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Lie Groups, Lie Algebras, and Some of Their Applications

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Preface

Only a century has elapsed since 1873, when Marius Sophus Lie began his research on what has evolved into one of the most fruitful and beautiful branches of modern mathematics—the theory of Lie groups. These researches culminated twenty years later with the publication of landmark treatises by S. Lie and F. Engel [1–3] between 1888 and 1893, and by W. Killing [1–4] from 1888 to 1890. Matrices and matrix groups had been introduced by A. Cayley, Sir W. R. Hamilton, and J. J. Sylvester (1850–1859) about twenty years before the researches of Lie and Engel began. At that time mathematicians felt that they had finally invented something of no possible use to natural scientists. However, Lie groups have come to play an increasingly important role in modern physical theories. In fact, Lie groups enter physics primarily through their finite- and infinite-dimensional matrix representations.

Certain natural questions arise. For example, just how does it happen that Lie groups play such a fundamental role in physics? And how are they used?

Lie groups found their way into physics even before the development of the quantum theory. They were useful for the description of pseudo-Riemannian (locally) homogeneous symmetric spaces, being used in particular in geometric theories of gravitation. But Lie groups were virtually forced into physics by the development of the modern quantum theory in 1925–1926. In this theory, physical observables appear through their hermitian matrix representatives, whereas processes producing transformations are described by their unitary or antiunitary matrix representations. Operators that close under commutation belong to a finite-dimensional Lie algebra; transformation processes described by a finite number of continuous parameters belong to a Lie group.

The kinds of applications of Lie group theory in modern physics fall into three distinct stages:

1. As symmetry groups (1929–1960). Symmetry implies degeneracy. The greater the symmetry, the greater the degeneracy. Assume that a Lie group G with Lie algebra $\mathfrak{g}$ commutes with a Hamiltonian $\mathscr{H}$:

$$G\mathscr{H}G^{-1} = \mathscr{H} \Leftrightarrow [\mathscr{H}, \mathfrak{g}] = 0$$

Then by Wigner's theorem the basis vectors spanning a fixed energy eigenspace carry a representation of G. For example, the three-dimensional isotropic harmonic oscillator whose Hamiltonian is

$$\mathscr{H} = \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 + 3/2)$$

where $$[a_i^\dagger, a_j] = -\delta_{ij}$$
$$[a_i, a_j] = [a_j^\dagger, a_i^\dagger] = 0$$

has spherical symmetry. Therefore, $\mathscr{H}$ commutes with the infinitesimal generators L_i of the rotation group $SO(3)$:

$$[\mathscr{H}, L_i] = 0 \qquad L_i \simeq a_j^\dagger a_k - a_k^\dagger a_j \qquad (i, j, k) = (1, 2, 3) \text{ cycl.}$$

The oscillator eigenfunctions therefore carry representations of the rotation group $SO(3)$.

However, the existence of an "accidental" degeneracy in this example gives a larger degeneracy than is demanded by the obvious geometric invariance group $SO(3)$. This suggests that a larger group, containing $SO(3)$ as a subgroup, may be a more useful symmetry group for this Hamiltonian. The group is $U(3)$, with Lie algebra U_{ij}:

$$[\mathscr{H}, U_{ij}] = 0 \qquad U_{ij} = a_i^\dagger a_j$$
$$\mathscr{H} = \hbar\omega \sum (a_i^\dagger a_i + 1/2); \qquad [U_{ij}, U_{rs}] = U_{is}\,\delta_{jr} - U_{rj}\,\delta_{si}$$

In fact, it is useful and even desirable from a calculational standpoint to label the oscillator eigenfunctions with $SU(3)$ representation labels (J. M. Jauch and E. L. Hill [1], J. P. Elliott [1]).

2. As nonsymmetry groups (1960–). Around 1960 physicists were gradually forced to realize that groups that do not commute with $\mathscr{H}$ can be even more useful than symmetry groups from a computational viewpoint. As an example, it is possible to find a 16-dimensional nonsymmetry group with generators $a_i^\dagger a_j$, $a_i^\dagger$, a_j, I

$$[\mathscr{H}, a_i^\dagger a_j] = 0$$
$$[\mathscr{H}, a_i^\dagger] = +\hbar\omega a_i^\dagger$$
$$[\mathscr{H}, a_j] = -\hbar\omega a_j$$
$$[\mathscr{H}, I] = 0$$

This nonsymmetry group is contracted from the noncompact group $U(3, 1)$. Using this noncompact algebra, any eigenstate can be obtained from any other by applying a sequence of elements in the Lie algebra. In particular, all excited states can be computed from the ground state, which, in turn, can be computed either by algebraic or by analytic (variational) methods. The hydrogen atom, superfluid and superconductor models, laser systems, and charged particles in external fields are some of the problems amenable to such treatment.

3. ? (1970–). Strictly speaking, the third class of applications is not yet known, although its appearance is probably around the corner. It now seems possible that Lie group theory, together with differential geometry, harmonic analysis, and some devious arguments, might be able to predict some of Nature's dimensionless numbers (α, m_p/m_e, m_μ/m_e, G^2/hc, ...). In retrospect, it seems clear that the application of group theory to physical problems represents the dividing line between kinematics and dynamics. The group theory gives the overall structure of the spectrum; the dynamics serves to define only the scale. We are looking forward to the day when Lie groups can be pushed to give also the dynamics, or scale, of a physical process. In terms of our model harmonic oscillator Hamiltonian, this means that we hope some day to be able to derive the scaling factor $\hbar\omega$ from fundamental group theoretical arguments.

The work presented here has evolved from a course on Lie groups and their physical applications which I taught several times at M.I.T. and at the University of South Florida. The course covered Lie groups and algebras, representation theory, realizations and special functions, and physical applications. Using the theory of Lie groups as a unifying vehicle, many different aspects of many fields of physics can be presented in an extremely economical way. A great number of calculations remain fundamentally unchanged from one field of physics to another; it is only the interpretation of the symbols and the language used which changes. Thus the Jahn-Teller effect and the Nilsson nucleus are but two aspects of the same phenomenology.

During the development of the course, I realized that a relatively small number of physicists have mastered the theory of Lie groups and are able to use it actively as a tool in their researches. These physicists spend their time primarily writing beautiful papers for one another. On the outside looking in are the relatively large number of physicists who would like to learn the material, who appreciate its power and usefulness, but who are hampered by the lack of an adequate text.

In this context, two established books deserve special mention and praise. These books may profitably be consulted by readers interested in alternative treatments of overlapping material. M. Hamermesh's book [1] has done yeoman service for the physics community during the last decade.

Unfortunately, it stops short of a thorough discussion of Lie group theory. S. Helgason's book [1], which has been equally important in the mathematics community, provides an excellent discussion of Lie group theory but is unfortunately beyond the grasp of most working physicists.

The purpose of this book is to bridge the gap between those who do not know Lie group theory and those who do know. In this sense, this work "fits between" the books of Hamermesh and Helgason.

It has been my intention throughout to present the material in such a way that it is accessible to physicists. I have tried to be as rigorous as possible. But when rigor and clarity have clashed, clarity has won out. There are a sufficient number of treatises on Lie groups by and for mathematicians, and the reader interested in complete rigor will have no trouble filling the gaps I have left.

This work has been aimed at the level of the graduate student. Problems of an illustrative nature have been worked out and included throughout the text. For a physicist it is not only desirable to understand the material, but necessary to be able to make calculations. It is hoped that the solved problems will lead more swiftly to this facility. Exercises have been included at the end of each chapter. Many of them are designed to bring on an awareness of how and where the mathematics presented finds its way into physics. Numerous figures—perhaps too many—have been included, in an attempt to foster easier understanding of the arguments presented in the text. This vice dates from many encounters with Professor I. M. Singer, who always managed to make an argument clearer with one or two telling sketches. The references within each chapter (superscript numbers) refer to the Notes and References section at the end of that chapter. The references in the closing section of each chapter refer to entries in the master bibliography at the end of the book.

The structure of this book resembles that of a concerto. The study develops (*allegro*) in Chapters 1 to 4, where the general properties of Lie groups and algebras are discussed. It continues and concludes (more *allegro*) in Chapters 7 to 10, which are principally devoted to the properties of the semisimple Lie groups. Chapters 5 and 6 provide a relief (*moderato*) from the development. In these chapters specific examples are used both to illustrate concepts developed earlier and to presage concepts to be dealt with subsequently.

Chapter 1, which is devoted to fundamental working definitions and notations, has been included to make the book as self-contained as possible. A cursory familiarity with modern algebra will allow the reader to bypass this chapter. I have tried to present here some of the basic concepts of modern mathematics in such a way that they are less mysterious to a student of physics.

Chapter 2 describes examples of Lie groups. In particular, the classical Lie groups are described following, to some extent, the treatment given by F. D. Murnaghan [1, 2]. This is a not altogether satisfying approach, and we return to the problem of enumerating all the real forms of the simple classical Lie algebras in Chapter 6, where a complete and elegant summary is presented.

In Chapter 3 we define, describe, and work with continuous groups and some of their properties. This treatment culminates in a definition of a Lie group, described more thoroughly in Chapter 4. In this chapter we display the relationship between Lie groups and Lie algebras; we also prove the three theorems of Lie. These theorems relate a Lie algebra to a Lie group by the linearization process. The converses to these three theorems—stated but not proved—relate Lie groups to Lie algebras by the inverse process, exponentiation.

Chapters 5 and 6 represent a watershed in our formal discussion of Lie groups and their algebras. Chapter 5, an elaboration of the concepts developed in the preceding chapters, takes the form of applications of the formal machinery to some of the classical groups—chiefly $SU(2)$. We indicate here also how this machinery can be applied to some useful physical problems. In Chapter 6 we describe more thoroughly the simple classical matrix groups and their algebras. The focal point of this chapter is the summary of all the real forms of the simple classical Lie algebras, and the coset spaces related to these real forms.

In Chapter 7 we resume our formal study of Lie groups and their algebras. All the major tools used in the classification theory of Lie algebras are trotted out one by one, dusted off, and applied to this classification problem. At the end of this chapter we present the commutation relations for all the classical complex simple Lie algebras in canonical form, using the concept of a root space diagram.

The canonical commutation relations are presented again at the beginning of Chapter 8 and are used in making a complete classification of all the root space diagrams. The completeness classification of B. L. van der Waerden [3] is used to construct the complete set of roots in any root space. E. B. Dynkin's approach [1], using Coxeter-Dynkin diagrams, then serves to furnish a convincing proof of the completeness of the classification.

Once all the root space diagrams have been classified, there remains only the problem of classifying the real forms which the complex simple algebras can have. This problem is treated in Chapter 9. The approach in itself leads to nothing surprising: all such real forms have already been encountered, using different arguments, in Chapter 6. This approach merely shows the completeness of the list in Chapter 6. The classification of the real forms used here involves a listing of the irreducible Riemannian symmetric spaces, which are cosets of a simple Lie group by a maximal compact subgroup. These spaces are interesting objects in their own right, and are of course intimately related to Lie groups. Moreover, the concepts and methods developed in Chapters 7 and 8 are applicable to the study of the irreducible Riemannian symmetric spaces. Application of these methods leads immediately to a complete listing of all the globally symmetric pseudo-Riemannian symmetric spaces.

The closing chapter is devoted to a study of how Lie groups and their algebras can be altered. We begin by studying the process of contraction, in which nonsemisimple Lie groups can be constructed from semisimple Lie groups by a limiting procedure. Chapter 10 closes with an indication of how the reverse process can take place; this is called group expansion.

Some terms appearing in this work are used in an unusual way. For example, the term "basis" is usually applied to an element in a linear vector space (basis vector); however, since I apply this term to analogous elements in a group, field, and algebra, the shortened term "basis" is much more appropriate. Such usage is designed to aid the understanding of the neophyte. In addition, I have not always used the same matrix structure to describe various Lie algebras, since I feel it is more useful to have several alternative descriptions of an algebra than one canonical one. I hope the cognoscenti will understand and appreciate these usages.

The physicists will be unhappy that so many important topics have been omitted. This work contains no systematic discussion of the representation theory of Lie groups and Lie algebras. Those interested in such material are urged to consult the books of H. Weyl [1, 2] and the works of É. Cartan [1, 24, 27, 28], the two classical giants in this field. Nor is there any systematic discussion of the theory of the special functions of mathematical physics. This material is treated in the books of N. Ja. Vilenkin [1], W. Miller [2], and J. D. Talman [1].

Finally, there is no systematic discussion of the applications of Lie group theory in modern physics. Such a systematic treatment, which would fill a volume in itself, could only be carried out after a treatment of the representation theory of Lie groups. In lieu of such treatment, numerous exercises indicating physical applications have been included at the end of each chapter. In addition, a number of physics papers dealing with every sort of application of Lie groups in physics have been placed in the bibliography. The interested reader has only to pick out some interesting-sounding titles from the bibliography and to follow them into the current literature. He is sure to be surrounded by Lie groups and unbelieveable applications in no time at all.

For the sins of omission and commission I am deeply sorry. I hope the former are somewhat compensated for by references in the bibliography. The latter I hope are few and far between.

I would like to thank my former students at M.I.T. and the students and faculty at U.S.F. for their many useful comments and suggestions. In addition, I would like to express my gratitute to Professors I. M. Singer and S. Helgason for useful discussions in the recent past, and for beautifully taught courses, which I had the privilege to attend, in the distant past. Thanks are also in order to Professor Peter Wolff for the key role he played in the preparation of the last half of the manuscript. An expression of gratitude is due to the programming policies of WCRB, which made the preparation of the manuscript reasonably pleasant, and to the staff of John Wiley & Sons for doing the same during the final stages of preparation. Finally, I would like to thank my wife for her patience during the preparation of this manuscript, as well as for typing large parts of it.

ROBERT GILMORE

Tampa, Florida
October 1973

Contents

CHAPTER 1

Introductory Concepts

At the present time, physicists find it convenient to try to describe the real world in terms of mathematics. Before we also explore the properties of the real world, we must have a firm grasp of the kinds of mathematical concepts that have been useful for physicists. These fundamental building blocks[1–3] are presented in this chapter, together with examples.

I. Basic Building Blocks

1. SET. A **set** is a collection of objects that do not necessarily have any additional structure or properties. For example, a collection of n oranges or bananas constitutes a set. So do n people. So do n points. The archetypical example of a set containing n (possibly infinite) objects is the set of n points.

2. GROUP. A **group** G is:

(a) a set $g_1, g_2, \ldots, g_n \in G$

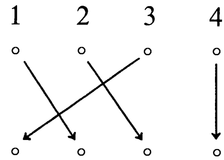
together with

(α) an operation, called group multiplication ($\circ$)

such that

1.	$g_i \in G,\ g_j \in G \Rightarrow g_i \circ g_j \in G$		closure
2.	$g_i \circ (g_j \circ g_k) = (g_i \circ g_j) \circ g_k$		associativity
3.	$g_1 \circ g_i = g_i = g_i \circ g_1$	for all g_i	existence of identity
4.	$g_k \circ g_l = g_l \circ g_k = g_1$		unique inverse $g_l = g_k^{-1}$ (1.1)

Example 1. The collection of all possible permutations of the points 1, 2, 3, 4 constitutes a group with 4! elements, or operations, called P_4

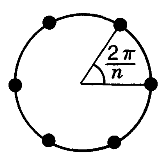


Before group operation

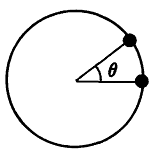
After group operation

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \tag{1.2}$$

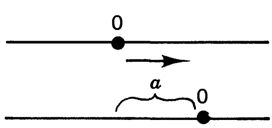
Example 2. The collection of rotations of the circle through multiples of $2\pi/n$ radians constitutes a group with n distinct operations. Such a (finite) group is said to be of **order** n.



Example 3. The collection of rotations of the circle through an angle θ ($0 \le \theta < 2\pi$) is an example of a continuous group. The group operations $g(\theta)$ exist in 1-1 correspondence with points on the interval $0 \le \theta < 2\pi$.



Example 4. The set T_a of rigid translations of the straight line through a distance a is another example of a continuous group. The group operations exist in 1-1 correspondence with the points on the line $-\infty < a < +\infty$.



Example 5. The set of real numbers, excluding 0, forms a group under the operation of multiplication. So do the complex numbers, provided we exclude 0. The identity operation in both groups is 1. But under the operation of addition, both the real and complex numbers form groups with identity element 0.

Example 6. The set of real $n \times n$ nonsingular matrices under matrix multiplication forms a group called $Gl\,(n, r)$. The subset of these matrices with determinant $+1$ forms a (sub)group called $Sl\,(n, r)$. The collection of $n \times n$ unitary matrices $U(n)$ also forms a group under matrix multiplication.

Comment. For the groups discussed in Examples 2 to 5, the order in which the group operations are applied is immaterial. A group that obeys a fifth postulate in addition to the four just listed is called an **abelian** or **commutative** group:

5. $g_i \circ g_j = g_j \circ g_i$ all $g_i, g_j \in G$ commutativity (1.1′)

In an abelian group it is customary to denote the group multiplication operation as $+$ instead of $\circ$. The groups of Examples 1 and 6 are not abelian.

3. FIELD. A **field** F is

(a) a set of elements $f_0, f_1, f_2, \ldots,$

together with two operations:

(α) $+$ called addition
(β) $\circ$ called scalar multiplication

such that Postulates A and B hold.

Postulate A. F is an abelian group under $+$, with f_0 the identity.

Postulate B

1.	$f_i \circ f_j \in F$		$\circ$ closure
2.	$f_i \circ (f_j \circ f_k)$	$= (f_i \circ f_j) \circ f_k$	$\circ$ associativity
3.	$f_i \circ 1$	$= 1 \circ f_i = f_i$	$\circ$ identity
4.	$f_i \circ f_i^{-1} = 1$	$= f_i^{-1} \circ f_i, f_i \neq f_0$	$\circ$ inverse, except for f_0
5.	$f_i \circ (f_j + f_k)$	$= f_i \circ f_j + f_i \circ f_k$	
	$(f_i + f_j) \circ f_k$	$= f_i \circ f_k + f_j \circ f_k$	distributive law (1.3)

If Postulate B-6 is also obeyed

6. $f_i \circ f_j = f_j \circ f_i$ commutativity (1.3′)

we say the field is commutative.

Only three fields are generally used by physicists. These are the real and complex numbers and the quaternions. The properties of the real numbers are assumed to be familiar.

Every complex number can be represented in the form

$$c = a1 + ib$$

where the units 1 and i ($=\sqrt{-1}$) obey

$$\begin{aligned} 1 \cdot 1 &= 1 \\ i \cdot 1 &= 1 \cdot i = i \\ i \cdot i &= -1 \end{aligned} \tag{1.4}$$

and a, b are arbitrary real numbers. Then we have

$$\begin{aligned} c_1 + c_2 &= (a_1 1 + ib_1) + (a_2 1 + ib_2) \\ &= (a_1 + a_2)1 + (b_1 + b_2)i \end{aligned} \tag{1.5}$$

$$\begin{aligned} c_1 c_2 &= (a_1 1 + b_1 i)(a_2 1 + b_2 i) \\ &= (a_1 a_2 - b_1 b_2)1 + (a_1 b_2 + b_1 a_2)i \end{aligned} \tag{1.6}$$

Every quaternion can be represented in the form

$$q = q_0 1 + q_1 \lambda_1 + q_2 \lambda_2 + q_3 \lambda_3 \tag{1.7}$$

where the q_i ($i = 0, 1, 2, 3$) are real numbers and the λ_i have multiplicative properties defined by

$$\begin{aligned} \lambda_0 \lambda_i &= \lambda_i \lambda_0 = \lambda_i \qquad i = 0, 1, 2, 3 \\ \lambda_i \lambda_i &= -\lambda_0 \\ \lambda_1 \lambda_2 &= -\lambda_2 \lambda_1 = \lambda_3 \\ \lambda_2 \lambda_3 &= -\lambda_3 \lambda_2 = \lambda_1 \\ \lambda_3 \lambda_1 &= -\lambda_1 \lambda_3 = \lambda_2 \end{aligned} \tag{1.8}$$

The sum and the product of two quaternions p and q are

$$\begin{aligned} p + q = (p_0 + q_0)\lambda_0 &+ (p_1 + q_1)\lambda_1 \\ &+ (p_2 + q_2)\lambda_2 + (p_3 + q_3)\lambda_3 \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} pq = (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3)\lambda_0 \\ + (p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2)\lambda_1 \\ + (p_0 q_2 + p_2 q_0 + p_3 q_1 - p_1 q_3)\lambda_2 \\ + (p_0 q_3 + p_3 q_0 + p_1 q_2 - p_2 q_1)\lambda_3 \end{aligned} \tag{1.10}$$

The set of eight elements $\pm\lambda_i$ forms a noncommutative group.

Complex conjugation can be defined for quaternions just as for complex numbers. Defining

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3)^* = (+\lambda_0, -\lambda_1, -\lambda_2, -\lambda_3) \tag{1.11q}$$

in direct analogy to

$$(1, i)^* = (+1, -i) \tag{1.11c}$$

we easily see

$$q^*q = \lambda_0 \left(\sum_{i=0}^{3} q_i^{\,2} \right) \tag{1.12}$$

The product of a quaternion with its conjugate is a real number which is ≥ 0. Also, $q^*q = 0$ implies that q is zero, in exact analogy with complex numbers.

4. LINEAR VECTOR SPACE. A **linear vector space** V consists of

(a) a collection $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \in V$, called vectors
(b) a collection $f_1, f_2, \ldots, \in F$, a field

together with two kinds of operations

(α) vector addition, $+$
(β) scalar multiplication, $\circ$

such that Postulates A and B hold.

Postulate A. $(V, +)$ is an abelian group.

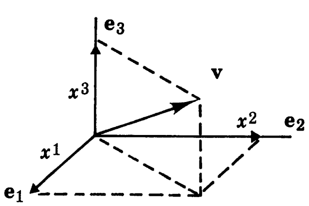
1.	$\mathbf{v}_i, \mathbf{v}_j \in V \Rightarrow \mathbf{v}_i + \mathbf{v}_j \in V$	closure
2.	$\mathbf{v}_i + (\mathbf{v}_j + \mathbf{v}_k) = (\mathbf{v}_i + \mathbf{v}_j) + \mathbf{v}_k$	associativity
3.	$\mathbf{v}_0 + \mathbf{v}_i = \mathbf{v}_i = \mathbf{v}_i + \mathbf{v}_0$	identity
4.	$\mathbf{v}_i + (-\mathbf{v}_i) = \mathbf{v}_0 = (-\mathbf{v}_i) + \mathbf{v}_i$	inverse
5.	$\mathbf{v}_i + \mathbf{v}_j = \mathbf{v}_j + \mathbf{v}_i$	commutativity

Postulate B

1.	$f_i \in F, \mathbf{v}_j \in V \Rightarrow f_i \mathbf{v}_j \in V$	closure′
2.	$f_i \circ (f_j \circ \mathbf{v}_k) = (f_i \circ f_j) \circ \mathbf{v}_k$	associativity′
3.	$1 \circ \mathbf{v}_i = \mathbf{v}_i = \mathbf{v}_i \circ 1$	identity′
4.	$f_i \circ (\mathbf{v}_k + \mathbf{v}_l) = f_i \circ \mathbf{v}_k + f_i \circ \mathbf{v}_l$	bilinearity
	$(f_i + f_j) \circ \mathbf{v}_k = f_i \circ \mathbf{v}_k + f_j \circ \mathbf{v}_k$	(1.13)

Example 1. The most primitive example of a vector is "something that points in some direction."

$$\mathbf{v} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \tag{1.14}$$



Example 2. If we associate V with F in the definition of a linear vector space, we see that the real and complex numbers and quaternions are linear vector spaces. The complex numbers form a vector space over the field of real numbers (basis 1, i) or complex numbers (basis 1); the quaternions form a vector space over the field of real numbers (bases $\lambda_0, \lambda_1, \lambda_2, \lambda_3$) or the quaternion field itself (basis 1).

Example 3. Let $\mathscr{L}$ be any linear differential or integral operator:

$$\mathscr{L}(\alpha\phi_1 + \beta\phi_2) = \alpha\mathscr{L}(\phi_1) + \beta\mathscr{L}(\phi_2) \tag{1.15}$$

Then if ϕ_1 and ϕ_2 are solutions to the equation

$$\mathscr{L}(\phi_i) = 0 \tag{1.16}$$

so also is any linear combination. The set of all solutions to the equation

$$\mathscr{L}(\phi) = 0 \tag{1.17}$$

is a linear vector space. Since a large class of the differential and integral operators of mathematical physics has this linearity property, a study of linear vector spaces and their properties is directly relevant for the physicist.

Example 4. The set of functions $f(\phi)$ defined on the circle $(0 \leq \phi < 2\pi)$ forms a linear vector space

$$f(\phi) = \sum_{-\infty}^{+\infty} a_m e^{im\phi} \tag{1.18}$$

where m is an integer.

For that matter, the set of functions defined on any set of points (either finite or infinite) forms a linear vector space.

Example 5. The set of all $N \times M$ matrices forms a vector space under matrix addition. In particular, the sets of $N \times 1$ and $1 \times N$ matrices form vector spaces, V_N and $V_N^\dagger$.

At this point it is convenient to introduce several concepts that are useful for describing the properties of vector spaces.

Definition. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are **linearly independent** if

$$\sum \alpha^i \mathbf{v}_i = 0 \Rightarrow \alpha^i = 0 \qquad i = 1, 2, \ldots, n \tag{1.19}$$

In Examples 1 and 4 above, we have

$$x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 = 0 \Rightarrow x^1 = x^2 = x^3 = 0 \tag{1.20}$$

$$\sum_{m=-\infty}^{+\infty} a_m e^{im\phi} = 0 \Rightarrow a_m = 0 \qquad m = 0, \pm 1, \pm 2, \ldots, \tag{1.21}$$

Therefore, the vectors $\mathbf{e}_i$ are linearly independent, as are the vectors $\mathbf{e}^{im\phi}$, $0 \le \phi < 2\pi$.

Definition. A vector space is N-**dimensional** if it is possible to find a set of N nonzero linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$, but every set of $N + 1$ nonzero vectors is linearly dependent.

Definition. Any such maximal set of vectors is called a **basis**, or **coordinate system**.

Then any vector $\mathbf{v}$ can be expanded in terms of a basis. For if

$$\beta\mathbf{v} + \sum_{i=1}^{N} \alpha^i \mathbf{v}_i = 0 \tag{1.22}$$

there is a nontrivial solution.

1. If β is zero,

$$\sum \alpha^i \mathbf{v}_i = 0 \Rightarrow \alpha^i = 0$$

and this is the trivial solution.

2. Therefore $\beta \neq 0$, and

$$\mathbf{v} = \sum_{i=1}^{N} \left(-\frac{\alpha^i}{\beta}\right)\mathbf{v}_i \tag{1.23}$$

is the unique expansion of $\mathbf{v}$ in terms of the basis $\mathbf{v}_i$.

By a fundamental[4] theorem of algebra, all N-dimensional vector spaces over the same field are isomorphic to each other. In particular, they are isomorphic to the canonical* N-dimensional vector space of $N \times 1$ matrices, with bases†

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \\ \vdots \\ 0 \end{pmatrix}; \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \\ \vdots \\ 0 \end{pmatrix}; \text{—}; \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{1.24}$$

Therefore, we can learn all the properties of any N-dimensional vector space merely by studying its faithful canonical representation V_N. The foregoing vector spaces with bases $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$, over the field of real numbers, complex numbers, and quaternions, are denoted R_N, C_N, and Q_N, respectively.

* Canonical means standard. A canonical form is one that has been standardized through use or convenience.

† The term "bases" is used in place of "basis vectors" whenever no confusion will result.

5. ALGEBRA. A **linear algebra** A consists of

(a) a collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \in V$, called vectors
(b) a collection $f_1, f_2, \ldots, \in F$, a field,

together with three kinds of operations

(α) vector addition, $+$
(β) scalar multiplication, $\circ$
(γ) vector multiplication, $\square$

such that we can state Postulates A to C.

Postulate A. Postulates $A1$ to $A5$ for a vector space hold.

Postulate B. Postulates $B1$ to $B4$ for a vector space hold.

Postulate C.

1. $\mathbf{v}_1, \mathbf{v}_2 \in V \Rightarrow \mathbf{v}_1 \square \mathbf{v}_2 \in V$ closure″ (1.25)
2. $(\mathbf{v}_1 + \mathbf{v}_2) \square \mathbf{v}_3 = \mathbf{v}_1 \square \mathbf{v}_3 + \mathbf{v}_2 \square \mathbf{v}_3$
   $\mathbf{v}_1 \square (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 \square \mathbf{v}_2 + \mathbf{v}_1 \square \mathbf{v}_3$ bilinearity′ (1.26)

Different varieties of algebras may be obtained, depending on which additional postulates are also satisfied.

3. $(\mathbf{v}_1 \square \mathbf{v}_2) \square \mathbf{v}_3 = \mathbf{v}_1 \square (\mathbf{v}_2 \square \mathbf{v}_3)$ associativity″
4. $\mathbf{v}_1 \square \mathbf{1} = \mathbf{v}_1$ existence of identity″; in general, this identity is not equal to the identity under $+$ or $\circ$
5. $\mathbf{v}_1 \square \mathbf{v}_2 = \pm \mathbf{v}_2 \square \mathbf{v}_1$ $\left\{\begin{matrix}\text{symmetric}\\ \text{antisymmetric}\end{matrix}\right\}$ under interchange
6. $\mathbf{v}_1 \square (\mathbf{v}_2 \square \mathbf{v}_3) = (\mathbf{v}_1 \square \mathbf{v}_2) \square \mathbf{v}_3 + \mathbf{v}_2 \square (\mathbf{v}_1 \square \mathbf{v}_3)$ derivative property

Example 1. The set of real $n \times n$ matrices forms a real n^2-dimensional vector space under matrix addition and scalar multiplication by real numbers. If we adjoin to this vector space the additional operation defined simply by matrix multiplication

$$(A \square B)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk} \tag{1.27}$$

this space becomes an associative algebra. The identity vector under $+$ is 0, the identity vector under $\square$ is the unit matrix I

$$(I)_{ik} = \delta_{ik}$$

The identity under $\circ$ is 1.

In addition to the postulates for an algebra, Example 1 satisfies Postulates C-3 and C-4; it is called a linear associative algebra with identity.

Example 2. The set of $n \times n$ real symmetric matrices, which obey

$$(S_{ij})^t = S_{ji} = +S_{ij} \qquad (S^t = S) \tag{1.28}$$

is a linear subspace of the vector space discussed in Example 1. However, if we adjoin the multiplication operation of Example 1, we do not satisfy Postulate C-1 for an algebra. That is, the product of two symmetric matrices is not in general a symmetric matrix:

$$\begin{aligned} (ST)^t &= (T)^t(S)^t = TS \neq ST \\ (S_{ij}T_{jk})^t &= T_{kj}S_{ji} \neq S_{kj}T_{ji} \end{aligned} \tag{1.29}$$

However, if we define the operation $\square$ by

$$\begin{aligned} S \square T = [S, T]_+ &= ST + TS \\ [S, \alpha T_1 + \beta T_2]_+ &= \alpha[S, T_1]_+ + \beta[S, T_2]_+ \end{aligned} \tag{1.30}$$

then both postulates C-1 and C-2 are satisfied. The real symmetric $n \times n$ matrices form an algebra under symmetrization, or **anticommutation**.

Example 3. The set of $n \times n$ real antisymmetric matrices

$$\begin{aligned} A^t &= -A \\ A_{ij} &= -A_{ji} \end{aligned} \tag{1.31}$$

is not closed under matrix multiplication either. But if we define the combinatorial operation $\square$ by antisymmetrization,

$$\begin{aligned} A \square B = [A, B] &= AB - BA \\ [A, \beta B + \gamma C] &= \beta[A, B] + \gamma[A, C] \end{aligned} \tag{1.32}$$

postulates C-1 and C-2 are satisfied and this system forms an algebra.

It is easily verified that this algebra in general has no identity, nor is it associative:

$$\begin{array}{ccl} A \square (B \square C) & = & ABC - ACB - BCA + CBA \\ \nparallel & & \qquad\quad \nparallel \\ (A \square B) \square C & = & ABC - BAC - CAB + CBA \end{array} \tag{1.33}$$

An algebra with the antisymmetric multiplication defined by the commutation relations (1.32) is called a **Lie algebra**, provided this combinatorial operation also obeys Postulate C-6:

$$A \square (B \square C) = (A \square B) \square C + B \square (A \square C)$$

This property, called a **derivation**, may be written more familiarly as

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]]$$

or

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0 \tag{1.34}$$

The latter form is called Jacobi's identity.

The process of accreting additional structure and complexity in going from a set to an algebra is shown schematically in Table 1.1. In general, the more highly structured a system is, the more we can prove about it. On the other hand, results that are true for a less structured system are also true, whenever applicable, in more highly structured systems.

TABLE 1.1

THE INCREASING COMPLEXITY OF THE VARIOUS MATHEMATICAL SYSTEMS OF USE TO A PHYSICIST

	Number and Kinds of Operations			
Number and Kinds of Elements	0	1 Group Multiplication	2 Group Multiplication for Abelian Groups + (Vector Addition); Scalar Multiplication ∘	3 Abelian Addition +, Scalar Multiplication ∘, Algebraic Multiplication □
1	Set	Group Section I.2 Postulates 1-4	Field Section I.3 Postulates $A1$–$A5$ Postulates $B1$–$B5$	
2			Vector Space Section I.4 Postulates $A1$–$A5$ Postulates $B1$–$B4$	Algebra Section I.5 Postulates $A1$–$A5$ Postulates $B1$–$B4$ Postulates $C1$, $C2$

II. Bases

Bases have been introduced in conjunction with linear vector spaces. This is a matter of convenience, since it is much easier to keep track of a small number of basis vectors than it is to account for every possible vector within a vector space.

The question of convenience transcends the concept of linear vector space; bases should exist in other systems as well. Their role, in a vector space, is as follows: every vector can be obtained by applying the operations pertinent to the system (vector addition and scalar multiplication) to the bases. This concept generalizes immediately, as we shall see.

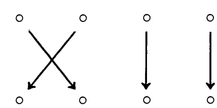
1. FOR A GROUP. For a group with elements $g_1, g_2, \ldots$, it is true that every element can be written in the form

$$g_k = g_{i_1} g_{i_2} \cdots g_{i_s} \tag{2.1}$$

The smallest number of distinct group operations which, multiplied in all possible combinations and powers, give all the distinct group operations, forms a basis for the group. These group operations are often called the **generators** of the group, for obvious reasons.

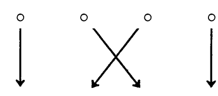
Example. The permutation group P_4, with $4! = 24$ distinct group elements, has as a basis the three interchanges:

$$P_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$



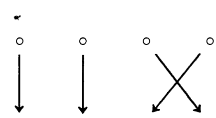
(2.2)

$$P_{23} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$



(2.3)

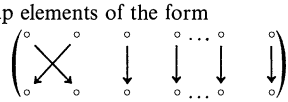
$$P_{34} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$



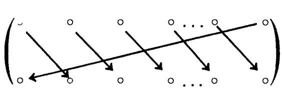
(2.4)

All other operations of P_4 can be obtained by applying these operations, and their inverses, in various orders.*

* In actual fact, all elements of the permutation group P_N can be obtained from only two group elements of the form



Single adjacent interchange



General shift

Hereafter, however, for the purposes of clarity, symmetry, and simplicity, we always consider the permutation groups P_N as having as generators the $N - 1$ "adjacent interchanges" $P_{i,\,i+1}$.

2. FOR A FIELD. For the fields discussed in Section I.3, a convenient set of bases is 1, $(1, i)$, $(\lambda_0 = 1, \lambda_1, \lambda_2, \lambda_3)$ for the real numbers, the complex numbers, and the quaternions, respectively. We have already seen that every quaternion is a linear superposition of real numbers $(q_i, i = 0, 1, 2, 3)$ multiplying these basis numbers:

$$q = q_0 1 + q_1\lambda_1 + q_2\lambda_2 + q_3\lambda_3$$

3. FOR A VECTOR SPACE. We now know how to choose a basis for a vector space. Because of the profound result that all N-dimensional vector spaces over the same field are equivalent (this is not true of groups with n elements or of algebras with the same dimension), it is necessary to study only one vector space of any dimension N in any detail. For the canonical N-dimensional vector space V_N of $N \times 1$ matrices, we have already chosen a canonical basis (1.24).

4. FOR AN ALGEBRA. Since an algebra is a vector space, a basis can be chosen for the vector space of any algebra. Since the algebraic operation is closed (C. 1), if $\mathbf{e}_i$ and $\mathbf{e}_j$ are any bases, we can write

$$\mathbf{e}_i \square \mathbf{e}_j = \sum_{k=1} C_{ij}{}^k \mathbf{e}_k \tag{2.5}$$

The $C_{ij}{}^k$ are numbers in the field of the vector space, called **structure constants**. These completely describe the structure of any algebra, for if

$$\begin{aligned} A &= \sum \alpha^i \mathbf{e}_i \qquad \in \text{Algebra} \\ B &= \sum \beta^j \mathbf{e}_j \end{aligned}$$

then, since $\square$ is bilinear, we have

$$\begin{aligned} A \square B &= (\sum \alpha^i \mathbf{e}_i) \square (\sum \beta^j \mathbf{e}_j) \\ &= \sum_i \sum_j \alpha^i \beta^j (\mathbf{e}_i \square \mathbf{e}_j) = \sum_i \sum_j \alpha^i \beta^j C_{ij}{}^k \mathbf{e}_k \end{aligned} \tag{2.5'}$$

To illustrate this concept, we compute the bases and structure constants with respect to these bases for the three examples treated in Section I.5.

Let $M_{ij}^{(n)}$ be the $n \times n$ matrix with $+1$ in the ith row and jth column and zeroes elsewhere:

$$M_{ij}^{(n)} = \begin{array}{c c} & j\text{th column} \\ i\text{th row} & \begin{pmatrix} & \vdots & \\ \cdots + & 1 & \cdots\cdots\cdots \\ & \vdots & \end{pmatrix} \end{array} \tag{2.6}$$

Then the $M_{ij}^{(n)}$ $(i, j = 1, 2, \ldots, n)$ are n^2 bases for the algebra of Example 1, (Section I.5). These bases obey

$$M_{ij}^{(n)} \square M_{kl}^{(n)} = M_{il}^{(n)} \delta_{jk} \tag{2.7}$$

The structure constants are then

$$C_{(ij),(kl)}^{(rs)} = \delta_{jk} \delta_i{}^r \delta_l{}^s$$

The bases for the algebra of Example 2 (Section I.5) are the symmetric matrices

$$S_{ij}^{(n)} = M_{ij}^{(n)} + M_{ji}^{(n)} = S_{ji}^{(n)} \tag{2.8s}$$

There are $\frac{1}{2} n(n+1)$ bases, with the structure constants determined from

$$[S_{ij}^{(n)}, S_{kl}^{(n)}]_+ = S_{il}^{(n)} \delta_{jk} + S_{jk}^{(n)} \delta_{il} + S_{ik}^{(n)} \delta_{jl} + S_{jl}^{(n)} \delta_{ik} \tag{2.9s}$$

Finally, for the algebra of real antisymmetric $n \times n$ matrices of Example 3 (Section I.5) we have $\frac{1}{2} n(n-1)$ bases

$$A_{ij}^{(n)} = M_{ij}^{(n)} - M_{ji}^{(n)} = -A_{ji}^{(n)} \tag{2.8a}$$

with commutation relations

$$[A_{ij}^{(n)}, A_{kl}^{(n)}] = A_{il}^{(n)} \delta_{jk} + A_{jk}^{(n)} \delta_{il} - A_{jl}^{(n)} \delta_{ik} - A_{ik}^{(n)} \delta_{jl} \tag{2.9a}$$

The bases $M_{ij}^{(n)}$, $S_{ij}^{(n)}$, $A_{ij}^{(n)}$ described previously are bases *for the vector space* of the algebras described in Examples 1 to 3 (Section I.5). In conjunction with our concept of what the term basis implies, we observe that the $2(n-1)$ matrices $M_{i,i+1}^{(n)}$ and $M_{i+1,i}^{(n)}$ $(i = 1, 2, \ldots, n-1)$ can generate the *entire* algebra of Example 1 under the combined operations of vector addition, scalar multiplication, and $\square$ = matrix multiplication. These $2(n-1)$ matrices are then called the *bases of the algebra* (or its generators), whereas all $M_{ij}^{(n)}$ are the bases for the vector space associated with the algebra—that is, before the additional operation $\square$ is defined.

Similarly, the $n-1$ matrices $S_{i,i+1}^{(n)}$ are bases for the algebra of Example 2, when $n > 2$, and the $n-1$ matrices $A_{i,i+1}^{(n)}$ are bases for the algebra of Example 3.

Any similarity between this last paragraph and the example of Section II.1 is not at all accidental, and is in fact profound.

III. Mappings, Realizations, Representations

The examples of the previous sections have probably been familiar. There is a reason for this—examples are supposed to be simple. There is another reason, however. Physicists prefer to work with mathematical structures that can be written down *explicitly* and on which calculations can be carried out. Thus, when confronted with an N-dimensional vector space, it is *convenient* to *map* it into the canonical vector space V_N (Section I.4) and then to perform the calculations on the appropriate matrices.

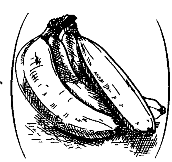
A mapping of one algebraic structure (group, field, etc.) into another similar algebraic structure is called a **homomorphism** if it preserves all combinatorial operations associated with that structure. If the mapping is in addition 1-1, or faithful, so that an inverse is well defined and exists, it is called an **isomorphism**.

If the mapping is into an algebraic structure that can be written down concretely and described analytically, it is called a **realization**. If it is into a set of matrices, it is called a **representation**.

1. SETS. Every set containing n objects is abstractly equivalent to every other set of n objects, in particular the set of n points $p_1, p_2, \ldots, p_n$. We can define a mapping f that maps a bunch of bananas $(b_1, b_2, \ldots, b_n)$ into a bunch of points $(p_1, \ldots, p_n)$ by

$$f(b_i) = p_i \qquad i = 1, 2, \ldots, n$$

$\cdot p_1$

f  $= p_2,\ p_3$

p_4

This mapping is faithful, or 1-1; thus an inverse exists

$$f^{-1}(p_i) = b_i$$

It is also possible to define unfaithful mappings of n objects into fewer than n points:

$$u(b_i) = p_1 \qquad i = 1, 2, \ldots, n$$

If a set contains a countably infinite set of objects, it can be put in 1-1 correspondence with the integers 1, 2, ..., which are points on the straight line R_1. If the objects in the set can be distinguished by 1, 2, ..., n real continuous variables, we can put them in 1-1 correspondence with a subset of points in the real Euclidean spaces $R_1, R_2, \ldots, R_n, \ldots$. Since these mappings are into $N \times 1$ matrices, they provide representations of sets.

2. GROUPS. The positive real numbers form a group under multiplication. We can find a faithful 1×1 matrix representation of the form

$$\begin{aligned} r = e^{\lambda} &\equiv \text{EXP } \lambda \\ r &> 0 \\ \infty > \lambda &> -\infty \\ \lambda &\in R_1 \end{aligned} \tag{3.1}$$

This representation has long been known and has been used to simplify the problem of multiplication

$$\begin{aligned} r_1 r_2 &= e^{\lambda_1} e^{\lambda_2} = e^{(\lambda_1 + \lambda_2)} \\ \ln (r_1 r_2) &= \ln r_1 + \ln r_2 \end{aligned} \tag{3.2}$$

The multiplicative group of positive numbers is isomorphic to the exponential of the real one-dimensional linear vector space R_1.

The complex numbers of modulus unity can be represented faithfully by a complex 1×1 matrix

$$C(\theta) \to e^{i\theta} \qquad -\pi \le \theta < \pi \tag{3.3}$$

From the group multiplication laws and the faithfulness of this representation, we can derive some simple trigonometric identities

$$\begin{aligned} (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) &= e^{i\theta} e^{i\phi} = e^{i(\theta + \phi)} \\ (\cos \theta \cos \phi - \sin \theta \sin \phi) \\ + i(\cos \theta \sin \phi + \sin \theta \cos \phi) &= \cos (\theta + \phi) + i \sin (\theta + \phi) \end{aligned} \tag{3.4}$$

$$\begin{aligned} \cos (\theta + \phi) &= \cos \theta \cos \phi - \sin \theta \sin \phi \\ \sin (\theta + \phi) &= \cos \theta \sin \phi + \sin \theta \cos \phi \end{aligned} \tag{3.4'}$$

In general, it is not possible to add or subtract group operations (group multiplication is the only operation defined). But *within a matrix representation for a group*, matrix addition is well defined. It is then possible to perform the operations

$$\lim_{\Delta\theta \to 0} \frac{e^{i\theta} e^{i\Delta\theta} - e^{i\theta}}{\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{e^{i\theta}(1 + i\Delta\theta) - e^{i\theta}}{\Delta\theta} \tag{3.5}$$

and to determine the derivatives of the following special functions:

$$\begin{aligned} \frac{d}{d\theta} (\cos \theta) &= -\sin \theta \\ \frac{d}{d\theta} (\sin \theta) &= \cos \theta \end{aligned} \tag{3.5'}$$

Equations (3.4) and (3.5) are prototype global and local identities which some special functions obey. All the special functions of mathematical physics[5,6] are related to representations of some particular groups. All the standard global and local identities can be obtained by the analogous processes carried out on the appropriate representations or realizations of these various groups.[7–9]

The group just discussed is in 1-1 correspondence with a subset of the space R_1. Whether a group is in 1-1 correspondence (by way of the exponential function) with a whole vector space or with a subset of a vector space depends on topological properties (noncompactness or compactness). That is, the matter depends on a study of the properties of the set of points on which the group multiplication structure is defined.

The two previous examples have related elements in a group to the exponential of an element in a vector space. This is a general property: the exponential of an element in a vector space on which a Lie algebra structure is defined is an element of some group. Defining

$$\text{EXP } \mathbf{X} = \sum_0^\infty \frac{(\mathbf{X})^n}{n!} \tag{3.6}$$

we will see, for example, that every $n \times n$ unitary matrix can be written as

$$U = \text{EXP } i\mathscr{H} \tag{3.7}$$

where $\mathscr{H}$ is an $n \times n$ hermitian matrix: $H_{ij} = H_{ji}^*$.

3. FIELD. The mapping that preserves the operations of a field,

$$\begin{aligned} \Gamma(c_1 + c_2) &= \Gamma(c_1) + \Gamma(c_2) \\ \Gamma(c_1 \circ c_2) &= \Gamma(c_1) \circ \Gamma(c_2) \end{aligned} \tag{3.8}$$

given, for the field of complex numbers, by

$$\begin{aligned} \Gamma(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \Gamma(i) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{3.9}$$

is a faithful representation for this field. From this representation we easily compute the multiplication structure for the complex numbers

$$\begin{array}{ccccc} (a1 + bi) & (c1 + di) & = & (ac - bd)1 + (ad + bc)i \\ \Big\downarrow \Gamma & \Big\downarrow \Gamma & & \Big\uparrow \Gamma^{-1} \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & \begin{pmatrix} c & d \\ -d & c \end{pmatrix} & = & \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix} \end{array} \tag{3.10}$$

A representation for the quaternions can be constructed similarly:

$$\Gamma(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Gamma(\lambda_j) = -i\sigma_j \tag{3.11}$$

where σ_j are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.12}$$

Under this representation

$$q = q_0 1 + q_1\lambda_1 + q_2\lambda_2 + q_3\lambda_3 \xrightarrow{\Gamma} \left(\begin{array}{c|c} q_0 - iq_3 & -iq_1 - q_2 \\ \hline -iq_1 + q_2 & q_0 + iq_3 \end{array}\right) \tag{3.13}$$

This representation can be used to work out the product of two quaternions. The results are presented in (1.10).

The quaternions are the only example of a noncommutative field to be encountered in this work. That is, the value of the product of two quaternions depends on the order in which the product is taken. If q' and q'' are two quaternions, then in general

$$q'q'' \neq q''q'$$

Although many of the familiar properties associated with commutative fields (the real and complex numbers) hold also for quaternions, some do not; as we will see in Chapter 2, this can lead to confusion.

Comment. It is clear that we can construct the complex numbers C from the real numbers R by introducing a quantity i_1 whose square is -1:

$$C = R + i_1 R \qquad (i_1{}^2 = -1)$$

Likewise, we can construct the quaternions Q from the complex numbers C in exactly the same way: by introducing another quantity i_2 ($\neq i_1$) whose square is -1

$$Q = C + i_2 C \qquad (i_2{}^2 = -1,\ i_1 i_2 = -i_2 i_1)$$

and which anticommutes with i_1. It is clear that we can continue in this way. But why are the fields constructed in this way not used in physics also? In going from the complex numbers to the quaternions, we lose the property of commutativity. In going from the quaternions to the next more complicated case (called the Cayley numbers), we lose the property of associativity. In going from the Cayley numbers to the next more complicated case, we lose the property of being a division ring. That is, if a and b are given elements in the field, the equation

$$ax + b = 0$$

has a unique solution for x when the field is a division ring. The answer to the question posed is provided graphically by the following chart:

Name of Field	Method of Construction $[i_j, i_k]_+ = -2\delta_{jk}$	Real Dimension	Division Ring, $ax + b = 0 \Rightarrow x$ unique	Associativity $(ab)c = a(bc)$	Commutativity, $ab = ba$
Real	R	2^0	Yes	Yes	Yes
Complex	$C = R + i_1 R$	2^1	Yes	Yes	Yes
Quaternion	$Q = C + i_2 C$	2^2	Yes	Yes	No
Cayley	$H = Q + i_3 Q$	2^3	Yes	No	No
	$J = H + i_4 H$	2^4	No	No	No
	$K = J + i_5 J$	2^5	No	No	No

4. VECTOR SPACES. Homomorphisms and isomorphisms between vector spaces preserve the operations of vector addition and scalar multiplication:

$$\Gamma(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = \alpha\Gamma(\mathbf{v}_1) + \beta\Gamma(\mathbf{v}_2) \tag{3.14}$$

Therefore, any such mapping is completely specified once its effect on the bases is known.

If A_N and B_N are two N-dimensional vector spaces over the same field, with bases $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N$, then the mapping $\mathbf{a}_i \to \Gamma(\mathbf{a}_i) = \mathbf{b}_i$, where $i = 1, 2, \ldots, N$, provides an isomorphism (equivalence) between A_N and B_N. The fact that an isomorphism can always be found between vector spaces of the same dimension over the same field, is responsible for the fundamental theorem referred to in Section I.4 that all N-dimensional vector spaces over a field F are equivalent to V_N over F.

If Γ is a mapping of A_N into B_M such that $\Gamma(\mathbf{a}) \in B_M$ for every $\mathbf{a} \in A_N$, then Γ is said to be **into**, or **injective**. If every element $\mathbf{b}$ of B_M is the image of some element $\mathbf{a}$ in A_N,

$$\mathbf{b} = \Gamma(\mathbf{a})$$

then Γ is **onto** or **surjective**. If

$$A_N \xrightarrow{\Gamma} B_M$$

and Γ is faithful, Γ is into and $M \geq N$. If Γ is onto and $M < N$, Γ is not faithful.

The terms into and onto and their consequences apply to mappings involving sets, groups, fields, and so on.

5. ALGEBRAS. We consider an algebra constructed from a linear vector space with bases A_{ij} $(i, j = 1, 2, \ldots, n)$ and

$$A_{ij} = -A_{ji} \tag{3.15}$$

In addition, we assume that the structure of the algebra is given by the commutation relations

$$[A_{ij}, A_{kl}] = A_{il}\,\delta_{jk} + A_{jk}\,\delta_{il} - A_{ik}\,\delta_{jl} - A_{jl}\,\delta_{ik} \tag{3.16}$$

A similar system was treated in Section II.4. The mapping of the abstract bases A_{ij} to the $n \times n$ antisymmetric matrices $A_{ij}^{(n)}$ given by (2.8a)

$$A_{ij} \xrightarrow{\Gamma} A_{ij}^{(n)} \tag{3.17}$$

is a representation for this algebra. The mapping R of the abstract bases A_{ij} to the concrete operators

$$X_{ij} = x^i\,\partial_j - x^j\,\partial_i$$

$$A_{ij} \xrightarrow{\text{R}} X_{ij} \tag{3.18}$$

provides a realization for this algebra. This algebra has additional matrix representations and operator realizations.

The exponential mapping maps the algebra of $n \times n$ antisymmetric matrices A onto the group of $n \times n$ orthogonal matrices with determinant $+1$ as follows:

$$A^{(n)} \xrightarrow{\text{EXP}} SO(n) \tag{3.19}$$

Résumé

This introductory chapter consists of three sections. In the first, we defined and discussed the algebraic structures which are often of interest to physicists: sets, groups, fields, linear vector spaces, and algebras. Each category involves somewhat more structure than its predecessors. Examples are given to illustrate each concept and structure and to indicate their usefulness to physicists.

In the second section, we treated the concept of basis and gave specific examples of bases for groups, fields, vector spaces, and algebras. Essentially, a basis consists of the minimal number of elements of an algebraic structure which can be employed to reconstruct the entire structure, using only the operations defined on that particular structure.

In the last section we defined some additional concepts applicable to the structures defined in the first section. These concepts include mappings, realizations, and (matrix) representations. The examples given in this section either anticipate or are prototypes for results obtained in subsequent chapters.

Exercises

1. Let $P_1, P_2, \ldots,$ be various physicists observing the universe around them. To do this, each sets up his own coordinate system $CS_1, CS_2, \ldots,$ and *represents* physical systems and concepts by mathematical expressions. To be able to communicate with one another, each physicist must be able to write down the observations of any other in terms of his own coordinate system. Let T_{ji} be the procedure which P_j uses to transform the data of P_i to his own reference frame.

(a) Show that all possible T_{ij} constitute a *set*.

(b) Show

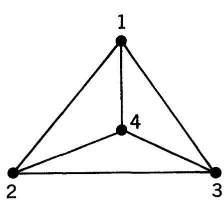
$$\begin{aligned} T_{ki} &= T_{kj} T_{ji} \\ T_{ki} &= T_{ik}^{-1} \\ T_{ii} &= \text{identity} \\ T_{ik} T_{ki} &= \text{identity} \end{aligned}$$

In other words, the set of all transformations from one physicist's coordinate system to another's coordinate system constitutes a group.

(c) Is this group of observer coordinate system transformations abelian?

(d) Does this set of coordinate system transformations have (or require) any other algebraic structure on it?

2. The vertices of a tetrahedron may be numbered as illustrated. Show that the group of all possible mappings of the tetrahedron into itself is isomorphic with the permutation group P_4. Some of the mappings in P_4 are *inversions* of the tetrahedron.



Such transformations cannot be made on a tetrahedron if it is carved out of a solid block of wood. How many group operations of P_4 can map a solid tetrahedron into itself?

(b) Show that these 12 operations are exactly the group operations whose 1×1 matrix representatives are $+1$ in the matrix representation generated by

$$\begin{aligned} P_{12} &= (1, 2) \to -1 \\ P_{23} &= (2, 3) \to -1 \\ P_{34} &= (3, 4) \to -1 \end{aligned}$$

3. Show that the collection of real $n \times n$ matrices $M^{(n)}$ constitutes:

(a) A set.

(b) A group under matrix multiplication when we exclude those matrices with $\det \|M^{(n)}\| = 0$.

(c) A linear vector space under matrix addition and scalar multiplication.

(d) An algebra under matrix addition, scalar multiplication, and matrix multiplication.

Show that the matrices $M^{(n)}$ do not form a field.

4. Show that the set of functions $f(x)$ defined pointwise on the real line is a linear vector space under scalar multiplication and pointwise addition

$$\psi(x) = \alpha f(x) + \beta g(x)$$

Show that this space is an algebra under pointwise multiplication of functions:

$$\Phi(x) = f(x) g(x)$$

5. Show that scalar-valued functions of $p \times q$ matrices (subsets of $V_{p \times q}$) form an algebra under the operations of Problem 4.

6. Show that all group operations in the permutation group P_4 can be constructed from the three adjacent interchanges [see (2.2)–(2.4)]. Now show that the three adjacent interchanges can themselves be constructed from just two operations: a single adjacent interchange and a general shift. Verify the comment in the footnote to (2.4) for arbitrary P_N.

7. Show the functions x^n ($n = 0, 1, 2, \ldots, \infty$) form bases for the linear vector space consisting of analytic functions defined on the straight line. This can easily be generalized: the functions

$$\prod_{i=1}^{p} \prod_{j=1}^{q} (z_{ij})^{r_{ij}} \qquad r_{ij} = 0, 1, 2, \ldots, \infty \text{ each } i, j$$

form bases for the algebra of scalar-valued analytic functions defined on the vector space $V_{p \times q}$. Can you generalize further?

8. Show that the functions $e^{im\phi}$ form bases for the set of all L^2 (square integrable) functions defined on the unit circle.

9. Let $p_1, p_2, \ldots, p_n$ be a finite set of points. Show that the space of functions defined on this finite set of points is a linear vector space. Show that the dimensionality of this vector space is n. Show that a suitable basis for this function space is the set of functions

$$f_i(p_j) = \delta_{ij}$$

Would it make any difference if the set of points p_j possessed some additional algebraic structure? For example, suppose they formed a finite group.

10. Let M be a set of $n \times n$ matrices. Consider the two mappings:

$$M \xrightarrow{\sigma} S = \tfrac{1}{2}(M + M^t)$$

$$M \xrightarrow{\alpha} A = \tfrac{1}{2}(M - M^t)$$

where $t =$ transpose.

Are σ, α isomorphisms, homomorphisms, or neither under the following conditions:

(a) They are considered as set mappings.

(b) They are considered as group mappings (det $\|M\| \neq 0$).

(c) They are considered as linear vector space mappings.

(d) They are considered as algebra mappings, where the third operation $\square$ is matrix multiplication in M, anticommutation in S, and commutation in A.

11. Show that the product of two hermitian matrices ($H = H^\dagger$ or $H_{ij} = H_{ji}^*$) is not in general hermitian. Also show that

$$[H_1, H_2]_+ = H_3$$

$$[H_1, H_2]_- = iH_4$$

where H_k are all hermitian matrices.

12. Use the isomorphism (3.13) of the quaternion field into the set of 2×2 complex matrices to compute explicitly the multiplication law for quaternions (1.10).

13. Consider the mapping of the space–time event (x, y, z, t) into the 2×2 matrix given by

$$(x, y, z, t) \rightarrow \sigma_\mu x^\mu = \left[\begin{array}{c|c} ct + z & x - iy \\ \hline x + iy & ct - z \end{array}\right] \qquad x^0 = ct, \qquad \sigma_0 = I_2$$

Show that this is an isomorphism and that

$$\det \|\sigma_\mu x^\mu\| = (ct)^2 - (x^2 + y^2 + z^2)$$

14. Verify by direct power series expansion using (3.6):

(a) If D is a diagonal matrix, then

$$\text{EXP } D = \text{EXP}\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} = \begin{bmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{bmatrix}$$

(b)
$$\text{EXP } \theta\begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(c)
$$\text{EXP } \theta\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

(d) If 0 is the identity (zero vector) in the linear vector space of $n \times n$ matrices, its image under the EXPonential mapping (3.6) is the identity group operation in the group of $n \times n$ nonsingular matrices.

15. Show that the 2×2 matrix representation **S** and the analytic realization **L**:

$$S_1 = \frac{i}{2}\sigma_1 \qquad L_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$S_2 = \frac{i}{2}\sigma_2 \qquad L_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$S_3 = \frac{i}{2}\sigma_3 \qquad L_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

have isomorphic commutation relations. Of what are **S** and **L** the representation and realization?

Notes and References

This chapter abstracts the necessary working tools used throughout the later chapters. These definitions and concepts are treated in greater detail in the first three references.

1. B. L. van der Waerden. [1]
2. G. Birkhoff, T. C. Bartee. [1]
3. N. Jacobson. [1]
4. I. M. Gel'fand. [1]
5. W. Magnus, F. Oberhettinger, R. P. Soni. [1]
6. H. Hochstadt. [1, 2]
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9. J. D. Talman. [1]

CHAPTER 2

The Classical Groups

In this chapter we enumerate all the classical matrix groups. We do this by introducing more and more structure on vector spaces until we are able to define all the groups.

In the first section we introduce the concept of change of basis[1–3] and associate with each change of basis in V_N an $N \times N$ nonsingular matrix. Such matrices belong to the General Linear Groups.

In the second section we introduce a number of new concepts: direct sum and product, reduction into subspaces, and in particular, fully symmetric and antisymmetric subspaces. The latter notion is used to define volume element in a vector space. The subgroups of the general linear groups which are volume-preserving are then defined: the Special Linear Groups.

The metric is presented in the third section. Various canonical metric forms are displayed, and the corresponding metric-preserving groups are defined.

Finally, the interrelationship of all these classical matrix groups—subgroups, isomorphic groups, and homomorphic groups—is indicated in the closing section.

I. General Linear Groups

1. CHANGE OF BASIS. The choice of a set of basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ in a vector space V_N is not unique. It is possible to choose other sets of basis vectors, or coordinate systems.

$$CS' = \{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_N\} \qquad CS'' = \{\mathbf{e}''_1, \mathbf{e}''_2, \ldots, \mathbf{e}''_N\}$$

The basis vectors for any coordinate system can be represented as a linear superposition of basis vectors in any other coordinate system, as in Fig. 2.1, where the transformation from coordinate system CS to CS', followed by the transformation from CS' to CS'' is easily computed in terms of the individual transformations involved:

$$\mathbf{e}_k'' = B_k{}^j\mathbf{e}_j' = B_k{}^jA_j{}^i\mathbf{e}_i = (BA)_k{}^i\mathbf{e}_i \tag{1.1}$$

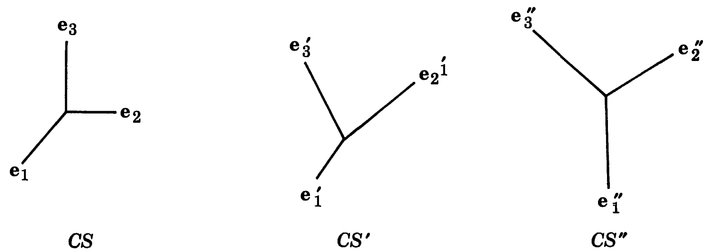


FIG. 2.1

$$\mathbf{e}_j' = A_j{}^i\mathbf{e}_i \qquad \mathbf{e}_k'' = B_k{}^j\mathbf{e}_j'$$

$$\mathbf{e}_k'' = C_k{}^i\mathbf{e}_i$$

Every set of basis vectors in V_N can be related to every other coordinate system by an $N \times N$ nonsingular matrix. The matrix must be nonsingular to possess an inverse.

The $N \times N$ matrix groups involved in changing bases in the vector spaces R_N, C_N, Q_N are called the **general linear groups** of $N \times N$ matrices over the reals, complex numbers, and quaternions, respectively: $Gl(N, r)$, $Gl(N, c)$, $Gl(N, q)$.

2. COVARIANCE AND CONTRAVARIANCE. The representation of a vector $\mathbf{v} \in V_N$ with respect to a coordinate system CS with basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ is

$$\mathbf{v} = \sum v^i\mathbf{e}_i \tag{1.2}$$

where the v^i are scalars in the field of the vector space V_N. The scalars v^i are called **coordinates** of the vector $\mathbf{v}$ with respect to the coordinate system CS.

If we choose a different coordinate system CS' with basis vectors $\{\mathbf{e}_1', \mathbf{e}_2', \ldots, \mathbf{e}_N'\}$ related to the original basis vectors by

$$\mathbf{e}_j' = A_j{}^i\mathbf{e}_i \tag{1.3}$$

the representation of the vector $\mathbf{v}$ in this new coordinate system is

$$\mathbf{v} = v'^j\mathbf{e}_j' = v'^jA_j{}^i\mathbf{e}_i = v^i\mathbf{e}_i$$

which can be expressed

$$(v'^jA_j{}^i - v^i)\mathbf{e}_i = 0$$

Since the $\mathbf{e}_i$ are linearly independent, the coefficient of each $\mathbf{e}_i$ must independently vanish:

$$v'^j A_j{}^i = v^i \tag{1.4}$$

We see that, in a very real sense, the basis vectors $\mathbf{e}_i$ and the components v^i transform in the opposite way from each other as we go from one coordinate system to another. Symbolically we have

$$\begin{aligned} \mathbf{e}' &= A\mathbf{e} \\ v &= v'A \qquad \text{or} \qquad v' = vA^{-1} \end{aligned} \tag{1.5}$$

We say that the basis vectors transform in a *co*variant (or *co*gredient) manner under the transformation group because the transformation is defined in terms of the basis vectors. The coordinates or components transform in a *contra*variant (or *contra*gredient) way.

In general, any pair of quantities that transform opposite to each other, as $\mathbf{e}' = A\mathbf{e}$, $v' = vA^{-1}$, are said to transform in a covariant and contravariant manner, depending on which (the $\mathbf{e}_i$ or the v^i) the transformation is defined on originally.[1–3]

II. Volume-preserving Groups

1. DIRECT SUMS. It is often desirable to construct a new vector space from two vector spaces $V_N{}^1$ and $V_{N'}{}^2$. In this section we give a prescription for constructing a **direct sum** vector space $V_N{}^1 \oplus V_{N'}{}^2$; in the next section we construct a **direct product** vector space $V_N{}^1 \otimes V_{N'}{}^2$.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ be basis vectors for $V_N{}^1$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{N'}\}$ be basis vectors for $V_{N'}{}^2$. Then the $N + N'$ basis vectors for the direct sum vector space $V_N{}^1 \oplus V_{N'}{}^2$ are $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N; \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{N'}\}$.

If $\mathbf{u} \in V_N{}^1$ and $\mathbf{v} \in V_{N'}{}^2$ have components $\{u^i\}$, $\{v^j\}$ with respect to the basis vectors $\{\mathbf{e}_i\}$, $\{\mathbf{f}_j\}$, then both $\mathbf{u}$ and $\mathbf{v}$ may be considered as vectors in the direct sum space $V_N{}^1 \oplus V_{N'}{}^2$ with components given by

$$\begin{array}{lll} \mathbf{u} \in V_N{}^1 & \mathbf{v} \in V_{N'}{}^2 & V_N{}^1 \oplus V_{N'}{}^2 \\ \{u^1, u^2, \ldots, u^N\} & & \rightarrow \{u^1, u^2, \ldots, u^N; 0, 0, \ldots, 0\} \\ & \{v^1, v^2, \ldots, v^{N'}\} & \rightarrow \{0, 0, \ldots, 0; v^1, v^2, \ldots, v^{N'}\} \end{array} \tag{2.1}$$

with respect to the basis vectors $\{\mathbf{e}_1 \cdots \mathbf{e}_N; \mathbf{f}_1 \cdots \mathbf{f}_{N'}\}$. Vector addition and scalar multiplication are defined in the usual way.

A change of basis in each space $V_N{}^1$ and $V_{N'}{}^2$,

$$\begin{aligned} \mathbf{e}'_i &= A_i{}^r \mathbf{e}_r \qquad & \mathbf{e}' &= A\mathbf{e} \\ \mathbf{f}'_j &= B_j{}^s \mathbf{f}_s \qquad & \mathbf{f}' &= B\mathbf{f} \end{aligned} \tag{2.2}$$

induces a change of basis in the space $V_N{}^1 \oplus V_{N'}{}^2$. With respect to the column vector bases

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \text{ } i\text{th row} \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ i\text{th row} \\ \\ \end{matrix}\right\} V_N{}^1 \text{ subspace} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} V_{N'}{}^2 \text{ subspace} \end{matrix} \tag{2.3}$$

$$\mathbf{f}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \text{ } j\text{th row} \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} V_N{}^1 \text{ subspace} \\ \left.\begin{matrix} \\ (N+j)\text{th row} \\ \\ \end{matrix}\right\} V_{N'}{}^2 \text{ subspace} \end{matrix}$$

the change of basis induced by the transformation (2.2) has the matrix structure

$$\begin{array}{c c} & \begin{matrix} \overbrace{}^{N} & \overbrace{}^{N'} \end{matrix} \\ \begin{matrix} N\{ \\ N'\{ \end{matrix} & \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \end{array} \tag{2.4}$$

Such a structure is called a **block-diagonal** structure.

If G_i is the group of possible changes of basis for V^i $(i = 1, 2)$, the most general change of basis in $V_N{}^1 \oplus V_{N'}{}^2$ induced by the transformations (2.2) is an element, called $A \oplus B$, of the direct sum group $G_1 \oplus G_2$. It is clear that the block-diagonal transformation (2.4) is a subgroup of the group of all possible changes of basis in $V_N{}^1 \oplus V_{N'}{}^2$.

The process of defining the direct sum of two vector spaces (or groups) is readily extended to the formation of direct sum spaces from three or more spaces.

Although the concept of direct sum is often used for vector spaces, it is never used for groups.

2. DIRECT PRODUCT. With the notation as before, the NN' basis vectors for the direct product space $V_N{}^1 \otimes V_{N'}{}^2$ are $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N'$. An arbitrary vector $\boldsymbol{\xi}$ in the direct product vector space $V_N{}^1 \otimes V_{N'}{}^2$ has components ξ^{ij} with respect to the basis vectors $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$:

$$\boldsymbol{\xi} = \sum_{i=1}^{N} \sum_{j=1}^{N'} \xi^{ij} \mathbf{e}_i \otimes \mathbf{f}_j \tag{2.5}$$

In general, $\boldsymbol{\xi}$ cannot be constructed as the direct product of a vector $\boldsymbol{\xi}^1 \in V_N{}^1$ and a vector $\boldsymbol{\xi}^2 \in V_{N'}{}^2$:

$$[\boldsymbol{\xi} \neq \boldsymbol{\xi}^1 \otimes \boldsymbol{\xi}^2 \qquad \text{in general}] \tag{2.6}$$

However, $\boldsymbol{\xi}$ can always be constructed as a *superposition* of such direct products.

Changes of bases $\mathbf{e}' = A\mathbf{e}$, $\mathbf{f}' = B\mathbf{f}$ in $V_N{}^1$ and $V_{N'}{}^2$, respectively, induce a change of basis in the direct product space:

$$\begin{aligned} \mathbf{e}'_i \otimes \mathbf{f}'_j &= (\textstyle\sum A_i{}^r \mathbf{e}_r) \otimes (\sum B_j{}^s \mathbf{f}_s) \\ &= \sum_{r=1}^{N} \sum_{s=1}^{N'} A_i{}^r B_j{}^s (\mathbf{e}_r \otimes \mathbf{f}_s) \end{aligned} \tag{2.7}$$

These transformations, which form a subgroup of the group of all possible transformations in this NN'-dimensional vector space, form a **direct product** group $G_1 \otimes G_2$. The group $G_1 \otimes G_2$ is also called a **Kronecker product** or **tensor product** group.

Definition. A **tensor** is a vector.

A tensor is an element in a direct product vector space. It has all the properties that any vector has. While we are making definitions, we present another:

Definition. A **spinor** is a vector in C_2, a complex, two-dimensional vector space. (We will later enlarge the concept of spinor significantly.)

A change of basis

$$\mathbf{e}'_i \otimes \mathbf{f}'_j = \sum_{r=1}^{N} \sum_{s=1}^{N'} A_i{}^r B_j{}^s (\mathbf{e}_r \otimes \mathbf{f}_s) \tag{2.7}$$

induces a change in the coordinates ξ^{ij} of a vector (tensor):

$$\xi^{rs} = \sum_i \sum_j \xi'^{ij} A_i{}^r B_j{}^s \tag{2.8}$$

Comparing (2.7) with (2.8), we see the bases $\mathbf{e}_i \otimes \mathbf{f}_j$ and coordinates ξ^{ij} transform in a covariant and a contravariant manner, respectively.

This process of forming new vector spaces or groups by taking direct products of two vector spaces or groups generalizes immediately to direct products involving three or more vector spaces or groups.

In physical applications we are frequently interested in taking the direct product of a vector space with *itself* two or more times. Following are the N^r basis vectors in $\underbrace{V_N \otimes V_N \otimes \cdots \otimes V_N}_{r \text{ times}}$:

$$\{\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_r}\}, \qquad i_j = 1, 2, \ldots, N; \qquad j = 1, 2, \ldots, r.$$

The transformation properties of these bases under changes of bases

$$\mathbf{e}'_j = A_j{}^i \mathbf{e}_i \qquad \mathbf{e}''_k = B_k{}^j \mathbf{e}'_j \tag{2.9}$$

in V_N are given by

$$\begin{aligned}
\mathbf{e}'_{j_1} \otimes \cdots \otimes \mathbf{e}'_{j_r} &= A_{j_1}{}^{i_1} \cdots A_{j_r}{}^{i_r} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_r} \\
\mathbf{e}''_{k_1} \otimes \cdots \otimes \mathbf{e}''_{k_r} &= B_{k_1}{}^{j_1} \cdots B_{k_r}{}^{j_r} \mathbf{e}'_{j_1} \otimes \cdots \otimes \mathbf{e}'_{j_r} \\
&= (BA)_{k_1}{}^{i_1} \cdots (BA)_{k_r}{}^{i_r} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_r}
\end{aligned} \tag{2.10}$$

An element in an rth-order direct or tensor product space $\underbrace{V_N \otimes \cdots \otimes V_N}_{r} \equiv (V_N)^r$ is called an rth-order tensor based on V_N, or more simply, an rth-order tensor. From (2.10), we see a change of basis A in V_N induces a change of basis in $(V_N)^r$. The group of basis transformations in V_N is closely related to the group $(G)^r$ of transformations induced in $(V_N)^r$. In particular the following are true.

1. The group multiplication property is preserved

$$\begin{array}{ll}
B \circ A \rightarrow & (B \circ A)^r \\
\text{in } G & \text{in } (G)^r \\
\text{on } V_N & \text{on } (V_N)^r
\end{array} \tag{2.11}$$

2. The transformations in $(G)^r$ are $N^r \times N^r$ matrix transformations.

Therefore, from (2.11) we know that $(G)^r$ is a representation of G. The tensor product space $(V_N)^r$ is said to be a **carrier space** for an rth-order tensor product representation $(G)^r$ of G. For a large class of groups (simple compact Lie groups), all possible representations can be constructed from the tensor product representations $(G)^r : r = 0, 1, 2, \ldots$.

Comment. We have used the process of forming direct sums or products to create new vector spaces and groups from two or more initial vector spaces or groups. The same process of taking direct sums and products can be used to create new sets, fields, or algebras, for example, from two or more initial sets, fields, or algebras.

3. SYMMETRIC REDUCTION IN TENSOR SPACE. In a second-order tensor space of the form $V_N \otimes V_N \equiv (V_N)^2$, it is possible to form linear combinations of the basis vectors $\mathbf{e}_i \otimes \mathbf{e}_j$ which are symmetric or antisymmetric under interchange of the order of the basis vectors:

$$\mathbf{s}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i \equiv \mathbf{e}_i \vee \mathbf{e}_j = +\mathbf{s}_{ji} \tag{2.12}$$

$$\mathbf{a}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i \equiv \mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{a}_{ji} \tag{2.13}$$

The $\frac{1}{2}N(N+1)$ and $\frac{1}{2}N(N-1)$ basis vectors $\mathbf{s}_{ij}$, $\mathbf{a}_{ij}$ have the following transformation properties

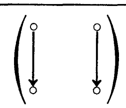
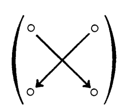
$$\begin{aligned} \mathbf{s}'_{ij} &= A_i{}^m A_j{}^n \mathbf{s}_{mn} \\ \mathbf{a}'_{ij} &= A_i{}^m A_j{}^n \mathbf{a}_{mn} \end{aligned} \tag{2.14}$$

under a change of basis $\mathbf{e}'_i = A_i{}^r \mathbf{e}_r$ in V_N.

Under application of the group operation (of P_2, the permutation group on two objects) interchanging the first and second index, the symmetrized bases $\mathbf{s}_{ij}$, $\mathbf{a}_{ij}$ are simply multiplied by ± 1. This property can be used in a procedure for constructing symmetrized bases. Specifically, if I represents the identity operation of P_2 and (12) represents the exchange operation, we have

$$\begin{aligned} \mathbf{s}_{ij} &= \{I + (+1)(12)\}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i \\ \mathbf{a}_{ij} &= \{I + (-1)(12)\}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i \end{aligned} \tag{2.15}$$

The effect of these two group operations on the basis vectors is shown in the following table:

		Basis Vector	
Group Operation		$\mathbf{s}_{ij}$	$\mathbf{a}_{ij}$
I		$+1$	$+1$
(12)		$+1$	-1

With respect to the symmetrized bases $\mathbf{e}_i \vee \mathbf{e}_j, \mathbf{e}_i \wedge \mathbf{e}_j$, a change of basis in $(V_N)^2$ induced by a change $\mathbf{e}_i = B_i{}^r\mathbf{e}_r$ assumes a direct sum structure

$$B \to \Gamma(B) = \left(\begin{array}{c|c} S(B) & 0 \\ \hline 0 & A(B) \end{array}\right) \qquad \left(\begin{array}{c} \mathbf{e}_i \vee \mathbf{e}_j \\ \hline \mathbf{e}_i \wedge \mathbf{e}_j \end{array}\right) \tag{2.16}$$

Here $S(B)$ is the symmetric second-order matrix representation of B; $A(B)$ is the antisymmetric second-order matrix representation of B.

The process of forming symmetrized subspaces in rth-order tensor spaces $(V_N)^r$ must await a thorough discussion of the permutation group P_r and its representations. However, two particular subspaces in $(V_N)^r$ can always be constructed easily. These are the fully symmetric and the fully antisymmetric subspaces, with basis vectors $\mathbf{e}_{i_1} \vee \mathbf{e}_{i_2} \vee \cdots \vee \mathbf{e}_{i_r}$ and $\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_r}$, respectively.

The procedure for constructing the symmetrized basis vectors is

$$\left.\begin{array}{l} \mathbf{e}_{i_1} \vee \mathbf{e}_{i_2} \vee \cdots \vee \mathbf{e}_{i_r} \\ \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_r} \end{array}\right\} = \sum_{\substack{\text{all elements} \\ \text{in } P_r}} \sigma(\pm 1)^\sigma \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_r} \tag{2.17}$$

where σ is a group operation of P_r. The σ can be constructed from the *bases*, the adjacent interchanges $(i, i+1) = P_{i,\,i+1}$ of P_r. By assigning to each basis of P_r the 1×1 matrix representative (± 1), we construct the fully symmetric and fully antisymmetric subspaces of $(V_N)^r$, respectively. Depending on whether the permutation operation σ is constructed from an even or an odd number of adjacent interchanges, $(-)^\sigma$ is $+1$ or -1.

For example, P_3 has two bases, or generators:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Other group elements of P_3 are constructed from these bases:

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad (a \quad b \quad c) & \\ = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad (a \quad c \quad b) & \\ = (c \quad a \quad b) & \\ = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad (a \quad b \quad c) & \end{aligned} \tag{2.18}$$

The other three operations can be written as follows:

$$\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}$$

Application of the operation in (2.17) to the basis vector $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ gives the linear combinations

$$(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k + \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_i + \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_j) \\ \pm (\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_k + \mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i) \tag{2.19}$$

for the symmetric (antisymmetric) subspaces of $(V_N)^3$.

Now we must ask how many independent basis vectors exist in the fully symmetric and antisymmetric subspaces of $(V_N)^r$. The question can be stated as follows: what are the dimensionalities for the carrier spaces of fully symmetric and antisymmetric rth-order tensor product representations of the general transformation of basis group in V_N?

It may be verified that this question, stated for the symmetric subspace, is equivalent to the "Bose-Einstein" counting problem: how many ways can r indistinguishable balls be put in N distinguishable boxes?[4] This number is calculated in Fig. 2.2.

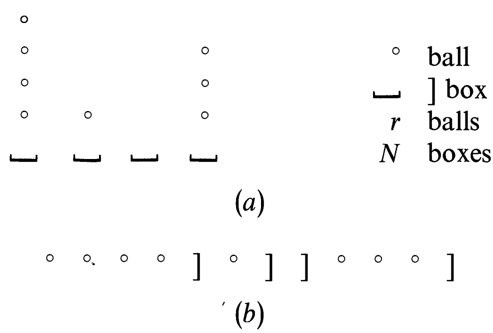


FIG. 2.2 THE BALL BOX PROBLEMS. THE NUMBER OF DISTINCT DIAGRAMS WITH A BOX AT THE FAR RIGHT IS $(N + r - 1)!/r!(N - 1)!$.

The same question, stated for antisymmetric bases, is equivalent to the "Fermi-Dirac" counting problem: how many ways can r indistinguishable balls be put in N distinguishable boxes, with at most one in each box? This number is

$$\binom{N}{r} = {}_N C_r = \frac{N!}{r!\,(N-r)!}$$

Both the fully symmetric and antisymmetric subspaces and representations have a great deal of intrinsic interest. In the following two sections we discuss applications of these representations.

4. FULLY SYMMETRIC SUBSPACES—EXPANSION OF FUNCTIONS. The direct product of a vector $\mathbf{x} = x^i\mathbf{e}_i \in V_N$ with itself: $\mathbf{x} \otimes \mathbf{x}$, lies entirely within the fully symmetric subspace of $(V_N)^2$:

$$\mathbf{x} \otimes \mathbf{x} = \sum \frac{1}{(1+\delta_{ij})!} x^i x^j \mathbf{e}_i \vee \mathbf{e}_j \tag{2.20}$$

The rth power of $\mathbf{x}$ lies within the symmetric subspace of $(V_N)^r$. The components of $(\mathbf{x})^r$ with respect to the bases $\mathbf{e}_{i_1} \vee \cdots \vee \mathbf{e}_{i_r}$ are of the form $(x^1)^{j_1}(x^2)^{j_2}\cdots(x^N)^{j_N}$, with $j_1 + j_2 + \cdots + j_N = r$. Such polynomials form a basis for the set of all analytic functions defined on V_N.

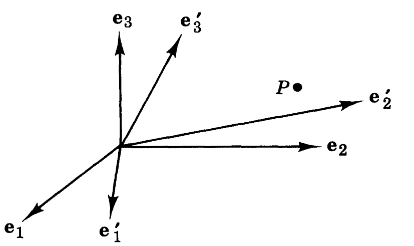


FIG. 2.3

Before proceeding, let us consider Fig. 2.3, where f is a function defined at every point p of a space R_N, and the value of f at p is $f(p)$. If the function is analytic, it may be expressed as a power-series expansion in the coordinates of the point p in some coordinate system S.

$$f(p) = f^S[x(p)]$$

In order for this function to have the same value at p, independent of coordinate system, its structure must change in going from coordinate system S to coordinate system S':

$$\begin{array}{c} f(p) \\ /\!/\!/ \qquad \backslash\!\backslash\!\backslash \\ f^S[x(p)] \equiv f^{S'}[x'(p)] \end{array}$$

Let $f(x^1, x^2, \ldots, x^N)$ be a real-valued, analytic scalar function defined on R_N, with respect to a coordinate system S. Since it is analytic, we can write

$$f^S(x) = f_0^S + x^i f_i^S + \frac{1}{2!} x^i x^j f_{ij}^S + \cdots \tag{2.21}$$

where x^i are the coordinates of the point p in R_N with respect to the bases $\mathbf{e}_i$. The x^i and their powers are also bases for the space of real analytic functions defined on R_N. The $f_{ijk\cdots}$ are then the coordinates multiplying the basis vectors $x^i x^j x^k \cdots$ in the various rth-order fully symmetric spaces. The coordinates $f_{ijk\cdots}^S$ may also be written

$$\begin{aligned} f_0{}^S &\equiv f^S(0) \\ f_{ijk\cdots}^S &\equiv \left.\frac{\partial^r f^S}{\partial x^i\, \partial x^j\, \partial x^k \cdots}\right|_{x=0} \end{aligned} \tag{2.22}$$

Suppose now we perform a change of basis in R_N given by

$$\mathbf{e}_i' = A_i{}^r \mathbf{e}_r \tag{2.23}$$

The function f is defined and has a particular value at the point $p \in R_N$, irrespective of the coordinate system (if any) we may choose in R_N. In coordinate system S', the *structure* of the function must change so that it assumes the *same value* at point p:

$$f^{S'}(p) = f_0{}^{S'} + x'^i f_i{}^{S'} + \frac{1}{2!} x'^i x'^j f_{ij}{}^{S'} + \cdots \tag{2.24}$$

The x^i transform in a contravariant way from the $\mathbf{e}_i$, the $f_{ij} \cdots$ transform contravariant to the $x^i x^j$, and therefore in the same way as (covariant with) the $\mathbf{e}_i \vee \mathbf{e}_j \vee \cdots$. Since

$$\begin{aligned} x^i &= x'^j A_j{}^i \\ A_j{}^i &= \frac{\partial x^i}{\partial x'^j} \end{aligned} \tag{2.25}$$

then we must also have

$$f_{ij\cdots}^{S'} = A_i{}^r A_j{}^s \cdots f_{rs\cdots}^S$$

In a more familiar form, this is written as

$$\left.\frac{\partial^r f^{S'}}{\partial x'^i \, \partial x'^j \cdots}\right|_0 = \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} \cdots \left.\frac{\partial^r f^S}{\partial x^r \, \partial x^s \cdots}\right|_0 \tag{2.26}$$

Proceeding through two successive changes of basis in R_N,

$$\mathbf{e}'_j = A_j{}^i \mathbf{e}_i \qquad x^i = x'^j A_j{}^i \qquad A_j{}^i = \frac{\partial x^i}{\partial x'^j} \tag{2.27}$$

$$\mathbf{e}''_k = B_k{}^j \mathbf{e}'_j \qquad x'^j = x''^k B_k{}^j \qquad B_k{}^j = \frac{\partial x'^j}{\partial x''^k} \tag{2.28}$$

we also find

$$\begin{aligned} f^{S''}_{ij\cdots} = \left.\frac{\partial^r f^{S''}}{\partial x''^i \, \partial x''^j \cdots}\right|_0 &= \left(\frac{\partial x'^k}{\partial x''^i} \frac{\partial x^u}{\partial x'^k}\right)\left(\frac{\partial x'^l}{\partial x''^j} \frac{\partial x^v}{\partial x'^l}\right) \\ &\qquad \cdots \left.\frac{\partial^r f^S}{\partial x^u \, \partial x^v \cdots}\right|_0 \\ &\equiv \frac{\partial x^u}{\partial x''^i} \frac{\partial x^v}{\partial x''^j} \cdots \left.\frac{\partial^r f^S}{\partial x^u \, \partial x^v \cdots}\right|_0 \end{aligned} \tag{2.29}$$

This is known as the chain rule in mathematical analysis.[5–7] Stated simply, we have

$$B_k{}^j A_j{}^i = C_k{}^i \Rightarrow \left.\frac{\partial x'^j}{\partial x''^k}\right|_0 \left.\frac{\partial x^i}{\partial x'^j}\right|_0 = \left.\frac{\partial x^i}{\partial x''^k}\right|_0$$

where 0 is the origin of coordinate systems S and S', and $(x^1, x^2, \ldots, x^N)$ and $(x'^1, x'^2, \ldots, x'^N)$ are the coordinates of the single point p in the systems S and S', respectively.

It often happens that we are interested in studying functions defined, not on a Euclidean space R_N, but on some other kind of space (e.g., the two-dimensional surface of a sphere in three-dimensional space). The standard procedure is to choose some point and to map this point and all points in the neighborhood (i.e., "nearby") of this point in a 1-1 manner onto points in the region of the origin of a space R_N of the appropriate dimensionality (Fig. 2.4). Basis vectors $\mathbf{e}_i$ can be chosen in R_N; the coordinates x^i of $\phi(p')$ can then be used to construct an expression for any function defined at p' in the vicinity of p. Since the $\partial/\partial x^i$ in the neighborhood of p have the same transformation properties as $\mathbf{e}_i$ in R_N, they are called "tangent vectors to the space at p'."

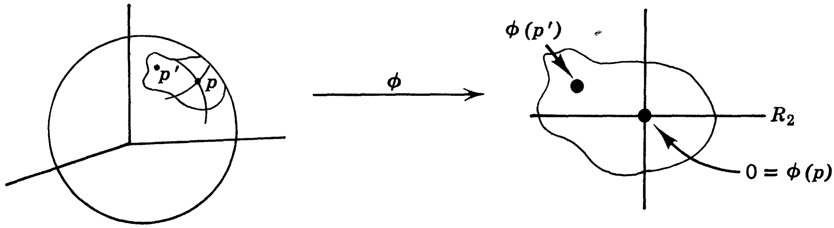


FIG. 2.4 THE MAPPING ϕ MAPS A NEIGHBORHOOD OF A POINT p ONTO A NEIGHBORHOOD OF THE ORIGIN OF R_N. THE MAPPING ϕ PRESERVES THE CONCEPT OF NEARNESS. IT IS A (LOCAL) ISOMORPHISM.

Two fundamental theorems of mathematical analysis are extensions to non-Euclidean spaces of the simple concepts we have encountered for linear vector spaces R_N, V_N. The **chain rule**

$$\frac{\partial x'^j}{\partial x''^k}\frac{\partial x^i}{\partial x'^j} = \frac{\partial x^i}{\partial x''^k}$$

is an expression of the group property of successive nonsingular transformations. And the **inverse function theorem** (or **implicit function theorem**)

$$x^i = x^i(x'^1, x'^2, \ldots, x'^N) \qquad \text{and} \qquad \left\| \frac{\partial x^i}{\partial x'^j} \right\|_p \neq 0 \qquad \text{implies}$$

that an inverse transformation $x'^j = x'^j(x^1, x^2, \ldots, x^N)$ exists (and can be constructed in the vicinity of p) is a statement that every group operation has an inverse.

5. FULLY ANTISYMMETRIC SUBSPACES—VOLUME ELEMENT. The completely antisymmetric subspace of $(V_N)^r$ is spanned by $\binom{N}{r} = N!/[r!(N-r)!]$ basis vectors. In particular, for $r = N$ there is only one basis vector

$$\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_N \tag{2.30}$$

in this subspace. This is fully antisymmetric: interchange of any two adjacent vectors multiplies this basis by (-1):

$$\mathbf{e}_2 \wedge \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \cdots \wedge \mathbf{e}_N = -\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \cdots \wedge \mathbf{e}_N. \tag{2.31}$$

The basis (2.30) is also called the **volume element** associated with the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ in V_N. Under an arbitrary change of basis

$$\mathbf{e}'_i = A_i{}^j\mathbf{e}_j \tag{2.32}$$

this new basis $\mathbf{e}_1' \wedge \mathbf{e}_2' \wedge \cdots \wedge \mathbf{e}_N'$ is a multiple of the basis $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_N$:

$$\mathbf{e}_1' \wedge \mathbf{e}_2' \wedge \cdots \wedge \mathbf{e}_N' = \det \|A\| \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_N \tag{2.33}$$

The multiplicative factor $\det \|A\|$ is called the **determinant** of the transformation.

$$\det \|A\| = \sum_{\substack{\text{all} \\ \sigma \in P_N}} (\pm)\sigma A_1{}^{i_1} A_2{}^{i_2} \cdots A_N{}^{i_N} \tag{2.34}$$

The sum extends over all group operations $\sigma \in P_N$. The σ permute the i_j among themselves. The $(\pm)$ is chosen depending on whether σ is constructed from an even or an odd number of the bases or generators of P_N.

The subset of elements of $Gl(N)$ which preserves volume,

$$\mathbf{e}_1' \wedge \mathbf{e}_2' \wedge \cdots \wedge \mathbf{e}_N' = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_N \tag{2.35}$$

forms a group defined by $\det \|A\| = +1$. These volume-preserving groups of transformations in R_N, C_N, are called the special linear groups; they are denoted $Sl(N, r)$ and $Sl(N, c)$.

The group $Gl(N, c)$ has two additional subgroups: $Sl_1(N, c)$, with real determinant, and $Sl_2(N, c)$, with complex determinant of modulus unity. The group $Gl(N, q)$ has only one unimodular subgroup $Sl(N, q)$ of matrices with determinant of modulus unity. This comes about because of the noncommutativity of the quaternion field.

III. Metric-preserving Groups

1. THE METRIC. To complete the enumeration and classification of the classical (matrix) groups, it is necessary to introduce one additional concept: the metric.

Definition. A **metric** function on a vector space V is a mapping of a pair of vectors into a number in the field F associated with the vector space.

$$(\mathbf{v}_1, \mathbf{v}_2) = f \qquad \mathbf{v}_1, \mathbf{v}_2 \in V, f \in F \tag{3.1}$$

This mapping obeys

$$(\mathbf{v}_1, \alpha\mathbf{v}_2 + \beta\mathbf{v}_3) = \alpha(\mathbf{v}_1, \mathbf{v}_2) + \beta(\mathbf{v}_1, \mathbf{v}_3) \tag{3.2}$$

and

$$(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_3)\alpha + (\mathbf{v}_2, \mathbf{v}_3)\beta \tag{3.2b}$$

or

$$(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_3)\alpha^* + (\mathbf{v}_2, \mathbf{v}_3)\beta^* \tag{3.2s}$$

Metrics obeying conditions (3.2) and (3.2b) are called **bilinear** metrics; those obeying (3.2) and (3.2s) are called **sesquilinear**. The metric is completely specified once its action on each pair of basis vectors is specified

$$
\begin{aligned}
(\mathbf{e}_i, \mathbf{e}_j) &= g_{ij} \\
(\mathbf{v}, \mathbf{u}) &= (v^i\mathbf{e}_i, u^j\mathbf{e}_j) = u^j g_{ij} v^{i(*)}
\end{aligned} \tag{3.3}
$$

The transformation properties of the metric function g_{ij} under a change of basis $\mathbf{e}'_i = A_i{}^r\mathbf{e}_r$ are given by

$$
\begin{aligned}
g'_{ij} &= (\mathbf{e}'_i, \mathbf{e}'_j) = (A_i{}^r\mathbf{e}_r, A_j{}^s\mathbf{e}_s) \\
&= A_j{}^s g_{rs} A_i^{r(*)}
\end{aligned} \tag{3.4}
$$

For real vector spaces, g'_{ij} has transformation properties identical with $\mathbf{e}'_i \otimes \mathbf{e}'_j$:

$$
\begin{aligned}
\mathbf{e}'_i \otimes \mathbf{e}'_j &= A_i{}^r A_j{}^s \qquad \mathbf{e}_r \otimes \mathbf{e}_s \\
g'_{ij} &= A_i{}^r A_j{}^s \qquad g_{rs}
\end{aligned} \tag{3.5}
$$

For real vector spaces the metric is a second-rank covariant tensor. For complex vector spaces, the bilinear metric also has the transformation properties of a second-rank covariant tensor.

Other definitions of metric often impose one or more additional properties which the metric function must satisfy. For example, a symmetry requirement is sometimes imposed:

$$(\mathbf{u}, \mathbf{v}) = \pm(\mathbf{v}, \mathbf{u}) \qquad g_{ij} = \pm g_{ji} \tag{3.6b}$$

$$(\mathbf{u}, \mathbf{v}) = \pm(\mathbf{v}, \mathbf{u})^* \qquad g_{ij} = \pm g_{ji}^* \tag{3.6s}$$

Sometimes the demand for positive definiteness is imposed:

$$(\mathbf{v}, \mathbf{v}) \geq 0 \qquad (\mathbf{v}, \mathbf{v}) = 0 \Rightarrow \mathbf{v} = 0 \tag{3.6$'$}$$

Both these requirements are too stringent for our purposes.

By including some sort of symmetry as a requirement which a metric must satisfy, we exclude a number of mathematical structures with possible physical applications. Most prominent among these is the Einstein theory of the nonsymmetric field. Furthermore, we become deeply concerned with metrics that are symmetric and metrics that are antisymmetric.

The positive definiteness requirement is altogether too restrictive for our purposes. It excludes the metric

$$
g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -c^2 \end{pmatrix} \tag{3.7}
$$

(with respect to the contravariant bases x, y, z, t) used in special relativity.

Bilinear metrics are often used to "raise and lower indices." We illustrate what we have in mind by an example. In coordinate system S with bases $\mathbf{e}_i$ we construct the quantity

$$x_i = x^j g_{ij}$$

In S' related to S by $\mathbf{e}'_i = A_i{}^r \mathbf{e}_r$, we have

$$\begin{aligned} x'_i &= x'^j g'_{ij} \\ &= x^t A^{-1}{}_t{}^j A_j{}^s g_{rs} A_i{}^r \\ &= (x^s g_{rs}) A_i{}^r \end{aligned}$$

The quantity x_i has covariant transformation properties. The metric g_{ij} has been used to transform the contravariant vector components x^j into covariant components $x_i = x^j g_{ij}$.

The inverse procedure can be carried out when the metric g_{ij} is nonsingular, that is, when

$$\det \|g_{ij}\| \neq 0$$

Under this circumstance the matrix g_{ij} has a unique inverse

$$(g^{-1})_{ij} = g^{ij}$$

The matrix elements g^{ij} of g^{-1} can be explicitly constructed to prove uniqueness. However, the uniqueness is more easily established by the following consideration: when $\|g\| \neq 0$, g, as a matrix, is an element in one of the general linear groups $Gl(N, -)$ and so has a unique inverse. Then it is easy to show that when x_i and x^j are related by

$$x^j = g^{ij} x_i$$

and x_i has covariant transformation properties, x^j transforms contravariantly.

2. KINDS OF METRICS. Once an additional structure, in the form of a metric, has been imposed on an N-dimensional vector space over a field F, it is no longer true that all N-dimensional spaces (over F) are essentially equivalent. There will be as many distinct N-dimensional spaces over F as there are distinct kinds of metrics.

An arbitrary metric can be written as the sum of a symmetric and an antisymmetric part

$$
\begin{aligned}
g_{ij} &= \overset{\vee}{g}_{ij} + \hat{g}_{ij} \\
\overset{\vee}{g}_{ij} &= \tfrac{1}{2}(g_{ij} + g_{ji}^{(*)}) = g_{ji}^{(*)} \\
\hat{g}_{ij} &= \tfrac{1}{2}(g_{ij} - g_{ji}^{(*)}) = -g_{ji}^{(*)}
\end{aligned}
\tag{3.8b(s)}
$$

Although an arbitrary metric cannot be put into a canonical form, there exist canonical forms for all symmetric sesquilinear metrics and nonsingular antisymmetric bilinear metrics.

1. It is always possible to find a transformation $\mathbf{e}_i' = S_i{}^r \mathbf{e}_r$ that brings a symmetric sesquilinear metric $\overset{\vee}{g}_{ij}$ into diagonal form

$$
\overset{\vee}{g}{}'_{ij} = S_j{}^s \overset{\vee}{g}_{rs} S_i{}^{r*} = \lambda_i \delta_{ij} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}
\tag{3.9}
$$

The λ_i are all real. If the bases $\mathbf{e}_i'$ are renormalized by the prescription

$$
\begin{aligned}
\mathbf{f}_i &= \frac{\mathbf{e}_i'}{\sqrt{|\lambda_i|}} \qquad && \lambda_i \neq 0 \\
\mathbf{f}_i &= \mathbf{e}_i' && \lambda_i = 0
\end{aligned}
\tag{3.10}
$$

the metric with respect to the $\mathbf{f}_i$ has the canonical form

$$
\begin{aligned}
\overset{\vee}{g}_{\text{canonical}} &= \left(\begin{array}{c|c|c} +1 I_{N_+} & & \\ \hline & 0 I_{N_0} & \\ \hline & & -1 I_{N_-} \end{array}\right) \\
&= (N_+, N_0, N_-)
\end{aligned}
\tag{3.11}
$$

where I_d is the $d \times d$ unit matrix, with $+1$ on the major diagonal and 0 elsewhere.

There are as many different N-dimensional vector spaces with symmetric sesquilinear metrics over a field F as there are distinct choices of canonical metrics (N_+, N_0, N_-), with $N_+ + N_0 + N_- = N$. This number is

$$
\frac{(N+2)!}{N!\,2!}
$$

If we demand that the metric be nonsingular, then $N_0 = 0$ and there are $N + 1$ distinct spaces V_N. If we demand that the metric be positive definite, $N_0 = N_- = 0$ and there is only one space V_N.

The process of going into a coordinate system in which the metric has a canonical form has a great deal of physical content. The transformation of coordinates from a real symmetric metric $g_{\mu\nu}(p)$ at space–time point p to the canonical form (3.7) at p is physically the process of going from an arbitrary local coordinate system around p to a coordinate system that is freely falling with respect to the local gravitational field, within the framework of the general theory of relativity.[3,9]

2. If $\hat{g}_{ij}$ is bilinear, antisymmetric, and nonsingular, then it also possesses a canonical form. But we can write

$$\|g\| = \|g^t\| = \|-g\| = (-)^N\|g\|. \tag{3.12}$$

where t denotes the matrix transpose $(M^t)_{ij} = M_{ji}$.

Therefore $\hat{g}$ is nonsingular only for N even. For N even $= 2M$, there exists a nonsingular transformation

$$\mathbf{e}'_i = A_i{}^r\mathbf{e}_r$$

which brings $\hat{g}$ into the block diagonal form:

$$\hat{g}'_{ij} = \begin{pmatrix} \begin{matrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{matrix} & & & \\ & \begin{matrix} & \lambda_2 \\ -\lambda_2 & \end{matrix} & & \\ & & & \\ & & & \begin{matrix} 0 & \lambda_M \\ -\lambda_M & 0 \end{matrix} \end{pmatrix} \tag{3.13}$$

Once again the metric can be brought into a canonical form by renormalizing the basis vectors.

$$(\mathbf{f}_M, \ldots, \mathbf{f}_2, \mathbf{f}_1, \mathbf{f}_{-1}, \mathbf{f}_{-2}, \ldots, \mathbf{f}_{-M})$$

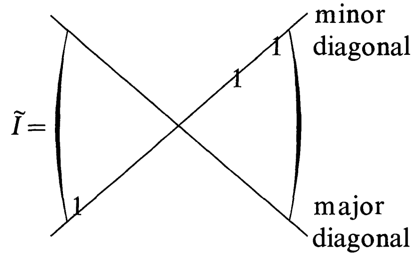
$$(\mathbf{f}_r, \mathbf{f}_{-r}) = \text{sign}\ (r) \qquad r = \pm 1, \pm 2, \ldots, \pm M$$

All other inner products vanish.

The canonical form for the nonsingular, bilinear, antisymmetric metric of V_{2M}, with respect to these bases, is

$$\hat{g}_{\text{canonical}} = \left(\begin{array}{c|c} 0 & +1\tilde{I}_M \\ \hline -1\tilde{I}_M & 0 \end{array}\right) \tag{3.14}$$

where $\tilde{I}_d$ is the $d \times d$ matrix with $+1$ on the *minor* diagonal and zero everywhere else:



(3.15)

Example. With respect to the metric in R_8

$$g = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& -1 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & 1 \\ &&&&&& -1 & 0 \end{pmatrix} \tag{3.16}$$

the inner product of vectors $x^i\mathbf{e}_i$ and $y^j\mathbf{e}_j$ is

$$(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4 + x^7y^8 - x^8y^7 \tag{3.17}$$

However, the inner product of a vector with itself is much simpler:

$$(\nabla, \nabla) = \frac{\partial}{\partial x_i} g_{ij} \frac{\partial}{\partial x_j} = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2} - \frac{\partial^2}{\partial x_4{}^2} \tag{3.18}$$

3. WEYL UNITARY TRICK. A real space with a signature (N_+, N_-) can be converted to a space with metric $(N_+ + N_-, 0)$ by choosing a new set of bases

$$(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{N_+}, \mathbf{e}_{N_++1}, \ldots, \mathbf{e}_{N_++N_-}) \rightarrow \\ (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{N_+}, i\mathbf{e}_{N_++1}, \ldots, i\mathbf{e}_{N_++N_-}) \tag{3.19}$$

Of course, we have to go outside the field of real numbers to perform this transformation. For example, the space–time of special relativity has metric $(+++-)$ with respect to the real contravariant bases (x, y, z, ct) but metric $(++++)$ with respect to (x, y, z, ict). This transformation from a mixed to a positive metric is called the **Weyl unitary trick**. It was apparently first used by Minkowski.

4. METRICS IN FUNCTION SPACES. The set of functions defined on a set (or space) of points itself forms a linear vector space. We are therefore at liberty to define a metric on such a space. If f, h are complex-valued functions on a real space R_N, a useful definition for the inner product of f and h is

$$(f, h) = \iint f^*(p)g(p, p')h(p') \; d``p"d``p'" \tag{3.20}$$

We can no longer use discrete indices on the metric g because the vector space is ∞ dimensional. A convenient set of bases are the delta functions

$$e_p = \delta(x - p) \tag{3.21}$$

It is usual to choose a diagonal metric $g(p, p') = \delta(p, p')$. With respect to some coordinate system in R_N, we put

$$(f, h) = \int \cdots \int f^*(x)h(x) \, dx^1 \cdots dx^N \tag{3.22}$$

However, it is occasionally useful to choose a symmetric though nondiagonal metric. An example of such a metric is

$$g(x, y) = e^{-\Sigma_1{}^N(x^i - y^i)^2/\lambda} \tag{3.23}$$

Under these conditions the metric function is nonlocal and has the analytic structure

$$(f, h) = \int d^N x \int d^N y f^*(x) e^{-(x-y)^2/\lambda} h(y) \tag{3.24}$$

Nonlocal metrics are called form factors[10,11] and have been used in the quantum theory.*

5. METRIC-PRESERVING GROUPS

THEOREM. *The subset of transformations of basis in V_N which preserves the mathematical structure of a metric forms a subgroup of $Gl\ (N. -)$.*

* For the original motivation for discussing such nonlocal metrics in the quantum theory, see Ref. 10. For further discussions of form factors, see Ref. 11.

Proof. We check that the elements of this subset obey the four group postulates.

(a) Closure. Let

$$
\begin{aligned}
g'_{kl} &= A_l{}^j g_{ij} A_k^{(*)i} = g_{kl} \\
g''_{mn} &= B_n{}^l g'_{kl} B_m^{(*)k} = g'_{mn} = g_{mn} \\
g''_{mn} &= B_n{}^l g'_{kl} B_m^{(*)k} \\
\| \quad & \quad \| \\
g'_{mn} &= B_n{}^l A_l{}^j g_{ij} A_k^{(*)i} B_m^{(*)k} \\
\| \quad & \quad \| \\
g_{mn} &= (BA)_n{}^j g_{ij} (BA)_m^{(*)i}
\end{aligned}
\tag{3.25}
$$

(b) Associativity. Associativity holds in the subset, since it holds in the original group.

(c) Identity. The identity is clearly an element of this subset.

(d) Unique inverse. Suppose A leaves the metric invariant (unchanged). Then A has a unique inverse B. We check that B also leaves the metric invariant.

$$
\begin{aligned}
g_{kl} &= A_l{}^j g_{ij} A_k^{(*)i} \\
B_s{}^l g_{kl} B_r^{(*)k} &= B_s{}^l A_l{}^j g_{ij} A_k^{(*)i} B_r^{(*)k} \\
&= (BA)_s{}^j g_{ij} (BA)_r^{(*)i} \\
&= \delta_s{}^j g_{ij}\ \delta_r{}^i = g_{rs}
\end{aligned}
\tag{3.26}
$$

This theorem is valid for all real and complex metric-preserving matrix groups. It is also valid for quaternion groups that preserve sesquilinear metrics, since two quaternions obey $(q'q'')^* = (q'')^*(q')^*$. It is not true for quaternion matrices and bilinear metrics, since two quaternions do not generally commute ($q'q'' \neq q''q'$), and as a result

$$A_k{}^i B_r{}^k \neq B_r{}^k A_k{}^i = (BA)_r{}^i$$

in the proof above.

Nevertheless, it is still possible to associate subgroups of $Gl(N, q)$ with groups that preserve bilinear metrics. This is done in the following way. Each quaternion in $Gl(N, q)$ is replaced by the corresponding 2×2 complex matrix using (3.13) of Chapter 1. The subset of matrices in this complex $2N \times 2N$ matrix representation of $Gl(N, q)$ that leaves invariant a bilinear metric forms a group, since the theorem is valid for bilinear metrics in complex linear vector spaces. We can associate an $N \times N$ quaternion-valued matrix with each $2N \times 2N$ complex-valued matrix in the resulting groups that preserve bilinear metrics in the space C_{2N}, which is a representation for the space Q_N. By abuse of the language, we refer to the groups constructed in this way simply as "bilinear metric-preserving quaternion groups." We note that we can construct a group preserving a non-singular antisymmetric metric in Q_N, for N odd as well as N even.

This theorem allows us to develop a rich structure of additional classical matrix groups. These are the matrix subgroups of $Gl(N, F)$ that preserve non-singular metrics. We now define these groups.

Definition. Groups preserving bilinear symmetric metrics are called **orthogonal**.

Definition. Groups preserving bilinear antisymmetric metrics are called **symplectic**.[12]

Definition. Groups preserving sesquilinear symmetric metrics are called **unitary**.

Notation. Orthogonal groups preserving metrics (N_+, N_-) in R_N, C_N, Q_N $(N = N_+ + N_-)$ are denoted $O(N_+, N_-; r)$, $O(N_+, N_-; c)$, and $O(N_+, N_-; q)$.

Notation. Symplectic groups on the spaces R_{2N}, C_{2N}, Q_N are denoted $Sp(2N, r)$, $Sp(2N, c)$, and $Sp(N, q)$.

Notation. Unitary groups preserving metrics (N_+, N_-) in R_N, C_N, Q_N $(N = N_+ + N_-)$ are denoted $U(N_+, N_-; r)$, $U(N_+, N_-; c)$, and $U(N_+, N_-; q)$.

At this point, we make a number of observations:

1. The groups $U(N_+, N_-; r)$ and $O(N_+, N_-; r)$ are identical, since there can be no difference between bilinear and sesquilinear metrics in a real vector space.

2. At a later stage, we will be working with the **unitary–symplectic** groups, which are the **intersection** of unitary groups and symplectic groups, both acting in the same vector space. That is, the elements of the unitary–symplectic groups are elements in both unitary groups and symplectic groups:

$$USp(2N_+, 2N_-) = U(2N_+, 2N_-; c) \cap Sp(2N; c) \qquad (N = N_+ + N_-)$$

These groups are isomorphic with unitary groups in a quaternion space:

$$USp(2N_+, 2N_-) \cong U(N_+, N_-; q)$$

The isomorphism comes about as follows. Each quaternion number can be represented by a 2×2 complex matrix. Under such a substitution, each quaternion-valued $N \times N$ matrix becomes a complex-valued $2N \times 2N$ matrix. In particular, elements in $U(N_+, N_-; q)$ become elements within $U(2N_+, 2N_-; c)$. The 2×2 complex matrices representing quaternions are not arbitrary, but have the particular form given by (3.13) in Chapter 1. The particular structure for these 2×2 submatrices tells us that the $2N \times 2N$ complex matrices also preserve an antisymmetric bilinear metric and thus belong also to $Sp(2N = 2N_+ + 2N_-; c)$.

3. The unitary groups preserving a positive definite metric obey the equations

$$1 = \sum_{r=1}^{N} A_i{}^r A_i{}^{r*} = \sum_{r=1}^{N} |A_i{}^r|^2$$

Each individual matrix element is bounded,

$$0 \le |A_i{}^r| \le 1$$

and so, in some sense, the entire group is bounded. Such "bounded" groups are called **compact**.

4. The relation between the bilinear and sesquilinear metrics and their symmetry or antisymmetry properties, on the one hand, and the classical groups, on the other, appear in Table 2.1. Although the table seems to indicate that an entire class of classical groups has been left out, this turns out to be incorrect: we see later that this list contains all the classical simple Lie groups.

TABLE 2.1

ALL THE CLASSICAL SIMPLE LIE GROUPS PRESERVE EITHER A BILINEAR OR A SESQUILINEAR METRIC, WHICH IS EITHER SYMMETRIC OR ANTISYMMETRIC

	Functional Form of Metric	
Symmetry Properties of Metric	Bilinear	Sesquilinear
Symmetric	$g_{ij} = +g_{ji}$ Orthogonal $O(N_+, N_-; r)$ $O(N_+, N_-; c)$ $O(N_+, N_-; q)$	$g_{ij} = +g_{ji}^*$ Unitary $U(N_+, N_-; c)$ $U(N_+, N_-; q) = USp(2N_+, 2N_-)$
Antisymmetric	$g_{ij} = -g_{ji}$ Symplectic $Sp(2N, r)$ $Sp(2N, c)$ $Sp(N, q)$	

5. The metric-preserving groups which are in addition volume-preserving are called the special metric-preserving groups and are denoted by an additional S:

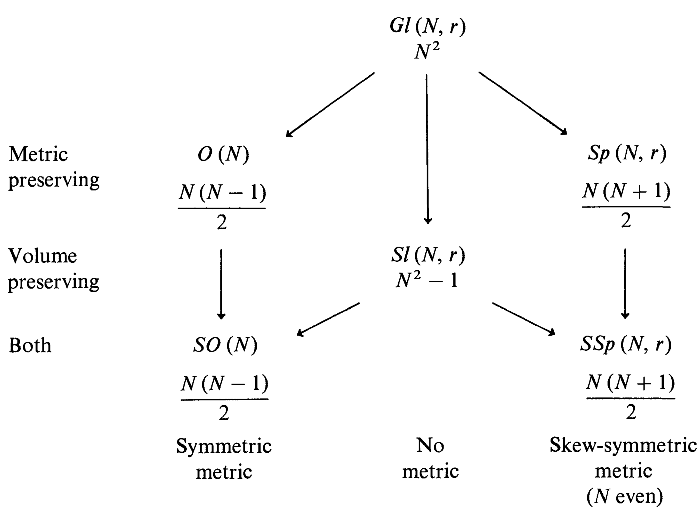
$$Sl(N, c) \cap U(N, c) = SU(N, c)$$

6. With each $N \times N$ quaternion matrix group we associate a $2N \times 2N$ complex matrix group using (3.13) of Chapter 1. We have the following identifications:

$$\begin{aligned} Sl(N, q) &= SU^*(2N) \\ U(N, q) &= USp(2N) \\ Sp(N, q) &= USp(2N) \\ O(N, q) &= SO^*(2N) \end{aligned}$$

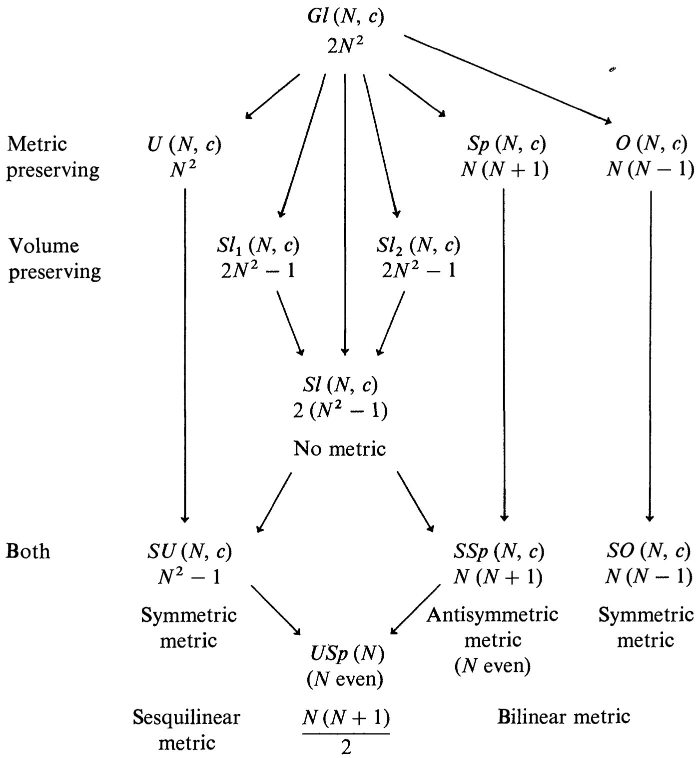
IV. Properties of the Classical Groups

1. RELATIONSHIPS BETWEEN THE CLASSICAL GROUPS. We have enumerated all the classical groups. The relationships between them are illustrated in Fig. 2.5. The restriction of a group G to a subgroup H is

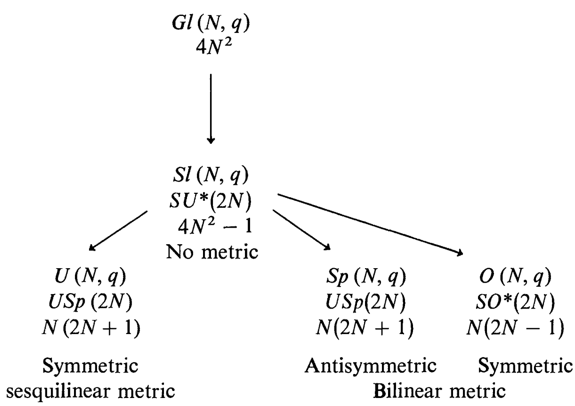


(a)

FIG. 2.5 THE RELATIONS AMONG VARIOUS KINDS OF CLASSICAL GROUPS AND THEIR DIMENSIONALITIES (a) REAL-VALUED GROUPS,



(b)



(c)

FIG. 2.5 (b) COMPLEX-VALUED GROUPS, (c) QUATERNION-VALUED GROUPS.

indicated by a downward arrow ($\swarrow$ or $\downarrow$ or $\searrow$) and denoted here and subsequently by $G \downarrow H$ or $\begin{matrix} G \\ \downarrow \\ H. \end{matrix}$ It is clear that the subfield restrictions $Q \downarrow C \downarrow R$ given by

$$\begin{array}{ccc} Q & & q_0 + q_1 i + q_2 j + q_3 k \\ \downarrow & & \downarrow \\ C & & q_0 + q_1 i \\ \downarrow & & \downarrow \\ R & & q_0 \end{array}$$

induce subspace restrictions $Q_N \downarrow C_N \downarrow R_N$, and that these in turn induce restrictions on the classical groups: $Gl(N, q) \downarrow Gl(N, c) \downarrow Gl(N, r)$. Thus there are additional subgroup restrictions possible, indicated schematically by Figs. 2.5$c \downarrow$ 2.5$b \downarrow$ 2.5a.

2. DIMENSIONS OF THE CLASSICAL GROUPS. Each of these matrix groups is a set of "points" (each "point" is a matrix) on which a combinatorial operation (matrix multiplication) is defined. The general procedure for studying the properties of such spaces has been indicated in Fig. 2.4. Specifically, we map the neighborhood of each point "p" in a 1-1 way into the neighborhood of the origin of R_η, for some η. Two "points" in a matrix group are close together, or in the same neighborhood, if each matrix element for one "point" is near the corresponding matrix element for the other group operation. That is, A and A' are "close together" if

$$|A_i{}^j - A_i'^j| \text{ is small} \qquad \text{for all } i, j$$

Since we are dealing with groups, we can move the neighborhood of any "p" into the neighborhood of the origin, merely by multiplying by p^{-1}. Thus we can learn a great deal about the group by studying in detail *only* the properties of the group near the origin. Leaving this study for the following chapter, we now merely compute the dimensionalities (the η in R_η) for these groups.

Let $f = 1, 2, 4$ be the dimensionality of the fields of real numbers, complex numbers, and quaternions when they are considered as vector spaces over the real field. The general linear groups consist of $N \times N$ matrices, in which each of the N^2 matrix elements is arbitrary. The dimensionality of the general linear groups is therefore fN^2.

TABLE 2.2

DIMENSIONALITIES OF THE SMALLER CLASSICAL GROUPS

	Real Groups				Complex Groups				
	$Gl(N, r)$	$Sl(N, r)$	$O(N, r)$ $SO(N, r)$	$Sp(N, r)$ $SSp(N, r)$ N even	$Gl(N, c)$	$Sl_1(N, c)$ $Sl_2(N, c)$	$Sl(N, c)$	$U(N, c)$	$SU(N, c)$
N	N^2	$N^2 - 1$	$\frac{N(N-1)}{2}$	$\frac{N(N+1)}{2}$	$2N^2$	$2N^2 - 1$	$2N^2 - 2$	N^2	$N^2 - 1$
1	1				2	1		1	
2	4	3	1	3	8	7	6	4	3
3	9	8	3		18	17	16	9	8
4	16	15	6	10	32	31	30	16	15
5	25	24	10		50	49	48	25	24
6	36	35	15	21	72	71	70	36	35

The remaining classical groups all occur as subgroups of the general linear groups. The subgroup restrictions assume the form of equations placing constraints on the possible values of the N^2 matrix elements belonging to the parent group $Gl(N, -)$. The special linear subgroups determined by

$$\det \|A_i{}^j\| \text{ has modulus unity}$$

have only one constraint placed on them. The dimensionalities of these subgroups are thus $fN^2 - 1$. The subgroup $Sl(N, c)$ of $Gl(N, c)$ consists of those matrices in $Gl(N, c)$ which obey

$$\det \|A_i{}^j\| = 1 + 0i$$

These matrices have restrictions placed both on the real and imaginary parts of the determinant. The dimensionality of this matrix group is therefore $2N^2 - 2$.

The metric-preserving groups have constraints placed on them of the form

$$g_{rs} = A_s{}^j \quad g_{ij} \quad A_r{}^i \qquad g_{ij} = \pm g_{ji}$$
$$g_{rs} = A_s{}^j \quad g_{ij} \quad A_r{}^{i*} \qquad g_{ij} = +g_{ji}^*$$

There is one set of constraints for each independent equation contained in the foregoing relations. It can be seen that the equations arising from the choice $r > s$ can be obtained from those with $r < s$ from the symmetry properties, and therefore contain no new information. The number of constraints placed on the matrix elements A_i^j by the $N(N-1)/2$ choices $r > s$ is

TABLE 2.2 (*Continued*)

	Complex Groups			Quaternion Groups				
	$Sp(N, c)$ $SSp(N, c)$ N even	$USp(N)$ N even	$O(N, c)$ $SO(N, c)$	$Gl(N, q)$	$Sl(N, q)$ $SU^*(2N)$	$U(N, q)$ $USp(2N)$	$Sp(N, q)$ $USp(2N)$	$O(N, q)$ $SO^*(2N)$
N	$N(N+1)$	$\frac{N(N+1)}{2}$	$N(N-1)$	$4N^2$	$4N^2-1$	$N(2N+1)$	$N(2N+1)$	$N(2N-1)$
1				4	3	3	3	1
2	6	3	2	16	15	10	10	6
3			6	36	35	21	21	15
4	20	10	12	64	63	36	36	28
5			20	100	99	55	55	45
6	42	21	30	144	143	78	78	66

$$\frac{fN(N-1)}{2}$$

We now investigate the additional number of constraints determined from the equations arising from g_{rr}. In the case of a bilinear antisymmetric metric over a commutative field, we easily find

$$0 = g_{rr} = A_r{}^j A_r{}^i g_{ij} = 0$$

Therefore, the equations arising from the diagonal members of the bilinear antisymmetric metric are identically satisfied when the field is commutative. For the noncommutative quaternion field this is no longer the case, and each diagonal equation places one constraint on the matrix element. The symplectic groups therefore have the dimensionalities

$$Sp(N, r):\quad N^2 - \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$$

$$Sp(N, c):\quad 2N^2 - \frac{2N(N-1)}{2} = N(N+1)$$

$$Sp(N, q):\quad 4N^2 - \frac{4N(N-1)}{2} - N = N(2N+1)$$

For the groups $O(N, r)$, $O(N, c)$, and $O(N, q)$, each diagonal equation provides one, two, and three constraints, respectively. But for the unitary groups, the same set of equations is necessarily real; thus each equation provides only one constraint. The dimensionalities of the orthogonal and unitary groups are

$$
\begin{aligned}
O(N,f)&: \quad fN^2-\frac{fN(N-1)}{2}-fN=\frac{fN(N-1)}{2}\\
O(N,q)&: \quad 4N^2-\frac{4N(N-1)}{2}-3N=\frac{2N(2N-1)}{2}\\
U(N,f)&: \quad fN^2-\frac{fN(N-1)}{2}-N=\frac{fN^2+fN-2N}{2}
\end{aligned}
$$

These dimensionalities have been listed in Fig. 2.5 and in Table 2.2.

LOCAL

3. ISOMORPHISMS AND HOMOMORPHISMS AMONG THE CLASSICAL GROUPS. Not all the classical groups are distinct. There are numerous isomorphisms and homomorphisms among them, particularly among those of lower dimensionality. Clearly, groups of different dimensionality cannot be isomorphic. Some of the existing isomorphisms and homomorphisms are given below. local

$$
\begin{array}{ll}
\dim = 3: & SU(2,c) \cong SO(3,r) \cong USp(2) \cong U(1,q) \cong Sl(1,q)\\
 & SU(1,1;c) \cong SO(2,1;r) \cong Sp(2,r) \cong Sl(2,r)\\
\dim = 6: & SO(4,r) \cong SU(2,c) \otimes SU(2,c)\\
 & SO^*(4) \cong SU(2,c) \otimes Sl(2,r)\\
 & SO(3,1;r) \cong Sl(2,c)\\
 & SO(2,2;r) \cong Sl(2,r) \otimes Sl(2,r)\\
\dim = 10: & SO(5,r) \cong USp(4)\\
 & SO(4,1;r) \cong USp(2,2)\\
 & SO(3,2;r) \cong Sp(4,r)\\
\dim = 15: & SO(6,r) \cong SU(4,c)\\
 & SO(5,1;r) \cong SU^*(4) \cong Sl(2,q)\\
 & SO^*(6) \cong SU(3,1;c)\\
 & SO(4,2;r) \cong SU(2,2;c)\\
 & SO(3,3;r) \cong Sl(4,r)
\end{array}
$$

Résumé

The first three sections of this chapter introduced concepts that are useful for the enumeration of the classical groups.

First, we presented the concepts of change of basis, covariance, and contravariance, to introduce the general linear groups.

In the second section, we brought in the concepts of direct sum and product, symmetric reduction in tensor space, and volume in order to describe the volume-preserving special linear groups.

The third section, dealing with metrics and their properties, served to introduce the metric-preserving groups, and the last section treated the interrelations between the classical groups.

Exercises

1. Compute the change of basis matrix transformation obtained by rotating the coordinate system in R_3 about the z-axis through an angle θ. Here (x, y, z) forms an orthogonal right-handed system of coordinates. What are the transformation properties of the components of a vector in R_3 ?

2. Show that the matrix transformation obtained in Problem 1 consists of a direct sum of a 2×2 matrix and a 1×1 matrix.

3. Let $\mathscr{F}$ be the vector space of analytic functions defined on the straight line R_1. Under a change of basis obtained by moving the origin to the point a ($\neq 0$), how do the basis vectors x^n transform? How do the coordinates $f^{(n)}(0)$ of any vector $f(x) \in \mathscr{F}$ transform?

4. Let $\mathscr{F}_x$ be the vector space of analytic functions on the x-axis R_x, and let $\mathscr{F}_y$ be the space of analytic functions on the y-axis R_y.

(a) Show that an analytic function $g(x, y) \in \mathscr{F}_{xy}$ defined on the direct product space $R_x \otimes R_y$ cannot in general be written as a direct product of two functions $f_1(x) \in \mathscr{F}_x$ and $f_2(y) \in \mathscr{F}_y$:

$$g(x, y) \neq f_1(x) f_2(y)$$

(b) Write down a complete set of basis vectors in $\mathscr{F}_{xy}$.

5. The spatial part of an electron (spin $= \frac{1}{2}$) wave function for fixed time t is a complex function defined over all space coordinates (x, y, z). The spin part of the electron wave function is a vector in C_2. Show that the total electron wave function is a vector in a tensor product space. Write down a complete set of basis vectors for this space.

6. Think about why a state containing r photons is described by a vector within a completely symmetric subspace, but a state containing r electrons is described by a vector within a completely antisymmetric subspace. See References 4 and 13.

7. Compute a 4×4 transformation matrix S with the property

$$S^t\, G(p)\, S = G_{(\text{flat})}$$

$$G(p) = \left[\begin{array}{c|ccc} c^2\left(1 + \dfrac{2\Phi(p)}{c^2}\right) & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \end{array}\right] \qquad G_{(\text{flat})} = \left[\begin{array}{c|ccc} c^2 & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \end{array}\right]$$

What is the physical interpretation of S? See Reference 9.

8. Let $e_m = e^{im\phi}$. The e_m form a complete set of basis vectors for the linear vector space of L^2 functions on the interval $[0, 2\pi)$. Introduce a metric on this space by

$$(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f^*(\phi)g(\phi)\,d\phi$$

Show that

$$(e_m, e_n) = \delta_{m,n} \qquad -\infty \leq m, n \leq +\infty$$

9. Let $f_1, f_2, \ldots, f_n$ $(,\ldots)$ be a complete set of basis vectors in a linear vector space of finite (infinite) dimensionality. Introduce an inner product in this space defined by

$$(f_i, f_j) = g_{ij}$$

(a) Show that the vector

$$f'_{k+1} = f_{k+1} - \sum_{i=1}^{k} f'_i \frac{(f'_i, f_{k+1})}{(f'_i, f'_i)}$$

is orthogonal to all vectors f'_j, $j < k + 1$.

(b) Show that $f'_i \neq 0$.

(c) The vectors $f''_i = f'_i/\sqrt{(f'_i, f'_i)}$, $i = 1, 2, \ldots, n$ $(\cdots)$ form an ortho*normal* set of basis vectors for this space.

(d) Compute the transformation matrix $M_i{}^j$:

$$f''_i = M_i{}^j f_j$$

Show that $M_i{}^j = 0$ for $j > i$, so that $M_i{}^j$ consists of lower triangular matrices. The procedure just outlined is called **Gramm-Schmidt** orthonormalization.

10. Consider the set of analytic L^2 functions on the interval (a, b) of the straight line R_1. Introduce an inner product in this vector space by

$$(f, g) = \int_a^b f^*(x)g(x)W(x)\,dx$$

where $W(x)$ is a distribution function that is positive almost everywhere. A nonorthogonal basis for this space consists of $f_k = x^k$, $k = 0, 1, 2, \ldots$. Using the result of Problem 9, show that a suitable set of polynomial bases can be constructed in the interval (a, b), with the properties that:

(a) They are bases for this space.

(b) They are orthonormal with respect to $W(x)$.

(c) $f''_n = P_n(x)$ is a polynomial of degree n.

These polynomial functions have the following names:[14]

Polynomial Function	$P_n(x)$	$W(x)$	Comment	Interval
Jacobi	$P_n^{\alpha,\beta}(x)$	$(1-x)^\alpha(1+x)^\beta$	$\alpha, \beta > -1$	$(-1, +1)$
Gegenbauer	$G_n^\gamma(x)$	$(1-x^2)^{\gamma-1/2}$	$\alpha = \beta = \gamma - \frac{1}{2}$	$(-1, +1)$
Tchebicheff	$T_n(x)$	$(1-x^2)^{-1/2}$	$\gamma = 0$	$(-1, +1)$
Legendre	$P_n(x)$	1	$\gamma = \frac{1}{2}$	$(-1, +1)$
Laguerre	$L_n^\alpha(x)$	$x^\alpha e^{-x}$	$\alpha > -1$	$(0, \infty)$
Hermite	$H_n(x)$	$e^{-1/2x^2}$	—	$(-\infty, +\infty)$

11. A quantum mechanical system with r internal degrees of freedom is described by a vector $|\psi\rangle$ in C_r. For convenience, such state vectors are normalized to unit length: $\langle\psi|\psi\rangle = +1$. Show that the largest group of transformations of this state space into itself, which preserves normalization, is $U(r, c)$.

12. Let M consist of $n \times n$ matrices over the field f, let I_n be the unit $n \times n$ matrix, and let λ be a number in the field f. Then

(a) $(\lambda I_n)M = M(\lambda I_n)$ if $f = r =$ reals
(b) $(\lambda I_n)M = M(\lambda I_n)$ if $f = c =$ complex numbers
(c) $(\lambda I_n)M \neq M(\lambda I_n)$ if $f = q =$ quaternions
(d) $(\lambda I_n)M = M(\lambda I_n)$ if $f = q =$ quaternions *but* λ is a real number

13. List those groups in Table 2.2 which have applications in physical theory.

14. (For those who already know the theory of finite groups.) Let G be a finite group of order vol G, with group operations $g_1 = \mathrm{Id}, g_2, g_3, \ldots, g_n$, where $n =$ vol G. Let Γ^λ be a unitary irreducible representation of G whose dimension is dim λ. Each matrix element $\Gamma_{ij}{}^\lambda(g_k)$ is a function defined on the set of group operations. According to Problem 9 of Chapter 1, the set of functions defined on this space of n points is a linear vector space of dimensionality n. An inner product is introduced in this function space as follows:

$$(\phi, \psi) = \sum_{i=1}^{n} \phi^*(g_i)\psi(g_i) \equiv \sum_{g \in G} \phi^*(g)\psi(g)$$

(a) The **orthogonality relation**

$$\sum_{g \in G} \sqrt{\dim \lambda/\mathrm{vol}\, G}\, \Gamma_{i'j'}^{\lambda'*}(g) \sqrt{\dim \lambda/\mathrm{vol}\, G}\, \Gamma_{ij}{}^\lambda(g) = \delta^{\lambda'\lambda}\, \delta_{i'i}\, \delta_{j'j}$$

shows that the functions $\sqrt{\dim \lambda/\mathrm{vol}\, G}\, \Gamma_{ij}{}^\lambda(g)$ are orthonormal in this linear vector space of functions.

(b) The **completeness relation**

$$\sum_\lambda \sum_i \sum_j \sqrt{\dim \lambda/\mathrm{vol}\, G}\, \Gamma_{ij}{}^{\lambda*}(g) \sqrt{\dim \lambda/\mathrm{vol}\, G}\, \Gamma_{ij}{}^\lambda(g') = \delta(g, g')$$

shows that these functions span this vector space.

(c) Introduce the notation

$$\sqrt{\dim \lambda/\mathrm{vol}\, G}\, \Gamma_{ij}{}^\lambda(g) \equiv \left\langle g \middle| \begin{matrix}\lambda \\ ij\end{matrix} \right\rangle$$

$$\sqrt{\dim \lambda/\mathrm{vol}\, G}\, \Gamma_{ij}{}^{\lambda*}(g) \equiv \left\langle \begin{matrix}\lambda \\ ij\end{matrix} \middle| g \right\rangle$$

and write the orthogonality and completeness relations in Dirac notation:

$$\sum_{g \in G} \left\langle \begin{matrix}\lambda' \\ i'j'\end{matrix} \middle| g \right\rangle \left\langle g \middle| \begin{matrix}\lambda \\ ij\end{matrix} \right\rangle = \left\langle \begin{matrix}\lambda' \\ i'j'\end{matrix} \middle| \begin{matrix}\lambda \\ ij\end{matrix} \right\rangle$$

$$\sum_\lambda \sum_i \sum_j \left\langle g' \middle| \begin{matrix}\lambda \\ ij\end{matrix} \right\rangle \left\langle \begin{matrix}\lambda \\ ij\end{matrix} \middle| g \right\rangle = \langle g' | g \rangle$$

How can each of the following be interpreted?

$$|g\rangle, \qquad \left|\begin{matrix}\lambda \\ ij\end{matrix}\right\rangle, \qquad \left\langle g\left|\begin{matrix}\lambda \\ ij\end{matrix}\right.\right\rangle$$

15. Prove that, if $S \to S_{i_1 i_2 \cdots i_m}^{j_1 j_2 \cdots j_n}$ is a tensor with m covariant and n contravariant indices, and if $T \to T_{k_1 k_2 \cdots k_{m'}}^{l_1 l_2 \cdots l_{n'}}$ is a tensor with m' covariant and n' contravariant indices, then

$$U = ST \to U_{i_1 i_2 \cdots i_m k_1 k_2 \cdots k_{m'}}^{j_1 j_2 \cdots j_n l_1 l_2 \cdots l_{n'}} = S_{i_1 \cdots i_m}^{j_1 \cdots j_n} T_{k_1 \cdots k_{m'}}^{l_1 \cdots l_{n'}}$$

is a tensor with $m + m'$ covariant and $n + n'$ contravariant indices. Prove that if $S \to S_{i_1 i_2 \cdots i_m}^{j_1 j_2 \cdots j_n}$ is a tensor with m covariant and n contravariant indices, then

$$S' = \sum_{l = j_1 = i_1} S_{l i_2 \cdots i_m}^{l j_2 \cdots j_n}$$

is a tensor with one less covariant and contravariant index. In particular, prove that if $T^{ijk_3 k_4 \cdots k_n}$ is a tensor of contravariant rank n, then $\sum_i \sum_j g_{ij} T^{ijk_3 \cdots k_n}$ is a contravariant tensor of rank $n - 2$.

Notes and References

1. B. L. van der Waerden. [1]
2. I. M. Gel'fand. [1]
3. A. Lichnerowicz. [1]
4. M. Born. [1]
5. W. Rudin. [2,3]
6. R. C. Buck. [1]
7. T. M. Apostol. [1]
8. I. M. Singer, J. A. Thorpe. [1]
9. H. P. Robertson, T. W. Noonan. [1]
10. H. A. Bethe. [1]
11. N. N. Bogoliubov, D. V. Shirkov. [1]
12. C. Chevalley. [1]
13. R. P. Feynman. [3]
14. See, for example, Table I in H. Hochstadt. [2]

CHAPTER 3

Continuous Groups—Lie Groups

This chapter is concerned with the global properties of continuous groups. In the first section we define the concepts of continuous group and continuous group of transformations. The concepts are illustrated by a concrete example in Section II. The third section is devoted to the introduction of additional new concepts of an algebraic and topological nature. These enable us to break any continuous group into its component pieces. Only these components must then be studied; all continuous groups are built up in a very definite way from these basic building blocks. These are discrete groups and connected continuous groups. The latter are essentially the Lie groups whose structure we study in the following chapters.

In the fifth and final section we construct a reasonable definition of integration over a space that has algebraic properties in addition to its properties as space. Two such definitions are found, and the measures associated with these definitions are constructed explicitly. The entire process is illustrated by example.

I. Topological Groups

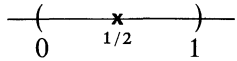
1. SOME BASIC DEFINITIONS. We begin our treatment of topological groups by presenting an adequate topological underpinning. Although this discussion of topological properties is rudimentary, it will be adequate for most of our purposes.

Let S be a **set** of points. If the point s_i is an element of the set S, we write $s_i \in S$. Now let S_1 and S_2 be two sets of points. We call S_1 a **subset** of S_2 if every point in S_1 is also contained in S_2. This is denoted by the inclusion symbol

$$S_1 \subset S_2$$

If there are points in S_2 not contained in S_1, S_1 is a **proper subset** of S_2. The set containing no points is called the **empty set** and is denoted by $\boldsymbol{\Phi}$.

Examples. The point $x = \frac{1}{2}$ is contained in the line segment connecting the points 0 and 1 of the real line. This segment is a subset of the real line



$$\tfrac{1}{2} \in (0, 1)$$
$$(0, 1) \subset R_1 = (-\infty, +\infty)$$

Definition. The **union** and the **intersection** of the set S_1 with the set S_2 are:

1. Union:

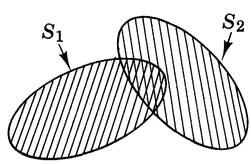
$$S_1 \cup S_2$$

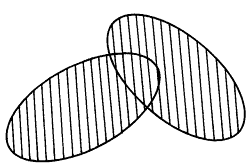
contains all points in *either* S_1 *or* S_2.

2. Intersection:

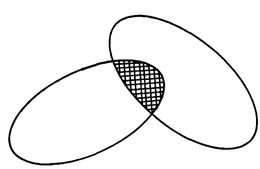
$$S_1 \cap S_2$$

contains all points in *both* S_1 *and* S_2.





$S_1 \cup S_2$



$S_1 \cap S_2$

Examples.

A topological space T is a set of points on which is placed a **topology** $\mathscr{T}$. The topology $\mathscr{T}$ is a choice (set) of subsets $S_1, S_2, \ldots$ of T:

$$S_i \subset T$$
$$S_i \in \mathscr{T}$$

The topology $\mathscr{T}$ obeys the following three axioms:

$\mathscr{T}1$: The empty set $\mathbf{\Phi}$ and the space T belong to $\mathscr{T}$. This is expressed symbolically as

$$\mathbf{\Phi} \in \mathscr{T}, \; T \in \mathscr{T}$$

$\mathscr{T}2$: Finite intersections of elements in $\mathscr{T}$ are elements in $\mathscr{T}$. Symbolically, we have

$$\bigcap_{i=1}^{\text{finite}} S_i \in \mathscr{T}$$

$\mathscr{T}3$: Arbitrary unions of elements in $\mathscr{T}$ are elements in $\mathscr{T}$. Symbolically, this is written as

$$\bigcup_{i=1}^{\text{finite orinfinite}} S_i \in \mathscr{T}$$

The elements S_i in the topology $\mathscr{T}$ are called **open sets**.

A topological space obeying the axioms $\mathscr{T}1$ to $\mathscr{T}3$ is too general for our present purposes. We must also assume some kind of **separability** axiom. We will assume that if p, q are distinct points of T, then open subsets S_p, S_q of T can be found that contain p and q, respectively, but which do not overlap. Symbolically:

$\mathscr{T}4$: If $p \in T$, $q \in T$, $p \neq q$, then there exist $S_p \in \mathscr{T}$, $S_q \in \mathscr{T}$ with the property $p \in S_p$, $q \in S_q$, $S_p \cap S_q = \mathbf{\Phi}$.

A topological space obeying the additional axiom $\mathscr{T}4$ is called a **Hausdorff** space.

Definition. An open set S_p containing p is called a **neighborhood** of p. Symbolically, we write $p \in S_p \in \mathscr{F}$.

Example. The two-dimensional plane R_2 is a set of points on which a standard topology is usually chosen. This topology is generated by the *interiors* of circles with arbitrary center and arbitrary nonzero radius. The topology consists of all such spheres together with all finite intersections and arbitrary unions of these sets. If, instead of circles, we choose squares with arbitrary center and edge length to generate the topology $\mathscr{T}$, we construct exactly the same topology generated by the earlier choice of circles. With this topology, R_2 is a Hausdorff space. Two distinct points p, q are separated by a nonzero distance $d(p, q)$. Then circles of radius $\frac{1}{3}\, d(p, q)$, centered on p and q, are open nonoverlapping subsets of R_2. Thus R_2 with the topology described previously, obeys axioms $\mathscr{T}1$ to $\mathscr{T}4$.

Example. The Euclidean space R_n (n finite) is a Hausdorff space under either the "sphere" or "cube" topology. The "sphere" topology is generated by the interior of spheres of arbitrary radius and arbitrary center. The topology consists of all such open sets, together with arbitrary unions and finite intersections. The "cube" topology is generated by interiors of cubes with arbitrary centers and edge lengths, together with all their finite intersections and arbitrary unions. The "sphere" and "cube" topologies are equivalent. It is easy to see that any two points can be separated. With the topology outlined earlier, R_n obeys the separability axiom $\mathscr{T}4$ and is therefore a Hausdorff space.

Question. Are the "sphere" and "cube" topologies described for R_n still equivalent for an infinite-dimensional space R_∞ ?

We now define three additional concepts: compactness, closure, and continuity.

Definition. A space T is **compact** if every infinite sequence of points t_1, $t_2, \ldots,$ ($t_i \in T$) contains a subsequence of points that (a) **converges** to a point and (b) this point is in T.

Example. The real line R_1 is not compact because the sequence of points $t_i = i$, $i = 1, 2, \ldots$ does not have a convergent subsequence. The circumference of the unit circle in R_2 is compact. The interior of the unit circle is not compact because the sequence of points $t_n = 1 - 1/n$, $n = 1, 2, \ldots,$ converges to a point on the circumference and therefore not in the original set.

Definition. A set T is **closed** if it contains all its **limit points**. The set T, together with all its limit points, is called the **closure** $\bar{T}$ of T.

Example. The closure of the interior of the unit circle in R_2 consists of the original set, together with its circumference.

Remark. If $\bar{S}$ is a closed set in T, then the complement of $\bar{S}$ in T is open. The **complement** of $\bar{S}$ in T consists of all points that are in T but not in $\bar{S}$.

Let ϕ be a mapping of the space T with topology $\mathscr{T}$, into the space U with topology $\mathscr{U}$. If $t_i \in T$, then the **image** of t_i is a point $u_i \in U$:

$$\phi(t_i) = u_i$$

Not every point in U is necessarily the image of some point in T. In addition, several different points in T may map onto one point in U. The set of all points $t_1, t_2, \ldots, \in T$ that map onto a particular point $u \in U$ is called the **inverse image** of u.

Definition. $\phi : (T, \mathscr{T}) \to (U, \mathscr{U})$ is **continuous** if the inverse image of *any* open set in U is an open set in T.

Example. In elementary calculus, a mapping $f: R_1 \to R_1$ is called a real-valued function of a single real variable. It is usual to demonstrate the continuity of f by showing that "for every δ there is an ε such that...." The δ describes an open set in the space $R_1 = (U, \mathscr{U})$, the ε describes an open set in the space $R_1 = (T, \mathscr{T})$. The usual "δ, ε" prescription for demonstrating the continuity of a function $f(x)$ is a special case of the definition of continuity given previously.

Definition. A **differentiable manifold** $\mathscr{M}$ consists of

1. A Hausdorff space $(T, \mathscr{T})$.
2. A collection Φ of mappings $\phi_p \in \Phi$:

$$\phi_p : T \to R_\eta \qquad p \in T$$

which obey the following properties

$\mathscr{M}1$. ϕ_p is a 1-1 mapping of an open set T_p $(p \in T_p)$ into an open set in R_η.

$\mathscr{M}2$. $\cup\, T_p = T$.

$\mathscr{M}3$. If $T_p \cap T_q$ is not empty, then $\phi_p(T_p \cap T_q)$ is an open set in R_η, and $\phi_q(T_p \cap T_q)$ is an open set in R_η which is different from $\phi_p(T_p \cap T_q)$. The mapping $\phi_p \circ \phi_q^{-1}$ must be continuous and differentiable.

$\mathscr{M}4$ (Maximality). The mappings $\phi_p \circ \phi_q^{-1}$ and $\phi_q \circ \phi_p^{-1}$, described in $\mathscr{M}3$, are mappings in Φ.

Explanation. The property $\mathscr{M}1$ allows us to construct a coordinate system in the neighborhood of any point p. We map p into the origin of R_η using ϕ_p. Any point q near p is mapped into a point $\phi_p(q)$ near $\mathbf{0} = \phi_p(p)$. We associate the coordinates $\phi_p{}^i(q)$ $(i = 1, 2, \ldots, \eta)$ of $\phi_p(q) \in R_\eta$, with the original point $q \in T$. This association provides a (local) coordinate system throughout T. (See Fig. 3.1*a*.)

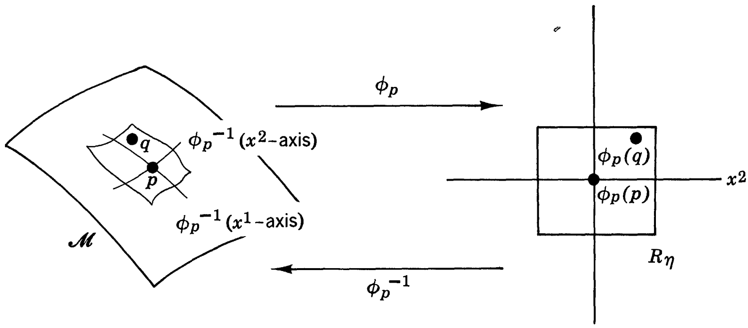
Axiom $\mathscr{M}2$ assures that such a local coordinate system can be established at any point in T.

Axiom $\mathscr{M}3$ involves the mapping

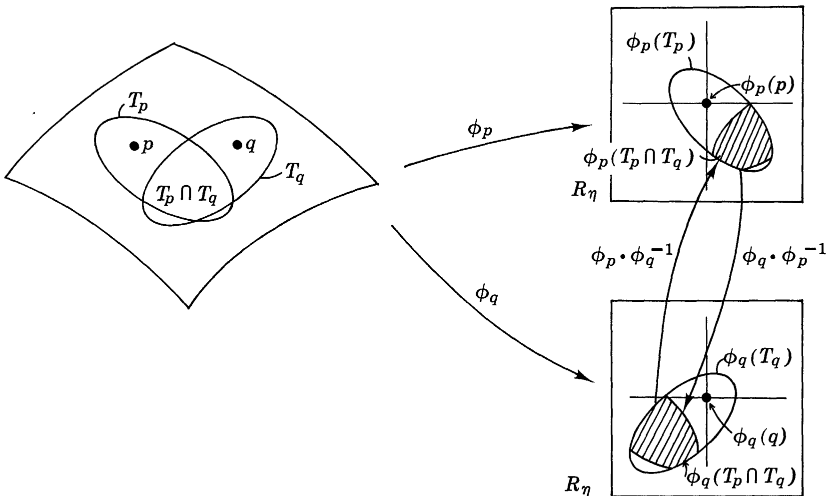
$$\phi_p \circ \phi_q^{-1} : R_\eta \to R_\eta$$

which can be studied using the standard techniques of the differential calculus. This axiom is illustrated in Fig. 3.1*b*.

Comment. Manifolds are important and useful because they are everywhere locally Euclidean. All concepts and methods of use in the study of the space R_η can be transferred by means of the axioms $\mathscr{M}1$ to $\mathscr{M}3$ to the



(a)



(b)

FIG. 3.1 MAPPINGS FOR p: (a) A NEIGHBORHOOD OF EVERY POINT p IS MAPPED BY ϕ_p ONTO THE INTERIOR OF THE CUBE WITH EDGE LENGTH 2ε, SURROUNDING THE ORIGIN ● IN R_η, FOR SOME η. $\phi_p(p) = 0$. THE INVERSE MAPPING ϕ_p^{-1} PROVIDES A COORDINATE SYSTEM IN THE NEIGHBORHOOD OF p. THE COORDINATES OF A POINT q ARE: $\phi_p^1(q)$, $\phi_p^2(q)$, ..., $\phi_p^\eta(q)$. (b) THE MAPPINGS ϕ_p, ϕ_q, MAP THE OPEN SETS T_p, T_q ONTO AN OPEN REGION SURROUNDING THE ORIGIN OF R_η. THE MAPPING $\phi_p \circ \phi_q^{-1}$ MAPS AN OPEN SET IN R_η INTO AN OPEN SET IN R_η. THEREFORE, $\phi_p \circ \phi_q^{-1}$ IS AMENABLE TO THE STANDARD STUDIES OF $f : R_\eta \to R_\eta$ BY MEANS OF ORDINARY DIFFERENTIAL CALCULUS. SIMILAR STATEMENTS HOLD FOR $\phi_q \circ \phi_p^{-1}$. BY $\mathcal{M}4$, BOTH THESE MAPPINGS ARE ELEMENTS IN Φ.

manifold. In particular, the dimensionality of the manifold $\mathcal{M}$ is the dimensionality of the space R_η, namely, η.

Example. The surface

$$x^2 + y^2 + z^2 = 1 \tag{1.1}$$

in R_3 is a two-dimensional manifold.

A topological group or a continuous group has two distinct kinds of structures on it. It has a topological structure, and it also has an algebraic structure. Algebraically, it is a group; it therefore obeys the axioms of a group [Chapter 1, (1.1)]. Topologically, it is a manifold. The algebraic (A) and topological (T) properties are combined by two additional axioms:

AT 1. The mapping $\quad \sigma \times \tau \to \sigma\tau \quad$ is continuous.

AT 2. The mapping $\quad \tau \to \tau^{-1} \quad$ is continuous.

The significance of these axioms is illustrated in Fig. 3.2. These are the *only* two axioms required to connect the algebraic with the topological properties of continuous groups. A rich and beautiful structure—the theory of Lie groups—results from the imposition of these axioms.

Definition. A **topological group** or a **continuous group**[4–8] consists of

1. An underlying η-dimensional manifold $\mathcal{M}$.
2. An operation ϕ mapping each pair of points (β, α) in the manifold into another point γ in the manifold.
3. In terms of coordinate systems around the points γ, β, α, we write

$$\gamma^\mu = \phi^\mu(\beta^1, \ldots, \beta^\eta; \alpha^1, \ldots, \alpha^\eta); \qquad \mu = 1, 2, \ldots, \eta \tag{1.2}$$

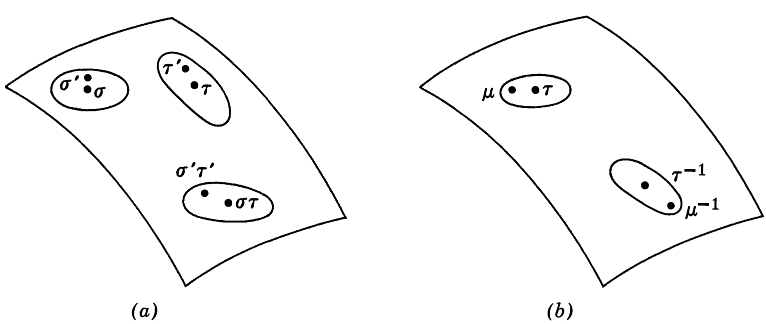


FIG. 3.2 THE SIGNIFICANCE OF THE AXIOMS COMBINING THE ALGEBRAIC WITH THE TOPOLOGICAL PROPERTIES OF A LIE GROUP. (*a*) AT 1: IF $\sigma \times \tau \to \sigma\tau$, THEN THE PRODUCT OF ANY GROUP ELEMENT NEAR σ WITH ANY GROUP ELEMENT NEAR τ IS A GROUP ELEMENT NEAR $\sigma\tau$. (*b*) AT 2: IF μ IS A GROUP ELEMENT NEAR τ THEN μ^{-1} IS A GROUP ELEMENT NEAR τ^{-1}.

The functions

$$\phi: \beta \times \alpha \rightarrow \gamma = \beta\alpha$$
$$\psi: \qquad \alpha \rightarrow \alpha^{-1}$$

must be continuous. The group multiplication properties may be transcribed into conditions on ϕ:

1. Closure:

$$\begin{array}{c} c = b \circ a \\ \Big\Downarrow \\ \gamma^{\mu} = \phi^{\mu}(\beta, \alpha) \qquad \alpha, \beta, \gamma \in \mathscr{M} \end{array} \tag{1.3}$$

2. Associativity:

$$\begin{array}{c} c \circ (b \circ a) = (c \circ b) \circ a \\ \Big\Downarrow \\ \phi^{\mu}(\gamma, \phi(\beta, \alpha)) \equiv \phi^{\mu}(\phi(\gamma, \beta), \alpha) \end{array} \tag{1.4}$$

3. Identity:

$$\begin{array}{c} e \circ a = a \;\; = a \circ e \\ \Big\Downarrow \\ \phi^{\mu}(\varepsilon, \alpha) = \alpha^{\mu} = \phi^{\mu}(\alpha, \varepsilon) \end{array} \tag{1.5}$$

4. Inverse:

$$\begin{array}{c} aa^{-1} = e \;\; = a^{-1}a \\ \Big\Downarrow \\ \phi^{\mu}(\alpha, \alpha^{-1}) = \varepsilon^{\mu} = \phi^{\mu}(\alpha^{-1}, \alpha) \end{array} \tag{1.6}$$

Definition. A **continuous group of transformations** consists of

(a) An underlying topological space $\mathscr{T}_{\eta}$, which is an η-dimensional manifold, together with a binary mapping ϕ:

$$\phi : \mathscr{T} \times \mathscr{T} \rightarrow \mathscr{T} \tag{1.7}$$

(b) A geometric space G_N, which is an N-dimensional manifold, and a mapping $f: \mathscr{T} \times G \rightarrow G$, which obey:

Postulate (a′): $\mathscr{T}_{\eta}$, ϕ obey the postulates of a topological group [(1.3)–(1.6)].

Postulate (b′): The function

$$y^i = f^i(\alpha^1, \ldots, \alpha^n; x^1, \ldots, x^N)$$

is continuous and in addition has the properties:

1. Closure′:

$$\begin{array}{c} \alpha \in \mathscr{T}, x \in G \Rightarrow \alpha x \in G: \\ \Big\Downarrow \\ y^i = f^i(\alpha^1, \ldots, \alpha^n, x^1, \ldots, x^N) \in G_N \end{array} \tag{1.3'}$$

2. Associativity′:

$$\begin{array}{c} \beta(\alpha x) = (\beta \circ \alpha)x \\ \Big\Downarrow \\ f^i(\beta, f(\alpha, x)) = f^i(\phi(\beta, \alpha), x) \end{array} \tag{1.4'}$$

3. Identity′:

$$\begin{array}{c} \varepsilon x = x \\ \Big\Downarrow \\ f^i(\varepsilon, x) = x^i \end{array} \tag{1.5'}$$

4. Inverse′:

$$\begin{array}{c} \alpha^{-1}(\alpha x) = \alpha(\alpha^{-1}x) = (\alpha\alpha^{-1})x = x \\ \Big\Downarrow \\ f^i(\alpha^{-1}, f(\alpha, x)) = f^i(\alpha, f(\alpha^{-1}, x)) = f^i(\phi(\alpha, \alpha^{-1}), x) = x^i \end{array} \tag{1.6'}$$

These definitions will be illustrated shortly by a useful example.

2. COMMENTS. In this and the following chapter, we distinguish the underlying "topological space" of the group from the "geometric space" on which the group acts. All properties of the topological space are indicated by Greek letters (the points α, β, ..., the mapping ϕ, the coordinate indices α^μ ...). All properties of the geometric space are indicated by Roman letters (the points x, y, ..., the mapping f, the coordinate indices x^i ...).

Every continuous group may be considered as a continuous group of transformations if we allow it to act on itself. We then have the identifications

$$G \equiv \mathscr{T} \qquad f \equiv \phi \tag{1.8}$$

Thus continuous groups are special cases of continuous groups of transformations.

3. ADDITIONAL COMMENTS. The starting point for Sophus Lie's study of continuous groups[7] was a study of partial differential equations. Let

$$\frac{\partial x^i}{\partial \tau} = u^i(x) \tag{1.9}$$

be the partial differential equation for a flow process. This equation has a solution of the form

$$x^i(\tau) = f^i(\tau; x_0) \tag{1.10}$$

This represents the position of a particular particle at time τ if its original position at $\tau = 0$ is x_0,

$$f^i(\tau = 0, x_0) = x_0{}^i \tag{1.11}$$

To each value of τ is associated a transformation of the space of points x into itself. The transformations for all possible values of τ form a group.

Our starting point for the study of continuous groups is the enumeration of all classical matrix groups. These groups were interpreted as changes of basis transformations within a vector space. It is also possible to interpret the classical groups as mappings of a space into itself in much the same way that (1.10) represents a mapping of a space into itself.

For the time being, it is useful to interpret transformations as changes of bases, rather than as mappings of a space into or onto itself. This is the "passive" interpretation of a transformation as opposed to the "active" interpretation. Under certain conditions, the "active" and "passive" interpretations are physically equivalent. The precise circumstances under which this equivalence occurs is governed by the *principle of equivalence*. It is this principle which puts physical content into the mathematical manipulations of group theoretical calculations.

II. An Example

In this section we consider an example[9,10] to clarify and elaborate on the definitions of the previous section.

1. THE TWO-PARAMETER GROUP OF COLLINEAR TRANSFORMATIONS ON THE STRAIGHT LINE R_1. On the straight line R_1, the two-parameter group of collinear transformations consists of:

(a) Those operations which change the length of the basis vector.
(b) Those operations which shift the origin.
(c) That operation which changes the orientation of the basis vector.

If p is any point in R_1, then with respect to some coordinate system S (choice of basis $\mathbf{e}_1$) it has coordinate $x(p)$.

(a) Under a stretch of the basis $\mathbf{e}_1 \to (1/\alpha^1)\mathbf{e}_1$ by α^1 ($\alpha^1 > 0$), the coordinate of p is multiplied by $\alpha^1 : x'(p) = \alpha^1 x(p)$.

(b) Under a motion of the origin from 0 to $-\alpha^2$, the coordinate of p becomes $x'(p) = x(p) + \alpha^2$.

(c) Under a reversal of the direction of $\mathbf{e}_1$, the coordinate of p becomes $x'(p) = -x(p)$.

These transformations are shown in Fig. 3.3.

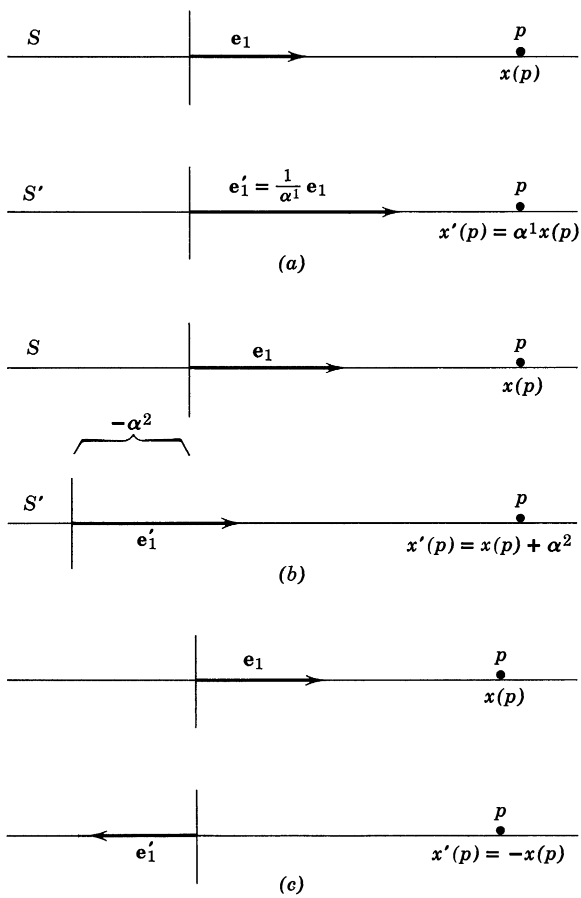


FIG. 3.3 THE THREE KINDS OF OPERATIONS THAT GENERATE THE TWO-PARAMETER GROUP OF TRANSFORMATIONS IN R_1.

