v})$ is called an isometry if

$$g\big(\rho(\mathbf{u}), \rho(\mathbf{v})\big) = g(\mathbf{u}, \mathbf{v}).$$

When the signature is lorentzian, the group of isometries is called the **Lorentz group**. This article uses the more generic term **isometry group**, which avoids committing to any particular signature. When the signature is (p, q), the isometry group is denoted $O(p, q)$. When $p = 0$ or $q = 0$, this is called an **orthogonal group**. When p and q are both nonzero, it's called an **indefinite orthogonal group** or **pseudo-orthogonal group**.

Every isometry is a composition of reflections.[13] These subgroups of $O(p, q)$ are of special interest:[14]

- $SO(p, q)$: an isometry belongs to this subgroup if and only if it may be expressed as a composition of an even number of reflections.[15] This is called the **(indefinite) special orthogonal group**.

- $SO^+(p, q)$: an isometry belongs to this subgroup if and only if it may be expressed as a composition of reflections consisting of an even number of reflections (possibly zero) along directions $\mathbf{r}$ with $g(\mathbf{r}, \mathbf{r}) > 0$ and an even number (possibly zero) along directions $\mathbf{r}$ with $g(\mathbf{r}, \mathbf{r}) < 0$.

If $p = 0$ or $q = 0$, then $SO^+(p, q) = SO(p, q)$. Otherwise, $SO^+(p, q)$ is a proper subgroup of $SO(p, q)$. The group $SO^+(p, q)$ is of special interest because it is connected[16] for every signature (p, q). In this article, it will be called the **connected isometry group**.[17]

[13] Article 39430, and statement (2.4.5) in Benn and Tucker (1989)

[14] The notation $SO^+(p, q)$ is common but not universal. Sources that use it include Borghini (2018), section V.1.2c, page 61. Other notations include $SO_+(p, q)$, $SO(p, q)^0$, and $SO^\uparrow(p, q)$.

[15] Varadarajan (2004), theorem 5.2.1

[16] Section 2 defined *connected*, and section 16 will show that $SO^+(p, q)$ has this property.

[17] It has also been called the **reduced special orthogonal group** (Harvey (1990), chapter 1, page 13).

5 Topology of the connected isometry group

The Lie group $SO^+(p,q)$ is connected.[18] This section reviews another basic fact about the topology of $SO^+(p,q)$:

> When $p \geq 2$ or $q \geq 2$ (or both), $SO^+(p,q)$ is not simply connected.

Sections 18-25 will offer some geometric intuition about why this is true. This section reviews some of the ingredients that can be used in a proper proof.

When a manifold M is not simply connected, its **fundamental group** $\pi_1(M)$ conveys information about how it fails to be simply connected.[19] Elements of this group are equivalence classes of closed paths that can be continuously morphed into each other, and the group structure (the rule for composing elements) is based on the idea of combining paths that cross each other.[20] If two closed paths in a manifold cross each other, then we can reinterpret them as a single closed path, just be reinterpreting the crossing as a pair of sharp turns that touch each other at their corners. This can be used to define a group in which each element is a class of closed paths that are all homotopic each other and in which the product is the act of merging two paths by reinterpreting a crossing as a pair of sharp turns. A connected manifold is simply connected if and only if its fundamental group is trivial – that is, if and only if all closed paths can be continuously deformed into each other even without reinterpreting crossings as sharp turns.

The fundamental group of a connected Lie group G is the same as the fundamental group of any **maximal compact subgroup** $K \subset G$, which is a compact subgroup that is not contained within any larger compact subgroup other than G

[18] Section 16

[19] The fundamental group of M still doesn't tell us everything about the topology of M. The fundamental group $\pi_1(M)$ is just one of an infinite family of **homotopy groups** $\pi_k(M)$, and even this whole infinite family still doesn't tell us everything about the topology of M.

[20] Hatcher (2001), section 1.1

itself. The identity $\pi_1(G) = \pi_1(K)$ follows from the **Cartan-Iwasawa-Malcev theorem**, which says[21] that that K is automatically connected (if G is), that the topology of K is uniquely determined by G,[22] and that G is topologically equivalent to $K \times \mathbb{R}^m$ for some $m \geq 0$.

To determine the fundamental group of $SO^+(p,q)$, we can use the fact that each of its maximal compact subgroups is equivalent to $SO(p) \times SO(q)$,[23] together with the isomorphism $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$[24,25] The fundamental group of $SO(2)$ is $\mathbb{Z}$, the additive group of integers. For $p \geq 3$, the fundamental group of $SO(p)$ is is $\mathbb{Z}_2$, the additive group of integers modulo 2, which has only two elements.[26,27] Using these ingredients, we can determine $\pi_1(SO^+(p,q))$ whenever $p \geq 2$ or $q \geq 2$. In particular, we can use these ingredients to deduce that $SO^+(p,q)$ is not simply connected if $p \geq 2$ or $q \geq 2$, as stated at the beginning of this section.[28]

This article will use more elementary methods to anticipate this basic property of $SO^+(p,q)$.

[21] Armstrong (2018), page 9

[22] G may have many different maximal compact subgroups, but they all have the same topology.

[23] Section 3 in Conrad (2018) lists two maximal compact subgroups for $SO(p,q)$. One of them is also a subgroup of $SO^+(p,q)$.

[24] Padgett (2014), lemma 2.27

[25] This is an example of a **Künneth formula** (Hatcher (2001), chapter 3, text below theorem 3.15).

[26] Fletcher (2022), section 4.1, above table 4.1

[27] Sections 19-21 will help explain, intuitively, why these statements about $\pi_1(SO(p))$ are true.

[28] For extra fun: section 4.1 in Fletcher (2022) tabulates the homotopy groups of a related family of Lie groups, which that author calls $GO(p,q)$. Those Lie groups include both isometries and dilations.

6 Review of Clifford algebra

To simplify the task of constructing a double cover of the connected isometry group, this article uses Clifford algebra. Article 03910 introduces Clifford algebra. This section gives a brief review.

Let $\mathbf{e}_1, ..., \mathbf{e}_d$ be a list of linearly independent vectors, so that every vector $\mathbf{v}$ can be written

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_d\mathbf{e}_d.$$

The **components** $v_1, v_2, ..., v_d$ of $\mathbf{v}$ are real numbers. Given this d-dimensional vector space over $\mathbb{R}$, the largest associative algebra generated by those vectors together with an identity element is called a **tensor algebra**. The tensor algebra has a basis consisting of the scalar 1, the vectors $\mathbf{e}_i$ (d of these), the products $\mathbf{e}_i\mathbf{e}_j$ (d^2 of these), the products $\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$ (d^3 of these), and so on. These are all linearly independent elements of the tensor algebra. In particular, $\mathbf{e}_j\mathbf{e}_k$ and $\mathbf{e}_k\mathbf{e}_j$ are not proportional to each other unless $j = k$. The scalar 1 serves as the multiplicative unit element. The product is associative, distributive, and linear. Imposing the additional relation

$$\mathbf{ab} + \mathbf{ba} = 2g(\mathbf{a}, \mathbf{b}). \tag{2}$$

for all vectors $\mathbf{a}, \mathbf{b}$ changes the tensor algebra to a **Clifford algebra**. The quantity $g(\mathbf{a}, \mathbf{b})$ on the right-hand side is defined by equation (1). Two vectors $\mathbf{a}, \mathbf{b}$ are orthogonal to each other if and only if $\mathbf{ab} = -\mathbf{ba}$ (in words: if and only if they anticommute with each other). According to equations (1) and (2), we can choose the basis vectors $\mathbf{e}_k$ so that

$$\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = \begin{cases} 2 & \text{if } j = k \in \{1, 2, ..., p\}, \\ -2 & \text{if } j = k \in \{p+1,\, p+2,\, ...,\, p+q\}, \\ 0 & \text{if } j \neq k. \end{cases} \tag{3}$$

Such a basis will be used often in this article.

