# Constrained Scalar Quantum Fields

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**Abstract**    This article uses the path integral formulation to construct toy models of multi-component scalar quantum fields whose equations of motion are implicitly nonlinear because of a constraint on the values of the field variables. The constraint can lead to interesting phenomena, like spontaneous symmetry breaking and asymptotic freedom, even though the equation of motion looks very simple.

This article constructs a few families of such models, using a path integral formulation that treats spacetime as a discrete lattice. Each family is characterized by a different type of **target space**, the space of possible values of the scalar field at each point in spacetime. In the $O(N)$ **models**, the target space is a sphere $S^{N-1}$. In the $Z_n$ **models** or **clock models**, the target space consists of $n$ equally-space points around a circle. In the **principal chiral models**, the target space is a Lie group.

For some of these models (the $O(2)$ model, $Z_n$ models, and principal chiral models), this article also explains how to derive a corresponding hamiltonian formulation. This is especially interesting in the case of the $Z_n$ models, because deriving a hamiltonian formulation requires taking a limit as time becomes continuous, but the field variables themselves are constrained to a discrete set of values. This article explains how a useful hamiltonian formulation can still be derived by taking a special kind of continuous-time limit. For the $Z_2$ model, the result is the hamiltonian formulation of the quantum Ising model, whose phase structure is studied in article 81040.

# Contents

# 1    Introduction to the $O(N)$ models

In quantum field theory (QFT), the collection of models that we know how to construct nonperturbatively[1] in continuous $d$-dimensional spacetime, without any mathematical ambiguity, is very limited when $d \geq 3$. The collection is much larger if we are willing to treat spacetime as a discrete structure, like a lattice.[2] This article considers a class of lattice-based models whose construction is especially simple. Each model is constructed using a path integral[3] whose action $S[\phi]$ is a functional of a scalar field $\phi(x)$ with one or more components at each point $x$ in spacetime:

$$\phi(x) = \big(\phi_1(x), \, ..., \, \phi_N(x)\big). \tag{1}$$

The norm of the field is constrained to satisfy the constraint

$$\phi^2(x) \equiv \sum_k \big(\phi_k(x)\big)^2 = R^2 \tag{2}$$

for a given constant $R > 0$. The action is required to have $O(N)$ rotational symmetry in this abstract $(N-1)$-dimensional space. This is an **internal symmetry**,[4] so it is exact even when spacetime is treated as a lattice.

In this article, the name $\boldsymbol{O(N)}$ **model**[5] refers to the simplest of these models with $O(N)$ symmetry, namely the one for which $(\partial\phi)^2$ is the only term in the lagrangian.

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[1] *Nonperturbatively* means without relying on any small-parameter approximations. Section 14.7 in Fradkin (2022a) says, "quantum field theories have properties that largely cannot be guessed from perturbation theory."

[2] Treating spacetime as a discrete structure is artificial, but we can chose the scale of the discreteness to be much finer than the finest resolution of any existing measurements.

[3] Article 63548

[4] Article 21916

[5] The name **N-vector model** is also used.

# 2   Motive

In a way, the constraint (2) is unnatural, because at resolutions that are sufficiently coarse compared to the spacetime lattice spacing, the predictions of a model that was constructed using this constraint become indistinguishable from the predictions of many other models that were constructed without the constraint.[6]

One motivation for considering models with the constraint (2) is that even the simplest such models have nonzero interactions. Without the constraint, the simplest[7] models are relatively boring, because they don't have any interactions. We can add interaction terms (terms involving higher-than-quadratic powers of the field variables),[8] but that introduces an unlimited number of coefficients, with no obviously-most-natural way of choosing their values. In contrast, among models satisfying the constraint (2), the simplest ones already include nonzero interactions. The constraint (2) and the symmetry requirement imply that the only possible non-constant terms in the lagrangian are terms with derivatives.[9] In the simplest models of this type, the only term in the lagrangian is $(\partial\phi)^2$, but the equation of motion is still implicitly nonlinear because of the constraint (2). As a result, even the simplest models of this type have nonzero interactions, and in some cases these interactions persist even in the continuous-spacetime limit.

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[6]This is an example of **universality** (article 10142). In particular, the constraint is not preserved by the momentum-shell renormalization group (article 22212).

[7]Here, *simplest* means that the action involves only terms that are quadratic in the field variables, which is the lowest order for which the model has any content at all (article 30983).