The basis $\mathbf{e}_1$ and the coordinate $x(p)$ of some fixed point p transform in a contravariant way. For our purposes, it is more convenient to define the operations of this group on the coordinate $x(p)$. We define the group operation (α^1, α^2) by $x' = (\alpha^1, \alpha^2)x = \alpha^1 x + \alpha^2$, $\alpha^1 \neq 0$. The underlying topological space $\mathscr{T}_2$ for this group is the plane R_2, excluding the straight line $\alpha^1 = 0$. The geometric space G_1 on which this group acts is R_1. The function f is given by

$$f(\alpha^1\alpha^2; x) = \alpha^1 x + \alpha^2 \tag{2.1}$$

The function ϕ is determined from the group multiplication:

$$x'' = f(\beta^1\beta^2; x') \qquad x' = f(\alpha^1\alpha^2; x) \tag{2.2}$$

$$x'' = \beta^1 x' + \beta^2 = \beta^1(\alpha^1 x + \alpha^2) + \beta^2 = f(\beta^1\alpha^1, \beta^1\alpha^2 + \beta^2; x)$$

$$\phi(\beta^1\beta^2, \alpha^1\alpha^2) = (\beta^1\alpha^1, \beta^1\alpha^2 + \beta^2) \tag{2.3}$$

We now verify that this is both a continuous group and a continuous group of transformations. First of all, $\mathscr{T}$ is a manifold, since each half-plane can be mapped in a 1-1 manner onto a part of R_2. Next,

$$\begin{aligned} \phi^1(\beta, \alpha) &= \beta^1\alpha^1 \\ \phi^2(\beta, \alpha) &= \beta^1\alpha^2 + \beta^2 \end{aligned} \tag{2.4}$$

are continuous functions and obey

1. Closure:

$$\phi(\beta, \alpha) \in \mathscr{T}_2 \qquad \text{and} \qquad \beta^1 \neq 0, \alpha^1 \neq 0 \Rightarrow \phi^1 = \beta^1\alpha^1 \neq 0 \tag{2.5}$$

2. Associativity:

$$\begin{array}{ccc} \phi(\gamma, \phi(\beta, \alpha)) & ? & \phi(\phi(\gamma, \beta), \alpha) \\ \| & & \| \\ \phi(\gamma^1, \gamma^2; \beta^1\alpha^1, \beta^1\alpha^2 + \beta^2) & & \phi(\gamma^1\beta^1, \gamma^1\beta^2 + \gamma^2; \alpha^1, \alpha^2) \\ \| & & \| \\ & (\gamma^1\beta^1\alpha^1, \gamma^1\beta^1\alpha^2 + \gamma^1\beta^2 + \gamma^2) & \end{array} \tag{2.6}$$

The ? can be replaced by = in this pentagon, and associativity holds.

3. Identity:

$$\phi(\alpha^1\alpha^2; 1, 0) = (\alpha^1, \alpha^2) = \phi(1, 0; \alpha^1\alpha^2) \tag{2.7}$$

4. Inverse:

$$\phi\left(\alpha^1, \alpha^2; \frac{1}{\alpha^1}, -\frac{\alpha^2}{\alpha^1}\right) = (1, 0) = \phi\left(\frac{1}{\alpha^1}, -\frac{\alpha^2}{\alpha^1}; \alpha^1, \alpha^2\right) \tag{2.8}$$

Similarly, we check that the postulates for a continuous group of transformations are satisfied. Clearly, $G_1 = R_1$ is a manifold and $f(\alpha^1\alpha^2; x) = \alpha^1 x + \alpha^2$ is continuous in all its arguments. Moreover, $(\mathscr{T}_2, \phi)$ is a continuous group, and f obeys

1. Closure′:

$$f(\alpha; x) \in R_1 \tag{2.5′}$$

2. Associativity′:

$$\begin{tikzcd}[column sep=small]
& f(\phi(\beta,\alpha),x) \arrow[dl, equal] & ? & f(\beta, f(\alpha,x)) \arrow[dr, equal] & \\
f(\beta^1\alpha^1, \beta^1\alpha^2 + \beta^2; x) \arrow[drr, equal] & & & & f(\beta^1, \beta^2; \alpha^1 x + \alpha^2) \arrow[dll, equal] \\
& & \beta^1\alpha^1 x + \beta^1\alpha^2 + \beta^2 & &
\end{tikzcd} \tag{2.6′}$$

The ? may be replaced by =, and Associativity′ holds.

3. Identity′:

$$f(1, 0; x) = x \tag{2.7′}$$

4. Inverse′:

$$f\left(\frac{1}{\alpha^1}, -\frac{\alpha^2}{\alpha^1}; \alpha^1 x + \alpha^2\right) = x \tag{2.8′}$$

2. SOME REALIZATIONS FOR THIS GROUP. The 1-1 mapping of $\mathscr{T}_2$ into the 2×2 nonsingular matrices given by

$$(\alpha^1, \alpha^2) \leftrightarrow \begin{pmatrix} \alpha^1 & \alpha^2 \\ 0 & 1 \end{pmatrix} \tag{2.9}$$

provides a convenient mechanism for defining the functions f and ϕ by matrix multiplication:

$$f\colon \begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha^1 & \alpha^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha^1 x + \alpha^2 \\ 1 \end{pmatrix} \tag{2.10}$$

$$\phi\colon \begin{pmatrix} \gamma^1 & \gamma^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta^1 & \beta^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^1 & \alpha^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta^1\alpha^1 & \beta^1\alpha^2 + \beta^2 \\ 0 & 1 \end{pmatrix} \tag{2.10′}$$

The mapping (2.9) is therefore a representation of this group in terms of 2×2 matrices.

Under $x \to x' = \alpha^1 x + \alpha^2$,

$$(x')^2 \to (\alpha^1 x + \alpha^2)^2 = (\alpha^1)^2 x^2 + 2\alpha^1\alpha^2 x + (\alpha^2)^2 1 \tag{2.11}$$

and in general

$$(x')^N \to (\alpha^1 x + \alpha^2)^N = \sum_{r=0}^{N} \binom{N}{r} (\alpha^1)^r (\alpha^2)^{N-r} x^r \tag{2.12}$$

Therefore, when N is a positive integer, the $N + 1$ homogeneous polynomials $(x^N, x^{N-1}, \ldots, x, 1)$ can be used as bases for an $(N + 1) \times (N + 1)$ matrix representation of this projective group. All these matrix representations are faithful. In fact, in (2.12) N need not be a positive integer. The representations are then ∞ dimensional.

Example. A faithful 4×4 matrix representation is

$$(\alpha^1, \alpha^2) \leftrightarrow \begin{pmatrix} (\alpha^1)^3 & 3(\alpha^1)^2\alpha^2 & 3\alpha^1(\alpha^2)^2 & (\alpha^2)^3 \\ 0 & (\alpha^1)^2 & 2\alpha^1\alpha^2 & (\alpha^2)^2 \\ 0 & 0 & \alpha^1 & \alpha^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.13}$$

The powers of $x : x^r$ (r is an integer ≥ 0) form bases for the space of analytic functions on R_1. This suggests that we look for realizations of this two-parameter group defined on functions $f(x)$. The mapping of $f(x)$ into another function defined on R_1 given by

$$(\alpha^1, \alpha^2) f(x) = f(\alpha^1 x + \alpha^2) = f'(x) \tag{2.14}$$

is a realization of this group, since it preserves the group operation.

It is often possible to find a subset of functions $f_i(x)$ (i may be a discrete or continuous index) that forms a basis for the space of all functions of the form $\alpha \circ f(x)$. Under these conditions, we can write

$$\begin{aligned} \alpha \circ f_i(x) &= f'_i(x) = f_i(\alpha \circ x) \\ &= \sum_j f_j(x) \Gamma_{ji}(\alpha) \end{aligned} \tag{2.15}$$

When this is true, the problem of studying all possible realizations of a group reduces to a study of all possible linear matrix representations of the group.

This property is true for any group, provided we consider enough of the representations of that group. For this reason the enumeration, classification and construction of matrix representations of continuous groups is a problem of great interest.

Another realization for this group is given by the mapping

$$\beta\phi(\alpha) = \phi'(\alpha) = \phi(\beta\alpha) \tag{2.14'}$$

III. Additional Necessary Concepts

1. TOPOLOGICAL CONCEPTS. A space is said to be **connected** if any two points in the space can be joined by a line and all the points of the line lie in the space.

The topological space $\mathscr{T}_2$ for our example group is not connected. Any line joining, say (1, 0) and (−1, 0), must contain the point $(\alpha^1 = 0, \alpha^2)$, which is not in $\mathscr{T}_2$.

If p is any point, we can look at the set of points connected with p. For example, all points in the left half-plane are connected to (−1, 0); all points in the right half-plane are connected to the identity (1, 0). A connected component of a continuous group (called a **sheet**) cannot itself be a group unless it contains the identity. An important theorem follows.

THEOREM. *The component of a continuous group that is connected with the identity is a group.*

In addition, all other sheets of the continuous group are isomorphic as manifolds, both to each other and to the connected component. A new group structure can also be defined on each sheet, so that it becomes isomorphic with the connected component.

A connected space is **simply connected** if a curve connecting any two points in the space can be continuously deformed into every other curve connecting the same two points. The connected component $G_{(1,0)}$ is simply connected, since any curve $(1, 0) - q - p$ can be continuously deformed to any other curve $(1, 0) - q' - p$, as shown in Fig. 3.4. However, a donut or torus is not simply connected, since no continuous mapping takes the major circumference into a minor circumference. This is illustrated in Fig. 3.5.

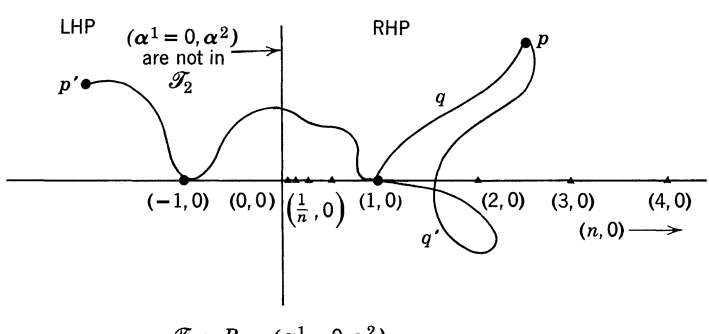


FIG. 3.4 THE TOPOLOGICAL PROPERTIES DEFINED IN SECTION III.1 ARE ILLUSTRATED IN THE PARTICULAR CASE OF THE TWO PARAMETER GROUP DESCRIBED IN SECTION II.

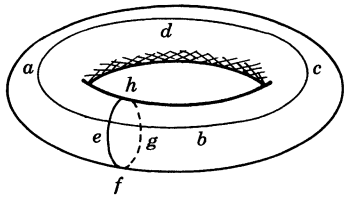


FIG. 3.5 THE TORUS IS NOT SIMPLY CONNECTED. THERE IS NO CONTINUOUS MAPPING OF THE MAJOR CIRCLE *abcd* ONTO THE MINOR CIRCLE *efgh*.

2. ALGEBRAIC CONCEPTS. Let H be a subgroup of G. Then for any fixed $g \in G$; the set of group operations

$$gHg^{-1} = \{gh_i g^{-1}, h_i \in H\} \tag{3.1}$$

also forms a group. This group is **conjugate** to H.

The one-dimensional abelian subgroup of $G_{(1,0)}$ given by

$$(\alpha^1 > 0, \alpha^2 = 0) \tag{3.2}$$

gives rise to the series of conjugate groups

$$(\beta^1, \beta^2)(\alpha^1, 0)(\beta^1, \beta^2)^{-1} = (\alpha^1, \beta^2(1 - \alpha^1)) \tag{3.3}$$

It is easily verified that this subset of group elements does in fact constitute a group. The subgroup conjugate to the abelian subgroup $(1, \alpha^2)$ is

$$(\beta^1, \beta^2)(1, \alpha^2)(\beta^1, \beta^2)^{-1} = (1, \beta^1\alpha^2) \tag{3.4}$$

Every element in this conjugate subgroup is in the original subgroup: we signify this by writing $gHg^{-1} = H$. Subgroups H that are self-conjugate for all elements $g \in G$, are called **invariant** subgroups or **normal** subgroups. The subgroup $H = (1, \alpha^2)$ is an abelian invariant subgroup. Abelian invariant subgroups are a prominent headache in the classification theory of Lie groups but, as if in compensation, they are exceedingly useful in the representation theory for the Lie groups that *cannot* be classified.

The fact that a group (vector space, algebra) has an invariant subgroup is exploited often in this and future chapters. We will exploit it using the following theorem.

THEOREM. *The connected component* G_0 *of a continuous group* G *is an invariant subgroup of* G.

Example. The component of the two parameter group that is connected with the identity forms an invariant subgroup in the two parameter group of collineations. In Fig. 3.4 the entire group is parameterized by the plane R_2 excluding the y axis. The right half-plane forms an invariant subgroup.

Direct sum and direct product are useful constructions for enriching the complexity of a given kind of algebraic structure. But "quotient" is the secret tool in the mathematician's arsenal for constructing new kinds of algebraic structures. For example, the direct sum and product of two integers m and n is yet another integer, whereas, their quotient m/n is not.

We look at the properties of the quotient G/H, where H is a subgroup of G. Such structures are called **cosets**. A right coset* is the set of group operations $c_0, c_1, \ldots, \in G$ with the property that

$$Hc_0 + Hc_1 + \cdots = G \tag{3.5}$$

and, furthermore, no element g of G is contained more than once in the sum on the left. In other words, the c_i are chosen in such a way that every element $g \in G$ can be written *uniquely*

$$g = h_i c_j \qquad \begin{array}{l} h_i \in H \\ c_j \in C_R \end{array} \tag{3.6r}$$

Analogously, left cosets C_L involve a unique decomposition

$$g = c_k h_l \qquad \begin{array}{l} h_l \in H \\ c_k \in C_L \end{array} \tag{3.6l}$$

Since the concept of coset is not straightforward and quite elusive, we illustrate it with several examples.

Example 1.

$$G = G_{(1,0)}, \qquad \begin{array}{l} H_1 = (\alpha^1 > 0, \alpha^2 = 0) \\ H_2 = (1, \alpha^2) \end{array}$$

$$\begin{array}{ccccccl}
(\gamma^1, \gamma^2) & \underset{=}{\text{unique}} & \underbrace{(\gamma^1, 0)} & \circ & \underbrace{\left(1, \dfrac{\gamma^2}{\gamma^1}\right)} & & (3.7) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\
G & & H_1 & \circ & C_R & & (3.7^1) \\
 & & C_L & \circ & H_2 & & (3.7^2)
\end{array}$$

These results are a reflection of two theorems.

THEOREM. *Left and right cosets* G/H *of a continuous group* G *by a closed subgroup* H *are manifolds of dimension*

$$\mathit{dim}\, \mathrm{G} - \mathit{dim}\, \mathrm{H}$$

* We call it a right coset because c is on the right of H. Other authors call this left coset because H is on the left of c.

THEOREM. *If* H *is an invariant subgroup of* G, *then the coset elements* $c_0, c_1, c_2, \ldots,$ *can be chosen in such a way that they are closed under multiplication and form a group called the* **factor group** G/H.

In the first example (3.7^1), it is fortuitous that G/H_1 is a group, since H_1 is not an invariant subgroup. But in the second decomposition (3.7^2) it is necessary that there be some choice of the C_L with a group structure, since H_2 is an invariant subgroup of G.

Example 2. The connected component of G, $G_{(1,0)}$, is an invariant subgroup. Every element of G can be written, for example, either

$$\begin{array}{ccc} g\circ(\tfrac{1}{2},0) & \text{or} & g\circ(-\tfrac{1}{2},0) \\ & & c_0=(\tfrac{1}{2},0) \\ g\in G_{(1,0)} & & c_1=(-\tfrac{1}{2},0) \end{array} \tag{3.8}$$

The coset representatives c_0, c_1 consist of one element from each sheet. However, this is not a very clever choice for coset representatives. A more astute choice for coset representatives is

$$(1,0) \qquad \text{and} \qquad (-1,0) \qquad (-1,0)^2=(1,0) \tag{3.9}$$

since these constitute the elements of a group of order 2. In general, a good choice for coset representatives for the factor group

$$\frac{G_{\text{continuous}}}{G_{\text{connected component}}}$$

would be the element in each sheet that becomes the identity under the mapping serving to turn that sheet into a group isomorphic with the connected component. This is illustrated in Fig. 3.6.

The crucial point for our study of continuous groups is the following theorem.

THEOREM. *The factor group of a continuous group* G *by its connected component* G_0 *is a discrete group* D *of dimension* 0:

$$\frac{\mathrm{G}}{\mathrm{G}_0}=\mathrm{D} \tag{3.10}$$

With this theorem, we can study all continuous groups if we study separately:

1. Only connected continuous groups.
2. Discrete groups.

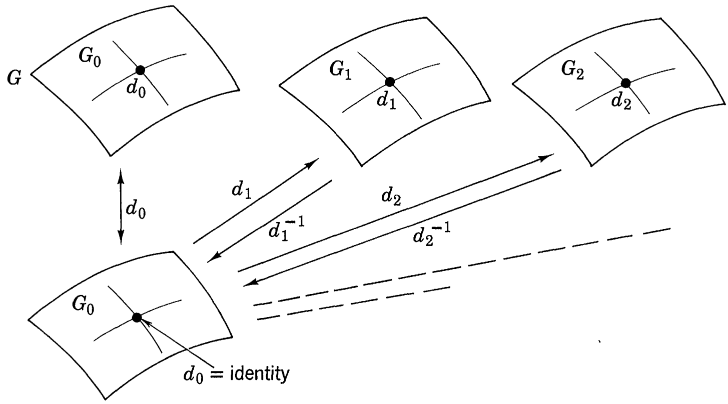


FIG. 3.6 A MANY-SHEETED CONTINUOUS GROUP G CAN BE WRITTEN AS A SUM OF GROUP OPERATIONS OF THE FORM $d_i G_0$: $G = \bigcup_{i=0} d_i G_0$, WHERE $d_i \in D$ AND G_0 IS THE CONNECTED COMPONENT. EACH SHEET IS TOPOLOGICALLY EQUIVALENT AS A MANIFOLD. ONLY THE SHEET $G_0 = d_0 G_0$ HAS A GROUP STRUCTURE. THE SHEET $G_i = d_i G_0$ CAN BE MADE ISOMORPHIC WITH THE SUBGROUP G_0 BY ACTING WITH d_i^{-1} ON THE LEFT.

If we know the structure of the connected component G_0 of a topological group, the structure of the discrete factor group D, and the structure of the mapping $DG_0 D^{-1} \to G_0$, the structure of the entire continuous group G can easily be constructed as follows. Let

$$g \in G \xrightarrow{\text{unique}} d_i \alpha \qquad d_i, d_j \in D \subset G$$

$$h \in G \xrightarrow{\text{unique}} d_j \beta \qquad \alpha, \beta \in G_0 \subset G$$

Then

$$\begin{aligned} g \circ h &= d_i \alpha \circ d_j \beta \\ &= d_i d_j (d_j^{-1} \alpha d_j) \beta \\ &= (d_k = d_i d_j)(\alpha' = d_j^{-1} \alpha d_j) \beta \end{aligned}$$

Since D has a group structure, the product $d_i \circ d_j = d_k \in D$. Since G_0 is an invariant subgroup, $d_j^{-1} \alpha d_j \in G_0$. Therefore, the group multiplication properties in the discrete factor group D and in the connected continuous component G_0, together with the mapping $\alpha \to \alpha' = d_j^{-1} \alpha d_j$ uniquely determine the group multiplication properties in the entire group G.

Let the elements in the discrete group D be

$$d_0 = \text{identity}, d_1, d_2, \ldots, d_n, \ldots$$

and let G_i be the sheet containing d_i. Then the sheet containing the identity $d_0 \in D$ is the connected component G_0. The mapping that converts any sheet G_i (which cannot be a group unless $i = 0$) into a group isomorphic with G_0 is given by

$$G_i = d_i G_0 \xrightarrow{d_i^{-1}} d_i^{-1} G_i = d_i^{-1} d_i G_0 = G_0$$

This process is illustrated in Fig. 3.6.

3. LOCAL CONCEPTS.[11-13]. Eventually we will be able to describe most of the (global) properties of a continuous group by knowing its properties only in the vicinity of the origin. To facilitate our study, we define an object that behaves essentially like the neighborhood of the identity element in a continuous group.

Definition. A **local continuous group** is a manifold $\mathscr{M}$ together with a binary operation ϕ which is defined on certain pairs of points $\beta, \alpha \in \mathscr{M}$ (in particular, in a neighborhood of the identity) with the following properties:

1. $\gamma = \phi(\beta, \alpha) \in \mathscr{M}$ when $\phi(\beta, \alpha)$ is defined, and ϕ is continuous where defined.
2. $\alpha \to \alpha^{-1}$ is continuous when defined.
3. When $\phi(\gamma, \beta)$ and $\phi(\beta, \alpha)$ are defined,

$$\phi[\phi(\gamma, \beta), \alpha] = \phi[\gamma, \phi(\beta, \alpha)]$$

if defined.

4. There is an identity element ε such that $\phi(\varepsilon, \alpha) = \phi(\alpha, \varepsilon) = \alpha$ is defined for $\alpha \in \mathscr{M}$.
5. If α^{-1} is defined, then

$$\phi(\alpha, \alpha^{-1}) = \phi(\alpha^{-1}, \alpha) = \varepsilon$$

It is clear that these are just the properties of a neighborhood of the identity ε of a continuous group. Since multiplication and inversion may correspond to points outside the neighborhood of ε, these are not always defined for elements in such a neighborhood. It is not always true that a local continuous group can be embedded in a global topological group.

Two local continuous groups $(\mathscr{M}, \phi)$ and $(\mathscr{M}', \phi')$ are **locally isomorphic** if

1. There is a 1-1 mapping $\mathscr{M} \leftrightarrow \mathscr{M}'$.
2. If $\alpha, \beta \in \mathscr{M}$ and $\alpha', \beta' \in \mathscr{M}'$, then $\alpha \circ \beta$ and α^{-1} are defined if and only if $\alpha' \circ \beta'$ and α'^{-1} are defined. In addition, the manifold isomorphism must preserve the group operation:

$$(\alpha \circ \beta)' = \alpha' \circ \beta'$$

Local continuous groups and local Lie groups arise when we linearize the group structure—that is, when we investigate the properties of the group only near its identity element.

Another local concept for which we have occasional use is the concept of local compactness. A space is **locally compact** if, around any point p, a neighborhood can be found whose closure (i.e., the neighborhood with its "boundary," all its limit points) lies within the set. Our two-parameter group is locally compact.

IV. Lie Groups

1. THE MOTIVATION. Let $\alpha_0 = \varepsilon$ (the identity) and let β be some other point in the connected component of a continuous group. Then ε and β can be joined by a line lying entirely within the group. Choose points

$$\alpha_0 = \varepsilon, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_\infty = \beta$$

on this line with the following properties:

1. α_i and α_{i+1} lie within a common neighborhood.
2. $\alpha_{i+1} \circ \alpha_i^{-1}$ lies inside some neighborhood of the identity ε for each value of i. This can always be done with a countable number of group operations α.

Then the group operation β can be written

$$\beta = \alpha_\infty \cdots (\alpha_3 \circ \alpha_2^{-1}) \circ (\alpha_2 \circ \alpha_1^{-1}) \circ (\alpha_1 \circ \alpha_0^{-1}) \circ \varepsilon \tag{4.1}$$

In other words, β is a product of group operations that are all close to the identity. The decomposition in (4.1) is illustrated in Fig. 3.7. We are then interested in studying the function

$$\phi(\alpha_{i+1}, \alpha_i^{-1}) = \phi(\alpha_i + \delta\alpha_i, \alpha_i^{-1})$$

with $\delta\alpha_i$ small. This clearly cries out for a Taylor series expansion. Therefore, we shall demand that ϕ be differentiable.

Definition. A **Lie group** is the connected component of a continuous group in which the composition function ϕ is analytic on its domain of definition.

Actually, the requirement for analyticity of ϕ is unnecessarily stringent; it may be dropped (Hilbert's fifth problem).

Lie groups of transformations and local Lie groups are defined in an analogous manner.

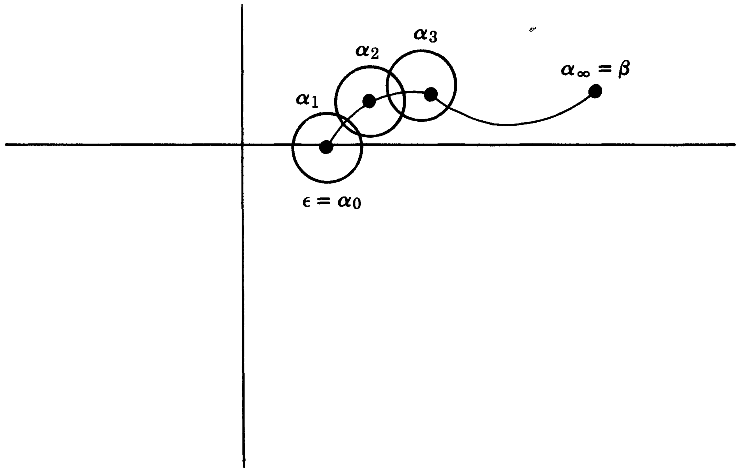


FIG. 3.7 ANY OPERATION β IN THE CONNECTED COMPONENT OF A CONTINUOUS GROUP CAN BE JOINED TO THE IDENTITY BY A LINE LYING ENTIRELY WITHIN THE GROUP. IT IS POSSIBLE TO CHOOSE A COUNTABLE NUMBER OF α_i, $\alpha_0 = \varepsilon$, $\alpha_\infty = \beta$ WITH THE PROPERTIES: (1) α_i, α_{i+1} LIE IN A COMMON NEIGHBORHOOD, AND (2) $\alpha_{i\mp 1} \circ \alpha_i^{-1}$ LIES IN A NEIGHBORHOOD OF THE IDENTITY FOR EACH i.

V. The Invariant Integral

We are about to embark on a detailed study of the properties of groups near the origin. Latèr we will want to return to a study of the global properties of groups—in particular to the functions defined on the group manifold. Such a study will require the use of an integral function defined over the group elements. The task of constructing a useful integral defined on a group manifold is more conveniently carried out now than later.

1. THE REARRANGEMENT PROPERTY. The operation of multiplication by a group element may be interpreted either as a mapping of the entire topological space onto itself, or as a change of basis within the topological space. If $f(\beta)$ is any (scalar-valued) function defined on the group—or equivalently topological space $\mathscr{T}$—then a reasonable requirement for a group integral is

$$\int f(\beta)\, d\mu(\beta) = \int f(\alpha\beta)\, d\mu(\beta) = \int f(\alpha\beta)\, d\mu(\alpha\beta) \tag{5.1}$$

This is reasonable because if the sum over β involves each β exactly once, the sum over $\alpha\beta$ involves each group operation also exactly once, although in a scrambled order. Since f is an arbitrary function, we demand of the measure defined on the group that it obey

$$d\mu_L(\beta) = d\mu_L(\alpha \circ \beta) \tag{5.2}$$

Measures with this property are called **left invariant** measures. **Right invariant** measures can also be defined:

$$d\mu_R(\beta) = d\mu_R(\beta \circ \alpha) \tag{5.3}$$

2. REPARAMETERIZATION OF G_0. We first define group integration on the connected component of a continuous group; then we extend it to the entire continuous group. We also demonstrate these concepts on the connected component of our example group.

Since it is customary to associate the origin of R_n with the group identity ε, we reparameterize the connected component G_0 according to

$$(\alpha^1, \alpha^2) \leftrightarrow \begin{pmatrix} e^{\alpha^1} & \alpha^2 \\ 0 & 1 \end{pmatrix} \tag{5.4}$$

Under this reparameterization,

$$\begin{aligned} (\beta^1, \beta^2) \circ (\alpha^1, \alpha^2) &= (\beta^1 + \alpha^1, e^{\beta^1}\alpha^2 + \beta^2) \\ (\alpha^1, \alpha^2)^{-1} &= (-\alpha^1, -e^{-\alpha^1}\alpha^2) \end{aligned} \tag{5.5}$$

Now let $\delta\alpha^1$, $\delta\alpha^2$ denote infinitesimal displacements in the topological space R_2. The volume at the identity enclosed by $\delta\alpha^1 \wedge \delta\alpha^2$ is shaded (Fig. 3.8). This volume element can be moved to a volume element around α either by left translation $\alpha \circ \delta V$ or by right translation $\delta V \circ \alpha$.

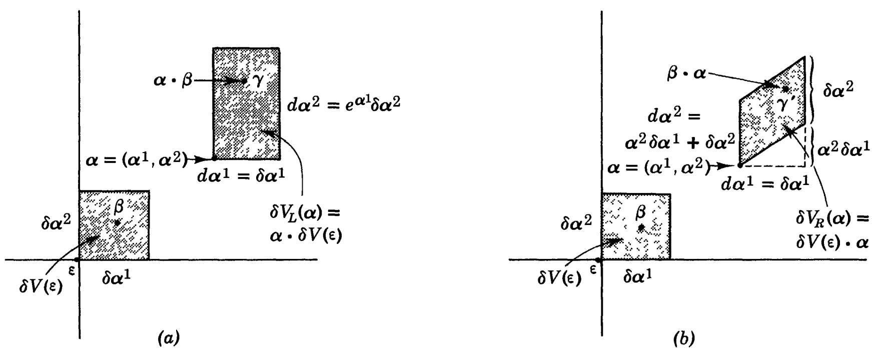


FIG. 3.8 A SMALL VOLUME ELEMENT NEAR THE IDENTITY IS CARRIED TO A SMALL VOLUME ELEMENT NEAR THE GROUP OPERATION α. IN EACH CASE, THE CHANGE IN THE ELEMENT OF VOLUME IS GIVEN BY THE JACOBIAN OF THE TRANSFORMATION: (a) UNDER A LEFT TRANSLATION BY α; (b) UNDER A RIGHT TRANSLATION BY α.

Let β be any point infinitesimally close to the identity and within the volume element $\delta V(\varepsilon)$ shown in Fig. 3.8*a*. Under a left translation by α $(\alpha \circ \beta)$ the volume element $\delta V(\varepsilon)$ around ε is moved to a volume element $\delta V_L(\alpha)$ around α. The coordinates of any point $\alpha \circ (\delta\alpha^1, \delta\alpha^2)$ are

$$(\alpha + d\alpha)^\mu = \phi^\mu(\alpha, \delta\alpha) = \phi^\mu(\alpha, 0) + \left.\frac{\partial \phi^\mu(\alpha, \beta)}{\partial \beta^\lambda}\right|_0 \delta\alpha^\lambda + \cdots \tag{5.61}$$

The Euclidean volume element $\delta V(\varepsilon)$ expands as it is moved to α by the factor

$$\alpha \circ \delta V(\varepsilon) = \delta V_L(\alpha) = d\alpha^1 \wedge d\alpha^2 = (\delta\alpha^1) \wedge (e^{\alpha^1}\, \delta\alpha^2) = e^{\alpha^1}\, \delta V(\varepsilon) \tag{5.71}$$

Therefore, we should weight the Euclidean volume element in the vicinity of α by a factor $e^{-\alpha^1}$, so that the product

$$\begin{pmatrix} \text{invariant} \\ \text{volume} \\ \text{element} \end{pmatrix} = (\text{density}) \times (\text{Euclidean volume element}) \tag{5.8}$$

is constant over the entire topological space.

Similar arguments (cf. Fig. 3.8*b*) involving right translations by α give a volume element

$$\begin{aligned} \delta V(\varepsilon) \circ \alpha = \delta V_R(\alpha) = d\alpha^1 \wedge d\alpha^2 &= (\delta\alpha^1) \wedge (\alpha^2\, \delta\alpha^1 + \delta\alpha^2) \\ &= \delta\alpha^1 \wedge \delta\alpha^2 = \delta V(\varepsilon) \end{aligned} \tag{5.7r}$$

On this group[9] the right invariant integral is defined by a uniform density, and the left invariant measure is defined by the nonuniform density $e^{-\alpha^1}$.

3. GENERAL LEFT AND RIGHT INVARIANT DENSITIES. We now turn to the general case. Let $\delta\alpha^1, \delta\alpha^2, \ldots, \delta\alpha^\eta$ be infinitesimal displacements in the η independent directions of R_η at the identity element 0. The volume they enclose is

$$dV(0) = \rho(0)\, \delta\alpha^1 \wedge \delta\alpha^2 \wedge \cdots \wedge \delta\alpha^\eta \tag{5.9}$$

Now move this volume element to an arbitrary group element α by left translation, and demand of the density function that

$$\rho(0)\, dV(0) = \rho_L(\alpha)\, dV_L(\alpha) \tag{5.10}$$

Here $dV_L(\alpha)$ is the Euclidean volume element $dV(0)$ after it has been moved from the vicinity of (0) to the vicinity of (α) by left translation with α. $dV_L(\alpha)$ is easy to compute:

$$dV_L(\alpha) = d\alpha^1 \wedge d\alpha^2 \wedge \cdots \wedge d\alpha^n \tag{5.11}$$

$$(\alpha + d\alpha)^\mu = \phi^\mu(\alpha, \delta\alpha) \tag{5.12}$$

$$= \phi^\mu(\alpha, 0) + \left.\frac{\partial\phi^\mu(\alpha, \beta)}{\partial\beta^\lambda}\right|_{\beta=0} (\delta\alpha)^\lambda + \cdots \tag{5.13}$$

Then we have

$$dV_L(\alpha) = \left.\frac{\partial\phi^1(\alpha, \beta)}{\partial\beta^{\lambda 1}}\right|_{\beta=0} \delta\alpha^{\lambda 1} \wedge \left.\frac{\partial\phi^2(\alpha, \beta)}{\partial\beta^{\lambda 2}}\right|_{\beta=0} \delta\alpha^{\lambda 2} \wedge \cdots \tag{5.14}$$

$$= \det \left\|\frac{\partial\phi^\mu(\alpha, \beta)}{\partial\beta^\lambda}\right\|_{\beta=0} dV(0) \tag{5.15}$$

The density function for left translations thus must satisfy (taking $\rho(0) = 1$)

$$\rho_L(\alpha) = \left\|\frac{\partial\phi(\alpha, \beta)}{\partial\beta}\right\|_{\beta=0}^{-1} \tag{5.16l}$$

An isomorphic calculation for right translations gives

$$\rho_R(\alpha) = \left\|\frac{\partial\phi(\beta, \alpha)}{\partial\beta}\right\|_{\beta=0}^{-1} \tag{5.16r}$$

These expressions may be used to verify (and in fact were computed) the density functions associated with Figs. 3.8. The invariant densities constructed in 5.16 are also called **Haar measures.**

4. EQUALITY OF LEFT AND RIGHT MEASURES. The left and right measures computed in Fig. 3.8 were not equal. Moreover, the integrals

$$\int \rho_L(\alpha)\, dV(\alpha), \qquad \int \rho_R(\alpha)\, dV(\alpha) \tag{5.17}$$

for this example do not converge, although any integral

$$\int f(\alpha)\rho_{L,R}(\alpha)\, dV(\alpha) \tag{5.18}$$

which did converge would be invariant. We ask: When are the left and right measures equal? When do the integrals of the measures converge?

When the group is compact, the measures are equal and the integrals converge. When the group is noncompact, the measures may be unequal but the integrals diverge. This property may be used as a test for compactness.

THEOREM. *The density functions $\rho_L(\alpha)$, $\rho_R(\alpha)$ giving the left and right invariant measures of a compact group are equal.*

Proof. We start with an element of volume $dV(0)$ at the identity group operation. Now we move this to an element of invariant volume at α by left translation. The result is

$$d\mu_L(\alpha) = \rho_L(\alpha)\, dV_L(\alpha) \tag{5.19}$$

If we now move this invariant volume back to the identity using right translation by α^{-1}, the result is a volume element at the identity, but with a possibly different size and shape from the original volume element, as shown in Fig. 3.9.

$$\begin{aligned} [\rho_L(\alpha)\, dV_L(\alpha)] \circ \alpha^{-1} &= \rho_L(\alpha)\, d\mu(0)\rho_R(\alpha^{-1}) \\ d\mu^1(0) &= [\rho_L(\alpha)\rho_R^{-1}(\alpha)]\, d\mu(0) \\ d\mu^1(0) &= f(\alpha)\, d\mu(0) \end{aligned} \tag{5.20}$$

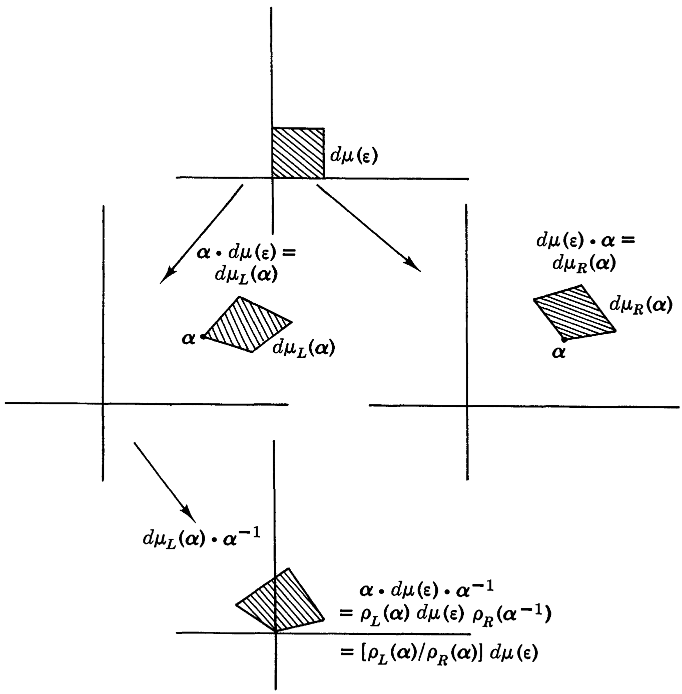


FIG. 3.9 UNDER LEFT AND RIGHT TRANSLATION BY α, THE INVARIANT VOLUME ELEMENT $d\mu(0)$ IS MAPPED INTO THE INVARIANT VOLUME ELEMENTS $d\mu_L(\alpha)$ AND $d\mu_R(\alpha)$ AT α. THESE GENERALLY HAVE DIFFERENT SIZES AND SHAPES. WHEN THE INVARIANT VOLUME ELEMENT $d\mu_L(\alpha)$ IS MAPPED BACK TO THE ORIGIN (IDENTITY ELEMENT) BY ACTING WITH α^{-1} ON THE RIGHT, THE NEW INVARIANT VOLUME ELEMENT AT THE IDENTITY MAY HAVE A DIFFERENT SIZE AND SHAPE FROM THE ORIGINAL VOLUME ELEMENT: $d\mu^1(0) = f(\alpha)\, d\mu(0)$.

Then we define recursively

$$d\mu^{n+1}(0) = \alpha \circ d\mu^{n}(0) \circ \alpha^{-1} = (\alpha)^{n+1} \circ d\mu(0) \circ (\alpha^{-1})^{n+1}$$

$$d\mu^{n+1}(0) = f(\alpha)\, d\mu^{n}(0) = f^{n+1}(\alpha)\, d\mu(0) \tag{5.21}$$

Since the group is compact, the limit, as $n \to \infty$, exists and is well defined:

$$\lim_{n\to\infty} (\alpha)^n = \beta \tag{5.22}$$

$$\begin{aligned} d\mu^{(\infty)}(0) &= \lim_{n\to\infty} \alpha^n\, d\mu(0)(\alpha^{-1})^n = \lim_{n\to\infty} [f(\alpha)]^n\, d\mu(0) \\ &= f(\beta)\, d\mu(0) \end{aligned} \tag{5.23}$$

If $f(\alpha) \gtrless 1$, then

$$\lim_{n\to\infty} [f(\alpha)]^n \begin{smallmatrix}\nearrow \infty \\ \searrow 0\end{smallmatrix} \tag{5.24}$$

But since β is an element of the group, $f(\beta)$ can be neither zero nor infinity, for then the operation $\beta \circ d\mu(0) \circ \beta^{-1}$ would be singular.

We conclude $f(\alpha) = 1$, thus $\rho_L(\alpha) = \rho_R(\alpha)$ and $d\mu_L(\alpha) = d\mu_R(\alpha)$:

$$\alpha \circ d\mu(0) \circ \alpha^{-1} = 1\, d\mu(0) \tag{5.25}$$

$$d\mu_L(\alpha) = \alpha \circ d\mu(0) = d\mu(0) \circ \alpha = d\mu_R(\alpha). \tag{5.26}$$

Thus the left and right measures on a compact group must be equal.

The left and right invariant measures are equal[8] on the following kinds of groups:

1. Finite groups
2. Discrete groups
3. (Locally compact) abelian groups
4. Compact groups
5. Simple groups
6. Semisimple groups
7. Real connected algebraic groups with determinant $+1$
8. Connected nilpotent Lie groups
9. Semidirect product groups, such as
   (a) Euclidean group $SO(3) \wedge T_3$ or $ISO(3)$
   (b) Poincaré group $SO(3, 1) \wedge T_{3,1}$ or $ISO(3, 1)$
10. Contractions of semisimple groups by maximal subgroups:
    (a) $ISO(p, q)$
    (b) $IU(p, q)$
    (c) $IUSp(2p, 2q)$
11. Lie groups for which $Ad(G)$ is unimodular[15]

5. EXTENSION TO CONTINUOUS GROUPS. The invariant integrals have so far been constructed for the connected components of a continuous group. To extend this to the other sheets we observe two things.

1. Every element $\alpha \in G$ can be written uniquely as $\alpha = d_i \circ \alpha_0$ where $d_i \in D$, a discrete factor group, and $\alpha_0 \in G_0$, the connected component.
2. Each sheet is topologically equivalent to every other sheet. The extension we seek, then, is

$$\int_{\alpha \in G} f(\alpha)\rho(\alpha)\, dV(\alpha) = \sum_{d_i \in D} \int_{\alpha_0 \in G_0} f(d_i \circ \alpha_0)\rho(\alpha_0)\, dV(\alpha_0) \tag{5.27}$$

In the event the connected component is of dimension 0, it contains only one point, the identity, and the invariant integral reduces to [with $\rho(\varepsilon) = 1$] the form familiar for discrete groups:

$$\int f(\alpha)\rho(\alpha)\, dV(\alpha) \rightarrow \sum_{d_i \in D} f(d_i) \tag{5.28}$$

Résumé

The first four sections of this chapter initiated our study of Lie groups. In the first section continuous groups were defined and described. In the second section an example of a topological group was treated in some detail. In the third section additional topological and algebraic concepts were introduced. From these, we realize it is not necessary to study all possible continuous groups: we need study only connected continuous groups and discrete groups. We also sense that it is not even necessary to study the entire connected component; only the group properties near the identity require detailed study. Lie groups were defined in the fourth section.

The fifth section dealt with invariant integration over a group. The general left and right invariant integrals were described, defined, and computed both abstractly and in terms of an example. The integral was extended over other sheets not connected to the identity. The form of the "integral" in the case of a discrete group was also given.

Exercises

1. Show that each of the classical matrix groups considered in Chapter 2 is a manifold. Show that each is a topological group. Of what dimensionality? How can each be coordinatized?

2. Compute how the group $SO(3)$ acts as a continuous group of transformations on the geometric space R_3. Compute how it acts as a topological group on its own underlying topological space. Is the underlying topological space locally isomorphic with R_3 ? Globally?

3. A representation of (2.9) acts on a linear vector space containing a basis vector $(x)^{N+k}$, where N is an integer greater than zero and k is not an integer. Show that this representation must also contain bases of the form $(x)^{n+k}$, where $N - n$ is an integer.

4. Consider the following manifolds: the forward light cone, the two-sheeted hyperboloid, the sphere surface, and the donut surface. Is each connected? Simply connected? Compact? Closed?

5. Prove that the set of all nonzero real rational numbers is a group under multiplication. Prove that it is not locally compact.

6. Prove that the group $O(3, r)$ contains two components. Also prove that the factor group $O(3, r)/SO(3, r)$ consists of the two elements $+I_3$ and $-I_3$. Show that the subset of $O(3, r)$ consisting of matrices with determinant -1 is topologically isomorphic with $SO(3, r)$. What mapping converts this subset into a group?

7. In general, how many components does $O(p; r)$ consist of? $O(p, q; r)$? What are they? In particular, how many components does the homogeneous Lorentz group $O(3, 1; r)$ have?

8. Let

$$f(\alpha^1, \alpha^2) = \frac{e^{-(\alpha 1)2}}{(\alpha^2)^2 + \omega^2} \qquad \text{in (5.1)}$$

Carry out the following integrals explicitly and show that they are equal:

(a) $$\int f(\alpha)\, d\mu_L(\alpha)$$

(b) $$\int f(\beta \circ \alpha)\, d\mu_L(\alpha)$$

Is this value equal to the following value?

(c) $$\int f(\alpha)\, d\mu_R(\alpha)$$

9. Let M be an $n \times n$ matrix with matrix elements $M_i{}^j$. Let $f^k(M)$ be a set of algebraic functions defined on these n^2 matrix elements. Under what conditions does the subset of $Gl(n)$ which leaves invariant the algebraic relations

$$f^k(M) = 0$$

form a group? Such a group is called an **algebraic group**.[14,15] Prove that all the classical groups discussed in Chapter 2 are algebraic groups.

10. The group $ISO(3)$ consists of rotations about the origin in three-dimensional space, together with all possible displacements of the origin. Show that this is an algebraic subgroup of $Sl(4, r)$. How many conditions are necessary to define this subgroup? What are they? What is the action of $ISO(3)$ on the coordinates (x, y, z) of a point in R_3 ?

11. The group $IU(n, f)$ consists of all elements in $U(n, f)$, together with all displacements of the origin in V_n (and all possible products). These groups are algebraic subgroups of $Gl(n+1, f)$. What are the algebraic relations? What is the matrix structure of these groups?

12. Enumerate other algebraic groups of importance in modern physical theory.

13. Construct the Haar measure on $O(3)$.

14. Let G be a discrete group and H a subgroup. When H is an invariant subgroup of G, prove that the coset representatives in G/H can be chosen in such a way that they are closed under multiplication. They then form a group. The coset representatives in G/H with this group structure form the *factor group* G/H.

Notes and References

The material presented here is a greatly expanded version of the material presented in the first few pages of Racah [1].

1. I. M. Singer, J. A. Thorpe. [1]
2. R. L. Bishop, R. J. Crittenden. [1]
3a. M. Spivak. [1]
3b. L. H. Loomis, S. Sternberg. [1]
4. C. Chevalley. [1]
5. L. Pontrjagin. [1]
6. G. Racah. [1]
7. S. Lie, F. Engel. [1–3]
8. A. J. Coleman. [1]
9. M. Hamermesh. [1], cf. especially pp. 313–317.
10. W. Miller. [3], cf. especially Part II.
11. W. Miller. [2]
12. P. M. Cohn. [1]
13. K. Nomizu. [1]
14. C. Chevalley. [1], p. 201.
15. S. Helgason. [1], cf. p. 366.

CHAPTER 4

Lie Groups and Lie Algebras

In this chapter we study the infinitesimal properties of Lie groups, that is, the properties of the group near the identity element. This leads in a natural way to the concepts of infinitesimal generator and Lie algebra.

Lie's three theorems provide a mechanism for constructing the Lie algebra associated with any Lie group. They also characterize the properties of a Lie algebra.

The converses of Lie's three theorems do the opposite: they supply a mechanism for associating a Lie group with any finite-dimensional Lie algebra. The correspondence is not 1-1; many different Lie groups may have the same Lie algebra. There is, however, a 1-1 correspondence between Lie algebras and simply connected Lie groups. All other Lie groups possessing the same algebra can be obtained from this simply connected group in a straightforward way, as factor groups of discrete invariant subgroups.

Finally, Taylor's theorem allows for the construction of a canonical analytic structure function $\phi(\beta, \alpha)$ from the Lie algebra.

These seven theorems—the three theorems of Lie and their converses, and Taylor's theorem—provide an essential equivalence between Lie groups and algebras. In our future work, we use either the group or the algebra, depending on convenience or suitability.

I. Infinitesimal Properties of Lie Groups

1. INFINITESIMAL GENERATORS FOR LIE GROUPS OF TRANSFORMATIONS. Let $(\mathscr{T}_n, \phi)$ be a Lie group that acts on the geometric space G_N by means of a transformation of coordinates $f(\alpha, x)$: in other words, a Lie group of transformations.

Now let $F(p)$ be any function defined on all points $p \in G$. Once we choose a coordinate system in G, we can assign each point p an N-tuple of coordinates.

$$p \xrightarrow[\text{system } S]{\text{in coordinate}} x^1(p), x^2(p), \ldots, x^N(p) \tag{1.1}$$

The function $F(p)$ can then be written in terms of the parameters $x^i(p)$ in coordinate system S:

$$F(p) = F^S[x^1(p), x^2(p), \ldots, x^N(p)] \tag{1.2}$$

In some other coordinate system S' the coordinates of p will change; thus the structural form of the function must also change to preserve the fixed value at point p

$$F(p) = F^{S'}[x'^1(p), x'^2(p), \ldots, x'^N(p)] \tag{1.3}$$

How do we determine $F^{S'}$ if we know F^S?

The coordinate systems S and S' in G are related by an element from the Lie group of transformations

$$x'^j(p) = f^j[\alpha, x(p)] \tag{1.4}$$

Since we want to know $F^{S'}$ in terms of $x'(p)$, and since we already know F^S in terms of $x(p)$, we must solve $x(p)$ in terms of $x'(p)$:

$$x^i(p) = f^i[\alpha^{-1}, x'(p)] \tag{1.5}$$

Then the complete solution to our problem is

$$\begin{aligned} F^{S'}&[x'^1(p), x'^2(p), \ldots, x'^N(p)] \\ &= F^S[f^1(\alpha^{-1}, x'(p)), f^2(\alpha^{-1}, x'(p)), \ldots, f^N(\alpha^{-1}, x'(p))] \end{aligned} \tag{1.6}$$

This solution is not in a particularly useful form. It is much more convenient to concentrate on transformations that are close to the identity. For a Lie group operation $\delta\alpha^\mu$ near the identity 0, the inverse is given by $(\delta\alpha^{-1})^\mu = -\delta\alpha^\mu$. Then we can write

$$\begin{aligned} x^j(p) &= f^j[(\delta\alpha^{-1}), x'(p)] = f^j[-\delta\alpha^\mu, x'(p)] \\ &= f^j[0, x'(p)] + \left.\frac{\partial f^j[\beta, x'(p)]}{\partial \beta^\mu}\right|_{\beta=0} (-\delta\alpha^\mu) + \cdots \\ &\cong x'^j(p) - \delta\alpha^\mu \left.\frac{\partial f^j[\beta, x'(p)]}{\partial \beta^\mu}\right|_{\beta=0} \end{aligned} \tag{1.7}$$

When this solution for $x(p)$ in terms of $x'(p)$ is employed in (1.6) we find

$$\begin{aligned} F^{S'}[x'(p)] &= F^S\left[x'^j(p) - \delta\alpha^\mu \left.\frac{\partial f^j[\beta, x'(p)]}{\partial \beta^\mu}\right|_{\beta=0}\right] \\ &\cong F^S[x'(p)] - \delta\alpha^\mu \left.\frac{\partial f^j[\beta, x'(p)]}{\partial \beta^\mu}\right|_{\beta=0} \frac{\partial}{\partial x'^j} F^S[x'(p)] \end{aligned} \tag{1.8}$$

To lowest order, then, the change in the structural form of F is

$$F^{S'}[x'(p)] - F^S[x'(p)] = \delta\alpha^\mu\left\{-\left.\frac{\partial f^j[\beta, x'(p)]}{\partial \beta^\mu}\right|_{\beta=0}\frac{\partial}{\partial x'^j}\right\}F^S[x'(p)] \qquad (1.9)$$

$$= \delta\alpha^\mu X_\mu(x')F^S(x') \qquad (1.10)$$

The quantities

$$X_\mu(x') = -\left.\frac{\partial f^j(\beta, x')}{\partial \beta^\mu}\right|_{\beta=0}\frac{\partial}{\partial x'^j} \qquad (1.11)$$

are called the generators of infinitesimal displacements of coordinate systems by $\delta\alpha^\mu$, or simply **generators**.

Finite displacements can be obtained by repeated applications of infinitesimal displacements.

Example. For our two-parameter example group,

$$f^1(\alpha^1, \alpha^2; x) = e^{\alpha^1}x + \alpha^2 \qquad (1.12)$$

$$X_1(x) = -\left.\frac{\partial f(\alpha, x)}{\partial \alpha^1}\right|_{\alpha=0}\frac{\partial}{\partial x} = -x\frac{\partial}{\partial x}$$

$$X_2(x) = -\left.\frac{\partial f(\alpha, x)}{\partial \alpha^2}\right|_{\alpha=0}\frac{\partial}{\partial x} = -\frac{\partial}{\partial x} \qquad (1.13)$$

Suppose we have a function which in S is (Fig. 4.1):

$$F^S = (x - c)^2 \qquad (1.14)$$

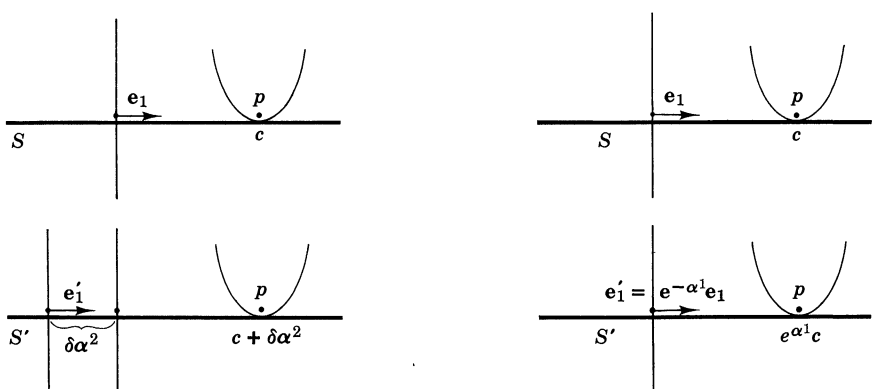


FIG. 4.1 BEHAVIOR OF $(x - c)^2$ IN TWO SITUATIONS: (a) UNDER A DISPLACEMENT OF THE ORIGIN DEFINED BY $x' = x + \delta\alpha^2$, THE STRUCTURE OF $(x - c)^2$ BECOMES $\{I + \delta\alpha^2 X_2(x')\}(x' - c)^2 = [x' - (c + \delta\alpha^2)]^2$. (b) UNDER A CHANGE OF LENGTH OF BASIS $\mathbf{e}_1' = \mathrm{e}^{-\alpha^1}\mathbf{e}_1$, THE FUNCTION $(x - c)^2$ BECOMES

$$\lim_{N\to\infty}\left\{I + \frac{\alpha^1}{N}X_1(x')\right\}^N(x' - c)^2 = [e^{-\alpha^1}x' - c]^2.$$

Then in S' related to S by

$$x' = x + \delta\alpha^2 \tag{1.15}$$

we want to see what the structure of the function is. We know it should be

$$F^{S'} = (x' - c - \delta\alpha^2)^2 \tag{1.16}$$

Calculating by means of (1.9), we find

$$\begin{aligned} F^{S'}(x') &= \{I + \delta\alpha^2 X_2(x')\}(x' - c)^2 \\ &= \left\{I - \delta\alpha^2 \frac{\partial}{\partial x}\right\}(x' - c)^2 \\ &= [x' - (c + \delta\alpha^2)]^2 - (\delta\alpha^2)^2 \\ &= [x' - (c + \delta\alpha^2)]^2 \end{aligned} \tag{1.17}$$

to first order.

The finite displacement

$$x' = x + \alpha^2 \tag{1.18}$$

is obtained by repeated application of infinitesimal displacements

$$\begin{aligned} F^{S'}(x') &= \lim_{N\to\infty} \left\{I + \frac{\alpha^2}{N} X_2(x')\right\}^N (x' - c)^2 \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \{\alpha^2 X_2(x')\}^n (x' - c)^2 \end{aligned} \tag{1.19}$$

$$\begin{aligned} &= \text{EXP } \alpha^2 X_2(x')(x' - c)^2 \\ &= e^{-\alpha^2\, \partial/\partial x}(x' - c)^2 \\ &= [x' - (c + \alpha^2)]^2 \end{aligned} \tag{1.20}$$

(The last equality will be recognized as simply Taylor's theorem.)

If we expand our coordinate system by shrinking the basis (Fig. 4.1*b*)

$$x' = e^{\delta\alpha 1}x = (1 + \delta\alpha^1)x$$

then we find, for infinitesimal displacements

$$F^{S'}(x') = \left\{I - \delta\alpha^1 x' \frac{\partial}{\partial x'}\right\}(x' - c)^2 = [(1 - \delta\alpha^1)x' - c]^2 \tag{1.21}$$

and, for finite displacements

$$\begin{aligned} F^{S'}(x') &= \lim_{N\to\infty} \left\{ I - \frac{\alpha^1}{N} x' \frac{\partial}{\partial x'} \right\}^N (x'-c)^2 \\ &= e^{-\alpha^1 x' \partial/\partial x'}(x'-c)^2 \\ &= [e^{-\alpha^1} x' - c]^2 \end{aligned} \tag{1.22}$$

2. INFINITESIMAL GENERATORS FOR A LIE GROUP. It is an easy matter to determine the infinitesimal generators for a Lie group, since it is a Lie group of transformations acting on itself. Under the identifications

$$\text{geometric space } G_N \to \text{topological space } \mathcal{T}_\eta$$

$$f(\alpha, x(p)) \to \phi(\alpha, \chi(p)) \tag{1.23}$$

$$F[x(p)] \to \Phi[\chi(p)] \tag{1.24}$$

the infinitesimal generators are

$$X_\mu(x') = -\left.\frac{\partial f^j(\beta, x')}{\partial \beta^\mu}\right|_{\beta=0} \frac{\partial}{\partial x'^j} \to X_\mu(\chi') = -\left.\frac{\partial \phi^\lambda(\beta, \chi')}{\partial \beta^\mu}\right|_{\beta=0} \frac{\partial}{\partial \chi'^\lambda} \tag{1.25}$$

Since a Lie group acting on itself is a nonsingular change of basis,

$$\det \left\| \left.\frac{\partial \phi^\lambda(\beta, \chi'(p))}{\partial \beta^\mu}\right|_{\beta=0} \right\| \neq 0 \tag{1.26}$$

Example. For our Lie group with parameterization [Chapter 3(5.4)] and multiplication law

$$\begin{aligned} \phi^1(\beta^1\beta^2; \chi^1\chi^2) &= \beta^1 + \chi^1 \\ \phi^2(\beta^1\beta^2; \chi^1\chi^2) &= e^{\beta^1}\chi^2 + \beta^2 \end{aligned} \tag{1.27}$$

$$\left.\begin{pmatrix} \dfrac{\partial \phi^1}{\partial \beta^1} = 1 & \dfrac{\partial \phi^2}{\partial \beta^1} = \chi^2 \\ \dfrac{\partial \phi^1}{\partial \beta^2} = 0 & \dfrac{\partial \phi^2}{\partial \beta^2} = 1 \end{pmatrix}\right|_{\beta=0} \tag{1.28}$$

Therefore

$$\begin{aligned} X_1(\chi) &= -\frac{\partial}{\partial \chi^1} - \chi^2 \frac{\partial}{\partial \chi^2} \\ X_2(\chi) &= \qquad\quad - \frac{\partial}{\partial \chi^2} \end{aligned} \tag{1.291}$$

Comment. The group operation may be interpreted in two ways.

1. The coordinates $\chi(p)$ in coordinate system S are given, in S' related to S by α, by the expressions

$$\chi'(p) = \phi(\alpha, \chi(p))$$

This interpretation is shown in Fig. 4.2.

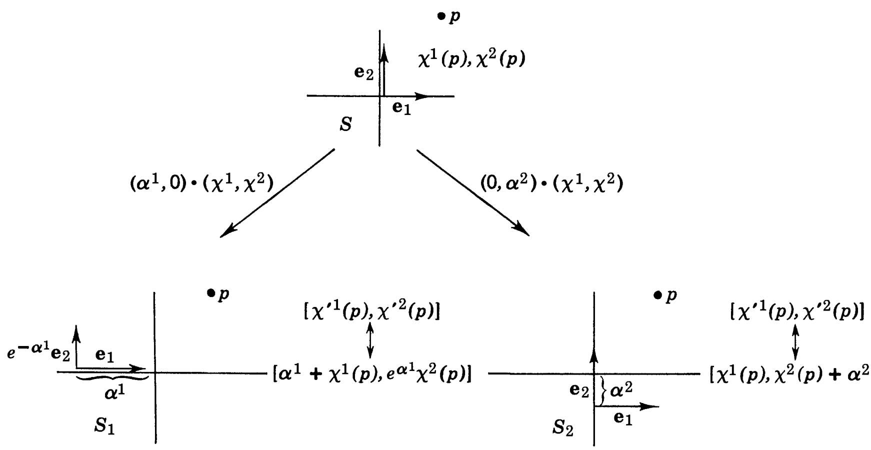


FIG. 4.2 UNDER LEFT TRANSLATION BY $(\alpha^1, 0)$, AND $(0, \alpha^2)$, THE COORDINATES $\chi^1(p)$, $\chi^2(p)$ OF p BECOME $[\alpha^1 + \chi^1(p), e^{\alpha^1}\chi^2(p)]$ AND $[\chi^1(p), \chi^2(p) + \alpha^2]$. THE EFFECT OF THESE TRANSFORMATIONS ON THE BASES IS SHOWN IN COORDINATE SYSTEMS S_1 AND S_2.

2. The coordinates $\alpha(p)$ in coordinate system S are given, in S' related to S by χ, by the expression

$$\alpha'(p) = \phi(\alpha(p), \chi)$$

Items 1 and 2 are called left translation by α and right translation by χ, respectively. The infinitesimal generators for right translations are given by

$$\begin{pmatrix} X_1(\alpha) \\ X_2(\alpha) \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & e^{\alpha^1} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \alpha^1} \\ \dfrac{\partial}{\partial \alpha^2} \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial}{\partial \alpha^1} \\ -e^{\alpha^1}\dfrac{\partial}{\partial \alpha^2} \end{pmatrix} \tag{1.29r}$$

3. INFINITESIMAL GENERATORS FOR MATRIX GROUPS. The generators for matrix groups are defined analogously. If $M(\alpha^1, \alpha^2, \ldots, \alpha^n)$ is

an element of a group of $r \times r$ matrices, we define the infinitesimal generators by

$$X_\mu(r \times r) = \lim_{\alpha^\mu \to 0} \frac{M(0, 0, \ldots, \alpha^\mu, 0, 0) - M(0, 0, \ldots, 0)}{\alpha^\mu} \tag{1.30}$$

The η generators constructed in this way are bases for a linear vector space, for $\lambda^\mu X_\mu(r \times r)$ is also an infinitesimal generator:

$$\lambda^\mu X_\mu(r \times r) = \lim_{\tau \to 0} \frac{M(\lambda^1\tau, \lambda^2\tau, \ldots, \lambda^n\tau) - M(0, 0, \ldots, 0)}{\tau}$$

In fact, the generators $X_\mu(x)$ and $X_\mu(\chi)$ are also bases for a linear vector space, for exactly the same reason. Furthermore, the X_μ are all linearly independent, since each group operation must possess a unique inverse.

Example. The generators for our 2×2 group of matrices are given by

$$\begin{aligned} X_1(2 \times 2) &= \frac{\partial}{\partial \alpha^1} \begin{pmatrix} e^{\alpha 1} & \alpha^2 \\ 0 & 1 \end{pmatrix}_{\alpha=0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ X_2(2 \times 2) &= \frac{\partial}{\partial \alpha^2} \begin{pmatrix} e^{\alpha 1} & \alpha^2 \\ 0 & 1 \end{pmatrix}_{\alpha=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{1.31}$$

In this manner, the generators for the $(N + 1) \times (N + 1)$ faithful representation of this group with bases $(x^N, x^{N-1}, \ldots, x^2, x^1, x^0)$ are constructed:

$$X_1 = \begin{pmatrix} N & & & & \bigcirc \\ & N-1 & & & \\ & & \ddots & & \\ & \bigcirc & & 1 & \\ & & & & 0 \end{pmatrix} \tag{1.32}$$

$$X_2(N + 1 \times N + 1) = \begin{pmatrix} 0 & N-1 & & & & & & & \\ & 0 & & & & & & \bigcirc & \\ & & 0 & & & & & & \\ & & & 0 & & & & & \\ & & & & 0 & & & & \\ & & & & & 0 & 3 & & \\ & \bigcirc & & & & & 0 & 2 & \\ & & & & & & & 0 & 1 \\ & & & & & & & & 0 \end{pmatrix}$$

An arbitrary group element can be determined by repeated application of infinitesimal displacements:

$$\mathrm{EXP}\{\alpha^1 X_1(2\times 2) + \alpha^2 X_2(2\times 2)\} = \mathrm{EXP}\begin{pmatrix} \alpha^1 & \alpha^2 \\ 0 & 0 \end{pmatrix} \tag{1.33}$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!}\begin{pmatrix} \alpha^1 & \alpha^2 \\ 0 & 0 \end{pmatrix}^N$$

$$= \begin{pmatrix} \sum_0^\infty \frac{(\alpha^1)^N}{N!} & \alpha^2 \sum_1^\infty \frac{(\alpha^1)^{N-1}}{N!} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\alpha 1} & \frac{\alpha^2}{\alpha^1}(e^{\alpha 1} - 1) \\ 0 & 1 \end{pmatrix} \tag{1.34}$$

This parameterization of our group is different from the one in Chapter 3, (5.4). The two parameterizations are called "analytically isomorphic" because both describe the same Lie group and because there is an analytic mapping from one to the other.

This suggests another problem in the study of Lie groups: how can we classify different Lie groups if even the same group may have two distinct analytic structures $\phi(\beta, \alpha)$ and $\phi'(\beta, \alpha)$ describing the group multiplication? The answer is simple. Both ϕ and ϕ' have the same local properties. This suggests that a canonical mapping ϕ can be constructed using the exponential mapping. This is often true; in fact, for the groups of interest to physicists, it is almost always true.

The analytic isomorphisms connecting canonical structures $\phi(\beta, \alpha)$ and various noncanonical mappings $\phi'(\beta, \alpha)$ are called **Baker-Campbell-Hausdorff**[8–10] **formulas.**

The group multiplication, defined by matrix multiplication of (1.34), is such that every straight line through the identity describes a one-parameter abelian subgroup:

$$(\alpha^1 t, \alpha^2 t) \circ (\alpha^1 s, \alpha^2 s) = (\alpha^1(t+s), \alpha^2(t+s)) \tag{1.35}$$

Furthermore, all subgroups with $\alpha^1 \neq 0$ are conjugate; the subgroup with $\alpha^1 = 0$ is invariant.

4. COMMUTATION RELATIONS. If we define a **commutator** by

$$[A, B] = AB - BA = -[B, A] \tag{1.36}$$

we observe that

$$[X_1(x), X_2(x)] = \left[-x\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}\right] = -\frac{\partial}{\partial x} = X_2(x) \tag{1.37a}$$

$$[X_1(\chi), X_2(\chi)] = \left[-\frac{\partial}{\partial \chi^1} - \chi^2\frac{\partial}{\partial \chi^2}, -\frac{\partial}{\partial \chi^2}\right] = -\frac{\partial}{\partial \chi^2} = X_2(\chi) \tag{1.37b}$$

$$[X_1(N+1), X_2(N+1)] = X_2(N+1) \tag{1.37c}$$

All realizations and representations of this Lie algebra have isomorphic commutation relations.

Clearly, this must be a fundamental property of Lie algebras. What is the significance of the commutation relations? If α and β are elements in a commutative (abelian) group, then

$$\alpha\beta\alpha^{-1} = \beta \tag{1.38c}$$

But if the group is not commutative, γ measures the amount by which $\alpha\beta\alpha^{-1}$ differs from β:

$$\alpha\beta\alpha^{-1} = \gamma\beta \tag{1.38nc}$$

The γ is a group element because

$$\alpha\beta(\beta\alpha)^{-1} = \gamma \tag{1.39}$$

The left-hand side of (1.39) is called the commutator of elements α, β *in a group.*

Now if α and β are close to the identity, we can expand them in terms of infinitesimal generators:

$$\begin{aligned} \delta\alpha &\to I + \delta\alpha^\mu X_\mu + \tfrac{1}{2}\,\delta\alpha^\mu X_\mu\,\delta\alpha^\nu X_\nu \\ \delta\beta &\to I + \delta\beta^\mu X_\mu + \tfrac{1}{2}\,\delta\beta^\mu X_\mu\,\delta\beta^\nu X_\nu \end{aligned} \tag{1.40}$$

Now forming the product $(\alpha\beta)\circ(\beta\alpha)^{-1}$ and keeping terms only to second order (and being careful of order), we write

$$\begin{aligned} (\alpha\beta)(\beta\alpha)^{-1} &= I + \delta\alpha^\mu\,\delta\beta^\nu(X_\mu X_\nu - X_\nu X_\mu) \\ &= I + \delta\alpha^\mu\,\delta\beta^\nu[X_\mu, X_\nu] \end{aligned} \tag{1.41}$$

This commutator must exist in the vector space of group generators, since $(\alpha\beta)(\beta\alpha)^{-1}$ is an element of the group. Therefore, the commutator may be expanded in terms of the bases X_λ:

$$[X_\mu, X_\nu] = C_{\mu\nu}{}^\lambda X_\lambda \tag{1.42}$$

The $C_{\mu\nu}{}^\lambda$ are called structure constants. They completely determine the structure of a Lie algebra and almost uniquely determine the structure of the Lie group with that Lie algebra.

Example. Take $\alpha = (t\alpha^1, 0)$ and $\beta = (0, t\alpha^2)$. Then we form the commutator of α with β.

$$\begin{pmatrix} e^{t\alpha^1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t\alpha^2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t\alpha^1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t\alpha^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (e^{t\alpha^1} - 1)t\alpha^2 \\ 0 & 1 \end{pmatrix} \quad (1.43)$$

Differentiation of the right-hand side with respect to t^2 yields the generator $X_2(2 \times 2)$. The computation carried out in (1.43) is illustrated in Fig. 4.3. It is clear that

$$\frac{\partial}{\partial t^2}\,\alpha(t)\beta(t)\alpha^{-1}(t)\beta^{-1}(t) = \alpha^1\alpha^2 X_2(2 \times 2)$$

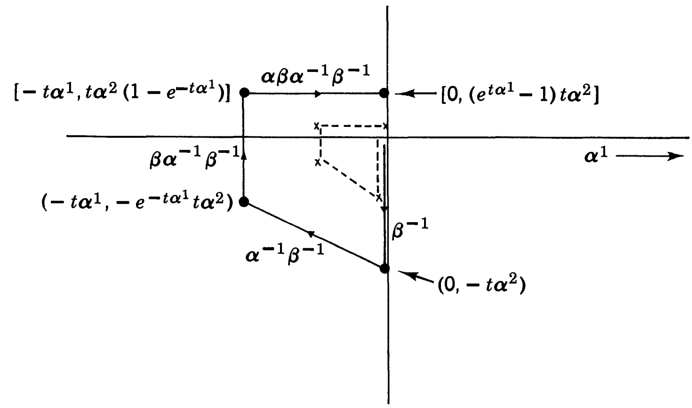


FIG. 4.3 IF $\alpha = (t\alpha^1, 0)$ AND $\beta = (0, t\alpha^2)$, THE FORMATION OF THE COMMUTATOR $\alpha\beta\alpha^{-1}\beta^{-1}$ IS SHOWN, FOR SUCCESSIVELY SMALLER VALUES OF THE PARAMETER t. THE CURVE

$$\alpha(t)\beta(t)\alpha^{-1}(t)\beta^{-1}(t)$$

LIES ON THE AXIS $\alpha^1 = 0$ CORRESPONDING TO THE INFINITESIMAL GENERATOR (WITH RESPECT TO t^2) $X_2(\alpha)$.

II. Lie's First Theorem

1. THEOREM. *If*[2]

$$x'^i(p) = f^i[\alpha^\mu, x(p)] \qquad \begin{array}{l} \mu = 1, 2, \ldots, \eta \\ i = 1, 2, \ldots, N \end{array} \quad (2.1)$$

is analytic, then

$$\frac{\partial x'^i}{\partial \alpha^\lambda} = \frac{\partial f^i[\alpha, x]}{\partial \alpha^\lambda} = \sum_{\kappa=1}^{\eta} \Psi_\lambda{}^\kappa(\alpha) u_\kappa{}^i(x') \quad (2.2)$$

where $u_\kappa{}^i(x')$ *is analytic.*

Proof. If we transform from coordinate system S to S' and then to S'',

$$S \xrightarrow{\alpha} S' \xrightarrow{\beta} S'' \tag{2.3}$$

the coordinates of any point p will change successively from $x^i(p)$ to $x'^i(p)$ to $x''^i(p)$. Suppose now that β is an operation close to the identity, say $\beta = \delta\alpha$. Then $x''^i(p)$ will be close to $x'^i(p)$, and the difference

$$dx'^i(p) = x''^i(p) - x'^i(p) \tag{2.4}$$

can be computed in two different ways:

$$\begin{aligned} x'^i(p) &= f^i[\alpha, x] \\ x''^i(p) &= (x' + dx')^i = f^i[\delta\alpha, x'] \\ dx'^i &= \delta\alpha^\lambda \left.\frac{\partial f^i[\beta, x']}{\partial \beta^\lambda}\right|_{\beta=0} \\ &= \delta\alpha^\lambda u_\lambda{}^i(x') \end{aligned} \tag{2.5a}$$

$$\begin{aligned} (\alpha + d\alpha)^\mu &= \phi^\mu(\delta\alpha, \alpha) \\ d\alpha^\mu &= \delta\alpha^\lambda \left.\frac{\partial \phi^\mu(\beta, \alpha)}{\partial \beta^\lambda}\right|_{\beta=0} \\ &= \delta\alpha^\lambda \Theta_\lambda{}^\mu(\alpha) \end{aligned} \tag{2.5b}$$

Since $\Theta_\lambda{}^\mu(\alpha)$ is an $\eta \times \eta$ nonsingular matrix, it has an inverse $\Psi(\alpha)$:

$$\Theta(\alpha)\Psi(\alpha) = I = \Psi(\alpha)\Theta(\alpha) \tag{2.6}$$

$$\delta\alpha^\lambda = d\alpha^\mu \Psi_\mu{}^\lambda(\alpha) \tag{2.7}$$

In terms of the displacements $d\alpha$ near α induced by the infinitesimal displacements $\delta\alpha$ at the identity, the displacement dx' at $x'(p)$ is given by

$$\begin{aligned} dx'^i &= d\alpha^\mu \Psi_\mu{}^\lambda(\alpha) u_\lambda{}^i(x') \\ \frac{\partial x'^i}{\partial \alpha^\mu} &= \sum_{\lambda=1}^{\eta} \Psi_\mu{}^\lambda(\alpha) u_\lambda{}^i(x') \end{aligned} \tag{2.8}$$

Since $f^i[\alpha, x]$ and $\phi^\mu(\beta, \alpha)$ are analytic,

$$\begin{aligned} u_\lambda{}^i(x') &= \left.\frac{\partial f^i[\beta, x']}{\partial \beta^\lambda}\right|_{\beta=0} \\ \Theta_\lambda{}^\mu(\alpha) &= \left.\frac{\partial \phi^\mu(\beta, \alpha)}{\partial \beta^\lambda}\right|_{\beta=0} \end{aligned} \tag{2.9}$$

are also analytic.

2. EXAMPLE. We reproduce the arguments of the theorem in terms of a concrete example. If $\alpha = (\alpha^1, \alpha^2)$ is a transformation and $\delta\alpha = (\delta\alpha^1, \delta\alpha^2)$ is close to the identity, then

$$x' = e^{\alpha 1}x + \alpha^2 \tag{2.10}$$

$$x' + dx' = e^{\delta\alpha 1}x' + \delta\alpha^2 \tag{2.11}$$

$$dx' = \delta\alpha^1 x' + \delta\alpha^2 \tag{2.12}$$

This computation is illustrated in Fig. 4.4.

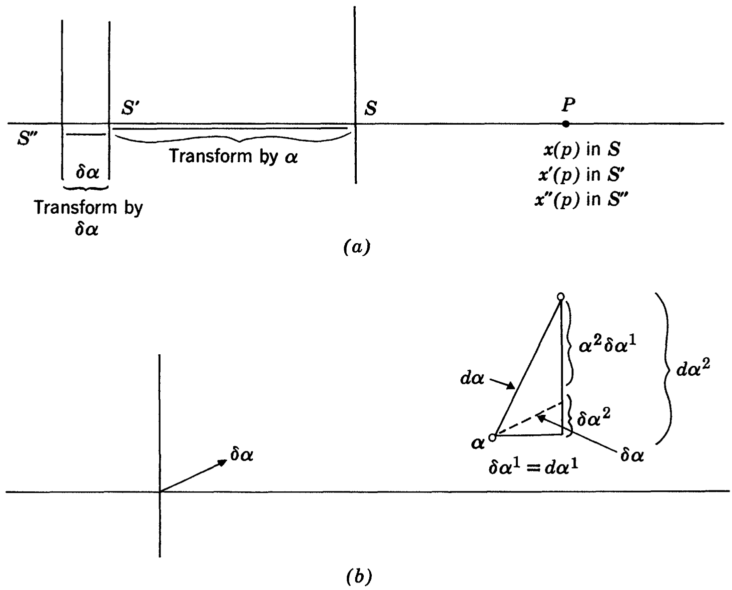


FIG. 4.4 THE TRANSFORMATION FROM $S \xrightarrow{\alpha} S' \xrightarrow{\delta\alpha} S''$ CAN BE COMPUTED IN TWO WAYS: (a) WE COMPUTE dx' IN TERMS OF $\delta\alpha$. (b) WE COMPUTE $d\alpha$ IN TERMS OF $\delta\alpha$ AND THEN INVERT TO FIND $\delta\alpha$ OF (a) IN TERMS OF THE $d\alpha$ IN (b).

Now, since $\delta\alpha \circ \alpha$ is near α, we can write

$$(\alpha + d\alpha)^\mu = \phi^\mu(\delta\alpha^1, \delta\alpha^2; \alpha^1, \alpha^2) \tag{2.13}$$

From Fig. 3.8*b*, we know that

$$\begin{aligned} d\alpha^1 &= \delta\alpha^1 \\ d\alpha^2 &= \alpha^2\,\delta\alpha^1 + \delta\alpha^2 \end{aligned} \qquad \text{or} \qquad \begin{pmatrix} d\alpha^1 \\ d\alpha^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha^2 & 1 \end{pmatrix} \begin{pmatrix} \delta\alpha^1 \\ \delta\alpha^2 \end{pmatrix} \tag{2.14}$$

The inverse of this matrix is easily constructed and used for determining dx' in terms of $d\alpha^{\mu}$:

$$\begin{pmatrix}\delta\alpha^1\\ \delta\alpha^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ \alpha^2 & 1\end{pmatrix}^{-1}\begin{pmatrix}d\alpha^1\\ d\alpha^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ -\alpha^2 & 1\end{pmatrix}\begin{pmatrix}d\alpha^1\\ d\alpha^2\end{pmatrix} \tag{2.15}$$

$$\begin{aligned} dx' &= x'\,\delta\alpha^1 + \delta\alpha^2\\ &\quad\delta\alpha \mapsto d\alpha\\ dx' &= x'\,d\alpha^1 + (-\alpha^2\,d\alpha^1 + d\alpha^2) \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we easily construct the differential equations satisfied by x':

$$\begin{aligned} \frac{\partial x'}{\partial\alpha^1} &= x' - \alpha^2\\ \frac{\partial x'}{\partial\alpha^2} &= 1 \end{aligned} \tag{2.17}$$

The formal expressions (2.8) constructed in the proof can also be used to determine these differential equations:

$$\Theta_\lambda{}^{\mu}(\alpha) = \left.\frac{\partial\phi^{\mu}(\beta,\alpha)}{\partial\beta^{\lambda}}\right|_{\beta=0} = \left.\begin{pmatrix}\dfrac{\partial\phi^1}{\partial\beta^1} & \dfrac{\partial\phi^2}{\partial\beta^1}\\[2ex] \dfrac{\partial\phi^1}{\partial\beta^2} & \dfrac{\partial\phi^2}{\partial\beta^2}\end{pmatrix}\right|_{\beta=0} = \begin{pmatrix}1 & \alpha^2\\ 0 & 1\end{pmatrix} \tag{2.18}$$

$$u_\lambda{}^1(x') = \left.\frac{\partial f(\beta,x')}{\partial\beta^{\lambda}}\right|_{\beta=0} = \left.\begin{pmatrix}\dfrac{\partial f}{\partial\beta^1}\\[2ex] \dfrac{\partial f}{\partial\beta^2}\end{pmatrix}\right|_{\beta=0} = \begin{pmatrix}x'\\ 1\end{pmatrix} \tag{2.19}$$

Then we have

$$\begin{pmatrix}\dfrac{\partial x'}{\partial\alpha^1}\\[2ex] \dfrac{\partial x'}{\partial\alpha^2}\end{pmatrix} = \begin{pmatrix}1 & -\alpha^2\\ 0 & 1\end{pmatrix}\begin{pmatrix}x'\\ 1\end{pmatrix} = \begin{pmatrix}x'-\alpha^2\\ 1\end{pmatrix} \tag{2.20}$$

3. COMMENT. In this theorem Lie decoupled partial differential equations into the product of two matrices. One matrix depends on the transformation parameters α^{μ} and the other depends on the initial conditions $x''(p)$.

This is a generalization of the problem of finding solutions for systems of simultaneous linear partial differential equations with constant coefficients,

$$\frac{\partial x^i}{\partial \tau} = A_j{}^i x^j(\tau) \tag{2.21}$$

which can be treated by standard algebraic techniques.

This study of simultaneous partial differential equations led Lie to investigate continuous transformation groups, from which the theory of Lie groups emerged. Lie groups have been studied so extensively in their own right that their connection with partial differential equations is often overlooked and forgotten. So it is sometimes quite a shock to learn that many of the differential equations of mathematical physics are expressions of the Casimir invariant of some Lie group in a particular representation and, moreover, that all the standard special functions of mathematical physics are simply related to matrix elements in the representations of a few of the simplest Lie groups.[3–5] It is safe to say that Lie group theory provides a unifying viewpoint for the study of *all* the special functions and *all* their properties.

III. Lie's Second Theorem

1. THEOREM. *The structure constants are constant. If* X_μ *are the generators of a Lie group, then the coefficients* $C_{\mu\nu}{}^\lambda$ *given by*

$$[X_\mu, X_\nu] = C_{\mu\nu}{}^\lambda X_\lambda \tag{3.1}$$

are constants.

Comment. The differential equations

$$\frac{\partial x^i}{\partial \alpha^\mu} = \sum_{\lambda=1}^{n} \Psi_\mu{}^\lambda(\alpha) u_\lambda{}^i(x) \tag{3.2}$$

do not always have a solution. The equations

$$\frac{\partial \psi}{\partial s} = t \qquad \frac{\partial \psi}{\partial t} = 0 \tag{3.3}$$

have no solution, for

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial s} = 1 \neq 0 = \frac{\partial}{\partial s}\frac{\partial \psi}{\partial t} \tag{3.4}$$

The necessary and sufficient conditions for the existence of a unique solution with initial conditions

$$x'^i = f^i[\alpha, x]\,|_{\alpha=0} = x^i \tag{3.5}$$

is that all mixed derivatives be equal:

$$\frac{\partial^2 x'^i}{\partial\alpha^\mu\,\partial\alpha^\nu} = \frac{\partial^2 x'^i}{\partial\alpha^\nu\,\partial\alpha^\mu} \tag{3.6}$$

Proof of Lie's Second Theorem. We apply the *integrability conditions* (3.6) to the equation (2.2) derived in Lie's first theorem:

$$\frac{\partial x^i}{\partial\alpha^\mu} = \sum \Psi_\mu{}^\kappa(\alpha) u_\kappa{}^i(x) \tag{3.7}$$

$$\begin{array}{ccc} \dfrac{\partial^2 x^i}{\partial\alpha^\mu\,\partial\alpha^\nu} & \underset{\text{conditions}}{\overset{\text{integrability}}{=}} & \dfrac{\partial^2 x^i}{\partial\alpha^\nu\,\partial\alpha^\mu} \\ \Big\downarrow & & \Big\downarrow \end{array} \tag{3.8}$$

$$\frac{\partial}{\partial\alpha^\mu}\{\Psi_\nu{}^\kappa(\alpha) u_\kappa{}^i(x)\} = \frac{\partial}{\partial\alpha^\nu}\{\Psi_\mu{}^\lambda(\alpha) u_\lambda{}^i(x)\} \tag{3.9}$$

Rearranging terms, (3.9) gives

$$\Psi_\nu{}^\kappa(\alpha)\frac{\partial u_\kappa{}^i(x)}{\partial\alpha^\mu} - \Psi_\mu{}^\lambda(\alpha)\frac{\partial u_\lambda{}^i(x)}{\partial\alpha^\nu} = \frac{\partial\Psi_\mu{}^\lambda(\alpha)}{\partial\alpha^\nu}u_\lambda{}^i(x) - \frac{\partial\Psi_\nu{}^\kappa(\alpha)}{\partial\alpha^\mu}u_\kappa{}^i(x) \tag{3.10}$$

Now we replace the terms $\partial u/\partial\alpha$ on the left by

$$\frac{\partial}{\partial\alpha^\mu}u_\kappa{}^i(x) = \frac{\partial x^j}{\partial\alpha^\mu}\frac{\partial u_\kappa{}^i(x)}{\partial x^j} = \Psi_\mu{}^\lambda(\alpha)u_\lambda{}^j(x)\frac{\partial u_\kappa{}^i(x)}{\partial x^j} \tag{3.11}$$

$$\frac{\partial}{\partial\alpha^\nu}u_\lambda{}^i(x) = \frac{\partial x^j}{\partial\alpha^\nu}\frac{\partial u_\lambda{}^i(x)}{\partial x^j} = \Psi_\nu{}^\delta(\alpha)u_\delta{}^j(x)\frac{\partial u_\lambda{}^i(x)}{\partial x^j} \tag{3.11'}$$

Equation (3.10) then becomes

$$\Psi_\nu{}^\delta(\alpha)\Psi_\mu{}^\lambda(\alpha)\left\{u_\lambda{}^j(x)\frac{\partial u_\delta{}^i(x)}{\partial x^j} - u_\delta{}^j(x)\frac{\partial u_\lambda{}^i(x)}{\partial x^j}\right\}$$
$$= \left\{-\frac{\partial\Psi_\nu{}^\kappa(\alpha)}{\partial\alpha^\mu} + \frac{\partial\Psi_\mu{}^\kappa(\alpha)}{\partial\alpha^\nu}\right\}u_\kappa{}^i(x) \tag{3.12}$$

The terms $\Psi(\alpha)$ on the left-hand side of this equation can be moved to the right-hand side. The remainder on the left-hand side is then only a function of x. On the right-hand side is only a function of α—almost. The term $u_\kappa{}^i(x)$ is a function of x and cannot be moved to the left-hand side, since it is an $\eta \times N$ matrix.

To get around this problem, we observe that these arguments hold for a Lie group of transformations. Isomorphic arguments hold for a Lie group. By making the identifications

$$\begin{aligned} f^i(\alpha, x) &\to \phi^\iota(\alpha, \chi) \\ u_\lambda{}^i(x) &\to \Theta_\lambda{}^\iota(\chi) \end{aligned} \tag{3.13}$$

(3.12) can be written in a form that is clearly separated:

$$\begin{aligned} &\Psi_\iota{}^\kappa(\chi)\left\{\Theta_\lambda{}^\varepsilon(\chi)\frac{\partial\Theta_\delta{}^\iota(\chi)}{\partial\chi^\varepsilon} - \Theta_\delta{}^\varepsilon(\chi)\frac{\partial\Theta_\lambda{}^\iota(\chi)}{\partial\chi^\varepsilon}\right\} \\ &\qquad = \Theta_\delta{}^\nu(\alpha)\Theta_\lambda{}^\mu(\alpha)\left\{-\frac{\partial\Psi_\nu{}^\kappa(\alpha)}{\partial\alpha^\mu} + \frac{\partial\Psi_\mu{}^\kappa(\alpha)}{\partial\alpha^\nu}\right\} = C_{\delta\lambda}{}^\kappa \end{aligned} \tag{3.14}$$

Since α and χ may be arbitrary group elements, each side of this equation must be a constant $C_{\delta\lambda}{}^\kappa$ by standard separation of variable arguments:

$$\begin{aligned} \Theta_\lambda{}^\varepsilon(\chi)\frac{\partial\Theta_\delta{}^\iota(\chi)}{\partial\chi^\varepsilon} - \Theta_\delta{}^\varepsilon(\chi)\frac{\partial\Theta_\lambda{}^\iota(\chi)}{\partial\chi^\varepsilon} &= C_{\delta\lambda}{}^\kappa\Theta_\kappa{}^\iota(\chi) \\ u_\lambda{}^j(x)\frac{\partial u_\delta{}^i(x)}{\partial x^j} - u_\delta{}^j(x)\frac{\partial u_\lambda{}^i(x)}{\partial x^j} &= C_{\delta\lambda}{}^\kappa u_\kappa{}^i(x) \end{aligned} \tag{3.15}$$

1. For a Lie group of transformations with infinitesimal generators given by

$$X_\lambda(x) = -\left.\frac{\partial f^i(\beta, x)}{\partial\beta^\lambda}\right|_{\beta=0}\frac{\partial}{\partial x^i} = -u_\lambda{}^i(x)\frac{\partial}{\partial x^i} \tag{3.16}$$

the commutation relations are

$$\begin{aligned} [X_\lambda(x), X_\delta(x)] &= \left[-u_\lambda{}^i(x)\frac{\partial}{\partial x^i}, -u_\delta{}^k(x)\frac{\partial}{\partial x^k}\right] \\ &= \left\{u_\lambda{}^i(x)\frac{\partial u_\delta{}^j(x)}{\partial x^i} - u_\delta{}^i(x)\frac{\partial u_\lambda{}^j(x)}{\partial x^i}\right\}\frac{\partial}{\partial x^j} \\ &= C_{\lambda\delta}{}^\kappa\left(-u_\kappa{}^j(x)\frac{\partial}{\partial x^j}\right) = C_{\lambda\delta}{}^\kappa X_\kappa \end{aligned} \tag{3.17}$$

2. For a Lie group with generators

$$X_\lambda(\alpha) = -\left.\frac{\partial\phi^\mu(\beta, \alpha)}{\partial\beta^\lambda}\right|_{\beta=0}\frac{\partial}{\partial\alpha^\mu} = -\Theta_\lambda{}^\mu(\alpha)\frac{\partial}{\partial\alpha^\mu} \tag{3.18}$$

the commutation relations are obtained in an isomorphic calculation:

$$
\begin{aligned}
[X_\lambda(\alpha), X_\delta(\alpha)] &= \left[-\Theta_\lambda{}^\varepsilon(\alpha)\frac{\partial}{\partial\alpha^\varepsilon}, -\Theta_\delta{}^\sigma(\alpha)\frac{\partial}{\partial\alpha^\sigma} \right] \\
&= \left\{ \Theta_\lambda{}^\varepsilon(\alpha)\frac{\partial\Theta_\delta{}^\sigma(\alpha)}{\partial\alpha^\varepsilon} - \Theta_\delta{}^\varepsilon(\alpha)\frac{\partial\Theta_\lambda{}^\sigma(\alpha)}{\partial\alpha^\varepsilon} \right\} \frac{\partial}{\partial\alpha^\sigma} \\
&= C_{\lambda\delta}{}^\kappa\left(-\Theta_\kappa{}^\sigma(\alpha)\frac{\partial}{\partial\alpha^\sigma} \right) = C_{\lambda\delta}{}^\kappa X_\kappa(\alpha)
\end{aligned}
\tag{3.19}
$$

This completes the proof that the structure constants are in fact constant, for a Lie group, for a Lie group of transformations, and for any of their realizations and representations.

2. EXAMPLE. The partial differential equations (2.20) for x in terms of α are

$$
\frac{\partial x}{\partial\alpha^1} = x - \alpha^2 \qquad \frac{\partial x}{\partial\alpha^2} = 1 \tag{3.20}
$$

These obey the integrability conditions

$$
\frac{\partial}{\partial\alpha^2}\frac{\partial x}{\partial\alpha^1} = \frac{\partial}{\partial\alpha^2}(x - \alpha^2) = \frac{\partial x}{\partial\alpha^2} - 1 = 1 - 1 = 0 \tag{3.21}
$$

$$
\frac{\partial}{\partial\alpha^1}\frac{\partial x}{\partial\alpha^2} = \frac{\partial}{\partial\alpha^1}(1) \qquad = 0 \tag{3.22}
$$

Therefore, we can find a unique solution to these partial differential equations with unique initial value

$$
f^1(\alpha = 0; x_0) = x_0
$$

3. COMMENT. The structure constants can be computed in terms of the group composition function ϕ. Since they are constant, they can be evaluated at any group element. At the identity element, $\Theta(\varepsilon) = \Psi(\varepsilon) = I$ and from (3.15) we can write

$$
\begin{aligned}
C_{\delta\lambda}{}^\kappa &= \frac{\partial\Theta_\delta{}^\kappa(\varepsilon)}{\partial\alpha^\lambda} - \frac{\partial\Theta_\lambda{}^\kappa(\varepsilon)}{\partial\alpha^\delta} \\
&= \left. \frac{\partial^2\phi^\kappa(\beta, \alpha)}{\partial\beta^\delta\,\partial\alpha^\lambda} - \frac{\partial^2\phi^\kappa(\beta, \alpha)}{\partial\beta^\lambda\,\partial\alpha^\delta} \right|_{\beta=\alpha=0}
\end{aligned}
\tag{3.23}
$$

In particular, for our example with

$$
\begin{aligned}
\phi^1(\beta, \alpha) &= \beta^1 + \alpha^1 \\
\phi^2(\beta, \alpha) &= e^{\beta^1}\alpha^2 + \beta^2
\end{aligned}
\tag{3.24}
$$

we easily compute that

$$C_{12}{}^{2} = \frac{\partial^2 \phi^2}{\partial \beta^1\, \partial \alpha^2} - \frac{\partial^2 \phi^2}{\partial \beta^2\, \partial \alpha^1} = 1 \tag{3.25}$$

IV. Lie's Third Theorem

1. THEOREM. *The structure constants obey*

$$\mathrm{C}_{\delta\lambda}{}^{\kappa} = -\mathrm{C}_{\lambda\delta}{}^{\kappa} \tag{4.1}$$

$$\mathrm{C}_{\alpha\beta}{}^{\sigma}\mathrm{C}_{\sigma\gamma}{}^{\rho} + \mathrm{C}_{\beta\gamma}{}^{\sigma}\mathrm{C}_{\sigma\alpha}{}^{\rho} + \mathrm{C}_{\gamma\alpha}{}^{\sigma}\mathrm{C}_{\sigma\beta}{}^{\rho} = 0 \tag{4.2}$$

Proof. From (3.23) we have

$$C_{\delta\lambda}{}^{\kappa} = \left.\frac{\partial^2 \phi^{\kappa}(\beta, \alpha)}{\partial \beta^{\delta}\, \partial \alpha^{\lambda}}\right|_{\beta=\alpha=0} - \left.\frac{\partial^2 \phi^{\kappa}(\beta, \alpha)}{\partial \beta^{\lambda}\, \partial \alpha^{\delta}}\right|_{\beta=\alpha=0} = -C_{\lambda\delta}{}^{\kappa} \tag{4.3}$$

Equation (4.2) is a trivial consequence of the Jacobi identity.

$$[[X_\alpha, X_\beta], X_\gamma] + [[X_\beta, X_\gamma], X_\alpha] + [[X_\gamma, X_\alpha], X_\beta] = 0 \tag{4.4}$$

Comment 1. The Jacobi identity may be proved by expanding each of the foregoing commutators. If products of the form $X_\alpha X_\beta X_\gamma$ are defined regardless of whether they are in the algebra, the proof is straightforward.

In general, the Jacobi identity follows from the associativity of the group multiplication

$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$$
$$\phi[\gamma, \phi(\beta, \alpha)] = \phi[\phi(\gamma, \beta), \alpha]$$

Upon expansion of γ, β, α near the identity and following by expansion of ϕ leads, after collection of terms ($\alpha \to I + X_\alpha$, $\beta \to I + X_\beta$, $\gamma \to I + X_\gamma$), to

$$[X_\alpha, [X_\beta, X_\gamma]] + [X_\beta, [X_\gamma, X_\alpha]] + [X_\gamma, [X_\alpha, X_\beta]] = 0$$

This follows readily, provided the individual operations are defined and associative:

$$[X_\alpha, X_\beta] = \underbrace{X_\alpha \circ X_\beta - X_\beta \circ X_\alpha}_{\text{defined}}$$

Thus, there is no difficulty establishing the Jacobi identity provided the operators X_α, X_β, X_γ belong to a matrix algebra or, more generally, to an associative algebra.

It can be shown[6] that every finite-dimensional Lie algebra has a faithful matrix representation. Equivalently, every finite-dimensional Lie algebra can be obtained from an associative matrix algebra. Therefore, the Jacobi identity for finite-dimensional Lie algebras *always* follows as a trivial consequence of associativity.

Skipping ahead a bit, we observe that the vector $[[X_\alpha, X_\beta], X_\gamma]$ is a basis vector for the two-dimensional representation of the permutation group P_3. In other words, it may be projected[7] from the product $X_\alpha X_\beta X_\gamma$ using standard projection operator techniques. If we try projecting out of $[[X_\alpha, X_\beta], X_\gamma]$ a basis for one of the one-dimensional representations of P_3, we must, of course, obtain zero. When these projections are carried out, one projection gives zero trivially and the other projection gives

$$2\{[[X_\alpha, X_\beta], X_\gamma] + [[X_\beta, X_\gamma], X_\alpha] + [[X_\gamma, X_\alpha], X_\beta]\} = 0 \tag{4.5}$$

Comment 2. The Jacobi identity may also be written

$$[X_\alpha, [X_\beta, X_\gamma]] = [[X_\alpha, X_\beta], X_\gamma] + [X_\beta, [X_\alpha, X_\gamma]] \tag{4.6}$$

In this form it bears a strong resemblance to the equation

$$\frac{d}{dx}(f(x)g(x)) = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx} \tag{4.6$'$}$$

It is for this reason that the Lie bracket [,] is sometimes called a derivative.

Example. Four of the eight structure constants for our elementary Lie group are necessarily zero by the antisymmetry requirement:

$$C_{11}{}^1 = C_{11}{}^2 = C_{22}{}^1 = C_{22}{}^2 = 0 \tag{4.7}$$

Only two of the remaining four are independent and must be computed

$$\begin{aligned} C_{12}{}^1 &= -C_{21}{}^1 = 0 \\ C_{12}{}^2 &= -C_{21}{}^2 = 1 \end{aligned} \tag{4.8}$$

2. STRUCTURE CONSTANTS AS MATRIX ELEMENTS. The structure constants for a Lie algebra provide a matrix representation for the algebra. In general, this representation is unfaithful. The representation is obtained by associating with X_μ ($\mu = 1, 2, \ldots, \eta$) an $\eta \times \eta$ matrix M_μ ($\eta \times \eta$) whose matrix elements are given by

$$(M_\mu)_\alpha{}^\beta = -C_{\mu\alpha}{}^\beta$$

THEOREM. $\mathrm{X}_\mu \rightarrow \mathrm{M}_\mu$, $(\mathrm{M}_\mu)_\alpha{}^\beta = -\mathrm{C}_{\mu\alpha}{}^\beta$ *is a representation for the Lie algebra of the* X_μ.

Proof. We show the X and the M have isomorphic commutation relations.

$$([M_\mu, M_\nu])_\alpha{}^\beta = (M_\mu)_\alpha{}^\gamma (M_\nu)_\gamma{}^\beta - (M_\nu)_\alpha{}^\gamma (M_\mu)_\gamma{}^\beta \tag{4.9}$$

$$= C_{\mu\alpha}{}^\gamma C_{\nu\gamma}{}^\beta - C_{\nu\alpha}{}^\gamma C_{\mu\gamma}{}^\beta \tag{4.10}$$

By the Jacobi identity

$$C_{\alpha\mu}{}^\gamma C_{\gamma\nu}{}^\beta + C_{\mu\nu}{}^\gamma C_{\gamma\alpha}{}^\beta + C_{\nu\alpha}{}^\gamma C_{\gamma\mu}{}^\beta = 0$$

$$C_{\mu\alpha}{}^\gamma C_{\nu\gamma}{}^\beta - C_{\nu\alpha}{}^\gamma C_{\mu\gamma}{}^\beta = C_{\mu\nu}{}^\gamma(-C_{\gamma\alpha}{}^\beta) \tag{4.11}$$

Then

$$[M_\mu, M_\nu]_\alpha{}^\beta = C_{\mu\nu}{}^\gamma (M_\gamma)_\alpha{}^\beta \tag{4.12}$$

Since matrix multiplication is well defined, these matrices trivially satisfy the Jacobi identity. They are also a basis for a linear vector space. Since the X_μ and M_μ have isomorphic commutation relations, the M_μ provide a representation of the X_μ.

Example. From (4.7) and (4.8) we construct a 2×2 matrix representation for our two-dimensional Lie group:

$$X_1 \to M_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad X_2 \to M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.13}$$

These representations obey the commutation relations, for

$$[M_1, M_2] = \left[\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = M_2 \tag{4.14}$$

Comment. This representation, sometimes called the **adjoint** or **regular representation**, will be used extensively when we study the properties of Lie algebras and, in particular, when we classify the various types of Lie algebras.

V. Converses of Lie's Three Theorems

The preceding three theorems provide a one-way connection between Lie groups and Lie algebras. In short, they show that for each Lie group there is a corresponding Lie algebra. In addition, they provide for a characterization of a Lie algebra by its structure constants. The Lie algebra is uniquely determined, up to a change of basis, by the Lie group.

We now wish to establish a two-way connection between Lie algebras and groups. That is, given a Lie algebra, can we associate with it a Lie group? The answer is supplied by the converses of the three preceding theorems. The theorems are presented without proof. The proofs may be found in Cohn [1].

1. THE CONVERSES

Converse to Lie's First Theorem. If the η functions $\gamma^\mu = \phi^\mu(\beta, \alpha)$ of 2η variables and N functions $x'^i = f^i(\beta, x)$ of $\eta + N$ variables obey

$$\frac{\partial x'^i}{\partial \alpha^\lambda} = \sum \Psi_\lambda{}^\kappa(\alpha) u_\kappa{}^i(x') \tag{5.1}$$

and

$$\Theta_\lambda{}^\kappa(\alpha) = \left.\frac{\partial \phi^\kappa(\beta, \alpha)}{\partial \beta^\lambda}\right|_{\beta=0} \xrightarrow{\alpha \to 0} \delta_\lambda{}^\kappa$$

and

$$\Psi_\lambda{}^\kappa(\alpha) = \Theta^{-1}(\alpha)_\lambda{}^\kappa$$

and

$$u_\kappa{}^i(x) = \left.\frac{\partial f^i(\beta, x)}{\partial \beta^\kappa}\right|_{\beta=0} \tag{5.2}$$

and if the functions $\phi^\mu(\beta, \alpha)$ and $f^i(\beta, x)$ and the derivatives $\Theta_\lambda{}^\kappa(\alpha)$ and $u_\lambda{}^k(x)$ are analytic around $\beta = \alpha = 0$ and $x = 0$, then

1. The η functions $\phi^\mu(\beta, \alpha)$ are the composition functions for a local Lie group with infinitesimal generators

$$X_\mu(\alpha) = -\Theta_\mu{}^\lambda(\alpha) \frac{\partial}{\partial \alpha^\lambda} \tag{5.3}$$

2. The N functions $f^i(\beta, x)$ are the composition functions for a local Lie group of transformations with infinitesimal generators

$$X_\mu(x) = -u_\mu{}^j(x) \frac{\partial}{\partial x^j} \tag{5.3'}$$

Converse to Lie's Second Theorem. Let $\mathfrak{g}$ be an η-dimensional Lie algebra of analytic infinitesimal transformations. Then there is an η-dimensional local Lie group G, unique up to a local analytic isomorphism, which has $\mathfrak{g}$ as its Lie algebra.

Comment. There are a very large number of analytic isomorphisms of the group composition function $\phi^\mu(\beta, \alpha)$. In the vicinity of the identity, all analytic isomorphisms correspond to a change of basis (linear transformation) in the Lie algebra. Thus local analytic isomorphisms at the identity exist in 1-1 correspondence with the group elements of $Gl(\eta, r)$.

It is the converse of the third theorem which allows us to discuss Lie groups and Lie algebras virtually interchangeably. This theorem is at the heart of the classification scheme of Lie groups according to their Lie algebras. It was first proved in a weak form by Lie, and subsequently in this very powerful form by Cartan.

Converse to Lie's Third Theorem. Let $\mathfrak{g}$ be an η-dimensional abstract Lie algebra over the field R of real numbers. Then there is a *simply connected* η-dimensional Lie group G whose Lie algebra is isomorphic with $\mathfrak{g}$. The group G is uniquely determined by $\mathfrak{g}$ up to local analytic isomorphism.

2. COMMENTS. This theorem tells us that there is *not* a 1-1 correspondence between Lie groups and Lie algebras. Many Lie groups may have the same algebra. But among all the groups with the same algebra, there is only one that is *simply connected.* This simply connected group is called the **universal covering group**.

All groups with the same Lie algebra can be obtained from the universal covering group in a simple way. If SG is a simply connected Lie group and D is one of its discrete invariant subgroups, such that

$$g\, d_i g^{-1} = d_j \qquad \begin{matrix} d_i, d_j \in D \\ g \in SG \end{matrix} \tag{5.4}$$

then the factor group

$$G \equiv \frac{SG}{D} \tag{5.5}$$

is a Lie group whose Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra of SG; G is multiply connected when D contains more than one element.

The enumeration of all possible Lie groups with the same Lie algebra then reduces to the problem of computing all possible discrete invariant subgroups of the simply connected group possessing that Lie algebra. In fact, since SG is connected and D is discrete, we can continuously shrink $g \to e$ in (5.4), obtaining

$$e\, d_i e^{-1} = d_j \Rightarrow d_j = d_i \tag{5.6}$$

Then (5.4) can be rewritten

$$g\, d_i = d_i g \tag{5.4'}$$

All possible discrete invariant subgroups of SG can be computed by determining the set of all discrete group operations d_i which commute with *every* element $g \in SG$. This set of d_i is a group D; any subgroup of D is a discrete invariant subgroup of SG, and all discrete invariant subgroups of SG are subgroups of D.

3. EXAMPLE. We have seen that a Lie algebra can be associated with our example group, and vice versa. We must inquire whether there are any other Lie groups, different from the simply connected two-parameter group, with the same Lie algebra. To answer this, we let

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \qquad a > 0 \tag{5.7}$$

represent an arbitrary group element g, and we look for group elements d

$$d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \tag{5.8}$$

that commute with g, for all possible g:

$$gd = dg \tag{5.9}$$

$$\downarrow$$

$$\begin{pmatrix} ad_{11} + bd_{21} & ad_{12} + bd_{22} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} d_{11}a & d_{11}b + d_{12} \\ d_{21}a & d_{21}b + d_{22} \end{pmatrix} \tag{5.10}$$

Since this equality must hold for all values of a ($\neq 0$) and b, we find

$$d_{12} = d_{21} = 0 \qquad d_{11} = d_{22} \tag{5.11}$$

Since d must itself be a group element, $d_{22} = 1$ and only the identity element commutes with all group operations g. Since SG has no discrete invariant subgroups (except the identity), there are no other Lie groups with the same Lie algebra

$$[X_1, X_2] = X_2 \tag{5.12}$$

In the following chapter we encounter numerous examples of Lie algebras that correspond to more than one group.

4. AN IMPORTANT COMMENT. The problem of classifying all Lie groups is now reduced, by the previous theorems, to two simpler problems:

(a) Classifying all possible Lie algebras.
(b) Constructing all possible discrete invariant subgroups associated with a given simply connected Lie group.

The relationship between Lie groups with the same Lie algebra, and that Lie algebra, is shown in Fig. 4.5.

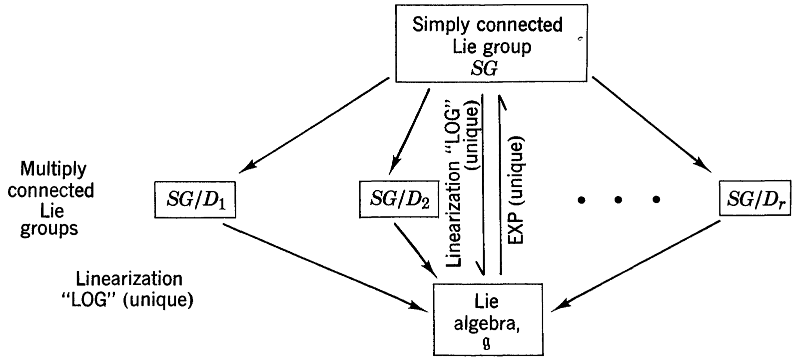


FIG. 4.5 THERE IS A UNIQUE, 1-1 CORRESPONDENCE $\mathfrak{g} \underset{\text{LOG}}{\overset{\text{EXP}}{\rightleftarrows}} SG$ BETWEEN A LIE ALGEBRA $\mathfrak{g}$ AND THE ASSOCIATED SIMPLY CONNECTED LIE GROUP SG. IF $D_1, D_2, \ldots, D_r$ ARE THE COMPLETE SET OF DISCRETE INVARIANT SUBGROUPS OF SG THEN THE COMPLETE SET OF LIE GROUPS WITH LIE ALGEBRA $\mathfrak{g}$ IS GIVEN BY $SG/D_1, SG/D_2, \ldots, SG/D_r$.

Remark. From Schur's Lemma,[3,5] it is possible to deduce that, if a matrix commutes with every element in an irreducible representation of a group, that matrix must be a multiple of the identity. Now the $N \times N$ matrices describing the classical matrix groups in V_N are irreducible. Furthermore, the discrete invariant subgroup D consists of $N \times N$ matrices. The conditions of the lemma are satisfied. Therefore, the discrete invariant subgroups of the classical matrix groups are easy to compute, for they consist of those multiples of the identity which are also elements in the group and have the form

$$d_i = \lambda_i I_N \qquad d_i \in D$$

We state a corollary to the converse of Lie's third theorem.

COROLLARY. *Two Lie groups with isomorphic Lie algebras are locally isomorphic and either*

(*a*) *Globally isomorphic, or*
(*b*) *Homomorphic images of a third group, the universal covering group*

Of the different unimodular classical groups discussed so far, only the groups $SU(n, c)$, $USp(2n)$, and $SO(n, r)$ are compact. The groups $SU(n)$ and $USp(2n)$ are simply connected, but the special orthogonal groups $SO(n)$ are doubly connected. Therefore, there is another compact Lie group whose Lie algebra is the same as that of $SO(n)$ and which is simply connected.

Definition. *Spin* (n) is the simply connected covering group of the compact doubly connected group $SO(n)$.

Example. *Spin* $(3) = SU(2)$.

Comment. Except in four cases (Chapter 9, Problem 8) the groups *Spin* (n) do not correspond to classical groups.

Notation. The simply connected universal covering group SG of a Lie group G is often denoted by placing a bar over G: i.e. $\bar{G} = SG$. Thus, $\overline{SO(3)} = Spin\ (3)$.

VI. Taylor's Theorem for Lie Groups

1. THE THEOREM. The previous sections establish a 1-1 correspondence between (simply connected) Lie groups and Lie algebras. They do not provide a method of constructing the analytic group multiplication $\gamma^{\mu} = \phi^{\mu}(\beta, \alpha)$ from the Lie algebra. The importance of the following theorem is that it provides a mechanism for constructing $\phi^{\mu}(\beta, \alpha)$ in a canonical way.

TAYLOR'S THEOREM FOR LIE GROUPS. *There exists an analytic mapping*

$$\gamma^{\mu} = \phi^{\mu}(\beta, \alpha) \tag{6.1}$$

in which every straight line through the origin is a one-dimensional abelian subgroup. The Lie group operation corresponding to the Lie algebra element $\alpha^{\mu}X_{\mu}$ *is*

$$\alpha^{\mu}X_{\mu} \rightarrow \mathrm{EXP}(-\alpha^{\mu}X_{\mu}) \tag{6.2}$$

Proof. Since

$$X_{\mu}(x) = -u_{\mu}{}^{j}(x)\frac{\partial}{\partial x^{j}} \tag{6.3}$$

we can write

$$\frac{\partial x^{i}}{\partial \alpha^{\lambda}} = \Psi_{\lambda}{}^{\sigma}(\alpha)u_{\sigma}{}^{i}(x) = -\Psi_{\lambda}{}^{\sigma}(\alpha)X_{\sigma}(x)x^{i} \tag{6.4}$$

When we look at the straight line

$$\alpha^{\mu}(\tau) = s^{\mu}\tau \tag{6.5}$$

through the origin of the Lie algebra, the x^i are functions of the single parameter τ:

$$\frac{dx^i(\tau)}{d\tau} = \frac{\partial x^i}{\partial \alpha^\lambda}\frac{d\alpha^\lambda}{d\tau} = -s^\lambda \Psi_\lambda{}^\sigma[\alpha = s\tau] X_\sigma(x) x^i(\tau) \tag{6.6}$$

Since the coordinates of the fixed point $[x^i(p)]$ now become functions of the single parameter τ, we can write

$$x^i(\tau) = x^j(0) T_j{}^i(\tau) \tag{6.7}$$

Then (6.6) reduces to

$$\frac{d}{d\tau} T_j{}^i(\tau) x^j(0) = -s^\lambda \Psi_\lambda{}^\sigma[\alpha = s\tau] X_\sigma[x(\tau)] T_j{}^i(\tau) x^j(0) \tag{6.8}$$

Since the $x^i(0)$ are arbitrary, we have the following matrix equation,

$$\frac{d}{d\tau} T_j{}^i(\tau) = -s^\lambda \Psi_\lambda{}^\sigma[\alpha = s\tau] X_\sigma[x(\tau)] T_j{}^i(\tau) \tag{6.9}$$

which is a first-order total differential equation for the matrix $T(\tau)$ with initial conditions

$$T_j{}^i(0) = \delta_j{}^i \tag{6.10a}$$

$$\begin{aligned} \left.\frac{d}{d\tau} T_j{}^i(\tau)\right|_{\tau=0} &= -s^\lambda \Psi_\lambda{}^\sigma(0) X_\sigma[x(0)] T_j{}^i(0) \\ &= -s^\sigma X_\sigma[x(0)] \delta_j{}^i \end{aligned} \tag{6.10b}$$

This total differential equation has the solution

$$\begin{aligned} T_j{}^i(\tau) &= \sum_{n=0}^{\infty} \frac{1}{n!} [-\tau s^\lambda X_\lambda[x(0)]\, \delta_j{}^i]^n \\ &= \mathrm{EXP} - \tau s^\lambda X_\lambda(x)\, \delta_j{}^i \\ &= \delta_j{}^i\, \mathrm{EXP} - \alpha^\lambda X_\lambda(x) \end{aligned} \tag{6.11}$$

The solution is unique and analytic if it converges. Also, since

$$\{\mathrm{EXP} - \tau_1 s^\lambda X_\lambda\}\{\mathrm{EXP} - \tau_2 s^\lambda X_\lambda\} = \mathrm{EXP} - (\tau_1 + \tau_2) s^\lambda X_\lambda \tag{6.12}$$

every straight line through the origin of the algebra maps into a one-dimensional abelian subgroup.

Example. For our two-dimensional group

1. If $\tau s^\lambda = \tau(\alpha^1, 0)$

$$T(1) = \text{EXP} - \alpha^1 X_1(x) = \text{EXP}\ \alpha^1 x \frac{\partial}{\partial x} \tag{6.13}$$

$$T(1)x = \left\{\text{EXP}\ \alpha^1 x \frac{\partial}{\partial x}\right\} x = e^{\alpha^1} x \tag{6.14}$$

2. If $\tau s^\lambda = \tau(0, \alpha^2)$

$$T(1) = \text{EXP} - \alpha^2 X_2 = \text{EXP}\ \alpha^2 \frac{\partial}{\partial x} \tag{6.15}$$

$$T(1)x = \{e^{\alpha^2 \partial/\partial x}\} x = x + \alpha^2 \tag{6.16}$$

3. Finally, if $\tau s^\lambda = \tau(\alpha^1, \alpha^2)$, we can write

$$T(\tau) = \text{EXP}\ \tau\left\{\alpha^1 x \frac{\partial}{\partial x} + \alpha^2 \frac{\partial}{\partial x}\right\} \tag{6.17}$$

$$\begin{aligned} T(\tau)x &= e^{\tau\{\alpha^1 x\, \partial/\partial x + \alpha^2\, \partial/\partial x\}} x \\ &= e^{\tau\alpha^1} x + \frac{\alpha^2}{\alpha^1}(e^{\tau\alpha^1} - 1) \end{aligned} \tag{6.18}$$

This mapping can easily be compared with (1.34)

$$\tau(\alpha^1, \alpha^2) \to \text{EXP}\ \tau\alpha^\mu X_\mu (2 \times 2) = \begin{pmatrix} e^{\tau\alpha^1} & \frac{\alpha^2}{\alpha^1}(e^{\tau\alpha^1} - 1) \\ 0 & 1 \end{pmatrix} \tag{6.19}$$

which acts on the basis vector

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \tag{6.20}$$

The actual group multiplication structure $\phi(\beta, \alpha)$ can be determined from this faithful representation.

2. AN AUXILIARY RESULT. Taylor's theorem for Lie groups is a strong result. But it does not assure us that we can obtain all Lie group elements by taking the exponential of some element in the Lie algebra. In view of the following theorem, Taylor's theorem is even stronger.

THEOREM. *Every element of a compact Lie group* G *lies on a one-dimensional abelian subgroup of* G *and can be obtained by exponentiating some element of the Lie algebra.*

Often, the theorems that hold for compact groups are no longer valid for noncompact groups. For this reason, the study of compact groups (and particularly the investigation of their representations) is substantially simpler than the corresponding study for noncompact groups. For noncompact groups, every element may be reached by exponentiating along a small number of straight lines in the Lie algebra.

3. COMMENT. Many analytic mappings $\phi(\beta, \alpha)$ may be associated with a given Lie group. All are analytically related. Infinitesimally, however, they are related by a local analytic isomorphism—that is, a linear transformation in $Gl(\eta, r)$. Thus any analytic $\phi(\beta, \alpha)$ can be used to determine the Lie algebra. From the Lie algebra a canonical $\phi(\beta, \alpha)$ can be constructed using Taylor's theorem. The canonical ϕ and the original ϕ may then be related, if necessary, by an analytic isomorphism.

Before we can put the study of Lie groups onto a completely canonical or standard basis, we must establish a canonical choice of basis vectors in the linear vector space associated with the group's Lie algebra. For a large class of groups (the semisimple groups) this can be done. The process is carried out in Chapter 8.

4. τ-ORDERED PRODUCTS. Our discussion of Taylor's theorem involved an integration in the Lie algebra along a straight line from the origin to some point $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^\eta)$. It is not necessary to integrate along a straight line; any curve will do. However, the corresponding group operation must then be expressed in a rather formal way.

Let $\alpha^\mu(\tau)$ be a curve in the algebra with the properties

$$\begin{aligned} \alpha^\mu(\tau = 0) &= (0, 0, \ldots, 0) \\ \alpha^\mu(\tau = 1) &= (\alpha^1, \alpha^2, \ldots, \alpha^\eta) \end{aligned} \tag{6.21}$$

Such a curve is illustrated in Fig. 4.6.

Then the differential equation (6.8) may be solved by iteration:

$$\frac{d}{d\tau} T(\tau) = -\frac{d\alpha^\lambda(\tau)}{d\tau} \Psi_\lambda{}^\sigma[\alpha(\tau)] X_\sigma(x) T(\tau) \tag{6.22}$$

$$\begin{aligned} T(\tau) - T(0) = \int_0^\tau d\tau' \Bigg\{ &-\frac{d\alpha^{\lambda'}(\tau')}{d\tau'} \Psi_{\lambda'}{}^{\sigma'}[\alpha(\tau')] X_{\sigma'}(x) \\ &\times \Bigg\{ T(0) + \int_0^{\tau'} -\frac{d\alpha^{\lambda''}(\tau'')}{d\tau''} \Psi_{\lambda''}^{\sigma''}[\alpha(\tau'')] X_{\sigma''}(x) \\ &\times \Bigg\{ T(0) + \int_0^{\tau''} -\frac{d\alpha^{\lambda'''}(t''')}{d\tau'''} \text{ etc.} \end{aligned} \tag{6.23}$$

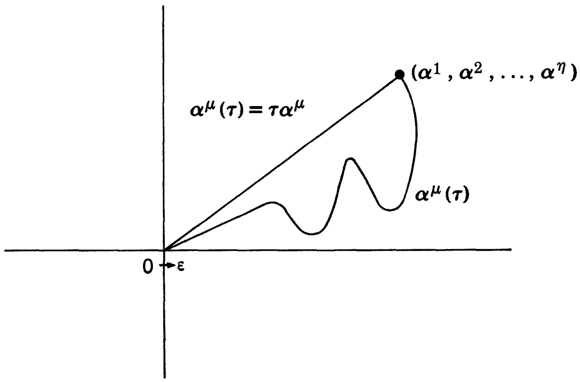


FIG. 4.6 IT IS POSSIBLE TO REACH THE GROUP OPERATION CORRESPONDING TO THE LIE ALGEBRA ELEMENT $(\alpha^1, \alpha^2, \ldots, \alpha^\eta)$ EITHER BY INTEGRATING ALONG THE STRAIGHT LINE $\alpha^\mu(\tau) = \tau\alpha^\mu$, AS IN SECTION VI.1, OR ALONG ANY OTHER CURVE $\alpha^\mu(\tau)$, AS IN SECTION VI.4.

Let us simplify this iterated solution by taking the following steps

1. Observe

$$T(0) = I \tag{6.24}$$

2. We relabel the integrand

$$-\frac{d\alpha^\lambda(\tau)}{d\tau}\Psi_\lambda{}^\sigma[\alpha(\tau)]X_\sigma(x) = -\frac{i}{\hbar}\mathscr{H}(\tau) \tag{6.25}$$

3. We observe (see Fig. 4.7, following) that the term coming from the second iteration can be rewritten

$$\begin{aligned}
&\int_0^\tau d\tau'\left(-\frac{i}{\hbar}\right)\mathscr{H}(\tau')\int_0^{\tau'} d\tau''\left(-\frac{i}{\hbar}\right)\mathscr{H}(\tau'') \\
&= \frac{1}{2!}\left\{\left(-\frac{i}{\hbar}\right)^2\int_0^\tau d\tau'\mathscr{H}(\tau')\int_0^{\tau'} d\tau''\mathscr{H}(\tau'')\right. \\
&\qquad \left. + \left(-\frac{i}{\hbar}\right)^2\int_0^\tau d\tau'\int_{\tau'}^\tau d\tau''\mathscr{H}(\tau'')\mathscr{H}(\tau')\right\} \\
&= \boldsymbol{\tau}\frac{1}{2!}\left(-\frac{i}{\hbar}\right)^2\int_0^\tau\int_0^\tau d\tau'\,d\tau''\mathscr{H}(\tau')\mathscr{H}(\tau'')
\end{aligned} \tag{6.26}$$

where $\boldsymbol{\tau}$ signifies an "ordering operator."

$$\boldsymbol{\tau}\mathscr{H}(\tau')\mathscr{H}(\tau'') = \begin{cases}\mathscr{H}(\tau')\mathscr{H}(\tau'') & \tau' > \tau'' \\ \mathscr{H}(\tau'')\mathscr{H}(\tau') & \tau'' > \tau'\end{cases} \tag{6.27}$$

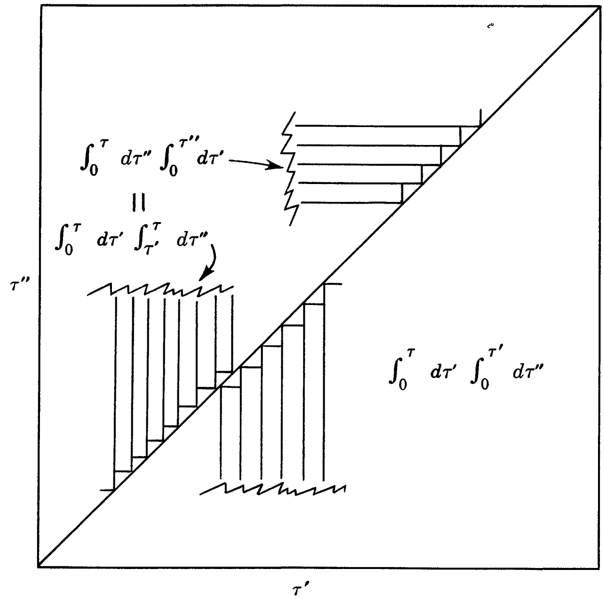


FIG. 4.7 THE INTEGRAL OVER THE SQUARE WITH EDGES OF LENGTH τ CAN BE BROKEN UP INTO TWO INTEGRALS OF THE KIND APPEARING IN (6.26). THE ORDER OF INTEGRATION MAY BE CHANGED IN ONE OF THESE INTEGRALS, WITH A CORRESPONDING CHANGE IN LIMITS. THIS CHANGE IN ORDERING IS COMPENSATED FOR BY INTRODUCING A τ ORDERING INTO THE INTEGRALS.

Since we can write

$$\boldsymbol{\tau}\mathscr{H}(\tau')\mathscr{H}(\tau'') = \mathscr{H}(\tau')\mathscr{H}(\tau'') + \theta(\tau'' - \tau')[\mathscr{H}(\tau''), \mathscr{H}(\tau')] \tag{6.28}$$

where the θ function is defined by

$$\theta(x) = \int_{-\infty}^{x} \delta(y)\, dy = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases} \tag{6.29}$$

it is clear that the $\boldsymbol{\tau}$-ordered product is not equal to the usual product unless the commutator

$$[\mathscr{H}(\tau''), \mathscr{H}(\tau')] \tag{6.30}$$

vanishes.

The third-order and higher-order iterations can be treated analogously. The complete iterative solution can be summed formally

$$T(\tau) = \boldsymbol{\tau}\ \mathrm{EXP} - \frac{i}{\hbar}\int_0^{\tau} \mathscr{H}(\tau')\, d\tau' \tag{6.31}$$

When we choose a straight-line path from 0 to α, this result should reduce to (6.11). For such a path we must compute $\Psi_\lambda{}^\sigma[\alpha(\tau)]$. This is simply done by observing

$$\phi^\mu[s\alpha^\mu, t\alpha^\mu] = (s + t)\alpha^\mu \tag{6.32}$$

$$\begin{aligned} \left.\frac{d\phi^\mu(s\alpha, t\alpha)}{ds}\right|_{s=0} &= \left.\frac{\partial\phi^\mu(\beta, t\alpha)}{\partial\beta^\lambda}\right|_{\beta=0} \left.\frac{d\beta^\lambda}{ds}\right|_{s=0} \\ &= \Theta^\mu{}_\lambda(t\alpha)\alpha^\lambda = \alpha^\mu \end{aligned} \tag{6.33}$$

Therefore, $\Psi_\lambda{}^\mu(\tau\alpha) = \delta_\lambda{}^\mu$, and for the straight-line path

$$T(\tau) = \tau \operatorname{EXP} \int_0^\tau d\tau'\{-\alpha^\lambda\, \delta_\lambda{}^\sigma X_\sigma(x)\} \tag{6.34}$$

Since the argument under the integral is not a function of τ', all commutators of the form (6.30) vanish. The τ indicating ordering in (6.31) may be dropped, and (6.34) reduces to the expression already derived for Taylor's theorem (6.11).

Résumé

The equivalence between Lie groups and Lie algebras provided by this chapter is important because a Lie algebra is a linear vector space on which there is an additional structure, the commutator. The algebra is therefore more amenable to detailed study than is the group.

A study of the local properties of a Lie group led to the concept of infinitesimal generators in Section I. The properties of infinitesimal generators, in particular the linear vector space which they span and the algebra which they form under commutation, was established by Lie's three theorems [Sections II–IV]. The converses of these theorems provide a connection between algebras and groups. There is a 1-1 correspondence between simply connected Lie groups (universal covering group) and Lie algebras; other Lie groups with isomorphic Lie algebras are simply obtained from the universal covering groups as factor groups by discrete invariant subgroups.

Finally, Taylor's theorem for Lie groups provided a mechanism for constructing, in a canonical way, the analytic group composition function $\phi(\beta, \alpha)$ from the Lie algebra.

In future chapters we freely jump between groups and algebras, both to prove theorems and to obtain physical results. Our choice—group or algebra—will be influenced by convenience or expedience. This can only be done because of the very close connection, established in this chapter, between the two.

Exercises

1. Consider the set of functions $f(x, y, z)$ defined on R_3. Construct the infinitesimal generators of $SO(3)$ on this set of functions.

2. Consider the set of functions $f(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ defined on R_n. Construct the infinitesimal generators of $SO(n)$ on this set of functions.

3. Construct the infinitesimal generators of $ISO(n)$ on the function space described in Problem 2.

4. Let p_i, $i = 1, 2, \ldots, n$ be a set of n points, and consider the space of complex functions defined on this point set. What is the dimensionality of this function space?

Let a metric be defined on this space by

$$(f, g) = \sum_{i=1}^{n} f^*(p_i)g(p_i)$$

Construct the infinitesimal generators of the metric-preserving group which acts in this space, using the method of Section I. In this case, the infinitesimal generators are $n \times n$ matrices.

5. Construct the generators of infinitesimal displacements for the abelian group of translations in R_n using the function space defined in Problem 2. From these generators, construct Taylor's theorem in R_n.

6. Using (1.40) in (1.39), derive (1.41).

7. (For those who know the theory of finite groups.) Let P_3 be the permutation group on three objects, and apply the projection operator $P^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$ to the objects A, B, C. Show

$$\begin{aligned} P_{12}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} ABC &= \frac{2}{3!} \sum_{g \in P_3} \Gamma_{21}^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(g^{-1}) gABC \\ &\cong [[A, C], B]. \end{aligned}$$

The projection operators $P_{11}^{\square\square\square}$, $P_{11}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ must project 0 from this state. Show that the projection operator $P_{11}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ applied to $[[A, C], B]$ immediately gives the Jacobi identity.

8. Compute the regular representation for $SO(3)$ using the infinitesimal generators constructed in Problem 1. Compare this with the regular representation for the Lie algebra of $SU(2)$, constructed in Problem 4.

9. Prove (assuming Schur's lemma) that the discrete invariant subgroups of $SU(n)$ are $e^{i2\pi k/r}I_n$, $k = 1, 2, \ldots, r$, and r divides n.

10. Compute the discrete invariant subgroups for $SO(2n)$ and $SO(2n + 1)$.

11. Show that EXPonentiation along a straight line in a Lie algebra (6.12) is tantamount to making an eigenstate decomposition in quantum field theory (6.31).

12. Show that the matrix

$$\begin{bmatrix} -\lambda & 0 \\ 0 & -1/\lambda \end{bmatrix}$$

in $Sl(2, r)$ cannot be reached by exponentiating along a straight line in the Lie algebra unless $\lambda = 1$. Show that it can easily be reached by traveling along a broken geodesic:

$$\begin{bmatrix} -\lambda & 0 \\ 0 & -1/\lambda \end{bmatrix} = \text{EXP}\,(A)\,\text{EXP}\,(B) \qquad (\lambda > 1)$$

Furnish explicit values for the 2×2 matrices A, B which give this group operation.

13. The plane or half-plane has been mapped by way of EXP onto the projective group in three different ways: (1.12), (1.34), and (5.7). Choose a particular operation in the *group*, and consider the coordinates of the point in each of the spaces R_2 which maps onto that group operation:

$$\begin{tikzcd}
& (\alpha^1, \alpha^2)_1 \arrow[d, "(1.34)"] & \\
& \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} & \\
(\alpha^1, \alpha^2)_2 \arrow[ur, "(1.12)"] & & (\alpha^1, \alpha^2)_3 \arrow[ul, "(5.7)"']
\end{tikzcd}$$

Each pair of coordinates $(\alpha^1, \alpha^2)_i$ is related to the others $(i + 1, i + 2 \bmod 3)$ by an analytic isomorphism. Compute these $6 = 3!$ analytic isomorphisms:

$$(\alpha^1, \alpha^2)_i = [f_{ij}{}^1(\alpha^1, \alpha^2)_j, f_{ij}{}^2(\alpha^1, \alpha^2)_j]$$

Analytic isomorphisms of Lie algebras (i.e., R_2) onto themselves which are brought about in this way are called Baker[8]-Campbell[9]-Hausdorff[10] formulas, or BCH formulas for short.

Notes and References

The spirit of this chapter follows closely the spirit of Chapter V in Cohn [1].

1. S. Lie, F. Engel. [1–3]
2. P. M. Cohn. [1]
3. N. Ja. Vilenkin. [1]
4. W. Miller. [1,2]
5. J. D. Talman. [1]
6. I. D. Ado. [1]
7. M. Hamermesh. [1]
8. H. F. Baker. [1–3]
9. J. E. Campbell. [1]
10. F. Hausdorff. [1]

CHAPTER 5

Some Simple Examples

In this chapter we present the abstract concepts developed in the previous chapter in a more concrete form. These concepts are elaborated in the framework of a number of useful examples. In addition, some new concepts are introduced and used in a concrete way.

The first section is devoted to a discussion of the three classical matrix groups $U(1, q)$, $SU(2, c)$, $SO(3, r)$ and their Lie algebras. In particular, the exponential mapping of these Lie algebras back onto their Lie groups is constructed explicitly.

Since the three Lie algebras are isomorphic, the Lie groups are closely related. By the converse to Lie's third theorem, these groups are locally isomorphic and either globally isomorphic or homomorphic images of a third group, their universal covering group. Since $SU(2, c)$ is simply connected, it is the covering group of $U(1, q)$ and $SO(3, r)$. By considering the parameter spaces for these three groups, we learn that $U(1, q)$ and $SU(2, c)$ are isomorphic, whereas $SO(3, r)$ is a $2 \to 1$ homomorphic image of both. This is shown in Section II.

In Section III the unitary irreducible representations of the group $SU(2, c)$ are constructed by symmetrizing the Nth-order tensor products $(C_2)^N$. The fully symmetric subspace of $(C_2)^N$ carries the $N + 1$ dimensional representation of $SU(2)$, usually called the $\mathscr{D}^j$ representation ($j = N/2$). Since $SU(2)$ is the covering group of $SO(3)$, all the representations of $SO(3)$ are contained among these representations of $SU(2)$. We set forth a mechanism for distinguishing between those representations of $SU(2)$ which are representations of $SO(3)$ [$j = l, l = 0, 1, 2, \ldots$] and those representations of $SU(2)$ which are not representations of $SO(3)$, the so-called "double-valued" representations. This double-valuedness is more thoroughly explored in Section V. Here we argue that the "double-valuedness" of the electron comes about because the electron has spin $\frac{1}{2}$, whereas the photon, with which we do geometry [$SO(3)$], has spin 1.

In Section IV we compute the universal covering group of $U(2, c)$, which is not simply connected. There are a number of homomorphic image groups.

We discuss possible unusual physical applications of one of these image groups, $U(2, c)$ itself. In particular, we hint how the representations of this group might give rise to a relation between isospin, baryon number, and strangeness, or how this group might give rise to the spin–statistics relationship.

Some BCH formulas for $SU(2)$ are presented and discussed in Section VI. These are used to rederive the matrix elements for the $\mathscr{D}^j$ representations of $SU(2)$ in a different way. The BCH formulas are also used to simplify computations involving the time evolution of two-level systems. A number of examples of laboratory experimental configurations are presented and discussed, and the data for one such experimental arrangement are furnished.

I. Relations between some Lie Algebras

1. 1×1 QUATERNION GROUPS. An arbitrary quaternion may be written

$$q = \sum_0^3 = q^i\lambda_i = q^0\lambda_0 + q^1\lambda_1 + q^2\lambda_2 + q^3\lambda_3$$

The q^i are all real and the λ_i obey the multiplication

$$\begin{array}{c|cccc} & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \hline \lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_1 & -\lambda_0 & \lambda_3 & -\lambda_2 \\ \lambda_2 & \lambda_2 & -\lambda_3 & -\lambda_0 & \lambda_1 \\ \lambda_3 & \lambda_3 & \lambda_2 & -\lambda_1 & -\lambda_0 \end{array} \qquad \lambda_1\lambda_2 = \lambda_3, \text{ etc.} \tag{1.1}$$

The identity under multiplication is $1\lambda_0$; the identity under addition is 0. If we exclude 0, the quaternions form a group under multiplication.

An arbitrary element in the vicinity of the identity is given by

$$1\lambda_0 + \delta q = (1 - \tfrac{1}{2}\,\delta\theta^0)\lambda_0 - \tfrac{1}{2}\,\delta\theta^i\lambda_i \tag{1.2}$$

The infinitesimal generators and their commutation relations are

$$[X_0, X_i] = 0$$

$$X_i = -\tfrac{1}{2}\lambda_i \qquad [X_i, X_j] = -\varepsilon_{ijk}X_k \tag{1.3}$$

The subgroup $Sl(1, q)$ of this quaternion group $Gl(1, q)$ consists of quaternions of modulus unity. Its infinitesimal generators are determined by

$$(1 + \delta q)^*(1 + \delta q) = 1 = (1 - \delta\theta^0)\lambda_0 + \mathscr{O}(\delta\theta)^2 \tag{1.4}$$

This subgroup corresponds to the Lie subalgebra with $\delta\theta^0 = 0$. Its infinitesimal generators are X_1, X_2, X_3.

These Lie algebras can be mapped in a canonical way onto the associated groups. First we map the three-dimensional subalgebra onto $Sl(1, q)$:

$$\sum_{i=1}^{3} \theta^i X_i \xrightarrow{\text{EXP}} Sl(1, q) = \text{EXP} \sum \theta^i X_i = \sum \frac{1}{n!} (\sum \theta^i X_i)^n \tag{1.5}$$

Since

$$(\sum \theta^i X_i)^2 = \sum (\theta^i)^2 (X_i)^2 + \frac{1}{2} \sum_{i \neq j}\sum \theta^i \theta^j (X_i X_j + X_j X_i)$$

$$= -\lambda_0 \left(\frac{1}{2}\right)^2 \theta^2$$

$$\theta^2 = \sum (\theta^i)^2; \qquad \hat{\theta}^i = \frac{\theta^i}{|\theta|} \tag{1.6}$$

the summation in (1.5) may be evaluated

$$\sum \frac{1}{n!} (\sum \theta^i X_i)^n = \lambda_0 \cos \frac{\theta}{2} - \hat{\theta}^i \lambda_i \sin \frac{\theta}{2} \tag{1.7}$$

The four-dimensional Lie algebra is mapped onto $Gl(1, q)$ in exactly the same way. Since

$$\text{EXP}\,(A + B) = \text{EXP}\,(A)\,\text{EXP}\,(B) \tag{1.8}$$

when $[A, B] = 0$, we can write immediately

$$\sum_{0}^{3} \theta^i X_i \xrightarrow{\text{EXP}} \text{EXP} \sum_{0}^{3} \theta^i X_i = \text{EXP}\, \theta^0 X_0 \;\text{EXP} \sum_{1}^{3} \theta^i X_i$$
$$= e^{-\theta^0/2} \left(\lambda_0 \cos \frac{\theta}{2} - \hat{\theta}^i \lambda_i \sin \frac{\theta}{2}\right) \tag{1.9}$$

This mapping is clearly onto the quaternion group. We see later that it is also simply connected. It is therefore the universal covering group corresponding to the Lie algebra (1.3). The examples of the following two sections are homomorphic images of this group.

2. 2×2 UNITARY GROUPS. The complex 2×2 matrices that preserve the matrix $g_{ij} = \delta_{ij}$ obey

$$\begin{array}{ccc} U_i^{*r} g_{rs} U_j^{s} & = & g_{ij} \\ \updownarrow & & \updownarrow \\ U^{\dagger} U & = & I \end{array} \tag{1.10}$$

An arbitrary unitary matrix close to the identity can be written $I + iM$. All the elements of the matrix M are infinitesimal, and in the first order the matrix obeys

$$M - M^\dagger = 0 \tag{1.11}$$

The most general 2×2 matrix with this symmetry is

$$\begin{aligned} M &= \frac{1}{2}\begin{pmatrix} \delta\theta^0 + \delta\theta^3 & \delta\theta^1 - i\,\delta\theta^2 \\ \delta\theta^1 + i\,\delta\theta^2 & \delta\theta^0 - \delta\theta^3 \end{pmatrix} \\ &= \frac{1}{2}\sum \delta\theta^i\sigma_i \end{aligned} \tag{1.12}$$

where σ_i are the usual Pauli spin matrices

$$\begin{array}{cccc} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \tag{1.13}$$

The infinitesimal generators $X_i = i/2\sigma_i$ obey

$$\begin{aligned} &[X_0, X_i] = 0 \\ &[X_i, X_j] = -\varepsilon_{ijk}X_k \qquad i, j, k = 1, 2, 3 \end{aligned} \tag{1.14}$$

These commutation relations are isomorphic with (1.3); the groups must therefore be closely related.

The three-dimensional algebra is mapped back onto the subgroup $SU(2, c)$ by the exponential mapping:

$$\sum_1^3 \theta^i X_i \xrightarrow{\text{EXP}} SU(2, c) = \text{EXP} \sum_1^3 \theta^i X_i$$

$$\left(\sum_1^3 \frac{i}{2}\sigma_i\theta^i\right)^2 = -\left(\frac{1}{2}\theta\right)^2 \tag{1.15}$$

The exponential function can be summed as follows:

$$\begin{aligned} \sum \frac{1}{n!}\left(\sum \frac{i}{2}\sigma_j\theta^j\right)^n &= \sum \frac{(-1)^n}{(2n)!}\left(\frac{1}{2}\theta\right)^{2n} \\ &\quad + \sum \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{2}\theta\right)^{2n}\left(\frac{i}{2}\theta^j\sigma_j\right) \\ &= \sigma_0 \cos\frac{\theta}{2} + i\hat{\theta}^j\sigma_j \sin\frac{\theta}{2} \end{aligned} \tag{1.16}$$

The four-dimensional Lie algebra is mapped onto $U(2, c)$ using (1.8):

$$\sum_{j=0}^{3} \theta^j X_j \xrightarrow{\text{EXP}} e^{i/2\theta^0}\left(\sigma_0 \cos\frac{\theta}{2} + i\hat{\theta}^j \sigma_j \sin\frac{\theta}{2}\right) \tag{1.17}$$

3. 3×3 ORTHOGONAL GROUPS. The Lie algebra for $SO(n)$ is determined in exactly the same way. These matrices obey

$$O^t O = I \tag{1.18}$$

Again designating a group operation infinitesimally close to the identity by $I + M$, we see that M obeys, to first order

$$M + M^t = 0 \tag{1.19}$$

For $SO(n, r)$, M is real. The most general real 3×3 matrix with the symmetry (1.19) is

$$M = \begin{bmatrix} 0 & \delta\theta^3 & -\delta\theta^2 \\ -\delta\theta^3 & 0 & \delta\theta^1 \\ \delta\theta^2 & -\delta\theta^1 & 0 \end{bmatrix} = \sum \delta\theta^i X_i \tag{1.20}$$

It is easily verified that the infinitesimal generators X_i obey

$$[X_i, X_j] = -\varepsilon_{ijk} X_k \qquad i, j, k = 1, 2, 3 \text{ cycl.} \tag{1.21}$$

The Lie algebra for $SO(3, r)$ is isomorphic to the algebras of $SU(2, c)$ and $Sl(1, q)$. The latter two groups are isomorphic, $SO(3, r)$ is a $2 \to 1$ homomorphic image of either.

The mapping of the Lie algebra into $SO(3, r)$ is a little more involved than in the previous two cases. Since $\sum \theta^j X_j$ is a 3×3 matrix, it satisfies its own third-order secular equation:

$$(\lambda + i\theta)(\lambda + 0\theta)(\lambda - i\theta) = 0 \tag{1.22}$$

This secular equation provides a relation between alternate powers of $\sum \theta^i X_i$:

$$(\sum \theta^i X_i)^3 = -\theta^2 \sum \theta^i X_i \tag{1.22$'$}$$

With this identity, all powers of $\theta^i X_i$ in the exponential sum may ultimately be related to the first and second powers:

$$\text{EXP} \sum \theta^i X_i = 1 + \sum_0^\infty \frac{(\theta^i X_i)^{2n+1}}{(2n+1)!} + \sum_0^\infty \frac{(\theta^i X_i)^{2n+2}}{(2n+2)!}$$

$$= 1 + \hat{\theta}^i X_i \sin\theta + (\sum \hat{\theta}^i X_i)^2 (1 - \cos\theta)$$

$$= \left[\begin{array}{c|c|c} \begin{gathered}\cos\theta \\ +\,\hat{\theta}_1^2(1-\cos\theta)\end{gathered} & \begin{gathered}\hat{\theta}_3 \sin\theta \\ +\,\hat{\theta}_1\hat{\theta}_2(1-\cos\theta)\end{gathered} & \begin{gathered}-\hat{\theta}_2 \sin\theta \\ +\,\hat{\theta}_1\hat{\theta}_3(1-\cos\theta)\end{gathered} \\ \hline \begin{gathered}-\hat{\theta}_3 \sin\theta \\ +\,\hat{\theta}_2\hat{\theta}_1(1-\cos\theta)\end{gathered} & \begin{gathered}\cos\theta \\ +\,\hat{\theta}_2^2(1-\cos\theta)\end{gathered} & \begin{gathered}\hat{\theta}_1 \sin\theta \\ +\,\hat{\theta}_2\hat{\theta}_3(1-\cos\theta)\end{gathered} \\ \hline \begin{gathered}\hat{\theta}_2 \sin\theta \\ +\,\hat{\theta}_3\hat{\theta}_1(1-\cos\theta)\end{gathered} & \begin{gathered}-\hat{\theta}_1 \sin\theta \\ \hat{\theta}_3\hat{\theta}_2(1-\cos\theta)\end{gathered} & \begin{gathered}\cos\theta \\ +\,\hat{\theta}_3^2(1-\cos\theta)\end{gathered} \end{array}\right] \tag{1.23}$$

This matrix can be written more graphically in terms of familiar vector and dyadic operations:

$$\text{EXP}\ \theta^i X_i = I_3 \cos\theta + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}(1 - \cos\theta) - \hat{\boldsymbol{\theta}} \times \sin\theta \tag{1.24}$$

The transformation matrix of (1.23) and (1.24) can be interpreted geometrically. It describes a change of orthonormal basis in a real three-dimensional space obtained by rotating the bases through an angle θ about the axis $\hat{\boldsymbol{\theta}}$ (cf. Chapter 4, Section I.2). If $\mathbf{v}$ is an arbitrary vector in S, the coordinates of the same vector $\mathbf{v}$ in S' are related to its coordinates in S by (1.24).

To compute these components, we decompose $\mathbf{v}$ into a part parallel to the rotation axis $\hat{\boldsymbol{\theta}}$ and a part which is perpendicular, as shown in Fig. 5.1.

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \tag{1.25}$$

$$\begin{aligned} \mathbf{v}_{\parallel} &= \hat{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}} \cdot \mathbf{v}) \\ \mathbf{v}_{\perp} &= \mathbf{v} - \hat{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}} \cdot \mathbf{v}) = \hat{\boldsymbol{\theta}} \times (\mathbf{v} \times \hat{\boldsymbol{\theta}}) \end{aligned} \tag{1.26}$$

$$\mathbf{v} = \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp} \tag{1.25$'$}$$

$$\begin{aligned} \mathbf{v}'_{\parallel} &= \mathbf{v}_{\parallel} \\ \mathbf{v}'_{\perp} &= \mathbf{v}_{\perp} \cos\theta - (\hat{\boldsymbol{\theta}} \times \mathbf{v}_{\perp}) \sin\theta \end{aligned} \tag{1.26$'$}$$

The total result is:

$$\begin{aligned} \mathbf{v}_{\text{in } S'} &= \hat{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}} \cdot \mathbf{v}) + [\mathbf{v} - \hat{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}} \cdot \mathbf{v})] \cos\theta - \hat{\boldsymbol{\theta}} \times \mathbf{v} \sin\theta \\ &= [I_3 \cos\theta + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}(1 - \cos\theta) - \hat{\boldsymbol{\theta}} \times \sin\theta]\mathbf{v} \end{aligned} \tag{1.27}$$

The operator on the right-hand side of (1.27) is exactly the operator (1.24).

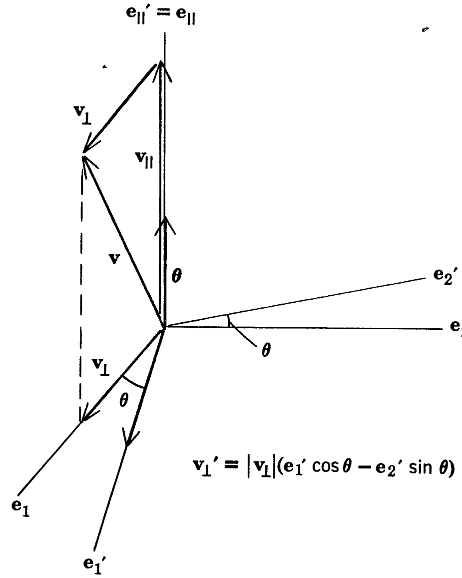


FIG. 5.1 THE MATRIX EXP $\theta^i X_i$ REPRESENTS A CHANGE OF ORTHOGONAL BASIS IN A REAL THREE-DIMENSIONAL VECTOR SPACE OBTAINED BY RIGID ROTATION AROUND AXIS $\hat{\theta}$ THROUGH AN ANGLE $|\theta|$. IF $\mathbf{v}$ IS AN ARBITRARY VECTOR, WITH COMPONENTS $\mathbf{v}_{\parallel}$, $\mathbf{v}_{\perp}$ IN S, ITS COMPONENTS $\mathbf{v}_{\parallel}$, $\mathbf{v}_{\perp}$ IN S' CAN EASILY BE EVALUATED. THE SAME RESULTS (COORDINATES) ARE OBTAINED IF THE COORDINATE SYSTEM IS LEFT FIXED AND THE VECTOR $\mathbf{v}$ IS ROTATED ABOUT THE AXIS $\hat{\boldsymbol{\theta}}$ THROUGH AN ANGLE $-|\theta|$.

In exactly the same way, if $f(\mathbf{x})$ is a scalar-valued function defined on R_3 with respect to coordinate system S, the structure of the same function in the new coordinate system S' related to S by EXP $\theta^i X_i$ is

$$f'(\mathbf{x}') = \text{EXP } \theta^i X_i(\mathbf{x}') f(\mathbf{x}') \tag{1.28}$$

The $X_i(\mathbf{x}')$ are differential operators

$$X_i(\mathbf{x}) = \varepsilon_{ijk} x^j \,\partial_k \qquad \begin{aligned} X_1 &= x^2\,\partial_3 - x^3\,\partial_2 \\ X_2 &= x^3\,\partial_1 - x^1\,\partial_3 \\ X_3 &= x^1\,\partial_2 - x^2\,\partial_1 \end{aligned} \tag{1.29}$$

and the summation represented by (1.28) is a Taylor series expansion.

II. Comparison of Lie Groups

1. PARAMETER SPACES FOR $Sl(1, q)$, $SU(2, c)$, $SO(3, r)$. To have a detailed understanding of these three-dimensional groups with isomorphic

algebras, we begin by associating with each element $\alpha^i X_i$ in the respective algebras the group element $(\hat{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}|)$:

$$\underset{\substack{\text{in}\\ \text{algebra}}}{\alpha^i X_i} \xrightarrow{\text{EXP}} \underset{\substack{\text{in}\\ \text{group}}}{\text{EXP}\, \alpha^i X_i} \equiv (\hat{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}|) \tag{2.1}$$

We draw a straight line through the origin of the algebra and in the direction $\hat{\boldsymbol{\alpha}}$ and follow this straight line away from the origin.

This straight line in the algebra maps onto a one-parameter subgroup ("geodesic") in the group. We say the EXPonential mapping "lifts" a straight line through the origin of the algebra onto a geodesic in the group. If every geodesic through the identity returns to the vicinity of the identity, the group is compact.

Since the exponential sums (1.7, 1.16, 1.23) involve periodic functions, the three groups $Sl(1, q)$, $SU(2, c)$, $SO(3, r)$ are compact.

We also observe:

$$\begin{array}{llll} \text{from (1.7)} & \text{for} \quad Sl(1, q) & (\hat{\boldsymbol{\alpha}}, \alpha + 4\pi) = (\hat{\boldsymbol{\alpha}}, \alpha) = -\ (\hat{\boldsymbol{\alpha}}, \alpha + 2\pi) \\ \text{from (1.16)} & \text{for} \quad SU(2, c) & (\hat{\boldsymbol{\alpha}}, \alpha + 4\pi) = (\hat{\boldsymbol{\alpha}}, \alpha) = -\ (\hat{\boldsymbol{\alpha}}, \alpha + 2\pi) \\ \text{from (1.23)} & \text{for} \quad SO(3, r) & (\hat{\boldsymbol{\alpha}}, \alpha + 2\pi) = (\hat{\boldsymbol{\alpha}}, \alpha) \end{array} \tag{2.2}$$

Equations (2.2) tell us that for $Sl(1, q)$ and $SU(2, c)$ we can choose as parameter space all points $\boldsymbol{\alpha}$ in the Lie algebra which obey $0 \leq |\boldsymbol{\alpha}| \leq 2\pi$. That is, the group operations are in 1-1 correspondence with elements of the algebra within a radius 2π of the origin. Since

$$(\hat{\boldsymbol{\alpha}}, 2\pi) = -(\hat{\boldsymbol{\alpha}}, 0) \begin{cases} \to -\lambda_0 & \text{for} \quad Sl(1, q) \\ \to -I_2 & \text{for} \quad SU(2, c) \end{cases} \tag{2.3}$$

we must also identify all points on the surface of this sphere with one group operation, $-$(identity).

This apparent degeneracy is not serious: it comes about from our desire to parameterize the entire group with one coordinate patch. This "degeneracy" could be lifted by introducing a second coordinate patch around the operation $-$(identity).

From (2.2) we see that the parameter space for $SO(3, r)$ consists of all points $\boldsymbol{\alpha}$ for which $0 \leq |\boldsymbol{\alpha}| \leq \pi$. Since

$$(\hat{\boldsymbol{\alpha}}, \pi) = (-\hat{\boldsymbol{\alpha}}, \pi) = (\hat{\boldsymbol{\alpha}}, -\pi) \qquad \text{for} \quad SO(3, r) \tag{2.4}$$

it is necessary to identify antipodal points in the algebra with the same group operation.

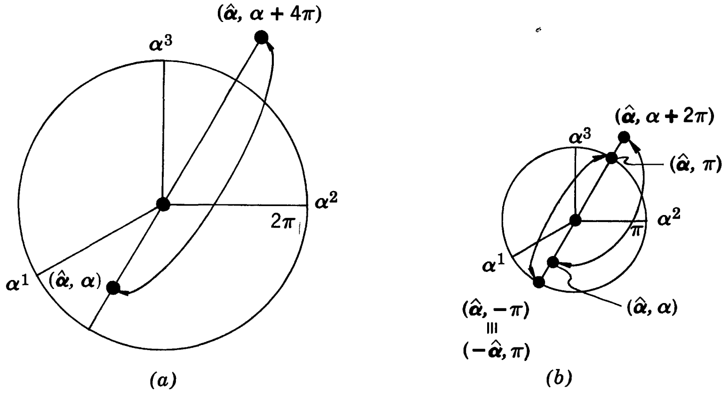


FIG. 5.2 THE ELEMENTS IN THE LIE ALGEBRA [IN (1.3), (1.14), AND (1.21)] MAP ONTO THE GROUP ELEMENTS MANY TIMES. THEREFORE, ONLY A PORTION OF THE LIE ALGEBRA IS NEEDED TO PARAMETERIZE THE GROUPS $Sl(1, q)$, $SU(2, c)$ (a) AND $SO(3, r)$ (b). FOR $SU(2, c)$, $(\boldsymbol{\alpha}, \alpha + 4\pi) = (\boldsymbol{\alpha}, \alpha)$, SO ONLY A SPHERE OF RADIUS 2π IN PARAMETER SPACE IS NEEDED. SINCE $(\boldsymbol{\alpha}, 2\pi) = -(\text{IDENTITY})$, ALL POINTS ON THE SURFACE ARE IDENTIFIED WITH THE SAME GROUP OPERATION. FOR $SO(3, r)$ ONLY A SPHERE OF RADIUS π IN PARAMETER SPACE IS REQUIRED. ALL ANTIPODAL POINTS MUST BE IDENTIFIED.

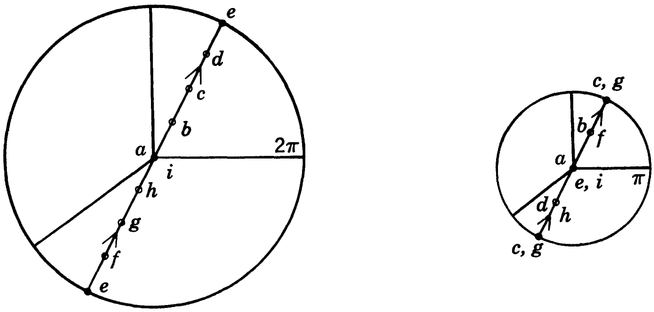
The parameter spaces for these groups are shown in Fig. 5.2. As we follow a straight line from the origin to $(\hat{\boldsymbol{\alpha}}, 2\pi)$ and then from $(\hat{\boldsymbol{\alpha}}, 2\pi) \equiv (-\hat{\boldsymbol{\alpha}}, 2\pi) \equiv (\hat{\boldsymbol{\alpha}}, -2\pi)$ back to the origin, we see that every pair of group operations in $Sl(1, q)$ or $SU(2, c)$ is associated with a single-group operation in $SO(3, r)$. Thus $SO(3, r)$ is a $2 \to 1$ homomorphic image of $SU(2, c)$

$$\begin{array}{ccc} (\hat{\boldsymbol{\alpha}}, \alpha) & & \\ & \searrow & \\ & & (\hat{\boldsymbol{\alpha}}, \alpha) \\ & \nearrow & \\ (-\hat{\boldsymbol{\alpha}}, 2\pi - \alpha) \equiv (\hat{\boldsymbol{\alpha}}, \alpha + 2\pi) & & \\ \text{in} \quad Sl(1, q) & & \text{in} \quad SO(3, r) \\ \text{or} \quad SU(2, c) & & \end{array} \tag{2.5}$$

The groups $Sl(1, q)$ and $SU(2, c)$ are themselves isomorphic because:

1. They have isomorphic Lie algebras.
2. Their group operations exist in 1-1 correspondence with the same portion of the Lie algebra, hence in 1-1 correspondence with each other.

Finally, in Fig. 5.3 we observe that when a geodesic in $SU(2, c)$ starts from the identity and reaches $-I_2$, the corresponding geodesic in $SO(3)$ has returned to the identity. As the geodesic in $SU(2)$ returns from $-I_2$ to the identity again, the geodesic in $SO(3)$ makes another round trip. This is just another facet of the $2 \to 1$ homomorphism that exists between these groups.



Sl (1, *q*); *SU* (2,*c*)　　　　*SO* (3, *r*)

FIG 5.3 AS WE FOLLOW A GEODESIC THROUGH THE IDENTITY OF $SU(2, c)$ AND $SO(3, r)$, IT RETURNS ONCE TO THE ORIGIN OF $SO(3, r)$ AS IT REACHES $-I_2$ IN $SU(2, c)$. IT RETURNS A SECOND TIME TO I_3 IN $SO(3, r)$ AS IT RETURNS TO THE IDENTITY I_2 IN $SU(2, c)$ FOR THE FIRST TIME.

It is also reflected in the fact that $SU(2, c)$ has a discrete invariant subgroup of order two: $(I_2, -I_2)$, but $SO(3)$ has only the identity in its discrete invariant subgroup.

2. CONNECTIVITY. It is clear that these Lie groups are connected. For if p, q are any two group operations or points in the parameterizing sphere, they can be connected by a line—in fact, by a straight line, lying completely within the parameter space.

To determine whether these groups are simply connected, we must see if any path beginning and ending at the origin can be continuously deformed to a point at the origin. If the path does not touch the parameter sphere surface, it can always be continuously deformed to the origin.

If the path does touch the sphere surface in the groups $Sl(1, q)$ and $SU(2, c)$, it can still be deformed continuously to the identity. Such a deformation is illustrated in Fig. 5.4. These two groups are therefore simply connected; consequently, they form the universal covering group with algebra (1.21).

If a path in $SO(3, r)$ cuts the sphere surface and returns to the identity, it cannot be continuously deformed to a point at the origin (Fig. 5.5), and $SO(3)$ is not simply connected.

We can define multiplication of paths as follows: given paths P_1 and P_2, starting and ending at the origin, we call their product $P_2 P_1$ the path that starts at the origin, follows P_1, returns to the identity, goes out along P_2, and again returns to the identity (Fig. 5.6). Under this multiplication the paths form a group called a homotopy group.[1] There are as many distinct homotopy group operations as there are classes of paths that cannot be deformed into each other.