7 Using Clifford algebra to describe reflections

Using Clifford algebra, the effect of a reflection along the direction $\mathbf{r}$ is described by the transformation $\mathbf{v} \to \rho_{\mathbf{r}}(\mathbf{v})$ with

$$\rho_{\mathbf{r}}(\mathbf{v}) \equiv -\frac{\mathbf{r}\mathbf{v}\mathbf{r}}{\mathbf{r}^2} \tag{4}$$

for all vectors $\mathbf{v}$. This is defined for any vector $\mathbf{r}$ with $\mathbf{r}^2 \neq 0$. To show that (4) really does describe a reflection, use equation (2) to write the numerator as

$$\mathbf{r}\mathbf{v}\mathbf{r} = -\mathbf{r}^2\mathbf{v} + 2g(\mathbf{v},\mathbf{r})\mathbf{r},$$

which gives

$$\rho_{\mathbf{r}}(\mathbf{v}) = \mathbf{v} - \frac{2g(\mathbf{v},\mathbf{r})}{g(\mathbf{r},\mathbf{r})}\,\mathbf{r}. \tag{5}$$

Equation (5) is the usual way to describe a reflection.[29]

Reflections preserve the scalar product:

$$g\big(\rho_{\mathbf{r}}(\mathbf{a}),\, \rho_{\mathbf{r}}(\mathbf{b})\big) = g(\mathbf{a},\mathbf{b}).$$

In other words, reflections are isometries.[30] Equations (2) and (4) make the proof easy:

$$\begin{aligned}
g\big(\rho_{\mathbf{r}}(\mathbf{a}),\, \rho_{\mathbf{r}}(\mathbf{b})\big) &= \frac{\mathbf{rar}\,\mathbf{rbr} + \mathbf{rbr}\,\mathbf{rar}}{2(\mathbf{r}^2)^2} \\
&= \frac{\mathbf{rabr} + \mathbf{rbar}}{2\mathbf{r}^2} = \frac{\mathbf{r}\,g(\mathbf{a},\mathbf{b})\,\mathbf{r}}{2\mathbf{r}^2} = g(\mathbf{a},\mathbf{b}).
\end{aligned}$$

[29] Article 39430

[30] Section 4

8 Isometries from reflections

Every isometry may be expressed as a composition of reflections.[31] Transformations in the connected isometry group may be expressed as a composition of reflections in which the numbers of reflections along directions with positive norm and negative norm are both even. The next few sections will describe a convenient way to package the composition of two reflections with the same norm, using Clifford algebra. This section enumerates the cases.

Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be mutually orthogonal vectors: $g(\mathbf{e}_1, \mathbf{e}_2) = 0$. These two vectors span a plane. The possibilities are:

- **Case 0:** The plane does not contain any self-orthogonal directions. This can be separated into two subcases:
  - **Case 0a:** The plane contains only directions $\mathbf{v}$ with $g(\mathbf{v}, \mathbf{v}) > 0$.
  - **Case 0b:** The plane contains only directions $\mathbf{v}$ with $g(\mathbf{v}, \mathbf{v}) < 0$.
- **Case 1:** The plane contains exactly one self-orthogonal direction. Again, this can be separated into two subcases:
  - **Case 1a:** The plane includes some directions $\mathbf{v}$ with $g(\mathbf{v}, \mathbf{v}) > 0$.
  - **Case 1b:** The plane includes some directions $\mathbf{v}$ with $g(\mathbf{v}, \mathbf{v}) < 0$.
- **Case 2:** The plane contains exactly two self-orthogonal directions, so it contains both positive-norm and negative-norm directions.
- **Case 3:** The plane contains only self-orthogonal directions.

For the study of isometries generated by reflections, only cases 0,1,2 are relevant. Case 3 is not relevant, because reflections along self-orthogonal directions are not defined.[32]

[31] Section 4

[32] Case 3 doesn't occur at all when the signature is euclidean or lorentzian, which are the signatures of greatest interest in physics. It occurs only when the signature (p, q) has both $p \geq 2$ and $q \geq 2$.

9 Isometries from reflections: case 0

In this case, the plane contains only positive-norm directions or only negative-norm directions. Within this plane, choose two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ that are orthogonal to each other and normalized so that $\mathbf{e}_1^2 = \mathbf{e}_2^2$. Consider a sequence of two reflections along any two directions in this plane. Without loss of generality, we can take one of the two directions to be $\mathbf{e}_1$, and we can take the other one to be

$$\mathbf{r} = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta.$$

The identity $\cos^2\theta + \sin^2\theta = 1$ implies $\mathbf{r}^2 = \mathbf{e}_k^2$.

Let $\rho(\mathbf{v})$ denote the result of reflecting $\mathbf{v}$ first along $\mathbf{r}$ and then along $\mathbf{e}_1$. With our choices of $\mathbf{e}_k$ and $\mathbf{r}$, equation (4) implies

$$\rho(\mathbf{v}) = R\mathbf{v}R^{-1}$$

with

$$R \equiv \mathbf{e}_1\mathbf{r} \qquad\qquad R^{-1} = \mathbf{r}\mathbf{e}_1.$$

The notation R^{-1} is appropriate because $R^{-1}R = RR^{-1} = 1$.

Define[33]

$$B \equiv \mathbf{e}_1\mathbf{e}_2.$$

Our assumptions about $\mathbf{e}_1$ and $\mathbf{e}_2$ imply $B^2 = -1$. We have not yet specified the sign of $\sigma \equiv \mathbf{e}_k^2$. The analysis is almost the same for either sign, but the cases $\sigma = 1$ and $\sigma = -1$ will be handled separately for clarity:[34,35]

$$\begin{aligned} R &\equiv \mathbf{e}_1\mathbf{r} = \cos\theta + B\sin\theta = e^{\theta B} && \text{if } \sigma = 1, \\ R &\equiv \mathbf{e}_1\mathbf{r} = -\cos\theta + B\sin\theta = -e^{-\theta B} && \text{if } \sigma = -1. \end{aligned}$$

[33] Mnemonic: B stands for *bivector*.

[34] In section 8, these were called case 0a and case 0b, respectively.

[35] As in article 18505, $e^{\theta B}$ is defined to be the unique function of θ that satisfies $\frac{d}{d\theta}e^{\theta B} = Be^{\theta B}$ and that equals 1 when $\theta = 0$.

These generalize **Euler's formula**. Writing R this way simplifies the task of calculating $\rho(\mathbf{v})$.