[8]Article 52890

[9]The derivatives $\partial\phi$ are really finite differences, because spacetime is treated as a lattice, so such terms make sense even in the single-component case ($N = 1$), where the constraint (2) implies $\phi(x) = \pm1$.

# 3  Framework

In the path integral formulation for scalar quantum fields, the state[10] at a given time[11] is represented by a complex-valued function $\Psi[\phi]$ of the field variables $\phi$ associated with all points in space at that time. The inner product of two states $\Psi_1[\phi]$ and $\Psi_2[\phi]$ is

$$\langle\Psi_1|\Psi_2\rangle \equiv \int [d\phi]\ \Psi_1^*[\phi]\Psi_2[\phi], \tag{3}$$

with an integration measure $[d\phi]$ that will be defined in the following sections. Only functions $\Psi[\phi]$ for which $\langle\Psi|\Psi\rangle$ is finite are used to represent states.

The relationship between an initial state $\Psi_i$ and a final state $\Psi_f$ can be written[12]

$$\Psi_f[\phi]_C \propto \int [d\phi]_{A\cup B}\ \exp\big(-S[\phi]_{A\cup B\cup C}\big)\,\Psi_i[\phi]_A \tag{4}$$

where $S$ is the (euclidean) action, $[\phi]_X$ denotes the collection of field variables associated with a region $X$ of spacetime, and the regions $A, B, C$ are defined like this:[13]

  $A$ is the set of spacetime points at time $t_i$,
  $B$ is the set of spacetime points with times in the range $t_i < t < t_f$,
  $C$ is the set of spacetime points at time $t_f$.

A point in $d$-dimensional spacetime will be denoted $x = (t, \mathbf{x})$, where $t$ is the time coordinate and $\mathbf{x}$ is the list of $D \equiv d-1$ spatial coordinates. The action $S$ and the measure $[d\phi]$ will be defined for the various models in the following sections.

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[10]In this article, *state* means an element of the Hilbert space. It it also called a *state-vector*, for the reasons explained in article 03431.

[11]This description uses the Schrödinger picture (article 22871).

[12]This article uses the euclidean path integral, from which the lorentzian path integral can be recovered by Wick rotation (article 63548).

[13]In this formulation, spacetime is treated as a lattice of finite extent in the spatial directions so that the number of integration variables $[\phi]_{A\cup B}$ is finite.

# 4   The $O(2)$ model: measure and action

For the $O(2)$ model,[14] the constraint (2) may be solved by writing

$$\phi(x) = \big(R\cos\theta(x),\ R\sin\theta(x)\big), \tag{5}$$

where a single angular variable $\theta(x)$ is associated with each lattice site $x$. In this case, the integration measure $[d\phi]$ in equations (3) and (4) is defined by

$$[d\phi] \equiv \prod_x d\theta(x) \qquad -\pi \le \theta(x) < \pi,$$

where the product is over all spacetime points $x$ at the specified time(s).[15]

   Let $\epsilon_0, \epsilon_1, \epsilon_2, ..., \epsilon_{d-1}$ be a set of $d$ basis vectors for the lattice, each with magnitude $\epsilon$. The action is

$$S[\phi]_X = \epsilon^d \sum_{x\in X} \frac{\big(\partial\phi(x)\big)^2}{2} = \epsilon^d \sum_{x\in X}\sum_{a,k} \frac{\big(\partial_a\phi_k(x)\big)^2}{2}$$

$$\equiv \epsilon^d \sum_{x\in X}\sum_{a,k} \frac{\big(\phi_k(x+\epsilon_a) - \phi_k(x)\big)^2}{2\epsilon^2}.$$

This looks superficially like the action for a free massless scalar field, but using (5) gives the manifestly non-quadratic action

$$S[\phi]_X = \epsilon^d R^2 \sum_{x\in X}\sum_a \frac{1 - \cos\big(\theta(x+\epsilon_a) - \theta(x)\big)}{\epsilon^2}. \tag{6}$$

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[14]Section 5 will generalize this construction to the $O(N)$ model for arbitrary $N$.

[15]In equation (3), the product is over all points in space at a single time. In equation (4), the product is over all $x \in A \cup B$.