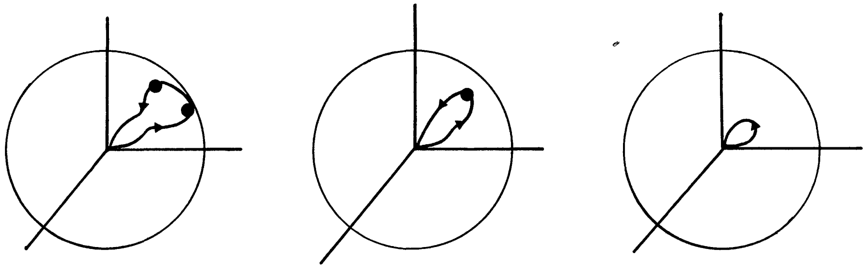


FIG. 5.4 ALL PATHS IN $SU(2, c)$ THAT START AND END AT THE ORIGIN AND CUT THE SURFACE OF THE PARAMETERIZING SPHERE ONCE, ARE CONTINUOUSLY DEFORMABLE TO A POINT AT THE ORIGIN. THE HOMOTOPY GROUP OF $SU(2)$ HAS ONLY ONE ELEMENT.[3]

For $SO(3)$, all paths that cut the sphere surface once are equivalent. All paths that cut the surface twice can be deformed continuously to the point at the origin (Fig. 5.7) and are thus equivalent to the identity element in the homotopy group. The homotopy group of $SO(3)$ has two elements: all paths that cut the surface an even number of times (the identity), and all other paths. The multiplication table for this group follows.

Multiplication Tables

For the Homotopy Group of $SO(3)$

	even	odd
even	even	odd
odd	odd	even

For the Discrete Invariant Subgroup of $SU(2)$

	I_2	$-I_2$
I_2	I_2	$-I_2$
$-I_2$	$-I_2$	I_2

(2.6)

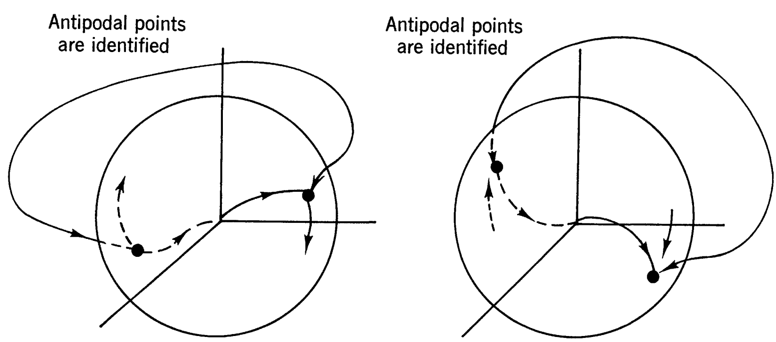


FIG. 5.5 ANY PATH IN $SO(3, r)$ THAT CUTS THE SURFACE OF THE PARAMETER SPHERE ONCE CANNOT BE CONTINUOUSLY DEFORMED TO THE POINT AT THE ORIGIN. AS SOON AS ONE ANTIPODAL POINT IS MOVED ONE WAY, THE OTHER MUST MOVE THE OTHER WAY IF IT IS TO REMAIN ANTIPODAL.

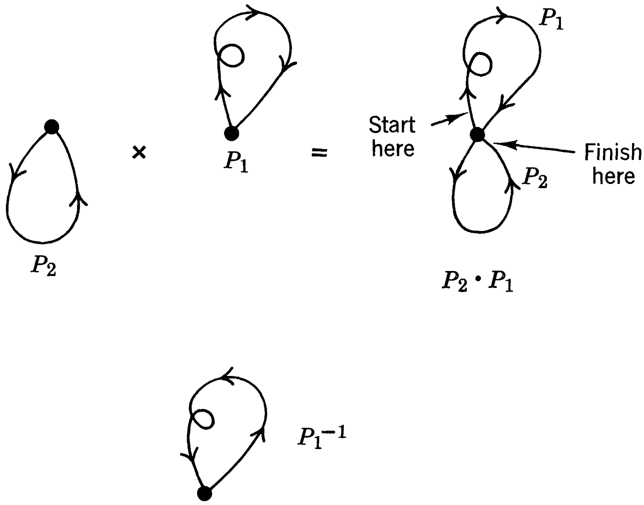


FIG. 5.6 THE PRODUCT OF PATHS $P_2 P_1$ IS THE PATH THAT STARTS FROM THE ORIGIN, GOES OUT ALONG P_1 AND RETURNS TO THE ORIGIN, DEPARTS AGAIN FROM THE ORIGIN ALONG P_2, AND RETURNS ONCE AGAIN ALONG P_2. THE INVERSE OF A HOMOTOPY GROUP OPERATION P_1 IS THE PATH P_1 TRAVERSED IN THE OPPOSITE DIRECTION.

We say that $SO(3, r)$ is doubly connected because its homotopy group has two operations; $Sl(1, q)$ and $SU(2, c)$ are simply, or singly, connected because their homotopy groups contain one element.

Comment 1. The simple connectedness of $SU(2, c)$ is very easily seen as follows: $SU(2, c)$ is isomorphic with $Sl(1, q)$, which consists of those quaternions

$$q = q^0\lambda_0 + q^1\lambda_1 + q^2\lambda_2 + q^3\lambda_3 \tag{2.7}$$

which obey the additional condition

$$(q^0)^2 + (q^1)^2 + (q^2)^2 + (q^3)^2 = 1 \tag{2.8}$$

The space of points that obey (2.8) is the three-dimensional sphere $S^3 \subset R_4$. The sphere S^N is simply connected when $N > 1$. The sphere S^1 (circle or torus T^1) is multiply connected.

Comment 2. We have defined geodesics on a group in a very natural way. Those passing through the identity are exponentials of straight lines through the origin of the Lie algebra. Those passing through the group operation β are left or right translations by β of those passing through the origin.

From Comment 1 we have been able to associate a Riemannian manifold S^3 with a Lie group. Therefore we have a canonical and natural mechanism

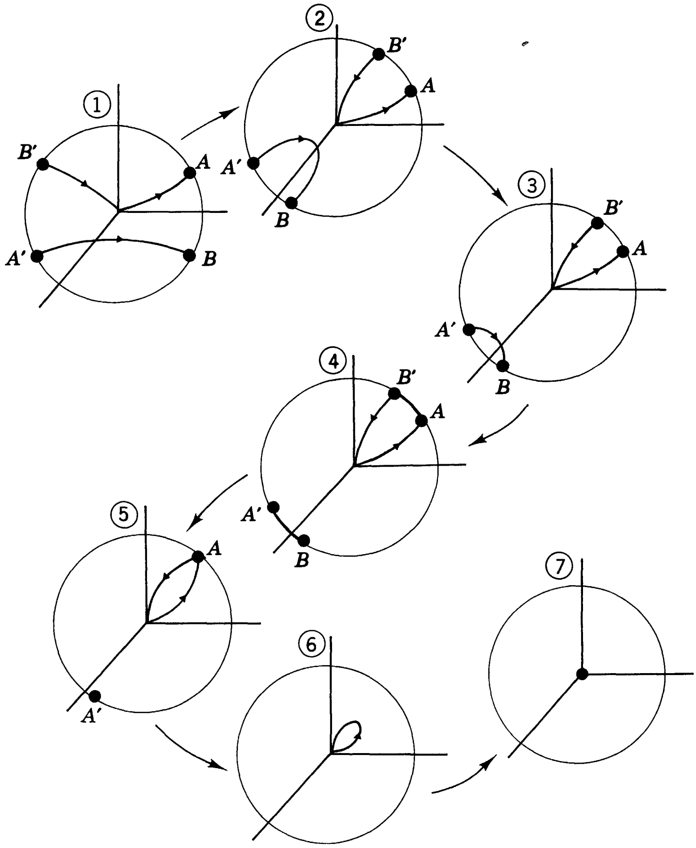


FIG. 5.7 A PATH IN $SO(3, r)$ STARTING AND ENDING AT THE ORIGIN, AND CUTTING THE SURFACE OF THE PARAMETERIZING SPHERE TWICE, CAN BE DEFORMED CONTINUOUSLY TO A POINT AT THE ORIGIN. THE HOMOTOPY GROUP[3] OF $SO(3)$ HAS TWO ELEMENTS: PATHS THAT CUT THE SURFACE AN EVEN NUMBER OF TIMES (THE IDENTITY) AND PATHS THAT CUT THE SURFACE AN ODD NUMBER OF TIMES (ODD).

for constructing geodesics on at least one Riemannian space. This is done introducing neither the Riemann-Christoffel nontensor $\Gamma_{\mu\nu}{}^{\lambda}$ nor the concept of parallel displacement.

We see later that all Riemannian symmetric (torsion-free) spaces can be obtained as coset spaces of Lie groups by particular subgroups. For such spaces geodesics can be naturally defined in terms of transformation groups acting on these spaces. Parallel displacements become a subsidiary concept, and the curvature tensor $R_{\mu\nu}{}^{\lambda}{}_{\kappa}$ is just the structure constant of a space, in exactly the same way the structure constants $C_{\mu\nu}{}^{\lambda}$ of a Lie group are components of a group's curvature tensor.

Comment 3. If we choose any point in S^2 (north pole) and look at geodesics, or lines of longitude, through this point, we find that they all converge again at a far distant point (south pole) (Fig. 6.2). (The same arguments hold for S^3 as well.) These points are special points in the associated groups: in $SU(2, c)$ they may be taken as the identity I_2 and $-(\text{identity}) = -I_2$. These "convergence points" or "recurring points" or "focal points" correspond to the elements in the discrete invariant subgroup.

3. HOMOTOPY AND DISCRETE INVARIANT SUBGROUPS. It is clear that there must be a very close connection between homotopy groups and discrete invariant subgroups. Here is the connection.

Let G be a Lie group with homotopy group H and Lie algebra $\mathfrak{g}$. Then there is exactly one simply connected Lie group $\overline{G} = SG$ with Lie algebra $\mathfrak{g}$, and G is a homomorphic image of SG. In fact:

$$G \cong SG/D \tag{2.9}$$

where $\cong$ indicates isomorphism and D is one of the discrete invariant subgroups of SG. Then

$$H \cong D \qquad (\text{cf. } 2.6) \tag{2.10}$$

Comment. This isomorphism quite unexpectedly provides a connection between topological and algebraic properties of Lie groups. We subsequently encounter several additional unexpected relations between the topological and algebraic properties of Lie groups. This means that either field may be used to provide results and information about the other. An effect of algebra on geometry is indicated in Comment 2 of the previous section. An effect of geometry on algebra is shown in Comment 3.

Caution. Let SG be a simply connected Lie group with maximal discrete invariant subgroup $D_{\max}$. If D_1 and D_2 are *distinct* but isomorphic subgroups of $D_{\max}$, then

$$\frac{SG}{D_1} \not\cong \frac{SG}{D_2}$$

but

$$H_1 \cong H_2 \tag{2.11}$$

Homotopy groups do not in general classify the different Lie groups with isomorphic algebras. For example, see Table 5.1 (p. 143).

Example 1. The simply connected group $SU(2, c)$ has a discrete invariant subgroup $D_{\max} = \{I_2, -I_2\}$. There are only two distinct groups with Lie algebra (1.21):

$$Sl(1, q) \cong SU(2, c) = \frac{SU(2, c)}{I_2} \qquad H = \{\text{point path}\}$$

$$SO(3, r) \cong \frac{SU(2, c)}{\{I_2, -I_2\}} \qquad H = \{\text{even, odd}\} \tag{2.12}$$

Example 2. The one-dimensional Lie algebra rX_1 has associated the simply connected Lie group of rigid translations of the straight line R_1. To find the discrete invariant subgroups of R_1, let x_0 be the operation in the discrete group nearest the identity. Then the discrete invariant subgroup contains also all points nx_0 $(n = 0, \pm 1, \pm 2, \ldots)$ and no others.[2] This group is isomorphic to the group of integers $Z = \{n\}$ under addition. There are then only two one-dimensional Lie groups:

$$R_1 \cong \frac{R_1}{\{0\}} \qquad \text{simply connected; noncompact}$$

$$T^1 \cong S^1 \cong \frac{R_1}{Z} = \frac{R_1}{\{n\}} \qquad \text{multiply connected; compact} \tag{2.13}$$

These are the rigid translations of the straight line R_1 and the rigid translations of the circle (one-dimensional sphere S^1 or torus T^1), respectively. The connection between these two one-dimensional Lie groups is illustrated in Fig. 5.8.

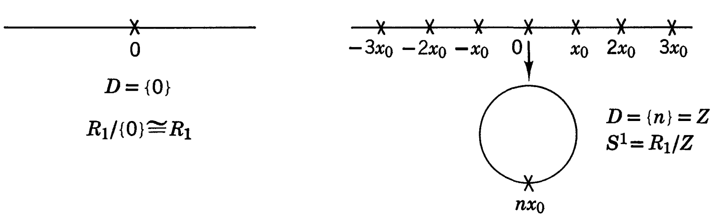


FIG. 5.8 THE SIMPLY CONNECTED GROUP R_1 HAS TWO DISTINCT DISCRETE INVARIANT SUBGROUPS. THERE ARE TWO GROUPS WITH LIE ALGEBRA $X_1 : R_1 \cong R_1/\{0\}$ OF DISPLACEMENTS OF THE REAL AXIS, AND $S^1 \cong R_1/\{n\}$ OF ROTATIONS OF THE CIRCLE. ALL ROTATIONS THROUGH $\theta + 2\pi n$ ARE IDENTIFIED.

III. Representations of $SU(2, c)$

1. GENERAL CONSIDERATIONS. Let $G \to \Gamma(G)$ be a representation of G

$$
\begin{array}{ccccc}
g_i & \circ & g_j & = & g_i \circ g_j \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(g_i) & \circ & \Gamma(g_j) & = & \Gamma(g_i \circ g_j)
\end{array}
\qquad g_i, g_j \in G \tag{3.1}
$$

where Γ are $n \times n$ matrices.

Suppose that $\gamma(H)$ is a representation of H, where H is a homomorphic image of G. Then $\gamma(G)$ is a representation of G, for

$$
\begin{array}{ccccc}
g_i & \circ & g_j & = & g_i \circ g_j \\
\downarrow & & \downarrow & & \downarrow \\
h_i & \circ & h_j & = & h_i \circ h_j \\
\downarrow & & \downarrow & & \downarrow \\
\gamma(h_i) & \circ & \gamma(h_j) & = & \gamma(h_i \circ h_j)
\end{array}
\qquad
\begin{array}{c}
g_i, g_j \in G \\ \\ h_i, h_j \in H \\ \\ \\
\end{array}
\tag{3.2}
$$

On the other hand, Γ may or may not be a representation of H. Let $g_0, g_1, g_2, \ldots, g_d$ be the elements of G which are mapped into the identity element h_0 of H. Then $g_0, g_1, \ldots, g_d$ form an invariant subgroup of G, and $H \simeq G/\{g_i\}$. If

$$\Gamma(g_0) = \Gamma(g_1) = \cdots = \Gamma(g_d)$$

then $\Gamma(G/\{g_i\} \cong H)$ is a representation of H; otherwise it is not.

That is, if

$$
\begin{array}{c}
g_0 \quad g_1 \cdots g_d \\
\searrow \ \downarrow \ \swarrow \\
h_0
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Gamma(g_0) \quad \cdots \Gamma(g_d) \\
\searrow \ \downarrow \ \swarrow \\
\Gamma(g_0)
\end{array}
\tag{3.3}
$$

then $\Gamma(H) \cong \Gamma(G/\{g_i\})$ is a representation of H.

For our purposes we replace G by SG and H by $G = SG/D$ in the foregoing arguments. Then if γ is a representation of G, it is a representation of SG as well. Those representations $\Gamma(SG)$ are also representations of G whenever

$$\Gamma(d_i) = \Gamma(\text{identity}) \qquad d_i \in D \tag{3.3'}$$

In short, once all representations of any simply connected Lie group have been determined, all representations of any other Lie group with isomorphic algebra have also been determined. And a simple test exists to weed out those representations of the simply connected group which are not representations of the homomorphic image group.

2. TENSOR PRODUCT REPRESENTATIONS. We have already seen a simple mechanism for constructing representations of a group: the formation of tensor products. Let $U(2)$ act in a complex two-dimensional space C_2 with bases

$$\xi = \begin{pmatrix}1\\0\end{pmatrix} = \left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle \qquad \eta = \begin{pmatrix}0\\1\end{pmatrix} = \left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle \tag{3.4}$$

Then an arbitrary metric-preserving change of basis in C_2 is given by a group operation in $U(2)$:

$$\alpha^\mu X_\mu \to \mathrm{EXP}\, \alpha^\mu X_\mu = (\alpha^\mu) = U(2) = \begin{bmatrix} a_1{}^1 & a_1{}^2 \\ a_2{}^1 & a_2{}^2 \end{bmatrix} \tag{3.5}$$

$$= \begin{bmatrix} e^{i\alpha_0/2}\left(\cos\frac{\alpha}{2} + i\hat{\alpha}_3 \sin\frac{\alpha}{2}\right) & e^{i\alpha_0/2}(i\hat{\alpha}_1 + \hat{\alpha}_2)\sin\frac{\alpha}{2} \\ e^{i\alpha_0/2}(i\hat{\alpha}_1 - \hat{\alpha}_2)\sin\frac{\alpha}{2} & e^{i\alpha_0/2}\left(\cos\frac{\alpha}{2} - i\hat{\alpha}_3 \sin\frac{\alpha}{2}\right) \end{bmatrix}$$

The $n + 1$ bases in the fully symmetric nth-order tensor product[4,5] may be taken as

$$\frac{\xi^{j+m}\eta^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} \equiv \left|\begin{matrix}j\\ m\end{matrix}\right\rangle \qquad \begin{aligned} j &= \frac{n}{2} \\ m &= -j, -j+1, \ldots, +j \end{aligned} \tag{3.6}$$

The change of bases (3.5) in C_2

$$\left|\begin{matrix}\frac{1}{2}\\ m\end{matrix}\right\rangle' = (\alpha^\mu)\left|\begin{matrix}\frac{1}{2}\\ m\end{matrix}\right\rangle = \left|\begin{matrix}\frac{1}{2}\\ m'\end{matrix}\right\rangle\left\langle\begin{matrix}\frac{1}{2}\\ m'\end{matrix}\right|(\alpha^\mu)\left|\begin{matrix}\frac{1}{2}\\ m\end{matrix}\right\rangle \tag{3.7}$$

induces a corresponding change of bases in the fully symmetric subspace of $(C_2)^{2j+1}$:

$$\left|\begin{matrix}j\\ m\end{matrix}\right\rangle' = (\alpha^\mu)\left|\begin{matrix}j\\ m\end{matrix}\right\rangle = \left|\begin{matrix}j\\ m'\end{matrix}\right\rangle\left\langle\begin{matrix}j\\ m'\end{matrix}\right|(\alpha^\mu)\left|\begin{matrix}j\\ m\end{matrix}\right\rangle \tag{3.8}$$

The matrix elements

$$\left\langle\begin{matrix}j\\ m'\end{matrix}\right|(\alpha^\mu)\left|\begin{matrix}j\\ m\end{matrix}\right\rangle$$

in the change of basis matrix are easily computed

$$\begin{aligned}\xi' &= a_1{}^1\xi + a_2{}^1\eta \\ \eta' &= a_1{}^2\xi + a_2{}^2\eta\end{aligned} \tag{3.9}$$

$$\left|\begin{matrix} j \\ m \end{matrix}\right\rangle' = \frac{(\xi')^{j+m}(\eta')^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} \tag{3.10}$$

The expression (3.10) can be expanded using the binomial theorem. After some algebra, it reduces to[5]

$$\left|\begin{matrix} j \\ m \end{matrix}\right\rangle' = \sum_{m'} \frac{(\xi)^{j+m'}(\eta)^{j-m'}}{\sqrt{(j+m')!\,(j-m')!}} \mathscr{D}_{m'm}{}^{j}\begin{pmatrix} a_1{}^1 & a_1{}^2 \\ a_2{}^1 & a_2{}^2 \end{pmatrix} \tag{3.11}$$

$$\begin{aligned}\left\langle\begin{matrix} j \\ m' \end{matrix}\right|(\alpha^\mu)\left|\begin{matrix} j \\ m \end{matrix}\right\rangle &= \mathscr{D}^j_{m'm}\begin{pmatrix} a_1{}^1 & a_1{}^2 \\ a_2{}^1 & a_2{}^2 \end{pmatrix} \\ &= [(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!]^{1/2} \\ &\quad \times \sum_\kappa \frac{(a_1{}^1)^{j+m-\kappa}(a_2{}^1)^\kappa(a_1{}^2)^{\kappa+m'-m}(a_2{}^2)^{j-m'-\kappa}}{(j+m-\kappa)!\,\kappa!\,(\kappa+m'-m)!\,(j-m'-\kappa)!}\end{aligned} \tag{3.12}$$

Since (3.12) provides a matrix representation for $U(2)$, it provides a representation for the subgroup $SU(2)$ as well. Furthermore, if we write

$$U(2) = e^{i\alpha_0/2}SU(2) \tag{3.13}$$

$$\left\langle\begin{matrix} j \\ m' \end{matrix}\right| U(2) \left|\begin{matrix} j \\ m \end{matrix}\right\rangle = e^{ij\alpha_0}\left\langle\begin{matrix} j \\ m' \end{matrix}\right| SU(2) \left|\begin{matrix} j \\ m \end{matrix}\right\rangle \tag{3.14}$$

Finally, we observe that the antisymmetric second-order tensor representation, with basis $(\xi\eta - \eta\xi)$, is one-dimensional:

$$(\xi\eta - \eta\xi) \to (\xi\eta - \eta\xi)' = e^{i\alpha_0}(\xi\eta - \eta\xi) \tag{3.15}$$

Under the subgroup restriction $U(2) \downarrow SU(2)$ this representation becomes the trivial identity representation.

Comment 1. The symbol $\xi^a\eta^b$ in (3.6) describes a symmetrized sum over all products of the form

$$\xi(i_1) \otimes \xi(i_2) \otimes \eta(i_3) \otimes \cdots \otimes \eta(i_{2j-1}) \otimes \xi(i_{2j}) \tag{3.16}$$

Here the arguments i_r describe which of the $2j$ identical vector spaces C_2 the basis vector $[\xi(i_r)$ or $\eta(i_r)]$ occurs in. The $i_1, i_2, \ldots, i_{2j}$ form a permutation of $1, 2, \ldots, 2j$. The symbol $\xi^a\eta^b$ is a shorthand notation for all sums of the form (3.16) having a basis vectors ξ and b basis vectors η appearing in all possible orders.

In a naturally induced metric

$$\begin{aligned}\langle \xi(i)\,|\,\xi(j)\rangle &= \langle \eta(i)\,|\,\eta(j)\rangle = \delta_{ij}\\ \langle \xi(i)\,|\,\eta(j)\rangle &= \langle \eta(j)\,|\,\xi(i)\rangle = 0\end{aligned} \tag{3.17}$$

the inner product of the tensors $\xi^a\eta^b$ is given by

$$\langle \xi^{a'}\eta^{b'}\,|\,\xi^a\eta^b\rangle = a!\,b!\,\delta_{a'a}\,\delta_{b'b} \tag{3.18}$$

To normalize the inner product to a canonical diagonal metric I_{2j+1} in the symmetric tensor product space C_{2j+1}, the factor

$$[(j+m)!\,(j-m)]^{-1/2}$$

has been included in (3.6). The induced transformation

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \middle| (\alpha^\mu) \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle$$

then preserves the diagonal metric I_{2j+1} and is therefore manifestly unitary.

Comment 2. These representations $\mathscr{D}^j_{m'm}[SU(2)]$ form a complete set[4] of unitary irreducible representations for the group $SU(2)$.

Comment 3. The 2×2 matrix generators of $SU(2)$ are given by

$$\begin{aligned}X_i(\tfrac{1}{2}) &= i\tfrac{1}{2}\sigma_i\\ [X_i, X_j] &= -\varepsilon_{ijk}X_k\end{aligned} \tag{3.19}$$

For future purposes, it is more convenient to work with a set of closely related but more familiar generators:

$$J_i = \frac{1}{i}X_i \tag{3.20}$$

$$\begin{aligned}[J_i, J_j] &= i\varepsilon_{ijk}J_k\\ e^{\alpha^i X_i} &= e^{i\alpha^i J_i}\end{aligned} \tag{3.21}$$

It is even more convenient to work with the "eigenoperator decomposition" of this algebra:

$$J_\pm = J_x \pm iJ_y \tag{3.22}$$

$$\begin{aligned}[J_3, J_\pm] &= \pm J_\pm\\ [J_+, J_-] &= 2J_3\end{aligned} \tag{3.23}$$

In the 2×2 matrix representation these operators have the form

$$\begin{array}{ccc} J_+(\tfrac{1}{2}) & J_-(\tfrac{1}{2}) & J_3(\tfrac{1}{2}) \\ \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \dfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{array} \tag{3.24}$$

The convenience of these particular operators is this: J_+ is an upper-triangular matrix, J_- is lower triangular, and J_3 is diagonal. Thus the representatives of these operators in a $(2j + 1) \times (2j + 1)$ matrix representation will be upper triangular, lower triangular, and diagonal, respectively.

In terms of the bases β, γ (*components* of a complex vector in C_2) these generators are given by:

$$J_+ = \beta \frac{\partial}{\partial \gamma}$$

$$J_- = \gamma \frac{\partial}{\partial \beta}$$

$$J_3 = \frac{1}{2}\left(\beta \frac{\partial}{\partial \beta} - \gamma \frac{\partial}{\partial \gamma}\right) \tag{3.25}$$

The effect of J_+ on a basis $\left|{j \atop m}\right\rangle$ is then easily computed:

$$\begin{aligned} J_+ \left| \begin{matrix} j \\ m \end{matrix} \right\rangle &= \beta \frac{\partial}{\partial \gamma} \frac{\beta^{j+m}\gamma^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} \\ &= \frac{\beta^{j+m+1}\gamma^{j-m-1}}{\sqrt{(j+m+1)!\,(j-m-1)!}} \\ &\quad \times (j-m)\left[\frac{(j+m+1)!\,(j-m-1)!}{(j+m)!\,(j-m)!}\right]^{1/2} \\ &= \left| \begin{matrix} j \\ m+1 \end{matrix} \right\rangle \left[\frac{(j+m+1)!\,(j-m)!}{(j+m)!\,(j-m-1)!}\right]^{1/2} \end{aligned} \tag{3.26}$$

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \right| J_+ \left| \begin{matrix} j \\ m \end{matrix} \right\rangle = \delta_{m',\,m+1} \left[\frac{(j+m')!\,(j-m)!}{(j+m)!\,(j-m')!}\right]^{1/2} \tag{3.27}$$

In this way, the matrix elements of $J_\pm$ are seen to come directly from the combinatorial (binomial) coefficients

$$\binom{2j}{j \pm m} = \frac{2j!}{(j+m)!\,(j-m)!}$$

arising from the decomposition of the $(2)^{2j}$-dimensional tensor product space $C_2{}^{2j}$ into its fully symmetric subspace $(C_2)^{2j+1}$ of dimension $2j+1$. The matrix elements of the bases J_+, J_-, J_3 are given by

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \middle| J_+^k \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle = \delta_{m',\,m+k} \left[\frac{(j+m')!\,(j-m)!}{(j+m)!\,(j-m')!} \right]^{1/2} \tag{3.28$_+$}$$

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \middle| J_-^k \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle = \delta_{m',\,m-k} \left[\frac{(j+m)!\,(j-m')!}{(j+m')!\,(j-m)!} \right]^{1/2} \tag{3.28$_-$}$$

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \middle| J_3^k \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle = \delta_{m',\,m} (m)^k \tag{3.28$_3$}$$

The calculation appearing in (3.26) is well defined regardless of whether $2j$ is integral. The factorials appearing in (3.27) and (3.28) must be replaced by gamma functions whenever $2j$ is not integral. The matrix elements then have the form

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \middle| J_+^k \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle = \delta_{m',\,m+k} \left[\frac{\Gamma(j+m'+1)\quad \Gamma(j-m+1)}{\Gamma(j+m+1)\quad \Gamma(j-m'+1)} \right]^{1/2} \tag{3.29$_+$}$$

and J_+^k has nonzero matrix elements for all values of k, and the representation is ∞ dimensional.

Comment 4. Equation (3.12), in addition to giving the matrix elements for the $SU(2)$ matrix (3.5) within the $(2j+1) \times (2j+1)$ unitary irreducible representation, can also be considered as a generating function for matrix elements of $J_\pm(j)$, $J_3(j)$. For we have

$$e^{\alpha^i X_i} \to e^{i(\alpha_+ J_+ + \alpha_- J_- + \alpha_3 J_3)}$$

$$\to \begin{bmatrix} \cos\theta + \dfrac{i\alpha_3}{2}\dfrac{\sin\theta}{\theta} & i\alpha_+ \dfrac{\sin\theta}{\theta} \\ i\alpha_- \dfrac{\sin\theta}{\theta} & \cos\theta - \dfrac{i\alpha_3}{2}\dfrac{\sin\theta}{\theta} \end{bmatrix} \tag{3.30}$$

$$\alpha_\pm = \tfrac{1}{2}(\alpha_x \mp i\alpha_y)$$

$$\theta^2 = \alpha_+ \alpha_- + \left(\frac{\alpha_3}{2}\right)^2 \tag{3.31}$$

In the vicinity of the identity in the group, or the 0 in the algebra, the matrix elements $a_i{}^j$ depend linearly on the algebra parameters (to lowest order):

$$\mathrm{EXP}\; i(\alpha_+ J_+ + \alpha_- J_- + \alpha_3 J_3) \to \begin{bmatrix} 1 + \dfrac{i\alpha_3}{2} & i\alpha_+ \\ i\alpha_- & 1 - \dfrac{i\alpha_3}{2} \end{bmatrix} \tag{3.32}$$

In the representation $\mathscr{D}^j_{m'm}$ we have

$$\mathscr{D}^j_{m'm} = \left\langle \begin{matrix} j \\ m' \end{matrix} \middle| \mathrm{EXP}\, i(\alpha_+ J_+ + \alpha_- J_- + \alpha_3 J_3) \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle \tag{3.33}$$

$$= \sum_{k=0}^{\infty} \left\langle \begin{matrix} j \\ m' \end{matrix} \middle| \frac{[i(\alpha_+ J_+ + \alpha_- J_- + \alpha_3 J_3)]^k}{k!} \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle \tag{3.34}$$

To compute the matrix elements of J^k_+, we must differentiate k times with $\partial/\partial i\alpha_+$ and evaluate the result at the identity ($\alpha = 0$):

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \middle| J^k_+ \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle = [(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!]^{1/2}$$
$$\times \sum_{\alpha \to 0} \frac{(\kappa + m' - m)!}{(\kappa + m' - m - k)!} \frac{(a_1{}^1)^{j+m-\kappa}(a_2{}^1)^{\kappa}(a_1{}^2)^{\kappa+m'-m-k}(a_2{}^2)^{j-m'-\kappa}}{(j+m-\kappa)!\,\kappa!\,(\kappa+m'-m)!\,(j-m'-\kappa)!}$$
$$\begin{bmatrix} a_1{}^1 \to 1 & a_1{}^2 \to 0 \\ a_2{}^1 \to 0 & a_2{}^2 \to 1 \end{bmatrix} \tag{3.35}$$

Since $a_1{}^2$, $a_2{}^1 \to 0$, the only term in the sum that can survive this limit occurs when both

$$\kappa = 0 \qquad \text{and} \qquad \kappa + m' - m - k = 0 \tag{3.36}$$

Therefore

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \middle| J^k_+ \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle = \delta_{m',\,m+k} \left[\frac{(j+m')!\,(j-m)!}{(j+m)!\,(j-m')!} \right]^{1/2} \tag{3.37}$$

The matrix elements of J_- and J_3 may be evaluated similarly.

3. REPRESENTATIONS OF $SO(3, r)$. The representations (3.12) for $SU(2)$ may or may not be representations for its homomorphic image $SO(3, r)$. They will be representations for $SO(3, r)$, provided

$$\mathscr{D}^j_{m'm}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathscr{D}^j_{m'm}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\delta_{m'm} \qquad = (-)^{2j}\,\delta_{m'm} \tag{3.38}$$

Only those representations of $SU(2, c)$ with $j = l$ (integer) are also representations of $SO(3)$. The representations $\mathscr{D}^l_{m'm}[SO(3)]$ form a complete set[5] of unitary irreducible representations of $SO(3, r)$.

IV. Quaternion Covering Group

1. DIRECT SUMS AND PRODUCTS. In (1.3) and (1.14) we found that $Gl(1, q)$ and $U(2, c)$ have isomorphic Lie algebras. Furthermore, the algebra splits into two mutually commuting subalgebras. One invariant subalgebra is spanned by X_0, the other by X_1, X_2, X_3. Such a Lie algebra decomposition is called a direct sum decomposition.

The simply connected covering group belonging to a direct sum algebra is the direct product of the simply connected covering groups for each component invariant subalgebra:

$$\begin{tikzcd}
\mathfrak{u}(2) = \mathfrak{u}(1) \arrow[d, "\mathrm{EXP}"] & \oplus & \mathfrak{su}(2) \arrow[d, "\mathrm{EXP}"] & \text{direct sum algebra} \arrow[d, "\mathrm{EXP}"'] \\
R_1 & \otimes & SU(2) \cong Gl(1, q) & \text{direct product group}
\end{tikzcd} \tag{4.1}$$

The quaternion group $Gl(1, q)$ is the covering group of $U(2)$.*

By listing all distinct discrete invariant subgroups of the universal covering group, we can list all distinct Lie groups with this direct sum algebra. The discrete invariant subgroups for the groups R_1 and $SU(2)$ have already been listed. Construction of the associated homomorphic images, or factor groups, is simplified by the following

THEOREM.

$$\frac{SG_1 \otimes SG_2}{D_1 \otimes D_2} \cong \left(\frac{SG_1}{D_1}\right) \otimes \left(\frac{SG_2}{D_2}\right) \tag{4.2}$$

To complete the listing of Lie groups with this algebra, we must find the discrete invariant subgroups of $Gl(1, q)$ which are not direct products of discrete invariant subgroups of R_1 and $SU(2)$. The only such subgroup is generated by powers of $\{e^{i\pi}, -I_2\}$. In Table 5.1 we list[2] the five distinct discrete invariant subgroups of $Gl(1, q)$, the associated Lie group, and the multiplication law for the group.

2. REPRESENTATIONS OF THE FACTOR GROUPS. The representations of R_1 are given by

$$aX_0 \rightarrow \mathrm{EXP}\ aX_0 \rightarrow e^{ika} = \Gamma_k(a) \tag{4.3}$$

* Here and below we follow the convention that the Lie algebra of a Lie group G is denoted by the corresponding lower case germanic letter $\mathfrak{g}$. Thus the Lie algebra of $SU(2)$ is denoted $\mathfrak{su}(2)$.

TABLE 5.1

$= \dfrac{R_1 \otimes SU(2)}{D}$	D Generated by	D Isomorphic to	Multiplication
$Gl(1, q)$ $= R_1 \otimes SU(2)$	$\{e^{\iota 0}, I_2\}$	$\{0\} \otimes I_2 = I$	$(e^{\alpha_2}\sigma_2)(e^{\alpha_1}\sigma_1) = e^{\alpha_2+\alpha_1}\sigma_2\sigma_1$
$T_1 \otimes SU(2)$	$\{e^{2\pi\iota}, I_2\}$	$Z \otimes I_2 = Z$	$(e^{\iota\alpha_2}\sigma_2)(e^{\iota\alpha_1}\sigma_1) = e^{\iota(\alpha_2+\alpha_1)}\sigma_2\sigma_1$
$R_1 \otimes SO(3)$	$\{e^{\iota 0}, -I_2\}$	$\{0\} \otimes Z_2 = Z_2$	$e^{\alpha_2}R(\sigma_2)e^{\alpha_1}R(\sigma_1) = e^{\alpha_2+\alpha_1}R(\sigma_2\sigma_1)$
$T_1 \otimes SO(3)$	$\{e^{2\pi\iota}, I_2\}$, $\{e^{\iota 0}, -I_2\}$	$Z \otimes Z_2$	$e^{\iota\alpha_2}R(\sigma_2)e^{\iota\alpha_1}R(\sigma_1) = e^{\iota(\alpha_2+\alpha_1)}R(\sigma_2\sigma_1)$
$U(2)$	$\{e^{\pi\iota}, -I_2\}$	Z	$(e^{\iota\alpha_2}\sigma_2)(e^{\iota\alpha_1}\sigma_1) = e^{\iota(\alpha_2+\alpha_1)}\sigma_2\sigma_1$

σ_ι are elements of $SU(2)$.
$R(\sigma_\iota)$ are the images of σ_ι in $SO(3)$.
$Z_2 = \{I_2, -I_2\}$

with k an arbitrary complex number. The unitary representations are given by

$$a \to e^{ika} \qquad (k \text{ real}) \tag{4.4}$$

For the factor group R_1/Z, which identifies all points $2\pi n$ with the identity, only the representations e^{ika} for which

$$e^{ik(2\pi n)} = e^{ik0} = 1 \tag{4.5}$$

are also representations of T_1. These representations are characterized by $k = $ integer.

The representation labels describing the unitary representations of the quaternion covering group are (j, k), with j integral or half-integral and k real.

The quantum numbers describing the unitary representations of the first four groups in Table 5.1 are straightforward to determine. We merely observe that the representations of a direct product group are the direct products of the representations of the individual groups. To determine the values which (j, k) may assume for representations of $U(2)$, we demand

$$\{e^{i\pi}, -I_2\} \to (e^{i\pi})^k \mathscr{D}^j_{m'm}(-I_2) = (e^{i0})^k \mathscr{D}^j_{m'm}(+I_2)$$
$$(-)^{k+2j} = +1 \tag{4.6}$$

Not only must k be integral, but its value is related to the value of j by

$$2j + k = 0 \bmod 2 = 2(\text{integer}) \tag{4.7}$$

The values of (j, k) describing the complete set of unitary irreducible representations for the five distinct groups with Lie algebra (1.3) are listed in Table 5.2.

TABLE 5.2

	Quantum Numbers				
	k		j		
Group	Real	Integer	Integer	Half-Integer	Comment
$Gl(1, q)$	X			X	
$T_1 \otimes SU(2)$		X		X	
$R_1 \otimes SO(3)$	X		X		
$T_1 \otimes SO(3)$		X	X		
$U(2)$		X		X	$(-)^{2j+k} = +1$

Comment. Although both $U(2)$ and $T_1 \otimes SU(2)$ have isomorphic discrete invariant subgroups—and, therefore, isomorphic homotopy groups—these groups are not themselves isomorphic. In this connection, see (2.11).

The relation between the Lie algebra, the universal covering group, the various homomorphic image groups, and the quantum numbers describing their representations, appears in Fig. 5.9.

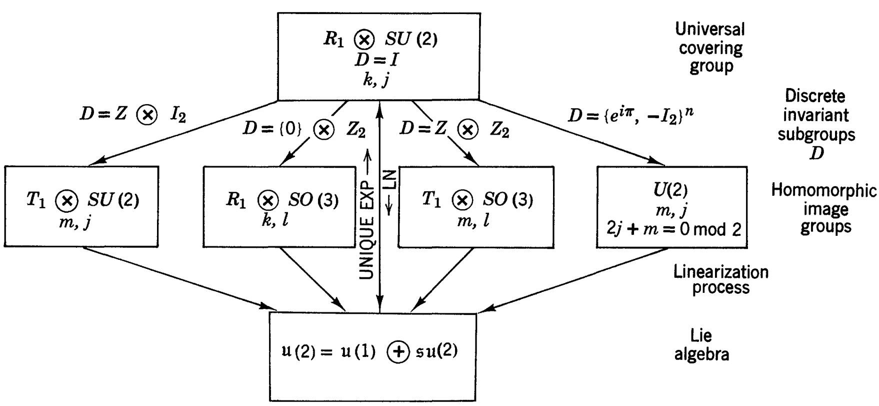


FIG. 5.9 ALL FIVE LIE GROUPS HAVE THE SAME LIE ALGEBRA. THERE IS A UNIQUE CORRESPONDENCE BETWEEN THE LIE ALGEBRA AND THE UNIVERSAL COVERING GROUP. ALL LIE GROUPS WITH THE SAME ALGEBRA ARE EASILY OBTAINED FROM THE UNIVERSAL COVERING GROUP AS HOMOMORPHIC IMAGES. THE QUANTUM NUMBERS LABELING THE COMPLETE SET OF UIR FOR THE COVERING GROUP ARE (j, k). THE SUBSET LABELING THE COMPLETE SET OF UIR FOR THE OTHER GROUPS IS DETERMINED BY THE REQUIREMENT

$$\Gamma^{j,k}(d_i) = \Gamma^{j,k}(\text{identity}), \qquad \text{where } d_i \in D.$$

3. TWO POSSIBLE PHYSICAL CONSEQUENCES. Invariances are connected with conservation laws. *If*:

1. The dynamical properties of a system are derivable from an *action principle, and*
2. A transformation group leaves the *action integral* invariant, *then*
3. The generators of the transformation group are conserved quantities.

Since the generators of a Lie group are local quantities, the conservation laws involving Lie groups derived in this way are local or differential statements.

If in addition we are discussing quantum mechanical systems, the transformation groups will enter the action principle primarily through their unitary irreducible representations. Then the invariants characterizing these representations are the quantum numbers in terms of which the possible quantum states must be described. These quantum numbers are also conserved.

We frequently encounter action integrals which are invariant under the transformation

$$\begin{aligned} \Psi &\to \Psi' = \mathscr{D}^j[SU(2)]\Psi \\ \Psi^\dagger &\to \Psi'^\dagger = \Psi^\dagger \mathscr{D}^j[SU(2)]^\dagger \end{aligned} \tag{4.8}$$

Then the generators of $SU(2)$ are conserved quantities, and the states Ψ are characterized by the quantum number j describing the unitary irreducible representations of $SU(2)$.

In addition, most Lagrangians are constructed to be unphased by the transformation

$$\begin{aligned} \Psi &\to \Psi' = e^{in\phi}\Psi \\ \Psi^\dagger &\to \Psi'^\dagger = \Psi^\dagger e^{-in\phi} \end{aligned} \tag{4.9}$$

These are the representations of the group $U(1)$. Its generator corresponds to a conserved quantity, and the states Ψ are described by the quantum number n.

If the group of the Lagrangian is the direct product group $U(1) \otimes SU(2)$, then the quantum numbers j and n are independent. But if the group is $U(2)$, j and n are related by

$$(-)^{2j+n} = +1 \tag{4.10}$$

Possibility A. If the subgroup $SU(2)$ acts on isospin space ($j \to t$), we would like to associate the quantum number n with hypercharge $\mathscr{Y} = B + S$. Then we find

$$(-)^{2t+\mathscr{Y}} = (-)^{2t_3+\mathscr{Y}} = +1 \tag{4.11}$$

$$2t_3 + \mathscr{Y} = 0 \bmod 2 = 2q \qquad (q \text{ integer}) \tag{4.12}$$

Solving the equation for q (charge)

$$q = t_3 + \tfrac{1}{2}(B + S) \tag{4.13}$$

This provides a relation[6] between the charge q, isospin t_3, and baryon number B and strangeness S of a particle.

Possibility B. If $SU(2)$ acts on ordinary spin components, it is customary to assume that the quantum number associated with $U(1)$ describes the charge state of a system. Since charge has already been related to other properties in Possibility A, and spin and charge seem to be dynamically unrelated, let us make the association $n \to s$ (statistics):

$$(-)^s = (-)^{2j} \begin{array}{lll} \nearrow & +1 & j \text{ integral} \\ \searrow & -1 & j \text{ half-integral} \end{array} \tag{4.14}$$

It is possible that this provides some relation between spin and statistics.

The usual connection[7] between spin and statistics is made by showing that:

1. Bose-Einstein statistics for half-integral spin particles is incompatible with the positive definiteness of the T_{44} component of the stress-energy-momentum tensor.
2. Fermi-Dirac statistics for integral spin particles is incompatible with causality.

In short, the usual assignment of statistics is compatible with two distinct dynamical principles.

It would be much more satisfying to find a purely kinematical, or group theoretical, relation between spin and statistics which would assume the following form, in a Fock representation:

$$a_{k'}a_k^\dagger \, (-)^{2j+1} a_k^\dagger a_{k'} = \delta_{k'k} \tag{4.15}$$

From this kinematical result alone would result both dynamical consequences:

1. Causality (for j integral).
2. Positive definiteness of the energy density (for j half-integral).

V. Spin and Double-Valuedness—DESCRIPTION OF THE ELECTRON. The electron at rest can exist in two independent states (or any linear combination of them) corresponding to "spin up" and "spin down." To describe a particle with two internal degrees of freedom, we must set up a two-component wave function to represent the particle

$$\psi(x) = u_1(x)\begin{pmatrix}1\\0\end{pmatrix} + u_2(x)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}u_1(x)\\u_2(x)\end{pmatrix} \tag{5.1}$$

The state space describing such a particle is then a tensor product space of the form

$$C_2 \otimes \mathscr{H}_4 \tag{5.2}$$

The C_2 is a complex two-dimensional space, called a spinor space, and $\mathscr{H}_4$ is a Hilbert space of complex-valued functions defined on four-dimensional space-time.

We can visualize the two independent internal states of the electron as in Fig. 5.10. If we now rotate the vector $|v\rangle$ in C_2 four successive times through $\pi/2$, we wind up where we started. In the meantime, our physical picture for the spin has rotated through 4π. These operations are illustrated in Fig. 5.11. The states $\pm|v_{1/2}\rangle$ both represent an electron in the spin up state; $\pm|v_{-1/2}\rangle$ both represent an electron in the spin down state. This is a manifestation of the $2 \rightarrow 1$ nature of the group homomorphism $SU(2, c) \rightarrow SO(3, r)$ in terms of the vector spaces on which these groups act.

A 2π rotation of the axes in R_3, which leaves R_3 unchanged, should have no effect on the measurement of the electron spin, even though it corresponds to a π rotation in C_2. This is true, since only the matrix elements of the spin are measurable:

$$\langle\phi|\boldsymbol{\sigma}|\psi\rangle \tag{5.3}$$

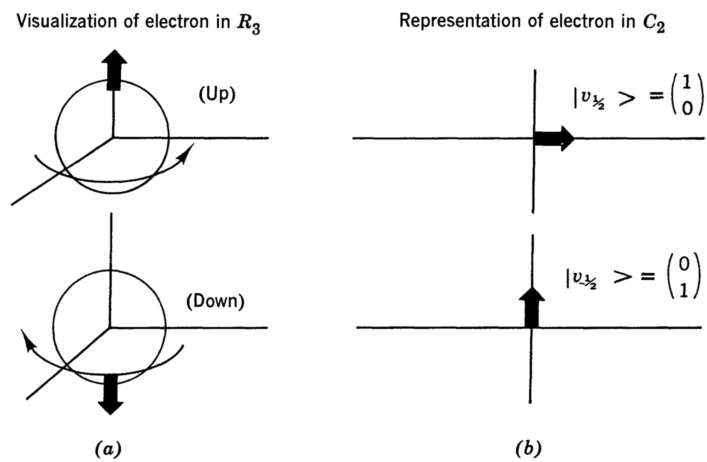


FIG. 5.10 WHEN THE ELECTRON IS IN A "SPIN UP" STATE, WE VISUALIZE IT AS IT APPEARS IN (*a*) (up) AND REPRESENT IT IN A COMPLEX TWO-DIMENSIONAL SPACE C_2, AS SHOWN IN (*b*). IN THE "SPIN DOWN" STATE, THE ELECTRON IS IMAGINED AS IN (*a*) (DOWN) IN A REAL THREE-DIMENSIONAL SPACE. SINCE THE SPIN UP AND DOWN STATES ARE ORTHOGONAL, IT IS REPRESENTED AS IN (*b*).

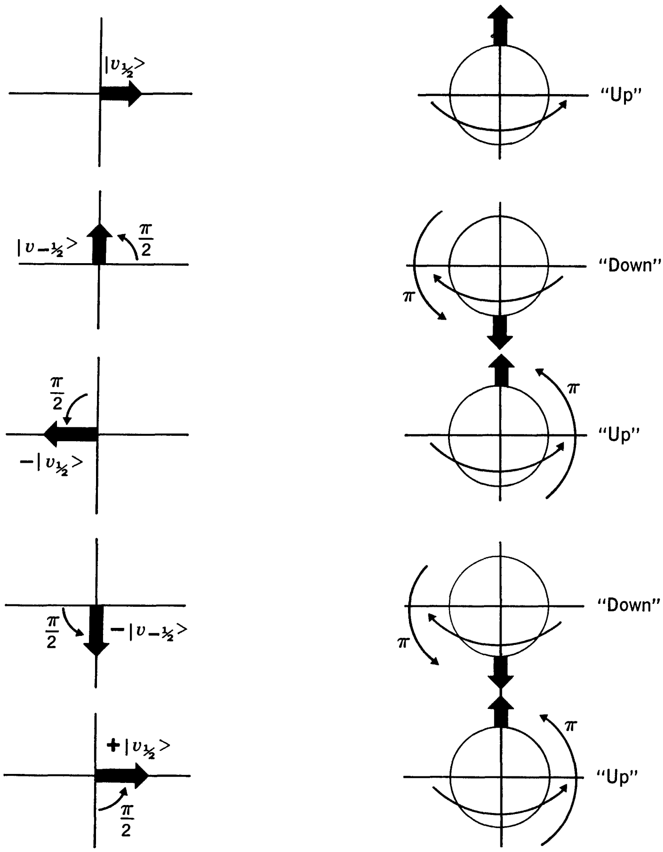


FIG. 5.11 AS WE ROTATE A STATE $|v\rangle$ IN C_2 SUCCESSIVELY FOUR TIMES THROUGH $\pi/2$ ABOUT ANY FIXED AXIS, WE WIND UP WHERE WE STARTED. IN THE MEANTIME, THE REPRESENTATIVE OF THE PHYSICAL STATE IN R_3 HAS RETURNED TWICE TO ITS ORIGINAL ORIENTATION. WE HAVE TRANSFERRED THE HOMOMORPHISM THAT EXISTS BETWEEN THE GROUPS $SU(2, c)$ AND $SO(3, r)$ ONTO THE VECTOR SPACES C_2 AND R_3 IN WHICH THESE GROUPS ACT AS CHANGES OF BASES.

Under a π rotation in C_2, the matrix elements become

$$\begin{gathered}\langle\phi|\{e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}\}^{\dagger}\boldsymbol{\sigma}\{e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}\}|\psi\rangle \\ \| \\ \langle\phi|\{-I_2\}\boldsymbol{\sigma}\{-I_2\}|\psi\rangle = \langle\phi|\boldsymbol{\sigma}|\psi\rangle\end{gathered} \tag{5.4}$$

It is exactly this quadratic transformation property of matrix elements which allows us to associate two unitary operations of $SU(2)$ with each physical rotation operation of $SO(3)$.

It is sometimes said that the j half-integral representations of $SU(2)$ are "double-valued" representations of $SO(3)$, allowed because of the nature of the measurement process. We prefer to think of it differently. The j half-integral representations are faithful representations of $SU(2)$. Moreover, $SO(3)$ is a homomorphic image of $SU(2)$, and thus some of the representations of $SU(2)$ are not representations of $SO(3)$.

The heart of the matter is that we are accustomed to thinking in terms of either a geometric three-dimensional space R_3 or a four-dimensional space-time R_4. All our concepts of geometry derive ultimately from the strong coupling of our senses (eyes) with photons (spin 1). We therefore interact strongly with the $\mathscr{D}^l$ representations—the half-integral representations are unfamiliar. If our eyes were constructed to interact with $j = 1/2$ particles, we would see a peculiar twofold degeneracy in those properties of the universe depending on integral spin particles.

VI. Noncanonical Parameterizations for $SU(2, c)$

1. BAKER-CAMPBELL-HAUSDORFF FORMULAS. So far in this chapter we have dealt principally with the group $SU(2)$. Moreover, we have discussed this group primarily in its canonical parameterization. This parameterization is obtained by the EXPonential mapping of the Lie algebra onto the Lie group. In this parameterization, every straight line through the origin of the algebra exponentiates onto a one-dimensional abelian subgroup.

To obviate the impression that noncanonical parameterizations are anathema, we propose now to deal with a number of them. This is not an empty academic exercise: we have a number of motivations for such a discussion.

1. Mathematical reasons. For one thing, it is often difficult to construct the canonical parameterization of a classical Lie group using the EXPonential mapping. It is even more difficult to construct canonical matrix representations of a group by the canonical mapping of the algebra's representations onto the group's representations with the EXPonential mapping.

2. Physical reasons. We will eventually want to associate physical operators with elements in Lie algebras and groups. For instance, it is often useful to associate shift-up and shift-down operators like J_+ and J_- with operators in a Hamiltonian which cause transitions to higher and lower energy levels. Then it becomes necessary to compute matrix elements of ordered operator products within particular representations. The existence of noncanonical parameterizations allows the construction of generating functions for products of operators in normal and symmetrized orderings.

An arbitrary element in the group $SU(2)$ is written

$$SU(2) \to \begin{bmatrix} a_1{}^1 & a_1{}^2 \\ a_2{}^1 & a_2{}^2 \end{bmatrix} \tag{6.1}$$

The matrix elements $a_i{}^j$ obey the metric-preserving condition (6.2) as well as the unimodular condition (6.3):

$$a_i{}^{r*}\,\delta_{rs} a_j{}^s = \delta_{ij} \tag{6.2}$$

$$\det\|a_i{}^r\| = +1 \tag{6.3}$$

It is easily verified that matrices of the form

$$\begin{bmatrix} x & iy \\ iy^* & x^* \end{bmatrix} \tag{6.1$'$}$$

$$x,\ y \text{ complex} \tag{6.2$'$}$$

$$x^*x + y^*y = 1 \tag{6.3$'$}$$

satisfy the conditions (6.2) and (6.3). Thus it is possible to describe every group element in $SU(2)$ by a pair of complex numbers x, y obeying (6.2′) and (6.3′). These group elements can equivalently be described by the parameters α_+, α_-, α_3 in the canonical mapping of the algebra onto the group

$$\text{EXP } i(\alpha_+^1 J_+ + \alpha_-^1 J_- + \alpha_3{}^1 J_3)$$

$$\to \begin{bmatrix} \cos\theta^1 + \dfrac{i\alpha_3{}^1}{2\theta^1}\sin\theta^1 & \dfrac{i\alpha_+^1}{\theta^1}\sin\theta^1 \\ \dfrac{i\alpha_-^1}{\theta^1}\sin\theta^1 & \cos\theta^1 - \dfrac{i\alpha_3{}^1}{2\theta^1}\sin\theta^1 \end{bmatrix}$$

$$\theta^1 = \sqrt{\alpha_+^1\alpha_-^1 + (\alpha_3{}^1/2)^2} \tag{6.4^1}$$

From the equality that must exist between (6.4) and (6.1′) it is possible to determine the reality conditions on the parameters α^1 and also the relation between the α^1's and x, y:

$$\alpha_3{}^1 \text{ real}$$

$$\alpha_-^1 = \alpha_+^{1*} \tag{6.5^1}$$

$$x = \cos\theta^1 + \frac{i\alpha_3{}^1}{2\theta^1}\sin\theta^1$$

$$y = \frac{\alpha_+^1}{\theta^1}\sin\theta^1 \tag{6.6^1}$$

For various reasons discussed earlier (and to be illustrated in the following sections), it is often desirable to map the algebra onto the group in a noncanonical way. For example, a coset decomposition is sometimes useful:

$$
\begin{array}{ccccc}
\mathrm{EXP}\; i(\alpha_+^2 J_+ + \alpha_-^2 J_-) & & e^{i\alpha_3{}^2 J_3} & = & \begin{bmatrix} x & iy \\ iy^* & x^* \end{bmatrix} \\
\downarrow & & \downarrow & & \downarrow \\
\begin{bmatrix} \cos\theta^2 & \dfrac{i\alpha_+^2}{\theta^2}\sin\theta^2 \\ \dfrac{i\alpha_-^2}{\theta^2}\sin\theta^2 & \cos\theta^2 \end{bmatrix} & & \begin{bmatrix} e^{i\alpha_3{}^2/2} & \\ & e^{-i\alpha_3{}^2/2} \end{bmatrix} & &
\end{array}
$$

$$
= \begin{bmatrix} \cos\theta^2\, e^{i\alpha_3{}^2/2} & \dfrac{i\alpha_+^2}{\theta^2}\sin\theta^2\, e^{-i\alpha_3{}^2/2} \\ \dfrac{i\alpha_-^2}{\theta^2}\sin\theta^2\, e^{i\alpha_3{}^2/2} & \cos\theta^2\, e^{-i\alpha_3{}^2/2} \end{bmatrix}
$$

$$
\theta^2 = \sqrt{\alpha_+^2 \alpha_-^2} \tag{6.4^2}
$$

From this we conclude once again that

$$
\alpha_3{}^2 \text{ real}
$$

$$
\alpha_-^2 = \alpha_+^{2*} \tag{6.5^2}
$$

$$
x = \cos\theta^2\, e^{i\alpha_3{}^2/2}
$$

$$
y = \frac{\alpha_+^2}{\theta^2}\sin\theta^2\, e^{-i\alpha_3{}^2/2} \tag{6.6^2}
$$

Now in general the point $(\alpha_+, \alpha_-, \alpha_3)$ in the Lie algebra will map into two *different* group operations under the mappings (6.4^1) and (6.4^2). Therefore, we look at *different* points in the Lie algebras involved in the parameterizations (6.4^1) and (6.4^2), which map onto the *same* group operation. In this way it is possible to find an analytic isomorphism relating the parameters $(\alpha_+^1, \alpha_-^1, \alpha_3{}^1)$ of (6.4^1) and $(\alpha_+^2, \alpha_-^2, \alpha_3{}^2)$ of (6.4^2). Then we can use this analytic isomorphism to go from either parameterization to the other.

Before presenting applications of these statements, we consider one more noncanonical parameterization in detail. This is given by

$$
\begin{array}{ccccc}
e^{i\alpha_+{}^4 J_+} & e^{i\alpha_3{}^4 J_3} & e^{i\alpha_-{}^4 J_-} & = & \begin{bmatrix} x & iy \\ iy^* & x^* \end{bmatrix} \\
\downarrow & \downarrow & \downarrow & & \downarrow \\
\begin{bmatrix} 1 & i\alpha_+^4 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} e^{i\alpha_3{}^4/2} & 0 \\ 0 & e^{-i\alpha_3{}^4/2} \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ i\alpha_-^4 & 1 \end{bmatrix} & = &
\end{array}
$$

$$
\begin{bmatrix} e^{i\alpha_3{}^4/2} - \alpha_+^4 \alpha_-^4\, e^{-i\alpha_3{}^4/2} & i\alpha_+^4\, e^{-i\alpha_3{}^4/2} \\ i\alpha_-^4\, e^{-i\alpha_3{}^4/2} & e^{-i\alpha_3{}^4/2} \end{bmatrix} \tag{6.4^4}
$$

$$\mathrm{EXP}\; i(\alpha_+^1 J_+ + \alpha_-^1 J_- + \alpha_3{}^1 J_3)$$
$$\|$$
$$\begin{bmatrix} \cos\theta^1 + \dfrac{i\alpha_3{}^1}{2\theta^1}\sin\theta^1 & \dfrac{i\alpha_+^1}{\theta^1}\sin\theta^1 \\ \dfrac{i\alpha_-^1}{\theta^1}\sin\theta^1 & \cos\theta^1 - \dfrac{i\alpha_3{}^1}{2\theta^1}\sin\theta^1 \end{bmatrix}$$

$$\mathrm{EXP}\; i(\alpha_+^2 J_+ + \alpha_-^2 J_-)\;\mathrm{EXP}\; i\alpha_3{}^2 J_3$$
$$\|$$
$$\begin{bmatrix} \cos\theta^2\, e^{i\alpha_3{}^2/2} & \dfrac{i\alpha_+^2}{\theta^2}\sin\theta^2\, e^{-i\alpha_3{}^2/2} \\ \dfrac{i\alpha_-^2}{\theta^2}\sin\theta^2\, e^{i\alpha_3{}^2/2} & \cos\theta^2\, e^{-i\alpha_3{}^2/2} \end{bmatrix}$$

$$\mathrm{EXP}\; i\alpha_3{}^3 J_3\;\mathrm{EXP}\; i(\alpha_+^3 J_+ + \alpha_-^3 J_-)$$
$$\|$$
$$\begin{bmatrix} e^{i\alpha_3{}^3/2}\cos\theta^3 & e^{i\alpha_3{}^3/2}\dfrac{i\alpha_+^3}{\theta^3}\sin\theta^3 \\ e^{-i\alpha_3{}^3/2}\dfrac{i\alpha_-^3}{\theta^3}\sin\theta^3 & e^{-i\alpha_3{}^3/2}\cos\theta^3 \end{bmatrix}$$

$$\begin{bmatrix} x & iy \\ iy^* & x^* \end{bmatrix}$$

$$e^{i\alpha_+{}^4 J_+}\, e^{i\alpha_3{}^4 J_3}\, e^{i\alpha_-{}^4 J_-}$$
$$\|$$
$$\begin{bmatrix} \begin{array}{c} e^{i\alpha_3{}^4/2} \\ -\alpha_+^4\alpha_-^4\, e^{-i\alpha_3{}^4/2} \end{array} & i\alpha_+^4\, e^{-i\alpha_3{}^4/2} \\ i\alpha_-^4\, e^{-i\alpha_3{}^4/2} & e^{-i\alpha_3{}^4/2} \end{bmatrix}$$

$$e^{i\alpha_-{}^5 J_-}\, e^{i\alpha_3{}^5 J_3}\, e^{i\alpha_+{}^5 J_+}$$
$$\|$$
$$\begin{bmatrix} e^{i\alpha_3{}^5/2} & i\alpha_+^5\, e^{i\alpha_3{}^5/2} \\ i\alpha_-^5\, e^{i\alpha_3{}^5/2} & \begin{array}{c} e^{-i\alpha_3{}^5/2} \\ -\alpha_+^5\alpha_-^5\, e^{i\alpha_3{}^5/2} \end{array} \end{bmatrix}$$

FIG. 5.12 FIVE DIFFERENT PARAMETERIZATIONS OF $SU(2.c)$ ARE ANALYTICALLY RELATED TO ONE ANOTHER THROUGH THE 2×2 MATRIX GROUP OPERATION IN THE MIDDLE.

In this parameterization we have

$$\alpha_3{}^4 \text{ complex}$$

$$\alpha_+^4 = \frac{y}{x^*}$$

$$\alpha_-^4 = \frac{y^*}{x^*} \tag{6.5^4}$$

$$x = e^{i\alpha_3{}^{4*}/2}$$

$$y = \alpha_+^4 e^{-i\alpha_3{}^4/2} \tag{6.6^4}$$

The canonical parameterization, the two noncanonical parameterizations, and two "dual" parameterizations are shown in Fig. 5.12. Each parameterization of the *algebra* is analytically related to the others through the intermediary of the matrix (6.1′) in the *group*. The reality conditions on the parameters $(\alpha_+, \alpha_-, \alpha_3)$, as well as their relation with the complex parameters, are listed in Table 5.3.

Example. The BCH formula connecting parameterizations 4 and 5 of Table 5.3 is

$$\alpha_+^4 = \alpha_-^{5*}$$

$$\alpha_-^4 = \alpha_+^{5*}$$

$$\alpha_3{}^4 = \alpha_3{}^{5*} \tag{6.7}$$

2. APPLICATION IN CONSTRUCTING REPRESENTATIONS

Remark. The BCH formulas developed in the preceding section relate different parameterizations of the defining 2×2 matrix representation of the group $SU(2, c)$. Since representations preserve the multiplication properties of the group, these BCH formulas are valid *within any representation* of $SU(2, c)$. We will use this observation to construct the unitary irreducible representations of $SU(2, c)$.

We wish to compute

$$\mathscr{D}^j_{m'm}\begin{bmatrix} x & iy \\ iy^* & x^* \end{bmatrix} = \left\langle \begin{matrix} j \\ m' \end{matrix} \right| \text{EXP } i(\alpha_+^1 J_+ + \alpha_3^1 J_3 + \alpha_-^1 J_-) \left| \begin{matrix} j \\ m \end{matrix} \right\rangle \tag{6.8^1}$$

$$= \left\langle \begin{matrix} j \\ m' \end{matrix} \right| e^{i\alpha_-{}^5 J_-} e^{i\alpha_3{}^5 J_3} e^{i\alpha_+{}^5 J_+} \left| \begin{matrix} j \\ m \end{matrix} \right\rangle \tag{6.8^5}$$

TABLE 5.3

DESCRIPTION OF RELATION BETWEEN THE FIVE SETS OF LIE ALGEBRA PARAMETERS $(\alpha_+, \alpha_-, \alpha_3)$ FOR THE FIVE PARAMETERIZATIONS IN FIG. 5.12, AND THE TWO COMPLEX NUMBERS x, y $(x^*x + y^*y = 1)$ DESCRIBING A GROUP ELEMENT IN $SU(2)$.

	1	2	3	4	5
x	$\cos\theta^1 + \frac{i\alpha_3^1}{2\theta^1}\sin\theta^1$	$\cos\theta^2\, e^{i\alpha_3{}^2/2}$	$e^{i\alpha_3{}^3/2}\cos\theta^3$	$e^{i\alpha_3{}^4/2}$ $-\alpha_+^4\alpha_-^4\, e^{-i\alpha_3{}^4/2}$	$e^{i\alpha_3{}^5/2}$
y	$\frac{\alpha_+^1}{\theta^1}\sin\theta^1$	$\frac{\alpha_+^2}{\theta^2}\sin\theta^2\, e^{-i\alpha_3{}^2/2}$	$e^{i\alpha_3{}^3/2}\frac{\alpha_+^3}{\theta^3}\sin\theta^3$	$\alpha_+^4\, e^{-i\alpha_3{}^4/2}$	$\alpha_+^5\, e^{i\alpha_3{}^5/2}$
Additional Comments	${\alpha_3}^1$ real	${\alpha_3}^2$ real	${\alpha_3}^3$ real	${\alpha_3}^4$ complex	${\alpha_3}^5$ complex
	$\alpha_-^1 = \alpha_+^{1*}$	$\alpha_-^2 = \alpha_+^{2*}$	$\alpha_-^3 = \alpha_+^{3*}$	$\alpha_+^4 = \frac{y}{x^*}$	$\alpha_+^5 = \frac{y}{x}$
	$\theta^1 = \sqrt{\alpha_+^1\alpha_-^1 + ({\alpha_3}^1/2)^2}$	$\theta^2 = \sqrt{\alpha_+^2\alpha_-^2}$	$\theta^3 = \sqrt{\alpha_+^3\alpha_-^3}$	$\alpha_-^4 = \frac{y^*}{x^*}$	$\alpha_-^5 = \frac{y^*}{x}$

The parameters (x, y), $(\alpha_+^1, \alpha_-^1, \alpha_3{}^1)$, and $(\alpha_+^5, \alpha_-^5, \alpha_3{}^5)$ are related as discussed in the previous section. The matrix elements (6.8^1) and (6.8^5) are equal for reasons noted in the opening remark. We will compute the matrix elements (6.8^5) explicitly.

Using the matrix elements (3.28) it is a simple matter to compute $e^{i\alpha_+ J_+}$:

$$\left\langle \begin{matrix} j \\ m \end{matrix} \middle| e^{i\alpha_+ J_+} \middle| \begin{matrix} j \\ m' \end{matrix} \right\rangle = \left\langle \begin{matrix} j \\ m \end{matrix} \middle| \sum_0^\infty \frac{(i\alpha_+)^k}{k!} (J_+)^k \middle| \begin{matrix} j \\ m' \end{matrix} \right\rangle$$

$$= \sum_{k=0}^{2j+1} \frac{(i\alpha_+)^k}{k!} \delta_{m,\, m'+k} \left[\frac{(j-m')!\,(j+m)!}{(j-m'-k)!\,(j+m-k)!}\right]^{1/2}$$

$$= \frac{(i\alpha_+)^{m-m'}}{(m-m')!} \left[\frac{(j-m')!\,(j+m)!}{(j-m)!\,(j+m')!}\right]^{1/2} \tag{6.9$_+$}$$

Now using this result and a similar one for $e^{i\alpha_- J_-}$ we compute the matrix elements (6.8^5):

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \middle| e^{i\alpha_- J_-} \middle| \begin{matrix} j \\ n' \end{matrix} \right\rangle \qquad \left\langle \begin{matrix} j \\ n' \end{matrix} \middle| e^{i\alpha_3 J_3} \middle| \begin{matrix} j \\ n \end{matrix} \right\rangle \qquad \left\langle \begin{matrix} j \\ n \end{matrix} \middle| e^{i\alpha_+ J_+} \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\frac{(i\alpha_-)^{n'-m'}}{(n'-m')!}\left[\frac{(j+n')!\,(j-m')!}{(j+m')!\,(j-n')!}\right]^{1/2} \quad \delta_{n'n} e^{in\alpha_3} \quad \frac{(i\alpha_+)^{n-m}}{(n-m)!}\left[\frac{(j+n)!\,(j-m)!}{(j+m)!\,(j-n)!}\right]^{1/2}$$

$$= \sum_{n=-j}^{+j} \left[\frac{(j-m')!\,(j-m)!}{(j+m')!\,(j+m)!}\right]^{1/2} \frac{(j+n)!}{(j-n)!} \frac{(i\alpha_+)^{n-m}}{(n-m)!} \frac{(i\alpha_-)^{n-m'}}{(n-m')!} e^{in\alpha_3} \tag{6.10}$$

This is essentially the expression we are looking for. We need now only make the replacements

$$\begin{aligned} i\alpha_+ e^{i\alpha_3/2} &\leftrightarrow iy \leftrightarrow -b^* = a_1{}^2 \\ i\alpha_- e^{i\alpha_3/2} &\leftrightarrow iy^* \leftrightarrow \quad b \;= a_2{}^1 \\ e^{i\alpha_3/2} &\leftrightarrow x \leftrightarrow \quad a \;= a_1{}^1 \end{aligned} \tag{6.11}$$

to have the representation matrix elements

$$\mathscr{D}^j_{m'm}\begin{pmatrix} x & iy \\ iy^* & x^* \end{pmatrix} = \mathscr{D}^j_{m'm}\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$$

given in terms of the original parameterization of the group element in terms of (x, y) or, equivalently, $(\alpha_+^1, \alpha_-^1, \alpha_3{}^1)$.

Since these representations have already been computed by a different technique (3.12), we will show the equivalence of the two expressions (6.10) and (3.12), repeated here for convenience:

$$\mathscr{D}^j_{m'm}\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} = [(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!]^{1/2}$$
$$\times \sum \frac{(a)^{j+m-\mu}(a^*)^{j-m'-\mu}(b)^{\mu}(-b^*)^{m'-m+\mu}}{(j+m-\mu)!\,(j-m'-\mu)!\,\mu!\,(m'-m+\mu)!} \tag{6.12}$$

Since the comparison is difficult, it will be carried out in a number of easy steps.

1. Replace the parameters appearing in (6.10) by the parameters a, b, $-b^*$ which appear in (6.12). This leads to

$$\mathscr{D}^j_{m'm}\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \to \sum_{n=-j}^{+j} \left[\frac{(j-m')!\,(j-m)!}{(j+m')!\,(j+m)!}\right]^{1/2} \frac{(j+n)!}{(j-n)!}$$
$$\times \frac{(-b^*)^{n-m}(b)^{n-m'}(a)^{m+m'}(1^2)^{j-n}}{(n-m)!\,(n-m')!} \tag{6.13}$$

2. Observe that the matrix elements given in the standard expression (6.12) are homogeneous polynomials of order $2j$ in the parameters a, a^*, b, $-b^*$. However, the matrix element appearing in (6.13)

(a) Is homogeneous in a, b, $-b^*$ of order

$$(n-m)+(n-m')+(m+m') = 2n$$

(b) Contains no powers of a^*.

Both deficiencies are easily remedied by the substitution

$$(1^2)^{j-n} = (1^2 = a^*a + b^*b)^{j-n} = \sum_{t=0}^{j-n} \binom{j-n}{t} (a^*a)^{j-n-t}(b^*b)^t \tag{6.14}$$

3. When the binomial (6.14) is expanded, the matrix elements (6.13) are expressed as a messy-looking double sum:

$$= \sum_{n=-j}^{+j} \sum_{t=0}^{j-n} \left[\frac{(j-m')!\,(j-m)!}{(j+m')!\,(j+m)!}\right]^{1/2} \frac{(j+n)!}{(j-n)!}$$
$$\times \frac{(-)^t(j-n)!}{t!\,(j-n-t)!} \frac{(a)^{j-n-t+m+m'}(a^*)^{j-n-t}(b)^{n-m'+t}(-b^*)^{n-m+t}}{(n-m')!\,(n-m)!} \tag{6.15}$$

4. We now "slice" the double summation in a different way. We set

$$n + t = \mu + m' \tag{6.16}$$

and hold μ fixed while summing over the remaining degree of freedom. With this substitution the expression (6.15) becomes

$$[(j-m')!\,(j+m')!\,(j-m)!\,(j+m)!]^{1/2}$$

$$\times \left\{\sum_{\mu} \frac{(a)^{j+m-\mu}(a^*)^{j-m'-\mu}(b)^{\mu}(-b^*)^{m'-m+\mu}}{(j+m')!\,(j+m)!\,(j-m'-\mu)!}\right\}$$

$$\times \left\{\sum_{n} \frac{(-)^{\mp\mu}(-)^{\pm(n-m')}(j+n)!}{(\mu+m'-n)!\,(n-m')!\,(n-m)!}\right\} \tag{6.17}$$

5. The expression within the last bracket can be written more simply

$$\frac{(-)^{-\mu}(j+m)!}{\mu!} \sum_{n} (-)^{n-m'} \binom{\mu}{n-m'} \binom{j+n}{j+m} \tag{6.18}$$

To evaluate this sum, we construct a generating function

$$(1+y)^d[1+x(1+y)]^a = \sum_{b} \binom{a}{b} (1+y)^{b+d} x^b$$

$$(1+y)^d[1+x+xy]^a = \sum_{b,\,c} \binom{a}{b} \binom{b+d}{c+d} y^{c+d} x^b \tag{6.19}$$

To obtain an alternating sum on the right, set $x = -1$. Then write

$$\sum_{k} (-)^a \binom{d}{k} y^{a+k} = \sum_{b,\,c} (-)^b \binom{a}{b} \binom{b+d}{c+d} y^{c+d} \tag{6.20}$$

Since the powers of y are linearly independent,

$$(-)^a \binom{d}{c+d-a} = \sum_{b} (-)^b \binom{a}{b} \binom{b+d}{c+d} \tag{6.21}$$

Applying this identity now to (6.18), we find

$$\frac{(-)^{-\mu}(j+m)!}{\mu!} \sum_{n} (-)^{n-m'} \binom{\mu}{n-m'} \binom{j+n}{j+m}$$

$$= \frac{(-)^{-\mu}(j+m)!}{\mu!} \frac{(-)^{\mu}(j+m')!}{(j+m-\mu)!\,(m'-m+\mu)!} \tag{6.18$'$}$$

On placing this expression in the last bracket appearing in (6.17), we recover (6.12).

Comment 1. The difficulty [(6.13)–(6.21)] in proving the equivalence between the expressions (6.10) and (6.12) should not mask the ease with which the matrix elements (6.10) were actually computed. This facility is a direct reflection of the particularly simple structure of the representing matrix elements in BCH parameterization 5 of Table 5.3: they are either upper triangular, lower triangular, or diagonal.

Comment 2. The construction of the matrix elements (6.10) presented in this section does not give any new information, since these matrix elements have been known for a long time. However, it does throw light on mechanisms for computing representation matrix elements for other groups in an explicit form. The difficulty is this: a Lie algebra representation can be mapped by the EXPonential mapping onto the corresponding representation of the associated Lie group. However, the matrix elements are generally hard-to-compute transcendental functions, since they are matrix elements of the transcendental function EXP. Using a BCH relation, it is possible to convert the EXP into the product of exponentials, each one involving *only* an upper triangular matrix, a lower triangular matrix, or a diagonal matrix. EXPonentials of diagonal matrices are easy to compute. EXPonentials of upper or lower triangular matrices are given by a terminating power series expansion. They are thus *algebraic* (rather than transcendental) functions and can be computed in closed form, involving only finite sums. The product of upper triangular, lower triangular, and diagonal matrices gives a finite (i.e., terminating) power series expansion for any matrix element in any finite-dimensional representation of any Lie group which is amenable to this treatment.

Lie groups that can be treated this way include:

1. All simple Lie groups.
2. All semisimple Lie groups.
3. All solvable Lie groups.
4. Semidirect product groups.
5. Contractions of a semisimple Lie group by a maximal subgroup (Chapter 10).

3. PHYSICAL APPLICATIONS. We consider now some physical situations in which the noncanonical parameterizations (Fig. 5-12) of $SU(2)$ are of value. Consider a free particle with two internal degrees of freedom. Such a system might be an electron with a spin "up" state and a spin "down" state. Or it might be an atom or molecule with only two internal energy levels of interest, the "ground" state and an "excited" state. Or it might be a nucleus with a spin up state and a spin down state. The Hamiltonian describing this system has the form

$$\mathscr{H}(x, p; t)\psi(x; t) = i\hbar \frac{\partial}{\partial t} \psi(x; t) \tag{6.22}$$

The continuous coordinates give rise to a solution of the form $e^{ikx}\psi(t)$; when the continuous coordinates have been removed, the expression (6.22) assumes the simple form

$$\mathscr{H}(t)\psi(t) = i\hbar \frac{\partial \psi(t)}{\partial t} \tag{6.22int}$$

Since the system has two internal degrees of freedom, $\psi(t)$ is a vector in C_2:

$$\psi(t) = \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = \psi_+(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_-(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.23}$$

The Hamiltonian is a time-dependent 2×2 hermitian matrix.

Since the state $\psi(t)$ is normalized to unity, the system states at times t and t_0 are related to each other by a 2×2 unitary transformation matrix:

$$\psi(t) = U(t, t_0)\psi(t_0) \tag{6.24}$$

The unitary transformation $U(t, t_0)$ obeys

$$\mathscr{H}(t)U(t, t_0)\psi(t_0) = i\hbar \frac{d}{dt} U(t, t_0)\psi(t_0)$$

$$\lim_{t \to t_0} U(t, t_0) = I_2 \tag{6.25}$$

Since the states

$$\begin{pmatrix} \psi_+(t_0) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_-(t_0) \end{pmatrix}$$

are linearly independent vectors in C_2, the unitary transformation itself obeys the equation of motion

$$\mathscr{H}(t)U(t, t_0) = i\hbar \frac{dU(t, t_0)}{dt} \tag{6.26}$$

This equation may be solved for U by an iterative procedure:

$$-\frac{i}{\hbar}\mathscr{H}(t)U(t, t_0) = \frac{U(t + \Delta t, t_0) - U(t, t_0)}{\Delta t} \tag{6.27}$$

$$U(t + \Delta t, t_0) = e^{-i/\hbar \mathscr{H}(t)\, \Delta t} U(t, t_0) \tag{6.28}$$

The determinant of $U(t + \Delta t, t_0)$ is related to the determinant of $U(t, t_0)$ by

$$\begin{aligned} \|U(t + \Delta t)\| &= \left\| \left(I_2 - \frac{i}{\hbar}\mathscr{H}(t)\, \Delta t \right) \right\| \, \|U(t)\| \\ &= \left(1 - \frac{i}{\hbar} \operatorname{tr} \mathscr{H}(t)\, \Delta t \right) \|U(t)\| \end{aligned} \tag{6.29}$$

In other words, if the 2×2 Hamiltonian matrix $\mathscr{H}(t)$ is traceless, the determinant of $U(t, t_0)$ is $+1$, for

$$\|U(t, t_0)\| = \|U(t_0, t_0)\| = \left\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\| = +1 \tag{6.30}$$

Since it is generally possible to choose $\mathscr{H}(t)$ to be traceless, it is also possible to restrict attention to unitary transformation matrices in the group $SU(2)$. The most general Hamiltonian and unitary transformation matrices have the form

$$\mathscr{H}(t) = \boldsymbol{\sigma} \cdot \mathbf{v}(t) = \begin{bmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{bmatrix} \qquad v_x \pm iv_y = v_\pm(t) \tag{6.31$\mathscr{H}$}$$

$$U(t) = \begin{bmatrix} x & iy \\ iy^* & x^* \end{bmatrix} \qquad \begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \tag{6.31U}$$

The first-order equation of motion for $U(t)$ leads directly to second-order equations of motion for the functions $x(t)$, $y(t)$:

$$\left\{\left(\hbar\frac{d}{dt}\right)^2 + v_-\left(\hbar\frac{d}{dt}\frac{1}{v_-}\right)\hbar\frac{d}{dt} + \left\{\mathbf{v}\cdot\mathbf{v} + i\hbar\frac{dv_z}{dt} + iv_-\,\hbar\left(\frac{d}{dt}\frac{1}{v_-}\right)v_z\right\}\right\}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{6.32a}$$

$$\text{Initial conditions:} \quad x(t = t_0) = 1 \qquad y(t = t_0) = 0$$

$$i\hbar\frac{dx}{dt}\bigg|_{t=t_0} = v_z \qquad i\hbar\frac{d(iy)}{dt}\bigg|_{t=t_0} = v_- \tag{6.32b}$$

The functions $x(t)$, $y(t)$ obey the same second-order equation of motion, but they have different initial conditions.

Comment 1. Equations (6.32) are not at all easy to solve unless they have constant coefficients. A necessary and sufficient two-part condition for the coefficients to be constant is

$$v_-\frac{d}{dt}\left(\frac{1}{v_-}\right) = -\frac{1}{v_-}\frac{dv_-}{dt} = \text{imaginary constant} = +i\omega$$

$$v_z(t) = \text{real constant} \tag{6.33}$$

Under these conditions (6.32) reduces to

$$\left[\left(\hbar\frac{d}{dt}\right)^2 + i\hbar\omega\left(\hbar\frac{d}{dt}\right) + \{v_x^2 + v_y^2 + v_z^2 + iv_z(+i\hbar\omega)\}\right]\begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{6.32$'$}$$

The solutions of (6.32′) have the form

$$e^{i\Omega_+ t},\ e^{i\Omega_- t}$$

$$\begin{aligned} \hbar\Omega_\pm &= -\frac{\hbar\omega}{2} \pm \sqrt{(v_z - \hbar\omega/2)^2 + (\mathbf{v}\cdot\mathbf{v} - v_z{}^2)} \\ &= -\frac{\hbar\omega}{2} \pm \frac{\hbar\Omega}{2} \end{aligned} \tag{6.34}$$

Linear combinations of these solutions satisfying the initial conditions for $x(t)$, $y(t)$ are

$$\begin{aligned} x(t) &= e^{-i\omega t/2}\left\{\cos\frac{\Omega t}{2} - i\frac{2(v_z - \hbar\omega/2)}{\hbar\Omega}\sin\frac{\Omega t}{2}\right\} \\ y(t) &= -e^{-i\omega t/2}\frac{2v_-}{\hbar\Omega}\sin\frac{\Omega t}{2} \end{aligned} \tag{6.35}$$

The solution for the unitary transformation $U(t, t_0) = U(t)$, under the conditions (6.33), is

$$\begin{aligned} U(t) &= \begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{bmatrix} \\ &\times \begin{bmatrix} \cos\frac{\Omega t}{2} - i\frac{2(v_z - \hbar\omega/2)}{\hbar\Omega}\sin\frac{\Omega t}{2} & -i\frac{2v_-}{\hbar\Omega}\sin\frac{\Omega t}{2} \\ -i\frac{2v_+}{\hbar\Omega}\sin\frac{\Omega t}{2} & \cos\frac{\Omega t}{2} + i\frac{2(v_z - \hbar\omega/2)}{\hbar\Omega}\sin\frac{\Omega t}{2} \end{bmatrix} \end{aligned} \tag{6.36}$$

Comment 2. It is quite apparent that the exact solution of (6.32), given formally (cf. Chapter 4, Section VI.4) by

$$U(t, t_0) = \tau\ \mathrm{EXP} - \frac{i}{\hbar}\int_{t_0}^{t} \mathscr{H}(\tau)\, d\tau \tag{6.37 formal}$$

can differ quite substantially from the transformation

$$\mathrm{EXP} - \frac{i}{\hbar}\int_{t_0}^{t} \mathscr{H}(\tau)\, d\tau \tag{6.37′}$$

The difference arises from the noncommutativity of $\mathscr{H}(t)$ with itself at different times

$$[\mathscr{H}(t_1), \mathscr{H}(t_2)] \neq 0 \tag{6.38}$$

The difference between (6.37) and (6.37′) for the example developed in Comment 1 is easily verified by direct computation:

$$\tau \text{ EXP } -\frac{i}{\hbar}\int_{t_0}^{t} \mathscr{H}(\tau)\, d\tau \neq \text{EXP } -\frac{i}{\hbar}\int_{t_0}^{t} \mathscr{H}(\tau)\, d\tau \tag{6.38$'$}$$

It is possible to establish laboratory[8] conditions that are essentially equivalent to the constant coefficient assumptions (6.33) of Comment 1. The 2×2 unitary transformation matrix (6.36) therefore contains a wealth of information. A large class of such experiments involves the interaction of a magnetic moment $\boldsymbol{\mu}$ with an external magnetic field $\mathbf{B}$. Resonance effects can be studied[9] by applying a small circularly polarized radio-frequency "tickling field" in a plane perpendicular to the magnetic field direction. The interaction between the magnetic moment and the magnetic field is given by

$$\mathscr{H} = -\boldsymbol{\mu} \cdot \mathbf{B} \tag{6.39}$$

The magnetic moment is proportional to the system's spin

$$\boldsymbol{\mu} = \gamma \hbar \mathbf{J} \tag{6.40}$$

For a spin half-system, $\mathbf{J} = \boldsymbol{\sigma}/2$. The gyromagnetic ratio is given by

$$\gamma = \frac{ge}{2mc} \tag{6.41}$$

where g is the anomalous magnetic moment of the spin. For a magnetic field applied along the z direction, we have

$$\mathscr{H}(t) = \begin{bmatrix} -\dfrac{ge\hbar}{4mc} B_z & 0 \\ 0 & +\dfrac{ge\hbar}{4mc} B_z \end{bmatrix} = \begin{bmatrix} \dfrac{\Delta E}{2} & 0 \\ 0 & -\dfrac{\Delta E}{2} \end{bmatrix} \tag{6.42}$$

A radio-frequency electromagnetic field of frequency $\omega/2\pi$ circularly polarized about an axis in the xy plane can be used to study resonance effects. Such a field gives an additional contribution to the Hamiltonian of the form

$$\mathscr{H}_{\text{rf}}(t) = \begin{bmatrix} 0 & a_- e^{-i\omega t} \\ a_+ e^{+i\omega t} & 0 \end{bmatrix} \qquad a_- = a_+^* \tag{6.43}$$

The Hamiltonian satisfies the conditions stated in Comment 1:

$$\mathscr{H}_{\text{tot}}(t) = \begin{bmatrix} \dfrac{\Delta E}{2} & a_- e^{-i\omega t} \\ a_+ e^{+i\omega t} & -\dfrac{\Delta E}{2} \end{bmatrix} \tag{6.44}$$

The unitary transformation matrix therefore has the structure (6.36), with

$$a_{\pm} e^{\pm i\omega t} = v_{\pm}(t), \qquad v_z = \frac{\Delta E}{2}$$

We give now a number of examples of the utility of (6.36).

Example 1. If the system starts in the ground state $\binom{0}{1}$ at time $t_0 = 0$, the probability of a transition to the excited state $\binom{1}{0}$ at time t is

$$\left| \left\langle \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right| U(t) \left| \begin{matrix} \frac{1}{2} \\ -\frac{1}{2} \end{matrix} \right\rangle \right|^2 = \left| -i \frac{2v_-}{\hbar\Omega} \sin \frac{\Omega t}{2} \right|^2$$

$$= \frac{4a_+ a_-}{(\Delta E - \hbar\omega)^2 + 4a_+ a_-} \sin^2 \frac{\Omega t}{2} \tag{6.45}$$

In particular, at the sample resonance frequency ω_L

$$\Delta E - \hbar\omega = 0 \qquad \omega = \omega_L \qquad \omega_L = \frac{\Delta E}{\hbar} \tag{6.46}$$

the probability of transition to the excited state is $+1$ after a "π-pulse"

$$\Omega t = \pi \qquad \text{or} \qquad t = \frac{\pi}{2} \frac{\hbar}{|a_{\pm}|} \tag{6.47}$$

We observe that the length of a π pulse at the resonance frequency ω_L depends on the amplitude of the driving field $|a_{\pm}|$.

Example 2. Classically speaking, the transition from the ground state to the excited state corresponds to a reorientation of the spin vector from the south pole to the north pole (Fig. 5.13). If this occurs during a π pulse at resonance, then during a $\pi/2$ pulse the spin vector should stop at the equator. To verify this, we can compute the direction cosines γ_i of the spin vector in the state $|\psi\rangle$. These are proportional to the expectation values

$$\gamma_i \simeq \langle \psi | \sigma_i | \psi \rangle \tag{6.48}$$

For $|\psi\rangle = \binom{a}{b}$ the expectation values are

$$\begin{aligned}
\gamma_x &= \begin{pmatrix} a \\ b \end{pmatrix}^{\dagger} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (a^*, b^*) \begin{pmatrix} b \\ a \end{pmatrix} = a^*b + b^*a = 2\text{Re}\, a^*b \\
\gamma_y &= \begin{pmatrix} a \\ b \end{pmatrix}^{\dagger} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (a^*, b^*) \begin{pmatrix} -ib \\ ia \end{pmatrix} = i(ab^* - a^*b) = 2\text{Im}\, a^*b \\
\gamma_z &= \begin{pmatrix} a \\ b \end{pmatrix}^{\dagger} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (a^*, b^*) \begin{pmatrix} a \\ -b \end{pmatrix} = a^*a - b^*b
\end{aligned} \tag{6.49}$$

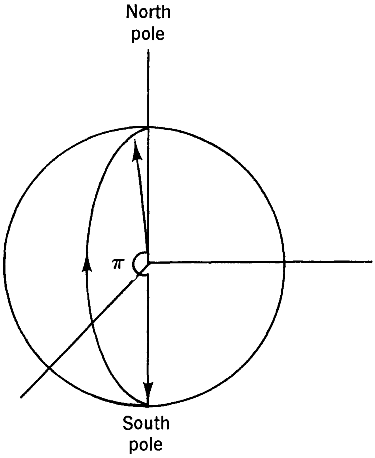


FIG. 5.13 THE TRANSITION FROM THE GROUND TO THE EXCITED STATE CORRESPONDS TO A REORIENTATION OF THE SPIN VECTOR FROM THE SOUTH POLE TO THE NORTH POLE.

The unitary transformation for a $\pi/2$ pulse at resonance is

$$U(t) = \begin{bmatrix} \text{EXP}\,\dfrac{-i\omega_L}{2}\dfrac{\pi\hbar}{4|a_\pm|} & 0 \\ 0 & \text{EXP}\,\dfrac{+i\omega_L}{2}\dfrac{\pi\hbar}{4|a_\pm|} \end{bmatrix} \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -i\dfrac{v_-}{|v_-|}\dfrac{1}{\sqrt{2}} \\ -i\dfrac{v_+}{|v_+|}\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix} \tag{6.50}$$

We will neglect the effect of the first matrix for the present. This just superposes an overall rotation by $\omega_L \pi / 2\Omega$ about the z-axis, as in the following example. The direction cosines for a spin evolving from the ground state under a $\pi/2$ pulse are

$$\begin{aligned} \gamma_x &= \text{Im } v_-/|v_-| \\ \gamma_y &= \text{Re } v_-/|v_-| \\ \gamma_z &= 0 \end{aligned} \tag{6.51}$$

The spin does "point" at the equator (Fig. 5.14).

Example 3. When the radio-frequency field is turned off ($a_\pm = 0$), the spin system will precess about the z-axis at the Larmor frequency (Fig. 5.15)

$$\omega_L = \frac{geB_z}{2mc}$$

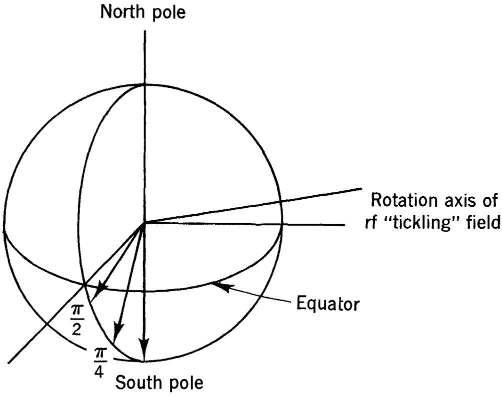


FIG. 5.14 IF THE SPIN VECTOR MOVES FROM THE SOUTH POLE TO THE NORTH POLE DURING A TIME INTERVAL Δt CALLED A π PULSE, DURING HALF THAT TIME INTERVAL $\Delta t/2$, CALLED A $\pi/2$ PULSE, THE SPIN VECTOR MOVES HALF AS FAR. IT STOPS AT THE EQUATOR.

If at time $t = 0$, $\psi = \binom{a}{b}$, then at any later time

$$\psi(t) = \begin{pmatrix} a & e^{-i\omega_L t/2} \\ b & e^{+i\omega_L t/2} \end{pmatrix}$$

The direction cosines of the spin are

$$\begin{aligned} \gamma_x(t) &= \text{Re } (ae^{-i\omega_L t/2})^*(be^{+i\omega_L t/2}) \\ &= \text{Re } (a^*b) \cos \omega_L t - \text{Im } (a^*b) \sin \omega_L t \\ &= \gamma_x(0) \cos \omega_L t - \gamma_y(0) \sin \omega_L t \end{aligned} \tag{6.52x}$$

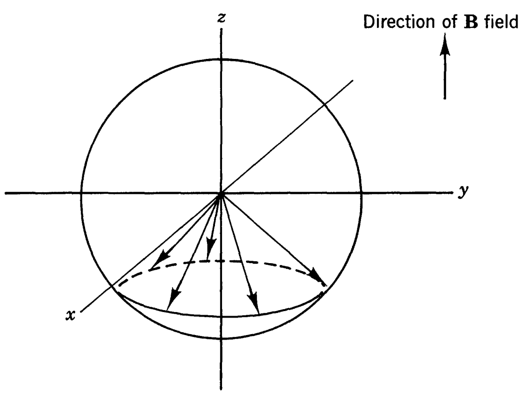


FIG. 5.15 IN THE ABSENCE OF A RADIO-FREQUENCY FIELD, THE SPIN WILL PRECESS ABOUT THE DIRECTION OF THE MAGNETIC FIELD AT THE LARMOR FREQUENCY $\omega_L = geB/2mc$.

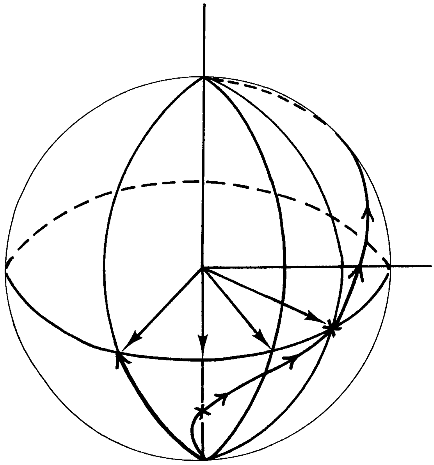


FIG. 5.16 DURING THE APPLICATION OF THE RADIO-FREQUENCY FIELD, THE SPIN MOVES UNDER THE COMBINATION OF TWO INFLUENCES: (1) THE RF FIELD ROTATES THE SPIN ABOUT THE FIELD POLARIZATION AXIS. (2) THE SPIN CONTINUES TO PRECESS ABOUT THE MAGNETIC FIELD DIRECTION AT THE LARMOR FREQUENCY. THE TIME SCALE OF (1) IS GOVERNED BY (INVERSELY PROPORTIONAL TO) THE AMPLITUDE OF THE RF FIELD. THE TIME SCALE OF (2) IS INVERSELY PROPORTIONAL TO THE EXTERNAL MAGNETIC FIELD STRENGTH.

Similarly,

$$\gamma_y(t) = \gamma_x(0) \sin \omega_L t + \gamma_y(0) \cos \omega_L t \tag{6.52y}$$

$$\gamma_z(t) = +\gamma_z(0) \tag{6.52z}$$

The meaning of the structure of the unitary transformations (6.36) and (6.50) is now clear. In each instance, the right-hand matrix describes the evolution of the system under the influence of the radio-frequency "tickling field" in the presence of the external magnetic field **B**. The left-hand matrix describes the free precession of the spin around the direction of the external field. These physical processes are illustrated in Fig. 5.16.

Example 4. We now investigate what happens to a state $\binom{a}{b}$ during a π pulse. To avoid complications we choose $v_\pm$ imaginary. The unitary transformation assumes the simple form

$$U\left(t = \frac{\pi}{2}\frac{\hbar}{|v_\pm|}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{6.53}$$

A simple calculation gives

$$\begin{aligned} \gamma_x &\to -\gamma_x \\ \gamma_y &\to +\gamma_y \\ \gamma_x &\to -\gamma_z \end{aligned} \tag{6.54}$$

In short, the π pulse rotates any spin through the angle π about the y-axis (Fig. 5.17).

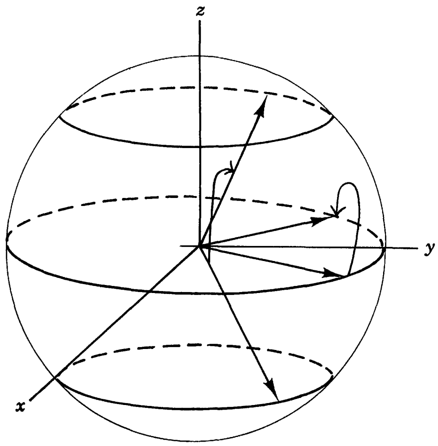


FIG. 5.17 A π PULSE ROTATES ANY SPIN THROUGH π RADIANS ABOUT THE POLARIZATION AXIS OF THE RF FIELD. HERE THE RF FIELD IS CIRCULARLY POLARIZED IN THE xz PLANE; THE y-AXIS IS THE POLARIZATION AXIS OF THE RF FIELD. MATHEMATICALLY, THIS IS EQUIVALENT TO CHOOSING $a_\pm$ IMAGINARY.

Example 5. The calculations carried out in Examples 1 to 4 are valid not only for spin one-half systems, but for systems of arbitrary integral or half-integral spin j. The only change is to replace the Pauli spin matrices by the appropriate J matrices everywhere

$$\mathbf{J}(\tfrac{1}{2}) = \tfrac{1}{2}\boldsymbol{\sigma} \to \mathbf{J}(j) \tag{6.55}$$

The Hamiltonian becomes

$$\begin{aligned}
\mathscr{H}(j = \tfrac{1}{2}) &= \Delta E J_z(\tfrac{1}{2}) + a_- e^{-i\omega t} J_+(\tfrac{1}{2}) + a_+ e^{+i\omega t} J_-(\tfrac{1}{2}) \\
&\qquad\qquad\Big\downarrow \\
\mathscr{H}(j; t) &= \Delta E J_z(j) + a_- e^{-i\omega t} J_+(j) + a_+ e^{+i\omega t} J_-(j)
\end{aligned} \tag{6.56}$$

The unitary evolution matrix $U(t, t_0)$ is a time-dependent $(2j + 1) \times (2j + 1)$ matrix which is easily obtained from the unitary 2×2 matrix

$$\begin{bmatrix} x & iy \\ iy^* & x^* \end{bmatrix} \xrightarrow{\frac{1}{2} \to j} \mathscr{D}^j_{mm'}\begin{bmatrix} x & iy \\ iy^* & x^* \end{bmatrix} \tag{6.57}$$

If a spin j system evolves from the state m', the transition probability to state m after time t is simply determined from (6.12):

$$\begin{aligned}
P_{m \leftarrow m'} &= [(j + m)!\,(j - m)!\,(j + m')!\,(j - m')!] \\
&\quad \times \left| \sum_\mu \frac{(x)^{j+m'-\mu}(x^*)^{j-m-\mu}(iy^*)^\mu(iy)^{m-m'+\mu}}{(j + m' - \mu)!\,(j - m - \mu)!\,(\mu)!\,(m - m' + \mu)!} \right|^2
\end{aligned} \tag{6.58}$$

The equations of motion for the spin j-system are identical to the equations of motion for the spin half-system.

Example 6. Evolution from the Ground State. The evolution of a system from the ground state is often of particular concern. Then, no matter what the field dependence is, the unitary evolution matrix $U(t)$ is well defined, although in some cases it may be difficult to compute. For a two-level system

$$U(t) = \begin{bmatrix} x & iy \\ iy^* & x^* \end{bmatrix} = \text{EXP}\left[i\hat{\mathbf{n}}(t)\cdot\boldsymbol{\sigma}\frac{\theta(t)}{2}\right] \tag{6.59}$$

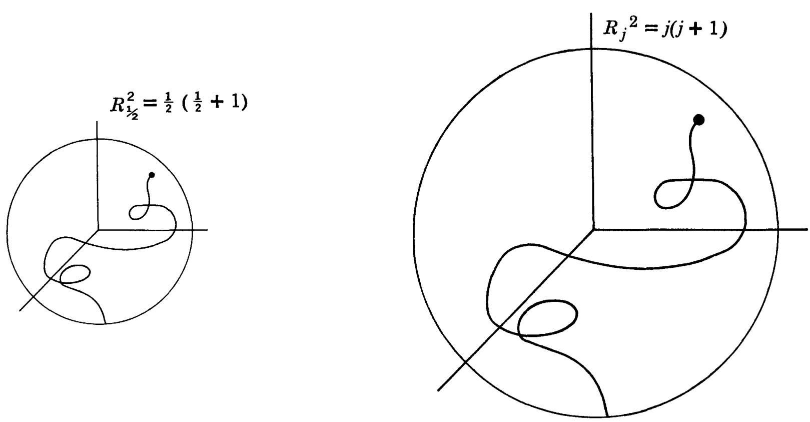
the unit vector $\hat{\mathbf{n}}(t)$ and the scalar $\theta(t)$ are well defined. This unitary transformation matrix can be reparameterized using a BCH relation:

$$e^{i(\alpha_+{}^1 J_+ + \alpha_3{}^1 J_3 + \alpha_-{}^1 J_-)} \leftrightarrow \text{EXP}\ i(\alpha_+^2 J_+ + \alpha_-^2 J_-)e^{i\alpha_3{}^2 J_3} \tag{6.60}$$

If the system evolves from the ground state, then that ground state can always be redefined by absorbing the unimodular function $e^{-ij\alpha_3}$:

$$\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle \to \left|\begin{matrix} j \\ -j \end{matrix}\right\rangle' = \left|\begin{matrix} j \\ -j \end{matrix}\right\rangle e^{-ij\alpha_3} \tag{6.61}$$

Under this redefinition, the evolution of the state is completely determined by the coset representative (Fig. 5.18) EXP $i(\alpha_+ J_+ + \alpha_- J_-)$. There is thus a 1-1 correspondence between the states evolving from the ground state under an *arbitrary* Hamiltonian (not subject to the restrictions stated in



(*a*) (*b*)

FIG. 5.18 THE EQUATIONS OF MOTION FOR A SPIN j SYSTEM ARE IDENTICAL WITH THE EQUATIONS OF MOTION FOR A SPIN HALF-SYSTEM. THE TRAJECTORY OF A SPIN j PARTICLE CAN BE OBTAINED BY PROJECTING THE TRAJECTORY OF A SPIN HALF-PARTICLE FROM THE SURFACE OF A SPHERE OF RADIUS $R_{1/2}^2 = \frac{1}{2}(\frac{1}{2}+1)$, AS IN ($a$), ONTO THE SURFACE OF A CONCENTRIC SPHERE OF RADIUS $R_j{}^2 = j(j+1)$, AS IN (b).

Comment 1) and the points on the surface of a sphere, provided only that the fields described by $\mathbf{B}$, $v_{\pm}(t)$ are classical.

To make this point more graphic we rewrite the parameters $\alpha_{\pm}$ in the BCH parameterization 2 [see (6.4^2)] in a geometric way. The south polar state is rotated to the point $\theta\phi$ on the surface of the sphere of radius $j(j+1)$ by rotating through an angle θ about an axis

$$\hat{\mathbf{n}} = (-\hat{\mathbf{i}}_x \sin\phi, \hat{\mathbf{i}}_y \cos\phi)$$

Then

$$\hat{\mathbf{n}} \cdot \mathbf{J}\theta = \alpha_+^2 J_+ + \alpha_-^2 J_-$$

$$\alpha_+^2 = -ie^{-i\phi}\frac{\theta}{2}$$

$$\alpha_-^2 = +ie^{+i\phi}\frac{\theta}{2} = \alpha_+^{2*} \tag{6.62}$$

The state $|\theta\phi\rangle$, which has evolved from the ground state after a time t and which exists in 1-1 correspondence with points on the sphere surface, is defined by

$$\left|\begin{matrix} j \\ \theta\phi \end{matrix}\right\rangle = \text{EXP } i(\alpha_+^2 J_+ + \alpha_-^2 J_-)\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle \tag{6.63}$$

The computation of the matrix elements in (6.63) is facilitated by the application of BCH formula (4, from Table 5.3):

$$\text{EXP } i(\alpha_+^2 J_+ + \alpha_-^2 J_-)\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle = e^{i\alpha_+^4 J_+} e^{i\alpha_3^4 J_3} e^{i\alpha_-^4 J_-}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle \tag{6.64}$$

Since J_- annihilates the state $|{}^{\,j}_{-j}\rangle$, we have

$$e^{i\alpha_-^4 J_-}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle = \left|\begin{matrix} j \\ -j \end{matrix}\right\rangle \tag{6.64$_-$}$$

The effect of the diagonal operator is also simply computed:

$$e^{i\alpha_3^4 J_3}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle = \left|\begin{matrix} j \\ -j \end{matrix}\right\rangle e^{-i\alpha_3^4 j} \tag{6.64$_z$}$$

Therefore, the state $|{}^{\,j}_{\theta\phi}\rangle$ is given (Figs. 5.19 and 5.20) by

$$\left|\begin{matrix} j \\ \theta\phi \end{matrix}\right\rangle = e^{i\alpha_+^4 J_+}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle e^{-i\alpha_3^4 j} \tag{6.64$_+$}$$

$$= \text{EXP } e^{-i\phi}\tan\frac{\theta}{2} J_+ \left|\begin{matrix} j \\ -j \end{matrix}\right\rangle \left(\cos\frac{\theta}{2}\right)^{2j} \tag{6.65}$$

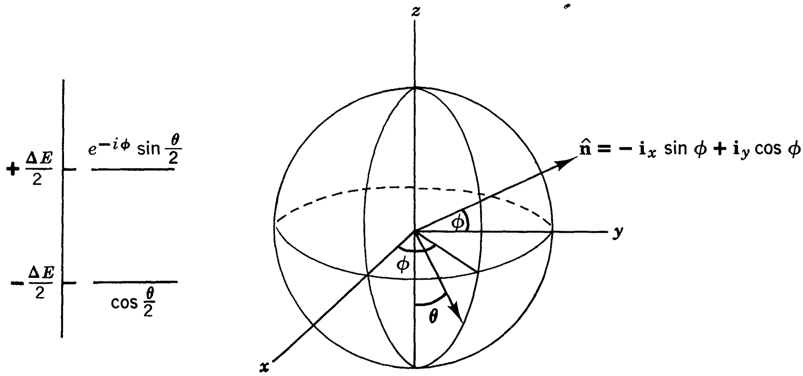


FIG. 5.19 A SPIN HALF-PARTICLE EVOLVING FROM THE GROUND STATE UNDER AN ARBITRARY HAMILTONIAN CAN BE DESCRIBED BY A POINT ON THE SURFACE OF A BLOCH SPHERE. THE STATE IS CONVENIENTLY DESCRIBED BY THE SPHERICAL COORDINATES θ, ϕ WHERE θ IS MEASURED FROM THE SOUTH POLE.

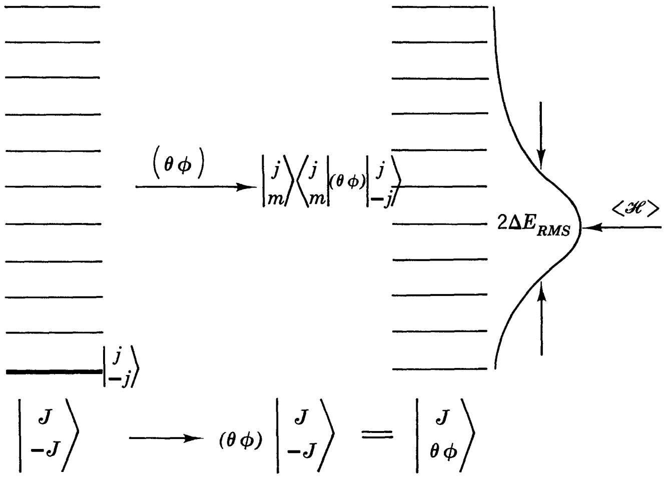


FIG. 5.20 A SPIN j SYSTEM EVOLVING FROM THE GROUND STATE $\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle$ IS ALSO UNIQUELY DESCRIBED BY A POINT ON THE SURFACE OF A SPHERE. IT IS ALSO DESCRIBED AS A SUPERPOSITION OF J_3 EIGENSTATES $\left|\begin{matrix} j \\ m \end{matrix}\right\rangle$ WITH COEFFICIENTS

$$\left\langle\begin{matrix} j \\ m \end{matrix}\right|(\theta\phi)\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle = \mathscr{D}^j_{m,\,-j}[\exp\,(i\hat{\mathbf{n}}(t)\cdot\boldsymbol{\sigma}\theta(t)/2)]$$

$$= \binom{2j}{j \pm m}^{1/2}\left(\cos\frac{\theta}{2}\right)^{j-m}\left(e^{-i\phi}\sin\frac{\theta}{2}\right)^{j+m}$$

The inner product between two such states is easily computed.[10] The states are nonorthogonal and overcomplete:

$$\left\langle \begin{matrix} j \\ \theta'\phi' \end{matrix} \middle| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle = \left[\cos\frac{\theta'}{2}\cos\frac{\theta}{2} + e^{-i(\phi-\phi')}\sin\frac{\theta'}{2}\sin\frac{\theta}{2}\right]^{2j} \tag{6.66}$$

Example 7, Bloch States. If each member of a system of N identical noninteracting spin one-half particles sees the same field, all will evolve in the same way. The total Hamiltonian is

$$H_{\text{tot}} = \sum_{i=1}^{N} h(i;t)$$

$$h(i;t) = \Delta E(i;t)J_3 + a_-(i;t)J_+ + a_+(i;t)J_- \tag{6.67}$$

Since each subsystem experiences the same external field, we can write

$$\Delta E(i;t) = \Delta E(t)$$

$$a_\pm(i;t) = a_\pm(t) \tag{6.68}$$

$$\begin{aligned} H_{\text{tot}}(t) = \Delta E(t)\sum J_3(\tfrac{1}{2}) + a_-(t)\sum J_+(\tfrac{1}{2}) \\ + a_+(t)\sum J_-(\tfrac{1}{2}) \end{aligned} \tag{6.69}$$

The sums of the single particle angular momentum matrices may be replaced by the total angular momentum matrices $\mathbf{J}(j = N/2)$:

$$H_{\text{tot}}(t) = \Delta E(t)J_3\left(\frac{N}{2}\right) + a_-(t)J_+\left(\frac{N}{2}\right) + a_+(t)J_-\left(\frac{N}{2}\right) \tag{6.70}$$

The system experiences exactly the same Hamiltonian as a single particle of total spin $j = N/2$. It therefore evolves exactly as computed in Example 6 (Fig. 5.21):

$$\left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle = \left| \begin{matrix} j \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} j \\ m \end{matrix} \right| \text{EXP } i(\alpha_+^2 J_+ + \alpha_-^2 J_-) \left| \begin{matrix} j \\ -j \end{matrix} \right\rangle \tag{6.71}$$

These states, describing the evolution of an ensemble of particles interacting under identical external influences, are called Bloch (coherent) states.

Since Bloch states describe systems containing many particles, it is useful to be able to compute correlation functions. These are expectation values of ordered products of operators, namely:

$$\left\langle \begin{matrix} j \\ \theta\phi \end{matrix} \right| J_+ J_+ J_- J_+ J_3 J_- J_- J_+ \left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle$$

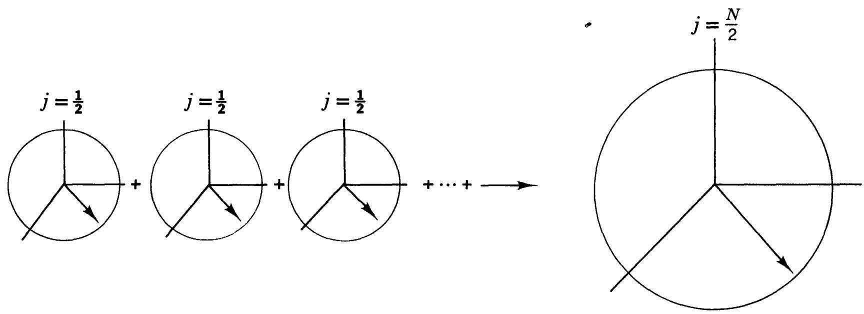


FIG. 5.21 IF ALL THE PARTICLES IN A SYSTEM OF N IDENTICAL PARTICLES EVOLVE FROM THE GROUND STATE UNDER IDENTICAL EXTERNAL FIELD CONDITIONS, THEY WILL ALL EVOLVE IN THE SAME WAY. THE N SPIN HALF-SUBSYSTEMS CAN BE REPLACED BY A SINGLE SYSTEM OF TOTAL SPIN $j = N/2$.

Three kinds of ordered products are of particular interest:[10]

Normal ordering: $\left\langle \begin{matrix} j \\ \theta\phi \end{matrix} \right| (J_+)^k (J_3)^m (J_-)^n \left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle$

Antinormal order: $\left\langle \begin{matrix} j \\ \theta\phi \end{matrix} \right| (J_-)^n (J_3)^m (J_+)^k \left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle$

Symmetrized ordering: $\left\langle \begin{matrix} j \\ \theta\phi \end{matrix} \right| P\{(J_+)^k (J_3)^m (J_-)^n\} \left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle$

In the last operator product, the sum extends over all possible distinct permutations of the operators J_+, J_-, and J_3.

The most convenient way for storing information about these correlation functions is through a generating function. Thus

$$\left\langle \begin{matrix} j \\ \theta\phi \end{matrix} \right| (J_+)^k (J_3)^m (J_-)^n \left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle = \left(\frac{\partial}{\partial i\alpha_+}\right)^k \left(\frac{\partial}{\partial i\alpha_3}\right)^m \left(\frac{\partial}{\partial i\alpha_-}\right)^n \times \left\langle \begin{matrix} j \\ \theta\phi \end{matrix} \right| e^{i\alpha_+ J_+} e^{i\alpha_3 J_3} e^{i\alpha_- J_-} \left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle \tag{6.72n}$$

The generating function in this case is

$$f^4(\theta\phi, \alpha_+^4 \alpha_-^4 \alpha_3{}^4) = \left\langle \begin{matrix} j \\ \theta\phi \end{matrix} \right| e^{i\alpha_+{}^4 J_+} e^{i\alpha_3{}^4 J_3} e^{i\alpha_-{}^4 J_-} \left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle \tag{6.73}$$

Generating functions for the antinormally ordered products and the fully symmetrized products are defined similarly. They are, of course, all related by BCH formulas:

$$
\begin{array}{ccc}
 & f^1(\theta\phi, \alpha_+^1\, \alpha_-^1\, \alpha_3{}^1) & \\
\swarrow\!\!\swarrow & & \searrow\!\!\searrow \\
f^4(\theta\phi, \alpha_+^4\, \alpha_-^4\, \alpha_3{}^4) & = & f^5(\theta\phi, \alpha_+^5\, \alpha_-^5\, \alpha_3{}^5)
\end{array}
$$

The generating functions can be computed explicitly with the help of the BCH relations. To show how, we compute the antinormal order generating function:

$$
\left\langle \begin{matrix} j \\ \theta\phi \end{matrix} \right| e^{i\alpha_- J_-} e^{i\alpha_3 J_3} e^{i\alpha_+ J_+} \left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle
$$

$$
= \left\{ \left(\cos\frac{\theta}{2}\right)^{2j} \operatorname{EXP} e^{-i\phi} \tan\frac{\theta}{2} J_+ \left| \begin{matrix} j \\ -j \end{matrix} \right\rangle \right\}^{\dagger} e^{i\alpha_- J_-} e^{i\alpha_3 J_3} e^{i\alpha_+ J_+}
$$

$$
\left\{ \operatorname{EXP} e^{-i\phi} \tan\frac{\theta}{2} J_+ \left| \begin{matrix} j \\ -j \end{matrix} \right\rangle \left(\cos\frac{\theta}{2}\right)^{2j} \right\} \tag{6.74}
$$

Since $e^{aJ_+} e^{bJ_+} = e^{(a+b)J_+}$, (6.74) simplifies considerably:

$$
\left(\cos^2\frac{\theta}{2}\right)^{2j} \left\langle \begin{matrix} j \\ -j \end{matrix} \right| \operatorname{EXP}\left(i\alpha_- + e^{i\phi}\tan\frac{\theta}{2}\right) J_- \, e^{i\alpha_3 J_3} \operatorname{EXP}\left(i\alpha_+ + e^{-i\phi}\tan\frac{\theta}{2}\right) J_+ \left| \begin{matrix} j \\ -j \end{matrix} \right\rangle \tag{6.75}
$$

The expression whose $-j$, $-j$ matrix element we seek can be rewritten with the order of the exponentials reversed:

$$
\left(\cos^2\frac{\theta}{2}\right)^{2j} \left\langle \begin{matrix} j \\ -j \end{matrix} \right| e^{i\alpha_+' J_+} e^{i\alpha_3' J_3} e^{i\alpha_-' J_-} \left| \begin{matrix} j \\ -j \end{matrix} \right\rangle \tag{6.76}
$$

The coefficients α_i' are related to α_i as shown in Fig. 5.12. However, the action of $e^{i\alpha_-' J_-}$ on $|{}^{\,j}_{-j}\rangle$ is trivial [cf. (6.64$_-$)], as is the action of $e^{i\alpha_+' J_+}$ on $\langle{}^{\,j}_{-j}|$. We are simply left with

$$
\left(\cos^2\frac{\theta}{2}\right)^{2j} \left\langle \begin{matrix} j \\ -j \end{matrix} \right| e^{i\alpha_3' J_3} \left| \begin{matrix} j \\ -j \end{matrix} \right\rangle \tag{6.77}
$$

As a result the antinormal order generating function that we are seeking is

$$
f^5(\theta\phi, \alpha_+^5\, \alpha_-^5\, \alpha_3{}^5) = \left(\cos^2\frac{\theta}{2}\right)^{2j} \Big[e^{-i\alpha_3{}^5/2} + e^{+i\alpha_3{}^5/2}
$$

$$
\times \left(i\alpha_-^5 + e^{i\phi}\tan\frac{\theta}{2}\right)\left(i\alpha_+^5 + e^{-i\phi}\tan\frac{\theta}{2}\right)\Big]^{2j} \tag{6.78}
$$

Since this is a polynomial of order $2j$ in $i\alpha_+^5$ and $i\alpha_-^5$, all expectation values involving more than $2j$ factors of either J_+ or J_- must automatically vanish.

It is useful at this point to observe that the generating function can be written in a much more convenient and easily remembered way, as follows. We let X be the 2×2 matrix (6.1′) and let M be the hermitian matrix

$$M = \begin{bmatrix} \sin^2 \dfrac{\theta}{2} & e^{-i\phi} \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} \\ e^{i\phi} \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} & \cos^2 \dfrac{\theta}{2} \end{bmatrix} \tag{6.79}$$

The matrix X can be written in terms of the parameters $(\alpha_+, \alpha_-, \alpha_3)$ in any of the noncanonical parameterizations. In parameterization 5, used for the computation of the generating function, we find

$$f^5(\theta\phi, \alpha_+^5 \alpha_-^5 \alpha_3^5) = (\mathrm{tr}\, X^5 M)^{2j} \tag{6.80^5}$$

Comment 1. The generating function for $2j = N$ particles is just the Nth power of the single particle generating function.

Comment 2. Since the generating function is written in terms of the group element X, it automatically describes *all possible* generating functions that can be related to each other, as in Fig. 5.12. Thus the normal order and the fully symmetrized order generating functions can be constructed by sight:

$$\begin{aligned} &f^5(\theta\phi, \alpha_+ \alpha_- \alpha_3) \\ &\quad \| \\ &(\mathrm{tr}\, X^5 M)^{2j} \\ &\begin{bmatrix} e^{i\alpha_3/2} \sin^2 \dfrac{\theta}{2} + i\alpha_- e^{i\alpha_3/2} e^{-i\phi} \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} \\ + i\alpha_+ e^{i\alpha_3/2} e^{i\phi} \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} + (e^{-i\alpha_3/2} + i\alpha_+ i\alpha_- e^{+i\alpha_3/2}) \cos^2 \dfrac{\theta}{2} \end{bmatrix}^{2j} \end{aligned} \tag{6.81^5}$$

$$\begin{aligned} &f^4(\theta\phi, \alpha_+ \alpha_- \alpha_3) \\ &\quad \| \\ &(\mathrm{tr}\, X^4 M)^{2j} \\ &\begin{bmatrix} (e^{i\alpha_3/2} + i\alpha_+ i\alpha_- e^{-i\alpha_3/2}) \sin^2 \dfrac{\theta}{2} + i\alpha_- e^{-i\alpha_3/2} e^{-i\phi} \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} \\ + i\alpha_+ e^{-i\alpha_3/2} e^{i\phi} \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} + e^{-i\alpha_3/2} \cos^2 \dfrac{\theta}{2} \end{bmatrix}^{2j} \end{aligned} \tag{6.81^4}$$

$$f^1(\theta\phi, \alpha_+ \alpha_- \alpha_3)$$
$$\|$$
$$(\mathrm{tr}\, X^1 M)^{2j}$$
$$\left[\begin{array}{l}\left(\cos|\alpha| + i\alpha_3 \dfrac{\sin|\alpha|}{2|\alpha|}\right)\sin^2\dfrac{\theta}{2} + i\alpha_- \dfrac{\sin|\alpha|}{|\alpha|} e^{-i\phi}\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} \\ + i\alpha_+ \dfrac{\sin|\alpha|}{|\alpha|} e^{i\phi}\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} + \left(\cos|\alpha| - i\alpha_3\dfrac{\sin|\alpha|}{2|\alpha|}\right)\cos^2\dfrac{\theta}{2}\end{array}\right]^{2j} \tag{6.81^1}$$

For purposes of clarity the superscripts (5, 4, and 1, respectively) have been omitted from the Lie algebra parameters α_+, α_-, α_3 appearing in (6.81).

Comment 3. Note that

$$\begin{gathered} \mathrm{tr}\, XM \neq 0 \\ \det \|XM\| = \|X\|\,\|M\| = (+1)(0) = 0 \end{gathered} \tag{6.82}$$

So in this case, at least, the trace function contains more information than the determinant function.

These generating functions can be used to compute the mean energy in a coherent state

$$\langle E \rangle = \hbar\omega_L \langle J_3 \rangle \tag{6.83}$$

$$\langle J_3 \rangle = \frac{\partial}{\partial i\alpha_3} f(\theta\phi, \alpha_+ \alpha_- \alpha_3) = -j\cos\theta \tag{6.84}$$

Similarly, the root mean square of the energy is given by

$$\begin{aligned} \frac{\Delta E_{RMS}^2}{(\hbar\omega_L)^2} &= \langle (J_3 - \langle J_3 \rangle)^2 \rangle \\ &= \left(\frac{\partial}{\partial i\alpha_3}\right)^2 f - \left(\frac{\partial f}{\partial i\alpha_3}\right)^2 \\ &= \tfrac{1}{2} j \sin^2\theta \\ \Delta E_{RMS} &= (j/2)^{1/2} \hbar\omega_L \sin\theta \end{aligned} \tag{6.85}$$

The term ΔE_{RMS} used in (6.85) describes the square root of the average square deviation of the energy E from its mean value $\langle E \rangle$. The term ΔE used elsewhere throughout this section describes the energy level splitting between the ground state and the excited state in a two-level system. The terms ΔE and ΔE_{RMS} describe different physical quantities, and should not be confused.

Example 8, Spin Echo Experiment.[11] Finally, we relax the assumption made in Example 7 that each spin subsystem evolves under the same Hamiltonian. It quite often happens, for example, that there is a slight inhomogeneity in the magnetic field establishing the energy level splitting ΔE in a sample of spins. Then the splitting is a function of the spin position within the magnet pole pieces

$$\Delta E(i) = \Delta E[\mathbf{B}(x)] = \Delta E(x) \tag{6.86}$$

Under these circumstances it is no longer justifiable to proceed from (6.67) to (6.70) in Example 7. Instead, we must treat each spin subsystem separately.

In a good laboratory magnet the field inhomogeneity is so small that each subsystem has approximately the same resonance frequency. We now consider an experiment in which, in each time interval, the Hamiltonian satisfies the conditions established in Comment 1, thus the results of Examples 2 to 4 can be applied directly (See Fig. 5.22).

Step 1 (Example 2). A $\pi/2$ pulse is applied to the system of spins in thermal equilibrium. If the temperature is sufficiently low ($kT \ll \Delta E$), most of the spins will be in the ground states. After the $\pi/2$ pulse, these spins will be pointing along a fixed line of longitude and toward the equator. There will be a slight dispersion rising to the equator because the spins will have a slight dispersion in resonance frequency, or duration of $\pi/2$ pulse, because of the magnetic field inhomogeneity. When $a_{\pm}$ is imaginary, the spins will scatter about the intersection of the x-axis with the Bloch sphere.

Step 2 (Example 3). The radio-frequency field is turned off, and the spins undergo free precession about the z-axis for a time length τ. During this time the spins spread out along the equator because their Larmor (precession) frequencies differ. A detector that measures the total magnetization of the sample (the components of each spin along a particular direction) will record a progressively degraded, or randomized, signal.

Step 3 (Example 4). A π pulse is now applied to the sample. This essentially rotates each spin about the y-axis. The slight field inhomogeneity has negligible effect in this step, as well as in step 1.

Step 4 (Example 3). The spins are allowed to precess freely again. The spins that had the largest precession rate (and were most "ahead of" the others in step 2) were put a corresponding distance behind by the π pulse of step 3. Since they catch up at the same rate in the final step, we would expect all spins to be pointing along the $-x$-axis a time τ after the π pulse has ended.

A detector should pick up a large signal at this time from the "unrandomized" system of spins in the inhomogeneous field. Such signals are called

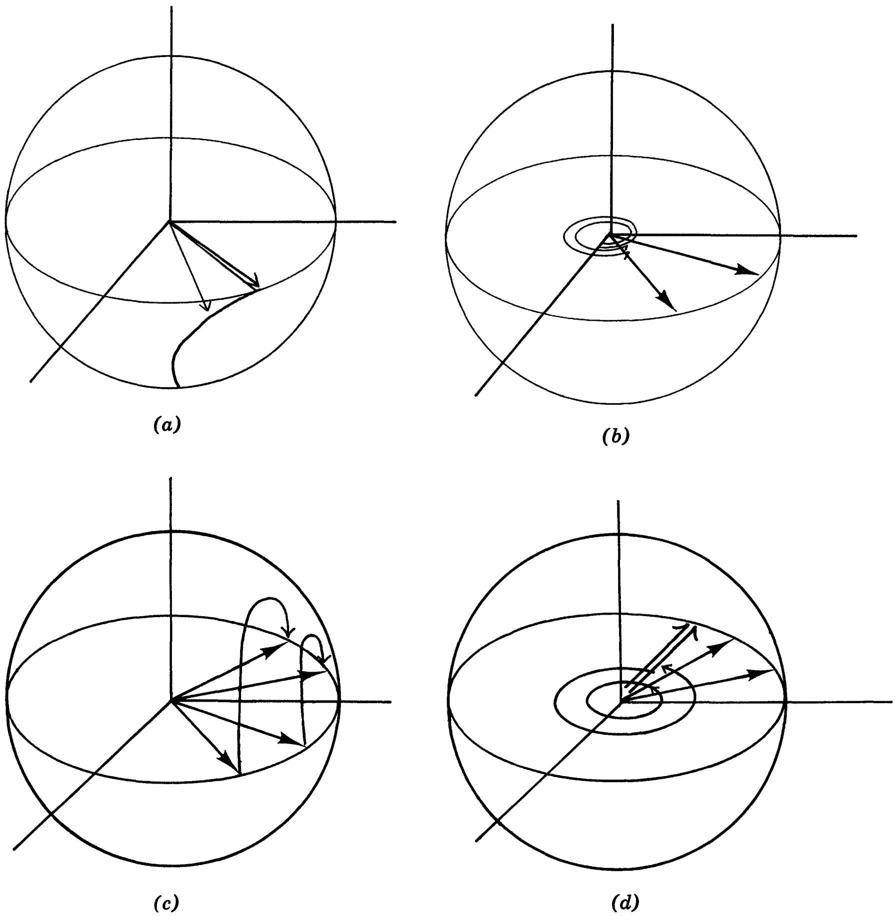


FIG. 5.22 SPIN ECHO EXPERIMENTS. (a) STEP 1: A $\pi/2$ PULSE IS APPLIED TO A SYSTEM OF SPINS IN THERMAL EQUILIBRIUM WITH $kT \ll \Delta E$. IN AN INHOMOGENEOUS FIELD, THE SPIN DIRECTIONS WILL SCATTER SLIGHTLY NEAR THE EQUATOR. (b) STEP 2: AFTER THE RF FIELD IS TURNED OFF, THE SPINS WILL PRECESS FREELY AT THE LOCAL LARMOR FREQUENCY. SPINS IN REGIONS OF STRONGER FIELDS WILL PRECESS AT A FASTER RATE THAN SPINS IN A REGION OF WEAKER FIELD. AFTER A SUFFICIENTLY LONG TIME, THE COHERENCE PROPERTIES OF SPINS IN DIFFERENT PARTS OF THE SAMPLE WILL APPEAR TO BE LOST. (c) STEP 3: AT TIME τ AFTER FREE PRECESSION HAS BEEN GOING ON, A π PULSE IS APPLIED. THE APPLICATION OF THIS π PULSE WILL ROTATE ALL SPINS ABOUT THE RF FIELD POLARIZATION AXIS. AFTER THE π PULSE, THE SPINS WILL AGAIN BE ORIENTED IN OR NEAR THE EQUATOR. THE SPINS MOST "IN FRONT" BEFORE THE π PULSE, WILL BE PUT EXACTLY AS "FAR BEHIND" AFTER THE π PULSE. (d) STEP 4: IN THE ENSUING PERIOD OF LENGTH τ DURING WHICH FREE PRECESSION IS TAKING PLACE, THE SPINS ARE "CATCHING UP" WITH ONE ANOTHER. AT THE END OF THE TIME INTERVAL τ THEY WILL ALL BE REGROUPED AND POINTING AT THE $-x$ AXIS. THE COHERENCE "LOST" IN STEP 2 IS REGAINED. THE SIGNAL IS CALLED AN ECHO.

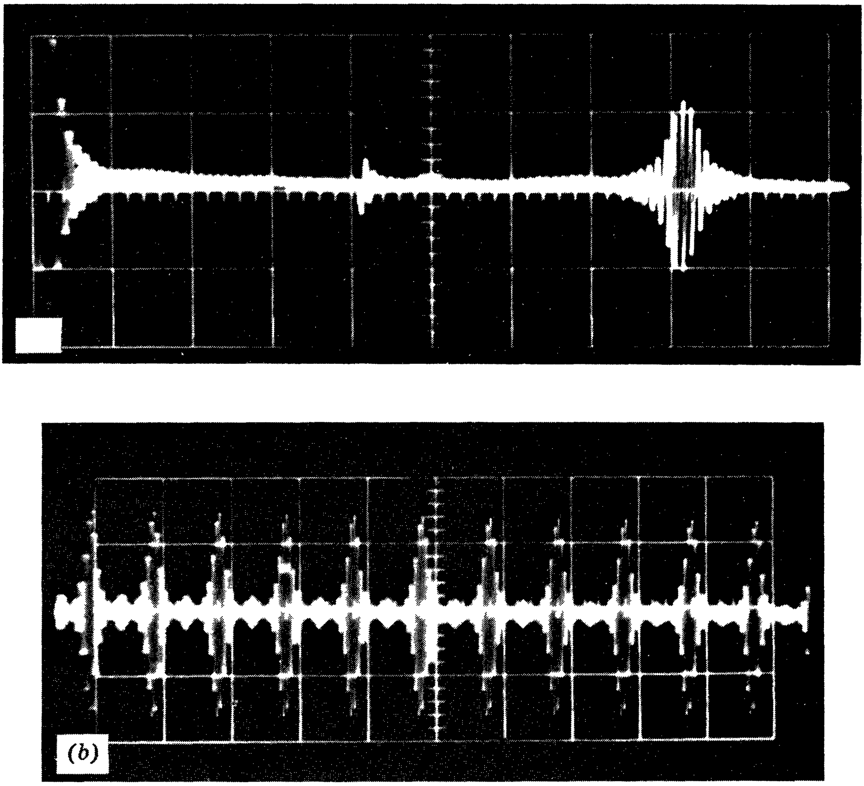


FIG 5 23 OSCILLOSCOPE TRACES REVEAL THE EXISTENCE OF SPIN ECHOES (REPRINTED BY KIND PERMISSION OF THE AUTHOR FROM A ABRAGAM *The Principles of Nuclear Magnetism* OXFORD CLAREDON PRESS 1961 PLATES III 8 III 9)

echoes The alternation of free precession periods and π pulses can be carried out many times in a single run The data appear in Fig 5 23

Resume

In this chapter we used in a concrete form many of the tools introduced in a more abstract way in Chapter 4

We compared the Lie algebras of $Gl(1\ q)$ and $U(2)$ also $Sl(1\ q)$ $SU(2)$ $SO(3)$ We also compared these groups by direct computation of the exponential mapping of the algebra onto the group The covering groups were computed explicitly as were their discrete invariant subgroups Homomorphic image groups with isomorphic algebras have been explicitly computed

Their homotopy groups were also computed and compared with appropriate discrete invariant subgroups.

The complete set of unitary irreducible representations for both covering groups $Gl(1, q)$ and $SU(2, c)$ were computed. The quantum numbers of the representations of the associated locally isomorphic groups were also computed. These results were summarized in Fig. 5.9, which graphically displays the covering group as a group that literally "covers" all other locally isomorphic groups and determines both their properties and their representations.

Some possible physical consequences were discussed. The $2 \to 1$ homomorphism between the transformation groups $SU(2)$ and $SO(3)$ was transferred to the spaces C_2 and R_3 in which they act. The consequence—"double-valuedness"—was discussed briefly.

The Baker-Campbell-Hausdorff formulas for $SU(2)$ were presented and used to derive the matrix elements for the representations of $SU(2)$ in a second way. Since the matrix elements so derived (6.10) differ in structure from the matrix elements derived from the more traditional approach (3.12), the two were shown to be equivalent. Finally, the BCH formulas served to derive time evolution operators and generating functions for physical systems which are available in the experimental laboratory.

Exercises

1. The parameter space for the rotation group $S^1 = SO(2)$ consists of the segment $[0, 2\pi)$ of the straight line, with the point 2π identified with 0. Show that the parameter space for $S^1 \otimes S^1$ consists of a square with sides of length 2π. Which boundary points on this square must be identified? Prove that this figure is topologically equivalent to a torus. Prove that it is compact. Prove that a straight line through (0, 0) returns to the vicinity of (0, 0) infinitely often, but only goes through (0, 0) if the tangent of the angle it makes with any side is a rational number. For irrational tangents, this line is called a skew-line on a torus (see Fig. 9.1).

2. Prove that the sphere $S^2 \subset R_3$ is simply connected. Prove that the circle S^1 and the torus $S^1 \otimes S^1$ are infinitely connected.

3. Prove that the only discrete invariant subgroup of the additive group R_1 of real numbers is isomorphic with the integers Z.

4. Show that "geodesics" in $SU(2)$ through the identity I_2 focus at the point $-I_2$. Show that "geodesics" in $SO(3)$ refocus again only back at the identity $+I_3$. Why does $SU(2)$ have two focal points while $SO(3)$ has only one?

5. Compute the foci in $SU(n)$. Show that these points all have the form $e^{2\pi ik/n}I_n$, where k is an integer. Prove that the discrete invariant subgroups in $SU(n)$ are all cyclic abelian groups of order r, having the structure

$$e^{2\pi ik/r}I_n, \qquad \frac{n}{r} = \text{integer}$$

6. Let G and H be groups and Γ a homomorphism: $G \xrightarrow{\Gamma} H$. Prove that the set of group elements in G which map onto the identity element in H forms an *invariant* subgroup of G.

7. If in Problem 6, G and H have the same Lie algebras, then the invariant subgroup Γ^{-1} (h_0 = identity in H) is a discrete invariant subgroup of G.

8. Prove that matrix elements for J_+, J_-, J_3 of the form (3.29) provide a representation for these generators even when $2j$ is not a nonnegative integer. If the hermiticity conditions

$$J_3^\dagger = J_3 \qquad J_+^\dagger = J_-$$

are changed to

$$J_3^\dagger = J_3 \qquad J_+^\dagger = -J_-$$

what conditions must be placed on j to make the following representations unitary?

$$\Gamma(J_3)^\dagger = \Gamma(J_3), \qquad \Gamma(J_+)^\dagger = +\Gamma(J_-)$$

Show that the condition that $2j$ be a nonnegative integer is required to make the representation (3.29) finite-dimensional.

9. Construct diagrams analogous to Fig. 5.9 for the groups $SU(3)$, $U(3)$, and $R \otimes SU(3)$.

10. Complete Fig. 5.12 by considering additional parameterizations that have not been presented in the text, for example, the various Euler angle parameterizations.

11. Prove the necessity and sufficiency of (6.33) in order that (6.32) have constant coefficients.

12. Verify (6.66).

13. Show that

$$\left\langle \begin{matrix} j \\ \theta\phi \end{matrix} \right| P\{(J_+)^k(J_3)^m(J_-)^n\} \left| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle$$

$$= \frac{(k+m+n)!}{k!\,m!\,n!}\left(\frac{\partial}{\partial i\alpha_+}\right)^k\left(\frac{\partial}{\partial i\alpha_3}\right)^m\left(\frac{\partial}{\partial i\alpha_-}\right)^n f^1(\theta\phi;\alpha_+\alpha_3\alpha_-)$$

14. Compute explicitly, in closed form, the matrix elements

$$\mathscr{D}^j_{m'm}(e^{i\beta J_y})$$

Show that they are given by the generating functions[10]

$$\mathscr{D}^J_{m,n}(\beta) = P^J_{m,n}(z) = \frac{(-)^{j-m}}{2^j(j-n)!}\left[\frac{(j-n)!\,(j+m)!}{(j+n)!\,(j-m)!}\right]^{1/2}$$

$$\times (1+z)^{-1/2(m+n)}(1-z)^{-1/2(m-n)}\left(\frac{d}{dz}\right)^{j-m}[(1-z)^{j-n}(1+z)^{j+n}]$$

where $z = \cos\beta$. Compare this with the weighting function for the Jacobi polynomials, given in Chapter 2, Problem 10.

15. Set $j = l$ (integer) in Problem 14, and choose *either* $m = 0$ or $n = 0$. Compare the generating function so obtained with the generating functions[10,12] for spherical harmonics and the associated Legendre polynomials. Also compare with Problem 10 of Chapter 2.

16. In Problem 14, set $j = l$ (integer) and *both* $m = n = 0$. You have obtained the generating function for Legendre polynomials.

Notes and References

1. D. Speiser. [1]
2. L. Michel. [1], pp. 139–146.
3. M. Hamermesh. [1], p. 330.
4. Ibid., pp. 348–357.
5. E. P. Wigner. [1], pp. 163–168.
6. See Ref. 2, pp. 153–154.
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8. F. Bloch, W. W. Hansen, M. Packard. [1]
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11. A. Abragam. [1]
12. N. Ja. Vilenkin. [1], Chapter 3.

CHAPTER 6

Classical Algebras

This short chapter serves two functions. First, a general procedure is described for computing the Lie algebra for any of the classical groups. Then the procedure is used to compute these algebras. Specifically, the general matrix structure for an arbitrary element in the classical algebras is presented. Next, we see that the classical groups fall into three basic classes—unitary, orthogonal, and symplectic. Variations within each class are essentially different reality conditions on the parameters of the Lie algebra. In other words, all the different groups within any particular class have the same complex extension. The commutation relations are written down explicitly only for the common complex extension.

Various decompositions of these algebras are made, corresponding to coset decompositions in the associated Lie group. For various related decompositions, the coset representatives are computed as the exponentials of a complementary vector subspace. They are also computed as surfaces in real spaces of higher dimensionality. We see that the study of cosets of Lie groups leads swiftly to a study of the so-called Riemannian symmetric spaces.

I. Computation of the Algebras

1. GENERAL PROCEDURES. An arbitrary element in the neighborhood of the identity of any of the classical matrix groups can be written in the form

$$I + \delta M \tag{1.1}$$

where δM is an infinitesimal matrix (i.e., all its elements are infinitesimals). The conditions defining the classical group may be transferred to statements about the properties of δM. We do this by writing down these conditions on the group operation (1.1), expanding, and keeping terms of the first order. Since δM describes a Lie algebra operation close to the origin, an arbitrary element M in the algebra has the same properties as the infinitesimal element δM.

The matrix groups which are volume-preserving obey the unimodular condition det ||group element|| = +1. In terms of the infinitesimal matrix δM, this condition becomes

$$\det \|I + \delta M\| \cong 1 + \operatorname{tr} \delta M = 1. \tag{1.2}$$

A general element in the Lie algebra of a volume-preserving group obeys

$$\operatorname{tr} M = 0 \tag{1.3}$$

In the neighborhood of the identity, the classical metric-preserving matrix groups obey

$$\begin{gathered} (I + \delta M)_i^{\ r(*)} g_{rs} (I + \delta M)_j^{\ s} = g_{ij} \\ \delta M_i^{\ r(*)} g_{rj} + g_{is}\, \delta M_j^{\ s} = 0 \end{gathered} \tag{1.4}$$

An arbitrary element in the Lie algebra then obeys the condition

$$\begin{gathered} M_i^{\ r(*)} = -g_{is} M_j^{\ s} g^{jr} \\ g_{rj} g^{js} = \delta_r^{\ s} \end{gathered} \tag{1.5}$$

By choosing canonical forms for the various metrics of interest, we will be able to put the Lie algebras into a canonical form.

Some of the classical groups satisfy more than one such condition. Thus $SU^*(2n)$ is a complex volume-preserving group that commutes with the canonical skew-symmetric metric (3.14 of Chapter 2) times the complex conjugation operation K. The algebras of such groups are the intersections of the algebras belonging to groups satisfying these conditions separately.

The Lie algebras computed in this way are defined on a real η-dimensional linear vector space R_η. The field of real numbers R_1 is not algebraically closed. That is, there are algebraic equations which do not have solutions in the field. For example,

$$x^2 + 1 = 0 \tag{1.6}$$

has no solutions in the field of real numbers. Since we later treat polynomial equations related to Lie algebras, it is useful to extend the field to an algebraically closed one. By allowing all real parameters in the vector spaces R_η to become complex, we form a complex extension of the Lie algebras over R_η to Lie algebras over C_η. Complex extensions of real Lie algebras will effect a significant simplification in our classification of these algebras.

2. UNITARY GROUPS. For $Gl(n, r)$ and $Gl(n, c)$, δM is an arbitrary real or complex matrix. The Lie algebras consist of all real or complex

matrices. A convenient basis for these algebras is the n^2 $n \times n$ matrices $E_{ij}^{(n)}$, with $+1$ in the ith row and the jth column and zeroes elsewhere:

$$E_{ij}^{(n)} = \begin{array}{cc} & \begin{array}{ccc} i & & j \end{array} \\ \begin{array}{c} \\ i \\ \\ j \end{array} & \left[\begin{array}{cccc} & & \vdots & \\ \cdots & \cdots & +1 & \cdots \\ & & \vdots & \\ & & \vdots & \end{array}\right] \end{array} \tag{1.7}$$

$$M = A^{ij} E_{ij}^{(n)} \tag{1.8}$$

The A^{ij} are arbitrary real or complex numbers. The complex extension of $Gl(n, r)$ is $Gl(n, c)$.

The volume-preserving subgroups $Sl(n, r)$, $Sl(n, c)$ are determined by $\operatorname{tr} M = \sum A^{ii} = 0$. The complex extension of $Sl(n, r)$ is $Sl(n, c)$.

The unitary groups $U(n, c)$ preserve the canonical metric $g_{ij} = \delta_{ij}$. The Lie algebra therefore obeys

$$M_i{}^{j*} = -M_j{}^i \qquad \text{or} \qquad M^\dagger = -M \tag{1.9}$$

It consists of all complex $n \times n$ matrices which are in addition antihermitian:

$$\begin{aligned} M &= A^{ij} E_{ij}^{(n)} \\ A^{ij} &= -A^{ji*} \end{aligned} \tag{1.10}$$

To find the complex extension of $U(n, c)$ we write A^{ij} in terms of its real and imaginary parts

$$\begin{aligned} A^{ij} &= a_{ij} + ib_{ij} \\ A^{ji} &= -a_{ij} + ib_{ij} \end{aligned} \tag{1.11}$$

Then we allow the real numbers a_{ij} and b_{ij} to become complex:

$$\begin{aligned} a_{ij} &\to a_{ij}^{(1)} + ia_{ij}^{(2)} \\ b_{ij} &\to b_{ij}^{(1)} + ib_{ij}^{(2)} \end{aligned} \tag{1.12}$$

Under this extension of base field, the Lie algebra becomes

$$M = [(a_{ij}^{(1)} - b_{ij}^{(2)}) + i(a_{ij}^{(2)} + b_{ij}^{(1)})]E_{ij}^{(n)} \tag{1.13}$$

The coefficients of the bases $E_{ij}^{(n)}$ now no longer obey (1.10) and (1.11) but instead are completely independent complex numbers. The complex extension of $U(n, c)$ is $Gl(n, c)$.

The unitary groups $U(p, q; c)$ preserve the canonical metric

$$g_{ij} = \varepsilon_i\,\delta_{ij} \qquad \varepsilon_i = \begin{cases} +1, & i = 1, 2, \ldots, p \\ -1, & i = p+1, \ldots, p+q \end{cases} \tag{1.14}$$

Since this metric is related to the canonical diagonal metric

$$g_{ij} = \delta_{ij} = I_{p+q} \tag{1.15}$$

by the Weyl unitary trick, the Lie algebras of $U(p + q; c)$ and $U(p, q; c)$ are related to each other by the Weyl unitary trick:

$$\begin{array}{cc} & \begin{array}{cc} \leftarrow p \rightarrow & \leftarrow q \rightarrow \end{array} \\ \begin{array}{c} p \\ q \end{array} & \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline -M_{12}^{\dagger} & M_{22} \end{array}\right] \end{array} \quad \begin{array}{c} \text{Weyl} \\ \text{unitary} \\ \rightarrow \\ \text{trick} \end{array} \quad \begin{array}{cc} & \begin{array}{cc} \leftarrow p \rightarrow & \leftarrow q \rightarrow \end{array} \\ \begin{array}{c} p \\ q \end{array} & \left[\begin{array}{c|c} M_{11} & iM_{12} \\ \hline -iM_{12}^{\dagger} & M_{22} \end{array}\right] \end{array}$$

$$\mathfrak{u}(p+q; c) \qquad\qquad\qquad \mathfrak{u}(p, q; c) \tag{1.16}$$

Here M_{11} and M_{22} are complex antihermitian matrices that generate the Lie algebra of the compact subgroups $U(p, c)$ and $U(q, c)$. If we set the subalgebras M_{11} and M_{22} equal to zero, the remaining matrix generators are antihermitian for the compact group $U(p + q;\ c)$ and hermitian for the noncompact group $U(p,\ q;\ c)$. We distinguish these cases by calling the generators compact and noncompact, respectively.

Since the algebras for $U(p + q; c)$ and $U(p, q; c)$ differ only by the Weyl unitary trick, they have identical complex extensions $Gl(p + q; c)$.

The subalgebras $\mathfrak{su}(p + q; c)$ and $\mathfrak{su}(p, q; c)$ are determined by the zero trace condition (1.3). These algebras have identical complex extensions: it is the algebra $\mathfrak{sl}(p + q; c)$.

The group $SU^*(2n)$ consists of all matrices in $Sl(2n, c)$ which commute with a real skew-symmetric matrix times the complex conjugation operator K. Then

$$\left(\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array}\right) K \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array}\right) K \tag{1.17}$$

and it is easily verified the Lie algebra of $SU^*(2n)$ has the general structure

$$\left(\begin{array}{c|c} A & B \\ \hline -B^* & +A^* \end{array}\right) \qquad \operatorname{tr}(A \oplus A^*) = 0 \tag{1.18}$$

where A and B are complex $n \times n$ matrices. Since $SU^*(2n)$ is a subgroup of $Sl(2n, c)$, the trace of this algebra is zero. It is easily verified that the complex extension of this algebra is identical to the Lie algebra of $Sl(2n, c)$.

The Lie algebras $\mathfrak{gl}(p + q; r)$, $\mathfrak{u}(p + q; c)$ and $\mathfrak{u}(p, q; c)$ all have the same complex extension $\mathfrak{gl}(p + q; c)$. The former algebras are called real forms of their common complex extension. In general, different real forms may have the same complex extension. The real forms differ in the reality properties imposed on the space of parameters A^{ij} (1.8). Real forms are related to each other essentially by analytic continuation. Isomorphic statements hold for $\mathfrak{sl}(p + q; r)$, $\mathfrak{su}(p + q; c)$, $\mathfrak{su}(p, q; c)$ and their common complex extension $\mathfrak{sl}(p + q; c)$; also for $\mathfrak{su}^*(2n)$ and $\mathfrak{sl}(2n, c)$.

Since these various real forms are related to a common complex extension, it is not necessary to write down the commutation relations for all these algebras. We need write down the commutation relations only for the common complex extension $\mathfrak{gl}(n, c)$:

$$[E_{ij}^{(n)}, E_{rs}^{(n)}] = E_{is}^{(n)}\, \delta_{jr} - E_{rj}^{(n)}\, \delta_{is} \tag{1.19}$$

3. ORTHOGONAL GROUPS. The real and complex orthogonal groups $SO(n, r)$ and $SO(n, c)$ preserve the canonical bilinear symmetric metric $g_{ij} = \delta_{ij}$. By (1.5), their algebras consist of real and complex antisymmetric matrices:

$$\begin{aligned} M &= A^{ij}E_{ij}^{(n)} \\ M^t &= -M \qquad A^{ij} = -A^{ji} \end{aligned} \tag{1.20}$$

The complex extension of $SO(n, r)$ is $SO(n, c)$.

The Lie algebra for $SO(p, q; r)$ is related to the Lie algebra for $SO(p + q; r)$ by the Weyl unitary trick:

$$\begin{array}{cc} & \begin{array}{cc} \longleftarrow p \longrightarrow & \longleftarrow q \longrightarrow \end{array} \\ \begin{array}{c} \uparrow \\ p \\ \downarrow \\ \uparrow \\ q \\ \downarrow \end{array} & \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline -M_{12}{}^t & M_{22} \end{array}\right] \\ & \mathfrak{so}(p + q; r) \end{array} \begin{array}{c} \text{Weyl} \\ \text{unitary} \\ \rightarrow \\ \text{trick} \end{array} \begin{array}{cc} & \begin{array}{cc} \longleftarrow p \longrightarrow & \longleftarrow q \longrightarrow \end{array} \\ \begin{array}{c} \uparrow \\ p \\ \downarrow \\ \uparrow \\ q \\ \downarrow \end{array} & \left[\begin{array}{c|c} M_{11} & iM_{12} \\ \hline -iM_{12}{}^t & M_{22} \end{array}\right] \\ & \mathfrak{so}(p, q; r) \end{array} \tag{1.21}$$

Here M_{11}, M_{22} are the compact subalgebras for $SO(p, r)$ and $SO(q, r)$. The subspace generated by the $p \times q$ matrices M_{12} and iM_{12} are compact generators for $SO(p + q;\ r)$ and noncompact generators for $SO(p,\ q;\ r)$, respectively.

The matrix elements of M_{12} are all real. Under complex extension,

$$SO(p, q; r) \xrightarrow[\text{extension}]{\text{complex}} SO(p, q; c) \tag{1.22}$$

It is easily verified that $SO(p + q; c)$ and $SO(p, q; c)$ have identical Lie algebras. Therefore, the groups $SO(p, q; r)$ are all real forms of the group $SO(p + q; c)$.

The group $SO^*(2n)$ is the subgroup of $SO(2n, c)$ which preserves the sesquilinear antisymmetric metric.[1] With respect to the antisymmetric metric

$$\left[\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array}\right] \tag{1.23}$$

it is easily verified that the Lie algebra of $2n \times 2n$ matrices for $SO^*(2n)$ has the structure

$$\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline -M_{12}^* & M_{11}^* \end{array}\right] \qquad \begin{array}{l} M_{11} \text{ skew symmetric} \\ M_{12} \text{ hermitian} \end{array} \tag{1.24}$$

It may be easily verified that the complex extension of this algebra is identical to the algebra of $SO(2n, c)$.

A convenient set of bases for the common complex extension is

$$O_{ij}^{(n)} = E_{ij}^{(n)} - E_{ji}^{(n)} = -O_{ji}^{(n)} \tag{1.25}$$

The commutation relations for the generators $O_{ij}^{(n)}$ are given by

$$[O_{ij}^{(n)}, O_{rs}^{(n)}] = O_{is}^{(n)}\,\delta_{jr} + O_{jr}^{(n)}\,\delta_{is} - O_{ir}^{(n)}\,\delta_{js} - O_{js}^{(n)}\,\delta_{ir} \tag{1.26}$$

4. SYMPLECTIC GROUPS. Symplectic groups leave invariant the bilinear antisymmetric metric whose canonical form is

$$\begin{array}{lll} g_{ij} = \sigma(i)\,\delta(i + j) & \sigma(i) = +1 & \text{when} \quad i > 0 \\ M_i{}^j = -\sigma(i)\sigma(j)M_{-j}{}^{-i} & \sigma(i) = -1 & \text{when} \quad i < 0 \\ i, j = \pm 1, \pm 2, \ldots, \pm n & & \end{array} \tag{1.27}$$

With respect to the bases $(\mathbf{f}_n, \ldots, \mathbf{f}_1, \mathbf{f}_{-1}, \ldots, \mathbf{f}_{-n})$ the metric has the structure [cf. (3.14) of Chapter 2]

$$\left[\begin{array}{c|c} & \tilde{I}_n \\ \hline -\tilde{I}_n & \end{array}\right] \tag{1.28}$$

and the Lie algebra has the form

$$M = \begin{array}{c} \\ \updownarrow n \\ \updownarrow n \end{array} \begin{array}{c} \leftarrow n \rightarrow \quad \leftarrow n \rightarrow \\ \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \end{array} \qquad \begin{array}{l} M^{\sim}_{12} = M_{12} \\ M^{\sim}_{21} = M_{21} \\ M^{\sim}_{22} = -M_{11} \end{array} \qquad \begin{array}{c} \text{Symplectic} \\ \text{symmetry} \end{array} \tag{1.29}$$

The matrix M has *symplectic symmetry*. That is, the transpose of this matrix about the *minor* diagonal (indicated by $M^{\sim}$) is closely related to the original matrix:

$$M^{\sim}_{12} = +M_{12}, M^{\sim}_{21} = +M_{21}, M^{\sim}_{11} = -M_{22}, M^{\sim}_{22} = -M_{11} \tag{1.30}$$

This symmetry may be compared with orthogonal symmetry, which involves a transpose with respect to the *major* diagonal:

$$M = -M^{t} \text{ (orthogonal symmetry)} \tag{1.31}$$

For $Sp(2n, r)$ all matrix elements in (1.29) are real; for $Sp(2n, c)$ they are complex, and $Sp(2n, c)$ is the complex extension of $Sp(2n, r)$.

The group $USp(2p, 2q)$ is the intersection of $Sp(2p + 2q; c)$ with $U(2p, 2q; c)$. Its Lie algebra possesses the symmetries of both the unitary and symplectic groups:

$$\begin{array}{c} \\ \updownarrow q \\ \updownarrow p \\ \updownarrow p \\ \updownarrow q \end{array} \begin{array}{c} \leftarrow q \rightarrow\leftarrow p \rightarrow\leftarrow p \rightarrow\leftarrow q \rightarrow \\ \left[\begin{array}{c|c|c|c} A_2 & C & D & B_2 \\ \hline C^{\dagger} & A_1 & B_1 & D^{\sim} \\ \hline D^{\dagger} & -B_1^{\dagger} & -A_1^{\sim} & -C^{\sim} \\ \hline -B_2^{\dagger} & D^{\dagger\sim} & -C^{\sim\dagger} & -A_2^{\sim} \end{array}\right] \end{array} \qquad \begin{array}{rcl} A_1^{\dagger} &=& -A_1 \\ A_2^{\dagger} &=& -A_2 \\ B_1^{\sim} &=& B_1 \\ B_2^{\sim} &=& B_2 \end{array} \tag{1.32}$$

The complex extension of this algebra is identical to the algebra of $Sp(2p + 2q; c)$.

A convenient choice of bases for these algebras is

$$\begin{aligned} Z_{ij} &= E^{(2n)}_{ij} - \sigma(i)\sigma(j)E^{(2n)}_{-j,\,-i} \\ Z_{i,\,-i} &= E^{(2n)}_{i,\,-i} \end{aligned} \tag{1.33}$$

The generators Z_{ij} then have the symplectic symmetry and commutation relations

$$Z_{ij} = -\operatorname{sign}(ij) Z_{-j,\,-i} \tag{1.34}$$

$$[Z_{ij}, Z_{rs}] = \operatorname{sign}(jr)\{Z_{is}\,\delta_{rj} + Z_{-j,\,-r}\,\delta_{is} + Z_{i,\,-r}\,\delta_{-j,\,s} + Z_{-j,\,s}\,\delta_{r,\,-i}\} \tag{1.35}$$

For convenience the structure of these classical matrix Lie algebras is summarized in Table 6.1.

5. BASES FOR THESE ALGEBRAS. Equations (1.19), (1.26), and (1.35) describe the commutation relations for the unitary, orthogonal, and symplectic groups and their various real forms. These generators are bases for the vector spaces associated with these groups. The algebras themselves have fewer bases.

Let us relabel the indices for the symplectic group as follows:

$$\begin{aligned} Z_{i'j'} \to Z_{ij} \qquad & i = 2i' \qquad (i' > 0) \\ & i = 2(-i') - 1 \qquad (i' < 0) \end{aligned} \tag{1.36}$$

Then the commutation relations and the symmetry properties of the generators for the unitary, orthogonal, and symplectic groups, as well as the generators P_{ij} of the permutation groups, can be written in this[2] economical way:

$$\begin{aligned} &[E_{ij}, E_{jr}] = E_{ir} && E_{ji} = E_{ij}^{\dagger} \\ &[O_{ij}, O_{jr}] = O_{ir} && O_{ji} = -O_{ij} \\ &[Z_{ij}, Z_{jr}] = Z_{ir} && Z_{-j',\,-i'} = -\sigma(i')\sigma(j')Z_{i'j'} \\ &P_{ij}P_{jr}P_{ji} = P_{ir} && P_{ji} = P_{ij}^{-1} \end{aligned} \tag{1.37}$$

where $i \le j \le r$.

The $n \times n$ matrix groups belonging to the unitary, orthogonal, and symplectic series have algebras that can be generated by only $n-1$ generators $E_{i,\,i+1}$; $O_{i,\,i+1}$; $Z_{i,\,i+1}$ together with the commutation relations and symmetry properties of (1.37). The permutation group P_n with $n!$ operations can be generated by the $n-1$ adjacent transpositions $P_{i,\,i+1}$.

We subsequently learn that we can do even better than this. Although it is necessary to use $n-1$ bases to generate the unitary algebras of $Sl(n, c)$, it is only necessary to use $[n/2]$ bases to generate the algebras of $SO(n)$ and $Sp(n)$. The entire permutation group P_n can be generated from just the two operations $(12) = P_{12}$ and $(12 \cdots n)$.