Every vector $\mathbf{v}$ may be written $\mathbf{v} = x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{p}$, where x, y, z are real numbers and $\mathbf{p}$ is orthogonal[36] to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Reflections are linear transformations, so after we know how the reflections affect each of these three terms, we automatically know how it affects $\mathbf{v}$. The effects are:

- $\mathbf{e}_1$ anticommutes with B, so

$$\begin{aligned}
\rho(\mathbf{e}_1) &= e^{\theta B}\mathbf{e}_1 e^{-\theta B} = e^{2\theta B}\mathbf{e}_1 = \mathbf{e}_1\cos(2\theta) - \mathbf{e}_2\sin(2\theta) && \text{if } \sigma = 1,\\
\rho(\mathbf{e}_1) &= e^{-\theta B}\mathbf{e}_1 e^{\theta B} = e^{-2\theta B}\mathbf{e}_1 = \mathbf{e}_1\cos(2\theta) - \mathbf{e}_2\sin(2\theta) && \text{if } \sigma = -1.
\end{aligned}$$

- $\mathbf{e}_2$ anticommutes with B, so

$$\begin{aligned}
\rho(\mathbf{e}_2) &= e^{\theta B}\mathbf{e}_2 e^{-\theta B} = e^{2\theta B}\mathbf{e}_2 = \mathbf{e}_2\cos(2\theta) + \mathbf{e}_1\sin(2\theta) && \text{if } \sigma = 1,\\
\rho(\mathbf{e}_2) &= e^{-\theta B}\mathbf{e}_2 e^{\theta B} = e^{-2\theta B}\mathbf{e}_2 = \mathbf{e}_2\cos(2\theta) + \mathbf{e}_1\sin(2\theta) && \text{if } \sigma = -1.
\end{aligned}$$

- $\mathbf{p}$ commutes with B, so

$$\begin{aligned}
\rho(\mathbf{p}) &= e^{\theta B}\mathbf{p} e^{-\theta B} = \mathbf{p} && \text{if } \sigma = 1,\\
\rho(\mathbf{p}) &= e^{-\theta B}\mathbf{p} e^{\theta B} = \mathbf{p} && \text{if } \sigma = -1.
\end{aligned}$$

This shows that the sign of σ does not affect the final result. We recognize the result as an ordinary rotation through angle 2θ in the $\mathbf{e}_1$-$\mathbf{e}_2$ plane. Notice that the rotation angle is twice the angle between the reflections. This simple fact will be essential for understanding the relationship between figures 1 and 2 in section 20.

Sections 10-11 analyze the remaining cases. Those analyses are similar to this one, so those sections will be more concise.

[36] Mnemonic: $\mathbf{p}$ stands for *perpendicular*.

10 Isometries from reflections: case 1

In this case, the plane contains exactly one self-orthogonal direction $\mathbf{h}$. Let $\mathbf{e}_1$ be any vector in the plane that is orthogonal to $\mathbf{h}$ and for which $\mathbf{e}_1^2 = \pm 1$. Consider a sequence of two reflections along any two non-self-orthogonal directions in this plane. Without loss of generality, we can take one of the two directions to be $\mathbf{e}_1$, and we can take the other one to be

$$\mathbf{r} = \mathbf{e}_1 + \beta\mathbf{h}.$$

The propertis $\mathbf{h}^2 = 0$ and $\mathbf{e}_1\mathbf{h} = -\mathbf{h}\mathbf{e}_1$ ensure that $\mathbf{r}^2 = \mathbf{e}_1^2$ for every value of β.

Let $\rho(\mathbf{v})$ denote the result of reflecting $\mathbf{v}$ first along $\mathbf{r}$ and then along $\mathbf{e}_1$. With our choices of $\mathbf{e}_k$ and $\mathbf{r}$, equation (4) implies

$$\rho(\mathbf{v}) = R\mathbf{v}R^{-1}$$

with $R \equiv \mathbf{e}_1\mathbf{r}$ and $R^{-1} = \mathbf{r}\mathbf{e}_1$, as before. Use the abbreviations $\sigma \equiv \mathbf{e}_1^2$ and $B \equiv \mathbf{e}_1\mathbf{h}$ like before, but now $B^2 = 0$. This gives

$$R = \sigma + \beta B \qquad\qquad R^{-1} = \sigma - \beta B.$$

Every vector $\mathbf{v}$ may be written $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{h} + z\mathbf{p}$, where x, y, z are real numbers and $\mathbf{p}$ is orthogonal to both $\mathbf{e}_1$ and $\mathbf{h}$. The effects on the three terms are:

- $\mathbf{e}_1$ anticommutes with B, so

$$\rho(\mathbf{e}_1) = R\mathbf{e}_1R^{-1} = R^2\mathbf{e}_1 = (1 - \beta^2 + 2\beta B)\mathbf{e}_1 = (1 - \beta^2)\mathbf{e}_1 - 2\beta\mathbf{h}.$$

- $B\mathbf{h} = \mathbf{h}B = 0$, so

$$\rho(\mathbf{h}) = R\mathbf{h}R^{-1} = \mathbf{h}.$$

- $\mathbf{p}$ commutes with B, so

$$\rho(\mathbf{p}) = R\mathbf{p}R^{-1} = \mathbf{p}.$$

This is called a **null rotation** in the $\mathbf{e}_1$-$\mathbf{h}$ plane.

11 Isometries from reflections: case 2

In this case, the plane contains both positive-norm directions and negative-norm directions. Within this plane, choose two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ that are orthogonal to each other and normalized so that $\mathbf{e}_1^2 = -\mathbf{e}_2^2$. Consider a sequence of two reflections along any two directions in this plane that are both have the same norm. Without loss of generality, we can take one of the two directions to be $\mathbf{e}_1$, and we can take the other one to be[37]

$$\mathbf{r} = \mathbf{e}_1 \cosh\theta + \mathbf{e}_2 \sinh\theta.$$

The identity $\cosh^2\theta - \sinh^2\theta = 1$ implies $\mathbf{r}^2 = \mathbf{e}_1^2$.

Let $\rho(\mathbf{v})$ denote the result of reflecting $\mathbf{v}$ first along $\mathbf{r}$ and then along $\mathbf{e}_1$. With our choices of $\mathbf{e}_k$ and $\mathbf{r}$, equation (4) implies

$$\rho(\mathbf{v}) = R\mathbf{v}R^{-1}$$

with $R \equiv \mathbf{e}_1\mathbf{r}$ and $R^{-1} = \mathbf{r}\mathbf{e}_1$, as before. Use the abbreviations $\sigma \equiv \mathbf{e}_1^2$ and $B \equiv \mathbf{e}_1\mathbf{e}_2$ like in section 9, but now $B^2 = 1$. This gives

$$R \equiv \mathbf{e}_1\mathbf{r} = \sigma\cosh\theta + B\sinh\theta = \sigma e^{\sigma\theta B}.$$

Every vector $\mathbf{v}$ may be written $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{p}$, where x, y, z are real numbers and $\mathbf{p}$ is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. The effects on these three terms are:

- $\mathbf{e}_1$ and $\mathbf{e}_2$ both anticommute with B, so

$$\begin{aligned}\rho(\mathbf{e}_1) &= R\mathbf{e}_1R^{-1} = R^2\mathbf{e}_1 = \mathbf{e}_1\cosh(2\theta) - \mathbf{e}_2\sinh(2\theta)\\ \rho(\mathbf{e}_2) &= R\mathbf{e}_2R^{-1} = R^2\mathbf{e}_2 = \mathbf{e}_2\cosh(2\theta) + \mathbf{e}_1\sinh(2\theta)\end{aligned}$$

- $\mathbf{p}$ commutes with B, so

$$\rho(\mathbf{p}) = R\mathbf{p}R^{-1} = \mathbf{p}.$$

We recognize the result as a boost with rapidity 2θ in the $\mathbf{e}_1$-$\mathbf{e}_2$ plane.

[37] The functions $\cosh\theta$ and $\sinh\theta$ are defined in article 77597.