# 5   The $O(N)$ model: measure and action

For any $N \geq 2$, the constraint (2) may be solved by writing

$$
\begin{aligned}
\phi_1(x) &= R \cos\big(\theta_1(x)\big) \\
\phi_2(x) &= R \sin\big(\theta_1(x)\big) \cos\big(\theta_2(x)\big) \\
\phi_3(x) &= R \sin\big(\theta_1(x)\big) \sin\big(\theta_2(x)\big) \cos\big(\theta_3(x)\big) \\
&\vdots \\
\phi_{N-1}(x) &= R \sin\big(\theta_1(x)\big) \cdots \sin\big(\theta_{N-2}(x)\big) \cos\big(\theta_{N-1}(x)\big) \\
\phi_N(x) &= R \sin\big(\theta_1(x)\big) \cdots \sin\big(\theta_{N-2}(x)\big) \sin\big(\theta_{N-1}(x)\big)
\end{aligned}
$$

with

$$
0 \leq \theta_k(x) < \pi \ \text{ if } \ k \leq N-2, \qquad -\pi \leq \theta_{N-1}(x) < \pi.
$$

The integration measure $[d\phi]$ in equations (3) and (4) is

$$
[d\phi] = \prod_x d\Omega(x)
$$

with[16]

$$
d\Omega(x) \equiv \prod_{k=1}^{N-1} \Big( \big(\sin\theta_k(x)\big)^{N-1-k} d\theta_k(x) \Big).
$$

Just like in section 4, the action is

$$
S[\phi]_X = \epsilon^d \sum_{x \in X} \frac{\big(\partial\phi(x)\big)^2}{2} \equiv \epsilon^d \sum_{x \in X} \sum_{a,k} \frac{\big(\phi_k(x+\epsilon_a) - \phi_k(x)\big)^2}{2\epsilon^2}. \tag{7}
$$

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[16]This is the usual measure on the sphere $S^{N-1}$, expressed using hyperspherical coordinates (Blumenson (1960)). This is the angular part of what the measure would have been without the constraint (2). This measure has $O(N)$ symmetry, even though that symmetry is obscured when the measure is written this way.

# 6   Another way to enforce the constraint

Consider a single lattice site $x$, and temporarily write $\phi_k$ as an abbreviation for $\phi_k(x)$. At this site, the constraint (2) is $\phi^2 = R^2$. Without that constraint, the quantity $R \equiv \sqrt{\phi^2}$ would be variable (not constant). Whether $R$ is variable or not, the $N$th component of $\phi$ may be written

$$\phi_N = \pm\varphi \qquad \varphi \equiv \sqrt{R^2 - \tilde{\phi}^2} \tag{8}$$

where $\tilde{\phi}$ includes only the first $N - 1$ components of $\phi$. If the value of $R$ were not fixed by the constraint (2), then the differential of (8) would be

$$d\varphi = \frac{R\,dR - \tilde{\phi}\cdot d\tilde{\phi}}{\varphi}, \tag{9}$$

and the integral of $f(\phi)$ over all $N$ independent components of $\phi$ could be written

$$\int d^N\phi\; f(\phi) = \int d\varphi \wedge d^{N-1}\tilde{\phi} \sum_{\text{signs}} f(\phi) = \int dR\, d^{N-1}\tilde{\phi} \sum_{\text{signs}} \frac{R}{\varphi} f(\phi).$$

The sum is over the two possible signs in (8). Now we can implement the constraint (2) just by omitting the factor $dR$ from the measure. Then the integral becomes

$$\int d^{N-1}\tilde{\phi} \sum_{\text{signs}} \frac{R}{\varphi} f(\phi).$$

This leads to an alternative way of writing the integration measure in equations (3) and (4), instead of using the approach that was described in section 5. The important message is that in this alternative formulation, we must include the factor $1/\varphi$ in the integrand. Pages 392-393 in Weinberg (1995) explain what would go wrong if this factor were excluded.[17]

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[17]That author writes the factor as $\exp\left(\log\left(R/\varphi\right)\right)$ so that the log term can be viewed as part of the action $S$.