TABLE 6.1

MATRIX STRUCTURES FOR THE LIE ALGEBRAS OF THE CLASSICAL GROUPS

Series	Group	Algebra	Component	Size	Symmetry: Unitary	Orthogonal	Symplectic	Real	Tr = 0	Comment
Unitary	$Gl(n, c)$	(M)		$n \times n$						Complex extension
	$Gl(n, r)$	(M)		$n \times n$				X		
	$U(p, q; c)$	$\left[\begin{array}{c\|c} A & B \\ \hline B^{\dagger} & C \end{array}\right]$	A	$p \times p$	X					
			C	$q \times q$	X					
			B	$p \times q$						
	$U(n, c)$	(A)		$n \times n$	X					Compact
Unitary Unimodular	$Sl(n, c)$	(M)		$n \times n$					X	Complex extension
	$Sl(n, r)$	(M)		$n \times n$				X	X	
	$SU(p, q; c)$	$\left[\begin{array}{c\|c} A & B \\ \hline B^{\dagger} & C \end{array}\right]$	$\left[\begin{array}{c\|c} A & \\ \hline & C \end{array}\right]$	$p \times p$ + $q \times q$	X				X	
			B	$p \times q$						
	$SU^*(2n)$	$\left[\begin{array}{c\|c} A & B \\ \hline -B^* & A^* \end{array}\right]$	$\left[\begin{array}{c\|c} A & \\ \hline & A^* \end{array}\right]$	$n \times n$ + $n \times n$					X	
			B	$n \times n$						
	$SU(n, c)$	(A)		$n \times n$	X				X	Compact

Orthogonal	$SO(n, c)$	(M)		$n \times n$		X			X	Complex extension
	$SO(p, q; r)$	$\left[\begin{array}{c\|c} A & B \\ \hline B^t & C \end{array}\right]$	$\left[\begin{array}{c\|c} A & \\ \hline & C \end{array}\right]$	$p \times p$ + $q \times q$		X		X	X	
			B	$p \times q$				X		
	$SO^*(2n)$	$\left[\begin{array}{c\|c} A & B \\ \hline -B^* & A^* \end{array}\right]$	A	$n \times n$		X			X	
			iB	$n \times n$	X					
	$SO(n, r)$	(A)		$n \times n$		X		X	X	Compact
Symplectic	$Sp(2n, c)$	$\begin{bmatrix} A & B \\ C & -\tilde{A} \end{bmatrix}$		$2n \times 2n$			X		X	Complex extension
	$Sp(2n, r)$	$\begin{bmatrix} A & B \\ C & -\tilde{A} \end{bmatrix}$		$2n \times 2n$			X	X	X	
	$USp(2p, 2q)$	$\left[\begin{array}{c\|c\|c\|c} A_q & E & F & B_q \\ \hline E^\dagger & A_p & B_p & \tilde{F} \\ \hline F^\dagger & -B_p^\dagger & -\tilde{A}_p & -\tilde{E} \\ \hline -B_q^\dagger & \tilde{F}^\dagger & -\tilde{E}^\dagger & -\tilde{A}_q \end{array}\right]$	$\left[\begin{array}{c\|c} A_p & B_p \\ \hline -B_p^\dagger & -\tilde{A}_p \end{array}\right]$	$2p \times 2p$	X		X		X	
			$\left[\begin{array}{c\|c} A_q & B_q \\ \hline -B_q^\dagger & -\tilde{A}_q \end{array}\right]$	$2q \times 2q$	X		X		X	
			E, F	$q \times p$						
	$USp(2n)$	$\left[\begin{array}{c\|c} A_n & B_n \\ \hline -B_n^\dagger & -\tilde{A}_n \end{array}\right]$		$2n \times 2n$	X		X		X	Compact

Symmetry about major diagonal: $M = M^t$.
Symmetry about minor diagonal: $M = \tilde{M}$.
Unitary symmetry: $m_{ij} = -m_{ji}^*$, $M = -M^\dagger$.
Orthogonal symmetry: $m_{ij} = -m_{ji}$, $M = -M^t$.
Symplectic symmetry: $m_{ij} = -\text{sign}(ij)m_{-j,-i}$.

The advantage of using $n-1$ generators in constructing these algebras and groups becomes apparent when one explicitly constructs their matrix representations.[2]

Since the various real forms of a common complex extension are related by reality properties or the Weyl unitary trick, it is possible to relate the matrix representations of one real form with those of another by looking at the analytic continuation of the matrix representative of just one generator. For example, in the analytic continuation $SU(p+q;c) \to SU(p,q;c)$, all generators $E_{i,i+1}$ remain unaffected except for

$$E_{p,p+1} \to iE_{p,p+1}$$

This suggests that representations of the noncompact groups $SU(p,q;c)$ can be obtained by investigating the analytic continuation of the generator $E_{p,p+1}$ (only) in a representation of the compact group $SU(p+q;c)$.

6. ORIGIN OF THE EMBEDDING GROUPS $SO^*(2n)$ AND $SU^*(2n)$. The existence of the two unfamiliar "embedding groups" $SO^*(2n)$ and $SU^*(2n)$ as real forms of $SO(2n,c)$ and $Sl(2n,c)$ often comes as a rude shock to aficionados of Lie group theory. The difficulty is further compounded by the lack of a simple explanation for their existence. We present one now.

The group $U(n,c)$ consists of those $n \times n$ complex matrices which preserve the canonical positive definite sesquilinear symmetric metric $g_{ij} = \delta_{ij}$. Each matrix element is a complex number; the Lie algebra obeys

$$M_i{}^j = -M_j{}^{i*}$$

There is a canonical representation of the complex numbers by real-valued 2×2 matrices [see Chapter 1, (3.10)]

$$re^{i\phi} \to r\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

$$\begin{array}{ccc} x+iy & & (x+iy)^* \\ \downarrow & & \downarrow \\ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} & & \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \end{array} \tag{1.38}$$

Under this representation of the complex numbers, every complex entry in $U(n,c)$ is replaced by a real 2×2 matrix. Since the $2n \times 2n$ *real* matrices so obtained preserve the metric

$$g'_{ij} = \delta_{ij} \otimes I_2 \tag{1.39c}$$

they form a subgroup of $SO(2n,r)$.

We investigate the Lie algebra of this subgroup under

$$u_i{}^j \to r_i{}^j + ic_i{}^j \tag{1.40}$$

$$
\begin{array}{rl}
\text{in } C_n & \begin{array}{cccccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & \end{array}\\
\text{in } R_{2n} & \begin{array}{ccccc} 1 & -1 & 2 & -2 & \end{array}\\
\begin{array}{r} 1 \\ \\ \delta M \to 2 \\ \\ \\ \end{array} \begin{array}{r} 1\\-1\\2\\-2\\ \\ \end{array} & \left[\begin{array}{cc|cc|c} 0 & c_1^1 & r_1^2 & c_1^2 & \cdots \\ -c_1^1 & 0 & -c_1^2 & r_1^2 & \\ \hline -r_1^2 & c_1^2 & 0 & c_2^2 & \cdots \\ -c_1^2 & -r_1^2 & -c_2^2 & 0 & \\ \hline \vdots & & \vdots & & \ddots \end{array}\right]
\end{array}
\tag{1.41}
$$

Since the Lie algebra consists of real antisymmetric matrices, it is clearly a Lie subalgebra of $\mathfrak{so}(2n, r)$.

It is useful at this point to rearrange the rows and columns of this matrix:

$$
\begin{array}{rl}
 & \begin{array}{cccccc} 1 & 2 & \cdots & -1 & -2 & \cdots \end{array}\\
\delta M \to \begin{array}{r} 1\\2\\ \vdots \\ -1 \\ -2 \\ \vdots \end{array} & \left[\begin{array}{ccc|ccc} 0 & r_1{}^2 & \cdots & c_1{}^1 & c_1{}^2 & \cdots \\ -r_1{}^2 & 0 & \cdots & c_1{}^2 & c_2{}^2 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \\ \hline -c_1{}^1 & -c_1{}^2 & \cdots & 0 & r_1{}^2 & \cdots \\ -c_1{}^2 & -c_2{}^2 & \cdots & -r_1{}^2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{array}\right]
\end{array}
\tag{1.42}
$$

$$
= \left[\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right] \qquad \begin{array}{l} B^t = +B \\ A^t = -A \\ A, B \quad \text{real} \end{array} \tag{1.43}
$$

The Lie algebra $\mathfrak{so}(2n, r)$ can be written as the direct sum of two vector subspaces $\mathfrak{k} \oplus \mathfrak{p}$:

$\mathfrak{k}$: The subspace of matrices (1.43), which form a $2n \times 2n$ real matrix representation of $\mathfrak{u}(n, c)$.

$\mathfrak{p}$: An orthogonal complementary subspace whose matrices have the general structure

$$\left[\begin{array}{c|c} C & D \\ \hline +D & -C \end{array}\right] \qquad \begin{array}{l} C^t = -C \\ D^t = -D \end{array} \tag{1.44}$$

In short, we have the decomposition

$$\begin{aligned} \mathfrak{so}(2n, r) &= \quad \mathfrak{u}(n, c) \quad \oplus \quad V_\perp \\ &= \left[\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right] \oplus \left[\begin{array}{c|c} C & D \\ \hline D & -C \end{array}\right] \\ &= \qquad \mathfrak{k} \qquad \oplus \qquad \mathfrak{p} \end{aligned} \tag{1.45}$$

It may be verified by direct calculation that the subspaces $\mathfrak{k}$, $\mathfrak{p}$ obey the commutation relations given symbolically by

$$\begin{aligned} [\mathfrak{k}, \mathfrak{k}] &\subset \mathfrak{k} \\ [\mathfrak{k}, \mathfrak{p}] &= \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] &= \mathfrak{k} \end{aligned} \tag{1.46}$$

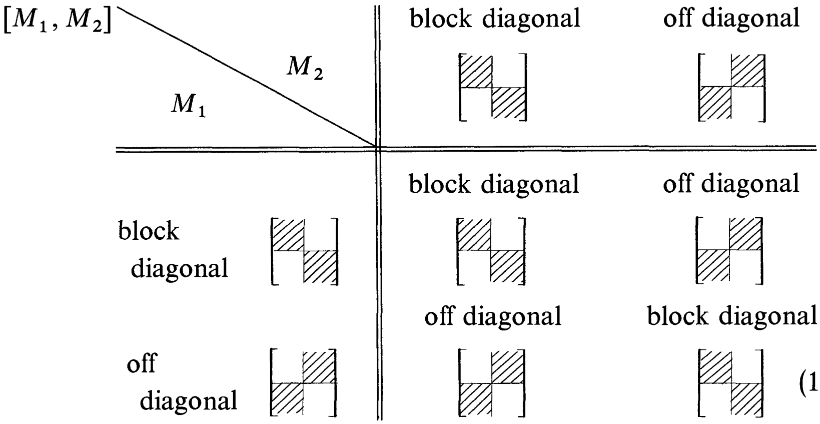
The commutation properties are most easily seen after making a similarity transformation using

$$S = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} +1I_n & -iI_n \\ \hline -iI_n & +1I_n \end{array}\right] \tag{1.47}$$

$$S \left[\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right] S^{-1} = \left[\begin{array}{c|c} A + iB & 0 \\ \hline 0 & A - iB \end{array}\right] \qquad \begin{array}{l} A^t = -A \\ B^t = +B \end{array} \tag{1.48k}$$

$$S \left[\begin{array}{c|c} C & D \\ \hline D & -C \end{array}\right] S^{-1} = \left[\begin{array}{c|c} 0 & D + iC \\ \hline D - iC & 0 \end{array}\right] \qquad \begin{array}{l} C^t = -C \\ D^t = -D \end{array} \tag{1.48p}$$

In this representation, the commutation properties of the matrix vector subspaces are especially easy to compute. The results are indicated in the following diagram:



(1.49)

Since $\mathfrak{so}(2n, r)$ is closed under commutation, any block diagonal submatrix arising from commutators belongs to the subalgebra $\mathfrak{k}$; any off-diagonal submatrix arising from commutators belongs to the subspace $\mathfrak{p}$.

If the Weyl unitary trick is now applied to the compact generators in the subspace $\mathfrak{p}$, they are converted to noncompact generators. Using (1.46), we find that the commutation relations obeyed by $\mathfrak{k}$ and $\mathfrak{p}^* = i\mathfrak{p}$ are

$$\begin{aligned}[][\mathfrak{k}, \mathfrak{k}] &\subset \mathfrak{k}\\ [\mathfrak{k}, \mathfrak{p}^*] &= \mathfrak{p}^*\\ [\mathfrak{p}^*, \mathfrak{p}^*] &= (-)\mathfrak{k}\end{aligned} \tag{1.46*}$$

Therefore, the matrices $\mathfrak{k} \oplus \mathfrak{p}^*$ are closed under commutation and form the Lie algebra of some noncompact group

$$\begin{aligned}\mathfrak{so}^*(2n) &= \quad \mathfrak{k} \quad \oplus i\mathfrak{p}\\ &= \mathfrak{u}(n, c) \oplus i[\mathfrak{so}(2n, r) \bmod \mathfrak{u}(n, c)]\end{aligned} \tag{1.50}$$

Comment 1. The subalgebra of matrices in $\mathfrak{k}$ is antihermitian and therefore maps onto a compact group under the EXPonential mapping. The matrices in the subspace $\mathfrak{p}(\mathfrak{p}^* = i\mathfrak{p})$ are antihermitian (hermitian) and therefore map onto compact (noncompact) cosets. The maximal compact subgroup of $SO^*(2n)$ is $U(n, c)$.

Comment 2. The Lie algebra of $SO^*(2n)$ satisfies the condition (1.24). The group thus obeys the condition giving rise to (1.24) and may be defined accordingly: $SO^*(2n)$ is the subgroup of $SO(2n, c)$ which preserves the sesquilinear antisymmetric metric.

The Lie algebra for the group $SU^*(2n)$ is derived in exactly the same way. The group $U(n, q)$ preserves the sesquilinear canonical metric $g_{ij} = \delta_{ij}$. The representation of quaternions in terms of 2×2 complex matrices [see Chapter 1, (3.13)]

$$q \to \begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix}$$

$$q^0\lambda_0 + q^1\lambda_1 + q^2\lambda_2 + q^3\lambda_3 \to \begin{bmatrix} q^0 - iq^3 & -iq^1 - q^2 \\ -iq^1 + q^2 & q^0 + iq^3 \end{bmatrix}$$

provides a representation of $U(n, q)$ in terms of $2n \times 2n$ complex matrices preserving the metric

$$g'_{ij} = \delta_{ij} \otimes I_2 \tag{1.39q}$$

In short, we have the inclusion

$$U(n, q) \subset U(2n, c)$$

The matrix Lie algebra $\mathfrak{k}$ of $U(n, q)$ consists of $2n \times 2n$ complex traceless antihermitian matrices. It is therefore contained within the Lie algebra $\mathfrak{g}$ of $SU(2n, c)$, consisting of *all* $2n \times 2n$ complex traceless antihermitian matrices. Once again, we have the vector subspace decomposition

$$\begin{aligned} \mathfrak{su}(2n, c) &= \mathfrak{u}(n, q) \oplus [\mathfrak{su}(2n, c) \bmod \mathfrak{u}(n, q)] \\ \mathfrak{g} &= \quad \mathfrak{k} \quad \oplus \mathfrak{p} \end{aligned} \tag{1.45q}$$

And again, the subalgebra $\mathfrak{k}$ and the subspace $\mathfrak{p}$ obey the commutation relations (1.46). By previous arguments, the matrices

$$\mathfrak{g}^* = \mathfrak{k} \oplus \mathfrak{p}^* \qquad (\mathfrak{p}^* = i\mathfrak{p}) \tag{1.46*q}$$

are closed under commutation. They form the Lie algebra for the noncompact classical group $SU^*(2n)$. The maximal compact subgroup of $SU^*(2n)$ is $U(n, q) \simeq USp(2n)$. Since the Lie algebra $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$ satisfies the (infinitesimal) condition (1.18), the corresponding group can be defined by the (global) condition (1.17).

Comment 3. Other noncompact groups may be constructed in the same way. The Lie algebras of $U(n, c)$, $SU(n, c)$, and $USp(2n)$ consist of complex matrices. In each case, the decomposition

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{p} \\ \mathfrak{k} &= \operatorname{Re} \mathfrak{g} \\ \mathfrak{p} &= i \operatorname{Im} \mathfrak{g} \end{aligned} \tag{1.51}$$

obeys the commutation relations (1.46). Then the matrix algebras

$$\mathfrak{g}^* = \mathfrak{k} \oplus \mathfrak{p}^* \qquad (\mathfrak{p}^* = i\mathfrak{p})$$

are closed under commutation; moreover, they consist of real matrices and give rise under EXP to the noncompact real groups $Gl(n, r)$, $Sl(n, r)$, and $Sp(2n, r)$.

7. SUMMARY OF THE REAL FORMS OF THE CLASSICAL GROUPS. The enumeration and description of the various real forms of the classical matrix groups has been somewhat involved. We present here a summary of how they arise, indicating that all real forms arise from one of three mechanisms. We defer until Chapter 9 the proof that these comprise *all* the real forms for the classical groups.

As usual, some descriptions are more easily stated in terms of the group, others, in terms of the algebra. We choose the path of least resistance.

Metric Preserving Condition. The $n \times n$ metric-preserving groups that obey the condition

$$\begin{aligned} &u_i^{\ r*} g_{rs} u_j^{\ s} = g_{ij} \\ &g_{ij} = \varepsilon(i)\, \delta_{ij} \\ &\varepsilon(i) = \begin{matrix} +1 \\ -1 \end{matrix} \quad \begin{cases} i = 1, 2, \ldots, p \\ i = p+1, \ldots, p+q = n \end{cases} \end{aligned} \tag{1.52}$$

are $U(p+q; f)$, where f describes the real, complex, and quaternion fields. These groups are

$$\begin{aligned} f &= R : O(p, q) \\ f &= C : U(p, q; c) \\ f &= Q : U(p, q; Q) \end{aligned} \tag{1.53}$$

For the real and complex fields, the unimodular restriction gives the classical groups $SO(p, q)$, $SU(p, q; c)$. The symplectic group $U(p, q; Q)$ is generally described by $2n \times 2n$ complex matrices, using the canonical embedding of quaternions into 2×2 complex matrices. As such, this group is denoted

$$U(p, q; Q) \to USp(2p, 2q) \tag{1.54}$$

Subfield Restriction. The subfield restrictions $Q \downarrow C \downarrow R$ induce the following compact subgroup restrictions:

$$\begin{array}{cc} U(n, Q) & Q \\ \downarrow & \downarrow \\ U(n, c) & C \\ \downarrow & \downarrow \\ O(n, r) & R \end{array} \tag{1.55}$$

Let $\mathfrak{g}$ be the Lie algebra for one of these compact groups [$U(n, Q)$, $U(n, c)$] and $\mathfrak{k}$ the subalgebra for the subgroup [$U(n, c)$, $\mathring{O}(n, r)$, resp.]. Then the algebra $\mathfrak{g}$ may be written as a direct sum of the subalgebra $\mathfrak{k}$ with a complementary subspace $\mathfrak{p}$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{1.56}$$

The subalgebra $\mathfrak{k}$ and the subspace $\mathfrak{p}$ obey the commutation relations

$$\begin{aligned} [\mathfrak{k}, \mathfrak{k}] &\subseteq \mathfrak{k} \\ [\mathfrak{k}, \mathfrak{p}] &\subseteq \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] &\subseteq \mathfrak{k} \end{aligned} \tag{1.57}$$

The algebra $\mathfrak{g}^*$ related to $\mathfrak{g}$ by the Weyl unitary trick

$$\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p} \tag{1.58}$$

is closed under commutation and describes a noncompact group. We find the following two cases

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{g}^*$	
$U(n, Q) \simeq USp(2n)$	$U(n, c)$	$Sp(2n, r)$	(1.59)
$SU(n, c)$	$SO(n)$	$Sl(n, r)$	

Embedding Groups. Embedding groups have been described in the previous section. Let $\mathfrak{k}$ be the Lie algebra for one of the compact groups $U(n, Q)$, $U(n, c)$. Then $\mathfrak{k}$ consists of $n \times n$ antihermitian matrices over the quaternion and complex fields. Using the canonical representations of one field into 2×2 matrices over the next lower field,

$$\begin{aligned} Q &\rightarrow 2 \times 2 \text{ matrices over } C \\ C &\rightarrow 2 \times 2 \text{ matrices over } R \end{aligned} \tag{1.60}$$

we can construct $2n \times 2n$ matrix representations of $\mathfrak{k}$ as follows:

$$\begin{aligned} \mathfrak{u}(n, Q) &\simeq \mathfrak{k}(n \times n, Q) \rightarrow \mathfrak{k}(2n \times 2n, C) \\ \mathfrak{u}(n, c) &\simeq \mathfrak{k}(n \times n, C) \rightarrow \mathfrak{k}(2n \times 2n, R) \end{aligned} \tag{1.61}$$

These matrices are antihermitian and closed under commutation; therefore, they are subalgebras of the Lie algebras for $SU(2n, c)$ and $SO(2n, r)$, respectively. The Lie algebras for these groups can be written in a direct sum decomposition:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \qquad\qquad\qquad\quad \oplus \mathfrak{p} \\ SU(2n, c): \mathfrak{g} &= \mathfrak{k}(2n \times 2n, C) \oplus \mathfrak{p} \\ SO(2n, r): \mathfrak{g} &= \mathfrak{k}(2n \times 2n, R) \oplus \mathfrak{p} \end{aligned} \tag{1.62}$$

The subalgebra $\mathfrak{k}$ and the subspace $\mathfrak{p}$ obey the commutation relations (1.57). The algebra $\mathfrak{g}^*$, related to $\mathfrak{g}$ by the Weyl unitary trick

$$\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p} \tag{1.63}$$

describes the embedding groups $SU^*(2n)$, $SO^*(2n)$, respectively. These results are summarized in Table 6.2.

TABLE 6.2

SUMMARY OF THE REAL FORMS OF THE CLASSICAL GROUPS

Process	Noncompact Group G^*	Maximal Compact Subgroup K	Associated Compact Group G	Dimension Coset Spaces G^*/K, G/K
Indefinite metric (p, q) preserving groups	$SO(p, q)$	$SO(p) \otimes SO(q)$	$SO(p+q)$	pq
	$SU(p, q)$	$S[U(p) \otimes U(q)]$	$SU(p+q)$	$2pq$
	$USp(2p, 2q)$ $\simeq U(p, q; Q)$	$USp(2p) \otimes USp(2q)$ $\simeq U(p; Q) \otimes U(q; Q)$	$USp(2p+2q)$ $\simeq U(p+q; Q)$	$4pq$
Subfield restriction	$Sl(n, r)$	$SO(n)$	$SU(n, c)$	$\frac{n(n+1)}{2} - 1$
	$Sp(2n, r)$	$U(n, c)$	$USp(2n)$ $\simeq U(n; Q)$	$n(n+1)$
Embedding groups	$SO^*(2n)$	$U(n, c)$	$SO(2n)$	$n(n-1)$
	$SU^*(2n)$	$USp(2n)$ $\simeq U(n, Q)$	$SU(2n)$	$\frac{2n(2n-1)}{2} - 1$

II. Topological Properties

1. CONNECTIVITY. Of the groups discussed in this chapter, only $Gl(n, r)$, $O(n, r)$, $O(n, c)$ and $SO(p, q; r)$ are not connected. The group $Gl(n, r)$ has two components, one with det > 0, the other with det < 0. The orthogonal groups $O(n)$ also have two components with det $= \pm 1$. The special orthogonal groups $SO(p, q; r)$ consist of two disconnected pieces when both p and q are odd.

From Table 6.1 we see that for each of the complex extensions there is a corresponding compact real form. The connectivity properties of the compact forms are well known.[3] $SU(n, c)$ is simply connected, as is $USp(2n)$. The groups $SO(n, r)$ are doubly connected except for $SO(2, r)$, which is multiply connected.

The discrete invariant subgroups of these compact forms are also easy to determine. We have already seen that all elements d_i of a discrete invariant subgroup commute with all the group operations g:

$$gd_i = d_i g \qquad d_i \in D, g \in G \tag{2.1}$$

It can be shown (using Schur's lemma) that the only matrices that commute with all these group operations are multiples of the identity. Only group operations of the form λI can be elements of the discrete invariant subgroup.

For the groups $SU(n, c)$,

$$\det \|\lambda I_n\| = \lambda^n I_n = +1 \tag{2.2}$$

Thus λ is an nth root of unity and the discrete invariant subgroup is the cyclic abelian group

$$Z_n = \{I_n, \lambda I_n, \lambda^2 I_n, \ldots, \lambda^{n-1} I_n\} \tag{2.3}$$

There are as many distinct discrete invariant subgroups of $SU(n, c)$ as there are subgroups of Z_n.

For the orthogonal groups $SO(n, r)$,

$$\lambda^n = +1 \tag{2.4}$$

Then λ must be $+1$ if n is odd, but it can be ± 1 if n is even. The orthogonal groups $SO(k)$ differ, depending on whether $k = 2n$ is even or $k = 2n + 1$ is odd:

$$k = 2n \qquad D_{\max} = Z_2 = \{I_{2n}, -I_{2n}\} \tag{2.5}$$

$$k = 2n + 1 \qquad D_{\max} = \{I_{2n+1}\} \tag{2.6}$$

The symplectic groups have a discrete invariant subgroup with two elements: $Z_2 = \{I_{2n}, -I_{2n}\}$.

2. COSETS. Each of the noncompact groups $SU(p, q; c)$, $SO(p, q; r)$, $USp(2p, 2q)$ has associated with it a noncompact algebra $\mathfrak{g}^*$. The vector space of these algebras decomposes quite naturally into two vector subspaces:

$$\mathfrak{g}^* = \mathfrak{k} \oplus \mathfrak{p}^* \tag{2.7}$$

where $\mathfrak{k}$ is the vector subspace consisting of all compact generators and $\mathfrak{p}^*$ consists of the remaining generators, all noncompact.

The vector space $\mathfrak{k}$ is closed under commutation and therefore forms a subalgebra. This compact subalgebra generates the maximal compact subgroups, namely, $S[U(p) \otimes U(q)]$, $SO(p) \otimes SO(q)$, $USp(2p) \otimes USp(2q)$.

The remaining subspace $\mathfrak{p}^*$ is naturally determined in these algebras. It may be regarded as the orthogonal complementary subspace to $\mathfrak{k}$, if we were to take the trouble to define a metric in $\mathfrak{g}^*$ in any reasonable way. With or without a metric, it is easily determined. Since $\mathfrak{p}^*$ is not closed under commutation, it does not form a subalgebra. The group operations EXP $\mathfrak{p}^*$ are elements of the noncompact group but do not themselves form a subgroup.

The naturally occurring decomposition (2.7) in the algebra has a counterpart in the Lie group. This is the coset decomposition. If G^* describes a noncompact Lie group and K the maximal compact subgroup, then we can write

$$G^* = P^*K \qquad \text{or} \qquad G^* = KP^* \tag{2.8}$$

$$\text{left coset decomposition} \qquad\qquad \text{right coset decomposition}$$

$$\begin{aligned} G^* &= \mathrm{EXP}\ \mathfrak{g}^* \\ K &= \mathrm{EXP}\ \mathfrak{k} \\ P^* &= \mathrm{EXP}\ \mathfrak{p}^* \end{aligned}$$

The left or right coset decompositions are unique: every group operation is uniquely written as the product of an element in the subgroup with an element in a coset. A natural choice for the coset representatives is given by EXP $\mathfrak{p}^*$. The space defined by EXP $\mathfrak{p}^*$ is noncompact. Geodesics through the identity are nonrecurring.

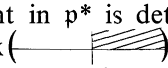
Associated with the noncompact real forms $SU(p, q)$, $SO(p, q)$, $USp(2p, 2q)$ are the compact real forms $SU(p + q)$, $SO(p + q)$, $USp(2p + 2q)$. Their Lie algebras may be written

$$(\mathfrak{g}^*)^* = \mathfrak{k} \oplus (\mathfrak{p}^*)^* = \mathfrak{k} \oplus i(i\mathfrak{p}) = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g} \tag{2.9}$$

The algebras $\mathfrak{g}^*$ and $\mathfrak{g}$ are simply related by the Weyl unitary trick. The space of cosets EXP $(\mathfrak{p}^*)^* \cong$ EXP $\mathfrak{p}$ will now be a compact space with recurring, and sometimes focusing, geodesics.

Example 1. The group $SU(1, 1)$ has maximal compact subgroup $S[U(1) \otimes U(1)] \cong U(1)$. Choosing X_3 as the generator of this maximal compact subgroup, we have the Lie algebra decomposition

$$\mathfrak{su}(1, 1) = \underbrace{\begin{bmatrix} \frac{i}{2}\alpha_3 & \\ & -\frac{i}{2}\alpha_3 \end{bmatrix}}_{\mathfrak{k}} \oplus \underbrace{\begin{bmatrix} & \frac{1}{2}(\alpha_1 - i\alpha_2) \\ \frac{1}{2}(\alpha_1 + i\alpha_2) & \end{bmatrix}}_{\mathfrak{p}^*} \tag{2.10}$$

An arbitrary element in $\mathfrak{p}^*$ is determined by the element in the upper right-hand submatrix , an arbitrary element in the coset space is determined by the values of the parameters in the same submatrix.

$$\text{EXP } \mathfrak{p}^* = \begin{bmatrix} \cosh\frac{\alpha}{2} & (\hat{\alpha}_1 - i\hat{\alpha}_2)\sinh\frac{\alpha}{2} \\ (\hat{\alpha}_1 + i\hat{\alpha}_2)\sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{bmatrix} \tag{2.11}$$

Since the matrix elements of $SU(1, 1)$ obey

$$u_i^{r*}g_{rs}u_j^{\ s} = g_{ij}$$
$$(u_1^{\ 1})^* u_1^{\ 1} - (u_1^{\ 2})^*(u_1^{\ 2}) = 1 \tag{2.12}$$

we can determine the structure of the coset surface by taking

$$x = \hat{\alpha}_1 \sinh\frac{\alpha}{2} \qquad \hat{\alpha}_1 = \frac{\alpha_1}{\alpha}$$

$$y = -\hat{\alpha}_2 \sinh\frac{\alpha}{2} \qquad \hat{\alpha}_2 = \frac{\alpha_2}{\alpha}$$

$$z = \cosh\frac{\alpha}{2} \qquad \alpha^2 = (\alpha_1)^2 + (\alpha_2)^2 \tag{2.13}$$

Then the coset space P^* is the subspace of R_3 determined by the equation

$$z^2 - x^2 - y^2 = 1 \tag{2.14}$$

The coset space of $SU(2)/U(1)$ is determined analogously. Since $\mathfrak{p}^*$ and $\mathfrak{p}$ are related by the unitary trick, so are the surfaces:

$$z^2 + x^2 + y^2 = 1 \tag{2.15}$$

The two coset surfaces are the hyperboloid H^2 and the sphere S^2. The former is noncompact, having nonrecurring geodesics. The latter surface is compact and has both recurring and focusing geodesics.

Example 2. The group $SO(2, 1) \cong SU(1, 1)$ may be treated isomorphically:

$$\mathfrak{so}(2, 1) = \underbrace{\left[\begin{array}{cc|c} 0 & \alpha_3 & 0 \\ -\alpha_3 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]}_{\mathfrak{k}} \oplus \underbrace{\left[\begin{array}{cc|c} 0 & 0 & -i\alpha_2 \\ 0 & 0 & i\alpha_1 \\ \hline i\alpha_2 & -i\alpha_1 & 0 \end{array}\right]}_{\mathfrak{p}^*} \tag{2.10$'$}$$

The exponential of $\mathfrak{p}^*$ may be read from (1.23) of Chapter 5:

$$\text{EXP } \mathfrak{p}^* = \left[\begin{array}{cc|c} & & -i\hat{\alpha}_2 \sinh\alpha \\ & & i\hat{\alpha}_1 \sinh\alpha \\ \hline i\hat{\alpha}_2 \sinh\alpha & -i\hat{\alpha}_1 \sinh\alpha & \cosh\alpha \end{array}\right] \tag{2.11$'$}$$

Since the subspace $\mathfrak{p}^*$ is defined by the upper right-hand submatrix, so also is the coset space. Since $SO(2, 1)$ preserves the metric

$$z^2 - x^2 - y^2 = 1 \tag{2.14'}$$

this coset space is the two-dimensional subspace H^2 of R_3, which satisfies

$$\begin{aligned} z &= \cosh \alpha \\ x &= \hat{\alpha}_1 \sinh \alpha \qquad\qquad z^2 - x^2 - y^2 = 1 \\ y &= -\hat{\alpha}_2 \sinh \alpha \end{aligned}$$

By the Weyl unitary trick, the coset space $SO(3)/SO(2) \cong S^2$. Geodesics through the identity are intersections of planes whose normal is in the xy plane, and the surfaces H^2 and S^2.

In Fig. 6.1 we illustrate how the hyperboloid H^2 originates as the coset $SU(1,1)/U(1) \cong SO(2,1)/SO(2)$. In Fig. 6.2 we illustrate how the sphere S^2 originates as the quotient $SU(2)/U(1) \cong SO(3)/SO(2)$. Figures 6.1 and 6.2 are related to each other by the Weyl unitary trick.

Example 3. It is not necessary to make a coset decomposition with respect to a compact subgroup. We can make the following vector subspace decomposition.

$$\mathfrak{su}(1, 1) = \begin{bmatrix} & \frac{1}{2}\alpha_1 \\ \frac{1}{2}\alpha_1 & \end{bmatrix} \oplus \begin{bmatrix} \frac{i}{2}\alpha_3 & -\frac{i}{2}\alpha_2 \\ \frac{i}{2}\alpha_2 & -\frac{i}{2}\alpha_3 \end{bmatrix} \tag{2.16}$$

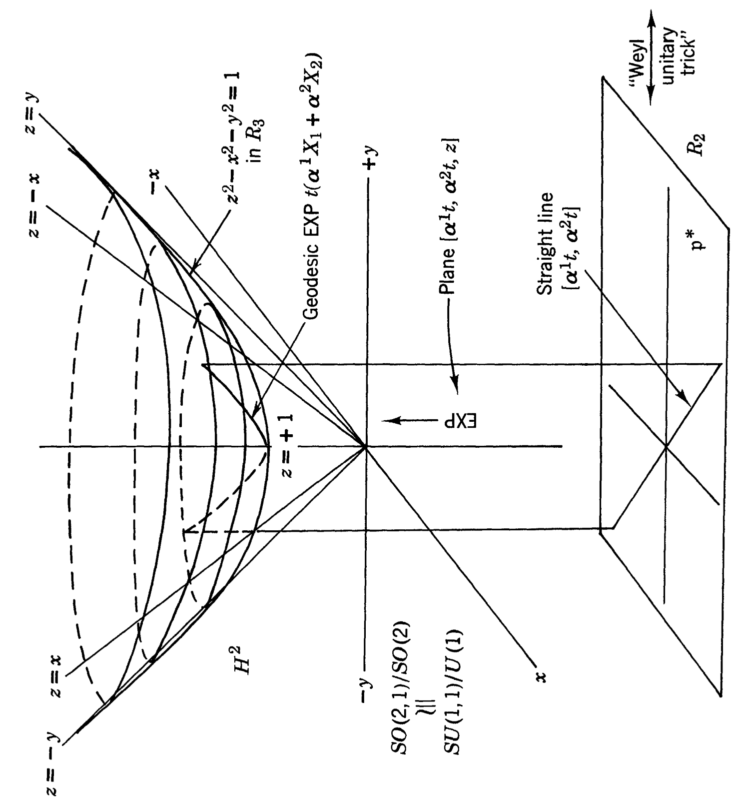
Then the subalgebra generated by X_1 generates the Lorentz group in the plane: $SO(1, 1)$. The remaining subspace maps onto

$$\mathrm{EXP} \begin{bmatrix} \frac{i}{2}\alpha_3 & -\frac{i}{2}\alpha_2 \\ \frac{i}{2}\alpha_2 & -\frac{i}{2}\alpha_3 \end{bmatrix} = \begin{bmatrix} \cos\frac{\phi}{2} + i\frac{\alpha_3}{\phi}\sin\frac{\phi}{2} & -i\frac{\alpha_2}{\phi}\sin\frac{\phi}{2} \\ i\frac{\alpha_2}{\phi}\sin\frac{\phi}{2} & \cos\frac{\phi}{2} - i\frac{\alpha_3}{\phi}\sin\frac{\phi}{2} \end{bmatrix} \tag{2.17}$$

$$z = \mathrm{Im}\, u_1{}^1 = \frac{\alpha_3}{\phi}\sin\frac{\phi}{2} \qquad \phi = \sqrt{(\alpha_3)^2 - (\alpha_2)^2}$$

$$y = \mathrm{Im} - u_1{}^2 = \frac{\alpha_2}{\phi}\sin\frac{\phi}{2}$$

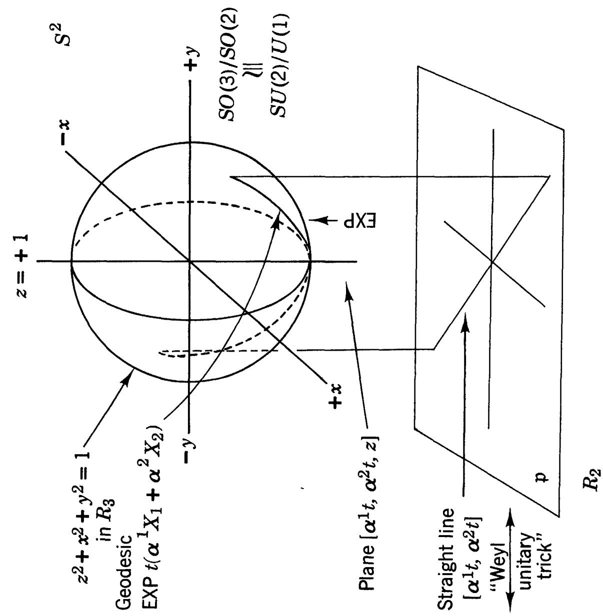
$$x = \mathrm{Re}\, u_1{}^1 = \cos\frac{\phi}{2} \qquad x^2 + z^2 - y^2 = 1 \tag{2.18}$$



$z = y$
$z = -x$
$z = x$
$z = -y$
$-x$
$z^2 - x^2 - y^2 = 1$ in R_3
Geodesic EXP $t(\alpha^1 X_1 + \alpha^2 X_2)$
$+y$
$-y$
x
$z = +1$
H^2
Plane $[\alpha^1 t, \alpha^2 t, z]$
EXP
$SO(2,1)/SO(2) \cong SU(1,1)/U(1)$
Straight line $[\alpha^1 t, \alpha^2 t]$
p^*
R_2
"Weyl unitary trick"

FIG. 6.1 THE COSET SPACE $P^* = \text{EXP}\ \mathfrak{p}^*$, OBTAINED FROM $SO(2,\ 1)/SO(2) \simeq SU(1,\ 1)/U(1)$, APPEARS AS THE SURFACE OF A HYPERBOLOID H^2 IN R_3. THE COSET $O(2,\ 1)/SO(2)$ ALSO INCLUDES THE SECOND (DISCONNECTED) SHEET OF THE HYPERBOLOID CONTAINING THE POINT $z = -1$.

	$SU(1, 1)$	$SO(2, 1)$	
	LEAVES INVARIANT THE METRIC $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	LEAVES INVARIANT THE METRIC $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	
$\mathfrak{su}(1, 1) \bmod \mathfrak{u}(1)$	$\frac{1}{2}\begin{bmatrix} 0 & (\alpha_1 - i\alpha_2) \\ (\alpha_1 + i\alpha_2) & 0 \end{bmatrix}$	$\left[\begin{array}{cc\|c} & & -i\alpha_2 \\ & & i\alpha_1 \\ \hline i\alpha_2 & -i\alpha_1 & \end{array}\right]$	$\mathfrak{so}(2, 1) \bmod \mathfrak{so}(2)$
$\downarrow$ EXP	$\downarrow$	$\downarrow$	$\downarrow$ EXP
$SU(1, 1)/U(1)$	$\begin{bmatrix} \cosh\frac{\alpha}{2} & (\hat{\alpha}_1 - i\hat{\alpha}_2)\sinh\frac{\alpha}{2} \\ (\hat{\alpha}_1 + i\hat{\alpha}_2)\sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{bmatrix}$	$\left[\begin{array}{cc\|c} & & -i\alpha_2 \sinh\alpha \\ & & i\alpha_1 \sinh\alpha \\ \hline i\hat{\alpha}_2 \sinh\alpha & -i\hat{\alpha}_1 \sinh\alpha & \cosh\alpha \end{array}\right]$	$SO(2, 1)/SO(2)$
	$\Vert$	$\Vert$	
	$\begin{bmatrix} z & x - iy \\ x + iy & z \end{bmatrix}$	$\left[\begin{array}{cc\|c} & & -iy \\ & & ix \\ \hline iy & -ix & z \end{array}\right]$	
	EQUATION SATISFIED BY INDEPENDENT COSET REPRESENTATIVE MATRIX ELEMENTS:	$z^2 + (-1)\{(x)^2 + (-y)^2\} = 1$	



S^2
$SO(3)/SO(2)$
$\cong$
$SU(2)/U(1)$
$z = +1$
$-x$
$+y$
$-y$
$+x$
EXP
$z^2 + x^2 + y^2 = 1$
in R_3
Geodesic
EXP $t(\alpha^1 X_1 + \alpha^2 X_2)$
Plane $[\alpha^1 t, \alpha^2 t, z]$
Straight line
$[\alpha^1 t, \alpha^2 t]$
"Weyl
unitary
trick"
p
R_2

FIG. 6.2 THE COSET SPACE $P = \mathrm{EXP}\ \mathfrak{p}$, OBTAINED FROM $SO(3)/SO(2) \simeq SU(2)/U(1)$, APPEARS AS THE SURFACE OF A SPHERE S^2 IN R_3.

	$SU(2)$	$SO(3)$	
	LEAVES INVARIANT THE METRIC $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	LEAVES INVARIANT THE METRIC $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
$\mathfrak{su}(2) \bmod \mathfrak{u}(1)$	$\frac{1}{2}\begin{bmatrix} 0 & (i\alpha_1 + \alpha_2) \\ (i\alpha_1 - \alpha_2) & 0 \end{bmatrix}$	$\left[\begin{array}{cc\|c} 0 & 0 & -\alpha^2 \\ 0 & 0 & \alpha^1 \\ \hline \alpha^2 & -\alpha^1 & 0 \end{array}\right]$	$\mathfrak{so}(3) \bmod \mathfrak{so}(2)$
$\downarrow$ EXP	$\downarrow$	$\downarrow$ EXP	$\downarrow$ EXP
$SU(2)/U(1)$	$\begin{bmatrix} \cos\frac{\alpha}{2} & (i\hat{\alpha}_1 + \hat{\alpha}_2)\sin\frac{\alpha}{2} \\ (i\hat{\alpha}_1 - \hat{\alpha}_2)\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix}$	$\left[\begin{array}{cc\|c} & & -\hat{\alpha}_2 \sin\alpha \\ & & \hat{\alpha}_1 \sin\alpha \\ \hline \hat{\alpha}_2 \sin\alpha & -\hat{\alpha}_1 \sin\alpha & \cos\alpha \end{array}\right]$	$SO(3)/SO(2)$
	$\Vert$	$\Vert$	
	$\begin{bmatrix} z & ix + y \\ ix - y & z \end{bmatrix}$	$\left[\begin{array}{cc\|c} & & -y \\ & & x \\ \hline y & -x & z \end{array}\right]$	
	EQUATION SATISFIED BY INDEPENDENT COSET REPRESENTATIVE MATRIX ELEMENTS:	$z^2 + (+1)\{(x)^2 + (-y)^2\} = 1$	

This surface is shown in Fig. 6.4. Only one geodesic is recursive. All group operations on this geodesic are saddle points on this surface, since the curvature is maximal in one direction and minimal in the other direction.

Example 4. The decompositions $\mathfrak{k} \oplus \mathfrak{p}$ corresponding to Examples 1 and 2 are given as follows:

$$\mathfrak{su}(2) = \mathfrak{u}(1) \oplus [\mathfrak{su}(2) \bmod \mathfrak{u}(1)]$$

$$\begin{bmatrix} \frac{i\alpha_3}{2} & \\ & \frac{-i\alpha_3}{2} \end{bmatrix} \oplus \frac{i}{2}\begin{bmatrix} & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \end{bmatrix}$$

$$\downarrow \text{EXP} \qquad\qquad \text{EXP} \downarrow$$

$$\begin{bmatrix} e^{i\alpha_3/2} & \\ & e^{-i\alpha_3/2} \end{bmatrix} \otimes \begin{bmatrix} \cos\frac{\alpha}{2} & i(\hat{\alpha}_1 - i\hat{\alpha}_2)\sin\frac{\alpha}{2} \\ i(\hat{\alpha}_1 + i\hat{\alpha}_2)\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix} \quad [2.19\ \mathrm{SU}(2)]$$

$$\mathfrak{so}(3) = \mathfrak{so}(2) \oplus [\mathfrak{so}(3) \bmod \mathfrak{so}(2)]$$

$$\begin{bmatrix} 0 & \alpha_3 & 0 \\ -\alpha_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & -\alpha_2 \\ 0 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{bmatrix}$$

$$\downarrow \text{EXP} \qquad \text{EXP} \downarrow$$

$$\left[\begin{array}{cc|c} \cos\alpha_3 & \sin\alpha_3 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \otimes \left[\begin{array}{cc|c} & & -\hat{\alpha}_2\sin\alpha \\ & & \hat{\alpha}_1\sin\alpha \\ \hline \hat{\alpha}_2\sin\alpha & -\hat{\alpha}_1\sin\alpha & \cos\alpha \end{array}\right] \quad [2.19\ \mathrm{SO}(3)]$$

The cosets $SU(2)/U(1)$ and $SO(3)/SO(2)$ are isomorphic with the sphere surface, using the arguments of (2.15). These cosets therefore have volume 4π.

The subgroups $U(1)$, $SO(2)$ are parameterized by the values of α_3:

$$\begin{array}{ll} U(1) & 0 \le \alpha_3 < 4\pi \\ SO(2) & 0 \le \alpha_3 < 2\pi \end{array} \qquad (2.20)$$

Since $U(1)$ and $SO(2)$ are abelian, they have constant invariant measure. Their volumes are therefore 4π and 2π, respectively. The invariant volumes of the groups $SU(2)$ and $SO(3)$ are

$$\begin{array}{cc}
V[SU(2)] & V[SO(3)] \\
\Vert & \Vert \\
V\left[\dfrac{SU(2)}{U(1)} \otimes U(1)\right] & V\left[\dfrac{SO(3)}{SO(2)} \otimes SO(2)\right] \\
\Vert & \Vert \\
V\left[\dfrac{SU(2)}{U(1)}\right] \times V[U(1)] & V\left[\dfrac{SO(3)}{SO(2)}\right] \times V[SO(2)] \\
\Vert & \Vert \\
16\pi^2 = 4\pi \times 4\pi & 4\pi \times 2\pi = 8\pi^2
\end{array} \tag{2.19'}$$

These volumes reflect the $2 \to 1$ nature of the homomorphism between $SU(2)$ and $SO(3)$:

$$\begin{gathered}
SU(2) \xrightarrow{2\to 1} SO(3) \\
V[SU(2)] = 2V[SO(3)]
\end{gathered}$$

3. CONTRACTION. The sphere and the hyperboloid are coset spaces of the form

$$\frac{SU(2)}{U(1)} \quad \text{and} \quad \frac{SU(1,1)}{U(1)} \tag{2.21}$$

respectively. They are related by the unitary trick. Moreover, as the characteristic parameter R describing these surfaces in R_3 becomes larger and larger

$$z^2 \pm (x^2 + y^2) = R^2 \to \infty \tag{2.22}$$

these surfaces look more and more like the plane R_2. We ask, how is the plane R_2 related to cosets?

The commutation relations for $SO(3)$ and $SO(2,1)$ may be summarized

$$\begin{aligned}
[X'_1, X'_2] &= -\lambda^2 X'_3 & X'_1 &= \lambda X_1 \\
[X'_2, X'_3] &= -X'_1 & X'_2 &= \lambda X_2 \\
[X'_3, X'_1] &= -X'_2 & X'_3 &= X_3
\end{aligned} \tag{2.23}$$

For λ real, $\neq 0$, the commutation relations may always be renormalized to the value $\lambda = 1$. This gives the algebra of $SO(3)$. For λ imaginary, $\neq 0$, these

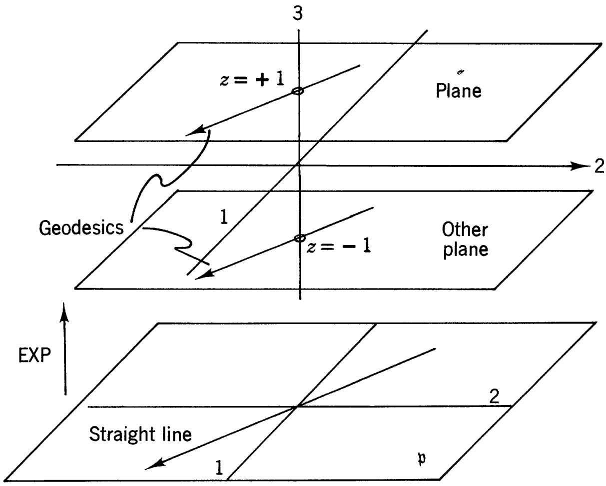


FIG. 6.3 THE PLANE IS THE LIMITING CASE (2.24) OF BOTH THE SPHERE AND THE HYPERBOLOID. THE COSET $IO(2)/SO(2)$ IS THE DISCONNECTED PAIR OF PLANES THROUGH THE POINTS $z = +1$ [WHICH IS ALSO THE COSET REPRESENTATIVE FOR $ISO(2)/SO(2)$] AND $z = -1$.

$ISO(2)$

LEAVES INVARIANT THE METRIC

$$\left[\begin{array}{cc|c} 0 & 0 & -\alpha_2 \\ 0 & 0 & \alpha_1 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$\mathfrak{iso}(2)$ mod $\mathfrak{so}(2)$

$$\downarrow \text{EXP}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \infty \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -\alpha_2 \\ 0 & 1 & \alpha_1 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$ISO(2)/SO(2)$

$$\left[\begin{array}{cc|c} & & -y \\ & & x \\ \hline 0 & 0 & z \end{array}\right]$$

EQUATION SATISFIED BY INDEPENDENT COSET REPRESENTATIVE MATRIX ELEMENTS:

$$z^2 + (0)[(x)^2 + (-y)^2] = 1$$
$$z = +1$$

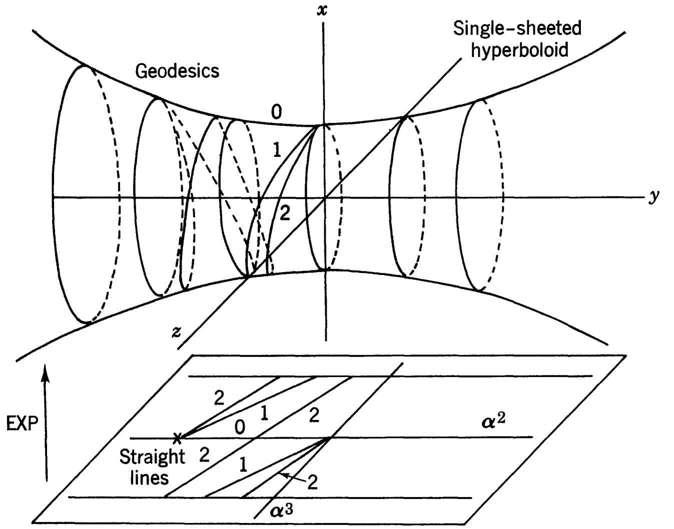


FIG. 6.4 THE SINGLE-SHEETED HYPERBOLOID IS OBTAINED AS THE COSET OF A NONCOMPACT GROUP BY A NONCOMPACT SUBGROUP: $SO(2, 1)/SO(1, 1) \simeq SU(1, 1)/R_1$.

$SO(2, 1)$

LEAVES INVARIANT THE METRIC

$$\left[\begin{array}{c|cc} 0 & \alpha^3 & -\alpha^2 \\ \hline -\alpha^3 & 0 & 0 \\ \alpha^2 & 0 & 0 \end{array}\right] \qquad \mathfrak{so}(2, 1) \bmod \mathfrak{so}(1, 1)$$

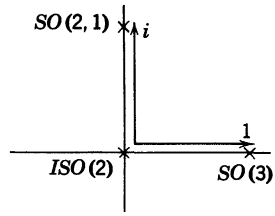
$$\Big\downarrow \text{EXP} \qquad\qquad \text{EXP} \Big\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \left[\begin{array}{c|cc} x & z & -y \\ \hline -z & & \\ y & & \end{array}\right] \qquad SO(2, 1)/SO(1, 1)$$

EQUATION SATISFIED BY INDEPENDENT COSET REPRESENTATIVE MATRIX ELEMENTS:

$$(x)^2 + (z)^2 - (-y)^2 = 1$$

commutations can be renormalized to the value $\lambda \to i$. This reproduces the algebra of $SO(2, 1)$.



(2.24)

This suggests that the commutation relations can be obtained from each other by following the path shown. For $\lambda = 0$ the commutation relations are formally those of the group of rigid translations and rotations in the plane: $ISO(2)$. The plane is isomorphic to the coset space $ISO(2)/SO(2)$. The relation between these three spaces is shown in (2.24) and in Fig. 6.1 ($\lambda = i$), Fig. 6.2 ($\lambda = 1$), and Fig. 6.3 ($\lambda = 0$). In Chapter 10 we describe the processes of contraction and expansion by means of which these metric-preserving homogeneous and inhomogeneous groups are related.

Résumé

In Section I of this chapter we presented the general structure for all the classical matrix groups. These groups fall essentially into three series: the unitary, the orthogonal, and the symplectic. Within each series are a number of different groups and algebras. But the distinctions between them deal with the reality properties on their Lie algebras. Within a series, all groups have a common complex extension; they may be regarded as analytic continuations of one another.

The commutation relations for the complex extensions were given. The commutation relations for the various real forms are easily obtained by making the appropriate reality restrictions.

The group generators are bases for the vector space of the Lie algebra. A smaller number of bases exist for the algebra. The entire algebra can be generated from these bases by taking appropriate commutation relations and symmetry properties. A very remarkable similarity exists between the bases for the unitary, orthogonal, symplectic, and symmetric groups. For the $n \times n$ matrix groups only $n - 1$ generators are enough to regenerate the entire algebra. This suggests the representation theories for these groups should all be closely related.

Various canonical decompositions of these Lie algebras, corresponding to coset decompositions, were discussed in Section II. The cosets for $SO(2, 1)/SO(2)$, and so on, were computed explicitly. These cosets easily receive an interpretation as subspaces in spaces of higher dimensionality. The study of these so-called Riemannian symmetric spaces as coset spaces of Lie groups leads to an understanding of the relationship between various classical surfaces (S^2, R_2, H^2). It also suggests strongly that we must eventually study some particular limiting processes for constructing new Lie groups from others (group contraction and group expansion).

Exercises

1. Let the infinitesimal transformation properties in V_N be given by

$$x^i \to x'^i = (I + \delta M)^i{}_j\, x^j$$

Compute the generator of infinitesimal displacements on the differential form dx^i. Show that

$$dx^i \to dx'^i = \delta M^i{}_j\, dx^j$$

2. Compute the generators of infinitesimal displacements on the forms

$$dx^i \wedge dx^j$$

3. Let

$$d\omega = f_{ijk\ldots}(x)\, dx^i \wedge dx^j \wedge dx^k \wedge \cdots$$

The generator of infinitesimal displacements on the form $d\omega$ is

$$[X(x) f_{ijk\ldots}(x)]\, dx^i \wedge dx^j \wedge dx^k \wedge \cdots + f_{ijk\ldots}(x) X(dx^i) \wedge dx^j \wedge dx^k \wedge \cdots$$
$$+ f_{ijk\ldots}(x)\, dx^i \wedge X(dx^j) \wedge dx^k \cdots + f_{ijk\ldots}(x)\, dx^i \wedge dx^j \wedge X(dx^k) \wedge \cdots + \cdots$$

4. Prove that, for any Lie algebra $\mathfrak{g}$ with a faithful matrix representation in which the subspace $\mathfrak{k}$ consists of block diagonal matrices and the subspace $\mathfrak{p}$ consists of off-diagonal matrices [see (1.49)], the subspace $\mathfrak{k}$ is closed under commutation, and therefore an algebra, whereas the subspace $\mathfrak{p}$ is not closed, and therefore not an algebra. It is assumed that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and that $[\mathfrak{p}, \mathfrak{p}] \neq 0$.

5. List the classical groups given in Table 6.1 for which there is a decomposition of the form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ are complex spaces as described in Problem 4, *and* $\mathfrak{k}$ is a maximal compact algebra. Show that the only two real groups (as distinguished from their complex extensions) that do not possess these properties are $Sl(n, r)$ and $SU^*(2n)$.

6. Let $D(A)$ be the matrix representation of the vector A in the defining matrix representation of one of the classical Lie algebras. Show that

$$(A, B) = \operatorname{tr} D(A)\, D(B)$$

obeys all the axioms of a metric, and can therefore be used to define a metric on any of the classical algebras.

7. Show that $(\mathfrak{k}, \mathfrak{p}) = 0$ under the metric defined in Problem 6, where $\mathfrak{k}$, $\mathfrak{p}$ are as described in Problem 4.

8. Show that $(\mathfrak{k}, \mathfrak{p}^*) = 0$, where $\mathfrak{p}^*$ is the dual to the subspace $\mathfrak{p}$ defined in Problem 7.

9. Write down the Lie algebra for $SO(4)$. Construct the three-dimensional subspace $\mathfrak{p}$ of 4×4 matrices

$$\mathfrak{so}(4) \bmod \mathfrak{so}(3)$$

Show that this subspace maps under EXP onto the surface of the unit sphere $S^3 \subset R_4$. Construct geodesics in S^3 explicitly by mapping a straight line in $\mathfrak{p}$ onto S^3. Do these geodesics focus?

10. Construct $H^3 \subset R_4$ by dualizing Problem 9. The elements in H^3 belong to the coset $SO(3, 1)/SO(3)$. Interpret these group operations physically as "boosts" from a coordinate system at rest to one moving with uniform velocity $\mathbf{v}$, in real physical Minkowski space time. Show that the product of two such "boosts" is not in general a boost, but differs from a boost by a rotation in three-dimensional space R_3. This rotation is called a **Wigner rotation**. When the two boosts are collinear their product is a pure boost. Obtain the familiar Einstein law for relativistic velocity addition in this way. The Wigner rotation is called the Thomas precession in this case.

11. Assume a physical system with 3 internal degrees of freedom evolves from the ground state at time t_0 under an arbitrary driving field which is classical. Show that its state at any time $t > t_0$ can be completely described by a point in the coset $SU(3)/U(2)$. Show that its state vector $\psi(t)$ evolves along a trajectory in this coset space. Compute the equations of motion in the coset in terms of the external classical driving field. Do this for a two-level system, and see that the results are contained in Chapter 5, Section VI. Generalize to a system with r internal degrees of freedom.

12. Show that the orthogonal, symplectic, and unitary ensembles of interest in the statistical theory of energy levels[4] correspond respectively to the cosets

$$\frac{U(n)}{SO(n)}, \qquad \frac{U(2n)}{USp(2n)}, \qquad \frac{U(n)}{\mathrm{Id}}$$

13. Let M be the matrix represented by (1.43) and let

$$G = \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix}.$$

Then

$$MG + GM^t = 0$$

so that the algebra $\mathfrak{u}(n)$ in $\mathfrak{so}(2n)$ preserves

(a) $g_{ij} = +g_{ji} = \delta_{ij}$

(b) $g'_{ij} = \text{sign}\,(i)\,\delta(i + j, 0); \qquad i, j = \pm 1, \pm 2, \ldots, \pm n.$

14. Let $\mathfrak{g}$ be a matrix Lie algebra and assume that $\mathfrak{g}$ has a direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

Prove that the vector subspaces $\mathfrak{k}$, $\mathfrak{p}$ obey the commutation relations (1.57) when

	$\mathfrak{k}$	$\mathfrak{p}$
(a)	$\mathfrak{g} + \mathfrak{g}^*$	$\mathfrak{g} - \mathfrak{g}^*$
(b)	$\mathfrak{g} - \mathfrak{g}^t$	$\mathfrak{g} + \mathfrak{g}^t$
(c)	$\mathfrak{g} - \mathfrak{g}^\dagger$	$\mathfrak{g} + \mathfrak{g}^\dagger$

provided in each case that $\mathfrak{k} \subset \mathfrak{g}$, $\mathfrak{p} \subset \mathfrak{g}$.

15. Let $\mathfrak{m}$ be the matrix Lie algebra obtained from $\mathfrak{gl}(n, q)$ by replacing each quaternion by the corresponding 2×2 complex matrix. Show that the subset of $2n \times 2n$ complex matrices $M \in \mathfrak{m}$ obeying either condition (a) or (b) or (c) below is a subalgebra of $\mathfrak{m}$. Show that these subalgebras are isomorphic with the classical algebras indicated:

(a)	$\mathrm{Tr}\, M = 0$	$\mathfrak{su}^*(2n) \simeq \mathfrak{sl}(n, q)$
(b)	$M^t + M = 0$	$\mathfrak{so}^*(2n) \simeq \mathfrak{so}(n, q)$
(c)	$M^\dagger + M = 0$	$\mathfrak{usp}(2n) \simeq \mathfrak{su}(n, q)$

Notes and References

1. S. Helgason. [1], pp. 339–346.
2. R. Gilmore. [4]
3. H. Weyl. [2]
4. F. J. Dyson. [1–3]

CHAPTER 7

Lie Algebras and Root Spaces

The reader more interested in results than methods may skip over this long and difficult chapter. The major results, the canonical commutation relations, are presented again at the beginning of the following chapter. Here, however, we seek a canonical form for the commutation relations of an arbitrary algebra. To this end we trot out the big guns one by one, and use them to demolish the structure of an algebra into its irreducible components.[1,2]

The first of the major tools, discussed in the first section, is the *regular* or *adjoint* representation of the algebra, equivalent to the structure constants. A study of the various possible subalgebras that an algebra may have leads to a semicanonical structure for an arbitrary algebra.

In the second section we present the secular equation and its roots, which lead to further information about the structure of an algebra. This information is summarized in the first criterion of solvability.

In the third section a metric is introduced (Cartan-Killing form) on the vector space associated with the Lie algebra. The metric presents yet more information about the structure of an algebra. This information is summarized in the second criterion of solvability.

These two criteria are molded into a single very powerful tool for analyzing any Lie algebra—the Cartan criterion, which is used in Section IV to prove the complete reducibility and faithfulness of the regular representation for semisimple algebras.

In Section V the root and metric concepts are exploited together to give a canonical structure to the commutation relations of an arbitrary semisimple algebra.

I. General Structure Theory for Lie Algebras

1. THE BASIC TOOLS. In this chapter we seek a canonical form for an arbitrary Lie algebra $\mathfrak{g}$. This canonical form is valid also for the Lie group G whose algebra is $\mathfrak{g}$. The mechanisms and proofs used to construct such a

canonical form may be applied either to the group or its algebra. Generally speaking, these proofs are related to each other by exponentiation or linearization

$$\text{proof for algebra} \underset{\ln}{\overset{\text{EXP}}{\rightleftarrows}} \text{proof for group} \tag{1.1}$$

Quite often, a proof for one involves niceties not entailed in the proof for the other. Since these proofs lead in the same general directions and to the same general place, we will follow the path of least resistance.

The properties of a Lie algebra are completely determined by its structure constants $C_{ij}{}^{k}$. With respect to a basis X_i for the linear vector space associated with the algebra, the $C_{ij}{}^{k}$ are defined by

$$[X_i, X_j] = C_{ij}{}^{k} X_k \tag{1.2}$$

Under a change of basis (1.3), the structure constants have the transformation properties of a third-order tensor (1.5), covariant of second rank, and contravariant of first:

$$Y_r = V_r{}^{i} X_i \tag{1.3}$$

$$[Y_r, Y_s] = D_{rs}{}^{t} Y_t \tag{1.4}$$

$$D_{rs}{}^{t} = V_r{}^{i} V_s{}^{j} C_{ij}{}^{k} (V^{-1})_k{}^{t} \tag{1.5}$$

The classification of Lie algebras involves the detailed study of this particular tensor. Such an investigation is facilitated by considering the tensor as a collection of matrices

$$-C_{ij}{}^{k} = (M_i)_j{}^{k} = \mathbf{R}(X_i)_j{}^{k} \tag{1.6}$$

We have already seen (Chapter 4, Section IV.2) that structure constants for an η-dimensional Lie algebra provide an $\eta \times \eta$ matrix representation for the algebra, called the regular or adjoint representation. The search for a canonical form for the commutation relations of a Lie algebra reduces to a search for a canonical form for the matrices $\mathbf{R}(X_i)$.

We can make a dent in the structure theory for $\mathbf{R}(X_i)$ by searching for subgroups of a Lie group. Let G be a Lie group with algebra $\mathfrak{g}$ and regular representation $\mathbf{R}(\mathfrak{g})$. The existence of a subgroup H implies very specific properties for the regular representation $\mathbf{R}(\mathfrak{h})$ of the subalgebra $\mathfrak{h}$. A systematic study of the subalgebras $\mathfrak{h}$, $\mathfrak{k}$, ... of an algebra $\mathfrak{g}$ leads to a systematic *reduction* and simplification in the structure of the regular representation. This entire procedure is carried out in Section I. It is as far as we can go without introducing additional concepts and more powerful techniques.

The regular representation is a linear transformation, or change of basis, in the linear vector space of a Lie algebra. All the powerful machinery of linear algebra may then be applied to this transformation in an attempt to put it into a canonical form. This is carried out in the usual way: the secular equation for the transformation $\mathbf{R}(X_i)$ is found and solved, and $\mathbf{R}(X_i)$ is transformed to the associated Jordan canonical form. This study is carried out in Section II.

It is possible to introduce a metric on the linear vector space of the algebra in a natural way. This is done in Section III. The metric reflects many of the properties of the algebra.

Finally, the eigenvalue techniques of Section II and the metric form of Section III may be used in conjunction to produce a powerful tool for distinguishing between the two essentially different kinds of Lie algebras (Section IV). The same procedure leads (Section V) to a canonical form for the commutation relations of one of these two classes (the semisimple algebras).

The canonical form developed in Section V may be exploited to provide a complete classification and description of all possible simple Lie algebras (Chapter 8).

2. THE REGULAR REPRESENTATION. The regular representation can always be defined for a group, field, linear vector space, algebra, and so on. A vector in a linear vector space can be associated with each element of the particular algebraic structure. The mappings

$$\begin{aligned} g &\to |g\rangle & g &\in \text{group } G \\ f &\to |f\rangle & f &\in \text{field } F \\ v &\to |v\rangle & v &\in \text{linear vector space } V \\ X_i &\to |X_i\rangle & X_i &\in \text{algebra } \mathfrak{g} \end{aligned} \tag{1.7}$$

associate vector spaces with distinct kinds of algebraic structures. A combinatorial operation in the algebraic structure will lead to a linear transformation in the associated vector space. For example, for a group and a Lie algebra, we can define the natural mappings

$$h|g\rangle = |gh^{-1}\rangle \tag{1.8g}$$

$$A|B\rangle = |[B, A]\rangle \tag{1.8a}$$

that map vectors in the vector space into other vectors that are naturally defined under the combinatorial operation. These linear transformations are called the **regular** or **adjoint** representation of the group and the algebra, respectively. The regular representation for a field can be defined similarly. The regular representation for a linear vector space is its canonical representation, studied in Chapter 1.

The regular representation for a Lie algebra has exactly as much information as the structure constants of the algebra in a more convenient form. For if X_i form a basis for the vector space of the algebra, then

$$\mathbf{R}(X_i)|X_j\rangle = |[X_j, X_i]\rangle = |-C_{ij}{}^k X_k\rangle = (-C_{ij}{}^k)|X_k\rangle \tag{1.9}$$

$$\mathbf{R}(X_i)_j{}^k = -C_{ij}{}^k \tag{1.10}$$

A study of either is equivalent to a study of the other.

The regular representation is not faithful in general. Suppose T_1 and T_2 are generators for the two-dimensional abelian group of rigid translations of the plane. Then we have

$$[T_1, T_2] = 0$$

$$\mathbf{R}(p^1 T_1 + p^2 T_2) = p^1 \mathbf{R}(T_1) + p^2 \mathbf{R}(T_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{1.11}$$

However, the regular representation is faithful *enough*. It may not distinguish between two distinct elements of one Lie algebra, but it does distinguish between different Lie algebras. Since we want to classify the various algebras, it is sufficient to study the regular representation.

The importance of the regular representation can be understood from the following three comments.

1. A multiple commutator can be associated with a product of matrices:

$$\mathbf{R}(A)\mathbf{R}(B)\mathbf{R}(C)|D\rangle = |[[[D, C], B], A]\rangle \tag{1.12}$$

This allows a correspondence between an associative algebra of $\eta \times \eta$ matrices that obey simple matrix multiplication and a Lie algebra with a more complex nonassociative combinatorial operation.

2. Matrices describe linear transformations of vector spaces. A great deal of machinery exists for the study of such transformations.

3. We can study both $\mathbf{R}(A)$ and A simultaneously, and use whichever is more convenient.

Example. The generators of the Galilean group are

$L_i = (\mathbf{r} \times \nabla)_i = \varepsilon_{ijk} x^j \partial_k$ generators of rotations about x^i

$v_i = t\nabla_i$ "boosts" along i direction

$p_i = \nabla_i$ displacements along i direction

$T = \dfrac{\partial}{\partial t}$ displacements along time axis (1.13)

The commutation relations are summarized by

$$
\begin{array}{llll}
\mathbf{L} \times \mathbf{L} = -\mathbf{L} & & \text{with} & \\
\mathbf{L} \times \mathbf{v} = -\mathbf{v} & & \mathbf{A} \times \mathbf{B} = -\mathbf{C} & \\
\mathbf{L} \times \mathbf{p} = -\mathbf{p} & & & \\
[T, \mathbf{v}] = \mathbf{p} & & [A_i, B_j] = -\varepsilon_{ijk} C_k & \qquad (1.14)
\end{array}
$$

All other commutators vanish. These may all be summarized in the regular representation as

$$
\boldsymbol{\theta} \cdot \mathbf{L} + \mathbf{a} \cdot \mathbf{v} + \mathbf{b} \cdot \mathbf{p} + \tau T \quad \xrightarrow{\text{Regular representation}} \tag{1.15}
$$

$$
\begin{array}{c|c}
 & \begin{array}{cccc} \mathbf{L} & \mathbf{v} & T & \mathbf{p} \end{array} \\
\begin{array}{c} \mathbf{L} \\ \mathbf{v} \\ T \\ \mathbf{p} \end{array} &
\left[\begin{array}{c|c|c|c}
-\boldsymbol{\theta} \cdot \boldsymbol{\Sigma} & -\mathbf{a} \cdot \boldsymbol{\Sigma} & & -\mathbf{b} \cdot \boldsymbol{\Sigma} \\ \hline
 & -\boldsymbol{\theta} \cdot \boldsymbol{\Sigma} & & \tau I_3 \\ \hline
 & & & -\mathbf{a} \\ \hline
 & & & -\boldsymbol{\theta} \cdot \boldsymbol{\Sigma}
\end{array}\right]
\end{array}
$$

$$
\mathbf{A} \cdot \boldsymbol{\Sigma} = \begin{bmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{bmatrix} \tag{1.16}
$$

We notice that the Galilean algebra has a very large number of subalgebras (any single set* of $\mathbf{L}$; $\mathbf{v}$; $\mathbf{p}$; T; any pair of sets except $T + \mathbf{v}$; any of the sets $\mathbf{L} + \mathbf{p} + \mathbf{v}$; $\mathbf{L} + \mathbf{p} + T$; $\mathbf{p} + \mathbf{v} + T$; but not $\mathbf{L} + \mathbf{v} + T$, which is not closed under commutation). We also notice that the regular representation has a surprisingly large number of zeroes. The existence of subalgebras and the presence of excess zeroes in the regular representation are closely related.

These considerations suggest that we make a detailed study of the kinds of subalgebras which a Lie algebra may possess.

3. SYSTEMATICS OF SUBALGEBRAS. Let $X_i, X_j, X_k, \ldots$ be the n generators for the subalgebra $\mathfrak{h}$ of an η-dimensional Lie algebra $\mathfrak{g}$, and let $X_\alpha, X_\beta, \ldots$ be the $\eta - n$ additional bases needed to span the algebra $\mathfrak{g}$. Then the regular representation for the generators X_α, X_i has the structure

* Technically speaking, we mean the *linear closure* of the set of generators indicated. We will henceforth refrain from speaking technically whenever possible.

$$\mathbf{R}(X_\alpha) \xrightarrow{\text{reg}} \begin{array}{c|cc} & \gamma & k \\ \hline \beta & \left[\begin{array}{c|c} * & * \\ C_{\alpha\beta}{}^{\gamma} & C_{\alpha\beta}{}^{k} \\ \hline * & * \\ C_{\alpha j}{}^{\gamma} & C_{\alpha j}{}^{k} \end{array}\right] \\ j & \end{array} = \left[\begin{array}{c|c} \mathbf{R}_{11}(X_\alpha) & \mathbf{R}_{12}(X_\alpha) \\ \hline \mathbf{R}_{21}(X_\alpha) & \mathbf{R}_{22}(X_\alpha) \end{array}\right] \tag{1.17}$$

$$\mathbf{R}(X_i) \xrightarrow{\text{reg}} \begin{array}{c|cc} & \gamma & k \\ \hline \beta & \left[\begin{array}{c|c} * & * \\ C_{i\beta}{}^{\gamma} & C_{i\beta}{}^{k} \\ \hline C_{ij}{}^{\gamma}=0 & * \\ \text{subalgebra} & C_{ij}{}^{k} \end{array}\right] \\ j & \end{array} = \left[\begin{array}{c|c} \mathbf{R}_{11}(X_i) & \mathbf{R}_{12}(X_i) \\ \hline \bigcirc & \mathbf{R}_{22}(X_i) \end{array}\right] \tag{1.17'}$$

The submatrices marked * contain nonzero elements in general.

The existence of a subalgebra leads to the presence of zero submatrices in the regular representatives of the subalgebra elements. This is abundantly verified by inspecting the regular representation (1.15) for the Galilean group.

The commutator of two elements can be computed from the regular representation:

$$\mathbf{R}([A, B]) = \mathbf{R}(A)\mathbf{R}(B) - \mathbf{R}(B)\mathbf{R}(A) \tag{1.18}$$

The commutator of two elements in the subalgebra spanned by the X_i is computed from (1.17′) and (1.18):

$$\mathbf{R}([X_i, X_j]) = \left[\begin{array}{c|c} \begin{array}{c} \mathbf{R}_{11}(X_i)\mathbf{R}_{11}(X_j) \\ -\mathbf{R}_{11}(X_j)\mathbf{R}_{11}(X_i) \end{array} & \begin{array}{c} \mathbf{R}_{11}(X_i)\mathbf{R}_{12}(X_j) + \mathbf{R}_{12}(X_i)\mathbf{R}_{22}(X_j) \\ -\mathbf{R}_{11}(X_j)\mathbf{R}_{12}(X_i) - \mathbf{R}_{12}(X_j)\mathbf{R}_{22}(X_i) \end{array} \\ \hline \bigcirc & \begin{array}{c} \mathbf{R}_{22}(X_i)\mathbf{R}_{22}(X_j) \\ -\mathbf{R}_{22}(X_j)\mathbf{R}_{22}(X_i) \end{array} \end{array}\right] \tag{1.19}$$

$$= \left[\begin{array}{c|c} \mathbf{R}_{11}([X_i, X_j]) & * \\ \hline \bigcirc & \mathbf{R}_{22}([X_i, X_j]) \end{array}\right] \tag{1.19'}$$

Not only do the $\eta \times \eta$ matrices $\mathbf{R}(X_i)$ form a representation of the subalgebra $\mathfrak{h}$, but so also do the $n \times n$ submatrices $\mathbf{R}_{22}(X_i)$ and the $(\eta - n) \times (\eta - n)$ submatrices $\mathbf{R}_{11}(X_i)$.

It should come as no surprise that the submatrices

$$\mathbf{R}_{22}(X_i) \rightarrow \mathbf{R}(X_i)_j{}^k = -C_{ij}{}^k \tag{1.20}$$

form a representation for $\mathfrak{h}$, since they *are* the structure constants of $\mathfrak{h}$. It is somewhat surprising that the submatrices $\mathbf{R}_{11}(X_i) \to -C_{i\beta}{}^{\gamma}$ also form a representation for $\mathfrak{h}$.

The vector space that carries the representation $\mathbf{R}_{22}(\mathfrak{h})$ consists of all elements X in the algebra $\mathfrak{h}$. The $(\eta - n)$-dimensional linear vector space that carries the representation $\mathbf{R}_{11}(\mathfrak{h})$ consists of those vectors of $\mathfrak{g}$ whose difference is a vector not in $\mathfrak{h}$. We denote this space as $\mathfrak{g}/\mathfrak{h}$ or $\mathfrak{g}$ mod $\mathfrak{h}$ or $\mathfrak{g} - \mathfrak{h}$. Two vectors in $\mathfrak{g}$ mod $\mathfrak{h}$ are identified if their difference lies in $\mathfrak{h}$:

$$c^{\alpha}X_{\alpha} + c^{i}X_{i} \equiv c^{\alpha}X_{\alpha} + d^{i}X_{i} \qquad \text{in} \quad \mathfrak{g} \bmod \mathfrak{h} \tag{1.21}$$

All vectors in $\mathfrak{h}$ project onto the vector 0 in $\mathfrak{g}$ mod $\mathfrak{h}$. This concept is illustrated in Fig. 7.1.

This "difference subspace" of a Lie algebra has an interpretation in terms of Lie groups as well. If

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\text{EXP}} G \\ \mathfrak{h} &\xrightarrow{\text{EXP}} H \\ \mathfrak{g} - \mathfrak{h} &\xrightarrow{\text{EXP}} \frac{G}{H} = \text{coset representatives} \end{aligned} \tag{1.22}$$

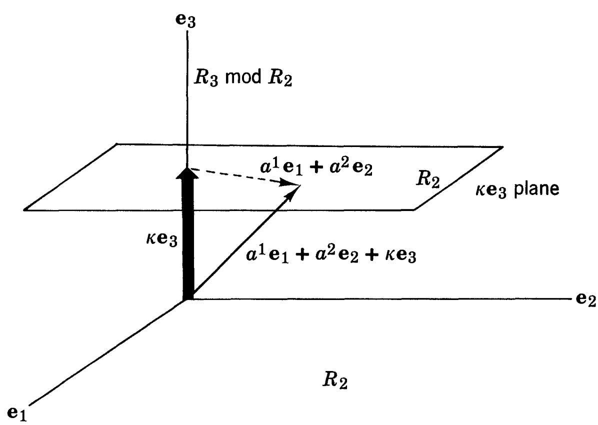


FIG. 7.1 EXAMPLE OF THE MODULUS CONCEPT. TWO VECTORS IN R_3 MOD R_2 ARE IDENTICAL IF THEIR DIFFERENCE LIES COMPLETELY WITHIN R_2. THE VECTORS $\kappa\mathbf{e}_3$ AND $a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + \kappa\mathbf{e}_3$ ARE IDENTICAL IN R_3 MOD R_2 BECAUSE THEIR DIFFERENCE $a^1\mathbf{e}_1 + a^2\mathbf{e}_2$ LIES COMPLETELY WITHIN THE SUBSPACE R_2. ALL VECTORS WITH THEIR "HEADS" IN THE $\kappa\mathbf{e}_3$ PLANE ARE IDENTIFIED WITH A SINGLE VECTOR IN R_3/R_2 OR $R_3 - R_2$. THE SPACE R_3 IS SLICED INTO A $(3 - 2 =)$ 1-DIMENSIONAL "MODULUS SPACE" OF PLANES R_2.

The matrix representation $\mathbf{R}_{11}(\mathfrak{h})$ is defined on the vector subspace of a Lie algebra which exponentiates onto the coset representatives G/H.

The matrix representation $\mathbf{R}(\mathfrak{h})$ is said to be **reducible** because it has the structure (1.17′), in which a submatrix ($\mathbf{R}_{21}$) vanishes. This is equivalent to the following statement: $\mathbf{R}(\mathfrak{h})$ maps the subspace $\mathfrak{h}$ into itself. The subspace $\mathfrak{h}$ is said to be **invariant** under a transformation when that transformation maps every vector back into that subspace.

Under what circumstances is the linear vector space $\mathfrak{g}$ mod $\mathfrak{h}$, spanned by the X_α, also closed under commutation and therefore a Lie algebra? Since we associate $X_i \in \mathfrak{h}$ with the vector 0 in $\mathfrak{g}/\mathfrak{h}$, we must have

$$[X_\alpha, X_j] = 0 \qquad \text{mod } \mathfrak{h} \tag{1.23}$$

That is, the commutator must be in the subalgebra $\mathfrak{h}$:

$$[X_\alpha, X_j] = C_{\alpha j}{}^k X_k \qquad C_{\alpha j}{}^\gamma = 0 \tag{1.24}$$

A subalgebra with this property is called an **invariant subalgebra** because it is mapped into itself by *all* elements of the algebra. Symbolically, we write

$$[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h} \tag{1.24′}$$

If $\mathfrak{h}$ is an invariant subalgebra of $\mathfrak{g}$, the regular representation has additional zero submatrices:

$$\mathbf{R}(X_\alpha) \rightarrow \begin{array}{c} \\ \beta \\ - \\ j \end{array} \begin{array}{c} \begin{array}{cc} \gamma & \quad k \end{array} \\ \left[\begin{array}{c|c} C_{\alpha\beta}{}^\gamma & C_{\alpha\beta}{}^k \\ \hline C_{\alpha j}{}^\gamma = 0 & C_{\alpha j}{}^k \end{array}\right] \end{array} = \left[\begin{array}{c|c} \mathbf{R}_{11}(X_\alpha) & \mathbf{R}_{12}(X_\alpha) \\ \hline \bigcirc & \mathbf{R}_{22}(X_\alpha) \end{array}\right] \tag{1.25}$$

$$\mathbf{R}(X_i) \rightarrow \begin{array}{c} \\ \beta \\ - \\ j \end{array} \begin{array}{c} \begin{array}{cc} \gamma & \quad k \end{array} \\ \left[\begin{array}{c|c} C_{i\beta}{}^\gamma = 0 & C_{i\beta}{}^k \\ \hline C_{ij}{}^\gamma = 0 & C_{ij}{}^k \end{array}\right] \end{array} = \left[\begin{array}{c|c} \bigcirc & \mathbf{R}_{12}(X_i) \\ \hline \bigcirc & \mathbf{R}_{22}(X_i) \end{array}\right] \tag{1.25′}$$

By a calculation analogous to (1.19), it is easily shown that both $\mathbf{R}_{11}$ and $\mathbf{R}_{22}$ are representations for the algebra $\mathfrak{g} - \mathfrak{h}$:

$$\mathbf{R}([X_\alpha, X_\beta]) = \left[\begin{array}{c|c} \mathbf{R}_{11}([X_\alpha, X_\beta]) & * \\ \hline \bigcirc & \mathbf{R}_{22}([X_\alpha, X_\beta]) \end{array}\right] \tag{1.26}$$

The matrices $\mathbf{R}_{11}$ are in fact just the structure constants for this algebra

$$\mathbf{R}_{11}(X_\alpha) \rightarrow \mathbf{R}(X_\alpha)_\beta{}^\gamma = -C_{\alpha\beta}{}^\gamma \tag{1.27}$$

$$\begin{aligned} [X_\alpha, X_\beta] &= C_{\alpha\beta}{}^\gamma X_\gamma + C_{\alpha\beta}{}^k X_k \\ &= C_{\alpha\beta}{}^\gamma X_\gamma \qquad \text{mod } \mathfrak{h} \end{aligned} \tag{1.28}$$

The presence of invariant subalgebras leads to reducibility of the regular representation. This suggests that we look for a mechanism for constructing invariant subalgebras. It is clear that

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^{(1)} \tag{1.29}$$

is an invariant subalgebra of $\mathfrak{g}$. For $\mathfrak{g}^{(1)}$ is closed under commutation, and the commutator of an element of $\mathfrak{g}^{(1)}$ with an element of $\mathfrak{g}$ is an element of $\mathfrak{g}^{(1)}$. Proceeding in this manner, we can construct an entire series of invariant subalgebras:

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \cdots \mathfrak{g}^{(n-1)} \supseteq \mathfrak{g}^{(n)} \supseteq \mathfrak{g}^{(n+1)} \cdots \tag{1.30}$$

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] \equiv \mathfrak{g}^{(i+1)} \tag{1.31}$$

It is clear that $\mathfrak{g}^{(i+1)}$ is an invariant subalgebra of $\mathfrak{g}^{(i)}$. In fact, it is an invariant subalgebra of $\mathfrak{g}^{(j)} (j \leq i)$.

Example. We show that $\mathfrak{g}^{(2)}$ is an invariant subalgebra of $\mathfrak{g}^{(0)}$. To do this, we must show

$$[\mathfrak{g}^{(0)}, \mathfrak{g}^{(2)}] \subseteq \mathfrak{g}^{(2)} \tag{1.32}$$

By the Jacobi identity

$$[\mathfrak{g}^{(0)}, [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]] = [\mathfrak{g}^{(1)}, [\mathfrak{g}^{(1)}, \mathfrak{g}^{(0)}]] + [\mathfrak{g}^{(1)}, [\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}]] \subseteq [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \equiv \mathfrak{g}^{(2)} \tag{1.33}$$

The general case

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subseteq \mathfrak{g}^{(k)} \qquad k \leq \max(i, j) \tag{1.34}$$

can be supplied by induction.

The Lie bracket is the generalization of the ordinary derivative [(4.6) of Chapter 4]. For this reason the algebra $\mathfrak{g}^{(i+1)}$ is said to be the **derivative** of $\mathfrak{g}^{(i)}$, or **derived** from $\mathfrak{g}^{(i)}$. The series of algebras (1.30) is called the **derived series**. For a finite-dimensional Lie algebra, we must eventually reach a value of n for which

$$\mathfrak{g}^{(n-1)} \neq \mathfrak{g}^{(n)} \equiv \mathfrak{g}^{(n+1)} = \cdots \tag{1.35}$$

If the derived series terminates with $\mathfrak{g}^{(n)}$ and $\mathfrak{g}^{(n)}$ contains only the element 0, then the algebra is **solvable**. If $\mathfrak{g}$ is solvable, $\mathfrak{g}^{(n-1)}$ is abelian.

Example. From the commutation relations (1.14) for the Galilean algebra, it is clear that $\mathfrak{g}^{(1)}$ is spanned by $\mathbf{L} + \mathbf{v} + \mathbf{p}$. In addition $\mathfrak{g}^{(1)} \equiv \mathfrak{g}^{(2)} \equiv \cdots$. The Galilean group is not solvable, due to the generators $\mathbf{L}$, which keep regenerating themselves as well as the generators **p** and **v**. If we look at the subalgebra $\mathfrak{r}$ of the Galilean group whose generators are **p**, **v**, T, then

$$\begin{array}{rcll} \mathfrak{r} \equiv \mathfrak{r}^{(0)} & \mathbf{v} \oplus \mathbf{p} \oplus T & \text{(nonabelian)} & \\ [\mathfrak{r}, \mathfrak{r}] \equiv \mathfrak{r}^{(1)} & \mathbf{p} & \text{(abelian)} & \\ [\mathfrak{r}^{(1)}, \mathfrak{r}^{(1)}] \equiv \mathfrak{r}^{(2)} & \{0\} & & (1.36) \end{array}$$

Then $\mathfrak{r}$ is a solvable subalgebra of $\mathfrak{g}$. It is also an invariant subalgebra of $\mathfrak{g}$. In fact, it is the *maximal solvable invariant subalgebra* of $\mathfrak{g}$ (sometimes called the **radical** of $\mathfrak{g}$). The factor algebra $\mathbf{L} = \mathfrak{g}/\mathfrak{r}$ has simple properties and is therefore called (semi-)simple.

The set of all group operations k which can be written as commutators

$$ghg^{-1}h^{-1} \equiv k \qquad \begin{array}{l} g, h \in G^{(i)} \\ k \in G^{(i+1)} \end{array} \tag{1.37}$$

is called the derived group $G^{(i+1)}$ of $G^{(i)}$. Since $G^{(i+1)}$ is an invariant subgroup, the coset $G^{(i)}/G^{(i+1)}$ has a group structure on it. In fact, the factor group

$$A^{(i)} \equiv G^{(i)}/G^{(i+1)} \tag{1.38}$$

is an abelian group, as we show shortly.

Corresponding to the series of derived Lie groups is the series of derived algebras

$$G^{(0)} \supset G^{(1)} \supset \cdots \supset G^{(n-1)} \supset G^{(n)} \equiv G^{(n+1)} \equiv \cdots \tag{1.39g}$$

$$\mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \cdots \supset \mathfrak{g}^{(n-1)} \supset \mathfrak{g}^{(n)} \equiv \mathfrak{g}^{(n+1)} \equiv \cdots \tag{1.39a}$$

Basis vectors for the vector space $\mathfrak{g}^{(i)}$ mod $\mathfrak{g}^{(i+1)}$ are the generators for the coset elements and factor group

$$\sum_{r=1}^{d_i} c^r X_r^{(i)} = \frac{\mathfrak{g}^{(i)}}{\mathfrak{g}^{(i+1)}} \xrightarrow{\text{EXP}} \frac{G^{(i)}}{G^{(i+1)}} \equiv A^i \tag{1.40}$$

The generators $X_r^{(i)}$ $(r = 1, 2, \ldots, d^{(i)})$ obey the commutation relations

$$[X_r^{(i)}, X_s^{(j)}] = C_{r\ s(k)}^{(i)(j)\ t} X_t^{(k)} \qquad k \geq \max(i, j) \tag{1.41}$$

In particular, for $i = j$ we have

$$[X_r^{(i)}, X_s^{(i)}] = 0 \qquad \text{mod } \mathfrak{g}^{(i)} \qquad (i < n) \tag{1.42}$$

That is, the generators for the factor group $G^{(i)}/G^{(i+1)} \equiv A^i$ commute among themselves. Therefore the groups $A^0, A^1, \ldots, A^{n-1}$ are all abelian. As a result, we conclude that solvable groups are constructed piecewise, in a step by step fashion, from abelian groups. A solvable algebra is a semidirect sum of abelian factor algebras obeying (1.40–1.42).

Example 1. The six-dimensional subalgebra of $\mathfrak{gl}(3, r)$ consisting of the upper triangular matrices with zero *below* the major diagonal is spanned by the generators $U_{ij}(1 \le i \le j \le 3)$ obeying

$$[U_{ij}, U_{rs}] = U_{is}\,\delta_{jr} \qquad \begin{matrix} i \le j \\ r \le s \end{matrix}$$

The regular representation of this six-dimensional Lie algebra has the block structure shown in (1.43) with respect to the ordering of the bases indicated

$$\sum_{1=i\le j}^{3} a^{ij} U_{ij} \xrightarrow{\text{reg}}$$

$$\begin{array}{c} \\ U_{11} \\ U_{22} \\ U_{33} \\ - \\ U_{12} \\ U_{23} \\ U_{13} \end{array}
\begin{array}{c} \begin{array}{cccccc} U_{11} & U_{22} & U_{33} & U_{12} & U_{23} & U_{13} \end{array} \\
\left[\begin{array}{ccc|cc|c}
 & & & -a^{12} & 0 & -a^{13} \\
 & \bigcirc & & +a^{12} & -a^{23} & 0 \\
 & & & 0 & +a^{23} & +a^{13} \\
\hline
 & \bigcirc & & a^{11} - a^{22} & 0 & -a^{23} \\
 & & & 0 & a^{22} - a^{33} & +a^{12} \\
\hline
 & \bigcirc & & & \bigcirc & a^{11} - a^{33}
\end{array}\right] \end{array} \tag{1.43}$$

Example 2. The three-dimensional subalgebra of $\mathrm{gl}(3, r)$, consisting of upper triangular matrices with zeroes *below and on* the major diagonal, is spanned by the generators $U_{ij}(1 \le i < j \le 3)$ obeying

$$[U_{ij}, U_{rs}] = U_{is}\,\delta_{jr} \qquad \begin{matrix} i < j \\ r < s \end{matrix}$$

The regular representation of this three-dimensional Lie algebra has the block structure shown in (1.43′) with respect to the ordering of the bases indicated

$$\sum_{1=i<j}^{3} a^{ij} U_{ij} \xrightarrow{\text{reg}} \begin{array}{c} U_{12} \\ U_{23} - \\ U_{13} \end{array}
\begin{array}{c} \begin{array}{ccc} U_{12} & U_{23} & U_{13} \end{array} \\
\left[\begin{array}{cc|c}
 & & -a^{23} \\
 & \bigcirc & +a^{12} \\
\hline
 & \bigcirc & \bigcirc
\end{array}\right] \end{array} \tag{1.43'}$$

Comment 1. The subalgebras of $\mathfrak{gl}(3, r)$ discussed in Examples 1 and 2 are both solvable. For Example 1 we have

$$\begin{array}{lll} \mathfrak{g}^{(0)} \bmod \mathfrak{g}^{(1)} & \text{spanned by} & U_{11}, U_{22}, U_{33} \\ \mathfrak{g}^{(1)} \bmod \mathfrak{g}^{(2)} & \text{spanned by} & U_{12}, U_{23} \\ \mathfrak{g}^{(2)} & \text{spanned by} & U_{13} \end{array} \tag{1.44}$$

It is clear from the structure of (1.43) that it is always possible to choose the bases for a solvable algebra in such a way that its regular representation has a block diagonal structure, with zero submatrices below the diagonal blocks. In fact, it is always possible to choose the bases within each subspace $\mathfrak{g}^{(i)}$ mod $\mathfrak{g}^{(i+1)}$ in such a way that the diagonal blocks are themselves also upper triangular, with zeroes below their major diagonal. These comments are summarized in Fig. 7.2.

Comment 2. The three-dimensional subalgebra described in Example 2 is the derived algebra of the six-dimensional algebra discussed in Example 1.

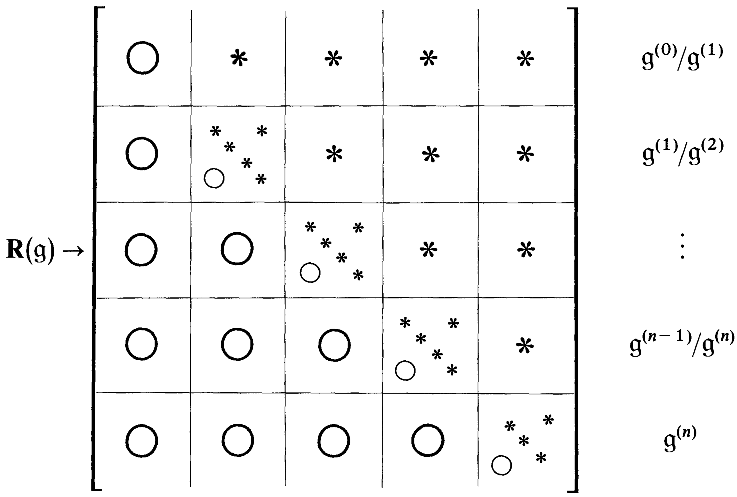


FIG. 7.2 REGULAR REPRESENTATION FOR SOLVABLE AND NILPOTENT ALGEBRAS. WITH RESPECT TO THE SUBSPACES $\mathfrak{g}^{(i)}/\mathfrak{g}^{(i+1)}$, ORDERED AS INDICATED, THE REGULAR REPRESENTATION ASSUMES THE BLOCK DIAGONAL STRUCTURE SHOWN. WITHIN EACH SUBSPACE $\mathfrak{g}^{(i)}/\mathfrak{g}^{(i+1)}$ IT IS ALSO POSSIBLE TO CHOOSE THE BASIS VECTORS SO THAT THE DIAGONAL SUBMATRIX AT THE INTERSECTION OF THE $\mathfrak{g}^{(i)}/\mathfrak{g}^{(i+1)}$ ROW AND COLUMN HAS AN UPPER TRIANGULAR STRUCTURE, WITH ZERO BELOW THE MAJOR DIAGONAL. THE NONZERO PARAMETERS ON THE MAJOR DIAGONAL ARE COORDINATES FOR ELEMENTS IN THE SUBSPACE $\mathfrak{g}^{(0)}/\mathfrak{g}^{(1)}$ ALONE. THE DIAGONAL SUBMATRIX AT THE INTERSECTION OF THE $\mathfrak{g}^{(0)}/\mathfrak{g}^{(1)}$ ROW AND COLUMN IS ZERO, SINCE $[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] = \mathfrak{g}^{(1)}$. WHEN $\mathfrak{g}$ IS NILPOTENT ALL ENTRIES ON THE MAJOR DIAGONAL ARE ZERO.

The regular representation (1.43′) can be obtained from (1.43) simply as follows:

1. Retain the submatrix whose bases span $\mathfrak{g}^{(1)}$.
2. Set equal to zero all parameters in this submatrix describing elements in the space $\mathfrak{g}^{(0)}/\mathfrak{g}^{(1)}$, which has been eliminated.

Since the only nonzero diagonal entries appearing in (1.43) describe elements in $\mathfrak{g}^{(0)}/\mathfrak{g}^{(1)}$, the regular representation for the derived algebra $\mathfrak{g}^{(1)}$ is upper triangular with zeroes below and on the major diagonal.

Let $\mathfrak{g}$ be a Lie algebra of $n \times n$ upper triangular matrices with zero below the major diagonal (i.e., $A_{ij} \in \mathfrak{g}$ implies $A_{ij} = 0$ when $i > j$). Then the product of any pair of matrices A, $B \in \mathfrak{g}$ is an upper triangular matrix, and in particular the commutator $[A, B]$ is upper triangular. Moreover, the matrix $[A, B]$ has only zeroes on the major diagonal, for

$$\begin{aligned} [A, B]_{ii} &= \sum_{j=1}^{n} A_{ij} B_{ji} - \sum_{j=1}^{n} B_{ij} A_{ji} \\ &= A_{ii} B_{ii} - B_{ii} A_{ii} = 0 \end{aligned} \tag{1.45s}$$

The algebra $\mathfrak{g}$ is therefore solvable.

$\Gamma(\mathfrak{g}) \rightarrow$

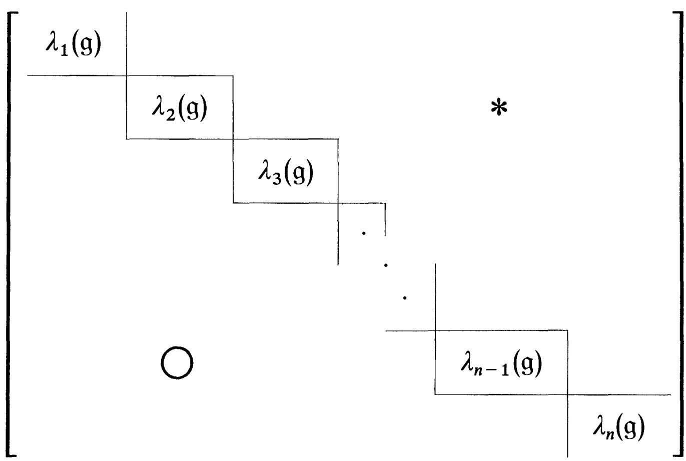


FIG. 7.3 EVERY REPRESENTATION FOR A SOLVABLE OR A NILPOTENT ALGEBRA $\mathfrak{g}$ CAN BE TRANSFORMED TO UPPER TRIANGULAR FORM, WITH ZEROES BELOW THE MAJOR DIAGONAL. SUCH A TRANSFORMATION IS ALLOWED BY LIE'S THEOREM AND EFFECTED BY THE ARGUMENTS LEADING UP TO (1.55). THE DIAGONAL MATRIX ELEMENTS ARE NONZERO *only* FOR NONZERO VECTORS IN $\mathfrak{g}/\mathfrak{g}^{(1)} : \lambda(\mathfrak{g}^{(1)}) = 0$.

Now let $\mathfrak{g}'$ be a Lie algebra of $n \times n$ upper triangular matrices with zero below and on the major diagonal. Then if $A \in \mathfrak{g}$

$$A^p = 0 \qquad p \geq n \tag{1.45n}$$

A matrix with the property $A^p = 0$, for some finite p, is said to be **nilpotent**. If every matrix in a matrix Lie algebra is nilpotent the algebra itself is called nilpotent. The algebra of Example 2 is nilpotent. Every nilpotent matrix Lie algebra can be transformed to upper triangular form, with zeroes below and on the major diagonal. We have the general result; the derived algebra of a solvable matrix Lie algebra is a nilpotent Lie algebra.

The properties of solvable and nilpotent matrix Lie algebras, and their representations, are summarized in Fig. 3.

4. LIE'S THEOREM. In this section we will prove Lie's theorem. This states essentially that every representation of a solvable algebra has a simultaneous eigenvector. To prove this theorem it is useful to learn some additional properties of solvable algebras and their representations. We do this by proving three preliminary theorems.

PRELIMINARY THEOREM 1. *Let* $\mathfrak{g}$ *be a solvable Lie algebra of dimension* n. *Then* $\mathfrak{g}$ *has a solvable subalgebra* $\mathfrak{h}$ *of dimension* n − 1.

Proof. Let $X_1, X_2, \ldots, X_r$ be bases for the vector space $\mathfrak{g}$ mod $\mathfrak{g}^{(1)}$. Then the linear vector space

$$\mathfrak{h} = \sum_{i=2}^{r} \alpha^i X_i \oplus \mathfrak{g}^{(1)} \qquad (\alpha^i \text{ complex})$$

spanned by $X_2, \ldots, X_r$ and $\mathfrak{g}^{(1)}$ is closed under commutation, since

$$[X_i, X_j] \in \mathfrak{g}^{(1)}$$
$$[X_i, \mathfrak{g}^{(1)}] \in \mathfrak{g}^{(1)}$$
$$[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(1)}$$

It is solvable, because every subalgebra of a solvable algebra is solvable.

Caution. The subalgebra $\mathfrak{h}$ is not generally the derived algebra $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$. In fact, we have the inclusions $\mathfrak{g} \supset \mathfrak{h} \supseteq \mathfrak{g}^{(1)}$.

PRELIMINARY THEOREM 2. *Assume the solvable algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *have the property dim* ($\mathfrak{g}$ *mod* $\mathfrak{h}$) = 1, *so that*

$$\mathfrak{g} = \alpha \mathrm{X} \oplus \mathfrak{h} \qquad (\alpha \textit{ complex}) \tag{1.46}$$

Assume also that $\mathbf{\Gamma}(\mathfrak{g})$ *is a representation for* $\mathfrak{g}$ *on the linear vector space* V *of dimension* p + 1. *Finally, assume that* $\mathbf{\Gamma}(\mathfrak{h})$ *has a simultaneous eigenvector* $|v\rangle$ *in* V. *Then* $\mathbf{\Gamma}(\mathfrak{h})$ *is a multiple of the identity on* V.

Proof. We can assume without loss of generality that the various powers of $\mathbf{\Gamma}(X)$, applied to $|v\rangle$, span V. If not, they span an invariant subspace $\tilde{V}$ of V and the theorem is valid for $\tilde{V}$. Define

$$|v_i\rangle = \mathbf{\Gamma}(X)|v_{i-1}\rangle$$
$$|v_0\rangle = |v\rangle$$

From the assumptions of the theorem

$$\mathbf{\Gamma}(H)|v\rangle = \lambda(H)|v\rangle \qquad H \in \mathfrak{h}$$

Now we show that $|v_i\rangle$ is an eigenvector of $\mathbf{\Gamma}(H)$ mod $\{|v_0\rangle, |v_1\rangle, \ldots, |v_{i-1}\rangle\}$. We assume this to be true for $|v_j\rangle, j = 0, 1, \ldots, i-1$ and prove it for $|v_i\rangle$.

$$\begin{aligned}\mathbf{\Gamma}(H)|v_i\rangle &= \mathbf{\Gamma}(H)\mathbf{\Gamma}(X)|v_{i-1}\rangle \\ &= \{\mathbf{\Gamma}(X)\mathbf{\Gamma}(H) + \mathbf{\Gamma}([H, X])\}|v_{i-1}\rangle\end{aligned}$$

Since $\mathfrak{g}$ is solvable, $[H, X] \in \mathfrak{h}$. Therefore

$$\begin{aligned}\mathbf{\Gamma}(H)|v_i\rangle &= \mathbf{\Gamma}(X)\lambda(H)|v_{i-1}\rangle + \lambda([H, X])|v_{i-1}\rangle \\ &= \lambda(H)\mathbf{\Gamma}(X)|v_{i-1}\rangle + \lambda([H, X])|v_{i-1}\rangle \\ &= \lambda(H)|v_i\rangle \qquad \text{mod } |v_0\rangle, |v_1\rangle, \ldots, |v_{i-1}\rangle\end{aligned} \tag{1.47}$$

In order to show that $\mathbf{\Gamma}(H)$ is actually diagonal on V we must show $\lambda([H, X]) = 0$. But

$$\begin{aligned}\mathbf{\Gamma}([H, X])|v_i\rangle &= \lambda([H, X])|v_i\rangle \qquad \text{mod } |v_0\rangle \cdots |v_{i-1}\rangle \\ \operatorname{tr} \mathbf{\Gamma}([H, X]) &= \lambda([H, X]) \dim V \\ \operatorname{tr} \{\mathbf{\Gamma}(H)\mathbf{\Gamma}(X) - \mathbf{\Gamma}(X)\mathbf{\Gamma}(H)\} &= \lambda([H, X])(p + 1)\end{aligned}$$

Since the trace of a commutator is zero, $\lambda([H, X]) = 0$ and from (1.47) we have the result

$$\mathbf{\Gamma}(H)|v_i\rangle = \lambda(H)|v_i\rangle$$

$$\mathbf{\Gamma}(H) \xrightarrow{\text{on } V} \lambda(H)I_{p+1} \tag{1.48}$$

Next, we prove a simple theorem for abelian algebras. This theorem is the model for Lie's theorem.

PRELIMINARY THEOREM 3. *Let* $\mathfrak{a}$ *be an abelian algebra with generators* $A_1, A_2, \ldots, A_r$ *and let* $\Gamma(\mathfrak{a})$ *be a representation of* $\mathfrak{a}$ *defined on the vector space* U. *Then* $\Gamma(\mathfrak{a})$ *has a simultaneous eigenvector in* U.

Proof. Every matrix has a nonzero eigenvector. Choose $|v_1\rangle \in U$ so that

$$\Gamma(A_1)|v_1\rangle = \lambda_1|v_1\rangle \tag{1.49a}$$

Then the subspace of U consisting of all vectors of the form

$$\Gamma(r^i A_i)|v_1\rangle \tag{1.50a}$$

is an invariant subspace and carries a representation of $\mathfrak{a}$. Every vector in this subspace is an eigenvector of $\Gamma(A_1)$, for

$$\begin{aligned} \Gamma(A_1)\{\Gamma(r^i A_i)|v_1\rangle\} &= \Gamma(r^i A_i)\Gamma(A_1)|v_1\rangle \qquad (\mathfrak{a}\text{ abelian}) \\ &= \lambda_1\{\Gamma(r^i A_i)|v_1\rangle\} \end{aligned} \tag{1.51a}$$

In this invariant subspace it is possible to find a vector $|v_{12}\rangle$ that is an eigenvector of $\Gamma(A_2)$:

$$\Gamma(A_2)|v_{12}\rangle = \lambda_2|v_{12}\rangle \tag{1.52a}$$

Then $\Gamma(r^i A_i)|v_{12}\rangle$ is an invariant subspace that carries a representation of $\mathfrak{a}$. Etc. We finally arrive at a subspace of vectors $|v_{12\ldots r}\rangle$ having the property

$$\begin{aligned} \Gamma(\mathbf{A})|v_{12\ldots r}\rangle &= \boldsymbol{\lambda}|v_{12\ldots r}\rangle \\ \Gamma(r^i A_i)|v_{12\ldots r}\rangle &= r^i\lambda_i|v_{12\ldots r}\rangle \end{aligned} \tag{1.53a}$$

where $\Gamma(\mathfrak{a})$ is diagonal on each vector in this subspace $U_{\boldsymbol{\lambda}}$.

Comment. The representation $\Gamma(\mathfrak{a})$ can be transformed to upper triangular form. With respect to the subspace $U_{\boldsymbol{\lambda}}$ and the complementary subspace

$$U^c = U \bmod U_{\boldsymbol{\lambda}}$$

the representation $\Gamma(r^i A_i)$ has the structure

$$\Gamma(r^i A_i) = \left[\begin{array}{c|c} * & * \\ \hline \bigcirc & r^i\lambda_i I \end{array}\right] \begin{array}{l} \} \ U \bmod U_{\boldsymbol{\lambda}} \\ \} \quad U_{\boldsymbol{\lambda}} \end{array} \tag{1.54a}$$

The subspace $U \bmod U_{\boldsymbol{\lambda}}$ carries a representation of $\mathfrak{a}$, and therefore preceding arguments can be used again verbatim. By induction, $\Gamma(r^i A_i)$ is upper triangular

$$\Gamma(r^i A_i) \rightarrow \begin{bmatrix} r^i\mu_i & & & * \\ & \ddots & & \\ & & r^i\nu_i & \\ O & & & r^i\lambda_i \end{bmatrix} \tag{1.55a}$$

Comment. This theorem is nothing more than a statement that all the irreducible representations of an abelian group or algebra are one-dimensional. The proof is greatly simplified if Schur's lemma is used.

We now turn to the proof of Lie's theorem. This theorem accomplishes for solvable algebras what preliminary theorem 3 does for abelian algebras. Not surprisingly, the proof of Lie's theorem is modeled on the proof of the previous theorem.

LIE'S THEOREM. *Let* $\mathfrak{g}$ *be a solvable algebra and* $\Gamma(\mathfrak{g})$ *be a representation for* $\mathfrak{g}$ *on the* $\mathrm{p}+1$ *dimensional vector space* V. *Then* $\Gamma(\mathfrak{g})$ *has a nonzero simultaneous eigenvector in* V.

Proof. The theorem is valid when $\mathfrak{g}$ is abelian. We assume it is valid for all solvable algebras whose dimensionality is less than the dimension of $\mathfrak{g}$. Then we prove the theorem for $\mathfrak{g}$. By induction, the theorem is then valid for any solvable algebra $\mathfrak{g}$.

1. By preliminary theorem 1

$$\mathfrak{g} = \alpha X \oplus \mathfrak{h} \qquad \alpha \text{ complex} \tag{1.46}$$

where $\mathfrak{h}$ is solvable and $X \notin \mathfrak{h}$. By assumption, the theorem is true for $\Gamma(\mathfrak{h})$, and therefore the conditions of preliminary theorem 2 are satisfied. Let $|v\rangle \in V$ be the simultaneous eigenvector of $\Gamma(\mathfrak{h})$ in V. Then

$$\Gamma(H)\,|\,v\rangle = \lambda(H)\,|\,v\rangle \qquad H \in \mathfrak{h} \tag{1.49s}$$

2. By preliminary theorem 2 the (sub)space of V consisting of all vectors of the form

$$\Gamma(X)^i\,|\,v_0\rangle = |\,v_i\rangle \tag{1.50s}$$

is an invariant subspace of $\mathfrak{h}$ and carries a representation of $\mathfrak{h}$. Every vector in this space is an eigenvector of $\Gamma(\mathfrak{h})$, for

$$\Gamma(H)\{\Gamma(X)|v_{i-1}\rangle\} \to \{\Gamma(X)\Gamma(H) + \Gamma([H, X])\}|v_{i-1}\rangle \qquad (\mathfrak{h} \text{ solvable})$$

$$\xrightarrow[\text{theorem 2}]{\text{Preliminary}} \lambda(H)\{\Gamma(X)\,|\,v_{i-1}\rangle\} \tag{1.51s}$$

In this invariant space V (under $\Gamma(\mathfrak{g})$) it is possible to find a nonzero eigenvector $|x\rangle$ of the *matrix* $\Gamma(X)$, since every matrix has a nonzero eigenvector.

$$\Gamma(\alpha X)\,|\,x\rangle = \lambda'(\alpha)\,|\,x\rangle \tag{1.52s}$$

The vector $|x\rangle$ is a simultaneous eigenvector of $\mathfrak{g}$, for

$$\Gamma(\alpha X + \mathfrak{h})\,|\,x\rangle = \{\lambda'(\alpha) + \lambda(\mathfrak{h})\}\,|\,x\rangle \tag{1.53s}$$

Comment. The representation $\Gamma(\mathfrak{g})$ can be transformed to upper triangular form. Let $|x\rangle$ span V^0. Then with respect to this subspace and the complementary subspace V mod V^0, the matrix $\Gamma(\mathfrak{g})$ has the structure

$$\Gamma(\mathfrak{g}) \to \begin{array}{c} \left[\begin{array}{c|c} * & * \\ \hline \bigcirc & \lambda_0(\mathfrak{g}) \end{array}\right] \end{array} \begin{array}{l} V \bmod V^0 \\ \\ V^0 \end{array} \tag{1.54s}$$

The subspace V mod V^0 carries a representation for $\Gamma(\mathfrak{g})$, so by Lie's theorem, there is a simultaneous eigenvector $|y\rangle$ in V mod V^0. Continuing in this way, we transform $\Gamma(\mathfrak{g})$ to upper triangular form

$$\Gamma(\mathfrak{g}) \to \begin{bmatrix} \lambda_p(\mathfrak{g}) & & & * & \\ & \lambda_{p-1}(\mathfrak{g}) & & & \\ & & \ddots & & \\ & \bigcirc & & \lambda_1(\mathfrak{g}) & \\ & & & & \lambda_0(\mathfrak{g}) \end{bmatrix} \begin{array}{c} V^p \bmod V^{p-1} \\ V^{p-1} \bmod V^{p-2} \\ \vdots \\ V^1 \bmod V^0 \\ V^0 \end{array} \tag{1.55s}$$

Comment. This corollary plays a key role in the development of a canonical form for *all* Lie algebras, the semisimple as well as the solvable and nonsemisimple.

Comment. By (1.45s), the commutator of two upper triangular matrices is an upper triangular matrix with zeroes on the major diagonal. If $|v\rangle$ is the simultaneous eigenvector for $\Gamma(\mathfrak{g}^{(0)})$, then

$$\Gamma([A, B])\,|\,v\rangle = 0 \qquad \text{all} \quad A, B \in \mathfrak{g}^{(0)}$$

Since every element in $\mathfrak{g}^{(1)}$ can be written as the commutator of two elements in $\mathfrak{g}^{(0)}$, we have

$$\Gamma(\mathfrak{g}^{(1)})\,|\,v\rangle = \Gamma([\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}])\,|\,v\rangle = \lambda(\mathfrak{g}^{(1)})\,|\,v\rangle = 0 \tag{1.56}$$

Example. The regular representation for the solvable algebra described in (1.43) has the requisite triangular structure with respect to the basis vectors $U_{ij} \to |\,U_{ij}\rangle$. The simultaneous eigenvector for the representations (1.43) and (1.43′) is $|\,U_{13}\rangle \simeq V^0$. In addition

$$\mathbf{R}(a^{ij}U_{ij})\,|\,U_{13}\rangle = -(a^{11} - a^{33})\,|\,U_{13}\rangle$$

It is clear that $\lambda(\mathfrak{g}^{(1)}) = 0$.

5. CLASSIFICATION OF LIE ALGEBRAS. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{u}$ be a solvable invariant subalgebra. Then $\mathfrak{g} - \mathfrak{u} = \mathfrak{g} \bmod \mathfrak{u}$ is a Lie algebra. If $\mathfrak{g} - \mathfrak{u}$ has a solvable invariant subalgebra $\mathfrak{v}$, then

$$(\mathfrak{g} - \mathfrak{u}) - \mathfrak{v} = \mathfrak{g} - (\mathfrak{u} \oplus \mathfrak{v})$$

is once again an algebra. Proceeding in this way, we can find a maximum solvable invariant subalgebra

$$\mathfrak{r} = \mathfrak{u} \oplus \mathfrak{v} \oplus \cdots \tag{1.57}$$

called the **radical** of $\mathfrak{g}$. Then $\mathfrak{g}/\mathfrak{r}$ no longer has any solvable invariant subalgebras and is called semisimple:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{s}\mathfrak{s} \oplus \mathfrak{r} \\ \mathfrak{s}\mathfrak{s} &= \mathfrak{g} - \mathfrak{r} \end{aligned} \tag{1.58}$$

The algebra $\mathfrak{s}\mathfrak{s}$ may have invariant subalgebras that are not solvable. If $\mathfrak{s}_1$ is such an invariant subalgebra, then $\mathfrak{s}\mathfrak{s}/\mathfrak{s}_1$ is a Lie algebra with no solvable invariant subalgebras. Proceeding thus, we have a canonical structure for the regular representation of a semisimple algebra [cf. (1.25)]

$$\mathfrak{s}\mathfrak{s} \xrightarrow{\text{reg}} \begin{bmatrix} \mathfrak{s}_1 & & * & * \\ & \mathfrak{s}_2 & & \\ & & \ddots & \\ 0 & & & \mathfrak{s}_n \end{bmatrix} \tag{1.59}$$

In fact, the generators of each invariant subalgebra can be chosen in such a way that the submatrix ** vanishes (see Section IV.3).

The regular representation for an arbitrary Lie algebra $\mathfrak{g}$ can now be described. Elements in the maximal solvable invariant subalgebra, or radical $\mathfrak{r}$, are given by

$$
\text{radical} \xrightarrow{\text{reg}}
\begin{array}{cc}
\begin{array}{cc} \qquad \mathfrak{ss} \qquad & \qquad \mathfrak{r} \qquad \end{array} & \\
\left[\begin{array}{c|c}
 & *_1 \\ \cline{2-2}
 & *_2 \\ \cline{2-2}
 O & \vdots \\ \cline{2-2}
 & *_n \\ \hline
 O & \begin{matrix} * & & & * \\ & * & & \\ & & \ddots & \\ O & & & * \end{matrix}
\end{array}\right]
& \begin{array}{c} \mathfrak{s}_1 \\ \mathfrak{s}_2 \\ \\ \mathfrak{s}_n \\ \mathfrak{r} \end{array}
\end{array}
\tag{1.60}
$$

Elements in the derived algebra of the maximum solvable invariant subalgebra have zeroes on and below the major diagonal

$$
\mathfrak{r}^{(1)} \xrightarrow{\text{reg}}
\begin{array}{cc}
\begin{array}{cc} \qquad \mathfrak{ss} \qquad & \qquad \mathfrak{r} \qquad \end{array} & \\
\left[\begin{array}{c|c}
 & *_1 \\ \cline{2-2}
 & *_2 \\ \cline{2-2}
 O & \vdots \\ \cline{2-2}
 & *_n \\ \hline
 O & \begin{matrix} 0 & & & * \\ & 0 & & \\ & & \ddots & \\ O & & & 0 \end{matrix}
\end{array}\right]
& \begin{array}{c} \mathfrak{s}_1 \\ \mathfrak{s}_2 \\ \\ \mathfrak{s}_n \\ \mathfrak{r} \end{array}
\end{array}
\tag{1.60'}
$$

The elements in the simple algebra $\mathfrak{s}_i$ belonging to the semisimple algebra $\mathfrak{ss} = \mathfrak{g} - \mathfrak{r}$ have regular representatives:

$$X_i \xrightarrow{\mathrm{reg}} \begin{array}{cc} \begin{array}{cccccc} \mathfrak{s}_1 & \mathfrak{s}_2 & \mathfrak{s}_i & \cdots & \mathfrak{s}_n & \mathfrak{r} \end{array} & \\ \left[\begin{array}{ccccc|c} \; & \; & \; & \; & \; & \; \\ & & & & & \\ & & \mathbf{R}_i(X_i) & & & **_i \\ & & & & & \\ & & & & & \\ \hline \multicolumn{5}{c}{0} & ** \end{array}\right] & \begin{array}{c} \mathfrak{s}_1 \\ \mathfrak{s}_2 \\ \mathfrak{s}_i \\ \vdots \\ \mathfrak{s}_n \\ \mathfrak{r} \end{array} \end{array} \tag{1.61}$$

This is as far as we can go in a classification of Lie algebras using just the properties of the algebras themselves. To proceed further we must introduce additional concepts besides that of the regular representation. This is done in the following two sections.

We summarize our classification of Lie algebras into three broad categories.

Definition. A **nonsemisimple** algebra has a solvable invariant subalgebra. In fact, it has an abelian invariant subalgebra.

Definition. A **semisimple** algebra has no solvable invariant subalgebra.

Definition. A **simple** algebra has no proper invariant subalgebras at all.

Comment. These definitions may be stated in terms of the reducibility properties of the regular representation:

1. A nonsemisimple algebra is reducible but not fully reducible.
2. A semisimple algebra is fully reducible.
3. A simple algebra is irreducible.

The terms "simple" and "irreducible" are often used interchangeably. The term "simple" is preferred for groups and algebras, "irreducible" for representations.

II. The Secular Equation

1. RANK. It is our objective to find a canonical form for the commutation relations of a Lie algebra. The canonical form we eventually arrive at should be independent of our initial particular choice of bases. To find such a canonical form, we use the techniques of linear algebra,[3] which have been highly developed toward this end.

Let $\mathfrak{g}$ be an η-dimensional Lie algebra with bases X_i, $i = 1, 2, 3, \ldots, \eta$. Then we can form the secular equation based on the regular representation of this algebra

$$\det \|\mathbf{R}(r^i X_i) - \lambda I_\eta\| = f(\lambda) = 0 \qquad (r^i \text{ real}) \tag{2.1}$$

This equation is invariant under a nonsingular transformation U:

$$\begin{aligned}\det &\|U\mathbf{R}(r^i X_i)U^{-1} - \lambda I_\eta\| \\ &= \det \|U\| \det \|\mathbf{R}(r^i X_i) - \lambda I_\eta\| \det \|U^{-1}\| = f(\lambda) = 0\end{aligned} \tag{2.2}$$

The secular equation is a polynomial in λ of order η, with real coefficients $\phi_j(r^i)$ depending on the choice of Lie algebra element

$$\det \|\mathbf{R}(r^i X_i) - \lambda I_\eta\| = \sum_{j=1}^{\eta} (-\lambda)^{\eta-j} \phi_j(r^i) = 0 \tag{2.3}$$

Not all the functions $\phi_j(r^i)$ are independent. For example,

$$\phi_0 = 1 \tag{2.4}$$

$$\phi_\eta = \det \|r^i \mathbf{R}(X_i)\| = 0 \tag{2.5}$$

The last result follows because

$$r^i \mathbf{R}(X_i) | r^k X_k\rangle = | [r^k X_k, r^i X_i]\rangle = 0 \tag{2.5$'$}$$

Since the linear transformation $\mathbf{R}(r^i X_i)$ annihilates a nonzero vector (corresponding to itself), it is singular and therefore must have vanishing determinant.

The number of functionally independent coefficients $\phi_j(r^i)$ of the secular equation (2.3) is called the **rank** of the algebra.

Explanation. It is often necessary, in classical analysis, to determine how many of the functions $f_1(x^i)$, $f_2(x^i)$, $\ldots$, $f_m(x^i)$ of the n variables x^i $(i = 1, 2, \ldots, n)$ are *functionally independent*. To carry this task out, it is useful to construct the $n \times m$ matrix M:

$$M_i^{\,j} = \left(\frac{\partial f_j}{\partial x^i}\right)$$

If every $(k+1) \times (k+1)$ submatrix of M has vanishing determinant, but there is a $k \times k$ submatrix with nonzero determinant, then k is:

1. the rank of the matrix M;
2. the number of functionally independent functions $f_j(x^i)$.

Example 1. The regular representation for a nilpotent algebra has the structure shown in Fig. 7.2. The secular equation is

$$\det \|\mathbf{R}(\text{Nilpotent}) - \lambda I_n\| = \det \left\| \begin{bmatrix} 0 & & & & * \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ \bigcirc & & & & 0 \end{bmatrix} - \lambda I_n \right\|$$

$$= (-\lambda)^n = 0 \tag{2.6}$$

The rank of a nilpotent algebra is 0.

Comment 1. If an algebra is solvable but not nilpotent, its rank is greater than zero.

Example 2. The regular representation for a semisimple algebra decomposes into the regular representations of the component simple invariant subalgebras. The secular equation is the direct product of the corresponding secular equations

$$\det \left\| \begin{bmatrix} \mathbf{R}_1(\mathfrak{s}_1) & & & & ** \\ & \mathbf{R}_2(\mathfrak{s}_2) & & & \\ & & \mathbf{R}_3(\mathfrak{s}_3) & & \\ & & & \ddots & \\ \bigcirc & & & & \mathbf{R}_n(\mathfrak{s}_n) \end{bmatrix} - \lambda I \right\|$$

$$= \prod_{i=1}^{n} \|\mathbf{R}_i(\mathfrak{s}_i) - \lambda I\| = 0 \tag{2.7}$$

This secular equation can be studied by studying each component secular equation separately.

Example 3. The dimensionalities of the classical groups $U(n)$ and $SU(n)$ are n^2 and $n^2 - 1$. The regular representations of their algebras, $\mathbf{R}[\mathfrak{u}(n)]$ and $\mathbf{R}[\mathfrak{su}(n)]$ consist of $n^2 \times n^2$ and $n^2 - 1 \times n^2 - 1$ matrices, respectively. Although the regular representation $\mathbf{R}[\mathfrak{su}(n)]$ is faithful, $\mathbf{R}[\mathfrak{u}(n)]$ is not faithful. This is most easily seen as follows. The algebra $\mathfrak{u}(n)$ can be written as a direct sum of the traceless subalgebra $\mathfrak{su}(n)$ and the abelian invariant subalgebra $\mathfrak{u}(1)$

$$\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathfrak{u}(1)$$

Therefore the $n^2 \times n^2$ matrix $\mathbf{R}[\mathfrak{u}(n)]$ consists of the $n^2 - 1 \times n^2 - 1$ submatrix $\mathbf{R}[\mathfrak{su}(n)]$, together with an additional row and column of zeroes.

The defining $n \times n$ matrix representations of both $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ are faithful. The regular and defining representations of $\mathfrak{su}(n)$ possess the same amount of information, since both are faithful. This is not the case for $\mathfrak{u}(n)$. The defining $n \times n$ representation carries more information than the regular representation, since the former is faithful while the latter is not. It is simpler to compute the secular equation for these algebras within the defining matrix representation than in the regular representation, since the defining representation consists of much smaller matrices. For $\mathfrak{u}(n)$ the secular equation computed within the faithful defining representation contains more information than the secular equation computed within the unfaithful regular representation. For $\mathfrak{su}(n)$ both secular equations have the same information content.

An arbitrary element in the defining representation for the algebra $\mathfrak{u}(n)$ is $a^{ij}M_{ij}^{(n)}$, with $M_{ij}^{(n)}$ given by (2.6) of Chapter 1. By an easy calculation the secular equation for this representation is

$$\det \|a^{ij}M_{ij}^{(n)} - \lambda I_n\| = (-\lambda)^{n-r}\phi_r(a^{ij}) = 0 \tag{2.8}$$

$$\phi_r(a^{ij}) = \frac{\varepsilon_{i_1 i_2 \cdots i_r i_{r+1} \cdots i_n} a^{i_1 j_1} \cdots a^{i_r j_r} \varepsilon_{j_1 j_2 \cdots j_r i_{r+1} \cdots i_n}}{r!\,(n-r)!} \tag{2.9}$$

$$a^{ij} = -a^{ji*} \tag{2.10}$$

There are n functionally independent coefficients ϕ_r for $\mathfrak{u}(n)$ within its defining matrix representation. The algebra of the subgroup $SU(n)$ has the additional constraint

$$\sum_{i=1}^{n} a^{ii} = \phi_1(a^{ij}) = 0 \tag{2.10$'$}$$

Thus $\mathfrak{su}(n)$ has $n - 1$ functionally independent coefficients within its defining representation.

Since the secular equations for $\mathfrak{su}(n)$ derived from the defining and regular representations have the same amount of information, we conclude that the secular equation derived from the regular representation of $\mathfrak{su}(n)$ has $n-1$ functionally independent coefficients. Therefore the rank of $\mathfrak{su}(n)$ (and also $SU(n)$) is $n-1$.

In the regular representation the secular equation of $\mathfrak{u}(n)$ is given by

$$\begin{aligned} \|\mathbf{R}[\mathfrak{u}(n)] - \lambda I_{n^2}\| &= \|\mathbf{R}[\mathfrak{su}(n) \oplus \mathfrak{u}(1)] - \lambda I_{n^2}\| \\ &= (-\lambda)\|\mathbf{R}[\mathfrak{su}(n)] - \lambda I_{n^2-1}\| \end{aligned}$$

This secular equation has exactly as many functionally independent coefficients as does the secular equation for $\mathfrak{su}(n)$. Therefore the rank of $\mathfrak{u}(n)$ is also $n-1$.

We note that it is possible to find n mutually commuting generators $u^i\,\partial_i$ ($i = 1, 2, \ldots, n$; no sum) in the algebra $\mathfrak{u}(n)$. This number is different from, and in fact greater than, the rank of $\mathfrak{u}(n)$. In general, the number of mutually commuting generators in the an algebra is greater than or equal to the rank of the algebra. The numbers are equal for semisimple algebras and groups. The algebra $\mathfrak{u}(n)$ is not semisimple.

Comment 2. The secular equation for a Lie algebra can be computed within any of its matrix representations. The number of functionally independent coefficients ϕ_r will be greater, the more faithful the representation is. All faithful representations have the same number of functionally independent coefficients. The rank of an algebra is defined to be the number of independent coefficients in the regular representation. For simple and semisimple algebras the regular representation is faithful. Therefore the rank, as well as the functionally independent coefficients ϕ_r themselves, can be computed within any faithful representation. A convenient representation for computational purposes is always the defining matrix representation.

Comment 3. The classical groups $SO(2n)$, $SO(2n+1)$, and $USp(2n)$ are semisimple. Their rank may be computed as in Example 3. In fact, they may be regarded as subgroups of $SU(2n)$ and $SU(2n+1)$, with additional constraints on the a^{ij} of (2.10). The functions ϕ_r can in each case be computed from the corresponding ϕ_r (2.9) of the unitary group. An easy calculation (Problem 6) shows that each of these classical groups has rank n.

Example 4. We now compute the rank of the six-dimensional algebra described in (1.43). The secular equation is given by

$$\|\mathbf{R}(a^{ij}U_{ij}) - \lambda I_6\| = (+\lambda)^3(a^{11} - a^{22} + \lambda)(a^{22} - a^{33} + \lambda)(a^{11} - a^{33} + \lambda) \tag{2.11}$$

This secular equation has only three nonvanishing coefficients $\phi_i(a^{ij})$, with $i = 1, 2, 3$. The 3×3 matrix $M_i^{\,j} = \partial\phi_i/\partial a^{jj}$ can easily be computed. This matrix has zero determinant and can, in fact, easily be shown to have rank 2. This can be seen more easily by observing that the coefficients appearing on the major diagonal in (1.43) are not all independent, but are related by the functional expression

$$(a^{11} - a^{22}) + (a^{22} - a^{33}) = (a^{11} - a^{33})$$

The algebra described in (1.43) has rank 2.

This result suggests that the solvable algebra $\mathfrak{r}$ (1.43) contains a nilpotent subalgebra $\mathfrak{n}$, of dimension $(6 - 2) = 4 = (3 + 1)$, which in turn contains the derived algebra $\mathfrak{r}^{(1)}$ (1.43′)

$$\mathfrak{r}^{(1)} \subset \mathfrak{n} \subset \mathfrak{r}$$

This is the case; $\mathfrak{n}$ has as bases

$$U_{12},\ U_{23},\ U_{13},\ U_{11} + U_{22} + U_{33}$$

Comment 4. The subalgebra of $\mathfrak{gl}(n, r)$ consisting of matrices with zero below the major diagonal has rank $n - 1$.

2. JORDAN CANONICAL FORM. The secular equation (2.3) cannot be solved in general. The reason is simple: since the field of real numbers is not an algebraically closed field, algebraic (polynomial) equations do not necessarily have real solutions.

By extending the field from real to complex numbers

$$r^i X_i \to c^i X_i$$

we guarantee that the ηth-order secular equation has exactly η solutions. The field extension does not affect the rank of the algebra. In the remaining part of this chapter and the next, we classify the complex extension algebras. We reserve a classification of their real forms until Chapter 9.

The secular equation can now be written as a product of η factors

$$\sum (-\lambda)^{\eta - j}\phi_j(c^i) = (\lambda - \alpha_0)^{d_0}(\lambda - \alpha_1)^{d_1}(\lambda - \alpha_2)^{d_2} \cdots \tag{2.12}$$

The α_i are the solutions of the secular equation, the d_i are the number of occurrences of the particular solution α_i. The α_i are complex, and the d_i are integers; $\alpha_0 = 0$ is always a solution.

Once the solutions, or **roots** α_i, of the secular equation have been found, it is possible to find a nonsingular transformation $U(c^i)$ which puts the transformation $\mathbf{R}(c^i X_i)$ into a particular canonical form, the Jordan canonical form:

$$U(c^i)\mathbf{R}(c^iX_i)U^{-1}(c^i) = \begin{bmatrix} M(\alpha_0) & & & & \\ & M(\alpha_1) & & & \\ & & M(\alpha_2) & & \\ & & & \ddots & \\ & & & & M(\alpha_r) \end{bmatrix} \tag{2.13}$$

This consists of block diagonal matrices $M(\alpha)$:

$$M(\alpha) = \begin{bmatrix} \alpha & 1 & & & & \\ & \alpha & 1 & & & \\ & & \alpha & 1 & & \\ & & & \alpha & 0 & \\ & & & & \alpha & 0 \\ & & & & & \alpha \end{bmatrix} \tag{2.14}$$

Each $M(\alpha_i)$ is a $d_i \times d_i$ matrix with α_i on the major diagonal, $+1$ or 0 on the diagonal just above it, and zeroes elsewhere.

The vector space on which the canonical form (2.13) acts breaks up into the direct sum of a number of subspaces, which are conveniently labeled by the roots α_i:

$$V = V(0) \oplus V(\alpha_1) \oplus V(\alpha_2) \oplus \cdots \oplus V(\alpha_r) \tag{2.15}$$

The dimension of $V(\alpha_i)$ is d_i, and these subspaces exhaust V.

Within any subspace $V(\alpha)$ we can choose bases $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d_\alpha}$. The action of $M(\alpha)$ on these bases is

$$\begin{bmatrix} \alpha & 1 & & & \\ & \alpha & 1 & & \\ & & \alpha & 1 & \\ & & & \alpha & 0 \\ & & & & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \mathbf{e}_5 \end{bmatrix} = \begin{bmatrix} \alpha\mathbf{e}_1 + \mathbf{e}_2 \\ \alpha\mathbf{e}_2 + \mathbf{e}_3 \\ \alpha\mathbf{e}_3 + \mathbf{e}_4 \\ \alpha\mathbf{e}_4 \\ \alpha\mathbf{e}_5 \end{bmatrix} \tag{2.16}$$

Within any subspace $V(\alpha)$ there is at least one eigenvector.

From (2.16) it is clear that the effect of the operator $\{M(\alpha) - \alpha I\}$ on the bases of $V(\alpha)$ is to annihilate some and to move others "up":

$$\{M(\alpha) - \alpha I\}^2 \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \mathbf{e}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_3 \\ \mathbf{e}_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.17}$$

In general, $\{M(\alpha_i) - \alpha_i I\}^{d\alpha_i}$ annihilates $V(\alpha_i)$, leaving $V(\alpha_j)$ invariant. Sometimes a smaller number m_i can be found such that $\{M(\alpha_i) - \alpha_i I\}^{m_i}$ annihilates $V(\alpha_i)$. Then $m_i - 1$ is the number of $+1$'s on the diagonal above the major diagonal, and $d_i - m_i$ is the number of zeroes. Both the secular polynomial (2.18s) and the minimal polynomial (2.18m) annihilate the vector space V:

$$\prod_{i=0}^{r} \{M(\alpha_i) - \alpha_i I\}^{d_i} \quad V = 0 \tag{2.18s}$$

$$\prod_{i=0}^{r} \{M(\alpha_i) - \alpha_i I\}^{m_i} \quad V = 0 \tag{2.18m}$$

Example. The group $ISO(3)$ of rotations and displacements of the origin in R_3 has generators $\mathbf{L} = \mathbf{r} \times \nabla$ and $\mathbf{p} = \nabla$. It is a subgroup of the Galilean group, with abelian invariant subalgebra $\mathbf{p}$ and simple factor algebra $\mathbf{L}$. The regular representation can be read directly from (1.15), and the secular equation computed:

$$\boldsymbol{\theta} \cdot \mathbf{L} + \mathbf{b} \cdot \mathbf{p} \xrightarrow{\text{reg} - \lambda I} \left\| \left[\begin{array}{c|c} \boldsymbol{\theta} \cdot \boldsymbol{\Sigma} - \lambda I_3 & \mathbf{b} \cdot \boldsymbol{\Sigma} \\ \hline 0 & \boldsymbol{\theta} \cdot \boldsymbol{\Sigma} - \lambda I_3 \end{array} \right] \right\| = 0 \tag{2.19}$$

$$\|\boldsymbol{\theta} \cdot \boldsymbol{\Sigma} - \lambda I_3\|^2 = \{\lambda(\lambda^2 + \boldsymbol{\theta} \cdot \boldsymbol{\theta})\}^2 = 0 \tag{2.20}$$

All roots of this equation are double:

$$\begin{aligned} \lambda &= 0, 0 \qquad |\theta|^2 = \boldsymbol{\theta} \cdot \boldsymbol{\theta} \\ \lambda &= +i|\theta|, +i|\theta| \\ \lambda &= -i|\theta|, -i|\theta| \end{aligned} \tag{2.21}$$

If we choose $\theta^3 = \theta$, $\theta^1 = \theta^2 = \mathbf{b} = 0$, then the subspace decomposition is

$$\begin{aligned} V_0 &= L_3 \oplus p_3 \\ V_{+i\theta} &= L_+ \oplus p_+ \qquad (A_\pm = A_1 \pm iA_2) \\ V_{-i\theta} &= L_- \oplus p_- \end{aligned} \tag{2.22}$$

We observe in this case that the roots are discrete and the subspaces $V_{\pm i\theta}$ are eigenspaces of the subspace V_0, which is abelian.

3. FIRST CRITERION OF SOLVABILITY. The presence of a solvable invariant subalgebra in a Lie algebra leads to fewer independent coefficients $\phi_i(c^j)$ and additional zero roots for the secular equation (2.3). This process reaches its logical conclusion for solvable algebras. We present the result formally as the first criterion of solvability.

FIRST CRITERION OF SOLVABILITY. *A Lie algebra is solvable if and only if its derived algebra has rank zero.*

Proof. If $\mathfrak{g}$ is solvable, then $\mathfrak{g}^{(1)}$ is nilpotent and by (2.6), $\mathfrak{g}^{(1)}$ has rank zero. Every algebra has a solvable subalgebra (N.B.: not necessarily invariant) of dimension ≥ 1. In particular, every one-dimensional subalgebra is solvable. Let $\mathfrak{m}$ be the maximal solvable subalgebra of the algebra $\mathfrak{g}^{(1)}$, and suppose rank $\mathfrak{g}^{(1)} = 0$. Then the regular representation of $\mathfrak{g}^{(1)}$, restricted to $\mathfrak{m}$, forms a representation of $\mathfrak{m}$. The subspace $\mathfrak{m}$ of $\mathfrak{g}^{(1)}$ forms an invariant subspace on which

$$\mathbf{R}(\mathfrak{g}^{(1)}) \downarrow \mathbf{R}(\mathfrak{m}) \tag{2.23}$$

acts. In other words, $\mathbf{R}(\mathfrak{m})$ maps $\mathfrak{m}$ into itself. Therefore a representation of $\mathfrak{m}$ is defined on the factor space $\mathfrak{g}^{(1)}$ mod $\mathfrak{m}$. By Lie's theorem, there is a vector $|v\rangle$ in $\mathfrak{g}^{(1)}$ mod $\mathfrak{m}$ which is a simultaneous eigenvector of $\mathbf{R}(\mathfrak{m})$:

$$\mathbf{R}(\mathfrak{m})\,|\,v\rangle = \lambda\,|\,v\rangle \tag{2.24}$$

Since $\mathfrak{g}^{(1)}$ has rank 0, $\lambda = 0$, and

$$\mathbf{R}(\mathfrak{m})\,|\,v\rangle = \,|[v, \mathfrak{m}]\rangle = 0 \bmod \mathfrak{m} \tag{2.25}$$

The commutator of $\mathfrak{g}^{(1)}$ mod $\mathfrak{m}$ with $\mathfrak{m}$ is an element of $\mathfrak{m}$; thus $\mathfrak{g}^{(1)}$ is solvable. Therefore $\mathfrak{g}$ itself is solvable.

4. PROPERTIES OF THE ROOT SUBSPACES. The subspaces V_{α_i} belonging to the roots α_i have a number of useful properties. Virtually all these properties flow from one key theorem.

KEY THEOREM. *Let* X_0 *be an arbitrary element of a Lie algebra, and let*

$$V = V_0 \oplus V_\alpha \oplus V_\beta \oplus V_\gamma \cdots \tag{2.26}$$

be a decomposition of the algebra with respect to the element X_0. *If*

$$X_\alpha \in V_\alpha, \qquad X_\beta \in V_\beta$$

then

$$[X_\alpha, X_\beta] \in V_{\alpha+\beta} \tag{2.27}$$

Proof. Both X_α and X_β obey equations of the form

$$\{\mathbf{R}(X_0) - \alpha I\}^{d_\alpha} X_\alpha = 0 \tag{2.28}$$

We must show that their commutator obeys a similar equation. First

$$\begin{aligned}
\{\mathbf{R}(X_0) - (\alpha+\beta)I\}\,|[X_\alpha, X_\beta]\rangle \\
&= |[[X_\alpha, X_\beta], X_0]\rangle - (\alpha+\beta)\,|[X_\alpha, X_\beta]\rangle \\
&= |[[X_\alpha, X_0], X_\beta] + [X_\alpha, [X_\beta, X_0]]\rangle - (\alpha+\beta)\,|[X_\alpha, X_\beta]\rangle \\
&= |[[X_\alpha, X_0] - \alpha X_\alpha, X_\beta] + [X_\alpha, [X_\beta, X_0] - \beta X_\beta]\rangle \\
&= \sum_0^1 \binom{1}{i} |[\{\mathbf{R}(X_0) - \alpha I\}^i X_\alpha, \{\mathbf{R}(X_0) - \beta I\}^{1-i} X_\beta]\rangle
\end{aligned} \tag{2.29}$$

Proceeding by induction, we write

$$\{\mathbf{R}(X_0) - (\alpha+\beta)I\}^k\,|[X_\alpha, X_\beta]\rangle = \sum_0^k \binom{k}{i} |[\{\mathbf{R}(X_0) - \alpha I\}^i X_\alpha, \{\mathbf{R}(X_0) - \beta I\}^{k-i} X_\beta]\rangle \tag{2.30}$$

For k sufficiently large ($k \geq d_\alpha + d_\beta$), all terms in the sum vanish, so the commutator belongs to the subspace $V_{\alpha+\beta}$. This result is written symbolically as

$$[V_\alpha, V_\beta] \subset V_{\alpha+\beta} \tag{2.27'}$$

In particular, we can make the following observations from this theorem:

1. $[V_0, V_0] \subset V_0$. The subspace V_0 is closed under commutation and therefore is a subalgebra.

2. The *subspace* V_0 carries a representation $\mathbf{R}_{V_0}(V_0)$ for the *subalgebra* V_0. This is just the regular representation of the entire algebra restricted to the subalgebra V_0 acting within V_0, and it obeys

$$\{\mathbf{R}(V_0) - 0I\}^{d_0} V_0 = 0 \tag{2.31}$$

Since the regular representation of V_0 has only zero eigenvalues, it has rank zero; V_0 is solvable and, in fact, nilpotent by the first criterion, stated earlier.

3. $[V_0, V_\alpha] \subset V_\alpha$. Each subspace V_α is invariant (mapped into itself) under the action of V_0. It therefore carries a representation of $V_0 : \mathbf{R}_\alpha(V_0)$. Then $\mathbf{R}_\alpha(V_0)$ is the regular representation of the algebra, restricted to the subalgebra V_0 acting in the subspace V_α.

4. Since V_0 is solvable, $\mathbf{R}_\alpha(V_0)$ has a simultaneous eigenvector (Lie's theorem).

$$\begin{aligned}
\{\mathbf{R}_\alpha(V_0) - \alpha I\}^1\,|E_\alpha\rangle &= 0 \\
\mathbf{R}_\alpha(V_0)\,|E_\alpha\rangle &= \alpha\,|E_\alpha\rangle
\end{aligned} \tag{2.32}$$

Within each subspace V_α can be found at least one vector E_α that is an eigenvector of $\mathbf{R}_\alpha(V_0)$. The importance of the subspace V_0 is now clear: the root forms are *linear* functions defined on this subspace.

5. The regular representation is *linear*. If $H_1, H_2, \ldots, H_{d_0}$ is a choice of bases for V_0,

$$\begin{aligned} \mathbf{R}(\lambda^i H_i)\,|\,E_\alpha\rangle &= \lambda^i \mathbf{R}(H_i)\,|\,E_\alpha\rangle \\ &= -\lambda^i \alpha_i |\,E_\alpha\rangle \end{aligned} \tag{2.33}$$

We can associate each root $\alpha(\lambda^i H_i)$ of a secular equation for $\mathbf{R}(\lambda^i H_i)$ with a *vector* $\boldsymbol{\alpha}$ having dim (V_0) components defined by

$$\mathbf{R}(H_i)|E_{\boldsymbol{\alpha}}\rangle = -\alpha_i|E_{\boldsymbol{\alpha}}\rangle$$

$$[\mathbf{H}, E_{\boldsymbol{\alpha}}] = +\boldsymbol{\alpha} E_{\boldsymbol{\alpha}} \tag{2.33$'$}$$

6. If an element of V_0 is contained in the derived algebra, then at least one root $\boldsymbol{\beta}$ is the negative of some root $\boldsymbol{\alpha}$. For if

$$[V_{\boldsymbol{\alpha}}, V_{\boldsymbol{\beta}}] \subset V_{\mathbf{0}}, \qquad \boldsymbol{\alpha} + \boldsymbol{\beta} = \mathbf{0} \tag{2.34}$$

7. Let $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ be a root and form the sum

$$S(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \sum V_{\boldsymbol{\beta}+n\boldsymbol{\alpha}} \tag{2.35}$$

This is called the $\boldsymbol{\alpha}$ chain containing $\boldsymbol{\beta}$. It is invariant under the actions of all algebra operations in $V_{\boldsymbol{\alpha}}$, $V_{-\boldsymbol{\alpha}}$, and $V_{\mathbf{0}}$. If $h \in V_{\mathbf{0}}$ can be written as the commutator of $X_{\boldsymbol{\alpha}}$ with $X_{-\boldsymbol{\alpha}}$, then we can compute the trace of the operation h in two ways:

$$\mathbf{R}_s(h) \sum V_{\beta+n\alpha} = \sum d_{\beta+n\alpha}\{\beta + n\alpha\} V_{\beta+n\alpha} \tag{2.36}$$

$$\operatorname{tr} \mathbf{R}_s(h) = \sum d_{\beta+n\alpha}\{\beta + n\alpha\} \tag{2.36$'$}$$

Each root form $\alpha(h)$, $\beta(h)$ is a function of the element $h \in V_0$. Since h is a commutator,

$$\operatorname{tr} \mathbf{R}_s(h) = \operatorname{tr}\,[\mathbf{R}_s(X_\alpha), \mathbf{R}_s(X_{-\alpha})] = 0 \tag{2.37}$$

$$\beta \sum d_{\beta+n\alpha} = -\alpha \sum n d_{\beta+n\alpha} \tag{2.38}$$

We conclude that all roots are real rational multiples of each other.

$$\beta(h) = \text{rational } (\beta, \alpha)\alpha(h) \tag{2.38$'$}$$

8. All these calculations have been done for an algebra decomposition (2.26) based on a particular choice of Lie algebra element X_0. All these procedures can be repeated for any other choice X'_0. We call an element X_0 **regular** provided

(a) Roots that are zero for X_0 are zero for all other choices of "starting point."

(b) The dimensionality of each $V_{\boldsymbol{\alpha}}$ is minimized; or conversely, the number of distinct roots is maximized.

When X_0 is a regular element, V_0 is called a **Cartan subalgebra**.

This is about as far as an investigation of root subspaces can take us. We summarize the properties of roots and their subspaces for convenience.

1. With respect to X_0 we have a decomposition

$$V = V_0 \oplus V_\alpha \oplus V_\beta \oplus V_\gamma \oplus \cdots$$

2. $\{\mathbf{R}(V_0) - \alpha I\}^{d_\alpha} V_\alpha = 0$
3. $[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$
4. V_0 is a solvable subalgebra.
5. In each V_α there is an eigenvector E_α

$$\mathbf{R}_\alpha(V_0)\,|\,E_\alpha\rangle = (\text{multiple})\ |\,E_\alpha\rangle$$

6. With respect to a basis $\mathbf{H}$ in V_0

$$[\mathbf{H}, E_\alpha] = \boldsymbol{\alpha} E_\alpha$$

7. Each root α can be described as a vector $\boldsymbol{\alpha}$ in a space of dimension $\dim(V_0)$.
8. β_i is a rational multiple of $\alpha_i \neq 0$, where $\beta_i = \beta(H_i)$.
9. Regular elements X_0 maximize the number of distinct roots $\boldsymbol{\alpha}$ and minimize the dimensionality of each subspace $V_{\boldsymbol{\alpha}}$. The subspace V_0 of a regular element X_0 is called a Cartan subalgebra.

Example. The regular elements in the Lie algebra (2.19) of $ISO(3)$ are those elements with $\boldsymbol{\theta} \neq 0$. Choosing as regular element

$$X_0 = \theta^3 L_3 \qquad (\theta^3 = \theta \neq 0)$$

the bases for the subalgebra V_0 are L_3 and p_3. Bases for $V_{+i\theta}$ are L_+, p_+; for $V_{-i\theta}$, L_- and p_-. With respect to these bases, the regular matrix representation for L_3 and p_3 is

$$\begin{array}{c} L_3 \\ \Big\downarrow{\scriptstyle \text{reg}} \\ -\left[\begin{array}{cc|cc|cc} & & & & & \\ & & & & & \\ \hline & & i & & & \\ & & & i & & \\ \hline & & & & -i & \\ & & & & & -i \end{array}\right] \end{array}
\begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} V_0 \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} V_{+i} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} V_{-i} \end{array}
\quad
\begin{array}{r} \left.\begin{array}{c} L_3 \\ p_3 \end{array}\right\{ \\ \left.\begin{array}{c} L_+ \\ - \\ p_+ \end{array}\right\{ \\ \left.\begin{array}{c} L_- \\ p_- \end{array}\right\{ \end{array}
\begin{array}{c} p_3 \\ \Big\downarrow{\scriptstyle \text{reg}} \\ \left[\begin{array}{cc|cc|cc} & & & & & \\ & & & & & \\ \hline & & 0 & i & & \\ & & & 0 & & \\ \hline & & & & 0 & -i \\ & & & & & 0 \end{array}\right] \end{array}$$

From this it is clear that, in the invariant (under V_0) subspace $V_{+i\theta}$, p_+ is a simultaneous eigenvector with two distinct eigenvalues

$$[L_3, p_+] = ip_+$$
$$[p_3, p_+] = 0p_+$$

The other basis L_+ for $V_{+i\theta}$ is not a simultaneous eigenvector of L_3 *and* p_3.

III. The Metric

1. MOTIVATION FOR CHOICE. In Chapter 2 metrics were introduced to construct a large variety of linear vector spaces. Now the shoe is on the other foot. We have at hand a large variety of linear vector spaces—the finite-dimensional Lie algebras. We must look for a mechanism for distinguishing between these various spaces.

This distinguishing mechanism is the Cartan[4]-Killing[5] metric function. Since it is to be a property of the algebra, it must be defined in terms of the structure constants, or equivalently, the regular representation. Let A and B be two elements of the Lie algebra. Then their inner product in the regular representation must be a function of the matrix representatives $\mathbf{R}(A)$ and $\mathbf{R}(B)$. We require the inner product $(A, B)_{\text{reg}}$ to be independent both of the original choice of bases X_i for the algebra, and the particular structure of reg. This immediately suggests that the metric be defined in terms of matrix invariants.

Two matrix invariants are easily accessible: the determinant and the trace.

$$\det \|UAU^{-1}\| = \det \|U\| \det \|A\| \det \|U\|^{-1} = \det \|A\| \tag{3.1}$$

$$\operatorname{tr} UAU^{-1} = \operatorname{tr} AU^{-1}U = \operatorname{tr} A \tag{3.2}$$

$$\operatorname{tr} A = \sum_{i=1}^{\eta} A_{ii} \tag{3.3}$$

Suppose we try to define the inner product by using a determinantal definition:

$$(A, B) = \|\mathbf{R}(A)\mathbf{R}(B)\| = \|\mathbf{R}(A)\| \, \|\mathbf{R}(B)\| = 0 \tag{3.4}$$

The determinant vanishes by (2.5). The inner product under this definition carries no information.

So we turn to a definition based on the trace function. Two possibilities present themselves:

$$g_{ij} = (X_i, X_j) \begin{cases} \nearrow \operatorname{tr} \mathbf{R}(X_i) \quad \operatorname{tr} \mathbf{R}(X_j) & \text{(3.5a)} \\ \searrow \operatorname{tr} \mathbf{R}(X_i)\mathbf{R}(X_j) & \text{(3.5b)} \end{cases}$$

The first possibility contains only η independent bits of information; the second $\eta(\eta+1)/2$. Since our game is to maximize the information content of this new tool, we adopt the definition:

$$(A, B) \quad = \operatorname{tr} \mathbf{R}(A)\mathbf{R}(B) \tag{3.6}$$

Comment 1. A determinantal definition would not be useful in any event. By exploiting the secular equation (2.3) we have already squeezed as much information out of the determinantal function as it is willing to give. Moreover, the η bits of information available from (3.5a) are also already available from a study of the secular equation.

Comment 2. The definition (3.6) may also be obtained from the following argument: the structure constants $C_{ir}{}^{s}$ are components of a mixed tensor. To form a second-rank covariant tensor g_{ij} from a pair of mixed tensors $C_{..}{}^{.}$, we need only contract both upper indices. The cross-contraction contains more information

$$(X_i, X_j)_{\text{reg}} = C_{ir}{}^{s} C_{js}{}^{r} \tag{3.6$'$}$$

Comment 3. Since the structure constants for all (real and complex extension) Lie algebras can be made real, g_{ij} is real and symmetric.

2. PROPERTIES AND EXAMPLES. The properties of the trace function can be transferred *in toto* to the metric function. If D, S, T are $n \times n$ matrices,

$$\begin{aligned} \operatorname{tr} ST &= \operatorname{tr} TS \\ \operatorname{tr} [S, T] &= 0 \\ \operatorname{tr} [S, D]T &= \operatorname{tr} S[D, T] \end{aligned} \tag{3.7}$$

The third property can be rearranged in a more suggestive way and compared with two other familiar relations:

$$\begin{aligned} d(f \circ g) &= (df) \circ g \quad + f \circ (dg) \\ [D, [S, T]] &= [[D, S], T] + [S, [D, T]] \\ 0 &= \operatorname{tr} [D, S]T + \operatorname{tr} S[D, T] \end{aligned} \tag{3.8}$$

This suggests that the metric is a constant or an invariant in some sense. Specifically, the metric defined on a group or its cosets is invariant under the group operations EXP D (i.e., under the entire group).

In summary, the properties of the metric are

$$\begin{aligned}(X,Y) &= (Y,X)\\ 0 &= ([D, X], Y) + (X, [D, Y])\\ (X, [Y, Z]) &= (Y, [Z, X]) = (Z, [X, Y])\end{aligned} \tag{3.7'}$$

A metric is said to be degenerate if there is a nonzero vector B orthogonal to all other vectors in the algebra:

$$(B, X)_{\text{reg}} = 0 \qquad \begin{array}{ll} & B \neq 0 \\ \text{all} & X \in \mathfrak{g}\end{array} \tag{3.9}$$

If B is chosen as a basis vector ($B = X_1$), then $g_{i1} = g_{1i} = 0$. The metric matrix has an entire row and column of zeroes, and so has vanishing determinant.

Example 1. Let $\mathfrak{g}$ be a nilpotent algebra. Then its regular representation is given by Fig. 7.2. If A, B are two elements of $\mathfrak{g}$,

$$(A, B)_{\text{reg}} = \text{tr}\begin{bmatrix} 0 & & & & & \\ & 0 & & & * & \\ & & 0 & & & \\ & & & 0 & & \\ & \bigcirc & & & 0 & \\ & & & & & 0 \end{bmatrix}\begin{bmatrix} 0 & & & & & \\ & 0 & & & * & \\ & & 0 & & & \\ & & & 0 & & \\ & \bigcirc & & & 0 & \\ & & & & & 0 \end{bmatrix} = 0 \tag{3.10}$$

If an algebra is nilpotent, the metric on the regular representation is completely degenerate.

Example 2. Let $\mathfrak{g}$ be a solvable algebra, and $\mathfrak{g}^{(1)}$ its derived algebra. Any element in the derived algebra is orthogonal to every element in $\mathfrak{g}$, for

$$(A, B)_{\text{reg}} = \text{tr}\underset{A \in \mathfrak{g},}{\begin{bmatrix} * & & & & & \\ & * & & & * & \\ & & * & & & \\ & & & \cdot & & \\ & \bigcirc & & & \cdot & \\ & & & & & * \end{bmatrix}}\underset{B \in \mathfrak{g}^{(1)}}{\begin{bmatrix} 0 & & & & & \\ & 0 & & & * & \\ & & 0 & & & \\ & & & \cdot & & \\ & \bigcirc & & & \cdot & \\ & & & & & 0 \end{bmatrix}} = 0 \tag{3.11}$$

Nonzero inner products can occur only between two nonzero vectors in $\mathfrak{g} \bmod \mathfrak{g}^{(1)}$. To give a concrete illustration, consider the vector $A = \sum_{1 \leq i \leq j}^{3} a^{ij}U_{ij}$ in the algebra described in (1.43). The inner product of A with itself is given by

$$(A, A)_{\text{reg}} = (a^{11} - a^{22})^2 + (a^{22} - a^{33})^2 + (a^{11} - a^{33})^2$$

Example 3. Let $\mathfrak{g}$ be an algebra and $\mathfrak{m}$ an invariant subalgebra, with $A \in \mathfrak{g}$, $B \in \mathfrak{m}$. Then

$$(A, B) = \operatorname{tr}\left[\begin{array}{c|c} \mathbf{R}_{\mathfrak{g}/\mathfrak{m}}(A) & * \\ \hline O & \mathbf{R}_{\mathfrak{m}}(A) \end{array}\right]\left[\begin{array}{c|c} O & * \\ \hline O & \mathbf{R}_{\mathfrak{m}}(B) \end{array}\right]$$
$$= \operatorname{tr} \mathbf{R}_{\mathrm{m}}(A)\mathbf{R}_{\mathrm{m}}(B) \tag{3.12}$$

If we decompose A as follows,

$$\begin{aligned} A &= A_{\perp} + A_{\|} \\ A_{\|} &\in \mathfrak{m} \\ A_{\perp} &\in \mathfrak{g} \bmod \mathfrak{m} \end{aligned} \tag{3.13}$$

then (3.12) is equivalent to the statement

$$(A_{\|} + A_{\perp}, B)_{\mathfrak{g}} = (A_{\|}, B)_{\mathfrak{m}} \tag{3.12$'$}$$

Example 4. With respect to the bases

$$O_{ij}^{(n)} = M_{ij}^{(n)} - M_{ji}^{(n)} \tag{3.14}$$

of $\mathfrak{so}(n)$, the metric is

$$g_{ij,rs} = -2(n-2)\delta_{ir}\delta_{js} \qquad \begin{array}{c} i<j \\ r<s \end{array} \tag{3.15}$$

The metric tensor for the orthogonal group is negative-definite. In general, the Cartan-Killing metric for compact groups is negative definite, and vice versa.

3. METRICS IN OTHER REPRESENTATIONS. The metric, like the rank, has been defined on the regular representation to facilitate the study of the structure constants. Like rank, the metric is very conveniently defined in terms of the regular representation but (like the secular equation) the regular representation by no means corners the market for the metric function. It can be defined within *any* representation $\boldsymbol{\Gamma}$.

To illustrate these comments, let $\mathfrak{ss}$ be a semisimple algebra. The regular representation for a semisimple algebra decomposes into the regular representations of the component simple invariant subalgebras

$$\mathfrak{ss} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_n \tag{3.16}$$

If $A \in \mathfrak{ss}$, then A can be written as the sum of vectors A_i:

$$A = A_1 + A_2 + \cdots + A_n$$
$$A_i \in \mathfrak{s}_i$$

The inner product of two vectors A, $B \in \mathfrak{ss}$ is given by an expression analogous to (2.7)

$$(A, B)_{\mathrm{reg}} = \operatorname{tr} \begin{bmatrix} \mathbf{R}_1(A_1) & & & * \\ & \mathbf{R}_2(A_2) & & \\ & & \ddots & \\ \bigcirc & & & \mathbf{R}_n(A_n) \end{bmatrix} \times \begin{bmatrix} \mathbf{R}_1(B_1) & & & * \\ & \mathbf{R}_2(B_2) & & \\ & & \ddots & \\ \bigcirc & & & \mathbf{R}_n(B_n) \end{bmatrix} \tag{3.17}$$

$$= \sum_{i=1}^{n} \operatorname{tr} \mathbf{R}_i(A_i)\mathbf{R}_i(B_i) = \sum_{i=1}^{n} (A_i, B_i)_{\mathbf{R}_i} \tag{3.18}$$

This inner product can be studied by studying each component inner product separately.

The inner product of vectors A, $B \in \mathfrak{ss}$ in the representation $\mathbf{\Gamma}$ is defined by the same trace condition

$$(A, B)_{\Gamma} = \operatorname{tr} \begin{bmatrix} \mathbf{\Gamma}_1(A_1) & & & * \\ & \mathbf{\Gamma}_2(A_2) & & \\ & & \ddots & \\ \bigcirc & & & \mathbf{\Gamma}_n(A_n) \end{bmatrix} \times \begin{bmatrix} \mathbf{\Gamma}_1(B_1) & & & * \\ & \mathbf{\Gamma}_2(B_2) & & \\ & & \ddots & \\ \bigcirc & & & \mathbf{\Gamma}_n(B_n) \end{bmatrix} \tag{3.19}$$

$$= \sum_{i=1}^{n} \operatorname{tr} \mathbf{\Gamma}_i(A_i)\mathbf{\Gamma}_i(B_i) = \sum_{i=1}^{n} (A_i, B_i)_{\mathbf{\Gamma}_i} \tag{3.20}$$

In general the inner products (3.17) and (3.19) are not proportional. But if $\mathfrak{s}$ is a *simple* Lie algebra, with no proper invariant subalgebras, then the inner product of two vectors A, $B \in \mathfrak{s}$ is a property of the algebra alone. The inner product within any representation $\mathbf{\Gamma}$ is related to an abstract inner product (A, B) by

$$\begin{aligned} (A, B)_{\Gamma} &= \operatorname{tr} \mathbf{\Gamma}(A)\mathbf{\Gamma}(B) = f(\mathbf{\Gamma})(A, B) \\ (A, B)_{\mathrm{reg}} &= \operatorname{tr} \mathbf{R}(A)\mathbf{R}(B) = f(\mathrm{reg})(A, B) \end{aligned} \tag{3.21}$$

The function $f(\mathbf{\Gamma})$ is a scalar, depending only on the representation $\mathbf{\Gamma}$. The inner product must carry exactly the same information for all faithful representations. The scalar function $f(\mathbf{\Gamma})$ is called the **index**[6] of the representation $\mathbf{\Gamma}$.