12 The spin groups

This section defines two Lie groups: the group $\text{Spin}(p,q)$, and a subgroup of $\text{Spin}(p,q)$ that will be denoted $\text{Spin}^+(p,q)$. Both will be called **spin groups**.[38] The group $\text{Spin}^+(p,q)$ is the focus of this article.[39]

Every element of the isometry group $O(p,q)$ is a composition of reflections. If the reflections are along the directions $\mathbf{r}_1, ..., \mathbf{r}_n$ with $\mathbf{r}_k^2 = \pm 1$, then the effect of their composition on a vector $\mathbf{v}$ is[40]

$$\mathbf{v} \to \begin{cases} R\mathbf{v}\tilde{R} & \text{if } n \text{ is even,} \\ -R\mathbf{v}\tilde{R} & \text{if } n \text{ is odd,} \end{cases} \tag{6}$$

with

$$R = \mathbf{r}_n \cdots \mathbf{r}_2\mathbf{r}_1 \qquad\qquad \tilde{R} = \mathbf{r}_1\mathbf{r}_2 \cdots \mathbf{r}_n. \tag{7}$$

The spin groups may be defined like this:

- $\text{Spin}(p,q)$ is the group of elements of the Clifford algebra that may be written in the form $R \equiv \mathbf{r}_n \cdots \mathbf{r}_2\mathbf{r}_1$ with $\mathbf{r}_k^2 = \pm 1$ and with an even number n of vectors in the product.[41] If $R \in \text{Spin}(p,q)$, then the transformation (6) belongs to $SO(p,q)$.

- $\text{Spin}^+(p,q)$ is the subgroup of $\text{Spin}(p,q)$ whose members R satisfy $\tilde{R}R = 1$. If $R \in \text{Spin}^+(p,q)$, then the transformation (6) belongs to $SO^+(p,q)$.

The definition of any group G includes a rule for composing two elements of G to get another element of G. In the spin groups, that rule is just the Clifford product: the composition of R and R' is RR', the Clifford product of R and R'.

[38] The name *spin group*, with no other qualifiers, usually refers to $\text{Spin}(p,q)$. The other group $\text{Spin}^+(p,q)$ has been called the **reduced spin group** (Harvey (1990), chapter 10, page 200).

[39] This is also the focus in Varadarajan (2004) (section 5.4, page 193, and also Varadarajan (2003), section 5.3), which uses the notation Spin for the group that I'm calling Spin^+. If $p = 0$ or $q = 0$, then $\text{Spin}(p,q) = \text{Spin}^+(p,q)$, so the distinction doesn't matter in that case.

[40] Section 7

[41] Figueroa-O'Farrill (2015), page 7

13 Another way to define the spin groups

Section 12 defined the spin groups in a relatively direct way. Much of the literature[42] uses an approach that is less direct. This section offers some guidance about how to relate the two approaches to each other.

The less-direct approach starts by defining the *Clifford group* $\Gamma(p,q)$. Then it defines $\mathrm{Pin}(p,q)$ as a special subgroup of $\Gamma(p,q)$,[43,44] and then it defines the spin groups as special subgroups of $\mathrm{Pin}(p,q)$. After unrolling these layers, the resulting definition says that an element R of the Clifford algebra belongs to $\mathrm{Spin}(p,q)$ if and only if it satisfies all of these conditions:

- R has a multiplicative inverse R^{-1}, and $R\mathbf{v}R^{-1}$ is a vector whenever $\mathbf{v}$ is a vector.

- R is an *even* element of the Clifford algebra, which means that it is a sum of products of vectors with an even numbers of vectors in each product.

- $R^{-1} = \pm\tilde{R}$, with $\tilde{R}$ defined to be the result of reversing the order of the vectors in each product in R.[45,46]

The definition used in section 12 clearly satisfies these conditions. The converse – that these conditions imply the definition in section 12 – is not so obvious. The key is to show that any R satisfying the first condition may be written as a single product of vectors, as required by equation (7). To show this,[47] use equation (2) to get $g(R\mathbf{a}R^{-1},\, R\mathbf{b}R^{-1}) = g(\mathbf{a},\mathbf{b})$, and then use the fact that every isometry is a composition of reflections.[48]

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[42]Examples include Lounesto (2001), section 17.2; and Benn and Tucker (1989), page 46

[43]Harvey (1990) defines the Pin and Spin groups directly (definitions 10.1 and 10.3), like in section 12.

[44]Borcherds (2012) contrasts two conflicting definitions of the Pin group (text below definition 195 on page 69).

[45]A map $R \to R^T$ is called an *anti-automorphism* if $(AB)^T = B^T A^T$ and $(A+B)^T = A^T + B^T$ and $(sA)^T = sA^T$ for all scalars s. The map $R \to \tilde{R}$ is defined to be the unique anti-automorphism satisfying $\tilde{\mathbf{v}} = \mathbf{v}$ for all vectors $\mathbf{v}$.

[46]$R\tilde{R}$ has been called the **spinor norm** of R (text above proposition 197 in section 14.1 in Borcherds (2012), which uses the notation R^T instead of $\tilde{R}$).

[47]Benn and Tucker (1989), page 44

[48]Section 4

14 Signature symmetry

For most pairs (p, q), the Clifford algebras with signatures (p, q) and (q, p) are different algebras.[49] Example: the Clifford algebra with signature $(0, 2)$ has three mutually anticommuting elements whose norms are all equal to -1, but the Clifford algebra with signature $(2, 0)$ does not.

In contrast, for every pair (p, q), the two spin groups $\text{Spin}(p, q)$ and $\text{Spin}(q, p)$ are always isomorphic to each other.[50] To prove this, let C be the Clifford algebra with signature (p, q), and let $\bar{C}$ be the Clifford algebra with signature (q, p). Let $\mathbf{e}_k$ be mutually orthogonal vectors in C satisfying

$$\mathbf{e}_k^2 = \begin{cases} \;\;1 & \text{if } k \in \{1, ..., p\}, \\ -1 & \text{if } k \in \{p+1, ..., p+q\}, \end{cases}$$

and let $\bar{\mathbf{e}}_k$ be mutually orthogonal vectors in $\bar{C}$ satisfying

$$\bar{\mathbf{e}}_k^2 = \begin{cases} -1 & \text{if } k \in \{1, ..., p\}, \\ \;\;1 & \text{if } k \in \{p+1, ..., p+q\}. \end{cases}$$

These are algebras over $\mathbb{R}$, the field of real numbers, but we can formally identify $\bar{\mathbf{e}}_k = i\mathbf{e}_k$ where i commutes with everything and satisfies $i^2 = -1$. The factors of i cancel in the spin group, because the spin group involves only products of even numbers of vectors. As a result, this formal identification has only one effect on the spin group, namely to replace $R \to -R$ whenever R is a product of an (even) number of vectors that is not a multiple of 4. This replacement doesn't affect the multiplication table, so $\text{Spin}(p, q)$ and $\text{Spin}(q, p)$ are isomorphic to each other.

The fact that $\text{Spin}(p, q)$ and $\text{Spin}(q, p)$ are isomorphic to each other implies that $SO(p, q)$ and $SO(q, p)$ are isomorphic to each other, because $SO(p, q)$ may be constructed (as an abstract group) from $\text{Spin}(p, q)$ by ignoring the difference between R and $-R$ for all $R \in \text{Spin}(p, q)$.

[49] They are different even as plain algebras (article 03910), not just as Clifford algebras.