# 7   The $O(2)$ model: single time-step, part 1

Section 10 will derive the hamiltonian for the $O(2)$ model that was defined in section 4. To prepare, consider the action for a single time-step. Write the variables associated with the past and future endpoints of the time-step as $\theta(\mathbf{x})$ and $\theta'(\mathbf{x})$, respectively, for each point $\mathbf{x}$ in space. Then, according to equation (6), the action for a single time-step may be written

$$s[\theta,\theta'] = \epsilon^d R^2 \left( \sum_{\mathbf{x}} \frac{1 - \cos\left(\theta'(\mathbf{x}) - \theta(\mathbf{x})\right)}{\epsilon^2} + \sum_{\{\mathbf{x},\mathbf{y}\}} \frac{1 - \cos\left(\theta(\mathbf{y}) - \theta(\mathbf{x})\right)}{\epsilon^2} \right), \quad (10)$$

where the sum over $\{\mathbf{x},\mathbf{y}\}$ is over all unordered pairs of nearest-neighbor points in the spatial lattice. To take the continuous-time limit, we must generalize the action so that the lattice spacing $dt$ in the time direction is independent of the lattice spacing $\epsilon$ in the $D \equiv d - 1$ spatial directions, like this:

$$s[\theta,\theta'] = R^2 \, dt \, \epsilon^D \left( \sum_{\mathbf{x}} \frac{1 - \cos\left(\theta'(\mathbf{x}) - \theta(\mathbf{x})\right)}{dt^2} + \sum_{\{\mathbf{x},\mathbf{y}\}} \frac{1 - \cos\left(\theta(\mathbf{y}) - \theta(\mathbf{x})\right)}{\epsilon^2} \right).$$
$$(11)$$

Use the abbreviations

$$\beta \equiv \frac{R^2 \, \epsilon^D}{dt} \quad (12)$$

and

$$V[\theta] \equiv R^2 \, \epsilon^D \sum_{\{\mathbf{x},\mathbf{y}\}} \frac{1 - \cos\left(\theta(\mathbf{y}) - \theta(\mathbf{x})\right)}{\epsilon^2} \quad (13)$$

so that

$$s[\theta,\theta'] = \beta \sum_{\mathbf{x}} \left[1 - \cos\left(\theta'(\mathbf{x}) - \theta(\mathbf{x})\right)\right] + dt \, V[\theta]. \quad (14)$$

Section 8 will use this to derive another way of writing the evolution equation (4) for a single time-step.

# 8   The $O(2)$ model: single time-step, part 2

If $\Psi[\theta]$ is the initial state, then the state after a single time-step is proportional to

$$\Psi'[\theta'] \equiv \int [d\theta] \ e^{-s[\theta,\theta']}\Psi[\theta]$$

where $[d\theta]$ denotes the integral over each $\theta(\mathbf{x})$ from $-\pi$ to $\pi$. Using (14), this becomes

$$\Psi'[\theta'] = \int \prod_{\mathbf{x}} \big(d\theta(\mathbf{x}) \ \exp\big(-\beta[1-\cos(\theta'(\mathbf{x})-\theta(\mathbf{x}))]\big)\big) f[\theta] \qquad (15)$$

with

$$f[\theta] \equiv e^{-dt\,V[\theta]}\Psi[\theta]. \qquad (16)$$

By shifting the integration variables, this may also be written

$$\Psi'[\theta'] = \int \prod_{\mathbf{x}} \big(d\theta(\mathbf{x}) \ \exp\big(-\beta[1-\cos(\theta(\mathbf{x}))]\big)\big) f[\theta+\theta'].$$

Now use the identity

$$f[\theta+\theta'] = \prod_{\mathbf{x}} \exp\left(\theta(\mathbf{x})\frac{\partial}{\partial\theta'(\mathbf{x})}\right) f[\theta']$$

to get this new way of writing the time-evolution equation for a single time-step:

$$\Psi'[\theta'] = \int \prod_{\mathbf{x}} \left(d\theta(\mathbf{x}) \ \exp\big(-\beta[1-\cos(\theta(\mathbf{x}))]\big)\exp\left(\theta(\mathbf{x})\frac{\partial}{\partial\theta'(\mathbf{x})}\right)\right) f[\theta']. \quad (17)$$

So far, no approximations have been made.

Section 10 will finish deriving the hamiltonian, after an interlude about unitarity in section 9.

# 9   The $O(2)$ model: is it unitary?

In quantum theory, time evolution should be unitary: it should preserve the norm of the state. This requirement is automatically satisfied if time evolution is generated by a self-adjoint operator (the hamiltonian). The time evolution equation (4) deviates from that condition in two ways. First, it treats time as discrete, so the usual concept of a *generator* of time evolution does not apply. Second, it uses the euclidean path integral, which clearly does not preserve the norm of the state because of the factor $e^{-S}$ in the integrand. That's okay, because the euclidean path integral is just a convenient technical device from which time evolution in lorentzian signature can be recovered by Wick rotation.[18] Section 10 will show that unitarity is restored after taking a continuous-time limit and Wick-rotating back to lorentzian signature.