Example. The representations for the group $SU(2)$ have been constructed already (Chapter 5, Section III.) The generators for the corresponding $\mathscr{D}^J$ representations of this algebra have also been constructed:

$$\begin{array}{cccc} J_3 \rightarrow \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} & \begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix} & \begin{array}{c} m\delta_{m'm} \\ -J \leq m,\ m' \leq +J \end{array} & \\ J = \frac{1}{2} & J = 1 & 2J = \text{integer} & \\ \text{defining} & \text{regular} & \mathscr{D}^J & \\ \text{represen-} & \text{represen-} & \text{represen-} & \\ \text{tation} & \text{tation} & \text{tation} & \end{array} \tag{3.22}$$

$$(J_3, J_3)_{\text{reg}} = (1)^2 + (0)^2 + (-1)^2 = 2 \tag{3.23}$$

$$(J_3, J_3)_J = \sum_{-J}^{+J} (m)^2 = \frac{1}{3} J(J+1)(2J+1) \tag{3.24}$$

The inner product of any two vectors in the algebra of $SU(2)$ is related to their inner product in the regular representation by

$$f^{-1}(J)(A, B)_J = (A, B) = f^{-1}(\text{reg})(A, B)_{\text{reg}}$$

$$(A, B)_J = \tfrac{1}{6} J(J+1)(2J+1) \quad (A, B)_{J=1} \tag{3.25}$$

Comment. The inner product (A, B) in any representation $\mathbf{\Gamma}$ can be related to the inner product in any other representation $\mathbf{\Gamma}'$ by computing the ratio $f(\mathbf{\Gamma})/f(\mathbf{\Gamma}')$. The computation can be performed by evaluating *one* inner product in both representations (provided the choice of A, B is not so silly that the inner product vanishes in *both* representations). This is the essence of the Wigner-Eckart theorem.[7,8] In physical applications the angular momentum algebra $\mathbf{L}$ is easy to work with. Any set of operators (say, $\mathbf{p}$) with the same transformation properties under rotations will have matrix elements scaled according to the matrix elements of $\mathbf{L}$.

4. SECOND CRITERION OF SOLVABILITY. If a Lie algebra is solvable, then the Cartan-Killing metric is completely degenerate on its derived algebra, and vice versa. These statements constitute the second criterion of solvability.

SECOND CRITERION OF SOLVABILITY. *An algebra is solvable if and only if its Cartan-Killing metric tensor is identically zero on its derived algebra.*

Proof. If $\mathfrak{g}$ is solvable, then $\mathfrak{g}^{(1)}$ has rank zero by the first criterion of solvability. Therefore, all the diagonal matrix elements in $\mathbf{R}(\mathfrak{g}^{(1)})$ are zero, and the Cartan-Killing metric vanishes by (3.10). Now suppose that $\mathfrak{g}$ is not solvable. Then the derived series terminates in an algebra

$$[\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}] = \mathfrak{g}^{(n+1)} = \mathfrak{g}^{(n)} \tag{3.26}$$

which does not vanish. We can find a Cartan subalgebra V_0 in this invariant subalgebra $\mathfrak{g}^{(n)}$. Since $\mathfrak{g}^{(n)}$ is not solvable, there is a nonzero root $\boldsymbol{\alpha}$, and a nonzero root $-\boldsymbol{\alpha}$, with

$$V_{\mathbf{0}} = \oplus \sum_{\boldsymbol{\alpha} \neq 0} [V_{\boldsymbol{\alpha}}, V_{-\boldsymbol{\alpha}}] \tag{3.27}$$

If H is any element of $V_{\mathbf{0}}$, then

$$\begin{aligned} 0 \underset{\text{assumption}}{\overset{\text{by}}{=}} (H, H)_{\mathfrak{g}^{(1)}} &\overset{(3.12)}{=} (H, H)_{\mathfrak{g}^{(n)}} \\ &= \operatorname{tr} \mathbf{R}_{\mathfrak{g}^{(n)}}(H)\mathbf{R}_{\mathfrak{g}^{(n)}}(H) \\ &= \sum_{\boldsymbol{\beta} \neq 0} d_{\boldsymbol{\beta}}(\boldsymbol{\beta})^2 \\ &= (\boldsymbol{\alpha})^2\{d_{\boldsymbol{\alpha}}1 + \sum_{\boldsymbol{\beta} \neq \boldsymbol{\alpha}} d_{\boldsymbol{\beta}}[\text{rational } (\boldsymbol{\beta}, \boldsymbol{\alpha})]^2\} \end{aligned} \tag{3.28}$$

The factor within the bracket can only vanish when $\boldsymbol{\alpha} = 0$. The factor $(\boldsymbol{\alpha})^2$ vanishes also only when $\boldsymbol{\alpha} = \mathbf{0}$. Then all roots are zero, $\mathfrak{g}^{(n)}$ is solvable and, therefore, $\mathfrak{g}^{(0)}$ is itself solvable.

Comment. Although the first and the second criterion of solvability differ in detail, both stem directly from the same source. This is the study of the diagonal matrix elements (eigenvalues) of the regular matrix representation of solvable and nilpotent algebras after they have been transformed to "Jordan canonical form."

Example. The six-dimensional solvable Lie algebra described in (1.43) has rank 2 (see 2.11) and inner product

$$\begin{aligned} (A, B)_{\text{reg}} = {} & (a^{11} - a^{22})(b^{11} - b^{22}) \\ & + (a^{22} - a^{33})(b^{22} - b^{33}) + (a^{11} - a^{33})(b^{11} - b^{33}) \end{aligned} \tag{3.29}$$

The Cartan-Killing metric is singular on $\mathfrak{g}$, and is identically zero on $\mathfrak{g}^{(1)}$. This algebra is solvable by the second criterion of solvability.

Comment. The inner product $(A, B)_{\text{reg}}$ above is faithful on $\mathfrak{g}/\mathfrak{n}$, where $\mathfrak{g}^{(1)} \subset \mathfrak{n} \subset \mathfrak{g}$ and $\mathfrak{n}$ is the maximal nilpotent subalgebra of $\mathfrak{g}$.

IV. Cartan's Criterion

1. THE CRITERION. The first criterion of solvability results from a study of the secular equation. The second criterion comes from the metric. Both are useful but limited.

Cartan synthesized both criteria into one which is much more useful. Not only does it distinguish solvable from nonsolvable algebras, but it also provides a tool for tearing apart an arbitrary algebra into its component pieces: the maximal solvable invariant subalgebra and the semisimple factor algebra.

CARTAN'S CRITERION

$$\|g\| = 0 \Leftrightarrow \textit{nonsemisimple} \tag{4.1a}$$

$$\Updownarrow$$

$$\|g\| \neq 0 \Leftrightarrow \textit{semisimple} \tag{4.1b}$$

Proof. If $\mathfrak{g}$ is nonsemisimple, then $\|g\| = 0$ by (1.60′). Now if g is singular, the equation

$$\sum r^i g_{ij} = 0$$

has nonzero solutions. The subspace $\mathfrak{u}$ of vectors of the form

$$\sum r^i X_i \in \mathfrak{u}, \qquad \sum r^i g_{ij} = 0$$

forms a subalgebra and, in fact, an invariant subalgebra; for if $Y \in \mathfrak{g}$ and X_1, $X_2 \in \mathfrak{u}$, then

$$\text{tr}\,[\mathbf{R}(X_1), \mathbf{R}(X_2)]\mathbf{R}(Y) = \text{tr}\,\mathbf{R}(X_1)[\mathbf{R}(X_2), \mathbf{R}(Y)] = 0 \tag{4.2}$$

In particular, every pair of vectors in this invariant subalgebra has vanishing inner product. By the second criterion, this invariant subalgebra is solvable, and thus $\mathfrak{g}$ is nonsemisimple.

We have demonstrated (4.1a) in the statement of Cartan's criterion. The statement of (4.1b) follows trivially from (4.1a).

2. COMMENTS AND EXAMPLES. Cartan's Criterion enables us to take apart an arbitrary nonsemisimple algebra into its component pieces: the semisimple subalgebra and the maximal solvable invariant subalgebra. An arbitrary nonsemisimple algebra has the semidirect sum structure

$$\begin{aligned} \mathfrak{nss} &= \mathfrak{ss} \oplus \mathfrak{r} & \qquad \mathfrak{ss} &= \text{semisimple} \\ & & \mathfrak{r} &= \text{radical} \end{aligned}$$

$$\begin{aligned} [\mathfrak{ss}, \mathfrak{ss}] &= \mathfrak{ss} \\ [\mathfrak{ss}, \mathfrak{r}] &\subseteq \mathfrak{r} \\ [\mathfrak{r}, \mathfrak{r}] &= \mathfrak{r}^{(1)} \qquad \mathfrak{r}^{(1)} \subseteq \mathfrak{n} \subseteq \mathfrak{r} \end{aligned} \tag{4.3}$$

where $\mathfrak{n}$ is the maximal nilpotent subalgebra of $\mathfrak{r}$. If $\mathfrak{r}$ is nilpotent, then $\mathfrak{r} = \mathfrak{n}$.

The Cartan-Killing metric is

(1) Nonsingular on $\mathfrak{ss}$.
(2) Nonsingular on $\mathfrak{r}$ mod $\mathfrak{n}$.
(3) Identically zero on $\mathfrak{n}$.

In addition, the Cartan-Killing metric g_{ij} is real and symmetric, and can therefore be brought to canonical diagonal form. These observations allow us to formulate a two-step procedure for computing $\mathfrak{ss}$ and $\mathfrak{r}$.

1. Bring the Cartan-Killing metric for $\mathfrak{nss}$ to canonical diagonal form

$$g \rightarrow \left[\begin{array}{c|c|c} +1I_{n_a} & & \\ \hline & -1I_{n_b} & \\ \hline & & 0I_{n_0'} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} \mathfrak{nss} \bmod \mathfrak{n} \\ \left.\vphantom{a}\right\} \mathfrak{n} \end{array} \tag{4.4-1}$$

The n_0' generators spanning the subspace on which g_{ij} is identically zero span $\mathfrak{n}$. The remaining generators span

$$\mathfrak{nss} \bmod \mathfrak{n} = \mathfrak{ss} \oplus (\mathfrak{r} \bmod \mathfrak{n})$$

2. Construct the Cartan-Killing metric for the factor algebra $\mathfrak{ss} \oplus (\mathfrak{r}$ mod $\mathfrak{n})$. Since $(\mathfrak{r}$ mod $\mathfrak{n})$ is abelian, the Cartan-Killing metric is identically zero on this subspace

$$g' \rightarrow \left[\begin{array}{c|c|c} +1I_{n_+} & & \\ \hline & -1I_{n_-} & \\ \hline & & 0I_{n_0} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} \mathfrak{ss} \\ \left.\vphantom{a}\right\} (\mathfrak{r} \bmod \mathfrak{n}) \end{array} \tag{4.4-2}$$

The metric g is nonsingular on the semisimple subalgebra $\mathfrak{ss}$. In addition, the n_- generators spanning the subspace on which g' is negative-definite are closed under commutation; in fact, they generate the maximal compact subgroup K associated with the algebra $\mathfrak{ss}$. The n_+ dimensional subspace on which g' is positive-definite is not closed under commutation. This subspace EXPonentiates onto the noncompact coset representatives SS/K.

If $(n_+ + n_-) = 0$, the algebra is solvable. If in addition $n_0 = 0$, the algebra is nilpotent. If $(n_0 + n_0') = 0$, the algebra is semisimple.

If the complex algebra $\mathfrak{g}$ is simple, it may have several different real forms. These are characterized by the different values that n_- and n_+ may assume (almost completely). Their difference is called the character χ of a real form of a complex algebra

$$\chi(\text{real form}) = n_+ - n_- \tag{4.5}$$

For semisimple algebras, the value of χ ranges from $-$(dimension of algebra) for the compact real form, to $+$(rank of algebra) for the least compact real form, called the normal form.

Example. We illustrate this two-step procedure by applying it to a nonsemisimple algebra. The simplest semisimple algebra has dimension 3, the simplest solvable algebra has dimension 2. Therefore, the simplest non-semisimple algebra has dimension 5. We choose as bases for this algebra J_3, $J_\pm$, K_3, and K_+:

$$\begin{aligned} &[J_3, J_\pm] = \pm J_\pm \\ &[J_+, J_-] = 2J_3 \qquad [J_i, K_j] = 0 \\ &[K_3, K_+] = +K_+ \end{aligned}$$

The Cartan-Killing metric for this algebra is

$$\left[\begin{array}{ccc|cc} 2 & 0 & 0 & & \\ 0 & 0 & 4 & & \\ 0 & 4 & 0 & & \\ \hline & & & 1 & 0 \\ & & & 0 & 0 \end{array}\right] \begin{array}{l} \left.\begin{array}{l} J_3 \\ J_+ \\ J_- \end{array}\right\} \mathfrak{ss} \\ \left.\begin{array}{l} K_3\}\ \mathfrak{r} \bmod \mathfrak{n} \\ K_+\}\ \mathfrak{n} = \mathfrak{r}^{(1)} \end{array}\right\} \mathfrak{r} \end{array} \tag{4.6-1}$$

We compute the metric on the algebra spanned by J_3, $J_\pm$, K_3 by performing step 2.

$$
\left[\begin{array}{ccc|c}
2 & 0 & 0 & \\
0 & 0 & 4 & \\
0 & 4 & 0 & \\
\hline
 & & & 0
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{l} J_3 \\ J_+ \\ J_- \end{array}\right\} \mathrm{ss} \\
K_3\}\mathrm{r} \bmod \mathrm{n}
\end{array}
\tag{4.6-2}
$$

The nonsemisimple groups and algebras of physical interest usually have a solvable invariant subalgebra that is also nilpotent. Then a single step is sufficient for the decomposition into the semisimple part and the radical. We illustrate this for the Galilean group in the following example.

Example. The metric tensor is easy to compute in general. Letting $a^i X_i$ and $a'^j X_j$ be two arbitrary vectors in an algebra, we write

$$\operatorname{tr} \mathbf{R}(a^i X_i)\mathbf{R}(a'^j X_j) = a^i g_{ij} a'^j \tag{4.7}$$

For the Galilean group, this gives, on inspection of (1.15),

$$(A, B) = -6\boldsymbol{\theta} \cdot \boldsymbol{\theta}' \tag{4.7$'$}$$

The metric tensor is completely determined by setting $a^i = 1$, $a'^j = 1$, all others are zero in (4.7):

$$
g_{ij} \xrightarrow[\text{group}]{\text{Galilean}}
\begin{array}{c}
\begin{array}{cccc} \mathbf{L} & \mathbf{v} & T & \mathbf{p} \end{array} \\
\left[\begin{array}{c|c|c|c}
-6I_3 & & & \\
\hline
 & \text{O} & & \\
\hline
 & & 0 & \\
\hline
 & & & \text{O}
\end{array}\right]
\end{array}
\tag{4.8}
$$

From the structure of (4.8) it is immediately apparent that $\mathbf{v}$, T, $\mathbf{p}$ generate the maximal solvable invariant subgroup and this subgroup is nilpotent; $\mathbf{L}$ generates a semisimple group, and this semisimple group is compact.

THEOREM. *The regular representation of a semisimple algebra is faithful.*

Proof. Let A, B be two elements in this algebra, with $\mathbf{R}(A) = \mathbf{R}(B)$. Then

$$\mathbf{R}(A) - \mathbf{R}(B) = \mathbf{R}(A - B) = 0 \tag{4.9}$$

Then $A - B$ is orthogonal to all elements in the algebra. Since the metric function of a semisimple algebra is nonsingular, $A - B$ must be the zero vector. Then

$$\mathbf{R}(A) = \mathbf{R}(B) \Leftrightarrow A = B \tag{4.10}$$

and reg is $1 - 1$.

Comment. With this theorem we have unveiled the last secret of the regular representation. It is faithful on every element of every semisimple algebra.

3. COMPLETE REDUCIBILITY OF SEMISIMPLE ALGEBRAS. Suppose that $\mathfrak{g} = \mathfrak{ss}$ is semisimple. Then $\|g_{ij}\| \neq 0$ and an inverse exists

$$g_{ij}g^{jk} = \delta_i^{\ k} = g^{kj}g_{ji} \tag{4.11}$$

These tensors are symmetric, nonsingular and dual, and they may be used to raise and lower indices. In particular, we can write

$$\begin{aligned} C_{ijk} &= C_{ij}{}^{l}g_{lk} \\ &= C_{ij}{}^{l}C_{lr}{}^{s}C_{ks}{}^{r} \\ &= -(C_{jr}{}^{l}C_{li}{}^{s} + C_{ri}{}^{l}C_{lj}{}^{s})C_{ks}{}^{r} \end{aligned} \tag{4.12}$$

This third-order tensor is completely antisymmetric:

$$\begin{array}{ccccccc} C_{ijk} = & & C_{jki} = & & C_{kij} & & \\ \| & & \| & & \| & = & C_{\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline \end{array}} \\ -C_{kji} = & & -C_{ikj} = & & -C_{jik} & & \end{array} \tag{4.13}$$

We now show that if a semisimple algebra $\mathfrak{ss}$ has an invariant subalgebra $\mathfrak{s}$, then the orthogonal complementary subspace $\mathfrak{s}_\perp$ forms an invariant subalgebra as well. The original algebra is simply the direct sum

$$\begin{aligned} &\mathfrak{ss} = \mathfrak{s} \oplus \mathfrak{s}_\perp \\ &\frac{\mathfrak{ss}}{\mathfrak{s}} = \mathfrak{s}_\perp \end{aligned} \tag{4.14}$$

THEOREM. *If the regular representation of a semisimple algebra is reducible, it is fully reducible.*

Proof. Let X_i, X_j, X_k, ..., span the invariant subalgebra $\mathfrak{s}$, and let X'_r, X'_s, X'_t, ..., round out the generators for $\mathfrak{ss}$. By a simple Schmidt orthogonalization, a new set of complementary bases can be chosen:

$$|X_r\rangle = \left\{ I - \sum_i \sum_j |X_i\rangle g^{ij} \langle X_j| \right\} |X'_r\rangle \tag{4.15}$$

where

$$g_{ij} = (X_i, X_j)_{\text{reg}} = \langle X_i | X_j \rangle = \text{tr}\, \mathbf{R}(X_i)\mathbf{R}(X_j) \tag{4.16}$$

is the nonsingular metric on $\mathfrak{s}$, and g^{ij} is its inverse. These bases X_r are orthogonal to the bases X_i. With respect to these bases, the metric tensor has a block diagonal structure

$$g = \left[\begin{array}{c|c} g_{rs} & \bigcirc \\ \hline \bigcirc & g_{ij} \end{array} \right] \tag{4.17}$$

Since the X_i span an invariant subalgebra,

$$C_{is}{}^{t} = 0 \qquad \text{(invariant)} \tag{4.18i}$$

$$C_{ij}{}^{t} = 0 \qquad \text{(subalgebra)} \tag{4.18s}$$

The structure constants $C_{\alpha s}{}^{k}$ $(\alpha = i, \ldots, ; r, \ldots)$ can be computed as follows:

$$\begin{array}{lll} C_{\alpha s}{}^{k} = & C_{\alpha s \beta}\, g^{\beta k} & \\ & \| & \\ & C_{\alpha s l}\, g^{lk} & \text{by (4.17)} \\ & \| & \\ & C_{i \alpha s}\, g^{ik} & \text{by (4.13)} \\ & \| & \\ & C_{i\alpha}{}^{t} g_{ts} g^{ik} & \text{by (4.12) and (4.17)} \\ & \| & \\ & 0 & \text{by (4.18)} \end{array} \tag{4.19}$$

The regular representation for the generators X_i and X_r is now

$$\mathbf{R}(X_i) = - \begin{array}{c} \\ s \\ \\ j \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} \quad t \quad & \quad k \quad \end{array} \\ \left[\begin{array}{c|c} \begin{array}{c} \bigcirc \\ C_{is}{}^{t} = 0 \end{array} & \begin{array}{c} \bigcirc \\ C_{is}{}^{k} = \\ C_{ji}{}^{t} g_{ts} g^{jk} \end{array} \\ \hline \begin{array}{c} \bigcirc \\ C_{ij}{}^{t} = 0 \end{array} & C_{ij}{}^{k} \neq 0 \end{array} \right] \end{array} \tag{4.20}$$

$$\mathbf{R}(X_r) = - \begin{array}{c} \\ s \\ \\ j \end{array} \begin{array}{c|c} t & k \\ \hline C_{rs}{}^{t} \neq 0 & \begin{matrix} \bigcirc \\ C_{rs}{}^{k} = \\ C_{lr}{}^{t} g_{ts} g^{lk} \end{matrix} \\ \hline \begin{matrix} \bigcirc \\ C_{rj}{}^{t} = 0 \end{matrix} & \begin{matrix} \bigcirc \\ C_{rj}{}^{k} = \\ -C_{lj}{}^{t} g_{tr} g^{lk} \end{matrix} \end{array} \tag{4.20'}$$

Continuing in this way, any semisimple algebra can be reduced to the direct sum of simple algebras. This reduction is unique, except for order:

$$\mathfrak{ss} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \mathfrak{s}_3 \oplus \cdots \tag{4.21}$$

Comment. For a nonsemisimple group $\|g\| = 0$, and the inverse g^{ij} is not defined. The calculation corresponding to (4.19) cannot be performed, and thus it is not possible to relate $C_{is}{}^{k}$ with $C_{ik}{}^{t}$ and $C_{rs}{}^{k}$ with $C_{ks}{}^{r}$. It is at this point that full reducibility, valid for semisimple groups, becomes invalid for nonsemisimple groups.

Summary. An arbitrary nonsemisimple Lie algebra has a semidirect sum structure

$$\begin{aligned} \mathfrak{nss} &= \mathfrak{ss} \quad \oplus \quad \mathfrak{r} \qquad && \mathfrak{r} = \text{radical} \\ & && \mathfrak{ss} = \mathfrak{nss} \bmod \mathfrak{r} \\ [\mathfrak{ss}, \mathfrak{ss}] &= \mathfrak{ss} \\ [\mathfrak{ss}, \mathfrak{r}] &\subseteq \mathfrak{r} \\ [\mathfrak{r}, \mathfrak{r}] &= \mathfrak{r}^{(1)} \subset \mathfrak{r} \end{aligned} \tag{4.22a}$$

The factor algebra $\mathfrak{ss}$ is essentially unique. Furthermore, $\mathfrak{ss}$ splits up into a direct sum of simple invariant subalgebras

$$\begin{aligned} \mathfrak{ss} &= \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \mathfrak{s}_3 \oplus \cdots \\ [\mathfrak{s}_i, \mathfrak{s}_i] &= \mathfrak{s}_i \\ [\mathfrak{s}_i, \mathfrak{s}_j] &= 0 \qquad i \neq j \end{aligned} \tag{4.22b}$$

The decomposition described in (4.22) is sometimes called a **Levi decomposition.**

V. Canonical Commutation Relations for Semisimple Algebras

1. STRUCTURE OF THE METRIC IN A ROOT SUBSPACE DECOMPOSITION. The root concepts developed in Section II and the metric concepts of Section III have individually led to some quite powerful

results. Together they yield a canonical structure for any semisimple Lie algebra.

We compute the Cartan-Killing metric tensor in a root space decomposition. The bases are $X_\alpha, X_\mu, \ldots$, which span the various root subspaces V_α, $V_\mu, \ldots$. Then

$$[V_\alpha, V_\mu] \subset V_{\alpha+\mu} \Rightarrow C_{\alpha\mu}{}^{\nu} = 0 \qquad \text{unless} \quad \alpha + \mu = \nu \tag{5.1}$$

The metric tensor is then

$$g_{\alpha,\beta} = C_{\alpha\mu}{}^{\nu} C_{\beta\nu}{}^{\mu} \tag{5.2}$$

and clearly has the structure

$$\begin{array}{cc}
 & \begin{array}{cccccc} V_0 & V_\alpha & V_{-\alpha} & V_\beta & V_{-\beta} & \end{array} \\
\begin{array}{r} V_0 \\ V_\alpha \\ g \to V_{-\alpha} \\ V_\beta \\ V_{-\beta} \\ \\ \end{array} &
\left[\begin{array}{cccccc} * & & & & & \\ & & * & & & \\ & * & & & & \\ & & & & * & \\ & & & * & & \\ & & & & & \ddots \end{array}\right]
\end{array} \tag{5.3}$$

Since g is nonsingular for a semisimple algebra:

1. The metric g restricted to the Cartan subspace V_0 is nonsingular

$$\| g \|_{\text{in } V_0} \equiv \| g_{ij}{}^{0} \| = \| h_{ij} \| \neq 0 \tag{5.4}$$

2. If α is a root, $-\alpha$ must be a root, and

$$\dim V_\alpha = \dim V_{-\alpha} = d_\alpha \tag{5.5}$$

3. The submatrix $g(V_\alpha, V_{-\alpha})$ can always be chosen to be the unit $d_\alpha \times d_\alpha$ matrix:

$$g(V_\alpha, V_{-\alpha}) = I_{d_\alpha} \tag{5.6}$$

This is true because the metric form is *linear* on each subspace V_α $(\alpha \neq 0)$. The subspaces V_α and $V_{-\alpha}$ are therefore *dual* to each other.

2. PROPERTIES OF THE CARTAN SUBALGEBRA

THEOREM. *The Cartan subalgebra of a semisimple algebra is commutative.*

Proof. Let H_r, H_s be any two elements in V_0. Then by (5.3) we write

$$([H_r, H_s], X_\alpha) = 0 \qquad X_\alpha \in V_\alpha, \boldsymbol{\alpha} \neq \mathbf{0} \tag{5.7}$$

Let H_t be any element in V_0. Then

$$\begin{aligned}([H_r, H_s], H_t) &= \operatorname{tr} \mathbf{R}([H_r, H_s])\mathbf{R}(H_t) \\ &= \operatorname{tr} \mathbf{R}(H_r)\mathbf{R}(H_s)\mathbf{R}(H_t) - \operatorname{tr} \mathbf{R}(H_s)\mathbf{R}(H_r)\mathbf{R}(H_t)\end{aligned} \tag{5.8}$$

The Cartan subalgebra is solvable. So by Lie's theorem the regular representation for V_0 can be transformed to upper-triangular form. The inner product is unchanged, since the trace function is independent of similarity transformations (changes of basis). Since the trace of a product of upper triangular matrices is independent of the order in which the product is taken, the difference in (5.8) vanishes. The commutator $[H_r, H_s]$ is orthogonal to every element in the algebra. Since $\|g\| \neq 0$ for a semisimple algebra, the commutator itself must be zero. Therefore, V_0 is abelian:

$$[H_r, H_s] = 0 \qquad H_r, H_s \in V_0 \tag{5.9}$$

Let E_α be an eigenvector in V_α and let $X_{-\alpha}$ be an element of $V_{-\alpha}$. These obey

$$[H_j, E_\alpha] = \alpha_j E_\alpha \tag{2.33$'$}$$

$$(E_\alpha, X_{-\alpha}) = 1 \tag{5.6$'$}$$

Then the commutator of these two vectors is a (possibly zero) vector in V_0:

$$[E_\alpha, X_{-\alpha}] = \alpha^i H_i \tag{5.10}$$

A simple relationship exists between the covariant root components α_j and the dual contravariant components α^i:

$$\begin{array}{ccc} (H_i, [E_\alpha, X_{-\alpha}]) & = & ([H_i, E_\alpha], X_{-\alpha}) \\ \| & & \| \\ (H_i, \alpha^j H_j) = \alpha^j h_{ij} & = \alpha_i = & (\alpha_i E_\alpha, X_{-\alpha}) \end{array} \tag{5.11}$$

The relation between the α_i and α^j allows us to reach these conclusions:

1. The number of functionally independent roots α_i is equal to (for regular elements of the algebra), or less than the number of functionally independent coefficients ϕ_i of the secular equation.

2.
$$\|h_{ij}\| = \|\text{tr}\, \mathbf{R}(H_i)\mathbf{R}(H_j)\|$$
$$= \left\| \sum_{\boldsymbol{\alpha} \neq \mathbf{0}} d_{\boldsymbol{\alpha}} \alpha_i \alpha_j \right\| \neq 0 \tag{5.12}$$

The number of independent α_i is equal to the dimension of V_0.

3. The number of independent contravariant roots α^j cannot exceed either dim V_0 or the number of independent α_i. Nor can it be less, for then it would be possible to find a nonzero subspace $\beta^i H_i$ of V_0 that is not formed by any commutators:

$$\beta^i H_i \not\subset \sum_{\boldsymbol{\alpha} \neq \mathbf{0}} [V_{\boldsymbol{\alpha}}, V_{-\boldsymbol{\alpha}}]$$

This cannot be, since every element of a semisimple algebra can be written as a commutator of other elements in the algebra.

These considerations are summarized in the Rank theorem.

RANK THEOREM. *Every semisimple algebra of rank l has exactly l independent:*
invariants ϕ_i, *called Casimir invariants*
bases H_i *in the Cartan subalgebra*
root forms $[\mathrm{H}_i, \mathrm{E}_{\boldsymbol{\alpha}}] = \alpha_i \mathrm{E}_\alpha$
dual root forms $[\mathrm{E}_{\boldsymbol{\alpha}}, \mathrm{X}_{-\boldsymbol{\alpha}}] = \alpha^i \mathrm{H}_i$

Comment. In general, the number of functionally independent roots is equal to the rank (except on nonregular elements). The dimensionality of the Cartan subalgebra is at least equal to the rank. Nilpotent algebras present the opposite extreme from semisimple algebras: their rank is zero and the Cartan "subalgebra" V_0 consists of the entire algebra.

3. FIRST CHAIN CONDITION. The $\boldsymbol{\alpha}$ chain (Fig. 7.4) containing $\mathbf{0}$ which is given by

$$S(\mathbf{0}, \boldsymbol{\alpha}) = \lambda E_{\boldsymbol{\alpha}} + V_0 + \sum_{n=1}^{\infty} V_{-n\boldsymbol{\alpha}} \qquad (\lambda \text{ complex}) \tag{5.13}$$

is mapped into itself under the actions of $E_{\boldsymbol{\alpha}}$ and $X_{-\boldsymbol{\alpha}}$. The $X_{-\boldsymbol{\alpha}}$ maps each subspace into the next lower one; V_0 maps each into itself, and $E_{\boldsymbol{\alpha}}$, which is the simultaneous eigenvector of V_0 in $V_{\boldsymbol{\alpha}}$, maps each subspace of (5.13) into the next higher one, while annihilating itself:

$$[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\alpha}}] = 0$$
$$[E_{\boldsymbol{\alpha}}, V_{-n\boldsymbol{\alpha}}] \subset V_{-(n-1)\boldsymbol{\alpha}}, \text{ etc.} \tag{5.14}$$

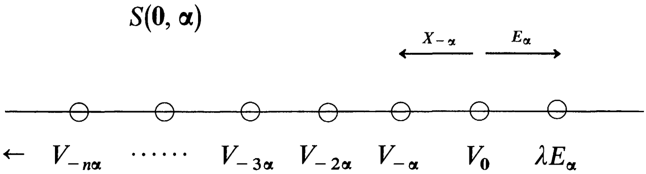


FIG. 7.4 THE MODIFIED $\boldsymbol{\alpha}$ CHAIN CONTAINING $\mathbf{0}$ STARTS WITH THE EIGENVECTOR E_α OF V_0. THE ACTION OF E_α IS TO MAP EACH SPACE $V_{-k\alpha}$ INTO THE NEXT HIGHER ONE $V_{-(k-1)\alpha}$. IN THE OTHER DIRECTION, THE MAPPING IS DONE BY $X_{-\alpha}$, RELATED TO E_α BY $(E_\alpha, X_{-\alpha}) = 1$. ALL ELEMENTS OF V_0 MAP EACH SUBSPACE INTO ITSELF.

This invariant subspace is closed under commutation and forms a finite-dimensional algebra. The regular representation $\mathbf{R}_s$ of this subalgebra is defined as usual. The trace of the operator (5.10) can be computed in two different ways:

$$\begin{array}{ccc} \operatorname{tr} \mathbf{R}_s(\alpha^i H_i) = & \left\{\sum_{i=1}^{l} \alpha^i \alpha_i\right\} & \left\{1 - \sum_{n=1}^{\text{finite}} n \dim V_{-n\alpha}\right\} \\ \| & \| & \\ \operatorname{tr} [\mathbf{R}_s(E_\alpha), \mathbf{R}_s(X_{-\alpha})] = & 0 & \end{array} \tag{5.15}$$

In the following section we show that, if $\boldsymbol{\alpha}$ is a nonzero root

$$\sum_{i=1}^{l} \alpha^i \alpha_i \neq 0 \qquad \boldsymbol{\alpha} \neq 0 \tag{5.16}$$

Assuming this, it is immediately apparent that:

1. If $\boldsymbol{\alpha} \neq 0$, $\dim V_\alpha = 1$.
2. If $\boldsymbol{\alpha}$ and $c\boldsymbol{\alpha}$ are nonzero roots, $c = \pm 1$.
3. If $\boldsymbol{\alpha} \neq 0$, then

$$E_\alpha, X_{-\alpha} \qquad \text{and} \qquad \alpha^i H_i = [E_\alpha, X_{-\alpha}] \tag{5.17}$$

forms a simple three-dimensional subalgebra of rank 1.

Since every root subspace V_α is one-dimensional, and since every V_α contains at least one simultaneous eigenvector of $\mathbf{H}$, we conclude: *every root subspace V_α is an eigenspace of V_0 to eigenvalue $\boldsymbol{\alpha}$.*

4. SECOND CHAIN CONDITION. We now choose the basis $E_\alpha \in V_\alpha$ in a canonical way. The inner product

$$(E_\alpha, E_{-\alpha}) = 1 \tag{5.6}$$

is not sufficient, since it is linear within each subspace. To fix the normalization, a nonlinear condition is necessary. Such a condition is

$$\mathbf{R}(E_{-\alpha}) = \mathbf{R}^{\dagger}(E_{\alpha}) = \mathbf{R}^{*t}(E_{\alpha}) \tag{5.18}$$

Here $\dagger$ is the adjoint, or complex conjugate (*) transpose (t). This process is well defined because it is carried out on a matrix representation.

The commutation relations between subspaces V_{α}, V_{β} ($\boldsymbol{\alpha} \neq \boldsymbol{\beta} \neq 0$) are given by

$$[E_{\alpha}, E_{\beta}] = N_{\alpha,\beta} E_{\alpha+\beta} \tag{5.19}$$

The structure constant $N_{\alpha,\beta} = 0$ if $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is not a root. The structure constants $N_{-\alpha,-\beta}$ may be computed through the regular representation, which is faithful

$$\begin{aligned}
[\mathbf{R}(E_{\alpha}), \mathbf{R}(E_{\beta})]^{\dagger} &= N^{*}_{\alpha,\beta}\, \mathbf{R}^{\dagger}(E_{\alpha+\beta}) \\
[\mathbf{R}(E_{-\beta}), \mathbf{R}(E_{-\alpha})] &= N^{*}_{\alpha,\beta}\, \mathbf{R}(E_{-\alpha-\beta}) = N_{-\beta,-\alpha}\, \mathbf{R}(E_{-\alpha-\beta}) \\
N_{-\alpha,-\beta} &= -N_{-\beta,-\alpha} = -N^{*}_{\alpha,\beta}
\end{aligned} \tag{5.20}$$

Since the regular representation for a complex Lie algebra may always be chosen to be real, the $N_{\alpha,\beta}$ obey

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta} \tag{5.20$'$}$$

These structure constants have additional symmetries. If $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are two nonzero roots whose sum is also a nonzero root, then we can find a root $\gamma(= -\boldsymbol{\alpha} - \boldsymbol{\beta})$:

$$\boldsymbol{\alpha} + \boldsymbol{\beta} + \gamma = 0 \tag{5.21}$$

Now we apply the Jacobi identity to the corresponding generators:

$$[E_{\alpha}, [E_{\beta}, E_{\gamma}]] + [E_{\beta}, [E_{\gamma}, E_{\alpha}]] + [E_{\gamma}, [E_{\alpha}, E_{\beta}]] = 0 \tag{5.22}$$

$$[E_{\alpha}, E_{-\alpha}]N_{\beta,\gamma} + [E_{\beta}, E_{-\beta}]N_{\gamma,\alpha} + [E_{\gamma}, E_{-\gamma}]N_{\alpha,\beta} = 0 \tag{5.23}$$

$$(\alpha^{i}N_{\beta,\gamma} + \beta^{i}N_{\gamma,\alpha} + \gamma^{i}N_{\alpha,\beta})H_{i} = 0 \tag{5.24}$$

The H_i are linearly independent, and thus each coefficient in (5.24) must vanish. Since only two of the three roots $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$,γ are independent, only two of the three contravariant roots are independent. The $N_{\alpha,\beta}$ obey

$$\begin{array}{ccc}
N_{\alpha,\beta} = & N_{\beta,\gamma} & = N_{\gamma,\alpha} \\
 & \| & \| \\
 & N_{\beta,-\alpha-\beta} & = N_{-\alpha-\beta,\alpha}
\end{array} \tag{5.25}$$

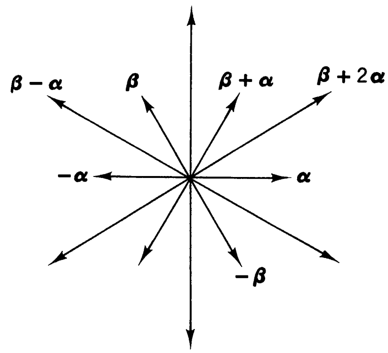


FIG. 7.5 THE α CHAIN CONTAINING β IS USED TO CONSTRUCT THE STRUCTURE CONSTANTS $N_{\alpha,\beta}$. FOR ANY ROOTS $\alpha \neq 0$ AND β THERE IS A UNIQUE α CHAIN CONTAINING β.

It is now time to compute these coefficients. For convenience the computation that follows has been summarized and drastically simplified in Fig. 7.6. Let α, $\beta \neq 0$ be independent roots, and let $S(\beta, \alpha)$ be the α chain containing β:

$$S(\beta, \alpha) = \sum_{-m}^{+n} V_{\beta+k\alpha} \tag{5.26}$$

This subspace belongs to an unbroken string of nonzero roots (Fig. 7.5). Moreover,

$$\beta + n\alpha \text{ is a root} \quad \text{but} \quad \beta + (n+1)\alpha \text{ is not} \tag{5.27u}$$

$$\beta - m\alpha \text{ is a root} \quad \text{but} \quad \beta - (m+1)\alpha \text{ is not} \tag{5.27l}$$

Both m and n are positive integers or zero. This subspace is invariant under the operations of the algebra $E_{\pm\alpha}$, $\alpha^i H_i$. We now apply the Jacobi identity (5.22) to the elements of this space, under the substitutions

$$\alpha \to \alpha$$
$$\gamma \to -\alpha$$
$$\beta \to \beta + k\alpha$$

$$[E_\alpha, E_{\beta+(k-1)\alpha}]N_{\beta+k\alpha,-\alpha} - [E_{\beta+k\alpha}, \alpha^i H_i] + [E_{-\alpha}, E_{\beta+(k+1)\alpha}]N_{\alpha,\beta+k\alpha} = 0 \tag{5.23$'$}$$

Performing the indicated commutation relations once again and rearranging terms gives

$$N_{\alpha,\beta+(k-1)\alpha}N_{\beta+k\alpha,-\alpha} + N_{-\alpha,\beta+(k+1)\alpha}N_{\alpha,\beta+k\alpha} = -\alpha^i(\beta + k\alpha)_i \tag{5.24$'$}$$

The symmetry properties (5.20′) and (5.25) can be exploited to turn this into a recursion relation:

$$N^2_{\alpha, \beta+(k-1)\alpha} = N^2_{\alpha, \beta+k\alpha} + \alpha^i(\beta + k\alpha)_i \tag{5.28}$$

This is straightforward to solve using the standard techniques. The indicial condition is

$$N_{\alpha, \beta+n\alpha} = 0 \tag{5.27u}$$

Working downward from there leads to

$$N^2_{\alpha, \beta+(k-1)\alpha} = (n - k + 1)\{\alpha^i\beta_i + \tfrac{1}{2}(n + k)\alpha^i\alpha_i\} \tag{5.29}$$

This recursion must terminate at the lower end of the chain:

$$N_{-\alpha, \beta-m\alpha} = 0 = -N_{\alpha, -\beta+m\alpha} = N_{\alpha, \beta-(m+1)\alpha} \tag{5.27l}$$

At $k = -m$ the recursion relation (5.29) yields

$$N^2_{\alpha, \beta-(m+1)\alpha} = 0 = (n + m + 1)\{\alpha^i\beta_i + \tfrac{1}{2}(n - m)\alpha^i\alpha_i\} \tag{5.30}$$

This provides a very important relationship between the inner products:

$$-n \leq \frac{2\alpha^i\beta_i}{\alpha^i\alpha_i} = -n + m \leq +m \tag{5.31}$$

This derivation, in particular (5.29) and (5.31), allows a number of very important conclusions to be drawn:

1. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are nonzero roots, then $2\alpha^i\beta_i/\alpha^i\alpha_i$ is an integer and

$$\boldsymbol{\beta}' = \boldsymbol{\beta} - \frac{2\alpha^i\beta_i}{\alpha^i\alpha_i}\boldsymbol{\alpha} \tag{5.32}$$

is a root.

2.

$$N_{\alpha, \beta} = \sqrt{n(m + 1)}\,\sqrt{\tfrac{1}{2}\alpha^i\alpha_i} \tag{5.33}$$

3. There is only one $\boldsymbol{\alpha}$ chain containing $\boldsymbol{\beta}$. If there were another,

$$\boldsymbol{\beta} - m'\boldsymbol{\alpha}, \boldsymbol{\beta} - (m' - 1)\boldsymbol{\alpha}, \ldots, \boldsymbol{\beta} + n'\boldsymbol{\alpha}$$

then m' and n' would obey (5.31). Such a chain could not lie wholly to the left or right of $\boldsymbol{\beta}$. It also cannot straddle $\boldsymbol{\beta}$:

$$\ldots, \boldsymbol{\beta} - \tfrac{1}{2}\boldsymbol{\alpha}, \boldsymbol{\beta} + \tfrac{1}{2}\boldsymbol{\alpha}, \ldots,$$

for then $\frac{1}{2}\boldsymbol{\alpha}$ would be a root, which it is not. Therefore, the $\boldsymbol{\alpha}$ chain containing $\boldsymbol{\beta}$ is unique.

4. $\alpha^i\alpha_i \neq 0$. For if it were 0, then E_{α} and $E_{-\alpha}$ would commute with all nonzero roots E_{β}. The Cartan subalgebra maps $E_{\pm\alpha}$ back onto themselves: $E_{\pm\alpha}$ forms an abelian invariant subalgebra. This is not possible, since the algebra is semisimple and contains no abelian invariant subalgebra.

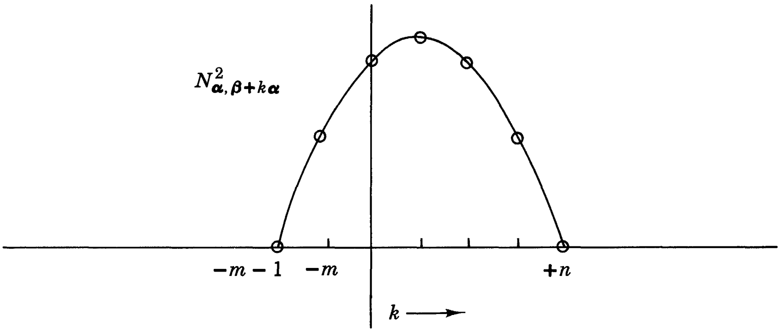


FIG. 7.6 THE COMPUTATION OF $N^2_{\alpha,\,\beta+k\alpha}$ PROCEEDS THROUGH THE RECURSION RELATION (5.24′). THIS HAS THE BOUNDARY VALUES (5.27u, 5.27l) AND A LINEAR DIFFERENCE (5.28). THE STRUCTURE CONSTANTS THUS HAVE A UNIQUELY DETERMINED FORM, PARABOLIC IN k AND VANISHING "AT THE EDGES." THEIR VALUE IS DETERMINED UP TO SCALE $\sqrt{\alpha^i\alpha_i/2}$. CONSIDERATIONS SUCH AS THESE FORM THE "DIMENSIONAL ANALYSIS OF REPRESENTATION THEORY" AND HAVE BEEN USED[10,11] TO CONSTRUCT EXPLICITLY THE REPRESENTATIONS OF MANY GROUPS.

Finally, we show that the Cartan-Killing metric, restricted to the Cartan subalgebra, is positive definite. Let

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i \tag{5.34}$$

be any vector in V_0. Then we compute the inner product of this vector with itself:

$$\begin{array}{ccc} (\alpha^i H_i, \alpha^j H_j) & = & \mathrm{tr}\ \mathbf{R}(\alpha^i H_i)\mathbf{R}(\alpha^j H_j) \\ \| & & \| \\ \alpha^i h_{ij} \alpha^j & = & \alpha^i \alpha^j\ \mathrm{tr}\ \mathbf{R}(H_i)\mathbf{R}(H_j) \end{array} \tag{5.35}$$

The trace on the right is easily computed: $\mathbf{R}(H_i)$ is diagonal; its diagonal elements are β_i, where the $\boldsymbol{\beta}$ are the nonzero roots

$$\alpha^i \alpha_i = \sum_{\boldsymbol{\beta} \neq 0} (\alpha^i \beta_i)(\alpha^j \beta_j) \tag{5.36}$$

From (5.31), $\alpha^i \beta_i$ is related to $\alpha^i \alpha_i$ by a real nonzero coefficient whenever $[E_{\pm\alpha}, E_{\boldsymbol{\beta}}] \neq 0$:

$$\alpha^i \alpha_i = (\alpha^j \alpha_j)^2 \sum_{\boldsymbol{\beta} \neq 0} \left(\frac{1}{2}\right)^2 [n_{\boldsymbol{\beta},\alpha} - m_{\boldsymbol{\beta},\alpha}]^2 \tag{5.37}$$

Since $\alpha^i \alpha_i \neq 0$, its value is

$$\alpha^i \alpha_i = \left\{ \sum_{\boldsymbol{\beta} \neq 0} \left(\frac{1}{2}\right)^2 [n_{\boldsymbol{\beta},\alpha} - m_{\boldsymbol{\beta},\alpha}]^2 \right\}^{-1} > 0 \tag{5.38}$$

Since any inner product of a nonzero vector in V_0 with itself is > 0, the metric is positive definite.

With this result, we have succeeded in transferring all the information contained in both the root concept and the Cartan-Killing metric into information contained in a much smaller space, the Cartan subspace or root space.

The dimensionality for this space is l, the rank of the algebra. This is generally far smaller than the dimensionality of the algebra itself. The independent roots of the secular equation are vectors in this subspace, and there are exactly l such independent vectors. All other vectors are linear combinations of these vectors with integral coefficients. The bases H_i of the Cartan subalgebra commute among themselves. Since the metric in the root space is positive definite, the generators H_i may be chosen to obey

$$(H_i, H_j) = \delta_{ij} \tag{5.39}$$

This endows the root space with a Euclidean metric.

This transferal of all the information contained in the separate root and metric concepts into a Euclidean space represents the final step in the reduction of the commutation relations for a Lie algebra to a canonical form. It represents a beautiful synthesis and an enormous simplification in the structure theory for semisimple algebras. The study of vectors obeying (5.31) leads, in fact, to a complete classification of all the simple complex Lie algebras.

5. THE CANONICAL COMMUTATION RELATIONS. It is useful to choose bases H_i in the Cartan subalgebra which are orthonormal. Recalling (5.39), we realize that the dual tensor h^{ij} has an identical structure, and the covariant and contravariant roots can be identified:

$$\alpha^i = \alpha_i \tag{5.40}$$

Since the metric tensor is given by (5.39), the roots are normalized:

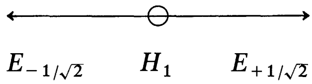
$$\delta_{ij} = \sum_{\boldsymbol{\alpha} \neq 0} \alpha_i \alpha_j \qquad \sum_{\boldsymbol{\alpha} \neq 0} \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = l \tag{5.41}$$

The commutation relations are given by

$$\begin{aligned} [H_i, H_j] &= 0 \\ [\mathbf{H}, E_{\boldsymbol{\alpha}}] &= \boldsymbol{\alpha} E_{\boldsymbol{\alpha}} \\ [E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] &= \boldsymbol{\alpha} \cdot \mathbf{H} \\ [E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] &= N_{\boldsymbol{\alpha}, \boldsymbol{\beta}} E_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \\ N^2_{\boldsymbol{\alpha}, \boldsymbol{\beta}} &= n(m+1)\tfrac{1}{2}(\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}) \end{aligned} \tag{5.42}$$

The structure constants obey also the symmetries (5.20′) and (5.25).

Example 1. There is only one simple complex Lie algebra of rank 1. If $\boldsymbol{\alpha} = \lambda \mathbf{e}_1$ is a nonzero root, $-\lambda \mathbf{e}_1$ is the other. Furthermore, from (5.41) the canonical choice for λ is $1/\sqrt{2}$.



$$[H, E_\pm] = \pm \frac{1}{\sqrt{2}} E_\pm$$

$$[E_+, E_-] = +\frac{1}{\sqrt{2}} H \tag{5.43}$$

Since the H_i are diagonal in the faithful regular representation, they can be taken diagonal in all representations. A 2×2 representation for this algebra is

$$\begin{array}{ccc} H & E_+ & E_- \\ \downarrow & \downarrow & \downarrow \\ \dfrac{1}{2\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \dfrac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \dfrac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array}$$

An arbitrary element in this 2×2 matrix algebra (which is faithful) is

$$c^1 H_1 + c^+ E_+ + c^- E_- = \frac{1}{2} \begin{bmatrix} \dfrac{c^1}{\sqrt{2}} & c^+ \\ c^- & -\dfrac{c^1}{\sqrt{2}} \end{bmatrix} \tag{5.44}$$

The c's are all complex. Real Lie algebras are obtained by reality restrictions on the c's. The restriction $c^1 \to i a_3$, $c^+ \to i a_1 + a_2$, $c^- = -(c^+)^*$ gives the compact real form $\mathfrak{su}(2)$:

$$\mathfrak{su}(2) \to \frac{i}{2} \begin{bmatrix} \dfrac{a_3}{\sqrt{2}} & a_1 - i a_2 \\ a_1 + i a_2 & \dfrac{-a_3}{\sqrt{2}} \end{bmatrix} \qquad a_i \text{ real} \tag{5.45c}$$

The restriction "c all real" gives the noncompact real form $\mathfrak{sl}(2, r) \simeq \mathfrak{su}(1, 1)$:

$$\mathfrak{sl}(2, r) \to \frac{1}{2} \begin{bmatrix} \dfrac{a^1}{\sqrt{2}} & a^+ \\ a^- & \dfrac{-a^1}{\sqrt{2}} \end{bmatrix} \qquad a\text{'s real} \tag{5.45n}$$

Example 2. There are a number of complex semisimple algebras of rank 2. One of these is shown in Fig. 7.7. The commutators can be read immediately just from the root space diagram. It is easily verified that all roots obey (5.41). In fact, all angle cosines are ± 1 or $\pm \frac{1}{2}$.

$$[E_1, E_2] = N_{1,2} E_3$$

$$N_{1,2} = \sqrt{(1)(0+1)}\sqrt{\tfrac{1}{2}\tfrac{1}{3}} = \sqrt{\tfrac{1}{6}}$$

$$N_{1,2} = N_{2,-3} = N_{-3,1} = -N_{-1,-2} = -N_{-2,3} = -N_{3,-1}$$

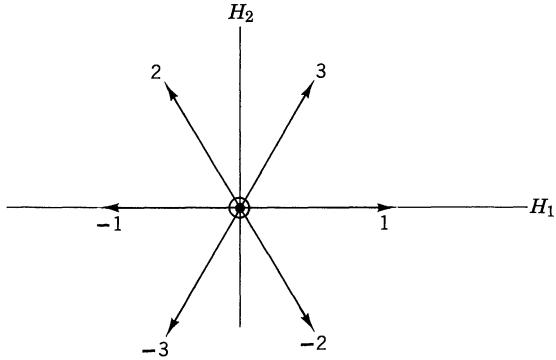


FIG. 7.7 ONE OF THE REAL FORMS FOR THIS ROOT SPACE DIAGRAM IS $SU(3)$. ALL ROOTS HAVE THE SAME LENGTH: $\sqrt{2/6}$. THE COMMUTATION RELATIONS ARE IMMEDIATE:

$$[H_1, H_2] = 0$$

$$[\mathbf{H}, E_{\pm 1}] = \pm \frac{1}{\sqrt{3}}(1, 0)E_{\pm 1}$$

$$[\mathbf{H}, E_{\pm 2}] = \pm \frac{1}{\sqrt{3}}\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)E_{\pm 2}$$

$$[\mathbf{H}, E_{\pm 3}] = \pm \frac{1}{\sqrt{3}}\left(+\frac{1}{2}, \frac{\sqrt{3}}{2}\right)E_{\pm 3}$$

$$[E_1, E_{-1}] = \frac{1}{\sqrt{3}} H_1$$

$$[E_2, E_{-2}] = \frac{1}{\sqrt{3}}\left(-\frac{1}{2}H_1 + \frac{\sqrt{3}}{2}H_2\right)$$

$$[E_3, E_{-3}] = \frac{1}{\sqrt{3}}\left(+\frac{1}{2}H_1 + \frac{\sqrt{3}}{2}H_2\right)$$

$$[E_1, E_2] = \frac{1}{\sqrt{6}}E_3, \; [E_2, E_{-3}] = \frac{1}{\sqrt{6}}E_{-1}, \; [E_{-3}, E_1] = \frac{1}{\sqrt{6}}E_{-2}$$

$$[E_{-1}, E_{-2}] = -\frac{1}{\sqrt{6}}E_{-3}, \; [E_{-2}, E_3] = -\frac{1}{\sqrt{6}}E_1, \; [E_3, E_{-1}] = -\frac{1}{\sqrt{6}}E_2$$

This algebra has two additional real forms besides $SU(3)$: $SU(2, 1)$ and $Sl(3, r)$. Their structures may be computed following the procedure outlined in Example 1.

Résumé

A number of powerful tools were used to construct a canonical form for an arbitrary Lie algebra. The results have not been completely satisfactory: only semicanonical forms exist for solvable and nonsemisimple algebras. The canonical form for a semisimple algebra was constructed in Section V.

In Section I the regular representation was exploited for a study of the structure constants and their properties under various circumstances. This representation is useful because it is easy to construct and contains exactly as much information as the structure constants.

In Section II, the secular equation was introduced, as well as the concepts of rank and root subspace decomposition. These were introduced primarily as an aid to the study of the regular representation, but they are valid concepts in all representations. The root spaces were seen to have a number of useful properties. In addition, the first criterion of solvability, which depends on a study of the secular equation, was stated and proved.

In Section III the Cartan-Killing metric was introduced, again primarily as an aid for the study of the regular representation. Like rank, it is a valid concept in other representations as well. The second criterion of solvability, which depends on a study of the metric, was stated and proved.

The Cartan criterion, a synthesis of the criteria in the two previous sections, was developed in Section IV and quickly used to prove the full reducibility of semisimple algebras and the faithfulness of the regular representation for semisimple algebras.

In the fifth and last section, all these tools were used to construct a canonical form for the commutation relations of a semisimple algebra. Moreover, all the information contained in the regular representation, the root subspace decomposition, and the Cartan-Killing form is transferred into a much smaller space, the root space, whose dimension is the rank of the algebra. Moreover, this space is endowed with a positive definite metric (the Cartan-Killing metric is never positive definite) and is therefore Euclidean. In addition, the nonzero vectors (roots) in this space obey a very simple relationship (5.31), which is exploited in the following chapter to classify all possible simple Lie algebras over the complex field (complex extension algebras).

Exercises

1. Prove that, in general, a Lie algebra is nonassociative but that its regular representation is always associative.

2. Using the notation of (1.17), prove that if $X_i, X_j, \ldots,$ span a subalgebra, then $C_{ij}{}^{\gamma} = 0$.

3. Prove that the difference linear vector spaces R, C, Q below are not closed under commutation. Prove also that under the EXPonential mapping, each maps onto the sphere whose dimensionality is indicated:

$$R\colon \mathfrak{so}(n) \bmod \mathfrak{so}(n-1) \xrightarrow{\text{EXP}} S^{n-1} \subset R_n$$

$$C\colon \mathfrak{u}(n) \bmod \mathfrak{u}(n-1) \xrightarrow{\text{EXP}} S^{2n-1} \subset R_{2n}$$

$$Q\colon \mathfrak{usp}(2n) \bmod \mathfrak{usp}(2n-2) \xrightarrow{\text{EXP}} S^{4n-1} \subset R_{4n}$$

Show that these results can be summarized by

$$\mathfrak{u}(n;f) \bmod \mathfrak{u}(n-1;f) \xrightarrow{\text{EXP}} S^{nf-1} \subset R_{fn}$$

where $f = (R, C, Q) = (1, 2, 4)$.

4. Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{g}^{(n)} = 0$, and $\mathfrak{g}^{(n-1)} \neq 0$. Prove that $\mathfrak{g}^{(n-1)}$ is abelian.

5. Prove that if the regular representation of a Lie algebra has zero on and below the major diagonal, then the algebra is nilpotent [use (1.12)].

6. The orthogonal groups $SO(n, r)$ may be obtained from the unitary groups $U(n, c)$ by suitable reality restrictions. Similarly, $USp(2n)$ is obtained from $SU(2n, c)$ by suitable reality restrictions.

(a) What are these restrictions in terms of the parameters a^{ij} [cf. (2.8)] in the Lie algebra?

(b) Write down the coefficients $\phi_r(a^{ij})$ explicitly for the algebras $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$, $\mathfrak{usp}(2n)$.

(c) Show that all odd coefficients

$$\phi_r,\ r = 1, 3, 5, \ldots, \begin{Bmatrix} 2n-1 \\ 2n+1 \\ 2n-1 \end{Bmatrix}$$

vanish identically. Express the even coefficients ϕ_r, $r = 2, 4, 6, \ldots, 2n$ explicitly using two Levi-Civita skew tensors $\varepsilon_{i_1 i_2 \cdots i_k}$ ($k = 2n$, $2n+1$, $2n$, resp.) as in (2.9).

(d) Prove that these even coefficients are all functionally independent.

(e) Conclude that the algebras $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$, $\mathfrak{usp}(2n)$ each has rank n.

(f) Show that $\phi_{2n}(a^{ij})$ for $\mathfrak{so}(2n)$ is a perfect square.

7. Replace the scalar parameters a^{ij} in the 3×3 matrix representation for $\mathfrak{so}(3)$ by the operators X_{ij}:

$$\begin{bmatrix} & a^{12} & a^{13} \\ -a^{12} & & a^{23} \\ -a^{13} & -a^{23} & \end{bmatrix} \to M = \begin{bmatrix} & X_{12} & X_{13} \\ -X_{12} & & X_{23} \\ -X_{13} & -X_{23} & \end{bmatrix}$$

(a) Show that

$$\operatorname{tr}(M)^2 = -2\{X_{12}{}^2 + X_{13}{}^2 + X_{23}{}^2\}$$

(b) Show that

$$[\mathbf{J}, \operatorname{tr}(M)^2] = [\mathbf{J}, -2\mathbf{J} \cdot \mathbf{J}] = 0$$

Use $J_1 = X_{23}$, $J_2 = X_{31}$, $J_3 = X_{12}$, and $[J_i, J_j] = -\varepsilon_{ijk} J_k$.

(c) Show that

$$\operatorname{tr}(M)^{2k+1} = 0$$
$$\operatorname{tr}(M)^{2k} = (-2)^k (J^2)^k$$

8. Since the result of Problem 7c depends only on the commutation relations in the algebra spanned by the generators **J**, prove that

(a) $[\mathbf{L}, \mathbf{L} \cdot \mathbf{L}] = 0 \qquad L_i = \varepsilon_{ijk} x^j \partial_k$

(b) $[\mathbf{S}, \mathbf{S} \cdot \mathbf{S}] = 0 \qquad S_i = \frac{i}{2} \sigma_i$

(c) $[\mathbf{J}, \mathbf{J} \cdot \mathbf{J}] = 0 \qquad [J_i, J_j] = -\varepsilon_{ijk} J_k$

9. Compute $\mathbf{J} \cdot \mathbf{J}$ within the $(2j+1) \times (2j+1)$ matrix representation of $\mathfrak{so}(3)$ and show that

$$\sum_{i=1}^{3} J_i{}^2 = -j(j+1) I_{2j+1}$$

10. Let

$$d\omega = \sum_{i=1}^{2} f_i(x)\, d\xi^i$$

be a differential form, where $f_i(x)$ is defined on R_3 and $\xi^i \in C_2$. The ξ^i form a basis on which $SU(2)$ acts, and the $x^k (k = 1, 2, 3)$ form a basis in R_3 on which the homomorphic image $\mathscr{D}^1[SU(2)]$ acts. Use the results of Problem 3, Chapter 6, to compute the generators of infinitesimal displacements on this form.

(a) Show that the three generators are given by

$$J_i = L_i + S_i$$

where

$$L_i = x^j \partial_k - x^k \partial_j \qquad (ijk = 1, 2, 3 \text{ cycl})$$

$$S_i = \frac{i}{2} \sigma_i \qquad (\sigma_i = \text{Pauli spin matrix})$$

(b) Show that

$$J^2 = (L + S)^2 = \mathbf{L} \cdot \mathbf{L} + \mathbf{S} \cdot \mathbf{S} + 2\mathbf{L} \cdot \mathbf{S}$$

(c) Interpret the cross term physically as the spin-orbit coupling term familiar from atomic physics.

11. Replace the Lie algebra parameters a^{ij} by the corresponding infinitesimal generators X_{ij} in (2.8) and (2.9) and call the resulting "matrix" M. Compute

$$\operatorname{tr} (M)^k$$

for the algebras $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$, and $\mathfrak{usp}(2n)$. Show that the trace vanishes for odd k, whereas for even k the first n nonvanishing trace operators are functionally independent.

12. Using (2.2) and (2.11d) of Chapter 8, show that the operators constructed in Problem 6 contain terms consisting only of elements in the Cartan subalgebra which have the following form:

$$\begin{aligned} &\mathfrak{u}(n): && \mathscr{C}_j \to \sum_{i_r \neq i_s} \prod_{r=1}^{j} H_{i_r} \\ &\mathfrak{so}(2n) && \\ &\mathfrak{so}(2n+1): && \mathscr{C}_{2j} \to \sum_{i_r \neq i_s} \prod_{r=1}^{j} H_{i_r}^{\,2} \\ &\mathfrak{usp}(2n) && \end{aligned}$$

13. General multilinear metrics can be defined on the vector space of a Lie algebra as follows:

$$(A_1, A_2, A_3, \ldots, A_r)_{\text{reg}} = \operatorname{tr} \mathbf{R}(A_1)\mathbf{R}(A_2)\cdots\mathbf{R}(A_r)$$

Show that for an r-linear metric on a simple Lie algebra

$$\frac{(A_1, A_2, \ldots, A_r)_\Gamma}{f_r(\boldsymbol{\Gamma})} = (A_1, A_2, \ldots, A_r) = \frac{(A_1, A_2, \ldots, A_r)_{\text{reg}}}{f_r(\text{reg})}$$

where $f_r(\boldsymbol{\Gamma})$ is called the r-index of the representation $\boldsymbol{\Gamma}$.

Prove

$$\frac{f_r(\boldsymbol{\Gamma})}{f_r(\text{def})} = \frac{\operatorname{tr} \{\boldsymbol{\Gamma}(A)\}^r}{\operatorname{tr} \{\text{def } (A)\}^r} = \frac{\dim (\boldsymbol{\Gamma})\mathscr{C}_r(\boldsymbol{\Gamma})}{\dim (\text{def})\mathscr{C}_r(\text{def})}$$

where $\mathscr{C}_r$ is the eigenvalue of the "rth Casimir operator" of the Lie algebra (computed in Problems 6, 7, 11, 12).

Show that for $\mathfrak{su}(2)$

$$f_2(j) = \tfrac{1}{6}\{(2j)(2j+1)(2j+2)\} f_2(j = \tfrac{1}{2})$$

14. Compute the Cartan-Killing metric tensor for the Lie algebras $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, $\mathfrak{usp}(2n)$ and for their related noncompact forms $\mathfrak{so}(p, q)$, $\mathfrak{su}(p, q)$, $\mathfrak{usp}(2p, 2q)$.

Notes and References

In this chapter we follow closely the presentation given in References 1 and 2,

1. N. Jacobson. [2]
2. D. Kleima, W. J. Holman, L. C. Biedenharn. [1]
3. I. M. Gel'fand. [1]
4. E. Cartan. [1]
5. W. Killing. [1–4]
6. E. B. Dynkin. [2]
7. E. P. Wigner. [1]
8. C. Eckart. [1]
9. G. Racah. [1]
10. I. M. Gel'fand, M. L. Tsetlein. [1,2]
11. R. Gilmore. [4]

CHAPTER 8

Root Spaces and Dynkin Diagrams

The opening section reviews the canonical commutation relations for a (complex) semisimple Lie algebra. The results are presented in a self-contained way for those who would rather bypass the structure theory developed in the previous chapter. Roots and their properties are also explained in the opening section.[1,2]

The remainder of Section I is devoted to a complete classification of all possible root spaces. The geometric properties of the roots are thoroughly exploited to this end. The procedure used is admirably designed to describe all the nonzero roots in each root space that is constructed. The same procedure could be used to show the completeness of the list by exhaustive analysis.[3]

The completeness is shown in Section III by other means. Specifically, systems of simple roots are defined and their properties enumerated. These properties can then be exploited to describe root spaces by a simple diagrammatic technique (Dynkin[4] diagrams). Dynkin diagrams are subject to so many restrictions that a complete and convincing list of all possible diagrams (and with it, of simple root spaces, Lie algebras, and Lie groups) is swiftly achieved.

In Section II the association is nailed down between the four unending chains of root spaces A_{n-1}; D_n, B_n, C_n and four series of classical groups $SU(n)$; $SO(2n)$, $SO(2n+1)$, $USp(2n)$. In the process, systems of simple roots are introduced, described, and used.

I. Classification of the Simple Root Spaces

1. REVIEW OF THE CANONICAL COMMUTATION RELATIONS. In the previous chapter we studied the structure of Lie algebras, breaking down an arbitrary Lie algebra into its constituent parts, namely, the solvable algebras and the semisimple algebras.

The solvable algebras are constructed in a stepwise fashion out of abelian algebras. The semisimple algebras are direct sums of simple algebras. The simple algebras regenerate themselves under commutation. There is not yet a canonical form for the commutation relations of solvable and nonsemisimple algebras. Nor is there a complete classification scheme for these algebras. Such a canonical form does exist for the semisimple algebras. It was developed in the previous chapter and is exploited here to classify all possible (complex extension) semisimple algebras.[1]

These commutation relations are most easily stated in terms of a Euclidean space R_l, on which exists a canonical positive definite metric. The nonzero vectors in this space are called roots. If $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are two root vectors in this space, they obey the very restrictive conditions[1,2]

$$\frac{2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} = \text{integer} \tag{1.1a}$$

$$\boldsymbol{\beta}' = \boldsymbol{\beta} - \frac{2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}\boldsymbol{\alpha} \qquad \text{a root also} \tag{1.1b}$$

The root $\boldsymbol{\beta}'$ is obtained by reflecting the root $\boldsymbol{\beta}$ in the hyperplane orthogonal to the root $\boldsymbol{\alpha}$:

$$\begin{aligned}
\boldsymbol{\beta} &= \boldsymbol{\beta}_{\parallel} + \boldsymbol{\beta}_{\perp} \\
&= \left(\boldsymbol{\beta} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\alpha}\boldsymbol{\alpha}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}\right) + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\alpha}\boldsymbol{\alpha}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} \\
|\beta'\rangle &= \left(|\beta\rangle - \frac{|\alpha\rangle\langle\alpha|}{\langle\alpha|\alpha\rangle}|\beta\rangle\right) - \frac{|\alpha\rangle\langle\alpha|}{\langle\alpha|\alpha\rangle}|\beta\rangle
\end{aligned} \tag{1.2}$$

The hyperplanes orthogonal to the roots are called **Weyl hyperplanes**; the operations of reflecting in these planes are called **Weyl reflections**. The totality of these reflections, and all their distinct products, constitutes the **Weyl group** for a root space diagram and its associated simple algebra.

There are two kinds of generators for any semisimple algebra. There is a maximal commuting subalgebra, of dimension l, called the Cartan subalgebra. The bases of this subalgebra are $H_1, H_2, \ldots, H_l$. The remaining generators are eigenoperators of the $\mathbf{H}$ to eigenvalue $\boldsymbol{\alpha}$. For each nonzero root $\boldsymbol{\alpha}$ there is exactly one corresponding eigengenerator $E_{\boldsymbol{\alpha}}$, and vice versa. If $\boldsymbol{\alpha}$ is a root, then $c\boldsymbol{\alpha}$ is a root only when $c = \pm 1, 0$. The independent roots $\boldsymbol{\alpha}$ span the root space R_l. There is a $1 \leftrightarrow 1$ correspondence between semisimple algebras of rank l and sets of roots in an l-dimensional space R_l. This correspondence is effected by the canonical commutation relations for the algebra, which are written in terms of roots $\boldsymbol{\alpha}$:

$$\begin{aligned}
[H_i, H_j] &= 0 \qquad 1 \le i, j \le l \\
[\mathbf{H}, E_{\boldsymbol{\alpha}}] &= \boldsymbol{\alpha} E_{\boldsymbol{\alpha}} \\
[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] &= \boldsymbol{\alpha}\cdot\mathbf{H} \\
[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] &= N_{\boldsymbol{\alpha},\boldsymbol{\beta}} E_{\boldsymbol{\alpha}+\boldsymbol{\beta}}
\end{aligned} \tag{1.3}$$

In addition to these basic commutation relations, there are some subsidiary (or constitutive) equations. The lengths of the nonzero roots are normalized by (1.4a). The structure constants $N_{\alpha,\beta}$ have the symmetries of (1.4b) and (1.4c). Their value is given explicitly by (1.4d).

$$\sum_{\alpha \neq 0} \alpha_i \alpha_j = \delta_{ij} \tag{1.4a}$$

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta} \tag{1.4b}$$

$$N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha} \tag{1.4c}$$

$$N^2_{\alpha,\beta} = n(1+m)\tfrac{1}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} \tag{1.4d}$$

$$\begin{array}{llll} \boldsymbol{\beta} + n\boldsymbol{\alpha} & \text{is a root,} & \boldsymbol{\beta} + (n+1)\boldsymbol{\alpha} & \text{is not} \\ \boldsymbol{\beta} - m\boldsymbol{\alpha} & \text{is a root,} & \boldsymbol{\beta} - (m+1)\boldsymbol{\alpha} & \text{is not} \end{array}$$

The rank l is:

1. The number of functionally independent coefficients in the secular equation of any faithful representation of the semisimple algebra.
2. The number of independent roots $\boldsymbol{\alpha}$ (of the secular equation).
3. The dimensionality of the root space R_l.
4. The dimension of the maximal commuting subalgebra, and the maximal number of mutually commuting generators $H_1, H_2, \ldots, H_l$.

The rank is generally a much smaller number than the dimension of the algebra ($l \simeq \mathcal{O}[\dim]^{1/2}$) and therefore furnishes a very convenient mechanism for keeping track of all the different generators and their commutation relations.

A semisimple algebra is reducible to a direct sum of simple algebras. Every element in one simple subalgebra commutes with every element in any other simple subalgebra. This property is transferred to their respective root space diagrams. Simple invariant subalgebras of a semisimple algebra have mutually orthogonal root space diagrams. The proof is immediate from (1.3).

2. THE RANK-2 ROOT SPACES.

Equations (1.1) provide a very restrictive set of conditions which every pair of roots in a root space diagram must obey. Since the root space is endowed with a standard Euclidean positive definite metric, the inner product receives an immediate and simple interpretation in terms of cosines. Specifically

$$0 \leq \cos^2(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\beta}\cdot\boldsymbol{\beta}} = \frac{n}{2}\frac{n'}{2} \leq 1 \tag{1.5}$$

where n and n' are integers.

Equation (1.5) puts a severe restraint on the range of the integers n, n' and the relative lengths of the roots $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$. All possibilities are summarized in Table 8.1. In this table, all nonzero roots in any of the root space diagrams can include only the few angles listed. The relative lengths of two roots is determined by the angle that they include.

TABLE 8.1

$\cos^2(\boldsymbol{\alpha}, \boldsymbol{\beta})$	$\theta(\boldsymbol{\alpha}, \boldsymbol{\beta})$	$\dfrac{2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} = n$	$\dfrac{2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\beta}\cdot\boldsymbol{\beta}} = n'$	$\dfrac{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}{\boldsymbol{\beta}\cdot\boldsymbol{\beta}} = \dfrac{n'}{n}$
1	0°, 180°	± 2	± 2	1
$\frac{3}{4}$	30°, 150°	± 3	± 1	3^{-1}
		± 1	± 3	3^{+1}
$\frac{2}{4}$	45°, 135°	± 2	± 1	2^{-1}
		± 1	± 2	2^{+1}
$\frac{1}{4}$	60°, 120°	± 1	± 1	1
0	90°	0	0	Undetermined

From (1.1) and the possibilities listed in Table 8.1 it is possible to list all root spaces of rank 2. These are constructed below.

1. $A_1 = B_1 = C_1$. There is only one root space of rank 1. If $\boldsymbol{\alpha}$ is the positive root, the only other nonzero root is $-\boldsymbol{\alpha}$, which is obtained by reflecting $\boldsymbol{\alpha}$ through the plane orthogonal to itself. The root space diagram is shown in Fig. 8.1. The length of $\boldsymbol{\alpha}$ is determined from (1.4a) to be $1/\sqrt{2}$.

2. G_2. If a root is added to the root space diagram of Fig. 8.1 making an angle of 30° with the root present, the new root must have a length $3^{\pm 1/2}$ of the original root. The root space is completed by reflecting in the Weyl planes. This process is demonstrated in Fig. 8.2. The roots in the completed root space diagram are labeled according to two conventions. One convention is obvious: $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal vectors along "the H_1 and H_2 axes" in R_2. Their normalization is given by (1.4a). The other convention is

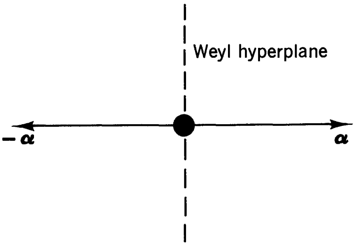


FIG. 8.1 THE THREE GENERATORS FOR THE COMPLEX ALGEBRAS $A_1 = B_1 = C_1$ ARE H_1 AND $E_{\pm\alpha}$.

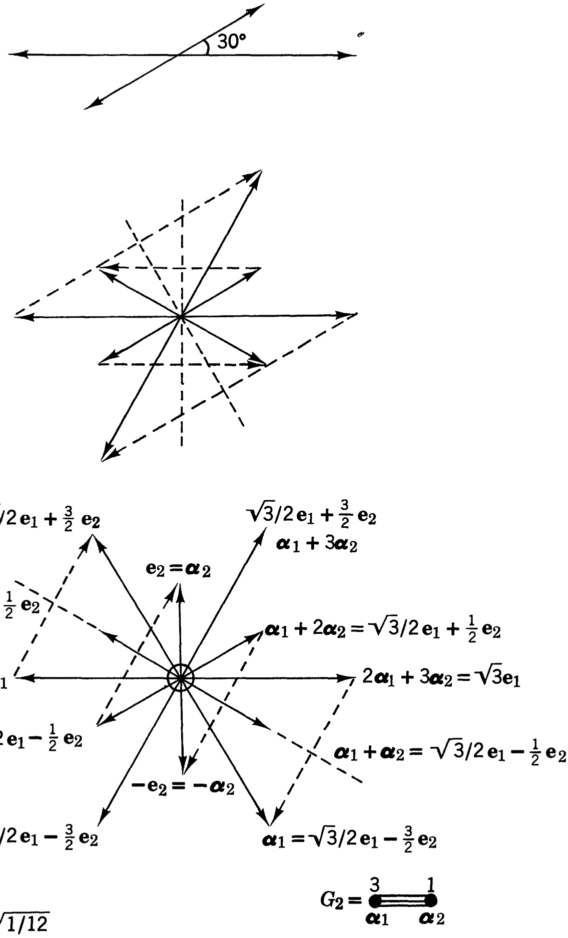


FIG. 8.2 THE ROOT SPACE DIAGRAM G_2 IS OBTAINED BY ADDING A VECTOR IN THE PLANE MAKING AN ANGLE OF 30° WITH THE ROOT ALREADY PRESENT FROM THE ROOT SPACE A_1. THE LENGTH OF THE NEW VECTOR IS $1/\sqrt{3}$ OF THE LENGTH OF THE ORIGINAL ROOT. TWO INDEPENDENT NUMBERING SYSTEMS HAVE BEEN USED FOR THE NONZERO ROOTS. FROM (1.4a), THE SUM OF THE SQUARE OF THE LENGTHS OF ALL ROOTS IS 2. THE LENGTHS OF THE SHORTEST ROOTS ARE $\sqrt{1/12}$.

discussed in Section III. The root space obtained by including a root of length $3^{+1/2}$ instead of $3^{-1/2}$ is related to the root space of Fig. 8.2 by rotation through 30°. The double circle at the center indicates the existence of two generators which are eigengenerators of H_1 and H_2 to eigenvalue 0 (namely, H_1 and H_2 themselves).

3. B_2. When the additional vector makes an angle of 45° with the original root, its relative length can be either $2^{\pm 1/2}$. If a root of length $\sqrt{2}$ is added, and the resulting diagram completed by the reflection process used in the previous example, the root space B_2 results. The nonzero roots are of the form

$$\pm\mathbf{e}_1 \pm \mathbf{e}_2, \qquad \pm\mathbf{e}_1, \qquad \pm\mathbf{e}_2 \tag{1.6B}$$

4. C_2. When the added vector has length $1/\sqrt{2}$, the nonzero roots are

$$\pm\mathbf{e}_1 \pm \mathbf{e}_2, \qquad \pm 2\mathbf{e}_1, \qquad \pm 2\mathbf{e}_2 \tag{1.6C}$$

The root spaces B_2 and C_2 are illustrated in Fig. 8.3.

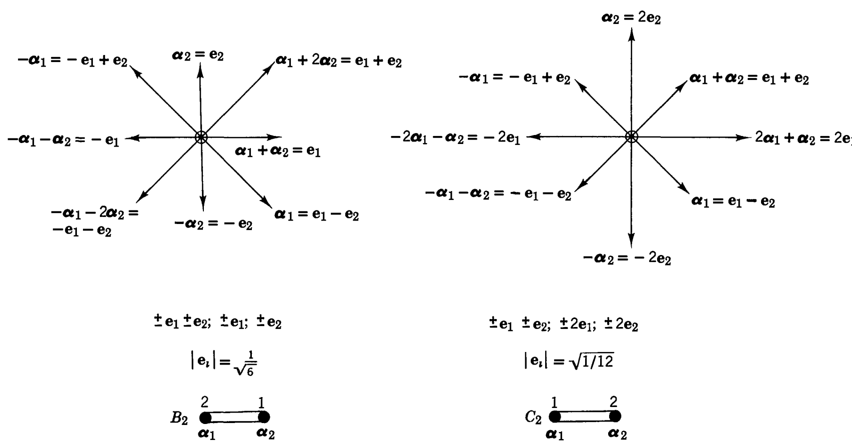


FIG. 8.3 THE ROOT SPACES FOR B_2 AND C_2 ARE RELATED TO EACH OTHER BY ROTATION THROUGH 45°.

5. A_2. A root added to Fig. 8.1 at an angle of 60° must have the same length as the roots already present. Upon completion, the root space A_2 is obtained (Fig. 8.4).

6. D_2. A root added to A_1 orthogonal to the roots already present has indeterminant length. The root space so formed consists of two mutually orthogonal root subspaces. The root space D_2 is semisimple and decomposes into two root spaces of type A_1. The root lengths in each subspace

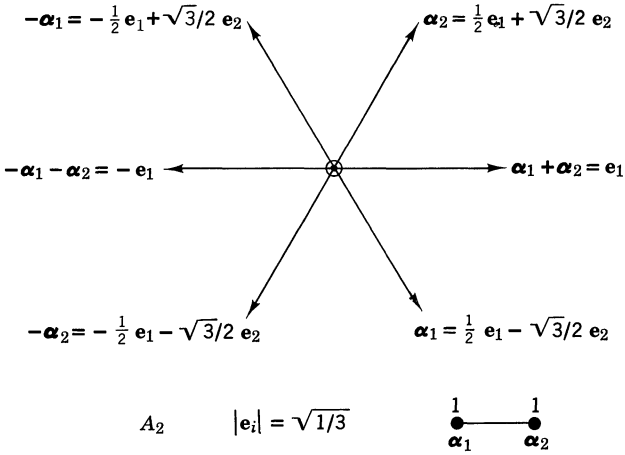


FIG. 8.4 THE ROOT SPACE DIAGRAM FOR A_2 HAS THE FAMILIAR HEXAGONAL SHAPE

may be determined separately. The semisimple root space D_2 is illustrated in Fig. 8.5.

7. A_2 and G_2 Revisited. The root space diagrams A_2 and G_2 have a lot more symmetry than is indicated by the algebraic expressions for their nonzero roots. To make this symmetry manifest in the algebraic root structure, it is convenient to consider A_2 and G_2 as two-dimensional subspaces of the three-dimensional space R_3. These subspaces are orthogonal to the vector

$$\mathbf{R} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \tag{1.7}$$

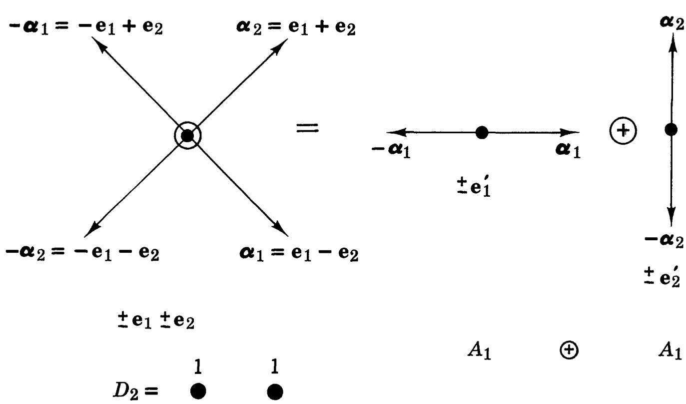


FIG. 8.5 THE ROOT SPACE D_2 IS SEMISIMPLE AND DECOMPOSES INTO THE DIRECT SUM OF TWO SIMPLE ROOT SPACES.

The nonzero roots for A_2 and G_2 then have a clear-cut symmetry:

$$A_2: \qquad \mathbf{e}_i - \mathbf{e}_j \qquad 1 \le i \ne j \le 3 \tag{1.6A}$$

$$\begin{aligned} G_2: \qquad & \mathbf{e}_i - \mathbf{e}_j \\ & \pm(\mathbf{e}_i + \mathbf{e}_j - 2\mathbf{e}_k) \qquad 1 \le i \ne j \ne k \le 3 \end{aligned} \tag{1.6G}$$

The corresponding hypersurfaces and nonzero roots are shown in Fig. 8.6. The generators of the algebras corresponding to the directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are H_1, H_2, H_3. The constraint (1.7) on the nonzero roots imposes a corresponding constraint on the H_i:

$$H_1 + H_2 + H_3 = 0 \tag{1.7'}$$

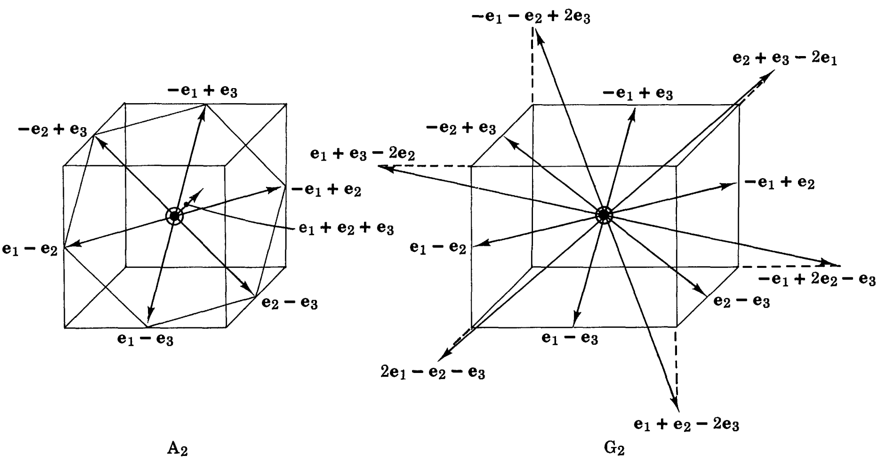


FIG. 8.6 THE ROOT SPACES A_2 AND G_2 CAN CONVENIENTLY BE CONSIDERED TO BE IN THE HYPERPLANE ORTHOGONAL TO THE VECTOR $\mathbf{R} + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ IN R_3. UNDER THESE CONDITIONS, THE ROOTS ARE ALGEBRAICALLY AS WELL AS GEOMETRICALLY SYMMETRIC.

3. CONSTRUCTION OF SIMPLE ROOT SPACES. The constructive procedure used in the last section can be used to construct all root spaces of rank 3 and higher. This is a rather laborious process.

Before we outline the workings of this process, it is useful to look for systems of roots that obey (1.1). The rank-2 root spaces suggest four possibilities:

$$\begin{aligned} A_l: \qquad & \mathbf{e}_i - \mathbf{e}_j && 1 \le i \ne j \le l+1 \\ D_l: \qquad & \pm\mathbf{e}_i \pm \mathbf{e}_j;\ \pm 0\mathbf{e}_i && \\ B_l: \qquad & \pm\mathbf{e}_i \pm \mathbf{e}_j;\ \pm 1\mathbf{e}_i && 1 \le i \ne j \le l \\ C_l: \qquad & \pm\mathbf{e}_i \pm \mathbf{e}_j;\ \pm 2\mathbf{e}_i && \mathbf{e}_i \cdot \mathbf{e}_j \simeq \delta_{ij} \end{aligned} \tag{1.8}$$

It is easily verified that these root systems satisfy (1.1) and are complete upon reflection in all Weyl hyperplanes. In each case, the roots belong to a rank-l root space. The extreme similarity between the root spaces D_l, B_l, C_l suggests the algebras are extremely closely related, and that it will be profitable, in the future, to treat them as siblings.

The existence of a root space G_2 suggests also that we look for root spaces, some of whose roots have the form

$$a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 + \cdots + a^l\mathbf{e}_l; \qquad a^i \neq 0, i = 1, 2, \ldots, l \tag{1.9}$$

Equations (1.1) place such a severe constraint on the components a^i of (1.9) that, in addition to the four unending chains of root spaces (1.8), only two additional chains can be found. The E chain contains three members, the F chain only one.

To construct all possible root spaces, a general procedure can be followed.[3] To the root space Σ_n we add an additional vector with the properties:

1. It coincides with none of the vectors already in Σ_n.
2. It obeys (1.1).

Then we complete the space by reflection. If all the vectors resulting from the completion still obey (1.1), the completion is a root space; otherwise, it is not.

This process is recursive, or inductive. All rank $n + 1$ root spaces can be constructed from the rank n root spaces. All the rank-2 root spaces are known and have been constructed from the rank-1 space. In this way all possible root spaces can be both enumerated and constructed.

Comment. The construction just described is called the process of exhaustion. This means that we proceed from each possible Σ_n to all possible Σ_{n+1} until we are exhausted.

We will not actually carry this process out in all possible detail. We will only carry it out far enough to construct the additional exceptional E and F series. This technique has the virtue of being explicitly constructive and the vice of not being thoroughly satisfying in its completeness of all possible root space diagrams. Since a very satisfying completeness demonstration exists using other techniques,[4] we employ this procedure just for its constructive value.

Example. The root space D_4 has nonzero roots of the form

$$\pm\mathbf{e}_i \pm \mathbf{e}_j \qquad 1 \leq i \neq j \leq 4 \tag{1.10}$$

There are $(2)^2\,{}_4C_2$ nonzero roots. We try to extend D_4 by adding an additional root of the form

$$r^1\mathbf{e}_1 + r^2\mathbf{e}_2 + r^3\mathbf{e}_3 + r^4\mathbf{e}_4 + r^5\mathbf{e}_5 \qquad r^2 = \sum (r^i)^2 \tag{1.11}$$

The space is completed by reflection. All resulting roots must obey (1.1).

Solutions A. First we assume the additional root lies completely within the four-dimensional space R_4 containing the roots of D_4 (i.e., $r^5 = 0$). Then the inner product of a root (1.10) with a root of the form (1.11) is

$$\frac{\pm r^i \pm r^j}{r\sqrt{2}} = 0, \qquad \pm\frac{1}{2}, \qquad \pm\frac{1}{\sqrt{2}}, \qquad \pm\frac{\sqrt{3}}{2} \tag{1.12}$$

The Diophantine equation (1.12) has only four solutions:

$$\begin{array}{lllll} B_4: & r^i = \pm 1, & r^j = 0; & r = 1 & (j \neq i) \\ C_4: & r^i = \pm 2, & r^j = 0; & r = 2 & (j \neq i) \\ F_4: & r^i = \pm 1, & & r = 2 & (i = 1, 2, 3, 4) \\ F_4: & r^i = \pm\frac{1}{2}, & & r = 1 & (i = 1, 2, 3, 4) \end{array}$$

The two solutions labeled F_4 can be rotated into each other in exactly the same way that B_2 and C_2 can be rotated into each other. It is easily verified that the $(2)^4$ roots

$$\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \tag{1.13}$$

obey (1.1). Completion demands also the roots $\pm 2\mathbf{e}_i$. The algebra corresponding to F_4 has the following generators:

$$\begin{array}{llrl} E_{\boldsymbol{\alpha}}: & \boldsymbol{\alpha} = \pm\mathbf{e}_i \pm \mathbf{e}_j & (2)^2\binom{4}{2} & = 24 \\ E_{\boldsymbol{\alpha}}: & \boldsymbol{\alpha} = \pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 & (2)^4 & = 16 \\ E_{\boldsymbol{\alpha}}: & \boldsymbol{\alpha} = \pm 2\mathbf{e}_i & 4\cdot 2 & = \ \ 8 \\ H_i: & & & = \ \ \underline{4} \\ & & & \ \ \ \ 52 \end{array} \tag{1.14}$$

Solutions B. If the root (1.11) does not lie completely within R_4, then $r^5 \neq 0$. The components r^i obey the same Diophantine equation (1.12) whose solutions are now

$$D_5: \quad r^i = \pm 1, \qquad r^j = 0\ (j \neq i), \qquad r^5 = \pm 1 \tag{1.15D}$$

$$E_5: \quad r^i = \pm\tfrac{1}{2}, \qquad i = 1, 2, 3, 4 \qquad r^5 = \pm\sqrt{2 - 4(\tfrac{1}{2})^2} \tag{1.15E}$$

In the root space E_5, the inner product of the additional vectors among themselves is

$$\frac{\pm\frac{1}{4}\pm\frac{1}{4}\pm\frac{1}{4}\pm\frac{1}{4}\pm 1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}\left(2, \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, -2\right) \tag{1.16}$$

Such inner products do not satisfy (1.1). If we demand that there be an even number of $+$ signs among the factors $\frac{1}{2}\mathbf{e}_i$,

$$\tfrac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4) \pm \sqrt{2 - 4(\tfrac{1}{2})^2}\,\mathbf{e}_5 \tag{1.16$'$}$$

then the alternate terms in (1.16) disappear, and the additional roots satisfy (1.1). There are no additional roots besides (1.10) and (1.16′). The generators of E_5 are

$$\begin{array}{llcl}
E_\alpha: & \alpha = (1.16') & 2\left[\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right] & = 16 \\
E_\alpha: & \alpha = (1.10) & (2)^2\,{}_4C_2 & = 24 \\
H_i: & & & = \underline{5} \\
 & & & \ 45
\end{array}$$

The algebra corresponding to D_5 has 45 generators also. In fact, the two root spaces are *isometric*:

$$D_5 \simeq E_5$$

4. THE E SERIES. The results of this example suggest we analyze the E series of root spaces. The nonzero roots are

$$E_{n+1}: \qquad \pm\mathbf{e}_i \pm \mathbf{e}_j \qquad 1 \le i \ne j \le n \tag{1.17a}$$

$$\tfrac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \cdots \pm \mathbf{e}_n) \pm \sqrt{2 - n(\tfrac{1}{2})^2}\,\mathbf{e}_{n+1} \tag{1.17b}$$

This series must terminate with the member E_8. Since all rank-2 root spaces have been constructed, the lowest nontrivial member is E_3.

E_3 : Possible nonzero roots for E_3 are

$$\pm\mathbf{e}_1 \pm \mathbf{e}_2 \qquad (2)^2\,{}_2C_2 \tag{1.18a}$$

$$\tfrac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2) \pm \sqrt{\tfrac{3}{2}}\,\mathbf{e}_3 \qquad (2)^3 \tag{1.18b}$$

The inner product of the roots (1.18b) with themselves is

$$\tfrac{1}{2}(\pm 2, \pm\tfrac{3}{2}, \pm 1) \tag{1.19}$$

Equation (1.1) is satisfied only if there is an even number of $+$ signs within the bracket in (1.18b). The roots $\pm(\mathbf{e}_1 - \mathbf{e}_2)$ are orthogonal to the remaining six nonzero roots, which all have the same length. Then E_3 is semisimple, and

$$E_3 \simeq A_2 \oplus A_1 \tag{1.20}$$

E_4 : Possible nonzero roots of E_4 are

$$\pm \mathbf{e}_i \pm \mathbf{e}_j \qquad 1 \leq i \neq j \leq 3 \tag{1.21a}$$

$$\tfrac{1}{2}(\pm \mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3) \pm \sqrt{2 - \tfrac{3}{4}}\,\mathbf{e}_4 \tag{1.21b}$$

The inner product of the roots (1.21b) among themselves is

$$\tfrac{1}{2}(\pm 2,\ \pm \tfrac{3}{2},\ \pm 1,\ \pm \tfrac{1}{2}) \tag{1.22}$$

Equation (1.1) is satisfied only when the roots (1.21b) contain an overall even number of plus signs. The generators are

$$\begin{array}{llll}
E_{\boldsymbol{\alpha}}: & \boldsymbol{\alpha} = (1.21\text{a}) & (2)^2\,{}_3C_2 & = 12 \\
E_{\boldsymbol{\alpha}}: & \boldsymbol{\alpha} = (1.21\text{b}) & \binom{4}{0} + \binom{4}{2} + \binom{4}{4} & = 8 \\
H_i: & & & = \underline{4} \\
 & & & \ 24
\end{array}$$

The algebra corresponding to root space A_4 has also 24 generators, and in fact these spaces are isometric:

$$E_4 \simeq A_4$$

E_5: This root space has already been treated (1.15E).

E_6 : By arguments identical to those for E_4, the nonzero roots of E_6 are

$$\pm \mathbf{e}_i \pm \mathbf{e}_j \qquad 1 \leq i \neq j \leq 5 \tag{1.23a}$$

$$\underbrace{\tfrac{1}{2}(\pm \mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \pm \mathbf{e}_5) \pm \sqrt{2 - \tfrac{5}{4}}\,\mathbf{e}_6}_{\text{overall even number of } + \text{ signs}} \tag{1.23b}$$

The generators for E_6 are

$$\begin{array}{llll}
E_{\boldsymbol{\alpha}}: & \boldsymbol{\alpha} = (1.23\text{a}) & (2)^2\binom{5}{2} & = 40 \\
E_{\boldsymbol{\alpha}}: & \boldsymbol{\alpha} = (1.23\text{b}) & \binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} & = 32 \\
H_i: & & & = \underline{6} \\
 & & & \ 78
\end{array}$$

E_7 : By arguments identical to those for E_5, the nonzero roots for E_7 are

$$\pm \mathbf{e}_i \pm \mathbf{e}_j \qquad 1 \le i \ne j \le 6 \tag{1.24a}$$

$$\underbrace{\tfrac{1}{2}(\pm \mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \pm \mathbf{e}_5 \pm \mathbf{e}_6)}_{\text{even number of + signs}} \pm \sqrt{2 - \tfrac{6}{4}}\,\mathbf{e}_7 \tag{1.24b}$$

In addition, completion under reflection demands the presence of two additional roots $\pm\sqrt{2}\,\mathbf{e}_7$. The generators of E_7 are

$$\begin{array}{llll}
E_{\alpha}: & \alpha = (1.24\text{a}) & (2)^2\binom{6}{2} & = \;\; 60 \\[1ex]
E_{\alpha}: & \alpha = (1.24\text{b}) & 2\left[\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6}\right] & = \;\; 64 \\[1ex]
E_{\alpha}: & \alpha = \pm\sqrt{2}\,\mathbf{e}_7 & & = \;\;\; 2 \\
H_i: & & & = \;\;\; 7 \\ \hline
 & & & \;\;\;\; 133
\end{array}$$

E_8 : Following the arguments for E_6, the nonzero roots for E_8 are

$$\pm \mathbf{e}_i \pm \mathbf{e}_j \qquad 1 \le i \ne j \le 7 \tag{1.25a}$$

$$\underbrace{\tfrac{1}{2}(\pm \mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \pm \mathbf{e}_5 \pm \mathbf{e}_6 \pm \mathbf{e}_7) \pm \tfrac{1}{2}\mathbf{e}_8}_{\text{overall even number of + signs.}} \tag{1.25b}$$

Completion demands the presence of the additional roots $\pm \mathbf{e}_i \pm \mathbf{e}_8$. The generators of E_8 are

$$\begin{array}{llll}
E_{\alpha}: & \alpha = (1.25\text{a}) & (2)^2\binom{7}{2} & = \;\; 84 \\[1ex]
E_{\alpha}: & \alpha = (1.25\text{b}) & \binom{8}{0} + \binom{8}{2} + \binom{8}{4} + \binom{8}{6} + \binom{8}{8} & = 128 \\[1ex]
E_{\alpha}: & \alpha = \pm \mathbf{e}_i \pm \mathbf{e}_8 & (2)^2 7 & = \;\; 28 \\
H_i: & & & = \;\;\; 8 \\ \hline
 & & & \;\; 248
\end{array}$$

The root spaces E_6, E_7, E_8 are not equivalent to any of the root spaces considered previously. They are also simple.

5. LIST OF THE SIMPLE ROOT SPACES. The list of the simple root space diagrams has now been completely exhausted. Their relationship with one another is summarized[3] in Figs. 8.7 and 8.8. The properties of these algebras are summarized in Table 8.2.

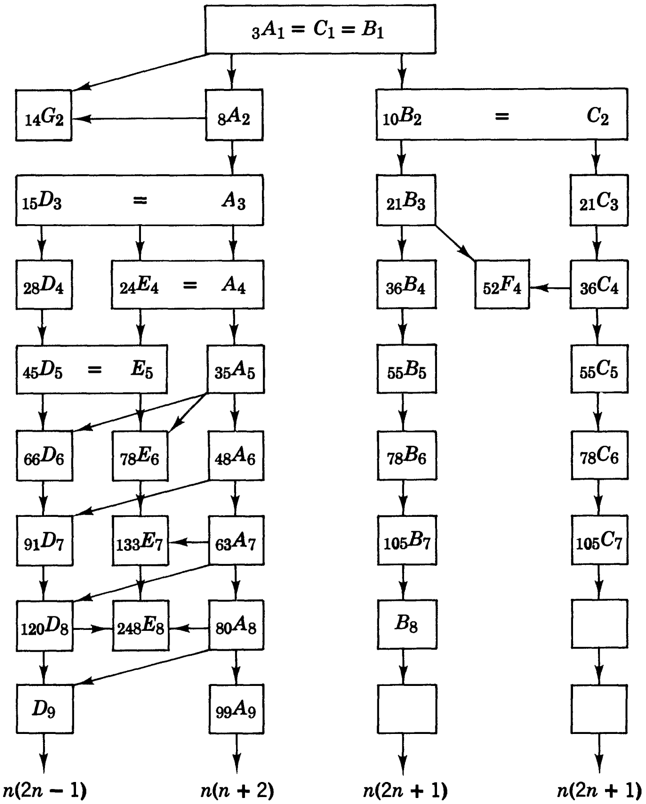


FIG. 8.7 THERE ARE FOUR UNENDING CHAINS OF LIE ALGEBRAS: THE A, B, C, AND D SERIES, AS SHOWN. THERE IS ALSO AN E SERIES, WHICH TERMINATES BEYOND E_8. THE DIMENSIONALITIES OF THESE SIMPLE LIE ALGEBRAS ALSO APPEAR.

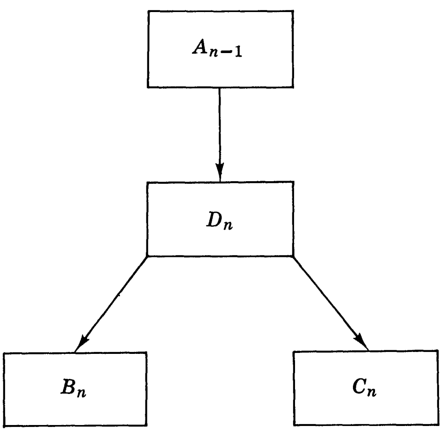


FIG. 8.8 THESE POSSIBLE ROOT SPACE EXTENSIONS EXIST BUT ARE NOT GENERALLY INDICATED IN FIG. 8.7.

TABLE 8.2

Root Space	Nonzero Roots	Number of Nonzero Roots	Rank	Total Number of Generators	Normalization $\lvert \mathbf{e}_i \rvert$
A_n	$\mathbf{e}_i - \mathbf{e}_j \quad 1 \le i \ne j \le n+1$	$n(n+1)$	n	$(n+1)^2 - 1$	$\frac{1}{\sqrt{2(n+1)}}$
D_n	$\pm\mathbf{e}_i \pm \mathbf{e}_j$	$2n(n-1)$	n	$\frac{2n(2n-1)}{2}$	
	$\pm 0\,\mathbf{e}_i$	0			$\frac{1}{\sqrt{2(2n-2)}}$
B_n	$\pm\mathbf{e}_i \pm \mathbf{e}_j$	$2n(n-1)$	n	$\frac{(2n+1)(2n)}{2}$	
	$\pm 1\mathbf{e}_i$	$2n$			$\frac{1}{\sqrt{2(2n-1)}}$
C_n	$\pm\mathbf{e}_i \pm \mathbf{e}_j$	$2n(n-1)$	n	$\frac{(2n+1)(2n)}{2}$	
	$\pm 2\mathbf{e}_i$	$2n$			$\frac{1}{\sqrt{2(2n+2)}}$
G_2	$\mathbf{e}_i - \mathbf{e}_j \quad 1 \le i \ne j \ne k \le 3$	6	2	14	$\frac{1}{\sqrt{24}}$
	$\pm(\mathbf{e}_i + \mathbf{e}_j) \mp 2\mathbf{e}_k$	6			
F_4	$\pm\mathbf{e}_i \pm \mathbf{e}_j \quad 1 \le i \ne j \le 4$	24			
	$\pm 2\mathbf{e}_i$	8	4	52	$\frac{1}{6}$
	$\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4$	16			
E_6	$\pm\mathbf{e}_i \pm \mathbf{e}_j \quad 1 \le i \ne j \le 5$	40			
	$\underbrace{\tfrac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \cdots \pm \mathbf{e}_5) \pm \sqrt{2 - 5/4}\,\mathbf{e}_6}_{\text{even number of + signs}}$	32	6	78	$\frac{1}{\sqrt{24}}$
E_7	$\pm\mathbf{e}_i \pm \mathbf{e}_j \quad 1 \le i \ne j \le 6$	60			
	$\pm\sqrt{2}\,\mathbf{e}_7$	2	7	133	$\frac{1}{6}$
	$\underbrace{\tfrac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \cdots \pm \mathbf{e}_6)}_{\text{even number of + signs}} \pm \sqrt{2 - 6/4}\,\mathbf{e}_7$	64			
E_8	$\pm\mathbf{e}_i \pm \mathbf{e}_j \quad 1 \le i \ne j \le 8$	112			
	$\underbrace{\tfrac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \cdots \pm \mathbf{e}_8)}_{\text{even number of + signs}}$	128	8	248	$\frac{1}{\sqrt{60}}$

II. Identification of the Classical Algebras

1. A_n AND $Sl(n + 1, c)$. It is only necessary to identify one classical Lie algebra with the commutation properties described by a root space. Then the complex extension of that algebra is described by the same root space diagram. All classical algebras with the same complex extension are also described by the same root space. The various real forms of a complex semisimple algebra differ among themselves only in the reality restrictions placed on the elements of the algebra.

The identification of a root space with an algebra proceeds most easily when the coordinates of all bases within the algebra are real. This particular reality restriction is characteristic of the *least* compact real form belonging to the common complex extension. The least compact real form of $Sl(n + 1, c)$ is $Sl(n + 1, r)$. Its generators can be given by the faithful $(n + 1) \times (n + 1)$ matrix representation or by the faithful analytic realization:

$$\begin{aligned} (M^{(n+1)}_{ij})_{rs} &= \delta_{ir}\,\delta_{js} \\ &= \begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{bmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{bmatrix} \end{array} \qquad (2.1) \\ M^{(\text{anal})}_{ij} &= u^i\,\partial_j \end{aligned}$$

The generators of the algebra and the generators corresponding to the root space are identified by

$$\begin{aligned} H_i &\leftrightarrow u^i\,\partial_i \\ E_{\mathbf{e}_i - \mathbf{e}_j} &\leftrightarrow u^i\,\partial_j \\ \sum_{i=1}^{n+1} H_i \leftrightarrow \sum_{i=1}^{n+1} u^i\,\partial_i &\leftrightarrow \sum_{i=1}^{n+1} M^{(n+1)}_{ii} = 0 \end{aligned} \qquad (2.2)$$

The root space A_n describes the commutation properties of the complex algebra $\mathfrak{sl}(n + 1, c)$ and all its real forms: $\mathfrak{sl}(n + 1, r)$, $\mathfrak{su}(n + 1)$, $\mathfrak{su}(n + 1 - p, p)$, $\mathfrak{su}^*(2n)$.

Example 1. When $i < j$, the root $\mathbf{e}_i - \mathbf{e}_j$ is positive. The matrix representatives for roots of the form $\pm(\mathbf{e}_i - \mathbf{e}_{i+1})$ in A_{n-1} are given in (2.3).

$$\sum_{1}^{n-1} \beta_i E_{\mathbf{e}_i - \mathbf{e}_{i+1}} + \sum_{1}^{n-1} \gamma_i E_{\mathbf{e}_{i+1} - \mathbf{e}_i} \xrightarrow[\Gamma]{\text{defining representation}}$$

$$\begin{array}{c@{}c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & & & & & n \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ \\ \\ \\ \\ n \end{array} &
\left[\begin{array}{cccccccc}
0 & \beta_1 & & & & & & \\
\gamma_1 & 0 & \beta_2 & & & & & \\
 & \gamma_2 & 0 & \beta_2 & & & & \\
 & & \gamma_3 & 0 & & & & \\
 & & & & 0 & & & \\
 & & & & & 0 & & \\
 & & & & & & 0 & \beta_{n-1} \\
 & & & & & & \gamma_{n-1} & 0
\end{array}\right]
\end{array} \tag{2.3}$$

$$\Gamma(E_{-\alpha}) = \Gamma^{\dagger}(E_{\alpha})$$

It is clear that, for $i < j$, the generator belonging to $\mathbf{e}_i - \mathbf{e}_j$ is obtained as a multiple commutator of generators belonging to the roots

$$\mathbf{e}_k - \mathbf{e}_{k+1}, \qquad k = i, i+1, \ldots, j-1 \tag{2.4}$$

From these considerations, we realize that the roots $\mathbf{e}_i - \mathbf{e}_{i+1}$ $(i = 1, 2, \ldots, n-1)$ can be used to generate the entire algebra A_{n-1}. They form a *basis for the algebra.*

2. THE SIBLING ALGEBRAS D_n, B_n, C_n. The root spaces D_n, B_n, C_n describe the commutation relations of the complex Lie groups $SO(2n, c)$, $SO(2n+1, c)$, $Sp(2n, c)$. To make this identification it is useful to work with the least compact real form of these algebras. The orthogonal groups preserve diagonal metrics of the form (2.5). These metrics are equivalent, under a similarity transformation, to metrics with elements on the minor diagonal only (2.5′).

$SO(n, n, r)$	$SO(n+1, n, r)$	$Sp(2n, r)$
$\left[\begin{array}{c\|c} I_n & \\ \hline & -I_n \end{array}\right]$	$\left[\begin{array}{c\|c\|c} I_n & & \\ \hline & 1 & \\ \hline & & -I_n \end{array}\right]$	

(2.5)

$$\begin{array}{c} 1\ 2 \cdots n \quad -n \cdots -1 \\ \left[\begin{array}{c|c} & \tilde{I}_n \\ \hline +\tilde{I}_n & \end{array}\right] \end{array} \qquad \begin{array}{c} 1\ 2 \cdots n \quad 0 \quad -n \cdots -1 \\ \left[\begin{array}{c|c|c} & & \tilde{I}_n \\ \hline & 1 & \\ \hline +\tilde{I}_n & & \end{array}\right] \end{array} \qquad \begin{array}{c} 1\ 2 \cdots n \quad -n \cdots -1 \\ \left[\begin{array}{c|c} & \tilde{I}_n \\ \hline -\tilde{I}_n & \end{array}\right] \end{array} \tag{2.5'}$$

$$g_{ij} = \chi_1(i)\,\delta_{i+j,\,0} \qquad g_{ij} = \chi_1(i)\,\delta_{i+j,\,0} \qquad g_{ij} = \chi_2(i)\,\delta_{i+j,\,0} \tag{2.5''}$$

As usual, $\tilde{I}_n$ is the $n \times n$ matrix with $+1$ on the minor diagonal and 0 elsewhere:

$$\tilde{I}_n = \begin{bmatrix} & & & & & +1 \\ & & & & +1 & \\ & & & +1 & & \\ & & +1 & & & \\ & +1 & & & & \\ +1 & & & & & \end{bmatrix} \tag{2.6}$$

The functions χ_1, χ_2, characteristic of the orthogonal and symplectic groups, are given by

$$\begin{array}{c|cc} & i \geq 0 & i < 0 \\ \hline \chi_1(i) & 1 & 1 \\ \chi_2(i) & 1 & -1 \end{array} \tag{2.7}$$

These matrix groups act in spaces of dimension $2n$ and $2n + 1$. The bases in these spaces are conveniently labeled

$$(x^0),\ x^{\pm 1},\ x^{\pm 2}, \ldots, x^{\pm n} \tag{2.8}$$

and arranged as indicated in (2.5′).

The Lie algebra for each of these three noncompact groups is easily computed:

$$\begin{aligned} (I + M)_i{}^r g_{rs} (I + M)_j{}^s &= g_{ij} \\ M_i{}^r g_{rj} + g_{ir} M_j{}^r &= 0 \\ \chi(i) M_i{}^k &= -\chi(k) M_{-k}{}^{-i} \end{aligned} \tag{2.9}$$

From (2.7) and (2.9) it is clear that all the generators of the orthogonal groups are antisymmetric under a reflection in the minor diagonal (indicated by $\sim$), whereas only about half of the generators of the symplectic group are.

The Lie algebras are given explicitly by

$$\begin{array}{ccc} \mathfrak{so}(n,n;r) & \mathfrak{so}(n+1,n;r) & \mathfrak{sp}(2n;r) \\ \left[\begin{array}{c|c} A & B \\ \hline C & -\tilde{A} \end{array}\right] & \left[\begin{array}{c|c|c} A & E & B \\ \hline F & & -\tilde{E} \\ \hline C & -\tilde{F} & -\tilde{A} \end{array}\right] & \left[\begin{array}{c|c} A & B \\ \hline C & -\tilde{A} \end{array}\right] \\ \tilde{B}=-B & \tilde{B}=-B & \tilde{B}=+B \\ \tilde{C}=-C & \tilde{C}=-C & \tilde{C}=+C \end{array} \tag{2.10}$$

where all matrices are real.

The $n \times n$ matrices B, C have only zeroes on the minor diagonal for the orthogonal groups. These minor diagonal elements are not necessarily zero for the symplectic group.

The mutually commuting generators for each of these algebras can be chosen diagonal

$$H_i = x^i \partial_i - x^{-i} \partial_{-i} \qquad i = 1, 2, \ldots, n \tag{2.11d}$$

When $i < j$, both roots $\mathbf{e}_i \pm \mathbf{e}_j$ are positive and described by upper triangular matrices. The corresponding generators are

$$E_{\mathbf{e}_i \mp \mathbf{e}_j} = x^i \partial_{\pm j} - \chi(\pm j) x^{\mp j} \partial_{-i} \qquad 1 \le i < j \le n \tag{2.11p}$$

The corresponding negative roots are defined in a manner analogous to (5.18) of Chapter 7:

$$\Gamma(E_{-\alpha}) = \Gamma^{\dagger}(E_{\alpha})$$

$$E_{-\mathbf{e}_i \pm \mathbf{e}_j} = x^{\pm j} \partial_i - \chi(\pm j) x^{-i} \partial_{\mp j} \qquad 1 \le i < j \le n \quad (2.11\text{n} = 2.11\text{p}\dagger)$$

Equations (2.11d), (2.11p), and (2.11n) provide a complete identification between the roots of D_n and the generators of $SO(n, n; r)$.

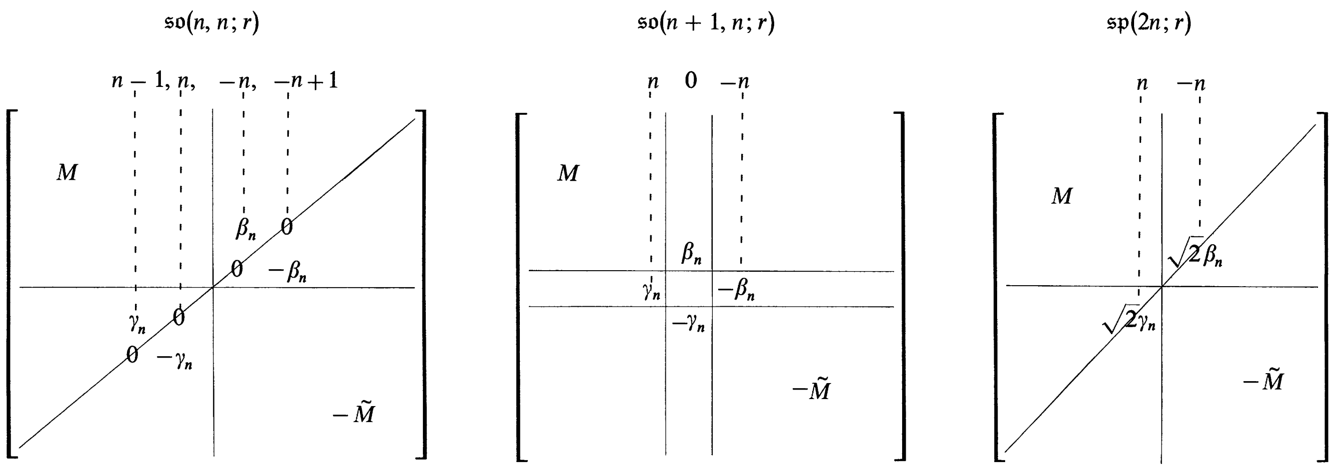
The root spaces B_n, C_n have $2n$ additional roots of the form $\pm\mathbf{e}_i$, $\pm 2\mathbf{e}_i$. The algebras $\mathfrak{so}(n + 1, n; r)$ and $\mathfrak{sp}(2n; r)$ have $2n$ additional generators corresponding to the elements in the row or column marked 0 and the elements on the minor diagonal, respectively. The identifications are

$$E_{+\mathbf{e}_i} = x^i \partial_0 - x^0 \partial_{-i} \tag{2.11B}$$

$$E^{\dagger}_{\mathbf{e}_i} = E_{-\mathbf{e}_i} = x^0 \partial_i - x^{-i} \partial_0 \qquad SO(n + 1, n; r)$$

$$E_{\pm 2\mathbf{e}_i} = \sqrt{2}\, x^{\pm i} \partial_{\mp i} \qquad Sp(2n; r) \tag{2.11C}$$

It is now a straightforward matter to verify that the generators of these algebras reproduce the commutation relations demanded by the corresponding root spaces.



$$\sum \beta_i E_{\alpha_i} + \sum \gamma_i E_{-\alpha_i}$$

FIG. 8.9 THE MATRIX REPRESENTATIVES OF THE POSITIVE ROOTS α_i (NEGATIVE ROOTS $-\alpha_i$) ARE UPPER (LOWER) TRIANGULAR AND LIE AS CLOSE TO THE MAJOR DIAGONAL AS POSSIBLE. BY THE ARGUMENTS OF EXAMPLE 1, ALL POSITIVE ROOTS ARE OBTAINED AS MULTIPLE COMMUTATORS OF THE BASIC ROOTS α_i. THE MATRIX M IS GIVEN BY (2.3), $\tilde{M}$ IS ITS TRANSPOSE ABOUT THE MINOR DIAGONAL.

Example 2. The set of positive roots

$$
\begin{aligned}
\alpha_1 &= \mathbf{e}_1 - \mathbf{e}_2 \\
\alpha_2 &= \qquad \mathbf{e}_2 - \mathbf{e}_3 \\
\alpha_3 &= \qquad\qquad \mathbf{e}_3 - \mathbf{e}_4 \\
&\vdots \\
\alpha_{n-1} &= \qquad\qquad\qquad \mathbf{e}_{n-1} - \mathbf{e}_n
\end{aligned}
\tag{2.12}
$$

forms a linearly independent set of roots in each of the spaces A_{n-1}; D_n, B_n, C_n. To form a basis one additional linearly independent root is needed for the rank n root spaces:

$$
\alpha_n = \begin{array}{cc}
— & A_{n-1} \\
\mathbf{e}_{n-1} + \mathbf{e}_n & D_n \\
\mathbf{e}_n & B_n \\
2\mathbf{e}_n & C_n
\end{array}
\tag{2.12$'$}
$$

The matrix representatives of these roots and their negatives is given in Fig. 8.9. For each of these algebras, generators corresponding to positive roots have matrix representatives which are upper triangular and thus can be written as multiple commutators of the basic (or simple) positive roots (2.12). All negative roots are the adjoints of the corresponding positive roots. The diagonal generators H_i can all be obtained from the commutator of a positive root generator with its adjoint. As for A_{n-1}, the entire algebra can be obtained from the generators corresponding to the positive roots (2.12), together with all their commutators. They form a *basis for the algebra.* It will be convenient, in the future, to treat just these basis generators, rather than the entire algebra.

III. Dynkin Diagrams

1. SIMPLE ROOTS. In Section I all the nonzero roots were given for each of the simple root spaces constructed. For any space, it is easy to check that the list of roots is complete. It is less easy to verify that the list of simple root spaces is itself complete, but Dynkin[4] has devised a very elegant mechanism for doing just this. The method suffers from the drawback that the roots within any space are not nearly as accessible; however, it turns out that all the simple root spaces have already been constructed and their roots enumerated. Thus Dynkin's method serves to demonstrate completeness convincingly and to show some isomorphisms among the spaces of lower dimensionality.

Within any rank n root space it is possible to choose any set of n linearly independent vectors as a basis. Then every root is expressible as a linear combination of the basis vectors. In Section I it was convenient to choose an orthogonal basis $\mathbf{e}_i$. These vectors were not actually roots themselves, except in the case B_n. Once the basis has been chosen, it is possible to order the various roots:

$$\mathbf{v} = \sum_1^n \beta^i \mathbf{e}_i \tag{3.1}$$

The root $\mathbf{v}$ is **positive** if the first nonvanishing coordinate β^i is larger than zero. The nonzero roots can be divided into those which are positive and those which are negative. The vector

$$\mathbf{R} = \frac{1}{2} \sum_{\boldsymbol{\alpha}>0} \boldsymbol{\alpha} \tag{3.2}$$

which is half the sum of all positive roots, plays an important role in representation theory. With respect to the bases $\mathbf{e}_i$, the coordinates R_i of $\mathbf{R}$ are

$$\begin{aligned}
&A_n: && R_i = \frac{n}{2} + 1 - i && 1 \le i \le n+1 \\
&D_n: && R_i = \frac{2n+0}{2} - i && \\
&B_n: && R_i = \frac{2n+1}{2} - i && \\
&C_n: && R_i = \frac{2n+2}{2} - i && 1 \le i \le n
\end{aligned} \tag{3.3}$$

It is useful to choose as bases in the n-dimensional root space n linearly independent *roots* $\boldsymbol{\alpha}_i (i = 1, 2, \ldots, n)$. Since the roots in a root space arise through the commutation of the associated generators, the choice of basis roots can always be made in such a way that if $\mathbf{v}$ is a root

$$\mathbf{v} = \sum \beta^i \boldsymbol{\alpha}_i \tag{3.4}$$

each of the coefficients β^i is an integer. Moreover, the choice can always be made in such a way that if $\mathbf{v} > 0$, each $\beta^i \ge 0$. A set of basis roots with this property is called a **simple system of roots**. The simple roots for the rank-2 algebras are given in Figs. 8.2 to 8.5. It is clear from these diagrams that all positive roots (in the right half-plane or on the upper half of the y-axis) have both properties: all β^i are integers and ≥ 0.

The simple roots for the algebras A_{n-1} and D_n, B_n, C_n have been given in (2.12). Their matrix representatives in the defining representation—together with those of the negative simple roots—are given in (2.3) and Fig. 8.9.

The n simple roots $\boldsymbol{\alpha}_i$ determine an $(n-1)$-dimensional hyperplane passing through their tips. This hyperplane passes one unit distance (called a level) from the origin. The remaining roots determine hyperplanes parallel to the original. All these planes are orthogonal to a vector $\mathbf{R}_\perp$. In general, $\mathbf{R}_\perp \neq \mathbf{R}$. In addition, all hyperplanes are spaced an integral distance from the origin. This distance is called the **level** of the root. Then if

$$\mathbf{v} = \sum_{i=1}^{n} \beta^i \boldsymbol{\alpha}_i$$

$$\text{level of } \mathbf{v} = \sum_{i=1}^{n} \beta^i \tag{3.5}$$

Level is another property of roots that, like $\mathbf{R}$, is of limited usefulness now but of fundamental importance in discussions of representations and of embeddings of one semisimple algebra in another. The vectors $\mathbf{R}$ and $\mathbf{R}_\perp$ for the root space C_2 are illustrated in Fig. 8.10.

We turn now to a more thorough discussion of the roots in a simple root space, and particularly to the simple roots and their properties. We do this by making ten simple observations:

1. First, we show that every nonzero root in a simple root space can be

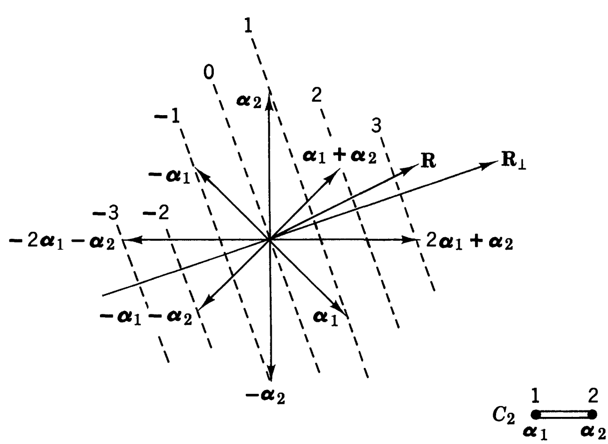


FIG. 8.10 FOR THE ROOT SPACE C_2, $\mathbf{R} = 2\mathbf{e}_1 + \mathbf{e}_2$ AND $\mathbf{R}_\perp = 3\mathbf{e}_1 + \mathbf{e}_2$. EACH NONZERO POSITIVE AND NEGATIVE ROOT IS DESCRIBED IN TERMS OF A SIMPLE BASIS. THE LEVEL OF EACH ROOT IS SHOWN ALONG THE DASHED HYPERPLANES THROUGH THE ROOT AND ORTHOGONAL TO $\mathbf{R}_\perp$.

written as the sum of two other nonzero roots (except in A_1). For suppose there is a root γ which cannot be written as the sum

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta} \qquad \boldsymbol{\alpha}, \boldsymbol{\beta} \neq 0$$

of nonzero roots. Then E_γ commutes with all eigenoperators E_α:

$$[E_\gamma, E_\alpha] = 0 \qquad \boldsymbol{\alpha} \neq 0, -\boldsymbol{\gamma} \tag{3.6}$$

If (3.6) did not hold, we would have

$$\begin{aligned}[E_\gamma, E_\alpha] &\neq 0 \\ &= N_{\gamma,\alpha} E_{\gamma+\alpha}\end{aligned}$$

In other words,

$$\boldsymbol{\gamma} + \boldsymbol{\alpha} = \boldsymbol{\beta} \Rightarrow \boldsymbol{\gamma} = (-\boldsymbol{\alpha}) + \boldsymbol{\beta}$$

Thus if γ cannot be written as the sum of nonzero roots, E_γ and $E_{-\gamma}$ commute with all E_α. So also does $\boldsymbol{\gamma} \cdot \boldsymbol{H} = [E_\gamma, E_{-\gamma}]$, for

$$\begin{aligned}[\boldsymbol{\gamma} \cdot \mathbf{H}, E_\alpha] &= [[E_\gamma, E_{-\gamma}], E_\alpha] \\ &= [E_\gamma, [E_{-\gamma}, E_\alpha]] + [E_{-\gamma}, [E_\alpha, E_\gamma]] = 0\end{aligned}$$

The three generators $E_\gamma, E_{-\gamma}, [E_\gamma, E_{-\gamma}]$ commute with all E_α, and therefore all $\boldsymbol{\alpha} \cdot \mathbf{H} = [E_\alpha, E_{-\alpha}]$ ($\boldsymbol{\alpha} \neq \pm\boldsymbol{\gamma}$). The subalgebra spanned by $E_\gamma, E_{-\gamma}, \boldsymbol{\gamma} \cdot \mathbf{H}$ and the subalgebra spanned by E_α, $E_{-\alpha}$, $\boldsymbol{\alpha} \cdot \mathbf{H}$ (all $\boldsymbol{\alpha} \neq \pm\boldsymbol{\gamma}$) are mutually commuting. The original algebra must therefore be semisimple, contrary to the assumption that it is simple.

2. We now introduce an ordering into a root space. For example, a lexigraphic ordering with respect to a Euclidean basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is given by

$$\boldsymbol{\alpha} = \sum_{i=1}^{n} \alpha^i \mathbf{e}_i$$

$$\boldsymbol{\alpha} > 0 \qquad \text{if} \quad \alpha^1 = 0, \alpha^2 = 0, \ldots, \alpha^{k-1} = 0 \qquad \text{but} \quad \alpha^k > 0 \tag{3.7}$$

Then the ordering among the roots is given by

$$\boldsymbol{\alpha} > \boldsymbol{\beta} \qquad \text{if} \quad \boldsymbol{\alpha} - \boldsymbol{\beta} > 0 \tag{3.8}$$

Such an ordering divides any root space Σ_n into three disjoint parts:

$$\begin{array}{cccccc} \Sigma_n = & \Sigma_n^+ & + & \Sigma_n^0 & + & \Sigma_n^- \\ & \text{all } \boldsymbol{\alpha} > 0 & & \boldsymbol{\alpha} = 0 & & \text{all } \boldsymbol{\alpha} < 0 \\ & & & \text{(Cartan subalgebra)} & & \end{array} \tag{3.9}$$

Comment 1. For our present purposes, the ordering (3.8) is not particularly significant; it is the decomposition (3.9) which is fundamentally important. An ordering of this form can easily be introduced without the intermediary of going through a more involved ordering (3.8) as follows: construct a hyperplane passing through the origin of Σ_n but not containing any nonzero roots. All roots on one side of the hyperplane belong in Σ_n^+; all roots on the other side are negatives of roots in Σ_n^+ and therefore belong to Σ_n^-.

Comment 2. The ordering (3.9) in root space has an important analog in the algebra. It corresponds to decomposing the regular representation, and in fact any faithful representation, into upper-triangular (Δ^+), lower-triangular (Δ^-), and diagonal matrices (d):

Root space: $$\Sigma_n = \Sigma_n^+ + \Sigma_n^0 + \Sigma_n^- \tag{3.10rs}$$

Faithful representation: $$\Delta = \Delta^+ + d \quad + \Delta^- \tag{3.10a,g}$$

The decomposition (3.10a,g) is ideally suited for applications of BCH formulas.

3. *Definition.* A **simple positive root** is a positive root that cannot be written as the sum of two other positive roots.

Comment 3. We will refer to the set of simple positive (negative) roots as S^+ (S^-).

Example. The simple positive roots in the rank-2 algebras are labeled $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ in Figs. 8.2 to 8.5.

4. We turn now to some properties of the simple positive roots. If $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are simple positive roots, then $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is not a root at all. Suppose that $\boldsymbol{\alpha} - \boldsymbol{\beta}$ were a root. Then we would have

$$\begin{aligned} \boldsymbol{\alpha} - \boldsymbol{\beta} &= \boldsymbol{\gamma} & \boldsymbol{\gamma} &\in \Sigma_n{}^{\pm} \\ \boldsymbol{\alpha} &= \boldsymbol{\gamma} + \boldsymbol{\beta} & \boldsymbol{\gamma} &\in \Sigma_n{}^{+} \\ \boldsymbol{\alpha} + (-\boldsymbol{\gamma}) &= \boldsymbol{\beta} & \boldsymbol{\gamma} &\in \Sigma_n{}^{-} \end{aligned}$$

In either case, we have a simple positive root expressed as the sum of two positive roots, which is a contradiction.

5. If $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are simple positive roots, then $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \leqq 0$. For consider the $\boldsymbol{\alpha}$ chain containing $\boldsymbol{\beta}$:

$$\boldsymbol{\beta} - m\boldsymbol{\alpha}, \ldots, \boldsymbol{\beta}, \ldots, \boldsymbol{\beta} + n\boldsymbol{\alpha}$$

$$\frac{2\boldsymbol{\beta} \cdot \boldsymbol{\alpha}}{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}} = m - n \tag{3.11}^*$$

* See (5.31) in Chapter 7.

By observation 4, $m = 0$; thus the inner product cannot be positive.

6. If $\boldsymbol{\beta}$ is a positive root,

$$\boldsymbol{\beta} = \sum k^i \boldsymbol{\alpha}_i \tag{3.12}$$

where $\boldsymbol{\alpha}_i$ are simple positive roots and the k^i are all nonnegative integers.

If $\boldsymbol{\beta}$ is a simple root, then (3.12) is trivial.

If $\boldsymbol{\beta}$ is not a simple positive root, then $\boldsymbol{\beta}$ is the sum of two positive roots (by observation 3)

$$\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 \qquad (\boldsymbol{\beta}_1 > 0, \boldsymbol{\beta}_2 > 0) \tag{3.13}$$

Applying the arguments leading to (3.13) inductively leads immediately to (3.12).

7. The simple positive roots are linearly independent. To understand this, suppose that $\boldsymbol{\alpha}_n$ could be written as a linear combination of the other simple positive roots:

$$\begin{aligned}
\boldsymbol{\alpha}_n &= \sum_{1}^{n-1} \lambda^r \boldsymbol{\alpha}_r \\
&= \sum_{i=1}^{n-1} \rho^i \boldsymbol{\alpha}_i + \sum_{j=1}^{n-1} \nu^j \boldsymbol{\alpha}_j \\
&= \mathbf{p} \qquad\quad + \mathbf{n}
\end{aligned}
\qquad
\left\{\begin{array}{l} \rho^i > 0 \\ \nu^j < 0 \\ \mathbf{p} > 0 \\ \mathbf{n} < 0 \end{array}\right. \tag{3.14}$$

We now form the inner product of $\boldsymbol{\alpha}_n$ with the positive vector $\mathbf{p}$ (which is not necessarily a root) in two ways:

$$\begin{aligned}
(\boldsymbol{\alpha}_n, \mathbf{p}) &= (\mathbf{p} + \mathbf{n}, \mathbf{p}) \\
&= (\mathbf{p}, \mathbf{p}) + (\mathbf{n}, \mathbf{p})
\end{aligned} \tag{3.15a}$$

Since the inner product is positive definite

$$(\mathbf{p}, \mathbf{p}) \geq 0, \qquad (\mathbf{p}, \mathbf{p}) = 0 \Rightarrow \mathbf{p} = 0$$

Also, the inner product $(\mathbf{n}, \mathbf{p})$ is nonnegative, for

$$(\mathbf{n}, \mathbf{p}) = \left(\sum \nu^j \boldsymbol{\alpha}_j, \sum \rho^i \boldsymbol{\alpha}_i\right) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \underbrace{\rho^i}_{(>0)} \; \underbrace{\nu^j}_{(<0)} \; \underbrace{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j)}_{(\leq 0)} \geq 0$$

Computing this inner product the second way, we find

$$(\boldsymbol{\alpha}_n, \mathbf{p}) = (\boldsymbol{\alpha}_n, \rho^i \boldsymbol{\alpha}_i) = \sum_{i=1}^{n-1} \rho^i (\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_n) \leq 0 \tag{3.15b}$$

Comparing these inner products, we have

$$\begin{array}{ccc} & (\boldsymbol{\alpha}_n, \mathbf{p}) & \\ \text{by (3.15a)} \swarrow & & \searrow \text{by (3.15b)} \\ \geq 0 & = & \leq 0 \Rightarrow \mathbf{p} = 0 \end{array} \tag{3.16}$$

Thus

$$\boldsymbol{\alpha}_n = \mathbf{p} + \mathbf{n} = \mathbf{0} + \mathbf{n} \tag{3.17}$$

Since $\boldsymbol{\alpha}_n$ is a positive root by assumption and $\mathbf{n}$ is negative by construction, we encounter a contradiction. Therefore, the positive simple roots are linearly independent.

8. The $\boldsymbol{\alpha}_i$ form a basis in Σ_n^+ and therefore in Σ_n.
   By observation 6 they span Σ_n^+.
   By observation 7 they are linearly independent.
9. If $\boldsymbol{\beta}$ is a positive root (not simple), there is an $\boldsymbol{\alpha}_i \in S_n^+$ with the property

$$\boldsymbol{\beta} - \boldsymbol{\alpha}_i \in \Sigma_n^+$$

There is an $\boldsymbol{\alpha}_i$ in S_n^+ for which $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} > 0$. If this were not true, then by observation 6, $\boldsymbol{\beta}$ and the $\boldsymbol{\alpha}_i \in S_n^+$ would be linearly independent, and $\boldsymbol{\beta}$ would be simple. Using (3.11) we write

$$\frac{2\boldsymbol{\beta} \cdot \boldsymbol{\alpha}}{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}} = m - n > 0 \Rightarrow m \geq 1 \qquad \text{for} \quad \text{some } \boldsymbol{\alpha}_i \in S_n^+) \tag{3.18}$$

where the value of m is at least $+1$ for some $\boldsymbol{\alpha}_i$, and $\boldsymbol{\beta} - \boldsymbol{\alpha}_i$ is a positive root.

10. The explicit construction of the simple positive root system S_n^+ is not difficult. We establish the construction in a faithful representation of the algebra and transfer the method into the root space.

The subalgebra $\mathfrak{g}^+$ whose generators correspond to all roots in Σ_n^+ is solvable. By Lie's Theorem (Chapter 7, Section I.4), in any representation the matrices representing $\mathfrak{g}^+$ may be taken to be upper triangular. We define the series of solvable subalgebras

$$\begin{aligned} \mathfrak{g}^+ &\equiv \mathfrak{g}^{+1} \\ [\mathfrak{g}^{+1}, \mathfrak{g}^{+1}] &= \mathfrak{g}^{+2} \\ [\mathfrak{g}^{+1}, \mathfrak{g}^{+2}] &= \mathfrak{g}^{+3} \\ \vdots \quad & \quad \vdots \\ [\mathfrak{g}^{+1}, \mathfrak{g}^{+k}] &= \mathfrak{g}^{+(k+1)} \end{aligned} \tag{3.19}$$

The subalgebra $\mathfrak{g}^{+k}$ is spanned by all generators whose roots lie on level k or greater. Then the factor algebra

$$\mathfrak{g}^{+k} \bmod \mathfrak{g}^{+(k+1)} \tag{3.20}$$

is spanned by all those generators whose roots lie on level k, and only by those roots. In particular, the simple roots all lie at level 1, and generate $\mathfrak{g}^{+1} \bmod \mathfrak{g}^{+2}$.

This constructive procedure for determining the roots at level k can now be transferred directly to the root space. The simple roots are obtained explicitly as follows:

1. Add all roots in Σ_n^+ together pairwise.
2. Remove sums of roots that are not themselves roots (the corresponding generators commute). The resulting system of roots is called Σ_n^{+2}.
3. All roots in Σ_n^{+1} but not Σ_n^{+2} are simple by definition:

$$S_n^+ = \Sigma_n^{+1} \bmod \Sigma_n^{+2} \tag{3.21}$$

Example. We compute the simple roots for various algebras, starting from a lexigraphic ordering (3.7).

A_n Positive roots: $\mathbf{e}_i - \mathbf{e}_j \qquad i<j$

Pairwise sums:

$$\begin{array}{lll} \mathbf{e}_i - \mathbf{e}_j & & 1 \le i < j \le n \\ & \mathbf{e}_j - \mathbf{e}_k & 2 \le j < k \le n+1 \\ \hline \mathbf{e}_i & \quad - \mathbf{e}_k & 1 \le i < k-1 \le n \end{array}$$

The only roots here that are not sums of positive roots are of the form $\mathbf{e}_i - \mathbf{e}_k$, $k = i+1$.

D_n Positive roots: $\mathbf{e}_i \pm \mathbf{e}_j \qquad 1 \le i < j \le n$

Pairwise sums:

$$\begin{array}{lll} \mathbf{e}_i \pm \mathbf{e}_j & & 1 \le i < j \le n-1 \\ & + \mathbf{e}_j \pm \mathbf{e}_k & 2 \le j < k \le n \\ \hline \mathbf{e}_i & \quad \pm \mathbf{e}_k & 1 \le i < k-1 \le n-1 \end{array}$$

The roots that are not obviously sums of positive roots are of the form $\mathbf{e}_i - \mathbf{e}_k$, $k = i+1$. There are $n-1$ such roots. The nth root necessary to span S_n^+ is $\mathbf{e}_{n-1} + \mathbf{e}_n$, which also cannot be written as the sum of two positive roots.

$$\begin{array}{llll} B_n & & + \mathbf{e}_i & \\ & \text{Positive roots: } \mathbf{e}_i \pm \mathbf{e}_j; & & i<j \\ C_n & & + 2\mathbf{e}_i & \end{array}$$

From the arguments used for D_n, the sums of roots of the form $\mathbf{e}_i \pm \mathbf{e}_j$ do not include $\mathbf{e}_i - \mathbf{e}_{i+1}$. The additional roots $+\mathbf{e}_i$ (for B_n) and $+2\mathbf{e}_i$ (for C_n) can also be written

$$\begin{aligned} +\mathbf{e}_i &= (\mathbf{e}_i - \mathbf{e}_j) + \mathbf{e}_j && i < j \\ +2\mathbf{e}_i &= (\mathbf{e}_i - \mathbf{e}_j) + (\mathbf{e}_i + \mathbf{e}_j) && i < j \end{aligned}$$

Therefore, all roots $\mathbf{e}_i$, $2\mathbf{e}_i$ can be written as the sum of two positive roots, for $1 \le i < n$. The simple roots for A_n; D_n, B_n, C_n which have just been constructed are

$$\begin{array}{ccc} & \mathbf{e}_n - \mathbf{e}_{n+1} & A_n \\ \mathbf{e}_i - \mathbf{e}_{i+1}; & \mathbf{e}_{n-1} + \mathbf{e}_n & D_n \\ & +\mathbf{e}_n & B_n \\ i = 1, 2, \ldots, n-1 & +2\mathbf{e}_n & C_n \end{array} \tag{3.22}$$

F_4 Positive roots:

$$\begin{array}{ll} \mathbf{e}_i \pm \mathbf{e}_j & 1 \le i < j \le 4 \\ \mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 & \\ \pm 2\mathbf{e}_i & \end{array}$$

Roots in F_4^+ that are not sums of two positive roots are

$$\begin{array}{c} \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 \\ \mathbf{e}_2 - \mathbf{e}_3 \\ \mathbf{e}_3 - \mathbf{e}_4 \\ 2\mathbf{e}_4 \end{array} \tag{3.23}$$

2. PROPERTIES OF THE DYNKIN DIAGRAMS. Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n$ be the simple roots spanning the root space for a semisimple algebra. Then

$$\frac{\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j}{\sqrt{\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_i}\sqrt{\boldsymbol{\alpha}_j \cdot \boldsymbol{\alpha}_j}} = 0, -\sqrt{\frac{1}{4}}, -\sqrt{\frac{2}{4}}, -\sqrt{\frac{3}{4}} \tag{3.24}$$

Because we are interested in the angles contained between the simple roots $\boldsymbol{\alpha}_i$, it is convenient to work with the unit vector $\mathbf{u}_i$ associated with $\boldsymbol{\alpha}_i$:

$$\mathbf{u}_i \equiv \frac{\boldsymbol{\alpha}_i}{|\boldsymbol{\alpha}_i|} \tag{3.25}$$

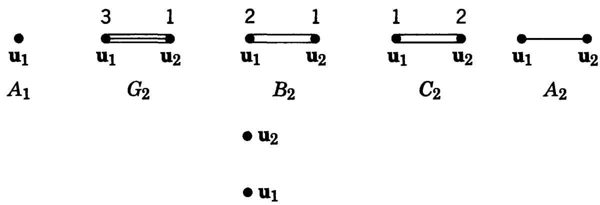
$$\mathbf{u}_i \cdot \mathbf{u}_j = -\sqrt{n/4} \qquad n = 0, 1, 2, 3 \tag{3.24'}$$

Since the entire algebra can be generated from the generators belonging to the simple roots, we should expect that root spaces can be completely classified by specifying the lengths and angles of the simple roots. This is true. An entire root space can be represented by the following procedure:

1. Represent each unit vector $\mathbf{u}_i$ by a dot $\cdot$;
2. Connect the dots for $\mathbf{u}_i$ and $\mathbf{u}_j$ by n lines (n is given by (3.24′));
3. Place the relative value of $\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_i$ above dot i.

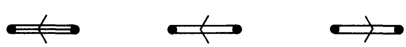
A diagram for a root space that is constructed according to these rules is called a **Dynkin diagram.**

Example. The Dynkin diagrams for the algebras of ranks 1 and 2 have been given in Figs. 8.2 to 8.5.



$$D_2 = A_1 \oplus A_1 \tag{3.26}$$

From these diagrams we can observe that B_2 and C_2 are equivalent under the exchange $\boldsymbol{\alpha}_1 \leftrightarrow \boldsymbol{\alpha}_2$. We also observe that the diagram for D_2 is disconnected and consists of two separate A_1 diagrams. Since the angle between vectors determines their relative lengths, it is not even necessary to place the value $\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_i$ over the ith dot. It is only necessary to specify which root is bigger by an arrow:



G_2 $\quad$ B_2 $\quad$ C_2 (3.26′)

All possible Dynkin diagrams can be classified, and the entire classification procedure is based on two observations so apparent that they hardly need explicit mention. Nevertheless for clarity we mention them.

Observation 1. The root space is positive definite. The inner product of any nonzero vector with itself is nonzero. For example,

$$\left(\sum_{i=1}^{n} \mathbf{u}_i, \sum_{j=1}^{n} \mathbf{u}_j\right) = \sum_{i=1}^{n} \mathbf{u}_i \cdot \mathbf{u}_i + 2\sum_{i>j} \mathbf{u}_i \cdot \mathbf{u}_j > 0 \tag{3.27}$$

Observation 2. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots,$ be an orthogonal system of unit vectors in the root space. Then $\mathbf{u} \cdot \mathbf{v}_i$ is the direction cosine of $\mathbf{u}$ with respect to $\mathbf{v}_i$.

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$$

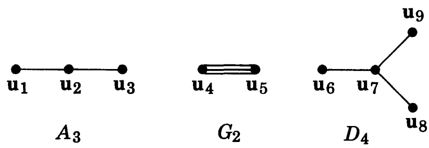
$$\sum (\mathbf{u} \cdot \mathbf{v}_i)^2 = \sum_{i=1} \cos^2 (\mathbf{u}, \mathbf{v}_i) \le 1 \tag{3.28}$$

When the equality holds, $\mathbf{u}$ can be expanded in terms of the $\mathbf{v}_i$. Then $\mathbf{u}$ is linearly dependent on the $\mathbf{v}_i$. The $\mathbf{v}_i$ and $\mathbf{u}$ cannot form a basis. When the inequality holds, $\mathbf{u}$ and the $\mathbf{v}_i$ can form (part of) a basis.

All properties of the Dynkin diagrams follow from (3.24) and Observations 1 and 2.

Property 1. If a diagram consists of several (dis)connected components, the corresponding root space and algebra are (semi)simple. Each connected component describes a simple algebra.

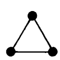
Example.



(3.29)

Proof. Every basis in one connected piece is orthogonal to every basis in every other piece. The bases span mutually orthogonal root subspaces, and the algebra consists of a number of commuting simple invariant subalgebras.

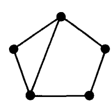
Property 2. There are no loops.

Example.  is not allowed.

Proof. If $\mathbf{u}_i$ and $\mathbf{u}_j$ (3.27) are connected at all, then $2\mathbf{u}_i \cdot \mathbf{u}_j \leq -1$. Equation (3.27) is then

$$n + 2 \sum_{1=i<j}^{n} \mathbf{u}_i \cdot \mathbf{u}_j > 0 \tag{3.30}$$

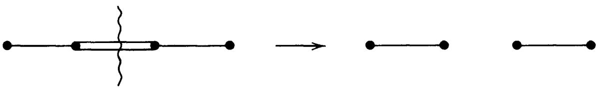
The number of connected pairs of roots must be less than n. A loop with n roots contains at least n lines:



(3.31)

From (3.30) it is also apparent that a diagram can contain no more than two double lines or one triple line.

Property 3. If the lines connecting any two $\mathbf{u}_i$ are severed, the resulting diagram is a Dynkin diagram.



(3.32)

Proof. Severing lines is equivalent to removing some of the root vectors from a root space. Those remaining span a subspace and generate a subalgebra.

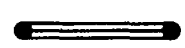
Property 4. The total number of lines connected to any vertex is at most 3.

Proof. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots,$ be connected to the vertex $\mathbf{u}$. Since there are no loops, the $\mathbf{v}_i$ are not connected to one another. They form an orthonormal system of unit vectors. By Observation 2 we can write

$$\sum_{i=1} (\mathbf{u} \cdot \mathbf{v}_i)^2 = \sum_{i=1} \frac{n_i}{4} < 1 \tag{3.33}$$

The number of lines $\sum n_i$ connected to $\mathbf{u}$ is < 4.

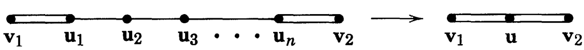
Comment 1. There is only one connected Dynkin diagram containing a triple line:



G_2 (3.34)

Property 5. Any set of dots $\mathbf{u}_i$ connected to one another by a simple chain can be shrunk to a single dot $\mathbf{u}$. The result is a diagram if the original one is.

Example.



(3.35)

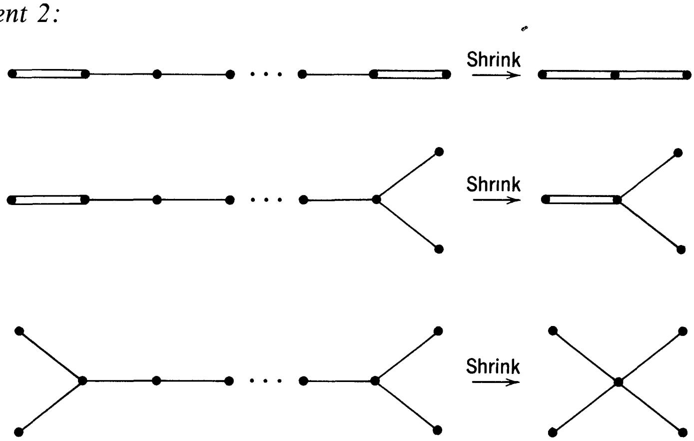
Proof. Let $\mathbf{u} = \sum_{i=1}^{n} \mathbf{u}_i$. Since the $\mathbf{u}_i$ are connected together by a simple chain,

$$\begin{aligned} 2\mathbf{u}_i \cdot \mathbf{u}_j &= 0 \qquad j > i+1 \\ 2\mathbf{u}_i \cdot \mathbf{u}_{i+1} &= -1 \end{aligned} \tag{3.36}$$

From (3.27) and (3.36), $\mathbf{u} \cdot \mathbf{u} = 1$. In addition,

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{u} &= \mathbf{v}_1 \cdot \sum_1^n \mathbf{u}_i = \mathbf{v}_1 \cdot \mathbf{u}_1 \\ \mathbf{v}_2 \cdot \mathbf{u} &= \mathbf{v}_2 \cdot \sum_1^n \mathbf{u}_i = \mathbf{v}_2 \cdot \mathbf{u}_n \end{aligned} \tag{3.37}$$

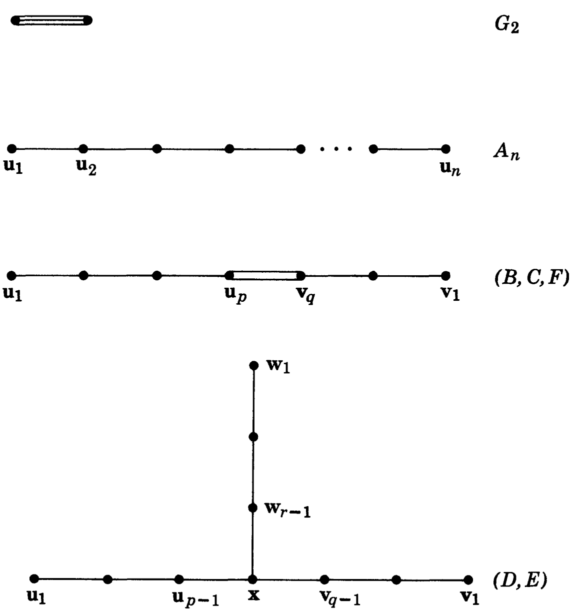
Comment 2:



(3.38)

Each of the shrunk diagrams violates Property 4. A diagram can contain at most one double line or split end.

Comment 3. The only remaining possibilities are:



(3.39)

Property 7. The only allowed values of $p \geq q$ in the diagram (B, C, F) of (3.39) are: p arbitrary, $q = 1$; and $p = q = 2$.

Proof. Put

$$\mathbf{u} = \sum_{i=1}^{p} i\mathbf{u}_i \qquad \mathbf{v} = \sum_{j=1}^{q} j\mathbf{v}_j \tag{3.40}$$

$$\mathbf{u}\cdot\mathbf{u} = \sum_{i=1}^{p}(i)^2 - \sum_{i=1}^{p-1}(i)(i+1) = \frac{1}{2}\,p(p+1)$$

$$\mathbf{v}\cdot\mathbf{v} = \qquad = \frac{1}{2}q(q+1)$$

$$\mathbf{u}\cdot\mathbf{v} = p\mathbf{u}_p\cdot q\mathbf{v}_q \qquad = pq\left(-\frac{1}{\sqrt{2}}\right) \tag{3.41}$$

From the Schwartz inequality we have

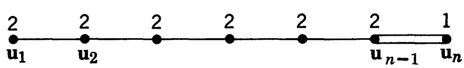
$$(\mathbf{u}\cdot\mathbf{v})^2 < (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v})$$

$$\frac{p^2q^2}{2} < \frac{1}{2}p(p+1)\frac{1}{2}q(q+1)$$

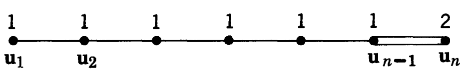
$$2 < \left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right) \tag{3.42}$$

The only solutions to this Diophantine equation are those stated.

Comment 4. The solution p arbitrary, $q = 1$ gives the two series of root spaces B_n, C_n:

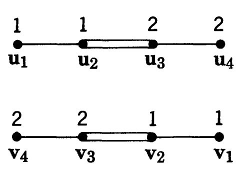


B_n



C_n (3.43)

The solution $p = q = 2$ also gives two diagrams that can be seen to be identical by relabeling the simple roots $\mathbf{u}_i \leftrightarrow \mathbf{v}_i$:



F_4

(3.44)

Property 8. The values of $p \geq q \geq r$ in (D, E) of (3.39) are: p arbitrary, $q = r = 2$; and the sets (3, 3, 2), (4, 3, 2), and (5, 3, 2).

Proof. Again define the vectors **u**, **v**, **w** as in (3.40). The direction cosine of **x** along **u** is (using $\mathbf{x} \cdot \mathbf{x} = 1$):

$$\cos^2(\mathbf{x}, \mathbf{u}) = \frac{(\mathbf{x} \cdot \mathbf{u})^2}{(\mathbf{u} \cdot \mathbf{u})} = \frac{\frac{1}{4}(p-1)^2}{\frac{1}{2}p(p-1)} = \frac{1}{2}\left(1 - \frac{1}{p}\right) \tag{3.45}$$

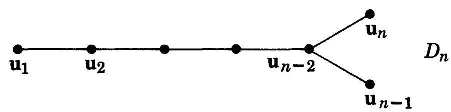
By Observation 2, the sum of the squares of the direction cosines of **x** along **u**, **v**, **w** is less than 1:

$$\frac{1}{2}\left(1 - \frac{1}{p} + 1 - \frac{1}{q} + 1 - \frac{1}{r}\right) < 1$$

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \tag{3.46}$$

The only solutions to this Diophantine equation are the ones stated.

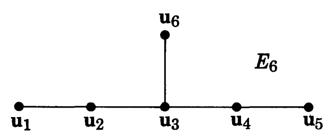
Comment 5. For the solution: p arbitrary, $q = r = 2$:



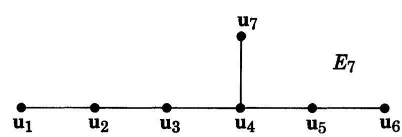
(3.47)

For the other three solutions we have:

p q r

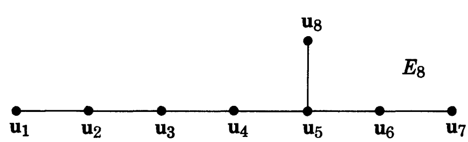


3 3 2 (Tetrahedral root space)



4 3 2 (Octahedral root space)

(3.48)



5 3 2 (Icosahedral root space)

These are exactly the exceptional algebras in the E series whose roots have already been determined by other means.

Comment 6. The nomenclature for these three root spaces has the following origin: to enumerate the regular Euclidean solids, let q be the number of regular p-gons that meet at any vertex. The sum of all angles at this vertex must be $< 2\pi$:

$$q\left(\pi - \frac{2\pi}{p}\right) < 2\pi$$

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2} \tag{3.49}$$

or

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{2} > 1 \tag{3.46$'$}$$

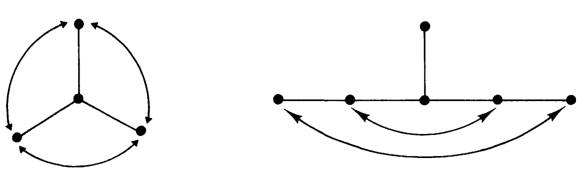
This Diophantine equation has solutions[5]

p	q	Regular Euclidean solids
3	3	Tetrahedron
3	4	Octahedron
4	3	Cube
3	5	Icosahedron
5	3	Pentagonal duodecahedron

(3.50)

These solutions come from essentially the same kind of equation that gives rise to the root spaces E_6, E_7, E_8. The cube and the octahedron are duals of each other (under $p \leftrightarrow q$) and are described by the same symmetry group. The icosahedron and the duodecahedron are similarly related. The tetrahedron is self-dual ($p = q$).

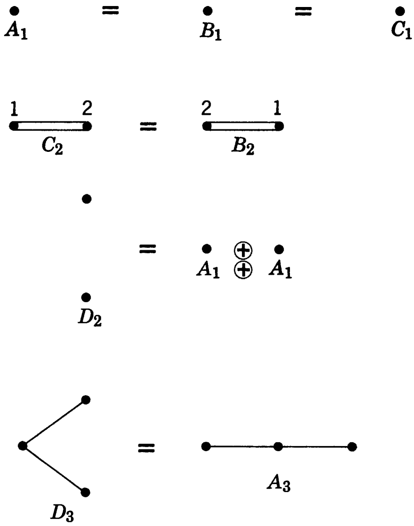
Comment 7. Two root spaces correspond to Dynkin diagrams with nontrivial symmetry:



D_4 $\qquad$ E_6 $\qquad$ (3.51)

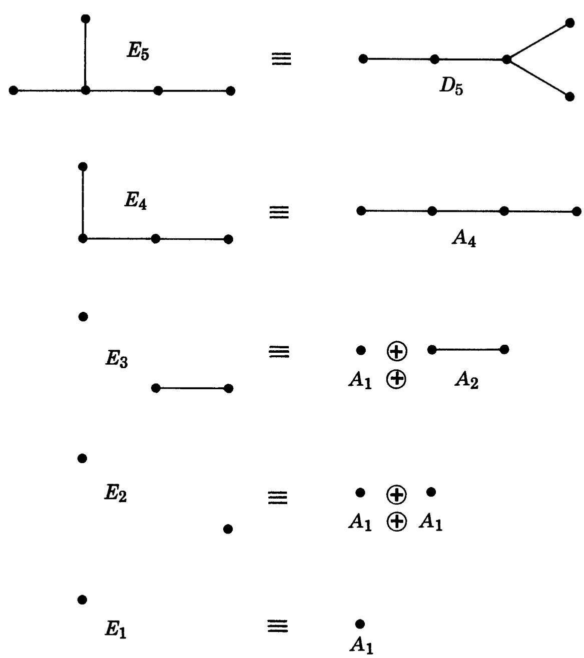
We would expect the algebras corresponding to these spaces to have some very unusual properties, arising from these symmetries, and not associated with the remaining members of their series. This is the case.

Comment 8. These diagrams make apparent at a glance the isomorphisms existing among the various root spaces:



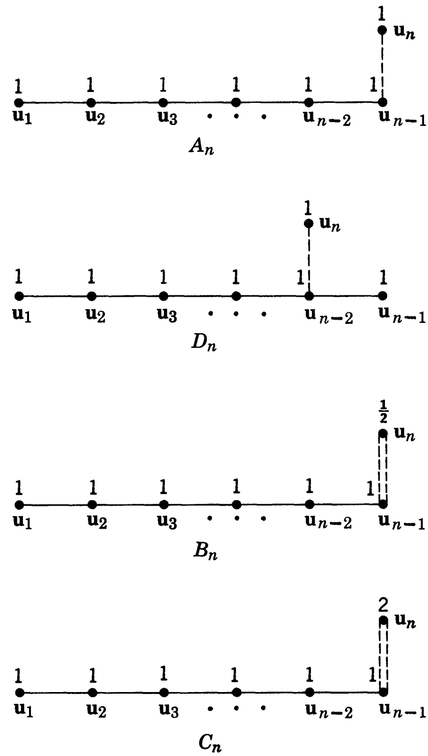
(3.52)

In addition, the isomorphisms between the lower members of the E series, which are all semisimple, are easily shown:



(3.53)

Comment 9. It has been observed that the simple bases for A_n, D_n, B_n, C_n differ only in their last basis (2.12)



(3.54)

This is made transparent by the diagrams.

Résumé

This chapter has completely listed and classified all possible simple Lie algebras by way of their root spaces. The procedure used in Section I is useful for describing all nonzero roots of the root spaces constructed. The procedure of Section III helps to demonstrate that the list in Table 8.2 is complete: no additional root spaces exist. In this sense, Section I and Section III are complementary.

Properties of the root spaces and in particular of the Dynkin diagrams were used to reveal the peculiarities of some of the algebras and the isomorphisms of others. Both the concept of root space and the Dynkin diagram can be extended to provide an economical mechanism for describing the representations of a simple algebra. The simple roots play a pivotal role in the explicit construction of the generators within any of the finite-dimensional representations.

Exercises

1. Compute dim [classical rank n algebra] and show

$$\frac{\dim [\text{alg}]}{\{\text{rank } [\text{alg}]\}^2} = 1 + \frac{2}{n} \xrightarrow{n \text{ large}} 1 \qquad A_n$$

$$2 - \frac{1}{n} \longrightarrow 2 \qquad D_n$$

$$2 + \frac{1}{n} \longrightarrow 2 \qquad B_n, C_n$$

2. Prove that although the classical root spaces B_2 and C_2 can be rotated into each other, B_3 and C_3 cannot be rotated into each other.

3. (a) Verify that the sets of roots of (1.8) satisfy all the properties required of the roots in a root space.

(b) Compute the Weyl group of reflections on each of these root spaces.

(c) Show that the Weyl group for A_{n-1} possesses $n!$ elements, that the Weyl groups for B_n and C_n possess $2^n n!$ elements each, and that for D_n the order of the Weyl group is $2^{n-1}n!$.