[50] Porteus (1995), chapter 16, page 146

15 Spin$^+(p,q)$ is a double cover of $SO^+(p,q)$

The group Spin(p,q) is a[51] double cover of $SO(p,q)$, and the group Spin$^+(p,q)$ is a double cover of $SO^+(p,q)$. This section focuses on the second assertion.[52,53]

The group Spin$^+(p,q)$ clearly covers $SO^+(p,q)$ at least twice, because changing the sign of a vector in the product (7) doesn't change anything in (6). In fact, Spin$^+(p,q)$ coveres $SO^+(p,q)$ exactly twice:[54] it is a double cover of $SO^+(p,q)$.

To prove this, let R be a product of an even number of vectors $\mathbf{r}$ with $\mathbf{r}^2 = 1$ and an even number of vectors $\mathbf{r}$ with $\mathbf{r}^2 = -1$ so that $R \in \text{Spin}^+(p,q)$. Suppose that R satisfies

$$R\mathbf{v}\tilde{R} = \mathbf{v} \qquad\qquad \text{for all vectors } \mathbf{v}. \tag{8}$$

Multiply both sides on the right by R to get $R\mathbf{v} = \mathbf{v}R$, which implies that any R satisfying (8) commutes with everything in the Clifford algebra (because vectors generate the whole Clifford algebra). This shows that R must be a real number. R is a product of vectors whose norms are ± 1, so $R^2 = \pm 1$. Combined with the fact that R is a real number, this implies $R = \pm 1$. This shows that exactly two elements of Spin$^+(p,q)$ correspond to the identity element of $SO^+(p,q)$, so Spin$^+(p,q)$ covers $SO^+(p,q)$ exactly twice.

[51] The indefinite article "a" is used here because a given Lie group may have more than one double cover (`https://math.stackexchange.com/questions/4609709/`). Every Lie group G has at least one double cover, namely $\mathbb{Z}_2 \times G$, where $\mathbb{Z}_2$ is the two-element group, but that double cover is not connected.

[52] Both assertions are acknowledged in Lounesto (2001), section 17.2; and in Porteus (1995), proposition 16.14.

[53] Footnote 39 in section 12

[54] Lounesto (2001), section 17.2; and Varadarajan (2004), section 5.4, page 193

16 Is Spin$^+(p,q)$ connected?

The group $\text{Spin}^+(1,1)$ is not connected.[55] To confirm this, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be mutually orthogonal vectors with $\mathbf{e}_1^2 = 1$ and $\mathbf{e}_2^2 = -1$. Every $R \in \text{Spin}^+(1,1)$ has the form $R = x + y\mathbf{e}_1\mathbf{e}_2$ with real coefficients x, y such that $R\tilde{R} = 1$. This implies $x^2 - y^2 = 1$, which describes a pair of hyperbolas in the x-y plane: one with $x > 0$, and one with $x < 0$. These are not connected to each other because x cannot be zero, so $\text{Spin}^+(1,1)$ is not connected.

If p or q is ≥ 2, then the group $\text{Spin}^+(p,q)$ is connected.[55] To deduce this, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be mutually orthogonal vectors with either $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$ or $\mathbf{e}_1^2 = \mathbf{e}_2^2 = -1$. Then $(\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta)\mathbf{e}_1$ is equal to ± 1 when $\theta \in \{0, \pi\}$, and the cases $\theta = 0$ and $\theta = \pi$ have opposite signs. This shows that -1, regarded as an element of $\text{Spin}^+(p,q)$, can be continuously morphed to 1. If $p \geq 1$, then let $\mathbf{r}_+$ be a vector with $g(\mathbf{r}_+, \mathbf{r}_+) = 1$. If $q \geq 1$, then let $\mathbf{r}_-$ be a vector with $g(\mathbf{r}_-, \mathbf{r}_-) = -1$. If p and q are both ≥ 1, then choose these two vectors to be orthogonal to each other so that

$$\mathbf{r}_+\mathbf{r}_- = -\mathbf{r}_-\mathbf{r}_+. \tag{9}$$

Starting with (7), each factor $\mathbf{r}_k$ with $g(\mathbf{r}_k, \mathbf{r}_k) > 0$ may be continuously morphed to either $\mathbf{r}_+$ or $-\mathbf{r}_+$, and each factor $\mathbf{r}_k$ with $g(\mathbf{r}_k, \mathbf{r}_k) < 0$ may be continuously morphed to either $\mathbf{r}_-$ or $-\mathbf{r}_-$. After those morphs, we can use equation (9) to rearrange the factors so that all factors of $\mathbf{r}_+$ are to the left of all factors of $\mathbf{r}_-$. The product involves an even number of each of these two vectors, so using $\mathbf{r}_\pm^2 = \pm 1$ reduces the whole product to ± 1. We already determined that -1 can be continuously morphed to 1, so this shows that the group $\text{Spin}^+(p,q)$ is connected.

This implies that $SO^+(p,q)$ is connected, too, including $SO^+(1,1)$, because $SO^+(p,q)$ is obtained (as an abstract group) from $\text{Spin}^+(p,q)$ by ignoring the difference between R and $-R$.

Unlike $\text{Spin}^+(p,q)$, the group $\text{Spin}(p,q)$ has two connected components when p and q are both nonzero.[56]

[55] Lounesto (2001), section 17.2; Varadarajan (2004), section 5.4, page 193; and Varadarajan (2003), section 5.3

[56] Figueroa-O'Farrill (2010), section 3.2, page 22

17 Is Spin$^+(p,q)$ simply connected?

If p or q is ≥ 2, then the group Spin$^+(p,q)$ is connected, but it may or may not be *simply connected*, depending on the signature (p,q). Here's a summary of the results:[57,58]

- If either p or q is ≤ 1 and the other one is ≥ 3, then Spin$^+(p,q)$ is simply connected.
- If either $p = 2$ or $q = 2$, then Spin$^+(p,q)$ is not simply connected.
- If $p \geq 3$ and $q \geq 3$, then Spin$^+(p,q)$ is not simply connected.

Some examples are listed here:

signature	simply con'd?
$(2,0)$	no
$(2,1)$	no
$(2,2)$	no
$(3,0)$	yes
$(3,1)$	yes
$(3,2)$	no
$(3,3)$	no

signature	simply con'd?
$(4,0)$	yes
$(4,1)$	yes
$(4,2)$	no
$(4,3)$	no
$(4,4)$	no

The results are the same if p and q are exchanged.[59] In the most physically relevant cases (euclidean signature with ≥ 3 dimensions, and lorentzian signature with ≥ 4 dimensions), the group Spin$^+(p,q)$ is simply connected even though $SO^+(p,q)$ is not.

The remaining sections explain how these results – and the results about $SO^+(p,q)$ that were summarized in section 5 – can all be anticipated intuitively, using the formulation that was described in sections 7 and (12).

[57] Sati and Shim (2015), section 3.3

[58] Spin$^+(1,1)$ is not connected, but the connected component that contains the identity element is simply connected.

[59] Section 14

18 Intuition for $(p, q) = (2, 0)$

The groups $G \equiv SO^+(2,0)$ and $\Sigma \equiv \text{Spin}^+(2,0)$ are not simply connected.[60]

To establish this, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be mutually orthogonal vectors normalized so that $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$. Every element of Σ may be written as

$$R = \mathbf{e}_1\mathbf{r}$$

with

$$\mathbf{r} \equiv \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta \tag{10}$$

for some θ. This says that Σ is topologically equivalent to a circle: every element of Σ corresponds to a point on the unit circle, specified by the angle θ.

The unit circle is the prototypical example of a manifold that is not simply connected. One example of a closed path on the unit circle is the one that goes from $\theta = 0$ to $\theta = 2\pi$, which wraps once around the unit circle. Another example is the path that stays at $\theta = 0$, which is a single point on the unit circle. These two paths are clearly not homotopic to each other: the first one cannot be continuously morphed to the second one without breaking it somewhere, because every point on the path must remain on the unit circle during the process. This shows that Σ is not simply connected.