Article 63548 showed that for a family of models without the constraint (2), time evolution is already unitary even without taking a continuous-time limit, but that's not the case for the models considered in this article. The rest of this section shows that the evolution defined by equations (15)-(16) for a single discrete time-step is not unitary, not even after Wick rotation from euclidean to lorentzian signature, not even after including an overall factor that equations (15)-(16) didn't bother to include because they're not unitary anyway.

To simplify the analysis, suppose that the spatial lattice has only one point. Then equation (15) reduces to

$$\Psi'(\theta') \propto \int_{-\pi}^{\pi} d\theta \ \exp\big(\beta\cos(\theta'-\theta)\big)f(\theta). \tag{18}$$

Now $\theta$ is a single real variable, and so is $\theta'$. Equation (18) can be expressed in terms of the Bessel functions $J_n(z)$ with integer order $n$, which are defined for arbitrary

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[18]Article 63548

complex-valued $z$. We can use the generating function[19]

$$\exp\left(\left(s - \frac{1}{s}\right)\frac{z}{2}\right) = \sum_{n=-\infty}^{\infty} J_n(z)s^n$$

as the definition of $J_n(z)$. Set $s = ie^{i\theta}$ to get the **Jacobi-Anger expansion**

$$\exp\left(iz\cos\theta\right) = \sum_{n=-\infty}^{\infty} J_n(z)i^n e^{in\theta}.$$

Use this in (18) with $z = -i\beta$ to get

$$\Psi'(\theta') \propto \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} d\theta \; J_n(-i\beta)i^n e^{in(\theta'-\theta)} f(\theta) \tag{19}$$

$$= \sum_{n=-\infty}^{\infty} J_n(-i\beta)i^n e^{in\theta'} f_n \tag{20}$$

where $f_n$ are the complex-valued coefficients in the Fourier transform of $f(\theta)$.

In order for the time evolution equation to be unitary, it must at least be reversible: if the norm of the initial state $\Psi$ is nonzero, then the norm the final state $\Psi'$ must also be nonzero. Using equation (20), we can say it like this: if any one of the quantities $f_n$ is nonzero, then the final state $\Psi'$ must also be nonzero. Otherwise, the time evolution equation is not even reversible, much less unitary.

The Bessel function $J_n(z)$ has an infinite number of zeros (values of $z$ for which $J_n(z) = 0$), and all of them occur at real values of $z$.[20] Equation (20) involves only imaginary values of $z$, but the question is whether time evolution is unitary after Wick rotation from euclidean to lorentzian signature but before taking a continuous-time limit. Wick rotation replaces $\beta \to -i\beta$.[21] After that replacement,

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[19]Equation (5.3.4) in Lebedev (1972), and equation (5) in Baring (2022)
[20]Lebedev (1972), section 5.13, theorem 1
[21]Article 63548

equation (20) becomes

$$\Psi'(\theta') \propto \sum_{n=-\infty}^{\infty} J_n(-\beta) i^n e^{in\theta'} f_n. \tag{21}$$

Now the theorem that was quoted before says that an infinite number of values of $\beta$ exist for which at least one of the quantities $f_n$ does not contribute to the sum because $J_n(-\beta)$ is zero for that $n$. For these values of $\beta$, the time evolution equation is not even reversible, much less unitary.

We can recover unitarity by taking a continuous-time limit, in which $\beta \to \infty$.[22] This will be done in section 10.

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[22]Equation (12)

# 10   The $O(2)$ model: hamiltonian

In the continuous-time limit $dt \to 0$, the coefficient $\beta$ defined by equation (12) becomes large. The quantity $1 - \cos(\theta)$ in equation (17) is nonnegative, so when $\beta$ is large, only values of $\theta$ for which $1 - \cos(\theta) \approx 0$ make a significant contribution to equation (17). This justifies using the approximation

$$1 - \cos(\theta) \approx \frac{\theta^2}{2}$$

in equation (17),[23] which gives

$$\Psi'[\theta'] \approx \int \prod_{\mathbf{x}} \left( d\theta(\mathbf{x}) \, \exp\left( -\beta \frac{\theta^2(\mathbf{x})}{2} \right) \exp\left( \theta(\mathbf{x}) \frac{\partial}{\partial \theta'(\mathbf{x})} \right) \right) f[\theta'].$$