(d) Show by direct computation that the product of the orders of the functionally independent Casimir invariants of the groups and algebras A_{n-1}; D_n, B_n, C_n is equal to the order of the Weyl group of reflections in these root spaces. The functionally independent Casimir invariants were computed in Problem 6, Chapter 7. Note that $\phi_{2n}(X_{ij})$ is a perfect square for D_n. The independent nth invariant for D_n is its square root, of order n (Problem 6f).

4. Show that the two solutions labeled F_4 following (1.12) can be rotated into each other.

5. Construct the mapping E_5 (1.16′) into D_5 explicitly. Construct the isometry E_4 (1.21) into A_4 explicitly.

6. Show that if V_1, V_2, ..., V_k, V_{k+1} are generators corresponding to positive vectors ($V_i \in \mathfrak{g}^+$) in a semisimple Lie algebra of rank n, then

$$[V_1, [V_2, [\ldots[V_k, V_{k+1}]\ldots]]] = 0$$

where k is the level of the "highest root."

7. Compute $\frac{1}{2}\sum_{\alpha>0}\alpha$ for each of the classical root spaces.

8. Project the roots of the root space G_2 onto the vector $\mathbf{R}_\perp$ (Fig. 8.10) and show that the projections have relative components

$$-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$$
$$-1, 0, 1$$

Conclude[6,7] that the algebra $\mathfrak{g}_2$ contains $\mathfrak{a}_1$ as a subalgebra and that under the subalgebra restriction $\mathfrak{g}_2 \downarrow \mathfrak{a}_1$, the regular representation of $\mathfrak{g}_2$ is reducible to the direct sum of representations of $\mathfrak{a}_1$ [or $\mathfrak{su}(2)$] with j values $j = 5, j = 1$.

9. Use (3.9) and (3.10) to indicate how BCH formulas (Chapter 5, Section VI) can be used to construct representations of semisimple Lie groups by EXPonentiating the corresponding Lie algebra representation, and having the EXP expansion terminate after a finite number of terms.

10. Prove that the factor algebra $\mathfrak{g}^{+k}$ mod $\mathfrak{g}^{+(k+1)}$ of (3.20) is abelian.

11. Property 2 (3.30) is more precisely stated as follows: there are no loops *in a connected diagram.* Show that only trivial modifications are necessary in the proof as stated, so that the proof holds for Property 2 essentially as stated.

12. Let Σ_n be the root space for a semisimple algebra of rank n. Consider a Cartan decomposition of this root space

$$\Sigma_n = \Sigma_n^+ + \Sigma_n^0 + \Sigma_n^-$$

(a) Show that the subalgebra corresponding to the root subspace $\Sigma_n^+ + \Sigma_n^0$ is solvable and has rank n.

(b) Show that the subalgebra corresponding to the root subspace Σ_n^+ is solvable and has rank 0 (nilpotent).

13. Show that the subset of roots in E_8 (1.25) orthogonal to the vector

$$\mathbf{R} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8$$

is isometric to the root space E_7 (1.24).

Notes and References

The material in this chapter closely parallels the corresponding material in References 1 and 2. The material in Section I follows Reference 3, whereas Section III was first presented in Reference 4.

1. N. Jacobson. [2]
2. D. Kleima, W. J. Holman, L. C. Biedenharn. [1]
3. B. L. van der Waerden. [3]
4. E. B. Dynkin. [1]
5. M. Hamermesh. [1], p. 49.
6. G. Racah. [1,2]
7. B. R. Judd. [1], Fig. 2, p. 198.

CHAPTER 9

Real Forms

We return now to a question introduced in Chapter 7. In order to determine the structure of the Lie algebras in terms of eigenvalue subspaces, we found it necessary to extend our algebraic field from the real to the complex numbers, since the reals are not algebraically closed and the complex numbers are. Under these extended circumstances, the entire classification scheme carried out in Chapter 8 has been for *complex* semisimple Lie algebras. Since every complex semisimple algebra generally has more than two real forms with the same complex extension, we have not yet answered the following questions. What are the real simple Lie algebras? How are they constructed? How are they characterized?

In the first section we develop all the algebraic machinery necessary for the real classification scheme. This involves nothing more than a study of the mappings of the compact Lie algebra into itself under mappings which obey $\sigma^2 = \text{Id}$. These are the so-called involutive automorphisms. Such mappings of a Lie algebra can be determined directly from the mappings of the associated Dynkin diagrams onto themselves.

Since the involutive automorphisms $\sigma^2 = \text{Id}$ have eigenvalues ± 1, the compact algebra $\mathfrak{g}$ decomposes naturally into two subspaces $\mathfrak{k}$, of eigenvalue $+1$, and $\mathfrak{p}$, of eigenvalue -1:

$$\begin{array}{ll} \mathfrak{g} \to \mathfrak{k} \oplus \mathfrak{p} & [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \\ \quad (\mathfrak{k}, \mathfrak{p}) = 0 & [\mathfrak{k}, \mathfrak{p}] = \mathfrak{p} \\ & [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \end{array}$$

where $\mathfrak{k}$ is called a maximal compact subalgebra and $\mathfrak{p}$ an orthogonal complementary subspace. By applying the Weyl unitary trick $\mathfrak{p} \to i\mathfrak{p}$, the algebra $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$ is closed under commutation and noncompact. All noncompact algebras $\mathfrak{g}^*$, which have the same complex root space as $\mathfrak{g}$, are obtained in this way.

The study of all involutive automorphisms σ (or, equivalently, all maximal subalgebras $\mathfrak{k}$ of $\mathfrak{g}$) leads to a completeness classification of the real forms of the simple Lie algebras. Tables of the real forms are presented in Section II, and in Section III a mechanism is introduced for distinguishing between the various real forms of a simple complex Lie algebra.

Now the fun begins. The automorphisms σ play a fundamental role in the Lie algebra decomposition just mentioned. Not only is the maximal subalgebra $\mathfrak{k}$ important, but so also are the complementary subspaces $\mathfrak{p}$, $i\mathfrak{p}$. For the spaces of points EXP $\mathfrak{p}$ are coset representatives that are endowed with a metric structure and in fact are globally symmetric Riemannian irreducible spaces.

These spaces can be studied by exactly the same techniques used for the study of Lie algebras. In Section IV we show how the familiar properties of Lie algebras can be transferred to these spaces: inner product, index, secular equation, eigenvalue decomposition, and rank.

In Section V we study these spaces in their own right, investigating in particular their analytic properties. This study rapidly leads to a wealth of information, not only about the spaces themselves, but also about the parent group. The distance and measure functions on these coset spaces lead directly to distance and measure functions on the parent group. The invariant measure on the classical matrix groups is computed both explicitly *and simply*; the volumes of the classical compact groups are also presented. These results, obtainable through a rank-1 coset decomposition, suggest a semicanonical way for describing the finite *group elements.* The triangular patterns introduced are dual to the Gel'fand-Tsetlein patterns used to describe *matrix elements* in the representations of these groups.

Not only can the symmetric coset spaces be studied using the techniques available (Chapters 7 and 8) for Lie algebras; they can be classified using these techniques (this chapter, first three sections) as well. Section VI is devoted to a study and classification of all the real forms of the (pseudo-) Riemannian globally symmetric coset spaces.

I. Algebraic Machinery

1. AUTOMORPHISMS. An **automorphism**[1,2] of an algebraic structure is a 1-1 mapping of the structure *onto* itself, which preserves all the defined algebraic operations. If S, T, ..., are automorphisms and x, y, ..., are elements in some algebraic structure in which an operation $\square$ is defined, then

$$\begin{aligned} S(x \square y) &= S(x) \square S(y) \\ TS(x \square y) &= T\{S(x) \square S(y)\} \\ &= TS(x) \square TS(y) \end{aligned} \tag{1.1}$$

The mapping of the algebraic element $S(x)$ back onto the original element x

$$S(x) \xrightarrow{S^{-1}} x \tag{1.2}$$

is the inverse of S, denoted S^{-1}. It is easily verified that the collection of automorphisms aut obeys the following properties:

$$\begin{array}{ll} S \in \text{aut},\ T \in \text{aut} \Rightarrow TS \in \text{aut} & \text{closure} \\ U(TS) = (UT)S & \text{associativity} \\ E(x) = x \Rightarrow E \in \text{aut} & \text{identity} \\ S \in \text{aut} \Rightarrow S^{-1} \in \text{aut} & \text{inverse} \end{array} \tag{1.3}$$

The collection of all automorphisms of an algebraic structure therefore forms a group, denoted "aut."

Example. The automorphisms of the additive group of real numbers obey

$$S(x + y) = S(x) + S(y) \qquad \begin{array}{r} x, y \in R \\ S(x), S(y) \in R \end{array} \tag{1.4}$$

It is clear that these automorphisms involve multiplication by a real number $\lambda(\neq 0)$:

$$S_\lambda(x) = \lambda x$$

It often happens that an algebraic structure is mapped *onto* itself under its own operations. Thus, for example, every element in a semisimple Lie algebra can be written as the commutator of two elements in the algebra:

$$[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a} \tag{1.5}$$

The equality holds for simple and semisimple Lie algebras.

The mappings of an algebraic structure onto itself induced by the structure itself, obey the properties of closure, associativity, identity, and inverse [see (1.3)] and therefore form a group also. This group is denoted "int," for it depends on (an in fact is isomorphic with) the internal structure of the system. It is also called the group of **inner automorphisms**, as opposed to aut, the group of **outer automorphisms**.

It is clear that int is a closed subgroup of aut. In fact, int is an invariant subgroup of aut. For if $g \in \text{aut}$, and if $a \in \text{int}$, then

$$gag^{-1} \in \text{algebra} = \text{int}$$

Thus we can write

$$(\text{aut})(\text{int} = \text{algebraic system})(\text{aut})^{-1} = (\text{int})$$

The space of cosets (aut)/(int) therefore has a group structure

$$\frac{\text{aut}}{\text{int}} = \text{factor group} \tag{1.6}$$

If $\mathfrak{g}$ is any simple Lie algebra, then we will show that the factor group

$$\frac{\text{aut}(\mathfrak{g})}{\text{int}(\mathfrak{g})} = \text{factor group} \tag{1.6G}$$

is discrete. Not only is it discrete, but it is finite as well, and in no case does it contain more than 3! elements. Furthermore, aut $(\mathfrak{g})$/int $(\mathfrak{g})$ can be determined directly by reference to the appropriate Dynkin diagram.

Let G be a simple or semisimple Lie group. We will show that int (G) is identical with the component of the group aut (G) which is connected to the identity:

$$\text{aut}_0\,(G) = \text{int}\,(G) \tag{1.7}$$

We do this by computing the Lie algebra of aut$_0$ (G) and showing it to be identical with the Lie algebra of G. Since int $(G) = G$, the result will be established.

All automorphisms of G are conjugations

$$x \to x' = dxd^{-1} \qquad x, x' \in G$$
$$d \in \text{aut}\,(G)$$

If we restrict ourselves to the component aut$_0$ (G) containing the identity, it is possible to determine the Lie algebra of aut$_0$ (G), and its effect on the Lie algebra of G. For operations d, x near the respective identities of aut$_0$ (G) and G, we have

$$\begin{aligned} d &= e^{\delta D} \simeq I + \delta D; && d \in \text{aut}_0\,(G), && D \in \text{aut}\,(G) \\ x &= e^{X} \simeq I + X; && x \in G, && X \in \mathfrak{g} \\ dxd^{-1} &\to e^{\delta D} e^{X} e^{-\delta D} \simeq I + X + \delta[D, X] \end{aligned} \tag{1.8}$$

The commutator $[D, X]$ represents a mapping of the Lie algebra onto itself. If X_i form a basis for the Lie algebra of G, then

$$[D, X_i] = \gamma_i{}^j(D) X_j \tag{1.9}$$

It is convenient to introduce a matrix representation for the Lie algebra of G at this point. If the representation is faithful, every calculation done within the representation is valid also for the abstract algebra. In particular, we will use mostly the regular representation, which is known to be faithful for semisimple algebras:

$$X_i \to \mathbf{R}(X_i)$$

Then the representative for the action of D on the Lie algebra of G is given by

$$\mathbf{R}([D, X_i]) = -\mathbf{R}_i{}^j(D)\mathbf{R}(X_j) \tag{1.10}$$

Since the automorphism group does not change the group multiplication properties of G, the structure constants for the group are left invariant. All properties of the group depending on the structure constants are also left invariant: in particular, the Cartan-Killing metric tensor remains invariant. For if X, Y are elements in the algebra, then

$$\begin{aligned} X \to X' &= X + \delta[D, X] \\ Y \to Y' &= Y + \delta[D, Y] \\ (X', Y') &= (X + \delta[D, X], Y + \delta[D, Y]) \\ &\simeq (X, Y) + \delta\{([D, X], Y) + (X, [D, Y])\} \end{aligned} \tag{1.11}$$

We have already seen [Chapter 7, (3.7′)] that for D in the Lie algebra of G, the combination within the brackets must vanish. This term must also vanish for every possible derivation D.

To show that D actually does have the properties of a derivation, we carry out a simple calculation within the faithful regular representation.

$$\begin{aligned} \mathbf{R}([D, [X, Y]]) &= [\mathbf{R}(D), [\mathbf{R}(X), \mathbf{R}(Y)]] \\ &= -[\mathbf{R}(Y), [\mathbf{R}(D), \mathbf{R}(X)]] - [\mathbf{R}(X), [\mathbf{R}(Y), \mathbf{R}(D)]] \\ &= [[\mathbf{R}(D), \mathbf{R}(X)], \mathbf{R}(Y)] + [\mathbf{R}(X), [\mathbf{R}(D), \mathbf{R}(Y)]] \\ &= \mathbf{R}([[D, X], Y] + [X, [D, Y]]) \end{aligned} \tag{1.12}$$

Since the regular representation is faithful, we have the result

$$[D, [X, Y]] = [[D, X], Y] + [X, [D, Y]] \tag{1.13}$$

Therefore, D is a derivation.

Comment. The foregoing calculation uses the Jacobi identity, which is valid for an arbitrary associative or Lie algebra. We cannot use the Jacobi identity directly on the commutator $[D, [X, Y]]$, since D is not necessarily in the Lie algebra of G. Therefore, we had to proceed by introducing the regular matrix representation. Introducing this associative matrix algebra, we can use the Jacobi identity with impunity.

We now show that, for any element D in the Lie algebra of $\text{aut}_0\,(G)$, we have

$$\mathbf{R}(D) = \mathbf{R}(K)$$

for some element K in the Lie algebra of G. We can then conclude that

$$\mathfrak{aut}_0\ (G) = \mathfrak{g}.$$

To show this, we construct the function

$$\Phi'(X) = \operatorname{tr} \mathbf{R}(D)\mathbf{R}(X) \qquad X \in \mathfrak{g} \tag{1.14$'$}$$

This function of X is linear in X. The function

$$\Phi(X) = \operatorname{tr} \mathbf{R}(K)\mathbf{R}(X) \tag{1.14}$$

is also linear in X. There is then a uniquely defined K in the Lie algebra of G with the property

$$\Phi(X) = \Phi'(X) \qquad \text{all} \quad X \in \mathfrak{g}$$

$$\operatorname{tr} \mathbf{R}(K)\mathbf{R}(X) = \Phi(X) = \Phi'(X) = \operatorname{tr} \mathbf{R}(D)\mathbf{R}(X) \tag{1.15}$$

We now define the operator

$$E = D - K \in \mathfrak{aut}_0\ (G)$$
$$\mathbf{R}(E) = \mathbf{R}(D) - \mathbf{R}(K) = 0$$

We show that E annihilates every vector Y in the Lie algebra of G.

$$\begin{aligned}([E, Y], Z) &= \operatorname{tr} \mathbf{R}(\mathbf{R}(E)Y)\mathbf{R}(Z)\\ &= \operatorname{tr} [\mathbf{R}(E), \mathbf{R}(Y)]\mathbf{R}(Z)\\ &= \operatorname{tr} \mathbf{R}(E)[\mathbf{R}(Y), \mathbf{R}(Z)]\\ &= 0\end{aligned} \tag{1.16}$$

The inner product vanishes because $\mathbf{R}(E) = 0$. This holds for an arbitrary vector Z, and thus we must have

$$[E, Y] = 0$$

since the Cartan-Killing metric for a semisimple algebra is nonsingular. In short, the transformation E must itself be the zero transformation:

$$\mathbf{R}(D) - \mathbf{R}(K) = 0 \Leftrightarrow D - K = 0 \tag{1.17}$$

The faithful representation is faithful not only on $\mathfrak{g}$, but on $\mathfrak{aut}_0\ (G)$ as well. Moreover, for every $D \in \mathfrak{aut}_0\ (G)$ there is a corresponding element $K \in \mathfrak{g}$. We have therefore shown that for semisimple $\mathfrak{g}$:

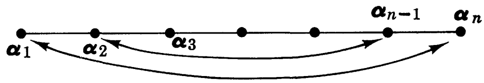
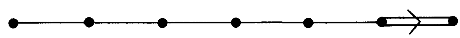
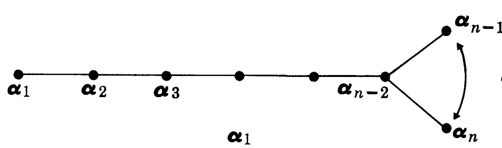
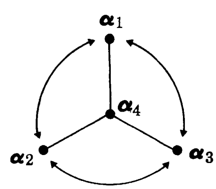
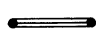
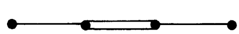
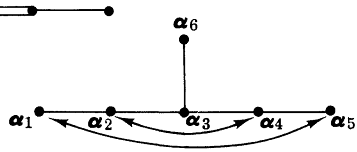
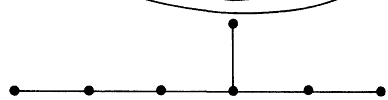
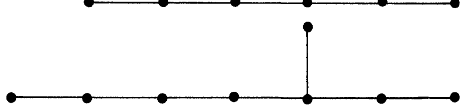
$$\mathfrak{aut}_0\ (G) = \mathfrak{g}$$
$$\mathrm{aut}_0\ (G) = G$$

Since the connected component of aut (G) is identical to G, we have

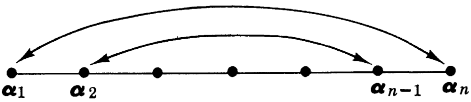
$$\frac{\mathrm{aut}\ (G)}{\mathrm{int}\ (G) = G} = \text{factor group}$$

and the factor group is discrete. The factor group preserves all commutation relations and all properties dependent on them. In particular, it preserves the metric properties of the semisimple algebras. All metric properties of the semisimple algebras are summarized by the associated Dynkin diagram. Thus the factor group for any simple algebra is exactly the finite group of mappings of the Dynkin diagram into itself, which preserves all inner products. These mappings are described below and collected in Table 9.1.

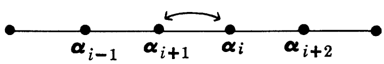
TABLE 9.1

Root System	Dynkin Diagram	Symmetry	Group
A_n		$\alpha_i \longleftrightarrow \alpha_{n+1-i}$	P_2
B_n			P_1
C_n			P_1
D_n		$\alpha_{n-1} \longleftrightarrow \alpha_n$	P_2
D_4		(e), (12), (23), (31) (123), (321)	P_3
G_2			P_1
F_4			P_1
E_6		$\alpha_i \longleftrightarrow \alpha_{6-i}$ $\alpha_6 \longleftrightarrow \alpha_6$	P_2
E_7			P_1
E_8			P_1

A_n. The only metric-preserving mapping, in addition to the identity, is the map $\alpha_i \leftrightarrow \alpha_{n+1-i}$:



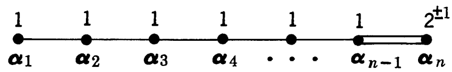
Transformations of the form $\alpha_i \leftrightarrow \alpha_j$



do not preserve inner products in general, for

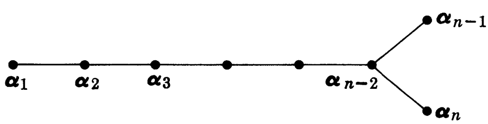
$$\cos(\alpha_{i-1}, \alpha_i) = -\tfrac{1}{2}$$
$$(\alpha'_{i-1}, \alpha'_i) = 0$$

B_n, C_n:

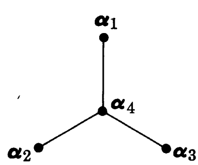


Any permutation of the simple roots will alter the inner product structure represented by the diagram. For these groups, the inner and outer automorphisms are identical.

D_n:



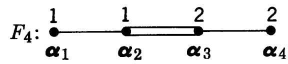
Interchange of the roots $\alpha_{n-1} \leftrightarrow \alpha_n$ does not change the metric properties represented by this diagram. The factor group is isomorphic with P_2 for D_n, $n > 4$. But for $n = 4$ the Dynkin diagram can be written:



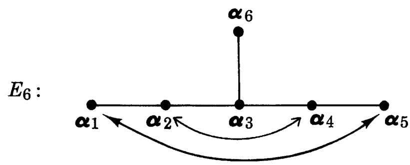
Any permutation of the three fundamental roots α_1, α_2, α_3 leaves invariant the metric properties represented by this diagram. Therefore, in this case the factor group is isomorphic with P_3.

G_2:

The only possible exchange here is $\alpha_1 \leftrightarrow \alpha_2$, which is an interchange of the long and short roots, and the metric structure as well.



There are no symmetries.



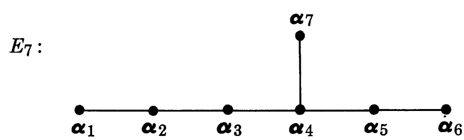
The only symmetries are:

$$\alpha_1 \leftrightarrow \alpha_5$$

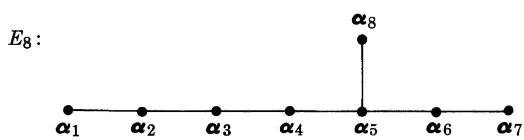
$$\alpha_2 \leftrightarrow \alpha_4$$

$$\alpha_3 \leftrightarrow \alpha_3 \qquad\qquad \alpha_6 \leftrightarrow \alpha_6$$

The factor group is isomorphic with P_2.



There are no discrete metric-preserving symmetries here.



There are no discrete metric-preserving symmetries here, either.

2. SOME RELATIONS BETWEEN ALGEBRAIC AND TOPOLOGICAL PROPERTIES. The concept of a Lie group involves a very close relationship between the topological properties of the group's underlying point space, on the one hand, and the group's algebraic properties, on the other. So close is this connection that it is possible to deduce some algebraic properties from the "corresponding" topological properties, and vice versa. In this section we state two such relationships; many additional ones are called on in discussing the theory of the special functions of mathematical physics.[3] In particular, we eventually prove the following theorem.

THEOREM. *A simple Lie group is compact if and only if the Cartan-Killing metric on its Lie algebra is negative definite.*

Before constructing a proof of this fundamental theorem, we discuss a number of examples.

Example 1. Suppose that the Cartan-Killing form on a Lie algebra is negative definite. Then we show that the algebra can be compact. The standard inner product for the bases H_i, E_α for a complex Lie algebra of rank n is given by

$$\begin{aligned} (H_i, H_j) &= \delta_{ij} \\ (E_\alpha, E_\beta) &= \delta_{\alpha+\beta,\,0} \end{aligned} \tag{1.18}$$

If the metric for a real form is to be negative definite, all the coordinates of the subalgebra spanned by the H_i must be purely imaginary

$$\begin{aligned} & ir^jH_j \qquad r^j \text{ real; no } j \text{ sum} \\ (ir^jH_j, ir^kH_k) &= -r^jr^k\,\delta_{jk} \end{aligned} \tag{1.19}$$

Then, since the regular matrix representatives of H_i are diagonal, of the form

$$H_i \xrightarrow[\text{representation}]{\text{regular}} \left[\begin{array}{c|ccc} \bigcirc & & & \\ \hline & \alpha_i & & \\ & & \beta_i & \\ & & & \gamma_i \end{array}\right] \begin{array}{l} H_1 \\ \vdots \\ H_n \\ E_\alpha \\ E_\beta \\ E_\gamma \\ \vdots \end{array} \tag{1.20a}$$

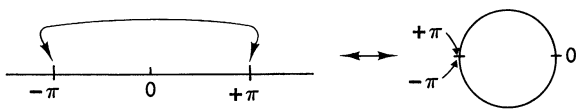
the subgroup generated by this subalgebra has diagonal matrix representatives:

$$e^{t(ir^JH_J)} \xrightarrow{\text{EXP}} \left[\begin{array}{c|ccc} I_n & & & \\ \hline & e^{itr^J\alpha_J} & & \\ & & e^{itr^J\beta_J} & \\ & & & \ddots \end{array}\right] \tag{1.20g}$$

If $\mathfrak{g}$ is the rank-1 simple algebra, (1.20g) reduces to

$$e^{t(iL_z)} \to \begin{array}{c} \left[\begin{array}{c|cc} 1 & & \\ \hline & e^{it} & \\ & & e^{-it} \end{array}\right] \end{array} \begin{array}{l} L_z \\ L_+ \\ L_- \end{array}$$

This matrix is clearly periodic (period 2π). The parameter space for this one-parameter subgroup is therefore a bounded interval $-\pi \leq t \leq +\pi$ on the straight line, with endpoints identified as follows:



That is, the parameter space is topologically equivalent to a circle S^1.

If the algebra is of rank 2, then the only nontrivial elements of the maximal abelian subgroup EXP $ti\lambda^j H_j$ have the form

$$e^{it\lambda^j H_j} \to e^{it\lambda^j \alpha_j} \tag{1.21}$$

All the α_j can be chosen to be integral ($\alpha_j = \pm 2, \pm 1, 0$ by construction of the root space diagrams for simple algebras). A typical one-dimensional subgroup appears in Fig. 9.1. If the λ^j are chosen so that all but one are 0, then the one-parameter subgroup is recursive, as before. The entire parameter space consists of a bounded line segment that is 2π long.

For arbitrary direction cosines $(\lambda^1, \lambda^2, \ldots)$, the one-parameter subgroup is described by a straight line $(t\lambda^1, t\lambda^2, \ldots)$ through the origin. Since the exponential function is periodic, each straight line segment in a square centered around (m, n) can be rigidly translated back to the equivalent line segment in the fundamental domain.[4] If the ratio λ^1/λ^2 is rational, the straight line will eventually pass through the point (p, q)

$$\left(\frac{p}{q} = \frac{\lambda^1}{\lambda^2}\right)$$

where p and q are integers. Beyond this point, the line segments in the fundamental domain begin repeating themselves.

When λ^1/λ^2 is irrational, no line segment falls on top of any other. In fact, the line segments "fill up" the fundamental domain, in the following sense.[5] Every infinite sequence of points on the line, when translated back into the

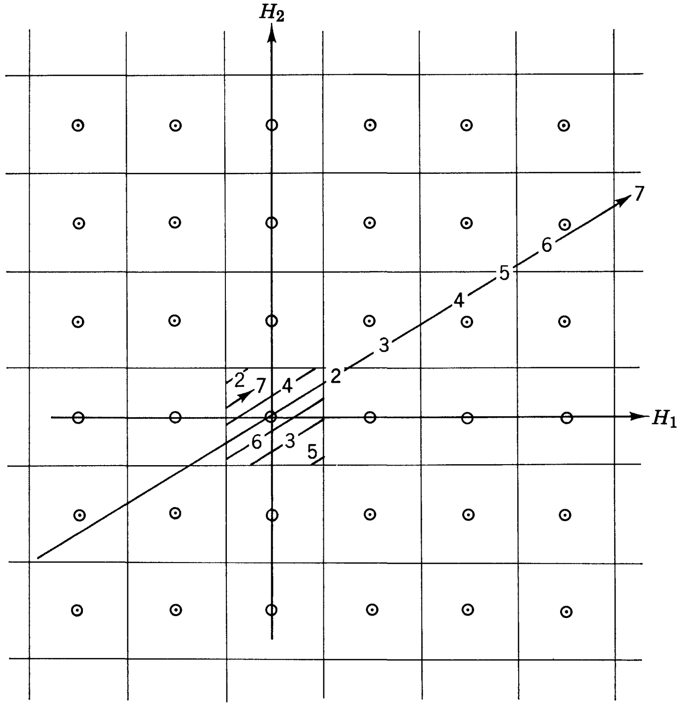


FIG. 9.1 AN ARBITRARY ONE-PARAMETER SUBGROUP exp $it\lambda^j\alpha_j$ IS DESCRIBED BY A STRAIGHT LINE THROUGH THE ORIGIN, WITH DIRECTION COSINES $(\lambda^1, \lambda^2, \ldots)$. BY THE PERIODICITY OF THE EXPONENTIAL FUNCTION, EVERY LINE SEGMENT WITHIN A SQUARE WHOSE CENTER IS AT (m, n) CAN BE RIGIDILY TRANSLATED TO A LINE SEGMENT WITHIN THE FUNDAMENTAL DOMAIN [SQUARE CENTERED AROUND $(0, 0)$]. WE HAVE ASSUMED HERE THAT $\alpha_1 = \alpha_2 = 1, \alpha_3 = \alpha_4 = \cdots = 0$.

fundamental domain, contains a subsequence that converges to some point within the fundamental domain or its boundary. The opposite is also true: every point in the fundamental domain is the limit point for such a sequence, regardless of whether that point lies on the line. Many points do not lie on any line segment. For example, it is easy to see that the points (x, x) lie on no such segment for x rational. Such a curve is said to be **dense**[5] in the space.

Points along the opposite boundaries of the fundamental domain are identical and must be identified. The domain is in fact topologically equivalent to a torus (Fig. 9.2). The line $(t\lambda^1, t\lambda^2)$ $(\lambda^1/\lambda^2$ irrational$)$ is usually called a **skew line on a torus**. This is a very popular curve among mathematicians—as a general rule of thumb, false conjectures fail first and fastest on this curve.

If the metric on the Lie algebra is not negative definite, then it is possible to find a coordinate system in which the coordinate of one of the diagonal

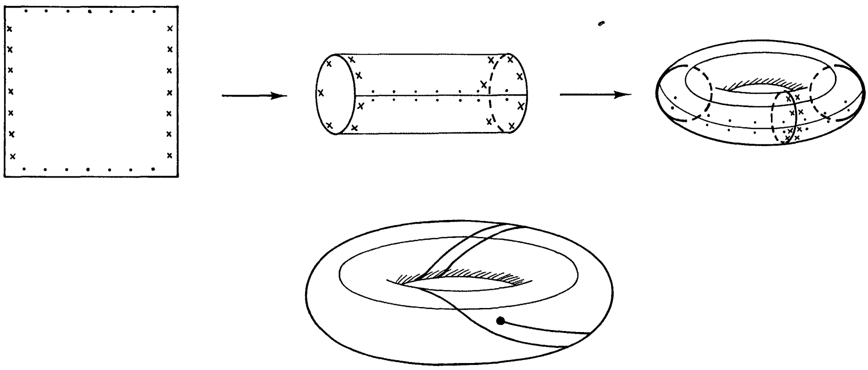


FIG. 9.2 POINTS ALONG OPPOSITE EDGES OF THE FUNDAMENTAL DOMAIN MUST BE IDENTIFIED BECAUSE THEY REPRESENT IDENTICAL GROUP OPERATIONS. UNDER THIS IDENTIFICATION THE DOMAIN IS GLOBALLY TOPOLOGICALLY EQUIVALENT TO A TORUS $S^1 \otimes S^1$. THE LINE $(\lambda^1 t, \lambda^2 t)$ ON THE FUNDAMENTAL DOMAIN IS IDENTICAL WITH THE SKEW LINE ON A TORUS WHEN λ^1/λ^2 IS IRRATIONAL.

operators, H_1, say, is real. Then in the regular representation

$$\mathrm{EXP}\; tH_1 \xrightarrow[\text{representation}]{\text{regular}} \left[\begin{array}{c|ccc} I_n & & & \\ \hline & e^{t\alpha_1} & & \\ & & e^{t\beta_1} & \\ & & & \ddots \end{array}\right] \tag{1.22}$$

with t, α, β, ..., all real. This matrix is nonrecursive. The one-parameter subgroup is isomorphic with the multiplicative group of real numbers. Since a simple Lie group with a noncompact (closed) subgroup must itself be noncompact, we see that a Lie algebra with a Cartan-Killing form that is not negative definite must exponentiate onto a noncompact group.

Example 2. The orthogonal group $SO(n)$ is compact, since the matrix elements $O_i{}^j$ obey the orthogonality relations

$$\sum_k O_i{}^k O_j{}^k = \sum_k O_k{}^i O_k{}^j = \delta_{ij}$$

$$0 \leq |O_i{}^k| \leq 1 \tag{1.23}$$

Since the domain of the $O_i{}^j$ is bounded, every infinite sequence of points has a limit point in the group.

To show that $SO(n)$ has a negative definite Cartan-Killing metric, we explicitly compute the metric. Instead of computing the inner product using the unwieldly regular representation, we recall that, if $\mathbf{D}(X)$ is a faithful representation of a semisimple algebra, then by (3.21) of Chapter 7, we can write

$$\frac{\operatorname{tr} \mathbf{D}(X)\mathbf{D}(Y)}{f(\mathbf{D})} = \frac{\operatorname{tr} \mathbf{R}(X)\mathbf{R}(Y)}{f(\mathbf{R})} \tag{1.24}$$

Once the constant of proportionality $f(\mathbf{R})/f(\mathbf{D})$ has been determined, all calculations may be carried out within the representation $\mathbf{D}$. Choosing $\mathbf{D}$ as the defining $n \times n$ matrix representation, it is not difficult to show that

$$\frac{(X, Y)_{\mathbf{D}}}{2} = \frac{(X, Y)_{\mathbf{R}}}{2(n-2)} \tag{1.25}$$

Within the defining representation $\mathbf{D}$, the infinitesimal generators are

$$\mathbf{D}(X_{i<j}) = \begin{array}{c c} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ \\ j \\ \\ \end{array} & \left[\begin{array}{ccccc} & & & & \\ & & & +1 & \\ & & & & \\ & & & & \\ & -1 & & & \\ & & & & \end{array}\right] \end{array}$$

It is a simple matter to calculate the inner product now.

$$(X_{ij}, X_{rs})_{\mathbf{D}} = \operatorname{tr} \mathbf{D}(X_{ij})\mathbf{D}(X_{rs}) = -2\delta_{ir}\,\delta_{js} \tag{1.26}$$

where $i < j; r < s$. The Cartan-Killing form is diagonal and negative definite.

We turn now to the noncompact real form $SO(p, q)$ related to the compact group $SO(p+q)$ by the Weyl unitary trick. Its generators have the structure

$$\begin{array}{ccccc} \mathbf{D}(X_{ij}) & & \mathbf{D}(X_{\alpha\beta}) & & \mathbf{D}(X_{i\beta}) \\ \left[\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & \\ & & \end{array}\right] & \begin{array}{c} \uparrow \\ p \\ \downarrow \\ \uparrow \\ q \\ \downarrow \end{array} & \left[\begin{array}{c|cc} & & \\ & & \\ \hline & & 1 \\ & -1 & \end{array}\right] & & \left[\begin{array}{c|c} & 1 \\ \hline 1 & \end{array}\right] \\ 1 \le i < j \le p & & p+1 \le \alpha < \beta \le p+q & & \end{array} \tag{1.27}$$

The inner product is again diagonal, with

$$\begin{aligned}(X_{ij}, X_{ij})_{\mathbf{D}} &= (X_{\alpha\beta}, X_{\alpha\beta})_{\mathbf{D}} = -2 \\ (X_{i\beta}, X_{i\beta})_{\mathbf{D}} &= +2\end{aligned} \tag{1.28}$$

Since the metric tensor is real and symmetric, bases can always be chosen in such a way that the metric has a canonical form, with ± 1 on the diagonal. In this case, such a choice simply involves a straightforward renormalization by $\sqrt{2}$ in the defining representation, and $\sqrt{2(n-2)}$ in the regular representation.

The trace of the canonical diagonal metric is then characteristic of the particular real form of a complex semisimple Lie algebra. This integer is called the **character** of the real form.

$$\begin{aligned}\text{Character} &= \begin{pmatrix}\text{number of}\\ \text{noncompact}\\ \text{generators}\end{pmatrix} - \begin{pmatrix}\text{number of}\\ \text{compact}\\ \text{generators}\end{pmatrix} \\ &= \begin{pmatrix}\text{dimension}\\ \text{of Lie}\\ \text{algebra}\end{pmatrix} - 2\begin{pmatrix}\text{dimension of}\\ \text{maximal compact}\\ \text{subalgebra}\end{pmatrix}\end{aligned} \tag{1.29}$$

In Table 9.2 we list the character of all the real inequivalent Lie groups $SO(p, q)$ related to the compact group $SO(p+q)$, for $p + q = 6$. Note that $SO(p, q) \simeq SO(q, p)$. We observe from the table that the character ranges

TABLE 9.2

a Real Form	*b* Maximal Compact Subgroup	*c* Dim D_3	*d* Dimension of the Maximum Compact Subgroup	*e* Character $(c) - 2(d)$
$SO(6)$	$SO(6)$	15	15	-15
$SO(5, 1) \simeq SO(1, 5)$	$SO(5)$	15	10	-5
$SO(4, 2) \simeq SO(2, 4)$	$SO(4) \otimes SO(2)$	15	$6 + 1$	$+1$
$SO(3, 3)$	$SO(3) \otimes SO(3)$	15	$3 + 3$	$+3$

from $-15 = -$(dimension of algebra) for the most compact real form, to $+3 = +$(rank of algebra) for the least compact real form. This is a general feature of the character function.

We return now to a proof of the theorem stated at the beginning of this section.

THEOREM. *The Cartan-Killing metric for a compact semisimple Lie group is negative definite, and conversely.*

Proof. We suppose first that the metric is negative definite, and choose bases X_i:

$$(X_i, X_j) = -\delta_{ij} \qquad 1 \le i, j \le \eta \tag{1.30}$$

If D is any element in the algebra, we can write

$$\begin{aligned}([D, X_i], X_j) + (X_i, [D, X_j]) &= 0 \\ \mathbf{R}_i{}^j(D) + \mathbf{R}_j{}^i(D) &= 0\end{aligned} \tag{1.31}$$

In other words, the regular matrix representative for an arbitrary element D is a skew symmetric matrix. The regular representation of the Lie algebra is a subalgebra of the defining representation of $\mathfrak{so}(\eta)$. The Lie group is a closed subgroup of $SO(\eta)$, which is compact. Therefore the group itself is compact.

Conversely, suppose the group is compact. The group is isomorphic with its group of inner automorphisms, which is therefore a compact transformation group acting on a real η-dimensional linear vector space; int (G) preserves both a positive and a negative definite metric. Choose a set of bases Y_i for which

$$(Y_i, Y_j) = +\delta_{ij} \qquad 1 \le i, j \le \eta \tag{1.30$'$}$$

This inner product is *not* the Cartan-Killing inner product. Then for an arbitrary X in the Lie algebra

$$\begin{aligned}([X, Y_i], Y_j) + (Y_i, [X, Y_j]) &= 0 \\ -\mathbf{R}_i{}^j(X) - \mathbf{R}_j{}^i(X) &= 0\end{aligned} \tag{1.31$'$}$$

The Cartan-Killing inner product is

$$\begin{aligned}(X, X) &= \operatorname{tr} \mathbf{R}(X)\mathbf{R}(X) \\ &= \sum_i \sum_j \mathbf{R}_i{}^j(X)\mathbf{R}_j{}^i(X) = -\sum_i \sum_j [\mathbf{R}_i{}^j(X)]^2 \le 0\end{aligned}$$

Since the regular representation is faithful, the equality holds only when X is 0. The Cartan-Killing metric is then negative definite.

Another theorem providing a deep connection between the algebraic and topological properties of Lie groups was first proved by Weyl.[6]

THEOREM. *If a Lie group is semisimple and compact, its universal covering group is compact.*

Comment. This theorem is both significant and nontrivial. The regular representation for any semisimple Lie algebra can always be constructed. By suitable reality restrictions (next section), the compact real form of this algebra can always be constructed. The EXPonential of this compact regular representation is a compact group, but it is the *most* multiply connected group with this compact algebra. By this theorem, the Lie group has a covering group which is compact. As an example of these comments, the groups $SU(2)$ and $SO(3)$ have a Lie algebra with a faithful 3×3 compact regular representation. The exponential of this representation is isomorphic with $SO(3)$—it is compact but doubly connected. By this theorem, $SO(3)$ has a compact simply connected covering group, which we know to be $SU(2)$.

The theorem is also nontrivial. The simplest of the rotation groups is $SO(2)$. This group is multiply connected and compact. Its covering group is isomorphic with the group of rigid displacements of the straight line R_1, which is simply connected but noncompact. Therefore, the theorem is not valid for $SO(2)$, the first of the rotation groups. This group, however, is not a simple group (namely, its Cartan-Killing metric is zero); thus it does not satisfy the conditions of the theorem. Generally speaking, the theorem is valid for the semisimple groups, since they are all finitely multiply connected. It fails for $SO(2)$ because it is infinitely connected. In other words, the discrete invariant subgroups are finite and infinite, respectively.

Comment. *Spin* $(n) = \overline{SO(n)}$ is compact for $n > 2$.

3. THE CLASSIFICATION MACHINERY. We turn now to a description of how the real forms are described and classified.

First of all, every complex Lie algebra has two easily accessible and uniquely defined real forms. These are:

1. The **normal** form of the complex algebra

$$\sum_{i=1}^{n} c^i H_i + \sum_{\alpha \neq 0} c^\alpha E_\alpha \qquad c \text{ complex} \tag{1.32}$$

consists of the subspace in which the complex coefficients c^i, c^α are all restricted to be real coefficients. We must show that this subspace is closed under commutation. But this is trivial, since all structure constants are real numbers. The Cartan-Killing form in terms of the bases H_i, E_α has the structure

$$g \rightarrow \begin{array}{c} \left[\begin{array}{cccc|cc|cc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & & 1 & & \\ & & & & 1 & & & \\ \hline & & & & & & & 1 \\ & & & & & & 1 & \end{array}\right] \end{array} \begin{array}{l} H_1 \\ H_2 \\ \vdots \\ H_n \\ E_\alpha \\ E_{-\alpha} \\ E_\beta \\ E_{-\beta} \end{array} \tag{1.33}$$

It is clear that the character χ of the normal form is equal to the rank n of the algebra:

$$\chi(\text{normal}) = +(\text{rank}) \tag{1.34n}$$

By a suitable choice of generators the metric tensor can be transformed to a suitable canonical diagonal form. With respect to the bases

$$H_i, \quad \frac{E_\alpha + E_{-\alpha}}{\sqrt{2}}, \quad \frac{E_\alpha - E_{-\alpha}}{\sqrt{2}} \tag{1.35n}$$

$$g \rightarrow \left[\begin{array}{cccc|cc|cc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & -1 & & \\ \hline & & & & & & 1 & \\ & & & & & & & -1 \end{array}\right] \begin{array}{l} H_1 \\ H_2 \\ \vdots \\ H_n \\ (E_\alpha + E_{-\alpha})/\sqrt{2} \\ (E_\alpha - E_{-\alpha})/\sqrt{2} \\ (E_\beta + E_{-\beta})/\sqrt{2} \\ (E_\beta - E_{-\beta})/\sqrt{2} \end{array}$$

2. The compact real form is obtained from the normal real form by a standard Weyl unitary trick. Then, with respect to the bases

$$iH_j, \quad \frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}, \quad \frac{(E_\alpha - E_{-\alpha})}{\sqrt{2}} \tag{1.35c}$$

the metric tensor has the canonical negative definite structure.

$$\chi(\text{compact}) = -(\text{dimension}) \tag{1.34c}$$

It remains to be shown that this collection of bases closes under commutation, with real structure constants:

$$\begin{aligned}
\left[iH_j, \frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}\right] &= -\alpha_j \frac{(E_\alpha - E_{-\alpha})}{\sqrt{2}} \\
\left[\frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}, \frac{E_\alpha - E_{-\alpha}}{\sqrt{2}}\right] &= -\boldsymbol{\alpha} \cdot i\mathbf{H} \\
\left[\frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}, \frac{E_\beta - E_{-\beta}}{\sqrt{2}}\right] &= \frac{i}{2}(N_{\alpha,\beta}E_{\alpha+\beta} - N_{-\alpha,-\beta}E_{-\alpha-\beta}) \\
&\quad - \frac{i}{2}(N_{\alpha,-\beta}E_{\alpha-\beta} - N_{-\alpha,\beta}E_{-\alpha+\beta}) \\
&= \frac{N_{\alpha,\beta}}{\sqrt{2}} \frac{i(E_{\alpha+\beta} + E_{-\alpha-\beta})}{\sqrt{2}} \\
&\quad - \frac{N_{\alpha,-\beta}}{\sqrt{2}} \frac{i(E_{\alpha-\beta} + E_{-\alpha+\beta})}{\sqrt{2}}
\end{aligned} \tag{1.36}$$

The remaining commutators can be computed analogously. Notice that we have made use of the symmetry $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ in proving the reality of the structure constants in the last commutation relation. Thus every complex semisimple Lie algebra has two readily accessible real forms—the normal and the compact—which are at opposite ends of the compactness spectrum. The normal real form is a real linear vector space spanned by the bases (1.35n); any element in the compact real form is a real linear combination of the basis vectors (1.35c). The complex extension of both the normal and the compact real forms is the complex linear vector space (1.32).

Let us analyze what we have done now, but in reverse order. Within the regular representation the matrices $\mathbf{R}(H_i)$, $\mathbf{R}(E_\alpha)$ are real, since they describe the (real) structure constants of the Lie algebra. Under the operation of complex conjugation, the generators of the compact algebra are mapped into themselves as follows

$$\begin{aligned}
\frac{E_\alpha - E_{-\alpha}}{\sqrt{2}} &\xrightarrow{\text{complex conjugation}} +1\,\frac{E_\alpha - E_{-\alpha}}{\sqrt{2}} \\
\frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}} &\xrightarrow{\qquad} -1\,\frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}} \\
iH_j &\xrightarrow{\qquad} -1\,iH_j
\end{aligned} \tag{1.37}$$

The complex conjugation operation splits the compact algebra into a subspace (subalgebra) with eigenvalue $+1$ and a complementary subspace (not an algebra) with eigenvalue -1. If we now perform the Weyl unitary trick on the subspace with eigenvalue -1, we recover the normal real form.

The complex conjugation operator is an example of an operator that

1. Maps the complex Lie algebra onto itself.
2. When applied twice, is the identity operator.

Any operator T that maps a Lie algebra onto itself is called an automorphism. Such automorphisms obey

$$(X, Y) = (TX, TY) \tag{1.38}$$

We have already constructed all automorphisms for all simple Lie algebras. Any automorphism with the property

$$T^2 = I \tag{1.39}$$

is called an **involutive automorphism.**[1,2] Such automorphisms have eigenvalues ± 1, for

$$(T - 1I)(T + 1I) = 0 \tag{1.40}$$

Such automorphisms split a Lie algebra into the subspaces with eigenvalues ± 1.

Let $\mathfrak{g}$ be a compact simple Lie algebra. We decompose $\mathfrak{g}$ into its eigensubspaces of T as follows

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{p} \\ T(\mathfrak{g}) &= T(\mathfrak{k}) \oplus T(\mathfrak{p}) \\ &= (+1)\mathfrak{k} \oplus (-1)\mathfrak{p} \end{aligned} \tag{1.41c}$$

It is easy to show that the subspaces $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal, for

$$(\mathfrak{k}, \mathfrak{p}) = (T\mathfrak{k}, T\mathfrak{p}) = (+\mathfrak{k}, -\mathfrak{p}) = -(\mathfrak{k}, \mathfrak{p}) = 0 \tag{1.42c}$$

Furthermore, we show that $\mathfrak{k}$ is closed under commutation and is thus a subalgebra:

$$\begin{aligned} ([K, K'], P) &= (T[K, K'], TP) = ([TK, TK'], TP) && K, K' \in \mathfrak{k} \\ &= ([+K, +K'], -P) = -([K, K'], P) = 0 && P \in \mathfrak{p} \end{aligned} \tag{1.43c}$$

Similarly, all other inner products of the form $([A, B], C)$ vanish when an odd number of arguments come from the subspace $\mathfrak{p}$ of eigenvalue -1. We have

$$\begin{aligned} [\mathfrak{k}, \mathfrak{k}] \perp \mathfrak{p} &\Rightarrow [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \\ [\mathfrak{k}, \mathfrak{p}] \perp \mathfrak{k} &\Rightarrow [\mathfrak{k}, \mathfrak{p}] = \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] \perp \mathfrak{p} &\Rightarrow [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \end{aligned} \tag{1.44c}$$

Thus $\mathfrak{k}$ forms a subalgebra, and $\mathfrak{p}$ forms the orthogonal complementary subspace.

If we now perform the Weyl unitary trick on the subspace $\mathfrak{p}$, we construct a new algebra $\mathfrak{g}^*$:

$$\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p} \tag{1.41n}$$

This space is closed under commutation with real structure constants, because $\mathfrak{g}$ has real structure constants and

$$\begin{aligned} &[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \\ &[\mathfrak{k}, i\mathfrak{p}] = i\mathfrak{p} \qquad\qquad \Rightarrow [\mathfrak{g}^*, \mathfrak{g}^*] = \mathfrak{g}^* \\ &[i\mathfrak{p}, i\mathfrak{p}] \subset (-)\mathfrak{k} \end{aligned} \tag{1.44n}$$

Furthermore, this new algebra $\mathfrak{g}^*$ is not compact unless $\mathfrak{k} = \mathfrak{g}$:

$$\chi(\mathfrak{g}^*) = \dim(\mathfrak{p}) - \dim(\mathfrak{k}) \tag{1.45}$$

The decomposition (1.41) is called a **Cartan decomposition**. The subspace $\mathfrak{k}$ is the maximal compact subalgebra of $\mathfrak{g}^*$. With respect to this decomposition, the regular representation of $\mathfrak{g}$ is

$$\mathbf{R}(\mathfrak{k}) = \begin{array}{cl} \left[\begin{array}{c|c} A & O \\ \hline O & B \end{array}\right] & \begin{array}{l} \}\,\mathfrak{k} \\ \}\,\mathfrak{p} \end{array} \end{array} \qquad \mathbf{R}(\mathfrak{p}) = \left[\begin{array}{c|c} O & C \\ \hline -C^t & O \end{array}\right]$$

$$\begin{aligned} A^t &= -A \qquad\qquad A, B, C \quad \text{all real} \\ B^t &= -B \end{aligned} \tag{1.46c}$$

Since $\mathfrak{g}$ is compact, the matrices $\mathbf{R}(\mathfrak{k})$ and $\mathbf{R}(\mathfrak{p})$ are skew symmetric and real. A similar decomposition for $\mathfrak{g}^*$ gives

$$\mathbf{R}(\mathfrak{k}) = \left[\begin{array}{c|c} A & O \\ \hline O & B \end{array}\right] \qquad \mathbf{R}(i\mathfrak{p}) = \left[\begin{array}{c|c} O & C \\ \hline +C^t & O \end{array}\right]$$

$$\begin{aligned} A^t &= -A \qquad\qquad A, B, C \quad \text{all real} \\ B^t &= -B \end{aligned} \tag{1.46n}$$

Again, $\mathbf{R}(\mathfrak{k})$ is skew. By removing a factor i from the bases of $i\mathfrak{p}$, the matrix $\mathbf{R}(i\mathfrak{p})$ becomes a real *symmetric* matrix.

By this construction it is possible to associate a noncompact real form $\mathfrak{g}^*$ with the compact form $\mathfrak{g}$ through the involutive automorphism T. It can be shown that this is a completeness construction as well: as T goes through all automorphisms obeying $T^2 = I$, $\mathfrak{g}^*$ runs through all real forms associated with the complex semisimple algebra of which $\mathfrak{g}$ is the known compact real form.[1,2,7]

To classify all real forms of all simple algebras, it only remains to find all automorphisms T obeying $T\mathfrak{g}T^{-1} = \mathfrak{g}$ and $T^2 = I$. Such T either commute or anticommute with the elements of the compact algebra. Since we have already constructed all automorphisms of the simple algebras, we have simply to search among them for those obeying the additional condition $T^2 = I$. There are only three such different automorphisms:

A. $T = K$ (complex conjugation)

B. $T = \left[\begin{array}{c|c} +I_p & \\ \hline & -I_q \end{array}\right] = I_{p,q}$

C. $T = \left[\begin{array}{c|c} & +I_p \\ \hline -I_p & \end{array}\right] = J_{p,p} \simeq \left[\begin{array}{c|c} & +\tilde{I}_p \\ \hline -\tilde{I}_p & \end{array}\right]$

II. Classification of the Real Forms

1. A_{n-1}. The complex simple root space A_{n-1} characterizes the classical group $Sl(n, c)$ and all its real forms. In the defining $n \times n$ matrix representation

$$\mathfrak{sl}(n, c) = (A) \qquad \operatorname{tr} A = 0,\ A \text{ complex} \tag{2.1}$$

Under restriction from complex to real variables, the normal real form has the structure

$$\mathfrak{sl}(n, r) = (A) \qquad \operatorname{tr} A = 0,\ A \text{ real} \tag{2.2}$$

The maximal compact subalgebra of $\mathfrak{sl}(n, r)$ consists of those real matrices A which are also antisymmetric:

$$\begin{aligned} \mathfrak{sl}(n, r) &= \mathfrak{k} \oplus i\mathfrak{p} \\ &= (A) \oplus (B) \\ &\quad A \quad \text{real antisymmetric} \\ &\quad B \quad \text{real symmetric;} \quad \operatorname{tr} B = 0 \end{aligned} \tag{2.3n}$$

Under the Weyl unitary trick we construct the Lie algebra for $SU(n, c)$:

$$\begin{aligned} \mathfrak{su}(n, c) &= \mathfrak{k} \oplus \mathfrak{p} \\ &= (A) \oplus i(B) \\ &\quad A \quad \text{real antisymmetric} \\ &\quad B \quad \text{real symmetric;} \quad \operatorname{tr} B = 0 \end{aligned} \tag{2.3c}$$

The algebra of $SU(n, c)$ consists of all traceless antihermitian matrices $(m_{ij} = -m_{ji}^*)$.

A. Under the involutive automorphism K we construct from the algebra of $SU(n, c)$ the algebra of $Sl(n, r)$. We have just used this procedure, in reverse order, to construct the compact real form $SU(n, c)$ of $Sl(n, c)$ from the normal real form $Sl(n, r)$ of $Sl(n, c)$.

B. Under the automorphism $T = I_{p,q}$ we find

$$\left[\begin{array}{c|c} I_p & \\ \hline & -I_q \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline -B^\dagger & C \end{array}\right]\left[\begin{array}{c|c} I_p & \\ \hline & -I_q \end{array}\right] = \left[\begin{array}{c|c} A & -B \\ \hline +B^\dagger & C \end{array}\right] \tag{2.4}$$

$$\mathfrak{k} = \left[\begin{array}{c|c} A & \\ \hline & C \end{array}\right] \qquad \mathfrak{p} = \left[\begin{array}{c|c} & B \\ \hline -B^\dagger & \end{array}\right] \tag{2.5}$$

$$A^\dagger = -A \qquad C^\dagger = -C \qquad B \text{ arbitrary complex}$$
$$\operatorname{tr}(A) + \operatorname{tr}(C) = 0$$

The action of the algebra $\mathfrak{g}^*$ on a basis vector can be described by

$$\left[\begin{array}{c|c} A & iB \\ \hline -iB^\dagger & C \end{array}\right]\left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_p \\ \hline u_{p+1} \\ \vdots \\ u_{p+q} \end{array}\right] \cong \left[\begin{array}{c|c} A & B \\ \hline +B^\dagger & C \end{array}\right]\left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_p \\ \hline iu_{p+1} \\ \vdots \\ iu_{p+q} \end{array}\right] \tag{2.6}$$

The algebra $\mathfrak{g}^*$ describes the group $SU(p, q; c)$, whose maximal compact subgroup, generated by $\mathfrak{k}$, is

$$S[U(p) \otimes U(q)] \simeq SU(p) \otimes SU(q) \otimes U(1) \tag{2.7}$$

We observe that the maximal compact subgroup has an abelian invariant subgroup.

C. Choose $T = J_{n,n}K$, acting on the defining $2n \times 2n$ representation of $\mathfrak{su}(2n, c)$:

$$\left[\begin{array}{c|c} & \tilde{I}_n \\ \hline -\tilde{I}_n & \end{array}\right] K \left[\begin{array}{c|c} A & B \\ \hline -B^\dagger & C \end{array}\right] K \left[\begin{array}{c|c} & -\tilde{I}_n \\ \hline \tilde{I}_n & \end{array}\right] = \left[\begin{array}{c|c} \tilde{I}C^*\tilde{I} & \tilde{I}B^{\dagger *}\tilde{I} \\ \hline -\tilde{I}B^*\tilde{I} & \tilde{I}A^*\tilde{I} \end{array}\right] \tag{2.8}$$

The $n \times n$ submatrices A and C are antihermitian and obey $\operatorname{tr}(A + C) = 0$, and B is complex. We have for any matrix M

$$\tilde{I}M^t\tilde{I} = +M^{\sim} \tag{2.9}$$

where $M^{\sim}$ indicates the reflection of M in its minor diagonal.

Then the transformation (2.8) is given by

$$\left[\begin{array}{c|c} A & B \\ \hline -B^{\dagger} & C \end{array}\right] \rightarrow \left[\begin{array}{c|c} -C^{\sim} & B^{\sim} \\ \hline (-B^{\sim})^{\dagger} & -A^{\sim} \end{array}\right] \tag{2.10}$$

The Cartan decomposition is

$$\begin{aligned} \mathfrak{k} &= \tfrac{1}{2}\left[\begin{array}{c|c} A - C^{\sim} & B + B^{\sim} \\ \hline -(B + B^{\sim})^{\dagger} & C - A^{\sim} \end{array}\right] = \left[\begin{array}{c|c} A_- & B_+ \\ \hline -B_+^{\dagger} & -A_-^{\sim} \end{array}\right] \\ \mathfrak{p} &= \tfrac{1}{2}\left[\begin{array}{c|c} A + C^{\sim} & B - B^{\sim} \\ \hline -(B - B^{\sim})^{\dagger} & C + A^{\sim} \end{array}\right] = \left[\begin{array}{c|c} A_+ & B_- \\ \hline -B_-^{\dagger} & A_+^{\sim} \end{array}\right] \end{aligned} \tag{2.11}$$

$$\begin{array}{ll} A^{\dagger} = -A & A_{\pm}^{\dagger} = -A_{\pm} \\ C^{\dagger} = -C & B_{\pm}^{\sim} = \pm B_{\pm} \\ B \text{ complex} & \operatorname{tr}(A_+) = 0 \end{array}$$

It is easily verified, by comparison with Table 6.1, that the maximal compact subalgebra $\mathfrak{k}$ generates the subgroup $USp(2n)$.

The noncompact algebra $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$ has the matrix structure

$$\left[\begin{array}{c|c} A_- + iA_+ & B_+ + iB_- \\ \hline -B_+^{\dagger} - iB_-^{\dagger} & \tilde{I}(-A_-^t + iA_+^t)\tilde{I} \end{array}\right] \tag{2.12}$$

Under the similarity transformation

$$\left[\begin{array}{c|c} I_n & \\ \hline & \tilde{I}_n \end{array}\right]$$

this matrix can be brought to the form

$$\left[\begin{array}{c|c} A_- + iA_+ & (B_+ + iB_-)\tilde{I} \\ \hline -[(B_+ + iB_-)\tilde{I}]^* & (A_- + iA_+)^* \end{array}\right] \tag{2.12'}$$

This is the Lie algebra of $SU^*(2n)$.

2. B_n. All real Lie algebras whose root space is B_n have the common complex extension $SO(2n+1, c)$, consisting of complex antisymmetric $(2n+1) \times (2n+1)$ matrices in the defining representation. The compact real form $SO(2n+1, r)$ consists of real antisymmetric matrices. Since the compact form is real, the action of K (automorphism type A, p. 339) is trivial. Moreover, $J_{n,n}$ (automorphism type C, p. 339) is not a symmetry, since it cannot act on (odd) $\times$ (odd) matrices.

B. The only nontrivial automorphism is $T = I_{p,q}$. We have

$$\left[\begin{array}{c|c} I_p & \\ \hline & -I_q \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline -B^t & C \end{array}\right]\left[\begin{array}{c|c} I_p & \\ \hline & -I_q \end{array}\right] = \left[\begin{array}{c|c} A & -B \\ \hline B^t & C \end{array}\right] \tag{2.13}$$

$$\mathfrak{k} = \left[\begin{array}{c|c} A & \\ \hline & C \end{array}\right] \qquad \mathfrak{p} = \left[\begin{array}{c|c} & B \\ \hline -B^t & \end{array}\right] \tag{2.14c}$$

$$A^t = -A$$
$$C^t = -C \qquad B \text{ real}$$

The Lie algebra for $\mathfrak{g}^*$ can be written

$$\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p} \to \left[\begin{array}{c|c} A & \\ \hline & C \end{array}\right] \oplus \left[\begin{array}{c|c} & B \\ \hline +B^t & \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline B^t & C \end{array}\right] \tag{2.14n}$$

$$A^t = -A$$
$$C^t = -C \qquad B \text{ real}$$

This is the algebra for $SO(p, q; r) \simeq SO(q, p; r)$. All inequivalent real forms of B_n are given by

$$SO(p+q) \to SO(p, q) \qquad p > q \quad \text{and} \quad p + q = 2n + 1. \tag{2.15B}$$

3. D_n. The arguments used for B_n can be used for D_n, with one modification.

B. The automorphism $I_{p,q}$ gives rise to the noncompact forms

$$SO(p+q) \to SO(p, q) \qquad p \geq q \quad \text{and} \quad p + q = 2n. \tag{2.15D}$$

C. In addition, the automorphism $J_{n,n}$ can be used, since the matrix algebra consists of (even) $\times$ (even) matrices. These transformations are given by

$$\left[\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline -B^t & C \end{array}\right]\left[\begin{array}{c|c} & -I_n \\ \hline I_n & \end{array}\right] = \left[\begin{array}{c|c} C & B^t \\ \hline -B & A \end{array}\right] \tag{2.16}$$

The Cartan decomposition is

$$\mathfrak{k} = \left[\begin{array}{c|c} A_+ & B_+ \\ \hline -B_+ & A_+ \end{array}\right] \qquad \mathfrak{p} = \left[\begin{array}{c|c} A_- & B_- \\ \hline B_- & -A_- \end{array}\right] \tag{2.17c}$$

$$(A_\pm)^t = -A_\pm \qquad (B_\pm)^t = \pm B_\pm$$

The Lie algebra for the noncompact real form is

$$\mathfrak{g}^* = \left[\begin{array}{c|c} A_+ + iA_- & B_+ + iB_- \\ \hline -B_+ + iB_- & A_+ - iA_- \end{array}\right] \tag{2.17n}$$

This is the Lie algebra for the group $SO^*(2n)$.

The structure of this algebra can be made more transparent by performing a similarity transformation

$$S = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c} 1 & i \\ \hline i & 1 \end{array}\right]\left[\begin{array}{c|c} I_n & \\ \hline & I_n \end{array}\right]:$$

as follows:

$$\mathfrak{g}^* \to S\mathfrak{g}^*S^{-1} = \left[\begin{array}{c|c} A_+ - iB_+ & O \\ \hline O & A_+ + iB_+ \end{array}\right] \oplus \left[\begin{array}{c|c} O & A_- + iB_- \\ \hline (A_- + iB_-)^\dagger & O \end{array}\right] \tag{2.17n$'$}$$

The compact subalgebra contains the antihermitian matrices $A_+ \pm iB_+$ on the diagonal. These generate the subgroup $U(n, c)$, which has an abelian invariant subgroup. The cosets are determined by the complex antisymmetric matrix $A_- + iB_-$.

4. C_n. All real Lie algebras whose root space is C_n have the common complex extension $Sp(2n, c)$. The compact real form of this algebra describes the group $USp(2n)$. The $2n \times 2n$ matrix algebra of this group has both unitary and symplectic symmetry:

$$\mathfrak{g} = \left[\begin{array}{c|c} A & B \\ \hline -B^\dagger & -A^\sim \end{array}\right] \qquad \begin{array}{l} A^\dagger = -A \\ B^\sim = +B \end{array} \tag{2.18c}$$

Since the automorphisms K and $J_{n,n}$ on this algebra produce identical results, we need only discuss the automorphisms K and $I_{2p,2q}$.

A. Under complex conjugation, the eigenspace of $\mathfrak{g}$ with eigenvalue $+1$ (-1) consists of all real (imaginary) matrices. Denoting by r, i the real and imaginary parts of A, B we have

$$\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p} = \left[\begin{array}{c|c} A_r & B_r \\ \hline -B_r{}^t & -A_r^{\sim} \end{array}\right] \oplus \left[\begin{array}{c|c} A_i & B_i \\ \hline B_i{}^t & -A_i^{\sim} \end{array}\right] \tag{2.18n}$$

These matrices maintain their symplectic symmetry, since the operation of reflecting about the minor diagonal commutes with the Weyl unitary trick. The algebra $\mathfrak{g}^*$ describes the noncompact group $Sp(2n, r)$.

The structure of this algebra can be made more transparent by performing a similarity transformation

$$S = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} 1 & i \\ \hline i & 1 \end{array}\right] \left[\begin{array}{c|c} I_n & \\ \hline & \tilde{I}_n \end{array}\right]$$

$$\mathfrak{g}^* \rightarrow S\mathfrak{g}^*S^{-1} = \left[\begin{array}{c|c} A' & O \\ \hline O & A'^* \end{array}\right] \oplus \left[\begin{array}{c|c} O & B' \\ \hline B'^{\dagger} & O \end{array}\right] \tag{2.18n$'$}$$

$$A'^{\dagger} = -A' \qquad\qquad B'^{t} = B'$$

The antihermitian matrices A' generate the maximal compact subgroup, $U(n, c)$, which has an abelian invariant subgroup. The cosets are determined by the complex symmetric matrix B'. We remark that the groups $SO^*(2n)$ and $Sp(2n, r)$ are very similar: the difference in their Lie algebra structure exists in the subspace $i\mathfrak{p}$ consisting of off-diagonal matrices B'. The complex matrices B' are antisymmetric (2.17n$'$) and symmetric (2.18n$'$) for the algebras $\mathfrak{so}^*(2n)$ and $\mathfrak{sp}(2n, r)$, respectively.

B. Finally, we apply the involutive automorphism $I_{2p,2q}$ to the algebra as follows:

$$\left[\begin{array}{cc|cc} I_p & & & \\ & -I_q & & \\ \hline & & -I_q & \\ & & & I_p \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline -B^{\dagger} & -A^{\sim} \end{array}\right] \left[\begin{array}{cc|cc} I_p & & & \\ & -I_q & & \\ \hline & & -I_q & \\ & & & I_p \end{array}\right] \tag{2.19}$$

The standard decomposition is

$$
\left[\begin{array}{c|c|c|c}
A_{11} & A_{12} & B_{11} & B_{12} \\
\hline
-A^{\dagger}_{12} & A_{22} & B_{21} & B^{\sim}_{11} \\
\hline
-B^{\dagger}_{11} & -B^{\dagger}_{21} & -A^{\sim}_{22} & -A^{\sim}_{12} \\
\hline
-B^{\dagger}_{12} & -B^{\sim\dagger}_{11} & A^{\sim\dagger}_{12} & -A^{\sim}_{11}
\end{array}\right]
$$

$$
= \left[\begin{array}{c|c|c|c}
A_{11} & & & B_{12} \\
\hline
 & A_{22} & B_{21} & \\
\hline
 & -B^{\dagger}_{21} & -A^{\sim}_{22} & \\
\hline
-B^{\dagger}_{12} & & & -A^{\sim}_{11}
\end{array}\right]
\oplus
\left[\begin{array}{c|c|c|c}
 & A_{12} & B_{11} & \\
\hline
-A^{\dagger}_{12} & & & B^{\sim}_{11} \\
\hline
-B^{\dagger}_{11} & & & -A^{\sim}_{12} \\
\hline
 & -B^{\sim\dagger}_{11} & A^{\sim\dagger}_{12} &
\end{array}\right]
\tag{2.20}
$$

The matrices on the major diagonal have unitary symmetry; those on the minor diagonal have symplectic symmetry:

$$
\begin{aligned}
A^{\dagger}_{11} &= -A_{11} \qquad & B^{\sim}_{12} &= +B_{12} \\
A^{\dagger}_{22} &= -A_{22} \qquad & B^{\sim}_{21} &= +B_{21}
\end{aligned}
$$

Performing the Weyl unitary trick and rearranging the submatrices gives

$$
\mathfrak{g}^* = \left[\begin{array}{cc|cc}
A_{11} & B_{12} & & \\
-B^{\dagger}_{12} & -A^{\sim}_{11} & \multicolumn{2}{c}{\raisebox{1ex}{0}} \\
\hline
 & & A_{22} & B_{21} \\
\multicolumn{2}{c|}{\raisebox{1ex}{0}} & -B^{\dagger}_{21} & -A^{\sim}_{22}
\end{array}\right]
\oplus i
\left[\begin{array}{cc|cc}
 & & A_{12} & B_{11} \\
\multicolumn{2}{c|}{\raisebox{1ex}{0}} & -B^{\sim\dagger}_{11} & A^{\sim\dagger}_{12} \\
\hline
-A^{\dagger}_{12} & B^{\sim}_{11} & & \\
-B^{\dagger}_{11} & -A^{\sim}_{12} & \multicolumn{2}{c}{\raisebox{1ex}{0}}
\end{array}\right]
\tag{2.21}
$$

The maximal compact subgroup is $USp(2p) \otimes USp(2q)$. The Lie algebra $\mathfrak{g}^*$ describes the group $USp(2p, 2q)$.

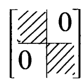
5. EXCEPTIONAL GROUPS. The five exceptional groups can be treated in exactly the same way. Since the calculations seem to be of academic interest to physicists, we content ourselves with presenting the results in the following section.

III. Discussion of Results

1. TABLES OF THE REAL FORMS. All but five of the simple root spaces correspond to classical matrix groups. The Lie algebras of these classical matrix groups correspond to the various real forms associated with the complex extension algebra.

Under the restriction of the faithful defining representation of a real matrix algebra $\mathfrak{g}^*$ to its maximal compact subalgebra $\mathfrak{k}$, either

1. The matrix structure for $\mathfrak{k}$ assumes a block diagonal form



(fully reducible), whereupon the subspace $i\mathfrak{p}$ assumes the form

$$\left[\begin{array}{c|c} 0 & B \\ \hline B^{\dagger} & 0 \end{array}\right]$$

or

2. The matrix structure for $\mathfrak{k}$ remains square and cannot be transformed to a block diagonal form (irreducible); then the subspace $i\mathfrak{p}$ consists also of square irreducible hermitian matrices of the form $[B']$.

Only two subspaces $i\mathfrak{p}$ satisfy the second condition. For $\mathfrak{sl}(n, r)$, the subspace $i\mathfrak{p}$ consists of real symmetric matrices. For $\mathfrak{su}^*(2n)$, the subspace $i\mathfrak{p}$ consists of hermitian antisymplectic matrices. The structure of the off-diagonal submatrix B describing the matrix subspace

$$i\mathfrak{p} = \left[\begin{array}{c|c} 0 & B \\ \hline +B^{\dagger} & 0 \end{array}\right]$$

is given in Table 9.3, which also lists the maximal compact subgroups and the character of the noncompact real form. All subspaces $\mathfrak{p}$, $i\mathfrak{p}$ of the classical algebras consist of traceless hermitian matrices.

The algebra $\mathfrak{d}_1$ describes the group $U(1) \simeq SO(2)$. D_1 is not a semisimple Lie group because its Cartan-Killing metric is zero. Subalgebras $\mathfrak{k}$ containing the abelian invariant subgroup D_1 are nonsemisimple. Since the subalgebra $\mathfrak{k}$ is embedded maximally in $\mathfrak{g}^*$, D_1 is the only abelian invariant subgroup which $\mathfrak{k}$ may possess.

When $\mathfrak{k}$ possesses the one-dimensional abelian invariant subgroup D_1, the space

$$P^{(*)} = \text{EXP}\,(i)\mathfrak{p} = \text{EXP}\,(\mathfrak{g}^{(*)} - \mathfrak{k}) = \frac{G^{(*)}}{K}$$

TABLE 9.3

Root Space	Compact Form	Associated Noncompact Form	Maximal Compact Subgroup	Structure of Matrix B, B'	Character χ of Noncompact Real Form	Rank of Coset	Dimension of Coset
A_{n-1}	$SU(n)$	$Sl(n, r)$	$SO(n)$	B', Hermitian symmetric	$n-1$	$n-1$	$\frac{n(n+1)}{2}-1$
	$SU(2n)$	$SU^*(2n)$	$USp(2n)$	B', Hermitian antisymplectic	$-2n-1$	$n-1$	$\frac{2n(2n-1)}{2}-1$
	$SU(p+q)$	$SU(p, q)$	$SU(p) \otimes SU(q) \otimes U(1)$	B Complex	$-(p-q)^2+1$	$\min(p, q)$	$2pq$
B_n	$SO(p+q)$	$SO(p, q)$	$SO(p) \otimes SO(q)$	B Real	$\frac{[(p+q)-(p-q)^2]}{2}$	$\min(p, q)$	pq
D_n	$SO(p+q)$	$SO(p, q)$	$SO(p) \otimes SO(q)$	B Real	$\frac{[(p+q)-(p-q)^2]}{2}$	$\min(p, q)$	pq
	$SO(2n)$	$SO^*(2n)$	$SU(n) \otimes U(1)$	B Complex antisymmetric	$-n$	$\left[\frac{n}{2}\right]$	$n(n-1)$
C_n	$USp(2n)$	$Sp(2n, r)$	$SU(n) \otimes U(1)$	B Complex symmetric	$+n$	n	$n(n+1)$
	$USp(2p+2q)$	$USp(2p, 2q)$	$USp(2p) \otimes USp(2q)$	B Complex symplectic (2.20)	$\frac{[-(2p+2q)-(2p-2q)^2]}{2}$	$\min(p, q)$	$4pq$
G_2	$G_{2(-14)}$	$G_{2(2)}$	$A_1 \oplus A_1$		$+2$	2	8
F_4	$F_{4(-52)}$	$F_{4(-20)}$	B_4		-20	1	16
	$F_{4(-52)}$	$F_{4(+4)}$	$C_3 \oplus A_1$		$+4$	4	28
E_6	$E_{6(-78)}$	$E_{6(-26)}$	F_4		-26	2	26
	$E_{6(-78)}$	$E_{6(-14)}$	$D_5 \oplus D_1$		-14	2	32
	$E_{6(-78)}$	$E_{6(2)}$	$A_5 \oplus A_1$		$+2$	4	40
	$E_{6(-78)}$	$E_{6(6)}$	C_4		$+6$	6	42
E_7	$E_{7(-133)}$	$E_{7(-25)}$	$E_6 \oplus D_1$		-25	3	54
	$E_{7(-133)}$	$E_{7(-5)}$	$D_6 \oplus A_1$		-5	4	64
	$E_{7(-133)}$	$E_{7(7)}$	A_7		$+7$	7	70
E_8	$E_{8(-248)}$	$E_{8(-24)}$	$E_7 \oplus A_1$		-24	4	112
	$E_{8(-248)}$	$E_{8(8)}$	D_8		$+8$	8	128

has very special properties: $P^{(*)}$ has even dimensionality and the $2n$ real parameters describing a point in $P^{(*)}$ may be considered as n complex parameters. Such a space is called a **hermitian symmetric** space.

2. THE CHARACTER FUNCTION. Inspection of Table 9.3 makes it clear that the character function defined by (1.29) provides a unique classification for all real forms of the exceptional algebras. For the classical algebras, the character function is sometimes degenerate; that is, two inequivalent real forms may have the same character. This can happen whenever the respective maximal compact subgroups have the same dimensionality. These degeneracies only occur within the systems A_n and D_n.

Degeneracies occur for $SU^*(2n)$ and $SU(2n - p, p)$ whenever

$$\frac{2n(2n+1)}{2} = (2n-p)^2 + p^2 - 1$$

The condition under which these are equal may be given as

$$(n+1) = 2(n-p)^2$$

A similar condition exists for $SO^*(2n)$ and $SO(2n - q, q)$:

$$n = (n-q)^2$$

The lowest values of the pairs (n, p), (n, q) for which this double degeneracy occurs are given in Table 9.4. We conclude from the table that, except for a

TABLE 9.4

	n	p	Groups	Character
A_{2n-1}	1	0	$SU^*(2) = SU(2)$	-3
	7	5	$SU^*(14)$; $SU(9, 5)$	-15
	17	14	$SU^*(34)$; $SU(20, 14)$	-35
	31	27	$SU^*(62)$; $SU(35, 27)$	-63
	49	44	$SU^*(98)$; $SU(54, 44)$	-99
	$2k^2 - 1$	$(2k+1)(k-1)$	$SU^*(4k^2 - 2)$; $SU(2k^2 + k - 1, 2k^2 - k - 1)$	$-4k^2 + 1$
D_n	4	2	$SO^*(8) = SO(6, 2)$	-4
	9	6	$SO^*(18)$; $SO(12, 6)$	-9
	16	12	$SO^*(32)$; $SO(20, 12)$	-16
	25	20	$SO^*(50)$; $SO(30, 20)$	-25
	36	30	$SO^*(72)$; $SO(42, 30)$	-36
	49	42	$SO^*(98)$; $SO(56, 42)$	-49
	k^2	$k^2 - k$	$SO^*(2k^2)$; $SO(k^2 + k, k^2 - k)$	$-k^2$

The cases included in this table are the only instances in which the character function does not provide a unique classification for the real forms of complex simple Lie groups. In the case $A_1 (k = 1)$ the groups $SU^*(2)$ and $SU(2)$ are isomorphic and the character function is unique. Also, in the case D_4 $(k = 2)$ the groups $SO^*(8)$ and $SO(6, 2)$ are isomorphic and the character function is unique.

TABLE 9.5

A_3	χ	D_3	Group Application	Reference
$SU(4)$	-15	$SO(6)$	$SU(4)$ is of interest in the Wigner supermultiplet theory.	1,2
$SU^*(4)$	-5	$SO(5, 1)$	$SO(5, 1)$ is a deSitter-like group, of interest in some cosmological models.	3
$SU(3, 1)$	-3	$SO^*(6)$	$SU(3, 1)$ is a transition operator algebra. All three-dimensional harmonic oscillator states belong to one unitary irreducible representation of $SU(3, 1)$.	4–6
$SU(2, 2)$	$+1$	$SO(4, 2)$	$SO(4, 2)$ is the conformal group. This is the largest group of transformations leaving Maxwell's equations structurally invariant.	7–9
			$SO(4, 2)$ is the largest known symmetry group of the hydrogen atom.	10–12
			$SO(4, 2)$ is one of the groups whose Green's functions may give information on the fine structure constant.	13,14
$Sl(4, r)$	$+3$	$SO(3, 3)$		

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set of measure zero (literally), the character function on the real forms is unique.

Finally, we use the character function to distinguish the real forms which exist in the series $A_3 \simeq D_3$. The real forms corresponding to each system [$SU(4)$ and $SO(6)$] are homomorphic because the root spaces are identical. We can make identifications of the real forms without comparing the commutation relations, since we know the corresponding characters. It is gratifying that many of these real forms have found applications in physics under a variety of circumstances. Such applications are indicated very briefly in Table 9.5.

IV. Properties of Cosets

1. MATRIX PROPERTIES. The Cartan decomposition of Lie algebras $\mathfrak{g}, \mathfrak{g}^*$ has a simple block diagonal structure within the regular representation:

$$\mathfrak{g} \xrightarrow[\text{rep}]{\text{reg}} \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] \oplus \left[\begin{array}{c|c} 0 & B \\ \hline -B^{\dagger} & 0 \end{array}\right] \begin{array}{l} \}\ \dim \mathfrak{k} \\ \}\ \dim \mathfrak{p} \end{array}$$

$$\mathfrak{g} \longrightarrow \mathfrak{k} \oplus \mathfrak{p} \tag{4.1c}$$

$$\mathfrak{g}^* \longrightarrow \mathfrak{k} \oplus i\mathfrak{p} \tag{4.1n}$$

$$\mathfrak{g}^* \longrightarrow \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] \oplus \left[\begin{array}{c|c} 0 & B \\ \hline +B^{\dagger} & 0 \end{array}\right]$$

The matrices A_1, A_2 are antihermitian, the representative of $\mathfrak{p}$ is also antihermitian, but the representative of $i\mathfrak{p}$ is hermitian. In fact, since the regular representation consists of real matrices, A_1 and A_2 are antisymmetric, B is real, and the representatives of $\mathfrak{p}$ and $i\mathfrak{p}$ are antisymmetric and symmetric, respectively.

The matrix representatives $\Gamma(\mathfrak{g})$ of the algebra $\mathfrak{g}$ can be exponentiated onto the associated matrix representatives Γ of the group as follows:

$$\begin{array}{cccl} \Gamma\{\text{EXP } \mathfrak{g}\} & \Gamma\{\text{EXP } \mathfrak{g}^*\} & & \text{representation of group} \\ \uparrow \text{EXP} & \uparrow \text{EXP} & & \\ \Gamma(\mathfrak{g}) & \Gamma(\mathfrak{g}^*) & & \text{representation of algebra} \end{array} \tag{4.2}$$

The representation $\Gamma(\mathfrak{g})$ of algebra $\mathfrak{g}$ gives rise to the associated representation $\Gamma^*(\mathfrak{g}^*)$ of the algebra $\mathfrak{g}^*$ as follows:

$$\Gamma(\mathfrak{g}) = \Gamma(\mathfrak{k} \oplus \mathfrak{p}) = \Gamma(\mathfrak{k}) \oplus \Gamma(\mathfrak{p}) \leftrightarrow \Gamma(\mathfrak{k}) \oplus i\Gamma(\mathfrak{p}) = \Gamma^*(\mathfrak{g}^*) \tag{4.3}$$

In general, within any finite-dimensional representation Γ, the matrix representatives of the subalgebra $\mathfrak{k}$ are antihermitian. The representatives $\Gamma(\mathfrak{p})$ are also antihermitian (since the entire algebra for a compact group consists of antihermitian matrices); the representatives $\Gamma(i\mathfrak{p}) = i\Gamma(\mathfrak{p})$ are then hermitian. Antihermitian ($M^\dagger = -M$) and hermitian ($M^\dagger = +M$) matrices map onto unitary and hermitian matrices, respectively, for

$$(e^M)^\dagger = e^{(M)^\dagger} = e^{\pm M} \begin{cases} \xrightarrow{+} (e^{+M})^{+1} & \text{hermitian} \\ \xrightarrow{-} (e^{+M})^{-1} & \text{unitary} \end{cases} \tag{4.4}$$

Comment 1. In the defining representation of the simple classical Lie groups, $\mathfrak{k}$ reduces in all but two cases to the block diagonal structure (4.1). The defining matrix representatives of the subgroup EXP $\mathfrak{k}$ and the coset representatives are given by

$$\text{EXP}\,\mathfrak{k} = \text{EXP}\left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] = \left[\begin{array}{c|c} e^{A_1} & 0 \\ \hline 0 & e^{A_2} \end{array}\right] \tag{4.5k}$$

$$\text{EXP}\,\mathfrak{p} = \text{EXP}\left[\begin{array}{c|c} 0 & B \\ \hline -B^\dagger & 0 \end{array}\right] = \left[\begin{array}{c|c} \cos\sqrt{BB^\dagger} & B\dfrac{\sin\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \\ \hline -\dfrac{\sin\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}B^\dagger & \cos\sqrt{B^\dagger B} \end{array}\right]$$

$$\xrightarrow{X = B\frac{\sin\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}} \left[\begin{array}{c|c} [I - XX^\dagger]^{1/2} & X \\ \hline -X^\dagger & [I - X^\dagger X]^{1/2} \end{array}\right] \tag{4.5p}$$

$$\text{EXP}\,i\mathfrak{p} = \text{EXP}\left[\begin{array}{c|c} 0 & B \\ \hline +B^\dagger & 0 \end{array}\right] = \left[\begin{array}{c|c} \cosh\sqrt{BB^\dagger} & B\dfrac{\sinh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \\ \hline B^\dagger\dfrac{\sinh\sqrt{BB^\dagger}}{\sqrt{BB^\dagger}} & \cosh\sqrt{B^\dagger B} \end{array}\right]$$

$$\xrightarrow{X = B\frac{\sinh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}} \left[\begin{array}{c|c} [I + XX^\dagger]^{1/2} & X \\ \hline +X^\dagger & [I + X^\dagger X]^{1/2} \end{array}\right] \tag{4.5p*}$$

The two cases in which the defining matrix representatives of $\mathfrak{k}$ remain irreducible [i.e., do not have the form shown in (4.1)] are $SU^*(2n) \downarrow USp(2n)$ and $Sl(n) \downarrow SO(n)$.

Comment 2. The Cartan decomposition for the Lie algebra $\mathfrak{gl}(n, c)$ is given by

$$\begin{pmatrix} \text{arbitrary} \\ \text{complex} \\ \text{matrix} \end{pmatrix} = \begin{pmatrix} \text{anti-} \\ \text{hermitian} \\ \text{matrix} \end{pmatrix} \oplus \begin{pmatrix} \\ \text{hermitian} \\ \text{matrix} \end{pmatrix} \tag{4.6}$$

The group elements in $Gl(n, c)$ can then generally be written

$$\begin{array}{ccccc} \text{EXP}\begin{pmatrix} \text{arbitrary} \\ \text{complex} \\ \text{matrix} \end{pmatrix} & = & \text{EXP}\begin{pmatrix} \text{antihermitian} \\ n \times n \\ \text{matrix} \end{pmatrix} & \otimes & \text{EXP}\begin{pmatrix} \text{hermitian} \\ n \times n \\ \text{matrix} \end{pmatrix} \\ \text{into but} \downarrow \text{not onto} & & \downarrow & & \downarrow \\ Gl(n, c) & = & (\text{unitary matrix}) & \otimes & (\text{hermitian matrix}) \end{array} \tag{4.7c}$$

This is a well-known result: an arbitrary nonsingular complex matrix can always be written as the product of a unitary and a hermitian matrix. The decomposition is unique, except for a set of "measure zero." Under the restriction from complex to real variables there is a similar result:

$$Gl(n, r) = (\text{orthogonal}) \otimes (\text{symmetric}) \tag{4.7r}$$

An arbitrary matrix $M \in Gl(n, c)$ can be uniquely decomposed according to (4.7c):

$$M = UH$$

The matrix H is easily determined by

$$M^\dagger M = (UH)^\dagger UH = H^\dagger(U^{-1}U)H = H^2 \tag{4.8}$$

Since $M^\dagger M$ has positive eigenvalues $d_i^{\,2}$, we can write

$$\begin{aligned} M^\dagger M = VD^2V^\dagger \qquad & V \text{ unitary, } D \text{ diagonal} \\ & D_{ij} = d_i\, \delta_{ij},\ d_i > 0 \end{aligned} \tag{4.9}$$

$$\begin{aligned} H = \sqrt{H^2} &= \sqrt{M^\dagger M} \\ &= \{(VDV^\dagger)(VDV^\dagger)\}^{1/2} \\ &= VDV^\dagger \end{aligned} \tag{4.10}$$

Since H is nonsingular, the unitary matrix U is given by

$$U = MH^{-1} = MVD^{-1}V^{-1} = M(M^\dagger M)^{-1/2} \tag{4.11}$$

2. INNER PRODUCTS AND INDEX. Since elements in the orthogonal subspaces $\mathfrak{p}$, $i\mathfrak{p}$ are also elements in the Lie algebras $\mathfrak{g}$, $\mathfrak{g}^*$, these subspaces can be studied by exactly the same techniques used in Chapter 7 for the study of the Lie algebras themselves. The only differences between the subspaces $\mathfrak{p}$, $i\mathfrak{p}$ and the algebras $\mathfrak{g}$, $\mathfrak{g}^*$ containing them, is that the complementary subspaces are not closed under commutation, whereas the algebras are. All concepts developed in Chapter 7, and not explicitly dependent on the closure under commutation, can be transferred directly from the Lie algebra to the vector subspaces $\mathfrak{p}$, $i\mathfrak{p}$.

For example, the Cartan-Killing inner product is defined between all pairs of vectors in a Lie algebra. In particular, the inner product between two elements in the subspace $\mathfrak{p}$ is well defined. In fact, the dual subspaces $\mathfrak{p}$, $i\mathfrak{p}$ have closely related metric tensors

$$\begin{array}{c}\text{in } \mathfrak{p} \leftrightarrow \text{in } i\mathfrak{p} \\ (X, Y) \leftrightarrow (iX, iY) = -(X, Y)\end{array} \tag{4.12}$$

Since the inner product on compact $\mathfrak{p}$ in $\mathfrak{g}$ is negative definite, the inner product in noncompact $i\mathfrak{p}$ in $\mathfrak{g}^*$ is positive definite. A choice of bases which diagonalizes the metric tensor in $\mathfrak{p}$ also diagonalizes it in $i\mathfrak{p}$.

Index is a concept that carries over from the algebra $\mathfrak{g}$ to the subspace $\mathfrak{p}$:

$$\frac{(X, Y)_{\Gamma}}{f(\Gamma)} = (X, Y) = \frac{(X, Y)_{\text{reg}}}{f(\text{reg})} \qquad X, Y \in \mathfrak{p} \tag{4.13c}$$

The inner product within any representation Γ^* of $i\mathfrak{p}$ is

$$+\,(iX, iY)_{\Gamma*} = \operatorname{tr} \Gamma(iX)\Gamma(iY) = -(X, Y)_{\Gamma} \tag{4.14}$$

The index for the space $i\mathfrak{p}$ is defined by

$$\frac{+(X, Y)_{\Gamma*}}{f(\Gamma^*)} = (X, Y) = \frac{(X, Y)_{\text{reg}*}}{f(\text{reg}^*)} \qquad X, Y \in i\mathfrak{p} \tag{4.13n}$$

Therefore, the indices for the dual spaces $\mathfrak{p}$, $i\mathfrak{p}$ are identical in the dual representations $\Gamma(\mathfrak{p})$ and $\Gamma^*(\mathfrak{p}) = \Gamma(i\mathfrak{p})$:

$$f(\Gamma) \equiv f(\Gamma^*) \tag{4.15}$$

Comment. It is now apparent why the coset spaces EXP$\mathfrak{p}$ are called **Riemannian globally symmetric spaces**. First of all, they are well-defined spaces with a topology derived from the underlying Lie group. They are globally symmetric because any point and its neighborhood can be moved to any other point and its neighborhood by a particular group operation.

The metric in the subspace $\mathfrak{p}$ ($i\mathfrak{p}$) can be identified with the metric on the coset EXP $\mathfrak{p}$ (EXP $i\mathfrak{p}$) at the identity group element:

$$g_{ij}\,(\text{in }\mathfrak{p}) \equiv g_{ij}\,(\text{identity in EXP }\mathfrak{p}) \tag{4.16}$$

The global symmetry can then be used to move the metric $g_{ij}(\text{Id})$ to other points on the space. Since the metric in $\mathfrak{p}$ is negative definite whereas in $i\mathfrak{p}$ it is positive definite, the metric on the coset is definite; therefore, it may be taken as positive definite. Thus the coset space is endowed with a positive definite metric which is naturally derived from the Cartan-Killing metric in the Lie algebra. This metric will be computed explicitly in Section V.5 for various classical off-diagonal coset spaces.

3. RANK. Rank can be defined for the subspaces $\mathfrak{p}$ just as it is defined for the algebra $\mathfrak{g}$. For $\mathfrak{g}$ it is defined as the maximal number of functionally independent solutions of the secular equation

$$\|\mathbf{R}(\mathfrak{k} \oplus \mathfrak{p}) - \lambda I_\eta\| = \|\mathbf{R}(\mathfrak{k}) \oplus \mathbf{R}(\mathfrak{p}) - \lambda I_\eta\| = 0 \tag{4.17}$$

The rank of the subspace $\mathfrak{p}$ is then the maximal number of functionally independent solutions of the equation

$$\|\mathbf{R}(\mathfrak{p}) - \lambda I_\eta\| = 0 \tag{4.18}$$

In the $\eta \times \eta$ regular matrix representation, $\mathbf{R}(\mathfrak{p})$ has the structure shown in (4.1). Therefore, we must construct the eigenvalue equation from

$$\left\| \left[\begin{array}{c|c} 0 & B \\ \hline \pm B^\dagger & 0 \end{array}\right] - \lambda I_\eta \right\| = 0 \qquad \begin{array}{c} + \leftrightarrow i\mathfrak{p} \\ - \leftrightarrow \mathfrak{p} \end{array} \tag{4.19}$$

$$\begin{array}{cc} & \dim \mathfrak{p} \\ \begin{array}{c} \uparrow \\ \dim \mathfrak{k} \\ \downarrow \end{array} & \left[\begin{array}{c} \\ B \\ \\ \end{array}\right] \end{array}$$

It may easily be verified that the nonzero eigenvalues of (4.19) are identical with the nonzero eigenvalues λ of

$$\|\pm B^\dagger B - (-\lambda)^2 I_{\dim \mathfrak{p}}\|, \qquad \|\pm B B^\dagger - (-\lambda)^2 I_{\dim \mathfrak{k}}\| \tag{4.20}$$

Once again, it is not necessary to compute either the rank or the secular equation from the regular representation $\mathbf{R}(\mathfrak{p})$: any faithful matrix representation will suffice. In general, the computation proceeds most easily in the defining matrix representation.

Example 1. The "Lie algebra" for the coset space $SO(p, 2)/SO(p) \otimes SO(2)$ is

$$
i\mathfrak{p} \leftrightarrow \left[\begin{array}{cccc|cc}
 & & & & a_1 & b_1 \\
 & & & & a_2 & b_2 \\
 & & \bigcirc & & \vdots & \vdots \\
 & & & & a_p & b_p \\
\hline
a_1 & a_2 & \cdots & a_p & & \\
b_1 & b_2 & \cdots & b_p & & \bigcirc
\end{array}\right] \tag{4.21}
$$

The nonzero eigenvalues of this matrix are found by solving

$$
\left\| \begin{bmatrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_p \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_p & b_p \end{bmatrix} - (-\lambda)^2 I_2 \right\| = \left\| \begin{bmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{bmatrix} - (-\lambda)^2 I_2 \right\| = 0 \tag{4.22}
$$

The eigenvalues are computed from the secular equation

$$
[(-\lambda)^2]^2 - (-\lambda)^2 \operatorname{tr} \begin{bmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{bmatrix} + \det \begin{bmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{bmatrix} = 0
$$

$$
(-\lambda_\pm)^2 = \tfrac{1}{2}\operatorname{tr} \pm [(\tfrac{1}{2}\operatorname{tr})^2 - \det]^{1/2} \tag{4.23}
$$

Since there are two functionally independent roots to the eigenvalue equation, the rank of $SO(p, 2)/SO(p) \otimes SO(2)$ is 2.

Example 2. The coset space $USp(2p, 2)/USp(2p) \otimes USp(2)$ is generated by the exponentials of the subspace

$$
i\mathfrak{p} \leftrightarrow \left[\begin{array}{cc|cc}
 & & a & b \\
 & \bigcirc & b^* & -a^* \\
\hline
a^\dagger & b^t & & \\
b^\dagger & -a^t & & \bigcirc
\end{array}\right] \tag{4.24}
$$

where (a), (b) are each $p \times 1$ submatrices.

The independent eigenvalues are determined from

$$
\left\| \left[\begin{array}{c|c} a^{t*} & b^t \\ \hline b^{t*} & -a^t \end{array}\right] \left[\begin{array}{c|c} a & b \\ \hline b^* & -a^* \end{array}\right] - (-\lambda)^2 I_2 \right\|
$$

$$
\left\| \left[\begin{array}{c|c} \mathbf{a}^*\cdot\mathbf{a} + \mathbf{b}^*\cdot\mathbf{b} & 0 \\ \hline 0 & \mathbf{a}^*\cdot\mathbf{a} + \mathbf{b}^*\cdot\mathbf{b} \end{array}\right] - (-\lambda)^2 I_2 \right\| = 0 \tag{4.25}
$$

Since there can be only one independent eigenvalue, the rank of this space is 1.

The ranks of the remaining coset spaces are easily determined:

1. The spaces

$$\frac{SO(r, s)}{SO(r) \otimes SO(s)}, \qquad \frac{U(r, s; c)}{U(r; c) \otimes U(s; c)}, \qquad \frac{U(r, s; q)}{U(r; q) \otimes U(s; q)}$$

are constructed by exponentiating the matrices

$$\begin{bmatrix} 0 & B \\ B^\dagger & 0 \end{bmatrix} \tag{4.26^1}$$

where B is an $r \times s$ matrix with arbitrary real, complex, and quaternion entries, respectively. The matrices

$$B^\dagger B, \qquad BB^\dagger$$

have min (r, s) independent eigenvalues.

2. The spaces

$$\frac{Sp(2n, r)}{U(n)}, \qquad \frac{SO^*(2n)}{U(n)}$$

are constructed from the subspace

$$i\mathfrak{p} \to \begin{bmatrix} 0 & M \\ M^\dagger & 0 \end{bmatrix} \qquad M^t = \pm M \tag{4.26^2}$$

The $n \times n$ matrices $M^\dagger M$ have n independent roots when M is symmetric. When M is antisymmetric, the nonzero roots are all doubled

$$\begin{array}{llll} & M^t = +M \nearrow & \text{rank } n & \dfrac{Sp(2n, r)}{U(n)} \\ M^\dagger M & & & \\ & M^t = -M \searrow & \text{rank } \left[\dfrac{n}{2}\right] & \dfrac{SO^*(2n)}{U(n)} \end{array}$$

3. The spaces

$$\frac{Sl(n, r)}{SO(n)}, \qquad \frac{SU^*(2n)}{USp(2n)}$$

are constructed by exponentiating the irreducible matrices

$$(A) \qquad \left[\begin{array}{c|c} A_+ & B_- \\ \hline B_-^\dagger & \tilde{A}_+ \end{array}\right] \tag{4.26^3}$$

$$\operatorname{tr}(A) = 0 \qquad \operatorname{tr}(A_+) = 0$$

The trace condition reduces the number of independent eigenvalues from n to $n - 1$ in each case.

Let $X_\alpha, X_\beta, \ldots,$ be the bases for the subalgebra $\mathfrak{k}$ and $X_i, X_j, \ldots,$ be bases for the subspace $\mathfrak{p}$. Then the coefficients $\phi_r(a^\alpha, a^i)$ of the secular equation for $\mathfrak{g}$ are determined from

$$\|\Gamma^{\text{faith}}(a^\alpha X_\alpha + a^i X_i) - \lambda I_n\| = \sum \lambda^{n-r}\phi_r(a^\alpha, a^i) = 0 \tag{4.27}$$

We have already remarked that the substitution

$$\begin{aligned} a^\alpha &\to X_\alpha \\ a^i &\to X_i \\ \phi_r(a^\alpha, a^i) &\to \mathscr{C}_r(X_\alpha, X_i) \end{aligned}$$

leads to homogeneous polynomial operators that commute with all generators X_β, X_i :

$$\begin{aligned} [\mathscr{C}_r(X_\alpha, X_i), X_\beta] &= 0 \\ [\mathscr{C}_r(X_\alpha, X_i), X_j] &= 0 \end{aligned} \tag{4.28}$$

Operators satisfying (4.28) are said to be *invariant*. These operators are known as **Casimir operators.**

When the Lie group $G = \text{EXP}\,\mathfrak{g}$ acts on the coset space $G/K = \text{EXP}\,\mathfrak{p}$, the operators X_α, X_i become first-order differential operators defined on the coordinates of the coset space:

$$X_\alpha \xrightarrow[\text{EXP}\,\mathfrak{p} = G/K]{\text{defined on}} X_\alpha\left(\frac{G}{K}\right) \tag{4.29α}$$

$$X_i \xrightarrow{\qquad\qquad} X_i\left(\frac{G}{K}\right) \tag{4.29i}$$

The Casimir operators are now invariant operators defined on the Riemannian symmetric space G/K. These operators[8–12] are called **Laplace-Beltrami operators Δ_r** :

$$\Delta_r(X_\alpha, X_i) \equiv \mathscr{C}_r\left(X_\alpha\left(\frac{G}{K}\right), X_i\left(\frac{G}{K}\right)\right) \tag{4.30}$$

The number of independent Casimir invariant operators for a group is equal to the rank of the group; the number of independent Laplace-Beltrami operators on a Riemannian symmetric coset space is equal to the rank of the space.

In a coset space of rank r, the first r Casimir invariants remain nontrivial and become the r-independent Laplace-Beltrami operators under (4.30). The remaining higher order Casimir operators, when restricted to the rank r coset space EXP $\mathfrak{p}$, are functionally dependent on the first r independent Laplace-Beltrami operators.

In a semisimple Lie group of rank n it is possible to find n mutually commuting, simultaneously diagonalizable bases H_i spanning the Cartan subalgebra. Since the subspace $\mathfrak{p}$ is not closed under commutation, and in fact, since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, the commutator of any two elements in $\mathfrak{p}$ must either lie in $\mathfrak{k}$ or be zero. The very close relationship between the rank of a simple algebra $\mathfrak{g}$ and the rank of the orthogonal complementary subspace $\mathfrak{p}$ suggests that it should be possible to find exactly as many mutually commuting generators among the bases of $\mathfrak{p}$, as the rank of $\mathfrak{p}$. This is true. We present the results explicitly.

1. The structure of the subspace $i\mathfrak{p}$ for the cosets

$$\frac{SO(r, s)}{SO(r) \otimes SO(s)}, \qquad \frac{U(r, s; c)}{U(r, c) \otimes U(s, c)}, \qquad \frac{U(r, s; q)}{U(r, q) \otimes U(s, q)}$$

is given in (4.26¹). The bases in each case are

$$f^i\, \partial_j \pm \text{adj} \qquad \begin{cases} + \text{noncompact coset} \\ - \quad \text{compact coset} \end{cases}$$
$$(f^i)^\dagger = \partial_i \qquad 1 \le i \le r$$
$$(\partial_i)^\dagger = f^i \qquad r+1 \le j \le r+s$$

$$\text{where } f = \begin{cases} r & \text{real groups} & SO(n, r) \\ c & \text{complex groups} & U(n, c) \\ q & \text{quaternion groups} & U(n, q) \simeq USp(2n) \end{cases}$$

The mutually commuting generators are

$$f^i\, \partial_{r+i} \pm f^{r+i}\, \partial_i \tag{4.31¹}$$

There are min (r, s) such generators.

2. The coset spaces for

$$\frac{Sp(2n, r)}{U(n)}, \qquad \frac{SO^*(2n)}{U(n)}$$

are given by (4.26²). The bases for $i\mathfrak{p}(\mathfrak{p})$ are

$$(u^i\,\partial_{-j} \pm u^j\,\partial_{-i}) \pm \text{adj} \qquad 1 \le i, j \le n$$

The $\pm$ within the bracket distinguishes between the two groups; the $\pm$ sign outside the bracket distinguishes between the noncompact real form and the compact real form

(±) \ ( ) ± adj	+	−
+	$Sp(2n, r)$	$USp(2n)$
−	$SO^*(2n)$	$SO(2n)$

The mutually commuting generators are:

$$\frac{Sp(2n, r);\ USp(2n)}{U(n)} \qquad u^i\,\partial_{-i} \pm u^{-i}\,\partial_i$$

$$\frac{SO^*(2n);\ SO(2n)}{U(n)} \qquad [u^i\,\partial_{-i} - u^{n+1-i}\,\partial_{-(n+1-i)}] \pm \text{adj} \qquad (4.31^2)$$

There are n and $[n/2]$ generators, respectively.

3. By entirely similar arguments, the commuting operators are

$$\frac{Sl(n, r)}{SO(n)} : \qquad x^i\,\partial_i \qquad \operatorname{tr}(x^i\,\partial_i) = 0$$

$$\frac{SU^*(2n)}{USp(2n)} : \qquad x^i\,\partial_i + x^{-i}\,\partial_{-i} \qquad \operatorname{tr}(x^i\,\partial_i) = 0 \qquad (4.31^3)$$

There are $(n - 1)$ commuting generators.

The properties of the classical Riemannian symmetric coset spaces, including their dimension, their rank, and the explicit form for their mutually commuting generators, are given explicitly in Table 9.6.

Comment. If the mutually commuting generators in $\mathfrak{p}(i\mathfrak{p})$ are called

in $\mathfrak{p}$	in $i\mathfrak{p}$
ih_i	h_i

then the group operations of the form

$$e^{(i)\lambda^i h_i t} \qquad \lambda^i \text{ fixed, real}$$

TABLE 9.6

Root Space	Noncompact Coset	Dimension $\mathfrak{p}$	Rank $\mathfrak{p}$	Rank $\mathfrak{g}$	Rank* $\mathfrak{k}$	Deficit = + Rank $\mathfrak{p}$ + Rank $\mathfrak{k}$ − Rank $\mathfrak{g}$	Diagonal Generators, $i\mathfrak{p}$, $\mathfrak{p}$
D_n	$\frac{SO(p,q)}{SO(p)\otimes SO(q)}$	pq	$\min(p,q)$	$\left[\frac{p+q}{2}\right]$	$\left[\frac{p}{2}\right]+\left[\frac{q}{2}\right]$	$\min(p,q)$ q even; $\min(p,q)-1$ q odd	$x^i\,\partial_{p+i}\pm x^{p+i}\,\partial_i$
B_n	$\frac{SO(p,q)}{SO(p)\otimes SO(q)}$	pq	$\min(p,q)$	$\left[\frac{p+q}{2}\right]$	$\left[\frac{p}{2}\right]+\left[\frac{q}{2}\right]$	$\min(p,q)$	$x^i\,\partial_{p+i}\pm x^{p+i}\,\partial_i$
A_n	$\frac{U(p,q)}{U(p)\otimes U(q)}$	$2pq$	$\min(p,q)$	$p+q$	$p+q$	$\min(p,q)$	$u^i\,\partial_{p+i}\pm u^{p+i}\,\partial_i$
C_n	$\frac{USp(2p,2q)}{USp(2p)\otimes USp(2q)}$	$4pq$	$\min(p,q)$	$p+q$	$p+q$	$\min(p,q)$	$q^i\,\partial_{p+i}\pm q^{p+i}\,\partial_i$
C_n	$\frac{Sp(2n,r)}{U(n)}$	$n(n+1)$	n	n	n	n	$u^i\,\partial_{-i}\pm u^{-i}\,\partial_i$
D_n	$\frac{SO^*(2n)}{U(n)}$	$n(n-1)$	$\left[\frac{n}{2}\right]$	n	n	$\left[\frac{n}{2}\right]$	$x^i\,\partial_{-i}-x^{n+1-i}\,\partial_{-(n+1-i)}\pm\text{adj}$
A_n	$\frac{Sl(n,r)}{SO(n,r)}$	$\frac{n(n+1)}{2}-1$	$n-1$	$n-1$	$\left[\frac{n}{2}\right]$	$\left[\frac{n}{2}\right]$	$x^i\,\partial_i$; $\text{tr}\,(x^i\,\partial_i)=0$
A_n	$\frac{SU^*(2n)}{USp(2n)}$	$\frac{2n(2n-1)}{2}-1$	$n-1$	$2n-1$	n	0	$x^i\,\partial_i+x^{-i}\,\partial_{-i}$; $\text{tr}\,(x^i\,\partial_i)=0$

G_2	$\dfrac{G_{2(+2)}}{A_1 \oplus A_1}$	8	2	2	$1 + 1$	2
F_4	$\dfrac{F_{4(-20)}}{B_4}$	16	1	4	4	1
	$\dfrac{F_{4(+4)}}{C_3 \oplus A_1}$	28	4	4	$3 + 1$	4
E_6	$\dfrac{E_{6(-26)}}{F_4}$	26	2	6	4	0
	$\dfrac{E_{6(-14)}}{D_5 \oplus D_1}$	32	2	6	$5 + 1$	2
	$\dfrac{E_{6(+2)}}{A_5 \oplus A_1}$	40	4	6	$5 + 1$	4
	$\dfrac{E_{6(+6)}}{C_4}$	42	6	6	4	4
E_7	$\dfrac{E_{7(-25)}}{E_6 \oplus D_1}$	54	3	7	$6 + 1$	3
	$\dfrac{E_{7(-5)}}{D_6 \oplus A_1}$	64	4	7	$6 + 1$	4
	$\dfrac{E_{7(+7)}}{A_7}$	70	7	7	7	7
E_8	$\dfrac{E_{8(-24)}}{E_7 \oplus A_1}$	112	4	8	$7 + 1$	4
	$\dfrac{E_{8(+8)}}{D_8}$	128	8	8	8	8

* In this Table we have assumed rank $D_1 = 1$.

1. Are geodesics in G^*/K (G/K).
2. Are elements in an abelian subgroup of G/K.

Since the curvature of an abelian subgroup is zero, the maximal dimension of a flat, totally geodesic submanifold of G/K or G^*/K is equal to the rank of this symmetric space.

To show the very close correspondence between the rank of a simple Lie group G and the rank of a Riemannian symmetric coset space G/K, we juxtapose their properties below:

The rank of a semisimple group G is:

1. The maximal number of functionally independent roots of the secular equation

$$\|\boldsymbol{\Gamma}(\mathfrak{g}) - \lambda I\| = 0$$

in the regular (or any faithful) representation.

2. The number of functionally independent coefficients $\phi_r(a^\alpha, a^i)$ in the secular equation.
3. The number of functionally independent Casimir invariants.
4. The maximal number of mutually commuting generators H_i in the Cartan subalgebra of $\mathfrak{g}$.
5. The number of linearly independent roots in the root space.

The rank of a Riemannian symmetric space $\mathfrak{p}$ is:

1. The maximal number of functionally independent roots of the secular equation

$$\|\boldsymbol{\Gamma}(\mathfrak{p}) - \lambda I\| = 0$$

in the regular (or any faithful) representation.

2. The number of functionally independent coefficients $\phi_r(a^\alpha = 0, a^i)$ in the secular equation.
3. The number of functionally independent Laplace-Beltrami operators defined on G/K.
4. The maximal number of mutually commuting generators h_i in the orthogonal complementary subspace $\mathfrak{p}$.
5. The maximal dimension of a flat (abelian) totally geodesic submanifold of G/K.

V. Analytical Properties of Cosets

1. STRUCTURE OF THE CLASSICAL COSETS. All but two of the classical cosets are exponentials of off-diagonal matrices

$$\left[\begin{array}{c|c} 0 & B \\ \hline \pm B^{\dagger} & 0 \end{array}\right] \tag{5.1}$$

To better understand the structure of these classical off-diagonal cosets, we first prove the following theorem.

THEOREM. *If* f(x) *is an analytic function and* M, N *are* m × n *and* n × m *matrices,*

$$\mathrm{Mf(NM) = f(MN)M}$$

whenever the n × n *matrix* f(NM) *and the* m × m *matrix* f(MN) *are defined.*

Proof. Since $f(x)$ is analytic

$$f(x) = \sum f^k(0)\frac{x^k}{k!}, \qquad f^k(0) = \left(\frac{d}{dx}\right)^k f(x)\bigg|_{x=0}$$

Then

$$\begin{aligned} Mf(NM) &= M\left\{\sum f^k(0)\frac{(NM)^k}{k!}\right\} \\ &= \left\{\sum f^k(0)\frac{(MN)^k}{k!}\right\}M \\ &= f(MN)M \end{aligned} \tag{5.2}$$

The exponentials of the off-diagonal linear vector subspaces $\mathfrak{p}$, $i\mathfrak{p}$ have been given in (4.5):

$$\text{EXP } i\mathfrak{p} = \text{EXP}\begin{bmatrix} 0 & B \\ +B^{\dagger} & 0 \end{bmatrix} = \left[\begin{array}{c|c} \cosh (BB^{\dagger})^{1/2} & B\dfrac{\sinh (B^{\dagger}B)^{1/2}}{(B^{\dagger}B)^{1/2}} \\ \hline B^{\dagger}\dfrac{\sinh (BB^{\dagger})^{1/2}}{(BB^{\dagger})^{1/2}} & \cosh (B^{\dagger}B)^{1/2} \end{array}\right] \tag{5.3}$$

The matrices $B^{\dagger}B$ and $BB^{\dagger}$ are hermitian with nonnegative eigenvalues. In fact, their nonzero eigenvalues are identical. Thus all square roots are well defined and can be computed explicitly.

The trigonometric identities valid for the hyperbolic functions are valid also for the submatrices appearing in (5.3). If B is an $m \times n$ matrix

$$\cosh^2 (B^\dagger B)^{1/2} - \left[B\frac{\sinh (B^\dagger B)^{1/2}}{(B^\dagger B)^{1/2}}\right]^\dagger \left[B\frac{\sinh (B^\dagger B)^{1/2}}{(B^\dagger B)^{1/2}}\right]$$

$$= \cosh^2 (B^\dagger B)^{1/2} - \frac{\sinh (B^\dagger B)^{1/2}}{(B^\dagger B)^{1/2}} B^\dagger B \frac{\sinh (B^\dagger B)^{1/2}}{(B^\dagger B)^{1/2}}$$

$$= \cosh^2 (B^\dagger B)^{1/2} - \sinh^2 (B^\dagger B)^{1/2} = I_n \tag{5.4}$$

Similarly,

$$\cosh^2 (BB^\dagger)^{1/2} - \left[B\frac{\sinh (B^\dagger B)^{1/2}}{(B^\dagger B)^{1/2}}\right]\left[B\frac{\sinh (B^\dagger B)^{1/2}}{(B^\dagger B)^{1/2}}\right]^\dagger = I_m \tag{5.5}$$

The relation (5.3) shows that the properties of the submatrix B, characterizing the complementary subspace $i\mathfrak{p}$ in the Lie algebra of $\mathfrak{g}^*$, are transferred directly to the coset representatives by the exponential map:

$$\begin{array}{ccc} \text{in } i\mathfrak{p} & & \text{in } G^*/K \\ \hline B & \xrightarrow{\text{EXP}} & B\dfrac{\sinh (B^\dagger B)^{1/2}}{(B^\dagger B)^{1/2}} \end{array} \tag{5.6}$$

Thus the description of B given in Table 9.3 carries over to the description of the coset representatives. But it is very inconvenient to use the relation (5.6) to describe the points in G^*/K, since (5.6) involves both a transcendental function and a square root. To avoid these difficulties, we simply describe the points in G^*/K by the $m \times n$ submatrix X:

$$X \equiv B\frac{\sinh (B^\dagger B)^{1/2}}{(B^\dagger B)^{1/2}} \tag{5.7}$$

Then by (5.4) and (5.5) the coset representatives are given, in terms of X, by

$$G^*/K = \left[\begin{array}{c|c} [I_m + XX^\dagger]^{1/2} & X \\ \hline +X^\dagger & [I_n + X^\dagger X]^{1/2} \end{array}\right] \tag{5.8n}$$

Since $X^\dagger X$ and $XX^\dagger$ have nonnegative eigenvalues, the diagonal submatrices are well defined and never singular.

A similar set of calculations can be carried out for the compact dual space $G/K = \text{EXP } \mathfrak{p}$ of G^*/K. The major change is to replace the hyperbolic by circular functions. The coset representatives are

$$G/K = \left[\begin{array}{c|c} [I_m - XX^\dagger]^{1/2} & X \\ \hline -X^\dagger & [I_n - X^\dagger X]^{1/2} \end{array}\right] \tag{5.8c}$$

The range of the parameters describing the submatrix X is determined by the demand that the coset representatives (5.8) remain group elements. More explicitly, this is written

$$0 \le X^\dagger X \le I_n \tag{5.9}$$

The inequality refers to the eigenvalues of $X^\dagger X$. Since $X^\dagger X$ and $XX^\dagger$ have the same nonzero set of eigenvalues

$$0 \le X^\dagger X \le I_n \Leftrightarrow 0 \le XX^\dagger \le I_m \tag{5.9$'$}$$

The domain of the parameters describing the submatrix X is bounded by the inequalities (5.9′).

Example.

$$\frac{SO(n+1)}{SO(n)} = \mathrm{EXP}\left[\begin{array}{ccc|c} & & & b_1 \\ & & & b_2 \\ & \bigcirc & & \\ & & & b_n \\ \hline -b_1 & & -b_n & \bigcirc \end{array}\right] = \mathrm{EXP}\left[\begin{array}{c|c} \bigcirc & \mathbf{b} \\ \hline -\mathbf{b}^t & \bigcirc \end{array}\right]$$

$$= \left[\begin{array}{c|c} [I_n - xx^t]^{1/2} & \mathbf{x} \\ \hline -\mathbf{x}^t & [1 - x^t x]^{1/2} \end{array}\right] \qquad x_i = b_i \frac{\sin\left(\sum b_j^2\right)^{1/2}}{\left(\sum b_j^2\right)^{1/2}}$$

The range of the parameters x_i is governed by

$$\sum_{i=1}^{n} x_i^2 \le 1$$

Setting $x_{n+1} = \pm\sqrt{1 - \sum x_i^2}$, the variables $x_i (i = 1, 2, \ldots, n+1)$ obey

$$x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2 = 1$$

The coset $SO(n+1)/SO(n)$ can be identified with the n-dimensional sphere embedded in $R_{n+1} : S^n \subset R_{n+1}$.

The two remaining classical Riemannian cosets can be constructed with similar ease.

1. $Sl(n, r)/SO(n, r)$. This coset is given by the exponential of hermitian symmetric (i.e., real) matrices e^H. These matrices obey

$$\begin{aligned} H^\dagger &= +H \\ H^t &= +H \qquad \mathrm{tr}\, H = 0 \end{aligned} \tag{5.10}$$

$$(e^H)^\dagger = e^{(H)^\dagger} = e^H$$
$$(e^H)^t = e^{(H)^t} = e^H \tag{5.11}$$

These matrices are unimodular, for

$$\|e^H\| = e^{\text{tr}\, H} = e^0 = +1 \tag{5.12}$$

The coset representatives of $Sl(n, r)/SO(n, r)$ consist of all real symmetric unimodular matrices.

2. $SU^*(2n)/USp(2n)$. These coset representatives are given by exponentials of hermitian antisymplectic matrices M [cf. (2.12′)] which obey

$$M^\dagger = +M$$
$$J_{n,n} M^t J_{n,n}^{-1} = +M \qquad \text{tr}\, M = 0 \tag{5.13}$$

The symmetry properties of the cosets e^M are inherited directly from the symmetry properties (5.13) in $i\mathfrak{p}$:

$$(e^M)^\dagger = e^{(M)^\dagger} = e^M$$
$$J_{n,n}(e^M)^t J_{n,n}^{-1} = e^{(J_{n,n} M^t J_{n,n}^{-1})} = e^{+M} \tag{5.14}$$

Once again, these matrices are unimodular

$$\|e^M\| = e^{\text{tr}\, M} = +1 \tag{5.15}$$

2. GLOBAL TRANSFORMATION PROPERTIES. The group G acts on the space of points G/K and maps it onto itself. In particular, the coset representatives $c \in G/K \subset G$ map the origin of G/K onto the point c in G/K. If c is any point in G/K, it can be mapped into any other point c' in G/K by some group operation:

$$(c'c^{-1})c \to c'$$
$$c'c^{-1} = g \in G \tag{5.16t}$$

Moreover, the only group operation which leaves *every* point c fixed is the identity

$$gc = c \qquad \text{for all } c \in G/K \qquad \Rightarrow \qquad g = \text{Id} \tag{5.16e}$$

Definition. A group G acts **transitively** on a space M if, for every pair of points $p, q \in M$, there is a group operation $g \in G$ with the property $g : p \to q$. We write this symbolically as

$$p \xrightarrow{g} q = gp \qquad \begin{array}{ll} \text{for every pair} & p, q \in M \\ \text{for some} & g \in G \end{array}$$

Definition. A group G acts **effectively** on a space M if the identity operation is the only group operation which leaves every point $p \in M$ fixed. We write this symbolically as

$$gp = p \qquad \text{for all } p \in M \qquad \Rightarrow \qquad g = \mathrm{Id}$$

Definition. The **orbit** of the point $p \in M$ under the group G is the set of all points $q \in M$ that can be reached by applying some group operation $g \in G$ to the point p.

Example. The north pole is mapped onto any point on the surface of a sphere by elements in the coset $SO(3)/SO(2)$

$$\left\{ \mathrm{EXP} \left[\begin{array}{cc|c} & & b_1 \\ & & b_2 \\ \hline -b_1 & -b_2 & \end{array} \right] \right\} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \left[\begin{array}{cc|c} & & x \\ & & y \\ \hline -x & -y & z \end{array} \right] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{5.17}$$

$$x^2 + y^2 + z^2 = 1$$

The sphere $S^2 \subset R_3$ is the **orbit** of the north pole under the operations of $SO(3)$. In fact, S^2 is the orbit of the north pole under the coset representatives $SO(3)/SO(2)$. We identify the sphere S^2 with the coset $SO(3)/SO(2)$.

The action of the group G on the space G/K is given simply by

$$gc = c'k \qquad \begin{cases} g \in G \\ c, c' \in \dfrac{G}{K} \\ k \in K \end{cases} \tag{5.18}$$

Since the product of two elements in G (namely, g and c) is an element in G, this element can be written uniquely in a coset decomposition. Then the mapping of G/K onto itself is given by

$$G : \frac{G}{K} \xrightarrow{\;g\;} \frac{G}{K}$$

$$c \xrightarrow{\;g\;} c' \qquad c, c' \in \frac{G}{K} \tag{5.19}$$

The subset of elements in G which leaves any point fixed forms a group, called the **stability group** $K(c)$ of c:

$$kc = c \qquad \begin{matrix} k \in K(c) \\ c \in \dfrac{G}{K} \end{matrix} \tag{5.20}$$

Since c is obtained by applying the group operation c to the origin Id of G/K, (5.20) can be expressed

$$c^{-1}kc \text{ Id} = \text{Id} \qquad \text{Id} \in \frac{G}{K}$$

$$k \in K(c)$$

$$c^{-1}kc \in K(\text{Id} = 0) \tag{5.21}$$

In other words, from (5.21) it is clear that the stability group $K(c)$ of any point c is conjugate to the stability group of the identity

$$K(c) = cK(\text{Id})c^{-1} \tag{5.22}$$

The stability group $K(\text{Id})$ is simple to compute:

$$K(\text{Id})\text{Id} = \text{Id} \in \frac{G}{K}$$

$$K(\text{Id}) = \text{Id} \times K \subset G \tag{5.23}$$

Therefore, $K(\text{Id})$ is just the group K itself.

Example. The stability subgroup of the north pole col (0, 0, ..., 0, 1) of $S^{n-1} \subset R_n$ is $SO(n-1)$

$$\left[\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & SO(n-1) & & \vdots \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The stability subgroup for any point on the sphere $S^{n-1} \subset R_n$ is conjugate:

$$K \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = c \left[\begin{array}{c|c} SO(n-1) & 0 \\ \hline 0 & 1 \end{array}\right] c^{-1}$$

$$c = \left[\begin{array}{ccc|c} & & & x_1 \\ & [I_{n-1} - xx^t]^{1/2} & & \vdots \\ & & & x_{n-1} \\ \hline -x_1 & \cdots & -x_{n-1} & [1 - x^t x]^{1/2} \end{array}\right]$$

We now determine explicitly the structure of the mapping of G/K onto itself under an arbitrary element of the group G. We will do this only for the off-diagonal cosets. In addition, we will carry the computation out for the noncompact real form G^*; the computation for the compact form G is related simply by a change of sign.

An arbitrary element of G^* has a block structure

$$g \in G^* = \begin{array}{cc} \overleftrightarrow{\;m\;} & \overleftrightarrow{\;n\;} \\ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array} \begin{array}{c} \updownarrow m \\ \updownarrow n \end{array} \tag{5.24G^*}$$

Various relations exist between the submatrices A, B, C, D depending on which classical group G^* is being investigated. A coset representative in G^*/K has the form

$$c \in \frac{G^*}{K} = \left[\begin{array}{c|c} W & X \\ \hline X^\dagger & Y \end{array}\right] \tag{5.24G^*/K}$$

$$\begin{aligned} W &= [I_m + XX^\dagger]^{1/2} \\ Y &= [I_n + X^\dagger X]^{1/2} \end{aligned} \tag{5.25}$$

Since the eigenvalues of $XX^\dagger$ are nonnegative, the square submatrices W, Y are hermitian and nonsingular; their inverses always exist.

The action of G on G/K, given abstractly in (5.19), is given concretely for off-diagonal cosets, by the matrix multiplication

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} W & X \\ \hline X^\dagger & Y \end{array}\right] = \left[\begin{array}{c|c} W' & X' \\ \hline X'^\dagger & Y' \end{array}\right] \left[\begin{array}{c|c} U(m) & 0 \\ \hline 0 & U(n) \end{array}\right] \tag{5.26}$$

Since the coset representatives are the submatrices X, it is necessary to solve (5.26) for X' in terms of X and A, B, C, D. This is done by solving the equation

$$\left[\begin{array}{c|c} * & AX + BY \\ \hline * & CX + DY \end{array}\right] = \left[\begin{array}{c|c} * & X'U(n) \\ \hline * & Y'U(n) \end{array}\right] \tag{5.27}$$

The solution of (5.27) is unpleasantly complicated because of the presence of the group operation $U(n)$.

To get around this difficulty we observe that there is a 1-1 correspondence between the coset representatives X and the submatrices Z defined by

$$
\begin{aligned}
Z &= XY^{-1} = X[I_n + X^\dagger X]^{-1/2} \\
X &= \qquad\quad = Z[I_n - Z^\dagger Z]^{-1/2}
\end{aligned} \tag{5.28}
$$

Since X and Z are related to each other in a 1-1 fashion, the Z may just as well be taken as the coset representatives. All calculations may be performed in terms of the variables Z instead of X.

For the noncompact cosets G^*/K, X is unbounded and Z is bounded, for

$$
\begin{gathered}
Y^\dagger Y - X^\dagger X = I_n \\
I_n - (XY^{-1})^\dagger (XY^{-1}) = (Y^{-1})^\dagger (Y^{-1}) \\
0 \le X^\dagger X < \infty \qquad \text{and} \qquad 0 \le Z^\dagger Z < 1
\end{gathered} \tag{5.29}
$$

For the dual compact cosets G/K, X is bounded and, therefore, Z is unbounded:

$$
0 \le X^\dagger X < 1 \Leftrightarrow 0 \le Z^\dagger Z < \infty
$$

The coset representatives are related projectively. Therefore the coordinates Z are called the **projective** coordinates of the coset G/K or G^*/K.

Example 1. The coset representatives of the sphere $S^2 \subset R_3$ are

$$
\text{EXP}\left[\begin{array}{cc|c} & & b_1 \\ & & b_2 \\ \hline -b_1 & -b_2 & \end{array}\right] \to \left[\begin{array}{cc|c} & & x_1 \\ & & x_2 \\ \hline -x_1 & -x_2 & x_3 \end{array}\right]
$$

$$
x_1{}^2 + x_2{}^2 + x_3{}^2 = 1 \qquad x_3 > 0
$$

The projective coordinates of the upper hemisphere are

$$
-\infty < z_i = \frac{x_i}{\sqrt{1 - x_1{}^2 - x_2{}^2}} < +\infty, \qquad i = 1, 2
$$

The space $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is called the real projective space PR_2 (Fig. 9.3).

Example 2. The coset representatives of the hyperboloid $H^2 \subset R_3$ are

$$
\text{EXP}\left[\begin{array}{cc|c} & & b_1 \\ & & b_2 \\ \hline b_1 & b_2 & \end{array}\right] \to \left[\begin{array}{cc|c} & & x_1 \\ & & x_2 \\ \hline x_1 & x_2 & x_3 \end{array}\right]
$$

$$
\begin{gathered}
x_3{}^2 - x_2{}^2 - x_1{}^2 = 1 \\
-\infty < x_1, x_2 < +\infty
\end{gathered}
$$

The projective coordinates of the hyperboloid are given by

$$-1 < z_i = \frac{x_i}{\sqrt{1 + x_1{}^2 + x_2{}^2}} < +1 \qquad i = 1, 2$$

The relation between coordinates x_i and z_i is shown in Fig. 9.4.

Example 3. The rank-1 cosets $U(n+1;f)/U(n;f) \otimes U(1;f)$ $(f = r, c, q)$ are given by

$$\text{EXP}\left[\begin{array}{c|c} & \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix} \\ \hline -b^\dagger & \end{array}\right] \rightarrow \left[\begin{array}{c|c} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \\ \hline -x^\dagger & r_{n+1} = (1 - x^\dagger x)^{1/2} \end{array}\right]$$

The coset representatives col $(x_1, x_2, \ldots, x_n)$ are bounded by

$$0 \leq x^\dagger x \leq 1$$

The projective coordinates z_i are unbounded

$$0 \leq z^\dagger z < \infty$$

The coset representatives z are points in the **real**, **complex**, or **quaternion projective spaces**:

$$\frac{U(n+1;f)}{U(n;f) \otimes U(1;f)} = \underset{\text{bounded}}{\left(X\right)} \rightarrow \underset{\text{unbounded}}{\left(Z\right) = \left(X\right)(Y)^{-1}} \begin{cases} \rightarrow PR_n & f = r, \text{ real} \\ \rightarrow PC_n & f = c, \text{ complex} \\ \rightarrow PQ_n & f = q, \text{ quaternion} \end{cases} \tag{5.30}$$

Example 4. The rank-1 cosets dual to those of the previous example are

$$\frac{U(n, 1;f)}{U(n;f) \otimes U(1;f)} \rightarrow \left[\begin{array}{c|c} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \\ \hline +x^\dagger & r_{n+1} = (1 + x^\dagger x)^{1/2} \end{array}\right]$$

The coset representatives X are unbounded; the projectively related coset representatives $Z = XY^{-1}$ are therefore bounded and exist in the **real**, **complex**, or **quaternion projective sphere**

$$\frac{U(n, 1; f)}{U(n; f) \otimes U(1; f)} = \begin{pmatrix} \\ X \\ \\ \end{pmatrix} \to \begin{pmatrix} \\ X \\ \\ \end{pmatrix} (Y)^{-1} = \begin{pmatrix} \\ Z \\ \\ \end{pmatrix} \begin{array}{l} \nearrow PRS^n \quad f = r, \text{ real} \\ \to PCS^n \quad f = c, \text{ complex} \\ \searrow PQS^n \quad f = q, \text{ quaternion} \end{array}$$

unbounded bounded

$$(5.31)$$

The noncompact cosets $U(n, 1; f)/U(n; f) \otimes U(1; f)$ are sometimes also called **real, complex**, and **quaternion hyperbolic spaces** when $f = r, c, q$.

The reason for introducing projective coset representatives should be apparent by inspection of (5.27). The transformation properties of the projective coset representatives can be computed explicitly. Not only that, but these transformation properties are surprisingly and unexpectedly simple:

$$\begin{aligned} (AX + BY)(CX + DY)^{-1} &= X'U(n)[Y'U(n)]^{-1} \\ \{(AXY^{-1} + B)Y\}\{(CXY^{-1} + D)Y\}^{-1} &= X'\{U(n)U^{-1}(n)\}Y'^{-1} \\ (AZ + B)(CZ + D)^{-1} &= Z' \end{aligned} \tag{5.32}$$

The projective coset representatives undergo fractional linear transformations under an arbitrary group operation (5.24).

Comment 1. These properties have been known for a long time and thoroughly studied for a single complex variable

$$z \simeq P\left\{\frac{SU(2)}{U(1)}\right\}$$

Equation (5.32) describes a step in the generalization of the study of a single complex variable to the theory of several complex variables. The projective mapping (5.28) is the well-known conformal mapping of the interior of the unit disc onto the upper half-plane.[13,14] The transformation (5.32) is analytic in the submatrices A, B, C, D in that it is a function only on these matrices, not their transposes, adjoints, or conjugates.

Comment 2. We emphasize that the projective transformation properties represented by (5.32) hold for *all* off-diagonal cosets. That is, they hold for all but two of the classical Riemannian symmetric spaces.

Although the transformation (5.32) is simple, it is nonlinear. The transformation properties of X, obtainable from (5.32) and (5.28), are very complicated. It is possible to describe the points in G^*/K in such a way that they undergo linear transformation properties under the action of G^*. However, this can only be done at the expense of a 1-1 relation between these "representatives" and the points in G^*/K.

Since each point in G^*/K is described uniquely by the submatrix X, it is also uniquely described by the matrix

$$T = \begin{array}{c c} \overset{\longleftarrow n \longrightarrow}{} & \\ \left[\begin{array}{c} X \\ \hline Y \end{array}\right] & \begin{array}{c} \updownarrow m \\ \updownarrow n \end{array} \end{array} \tag{5.33a}$$

$$Y = [I_n + X^\dagger X]^{1/2} \tag{5.33b}$$

The submatrix T obeys whatever conditions it inherits from G^*. For example, for $G^* = U(m, n; f)$,

$$\left[\begin{array}{c} \\ \hline T^\dagger \end{array}\right] \left[\begin{array}{c|c} I_m & \\ \hline & -I_n \end{array}\right] \left[\begin{array}{c|c} & T \end{array}\right] = \left[\begin{array}{c|c} I_m & \\ \hline & -I_n \end{array}\right]$$

$$T^\dagger I_{m,n} T = X^\dagger X - Y^\dagger Y = -I_n \tag{5.34}$$

It is clear that if T satisfies (5.34), then so also does gT [g as in (5.24)], for

$$T^\dagger(g^\dagger I_{m,n} g)T = T^\dagger I_{m,n} T = -I_n \tag{5.35}$$

If we now relax the constraint (5.33b), there is a many $\rightarrow$ 1 correspondence between the matrices T and the points in G^*/K. In compensation for this nonuniqueness, the transformation properties of T are linear:

$$T = \begin{bmatrix} X \\ Y \end{bmatrix} \rightarrow T' = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X' = AX + BY \\ Y' = CX + DY \end{bmatrix} \tag{5.36}$$

Comment. The coordinates $T = \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ are called **homogeneous coordinates** of a point in G^*/K. The coordinates $Z = XY^{-1}$ are therefore sometimes called **inhomogeneous coordinates**.

There is yet a fourth mechanism for labeling the points in the space G^*/K. When applicable, it is *very* useful, for it has direct application to physical problems. To see what these might be, it is only necessary to realize that we need not be restricted to the block submatrices in describing the coset representatives.

Since the coordinates T give a many $\rightarrow$ 1 correspondence with the points in G^*/K, we can find a 1-1 correspondence by choosing a subset of elements in T. One obvious choice is the submatrix X. Other choices include several rows (p) from the submatrix X, and additional rows ($m - p$) from Y. These rows must be chosen in such a way that a 1-1 correspondence between the new coordinates and the X or Z is preserved. To show how useful such a parameterization of G^*/K can actually be, we consider a concrete example.

Example. The coset $SO(n, 2)/SO(n) \otimes SO(2)$ is given by

$$\left[\begin{array}{c|cc} & x_1 & y_1 \\ & x_2 & y_2 \\ * & \vdots & \vdots \\ & x_n & y_n \\ \hline * & x_{n+1} & y_{n+1} \\ & x_{n+2} & y_{n+2} \end{array}\right] \tag{5.37}$$

$$\begin{bmatrix} x_{n+1} & y_{n+1} \\ x_{n+2} & y_{n+2} \end{bmatrix} = \left[I_2 + \begin{pmatrix} \mathbf{x} \cdot \mathbf{x} & \mathbf{x} \cdot \mathbf{y} \\ \mathbf{y} \cdot \mathbf{x} & \mathbf{y} \cdot \mathbf{y} \end{pmatrix} \right]^{1/2} \tag{5.38}$$

Parameterization 1: The coset representatives are given by

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \begin{array}{l} -\infty < x_i < +\infty \\ -\infty < y_i < +\infty \end{array} \tag{5.39}$$

Parameterization 2: The representatives are

$$Z = XY^{-1} = \begin{bmatrix} z_1{}^1 & z_1{}^2 \\ z_2{}^1 & z_2{}^2 \\ \vdots & \vdots \\ z_n{}^1 & z_n{}^2 \end{bmatrix} \qquad \begin{array}{c} -1 < z_i{}^j < +1 \\ 0 \le Z^t Z < +1 \end{array} \tag{5.40}$$

Parameterization 3: The representatives are

$$\left[\begin{array}{cc} \mathbf{x} & \mathbf{y} \\ \hline \tilde{x} & \tilde{y} \end{array}\right] \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \tilde{x} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} \tag{5.41}$$

The orthogonality properties inherited by T from the parent $SO(n, 2)$ group are

$$\begin{aligned} \tilde{x} \cdot \tilde{x} - \mathbf{x} \cdot \mathbf{x} &= 1 \\ \tilde{y} \cdot \tilde{y} - \mathbf{y} \cdot \mathbf{y} &= 1 \\ \tilde{x} \cdot \tilde{y} - \mathbf{x} \cdot \mathbf{y} &= 0 \end{aligned} \tag{5.42}$$

These three conditions can be simplified by considering the pairs of real variables (x_i, y_i) as a single complex variable

$$t_i = x_i + iy_i \qquad 1 \le i \le n + 2$$

Then (5.42) is summarized by

$$\begin{aligned} -t^{\dagger}I_{n,2}t &= 2 > 0 \\ t^{t}I_{n,2}t &= 0 \end{aligned} \tag{5.43}$$

where t is an $(n+2) \times 1$ column vector with complex entries.

Parameterization 4: Finally, we wish to choose some matrix elements (x_i, y_i) from the $n \times 2$ matrix X (5.39), and the remainder from the symmetric 2×2 matrix $Y = [I_2 + X^t X]^{1/2}$ (5.38). In particular, we will choose as coset representatives the elements (x_i, y_i), $2 \leq i \leq n+1$. The n parameters x_i now occur *asymmetrically* since x_{n+1} is a diagonal matrix element in $SO(n, 2)$. In particular, $x_{n+1} \geq 1$. Since none of the parameters y_i $(2 \leq i \leq n+1)$ occurs as a diagonal matrix element in $SO(n, 2)$, there are no analogous constraints on the y_i. These coset representatives obey

$$\begin{aligned} -\infty < x_i &< +\infty \qquad 2 \leq i \leq n \\ -\infty < y_j &< +\infty \qquad 2 \leq j \leq n+1 \\ 1 &\leq x_{n+1} \end{aligned}$$

At this point it is useful to look for a projective realization of the coset representatives (x_i, y_i), with $2 \leq i \leq n+1$. Using (5.38)–(5.40) as a model, such a projective realization would be constructed by multiplying the $n \times 2$ matrix of coset representatives by the inverse of the "left over" submatrix:

$$\begin{array}{r} \\ \text{coset} \\ \text{represen-} \\ \text{tatives} \\ \\ \end{array} \left\{ \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_{n+1} & y_{n+1} \\ x_{n+2} & y_{n+2} \end{bmatrix} \right. \quad \begin{array}{c} \text{remaining} \\ 2 \times 2 \\ \text{submatrix} \end{array} \rightarrow \begin{bmatrix} x_2 & y_2 \\ \vdots & \vdots \\ x_{n+1} & y_{n+1} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_{n+2} & y_{n+2} \end{bmatrix}^{-1}$$

But whereas the submatrix Y of (5.38) is nonsingular and has a well defined inverse leading to the bounded space given in (5.40), the submatrix

$$\begin{bmatrix} x_1 & y_1 \\ x_{n+2} & y_{n+2} \end{bmatrix}$$

may be singular. We must, therefore, find another way to construct a projective realization of these coset representatives.

This is done most easily by considering each pair of real variables (x_i, y_i) to be a single complex variable

$$t_j = x_j + iy_j \qquad 1 \leq j \leq n+2$$

Since Y (5.38) is symmetric, $y_{n+1} = x_{n+2}$ and

$$\begin{aligned} t_{n+1} - it_{n+2} &= (x_{n+1} + iy_{n+1}) - i(x_{n+2} + iy_{n+2}) \\ &= x_{n+1} + y_{n+2} = \operatorname{tr} Y \geq 2 \end{aligned}$$

Moreover, the imaginary part of the ratio t_{n+1}/t_{n+2} is less than zero, for

$$\begin{aligned}\operatorname{Im} t_{n+1}/t_{n+2} &= \operatorname{Im}\frac{x_{n+1}+iy_{n+1}}{x_{n+2}+iy_{n+2}}\\ &= \operatorname{Im}\frac{(x_{n+1}+iy_{n+1})(y_{n+1}-iy_{n+2})}{|y_{n+1}+iy_{n+2}|^2}\\ &= \frac{-\det Y}{|x_{n+2}+iy_{n+2}|^2}<0\end{aligned}$$

In terms of the variables t_j, a projective realization analogous to (5.40) is obtained using the variables

$$\alpha_j = \frac{t_j}{t_{n+1}-it_{n+2}} \qquad 1 \le j \le n$$

The α_j are bounded. The conditions that they obey can be determined from

$$\begin{aligned}\sum_{j=1}^{n}\alpha_j\alpha_j = \frac{\sum_{j=1}^{n} t_j{}^2}{(t_{n+1}-it_{n+2})^2} &= \frac{t_{n+1}^2+t_{n+2}^2}{(t_{n+1}-it_{n+2})^2}\\ &= \frac{t_{n+1}+it_{n+2}}{t_{n+1}-it_{n+2}}\\ &= \frac{(t_{n+1}/t_{n+2})+i}{(t_{n+1}/t_{n+2})-i}\end{aligned}$$

Since $\operatorname{Im} t_{n+1}/t_{n+2} < 0$, the ratio above has absolute value less than $+1$:

$$\left|\sum_{j=1}^{n}\alpha_j^2\right| < 1$$

In addition

$$\begin{aligned}1+\left|\sum_{j=1}^{n}\alpha_j\alpha_j\right|^2 &= 1+\left|\frac{t_{n+1}+it_{n+2}}{t_{n+1}-it_{n+2}}\right|^2\\ &= 2\frac{t_{n+1}^*t_{n+1}+t_{n+2}^*t_{n+2}}{|t_{n+1}-it_{n+2}|^2}\\ &> 2\frac{\sum_{j=1}^{n} t_j^*t_j}{|t_{n+1}-it_{n+2}|^2} = 2\sum_{j=1}^{n}\alpha_j^*\alpha_j\end{aligned}$$

As a result, the α_j obey the additional condition

$$1+\left|\sum_{j=1}^{n}\alpha_j\alpha_j\right|^2 - 2\sum_{j=1}^{n}\alpha_j^*\alpha_j > 0$$

The bounded subset of complex n-dimensional space C_n that obeys the conditions

$$|\alpha^t\alpha| < 1$$
$$1 + |\alpha^t\alpha|^2 - 2\alpha^\dagger\alpha > 0$$

is called a **polydisc** D^n. The points in the polydisc D^n provide a bounded realization of the coset $SO(n, 2)/SO(n) \otimes SO(2)$ which is equivalent to, but not equal to, the realization of (5.40)

$$\begin{bmatrix} z_1^1 & z_1^2 \\ \vdots & \vdots \\ z_n^1 & z_n^2 \end{bmatrix} Y \simeq \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \operatorname{tr} Y$$

An unbounded realization can be obtained by applying the Weyl unitary trick to the previous arguments. We define

$$\beta_k = \frac{t_k}{t_1 - t_{n+2}} \qquad 2 \le k \le n+1$$

The complex parameters β_k obey

$$\beta_2^2 + \cdots + \beta_n^2 - \beta_{n+1}^2 = \frac{t_{n+2}^2 - t_1^2}{(t_{n+2} - t_1)^2} = \frac{t_{n+2} + t_1}{t_{n+2} - t_1}$$

$$\beta_2^*\beta_2 + \cdots + \beta_n^*\beta_n - \beta_{n+1}^*\beta_{n+1} = \frac{-2}{|t_{n+2} - t_1|^2} + \frac{t_{n+2}^* t_{n+2} - t_1^* t_1}{|t_{n+2} - t_1|^2}$$
$$> \frac{1}{2}\frac{(t_{n+2} + t_1)}{(t_{n+2} - t_1)} + \frac{1}{2}\frac{(t_{n+2} + t_1)^*}{(t_{n+2} - t_1)^*}$$

As a result

$$(\beta_2 - \beta_2^*)^2 + \cdots + (\beta_n - \beta_n^*)^2 - (\beta_{n+1} - \beta_{n+1}^*)^2 = \frac{4}{|t_{n+2} - t_1|^2} > 0$$

Finally, this result can be rewritten

$$-(\operatorname{Im}\beta_2)^2 - \cdots - (\operatorname{Im}\beta_n)^2 + (\operatorname{Im}\beta_{n+1})^2 > 0 \tag{5.44}$$

This is the only condition which the n complex parameters β_k ($2 \le k \le n+1$) obey.

Equation (5.44) tells us that, while the real part of the n complex parameters β_k may be arbitrary, the complex part of the β is not arbitrary and is, in fact, confined to the interior part of the forward light cone (since $\operatorname{Im}\beta_{n+1} > 0$) in a real Minkowski space $M_{n-1,1}$ with signature $(n-1, 1)$. Symbolically, we may write for the range of the variables

$$\beta \subset R_n + iT_n$$

where R_n is the real Euclidean n-dimensional space, and T_n is the interior part of the forward light cone in $M_{n-1,1}$.

The parameterizations α_j and β_k of the coset representatives $SO(n, 2)/SO(n) \otimes SO(2)$ are extremely useful from a physical point of view. It is always possible to map the interior of the forward light cone in $M_{n-1,1}$, represented by the imaginary part of the coordinates β_k, into the interior of the unit polydisc D^n, described by the coordinates α_j. There is, moreover, a complex structure on the space D^n, since it is a hermitian symmetric space and thus a Kählerian manifold. In addition, even though the interior of D^n is not compact, it is a bounded subset of C_n. As a result, it is possible to bring the rich variety of powerful mathematical tools that have been developed for the study of bounded complex manifolds, to bear on the study of physics in the interior of and, in the limit, on the boundary of the light cone in an n-dimensional Minkowski space $M_{n-1,1}$.

From the foregoing examples and parameterizations, we see that there is available a large variety of mechanisms for describing coset representatives of compact spaces G/K and of noncompact spaces G^*/K. In general, there is a richer variety of different parameterizations available for the noncompact spaces than for the compact spaces. An indication of this variety can be seen by referring to Fig. 9.3 and Fig. 9.4, describing the cosets $SO(3)/SO(2)$ and $SO(2, 1)/SO(2)$, respectively.

The coset representatives in Fig. 9.3 may be chosen in a number of distinct but equivalent ways:

1. They may be chosen as the submatrix elements $\binom{x}{y}$. As such, they lie in the interior of, and on the boundary of, the unit disc (shown in the $z = 0$ plane).
2. The point in the disc (1) can be projected onto the point (x, y, z) on the sphere of radius $+1$.
3. If the point on the sphere surface is illuminated by a light source at the origin of the sphere, it casts a shadow on the plane $z = 1$. The coordinates of the shadow in $R_2(z = 1)$ are the projective coordinates $(x/z, y/z)$.

The coset representatives in Fig. 9.4 may be chosen in a larger variety of ways. It is possible to pass from one description of the coset to another through the intermediary of the hyperboloid:

1. The coset representatives may be taken as the point $\binom{x}{y}$ in R_2. These coset representatives are unbounded.
2. The homogeneous coordinates (x, y, z) may also be used. These are the coordinates of the uniquely defined point on the upper sheet of the hyperboloid which lies over the coset representative (x, y) of description 1.

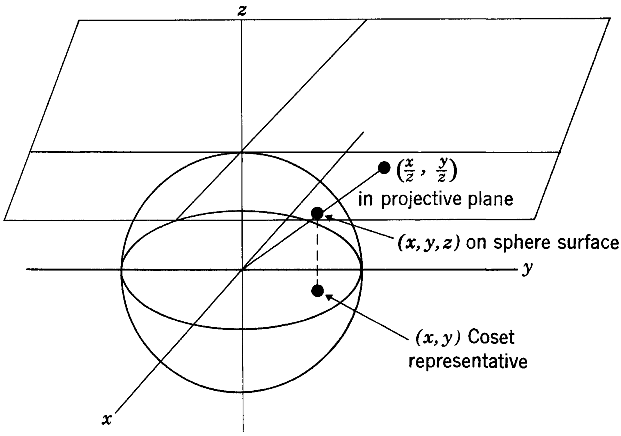


FIG. 9.3

$$\begin{bmatrix} \left[I_2 - \begin{pmatrix} x \\ y \end{pmatrix} (x, y) \right]^{1/2} & \begin{matrix} x \\ y \end{matrix} \\ \hline -x \quad -y & \left[1 - (x, y) \begin{pmatrix} x \\ y \end{pmatrix} \right]^{1/2} \end{bmatrix}$$

THE COSET REPRESENTATIVES OF THE SPHERE $S^2 = SO(3)/SO(2)$ CAN BE CHOSEN IN SEVERAL DIFFERENT WAYS: (1) AS THE POINT (x, y) IN THE INTERIOR OF, OR ON THE BOUNDARY OF, THE UNIT CIRCLE IN THE $z = 0$ PLANE; (2) AS THE POINT (x, y, z) ON THE SURFACE OF THE UNIT SPHERE (HOMOGENEOUS COORDINATES); AND (3) AS THE POINT $(x/z, y/z)$ IN THE $z = 1$ PLANE (INHOMOGENEOUS OR PROJECTIVE COORDINATES).

3. If a line is drawn between the point (x, y, z) on the hyperboloid (from parameterization 2) and the origin (000), this line intersects the plane $z = 1$ in a point whose coordinates in $R_2(z = 1)$ are $(x/z, y/z)$. These are the projective coordinates of the coset representatives $SO(2, 1)/SO(2)$. Since the parameters (x, y) of 1 are unbounded, the projective coordinates $\binom{x}{y} z^{-1}$ are bounded and in fact lie in the interior of the unit disc.

4. If the coordinates (x, y, z) of 2 are now projected parallel to the X-axis onto the YZ plane $x = 0$, then the (y, z) coordinates of the image point obey

$$z^2 - y^2 \geq 1 \qquad z > |y|$$

This image point belongs to the interior or boundary of the "mass-1 light cone."

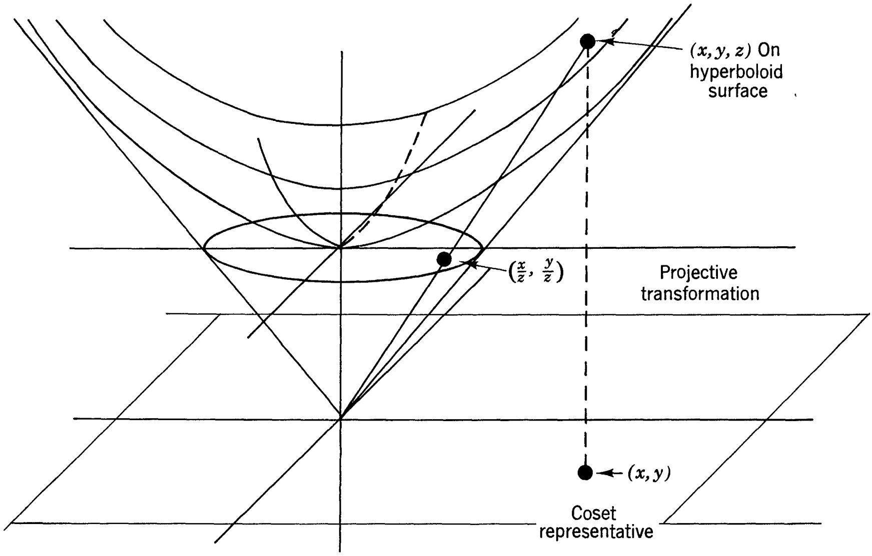


FIG. 9.4

$$\left[\begin{array}{c|c} \left[I_2 + \begin{pmatrix} x \\ y \end{pmatrix}(x, y)\right]^{1/2} & \begin{matrix} x \\ y \end{matrix} \\ \hline x \quad y & \left[1 + (x, y)\begin{pmatrix} x \\ y \end{pmatrix}\right]^{1/2} \end{array}\right]$$

THE COSET REPRESENTATIVES OF THE HYPERBOLOID $H^2 = SO(2, 1)/SO(2)$ CAN BE CHOSEN IN SEVERAL DIFFERENT WAYS: (1) AS THE POINT (x, y) IN THE $z = 0$ PLANE; (2) AS THE POINT (x, y, z) ON THE SURFACE OF THE UNIT HYPERBOLOID (HOMOGENEOUS COORDINATES); (3) AS THE POINT $(x/z, y/z)$ IN THE INTERIOR OF THE UNIT CIRCLE IN THE $z = 1$ PLANE (INHOMOGENEOUS OR PROJECTIVE COORDINATES); (4) AS THE PROJECTION OF THE POINT (x, y, z) ONTO THE $x = 0$ PLANE. THE POINT $(0, y, z)$ OBEYS $z^2 - y^2 \geq 1$; AND (5) AS THE INHOMOGENEOUS COORDINATES $(z/|x|, y/|x|)$, WHICH EXIST IN THE INTERIOR OF THE FORWARD LIGHT CONE, AND WHICH OBEY $(z/|x|)^2 - (y/|x|)^2 > 0$.

5. If the projective coordinates of parameterization 4 are used, these are $(y, z)/|x|$. These coordinates obey

$$\left(\frac{z}{|x|}\right)^2 - \left(\frac{y}{|x|}\right)^2 > 0 \qquad \frac{z}{|x|} > \frac{|y|}{|x|}$$

These points belong to the interior of the forward light cone.

Larger, more complicated groups have an even richer spectrum of distinct but equivalent coset representatives.

3. INFINITESIMAL TRANSFORMATIONS. In the previous section we have computed the global transformation properties of the group G acting on the manifold G/K. In this section we compute the generators of the

infinitesimal displacements of G acting on the underlying geometric space G/K.

These generators can be computed on the coset representatives X, the projective representatives $Z = XY^{-1}$, the homogeneous coordinates $T = \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, or on any other coset representatives related to these by BCH formulas. The infinitesimal generators for these domains are, of course, all related to one another.

It is simple to compute the infinitesimal generators for the projective domain Z. Under an arbitrarily small displacement in G

$$g \to \delta g = \left[\begin{array}{c|c} I_m + \delta A & \delta B \\ \hline \delta C & I_n + \delta D \end{array}\right] \tag{5.45}$$

the coordinates of Z' are given by

$$\begin{aligned} Z \to Z' &= Z + dZ \\ &= [(I_m + \delta A)Z + \delta B][\delta C Z + (I_n + \delta D)]^{-1} \\ &\simeq [Z + \delta A Z + \delta B][I_n - \delta C Z - \delta D] \end{aligned} \tag{5.46}$$

$$dZ = \delta A Z + \delta B - Z\,\delta C Z - Z\,\delta D \tag{5.47}$$

Since the infinitesimals are also given by $dZ = X(\delta g)Z$ the infinitesimal generators $X(\delta g)$ can be written concisely as

$$X\left[\begin{array}{c|c} \delta A & \delta B \\ \hline \delta C & \delta D \end{array}\right] = \operatorname{tr}\,(dZ)(\partial_Z)^t \tag{5.48}$$

$$\partial_Z = \nabla_Z = \begin{bmatrix} \partial_1^{m+1} & \partial_1^{m+2} & \cdots & \partial_1^{m+n} \\ \partial_2^{m+1} & \partial_2^{m+2} & \cdots & \partial_2^{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_m^{m+1} & \partial_m^{m+2} & \cdots & \partial_m^{m+n} \end{bmatrix} \tag{5.49}$$

Example 1. We determine the infinitesimal generators of $SO(3)$ acting in the real projective plane PR_2. A group operation near the identity of $SO(3)$ is given by

$$\delta g = I_3 + \left[\begin{array}{cc|c} 0 & \delta a^3 & -\delta a^2 \\ -\delta a^3 & 0 & \delta a^1 \\ \hline \delta a^2 & -\delta a^1 & 0 \end{array}\right]$$

$$\begin{bmatrix} z'^1 \\ z'^2 \end{bmatrix} = \begin{bmatrix} z^1 + dz^1 \\ z^2 + dz^2 \end{bmatrix} = \frac{\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \delta a^3 \\ -\delta a^3 & 0 \end{bmatrix}\right\}\begin{bmatrix} z^1 \\ z^2 \end{bmatrix} + \begin{bmatrix} -\delta a^2 \\ \delta a^1 \end{bmatrix}}{(\delta a^2, -\delta a^1)\begin{bmatrix} z^1 \\ z^2 \end{bmatrix} + 1} \tag{5.50}$$

With

$$\partial = \begin{bmatrix} \partial_1 \\ \partial_2 \end{bmatrix} \qquad \text{and} \qquad \partial^t = (\partial_1, \partial_2),$$

the infinitesimal displacements are obtained from

$$\begin{aligned}
\operatorname{tr} dZ\, \partial_Z{}^t &= dz^1 \frac{\partial}{\partial z^1} + dz^2 \frac{\partial}{\partial z^2} \\
&= \{\delta a^3 z^2 - \delta a^2 - z^1(\delta a^2 z^1 - \delta a^1 z^2)\} \frac{\partial}{\partial z^1} \\
&\quad + \{-\delta a^3 z^1 + \delta a^1 - z^2(\delta a^2 z^1 - \delta a^1 z^2)\} \frac{\partial}{\partial z^2}
\end{aligned} \tag{5.51}$$

The generators of infinitesimal displacements can be read off immediately from the generating function (5.51) for the infinitesimal displacements:

$$\operatorname{tr} dZ\, \partial_Z{}^t = \sum_{i=1}^{3} \delta a^i X_i \tag{5.52}$$

$$\begin{aligned}
X_1 &= \partial_2 + z^2(z^1 \partial_1 + z^2 \partial_2) = \partial_2 + z^2 \mathbf{z} \cdot \mathbf{\nabla} \\
X_2 &= -\partial_1 - z^1(z^1 \partial_1 + z^2 \partial_2) = -\partial_1 - z^1 \mathbf{z} \cdot \mathbf{\nabla} \\
X_3 &= z^2 \partial_1 - z^1 \partial_2 = -(\mathbf{z} \times \mathbf{\nabla})_3
\end{aligned} \tag{5.53}$$

Example 2. We will compute the infinitesimal generators of the group $SO(p, q)$, acting in the geometric space $SO(p, q)/SO(p) \otimes SO(q)$ given by the projective coordinates $Z = XY^{-1}$. This is a bounded domain, and for p or $q = 2$ it is the generalization of the unit disc $D^{(q \text{ or } p)}$.

$$\begin{aligned}
X_i{}^j = X \left[\begin{array}{cc|c} -\delta a_i{}^j & \delta a_i{}^j & \\ \hline & & \end{array}\right] &= \sum_k \sum_l \sum_r (\delta A)_k{}^l\, z_l^{m+r}\, \partial^k_{m+r} \\
&\to z_j^{m+r}\, \partial^i_{m+r} - z_i^{m+r}\, \partial^j_{m+r}
\end{aligned} \tag{5.53i}$$

$$\begin{aligned}
X^{m+s}_{m+r} = X \left[\begin{array}{c|cc} & & \\ \hline & -\delta a^{m+s}_{m+r} & \delta a^{m+s}_{m+r} \end{array}\right] &= -z_k^{m+t} (\delta D)^{m+u}_{m+t}\, \partial^k_{m+u} \\
&= -z_k^{m+r}\, \partial^k_{m+s} + z_k^{m+s}\, \partial^k_{m+r}
\end{aligned} \tag{5.53ii}$$

$$\begin{aligned}
X_i^{m+r} = X \left[\begin{array}{c|c} & \delta a_i^{m+r} \\ \hline \pm \delta a_i^{m+r} & \end{array}\right] &= (\delta B)_j^{m+s}\, \partial^j_{m+s} - z_j^{m+s} (\delta C)^k_{m+s} z_k^{m+t}\, \partial^j_{m+t} \\
&\to \partial^i_{m+r} \mp z_j^{m+r} z_i^{m+t}\, \partial^j_{m+t} \\
&= \pm X^i_{m+r}
\end{aligned} \tag{5.53iii}$$

where $1 \leq i, j, k, l \leq m$ and $1 \leq r, s, t, u \leq n$, and for purposes of symmetry and convenience, we have set $\partial^k_{m+r} \equiv \partial^{m+r}_k$.

The generators for $SO(p, q)/SO(p) \otimes SO(q)$ and $SO(p+q)/SO(p) \otimes SO(q)$ are closely related. They differ only in the last two lines of (5.53iii). The upper sign holds for $SO(p, q)/SO(p) \otimes SO(q)$, the lower for $SO(p+q)/SO(p) \otimes SO(q)$.

4. INFINITESIMAL'S TRANSFORMATIONS. Under an arbitrary group operation, an infinitesimal displacement at one point will be mapped into an infinitesimal displacement at another point. The subgroup of operations leaving a point fixed will then map the infinitesimal displacements at a point onto the infinitesimal displacements at the same point.

It is convenient to choose as a basis for the infinitesimal displacements in the independent directions in G^*/K, or $i\mathfrak{p}$, the submatrix (dx):

$$\left[\begin{array}{c|c} I_m & dx_i^{m+r} \\ \hline dx^\dagger & I_n \end{array}\right] \qquad (dx) = dx_i^{m+r} \tag{5.54}$$

Then an arbitrary infinitesimal displacement is given by

$$\begin{aligned} dx &= \operatorname{tr}(A)(dx)^t = \operatorname{tr}(A)^t(dx) \\ &= A^i_{m+r}\, dx_i^{m+r} = A_i^{m+r}\, dx^i_{m+r} \end{aligned} \tag{5.55}$$

Since the submatrices of the form A form a linear vector space, so also do the infinitesimal displacements at a point:

$$\begin{aligned} dx_1 + dx_2 &= \operatorname{tr}(A_1)(dx)^t + \operatorname{tr}(A_2)(dx)^t \\ &= \operatorname{tr}(A_1 + A_2)(dx)^t \\ c\, dx_1 &= c \operatorname{tr}(A)(dx)^t = \operatorname{tr}(cA)(dx)^t \end{aligned} \tag{5.56}$$

The dimensionality of this linear vector space is equal to the dimensionality of the vector subspace $i\mathfrak{p}$ and the Riemannian space G^*/K. The vector space of infinitesimal displacements actually describes the independent directions a geodesic through the origin may have.