To show that G is also not simply connected, use the fact that two points that differ by $\delta\theta = \pi$ in Σ are mapped to the same point in G (because the corresponding vectors (10) are equal to each other except for the overall sign), so G is still topologically a circle – still not simply connected.

[60] The abbreviations G and Σ are specific to this section. Subsequent sections will recycle the symbols G and Σ for other signatures.

19 Intuition for $(p,q)=(3,0)$

The group $G\equiv SO^+(3,0)$ is not simply connected, but $\Sigma\equiv \mathrm{Spin}^+(3,0)$ is.[61] This section explains how to anticipate those results intuitively.

Every transformation in G may be written as in equation (6) with R equal to the product of two unit vectors. The group Σ consists of all such two-vector products. Explicitly, every element of Σ has the form $R=\mathbf{r}'\mathbf{r}$ with unit vectors $\mathbf{r}'$ and $\mathbf{r}$. Suppose that $\mathbf{r}'(\lambda)$ and $\mathbf{r}(\lambda)$ both trace out closed paths on the unit sphere as λ goes from 0 to 1. Then $R(\lambda)\equiv \mathbf{r}'(\lambda)\mathbf{r}(\lambda)$ traces out a closed path in Σ as λ goes from 0 to 1. From here, the fact that every closed path on the unit sphere is homotopic to a point (which should be intuitively clear) immediately implies that every closed path in Σ is also homotopic to a point. In other words, Σ is simply connected.

What about G? A path that is closed in Σ is also closed in G, so if $\mathbf{r}'$ and $\mathbf{r}$ both trace out closed paths on the unit sphere, then the corresponding path in G (defined by equation (6) with $R=\mathbf{r}'\mathbf{r}$) is also homotopic to a point in G, just like it is in Σ. However, the group G hosts other closed paths that cannot be described this way, and those other closed paths are *not* homotopic to a point.

For one example, consider any path in Σ for which R goes from 1 to -1. This path is not closed in Σ, but it is closed in G. If we continuously morph the path, then it remains closed in G only if it always includes two points whose Rs are each other's negatives. In other words, the path must always have two endpoints that are on opposite sides of the manifold Σ. With that restriction, the path cannot be continuously morphed to a point in Σ, which implies that it can't be continuously morphed to a point in G, either. This doesn't revoke Σ's simply-connected status, because this path isn't closed in Σ. It it closed in G, though, so the fact that it can't be continuously morphed to a point means that G is not simply connected.

[61] Remember footnote 60 in section 18.

20 Visualizing path-morphing as movie-morphing

Use the abbreviations $G \equiv SO^{+}(3,0)$ and $\Sigma \equiv \text{Spin}^{+}(3,0)$ again, like in section 19.

A path in G is a continuous sequence of rotations in 3d euclidean space, so the path may be depicted as a movie – a continuous sequence of object-orientations obtained by applying each rotation to the object's initial orientation. The act of continuously morphing the path may then be depicted as the act of continuously morphing the original movie to a different movie.

For an example, let $\mathbf{e}_k$ with $k \in \{1,2,3\}$ be a set of mutually orthogonal unit vectors, and consider rotations of the form (6) with

$$
\begin{aligned}
R &= \mathbf{r}(0,\phi)\mathbf{r}(\theta,\phi) \\
\mathbf{r}(\theta,\phi) &\equiv \mathbf{e}_1\cos\phi + (\mathbf{e}_2\cos\theta + \mathbf{e}_3\sin\theta)\sin\phi.
\end{aligned}
\tag{11}
$$

For any given value of ϕ, the vector $\mathbf{r}(\theta,\phi)$ traces out a closed path on the unit sphere as θ goes from 0 to 2π, so R also traces out a closed path in Σ. If $\phi = 0$, then this closed path is a single point, corresponding to just the identity rotation in 3d euclidean space. This movie shows a motionless object. If $\phi = \pi/2$, then the closed path corresponds to a continuous sequence of rotations in 3d euclidean space, all in the $\mathbf{e}_2$-$\mathbf{e}_3$ plane – all about the $\mathbf{e}_1$ axis in colloquial terms – so this movie shows an object rotating about a single axis from 0 to 4π.[62] As ϕ morphs from 0 to $\pi/2$, the first movie (a motionless object) morphs into the second movie (an object executing a 4π roll about a single axis).

Figure 1 illustrates the reflection-vectors for a series of these closed paths, and figure 2 illustrates the corresponding series of rotating-object movies. This shows how a motionless-object movie can be continuously morphed into a 4π-roll movie, respecting the constraint that the final orientation equal to the initial orientation in each movie (so the corresponding paths in G are closed). In contrast, a motionless-object movie cannot be continuously morphed into a 2π-roll movie without violating that constraint: in G, the 2π-roll path is not homotopic to a point.

[62] Recall section 9: as the angle between the two reflections goes from 0 to 2π, the resulting rotation angle goes from 0 to 4π.