Only small values of $\theta(\mathbf{x})$ contribute when $\beta$ is large, so we might as well extend the limits of integration to $\pm\infty$. Then the integrals over $\theta(\mathbf{x})$ are Gaussian, and evaluating them gives

$$\Psi'[\theta'] \propto \prod_{\mathbf{x}} \exp\left( \frac{1}{2\beta} \left( \frac{\partial}{\partial \theta'(\mathbf{x})} \right)^2 \right) f[\theta'].$$

Now use (12), (13), and (16) to get

$$\Psi'[\theta'] \propto e^{-H\, dt} \, \Psi[\theta']$$

where the hamiltonian is

$$H = -\epsilon^D \sum_{\mathbf{x}} \frac{1}{2} \left( \frac{1}{R\,\epsilon^D} \frac{\partial}{\partial \theta(\mathbf{x})} \right)^2 + R^2\, \epsilon^D \sum_{\{\mathbf{x},\mathbf{y}\}} \frac{1 - \cos\left( \theta(\mathbf{y}) - \theta(\mathbf{x}) \right)}{\epsilon^2}. \tag{22}$$

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[23]The simplicity of this justification is one advantage of the euclidean path integral instead of the lorentzian version (footnote 12 in section 3).

# 11    Another way to write the hamiltonian

Use the identity

$$\cos\left(\theta(\mathbf{y}) - \theta(\mathbf{x})\right) = \frac{e^{i\theta(\mathbf{y})}e^{-i\theta(\mathbf{x})} + \text{complex conjugate}}{2}$$

to see that the hamiltonian (22) may also be written[24]

$$H = \epsilon^D \sum_{\mathbf{x}} H_0(\mathbf{x}) + R^2\,\epsilon^{D-2} \sum_{\{\mathbf{x},\mathbf{y}\}} \left(1 - \frac{Z(\mathbf{y})Z^\dagger(\mathbf{x}) + Z^\dagger(\mathbf{x})Z(\mathbf{y})}{2}\right) \qquad (23)$$

where the operators $H_0(\mathbf{x})$ and $Z(\mathbf{x})$ are defined by

$$H_0(\mathbf{x})\Psi[\theta] = -\frac{1}{2}\left(\frac{1}{R\,\epsilon^D}\,\frac{\partial}{\partial\theta(\mathbf{x})}\right)^2 \Psi[\theta] \qquad (24)$$

and

$$Z(\mathbf{x})\Psi[\theta] = e^{i\theta(\mathbf{x})}\Psi[\theta]. \qquad (25)$$

The operators $Z(\mathbf{x})$ and $Z^\dagger(\mathbf{x})$ commute with $H_0$ when $\mathbf{y} \neq \mathbf{x}$, but not when $\mathbf{y} = \mathbf{x}$.

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[24]Throughout this article, a sum over $\{\mathbf{x}, \mathbf{y}\}$ means the sum over all unordered nearest-neighbor pairs of points in the spatial lattice, as in section 7.

# 12   The $Z_n$ model

In the $\boldsymbol{Z_n}$ **model**, the field is given by

$$\phi(x) = \big(R\cos\theta(x),\ R\sin\theta(x)\big) \tag{26}$$

like in the $O(2)$ model, but now the angles $\theta(x)$ are restricted to integer multiples of $2\pi/n$:

$$\theta(x) = 2\pi k(x)/n \qquad\qquad k(x) \in \{0, 1, 2, ..., n-1\}. \tag{27}$$

The integration measure $[d\phi]$ in equations (3) and (4) is simply a sum over the integers $k(x)$ at each spacetime point $x$. The action is the same as the action for the $O(2)$ model,[25]

$$
\begin{aligned}
S[\phi]_X &= \epsilon^d \sum_{x\in X}\sum_{a,j} \frac{\big(\phi_j(x+\epsilon_a) - \phi_j(x)\big)^2}{2\epsilon^2} \\
&= \epsilon^d R^2 \sum_{x\in X}\sum_{a} \frac{1 - \cos\big(\theta(x+\epsilon_a) - \theta(x)\big)}{\epsilon^2},
\end{aligned} \tag{28}
$$

but with the angles $\theta(x)$ restricted to the discrete set of values (27).