The mapping of this linear vector space at the identity coset onto itself under the action of the stability subgroup K is given simply by

$$\begin{gathered} \left[\begin{array}{c|c} U(m) & \\ \hline & U(n) \end{array}\right] \left[\begin{array}{c|c} I_m & dx \\ \hline dx^\dagger & I_n \end{array}\right] = \left[\begin{array}{c|c} I_m & dx' \\ \hline dx'^\dagger & I_n \end{array}\right] \left[\begin{array}{c|c} U'(m) & \\ \hline & U'(n) \end{array}\right] \\ \left[\begin{array}{c|c} U(m) & U(m)\, dx \\ \hline U(n)\, dx^\dagger & U(n) \end{array}\right] = \left[\begin{array}{c|c} U'(m) & dx'\, U'(n) \\ \hline dx'^\dagger U'(m) & U'(n) \end{array}\right] \end{gathered} \tag{5.57}$$

$$dx' = U(m)\, dx U^{-1}(n) \tag{5.58}$$

Example. The independent infinitesimal displacements at the north pole of $S^n \subset R_{n+1}$ are given by

$$\begin{bmatrix} dx^1 \\ dx^2 \\ \vdots \\ dx^n \end{bmatrix}$$

Their transformation properties under an element in the stability subgroup $SO(n)$ are

$$\begin{bmatrix} dx'^1 \\ \vdots \\ dx'^n \end{bmatrix} = SO(n) \begin{bmatrix} dx^1 \\ \vdots \\ dx^n \end{bmatrix} SO(1)^{-1} = SO(n) \begin{bmatrix} dx^1 \\ \vdots \\ dx^n \end{bmatrix}$$

Under a translation by the coset element $c \in G^*/K$, the infinitesimals at the origin are moved to infinitesimals at c according to

$$\left[\begin{array}{c|c} W & X \\ \hline X^\dagger & Y \end{array}\right] \left[\begin{array}{c|c} I_m & dx(0) \\ \hline dx(0)^\dagger & I_n \end{array}\right] = \left[\begin{array}{c|c} W' & X' \\ \hline X'^\dagger & Y' \end{array}\right] \left[\begin{array}{c|c} U(m) & \\ \hline & U(n) \end{array}\right] \tag{5.59}$$

$$dx(c) = X' - X$$

The explicit form of the infinitesimal $dx(c)$ is computed by first constructing products of the form $X'Y'^\dagger$ to eliminate the compact subgroup element $U(n)$

$$(X + W\,dx(0))(Y + X^\dagger\,dx(0))^\dagger = X'Y'^\dagger$$

Then each side is multiplied by its own adjoint and the result is expanded. After simplification, the result is

$$W\,dx(0)YY + XX^\dagger W\,dx(0) = dx(c)YY + XX^\dagger\,dx(c) \tag{5.60}$$

In general, it is difficult to solve this equation for the infinitesimals $dx(c)$ in terms of the coset representatives

$$\begin{bmatrix} W & X \\ X^\dagger & Y \end{bmatrix}$$

and the infinitesimals $dx(0)$ at the origin of coordinates. The difficulty arises from the noncommutativity of $dx(c)$ and both of the matrices Y^2 and $XX^\dagger$ on the right-hand side of (5.60). When $dx(c)$ commutes with either Y^2 or $XX^\dagger$, then (5.60) can easily be solved.

When G^*/K is a rank one coset, then either Y^2 $(= I_n + X^\dagger X)$ or $XX^\dagger$ must be a scalar multiple of the identity matrix. This insures commutativity for at least one term on the right of (5.60). If we assume

$$[Y^2, dx] = 0$$

then the solution is given by

$$dx(c) = W\,dx(0) \tag{5.61}$$

5. GEODESICS, DISTANCE, METRIC. Geodesics through the identity element in the space of coset representatives are obtained by exponentiating straight lines through the origin of the Lie algebra. In the case of off-diagonal cosets, these straight lines in the algebra have the structure

$$t\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \tag{5.62}$$

The geodesics have coordinates given by

$$\left[\begin{array}{c|c} \cosh t\sqrt{AB} & A\dfrac{\sinh t\sqrt{BA}}{\sqrt{BA}} \\ \hline B\dfrac{\sinh t\sqrt{AB}}{\sqrt{AB}} & \cosh t\sqrt{BA} \end{array}\right] \tag{5.63}$$

When A and B are related by $B = \pm A^{\dagger}$, the only independent coordinates occurring in the matrices (5.62) and (5.63) are in the upper right-hand corner. The geodesic coordinates are then given by

$$\begin{aligned} X(t) &= A\frac{\sinh t(A^{\dagger}A)^{1/2}}{(A^{\dagger}A)^{1/2}} \qquad B = +A^{\dagger} \\ &= A\frac{\sin t(A^{\dagger}A)^{1/2}}{(A^{\dagger}A)^{1/2}} \qquad B = -A^{\dagger} \end{aligned} \tag{5.64}$$

Example. The coordinates of a point on a geodesic passing through the identity coset representative, or north pole, in $SO(n+1)/SO(n)$, are given by

$$\left[\begin{array}{c} a_1 \sin t \\ a_2 \sin t \\ \vdots \\ a_n \sin t \\ \hline \cos t \end{array}\right] \qquad \mathbf{a}\cdot\mathbf{a} = 1 \tag{5.65}$$

This geodesic leaves the north pole with a "velocity" vector (first derivative) in the direction

$$\left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \\ \hline 0 \end{array}\right] \tag{5.66}$$

This direction lies in the tangent plane to the north pole. The geodesic lies in a plane determined by the origin of the sphere $S^n \subseteq R_{n+1}$, the north pole, and the vector (5.66) at the north pole. In fact, the geodesic orbit is the intersection of this plane with the unit sphere.

Since the geodesic trajectory lies in a two-dimensional plane, every vector in R_{n+1} drawn between the origin col $(0, 0, \ldots, 0, 0)$ of R_{n+1} and a point on the geodesic, is orthogonal to $(n + 1) - 2$ vectors in R_{n+1}. If the geodesic (5.65) were defined by

$$\left[\begin{array}{c} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_n = 1 \\ \hline a_{n+1} = 0 \end{array}\right] \tag{5.67}$$

these vectors would be

$$\mathbf{v}_1 = \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \end{array}\right] \qquad \mathbf{v}_2 = \left[\begin{array}{c} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \end{array}\right] \qquad \cdots \qquad \mathbf{v}_{n-1} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \hline 0 \end{array}\right] \tag{5.68}$$

In fact, this may be used as an alternative definition for geodesic orbit on S^n.

To construct a geodesic passing through points other than the north pole in S^n, we apply the same group operation γ to each point $g(t)$ on a geodesic through the origin. Since $SO(n + 1)$ is a metric-preserving group, we can write

$$(\gamma g(t), \gamma \mathbf{v}_i) = (g(t), \mathbf{v}_i) = 0$$

In short, if the geodesic $g(t)$ is orthogonal to $n - 1$ independent vectors $\mathbf{v}_i$, the curve $\gamma g(t)$ in $SO(n + 1)/SO(n)$ is orthogonal to the $n - 1$ independent vectors $\gamma \mathbf{v}_i$. Thus $\gamma g(t)$ is a geodesic also.

When the group operation γ belongs to the subgroup $SO(n)$, then the geodesic still passes through the north pole; but it lies in a different plane, specifically, the plane determined by

$$\left[\begin{array}{c|c} SO(n) & \\ \hline & 1 \end{array}\right] \left[\begin{array}{c} a_1 \sin t \\ \vdots \\ a_n \sin t \\ \hline \cos t \end{array}\right] = \left[\begin{array}{c|c} * & SO(n) \begin{bmatrix} a_1 \sin t \\ \vdots \\ a_n \sin t \end{bmatrix} \\ \hline * & \cos t \end{array}\right] \tag{5.69}$$

The properties of geodesics on the sphere $SO(n+1)/SO(n)$, described in the foregoing example, hold in general for geodesics on any of the Riemannian coset spaces. These properties are summarized as follows:

1. The geodesic $g(t)$ through the identity coset element in G/K is the exponential of a straight line Vt in the orthogonal subspace $\mathfrak{p}$:

$$g(t) = \text{EXP } Vt \qquad V \in \mathfrak{p}$$
$$-\infty < t < +\infty$$

2. Operations in the stability subgroup $K(\text{Id})$ map a geodesic $g(t)$ through the identity, into another geodesic through the identity. If $k = \text{EXP } V_1$, then

$$\begin{aligned} &g(t) \xrightarrow{k} g'(t) \qquad k \in K(\text{Id}) \\ &g'(t) = \text{EXP } V_1 g(t) \text{ EXP } -V_1 \end{aligned} \tag{5.70}$$

3. The family of geodesics through the point c in G/K is obtained by applying the coset operation c to all the group operations $g(t)$, where $g(t)$ is any geodesic through the identity:

$$\begin{array}{cccc} \text{Family of Geodesics} & & \text{Family of Geodesics} & \\ \text{through identity} & & \text{through } c & \\ \hline g(t) & \xrightarrow{c} & g'(t) & \\ cg(t) & = & g'(t)k & c \in G/K \\ & & & k \in K(Id) \end{array} \tag{5.71}$$

Warning. An arbitrary straight line in the subspace $\mathfrak{p}$ maps onto a geodesic in G/K under the EXPonential map *only* when it passes through the origin of $\mathfrak{p}$.

Since it is possible to construct geodesics in G/K, it is possible to define distance in G/K. First, we define the distance between any point in G/K and the identity coset element, or origin, of G/K.

If c is any point in G/K, it can be connected to the origin by a geodesic

$$\begin{aligned} &g(t) = \text{EXP } tV \qquad V \in \mathfrak{p} \\ &g(0) = \text{Id} \\ &g(1) = c \end{aligned} \tag{5.72}$$

where V is a vector in the linear vector space $\mathfrak{p}$, which is endowed with a (positive or negative) definite metric derived from the Cartan-Killing metric on $\mathfrak{g}$, which is restricted to the subspace $\mathfrak{p}$. Therefore, V has a well-defined length in $\mathfrak{p}$:

$$V = a^i X_i \qquad X_i \text{ bases for } \mathfrak{p}$$
$$\|V\| = [\,|(a^i X_i, a^j X_j)|\,]^{1/2} \tag{5.73}$$

We now identify the length of the vector V in $\mathfrak{p}$ with the length of the geodesic EXP tV, connecting Id and $c = \text{EXP } V$ in G/K:

$$d(\text{Id}, c) \equiv \|V\| \tag{5.74}$$

Example. We compute the distance between the north pole of $S^n \subset R_{n+1}$, and any point in S^n. The metric on $\mathfrak{p}$ is derived directly from the Cartan-Killing metric in the Lie algebra of $SO(n+1)$:

$$(X_{i,n+1}, X_{j,n+1}) = -\delta_{ij}$$

This metric may just as well be taken to be positive definite. Then, since

$$a^i X_{i,n+1} = \left[\begin{array}{ccc|c} & & & a^1 \\ & & & a^2 \\ & & & \vdots \\ & & & a^n \\ \hline -a^1 & \ldots & -a^n & \end{array}\right] \xrightarrow{\text{EXP}} \left[\begin{array}{c|c} & x^1 = a^1 \dfrac{\sin at}{a} \\ * & x^2 = a^2 \dfrac{\sin at}{a} \\ & \vdots \\ & x^n = a^n \dfrac{\sin at}{a} \\ \hline * & x^{n+1} = \cos at \end{array}\right]$$

$$a = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

the geodesic distance between

$$\begin{array}{ccc} NP & & \mathbf{x} \\ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} & \text{and} & \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \\ x^{n+1} \end{bmatrix} \end{array}$$

is

$$d(NP, \mathbf{x}) = at$$

In particular, the geodesic distance between the north pole and any point on the equator ($\cos at = 0$) is $\pi/2$; the distance between the north and the south pole along a line of longitude is π.

The distance between two points is the length of the shortest geodesic connecting them. Since any geodesic in G/K can be mapped onto any other geodesic in G/K, we have a mechanism for determining the distance between *any* two points A, B in G/K:

$$\begin{array}{c} d(A,B) \\ /\!/\!/ \qquad \backslash\!\backslash\!\backslash \\ d(\mathrm{Id}, A^{-1}B) \equiv d(B^{-1}A, \mathrm{Id}) \end{array} \tag{5.75}$$

Example. The distance between *any* point in $S^n \subset R_{n+1}$ and its equator is $\pi/2$. The distance between any two points that are poles apart is π.

Comment. In noncompact spaces of the form G^*/K it is not always possible to join every point to the identity by a single geodesic. However, the joining can always be done by a finite number, and in fact, by a small number, of geodesics. The distance between the point and the origin is then simply the sum of the geodesic lengths.

Example. The point

$$\begin{bmatrix} -e^{\lambda} & 0 \\ 0 & -e^{-\lambda} \end{bmatrix}$$

in $Sl(2, r)/\mathrm{Id}$ cannot be written as the exponential of a point in the Lie algebra of $Sl(2, r)$. But it can be written

$$\begin{bmatrix} -e^{\lambda} & 0 \\ 0 & -e^{-\lambda} \end{bmatrix} = \mathrm{EXP}\ \pi \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \mathrm{EXP}\ \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$d\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -e^{\lambda} & 0 \\ 0 & -e^{-\lambda} \end{bmatrix}\right\} = -\pi + \lambda \tag{5.76}$$

The metric in $Sl(2, r)$ is indefinite.

Finally, we determine the metric tensor $g_{ij}(c)$ on G/K in terms of the Cartan-Killing metric tensor (X_i, X_j) on $\mathfrak{p}$. To begin, we note that the metric at the identity in G/K can be identified with the metric $g_{ij} = (X_i, X_j)$ in $\mathfrak{p}$. The distance between the identity Id and the coset representative infinitesimally close by

$$dc = \mathrm{EXP}\ dx^i X_i \qquad X_i \quad \text{basis for } \mathfrak{p}$$

is given by

$$ds^2(\text{at identity}) = |d(\text{Id}, dc)|^2$$
$$g_{ij}(\text{Id})\, dx^i\, dx^j = (dx^i X_i, dx^j X_j) = g_{ij}(\text{in}\, \mathfrak{p})\, dx^i\, dx^j$$

Therefore we have the identification

$$g_{ij}(\text{Id in } G/K) \equiv g_{ij}(\text{in } \mathfrak{p}) \tag{5.77}$$

The metric tensor $g_{ij}(c)$ at any point c in G/K is determined by observing that an infinitesimal length is invariant under displacement:

$$\begin{array}{ccc} ds^2(0) & \xrightarrow[=]{\text{Id}\to c} & ds^2(c) \\ \| & & \| \\ g_{ij}(0)\, dx^i(0)\, dx^j(0) & = & g_{ij}(c)\, dx^i(c)\, dx^j(c) \end{array} \tag{5.78}$$

The infinitesimal displacements at c are related to the infinitesimal displacements at (0) by a linear transformation that depends on the coset representative c and can be determined from (5.60) in principle:

$$dx^r(c) = J_i{}^r(c)\, dx^i(0)$$
$$J_i{}^r(c) = \frac{\partial x^r(c)}{\partial x^i(0)} \tag{5.79}$$

Therefore, the metrics are related by

$$g_{rs}(c) = g_{ij}(0) \frac{\partial x^i(0)}{\partial x^r(c)} \frac{\partial x^j(0)}{\partial x^s(c)} \tag{5.80}$$

Example. We compute the metric tensor $g_{ij}(c)$ over the sphere $S^n \subset R_{n+1}$. The metric at the identity is

$$g_{ij}(0) = (X_i, X_j) \to \delta_{ij}$$

The transformation properties of the infinitesimal displacements $dx^i(0)$ under an arbitrary coset representative are given by (5.61), since the sphere is a rank-1 space:

$$dx(c) = W\, dx(0)$$
$$W = \left[I_n - \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} (x^1, x^2, \ldots, x^n) \right]^{1/2}$$

The metric tensor $g_{ij}(x)$ is given explicitly by

$$\begin{aligned} g_{rs}(x) &= g_{ij}(0)(W^{-1})_r{}^i (W^{-1})_s{}^j \\ &= (W^{-1})_r{}^i (W^{-1})_s{}^i = (W^{-2})_{rs} \\ &= [I_n - xx^t]_{rs}^{-1} \end{aligned} \tag{5.81}$$

We now use this metric to compute the distance between the north pole and the equator along the line $x^1 = t$, $x^i = 0$ $(2 \le i \le n)$.

$$g_{rs}(x) = \frac{1}{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \left[\begin{array}{c|c} t^2 & \bigcirc \\ \hline \bigcirc & \bigcirc \end{array}\right]}$$

$$= \begin{bmatrix} \frac{1}{1-t^2} & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$\begin{aligned} d(NP, \text{equator}) &= \int_{\text{NP}}^{\text{eq}} ds = \int_0^1 \sqrt{g_{ij}(x)\, dx^i\, dx^j} \\ &= \int_{t=0}^{t=1} \frac{dx^1 = dt}{\sqrt{1-t^2}} = \frac{\pi}{2} \end{aligned} \tag{5.82}$$

6. MEASURE AND VOLUME ON COSETS. The invariant measure on a coset is related to the invariant measure on the parent group. We will use this observation to compute the invariant volumes of all compact classical groups.

If an element of volume at the identity

$$dV(\text{Id}) = dx^1(\text{Id}) \wedge dx^2(\text{Id}) \wedge \cdots \wedge dx^n(\text{Id}) \tag{5.83}$$

is moved from Id to c by means of the group (coset) operation c, then the Euclidean volume element is

$$\begin{aligned} dV(c) &= dx^1(c) \wedge dx^2(c) \wedge \cdots \wedge dx^n(c) \\ &= \|J(c)\|\, dx^1(0) \wedge dx^2(0) \wedge \cdots \wedge dx^n(0) \end{aligned} \tag{5.84}$$

In other words, the *Euclidean* volume $dV(c)$, given in terms of the displacements $dx^i(c)$ over the coset G/K, varies from place to place on the coset.

We define an invariant measure on the coset exactly the same as we defined an invariant measure on a group and an invariant metric on a coset:

$$\begin{array}{ccc} d\mu(\text{Id}) & \xrightarrow[=]{\text{Id} \to c} & d\mu(c) \\ \| & & \| \\ \rho(\text{Id})\, dV(\text{Id}) & = & \rho(c)\, dV(c) \end{array} \tag{5.85}$$

Then from (5.85) we find

$$\rho(\mathrm{Id})\, dV(\mathrm{Id}) = \rho(c)\, dV(c) = \rho(c)\|J(c)\|\, dV(\mathrm{Id})$$

The invariant measure on a coset is then given by

$$\begin{aligned} d\mu(0) &= d\mu(c) \\ d\mu(c) &= \rho(c)\, dV_{\mathrm{Euclid}}(c) \\ &= \frac{\rho(0)}{\|J(c)\|}\, dV_{\mathrm{Euclid}}(c) \end{aligned} \tag{5.86}$$

With this Haar measure on a coset, it is possible to compute the invariant volumes of the classical compact cosets.

Example 1. The volume of the upper hemisphere of $SO(n+1)/SO(n)$ is

$$\underset{\substack{0 \le x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2 \le 1 \\ x_{n+1} > 0}}{\int \int \cdots \int} \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{\left\| \left[I_n - \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} (x_1 \cdots x_n) \right] \right\|^{1/2}} \tag{5.87r}$$

Using the relation

$$\det \|I_m - XX^\dagger\| = \det \|I_n - X^\dagger X\| \tag{5.88r}$$

this integral reduces to

$$\underset{\substack{0 \le x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2 \le 1 \\ x_{n+1} \ge 0}}{\int \int \cdots \int} \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{\sqrt{1 - x_1{}^2 - x_2{}^2 - \cdots - x_n{}^2}} = \frac{1}{2} V[S^n \subset R_{n+1}] \tag{5.89r}$$

$$\underset{x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2 + x_{n+1}{}^2 = 1}{\int \int \cdots \int} \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{\pm x_{n+1}} = V[S^n \subset R_{n+1}] \tag{5.90r}$$

The integral on the left-hand side of (5.90r) can be written much more simply

$$= \underset{x_1 x_2 \cdots\ x_n x_{n+1}}{\int \int \cdots \int \int} \delta\left(\sqrt{x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2 + x_{n+1}^2} - 1\right)$$

$$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \wedge dx_{n+1} \tag{5.91r}$$

The simplest way to compute the surface area of a sphere of radius 1 in R_{n+1} is the following. The integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi} \tag{5.92}$$

is well known. The product of n such integrals can be computed explicitly and also rewritten in terms of spherical coordinates in R_n:

$$\int_0^\infty e^{-r^2} r^{n-1}\, dr \int d\Omega\,(S^{n-1} \subset R_n) = \prod_{i=1}^{n} \int_{-\infty}^{+\infty} e^{-x_i^2}\, dx_i = (\pi)^{n/2} \tag{5.93}$$

The integral over r on the left-hand side is given in terms of the Γ function

$$\int_0^\infty e^{-r^2} r^{n-1}\, dr = \frac{1}{2}\Gamma\left(\frac{n}{2}\right) \tag{5.94}$$

Therefore

$$V[S^{n-1} \subset R_n] = \int d\Omega(S^{n-1} \subset R_n) = \frac{2(\pi)^{n/2}}{\Gamma(n/2)} \tag{5.95r}$$

The volume of $S^n \subset R_{n+1}$ is

$$V[S^n \subset R_{n+1}] = \int_{x_1}\int_{x_2} \cdots \int_{x_{n+1}} \delta\left(\sqrt{x_1{}^2 + \cdots + x_{n+1}^2} - 1\right) dx_1 \cdots dx_{n+1} = \frac{2(\pi)^{n+1/2}}{\Gamma[(n+1)/2]} \tag{5.96r}$$

Example 2. The coset $U(n+1)/U(n) \otimes U(1)$ is given by

$$\mathrm{EXP}\left[\begin{array}{c|c} & \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix} \\ \hline -b^\dagger & \end{array}\right] \to \left[\begin{array}{c|c} & \begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{matrix} \\ \hline -u^\dagger & r_{n+1} \end{array}\right] \tag{5.97}$$

The coordinate $r_{n+1} = \sqrt{1 - u^\dagger u}$ is real; the u_i are all complex. These coordinates obey

$$u_1^* u_1 + u_2^* u_2 + \cdots + u_n^* u_n + r_{n+1}^2 = 1 \tag{5.98c}$$

When we write each complex u_i in terms of its real and imaginary parts

$$u_i = r_i + i\alpha_i \tag{5.99c}$$

the constraint reduces to

$$\sum_{i=1}^{n} r_i{}^2 + \sum_{i=1}^{n} \alpha_i{}^2 + r_{n+1}^2 = 1 \tag{5.100c}$$

Following a calculation entirely analogous to that in Example 1, the volume of the coset $U(n+1)/U(n)\otimes U(1)$ is

$$V\left[\frac{U(n+1)}{U(n)\otimes U(1)}\right]=\int\cdots\int\frac{d^2u_1\wedge\cdots\wedge d^2u_n}{\sqrt{1-\sum u_i^*u_i}}$$

$$d^2u_i=dr_i\wedge d\alpha_i \tag{5.89c}$$

$$V\left[\frac{U(n+1)}{U(n)\otimes U(1)}\right]$$

$$=\int_{r_1}\cdots\int_{r_n}\int_{\alpha_1}\cdots\int_{\alpha_n}\int_{r_{n+1}}\delta\left(\sqrt{\sum r_i{}^2+\sum\alpha_i{}^2+r_{n+1}^2}-1\right)dr_i\,d\alpha_i\,dr_{n+1} \tag{5.91c}$$

$$=V[S^{2n}\subset R_{2n+1}]=\frac{2(\pi)^{2n+1/2}}{\Gamma[(2n+1)/2]} \tag{5.96c}$$

Example 3. The coset $USp(2n+2)/USp(2n)\otimes USp(2)$ is obtained by replacing the complex numbers by quaternions in (5.97):

$$\begin{aligned}u_i\to q_i&\qquad i=1,2,\ldots,n\\&=r_i+i\alpha_i+j\beta_i+k\gamma_i\end{aligned} \tag{5.99q}$$

$$d^2u_i\to d^4q_i=dr_i\wedge d\alpha_i\wedge d\beta_i\wedge d\gamma_i \tag{5.100q}$$

The coset $USp(2n+2)/USp(2n)\otimes USp(2)$ is identified with the sphere $S^{4n}\subset R_{4n+1}$:

$$V\left[\frac{USp(2n+2)}{USp(2n)\otimes USp(2)}\right]=\frac{2(\pi)^{4n+1/2}}{\Gamma[(4n+1)/2]} \tag{5.96q}$$

Comment. The three classical compact rank-1 cosets may be summarized as follows:

$$\frac{U(n+1;f)}{U(n;f)\otimes U(1;f)}\qquad f=\begin{array}{rl}1 & \text{real}\\ 2 & \text{complex}\\ 4 & \text{quaternion}\end{array} \tag{5.101}$$

The volumes of these cosets are

$$V\left[\frac{U(n+1;f)}{U(n;f)\otimes U(1;f)}\right]=\frac{2(\pi)^{nf+1/2}}{\Gamma[(nf+1)/2]} \tag{5.102}$$

Example 4. The homogeneous Lorentz group $SO(1, 3)$ leaves invariant the metric

$$E^2 - (\mathbf{p}c) \cdot (\mathbf{p}c) = (mc^2)^2$$

$$p^\mu c = \begin{bmatrix} p^0 c = E \\ p^1 c \\ p^2 c \\ p^3 c \end{bmatrix} \tag{5.103}$$

The stability subgroup of the vector col $(mc^2, 0, 0, 0)$ is $SO(3)$. That is, the subgroup of $SO(1, 3)$ which leaves unchanged the state of a particle at rest is the rotation group, for

$$\left[\begin{array}{c|c} 1 & \\ \hline & SO(3) \end{array}\right]\left[\begin{array}{c} mc^2 \\ \hline \mathbf{0} \end{array}\right] = \left[\begin{array}{c} mc^2 \\ \hline SO(3)\,\mathbf{0} \end{array}\right] = \left[\begin{array}{c} mc^2 \\ \hline \mathbf{0} \end{array}\right]$$

The state of a particle in motion with a uniform velocity $\mathbf{v}$ is obtained from the state of a particle at rest by applying a group operation from the coset $SO(1, 3)/SO(3)$. It is then necessary to consider invariant measure on these cosets. This invariant measure is

$$d\mu\left[\frac{SO(1, 3)}{SO(3)}\right] = \delta(\sqrt{E^2 - (\mathbf{p}c) \cdot (\mathbf{p}c)} - mc^2)\, dE\, dp^1\, dp^2\, dp^3$$

$$\to \frac{dp^1\, dp^2\, dp^3}{|E|}; \qquad |E| = \sqrt{(\mathbf{p}c) \cdot (\mathbf{p}c) + (mc^2)^2} \tag{5.104}$$

7. MEASURE AND VOLUME ON THE CLASSICAL GROUPS. The methods of the preceding section provide a constructive procedure for determining the measure and volume of the classical groups. The measure on a group is nothing more than the product of measures on the cosets of a nested sequence of subgroups.

Let G be a group and H a subgroup with the usual coset decomposition

$$g = ch \qquad \begin{cases} g \in G \\ h \in H \\ c \in \dfrac{G}{H} \end{cases} \tag{5.105g}$$

Invariant integrals of the form

$$I = \int_{g \in G} f(g)\, d\mu(g) \tag{5.106}$$

frequently occur. If the function $f(g)$ is such that its dependence on the subgroup H and the coset G/H may be separated (separation of variables), then the function may be decomposed

$$f(g) = f(ch) = f_1(c) f_2(h) \tag{5.105f}$$

Under these conditions, the integral itself may be written as the product of two integrals. This is nothing more than the decomposition of multiple integrals in a group, rather than the more familiar Euclidean space:

$$I = \int_{c \in G/H} f_1(c)\, d\mu(c) \int_{h \in H} f_2(h)\, d\mu(h) \tag{5.107}$$

Various choices for the functions $f_1(c)$ and $f_2(h)$ yield different aspects of information about the group G:

$$
\begin{tikzcd}
I = \arrow[dd, "f(g)=1"'] & & \int_{c \in G/H} f_1(c)\, d\mu(c) \arrow[dl, "f_1(c)=1"'] & \int_{h \in H} f_2(h)\, d\mu(h) \arrow[dr, "f_2(h)=1"] & & (5.107) \\
& \mathrm{vol}\left(\frac{G}{H}\right) \int_{h \in H} f_2(h)\, d\mu(h); \arrow[dr, "f_2(h)=1"'] & & & \int_{c \in G/H} f_1(c)\, d\mu(c)\, \mathrm{vol}\,(H) \arrow[dl, "f_1(c)=1"] & (5.108) \\
\mathrm{vol}\,(G) = & & \mathrm{vol}\left(\frac{G}{H}\right) & \mathrm{vol}\,(H) & & (5.109)
\end{tikzcd}
$$

Equation (5.109) allows us to construct the measure and volume of the classical groups from the measure and volume on their cosets, which have been computed already.

Before actually constructing these volumes, we note that the group volumes are finite *only* for compact groups. If G is noncompact, both sides of (5.109) will diverge. This is easily verified in a particular case by integrating over the invariant measure (5.104). These divergence problems can sometimes be circumvented by defining a *normalized* measure

$$d\mu_N(G) = \frac{d\mu(G)}{\mathrm{vol}\,(G)} \tag{5.110}$$

The normalized measure is an averaging measure

$$\langle f \rangle = \frac{\int f(g)\, d\mu(g)}{\int d\mu(g)} = \int f(g)\, d\mu_N(g) \tag{5.111}$$

For compact groups, (5.111) differs in no essential way from (5.106). For noncompact groups, both numerator and denominator in (5.111) diverge. However, the limits on the integrals involved can often be taken in such a way that the ratio is always well defined, even though neither numerator nor denominator exists.

The normalized measure on a group G provides a normalized measure on a subgroup H and the coset G/H

$$\begin{tikzcd}
 & \dfrac{\int f_2(h)\, d\mu(h)}{\int d\mu(h)} \\
\dfrac{\int f_1(c) f_2(h)\, d\mu(ch)}{\int d\mu(c) \int d\mu(h)} \arrow[ur, "f_1 = 1"] \arrow[dr, "f_2 = 1"'] & \\
 & \dfrac{\int f_1(c)\, d\mu(c)}{\int d\mu(c)}
\end{tikzcd} \tag{5.112}$$

The normalized measure on G furnishes a normalized measure for essentially anything embedded in G. If G_1 is a subgroup of G and H_1 a subgroup of H ($H \subset G$, $H_1 \subset G_1$), and if G_1/H_1 is embedded in G/H, we can write

$$\begin{tikzcd}
 & G \arrow[dl] \arrow[d] \arrow[dr] & \\
H \arrow[dd] & & G/H \arrow[dd] \\
 & G_1 \arrow[dl] \arrow[dr] & \\
H_1 & & G_1/H_1
\end{tikzcd} \tag{5.113}$$

$$d\mu_N(G) \xrightarrow{G \downarrow G_1} \frac{\text{vol}\,(G_1)}{\text{vol}\,(G)}\, d\mu_N(G_1) \tag{5.114}$$

$$d\mu_N(G) \xrightarrow{G \downarrow H_1} \frac{\text{vol}\,(G_1)}{\text{vol}\,(G)} \frac{\text{vol}\,(H_1)}{\text{vol}\,(G_1)}\, d\mu_N(H_1) \tag{5.115}$$

$$d\mu_N\left(\frac{G}{H}\right) \xrightarrow{G/H \downarrow G_1/H_1} \frac{\text{vol}\,(G_1/H_1)}{\text{vol}\,(G/H)}\, d\mu_N\left(\frac{G_1}{H_1}\right) \tag{5.116}$$

Comment. Equation (5.116) holds when applied to the coset representatives X or to any coset representatives related by BCH formulas. In particular, it holds for the projectively related coset representatives $Z = XY^{-1}$.

Example. An invariant measure can be defined on $SO(5, 2)$. This provides a measure on the subgroup $SO(4, 2)$, the various maximal compact subgroups, and the coset spaces:

$$\begin{tikzcd}
d\mu_N[SO(n,2)] = d\mu_N\left[\dfrac{SO(n,2)}{SO(n)\otimes SO(2)}\right] \arrow[d, "Z"', "= XY^{-1}"] & d\mu_N[SO(n)\otimes SO(2)] \arrow[d] \\
d\mu_N[D^n] \quad \times & d\mu_N[SO(n)\otimes SO(2)]
\end{tikzcd} \tag{5.117}$$

The measures induced from $d\mu_N[SO(5,2)]$ are

$$d\mu_N[SO(5,2)] \xrightarrow[\substack{SO(5,2)\\ \downarrow \\ SO(4,2)}]{} \frac{\text{vol }(D^4)}{\text{vol }(D^5)}\, d\mu_N(D^4)$$

$$\times \frac{\text{vol }[SO(4)\otimes SO(2)]}{\text{vol }[SO(5)\otimes SO(2)]}\, d\mu_N[SO(4)\otimes SO(2)] \tag{5.118}$$

The Euclidean volumes of the bounded domains, the polydiscs D^n, are

$$\text{vol }(D^n) = \frac{\pi^n}{2^{n-1}n!} \tag{5.119}$$

The invariant volumes of the domains are, of course, infinite. The ratios of the volumes of the maximal compact subgroups are determined from

$$\frac{\text{vol }[SO(n)\otimes SO(2)]}{\text{vol }[SO(n-1)\otimes SO(2)]} = \text{vol}\left[\frac{SO(n)\otimes SO(2)}{SO(n-1)\otimes SO(2)}\right] = \text{vol}\left[\frac{SO(n)}{SO(n-1)}\right]$$

$$= \text{vol }[S^{n-1} \subset R_n] = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{5.120}$$

where $n = 5$.

Each of the classical compact groups can be written recursively as a product of coset representatives

$$U(n;f) = \frac{U(n;f)}{U(n-1;f)} \otimes \left\{ U(n-1;f) = \frac{U(n-1;f)}{U(n-2;f)} \otimes \left\{ U(n-2;f) \cdots \right.\right.$$

$$= \frac{U(n;f)}{U(n-1;f)} \otimes \frac{U(n-1;f)}{U(n-2;f)} \otimes \cdots \otimes \frac{U(2;f)}{U(1;f)} \otimes \frac{U(1;f)}{\text{Id}}$$

$$= \prod_{k=1}^{n} \frac{U(k;f)}{U(k-1;f)} \tag{5.121}$$

The measure on $U(n;f)$ is the product of the measures on the cosets; the volume on $U(n;f)$ is the product of the volumes of these cosets. In the preceding section we have computed the volumes of the cosets

$$\frac{U(n;f)}{U(n-1;f)\otimes U(1;f)}$$

We compute here the volumes of the cosets

$$\frac{U(n;f)}{U(n-1;f)}$$

which act on the polar vector col (0, 0, ..., 0, 1) and map it into the vector

$$\left[\begin{array}{c|c} & x_1 \\ & x_2 \\ & \vdots \\ & x_{n-1} \\ \hline -x^\dagger & x_n \end{array}\right]\left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ \hline x_n \end{array}\right] \qquad x = \begin{array}{l} r\colon SO(n) \\ c\colon U(n) \\ q\colon USp(2n) \end{array}$$

$$x_1^* x_1 + x_2^* x_2 + \cdots + x_{n-1}^* x_{n-1} + x_n^* x_n = 1$$

There is a 1-1 correspondence between the coset representatives $U(n;f)/U(n-1;f)$ and the points on the surface of an $(nf-1)$-dimensional sphere embedded in R_{nf}. Thus

$$\operatorname{vol}\left[\frac{U(n;f)}{U(n-1;f)}\right] = \operatorname{vol}\,[S^{nf-1} \subset R_{nf}] = \frac{2\pi^{nf/2}}{\Gamma(nf/2)} \tag{5.122}$$

The volumes of the classical groups are

$$\operatorname{vol}\,[U(n;f)] = \prod_{k=1}^{n} \frac{2\pi^{kf/2}}{\Gamma(kf/2)} \tag{5.123}$$

More specifically, the volumes of the classical compact groups are given explicitly by

$$\operatorname{vol}\,[SO(r)] = \frac{2^{\left[\frac{r-1}{2}\right]}(2\pi)^{\left[\left(\frac{r}{2}\right)^2\right]}}{(r-2)!\,(r-4)!\cdots \begin{smallmatrix}\nearrow 4!\,2! \\ \searrow 5!\,3!\,1!\end{smallmatrix}}$$

$$\operatorname{vol}\,[SO(2n)] = \frac{2^{n-1}(2\pi)^{n^2}}{(2n-2)!\,(2n-4)!\cdots!\,4!\,2!} \qquad \operatorname{vol}\,[SO(2n+1)] = \frac{2^{n}(2\pi)^{n(n+1)}}{(2n-1)!\,(2n-3)!\cdots!\,5!\,3!\,1!}$$

$$\operatorname{vol}\,[U(n,c)] = \frac{2^{n}(\pi)^{n(n+1)/2}}{(n-1)!\,(n-2)!\cdots!\,3!\,2!\,1!} \qquad \operatorname{vol}\,[U(n,q) = USp(2n)] = \frac{2^{n}(\pi)^{n(n+1)}}{(2n-1)!\,(2n-3)!\cdots!\,5!\,3!\,1!}$$

Comment. The volume of the coset $U(n;\, f)/U(n-1;\, f)$ can be determined from the volume of $U(n;f)/[U(n-1;f) \otimes U(1;f)]$. The action of $U(n;f)/U(n-1;f)$ on a point in the coset space is given by

$$\underset{\substack{U(1;f)\\ |y_n|=1}}{\left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \hline & & & & y_n \end{array}\right]} \underset{\dfrac{U(n;f)}{U(n-1;f)\otimes U(1;f)}}{\left[\begin{array}{c|c} & x_1 \\ * & x_2 \\ & \vdots \\ & x_{n-1} \\ \hline * & r_n \end{array}\right]} \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ \hline x_n \end{array}\right] \tag{5.124}$$

where $x_n = r_n y_n$. The invariant measure is determined most easily by computing the measure on the orbit of the north pole col $(0, 0, \cdots, 0, 1)$ under the action of the coset $U(n;\, f)/[U(n-1;\, f) \otimes U(1;\, f)]$, followed by the action of the group $U(1;\, f)$. The result is

$$\delta(r_n - \sqrt{x_n^* x_n})\, d^f x_n$$
$$\delta(\sqrt{x_1^* x_1 + \cdots + x_{n-1}^* x_{n-1} + r_n{}^2} - 1)\, d^f x_1 \wedge d^f x_2 \cdots \tag{5.125}$$

The total volume is then the integral over all coordinates appearing in (5.125). The simplest way to compute the volume described by (5.125) is to perform the dr_n integral first:

$$\delta(\sqrt{x_1^* x_1 + \cdots + x_n^* x_n} - 1)\, d^f x_1 \wedge \cdots \wedge d^f x_n = d\mu\left[\frac{U(n;f)}{U(n-1;f)}\right] \tag{5.126}$$

The integral can also be performed by computing the volumes of $U(1;f)$ and the coset $U(n;\, f)/U(n-1;\, f) \otimes U(1;\, f)$ in (5.125). These are the surface areas of spheres of radius y in R_f and of radius x in $R_{(n-1)f}$. The radii x and y are not independent, but are related by the condition that their squares sum to 1:

$$\iint \left\{\frac{2\pi^{f/2}}{\Gamma(f/2)} y^{f-1}\right\}\left\{\frac{2\pi^{(n-1)f/2}}{\Gamma[(n-1)f/2]} x^{(n-1)f-1}\right\} \delta(\sqrt{x^2+y^2} - 1)\, dx\, dy \tag{5.127}$$

The integral (5.127) is computed by first integrating over y

$$\frac{2\pi^{f/2}}{\Gamma(f/2)} \frac{2\pi^{(n-1)f/2}}{\Gamma[(n-1)f/2]} \int_{x=0}^{x=1} [1 - x^2]^{f-2/2} x^{(n-1)f-1}\, dx \tag{5.128}$$

The integral in (5.128) is a well-known Beta[15] function

$$\int_{x=0}^{x=1} x^m (1-x^2)^{n/2}\, dx = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+2}{2}\right)$$
$$= \frac{1}{2} \frac{\Gamma[(m+1)/2]\Gamma[(n+2)/2]}{\Gamma[(m+n+3)/2]} \tag{5.129}$$

The result is given simply by (5.122).

Example. The Lorentz group $SO(1, 3)$ is decomposed in a coset decomposition according to

$$
\begin{bmatrix} 0 & b_3{}^4 & b_2{}^4 & b_1{}^4 \\ b_3{}^4 & 0 & 0 & 0 \\ b_2{}^4 & 0 & 0 & 0 \\ b_1{}^4 & 0 & 0 & 0 \end{bmatrix}
\oplus
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_2{}^3 & b_1{}^3 \\ 0 & -b_2{}^3 & 0 & 0 \\ 0 & -b_1{}^3 & 0 & 0 \end{bmatrix}
\oplus
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1{}^2 \\ 0 & 0 & -b_1{}^2 & 0 \end{bmatrix}
$$

$$
\Big\downarrow \text{EXP} \qquad\qquad \Big\downarrow \text{EXP} \qquad\qquad \Big\downarrow \text{EXP}
$$

$$
\left[\begin{array}{c|c} \sqrt{1 + \mathbf{x}^4\mathbf{x}^{4T}} & \mathbf{x}^4 \\ \hline \mathbf{x}^{4T} & * \end{array}\right]
\otimes
\left[\begin{array}{c|c|c} 1 & & \\ \hline & \sqrt{1 - \mathbf{x}^3\mathbf{x}^{3T}} & \mathbf{x}^3 \\ \hline & -\mathbf{x}^{3T} & * \end{array}\right]
\otimes
\left[\begin{array}{cc|c|c} 1 & & & \\ & 1 & & \\ \hline & & \sqrt{1 - \mathbf{x}^2\mathbf{x}^{2T}} & \mathbf{x}^2 \\ \hline & & -\mathbf{x}^{2T} & * \end{array}\right]
$$

The measure on the Lorentz group $SO(1, 3)$ may be read directly from the matrix element occurring at the intersection of the "rank-1 row and column":

$$
\frac{dx_3{}^4 \wedge dx_2{}^4 \wedge dx_1{}^4}{\sqrt{1 + (x_3{}^4)^2 + (x_2{}^4)^2 + (x_1{}^4)^2}} \wedge \frac{dx_2{}^3 \wedge dx_1{}^3}{\sqrt{1 - (x_2{}^3)^2 - (x_1{}^3)^2}} \wedge \frac{dx_1{}^2}{\sqrt{1 - (x_1{}^2)^2}} \tag{5.130}
$$

In general, the invariant measure for a rank-1 coset is the reciprocal of the matrix element that occurs at the intersection of the "rank-1 row" with the "rank-1 column".

8. CANONICAL COSET PARAMETERIZATION OF THE CLASSICAL GROUPS. The classical matrix Lie groups can be constructed by the canonical mapping of the algebra onto the matrix group under EXP, or by means of numerous noncanonical parameterizations. These parameterizations are all related to each other by Baker-Campbell-Hausdorff formulas (the prototypes of these were treated in Chapter 5, Section VI). The most convenient parameterization for our purposes is the canonical coset decomposition

$$
U(n;f) = \frac{U(n;f)}{U(n-1;f)} \otimes \frac{U(n-1;f)}{U(n-2;f)} \otimes \cdots \otimes \frac{U(2;f)}{U(1;f)} \otimes \frac{U(1;f)}{\text{Id}}
$$

Each of these rank-1 cosets can be determined explicitly from the coordinates in the vector space

$$\mathfrak{p} = \mathfrak{u}(k;f) \bmod \mathfrak{u}(k-1;f)$$

For purposes that become apparent later, it is useful to write the elements in the Lie algebra, not in terms of square matrices, but in the closely related triangular arrays:

$$\begin{array}{c} n \\ \vdots \\ 2 \\ 1 \end{array}\begin{bmatrix} b_n{}^n & \cdots & b_n{}^2 & b_n{}^1 \\ \vdots & \ddots & \vdots & \vdots \\ b_2{}^n & \cdots & b_2{}^2 & b_2{}^1 \\ b_1{}^n & \cdots & b_1{}^2 & b_1{}^1 \end{bmatrix} \to \begin{array}{ccccccc} b_1{}^n & b_2{}^n & \cdots & b_n{}^n & \cdots & b_n{}^2 & b_n{}^1 \\ & & & \vdots & & & \\ & & b_1{}^2 & b_2{}^2 & b_2{}^1 & & \\ & & & b_1{}^1 & & & \end{array} \tag{5.131}$$

Such a triangular array describes a well-defined group operation under the semicanonical mapping

$$\begin{bmatrix} b_n{}^n & \cdots & b_n{}^2 & b_n{}^1 \\ \vdots & \ddots & \vdots & \vdots \\ b_2{}^n & \cdots & b_2{}^2 & b_2{}^1 \\ b_1{}^n & \cdots & b_1{}^2 & b_1{}^1 \end{bmatrix} \tag{5.132}$$

$$\mathrm{EXP}\ \big\downarrow\ \text{(semicanonical parameterization)}$$

$$\mathrm{EXP}\left[\begin{array}{c|ccc} b_n{}^n & \cdots & b_n{}^2 & b_n{}^1 \\ \hline \vdots & & & \\ b_2{}^n & & & \\ b_1{}^n & & & \end{array}\right] \cdots \mathrm{EXP}\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & b_2{}^2 & b_2{}^1 \\ \hline & & b_1{}^2 & \end{array}\right] \mathrm{EXP}\left[\begin{array}{cc} & \\ & \\ & \\ & \boxed{b_1{}^1} \end{array}\right]$$

This parameterization is actually suggested by the last example in the preceding section (5.130).

Comment 1. These triangular patterns contain all elements describing the coset $U(k;f)/U(k-1;f)$ in the kth row. In this form, the triangular patterns describing *group elements* are dual to the triangular patterns, called Gel'fand-Tsetlein patterns, which describe *matrix elements* within the irreducible representations of these groups.[16–19] In fact, the representations are constructed explicitly using this duality.

Comment 2. The elements within the group can be described by triangular patterns (5.131) related to square matrices in *either* the group or the algebra. The classical groups are described by *nonlinear* constraints, whereas their algebras, which result from a linearization of these constraints near the identity, are prescribed by *linear* constraints. The symmetry properties of the group operations are brought out more clearly when these elements are described by the element in the algebra that maps onto the associated group operation under the semicanonical parameterization (5.132).

The symmetry properties of the triangular patterns (5.131) are further enhanced by a slight relabeling of the matrix element entries. We let the upper index describe the row in which the entry occurs in the triangular pattern. The lower index describes how far to the right of the principal diagonal the entry occurs. Equivalently, this is the distance above the major diagonal in the square matrix array.

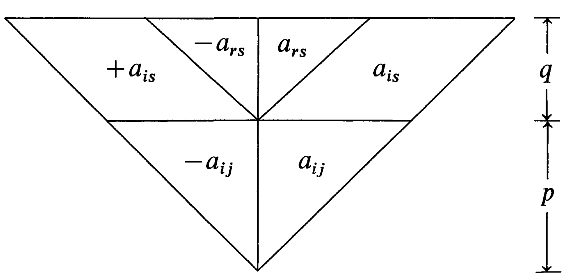
Example.

$$\begin{bmatrix} b_3{}^3 & b_3{}^2 & b_3{}^1 \\ b_2{}^3 & b_2{}^2 & b_2{}^1 \\ b_1{}^3 & b_1{}^2 & b_1{}^1 \end{bmatrix}$$

$$\begin{array}{ccccc} b_1{}^3 & b_2{}^3 & b_3{}^3 & b_3{}^2 & b_3{}^1 \\ & b_1{}^2 & b_2{}^2 & b_2{}^1 & \\ & & b_1{}^1 & & \end{array} \longleftrightarrow \begin{array}{ccccc} b_{-2}{}^3 & b_{-1}{}^3 & b_0{}^3 & b_{+1}{}^3 & b_{+2}{}^3 \\ & b_{-1}{}^2 & b_0{}^2 & b_{+1}{}^2 & \\ & & b_0{}^1 & & \end{array} \tag{5.133}$$

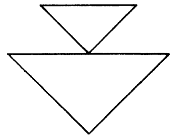
The usefulness of these patterns in describing elements within the classical groups is now indicated.

Example 1. These arrays for the orthogonal groups $SO(p, q)$ have the structure

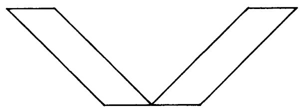


$$1 \leq i < j \leq p$$
$$p + 1 \leq r < s \leq p + q$$

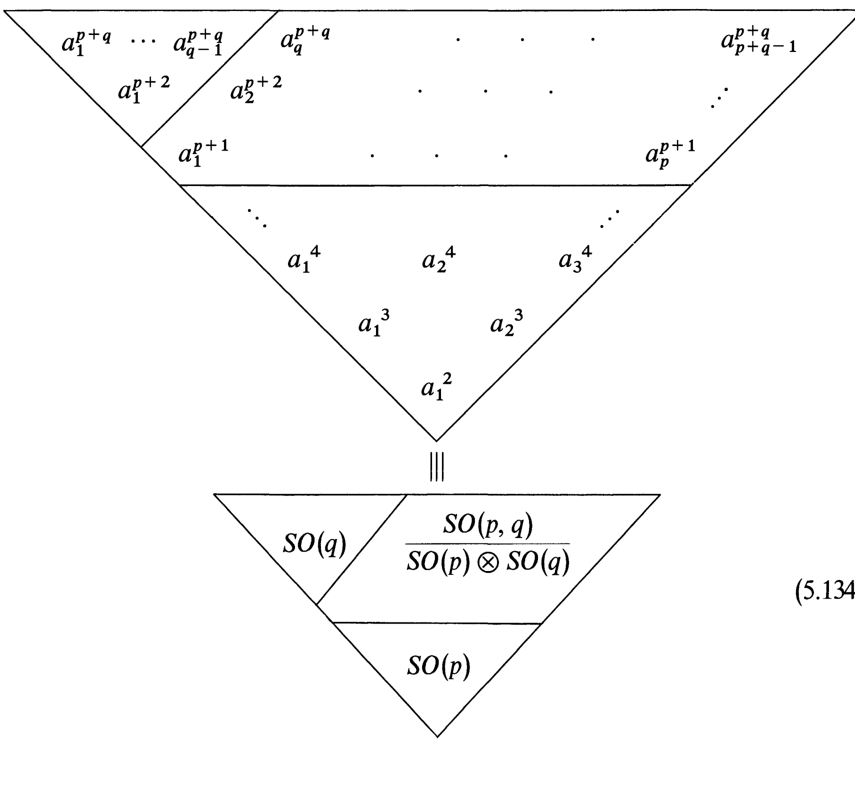
The two triangular pieces



describe elements in the maximal compact subgroup $SO(p) \otimes SO(q)$. The parallelopipeds



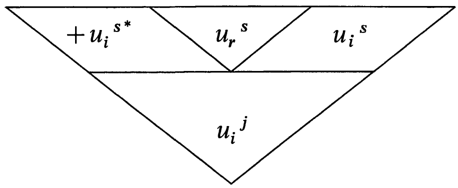
describe elements in the coset $SO(p, q)/SO(p) \otimes SO(q)$. It is clear from the diagram that the triangular pattern contains redundant information. The pattern can be simplified by omitting the left half:



(5.134)

The number of elements occurring on the vertical diagonal in (5.134) is $[p + q/2]$ (i.e., the rank of the group). These are exactly the parameters which describe the *class*[20] of a group operation. They are dual to the diagonal operators H_i of $SO(n)$.

Example 2. The unitary groups can be treated in exactly the same way:



(5.135)

The elements in the triangular structures are antihermitian

$$\begin{aligned} u_i{}^{j*} &= -u_j{}^i \qquad 1 \le i, j \le p \\ u_r{}^{s*} &= -u_s{}^r \qquad p+1 \le r, s \le p+q \end{aligned}$$

The elements within the parallelograms are hermitian

$$u_i{}^{r*} = +u_r{}^i$$

Once again, the elements on the diagonal in this pattern describe the classes of $U(p, q; c)$. The subgroup $SU(p, q; c)$ is subject to the constraint

$$\operatorname{tr} u_i{}^i = 0 \qquad \text{or} \qquad \operatorname{tr} u_0{}^i = 0$$

The sum of the elements in the class column is zero. Once again, the number of independent elements on the main vertical is equal to the rank of the group. These elements are dual to the generators H_i. They characterize the classes of $SU(p, q)$.

Example 3. The quaternion group $U(r, s; q)$ can be treated exactly the same way the unitary group $U(r, s; c)$ has been treated. The only change is to replace the complex numbers by quaternions

$$u_i{}^j \to q_i{}^j$$

If the symplectic groups are to be written in terms of complex numbers rather than quaternions, each q is replaced by a 2×2 complex matrix. The pattern can still be written in a triangular form:

$$q_i^{\ j} = -q_j^{\ i*}$$

$$\left[\begin{array}{c|c} q_2^{\ 2} & q_2^{\ 1} \\ \hline q_1^{\ 2} & q_1^{\ 1} \end{array}\right] \to \left[\begin{array}{cc|cc} x_2^{\ 2} & y_2^{\ 2} & x_1^{\ 2} & y_1^{\ 2} \\ -y_2^{*2} & x_2^{*2} & -y_1^{*2} & x_1^{*2} \\ \hline -x_1^{*2} & y_1^{\ 2} & x_1^{\ 1} & y_1^{\ 1} \\ -y_1^{*2} & -x_1^{\ 2} & -y_1^{*1} & x_1^{*1} \end{array}\right] \quad \begin{array}{c} USp(4) \simeq U(2;q) \\ x_i^{\ i} = -x_i^{*i} \end{array} \tag{5.136}$$

$$\begin{array}{ccccccc} -y_1^{*2} & -x_1^{*2} & -y_2^{*2} & x_2^{\ 2} & y_2^{\ 2} & x_1^{\ 2} & y_1^{\ 2} \\ & -x_1^{\ 2} & y_1^{\ 2} & x_2^{*2} & -y_1^{*2} & x_1^{*2} & \\ & & -y_1^{*1} & x_1^{\ 1} & y_1^{\ 1} & & \\ & & & x_1^{*1} & & & \end{array} \qquad \begin{array}{l} \operatorname{Re} x_1^{\ 1} = 0 \\ \operatorname{Re} x_2^{\ 2} = 0 \end{array}$$

Note that tr is zero, signifying that the symplectic groups are unimodular.

Comment. In each of these examples, the number of independent elements on the major vertical is equal to the rank of the group. These elements are dual to the diagonal operators H_i. As such, they characterize the classes of these groups. Since these statements hold also for the complex extensions of the compact algebras, they hold for all real forms.

VI. Real Forms of the Symmetric Spaces

1. ALGEBRAIC MACHINERY. From Section V it is clear that the symmetric spaces have an independent existence in their own right. From Section IV we learned that the full power of the techniques used for the study of Lie groups (Chapters 7 and 8) can be brought to bear on the study of the symmetric spaces. The symmetric spaces are classified here using the techniques presented in Section I of this chapter. The inequivalent real forms of the symmetric spaces are presented in a generalization of Table 9.3 presented in Section II of this Chapter. Finally, the real forms of these symmetric spaces can be distinguished, almost uniquely, in exactly the same way the real forms of the simple Lie algebras have been distinguished from each other (see Section III).

The symmetric spaces considered so far have been exclusively cosets of the form G/K or G^*/K. Here G is the compact real form of the complex extension group G^C, and G^* is the real form of G^C, dual to G; K is the maximal compact subgroup of noncompact G^*. The symmetric coset space G/K is endowed with a negative definite metric derived from the Cartan-Killing form on $\mathfrak{g}$. The dual space G^*/K is endowed with an identical but positive definite metric, derived from the Cartan-Killing form on $\mathfrak{g}^*$, restricted to the subspace $i\mathfrak{p}$ orthogonal to $\mathfrak{k}$. Since the metric is definite in both cases, it has been considered to be positive definite: thus the symmetric coset spaces are also Riemannian spaces.

For Lie groups there is a rich structure of real forms of G^C between the two extremes—the two most easily accessible real forms, which are the compact and the normal real forms. Therefore, for the coset spaces, as well, there is a rich structure of "real forms" of the "complex extension" coset spaces

$$\left(\frac{G}{K}\right)^C \simeq \frac{G^C}{K^C} \tag{6.1}$$

Within this structure the two spaces G/K and G^*/K appear as the "boundary spaces," endowed with negative and positive definite metric. All intermediate real forms of G^C/K^C possess an indefinite metric: they are pseudo-Riemannian symmetric spaces.

Example. The sphere and the hyperboloid are the Riemannian symmetric spaces associated with the group $SO(3)$:

$$S^2 \simeq \frac{SO(3)}{SO(2)} \qquad H^2 \simeq \frac{SO(2,1)}{SO(2)}$$

From Fig. 6.4, it is clear that there is yet another symmetric coset space associated with a real form of $SO(3)$. This is the single-sheeted hyperboloid. In this series of three symmetric spaces, the two Riemannian spaces occur as endpoints in the series of all symmetric spaces related to real forms of G^C/K^C:

Sphere (Fig. 6.2)	Single-sheeted hyperboloid (Fig. 6.4)	Double-sheeted hyperboloid (Fig. 6.1)	
$S^2 \simeq \frac{SO(3)}{SO(2)}$	$\frac{SO(2,1)}{SO(1,1)}$	$H^2 \simeq \frac{SO(2,1)}{SO(2)}$	
Negative definite metric	Indefinite metric	Positive definite metric	
"Riemannian"	"pseudo-Riemannian"	Riemannian	(6.2)

It is clear that the pseudo-Riemannian symmetric spaces are cosets of a noncompact group by a maximal [in the sense of (1.44)] noncompact subgroup. The maximality of a subgroup is intimately related to the existence of involutive automorphisms. Let $\mathfrak{g}$ be a compact simple Lie algebra, $\mathfrak{g}^C$ its complex extension, and $\mathfrak{g}^*$ a real form of $\mathfrak{g}^C$. Then it is possible to find an involutive automorphism τ of $\mathfrak{g}^*$ with the properties:[21]

$$\mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{m} \tag{6.3}$$

$$\begin{aligned} \tau(\mathfrak{g}^*) &= \tau(\mathfrak{h}) \oplus \tau(\mathfrak{m}) \\ &= (1)\mathfrak{h} \oplus (-1)\mathfrak{m} \end{aligned} \tag{6.4}$$

The decomposition (6.4) is exactly analogous to the decomposition of (1.41) to (1.44). By arguments isomorphic with those used in Section I, we can write

$$(\mathfrak{h}, \mathfrak{m}) = 0 \tag{6.5}$$

Inner products of the form

$$([X, Y], Z) \qquad X, Y, Z \qquad \text{in} \quad \textit{either} \quad \mathfrak{h} \quad \textit{or} \quad \mathfrak{m}$$

vanish if an odd number of the vectors belong to the subspace $\mathfrak{m}$ of eigenvalue -1 under τ. Since $\mathfrak{g}^*$ is simple, we have, following (1.44),

$$\begin{aligned} ([\mathfrak{h}, \mathfrak{h}], \mathfrak{m}) &= 0 \qquad & [\mathfrak{h}, \mathfrak{h}] &\subseteq \mathfrak{h} \\ ([\mathfrak{h}, \mathfrak{m}], \mathfrak{h}) &= 0 \qquad & [\mathfrak{h}, \mathfrak{m}] &= \mathfrak{m} \\ ([\mathfrak{m}, \mathfrak{m}], \mathfrak{m}) &= 0 \qquad & [\mathfrak{m}, \mathfrak{m}] &\subseteq \mathfrak{h} \end{aligned} \tag{6.6}$$

The Lie algebra $\mathfrak{g}^*$ is obtained from $\mathfrak{g}$ using the involutive automorphism σ:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{6.7}$$

$$\begin{aligned} \sigma(\mathfrak{g}) &= \sigma(\mathfrak{k}) \oplus \sigma(\mathfrak{p}) \\ &= (+1)\mathfrak{k} \oplus (-1)\mathfrak{p} \end{aligned} \tag{6.8}$$

$$\mathfrak{g}^\sigma = \mathfrak{k} \oplus i\mathfrak{p} \tag{6.9}$$

The analogs of (6.5) and (6.6), given by (1.42) and (1.44), hold for $\mathfrak{g}$ and $\mathfrak{g}^*$.

Just as all the Riemannian symmetric spaces are

$$\frac{G}{K_\sigma} = \text{EXP}\ \mathfrak{g} \bmod \mathfrak{k} = \text{EXP}\ \mathfrak{p}$$

$$\frac{G^*}{K} = \frac{G^\sigma}{K_\sigma} = \text{EXP}\ \mathfrak{g}^* \bmod \mathfrak{k} = \text{EXP}\ i\mathfrak{p} \tag{6.10}$$

all the pseudo-Riemannian symmetric coset spaces are

$$\begin{aligned} \frac{G^\sigma}{H_\tau} &= \mathrm{EXP}\ \mathfrak{g}^* \bmod \mathfrak{h} &&= \mathrm{EXP}\ \mathfrak{m} \\ \left(\frac{G^\sigma}{H}\right)^\tau = \frac{G^{\sigma\tau}}{H_\tau} &= \mathrm{EXP}\ (\mathfrak{g}^*)^{*'} \bmod \mathfrak{h} &&= \mathrm{EXP}\ i\mathfrak{m} \end{aligned} \tag{6.11}$$

The spaces EXP $\mathfrak{m}$ and EXP $i\mathfrak{m}$ are dual.

Since σ splits $\mathfrak{g}$, any vector in $\mathfrak{k}(i\mathfrak{p})$ is an eigenvector of σ to eigenvalue $+1(-1)$. Since τ splits $\mathfrak{g}^*$, any vector in $\mathfrak{h}(\mathfrak{m}, i\mathfrak{m})$ is an eigenvector of τ to eigenvalue $+1(-1)$. The four subspaces $\mathfrak{g}_{\sigma\tau}$

		Eigenvalue of τ	
		$+$	$-$
Eigenvalue of σ	$+$	$\mathfrak{k} \cap \mathfrak{h} \equiv \mathfrak{g}_{++}$	$\mathfrak{k} \cap \mathfrak{m} \equiv \mathfrak{g}_{+-}$
	$-$	$\mathfrak{p} \cap i\mathfrak{h} \equiv \mathfrak{g}_{-+}$	$\mathfrak{p} \cap i\mathfrak{m} \equiv \mathfrak{g}_{--}$

(6.12)

exhaust $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{++} \oplus \mathfrak{g}_{+-} \oplus \mathfrak{g}_{-+} \oplus \mathfrak{g}_{--} \tag{6.13}$$

Since any vector in any of the subspaces $\mathfrak{g}_{\sigma\tau}$ of (6.12) is an eigenvector of *both* σ and τ, and since the subspaces $\mathfrak{g}_{\sigma\tau}$ exhaust $\mathfrak{g}$, the mappings σ and τ commute, and their simultaneous eigenvalues can be used to describe $\mathfrak{g}$, as has been done in (6.13).

$$\begin{aligned} \sigma\tau = \tau\sigma; &\qquad \sigma^2 = \tau^2 = \mathrm{Id} \\ (\sigma\tau)^2 = \sigma\tau\sigma\tau &= \sigma\sigma\tau\tau = \mathrm{Id} \end{aligned} \tag{6.14}$$

Therefore, σ, τ and $\sigma\tau$ are involutive automorphisms of $\mathfrak{g}$.

Applying arguments akin to (6.5) to the decomposition (6.13), we find

$([\mathfrak{g}_{\sigma\tau}, \mathfrak{g}_{\sigma'\tau'}], \mathfrak{g}_{\sigma''\tau''})$	$\mathfrak{g}_{\sigma''\tau''}$			
$\sigma\sigma', \tau\tau'$	$++$	$+-$	$-+$	$--$
$++$		0	0	0
$+-$	0		0	0
$-+$	0	0		0
$--$	0	0	0	

(6.15)

$$[\mathfrak{g}_{\sigma\tau}, \mathfrak{g}_{\sigma'\tau'}] \subseteq \mathfrak{g}_{\sigma\sigma', \tau\tau'} \tag{6.16}$$

From (6.16) it is possible to conclude that various subspaces of $\mathfrak{g}$ are closed under commutation:

Eigenspace to Eigenvalue +1 of	Closed under Commutation	
1. Id, σ, τ, $\sigma\tau$	$\mathfrak{g}_{++}$	
2. Id, σ	$\mathfrak{g}_{++} \oplus \mathfrak{g}_{+-}$	
Id, τ	$\mathfrak{g}_{++} \oplus \mathfrak{g}_{-+}$	
Id, $\sigma\tau$	$\mathfrak{g}_{++} \oplus \mathfrak{g}_{--}$	
3. Id	$\mathfrak{g}_{++} \oplus \mathfrak{g}_{+-} \oplus \mathfrak{g}_{-+} \oplus \mathfrak{g}_{--}$	(6.17)

From (6.17) we conclude that all three involutive automorphisms σ, τ, $\sigma\tau$ occur in this discussion on an equal footing.

2. TABLES OF THE REAL FORMS. All distinct mappings σ have already been determined and are listed in Table 9.3. It remains only to determine the mappings τ. But since the involutive automorphisms σ and τ (as well as $\sigma\tau$) occur on an equal footing in (6.12) to (6.17), the possibilities available for the mappings τ are exactly and only those which are available for the mappings σ. Therefore the classification of all pseudo-Riemannian spaces proceeds by applying all possible mappings τ, listed in Table 9.3, to all possible real forms, also listed in Table 9.3. Thus, in a very real sense, the Table of Real Forms of the symmetric coset spaces [21–24] (Table 9.7) is the "square of Table 9.3."

In terms of the decomposition (6.13), the symmetric space G^σ/H_τ is

$$\begin{aligned}\frac{G^\sigma}{H_\tau} &= \mathrm{EXP}\,\{(\mathfrak{g}_{++} \oplus \mathfrak{g}_{+-}) \oplus i(\mathfrak{g}_{-+} \oplus \mathfrak{g}_{--}) \bmod (\mathfrak{g}_{++} \oplus i\mathfrak{g}_{-+})\} \\ &= \mathrm{EXP}\,(\mathfrak{g}_{+-} \oplus i\mathfrak{g}_{--}) \end{aligned} \tag{6.18}$$

$$\left(\frac{G^\sigma}{H_\tau}\right)^\tau = \frac{G^{\sigma\tau}}{H_\tau} = \mathrm{EXP}\; i\mathfrak{m} = \mathrm{EXP}\,\{i\mathfrak{g}_{+-} \oplus (i)^2\mathfrak{g}_{--}\} \tag{6.19}$$

Since the construction of the symmetric spaces amounts to the exponentiation of an orthogonal complementary subspace, the construction (6.19) can be indicated schematically by

$$\begin{array}{ccc} \mathfrak{g} \xrightarrow{\ \tau\ } \mathfrak{g}^\tau = & (\mathfrak{g}_{++} \oplus \mathfrak{g}_{-+}) \oplus & i(\mathfrak{g}_{+-} \oplus \mathfrak{g}_{--}) \\ & \sigma: \Big\downarrow\ \mathfrak{k}_\tau \to \mathfrak{h}_\tau & \sigma: \Big\downarrow\ \dfrac{G^\tau}{K_\tau} \to \dfrac{G^{\sigma\tau}}{H_\tau} \\ & (\mathfrak{g}_{++} \oplus i\mathfrak{g}_{-+}) \oplus & i(\mathfrak{g}_{+-} \oplus i\mathfrak{g}_{--}) \\ & \mathfrak{h} \quad \oplus & i \quad \mathfrak{m} \end{array} \tag{6.20}$$

In short, the real forms of the symmetric spaces are completely determined by a compact group G and by two involutive automorphisms σ, τ. One automorphism τ serves to single out a compact subgroup $K_\tau = \text{EXP}\,(\mathfrak{g}_{++} \oplus \mathfrak{g}_{-+})$ and a symmetric space $\text{EXP}\,(i)(\mathfrak{g}_{+-} \oplus \mathfrak{g}_{--})$ with a definite metric. The other automorphism σ serves to convert the compact subgroup to a noncompact subgroup

$$K_\tau = \text{EXP}\,(\mathfrak{g}_{++} \oplus \mathfrak{g}_{-+}) \xrightarrow{\sigma} \text{EXP}\,(\mathfrak{g}_{++} \oplus i\mathfrak{g}_{-+}) = H_\tau = K_\tau{}^\sigma$$

and to convert the symmetric space with a definite metric to one with an indefinite metric

$$\frac{G^{(*)}}{K_\tau} = \text{EXP}\,(i)(\mathfrak{g}_{+-} \oplus \mathfrak{g}_{--}) \xrightarrow{\sigma} \text{EXP}\,(i)(\mathfrak{g}_{+-} \oplus i\mathfrak{g}_{--}) = \frac{G^{(\sigma)\tau}}{K_\tau{}^\sigma} = \frac{G^{(\sigma)\tau}}{H_\tau}$$

Other real forms related to (6.20) are obtained by interchanging the order in which σ and τ are applied and by replacing either σ or τ by $\sigma\tau$.

The Riemannian symmetric spaces G/K, G^*/K, the endpoints in the series of real forms obtained from G and σ, τ, are exactly those spaces obtained under the condition of degeneracy

$$\sigma = \tau^{-1} = \tau \qquad \text{or} \qquad \sigma\tau = \text{Id} \tag{6.21}$$

Table 9.3 was designed to classify and enumerate all the real forms of the simple Lie groups. The Riemannian coset spaces obtained from this table were merely a by-product. Because of its original function, Table 9.3 does not include all possible Riemannian symmetric spaces, although it is complete in its listing of the real forms of the simple Lie groups. The Riemannian symmetric coset spaces clearly missing from Table 9.3 have been restored to Table 9.7. These include the compact Lie group G itself, as well as the space dual to it, G^C/G.

All real forms $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}^C$ have the same complex extension

$$\mathfrak{g}^C = \mathfrak{g}^* \oplus i\mathfrak{g}^* \tag{6.22}$$

Clearly, σ is an involutive automorphism if

$$\begin{aligned}\sigma(\mathfrak{g}^C) &= \sigma(\mathfrak{g}^*) \oplus \sigma(i\mathfrak{g}^*)\\ &= (+1)\mathfrak{g}^* \oplus (-1)(i\mathfrak{g}^*)\end{aligned} \tag{6.23}$$

The symmetric spaces obtained from this mapping have also been included in Table 9.7.

TABLE 9.7

Automorphism Type: σ	Group G	Automorphism Type: τ	Maximal Subgroup H	Number of Generators of G: Compact	Number of Generators of G: Noncompact	Number of Generators of H: Compact	Number of Generators of H: Noncompact
—	$SU(n)$	$A\,0$	Identity	n^2-1	0	0	0
	$SU(n)$	$A\,\mathrm{I}$	$SO(n)$	n^2-1	0	$\frac{n(n-1)}{2}$	0
	$SU(2n)$	$A\,\mathrm{II}$	$USp(2n)$	$(2n)^2-1$	0	$\frac{2n(2n+1)}{2}$	0
	$SU(p+q)$	$A\,\mathrm{III}$	$SU(p)\otimes SU(q)\otimes U(1)$	$(p+q)^2-1$	0	p^2+q^2-1	0
	$SU(n)$	$A\,0$	$SU(n)$	n^2-1	0	n^2-1	0
—	$Sl(n,c)$	$A\,\mathrm{I}$	$SO(n,c)$	n^2-1	n^2-1	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$
	$Sl(2n,c)$	$A\,\mathrm{II}$	$Sp(2n,c)$	$(2n)^2-1$	$(2n)^2-1$	$\frac{2n(2n+1)}{2}$	$\frac{2n(2n+1)}{2}$
	$Sl(p+q,c)$	$A\,\mathrm{III}$	$Sl(p,c)\otimes Sl(q,c)\otimes Gl(1,c)$	$(p+q)^2-1$	$(p+q)^2-1$	p^2+q^2-1	p^2+q^2-1
$A\,0$	$Sl(n,c)$	$A\,0$	$SU(n)$	n^2-1	n^2-1	n^2-1	0
	$Sl(n,c)$	$A\,\mathrm{I}$	$Sl(n,r)$	n^2-1	n^2-1	$\frac{n(n-1)}{2}$	$\frac{(n-1)(n+2)}{2}$
	$Sl(2n,c)$	$A\,\mathrm{II}$	$SU^*(2n)$	$(2n)^2-1$	$(2n)^2-1$	$n(2n+1)$	$(n-1)(2n+1)$
	$Sl(p+q;c)$	$A\,\mathrm{III}$	$SU(p,q)$	$(p+q)^2-1$	$(p+q)^2-1$	p^2+q^2-1	$2pq$
$A\,\mathrm{I}$	$Sl(n,r)$	$A\,\mathrm{I}$	$SO(n)$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}-1$	$\frac{n(n-1)}{2}$	0
	$Sl(p+q;r)$	$A\,\mathrm{I}$	$SO(p,q)$	$\frac{(p+q)(p+q-1)}{2}$	$\frac{(p+q)(p+q+1)}{2}-1$	$\frac{p(p-1)+q(q-1)}{2}$	pq
	$Sl(2n,r)$	$A\,\mathrm{II}$	$Sp(2n,r)$	$\frac{2n(2n-1)}{2}$	$\frac{2n(2n+1)}{2}-1$	n^2	$n(n+1)$
	$Sl(p+q;r)$	$A\,\mathrm{III}$	$Sl(p)\otimes Sl(q)\otimes SO(1,1)$	$\frac{(p+q)(p+q-1)}{2}$	$\frac{(p+q)(p+q+1)}{2}-1$	$\frac{p(p-1)+q(q-1)}{2}$	$\frac{p(p+1)}{2}+\frac{q(q+1)}{2}-1$
	$Sl(2n,r)$	$A\,\mathrm{III}$	$Sl(n,c)\otimes SO(2)$	$\frac{2n(2n-1)}{2}$	$\frac{2n(2n+1)}{2}-1$	n^2	n^2-1

A II	$SU^*(2n)$	A I	$SO^*(2n)$	$\frac{2n(2n+1)}{2}$	$(n-1)(2n+1)$	n^2	$n(n-1)$
	$SU^*(2n)$	A II	$USp(2n)$	$\frac{2n(2n+1)}{2}$	$(n-1)(2n+1)$	$\frac{2n(2n+1)}{2}$	0
	$SU^*(2p+2q)$	A II	$USp(2p, 2q)$	$(p+q)(2p+2q+1)$	$(p+q-1) \times (2p+2q+1)$	$p(2p+1) + q(2q+1)$	$4pq$
	$SU^*(2p+2q)$	A III	$SU^*(2p) \otimes SU^*(2q) \otimes R$	$(p+q)(2p+2q+1)$	$(p+q-1) \times (2p+2q+1)$	$p(2p+1) + q(2q+1)$	$(p-1)(2p+1) + (q-1)(2q+1) + 1$
	$SU^*(2n)$	A III	$Sl(n, c) \otimes SO(2)$	$\frac{2n(2n+1)}{2}$	$(n-1)(2n+1)$	n^2	$n^2 - 1$
A III	$SU(p, q)$	A I	$SO(p, q)$	$p^2 + q^2 - 1$	$2pq$	$\frac{p(p-1) + q(q-1)}{2}$	pq
	$SU(n, n)$	A I	$SO^*(2n)$	$2n^2 - 1$	$2n^2$	n^2	$n(n-1)$
	$SU(2p, 2q)$	A II	$USp(2p, 2q)$	$(2p)^2 + (2q)^2 - 1$	$2(2p)(2q)$	$p(2p+1) + q(2q+1)$	$(2p)(2q)$
	$SU(n, n)$	A II	$Sp(2n, r)$	$2n^2 - 1$	$2n^2$	n^2	$n(n+1)$
	$SU(p, q)$	A III	$SU(p) \otimes SU(q) \otimes U(1)$	$p^2 + q^2 - 1$	$2pq$	$p^2 + q^2 - 1$	0
	$SU(p, q)$	A III	$SU(h, k) \otimes SU(p-h, q-k) \otimes U(1)$	$p^2 + q^2 - 1$	$2pq$	$h^2 + (p-h)^2 k^2 + (q-k)^2 - 1$	$2hk + 2(p-h)(q-k)$
	$SU(n, n)$	A III	$Sl(n, c) \otimes SO(1, 1)$	$2n^2 - 1$	$2n^2$	$n^2 - 1$	n^2
	$SO(n)$	BD 0	Identity	$\frac{n(n-1)}{2}$	0	0	0
	$SO(p+q)$	BD I	$SO(p) \otimes SO(q)$	$\frac{(p+q)(p+q-1)}{2}$	0	$\frac{p(p-1) + q(q-1)}{2}$	0
	$SO(2n)$	D III	$U(n)$	$\frac{2n(2n-1)}{2}$	0	n^2	0
	$SO(n)$	BD 0	$SO(n)$	$\frac{n(n-1)}{2}$	0	$\frac{n(n-1)}{2}$	0
	$SO(p+q; c)$	BD I	$SO(p, c) \otimes SO(q, c)$	$\frac{(p+q)(p+q-1)}{2}$	$\frac{(p+q)(p+q-1)}{2}$	$\frac{p(p-1) + q(q-1)}{2}$	$\frac{p(p-1) + q(q-1)}{2}$
	$SO(2n, c)$	D III	$Gl(n, c)$	$\frac{2n(2n-1)}{2}$	$\frac{2n(2n-1)}{2}$	n^2	n^2

Continued

TABLE 9.7—*Continued*

Automorphism Type: σ	Group G	Automorphism Type: τ	Maximal Subgroup H	Number of Generators of G		Number of Generators of H	
				Compact	Noncompact	Compact	Noncompact
BD 0	$SO(n, c)$	BD 0	$SO(n, r)$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	0
	$SO(p+q; c)$	BD I	$SO(p, q; r)$	$\frac{(p+q)(p+q-1)}{2}$	$\frac{(p+q)(p+q-1)}{2}$	$\frac{p(p-1)+q(q-1)}{2}$	pq
D 0	$SO(2n, c)$	D III	$SO^*(2n)$	$\frac{2n(2n-1)}{2}$	$\frac{2n(2n-1)}{2}$	n^2	$n(n-1)$
BD I	$SO(p, q)$	BD I	$SO(p) \otimes SO(q)$	$\frac{p(p-1)+q(q-1)}{2}$	pq	$\frac{p(p-1)+q(q-1)}{2}$	0
	$SO(p, q)$	BD I	$SO(h, k) \otimes SO(p-h, q-k)$	$\frac{p(p-1)+q(q-1)}{2}$	pq	$\frac{h(h-1)+k(k-1)}{2} + (p-h)\frac{(p-h-1)}{2} + (q-k)\frac{(q-k-1)}{2}$	$hk + (p-h)(q-k)$
D I	$SO(n, n)$	D 0	$SO(n, c)$	$\frac{2n(n-1)}{2}$	n^2	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$
	$SO(2p, 2q)$	D III	$SU(p, q) \otimes U(1)$	$\frac{2p(2p-1)}{2} + \frac{2q(2q-1)}{2}$	$(2p)(2q)$	p^2+q^2-1	$2pq$
	$SO(n, n)$	D III	$Sl(n, r) \otimes SO(1, 1)$	$\frac{2n(n-1)}{2}$	n^2	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}$
D III	$SO^*(2n)$	D 0	$SO(n, c)$	n^2	$n(n-1)$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$
	$SO^*(2p+2q)$	D I	$SO^*(2p) \otimes SO^*(2q)$	$(p+q)^2$	$(p+q)(p+q-1)$	p^2+q^2	$p(p-1)+q(q-1)$
	$SO^*(2n)$	D III	$U(n)$	n^2	$n(n-1)$	n^2	0
	$SO^*(2p+2q)$	D III	$SU(p, q) \otimes U(1)$	$(p+q)^2$	$(p+q)(p+q-1)$	p^2+q^2-1	$2pq$
	$SO^*(4n)$	D III	$SU^*(2n) \otimes SO(1, 1)$	$(2n)^2$	$2n(2n-1)$	$\frac{2n(2n+1)}{2}$	$n(2n-1)$

	$USp(2n)$	C 0	Identity	$\frac{2n(2n+1)}{2}$	0	0	0
	$USp(2n)$	C I	$U(n)$	$\frac{2n(2n+1)}{2}$	0	n^2	0
	$USp(2p+2q)$	C II	$USp(2p) \otimes USp(2q)$	$(p+q)(2p+2q+1)$	0	$\frac{2p(2p+1)}{2} + \frac{2q(2q+1)}{2}$	0
	$USp(2n)$	C 0	$USp(2n)$	$\frac{2n(2n+1)}{2}$	0	$\frac{2n(2n+1)}{2}$	0
	$Sp(2n, c)$	C I	$Gl(n, c)$	$\frac{2n(2n+1)}{2}$	$\frac{2n(2n+1)}{2}$	n^2	n^2
	$Sp(2p+2q; c)$	C II	$Sp(2p, c) \otimes Sp(2q, c)$	$(p+q)(2p+2q+1)$	$(p+q)(2p+2q+1)$	$\frac{2p(2p+1)}{2} + \frac{2q(2q+1)}{2}$	$\frac{2p(2p+1)}{2} + \frac{2q(2q+1)}{2}$
C 0	$Sp(2n, c)$	C 0	$USp(2n)$	$\frac{2n(2n+1)}{2}$	$\frac{2n(2n+1)}{2}$	$\frac{2n(2n+1)}{2}$	0
	$Sp(2n, c)$	C I	$Sp(2n, r)$	$\frac{2n(2n+1)}{2}$	$\frac{2n(2n+1)}{2}$	n^2	$n(n+1)$
	$Sp(2p+2q; c)$	C II	$USp(2p, 2q)$	$(p+q)(2p+2q+1)$	$(p+q)(2p+2q+1)$	$p(2p+1)+q(2q+1)$	$4pq$
C I	$Sp(4n, r)$	C 0	$Sp(2n, c)$	$(2n)^2$	$(2n)(2n+1)$	$\frac{2n(2n+1)}{2}$	$\frac{2n(2n+1)}{2}$
	$Sp(2n, r)$	C I	$U(n)$	n^2	$n(n+1)$	n^2	0
	$Sp(2p+2q; r)$	C I	$U(p, q)$	$(p+q)^2$	$(p+q)(p+q+1)$	p^2+q^2-1	$2pq$
	$Sp(2n, r)$	C I	$Sl(n, r) \otimes R$	n^2	$n(n+1)$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}$
	$Sp(2p+2q; r)$	C II	$Sp(2p, r) \otimes Sp(2q, r)$	$(p+q)^2$	$(p+q)(p+q+1)$	p^2+q^2	$p(p+1)+q(q+1)$
C II	$USp(2n, 2n)$	C 0	$Sp(2n, c)$	$2n(2n+1)$	$4n^2$	$\frac{2n(2n+1)}{2}$	$\frac{2n(2n+1)}{2}$
	$USp(2p, 2q)$	C I	$U(p, q)$	$p(2p+1)+$ $q(2q+1)$	$4pq$	p^2+q^2	$2pq$
	$USp(2n, 2n)$	C I	$SU^*(2n) \otimes SO(1,1)$	$2n(2n+1)$	$4n^2$	$\frac{2n(2n+1)}{2}$	$\frac{2n(2n-1)}{2}$
	$USp(2p, 2q)$	C II	$USp(2p) \otimes USp(2q)$	$p(2p+1)+q(2q+1)$	$4pq$	$p(2p+1)+q(2q+1)$	0
	$USp(2p, 2q)$	C II	$USp(2h, 2k) \otimes$ $USp(2p-2h, 2q-2k)$	$p(2p+1)+q(2q+1)$	$4pq$	$h(2h+1)+k(2k+1)$ $+(p-h)(2p-2h+1)$ $+(q-k)(2q-2k+1)$	$4hk+$ $4(p-h)(q-k)$

Continued

TABLE 9.7—*Continued*

Automorphism Type: σ	Group G	Automorphism Type: τ	Maximal Subgroup H	Number of Generators of G Compact	Number of Generators of G Noncompact	Number of Generators of H Compact	Number of Generators of H Noncompact
	$G_{2(-14)}$	G_2 0	Identity	14	0	0	0
	$G_{2(-14)}$	G_2 I	$SU(2) \otimes SU(2)$	14	0	3 + 3	0
	$G_{2(-14)}$	G_2 0	$G_{2(-14)}$	14	0	14	0
	$G_2{}^C$	G_2 I	$Sl(2, c) \otimes Sl(2, c)$	14	14	3 + 3	3 + 3
G_2 0	$G_2{}^C$	G_2 0	$G_{2(-14)}$	14	14	14	0
	$G_2{}^C$	G_2 I	$G_{2(+2)}$	14	14	6	8
G_2 I	$G_{2(+2)}$	G_2 I	$SU(2) \otimes SU(2)$	6	8	6	0
	$G_{2(+2)}$	G_2 I	$SU(1, 1) \otimes SU(1, 1)$	6	8	1 + 1	2 + 2
	$F_{4(-52)}$	F_4 0	Identity	52	0	0	0
	$F_{4(-52)}$	F_4 I	$USp(6) \otimes USp(2)$ $C_3 \oplus C_1$	52	0	21 + 3	0
	$F_{4(-52)}$	F_4 II	$SO(9) = B_4$	52	0	36	0
	$F_{4(-52)}$	F_4 0	$F_{4(-52)}$	52	0	52	0
	$F_4{}^C$	F_4 I	$Sp(6, c) \otimes Sp(2, c)$	52	52	21 + 3	21 + 3
	$F_4{}^C$	F_4 II	$SO(9, c)$	52	52	36	36
F_4 0	$F_4{}^C$	F_4 0	$F_{4(-52)}$	52	52	52	0
	$F_4{}^C$	F_4 I	$F_{4(+4)}$	52	52	24	28
	$F_4{}^C$	F_4 II	$F_{4(-20)}$	52	52	36	16
F_4 I	$F_{4(+4)}$	F_4 I	$USp(6) \otimes SU(2)$	21 + 3	28	21 + 3	0
	$F_{4(+4)}$	F_4 I	$USp(4, 2) \otimes SU(2)$	21 + 3	28	(10 + 3) + 3	8
	$F_{4(+4)}$	F_4 I	$Sp(6, r) \otimes SU(1, 1)$	21 + 3	28	9 + 1	12 + 2
	$F_{4(+4)}$	F_4 II	$SO(5, 4)$	21 + 3	28	10 + 6	20
F_4 II	$F_{4(-20)}$	F_4 I	$USp(4, 2) \otimes SU(2)$	36	16	(10 + 3) + 3	8
	$F_{4(-20)}$	F_4 II	$SO(9)$	36	16	36	0
	$F_{4(-20)}$	F_4 II	$SO(8, 1)$	36	16	28	8

	$E_{6(-78)}$	E_6 0	Identity	78	0	0	0
	$E_{6(-78)}$	E_6 I	$USp(8) = C_4$	78	0	36	0
	$E_{6(-78)}$	E_6 II	$SU(6) \otimes SU(2) = A_5 \oplus A_1$	78	0	35 + 3	0
	$E_{6(-78)}$	E_6 III	$SO(10) \otimes SO(2)$	78	0	45 + 1	0
			$D_5 \oplus D_1$				
	$E_{6(-78)}$	E_6 IV	$F_{4(-52)}$	78	0	52	0
	$E_{6(-78)}$	E_6 0	$E_{6(-78)}$	78	0	78	0
	$E_6{}^C$	E_6 I	$Sp(8, c)$	78	78	36	36
	$E_6{}^C$	E_6 II	$Sl(6, c) \otimes Sl(2, c)$	78	78	35 + 3	35 + 3
	$E_6{}^C$	E_6 III	$SO(10, c) \otimes SO(2, c)$	78	78	45 + 1	45 + 1
	$E_6{}^C$	E_6 IV	$F_4{}^C$	78	78	52	52
E_6 0	$E_6{}^C$	E_6 0	$E_{6(-78)}$	78	78	78	0
	$E_6{}^C$	E_6 I	$E_{6(+6)}$	78	78	36	42
	$E_6{}^C$	E_6 II	$E_{6(+2)}$	78	78	35 + 3	40
	$E_6{}^C$	E_6 III	$E_{6(-14)}$	78	78	45 + 1	32
	$E_6{}^C$	E_6 IV	$E_{6(-26)}$	78	78	52	26
E_6 I	$E_{6(+6)}$	E_6 I	$USp(8)$	36	42	36	0
	$E_{6(+6)}$	E_6 I	$USp(4, 4)$	36	42	20	16
	$E_{6(+6)}$	E_6 I	$Sp(8, r)$	36	42	16	20
	$E_{6(+6)}$	E_6 II	$SU^*(6) \otimes SU(2)$	36	42	21 + 3	14
	$E_{6(+6)}$	E_6 II	$Sl(6, r) \otimes Sl(2, r)$	36	42	15 + 1	20 + 2
	$E_{6(+6)}$	E_6 III	$SO(5, 5) \otimes SO(1, 1)$	36	42	20	25 + 1
	$E_{6(+6)}$	E_6 IV	$F_{4(+4)}$	36	42	24	28
E_6 II	$E_{6(+2)}$	E_6 I	$USp(6, 2)$	38	40	24	12
	$E_{6(+2)}$	E_6 I	$Sp(8, r)$	38	40	16	20
	$E_{6(+2)}$	E_6 II	$SU(6) \otimes SU(2)$	38	40	35 + 3	0
	$E_{6(+2)}$	E_6 II	$SU(4, 2) \otimes SU(2)$	38	40	19 + 3	16
	$E_{6(+2)}$	E_6 II	$SU(3, 3) \otimes SU(1, 1)$	38	40	17 + 1	18 + 2
	$E_{6(+2)}$	E_6 III	$SO(6, 4) \otimes SO(2)$	38	40	21 + 1	24
	$E_{6(+2)}$	E_6 III	$SO^*(10) \otimes SO(2)$	38	40	25 + 1	20
	$E_{6(+2)}$	E_6 IV	$F_{4(+4)}$	38	40	24	28
E_6 III	$E_{6(-14)}$	E_6 I	$USp(4, 4)$	46	32	20	16
	$E_{6(-14)}$	E_6 II	$SU(4, 2) \otimes SU(2)$	46	32	19 + 3	16
	$E_{6(-14)}$	E_6 II	$SU(5, 1) \otimes SU(1, 1)$	46	32	25 + 1	10 + 2
	$E_{6(-14)}$	E_6 III	$SO(10) \otimes SO(2)$	46	32	45 + 1	0
	$E_{6(-14)}$	E_6 III	$SO(8, 2) \otimes SO(2)$	46	32	29 + 1	16
	$E_{6(-14)}$	E_6 III	$SO^*(10) \otimes SO(2)$	46	32	25 + 1	20
	$E_{6(-14)}$	E_6 IV	$F_{4(-20)}$	46	32	36	16

Continued

TABLE 9.7—*Concluded*

Automorphism Type: σ	Group G	Automorphism Type: τ	Maximal Subgroup H	Number of Generators of G: Compact	Number of Generators of G: Noncompact	Number of Generators of H: Compact	Number of Generators of H: Noncompact
E_6 IV	$E_{6(-26)}$	E_6 I	$USp(6, 2)$	52	26	24	12
	$E_{6(-26)}$	E_6 II	$SU^*(6) \otimes SU(2)$	52	26	21 + 3	14
	$E_{6(-26)}$	E_6 III	$SO(9, 1) \otimes SO(1, 1)$	52	26	36	9 + 1
	$E_{6(-26)}$	E_6 IV	$F_{4(-52)}$	52	26	52	0
	$E_{6(-26)}$	E_6 IV	$F_{4(-20)}$	52	26	36	16
	$E_{7(-133)}$	E_7 0	Identity	133	0	0	0
	$E_{7(-133)}$	E_7 I	$SU(8) = A_7$	133	0	63	0
	$E_{7(-133)}$	E_7 II	$SO(12) \otimes SO(3) = D_6 \oplus B_1$	133	0	66 + 3	0
	$E_{7(-133)}$	E_7 III	$E_{6(-78)} \otimes SO(2)$	133	0	78 + 1	0
	$E_{7(-133)}$	E_7 0	$E_{7(-133)}$	133	0	133	0
	$E_7{}^C$	E_7 I	$Sl(8, c)$	133	133	63	63
	$E_7{}^C$	E_7 II	$SO(12, c) \otimes SO(3, c)$	133	133	66 + 3	66 + 3
	$E_7{}^C$	E_7 III	$E_6{}^C \otimes SO(2, c)$	133	133	78 + 1	78 + 1
E_7 0	$E_7{}^C$	E_7 0	$E_{7(-133)}$	133	133	133	0
	$E_7{}^C$	E_7 I	$E_{7(+7)}$	133	133	63	70
	$E_7{}^C$	E_7 II	$E_{7(-5)}$	133	133	69	64
	$E_7{}^C$	E_7 III	$E_{7(-25)}$	133	133	79	54
E_7 I	$E_{7(+7)}$	E_7 I	$SU(8)$	63	70	63	0
	$E_{7(+7)}$	E_7 I	$SU(4, 4)$	63	70	31	32
	$E_{7(+7)}$	E_7 I	$Sl(8, r)$	63	70	28	35
	$E_{7(+7)}$	E_7 I	$SU^*(8)$	63	70	36	27
	$E_{7(+7)}$	E_7 II	$SO(6, 6) \otimes SO(2, 1)$	63	70	30 + 1	36 + 2
	$E_{7(+7)}$	E_7 II	$SO^*(12) \otimes SO(3)$	63	70	36 + 3	30
	$E_{7(+7)}$	E_7 III	$E_{6(+2)} \otimes SO(2)$	63	70	38 + 1	40
	$E_{7(+7)}$	E_7 III	$E_{6(+6)} \otimes SO(1, 1)$	63	70	36	42 + 1
E_7 II	$E_{7(-5)}$	E_7 I	$SU(6, 2)$	69	64	39	24
	$E_{7(-5)}$	E_7 I	$SU(4, 4)$	69	64	31	32
	$E_{7(-5)}$	E_7 II	$SO(12) \otimes SO(3)$	69	64	66 + 3	0
	$E_{7(-5)}$	E_7 II	$SO(8, 4) \otimes SO(3)$	69	64	34 + 3	32
	$E_{7(-5)}$	E_7 II	$SO^*(12) \otimes SO(2, 1)$	69	64	36 + 1	30 + 2
	$E_{7(-5)}$	E_7 III	$E_{6(-14)} \otimes SO(2)$	69	64	46 + 1	32
	$E_{7(-5)}$	E_7 III	$E_{6(+2)} \otimes SO(2)$	69	64	38 + 1	40

E_7 III	$E_{7(-25)}$	E_7 I	$SU(6, 2)$	79	54	39	24
	$E_{7(-25)}$	E_7 I	$SU^*(8)$	79	54	36	27
	$E_{7(-25)}$	E_7 II	$SO(10, 2) \otimes SO(2, 1)$	79	54	46 + 1	20 + 2
	$E_{7(-25)}$	E_7 II	$SO^*(12) \otimes SO(3)$	79	54	36 + 3	30
	$E_{7(-25)}$	E_7 III	$E_{6(-78)} \otimes SO(2)$	79	54	78 + 1	0
	$E_{7(-25)}$	E_7 III	$E_{6(-26)} \otimes SO(1, 1)$	79	54	52	26 + 1
	$E_{7(-25)}$	E_7 III	$E_{6(-14)} \otimes SO(2)$	79	54	46 + 1	32
	$E_{8(-248)}$	E_8 0	Identity	248	0	0	0
	$E_{8(-248)}$	E_8 I	$SO(16) = D_8$	248	0	120	0
	$E_{8(-248)}$	E_8 II	$E_{7(-133)} \otimes SU(2)$	248	0	133 + 3	0
	$E_{8(-248)}$	E_8 0	$E_{8(-248)}$	248	0	248	0
	$E_8{}^C$	E_8 I	$SO(16, c)$	248	248	120	120
	$E_8{}^C$	E_8 II	$E_7{}^C \otimes Sl(2, c)$	248	248	133 + 3	133 + 3
E_8 0	$E_8{}^C$	E_8 0	$E_{8(-248)}$	248	248	248	0
	$E_8{}^C$	E_8 I	$E_{8(+8)}$	248	248	120	128
	$E_8{}^C$	E_8 II	$E_{8(-24)}$	248	248	136	112
E_8 I	$E_{8(+8)}$	E_8 I	$SO(16)$	120	128	120	0
	$E_{8(+8)}$	E_8 I	$SO(8, 8)$	120	128	56	64
	$E_{8(+8)}$	E_8 I	$SO^*(16)$	120	128	64	56
	$E_{8(+8)}$	E_8 II	$E_{7(-5)} \otimes SU(2)$	120	128	69 + 3	64
	$E_{8(+8)}$	E_8 II	$E_{7(+7)} \otimes SU(1, 1)$	120	128	63 + 1	70 + 2
E_8 II	$E_{8(-24)}$	E_8 I	$SO(12, 4)$	136	112	72	48
	$E_{8(-24)}$	E_8 I	$SO^*(16)$	136	112	64	56
	$E_{8(-24)}$	E_8 II	$E_{7(-133)} \otimes SU(2)$	136	112	133 + 3	0
	$E_{8(-24)}$	E_8 II	$E_{7(-25)} \otimes SU(1, 1)$	136	112	79 + 1	54 + 2
	$E_{8(-24)}$	E_8 II	$E_{7(-5)} \otimes SU(2)$	136	112	69 + 3	64

The simple Lie groups appearing in Table 9.7 are classed according to the root space of $\mathfrak{g}^*$. Within each root space classification, the entries are divided into a number of subsets:

1. The first subset consists of the compact group G, and all possible maximal compact subgroups K, including the improper subgroups $K = \text{Id}$ and $K = G$. The symmetric space associated with the first improper subgroup

$$\frac{G}{\text{Id}} = G \tag{6.24}$$

is the compact group G itself. The other improper subgroup plays a role more easily discerned in the algebra:

$$\mathfrak{g} = (\mathfrak{k} = \mathfrak{g}) \oplus (\mathfrak{p} = 0) \to \mathfrak{g}^* = \mathfrak{g} \oplus i0 = \mathfrak{g} \tag{6.25}$$

That is, this is the improper maximal subgroup decomposition mapping the compact form into itself: $\mathfrak{g}^* = \mathfrak{g}$. The related coset

$$\frac{G}{G} = \text{identity} \tag{6.26}$$

is the space consisting of a single point, of dimensionality 0. These two improper decompositions have been labeled "0" within the first subset, since they are so closely related. The remaining proper decompositions within the first subset, which occur between the extremes (6.24) and (6.25), are listed between these extremes and are labeled as they appear in Table 9.3.

2. The second subset of terms consists of the complex extension G^C and the complex extension K^C of all proper subgroups K. These are immediately determined from the "sandwiched" entries of subset 1.

3. The third subset of entries consists of the complex extension G^C and all proper subgroups G^* of real forms of G^C, under the decomposition of (6.22) and (6.23). All spaces G^C/G^* have the same dimensionality as G. The end-points in this series of symmetric spaces are

$$\frac{G^C}{G} \qquad \text{which is dual to} \qquad \frac{G}{\text{Id}}$$

4, 5, ..., . The fourth subset consists of the most noncompact real form of G^C, with all possible subgroups H, taken in order. The fifth subset consists of the next most noncompact form, with all possible subgroups, and so on, until all noncompact forms G^* of G^C have been exhausted. This series can be regarded as a thorough analysis of all possible real restrictions of the complex extension of the compact groups G and K, as shown in (6.27).

$$
\begin{tikzcd}[column sep=small]
 & \mathfrak{g}^C = \mathfrak{k}^C \oplus \mathfrak{p}^C \arrow[dl, shift left, "\text{Compact restriction}"'] \arrow[dr, shift left, "\text{Real restriction}"] & \\
\underset{\text{Compact real form}}{\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}} \arrow[ur, shift left, "\text{Complex extension}"] \arrow[rr, shift left, "\text{“analytic”}"] & & \underset{\text{Noncompact real form}}{\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*} \arrow[ul, shift left] \arrow[ll, shift left, "\text{continuation}"]
\end{tikzcd}
\tag{6.27}
$$

Comment. The symmetric spaces of the form

$$\frac{G^*}{G^*} = \mathrm{Id}; \qquad \frac{G^*}{\mathrm{Id}} = G^* \tag{6.28}$$

have been omitted from all but the first (compact) subset. Were these spaces to be included, they would "sandwich" the entries within each subset, as they do in the first subset of entries. These omitted spaces are exactly the noncompact real Lie groups listed in Table 9.3 and considered not as Lie groups, but as real forms of the symmetric spaces.

3. CHARACTERS OF THE REAL FORMS. Once again, the real forms can be classified almost uniquely by their character functions. This time the real forms under consideration are not Lie groups, but rather cosets of a Lie group by a Lie subgroup. Since the classification scheme for a Lie group uses one character function, the classification scheme for the symmetric spaces, involving two Lie groups, must use two character functions. These can conveniently be chosen as the characters of the group and of the subgroup; they can also be chosen in other convenient ways. The necessary information for constructing characters of the real forms is included in Table 9.7. A useful guideline in the construction of characters for real forms of the symmetric spaces is that the characters of dual spaces be the negatives of each other. As in the case of the real forms of the simple Lie groups, the characters of the symmetric spaces provide an almost unique classification of these spaces. They are nondegenerate except on a set of measure zero.

Example. We compute the character of $U(p, q)/U(p-h, q-k) \otimes U(h, k)$. The compact and noncompact generators of this space are as follows:

	Number of Compact Generators	Number of Noncompact Generators
$U(p, q)$	$p^2 + q^2$	$2pq$
$U(p-h, q-k)$	$(p-h)^2 + (q-k)^2$	$2(p-h)(q-k)$
$U(h, k)$	$h^2 + k^2$	$2hk$
$\dfrac{U(p, q)}{U(p-h, q-k) \otimes U(h, k)}$:	$2h(p-h) + 2k(q-k)$	$2h(q-k) + 2k(p-h)$

The character of this space is simply determined from

$$\chi\left(\frac{G}{H}\right) = \chi[U(p, q)] \qquad\qquad = -(p-q)^2$$

$$\chi(H) = \chi[U(p-h, q-k) \otimes U(h, k)] = -(p-q-h+k)^2 - (h-k)^2$$

Résumé

We set out, in this chapter, to study and classify the real forms of the complex semisimple root spaces. The methods designed to accomplish this classification provide a tool for taking apart a Lie algebra into its component pieces: a maximal subalgebra and its orthogonal complementary subspace. The latter subspaces have a great deal of interest in their own right. They are endowed with a metric and are globally symmetric. Moreover, they can be studied by exactly the same techniques used earlier in the study of the parent Lie algebras themselves. Thus it was a simple matter to map these concepts (secular equation, eigensubspace decomposition, rank, index, metric) from the algebra study onto the coset space study.

The techniques of the first half of this chapter were developed for the study and classification of the real forms of the Lie algebras. However, these techniques—which led to and, in fact, forced the study of the coset spaces on us—are used again (application "squared") to study and classify all the real forms of the coset spaces themselves.

The tools required for the classification process involve essentially only the involutive automorphisms possessed by a Lie algebra. Such involutive automorphisms can be determined, in the case of each simple Lie algebra, simply by inspection of the root space's Dynkin diagram.

The appropriate tool for the identification of the various real forms is the character function. The character provides a unique identification of all the real forms, except on a set of measure zero. For the classification of the real simple Lie algebras, one character function is sufficient. For the classification of the real forms of the (pseudo-) Riemannian globally symmetric spaces, two character functions are necessary, since these spaces are cosets of a real simple Lie group by a maximal real Lie subgroup.

Exercises

1. Prove (1.34) by showing

(a) When the submatrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is diagonalized, the submatrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ results.

(b) The trace of a matrix is invariant under similarity transformations.

2. Prove that the generators (1.35c) close under commutation. You must show this for the commutators not explicitly carried out in (1.36).

3. The Hamiltonian for a one-dimensional harmonic oscillator is

$$\mathscr{H} = \frac{p_x^{\,2}}{2m} + \frac{1}{2}kx^2$$

Construct the creation and annihilation operators

$$a_x^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\, x - \frac{ip}{\sqrt{m\omega}}\right)$$

$$a_x = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\, x + \frac{ip}{\sqrt{m\omega}}\right)$$

where $\omega = \sqrt{k/m}$.

(a) Use the relation $[p_x, x] = \hbar/i$ to show $[a, a^\dagger] = +1$.

(b) Use the derivative property of the Lie bracket

$$[A, BC] = [A, B]C + B[A, C]$$

to prove

$$[a^\dagger a, a^\dagger] = +a^\dagger$$
$$[a^\dagger a, a] = -a$$

(c) Prove

$$\mathscr{H} = \tfrac{1}{2}\hbar\omega(a^\dagger a + aa^\dagger) = \hbar\omega(a^\dagger a + \tfrac{1}{2})$$

(d) Prove that the generators $a^\dagger a = n$ (number operator), $a^\dagger$ (creation operator), and a (annihilation operator) close under commutation and form a Lie algebra, provided a fourth generator I is included. The four-dimensional nonsemisimple Lie algebra is commonly called the harmonic oscillator algebra and is usually denoted $\mathfrak{h}_4$.

4. The Hamiltonian for a three-dimensional isotropic harmonic oscillator is

$$\mathscr{H} = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + \frac{1}{2}kr^2 \qquad (r^2 = x^2 + y^2 + z^2)$$

(a) Prove that $[\mathscr{H}, L_i] = 0$

$$L_i = x^j\,\partial_k - x^k\,\partial_j \qquad i, j, k = 1, 2, 3 \text{ cycl.}$$

In other words, the oscillator Hamiltonian is "invariant" under the infinitesimal generators of the rotation group. It is therefore "invariant" under the group operations constructed from these generators (i.e., under the operations of the rotation group itself). What does this mean?

(b) Show that if

$$e^{\boldsymbol{\theta}\cdot\mathbf{L}}\mathscr{H}(x)e^{-\boldsymbol{\theta}\cdot\mathbf{L}} = \mathscr{H}\{x'^r = x^i\mathscr{D}^j(e^{\boldsymbol{\theta}\cdot\mathbf{L}})_i^{\,r}\} = \mathscr{H}(x)$$

then

$$\lim_{\boldsymbol{\theta}\to 0} e^{\boldsymbol{\theta}\cdot\mathbf{L}}\mathscr{H}(x)e^{-\boldsymbol{\theta}\cdot\mathbf{L}} = \mathscr{H}(x) + \boldsymbol{\theta}\cdot[\mathbf{L}, \mathscr{H}(x)] = \mathscr{H}(x)$$

so that

$$[\mathbf{L}, \mathscr{H}(x)] = 0$$

(c) Prove that

$$\mathscr{H} = \hbar\omega\left(\sum_{i=1}^{3} a_i^\dagger a_i + \frac{3}{2}\right)$$

where $a_i^\dagger$, a_i are the creation and annihilation operators for excitations along the ith $(i = x, y, z)$ axis.

(d) Prove that

$$[\mathscr{H}, a_i^\dagger a_j] = 0$$

(e) Prove that the $(3)^2 = 9$ operators $a_i^\dagger a_j$ close under commutation to form the algebra u(3). In particular, choose

$$a_i^\dagger a_j = u^i\, \partial_j = E_{\mathbf{e}_i - \mathbf{e}_j}, \qquad i \neq j$$
$$a_i^\dagger a_i = u^i\, \partial_i = H_i$$

(f) Show that

$$\mathscr{H} = \hbar\omega \mathbf{H} \cdot \mathbf{R}$$
$$\mathbf{R} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$$

where $\mathscr{H}$ is the Hamiltonian for the three-dimensional harmonic oscillator and $\mathbf{H} = (H_1, H_2, H_3)$ are the three bases for the Cartan subalgebra of u(3).

(g) Conclude that the Hamiltonian generates an abelian invariant subgroup of $U(3)$. What does this say about the time evolution of harmonic oscillator states?

(h) Interpret these nine generators $a_i^\dagger a_j$ as follows:

$\sum_{i=1}^{3} a_i^\dagger a_i = \mathscr{H}$	Energy (a scalar)
$a_i^\dagger a_j - a_j^\dagger a_i = L_k$	Angular momentum operators, a second-order skew-symmetric tensor in R_3
$a_i^\dagger a_j + a_j^\dagger a_i - \frac{2}{3}\mathscr{H}\, \delta_{ij}$	Quadrupole moment operators, a second-order traceless symmetric tensor in R_3

5. The n-dimensional harmonic oscillator Hamiltonian is

$$\mathscr{H} = \sum_{i=1}^{n} \left\{\frac{p_i^2}{2m} + \frac{1}{2} k_i x_i^2\right\}$$

(a) Show that this Hamiltonian can be written[25]

$$\mathscr{H} = \sum_{i=1}^{n} \hbar\omega_i (a_i^\dagger a_i + \tfrac{1}{2})$$

where the $a_i^\dagger$, a_i are as in Problem 3.

(b) When $\omega_i = \omega$ ($i = 1, 2, \ldots, n$), show that

$$[\mathscr{H}, a_i^\dagger a_j] = 0$$

(c) Partition and interpret these generators as follows:

$\mathscr{H} = \sum_{i=1}^{n} H_i = \sum_{i=1}^{n} a_i^\dagger a_i$	Energy operator (a scalar); this generates the abelian invariant subgroup
$L_{ij} = a_i^\dagger a_j - a_j^\dagger a_i$	A second-order skew-symmetric tensor in R_n; these are the infinitesimal generators of the subgroup $SO(n)$ of $SU(n)$
$T_{ij} = a_i^\dagger a_j + a_j^\dagger a_i - \frac{2}{n}\{\sum a_k^\dagger a_k\}\,\delta_{ij}$	A second-order traceless symmetric tensor in R_n; these are the "quadrupole" or "bulge displacement" operators and do not close under commutation

(d) Identify the generators T_{ij} as the infinitesimal generators for the irreducible coset space $SU(n)/SO(n)$.

(e) Let annihilation operators b_i be related to the a_j by a unitary transformation matrix U:

$$b_i = U_i{}^j a_j; \qquad U^\dagger = U^{-1}$$

Prove

$$\mathscr{H} = \sum \hbar\omega a_i^\dagger a_i = \sum \hbar\omega b_j^\dagger b_j$$

and

$$[a_i, a_j^\dagger] = [b_i, b_j^\dagger] = \delta_{ij}$$

(f) Prove

$$[\mathscr{H}, a_i^\dagger] = +\hbar\omega a_i^\dagger$$

$$[\mathscr{H}, a_j] = -\hbar\omega a_j$$

Show that the generators $a_i^\dagger a_j$, $a_i^\dagger$, a_j close on themselves and form a Lie algebra, provided the operator I is also included (because $[a_i, a_j^\dagger] = +\delta_{ij} I$).

(g) Show that the zeroth and first-order operators span a solvable invariant subalgebra in the algebra described in part (f).

6. Let $q_1, q_2, \ldots, q_n$ be canonical coordinates and let $p_1, p_2, \ldots, p_n$ be their canonical momenta:

$$[p_r, q_s] = \frac{\hbar}{i}\delta_{rs}$$

(a) Prove that these canonical commutation relations are satisfied

in the q representation by $q_r \to q_r$; $\quad p_s \to \dfrac{\hbar}{i}\dfrac{\partial}{\partial q_s}$

in the p representation by $p_s \to p_s$; $\quad q_r \to \dfrac{-\hbar}{i}\dfrac{\partial}{\partial p_r}$

(b) Let M be the operator

$$M = \sum_{r=1}^{n} \alpha^r q_r + \sum_{s=1}^{n} \beta^s p_s$$

where α^j, β^k are not operators but, rather, arbitrary complex numbers. Show that M is hermitian ($M^\dagger = M$) if and only if all coefficients α^r, β^s are real.

(c) Write

$$M = \begin{bmatrix} \alpha^1 q_1 \\ \alpha^2 q_2 \\ \vdots \\ \alpha^n q_n \\ \hline \beta^1 p_1 \\ \beta^2 p_2 \\ \vdots \\ \beta^n p_n \end{bmatrix}$$

Show that the canonical commutation relations are concisely summarized by

$$M^t \left[\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array}\right] M = \sum_{i=1}^{n} \alpha^i \beta^i (q_i p_i - p_i q_i) = i\hbar \sum_{i=1}^{n} \alpha^i \beta^i$$

(d) Let S be a linear transformation of the operators $\mathbf{q}$, $\mathbf{p}$ into themselves. Then $M' = SM$. If the new coordinates $\mathbf{q}'$, $\mathbf{p}'$ are canonical, show that the group of transformations S is the real symplectic group:[26]

$$S^t \left[\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array}\right] S = \left[\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array}\right] \Leftrightarrow S \in Sp(2n, r)$$

(e) Show that the collection of generators

$$a^{ij} p_i p_j, \qquad b^{ij}(p_i q_j + q_j p_i), \qquad c^{ij} q_i q_j$$

closes on itself to form $\mathfrak{sp}(2n, r)$.

(f) Show that the maximal compact subalgebra is spanned by the bilinear combinations

$$a_i^\dagger a_j \qquad \begin{cases} a_i = \sqrt{\tfrac{1}{2}}\,(q + ip) \\ \\ a_i^\dagger = \sqrt{\tfrac{1}{2}}\,(q - ip) \end{cases} \qquad (\hbar = 1)$$

Construct the generators of the coset $Sp(2n, r)/U(n, c)$ and interpret them physically.

(g) Show that the set of all bilinear products containing q_i, p_j, all linear generators, and the identity operator closes on itself under commutation. Show that the zeroth and first-order operators span an invariant solvable subalgebra.

7. Cartan demonstrated that the space G/H is simply connected when G is simple, compact, and simply connected, and when H is maximal.

(a) Show that all spheres $S^n \subset R_{n+1}$ are simply connected, $n > 1$.

(b) S^1 is multiply connected. Which property of the foregoing theorem is not satisfied for $S^1 = SO(2)/\mathrm{Id}$?

(c) The maximal compact subgroup of $SO^*(2n)$ and $Sp(2n, r)$ is $U(n, c)$ in each case. Show that the connectivity of $SO^*(2n)$, $Sp(2n, r)$ is just the connectivity of $U(n, c)$. Show the connectivity of $U(n, c)$ is the same as the connectivity of $SU(n, c) \otimes U(1, c)$. Conclude that the noncompact groups $SO^*(2n)$, $Sp(2n, r)$ have the same connectivity as the circle S^1; namely, they are infinitely connected.

(d) Are the homogeneous spaces

$$\frac{SO^*(2n)}{U(n, c)} \qquad \frac{Sp(2n, r)}{U(n, c)}$$

multiply or simply connected? Why? What space are they globally topologically equivalent to?

8. Prove:

$$\begin{aligned} Spin\ (3) &= SU(2) \\ Spin\ (4) &= SU(2) \otimes SU(2) \\ Spin\ (5) &= USp(4) \\ Spin\ (6) &= SU(4) \end{aligned}$$

Prove that *Spin* (n) is not isomorphic with a classical group for $n > 6$.

9. Prove by construction that each of the classical Lie algebras $\mathfrak{u}(n; f)$ has a realization in terms of bilinear combinations of n independent creation and annihilation operators.

10. The hydrogen atom, 1925:[27]

Verify by direct computation that the generators L_i of infinitesimal rotations in R_3 commute with the Hamiltonian for the hydrogen atom

$$\mathscr{H} = \frac{p^2}{2m} + V(r)$$

$$V(r) = -\frac{e^2}{r}$$

Prove that the commutators $[\mathscr{H}, L_i]$ are zero for *any* spherically symmetric potential $V(r)$. Spherical symmetry means that the potential depends only on the magnitude $|\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2}$, not on the orientation.

11. The hydrogen atom, 1926:[28,29]
Verify by direct computation that the infinitesimal generators

$$\mathbf{L} = \frac{1}{2}(\mathbf{r} \times \mathbf{p} - \mathbf{p} \times \mathbf{r})$$

$$\mathbf{M} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{Ze^2\mathbf{r}}{r}$$

obey the commutation relations:

$$\begin{aligned}
[\mathscr{H}, L_i] &= 0 \\
[\mathscr{H}, M_i] &= 0 \\
[L_j, L_k] &= i\hbar\varepsilon_{jkm} L_m \\
[L_j, M_k] &= i\hbar\varepsilon_{jkm} M_m \\
[M_j, M_k] &= -i\hbar\varepsilon_{jkm} L_m\left(\frac{2}{m}\mathscr{H}\right)
\end{aligned}$$

Thus show (*a*) that the operators **L**, **M** are infinitesimal generators for the group $SO(4)$ when acting on eigenstates of $\mathscr{H}$ with $E < 0$ (the bound states); (*b*) that they are generators for $SO(3, 1)$ on eigenstates of $\mathscr{H}$ with $E > 0$ (the so-called scattering states); and (*c*) that they are generators for $ISO(3)$ when operating on the eigenstates of zero energy $E = 0$. Here m is the reduced mass of the proton–electron system:

$$m = \frac{m_e M_p}{m_e + M_p} = \frac{m_e}{1 + (m_e/M_p)}$$

where $M_p/m_e = 6\pi^5$.

12. The hydrogen atom, 1965:[30]
Let (p_1, p_2, p_3, p_4) be the canonically conjugate variables to (x, y, z, t) and let the variable $(\xi_1, \xi_2, \xi_3, \xi_4)$ be related to the p_μ by a projective transformation:

$$\begin{aligned}
\boldsymbol{\xi} &= 2p_4(\mathbf{p}\cdot\mathbf{p} + p_4{}^2)^{-1}\mathbf{p} \\
\xi_4 &= \quad (\mathbf{p}\cdot\mathbf{p} - p_4{}^2)(\mathbf{p}\cdot\mathbf{p} + p_4{}^2)^{-1}
\end{aligned}$$

Show that the six generators of Problem 11:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\mathbf{A} = \frac{1}{\sqrt{-2m\mathscr{H}}}\left\{\frac{1}{2}(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}) + \frac{Zme^2}{r}\mathbf{r}\right\}$$

generate $SO(4)$ on bound states. Show that four additional operators, a three-vector and a scalar given by

$$\mathbf{B} = \mathbf{A}\xi_4 + \mathbf{L} \times \boldsymbol{\xi} - \frac{3}{2}i\boldsymbol{\xi} = \frac{i}{2}[\boldsymbol{\xi}, L^2 + A^2]$$

$$S = \mathbf{A}\cdot\boldsymbol{\xi} \qquad + \frac{3}{2}i\xi_4 = \frac{i}{2}[\xi_4, L^2 + A^2]$$

close under commutation with each other and the previous six operators to give the Lie algebra $\mathfrak{so}(4, 1)$.

13. The hydrogen atom, 1965–1972:[31–33]

(a) Show that the ten generators:

$$\begin{aligned}
\mathbf{J} &= \mathbf{r} \times \mathbf{p} \\
\mathbf{A} &= \tfrac{1}{2}\mathbf{r}(\mathbf{p} \cdot \mathbf{p}) - \mathbf{p}(\mathbf{r} \cdot \mathbf{p}) - \tfrac{1}{2}\mathbf{r} \\
\mathbf{M} &= \tfrac{1}{2}\mathbf{r}(\mathbf{p} \cdot \mathbf{p}) - \mathbf{p}(\mathbf{r} \cdot \mathbf{p}) + \tfrac{1}{2}\mathbf{r} \\
M_4 &= \mathbf{r} \cdot \mathbf{p} - i
\end{aligned}$$

close under commutation to form the Lie algebra $\mathfrak{so}(4, 1)$.

(b) Show that the five additional generators

$$\begin{aligned}
\boldsymbol{\Gamma} &= r\mathbf{p} \\
\Gamma_4 &= \tfrac{1}{2}\{r(\mathbf{p} \cdot \mathbf{p}) - r\} \\
\Gamma_5 &= \tfrac{1}{2}\{r(\mathbf{p} \cdot \mathbf{p}) + r\} \qquad (r = |\mathbf{r}|)
\end{aligned}$$

close under commutation with each other and the ten operators of (a) to form the algebra $\mathfrak{so}(4, 2)$.

(c) Show that the three generators M_4, Γ_4, Γ_5 span an $\mathfrak{so}(2, 1)$ noncompact subalgebra. Show also that these three generators commute with the three angular momentum generators $\mathbf{J}$.

(d) Show that these 15 operators and their commutators are adequately summarized in the following matrix array:

$$\begin{array}{cc}
\begin{matrix} + & + & + & + & - & - \end{matrix} & \\
\left[\begin{array}{cccc|cc}
0 & J_3 & -J_2 & A_1 & M_1 & \Gamma_1 \\
 & 0 & J_1 & A_2 & M_2 & \Gamma_2 \\
 & & 0 & A_3 & M_3 & \Gamma_3 \\
 & & & 0 & M_4 & \Gamma_4 \\
\hline
 & & & & 0 & \Gamma_5 \\
 & & & & & 0
\end{array}\right] & \begin{matrix} + \\ + \\ + \\ + \\ - \\ - \end{matrix}
\end{array}$$

In some sense the vector $\mathbf{M}$ is dual to the Runge-Lenz vector $\mathbf{A}$, and the generators Γ_4, Γ_5 are "dual" to each other, too. The generators $\mathbf{J}$ are the standard angular momentum operators.

(e) How many interesting subalgebras of $\mathfrak{so}(4, 2)$ can be constructed from these generators?

14. The Schrödinger, Klein-Gordon, and Dirac Hamiltonians are given by:[33]

$$\mathscr{H}_{\mathrm{S}} = \frac{\pi^2}{2m} - \frac{\alpha}{r}, \qquad \alpha = \frac{e^2}{\hbar c} \cong \frac{1}{137}$$

$$\mathscr{H}_{\mathrm{KG}} = \{\pi^2 + m^2\}^{1/2} - \frac{\alpha}{r}$$

$$\mathscr{H}_{\rm D} = \boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \gamma^0 m - \frac{\alpha}{r}$$

$$\mathscr{H}_{\rm D;\,FW} = \left\{\pi^2 + m^2 - i\alpha\boldsymbol{\alpha} \cdot \frac{\hat{\mathbf{r}}}{r^2}\right\}^{1/2} - \frac{\alpha}{r}$$

where $\mathscr{H}_{\rm D;\,FW}$ is the Dirac Hamiltonian after a first-order Foldy-Wouthuysen transformation has been performed on it.

(a) For the Schrödinger Hamiltonian show

$$\left.\begin{aligned} M_4 &= \mathbf{r} \cdot \boldsymbol{\pi} - i \\ \Gamma_4 &= \tfrac{1}{2}(r\pi^2 - r) \\ \Gamma_5 &= \tfrac{1}{2}(r\pi^2 + r) \end{aligned}\right. \qquad \text{span } \mathfrak{so}(2, 1)$$

$$\Theta = r(\mathscr{H}_{\rm S} - E) = \frac{1}{2m}(\Gamma_5 + \Gamma_4) - E(\Gamma_5 - \Gamma_4) - \alpha$$

(b) For the Klein-Gordon Hamiltonian show

$$\left.\begin{aligned} M_4 &= \mathbf{r} \cdot \boldsymbol{\pi} - i \\ \Gamma_4 &= \frac{1}{2}\left(r\pi^2 - r - \frac{\alpha^2}{r}\right) \\ \Gamma_5 &= \frac{1}{2}\left(r\pi^2 + r - \frac{\alpha^2}{r}\right) \end{aligned}\right. \qquad \text{span } \mathfrak{so}(2, 1)$$

$$\begin{aligned} \Theta &= r\left\{\left(\mathscr{H}_{\rm KG} + \frac{\alpha}{r}\right)^2 - \left(E + \frac{\alpha}{r}\right)^2\right\} \\ &= \Gamma_5 + \Gamma_4 - (E^2 - m^2)(\Gamma_5 - \Gamma_4) - 2\alpha E \end{aligned}$$

(c) For the Dirac Hamiltonian show

$$\left.\begin{aligned} M_4 &= \mathbf{r} \cdot \boldsymbol{\pi} - i \\ \Gamma_4 &= \frac{1}{2}\left\{r\pi^2 - r - \frac{\alpha^2}{r} - \frac{i\alpha\boldsymbol{\alpha} \cdot \mathbf{r}}{r^2}\right\} \\ \Gamma_5 &= \frac{1}{2}\left\{r\pi^2 + r - \frac{\alpha^2}{r} - \frac{i\alpha\boldsymbol{\alpha} \cdot \mathbf{r}}{r^2}\right\} \end{aligned}\right. \qquad \text{span } \mathfrak{so}(2, 1)$$

$$\begin{aligned} \Theta &= r\left\{\left(\mathscr{H}_D + \frac{\alpha}{r}\right)^2 - \left(E + \frac{\alpha}{r}\right)^2\right\} \\ &= \Gamma_5 + \Gamma_4 - (E^2 - m^2)(\Gamma_5 - \Gamma_4) - 2\alpha E \end{aligned}$$

(d) Conclude that the operators governing the time evolution of these systems can be written as a linear superposition of the generators of an $\mathfrak{so}(4, 2)$ subalgebra.

15. Prove that the nonzero eigenvalues of (4.19) are given by the expressions (4.20). What are the missing eigenvalues?

16. Show that the sphere $U(n;f)/U(n-1;f) = S^{fn-1} \subset R_{fn}$ is the orbit of the "north pole" under $U(n;f)$. How is the "north pole" suitably defined?

17. Prove (5.28).

18. Define an analytic function of a complex matrix argument in a reasonable way. Make sure your definition is valid for the case of a single complex variable $f(z) = f[SO(2, 1)/SO(2)]$. Show that the function described by (5.32) satisfies your definition.

19. Simplify the expression (5.53) even further by using a vector cross-product notation for **X**.

20. Verify that the arguments following (5.66) are applicable to the spheres $U(n+1; f)/U(n; f)$.

21. Describe the properties of the geodesics which exist in a classical rank-1 space. Show that the geodesics are recurring in a compact space, nonrecurring in its dual, and may be recurring or nonrecurring in the intermediate pseudo-Riemannian symmetric spaces. How many focusing points do geodesics have as a function of the connectivity properties of simply connected compact G and maximal subgroup H?

22. The homogeneous pseudo-Riemannian symmetric spaces are of intrinsic physical interest.

(a) What physical interpretation does each of these properties receive:
homogeneous?
(pseudo-) Riemannian?
symmetric?

(b) Compare these properties with the assumptions of the "perfect cosmological principle": "the universe looks the same to *any* observer in natural motion, regardless of either his position or his time."

(c) Which of the mathematical homogeneous pseudo-Riemannian symmetric spaces might be a suitable model for our universe?

23. Robertson-Walker metric on $SO(n+1)/SO(n) = S^n \subset R_{n+1}$: On spheres S^n it is often useful to transform from the coordinate system given by (5.8c) to a "radial-spherical" coordinate system. The line element in this new coordinate system may be called a Robertson-Walker line element. This is easily constructed in a number of simple steps.

(a) The metric tensor on S^n is given by

$$g_{ij}(X) = \{W^{-2}\}_{ij} = \left\{\frac{1}{I_n - XX^t}\right\}_{ij}$$

where X is an $n \times 1$ matrix and is a coset representative in $SO(n+1)/SO(n)$.

(b) The line element on S^n is given by (5.78) through (5.81)

$$ds^2(X) = dx^t \frac{1}{I_n - XX^t} dx$$

(c) A transformation from the coordinate system X to the radial-spherical coordinate system ϕ^μ is given by the *nonlinear* transformation $\phi^\mu = f^\mu(x^j)$, $x^j = [f^{-1}(\phi^\mu)]^j$:

$$\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^{n-1} \\ x^n \end{bmatrix} = \begin{bmatrix} \phi^1 \cos\phi^2 & & \\ \phi^1 \sin\phi^2 \cos\phi^3 & & \\ \vdots & & \\ \phi^1 \sin\phi^2 \sin\phi^3 & \cdots & \sin\phi^{n-1} \cos\phi^n \\ \phi^1 \sin\phi^2 \sin\phi^3 & \cdots & \sin\phi^{n-1} \sin\phi^n \end{bmatrix}$$

Show that the coordinate system ϕ^μ is an orthogonal coordinate system. In particular:

$$X^t X = (\phi^1)^2$$

(d) Conclude that ϕ^1 is the radial coordinate and ϕ^μ, $\mu = 2, 3, \ldots, n$ are angular coordinates. We occasionally write $\phi^1 = r$.

(e) Show that the line $\phi^1 =$ arbitrary, $\phi^2, \phi^3, \ldots, \phi^n$ all constant, is a geodesic in S^n.

(f) The infinitesimal displacements dx^i are related to the infinitesimal displacements $d\phi^\mu$ by a *linear* transformation:

$$dx^i = \frac{\partial x^i}{\partial \phi^\mu} d\phi^\mu = M_\mu{}^i \, d\phi^\mu$$

Compute the $n \times n$ matrix $M_\mu{}^i$.

(g) Prove that the columns of M are orthogonal:

$$\begin{aligned} (M^t M)_{\mu\nu} &= \sum_{i=1}^{n} M_\mu{}^i M_\nu{}^i = \delta_{\mu\nu} \left(\phi^1 \prod_{\lambda=2}^{\mu-1} \sin\phi^\lambda \right)^2 & \mu > 1 \\ &= \delta_{\mu\nu} & \mu = 1 \end{aligned}$$

(h) Observe that the first column of M is given by

$$M_1{}^i = \frac{1}{r} x^i$$

This means that

$$X^t M = (r, 0, 0, \ldots, 0)$$

since the columns of M are orthogonal.

(i) Compute

$$\begin{aligned} ds^2(\phi^\mu) &= (M\, d\phi)^t \frac{1}{I_n - XX^t} (M\, d\phi) \\ &= d\phi^t M^t M\, d\phi + d\phi^t \sum_{k=0}^{\infty} (M^t X)(X^t X)^k (X^t M)\, d\phi \end{aligned}$$

(j) Conclude:

$$\begin{aligned} ds^2(\phi^\mu) &= \frac{dr^2}{1 - r^2} + r^2 \sum_{m=2}^{n} \left(\prod_{k=2}^{m-1} \sin\phi^k \, d\phi^m \right)^2 \\ &= \frac{dr^2}{1 - r^2} + r^2 \Big\{ (d\phi^2)^2 + (\sin\phi^2 \, d\phi^3)^2 + \cdots \\ &\qquad + (\sin\phi^2 \sin\phi^3 \cdots \sin\phi^{n-1} \, d\phi^n)^2 \Big\} \end{aligned}$$

24. Analysis in radial-spherical coordinates:

(a) Use the diagonal metric tensor $g_{\mu\nu}(\phi^\lambda)$ computed in the previous problem to compute the volume elements on the sphere:

$$\text{vol}\,(\boldsymbol{\phi}) = \sqrt{\|g\|}\, d\phi^1 \wedge d\phi^2 \wedge \cdots \wedge d\phi^n$$

$$= \frac{1}{\sqrt{1-r^2}} r^{n-1} \sin^{n-2}\phi^2 \sin^{n-3}\phi^3 \cdots$$

$$\sin^{n-k}\phi^k \cdots \sin^1\phi^{n-1} \sin^0\phi^n\, dr \wedge d\phi^2 \wedge d\phi^3 \wedge \cdots \wedge d\phi^{n-1} \wedge d\phi^n$$

(b) Compute the second-order Laplace-Beltrami operator using

$$\Delta = \sum_{\mu\nu} \frac{1}{\sqrt{\|g\|}} \partial_\mu g^{\mu\nu} \sqrt{\|g\|}\, \partial_\nu\,; \qquad \partial_\lambda = \frac{\partial}{\partial\phi^\lambda}$$

Here $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$:

$$g^{\mu\nu} g_{\nu\lambda} = \delta_\lambda{}^\mu$$

(c) Compare this with the Casimir operator

$$\mathscr{C}_2\left[\frac{SO(n+1)}{SO(n)}\right] = \sum_{r,s=1}^{n+1} X_{r,s}^2(\phi)$$

(d) Show that the Laplace-Beltrami operators on spheres can be written recursively as follows:

$$\Delta(S^n) = \partial_n(f_1(\phi)\,\partial_n) + f_2(\phi)\,\Delta(S^{n-1})$$

Compute $f_1(\phi)$ and $f_2(\phi)$.

25. Compute the second- and fourth-order Laplace-Beltrami operators on D^n $(n = 4, 5)$ [see (5.40)].

26. Let $g_{ij}(x)$ be the nonsingular metric defined over a (pseudo-) Riemannian symmetric space.

(a) Show that the generators of infinitesimal displacements are given by

$$X_{rs} = g_{rt} x^t\, \partial_s - g_{st} x^t\, \partial_r$$

(b) Show that these generators obey

$$[X_{ab}, X_{rs}] = X_{as} g_{rb} - X_{ar} g_{sb} - X_{rb} g_{as} + X_{sb} g_{ar}$$

(c) Show that

$$[X_{ab}, \Delta] = 0$$

where $\Delta = G^{ab;\,rs} X_{ab} X_{rs}$ and $G^{ab;\,rs}$ is the inverse of the Cartan-Killing metric on the group of isometries of the Riemannian space:

$$G_{ab;\,rs} = \text{tr}\,\{\text{def}\,(X_{ab})\,\text{def}\,(X_{rs})\}.$$

(d) Show that Δ consists of terms that are both quadratic and linear in the operators ∂_i, and that

$$\Delta = g^{rs}\,\partial_r\,\partial_s - g^{rs}\Gamma_{rs}{}^{t}\,\partial_t$$

where the function $\Gamma_{rs}{}^{t}$ is *not* a tensor and is given by

$$\Gamma_{rs}{}^{t} = \frac{1}{2}\sum_u g^{tu}(\partial_s g_{ru} + \partial_r g_{su} - \partial_u g_{rs})$$

and $\Gamma_{rs}{}^{t}$ is the so-called Riemann-Christoffel symbol.

27. Use the results of Problem 26d to compute the coefficients of the connection $\Gamma_{rs}{}^{t}$ from the metric tensor as given in Problem 23j. In particular, show that

$$\Gamma_{rs}{}^{t} = \begin{array}{ll} f_r & r < s = t \\ f_s & s < r = t \\ -g^{tt}g_{rr}f_t & t < r = s \\ \dfrac{\lambda r}{1-\lambda r^2} & r = s = t = 1 \qquad (\lambda = +1) \\ 0 & \text{otherwise} \end{array}$$

where

$$f_k = \frac{1}{r} \qquad k = 1$$

$$f_k = \cot \phi^k \qquad k > 1$$

28. (a) Show that the group preserving the real symmetric inner product in the real linear vector space with signature

$$(+ + \cdots +, \lambda^{-1}) \qquad \lambda = \pm 1, 0$$

is

$$\lambda = \begin{array}{c} +1 : SO(n+1) \\ 0 : ISO(n) \\ -1 : SO(n, 1) \end{array}$$

(b) Show that the invariant metric tensor in the coordinate system of (5.8) and (5.81) is given by

$$g_{rs} = \{I_n - \lambda x x^t\}^{-1}_{rs}$$

(c) Use the relations

$$g^{cd}(\partial_a g_{bc}) = -g_{bc}(\partial_a g^{cd})$$
$$\partial_a g^{cd} = -\lambda(\delta_a{}^{c}x^d + \delta_a{}^{d}x^c)$$

to compute the coefficients of the Riemann-Christoffel connection and to show that

$$\Gamma_{rs}{}^{t} = \lambda g_{rs} x^t$$

Notes and References

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CHAPTER 10

Contractions and Expansions

Thus far most of our efforts have been bent toward a classification and description of the semisimple Lie algebras and their basic building blocks, their irreducible components, the simple algebras. We turn in this chapter to a discussion of some nonsemisimple groups that arise in physical applications.

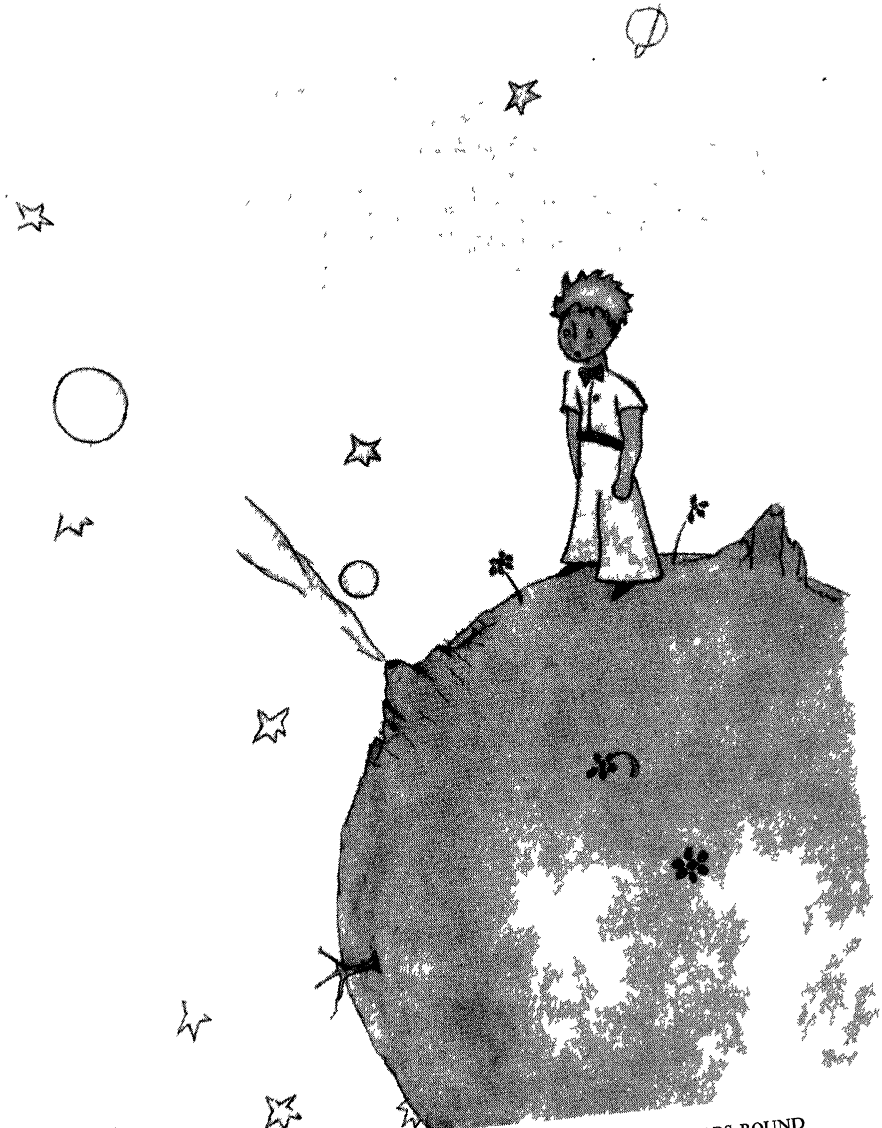
Specifically, we study the transformations of a Lie algebra which alter the structure of the algebra. Since the structure constants transform like a tensor (first-order contravariant, second-order antisymmetric covariant) under a nonsingular linear transformation, we must go beyond the usual change of basis mappings. We study the effects of a linear change of basis that becomes singular in a particular limit. When the transformed structure constants approach a well-defined limit as the transformation becomes singular, a new Lie algebra results. We call this new algebra the contracted limit of the original algebra. The contracted algebra is always nonsemisimple.

The converse problem is also treated. Given a nonsemisimple algebra $\mathfrak{g}'$, under what circumstances can a semisimple algebra $\mathfrak{g}$ be found for which $\mathfrak{g}'$ is the contracted limit? Clearly, we must go beyond either singular or nonsingular *linear* transformations in $\mathfrak{g}'$. A simple nonlinear transformation in $\mathfrak{g}'$ allows the reconstruction of $\mathfrak{g}$ under a large class of physically interesting conditions. The general expansion problem is not yet solved.

I. Simple Contractions

1. *THE LITTLE PRINCE.* If we look out into our solar system in a certain direction, following the calculations of the Turkish astronomer,[1] we will be able to find Asteroid B-612. We will also see that it is round (Fig. 10.1*a*).

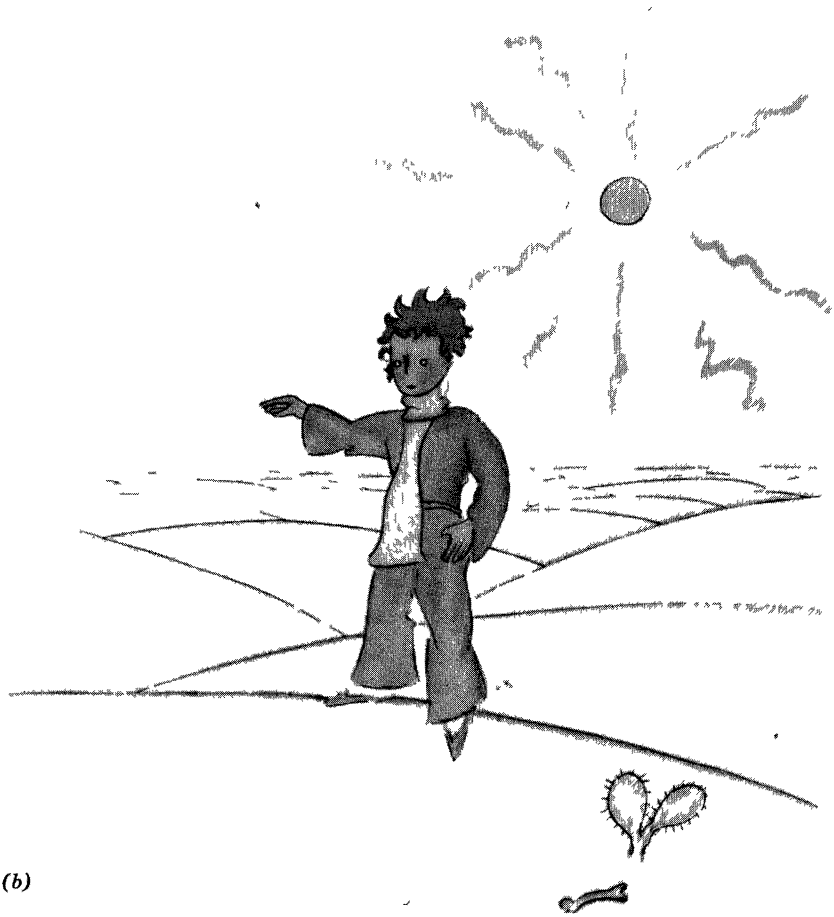
Asteroid B-612 is a very special asteroid, for it is the home of the shy Little Prince. How can he tell that his home is round? That's easy enough: he can see that it is so. He can walk, in a few short steps from one end to the other, and he can continue on in a straight line to his starting point.



(a)

FIG 10 1a TO THE LITTLE PRINCE HIS HOME ASTEROID B 612 APPEARS ROUND

The Little Prince might find great difficulty in figuring out the size and shape of our home planet Earth, though. Since it looks flat on the Great Plains of Africa (Sahara), it might seem that the earth is flat, at least locally (Fig. 10.1*b*). But if the Little Prince reaches this conclusion, he is easily forgiven: mankind believed that this was so until only relatively few centuries ago.



(*b*)

FIG 10 1*b* TO THE LITTLE PRINCE, OUR HOME, THE EARTH, APPEARS FLAT FROM *The Little Prince* BY ANTOINE DE SAINT-EXUPERY, COPYRIGHT, 1943, BY HARCOURT BRACE JOVANOVICH, INC.; RENEWED, 1971, BY CONSUELO DE SAINT-EXUPERY. REPRODUCED BY PERMISSION OF HARCOURT BRACE JOVANOVICH, INC

If you were a prince, how would you go about determining the size and shape of your home? In the journey above, we have encountered two characteristic sizes: the size characteristic of the Prince d (e.g., his height) and a size characteristic of his home R (e.g., its radius). It is very simple to determine the size and shape of the surface when $d/R \cong 1$, but it is an altogether different matter to determine the size and shape when $d/R \ll 1$.

Let's look now more carefully at how the Prince could learn the size and shape of asteroids of larger and larger radius.

He could first set up a coordinate system in his own vicinity, say, at the north pole. Then he could lay off a distance d—his own body length—along a straight line in the $-y$ direction, followed by a displacement a distance d along a straight line (geodesic) perpendicular to the first. From outside the asteroid (Fig. 10.2) we can see that what he is doing is completely equivalent to the following two-step process:

1. Rotating the point at the north pole about the x-axis through an angle $+(d/R)$.
2. Rotating this point about an axis

$$\hat{\mathbf{n}} = \hat{\mathbf{i}}_y \cos\theta + \hat{\mathbf{i}}_z \sin\theta \qquad \theta = \frac{d}{R} \tag{1.1}$$

through some angle $\theta = d/R$.

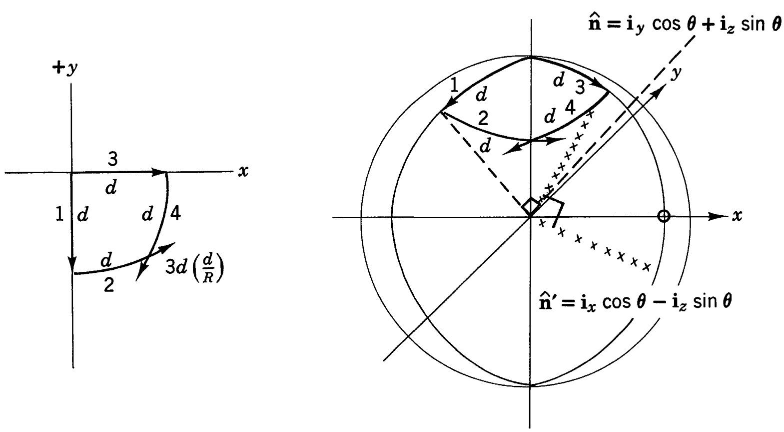


FIG. 10.2 THE LITTLE PRINCE CAN MAKE ORTHOGONAL GEODESIC DISPLACEMENTS OF LENGTH d IN TWO WAYS, DESCRIBED BY (1.4) AND (1.4′). THE DISPLACED POINTS IN GENERAL DO NOT COINCIDE. THE DISTANCE THAT SEPARATES THEM YIELDS INFORMATION ON THE CHARACTERISTIC SIZE OF THE SPACE: IN THIS CASE, THE SPHERE RADIUS. SINCE THE DISPLACEMENT PATHS CROSS, THE SURFACE "CLOSES UP" AS WE PROCEED AWAY FROM THE NORTH POLE. IF THE SPACE POSSESSES THE CHARACTERISTIC OF "CLOSING UP" AS WE PROCEED AWAY FROM EVERY POINT, IT WILL CLOSE UP ON ITSELF AND BE COMPACT.

We shall assume that θ is small. We have many justifications for this assumption, including the following:

1. If $\theta = d/R \cong 1$, then the Little Prince is of a size comparable to his asteroid and could measure its size and shape directly merely by looking at or walking around it. He would not have to resort to indirect measurement techniques.
2. We wish to avoid obfuscating topological niceties: a man walks one mile south, one mile east, and one mile north. What color is the bear? How many penguin's toes does he step on?

These rotation operations are expressed simply. The first operation is given by EXP θJ_x:

$$e^{\theta J_x}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cong \{I_3 + \theta(y\,\partial_z - z\,\partial_y)\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y - \theta z \\ z + \theta y \end{bmatrix} \tag{1.2}$$

This equation expresses the fact that the north pole [col (0, 0, R)] is moved in the negative y direction by a rotation around the x-axis:

$$e^{\theta J_x}\begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix} = \begin{bmatrix} 0 \\ -\theta R = -d \\ R \end{bmatrix} \qquad d \ll R$$

The other rotation operations are treated analogously. The product of displacements 1 followed by 2 [see (1.1)] is given by

$$\text{EXP } \theta(J_y \cos\theta + J_z \sin\theta)\, e^{\theta J_x} \tag{1.3}$$

Assuming θ to be small, this group operation can be expanded to terms of order θ^2:

$$\left\{I + \theta J_y + \theta^2 J_z + \frac{(\theta J_y)^2}{2!} + \cdots\right\}\left\{I + \theta J_x + \frac{(\theta J_x)^2}{2!} + \cdots\right\}$$

$$= I + \theta(J_y + J_x) + \theta^2\left(\underline{J_z} + \frac{1}{2}J_y{}^2 + \frac{1}{2}J_x{}^2 + J_y J_x\right) + \cdots \tag{1.4}$$

There is nothing sacrosanct about the order of displacements indicated earlier. These mappings can be accomplished in reverse order:

3. Rotate the north pole about the y-axis through an angle θ.
4. Rotate the resulting point about the axis

$$\hat{\mathbf{n}}' = \hat{\mathbf{i}}_x \cos\theta - \hat{\mathbf{i}}_z \sin\theta$$

through the same angle θ.

The product of these two transformations is again a transformation. The mapping is given by

$$\mathrm{EXP}\ \theta(J_x \cos\theta - J_z \sin\theta)\, e^{\theta J_y} \tag{1.1'}$$

Once again, to second order this is

$$\left\{I + \theta J_x - \theta^2 J_z + \frac{(\theta J_x)^2}{2!} + \cdots\right\}\left\{I + \theta J_y + \frac{(\theta J_y)^2}{2!} + \cdots\right\}$$

$$= I + \theta(J_x + J_y) + \theta^2\left(-\underline{J_z} + \frac{1}{2}J_x{}^2 + \frac{1}{2}J_y{}^2 + J_x J_y\right) + \cdots \tag{1.4'}$$

The images of the north pole under the two distinct group operations 1 followed by 2 [see (1.4)] and 3 followed by 4 [see (1.4′)] fall at different points. The distance by which they are separated is a measure of the radius R of the sphere surface. When the images occur at the same spot, the Little Prince is on a plane surface. The distance separating the images is computed by subtracting (1.4′) from (1.4)

$$\begin{aligned} \text{Eq. } (1.4) - \text{Eq. } (1.4') &= 2\theta^2 J_z + \theta^2(J_y J_x - J_x J_y) \\ &= 3\theta^2 J_z = 3\left(\frac{d}{R}\right)^2 J_z \end{aligned} \tag{1.5}$$

The angular separation between the two images of the north pole, as measured from the center of the asteroid, is

$$3\left(\frac{d}{R}\right)^2$$

The actual surface distance separating these images is therefore

$$3d\left(\frac{d}{R}\right)$$

The angular separation between these points, as measured from the north pole, is

$$\frac{3}{\sqrt{2}}\left(\frac{d}{R}\right)$$

The rotation direction is negative. It is clear that a knowledge of the degree of noncommutativity of orthogonal geodesic displacements of equal length is sufficient to determine the radius R of the asteroid, once the scaling length d is known.

For an earth-based astronomer, it is clearly more convenient to measure the motions of the asteroid by specifying angles of rotation about the x, y, z

axes through the center. For the Little Prince, it is just as clearly more convenient to specify motions by indicating angle of rotation about the north pole, together with the displacement distance (for small displacements) in the x and y directions. Once again, we emphasize that the choice of a coordinate system, or basis, is purely a matter of convenience, subject to the whims and feelings of the particular observer.

According to an earth-based observer, every finite reorientation of the asteroid can be described by a group operation

$$\mathrm{EXP}\,\{\theta_x J_x + \theta_y J_y + \theta_z J_z\}$$

Here J_x, J_y, J_z are the generators of infinitesimal rotations around the x, y, z axes, and θ_x, θ_y, θ_z are the angular coordinates of a displacement as seen by an earth-based astronomer. Since J_y induces displacements in the $+x$ direction that can be measured in units of the distance d, we can write

$$\begin{aligned} \alpha d &= R\theta_y \\ \theta_y J_y &= \alpha P_x = \alpha\left(\frac{d}{R} J_y\right) \end{aligned} \tag{1.6}$$

where P_x is the Little Prince's generator of infinitesimal displacements in the $+x$ direction. Similarly, the generators J_x induce displacements in the $-y$ direction:

$$\begin{aligned} \beta d &= -R\theta_x \\ \theta_x J_x &= \beta P_y = \beta\left(\frac{-d}{R} J_x\right) \end{aligned} \tag{1.7}$$

The operator J_z induces rotations around the north pole, which are conveniently measured by the Little Prince, since they leave the north pole fixed.

The commutation relations for this new set of generators are easily obtained

$$\begin{aligned} [J_z, P_x] &= \left[J_z, \frac{d}{R} J_y\right] && = \frac{d}{R} J_x && = -P_y \\ [J_z, P_y] &= \left[J_z, \frac{-d}{R} J_x\right] && = +\frac{d}{R} J_y = +P_x \\ [P_x, P_y] &= \left[\frac{d}{R} J_y, \frac{-d}{R} J_x\right] = && && = -\left(\frac{d}{R}\right)^2 J_z \end{aligned} \tag{1.8}$$

We note that the first two Lie brackets are independent of the ratio d/R, whereas the last is not.

The relation between these two different coordinate systems of convenience is given simply by the transformation

$$\begin{bmatrix} P_y \\ P_x \\ J_z \end{bmatrix} = \begin{bmatrix} \frac{-d}{R} & 0 & 0 \\ 0 & \frac{d}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J_x \\ \\ J_z \end{bmatrix} \tag{1.9}$$

The Lie group parameters are given by the inverse transformation

$$(\beta, \alpha, \theta_z) = (\theta_x, \theta_y, \theta_z) \begin{bmatrix} \frac{-d}{R} & 0 & 0 \\ 0 & \frac{d}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \tag{1.10}$$

Again we emphasize that the choice of coordinate system is purely a matter of convenience. We have seen this long ago when we observed that the structure constants $c_{ij}{}^k$ transform as a third-order tensor under a nonsingular change of basis.

We have noted that the commutation relations depend on the ratio d/R. Thus when d is fixed and $R \to \infty$, the commutation relations (1.8) approach more and more closely the commutation relations of the Euclidean group $E(2) \cong ISO(2)$. We can formulate a different interpretation, though. We can assume that R is fixed and allow $d \to 0$. Thus we are *contracting* the observer. Once again, the commutation relations approach those of the Euclidean group, since only the ratio d/R is involved.

For $d/R \neq 0$, an arbitrary element in the Little Prince's Lie algebra can be written

$$\begin{aligned} \alpha P_x + \beta P_y + \theta_z J_z &= \alpha\left(\frac{d}{R} J_y\right) + \beta\left(\frac{-d}{R} J_x\right) + \theta_z J_z \\ &= \left(\alpha \frac{d}{R}\right) J_y + \left(-\beta \frac{d}{R}\right) J_x + \theta_z J_z \end{aligned} \tag{1.11}$$

As $d/R \to 0$, all finite coordinates α, β come from a region of the original algebra which approaches closer and closer to the identity

$$\theta_x = -\beta \frac{d}{R} \to 0; \qquad \theta_y = \alpha \frac{d}{R} \to 0$$

Therefore, the algebra $\mathfrak{iso}(2)$ is parameterized by a "contracted" region of the original algebra $\mathfrak{so}(3)$. Since the parameters θ_x, $\theta_y \to 0$, the corresponding generators P_y, P_x must commute

$$[P_y, P_x] = 0$$

The contracted generators span an abelian invariant subalgebra.

Comment 1. Notice that in (1.1) to (1.4′) we have used an active interpretation of the group's infinitesimal generators. Thus they describe a mapping of the space onto itself, rather than a change of coordinate system within the space.

Comment 2. The unfamiliar factor 3 appearing in (1.5) is correct. Its presence can be explained as follows. In describing the curvature of a group we have always compared the noncommutativity properties of two group operations of the form EXP $\alpha^\mu X_\mu$; that is,

$$e^{\theta_x J_x} e^{\theta_y J_y} \qquad \text{and} \qquad e^{\theta_y J_y} e^{\theta_x J_x} \tag{1.12}$$

Under the conditions described here, we are working with geodesics and displacements on the coset space. Since geodesics on a sphere are great circles, we must be sure that all displacements are along great circles. This is why the second displacement operations, 2 and 4, are given by

$$\text{EXP } \theta\{J_y \cos\theta + J_z \sin\theta\} \qquad \text{instead of} \quad \text{EXP } \theta J_y$$

and

$$\text{EXP } \theta\{J_x \cos\theta - J_z \sin\theta\} \qquad \text{instead of} \quad \text{EXP } \theta J_x$$

respectively.

The underlined terms in (1.4) and (1.4′) then give the extra two factors of $\theta^2 J_z$. The commutator in (1.5) is the same commutator that occurs in the discussion of the noncommutativity of the group operations (1.12).

Comment 3. The procedure for determining the size R of the sphere through the noncommutativity of the geodesic displacements, outlined in (1.1) to (1.5) and in Fig. 10.2, gives information about the shape of the Riemannian surface, as well. Since the geodesic displacements cross, the space "closes up" as we move out from our origin of coordinates on the sphere surface. If orthogonal geodesics everywhere on the surface have this property, the space must close upon itself. On the other hand, if the orthogonal geodesics do not cross, they exist on a hyperbolic surface (Fig. 10.3). Again, the distance separating two images of a single point under noncommuting orthogonal geodesic trajectories gives a measure of

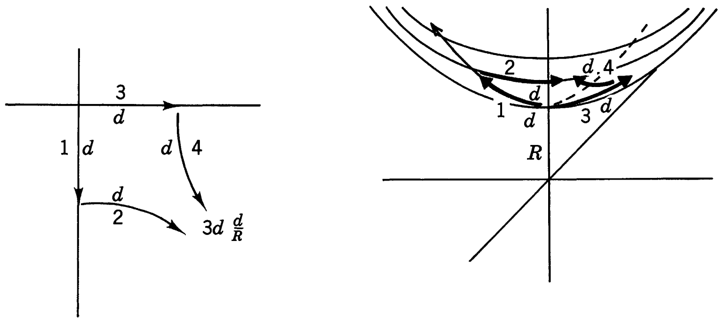


FIG. 10.3 GEODESIC DISPLACEMENTS ANALOGOUS TO (1.4) AND (1.4′) CAN ALSO BE CARRIED OUT ON THE HYPERBOLOID $H^2 \cong SO(2, 1)/SO(2)$. THESE ARE AGAIN SUCCESSIVE DISPLACEMENTS OF LENGTH d ALONG ORTHOGONAL FAMILIES OF GEODESICS. THE DISPLACED POINTS DO NOT COINCIDE HERE EITHER, INDICATING THAT THE SPACE IS NOT FLAT (ABELIAN). THE DISTANCE SEPARATING THE DISPLACED IMAGES YIELDS INFORMATION ON THE CURVATURE R OF THIS SPACE. SINCE THE DISPLACEMENT PATHS DO NOT CROSS, THIS SPACE "OPENS UP" AS WE PROCEED AWAY FROM THE ORIGIN. THE SPACE IS NOT COMPACT.

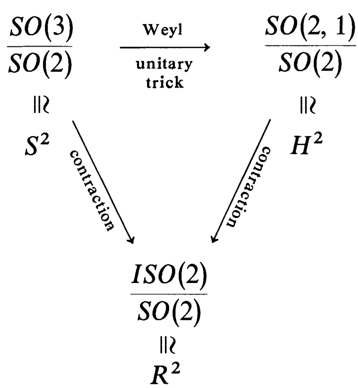


FIG. 10.4. THE SPHERE S^2, THE HYPERBOLOID H^2, AND THE EUCLIDEAN PLANE R^2 CAN ALL BE REALIZED AS COSET SPACES OF A LIE GROUP BY A SUBGROUP OF ROTATIONS ABOUT THE z-AXIS. THE GROUPS $SO(3)$ AND $SO(2, 1)$ ARE DIFFERENT REAL FORMS OF THE COMPLEX GROUP WITH ALGEBRA $\mathfrak{a}_1$. THEY ARE RELATED BY THE WEYL UNITARY TRICK. BOTH CAN BE CONTRACTED TO THE EUCLIDEAN GROUP $E(2)$:

$$\begin{tikzcd} SO(3) \arrow[dr] & \\ & ISO(2) \\ SO(2,1) \arrow[ur] & \end{tikzcd}$$

the space's curvature. Finally, if the two image points coincide, the space is (locally) flat and without curvature—that is, Euclidean. All three spaces can be realized as coset spaces of a Lie group by a particular subgroup consisting of rotations about the z-axis (Fig. 10.4).

2. INÖNÜ–WIGNER CONTRACTION. The contraction procedure discussed in the preceding section involves a sequence of change of basis transformations depending on the parameter d/R. The transformation becomes singular in the limit $d/R \rightarrow 0$. Nevertheless, the Lie bracket exists and is well defined in this singular limit. The original and contracted algebras are not isomorphic.

Nonsingular changes of bases can never lead to new algebras. The reason is simple. The structure constants ${c_{ij}}^k$ transform like a second-rank covariant, first-rank contravariant tensor under a nonsingular change of basis. Under such a transformation the new structure constant tensor possesses exactly as much information as the original. To study the contraction process further, we must study singular changes of bases.

These studies can proceed along two distinct lines; that is, in a coordinate-dependent version or in a coordinate-free version.

A. The Coordinate-Dependent Version (Inönü–Wigner Contraction).[2,3] Let X_i, $i = 1, 2, \ldots, \eta$, be a set of basis vectors for a Lie algebra $\mathfrak{g}$. Let a new set of bases Y_j be related to the X_i by

$$\begin{aligned} Y_j &= U(\varepsilon)_j{}^i X_i \\ U(\varepsilon = 1)_j{}^i &= \delta_j{}^i \\ \det \|U(\varepsilon = 0)\| &= 0 \end{aligned} \tag{1.13}$$

That is, for $\varepsilon = 1$, $U(1)$ is the identity transformation but for $\varepsilon = 0$, $U(0)$ is singular. Then the structure constants with respect to the new bases are given by

$$\begin{aligned} [Y_i, Y_j] &= c_{ij}{}^k(\varepsilon) Y_k \\ [U(\varepsilon)_i{}^r X_r, U(\varepsilon)_j{}^s X_s] &= c_{ij}{}^k(\varepsilon) U(\varepsilon)_k{}^t X_t \\ U(\varepsilon)_i{}^r U(\varepsilon)_j{}^s c_{rs}{}^t(1) U^{-1}(\varepsilon)_t{}^k &= c_{ij}{}^k(\varepsilon) \end{aligned} \tag{1.14}$$

As long as $U(\varepsilon)$ remains nonsingular ($\varepsilon \neq 0$), the structure constants have the usual tensor properties. When $\varepsilon \rightarrow 0$, the limit $c_{ij}{}^k(\varepsilon)$ may or may not exist. When the limit

$$\lim_{\varepsilon \rightarrow 0} c_{ij}{}^k(\varepsilon) = c_{ij}{}^k(0) = c'_{ij}{}^k \tag{1.15}$$

exists and is well defined, the new structure constants $c'_{ij}{}^k$ characterize a Lie algebra that may or may not be isomorphic with the original algebra.