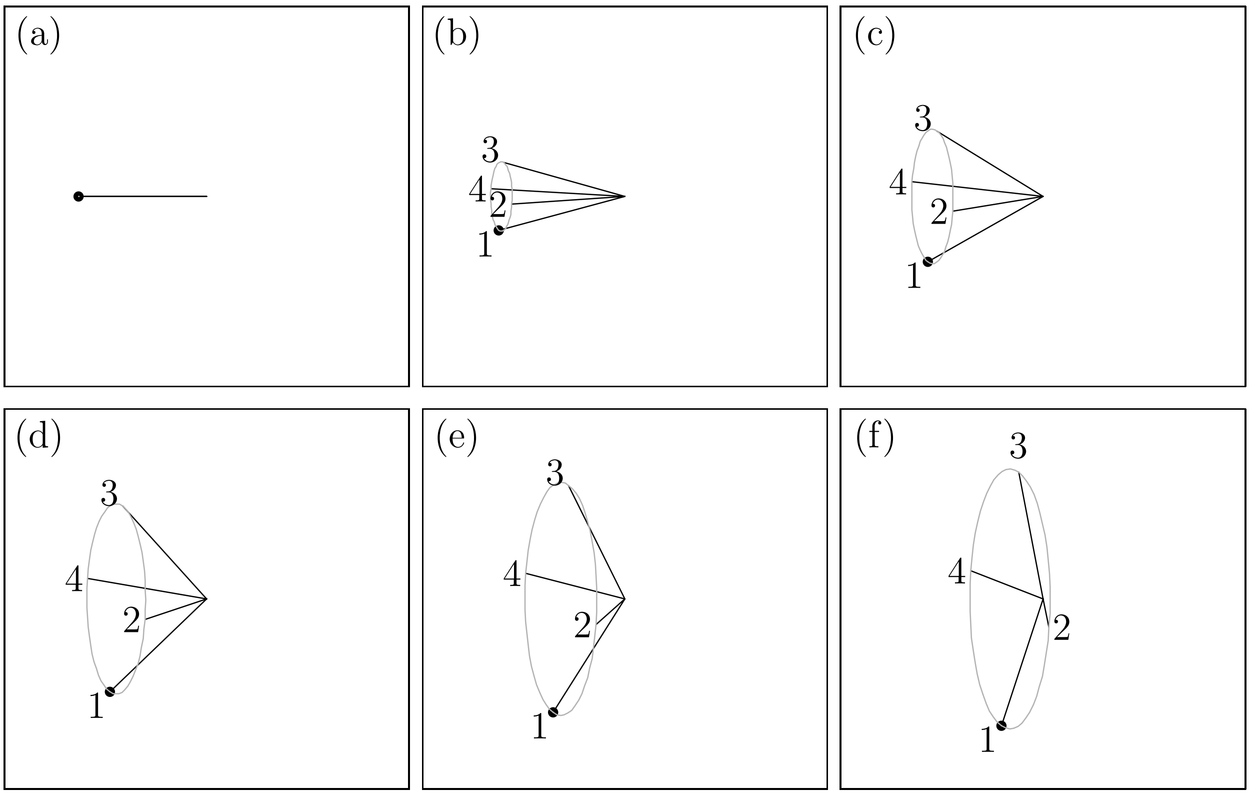

Figure 1 – Six of the closed paths in this series that was described in section 20, namely $\phi = n\pi/12$ with $n \in \{0, 1, 2, 3, 4, 5\}$ (pictures a,b,c,d,e,f). In these pictures, $\mathbf{e}_1$ points to the left, $\mathbf{e}_2$ points down, and $\mathbf{e}_3$ points out of the page. The gray circle traces out the continuous closed path, and the four spokes depict four representative unit vectors $\mathbf{r}(\phi, \theta)$ on that path, namely those with $\theta \in \{0,\ \pi/2,\ \pi,\ 3\pi/2\}$. The plane spanned by the two reflection-vectors $\mathbf{r}(0, \phi)$ and $\mathbf{r}(\theta, \phi)$ is the plane of the rotation defined by using equation (11) in equation (6). The case $\theta = 0$ gives the identity rotation. Figure 2 shows the corresponding series of rotating-object movies.

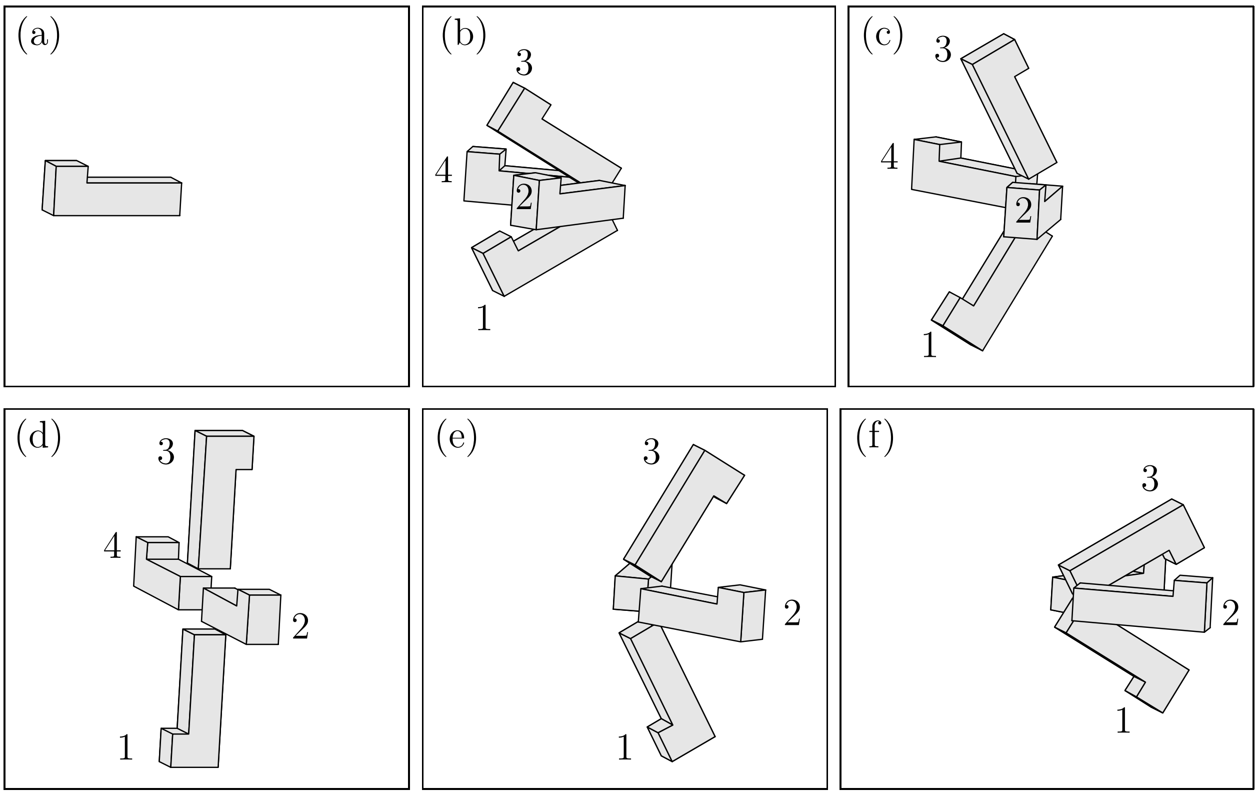

Figure 2 – The movies corresponding to the closed paths in figure 1. In each of the six movies, the reference object starts with the orientation labelled 1 and passes through a continuous series of orientations, represented in these pictures by the snapshots labelled 1,2,3,4, finally returning to the initial orientation 1. In each movie, the orientation labelled k (with $k \in \{1, 2, 3, 4\}$) is obtained from the reference orientation (labelled 1) by first reflecting along the vector labelled k in figure 1 and then reflecting along the vector labelled 1 in figure 1, as prescribed in equations (6) and (11). When comparing this figure to figure 1, remember the angle-doubling phenomenon that was highlighted in section 9.

21 Intuition for $(p,q) = (p,0)$ with $p \geq 3$

When $p \geq 3$, the group $G \equiv SO^+(p,0)$ is not simply connected, but its double cover $\Sigma \equiv \text{Spin}^+(p,0)$ is simply connected.[63]

Section 19 already explained this for $p = 3$. The intuition for $p > 3$ is the same, except that now the *unit sphere* is S^{p-1}, the $(p-1)$-dimensional "surface" of a p-dimensional ball. As long as $p \geq 3$, any closed path on S^{p-1} can be continuously morphed to a point, so Σ is simply connected.

The connected isometry group G is still not simply connected: a path in Σ that starts at $R = 1$ and ends at $R = -1$ is closed in G but not in Σ, so it cannot be continuously morphed to a point without breaking it in G.

[63] Remember footnote 60 in section 18.

22 Intuition for $(p, q) = (1, 1)$

In this case, the connected isometry group $G \equiv SO^+(1,1)$ is simply connected, but its double cover $\Sigma \equiv \mathrm{Spin}^+(1,1)$ is not even connected.[64] Section 16 already showed that Σ is not connected.

To show that G is simply connected, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be mutually orthogonal vectors with $\mathbf{e}_1^2 = 1$ and $\mathbf{e}_2^2 = -1$. Every $R \in \Sigma$ has the form $R = x + y\mathbf{e}_1\mathbf{e}_2$ with real coefficients satisfying $x^2 - y^2 = 1$ so that $R\tilde{R} = 1$. The components of R with $x > 0$ and $x < 0$ are not connected to each other, but the sign of x makes no difference in G (equation (6)), so for G we only need to consider the case $x > 0$. The pair of conditions $x^2 - y^2 = 1$ and $x > 0$ describes a single connected hyperbola in the x-y plane. The hyperbola is topologically equivalent to an infinite line, which is the prototypical example of a simply connected manifold. This shows that G is simply connected.