The $Z_n$ model gets its name from some of its internal symmetries:[26] any transformation that adds the same integer to each $k(x)$ is a symmetry, and adding an integer multiple of $n$ has no effect at all. These transformations constitute the group called $Z_n$, the additive group of integers modulo $n$. This model is also called the $\boldsymbol{n}$**-clock model**,[27] because the allowed values of the field (26) are $n$ equally-spaced points around a circle, like the tick-marks on a clock. The $O(2)$ model can be recovered by taking $n\to\infty$.

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[25]Section 4

[26]When $n \geq 3$, the group of internal symmetries is larger than $Z_n$, because the transformation that reverses the sign of every $k(x)$ is also a symmetry. When $n = 2$, the second component of $\phi(x)$ is always zero, and reversing the sign of every $k(x)$ has the same effect as adding 1 to every $k(x)$.

[27]The letter $p$ or $q$ is often used instead of $n$.

# 13    Hamiltonian for the $Z_n$ model: perspective

Starting with the path integral formulation of the $Z_n$ model (section 12), we can take $n \to \infty$ to get the path integral formulation of the $O(2)$ model. This is relatively straightforward. From there, sections 7-10 explained how we can take a continuous-time limit $(dt \to 0)$ to get the hamiltonian formulation of the $O(2)$ model. That sequence of limits is depicted in this diagram:

$$
\begin{array}{l}
Z_n \text{ path integral} \\
\quad {\scriptstyle n\to\infty} \Big\downarrow \\
O(2) \text{ path integral} \xrightarrow[\;dt\to 0\;]{} O(2) \text{ hamiltonian}
\end{array}
$$

The next few sections explain how to implement the two new arrows in this diagram:

$$
\begin{array}{lcl}
Z_n \text{ path integral} & \xrightarrow[\;]{dt\to 0} & Z_n \text{ hamiltonian} \\
\quad {\scriptstyle n\to\infty}\Big\downarrow & & \Big\downarrow{\scriptstyle n\to\infty} \\
O(2) \text{ path integral} & \xrightarrow[\;dt\to 0\;]{} & O(2) \text{ hamiltonian}
\end{array}
$$

Starting with the path integral formulation of the $Z_n$ model, sections 14-17 will explain how take a continuous-time limit $(dt \to 0)$ to get a hamiltonian formulation of the $Z_n$ model. To check that this hamiltonian formulation of the $Z_n$ model is natural,[28] section 21 will show that it reproduces the hamiltonian formulation of the $O(2)$ model when $n \to \infty$.

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[28]This is worth checking, because the $dt \to 0$ limit represented by the top arrow is a non-obvious generalization of the $dt \to 0$ limit represented by the bottom arrow.

# 14   The $Z_n$ model: single time-step

The hamiltonian generates evolution in continuous time. We need to allow the coefficients of the time-derivative and spatial-derivative terms to be different so that we can take a continuous-time limit while retaining a finite lattice spacing in the spatial directions. For this purpose, write the action for a single time-step as

$$s[\theta, \theta'] = s_0[\theta, \theta'] + s_1[\theta] + \text{constant} \tag{29}$$

with

$$s_0[\theta, \theta'] = 2\kappa \sum_{\mathbf{x}} \Big(1 - \cos\big(\theta(\mathbf{x}) - \theta'(\mathbf{x})\big)\Big) \tag{30}$$

$$s_1[\theta] = -dt\, \epsilon^{D-2} R^2 \sum_{\{\mathbf{x},\mathbf{y}\}} \cos\big(\theta(\mathbf{x}) - \theta(\mathbf{y})\big). \tag{31}$$

$\theta(\mathbf{x})$ denotes the variables at one time, $\theta'(\mathbf{x})$ denotes the variables at the other time, and $D \equiv d-1$ is the number of spatial dimensions. The relationship between $2\kappa$ and the other coefficients will be specified in section 16. When $dt = \epsilon$ and $2\kappa = \epsilon^{d-2} R^2$, this reduces to the single time-step version of the original action (28).

If $\Psi[\theta]$ is the initial state, then the state after a single time step is[29]

$$\Psi'[\theta'] = \sum_{[\theta]} e^{-s[\theta,\theta']} \Psi[\theta]. \tag{32}$$

We can think of this as a matrix equation in which $[\theta]$ is a single index,[30] $\Psi$ and $\Psi'$ are both single-column matrices, and $T[\theta', \theta] \equiv e^{-s[\theta,\theta']}$ are the components of a square matrix $T$ called the **transfer matrix**. Written in matrix notation, equation (32) is $\Psi' = T\Psi$.