B. The Coordinate-Free Version (Saletan Contraction).[4] We can also consider $U(\varepsilon)$ to be a 1-1 mapping of one linear vector space onto another that preserves the commutation bracket. In other words, it is a 1-1 mapping of an algebra $\mathfrak{g}$ onto an algebra $\mathfrak{g}'$ ($\varepsilon \neq 0$) (Fig. 10.5). The Lie bracket in V can be computed from that in V':

$$\begin{aligned} [a', b'] &= c' \\ [Ua, Ub] &= Uc \\ U^{-1}(\varepsilon)[U(\varepsilon)a, U(\varepsilon)b] &= c \end{aligned} \tag{1.16}$$

The bracket is ε-dependent. As long as $U(\varepsilon)$ remains nonsingular ($\varepsilon \neq 0$), the bracket exists and is well defined. When $\varepsilon \to 0$, the limit above may or may not exist. When the limit

$$\lim_{\varepsilon \to 0} U^{-1}(\varepsilon)[U(\varepsilon)a, U(\varepsilon)b] = [a, b]' \tag{1.17}$$

exists, it may define a new algebraic structure on the vector space V. This new algebraic structure may or may not be isomorphic with the original algebraic structure.

Comment. We denote the contracted algebra by

$$\mathfrak{g}' = U\mathfrak{g} \tag{1.18}$$

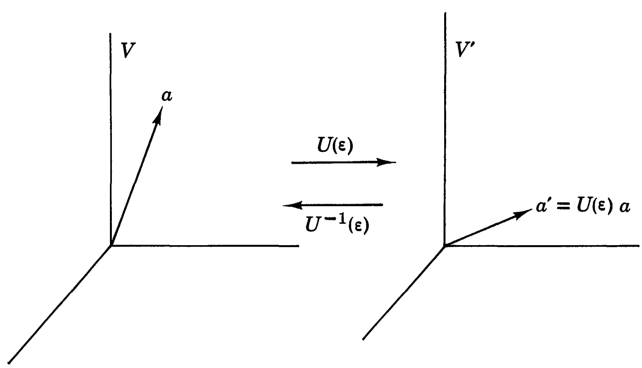


FIG. 10.5 THE LINEAR TRANSFORMATION U IS A NONSINGULAR TRANSFORMATION ($\varepsilon \neq 0$) MAPPING THE LINEAR VECTOR SPACE V (WHICH IS ALSO A LIE ALGEBRA) INTO THE LINEAR VECTOR SPACE V'. AS LONG AS U REMAINS NONSINGULAR, $U^{-1}(\varepsilon)$ EXISTS, AND THE VECTOR SPACES V, V' HAVE ISOMORPHIC COMMUTATION RELATIONS, AND ARE THUS IDENTICAL AS LIE ALGEBRAS. WHEN U BECOMES SINGULAR ($\varepsilon \to 0$), U ANNIHILATES A LINEAR VECTOR SUBSPACE OF V, U^{-1} FAILS TO EXIST, AND V' MAY OR MAY NOT HAVE A WELL-DEFINED LIE ALGEBRA STRUCTURE ON IT.

Then

$$\dim \mathfrak{g}' = \dim V = \dim \mathfrak{g}$$

All contraction processes come down, ultimately, to a study of the properties of the Lie bracket under singular transformations $U(\varepsilon)$. Such a study can be made explicitly coordinate dependent, or it may be made in a coordinate-free version. In the following section we study the general coordinate-free Saletan contractions. Therefore, without loss of subsequent generality, we study a restricted class of contractions in an explicit basis.

Let $\mathfrak{g}$ be a Lie algebra with bases X_μ, $\mu = 1, 2, \ldots, \eta$. We write $\mathfrak{g}$ (as a vector space) as the direct sum of two vector spaces

$$\mathfrak{g} = V_R \oplus V_N \tag{1.19}$$

We also choose $U(\varepsilon)$ to have the simple structure

$$U(\varepsilon) = \begin{array}{cc} V_R & V_N \\ \left[\begin{array}{c|c} I & 0 \\ \hline 0 & \varepsilon I \end{array}\right] \end{array} \tag{1.20}$$

When we select

$$X_\alpha, X_\beta, X_\gamma, \ldots, \qquad \text{and} \qquad X_i, X_j, X_k, \ldots,$$

as the bases for V_R and V_N, respectively, the transformation $U(\varepsilon)$ has the form

$$\begin{aligned} U(\varepsilon)X_\alpha &= X_\alpha = Y_\alpha \\ U(\varepsilon)X_i &= \varepsilon X_i = Y_i \end{aligned} \tag{1.21}$$

From (1.14) and from the simple diagonal structure (1.20) of $U(\varepsilon)$, it is clear that the structure constants undergo the transformation

$$c_{\mu\nu}{}^{\lambda}(\varepsilon) = \varepsilon^{\text{power}} c_{\mu\nu}{}^{\lambda}(1)$$

where power = number of covariant Latin indices
$-$number of contravariant Latin indices. (1.22)

The only convergence problem, therefore, occurs for one contravariant Latin index and two covariant (early) Greek indices; that is, for commutators of the form

$$\begin{aligned} [Y_\alpha, Y_\beta] &= c(\varepsilon)_{\alpha\beta}{}^{k} Y_k \\ c_{\alpha\beta}{}^{k}(\varepsilon) &= \varepsilon^{-1} c_{\alpha\beta}{}^{k}(1) \end{aligned} \tag{1.23}$$

This particular structure constant will converge if and only if $c_{\alpha\beta}{}^{k}(1) = 0$ to begin with.

When $c_{\alpha\beta}{}^{k}(1) = 0$, all structure constants converge in the limit $\varepsilon \to 0$. The contracted structure constants c' are in fact simply related to the original structure constants

$c'_{\alpha\beta}{}^{\gamma} = c_{\alpha\beta}{}^{\gamma}$ $c'_{\alpha\beta}{}^{k} = c_{\alpha\beta}{}^{k} \quad = 0$	The X_α and Y_α are closed under commutation, and therefore span subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively
$c'_{\alpha j}{}^{\gamma} = \varepsilon^1 c_{\alpha j}{}^{\gamma} \to 0$ $c'_{\alpha j}{}^{k} = c_{\alpha j}{}^{k}$	The Y_i span an invariant subalgebra of $\mathfrak{g}'$
$c'_{ij}{}^{\gamma} = \varepsilon^2 c_{ij}{}^{\gamma} \to 0$ $c'_{ij}{}^{k} = \varepsilon^1 c_{ij}{}^{k} \to 0$	The invariant subalgebra spanned by the Y_i is also abelian

From these structure constants we immediately reach the following conclusions:

1. V_R is closed under commutation in $\mathfrak{g}^{(0)}$.
2. V_R also forms a subalgebra in $\mathfrak{g}^{(1)}$.
3. V_N forms a subspace in $\mathfrak{g}^{(0)}$.
4. V_N is an abelian invariant subalgebra in $\mathfrak{g}^{(1)}$.

The algebra $\mathfrak{g}^{(0)}$ may be simple, semisimple, or nonsemisimple. The contracted algebra $\mathfrak{g}^{(1)}$ is necessarily nonsemisimple. These results are summarized by the theorem, valid for Inönü-Wigner contractions.

THEOREM. *Let* $\mathfrak{g}^{(0)}$ *be a Lie algebra and* $\mathrm{U}(\varepsilon)$ *a singular transformation when* $\varepsilon \to 0$, *with the following properties*:

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathrm{V_R} \oplus \mathrm{V_N} \\ \mathrm{U}(0)\mathrm{V_R} &= \mathrm{V_R} \\ \mathrm{U}(0)\mathrm{V_N} &= 0 \end{aligned} \tag{1.24}$$

Then $\mathfrak{g}^{(0)}$ *can be contracted with respect to* $\mathrm{V_R}$ *if and only if* $\mathrm{V_R}$ *is closed under commutation. Then* $\mathrm{V_R}$ *forms a subalgebra in both* $\mathfrak{g}^{(0)}$ *and* $\mathfrak{g}^{(1)}$; $\mathrm{V_N}$ *is an abelian invariant subalgebra of* $\mathfrak{g}^{(1)}$, *and* $\mathfrak{g}^{(1)}$ *is thus nonsemisimple.*

Once again, it is easy to see why an abelian invariant subalgebra exists in $\mathfrak{g}^{(1)}$. Writing for an arbitrary element in the Lie algebra $\mathfrak{g}^{(1)}$

$$\begin{aligned} b^\alpha Y_\alpha + b^i Y_i &= (b^\alpha, b^i)\begin{bmatrix} Y_\alpha \\ Y_i \end{bmatrix} \\ &= (b^\alpha, b^i)\left[\begin{array}{c|c} I & \\ \hline & \varepsilon I \end{array}\right]\begin{bmatrix} X_\alpha \\ X_i \end{bmatrix} \\ &= (a^\alpha, a^i)\begin{bmatrix} X_\alpha \\ X_i \end{bmatrix} \end{aligned} \tag{1.25}$$

we see that the coordinates a, b are related by

$$(b^\alpha, \varepsilon b^i) = (a^\alpha, a^i)$$

The entire region in $\mathfrak{g}'$

$$-\infty < b^i < +\infty$$

is parameterized by the neighborhood infinitesimally close to the origin in $\mathfrak{g}^{(0)}$

$$a^i \to 0 \tag{1.26}$$

Therefore, the related generators Y_i must commute.

An algebra that has been contracted once can be further contracted. If it is contracted with a different U of the form (1.20), (i.e., with respect to a different subspace and subalgebra V_R), then a nonisomorphic algebra may result. If it is contracted with respect to the same subalgebra V_R, an isomorphic algebra is produced. If an algebra is contracted with respect to the subalgebra $V_R = 0$, an abelian algebra results.

3. SOME USEFUL CONTRACTIONS. In this section we consider a number of examples of contractions that are useful from a physical point of view.

Example 1. The group $SO(n+1)$ maps the surface of the n-dimensional sphere S^n embedded in R_{n+1} onto itself. The equation for S^n is

$$\mathbf{x}\cdot\mathbf{x} + x_{n+1}^2 = R^2$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \tag{1.27s}$$

The infinitesimal generators X_{ij} for $SO(n+1)$ obey the usual commutation relations

$$X_{ij} = x^i\,\partial_j - x^j\,\partial_i \tag{1.28s}$$

$$[X_{ij}, X_{rs}] = X_{is}\,\delta_{jr} + X_{jr}\,\delta_{is} - X_{ir}\,\delta_{js} - X_{js}\,\delta_{ir} \tag{1.29s}$$

In the limit $R \to \infty$ and at the north pole, we have

$$\mathbf{x} = \mathbf{0}, \qquad x_{n+1} = R \tag{1.29}$$

It is useful to use an alternative set of infinitesimal generators given by

$$Y_{ij} = \lim_{R\to\infty} X_{ij} = x^i\,\partial_j - x^j\,\partial_i$$

$$P_i = \lim_{R\to\infty} \frac{1}{R} X_{i,\,n+1} = \lim_{R\to\infty} \frac{1}{R}(x^i\,\partial_{n+1} - x^{n+1}\,\partial_i) \to -\partial_i \tag{1.30s}$$

where $1 \le i, j \le n$.

The generators $X_{ij} = Y_{ij}$ span V_R and $P_i = \varepsilon X_{i,n+1}$ span V_N. The commutation relations for the contracted algebra are

$$\begin{aligned} &[Y_{ij}, Y_{rs}] = \mathfrak{so}(n) \\ &[Y_{ij}, P_r] = P_i\,\delta_{jr} - P_j\,\delta_{ir} \\ &[P_r, P_s] = 0 \end{aligned} \tag{1.31s}$$

where $1 \le i, j, r, s \le n$.

The group whose algebra is given by (1.31) is the inhomogeneous Euclidean group $E(n)$ or $ISO(n)$:

$$SO(n+1) \to E(n) = ISO(n) \tag{1.32s}$$

Example 2. The group $SO(n, 1)$ maps the surfaces of the n-dimensional hyperboloid H^n embedded in R_{n+1} onto itself. The equation for H^n is

$$\mathbf{x} \cdot \mathbf{x} - x_{n+1}^2 = -R^2 \tag{1.27h}$$

The infinitesimal generators X_{ij} for $SO(n, 1)$ are

$$\begin{aligned} X_{ij} &= x^i\,\partial_j - x^j\,\partial_i \\ X_{i,n+1} &= x^i\,\partial_{n+1} + x^{n+1}\,\partial_i \end{aligned} \tag{1.28h}$$

where $1 \le i, j \le n$.

The commutation relations are

$$[X_{ij}, X_{rs}] = X_{is}g_{jr} + X_{jr}g_{is} - X_{ir}g_{js} - X_{js}g_{ir} \tag{1.29h}$$

Once again, it is convenient to redefine the generators in the vicinity of the "north pole" $(\mathbf{0}, R)$, by defining

$$Y_{ij} = \lim_{R\to\infty} X_{ij} = x^i\,\partial_j - x^j\,\partial_i$$

$$Q_i = \lim_{R\to\infty} \frac{1}{R} X_{i,n+1} = \lim_{R\to\infty} \frac{1}{R}(x^i\,\partial_{n+1} + x^{n+1}\,\partial_i) \to \partial_i \tag{1.30h}$$

The commutation relations are again

$$\begin{aligned} &[Y_{ij}, Y_{rs}] = \mathfrak{so}(n) \\ &[Y_{ij}, Q_r] = Q_i\,\delta_{jr} - Q_j\,\delta_{ir} \\ &[Q_r, Q_s] = 0 \end{aligned} \tag{1.31h}$$

Thus we also have the contraction

$$SO(n, 1) \to E(n) = ISO(n) \tag{1.32h}$$

Example 3. We could have started with the hyperboloid

$$\mathbf{x}\cdot\mathbf{x} - x_{n+1}^2 = +R^2$$

instead of (1.27h). If we had chosen then to redefine our generators in the vicinity of the "west pole" $(R, 0, 0, \ldots, 0)$, the contraction procedure would have given

$$SO(n, 1) \rightarrow ISO(n-1, 1) = E(n-1, 1) \tag{1.32h$'$}$$

It is clear that possibilities exist for contracting the simple groups $SO(p, q)$ to the inhomogeneous groups:

$$SO(p, q) \begin{array}{l} \nearrow ISO(p, q-1) \\ \searrow ISO(p-1, q) \qquad p \neq q \end{array} \tag{1.33}$$

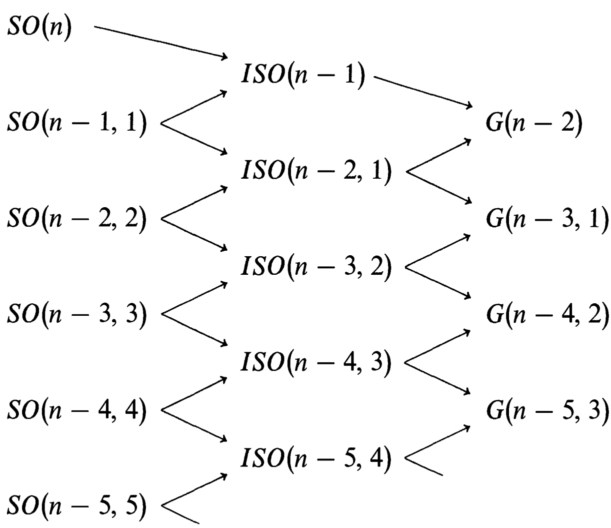


FIG. 10.6. UNDER A SIMPLE CONTRACTION AS GIVEN BY (1.27) TO (1.33), THE SEMISIMPLE LIE GROUP $SO(p, q)$ CAN BE CONTRACTED TO EITHER OF THE INHOMOGENEOUS LIE GROUPS $ISO(p, q-1)$ OR $ISO(p-1, q)$. CONVERSELY, THE INHOMOGENEOUS GROUP $ISO(p, q)$ CAN BE OBTAINED BY CONTRACTION *only* FROM THE SEMISIMPLE GROUPS $SO(p+1, q)$ AND $SO(p, q+1)$. THE GALILEAN GROUP $G(n_1, n_2)$ CAN BE OBTAINED BY THE CONTRACTION OF TYPE (1.43) APPLIED TO EITHER OF THE INHOMOGENEOUS GROUPS $ISO(n_1+1, n_2)$ OR $ISO(n_1, n_2+1)$. THE GALILEAN GROUP $G(n_1, n_2)$ CAN ALSO BE OBTAINED DIRECTLY FROM THE SEMISIMPLE GROUP $SO(n_1+1, n_2+1)$ BY A SINGLE CONTRACTION (1.49).

These possibilities are sketched in Fig. 10.6. What is less clear, but also true, is that the inhomogeneous group $ISO(p, q)$ can be obtained by the contraction of *only* two simple groups

$$\begin{array}{ccc} SO(p+1, q) & \searrow & \\ & & ISO(p, q) \\ SO(p, q+1) & \nearrow & \end{array} \tag{1.33'}$$

The laws of physics seem to be invariant under the Poincaré group or the inhomogeneous Lorentz group $ISO(3, 1)$, at least locally. We may suppose that the local "flatness" of the universe is due to an enormously large radius R for the universe. If this is so and the universe is homogeneous (i.e., every point looks like every other point), then the global symmetry of the universe is governed by either of the two deSitter groups ($R < \infty$)

$$\begin{array}{ccc} SO(4, 1) & \searrow & \\ & & ISO(3, 1) \\ SO(3, 2) & \nearrow & \end{array}$$

Example 4. In Examples 1 and 2, the abelian invariant subalgebras spanned by P_i and Q_i generate abelian invariant subgroups isomorphic with R_n. Therefore, the coset spaces have a group structure, and in fact

$$\begin{aligned} \frac{ISO(n)}{R_n} &\cong SO(n) \\ \frac{ISO(n-1, 1)}{R_n} &\cong SO(n-1, 1) \end{aligned} \tag{1.34}$$

This is general and straightforward. Since V_N is an abelian invariant subalgebra in $\mathfrak{g}^{(1)}$, the coset space possesses a group structure; indeed, we can write

$$\frac{\text{EXP } \mathfrak{g}^{(1)}}{\text{EXP } V_N} = \text{EXP } \mathfrak{g}^{(1)} \bmod V_N = \text{EXP } V_R \tag{1.35}$$

What is less obvious but prettier is that the surfaces S^n, H^n, R^n are also coset spaces. We show this as follows. In the defining $(n+1) \times (n+1)$ matrix representations for the Lie algebras of the groups, we have

$$
\begin{array}{rcccc}
\mathfrak{g} = & V_R & \oplus & V_N & \\
\mathfrak{so}(n+1) = & \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array}\right] & \oplus & \left[\begin{array}{c|c} 0 & A \\ \hline -A^t & 0 \end{array}\right] & \\
\mathfrak{so}(n,1) = & \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array}\right] & \oplus & \left[\begin{array}{c|c} 0 & A \\ \hline +A^t & 0 \end{array}\right] &
\end{array} \tag{1.36}
$$

where M is an $n \times n$ skew-symmetric matrix and A is a real $n \times 1$ column vector.

The coset spaces are isomorphic with EXP V_N. In particular,

$$
\frac{SO(n+1)}{SO(n)} = \text{EXP}\left[\begin{array}{c|c} & A \\ \hline -A^t & \end{array}\right] = \left[\begin{array}{c|c} \cos\,(AA^t)^{1/2} & A\,\dfrac{\sin\,(A^tA)^{1/2}}{(A^tA)^{1/2}} \\ \hline -\dfrac{\sin\,(A^tA)^{1/2}}{(A^tA)^{1/2}}\,A^t & \cos\,(A^tA)^{1/2} \end{array}\right]
$$

$$
\frac{SO(n,1)}{SO(n)} = \text{EXP}\left[\begin{array}{c|c} & A \\ \hline A^t & \end{array}\right] = \left[\begin{array}{c|c} \cosh\,(AA^t)^{1/2} & A\,\dfrac{\sinh\,(A^tA)^{1/2}}{(A^tA)^{1/2}} \\ \hline \dfrac{\sinh\,(A^tA)^{1/2}}{(A^tA)^{1/2}}\,A^t & \cosh\,(A^tA)^{1/2} \end{array}\right] \tag{1.37}
$$

We let $(\mathbf{x}, x_{n+1})$ describe elements in the last row or column of the matrices just given. Since these matrices are also group elements in the metric-preserving groups $SO(n+1)$ and $SO(n, 1)$, they obey the condition

$$
O_i{}^r g_{rs} O_j{}^s = g_{ij} \tag{1.38}
$$

With the metrics chosen $(++\cdots+, \pm)$ this condition is realized more concretely as

$$
\mathbf{x}\cdot\mathbf{x} + x_{n+1}^2 = +1 \qquad SO(n+1) \tag{1.27s}
$$

$$
\mathbf{x}\cdot\mathbf{x} - x_{n+1}^2 = -1 \qquad SO(n,1) \tag{1.27h}
$$

To emphasize the contraction process, we replace x_{n+1} by Rx_{n+1}:

$$
\mathbf{x}\cdot\mathbf{x} \pm (Rx_{n+1})^2 = \pm R^2 \tag{1.39}
$$

Finally, renormalizing by R, the equation for the contracted surface is

$$\pm\frac{1}{R^2}\mathbf{x}\cdot\mathbf{x}+x_{n+1}^2=1 \qquad R\to\infty \tag{1.40}$$

Specifically, the solution is $x_{n+1} = +1$, $\mathbf{x}$ arbitrary, and it is the equation for a flat Euclidean space R_n. These cases are illustrated in Figs. 6.1–6.3.

To summarize, we have explicitly constructed the relationships

$$\begin{tikzcd}
\dfrac{SO(n+1)}{SO(n)} \arrow[rr, leftrightarrow, "\text{Weyl}", "\text{unitary trick}"'] \arrow[d, phantom, "\wr\|"] \arrow[ddr, "\text{contraction}"] & & \dfrac{SO(n,1)}{SO(n)} \arrow[d, phantom, "\wr\|"] \arrow[ddl, "\text{contraction}"'] \\
S^n & & H^n \\
& \dfrac{ISO(n)}{SO(n)} \arrow[d, phantom, "\wr\|"] & \\
& R^n &
\end{tikzcd}$$

Example 5. We now give an example of an algebra obtained by two successive contractions with respect to two different subspaces V_{R_1} and V_{R_2}. Let $\mathfrak{g}^{(0)} = \mathfrak{so}(n+2)$ with generators given as in (1.29s). Choosing

$$\begin{aligned} V_{R_1} &= X_{ij} \\ V_{N_1} &= X_{i,n+2} \qquad 1\le i,j\le n+1 \end{aligned} \tag{1.41}$$

and contracting with the parameter $\varepsilon_1 = c/R \to 0$, we immediately get the algebra $\mathfrak{g}^{(1)} = \mathfrak{iso}(n+1)$. The bases for this algebra are

$$\begin{aligned} X_{ij} &= x^i\,\partial_j - x^j\,\partial_i \qquad\qquad 1\le i,j\le n+1 \\ P_i' &= \lim_{c/R\to 0}\frac{c}{R}X_{i,n+2} \end{aligned} \tag{1.42}$$

Now we contract a second time using the parameter $\varepsilon_2 = 1/c \to 0$, but with respect to the decomposition

$$\begin{aligned} V_{R_2} &= X_{ij} \qquad\qquad 1\le i,j\le n \\ &\quad P_{n+1}' = T \\ V_{N_2} &= X_{i,n+1} \\ &\quad P_i' \qquad\qquad 1\le i,j\le n \end{aligned} \tag{1.43}$$

Choosing $x^{n+2} = R$ and $x^{n+1} = ct$, the bases for the contracted algebra $\mathfrak{g}^{(2)}$ are given by

$$
\begin{aligned}
X_{ij} &= x^i \, \partial_j - x^j \, \partial_i \\
T &= \lim_{c/R \to 0} \frac{c}{R} (x^{n+1} \, \partial_{n+2} - x^{n+2} \, \partial_{n+1}) \to -\partial_t \\
V_i &= \lim_{1/c \to 0} \frac{1}{c} (x^i \, \partial_{n+1} - x^{n+1} \, \partial_i) \to -t \, \partial_i \\
P_i &= \lim_{1/c \to 0} \lim_{c/R \to 0} \frac{1}{c} \frac{c}{R} (x^i \, \partial_{n+2} - x^{n+2} \, \partial_i) \to -\partial_i \\
& \qquad 1 \leq i, j \leq n
\end{aligned}
\tag{1.44}
$$

Then the commutation relations for the contracted algebra $\mathfrak{g}^{(2)}$ are

$$
\begin{aligned}
&\left.\begin{aligned}
[X_{ij}, X_{rs}] &= \mathfrak{so}(n) \\
[X_{ij}, P_k] &= P_i \, \delta_{jk} - P_j \, \delta_{ik}
\end{aligned}\right\} \mathfrak{iso}(n) \\
&[X_{ij}, V_k] = V_i \, \delta_{jk} - V_j \, \delta_{ik} \\
&[X_{ij}, T] = 0 \\
&[P_i, P_j] = 0 \\
&[P_i, V_j] = 0 \\
&[P_i, T] = 0 \\
&[V_i, V_j] = 0 \\
&[V_i, T] = P_i
\end{aligned}
\tag{1.45}
$$

These are exactly the commutation relations for the Galilean group $G(n)$ when $n = 3$. The easiest way to see this is as follows. The Galilean group $G(3)$ consists of the operations described in Table 10.1. These group operations have the infinitesimal generators shown also in this table. A comparison of the generators in the table with the infinitesimal generators given in (1.44) reveals that the Galilean group corresponds to the special case $n = 3$ in this example.

Comment 1. The Galilean group is the nonrelativistic limit of the Poincaré group ($c \to \infty$). Since the Galilean group actually has a more complicated structure than the Poincaré group, the laws of physics actually have a simpler structure in their relativistically covariant form than in their nonrelativistic limit.

TABLE 10.1

Group Operations in the Galilean Group	Infinitesimal Generator	Structure of Generator
Rotations about a fixed point in the three spatial directions $i = 1, 2, 3$	$L_i = \varepsilon_{ijk} X_{jk}$	$L_i = x^j\,\partial_k - x^k\,\partial_j$ $i, j, k = 1, 2, 3$ cycl.
Displacements of the origin of coordinates in the three spatial directions	P_i	$P_i = -\partial_i$
Displacement of the origin of the time coordinate	T	$T = -\partial_t$
Boosts at fixed velocity in the three spatial directions	V_i	$V_i = -t\,\partial_i$

Comment 2. Galilean groups in pseudo-Euclidean spaces $G(n_1, n_2)$ can be constructed by contracting the inhomogeneous groups $ISO(n_1 + 1, n_2)$ or $ISO(n_1, n_2 + 1)$ when $n_1 \neq n_2$. (Fig. 10.6).

Example 6. The construction by contraction of the Galilean group, carried out in Example 5, is rather complicated because it involves two contractions. Physically, these correspond to the two limits

1. Noncosmological limit: We let the radius of the universe $R \to \infty$ in such a way that $c/R \to 0$:

$$\begin{tikzcd} SO(4,1) \arrow[dr] & \\ & ISO(3,1) \\ SO(3,2) \arrow[ur] & \end{tikzcd}$$

2. Nonrelativistic limit: We let the velocity of light $c \to \infty$:

$$\begin{tikzcd} ISO(3,1) \arrow[dr] & \\ & G(3) \\ ISO(4) \arrow[ur] & \end{tikzcd}$$

We ask, is there a simpler way to construct the Galilean group from a simple Lie group? In particular, is it possible to construct $G(n)$ from $SO(n + 2)$, or one of its related real forms, by means of a single Inönü-Wigner like contraction?

To answer this, we write down the commutation relations of the bases for $\mathfrak{so}(n+2)$ and present them, in (1.46), in the format of (1.45):

$$
\begin{array}{lrl}
 & [X_{ij}, X_{rs}] &= \mathfrak{so}(n) \\
 & [X_{ij}, X_{k,n+2}] &= X_{i,n+2}\,\delta_{jk} - X_{j,n+2}\,\delta_{ik} \\
 & [X_{ij}, X_{k,n+1}] &= X_{i,n+1}\,\delta_{jk} - X_{j,n+1}\,\delta_{ik} \\
 & [X_{ij}, X_{n+1,n+2}] &= 0 \\
(*) & [X_{i,n+2}, X_{j,n+2}] &= -X_{ij} \\
(*) & [X_{i,n+2}, X_{j,n+1}] &= +X_{n+1,n+2}\,\delta_{ij} \\
(*) & [X_{i,n+2}, X_{n+1,n+2}] &= -X_{i,n+1} \\
(*) & [X_{i,n+1}, X_{j,n+1}] &= -X_{ij} \\
 & [X_{i,n+1}, X_{n+1,n+2}] &= X_{i,n+2}
\end{array}
\tag{1.46}
$$

The commutators in (1.46) which differ from the corresponding commutators in (1.45) are labeled (*). The corresponding commutators in (1.45) are all zero.

The nonzero commutators in (1.46) labeled (*) can be made to vanish if we contract at least one of the arguments inside their Lie bracket. Since $[X_{i,n+1}, X_{j,n+1}] \neq 0$, we must contract the $X_{i,n+1}$. Similar arguments apply for $X_{i,n+2}$. If such a contraction is performed, three of the four equations (*) in (1.46) become zero, but

$$\lim_{\varepsilon \to 0} [\varepsilon X_{i,n+2}, X_{n+1,n+2}] = \lim_{\varepsilon \to 0} (-\varepsilon X_{i,n+1}) \neq 0 \tag{1.47}$$

Therefore we must also contract $X_{n+1,n+2}$. But then

$$\lim_{\varepsilon \to 0} [\varepsilon X_{i,n+1}, \varepsilon X_{n+1,n+2}] = \lim_{\varepsilon \to 0} \varepsilon(\varepsilon X_{i,n+2}) \to 0 \tag{1.48}$$

and we forfeit the commutation relations of the Galilean group.

From these arguments it is clear that a simple Inönü-Wigner contraction with respect to some generators X_{ij} of $\mathfrak{so}(n+2)$ does not give the Galilean algebra $\mathfrak{g}(n)$. Nor does it help to contract linear combinations such as $X_{i,n+1} \pm (i)X_{i,n+2}$. In spite of these arguments, the commutation relations (1.46) are so similar to (1.45), particularly after contraction of $X_{i,n+1}$ and $X_{i,n+2}$, that there should be a simple way to construct the Galilean algebra $\mathfrak{g}(n)$ from the algebra $\mathfrak{so}(n+2)$ by contraction.

The mechanism we desire has, in fact, already been suggested by the physical considerations in (1.42) and (1.43). It will easily be seen that a contraction of the form

$$\begin{aligned} X_{ij} &\to 1X_{ij} \\ X_{i,n+1} &\to \lim_{\varepsilon\to 0} \varepsilon X_{i,n+1} = V_i \\ X_{n+1,n+2} &\to \lim_{\varepsilon\to 0} \varepsilon X_{n+1,n+2} = T \\ X_{i,n+2} &\to \lim_{\varepsilon\to 0} \varepsilon^2 X_{i,n+2} = P_i \end{aligned} \tag{1.49}$$

immediately converts the commutation relations (1.46) for $\mathfrak{so}(n+2)$ into the commutation relations (1.45) for $\mathfrak{g}(n)$.

Such a contraction is not an Inönü-Wigner contraction because of the presence of the $\varepsilon^2 I_n$ term in the singular transformation $U(\varepsilon)$. However, since the singular transformation $U(\varepsilon)$ involved in this contraction has such a simple form, it is sometimes called a generalized Inönü-Wigner contraction.

Generalized Inönü-Wigner contractions may be studied in detail just as Inönü-Wigner contractions have been studied. The properties of such contractions are more complicated than those of the simpler Inönü-Wigner contractions, as there are many more "degrees of freedom" available. We will not study the properties of generalized Inönü-Wigner contractions any further, since we will study the general properties of contractions in a coordinate independent way in Section II.

Comment 1. The bases X_{ij}, P_i span an algebra isomorphic with $\mathfrak{iso}(n)$. The generators X_{ij}, V_i span a dual algebra, also isomorphic with $\mathfrak{iso}(n)$. This "duality" is often considered as a very surprising feature of the conformal group generators. It is, in fact, quite transparent from (1.46) how this comes about. The n generators $X_{i,n+1}$ and the n generators $X_{j,n+2}$ ($1 \leq i$, $j \leq n$) both transform like the components of an n-vector under commutation with the generators X_{ij} ($1 \leq i, j \leq n$), which span the subalgebra $\mathfrak{so}(n)$. The commutation relations with this subalgebra are not affected by the contraction process.

Comment 2. This contraction procedure can be carried out on noncompact real forms of $SO(n+2)$. Galilean groups in mixed-metric spaces can then result. The inhomogeneous indefinite-metric-preserving groups and the mixed-metric Galilean groups which can result from the contraction of $\mathfrak{so}(p, q)$ are summarized in Fig. 10.6.

Comment 3. A group whose algebra is $\mathfrak{g}(3, 1)\oplus\mathfrak{d}_0$ has recently been proposed as a candidate for a new dynamical group for relativistic quantum mechanics. It is easily verified that the algebra $\mathfrak{g}(3, 1)$ contains the algebras $\mathfrak{g}(3)$ and $\mathfrak{iso}(3, 1)$ as subalgebras. The actual group associated with the algebra is essentially $G(3, 1) \otimes D_0$, and is obtained by contraction from the universal covering group of $D_0 \otimes SO(4, 2) \cong U(2, 2)$.

4. THE BAKER–CAMPBELL–HAUSDORFF FORMULA. Let us choose the generators of $U(2)$ as follows:

$$\begin{aligned} [J_3, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= 2J_3 \\ [\mathbf{J}, \xi] &= 0 \end{aligned} \tag{1.50}$$

A convenient faithful 2×2 matrix representation is given by

$$\begin{array}{cccc} J_3 & \xi & J_+ & J_- \\ \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array} \tag{1.51}$$

These generators can be exponentiated to the group $Gl(2, c)$ and all its subgroups and real forms.

It is very useful[6] to perform the following change of basis

$$\begin{bmatrix} h_+ \\ h_- \\ h_3 \\ I \end{bmatrix} \cdot= \begin{bmatrix} a & & & \\ & a & & \\ & & 1 & +\dfrac{b}{2a^2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} J_+ \\ J_- \\ J_3 \\ \xi \end{bmatrix} \tag{1.52}$$

The commutation properties of the generators $\mathbf{h}$, I are

$$\begin{aligned} [h_3, h_\pm] &= \pm h_\pm \\ [h_+, h_-] &= 2a^2h_3 - bI \\ [\mathbf{h}, I] &= 0 \end{aligned} \tag{1.53}$$

In the limit $a \to 0$, the commutation relations are well defined, even though the transformation is singular. When $b = 0$, these are the commutation relations of the group $E(2) \otimes R$. When $b \neq 0$, it can always be renormalized to ± 1.

The algebra with the commutation relations $a = 0, b = 1$ is called $\mathfrak{h}_4$. The algebra $\mathfrak{h}_4$ occurs frequently in physics. We give two instances:

1. The single-mode photon number operator ($N = a^{\dagger}a$) and the creation ($a^{\dagger}$) and annihilation (a) operators obey the commutation relations

$$\begin{aligned} [N, a^{\dagger}] &= +a^{\dagger} \\ [N, a] &= -a \\ [a^{\dagger}, a] &= -I \end{aligned} \tag{1.54}$$

The algebra with generators N, $a^{\dagger}$, a, I is closed under commutation and isomorphic with $\mathfrak{h}_4$

$$\begin{aligned} h_3 &\leftrightarrow N = a^{\dagger}a \\ h_+ &\leftrightarrow a^{\dagger} \\ h_- &\leftrightarrow a \\ I &\leftrightarrow I \end{aligned} \tag{1.55}$$

2. The quantum mechanical position and momentum operators Q, P obey

$$[Q, P] = i\hbar I$$

The algebra generated by P, Q, I is isomorphic with a three-dimensional subalgebra of $\mathfrak{h}_4$:

$$\begin{aligned} h_+ &\leftrightarrow \frac{Q - iP}{\sqrt{2\hbar}} \\ h_- &\leftrightarrow \frac{Q + iP}{\sqrt{2\hbar}} \\ I &\leftrightarrow I \end{aligned} \tag{1.56}$$

In terms of P, Q the "number" operator h_3 is given by

$$h_3 = \frac{P^2 + Q^2}{2\hbar} \tag{1.57}$$

The close association of this algebra with the harmonic oscillator leads to the widespread use of $\mathfrak{h}_4$ in physics and in fact accounts for the isomorphism described in item 1.

We now return for a moment to the generators $\mathbf{J}$, ξ and prove the identity

$$\text{EXP } t(J_+ + J_-) = \text{EXP } (\tanh tJ_-) \text{ EXP } (2 \ln \cosh t)J_3 \text{ EXP } (\tanh tJ_+) \tag{1.58}$$

The proof of this within the 2×2 matrix representation (1.51) is simple. On the left-hand side we write

$$\text{EXP } t\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \tag{1.59}$$

On the right we have

$$\text{EXP}\begin{bmatrix}\ln\cosh t & 0\\ 0 & -\ln\cosh t\end{bmatrix} = \begin{bmatrix}\cosh t & 0\\ 0 & \dfrac{1}{\cosh t}\end{bmatrix} \tag{1.60}$$

$$\{\text{EXP}\tanh tJ_-\}^\dagger =$$

$$\text{EXP}\tanh tJ_+ = \text{EXP}\begin{bmatrix}0 & \tanh t\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & \tanh t\\ 0 & 1\end{bmatrix} \tag{1.61}$$

Multiplication of the terms appearing on the right-hand side gives immediately the left-hand side.

The identity (1.58) is valid within the 2×2 matrix representation, which is the defining representation for the group. Any identity in the group elements is valid within all representations of the group. The exponentials of the generators are group elements. Therefore, (1.58) is an identity within all representations of this group.

We now contract the identity (1.58) using (1.52) and assuming

$$\lim_{a\to 0}\frac{t}{a} = \beta \tag{1.62}$$

On the left we have

$$\text{EXP}\, t(J_+ + J_-) \to \text{EXP}\,\frac{t}{a}(aJ_+ + aJ_-) \to \text{EXP}\,\beta(h_+ + h_-) \tag{1.59$'$}$$

Similarly, we have the limit

$$\begin{aligned} 2\ln\cosh tJ_3 &\to \lim_{a\to 0}\ln\cosh^2 \beta a\left(h_3 - \frac{1}{2a^2}I\right)\\ &= \lim_{a\to 0}\ln\,(1 + a^2\beta^2)\left(h_3 - \frac{1}{2a^2}I\right)\\ &= \lim_{a\to 0} a^2\beta^2\left(h_3 - \frac{1}{2a^2}I\right)\\ &= -\frac{\beta^2}{2}I \end{aligned} \tag{1.60$'$}$$

The identity (1.58), which is valid for $Gl(2)$, contracts to the following identity, valid for H_4:

$$\text{EXP}\,\beta(h_+ + h_-) = \text{EXP}\,\beta h_-\;\; \text{EXP}\,\frac{\beta^2}{2}[h_+, h_-]\;\text{EXP}\,\beta h_+ \tag{1.58$'$}$$

Now let A, B be any two operators whose commutator $[A, B] = C$ commutes with both A and B

$$[A, B] = C$$
$$[A, C] = [B, C] = 0 \tag{1.63}$$

The operators A, B, C are closed under commutation and span an algebra isomorphic with a subalgebra of $\mathfrak{h}_4$:

$$h_+ \leftrightarrow A$$
$$h_- \leftrightarrow B$$
$$[h_+, h_-] = -I \leftrightarrow C = [A, B] \tag{1.64}$$

For the operators A, B, C this identity assumes the form

$$e^{\beta(A+B)} = e^{\beta B} e^{\beta^2/2[A, B]} e^{\beta A} \tag{1.65}$$

This formula is known as the Baker-Campbell-Hausdorff formula.

Comment. Other BCH formulas can be constructed in the same way, merely by starting from identities of the form of (1.58), satisfied by the generators of (simple) Lie algebras, and performing appropriate contractions. A large class of BCH relations for the group H_4 can be constructed easily by contracting the $Gl(2, c)$ BCH formulas described in Fig. 5.12.

II. Saletan[4] Contractions

In Section I of this chapter we studied a large class of simple contractions in an explicitly coordinate-dependent way. However, not all the nonsemisimple algebras used by physicists can be obtained from the semisimple algebras by an Inönü-Wigner contraction. One such example is the contraction

$$U(2) \rightarrow H_4$$

leading to the Baker-Campbell-Hausdorff formula.

We return now to a study of contractions from a coordinate-independent point of view. These are the so-called Saletan contractions; we shall follow closely Saletan's approach.[4]

1. THE FIRST CONTRACTION

A. Structure of U. Contraction is the study of the Lie bracket under a singular change of basis. Let V be a linear vector space on which a commutation is defined, such that $[V, V] \subset V$. That is, V is also a Lie algebra. Let U be a singular mapping of V into itself. Since U is singular, U annihilates a linear vector subspace V_1 of V. Let

$$\begin{aligned} V &= V_\perp{}^1 \oplus V_1 \\ UV_1 &= 0 \end{aligned} \tag{2.1}$$

Repeating these arguments, we see that U annihilates a linear vector subspace of UV:

$$V = V_\perp{}^2 \oplus V_2 \qquad U^2 V_2 = 0 \tag{2.2}$$

In general, we define

$$\begin{aligned} V &= V_\perp{}^m \oplus V_m \\ U^m V_m &= 0 \end{aligned} \tag{2.3}$$

Every time U acts on the space $U^m V$, the dimensionality of the subspace $U^{m+1} V$ either decreases or remains constant. Since V is finite dimensional, for m sufficiently large ($m = p$) this process must come to a halt, and we have

$$\begin{aligned} U^{k+p} V &= U^p V = V_R \\ V_{k+p} &= V_p = V_N \end{aligned} \tag{2.4}$$

where V_R is the subspace of V Remaining after U has been applied sufficiently many times and V_N is the Null part of V, the subspace annihilated by applying U sufficiently often.

The mapping U is 1-1 on V_R, for if it were not, it would annihilate a subspace of V_R. Since $U|_R$ (read: U restricted to V_R) is 1-1, $U|_R$ is faithful.

The action of U on V is revealed schematically in Fig. 10.7, which also indicates the general structure U may assume in a suitable choice of bases.

B. Necessary and Sufficient Conditions. We return to a study of the properties of the Lie bracket under an ε-dependent transformation which becomes singular in the limit $\varepsilon \to 0$.

$$\lim U^{-1}(\varepsilon)[U(\varepsilon)a,\ U(\varepsilon)b] = [a, b]' \tag{2.5}$$

Choose $U(\varepsilon)$ to have the structure

$$U(\varepsilon) = \varepsilon I + (1 - \varepsilon)U \tag{2.6}$$

We assume that U is as given in Fig. 10.7, and that $U^{-1}(\varepsilon)$ exists in the interval $(0, 1]$:

$$\begin{aligned} U^{-1}(\varepsilon) &= \left\{(1 - \varepsilon)\left[\frac{\varepsilon}{1 - \varepsilon} I + U\right]\right\}^{-1} \\ &= \frac{1}{1 - \varepsilon}(\kappa + U)^{-1} \end{aligned} \tag{2.7}$$

where $\kappa = \varepsilon I/(1 - \varepsilon)$.

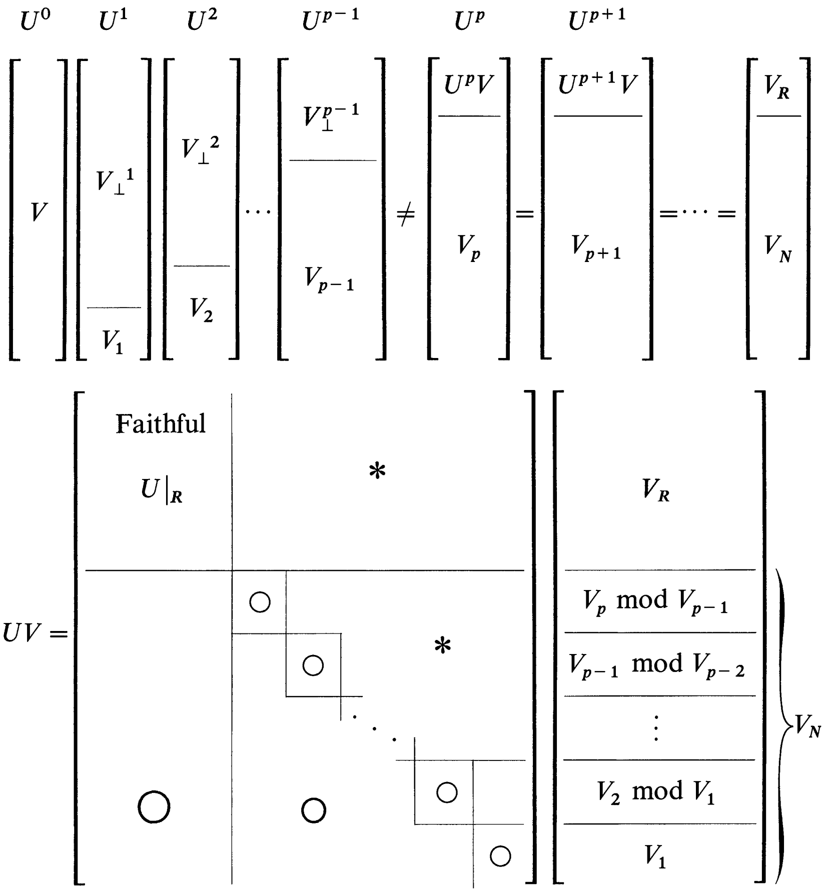


FIG. 10.7. THE ACTION OF THE LINEAR TRANSFORMATION U ON THE LINEAR VECTOR SPACE V IS SHOWN HERE. V IS GIVEN IN A CANONICAL MATRIX REPRESENTATION; THE BASES HAVE BEEN CHOSEN IN SUCH A WAY THAT THOSE NEAREST THE BOTTOM (IN THIS MATRIX REPRESENTATION) ARE ANNIHILATED SOONER. THOSE REMAINING AFTER REPEATED APPLICATION OF U FORM THE LINEAR VECTOR SUBSPACE V_R, AT THE TOP IN THIS PARTICULAR REPRESENTATION. THE TRANSFORMATION $U|_R$ IS FAITHFUL WHILE $U|_N$ IS NILPOTENT.

With this change in notation, the foregoing commutator becomes

$$\lim_{\varepsilon,\,\kappa\to 0} (1-\varepsilon)(\kappa+U)^{-1}[(\kappa+U)a, (\kappa+U)b] = [a, b]' \tag{2.8}$$

To study this equation more carefully, we decompose the commutator into its components within V_R and V_N

$$[(\kappa+U)a, (\kappa+U)b] = [(\kappa+U)a, (\kappa+U)b]_R + [(\kappa+U)a, (\kappa+U)b]_N \tag{2.9}$$

Then on the piece $[\ ,\ ]_R$, U is nonsingular, and we have

$$(\kappa+U)^{-1}|_R = U^{-1}|_R(\kappa U^{-1}|_R + I)^{-1} \tag{2.10}$$

$$\lim_{\kappa\to 0} (1-\varepsilon)(\kappa+U)^{-1}[(\kappa+U)a, (\kappa+U)b]_R \to U^{-1}[Ua, Ub]_R \tag{2.11}$$

On the component $[\ ,\ ]_N$ in V_N, U is singular; thus the expansion of (2.10) is not valid. We must use another expansion:

$$(\kappa+U)^{-1} = \kappa^{-1}\left(I+\frac{U}{\kappa}\right)^{-1} = \frac{1}{\kappa}\sum\left(-\frac{U}{\kappa}\right)^k \tag{2.12}$$

With this expansion the limit (2.8) can be carried out:

$$\begin{aligned}\lim_{\kappa\to 0}\frac{1}{\kappa}\sum\left(-\frac{U}{\kappa}\right)^k &\{\kappa^2[a, b] + \kappa[Ua, b] + \kappa[a, Ub] + [Ua, Ub]\}_N \\ &= -U[a, b]_N + [Ua, b]_N + [a, Ub]_N \\ &\quad + \left(\frac{1}{\kappa} - \frac{U}{\kappa^2} + \frac{U^2}{\kappa^3}\cdots\right)\{[Ua, Ub] - U[Ua, b] \\ &\quad - U[a, Ub] + U^2[a, b]\}_N \end{aligned} \tag{2.13}$$

The limit exists if and only if the terms within the curly brackets vanish.

The commutation relations (CR) and the necessary and sufficient conditions (NSC) are then given by

$$\begin{aligned}[a, b]^{(1)} &= U^{-1}[Ua, Ub]_R + \nabla_U[a, b]_N && \text{CR}\\ 0 &= [Ua, Ub]_N - U\nabla_U[a, b]_N && \text{NSC}\\ -\nabla_U[a, b]_N &\equiv U[a, b]_N - [Ua, b]_N - [a, Ub]_N\end{aligned}$$

Comment. The NSC can be rewritten

$$U^2[a, b]_N - U[Ua, b]_N = U[a, Ub]_N - [Ua, Ub]_N \tag{2.14}$$

This equation is valid when arbitrary powers of U are applied on both sides:

$$\begin{aligned} U^r[a, b] - U^{r-1}[Ua, b] &= U^{r-1}[a, Ub] - U^{r-2}[Ua, Ub] \\ U^{r-1}[a, Ub] - U^{r-2}[Ua, Ub] &= U^{r-2}[a, U^2b] - U^{r-3}[Ua, U^2b] \\ &\vdots \\ &= U^{r-t}[a, U^tb] - U^{r-t-1}[Ua, U^tb] \end{aligned} \tag{2.15}$$

Interchanging the middle terms and proceeding again by induction leads to

$$\begin{aligned} U^r[a, b] - U^{r-t}[a, U^tb] &= U^{r-1}[Ua, b] - U^{r-t-1}[Ua, U^tb] \\ &\vdots \\ &= U^{r-p}[U^pa, b] - U^{r-t-p}[U^pa, U^tb] \end{aligned} \tag{2.16}$$

The result is a very elegant statement for the necessary and sufficient conditions:

$$U^{p+q}[a, b]_N - U^p[a, U^qb]_N = U^q[U^pa, b]_N - [U^pa, U^qb]_N \tag{2.17}$$

C. Structure of $\mathfrak{g}^{(0)}$. We will prove two simple theorems about the structure of $\mathfrak{g}^{(0)}$.

THEOREM. *If* U *contracts* $\mathfrak{g}^{(0)}$, *then* U^m *does also.*

Proof. We must show that U^m satisfies the NSC. Putting $p = q = m$ in (2.17) gives

$$(U^m)^2[a, b]_N - (U^m)[U^ma, b]_N - (U^m)[a, U^mb]_N + [U^ma, U^mb]_N = 0 \tag{2.18}$$

Since the NSC are satisfied for U^m, the commutation relations are given by

$$[a, b]^{(m)} = U^{-m}[U^ma, U^mb]_R + \nabla_{U^m}[a, b]_N \tag{2.19}$$

Symbolically, we write

$$\mathfrak{g}^{(m)} = U^m \cdot \mathfrak{g}^{(0)} \tag{2.20}$$

THEOREM. $\mathrm{U}^1\mathrm{V}$ *is a subalgebra of* $\mathfrak{g}^{(0)}$.

Proof. Choose vectors a', b' in U^1V. Then there are vectors a, b in V such that $a' = Ua$, $b' = Ub$.

$$\begin{aligned} &[Ua, Ub] = [Ua, Ub]_R + [Ua, Ub]_N \\ &[Ua, Ub]_R \in V_R \subset U^1V \\ &[Ua, Ub]_N = U(\nabla_U[a, b]_N) \subset U^1V \end{aligned} \tag{2.21}$$

Therefore, $U^1 V$ is closed under commutation. By the same token, $U^m V$ is a subalgebra of $\mathfrak{g}^{(0)}$. In particular, V_R is a subalgebra of $\mathfrak{g}^{(m)}$, for every value of m.

D. Structure of $\mathfrak{g}^{(1)}$

THEOREM. V_1 *is an invariant subalgebra of* $\mathfrak{g}^{(1)}$.

Proof. We must show that for any vector $a \in V_1$ and any arbitrary vector $b \in \mathfrak{g}^{(1)}$, $[a, b]^{(1)} \in V_1$. But

$$\begin{aligned}[a, b]^{(1)} &= U^{-1}[Ua, Ub]_R - U[a, b]_N + [Ua, b]_N + [a, Ub]_N \\ &= \qquad\qquad\qquad\quad - U[a, b]_N \qquad\qquad\quad + [a, Ub]_N \end{aligned} \tag{2.22}$$

If the commutator is in V_1, then U^1 must annihilate it. Applying U to the commutator and using NSC, we find

$$\begin{aligned} U[a, b]^{(1)} &= -U^2[a, b]_N + U[a, Ub]_N \\ &= -U[Ua, b]_N + [Ua, Ub]_N = 0 \end{aligned} \tag{2.23}$$

THEOREM. *The subalgebra* $V_1 \subset \mathfrak{g}^{(1)}$ *is* solvable.

Proof. We must show that the sequence of derived algebras ends in zero. Let a, b be in V_1. Then $Ua = Ub = 0$.

$$\begin{aligned}[a, b]^{(1)} &= U^{-1}[Ua, Ub]_R - U[a, b]_N + [Ua, b]_N + [a, Ub]_N \\ &= \qquad\qquad\qquad\quad - U[a, b]_N \end{aligned} \tag{2.24}$$

Since V_1 is an invariant subalgebra, $[a, b]_N^{(1)} \in V_1$. Therefore, the derived algebra is in both V_1 and UV:

$$V_1^{(1)} = [V_1, V_1]' \subset V_1 \cap UV = UV_2 \tag{2.25}$$

Now choose $a, b \in UV_2 \subset UV$. Then U annihilates both a and b:

$$[a, b]^{(1)} = -\underbrace{[a, b]_N}_{UV} \in V_1 \cap U^2 V = U^2 V_3 \tag{2.26}$$

Proceeding in this way, choose $a, b \in U^k V_{k+1} \subset U^k V$. Then U annihilates both a and b:

$$[a, b]^{(1)} = -U\underbrace{[a, b]_N}_{U^k V} \in V_1 \cap U^{k+1} V = U^{k+1} V_{k+2} \tag{2.27}$$

In short, we have a sequence of derived algebras

$$
\begin{aligned}
(V_1)' &\subset U^1 V_2 \\
(U^1 V_2)' &\subset U^2 V_3 \\
(U^2 V_3)' &\subset U^3 V_4 \\
&\vdots \\
(U^{p-1} V_p)' &\subset U^p V_{p+1} = U^p V_p = 0
\end{aligned} \tag{2.28}
$$

The sequence of derived algebras ends with zero, since for sufficiently large p, $V_{p+1} = V_p$. Therefore, V_1 is solvable. Since it is also an invariant subalgebra of $\mathfrak{g}^{(1)}$, $\mathfrak{g}^{(1)}$ is nonsemisimple.

2. FURTHER CONTRACTIONS

A. Commutativity of Contractions. We will now show that the algebra $\mathfrak{g}^{(j)}$ can be contracted by U^i. The result is $\mathfrak{g}^{(i+j)}$:

$$U^i \mathfrak{g}^{(j)} = U^{i+j} \mathfrak{g}^{(0)} = U^j \mathfrak{g}^{(i)} = \mathfrak{g}^{(i+j)} \tag{2.29}$$

First we show that the NSC are satisfied. Then we show that the algebra $U^i \mathfrak{g}^{(j)}$ has CR isomorphic with $\mathfrak{g}^{(i+j)}$.

Since the verification that the NSC are satisfied is somewhat involved, we set it up in the following way. The four terms whose sum must vanish are listed in the left-hand column. They are written out explicitly, using CR, in the matrix array. The column sums of the matrix are indicated below each column of the matrix. Since each column sum vanishes separately, by (2.17), the NSC are satisfied.

$$
\begin{array}{rcllll}
U^{2i}[a, b]_N^{(j)} & = & U^{2i}\{-U^j[a, b] & + [U^j a, b] & & + [a, U^j b]\}_N \\
+ & & & & & \\
-U^i[U^i a, b]_N^{(j)} & = & -U^i\{-U^j[U^i a, b] & + [U^{i+j} a, b] & & + [U^i a, U^j b]\}_N \\
+ & & & & & \\
-U^i[a, U^i b]_N^{(j)} & = & -U^i\{-U^j[a, U^i b] & + [U^j a, b] & & + [a, U^{i+j} b]\}_N \\
+ & & & & & \\
[U^i a, U^i b]_N^{(j)} & = & \{-U^j[U^i a, U^i b] & + [U^{i+j} a, U^i b] & & + [U^i a, U^{i+j} b]\}_N \\
\| & & \| & \| & & \| \\
0 & = & -U^j f(a, b) & + f(U^j a, b) & & + f(a, U^j b)
\end{array}
$$

$$f(a, b) \equiv U^{2i}[a, b]_N - U^i[U^i a, b]_N - U^i[a, U^i b]_N + [U^i a, U^i b]_N = 0 \tag{2.30}$$

The CR are verified with about the same amount of difficulty.

$$\{[a, b]^j\}^{(i)} = U^{-i}[U^i a, U^i b]_R^{(j)} - U^i[a, b]_N^{(j)} + [U^i a, b]_N^{(j)} + [a, U^i b]_N^{(j)} \tag{2.31}$$

For the term in V_R we have

$$\begin{aligned} U^{-i}[U^i a, U^i b]_R{}^j &= U^{-i}\{U^{-j}[U^j(U^i a), U^j(U^i b)]\}_R \\ &= U^{-(i+j)}[U^{i+j}a, U^{i+j}b]_R \end{aligned} \tag{2.32}$$

For the terms in V_N we find

$$\begin{aligned} -U^i[a, b]_N{}^j &= \underline{U^{i+j}[a, b]_N} \quad - \overline{U^i[U^j a, b]_N} - \underline{U^i[a, U^j b]_N} \\ +[U^i a, b]_N{}^j &= -\underline{U^j[U^i a, b]_N} + [U^{i+j}a, b]_N + \underline{[U^i a, U^j b]_N} \\ +[a, U^i b]_N{}^j &= -\overline{U^j[a, U^i b]_N} + \overline{[U^j a, U^i b]_N} + [a, U^{i+j}b]_N \end{aligned} \tag{2.33}$$

The underlined terms vanish by (2.17). The overlined terms give $-U^{i+j}[a, b]_N$ for the same reason. The result is

$$\begin{aligned} \{[a, b]^j\}^{(i)} = U^{-(i+j)}[U^{i+j}a, U^{i+j}b]_R \\ -U^{i+j}[a, b]_N + [U^{i+j}a, b]_N + [a, U^{i+j}b]_N \end{aligned} \tag{2.34}$$

Therefore, contractions commute.

B. Structure of $\mathfrak{g}^{(i)}$. In this section we prove three simple results shedding light on the contracted algebras $\mathfrak{g}^{(i)}$.

THEOREM. V_i *is an invariant subalgebra of* $\mathfrak{g}^{(i+j)}$ $(j \geq 0)$.

Proof. Let $a \in V_i$. We need to show that, for arbitrary $b \in V$, $[a, b]^{(m)} \in V_i$ when $m \geq i$. But

$$\begin{aligned} U^i[a, b]^{(m)} &= U^i U^{-m}[U^m a, U^m b]_R - U^{m+i}[a, b]_N + U^i[U^m a, b]_N + U^i[a, U^m b]_N \\ &= -U^{m+i}[a, b]_N + U^i[a, U^m b]_N \\ &= -U^m[U^i a, b]_N + [U^i a, U^m b]_N = 0 \end{aligned} \tag{2.35}$$

We denote the invariant subalgebra V_i in $\mathfrak{g}^{(i+j)}$ by $\mathfrak{g}_i^{(i+j)}$.

THEOREM. *The derived algebra of* $\mathfrak{g}_i^{(i+j)}$ *is contained in* V_{i-j}.

Proof. Let $a, b \in \mathfrak{g}_i^{(i+j)}$. We must show U^{i-j} annihilates their commutator:

$$[a, b]^{(i+j)} = -U^{i+j}[a, b]_N \tag{2.36}$$

Applying U^{i-j} we have

$$-U^{2i}[a, b]_N = -[U^i a, b]_N - [a, U^i b]_N + [U^i a, U^i b]_N = 0 \tag{2.37}$$

THEOREM. $\mathfrak{g}_i^{(2i+k)}$ *is abelian*, $k \geq 0$.

Proof. Put $j = i + k$ in the previous theorem. Then $(\mathfrak{g}_i^{(2i+k)})' \subset V_{-k} = 0$. Since the derived algebra is zero, the algebra is abelian.

C. Termination of Contractions. Let p be the smallest integer for which $V_R = U^p V$. Then we will prove that the contraction process terminates beyond p. That is,

$$\mathfrak{g}^{(p)} \cong \mathfrak{g}^{(p+i)} \qquad i \geq 0 \tag{2.38}$$

To do this, we must compare commutation relations in $\mathfrak{g}^{(p)}$ and $\mathfrak{g}^{(p+i)}$. We do this in three stages.

1. Let $a, b \in V_R$. Then since V_R is a subalgebra in $\mathfrak{g}^{(p+i)}$, we have

$$[a, b]^{(p+i)} = U^{-i}U^{-p}[U^p(U^i a), U^p(U^i b)]_R \tag{2.39}$$

2. Let $a \in V_R$, $b \in V_N$. Then

$$\begin{aligned} [a, b]^{(p+i)} &= -U^{p+i}[a, b]_N + [U^{p+i}a, b]_N \\ &= [U^p(U^i a), b]_N \end{aligned} \tag{2.40}$$

3. Let $a, b \in V_N$. Then

$$[a, b]^{(p+i)} = -U^{p+i}[a, b]_N = 0 \tag{2.41}$$

Since U is 1-1 on V_R, so are all integral powers of U. Under the isomorphism

in $\mathfrak{g}^{(p+i)}$		in $\mathfrak{g}^{(p)}$
$U^i a$	$\leftrightarrow$	a
$U^i b$	$\leftrightarrow$	b

(2.42)

the commutation relations in $\mathfrak{g}^{(p)}$ and $\mathfrak{g}^{(p+i)}$ are identical. Therefore, all algebras contracted by U beyond $\mathfrak{g}^{(p)}$ are isomorphic. The only nonisomorphic algebras that can be constructed by contracting with U are

$$\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, \ldots, \mathfrak{g}^{(p)}$$

The sequence of distinct contracted algebras terminates beyond $\mathfrak{g}^{(p)}$. All algebras $\mathfrak{g}^{(i)}$ have the same dimension. All are nonsemisimple with the possible exception of the starting algebra $\mathfrak{g}^{(0)}$. It can be verified that the algebra $\mathfrak{g}^{(p)}$ can be obtained directly from $\mathfrak{g}^{(0)}$ by a simple Inönü-Wigner contraction using

$$\begin{aligned} U(\varepsilon)V_R &\to V_R \\ U(\varepsilon)V_N &\to \varepsilon V_N \end{aligned}$$

D. Application. We choose as algebra $\mathfrak{g}^{(0)}$ the simple algebra $\mathfrak{so}(n+2)$. It is convenient to label the bases for the underlying vector space of the algebra as follows

$$\begin{aligned} X_{ij} &= x^i \partial_j - x^j \partial_i \\ V_i &= X_{i,n+1} + iX_{i,n+2} \\ P_i &= X_{i,n+1} - iX_{i,n+2} \\ T &= \frac{1}{2i} X_{n+1,n+2} \qquad 1 < i, j < n \end{aligned} \tag{2.43}$$

Then we contract $\mathfrak{so}(n+2)$ with the mapping

$$\begin{aligned} UX_{ij} &\to X_{ij} \\ UV_i &\to P_i \\ UP_i &\to 0 \\ UT &\to 0 \end{aligned} \tag{2.44}$$

The subspace V_1 is spanned by the generators P_i and T.

The subspace V_2 mod V_1 is spanned by the generators V_i.

The subspace V_N is the linear closure of V_i, P_i, and T, while the subspace V_R is spanned by the X_{ij}.

It may be verified by direct computation that NSC are satisfied. Therefore $\mathfrak{so}(n+2)$ may be contracted by either U or U^2. Contraction by U^2 is equivalent to Inönü-Wigner contraction. Therefore our interest centers on $\mathfrak{g}^{(1)} = U \cdot \mathfrak{g}^{(0)} = U \cdot \mathfrak{so}(n+2)$. The commutation relations which the generators (2.43) obey may be computed directly using CR. In fact, in $\mathfrak{g}^{(1)}$ (but not $\mathfrak{g}^{(0)}$ or $\mathfrak{g}^{(2)}$) the generators (2.43) obey the commutation relations (1.45). Therefore $U \cdot \mathfrak{so}(n+2)$ is isomorphic with the Galilean algebra $\mathfrak{g}(n)$.

It may seem to the reader that we have rather belabored the Galilean group. This feeling is justified; but our belaboring is justified, too. The Galilean group is the most complicated nonsemisimple group familiar to physicists. Most groups appearing in physical applications are either (semi-) simple or can be obtained from simple groups by a simple Inönü-Wigner contraction.

3. SALETAN CONTRACTIONS IN A KUPCZYNSKI[7] BASIS. The results for Saletan contractions have been presented in a coordinate-free version in the previous sections. In down-to-earth applications, it is useful to display the commutation relations for the contracted algebras in a concrete, explicitly coordinate-dependent form. To this end, we must choose a suitable coordinate system in V. In terms of this system, the commutation relations

in $\mathfrak{g}^{(0)}$, as well as all contracted algebras $\mathfrak{g}^{(i)}$, assume a very simple form. We follow closely the procedure used by Kupczynski.[7]

A. Choice of a Basis. First of all, we choose a basis for V_R. From previous results, we know that U is 1-1 on V_R and, in addition, does not change the algebraic structure of V_R. Therefore, without loss of generality we can choose $U|_R$ as the identity operator I_R. For bases $X_r, X_s, \ldots, X_n$ spanning V_R,

$$UX_r = X_r$$

Next, we can choose $U|_N$ to have a Jordan canonical form

$$U|_N \xrightarrow[\text{form}]{\substack{\text{Jordan}\\ \text{canonical}}} \begin{bmatrix} M(\lambda_1) & & & \\ & M(\lambda_2) & & \\ & & M(\lambda_3) & \\ & & & \ddots \end{bmatrix} \tag{2.45}$$

Each of the submatrices on the diagonal has the structure

$$M(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \lambda_i & & \\ & & & & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix} \tag{2.46}$$

Since $U|_N$ annihilates V_N after suitably many applications (say, p), U is nilpotent on V_N and has only zero eigenvalues. Then each $M(\lambda_i = 0)$ has the following effect on the bases spanning the appropriate subspace.

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} X_{\alpha,1} \\ X_{\alpha,2} \\ \vdots \\ X_{\alpha,\alpha-1} \\ X_{\alpha,\alpha} \end{bmatrix} = \begin{bmatrix} X_{\alpha,2} \\ X_{\alpha,3} \\ \vdots \\ X_{\alpha,\alpha} \\ 0 \end{bmatrix} \tag{2.47}$$

$$U^r X_{\alpha,s} = X_{\alpha,s+r}$$
$$X_{\alpha,\alpha+1} = X_{\alpha,\alpha+2} = \cdots = 0 \tag{2.48}$$

Here α is to be interpreted as the dimensionality of an invariant subspace of V_N on which U acts. In short, V_N is partitioned into invariant subspaces of dimensionality $\alpha, \beta, \gamma, \ldots$. The subspace of largest dimensionality has dimension p. Clearly

$$\begin{array}{ll} X_{\alpha,\alpha}, X_{\beta,\beta}, \ldots, X_{\omega,\omega} & \text{span } V_1 \\ X_{\alpha,\alpha-1}, X_{\beta,\beta-1}, \ldots, X_{\omega,\omega-1} & \text{span } V_2 \bmod V_1 \end{array}$$

With respect to such a choice of basis vectors, the transformation U has the structure shown in Fig. 10.8. It is therefore easy to see that

$$\dim V_1 \geq \dim V_2 \bmod V_1 \geq \dim V_3 \bmod V_2 \geq \cdots$$
$$\dim V_{i+1} \bmod V_i = \dim V_{i+1} - \dim V_i \tag{2.49}$$

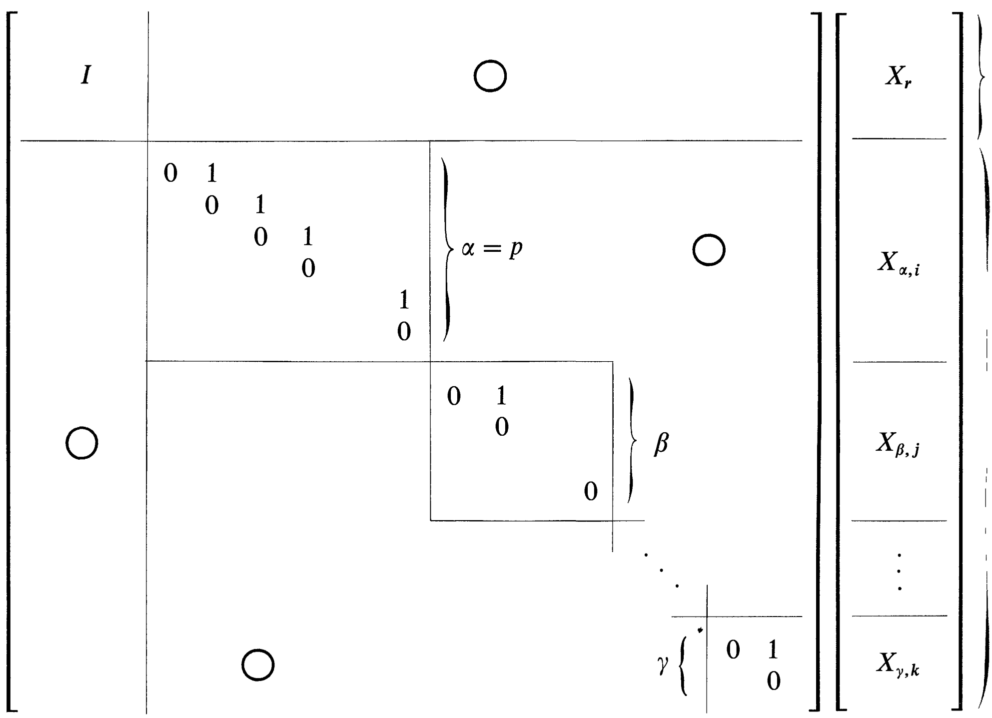


FIG. 10.8. THE SINGULAR TRANSFORMATION U CAN ALWAYS BE TRANSFORMED, BY A SUITABLE CHOICE OF BASES, TO THE CANONICAL FORM ILLUSTRATED, WHERE

$U|_R$ IS THE IDENTITY, AND $UX_r = X_r$, AND

$U|_N$ IS NILPOTENT, AND $UX_{\alpha,\iota} = X_{\alpha,\iota+1}$.

With respect to this canonical choice of bases, the structure constants are defined by

$$
\begin{aligned}
[X_r, X_s]^{(0)} &= c_{r,s}^{(0)t} X_t + c_{r,s}^{(0)\gamma k} X_{\gamma k} \\
[X_r, X_{\beta j}]^{(0)} &= c_{r;\beta j}^{(0)t} X_t + c_{r;\beta j}^{(0)\gamma k} X_{\gamma k} \\
[X_{\alpha i}, X_{\beta j}]^{(0)} &= c_{\alpha i;\beta j}^{(0)t} X_t + c_{\alpha i;\beta j}^{(0)\gamma k} X_{\gamma k}
\end{aligned} \tag{2.50}
$$

B. Necessary and Sufficient Conditions. The structure constants given in (2.50) are not all independent. They must be such that $\mathfrak{g}^{(0)}$ can be contracted by U. In other words, they must obey the NSC, transcribed into a coordinate-dependent form. For example, let $a \in V_R$ and $b \in V_N$ as follows:

$$a = X_r \qquad \text{and} \qquad b = X_{\beta j}$$

Then NSC becomes

$$
\begin{aligned}
U^2[X_r, X_{\beta j}]_N - U[UX_r, X_{\beta j}]_N - U[X_r, UX_{\beta j}]_N + [UX_r, UX_{\beta j}]_N = 0 \\
c_{r;\beta j}^{(0)\gamma k} X_{\gamma, k+2} - c_{r;\beta j}^{(0)\gamma k} X_{\gamma, k+1} - c_{r;\beta j+1}^{(0)\gamma k} X_{\gamma, k+1} + c_{r;\beta j+1}^{(0)\gamma k} X_{\gamma k} = 0
\end{aligned} \tag{2.51}
$$

Since the bases $X_{\gamma,k}$ are linearly independent, (2.51) gives a relation between the structure constants. This expression can be rearranged in a form in which it resembles a recursion relation on the indices j, k:

$$c_{r;\beta,j+1}^{(0)\gamma,k+1} - c_{r;\beta j}^{(0)\gamma,k+0} = c_{r;\beta,j+1}^{(0)\gamma,k+2} - c_{r;\beta j}^{(0)\gamma,k+1} \tag{2.52}$$

With $k = \gamma$, the right-hand side vanishes because $X_{\gamma,\gamma+1} \equiv 0$. Then (2.52) reduces to

$$-c_{r;\beta j}^{(0)\gamma,\gamma} = 0 \tag{2.53}$$

With $k = \gamma - 1$, the right-hand side of (2.52) still vanishes because of (2.53).

$$c_{r;\beta,j+1}^{(0)\gamma,\gamma} - c_{r;\beta j}^{(0)\gamma,\gamma-1} = 0 \tag{2.54}$$

Continuing along these lines, we obtain the condition

$$c_{r;\beta,j+1}^{(0)\gamma,k+1} - c_{r;\beta j}^{(0)\gamma,k} = 0 \tag{2.55}$$

Similar conditions can be explicitly worked out for the cases $a, b \in V_R$ and $a, b \in V_N$. The results are

$$c_{r,s}^{(0)\gamma k} = 0 \tag{2.56a}$$

$$c_{r;\beta,j+1}^{(0)\gamma,k+1} = c_{r;\beta j}^{(0)\gamma k} \tag{2.56b}$$

$$c_{\alpha,i+1;\beta,j+1}^{(0)\gamma,k+2} - c_{\alpha,i;\beta,j+1}^{(0)\gamma,k+1} - c_{\alpha,i+1;\beta,j}^{(0)\gamma,k+1} + c_{\alpha,i;\beta,j}^{(0)\gamma,k} = 0 \tag{2.56c}$$

The first condition is clear: V_R is closed under commutation and forms a subalgebra in $\mathfrak{g}^{(m)}$, $m = 0, 1, 2, \ldots, p$.

Since the NSC for further contractions by U all arise directly from the conditions [NSC or (2.17)], these further conditions add no new information to the constraints given in (2.56).

C. The Contracted Commutation Relations. The structure constants for the first contraction can now be computed in a straightforward way.

1. $a, b \in V_R$

$$[X_r, X_s]^{(1)} = U^{-1}[UX_r, UX_s]_R - U[X_r, X_s]_N + [UX_r, X_s]_N + [X_r, UX_s]_N$$

Since V_R is closed under commutation, the projection of these commutators in V_N is zero.

$$\begin{aligned} c_{r,s}^{(1)t} &= c_{r,s}^{(0)t} \\ c_{r,s}^{(1)\gamma k} &= 0 \end{aligned} \tag{2.57}$$

2. $a \in V_R$, $b \in V_N$

$$\begin{aligned}[X_r, X_{\beta j}]^{(1)} &= U^{-1}[UX_r, UX_{\beta j}]_R - U[X_r, X_{\beta j}]_N + [UX_r, X_{\beta j}]_N \\ &\quad + [X_r, UX_{\beta j}]_N \\ &= c_{r;\beta,j+1}^{(0)t} X_t - c_{r;\beta j}^{(0)\gamma k} X_{\gamma,k+1} + (c_{r;\beta j}^{(0)\gamma k} + c_{r;\beta,j+1}^{(0)\gamma k}) X_{\gamma k} \end{aligned} \tag{2.58}$$

The structure constants are then

$$\begin{aligned} c_{r;\beta j}^{(1)t} &= c_{r;\beta,j+1}^{(0)t} \\ c_{r;\beta j}^{(1)\gamma k} &= (c_{r;\beta,j+1}^{(0)\gamma k} - c_{r;\beta j}^{(0)\gamma,k-1}) + c_{r;\beta j}^{(0)\gamma k} = c_{r;\beta j}^{(0)\gamma k} \end{aligned} \tag{2.59}$$

The term in parentheses vanishes by (2.55).

3. $a, b \in V_N$

$$\begin{aligned}[X_{\alpha i}, X_{\beta j}]^{(1)} &= U^{-1}[UX_{\alpha i}, UX_{\beta j}]_R - U[X_{\alpha i}, X_{\beta j}]_N \\ &\quad + [UX_{\alpha i}, X_{\beta j}]_N + [X_{\alpha i}, UX_{\beta j}]_N \\ &= c_{\alpha,i+1;\beta,j+1}^{(0)t} X_t + \{-c_{\alpha i;\beta j}^{(0)\gamma,k-1} + c_{\alpha,i+1;\beta j}^{(0)\gamma k} + c_{\alpha i;\beta,j+1}^{(0)\gamma k}\} X_{\gamma k} \\ &= c_{\alpha,i+1;\beta,j+1}^{(0)t} X_t + c_{\alpha,i+1;\beta,j+1}^{(0)\gamma,k+1} X_{\gamma k} \end{aligned} \tag{2.60}$$

The structure constants are

$$\begin{aligned} c_{\alpha i,\beta j}^{(1)t} &= c_{\alpha,i+1;\beta,j+1}^{(0)t} \\ c_{\alpha i,\beta j}^{(1)\gamma k} &= c_{\alpha,i+1;\beta,j+1}^{(0)\gamma,k+1} \end{aligned} \tag{2.61}$$

The structure constants for $\mathfrak{g}^{(m)}$ can be obtained by replacing U by U^m everywhere in the calculations above. We summarize the results

$$c^{(m)t}_{r,s} = c^{(0)t}_{r,s}; \qquad c^{(m)\gamma k}_{r,s} = c^{(0)\gamma k}_{r,s} = 0 \tag{2.62a}$$

$$c^{(m)t}_{r;\beta j} = c^{(0)t}_{r;\beta j+m}; \qquad c^{(m)\gamma k}_{r;\beta j} = c^{(0)\gamma k}_{r;\beta j} \tag{2.62b}$$

$$c^{(m)t}_{\alpha i;\beta j} = c^{(0)t}_{\alpha, i+m;\beta, j+m}; \qquad c^{(m)\gamma k}_{\alpha i;\beta j} = c^{(0)\gamma, k+m}_{\alpha, i+m;\beta, j+m} \tag{2.62c}$$

The second relation of (2.62c) is valid only when $k + m \leq \gamma$. When $k + m > \gamma$, the result is

$$c^{(m)\gamma, \gamma-c}_{\alpha i;\beta j} = c^{(0)\gamma, \gamma}_{\alpha, i+m;\beta, j+c} + c^{(0)\gamma, \gamma}_{\alpha, i+c;\beta, j+m} - c^{(0)\gamma, \gamma-m+c}_{\alpha, i+c;\beta, j+c} \tag{2.62c$'$}$$

It is clear that for $m \geq p$

$$c^{(m)t}_{\alpha i,\beta j} = 0 \qquad \text{and} \qquad c^{(m)\gamma k}_{\alpha i,\beta j} = 0 \tag{2.62c$''$}$$

That is, V_N is an abelian invariant subalgebra.

Various additional relationships exist between the structure constants for $\mathfrak{g}^{(0)}$. These have been given by Kupczynski,[7] and they have the form, for example, of

$$c^{(0)\gamma k}_{r;\beta j} = \begin{cases} c^{(0)\gamma, k-j+1}_{r;\beta 1} & k \geq j \\ 0 & k < j \end{cases}$$

$$c^{(0)\gamma, k-j-1}_{r;\beta, \beta-j} = 0$$

The similar relations can all be obtained by starting from (2.17).

III. Expansions

There is a kind of duality in the information that is available about different kinds of Lie groups. There is available a complete classification scheme for the semisimple groups, but it is not yet known how to construct complete systems of irreducible representations for all their noncompact real forms. On the other hand, a construction process for all inequivalent representations of nilpotent groups exists, although there is as yet no classification scheme for all possible nilpotent and nonsemisimple groups.

Briefly, this means that it is sometimes more convenient to work with semisimple groups, whereas at other times it is simpler to work with nonsemisimple groups. In the earlier part of this chapter we have seen how to construct nonsemisimple groups from semisimple ones by contraction. Is it possible, perhaps, to construct semisimple groups from nonsemisimple groups by "expansion"? By expansion we mean the replacement of some of the infinitesimal generators in the subspace V_N of a nonsemisimple group by functions of the generators of the group. If these function-generators, together with the unaltered generators in V_R, close under commutation to a simple algebra, then we have "expanded" the original group.

We illustrate the expansion procedure by a number of concrete examples before giving a general discussion.

1. EXAMPLES OF USEFUL EXPANSIONS

A. The Little Prince Again. The Little Prince may realize that his home looks flat merely because his size is small compared with its characteristic dimensions ($d/R \ll 1$). He might then ask himself "What curved spaces will look to me like R^2 when I look at a small enough region?"

To put it differently, the Little Prince realizes that the isometries of his space have infinitesimal generators

P_1	displacements in the x direction
P_2	displacements in the y direction
Z	rotations about the z-axis

$$\begin{aligned} [Z, P_1] &= -P_2 \\ [Z, P_2] &= +P_1 \\ [P_1, P_2] &= 0 \end{aligned} \tag{3.1}$$

The group operations corresponding to P_1, P_2 leave no point fixed, the rotations around the z-axis leave his origin of coordinates fixed. The plane R_2 is a coset space $ISO(2)/SO(2)$. To find a curved space which looks like R_2 in the limit d small, we look for a coset $G/SO(2)$. We demand that G be (semi-)simple. Clearly, $SO(2) \subset G$. In addition,

$$\dim \left\{\frac{G}{SO(2)}\right\} = 2 \qquad \dim G = 3 \tag{3.2}$$

Since we already know that the only simple algebra of dimension 3 is $\mathfrak{a}_1$, we know $G = SU(2) \cong SO(3)$ or $G = SU(1, 1) \cong SO(2, 1)$. The question now is, can we find functions $f_i(Z, \mathbf{P})$ $(i = 1, 2)$ which, together with Z, close under commutation to give a real form of A_1 ?

To answer this question, we observe first that

$$\mathscr{C}_2' = P_1{}^2 + P_2{}^2 \tag{3.3}$$

commutes with Z, P_1, and P_2, for

$$\begin{aligned} [P_1{}^2 + P_2{}^2, Z] &= [P_1{}^2, Z] + [P_2{}^2, Z] \\ &= \{P_1[P_1, Z] + [P_1, Z]P_1\} + \{P_2[P_2, Z] + [P_2, Z]P_2\} \\ &= 0 \end{aligned} \tag{3.4}$$

Now we form the commutators

$$
\begin{aligned}
[Z^2, P_i] \equiv Y_i &= Z[Z, P_i] + [Z, P_i]Z = \{Z, [Z, P_i]\} \\
Y_1 &= -ZP_2 - P_2 Z \\
Y_2 &= +ZP_1 + P_1 Z
\end{aligned}
\tag{3.5}
$$

It is an easy matter to check that the following commutation relations are satisfied:

$$
\begin{aligned}
[Z, Y_1] &= -Y_2 \\
[Z, Y_2] &= +Y_1 \\
[Y_1, Y_2] &= +2Z(P_1{}^2 + P_2{}^2) + 2(P_1{}^2 + P_2{}^2)Z
\end{aligned}
\tag{3.6}
$$

Since the combination $(P_1{}^2 + P_2{}^2)$ commutes with everything in sight, we can define

$$
Y'_i = \frac{Y_i}{2\sqrt{\mp(P_1{}^2 + P_2{}^2)}}
\tag{3.7}
$$

The commutation relations satisfied by Z, Y'_i are then

$$
\begin{aligned}
[Z, Y'_1] &= -Y'_2 \\
[Z, Y'_2] &= +Y'_1 \\
[Y'_1, Y'_2] &= \mp Z
\end{aligned}
\tag{3.8}
$$

These are the commutation relations for the simple groups $SO(2 \pm 1)$, respectively.

B. More General Orthogonal Groups.[8–11] The CR for the groups $SO(n+1)$ and $ISO(n)$ have been given in (1.29) and (1.31). In the algebra $SO(n+1)$ it is not difficult to construct an operator that commutes with all generators X_{ij}:

$$
\begin{aligned}
\mathscr{C}_2(X) &= \sum_{i<j}^{n+1} X_{ij}{}^2 \\
[\mathscr{C}_2(X), X_{rs}] &= 0
\end{aligned}
\tag{3.9}
$$

We delay the verification for later [see (3.29)]. Such an operator is called a Casimir invariant. The contracted Casimir invariant is simply

$$
\begin{aligned}
\mathscr{C}'_2(X, P) &= \lim_{\varepsilon \to 0} \varepsilon^2 \mathscr{C}_2(X) \\
&= \underbrace{\lim_{\varepsilon \to 0} \sum_{i<j}^{n} (\varepsilon X_{ij})^2}_{\downarrow\; 0} + \underbrace{\lim_{\varepsilon \to 0} \sum_{i}^{n} (\varepsilon X_{i,n+1})^2}_{\downarrow\; \sum_{i=1}^{n} P_i{}^2} \\
&= 0 + \sum_{i=1}^{n} P_i{}^2
\end{aligned}
\tag{3.10}
$$

The contracted invariant $\sum_{i=1}^{n} P_i{}^2$ commutes with all generators of the contracted group

$$[\mathscr{C}'_2(X, P), X_{ij}] = 0$$
$$[\mathscr{C}'_2(X, P), P_k] = 0 \tag{3.11}$$

The invariance of $\mathscr{C}'_2(X, P)$ stems directly from (3.9) and (3.10). In other words, these invariance and contraction properties commute.

The kernel* of the limit (3.10) is the Casimir invariant $\mathscr{C}_2[SO(n)]$

$$\mathscr{C}_2[SO(n)] = \sum_{i<j}^{n} X_{ij}{}^2 \tag{3.12}$$

This clearly commutes with the generators X_{ij} of the subgroup $SO(n)$ of $ISO(n)$. However, it does not commute with the generators P_i. We define

$$Y_i = \frac{1}{2}[P_i, \mathscr{C}_2[SO(n)]]$$
$$= \frac{1}{2}\sum_j \{P_j, X_{ij}\} \tag{3.13}$$

It can now be verified by brute strength that

$$[X_{rs}, Y_i] = Y_r\, \delta_{is} - Y_s\, \delta_{ir}$$
$$[Y_i, Y_j] = X_{ij} \sum_{r=1}^{n} P_r{}^2 \tag{3.14}$$

Therefore, by defining

$$Z_{ij} = X_{ij}$$
$$Z_{i,\, n+1} = \frac{Y_i}{(-\sum_1^n P_r{}^2)^{1/2}} \qquad 1 \leq i, j \leq n \tag{3.15}$$

we see that the $Z_{k,\, l}$ $(1 \leq k, l \leq n + 1)$ close under commutation and generate the group $SO(n + 1)$.

Comment 1. If we define

$$Z'_{i,\, n+1} = \frac{Y_i}{(\sum_1^n P_r{}^2)^{1/2}} \tag{3.15'}$$

then the generators Z' close on the algebra of $SO(n, 1)$.

* Kernel means the part that is annihilated in the mapping.

Comment 2. By using the unitary trick we can convert the algebra of $ISO(n)$ to the algebra of $ISO(p, q)$ with $p + q = n$. Then we have the following possible expansions:

$$ISO(p, q) \begin{cases} \nearrow SO(p + 1, q) \\ \searrow SO(p, q + 1) \qquad p \neq q \end{cases} \tag{3.16}$$

C. The Unitary Groups.[8–15] The infinitesimal generators and the Casimir invariant for the groups $Gl(n + 1)$ and its real forms are (using $U^i{}_j = u^i\,\partial_j$)

$$\begin{aligned} [U^i{}_j, U^r{}_s] &= U^i{}_s\,\delta_j{}^r - U^r{}_j\,\delta_s{}^i \\ \mathscr{C}_2(U) &= \sum_{i,j} U^i{}_j U^j{}_i \end{aligned} \tag{3.17}$$

The subgroup $Sl(n + 1)$, with algebra $\mathfrak{a}_n$, obeys an additional traceless condition, which is written most conveniently for our purposes as follows:

$$U^{n+1}_{n+1} = -\sum_1^n U^i{}_i \tag{3.18}$$

We now contract with respect to the decomposition

$$\begin{aligned} V_R &: U^i{}_j \qquad i \leq i, j \leq n \\ V_N &: U^i_{n+1}, U^{n+1}_i \end{aligned} \tag{3.19}$$

$$\begin{aligned} P^i &= \lim_{\varepsilon \to 0} \varepsilon U^i_{n+1} \\ P_i &= \lim_{\varepsilon \to 0} \varepsilon U^{n+1}_i \end{aligned} \tag{3.20}$$

The commutation relations for the resulting algebra, which can be called $IU(n)$, are

$$\begin{aligned} [U^i{}_j, U^r{}_s] &= \mathfrak{gl}(n) \\ [U^i{}_j, P^k] &= P^i\,\delta_j{}^k \\ [U^i{}_j, P_k] &= -P_j\,\delta_k{}^i \\ [P^i, P^j] = [P^i, P_j] &= [P_i, P_j] = 0 \\ \mathscr{C}'_2(U^i{}_j, P^i, P_j) &= \sum_1^n P^i P_i \end{aligned} \tag{3.21}$$

Proceeding as before, we construct the commutator of the abelian invariant subalgebra V_N with the kernel of the contracted Casimir operator

$$\mathscr{C}_2[\mathfrak{gl}(n)] = \sum_1^n U^i{}_j U^j{}_i \tag{3.22}$$

$$\begin{aligned} Y^i &= \tfrac{1}{2}[P^i, \mathscr{C}_2[\mathfrak{gl}(n)]] = -\tfrac{1}{2}\{P^s, U^i{}_s\} \\ Y_j &= \tfrac{1}{2}[P_j, \mathscr{C}_2[\mathfrak{gl}(n)]] = \tfrac{1}{2}\{P_s, U^s{}_j\} \end{aligned} \tag{3.23}$$

It is easy enough to check the commutation properties of the generators $U^i{}_j$, Y^i, Y_j:

$$\begin{aligned} [U^i{}_j, U^r{}_s] &= \mathfrak{gl}(n) \\ [U^i{}_j, Y^r] &= Y^i \delta_j{}^r \\ [U^i{}_j, Y_s] &= -Y_j \delta_s{}^i \\ [Y_i, Y^j] &= -U^j{}_i \sum P^r P_r - \tfrac{1}{2}\{P^r P_s, U^s{}_r\} \delta_i{}^j \\ [Y_i, Y_j] &= [Y^i, Y^j] = 0 \end{aligned} \tag{3.24}$$

Once again, we can assure this collection of generators closes under commutation by defining

$$\begin{aligned} Z^i{}_j &= U^i{}_j \\ Z^i_{n+1} &= \frac{Y^i}{\sqrt{\sum P^r P_r}} \\ Z^{n+1}_j &= \frac{Y_j}{\sqrt{\sum P^r P_r}} \\ Z^{n+1}_{n+1} &= \frac{-\frac{1}{2}\{P^r P_s, U^s{}_r\}}{\sum P^r P_r} \end{aligned} \tag{3.25}$$

The $Z^i{}_j$ are the generators for the algebra $\mathfrak{sl}(n+1)$. In particular, this calculation holds for the various real forms possessing the subgroup $Gl(n)$ or one of its forms. Thus we have the expansions

$$IU(p, q) \begin{cases} \nearrow SU(p+1, q) \\ \searrow SU(p, q+1) \end{cases} \tag{3.26}$$

D. The Symplectic Groups.[16,17] The symplectic groups can be treated in the same way.

1. We can contract $USp(2p, 2q)$ with respect to either of its maximal subgroups:

$$V_R = USp(2p-2, 2q) \otimes USp(2) \xrightarrow{\text{contr}} IUSp[2(p-1), 2q]$$
$$V_R' = USP(2p, 2q-2) \otimes USp(2) \xrightarrow{\text{contr}} IUSp[2p, 2(q-1)] \tag{3.27c}$$

2. We can expand $IUSp(2p, 2q)$ into either of the real forms

$$IUSp(2p, 2q) \xrightarrow{\text{expand}} \check{U}Sp[2(p+1), 2q]$$
$$IUSp(2p, 2q) \xrightarrow{\text{expand}} USp[2p, 2(q+1)] \tag{3.27}$$

Since the verification of the commutation relations is rather messy for these groups, we pass now to a more general study of the expansion process.

2. EXPANSIONS OF RANK–1 SPACES.[18] Because the arguments involved are not exactly transparent, we break this section down into a number of simpler subsections.

A. A Casimir Invariant for Semisimple Groups. Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$, generators $X_\mu, \ldots$, structure constants $c_{\mu\kappa}{}^{\lambda}$, and metric tensor

$$g_{\mu\nu} = c_{\mu\kappa}{}^{\lambda} c_{\nu\lambda}{}^{\kappa} \tag{3.28}$$

Since $\mathfrak{g}$ is semisimple, $g_{\mu\nu}$ is nonsingular and has a uniquely defined inverse $g^{\mu\nu}$. The homogeneous second-order polynomial operator constructed from $g^{\mu\nu}$ commutes with all group generators:

$$\begin{aligned}[g^{\mu\nu} X_\mu X_\nu, X_\lambda] &= g^{\mu\nu} c_{\mu\lambda}{}^{\tau} X_\tau X_\nu + g^{\mu\nu} c_{\nu\lambda}{}^{\tau} X_\mu X_\tau \\ &= (g^{\gamma\tau} g^{\mu\nu} c_{\gamma\mu\lambda})(X_\tau X_\nu + X_\nu X_\tau) \\ &= 0\end{aligned} \tag{3.29}$$

For simple Lie groups, the second-order Casimir operator

$$\mathscr{C}_2(X) = g^{\mu\nu} X_\mu X_\nu \tag{3.30}$$

is unique up to a numerical scaling factor. Therefore, any homogeneous second-order polynomial operator that is constructed from the generators and commutes with the generators must be proportional to the Casimir invariant.

B. Properties of a Cartan Decomposition. Let G be a simple Lie group and K a subgroup, with corresponding algebras $\mathfrak{g}$, $\mathfrak{k}$. Let a Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$ be given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{3.31}$$

The linear vector subspace $\mathfrak{p}$ is orthogonal to $\mathfrak{k}$ under the Cartan-Killing metric tensor. Moreover, we have

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \tag{3.32a}$$

$$[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p} \tag{3.32b}$$

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \tag{3.32c}$$

$$(\mathfrak{k}, \mathfrak{p}) = 0 \tag{3.32d}$$

Generally speaking, in a Cartan decomposition $\mathfrak{k}$ is the maximal compact subalgebra of $\mathfrak{g}$, and the subspace $\mathfrak{p}$ consists of the noncompact generators of $\mathfrak{g}$. Then the related algebra

$$\mathfrak{g}^* = \mathfrak{k} \oplus \mathfrak{p}^* \qquad \mathfrak{p}^* = i\mathfrak{p} \tag{3.33}$$

is simple and compact, and $\mathfrak{p}^*$ is dual to $\mathfrak{p}$. The coset space $P = G/K$ is given by

$$\frac{G}{K} = \frac{\text{EXP } \mathfrak{g}}{\text{EXP } \mathfrak{k}} = \text{EXP } \mathfrak{p} = P \tag{3.34}$$

For our purposes, we need not assume that $\mathfrak{k}$ is compact, but just that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal and obey (3.32d).

When $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$, the subalgebra $\mathfrak{k}$ is semisimple because it reproduces itself under commutation. If $[\mathfrak{k}, \mathfrak{k}] \neq \mathfrak{k}$, then it can be shown[19] that $\mathfrak{k}$ contains an abelian invariant subalgebra of dimension at most 1 (i.e., exactly 1). Under these circumstances, $\mathfrak{p}$ has an almost complex structure on it (is a Kählerian manifold) and as such is a hermitian symmetric space. Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is simple, $[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p}$. That is, $\mathfrak{k}$ maps $\mathfrak{p}$ *onto* itself. Moreover, the action of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible. The linear vector subspace $\mathfrak{p}$, in other words, carries an irreducible representation of $\mathfrak{k}$. Finally, $\mathfrak{p}$ closes in $\mathfrak{k}$ under commutation.

Let basis vectors for $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ be denoted as follows:

$X_\mu, X_\nu, \ldots,$	for	$\mathfrak{g}$ (late Greek indices)
$X_\alpha, X_\beta, \ldots,$	for	$\mathfrak{k}$ (early Greek indices)
$X_i, X_j, \ldots,$	for	$\mathfrak{p}$ (Roman indices)

Then the commutation relations are expressed

$$[X_\alpha, X_\beta] = c_{\alpha\beta}{}^\gamma X_\gamma + 0 \tag{3.35a}$$

$$[X_\alpha, X_j] = 0 \qquad + c_{\alpha j}{}^k X_k \tag{3.35b}$$

$$[X_i, X_j] = c_{ij}{}^\gamma X_\gamma \ + 0 \tag{3.35c}$$

The regular representation has a very convenient block structure

$$\mathbf{R}(X_\alpha) = \begin{array}{c c} & \begin{array}{cc} \gamma & k \end{array} \\ \begin{array}{c} \beta \\ - \\ j \end{array} & \left[\begin{array}{c|c} c_{\alpha\beta}{}^{\gamma} & 0 \\ \hline 0 & c_{\alpha j}{}^{k} \end{array}\right] \end{array} \tag{3.36a}$$

$$\mathbf{R}(X_i) = \begin{array}{c c} & \begin{array}{cc} \gamma & k \end{array} \\ \begin{array}{c} \beta \\ - \\ j \end{array} & \left[\begin{array}{c|c} 0 & c_{i\beta}{}^{k} \\ \hline c_{ij}{}^{\gamma} & 0 \end{array}\right] \end{array} \tag{3.36b}$$

From this block structure the orthogonality of $\mathfrak{k}$ and $\mathfrak{p}$ is easily seen

$$(X_\alpha, X_i) = \operatorname{tr}\left(\begin{array}{c|c} & 0 \\ \hline 0 & \end{array}\right)\left(\begin{array}{c|c} 0 & \\ \hline & 0 \end{array}\right) = \operatorname{tr}\left(\begin{array}{c|c} 0 & \\ \hline & 0 \end{array}\right) = 0 \tag{3.36c}$$

The nonzero components of the metric tensor are

$$g_{\alpha\beta} = c_{\alpha\mu}{}^{\nu} c_{\beta\nu}{}^{\mu} = c_{\alpha\gamma}{}^{\delta} c_{\beta\delta}{}^{\gamma} + c_{\alpha k}{}^{l} c_{\beta l}{}^{k} \tag{3.37a}$$

$$g_{ij} = c_{i\mu}{}^{\nu} c_{j\nu}{}^{\mu} = c_{ik}{}^{\gamma} c_{j\gamma}{}^{k} + c_{i\gamma}{}^{k} c_{jk}{}^{\gamma} \tag{3.37b}$$

Since $\mathfrak{g}$ is simple, it is possible to write

$$0 \neq \|g_{\mu\nu}\| = \det \left[\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & g_{ij} \end{array}\right] = \|g_{\alpha\beta}\|\,\|g_{ij}\| \tag{3.38}$$

We conclude

1. The metric tensor $g_{\alpha\beta}$ induced on $\mathfrak{k}$ by the Cartan-Killing form on $\mathfrak{g}$ is nonsingular.
2. The metric tensor g_{ij} induced on $\mathfrak{p}$ by the Cartan-Killing form on $\mathfrak{g}$ is nonsingular.

These are important and nontrivial conclusions. The metric tensor $k_{\alpha\beta}$ on $\mathfrak{k}$, computed from the structure of $\mathfrak{k}$ as an algebra, is

$$k_{\alpha\beta} = c_{\alpha\gamma}{}^{\delta} c_{\beta\delta}{}^{\gamma} \neq g_{\alpha\beta} \tag{3.39}$$

Since it is possible that $\mathfrak{k}$ is nonsemisimple [$V = SU(n+1)$, $V_R = U(n)$ in (3.19)], $k_{\alpha\beta}$ may be singular. However, $g_{\alpha\beta}$ is nonsingular and is the metric on $\mathfrak{k}$ that is used in this work.

The nonsingularity of the metric g_{ij} endows the coset space P with a (pseudo)-Riemannian structure.

C. Additional Properties of a Cartan Decomposition. Since the metric $g_{\mu\nu}$ has a block diagonal structure (3.38), so also does its inverse $g^{\mu\nu}$. The Casimir invariant $\mathscr{C}_2$ then has the structure

$$\mathscr{C}_2(X_\alpha, X_i) = g^{\gamma\delta}X_\gamma X_\delta + g^{kl}X_k X_l \tag{3.38$'$}$$

We observe that the commutator of $\mathscr{C}_2$ with X_α can be written

$$[X_\alpha, g^{\gamma\delta}X_\gamma X_\delta] = 0 = -[X_\alpha, g^{kl}X_k X_l] \tag{3.40a}$$

The left-hand side is a homogeneous second-order polynomial in the bases of $\mathfrak{k}$, the right-hand side is a second-order polynomial in the bases of $\mathfrak{p}$. By separation of variables arguments, both are equal to 0 separately.

This argument does not work for the bases of $\mathfrak{p}$

$$[X_i, g^{\gamma\delta}X_\gamma X_\delta] = -[X_i, g^{kl}X_k X_l] \neq 0 \tag{3.40b}$$

because both sides are second-order polynomials linear in both $\mathfrak{k}$ and $\mathfrak{p}$. However, from this expression we can derive a well-known relation between the structure constants:

$$g^{\gamma\delta}c_{\gamma i}{}^{k} = g^{kl}c_{il}{}^{\delta} \tag{3.40b$'$}$$

We will also find it convenient at this point to derive another relation existing between the structure constants

$$\begin{aligned} c_{rs}{}^{\gamma}c_{\gamma\beta}{}^{\delta} &= c_{rs}{}^{\mu}c_{\mu\beta}{}^{\delta} \\ &= -\left(c_{\beta r}{}^{\mu}c_{\mu s}{}^{\delta} + c_{s\beta}{}^{\mu}c_{\mu r}{}^{\delta}\right) \\ &= -\left(c_{\beta r}{}^{t}c_{ts}{}^{\delta} + c_{s\beta}{}^{t}c_{tr}{}^{\delta}\right) \end{aligned} \tag{3.41}$$

This equality results directly from the Jacobi identity.

D. A Useful Property of Rank-1 Algebras. The curvature of a Riemannian space can be defined both in a coordinate-free[20] and in a coordinate-dependent[21] way:

$$\begin{array}{ccc} & R(X_\mu, X_\nu)X_\lambda & \\ // & & \backslash\backslash \\ -[[X_\mu, X_\nu], X_\lambda] & = & R_{\mu\nu,\lambda}{}^{\kappa}X_\kappa \end{array} \tag{3.42}$$

For the Riemannian symmetric spaces $P = \text{EXP}\ \mathfrak{p}$ with metric g_{ij}, the generators are $X_i, X_j, \ldots$, and (3.42) becomes

$$-c_{ij}{}^{\alpha}c_{\alpha k}{}^{l} = R_{ij,k}{}^{l} \tag{3.43}$$

The classical compact coset spaces of rank 1 have a very useful property. They are all isomorphic with spheres. Spheres are spaces with constant curvature. For spaces with constant sectional curvature we have a generalization[22] of the vector triple-cross-product rule $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$:

$$
\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \\
R(X, Y)Z &= \rho\{g(Z, Y)X - g(Z, X)Y\}
\end{aligned}
\tag{3.44}
$$

The sectional curvature ρ is given by[22]

$$
\begin{aligned}
-\rho &= (R(X_i, X_j)X_i, X_j) \\
&= -c_{ij}{}^{\gamma} c_{\gamma i}{}^{k} g_{kj} \\
&= -\sum_{\gamma, \xi} c_{ij}{}^{\gamma} c_{ij}{}^{\xi} g_{\gamma\xi} \qquad i \neq j \text{ fixed}
\end{aligned}
\tag{3.45}
$$

In summary, we have for coset spaces P of rank 1 the following relations between the structure constants:[18]

$$
\begin{aligned}
R_{ij,k}{}^{l} &= -c_{ij}{}^{\alpha} c_{\alpha k}{}^{l} \qquad \text{valid for rank } n \\
&= \rho\{g_{jk}\, \delta_i{}^{l} - g_{ki}\, \delta_j{}^{l}\} \qquad \text{valid for rank 1} \\
\rho &= \sum_{\gamma, \xi} c_{ij}{}^{\gamma} c_{ij}{}^{\xi} g_{\gamma\xi}
\end{aligned}
\tag{3.46}
$$

In particular, we have the following useful identity for rank-1 cosets:[18]

$$
c_{ij}{}^{\alpha} c_{\alpha k}{}^{l} = \rho\{g_{ki}\, \delta_j{}^{l} - g_{kj}\, \delta_i{}^{l}\}
\tag{3.47}
$$

E. Contraction of a Cartan Decomposition. We now contract the algebra $\mathfrak{g}$ to the algebra $\mathfrak{g}'$ using a simple Inönü-Wigner contraction and a Cartan decomposition:

$$
\begin{aligned}
V_R &= \mathfrak{k} \qquad X_\alpha \to Y_\alpha \\
V_N &= \mathfrak{p} \qquad X_i \to \varepsilon X_i = Y_i
\end{aligned}
\tag{3.48}
$$

The structure constants of $\mathfrak{g}'$ are

$$
[Y_\alpha, Y_\beta] = d_{\alpha\beta}{}^{\gamma} Y_\gamma + 0 \qquad d_{\alpha\beta}{}^{\gamma} = c_{\alpha\beta}{}^{\gamma} \tag{3.49a}
$$

$$
[Y_\alpha, Y_j] = 0 \qquad + d_{\alpha j}{}^{k} Y_k \qquad d_{\alpha j}{}^{k} = c_{\alpha j}{}^{k} \tag{3.49b}
$$

$$
[Y_i, Y_j] = d_{ij}{}^{\gamma} Y_\gamma + 0 \qquad d_{ij}{}^{\gamma} = \varepsilon^2 c_{ij}{}^{\gamma} \to 0 \tag{3.49c}
$$

In terms of the regular representation we have

$$
Y_\alpha \to \begin{array}{c} \\ \beta \\ - \\ j \end{array}
\begin{array}{c} \begin{array}{cc} \gamma & \quad k \end{array} \\ \left[\begin{array}{c|c} c_{\alpha\beta}{}^{\gamma} & 0 \\ \hline 0 & c_{\alpha j}{}^{k} \end{array}\right] \end{array}
\tag{3.50a}
$$

$$
Y_i \to \begin{array}{c} \\ \beta \\ - \\ j \end{array}
\begin{array}{c} \begin{array}{cc} \gamma & \quad k \end{array} \\ \left[\begin{array}{c|c} 0 & c_{i\beta}{}^{k} \\ \hline \varepsilon^2 c_{ij}{}^{\gamma} \to 0 & 0 \end{array}\right] \end{array}
\tag{3.50b}
$$

The only information lost in this contraction is contained in the structure constants $c_{ij}{}^{\gamma}$.

Although the contracted Casimir operator $\mathscr{C}_2(Y_\alpha, Y_i)$ is not well defined as $\varepsilon \to 0$, because $g^{ij} \cong \varepsilon^{-2}$, the following limit exists:

$$\mathscr{C}'_2(Y_\alpha, Y_i) = \lim_{\varepsilon \to 0} \varepsilon^2 \mathscr{C}_2(Y_\alpha, Y_i)$$

$$= \lim_{\varepsilon \to 0} g^{\alpha\beta}(\varepsilon X_\alpha)(\varepsilon X_\beta) + \lim_{\varepsilon \to 0} g^{ij}(\varepsilon X_i)(\varepsilon X_j) \to g^{ij} Y_i Y_j \tag{3.51}$$

Moreover, this contracted operator is an invariant of $\mathfrak{g}'$ because $\mathscr{C}_2$ is an invariant of $\mathfrak{g}$:

$$[\mathscr{C}'_2(Y_\alpha, Y_i), Y_\beta] = [\mathscr{C}'_2(Y_\alpha, Y_i), Y_j] = 0 \tag{3.52}$$

Every invariant operator of $\mathfrak{g}'$ is contracted from an invariant of $\mathfrak{g}$. But all invariant operators on $\mathfrak{g}$ are known; $\mathfrak{g}$ possesses one unique second-order homogeneous polynomial operator. Therefore, so also does $\mathfrak{g}'$.

From the structure constants d on $\mathfrak{g}'$ it is possible to reconstruct the metric $g_{\alpha\beta}$ but not g_{ij}, for

$$g_{\alpha\beta} = d_{\alpha\gamma}{}^{\delta}\, d_{\beta\delta}{}^{\gamma} + d_{\alpha k}{}^{l}\, d_{\beta l}{}^{k} \tag{3.53a}$$

$$g_{ij} = d_{i\gamma}{}^{t}\, d_{jt}{}^{\gamma} + d_{it}{}^{\gamma}\, d_{j\gamma}{}^{t} \to 0 \tag{3.53b}$$

However, the uniqueness of $\mathscr{C}'_2(Y_\alpha, Y_i)$ means that once an operator $\mathscr{C}'_2$ has been constructed which commutes with all generators of $\mathfrak{g}'$, this invariant has the form

$$\mathscr{C}'_2(Y_\alpha, Y_i) = h g^{kl} Y_k Y_l \tag{3.54}$$

where h is a nonzero scaling constant. Without loss of generality, we can choose $h = +1$. The choice $h = -1$ gives the metric tensor on the dual P^* to P: G^*/K.

We conclude that no information is lost in the contraction $\mathfrak{g} \to \mathfrak{g}'$, except for the sign of h. For the structure constants $c_{ij}{}^{\gamma}$, which are not known from the commutation properties of $\mathfrak{g}'$, are given by

$$c_{ij}{}^{\gamma} = g^{\gamma\beta}\, d_{\beta i}{}^{k} g_{kj} \tag{3.55}$$

1. The $d_{\beta i}{}^{k}$ are known from $\mathfrak{g}'$.
2. The $g^{\gamma\beta}$ are known by (3.53a).
3. The g_{kj} are known by (3.54).

F. Expansion of Rank-1 Contracted Spaces. Since we have complete information about the structure constants c of $\mathfrak{g}$ from the structure constants d of $\mathfrak{g}'$, together with the Casimir invariant $\mathscr{C}'_2$, it should be possible to expand $\mathfrak{g}'$ to $\mathfrak{g}$.

We define

$$\begin{aligned} Z_\alpha &= Y_\alpha \\ Z_i &\equiv [Y_i, g^{\gamma\beta} Y_\gamma Y_\beta] \\ &= g^{\gamma\beta}\, d_{i\gamma}{}^t \{Y_t, Y_\beta\} \end{aligned} \tag{3.56}$$

We show that the generators Z are suitable candidates for expanding $\mathfrak{g}'$ back to $\mathfrak{g}$. The first two commutation relations are simple to verify:

$$\begin{aligned} [Z_\alpha, Z_\beta] &= c_{\alpha\beta}{}^\gamma Z_\gamma \\ [Z_\alpha, Z_j] &= [Y_\alpha, [Y_j, g^{\gamma\beta} Y_\gamma Y_\beta]] \\ &= -[g^{\gamma\beta} Y_\gamma Y_\beta, [Y_\alpha, Y_j]] - [Y_j, [g^{\gamma\beta} Y_\gamma Y_\beta, Y_\alpha]] \\ &= d_{\alpha j}{}^k [Y_k, g^{\gamma\beta} Y_\gamma Y_\beta] + 0 \qquad [\text{by (3.40a)}] \\ &= c_{\alpha j}{}^k Z_k \end{aligned} \tag{3.57}$$

Finally, we must compute the commutator of the generators Z_i and Z_j

$$[Z_i, Z_j] = g^{\gamma\beta}\, d_{i\gamma}{}^r g^{\xi\eta}\, d_{j\xi}{}^s [\{Y_r, Y_\beta\}, \{Y_s, Y_\eta\}] \tag{3.58}$$

The expansion of the anticommutators $\{\ ,\ \}$ within the commutator $[\ ,\ ]$ leads to four kinds of terms discussed symbolically below, where l, l' are Latin indices and g are early Greek indices

$$\begin{aligned} &1.\quad [Y_r Y_\beta, Y_s Y_\eta] \to A_1^{ll'g} Y_l Y_{l'} Y_g \\ &2.\quad [Y_r Y_\beta, Y_\eta Y_s] \to A_2^{ll'g} Y_l Y_g Y_{l'} \\ &3.\quad [Y_\beta Y_r, Y_s Y_\eta] \to A_3^{ll'g} Y_{l'} Y_g Y_l \\ &4.\quad [Y_\beta Y_r, Y_\eta Y_s] \to A_4^{ll'g} Y_g Y_l Y_{l'} \end{aligned} \tag{3.59}$$

A moment's consideration will reveal that for any fixed indices (l, l', g)

$$A_1^{ll'g} = A_2^{ll'g} = A_3^{ll'g} = A_4^{ll'g} \tag{3.60}$$

The first and fourth terms together form a symmetrized combination, as do the second and third. Moreover, we have

$$\begin{aligned} Y_l Y_g Y_{l'} &= Y_l([Y_g, Y_{l'}] + Y_{l'} Y_g) = Y_l Y_{l'} Y_g + Y_l c^t_{gl'} Y_t \\ Y_{l'} Y_g Y_l &= ([Y_{l'}, Y_g] + Y_g Y_{l'}) Y_l = Y_g Y_l Y_{l'} + Y_t c^t_{l'g} Y_l \end{aligned} \tag{3.61}$$

In bringing the terms of structure types 2 and 3 into a normally ordered form with both Latin indices together on the left or right, we see the quadratic terms of the form $Y_t Y_l$ cancel out in pairs. The cubic terms $Y_g Y_l Y_{l'}$ and $Y_l Y_{l'} Y_g$ are identical to those appearing in 1 and 4. Therefore, we need compute explicitly only the terms arising from 1 of (3.59), which are

$$[Y_r Y_\beta, Y_s Y_\eta] = Y_r Y_t\, d_{\beta s}{}^t Y_\eta + Y_t Y_s\, d_{r\eta}{}^t Y_\beta + Y_r Y_s\, d_{\beta\eta}{}^\lambda Y_\lambda \tag{3.62}$$

Then the commutator (3.58) has the structure

$$P_{ij}^{rt,\,\eta} Y_r Y_t Y_\eta + Q_{ij}^{ts,\,\beta} Y_t Y_s Y_\beta + R_{ij}^{rs,\,\lambda} Y_r Y_s Y_\lambda \tag{3.63}$$

We now compute these three tensors, term by term

$$\begin{aligned}
P_{ij}^{rt,\,\eta} &= \underline{g^{\gamma\beta}}\, d_{i\gamma}{}^r g^{\xi\eta}\, d_{j\xi}{}^s\, \underline{d_{\beta s}{}^t} \\
&\overset{(55)}{=} g^{\xi\eta}\, d_{j\xi}{}^s\, \underline{d_{i\gamma}{}^r (c_{st'}^{\gamma} g^{t't})} \\
&\overset{(47)}{=} g^{\xi\eta}\, d_{j\xi}{}^s (\rho(g_{t'i}\, \delta_s{}^r - g_{is}\, \delta_{t'}{}^r)) g^{t't} \\
&= -\,\underline{\rho g^{\xi\eta}\, d_{j\xi}{}^s g_{si}} g^{rt} + \rho g^{\xi\eta}\, d_{j\xi}{}^r\, \delta_i{}^t
\end{aligned} \tag{3.64a}$$

Similarly,

$$Q_{ij}^{ts,\,\beta} = \underline{\rho g^{\gamma\beta}\, d_{i\gamma}{}^r g_{rj}} g^{ts} - \rho g^{\gamma\beta}\, d_{i\gamma}{}^s\, \delta_j{}^t \tag{3.64b}$$

Finally, we come to the most involved calculation

$$\begin{aligned}
R_{ij}^{rs,\,\lambda} &= \underline{g^{\gamma\beta}\, d_{i\gamma}{}^r}\, \underline{g^{\xi\eta}\, d_{j\xi}{}^s}\, \underline{d_{\beta\eta}{}^\lambda} \\
&\overset{(55)^2}{=} +\, (g^{r'r} c_{r'i}^{\beta})\underline{(g^{s's} c_{s'j}^{\eta})(-c_{\eta\beta}{}^\lambda)} \\
&\overset{(41)}{=} g^{r'r} g^{s's} \underline{\underline{c_{r'i}^{\beta}}}(\underline{c_{\beta s'}^{t}} c_{tj}{}^\lambda + \underline{c_{j\beta}{}^t} c_{ts'}^{\lambda}) \\
&\overset{(47)^2}{=} \rho g^{r'r} g^{s's} \{c_{tj}{}^\lambda (g_{s'r'}\, \delta_i{}^t - g_{s'i}\, \delta_{r'}{}^t) \\
&\quad - c_{ts'}^{\lambda} (g_{jr'}\, \delta_i{}^t - g_{ji}\, \delta_{r'}{}^t)\} \\
&= \rho c_{ij}{}^\lambda g^{rs} - \rho g^{rt} c_{tj}{}^\lambda\, \delta_i{}^s - \rho g^{ss'} c_{is'}^{\lambda}\, \delta_j{}^r \\
&\quad + \rho g^{r'r} g^{s's} c_{r's'}^{\lambda} g_{ji}
\end{aligned} \tag{3.64c}$$

Summing these results for the tensors P, Q, R we have

$$(P_{ij}^{rs,\,\lambda} + Q_{ij}^{rs,\,\lambda} + R_{ij}^{rs,\,\lambda}) Y_r Y_s Y_\lambda = -\rho c_{ij}{}^\lambda (g^{rs} Y_r Y_s) Y_\lambda \tag{3.64}$$

Thus

$$\begin{aligned}
[Z_i, Z_j] &= -2\rho c_{ij}{}^\lambda \{Y_\lambda, g^{rs} Y_r Y_s\} \\
&= (-4\rho g^{rs} Y_r Y_s) c_{ij}{}^\lambda Z_\lambda
\end{aligned} \tag{3.58'}$$

We are now in a position to reconstruct the algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ from $\mathfrak{g}'$

$\mathfrak{g}$:

$$\begin{aligned}
Z'_\alpha &= Z_\alpha \\
Z'_j &= Z_j \{-4\rho g^{rs} Y_r Y_s\}^{-1/2}
\end{aligned} \tag{3.65g}$$

$\mathfrak{g}^*$:

$$
\begin{aligned}
Z'_\alpha &= Z_\alpha \\
Z'_j &= Z_j\{4\rho g^{rs} Y_r Y_s\}^{-1/2}
\end{aligned}
\tag{3.65g*}
$$

G. Comments

1. We see that either of the dual algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ can be contracted to $\mathfrak{g}'$. Conversely, $\mathfrak{g}'$ (of rank 1) can be expanded to either of the dual algebras $\mathfrak{g}$ or $\mathfrak{g}^*$.

2. This expansion procedure depends on the structure constants of the algebra, rather than on the real form chosen. The construction, therefore, works for noncompact $\mathfrak{k}$ and $\mathfrak{p}$ of rank 1. Then we have the following expansion possibilities:

$$
\begin{tikzcd}
 & SO(p+1, q) \\
ISO(p, q) \arrow[ur] \arrow[dr] & \\
 & SO(p, q+1)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & SU(p+1, q) \\
IU(p, q) \arrow[ur] \arrow[dr] & \\
 & SU(p, q+1)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & USp(2(p+1), 2q) \\
IUSp(2p, 2q) \arrow[ur] \arrow[dr] & \\
 & USp(2p, 2(q+1))
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & F_{4(-20)} \\
(F_4 \bmod B_4) \wedge SO(9) \arrow[ur] \arrow[dr] & \\
 & F_{4(-52)}
\end{tikzcd}
$$

3. From (3.57) we see that any operator invariant on $\mathfrak{k}$ but not on $\mathfrak{g}'$ is a possible candidate for use in (3.56). Such an operator automatically gives

$$
[Z_\alpha, [Y_j, \text{inv}]] = c_{\alpha j}{}^k [Y_k, \text{inv}] \tag{3.66}
$$

by the Jacobi identity. Although the Jacobi identity is, strictly speaking, valid only within the algebra, a more detailed analysis reveals that (3.57) is always valid for polynomial invariants.

4. Since no information is lost in the contraction $\mathfrak{g} \to \mathfrak{g}'$, it should be possible to expand $\mathfrak{g}'$ to $\mathfrak{g}^{(*)}$ regardless of the rank of $\mathfrak{p}^{(*)}$. This has not been accomplished so far. It seems reasonable that the procedure, when and if found, will involve use of the appropriate invariant in (3.56). All irreducible polynomial invariants on $\mathfrak{k}$ can be constructed directly from the Casimir

invariants on $\mathfrak{g}$, which are all known. The construction of these invariants for the classical matrix groups has already been outlined in Problem 11 of Chapter 7.

Résumé

The contraction process has been studied in both a coordinate-dependent way and in a coordinate-free way. In the coordinate-dependent investigation we chose a particularly simple class of contractions, the Inönü-Wigner contractions. The necessary and sufficient conditions for such contractions to exist are easily worked out. Such contractions exist for any Cartan decomposition of any semisimple algebra.

The coordinate-free study of singular changes of basis was more difficult because it is more general. However, the necessary and sufficient conditions for such (Saletan) contractions to exist were also derived explicitly. A large number of properties of both the original and the contracted algebras can be stated succinctly.

Using the coordinate-free results as a guideline, it was possible to choose, in a canonical way, a set of basis vectors for the algebra $\mathfrak{g}^{(0)}$ (with respect to the singular transformation U) in such a way that the structure of all possible distinct contracted algebras can be immediately determined from the structure constants of the algebra $\mathfrak{g}^{(0)}$.

Finally, we determined under what conditions a nonsemisimple algebra can be expanded to a simple algebra under a nonlinear change of basis. Although the procedure described should be valid for cosets of arbitrary rank n, the explicit construction we have proposed is valid only for rank-1 expansions.

Exercises

1. Prove that if an algebra is contracted by means of an Inönü-Wigner contraction with respect to the subalgebra $\mathfrak{m} = 0$, an abelian algebra results.

2. Prove that the use of the Jacobi identity is valid in (3.66). This can be accomplished by showing that the collection of all possible products of generators of a Lie algebra, taken in all possible orders, is an associative algebra. Then the Jacobi identity is valid in this associative algebra. This infinite-dimensional associative algebra is called the **universal enveloping algebra** of the underlying Lie algebra.

3. Compute the invariant necessary in (3.66) to perform a rank-r expansion of the type discussed in Section III.

4. List all possible rank-1 pseudo-Riemannian symmetric spaces contained in Table 9.7. For each, list the group of isometries of the associated contracted space. Show that this group is the semidirect product of a pseudo-Euclidean space with the maximal subgroup.

5. Let G be a simple matrix Lie group with maximal compact subgroup $K = K_1 \otimes K_2$:

$$G \downarrow \left[\begin{array}{c|c} K_1 & 0 \\ \hline 0 & K_2 \end{array}\right]$$

(a) Show that the cosets have the structure on the left:

$$\left[\begin{array}{c|c} \{I_p + XX^\dagger\}^{1/2} & X \\ \hline X^\dagger & \{I_q + X^\dagger X\}^{1/2} \end{array}\right] \xrightarrow[\text{Wigner contraction}]{\text{Inönü-}} \left[\begin{array}{c|c} I_p & X \\ \hline 0 & I_q \end{array}\right]$$

(b) Show that these cosets contract to the subgroup shown above under a standard Inönü-Wigner contraction with respect to K.

(c) Use the group multiplication law in G to construct the group multiplication law in $G' = K \wedge X$:

$$(\mathrm{Id}, X) \otimes (K_1 \otimes K_2, 0) = (K_1 \otimes K_2, XK_2)$$
$$(K_1 \otimes K_2, 0) \otimes (\mathrm{Id}, X) = (K_1 \otimes K_2, K_1 X)$$

(d) Show that the multiplication law in G' is

$$(K_1 \otimes K_2; X) \otimes (K_1' \otimes K_2'; X') = (K_1 K_1' \otimes K_2 K_2'; K_1 X' + XK_2')$$

(e) Apply these results to construct the group multiplication laws in the following groups:

$ISO(3)$:	The group of rigid rotations and displacements of R_3.
$ISO(3, 1)$:	The Poincaré group, or the inhomogeneous Lorentz group (IHLG)
$ISO(p, q)$:	The group of rigid rotations and translations of $R_{p,q}$
$IU(p, q)$:	The inhomogeneous group in $C_{p,q}$

6. Let a Lie algebra $\mathfrak{m}$ be given symbolically by

$$\mathfrak{m} = \left[\begin{array}{c|c} M_{ij} & M_{i\beta} \\ \hline M_{\alpha j} & M_{\alpha\beta} \end{array}\right]$$

The commutation properties of this algebra are described by the block diagonal matrix structure of this algebra; for example,

$$[M_{i\beta}, M_{\alpha j}] \subset M_{ij} \oplus M_{\alpha\beta}, \text{ etc.}$$

Now perform the following contraction:

$$\left[\begin{array}{c|c} M_{ij} & M_{i\beta} \\ \hline M_{\alpha j} & M_{\alpha\beta} \end{array}\right] \to \left[\begin{array}{c|c} N_{ij} & N_{i\beta} \\ \hline N_{\alpha j} & N_{\alpha\beta} \end{array}\right] = \left[\begin{array}{c|c} M_{ij} & \varepsilon M_{i\beta} \\ \hline \varepsilon M_{\alpha j} & M_{\alpha\beta} - \dfrac{I}{\varepsilon^2} \end{array}\right]$$

(a) Show that the result of this singular transformation is an algebra if and only if:

All generators $M_{\alpha\beta}$ commute; and
The algebra $\mathfrak{m}$ contains a generator I with the property

$$[\mathfrak{m}, I] = 0$$

(b) Show that this contraction leads to the "harmonic oscillator algebra" $\mathfrak{h}_4$ when $\mathfrak{m} = \mathfrak{u}(2)$ and $M_{ij} \oplus M_{\alpha\beta} = \mathfrak{u}(1) \oplus \mathfrak{u}(1)$. Show that the contracted commutation relations are realized by the four operators

$$a^\dagger a, \qquad a^\dagger, \qquad a, \qquad I$$

(c) Show that this contraction leads to an algebra when

$$\mathfrak{m} = \mathfrak{u}(n+1) \qquad \text{and} \qquad M_{ij} \oplus M_{\alpha\beta} = \mathfrak{u}(n) \oplus \mathfrak{u}(1).$$

Show that the contracted commutation relations are realized by the $(n+1)^2$ operators

$$a_i^\dagger a_j; \qquad a_i^\dagger; \qquad a_j; \qquad I \qquad (1 \le i, j \le n)$$

This algebra can thus be called the "multiphoton mode algebra."

(d) Show that this contraction does not lead to an algebra in the following two cases:

$$\mathfrak{m} = \mathfrak{so}(n+2) \qquad M_{ij} \oplus M_{\alpha\beta} = \mathfrak{so}(n) \oplus \mathfrak{so}(2)$$
$$\mathfrak{m} = \mathfrak{usp}(2n+2) \qquad M_{ij} \oplus M_{\alpha\beta} = \mathfrak{usp}(2n) \oplus \mathfrak{usp}(2)$$

(e) Let $\mathfrak{i}$ be an operator with the property $[\mathfrak{m}, \mathfrak{i}] = 0$. Then the contraction

$$\mathfrak{m} \oplus \mathfrak{i} = \mathfrak{so}(n+2) \oplus \mathfrak{i}, \qquad M_{ij} \oplus M_{\alpha\beta} = \{\mathfrak{so}(n) \oplus \mathfrak{i}\} \oplus \mathfrak{so}(2)$$

leads to a Lie algebra.

(f) With $\mathfrak{i}$ as previously,

$$\mathfrak{m} \oplus \mathfrak{i} = \mathfrak{usp}(2n+2) \oplus \mathfrak{i}; \qquad M_{ij} \oplus M_{\alpha\beta} = \{\mathfrak{usp}(2n) \oplus \mathfrak{i}\} \oplus H_1$$

gives a Lie algebra. Here H_1 is the basis for the one-dimensional Cartan subalgebra of $\mathfrak{usp}(2)$.

Note that the algebra constructed in this way is not contracted from $\mathfrak{usp}(2n+2)$. Rather, it is contracted from a linear vector subspace of $\mathfrak{usp}(2n+2) \oplus \mathfrak{i}$. This subspace is spanned by $\mathfrak{i}$ and all the basis vectors of $\mathfrak{usp}(2n+2)$ except for the two bases $E_\pm$ which are eigenvectors of H_1 in the subalgebra $\mathfrak{usp}(2)$.

7. Under the contraction $SO(3) \to ISO(2)$ the representations of $SO(3)$ contract to representations of $ISO(2)$. Since $ISO(2)$ is a noncompact group, it has no faithful finite-dimensional unitary representations. We therefore consider the following limit:

$$\lim a \downarrow 0 \qquad aJ_\pm \to P_\pm \qquad al \to p(\text{finite})$$
$$l \uparrow \infty \qquad J_3 \to J_3 \qquad \left|\begin{matrix} l \\ m \end{matrix}\right\rangle \to \left|\begin{matrix} p \\ m \end{matrix}\right\rangle$$
$$p\frac{\beta}{a} = l\beta = x(\text{finite})$$

(a) Compute the matrix elements

$$\left\langle \begin{matrix} l \\ m' \end{matrix} \right| aJ_\pm \left| \begin{matrix} l \\ m \end{matrix} \right\rangle \xrightarrow{\lim} \left\langle \begin{matrix} p \\ m' \end{matrix} \right| P_\pm \left| \begin{matrix} p \\ m \end{matrix} \right\rangle$$

$$a[(l \mp m)(l \pm m + 1)]^{1/2}\, \delta_{m', m\pm 1} \xrightarrow{\lim} p\, \delta_{m', m\pm 1}$$

(b) Compute the contracted limit of the Jacobi polynomials (Chapter 5, Problem 14). Show that

$$\lim P^l_{mn}\left(\cos\frac{x}{l}\right) = (-)^{m-n} J_{m-n}(x)$$

where $J_k(x)$ is the kth Bessel function.

(c) Contract the spherical harmonics, and show that

$$\lim \sqrt{2\pi/l}\; Y^l_m\left(\beta = \frac{x}{l}\right) \to J_m(x)$$

(d) Contract the Legendre polynomials and show that

$$\lim P^l\left(\cos\beta = \frac{x}{l}\right) \to J_0(x)$$

(e) Contract the orthogonality and completeness relations for the spherical harmonics, and show that they become the orthogonality and completeness relations for the Bessel functions.

(f) Contract the generating function (Chapter 5, Problems 14 and 15) for spherical harmonics, and show that it becomes a generating function for the Bessel functions.

(g) In the expression

$$\begin{array}{ccc} & e^{\alpha_+ J_+} Y^l_m(\theta, \phi) & \\ & // \qquad \backslash\backslash & \\ \sum_k A^l_{m;\,k}\, Y^l_{m+k}(\theta, \phi) & = & Y^l_m(\theta', \phi') \end{array}$$

compute the coefficients $A^l_{m;\,k}$ and the arguments θ', ϕ' (complex in general) explicitly. Now contract these results to construct classical generating functions for the Bessel functions.

(h) Show that the Casimir invariant on the spherical harmonics [more correctly, the Laplace-Beltrami operator on the eigenfunctions over $SO(3)/SO(2)$] contracts to the Bessel equation.

8. Under the contraction $\mathfrak{u}(2) \to \mathfrak{h}_4$ the representations of $U(2)$ contract to representations of H_4. Since $\mathfrak{h}_4$ is a noncompact algebra, it has no faithful finite-dimensional antihermitian representations. We therefore consider the following limit:

$$\begin{array}{lll} \lim a \to 0, & aJ_\pm \to h_\pm, & 2ja^2 \to 1 \\[2ex] j\uparrow +\infty, \quad m\downarrow -\infty, & J_3 + \dfrac{1}{2a^2} \to h_3, & \left|\begin{matrix} j \\ m \end{matrix}\right\rangle \to \left|\begin{matrix} \infty \\ n \end{matrix}\right\rangle \\[2ex] j + m \to n \text{ (finite)} & \xi \to I & \end{array}$$

$$\theta \to \frac{\pi}{2} - \sqrt{2}\, ax$$

(a) Compute the matrix elements

$$\left\langle \begin{matrix} j \\ m' \end{matrix} \middle| aJ_\pm \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle \xrightarrow{\lim} \left\langle \begin{matrix} \infty \\ n' \end{matrix} \middle| h_\pm \middle| \begin{matrix} \infty \\ n \end{matrix} \right\rangle$$

$$a[(j \mp m)(j \pm m + 1)]^{1/2}\, \delta_{m', m\pm 1} \xrightarrow{\lim} \begin{matrix} \sqrt{n+1}\, \delta_{n', n+1} \\ \sqrt{n}\ \ \delta_{n', n-1} \end{matrix}$$

(b) Contract the spherical harmonics and show

$$(l)^{1/4} P^l_{n-l,\,0}\left[\frac{\pi}{2} - \sqrt{2}\, ax\right] \xrightarrow{\lim} \psi_n(x) = N_n H_n(x) e^{-1/2x^2}$$

where $H_n(x)$ is the nth Hermite polynomial and $N_n = 1/\sqrt{2^n n!\sqrt{\pi}}$ is the usual normalizing coefficient.

(c) Carry out the steps (c) to (h) of Problem 7. The results are obtained by making the appropriate changes:

Bessel function → harmonic oscillator eigenfunction

Bessel equation → Schrödinger equation for harmonic oscillator.

Except for these minor details, the study of the spherical harmonics outlined in Chapter 5, Problems 14 to 16, can be applied directly to study the properties of the Bessel functions (previous problem) and the oscillator eigenfunctions (this problem).

9. In statistical physics it is often necessary to compute the thermal average of an operator $\mathcal{O}$. This is an expectation value of the form

$$\langle \mathcal{O} \rangle_{\text{th}} = \frac{\operatorname{tr} e^{-\beta \mathcal{H}} \mathcal{O}}{\operatorname{tr} e^{-\beta \mathcal{H}}}$$

We show in this problem how to compute certain such expectation values for the harmonic oscillator. Consider the operators

$$e^{-\beta E J_3}, \qquad \beta = \frac{1}{kT}, \qquad E = \hbar\omega \qquad \text{and} \qquad \text{EXP } i(\alpha_+ J_+ + \alpha_- J_-), \qquad \alpha_\pm \text{ complex}$$

(a) Compute $\langle e^{i(\alpha_+ J_+ + \alpha_- J_-)} \rangle_{\text{th}}$ in the 2×2 $j = \frac{1}{2}$ matrix representation of $Sl(2)$ and show

$$\langle \text{EXP } i(\alpha_+ J_+ + \alpha_- J_-) \rangle_{\text{th}} = \frac{\operatorname{tr} \begin{bmatrix} \cos\theta\, e^{-1/2\beta E} & i\alpha_+ \dfrac{\sin\theta}{\theta} e^{-1/2\beta E} \\ i\alpha_- \dfrac{\sin\theta}{\theta} e^{1/2\beta E} & \cos\theta\, e^{1/2\beta E} \end{bmatrix}}{\operatorname{tr} \begin{bmatrix} e^{-1/2\beta E} & 0 \\ 0 & e^{1/2\beta E} \end{bmatrix}}$$

$$\theta^2 = \alpha_+ \alpha_-$$

(b) The numerator is not generally a hermitian matrix, and thus, in general, it cannot be diagonalized. Show that it *can* be transformed to upper-triangular form and that its eigenvalues are given by

$$\lambda^2 - \lambda \cos\theta[e^{1/2\beta E} + e^{-1/2\beta E}] + 1 = 0$$

$$\lambda_\pm = \frac{\cos\theta[e^{1/2\beta E} + e^{-1/2\beta E}] \pm \{\cos^2\theta(e^{1/2\beta E} + e^{-1/2\beta E})^2 - 4\}^{1/2}}{2}$$

$$\lambda_- = \lambda_+^{-1}$$

(c) Show that the trace in the numerator is

$$\lambda_+ + \lambda_- = \lambda_+ + \lambda_+^{-1} = \sum_{m=-1/2}^{m=+1/2} (\lambda_+)^{2m} = \frac{(\lambda_+)^2 - (\lambda_-)^2}{\lambda_+ - \lambda_-}$$

(d) Show that the trace in the denominator is obtained from the numerator trace by setting $\alpha_+ = \alpha_- = 0$.

(e) Compute the thermal average within the $(2j+1) \times (2j+1)$ matrix representation:

$$\langle \text{EXP}\ i(\alpha_+ J_+ + \alpha_- J_-)\rangle_{\text{th}} = \frac{\lambda_+^{2j+1}(\alpha) - \lambda_-^{2j+1}(\alpha)}{\lambda_+^{2j+1}(0) - \lambda_-^{2j+1}(0)} \frac{\lambda_+(0) - \lambda_-(0)}{\lambda_+(\alpha) - \lambda_-(\alpha)}$$

(f) Contract the algebra $\mathfrak{u}(2)$ to the harmonic oscillator algebra $\mathfrak{h}_4$ using (1.52) and

$$EJ_3 \to E\left(h_3 - \frac{1}{2c^2}\right)$$

$$\frac{\alpha_\pm}{c} \to \gamma_\pm \qquad cJ_+ \to a^\dagger \qquad cJ_- \to a$$

Then show that

$$\langle \text{EXP}\ i(\alpha_+ J_+ + \alpha_- J_-)\rangle_j \xrightarrow[\mathfrak{u}(2)\to\mathfrak{h}_4]{\lim} \langle \text{EXP}\ i(\gamma_+ a^\dagger + \gamma_- a)\rangle_{\mathfrak{h}_4}$$

(g) Compute the thermal average in $\mathfrak{h}_4$ by contracting the thermal average in (e). In particular, show

$$\begin{aligned}\langle \text{EXP}\ i(\alpha_+ J_+ + \alpha_- J_-)\rangle_j &\to \lim_{j\uparrow\infty} \left[\frac{\lambda_+(\alpha)}{\lambda_+(0)}\right]^{2j+1} \\ &= \lim_{c\downarrow 0} \left\{1 - c^2 \frac{\gamma_+\gamma_-}{2} \coth \tfrac{1}{2}\beta E\right\}^{1/c^2} \\ &= \text{EXP}\ -\tfrac{1}{2}\gamma_+\gamma_- \coth \tfrac{1}{2}\beta E\end{aligned}$$

10. The expression for the harmonic oscillator thermal average computed in Problem 9 used the two group theoretical procedures: the Baker-Campbell-Hausdorff formula, and group contraction.

Use the expression of Problem 9 to compute the intensity distribution of X-rays scattered from the surface of an ordered crystal lattice. You must make the following assumptions:

(a) The instantaneous X-ray distribution is given by

$$I_0 = |f_0|^2 \sum_{l,\,l'} \text{EXP } i\mathbf{k} \cdot \{\mathbf{r}(l) - \mathbf{r}(l')\}$$

where $\mathbf{r}(l)$ is the instantaneous position operator of the atom at lattice equilibrium site $\mathbf{x}(l)$, f_0 is the single-particle atomic scattering factor, and $\mathbf{k}$ is the scattering wave vector

$$\mathbf{k} = \frac{2\pi}{\lambda}(\mathbf{s} - \mathbf{s}_0)$$

where λ is the wavelength of the incident and elastically scattered beam, and $\mathbf{s}_0$, $\mathbf{s}$ are the unit vectors in the direction of the incident and scattered beams, respectively.

(b) Replace $\mathbf{r}(l) = \mathbf{x}(l) + \mathbf{u}(l)$, where $\mathbf{u}(l)$ is the displacement of the atom from its equilibrium site $\mathbf{x}(l)$. Show that the set $\mathbf{u}(l)$ can be transformed to normal coordinates, described by decoupled harmonic oscillator Hamiltonians.

(c) Assume that the exposure lasts much longer than any phonon characteristic time. By using an ergodic theorem, show that the time average of the instantaneous scattered X-ray intensity spectrum can be replaced by the thermal average:

$$\langle \text{EXP } i\mathbf{k} \cdot \{\mathbf{u}(l) - \mathbf{u}(l')\}\rangle_{\text{time}} \to \langle \text{EXP } i\mathbf{k} \cdot \{\mathbf{u}(l) - \mathbf{u}(l')\}\rangle_{\text{thermal}}$$

(d) After transforming to the normal coordinates, which obey *decoupled* harmonic oscillator Hamiltonians, compute the thermal average in (c) using the result of Problem 9. Show that the usual hyperbolic cotangent law results.[6,23]

11. Use the contraction procedure indicated in (1.52) and a suitable Baker-Campbell-Hausdorff parameterization from Fig. 5.12 to prove the following result:

$$e^{\lambda J_3} \text{ EXP } (\alpha_+ J_+ + \alpha_- J_-) = \text{EXP } (\lambda' J_3 + \alpha'_+ J_+ + \alpha'_- J_-)$$

$$e^{\lambda a^\dagger a} \text{ EXP } (\gamma_+ a^\dagger + \gamma_- a) = \text{EXP } \left[-\tfrac{1}{4}\lambda\gamma_+\gamma_- \left\{ 1 + \frac{2}{\lambda}\coth\frac{\lambda}{2} - \coth^2\frac{\lambda}{2} \right\} \right]$$

$$\times \text{ EXP } [\lambda a^\dagger a + (\tfrac{1}{2}\lambda)(\sinh\tfrac{1}{2}\lambda)^{-1}\{\gamma_+ e^{1/2\lambda} a^\dagger + \gamma_- e^{-1/2\lambda} a\}]$$

$$\gamma_\pm = \lim_{\varepsilon \downarrow 0} \frac{\alpha_\pm}{\varepsilon}$$

12. Contract S^3 to R_3 and show that the line element and the volume element on S^3, given in Problems 23 and 24 of Chapter 9, contract to the usual line and volume elements on R_3:

$$ds^2 = dr^2 + (r\, d\theta)^2 + (r \sin\theta\, d\phi)^2$$
$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

where $\phi^1 = r$, $\phi^2 = \theta$, and $\phi^3 = \phi$.

13. Follow the procedures indicated in Problems 23 and 24 of Chapter 9, to compute the metric and volume on $H^n \subset R_{n+1}$. This is most easily accomplished using the Weyl unitary trick, where applicable. Contract this result to the corresponding result in R_n. Show that the only differences in the three coset spaces are governed by the coefficients of dr^2:

$$\frac{dr^2}{1-\lambda r^2} \qquad \lambda = \begin{array}{rl} +1 & S^n = \dfrac{SO(n+1)}{SO(n)} \\[2ex] 0 & R_n = \dfrac{ISO(n)}{SO(n)} \\[2ex] -1 & H^n = \dfrac{SO(n,1)}{SO(n)} \end{array}$$

14. Let $f_i^\dagger, f_j$ be the creation and annihilation operators for n $(1 \le i, j \le n)$ *Fermions*. These operators obey the following *anti*commutation relations:

$$\begin{aligned}
\{f_i^\dagger, f_j\} &= [f_i^\dagger, f_j]_+ = f_i^\dagger f_j + f_j f_i^\dagger = \delta_{ij} \\
\{f_i, f_j\} &= [f_i, f_j]_+ = f_i f_j + f_j f_i = 0 \\
\{f_i^\dagger, f_j^\dagger\} &= [f_i^\dagger, f_j^\dagger]_+ = f_i^\dagger f_j^\dagger + f_j^\dagger f_i^\dagger = 0
\end{aligned}$$

In particular, $f_i^\dagger f_i^\dagger = f_j f_j = 0$, which says that no more than one Fermion may occupy a given state (i).

(a) Compare these *anti*commutation relations with the commutation relations obeyed by Boson operators $b_i^\dagger$, b_j:

$$\begin{aligned}
[b_i^\dagger, b_j]_- &= b_i^\dagger b_j - b_j b_i^\dagger = -\delta_{ij} \\
[b_i^\dagger, b_j^\dagger]_- &= [b_i, b_j]_- = 0
\end{aligned}$$

In particular, show that

$$f_i^\dagger f_i^\dagger = 0 \qquad \text{but} \qquad b_i^\dagger b_i^\dagger \neq 0$$

(b) Show that the bilinear combinations of the Fermion operators $f_i^\dagger f_j$ obey the following *commutation* relations:

$$\begin{aligned}
[f_i^\dagger f_j, f_k^\dagger]_- &= +f_i^\dagger \, \delta_{jk} \\
[f_i^\dagger f_j, f_m]_- &= -f_j \, \delta_{im}
\end{aligned}$$

Compare these commutation relations with the corresponding Boson commutation relations, and show that there is no difference.

(c) Show that the bilinear combinations of number preserving operators obey the commutation relations:

$$[a_i^\dagger a_j, a_k^\dagger a_m]_- = a_i^\dagger a_m \, \delta_{jk} - a_k^\dagger a_j \, \delta_{mi}$$

where the operators a may be *either* Fermion operators (f) or Boson operators (b).

(d) Show that the number preserving bilinear combinations $f_i^\dagger f_j$, $b_i^\dagger b_j$ span an algebra of type u(n), and make the associations

$$\begin{aligned} a_i^\dagger a_i &= H_i \\ a_i^\dagger a_j &= E_{\mathbf{e}_i - \mathbf{e}_j} \qquad a = b, f \end{aligned}$$

(e) Show that the set of *all* bilinear combinations of creation and annihilation operators, number nonpreserving ($a^\dagger a^\dagger$, aa) as well as number preserving ($a^\dagger a$), close under commutation, and that their collective commutation relations are described by a rank-n root space. In particular, make the following identifications:

$$\begin{array}{llll} f_i^\dagger f_i & b_i^\dagger b_i & H_i & \\ f_i^\dagger f_j & b_i^\dagger b_j & E_{\mathbf{e}_i - \mathbf{e}_j} & \\ f_i^\dagger f_j^\dagger & b_i^\dagger b_j^\dagger & E_{\mathbf{e}_i + \mathbf{e}_j} & i \neq j \\ f_i f_j & b_i b_j & E_{-\mathbf{e}_i - \mathbf{e}_j} & i \neq j \\ & b_i^\dagger b_i^\dagger & E_{+2\mathbf{e}_i} & \\ & b_i b_i & E_{-2\mathbf{e}_i} & \end{array}$$

In particular, draw the inference that the Boson operators span a root space of type C_n, whereas the Fermion operators span a root space of type D_n.

(f) Show that the roots $\pm 2\mathbf{e}_i$, present for the Boson case, are absent for the Fermion case just because

$$f_i^\dagger f_i^\dagger = 0 \qquad \text{but} \quad b_i^\dagger b_i^\dagger \neq 0 \qquad (+2\mathbf{e}_i)$$

(g) Let M be the operator

$$M = \sum_{i=1}^{n} \alpha^i a_i^\dagger + \sum_{j=1}^{n} \beta^j a_j \qquad a = b, f$$

Then show that M is hermitian ($M^\dagger = +M$) if and only if

$$\beta^i = \alpha^{i*}$$

(h) Form the linear combinations $(a_i^\dagger + a_i)$, $i(a_i^\dagger - a_i)$ $(a = b, f)$ and show that a *real* linear superposition of these $2n$ operators is hermitian:

$$M = \sum_i r^i(a_i^\dagger + a_i) + \sum_j s^j(i)(a_j^\dagger - a_j)$$

(i) Write

$$M = \begin{bmatrix} r^1(a_1^\dagger + a_1) \\ s^1(i)(a_1^\dagger - a_1) \\ \vdots \\ \vdots \\ r^n(a_n^\dagger + a_n) \\ s^n(i)(a_n^\dagger - a_n) \end{bmatrix},$$

Show that the commutation and anticommutation relations are concisely summarized by the matrix statement

$$M^t\{I_n \otimes D_\pm\}M = \sum_{i=1}^{n} (r_i^2 + s_i^2)[f_i, f_i^\dagger]_+ \qquad (+ = f)$$

$$\sum_{j=1}^{n} 2ir_j s_j [b_j, b_j^\dagger]_- \qquad (- = b)$$

where

$$I_n \otimes D = \begin{bmatrix} D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{bmatrix}$$

and $D_{+\text{ or }f} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, whereas $D_{-\text{ or }b} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(j) Let T be a mapping of the annihilation operators into themselves:

$$a_i \to a_i' = T_i{}^j a_j$$

Then T effects a transformation on M:

$$M \to M' = SM$$

Show that S preserves the hermiticity of M if and only if S is real and

$$S^t\{I_n \otimes D_\pm\}S = I_n \otimes D_\pm$$

Since S preserves a real bilinear

$$\begin{Bmatrix} \text{symmetric} \\ \text{antisymmetric} \end{Bmatrix} \text{metric in } R_{2n} \text{ for } \begin{Bmatrix} \text{Fermions, } + \\ \text{Bosons, } - \end{Bmatrix}$$

conclude that

$$S \in SO(2n, r) \qquad \text{for} \quad +, \text{ Fermions}$$

$$S \in Sp(2n, r) \qquad \text{for} \quad -, \text{ Bosons}$$

(k) The groups computed in (j) are groups of outer automorphisms aut $(a^\dagger, a)$. Since int $(a^\dagger, a) \subset$ aut $(a^\dagger, a)$, we conclude that the generators described in (e) span the Lie algebras $\mathfrak{so}(2n, r)$ and $\mathfrak{sp}(2n, r)$.

(l) Show that these results can be summarized by the following table:

	Fermion Case $a = f$	Boson Case $a = b$
Number preserving		
$a_i^\dagger a_j$	$\mathfrak{u}(n)$	$\mathfrak{u}(n)$
Number nonpreserving		
$a_i^\dagger a_j$ together with $a_i^\dagger a_j^\dagger$, $a_j a_i$	$\mathfrak{so}(2n)$	$\mathfrak{sp}(2n, r)$

15. Thomas Commutation Relations: Let $\mathfrak{g}$ be a Lie algebra with a decomposition of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \qquad \begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subseteq \mathfrak{h} \\ [\mathfrak{h}, \mathfrak{m}] &\subseteq \mathfrak{m} \\ [\mathfrak{m}, \mathfrak{m}] &\subseteq \mathfrak{h} \end{aligned}$$

These commutation relations can be called "Thomas commutation relations" because they give rise to Thomas precessionlike effects.

(a) Show that every semisimple algebra has a "Thomas decomposition." In particular, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ obeys the Thomas commutation relations.

(b) Let X, X' be two vectors in the subspace $\mathfrak{m}$. Then in general the product of the associated coset representatives is *not* a coset representative. Show that such a product can always be written in the form

$$e^{X'}e^{X} = e^{Y}e^{R} \qquad \text{with} \quad \begin{aligned} X', X, Y &\in \mathfrak{m} \\ R &\in \mathfrak{h} \end{aligned}$$

(c) If X and $X' = X + \delta X$ are "close together," the product $e^{X'}e^{-X}$ is near the identity and can be written

$$e^{X'}e^{-X} = e^{\delta Y}e^{\delta R}$$

where δY, δR are vectors infinitesimally close to the origin in $\mathfrak{m}$ and $\mathfrak{h}$, respectively.

(d) Assume that X and X' are also small and show by expansion of the exponential to second order and comparison of the lowest order nonvanishing terms that

$$\begin{aligned} \delta X = X' - X \quad &= \delta Y \\ \tfrac{1}{2}[X, \delta X] = \tfrac{1}{2}[X', -X] &= \delta R \end{aligned}$$

16. Thomas Precession

(a) Compute the Lie algebra of the homogeneous Lorentz group $SO(3, 1)$. This group leaves invariant the inner product

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2, \qquad x^4 = ct$$

(b) Show that the infinitesimal generators X_{ij} span the maximal compact subalgebra $SO(3)$, whereas the generators X_{i4} exponentiate onto the set of "boosts" $SO(3, 1)/SO(3)$. Interpret these boost operations $B(\mathbf{v})$ as transformations from a frame at rest to one moving with fixed velocity $\mathbf{v}$ and axes parallel to a frame at rest. ($1 \leq i$, $j \leq 3$.)

(c) Show that the decomposition

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} & &\oplus \mathfrak{p} \\ \mathfrak{so}(3, 1) &= \mathfrak{so}(3) & &\oplus \{\mathfrak{so}(3, 1) \bmod \mathfrak{so}(3)\} \\ &= \textstyle\sum \theta^{ij}X_{ij} & &\oplus \textstyle\sum v^i X_{i4} \end{aligned}$$

obeys the Thomas commutation relations.

(d) Show that, for nonrelativistic velocities $|\mathbf{v}|/c \ll 1$, the vector in $\mathfrak{p}$ that maps onto $B(\mathbf{v})$ is

$$B(\mathbf{v}) = \mathrm{EXP}\,\frac{\mathbf{v}}{c}\cdot\mathbf{V}, \qquad V_i = X_{i4}$$

Show that, for arbitrary velocities $|\mathbf{v}| < c$, the vector in $\mathfrak{p}$ that maps onto the group operation $B(\mathbf{v})$ is $\boldsymbol{\gamma}\cdot\mathbf{V}$, where

$$\boldsymbol{\gamma}\cdot\mathbf{V} = \frac{\mathrm{Tanh}^{-1}\,|\mathbf{v}|/c}{|\mathbf{v}|/c}\,\frac{\mathbf{v}}{\mathrm{c}}\cdot\mathbf{V}$$

Show that the product of two successive boosts, $B(\mathbf{v}_1)$ followed by $B(\mathbf{v}_2)$, is itself a boost if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. Use the addition of the arguments $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ in the Lie algebra to construct the matrix product:

$$\begin{array}{ccccc} B(\mathbf{v}_2) & B(\mathbf{v}_1) & = & B(\mathbf{v}') \\ \updownarrow & \updownarrow & & \updownarrow \\ \mathrm{EXP}\,\boldsymbol{\gamma}_2\cdot\mathbf{V} & \mathrm{EXP}\,\boldsymbol{\gamma}_1\cdot\mathbf{V} & = & \mathrm{EXP}\,(\boldsymbol{\gamma}_2+\boldsymbol{\gamma}_1)\cdot\mathbf{V} \end{array}$$

In particular, derive the **Einstein velocity addition law**

$$\mathbf{v}' = \frac{\mathbf{v}_1 + \mathbf{v}_2}{1 + \dfrac{\mathbf{v}_1}{c}\cdot\dfrac{\mathbf{v}_2}{c}}$$

(e) The rest frame of a particle with velocity $\mathbf{v}$ is obtained by applying $B(\mathbf{v})$ to an observer's rest frame. The rest frame for a particle of velocity $\mathbf{v} + \delta\mathbf{v}$ *cannot* be obtained by applying the operation $B(\delta\mathbf{v})$ to the frame moving with velocity $\mathbf{v}$, unless $\delta\mathbf{v}$ and $\mathbf{v}$ happen to be parallel. Instead, the rest frame of the particle with velocity $\mathbf{v} + \delta\mathbf{v}$ must be obtained by

(i) First going from the rest frame of the particle moving with velocity $\mathbf{v}$ to the observer's rest frame using $B(-\mathbf{v}) = B^{-1}(\mathbf{v})$;

(ii) then going from the observer's rest frame to a frame moving with velocity $\mathbf{v} + \delta\mathbf{v}$ using $B(\mathbf{v} + \delta\mathbf{v})$.

The result is

$$B(\mathbf{v} + \delta\mathbf{v})B(-\mathbf{v}).$$

(f) Show that this product is equivalent to

$$\mathrm{EXP}\left(\frac{\mathbf{v}+\delta\mathbf{v}}{c}\right)\cdot\mathbf{V}\;\mathrm{EXP}\left(\frac{-\mathbf{v}}{c}\right)\cdot\mathbf{V} = \mathrm{EXP}\,\frac{\delta\mathbf{v}}{c}\cdot\mathbf{V}\;\mathrm{EXP}\,\delta\boldsymbol{\theta}_T\cdot\mathbf{L}$$

The Thomas rotation is given by

$$\begin{aligned}\delta\boldsymbol{\theta}_T\cdot\mathbf{L} &= \frac{1}{2}\left[\left(\frac{\mathbf{v}+\delta\mathbf{v}}{c}\right)\cdot\mathbf{V}, \left(\frac{-\mathbf{v}}{c}\right)\cdot\mathbf{V}\right] \\ &= \frac{1}{2}\frac{\mathbf{v}\times\delta\mathbf{v}}{c^2}\cdot\mathbf{L}\end{aligned}$$

Using $\delta \mathbf{v} = \mathbf{a}\, \delta t$, compute the Thomas precession:

$$\frac{\delta \boldsymbol{\theta}_T}{\delta t} = \boldsymbol{\omega}_T = \frac{\mathbf{v} \times \mathbf{a}}{2c^2}$$

17. Consider the three-dimensional Lie algebra spanned by the generators X, Y, Z with commutation relations

$$[X, Y] = Z$$
$$[Z, X] = [Z, Y] = 0$$

(a) Show that the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \qquad \begin{array}{r} X, Y \text{ span } \mathfrak{m} \\ Z \text{ spans } \mathfrak{h} \end{array}$$

is a Thomas decomposition.

(b) For $\lambda \to 0$, show

$$\begin{aligned} e^{(X+\lambda Y)} e^{-X} &= e^{\lambda Y} e^{1/2[X+\lambda Y,\, -X]} \\ &= e^{\lambda Y} e^{\lambda/2[X,\, Y]} \end{aligned}$$

(c) Show that the order of the powers of Y and $Z = [X, Y]$ on the right-hand side is immaterial. Conclude that the foregoing expression is valid for *arbitrary* values of the parameter λ. In particular, set $\lambda = 1$ and recover the BCH formula:

$$e^X e^Y = e^{(X+Y)} e^{1/2[X,\, Y]}$$

(d) Show that the factor of $\frac{1}{2}$ that occurs in the BCH relation just given, and the factor of $\frac{1}{2}$ that occurs in the Thomas precession, arise from the same source. In particular, show that the latter arises simply from the Thomas commutation relations. Finally, show that this factor $\frac{1}{2}$ is exactly the factor $1/2!$ which is the coefficient of the quadratic term in the power series expansion of the exponential function:

$$e^X = 1 + X + \underset{\substack{\uparrow \\ \text{Thomas factor of } \frac{1}{2}}}{\frac{1}{2!}} (X)^2 + \cdots$$

18. Contract the algebra spanned by J_3, $J_\pm$:

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_3$$

with respect to the subalgebra spanned by J_-. Use a simple Inönü-Wigner contraction and define

$$\lim_{\varepsilon \to 0} \varepsilon(2J_3) \to P$$

$$\lim_{\varepsilon \to 0} \varepsilon \quad J_+ \to T$$

$$\lim_{\varepsilon \to 0} \quad J_- \to V$$

Show in particular that the commutation relations in the contracted algebra are isomorphic to the commutations relations of the operators

$$P' = \partial_x$$
$$T' = \partial_t$$
$$V' = t\,\partial_x$$

Conclude that if the algebra $\mathfrak{a}_1$ is contracted with respect to a nilpotent subalgebra, the Galilean algebra $\mathfrak{g}(1)$ results. Why is it that the Galilean algebra $\mathfrak{g}(n)$ cannot be constructed by simple Inönü-Wigner contraction from a simple algebra when $n > 1$?

19. Show that the operators x^i and $\partial_j = \partial/\partial x^j$ obey the same commutation relations as do the Boson creation and annihilation operators ($1 \le i, j \le n$):

$$[x^i, \partial_j] = -\delta_{ij} = [b_i^\dagger, b_j]$$

Conclude that the generators for any Lie group of transformations can be modeled by Boson creation and annihilation operators. Construct a realization for each of the root spaces A_{n-1}; D_n, B_n, C_n in terms of the appropriate Boson operators. Use the results of Chapter 8, Section II for these purposes.

20. Compute the commutation relations of the operators

$$b^\dagger b^\dagger b, \; b^\dagger b, \; -b$$

where $b^\dagger$ and b are the Boson creation and annihilation operators obeying $[b^\dagger, b] = -1$. Show that these three operators span an algebra isomorphic with $\mathfrak{a}_1$ under the association

$$J_+ = b^\dagger b^\dagger b$$
$$J_3 = b^\dagger b$$
$$J_- = -b.$$

21. Let $\mathfrak{m}$ be a Lie algebra of $n \times n$ matrices, with $R, S, T, U, \cdots \in \mathfrak{m}$. Let $a^\dagger$ be the $1 \times n$ row matrix of creation operators $a^\dagger = (a_1^\dagger, a_2^\dagger, \cdots, a_n^\dagger)$, where the $a_i^\dagger$ can represent creation operators for the ith mode of *either* a fermion *or* a boson system. Finally, let $\mathscr{R} = a^\dagger R a = \sum_i \sum_j a_i^\dagger R_{ij} a_j$, $\mathscr{S} = a^\dagger S a$, etc. Then show that

(i) $[R, S] = T \Leftrightarrow [\mathscr{R}, \mathscr{S}] = \mathscr{T}$
(ii) $e^R e^S = e^U \Leftrightarrow e^{\mathscr{R}} e^{\mathscr{S}} = e^{\mathscr{U}}$

Notes and References

The material in Section I.2 closely follows reference [3]. The material in Section II.1 and II.2 follows essentially the presentation given by Saletan in reference [4]. The material in Section II.3 follows reference [7]. In Section III.1 we review the material presented in references [8] to [17], presenting it in considerably simplified form. The presentation of the rank-1 expansion procedure is a slight enlargement of the material in reference [18].

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Author Index

Subject Index

References to definitions in **boldface**; references to problems in *italics*.