[64] Remember footnote 60 in section 18.

23 Intuition for $(p, q) = (2, 1)$

The groups $G \equiv SO^+(2,1)$ and $\Sigma \equiv \text{Spin}^+(2,1)$ are not simply connected.

This can be understood intuitively using a generalization of the intuition that was used in section 18. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be mutually orthogonal vectors normalized so that $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$ and $\mathbf{e}_3^2 = -1$. Consider a path in R of the form $R = \mathbf{e}_1\mathbf{r}(\lambda)$ with $\mathbf{r}^2(\lambda) = 1$. For all λ, the vectors $\mathbf{r}(\lambda)$ are restricted topologically to the two-dimensional surface of a hyperboloid, because they must have the form

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

with $x^2 + y^2 - z^2 = 1$. This hyperboloid is topologically equivalent to (the surface of) an infinitely long cylinder. A closed path that wraps around the cylinder n times cannot be continuously morphed to one that wraps around the cylinder a different number of times. More generally, the first factor $\mathbf{e}_1$ can be morphed into a closed path, but that can't undo the wrapping of $\mathbf{r}(\lambda)$, so Σ admits infinitely many different classes of closed paths that cannot be continuously morphed into each other. This shows that Σ is not simply connected.

The map from Σ to G is two-to-one, defined by ignoring the difference between R and $-R$. That's not enough to make G simply connected, for essentially the same reason as in section 18.

24 Intuition for $(p, q) = (p, 1)$ with $p \geq 3$

When $p \geq 3$, the group $G \equiv SO^+(p, 1)$ is not simply connected, but its double cover $\Sigma \equiv \text{Spin}^+(p, 1)$ is simply connected.

The intuition is essentially the same as in section 21, because, topologically, the only effect of the extra dimension (the "1" in $(p, 1)$) is to replace the sphere S^{p-1} with the cylinder $S^{p-1} \times \mathbb{R}$. When $p \geq 3$, this is simply connected just like S^{p-1} is. This can be used to show that Σ is simply connected, just like in section 21.

In contrast, G is not simply connected, again for the same reason as in section 21: G has closed paths that are not closed in Σ. This occurs when the path's endpoints in Σ are on opposite sides of the sphere S^{p-1}, and the path cannot be continuously morphed to a point without violating that condition – without breaking the closed path in G. This shows that G is not simply connected.

25 Intuition for (p, q) with p and q both ≥ 2

When p and q are both ≥ 2, the groups $G \equiv SO^+(p,q)$ and $\Sigma \equiv \text{Spin}^+(p,q)$ are not simply connected. For G, the intuition is just like in section 24. For Σ, the intuition involves a new phenomenon that does not occur when either p or q is less than 2. This section describes that new phenomenon.

Let $\mathbf{e}_1, ..., \mathbf{e}_p$ be mutually orthogonal vectors with $\mathbf{e}_k^2 = 1$, let $\mathbf{f}_1, ..., \mathbf{f}_q$ be mutually orthogonal vectors with $\mathbf{f}_k^2 = -1$, and consider the product

$$R(\theta) = \mathbf{e}_1\mathbf{r}(\theta)\mathbf{f}_1\mathbf{s}(\theta) \tag{12}$$

with

$$\begin{aligned}\mathbf{r}(\theta) &= \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta \\ \mathbf{s}(\theta) &= \mathbf{f}_1 \cos\theta + \mathbf{f}_2 \sin\theta.\end{aligned} \tag{13}$$

The quantity $R(\theta)$ belongs to $\Sigma \equiv \text{Spin}^+(p,q)$, because each of the first two vectors in the product (12) has norm 1, and each of the last two has norm -1. Equation (12) implies $R(\pi) = R(0)$, so $R(\theta)$ traces out a closed path in Σ as θ goes from 0 to π. The remaining task is to show that this closed path cannot be continuously morphed to a point, which implies that Σ is not simply connected.

Let V_+ and V_- be the sets of vectors whose norms are positive and negative, respectively. A vector $\mathbf{v}$ in either of these sets may be written as $\mathbf{v} = \mathbf{x} + \mathbf{y}$, where

- $\mathbf{x}$ is a linear combination of the $\mathbf{e}_k$s,
- $\mathbf{y}$ is a linear combination of the $\mathbf{f}_k$s.

The vectors $\mathbf{x}$ and $\mathbf{y}$ anticommute with each other, so $\mathbf{v}$ belongs to V_+ if $|\mathbf{x}^2| - |\mathbf{y}^2| > 0$, and it belongs to V_- if $|\mathbf{x}^2| - |\mathbf{y}^2| < 0$. To describe the topology of V_+ and V_-, let M^n denote $\mathbb{R}^n$ with an n-dimensional ball deleted. Then V_+ is topologically equivalent to $M^p \times \mathbb{R}^q$, where the factor $\mathbb{R}^q$ is parameterized by the vector $\mathbf{y}$, and the factor M^p is parameterized by $\mathbf{x}$, which is a linear combination of the $\mathbf{e}_k$s subject to the constraint $|\mathbf{x}^2| > |\mathbf{y}^2|$. Similarly, V_- is topologically equivalent to $\mathbb{R}^p \times M^q$.

Topologically, when the quantity $R(\theta)$ defined above goes from $\theta = 0$ to $\theta = \pi$, the factor $\mathbf{r}(\theta)$ goes from one side of the deleted p-dimensional ball in $M^p \times \mathbb{R}^q$ to the opposite side of that deleted ball, and the factor $\mathbf{s}(\theta)$ goes from one side of the deleted q-dimensional ball in $\mathbb{R}^p \times M^q$ to the opposite of that deleted ball. The corresponding closed path in Σ cannot be continuously morphed to a point without violating those conditions, which would correspond to breaking the path in Σ. This shows that Σ is not simply connected.

This phenomenon requires that p and q are both ≥ 2. When p or q is ≤ 1, products R of the form described at the beginning of this section don't exist.

26 Other topological properties

This article focused on just two basic topological properties: whether the given Lie group is connected, and whether it is also simply connected. We can get more information about the topology of a Lie group M (using the letter M for manifold) by determining the fundamental group $\pi_1(M)$, as illustrated in section 5. Saying that the manifold M is simply connected means that the fundamental group $\pi_1(M)$ is trivial, but when $\pi_1(M)$ is not trivial, it can be non-trivial in different ways, so this gives us more information about the topology of manifolds that are not simply connected.

We can get even more information about the topology of M by considering the higher homotopy groups $\pi_2(M)$, $\pi_3(M)$, and so on. Roughly,[65] $\pi_k(M)$ is defined by considering equivalence classes of embeddings of the sphere S^k in M, just like the fundamental group $\pi_1(M)$ is defined by considering equivalence classes of embeddings of the circle S^1 in M – which are closed paths in M. A manifold M may have some non-trivial higher homotopy groups $\pi_k(M)$ even if its fundamental group $\pi_1(M)$ is trivial. As an example, consider $\Sigma \equiv \text{Spin}^+(p, 0)$ with $p \geq 3$. Then Σ simply connected ($\pi_1(\Sigma)$ is trivial), and the second homotopy group $\pi_2(\Sigma)$ also turns out to be trivial,[66] but the third homotopy group $\pi_3(\Sigma)$ is nontrivial.[67]

[65] Section 4.5 in Nakahara (1990) gives a precise definition.

[66] $\pi_2(M)$ is trivial for every connected Lie group M (Santos (2018), theorem 3.1, and `https://mathoverflow.net/questions/8957/`).

[67] Borcherds (2012), section 15.5

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