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[29]The square brackets in $\sum_{[\theta]}$ are a reminder that the sum is over an enormous number of variables (27), including one discrete variable $\theta(\mathbf{x})$ for each site $\mathbf{x}$ of the spatial lattice.

[30]This index can take $n^m$ different values, where $m$ is the number of sites in the spatial lattice (footnote 29).

# 15   The $Z_n$ model: is it unitary?

Section 17 will use a continuous-time limit to obtain a hamiltonian formulation of the $Z_n$ model. That will implicitly show that the $Z_n$ model is unitary when time is continuous, after Wick rotation from euclidean signature to lorentzian siguature as explained in section 9.

   Using the $Z_2$ model as an example, this section shows that time evolution is not unitary when time is discrete. This is similar to the conclusion in section 9 about the $O(2)$ model, but here the analysis is easier.

   As in section 9, suppose that the spatial lattice has only one point. Then the function $\Psi[\theta]$ in equation (32) is just a pair of complex numbers, $\Psi[0]$ and $\Psi[\pi]$, and the size of the transfer matrix is only $2 \times 2$. Explicitly, the transfer matrix is

$$T = \begin{bmatrix} e^{-s[0,0]} & e^{-s[\pi,0]} \\ e^{-s[0,\pi]} & e^{-s[\pi,\pi]} \end{bmatrix} = \begin{bmatrix} e^{-s_0[0,0]} & e^{-s_0[\pi,0]} \\ e^{-s_0[0,\pi]} & e^{-s_0[\pi,\pi]} \end{bmatrix} \begin{bmatrix} e^{-s_1[0]} & 0 \\ 0 & e^{-s_1[\pi]} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & e^{-2\kappa} \\ e^{-2\kappa} & 1 \end{bmatrix} \begin{bmatrix} e^{-s_1[0]} & 0 \\ 0 & e^{-s_1[\pi]} \end{bmatrix}$$

After Wick rotation back to lorentzian signature, this becomes

$$T = \begin{bmatrix} 1 & e^{i2\kappa} \\ e^{i2\kappa} & 1 \end{bmatrix} \begin{bmatrix} e^{is_1[0]} & 0 \\ 0 & e^{is_1[\pi]} \end{bmatrix}.$$

This is invertible (so time evolution is reversible) for all nonzero $\kappa$, but it's not unitary for most values of $\kappa$, not even after adjusting the overall normalization.

# 16   The $Z_n$ model: continuous-time limit

The action for a single time-step, equations (29)-(31), has the same form as it did in the $O(2)$ model, equation (11). To derive the hamiltonian for the $O(2)$ model, we took the coefficient $\kappa$ to be proportional to $1/dt$. In that case, as $dt \to 0$, the factor $e^{-s_0[\theta,\theta']}$ pushes all components of the transfer matrix to zero except those in which the differences $\theta(\mathbf{x}) - \theta'(\mathbf{x})$ approach zero at least as fast as $dt$ does. That's fine when the allowed values of these variables form a continuum, like they do in the $O(2)$ model, because then the differences can *approach* zero without being *equal* to zero.

In the $Z_n$ model, though, the variables are discrete, so the differences cannot approach zero unless they all become *equal* to zero when the magnitude of $dt$ drops below some finite threshold. As a result, if we required $\kappa \propto 1/dt$ in the $Z_n$ model, then all off-diagonal components[31] of the transfer matrix would approach zero exponentially (like $e^{-1/dt}$) as $dt \to 0$. The transfer matrix would not have the form $T = e^{-H\,dt}$ when $dt$ is small, which is one of the conditions that is normally used to define the hamiltonian $H$.

We can fix this by using a different relationship between $\kappa$ and $dt$. To motivate the relationship, write the components of the transfer matrix as

$$T[\theta', \theta] = T_0[\theta', \theta]e^{-s_1[\theta]}, \tag{33}$$

where $T_0$ is the matrix with components

$$T_0[\theta', \theta] \equiv e^{-s_0[\theta,\theta']} = \prod_{\mathbf{x}} \exp\left(-2\kappa\Big[1 - \cos\big(\theta(\mathbf{x}) - \theta'(\mathbf{x})\big)\Big]\right). \tag{34}$$

The components of $T_0$ with the largest magnitude are the diagonal components (those with $\theta'(\mathbf{x}) = \theta(\mathbf{x})$ for all $\mathbf{x}$): those components are equal to 1. The components with the second-largest magnitude are equal to

$$\exp\left(-2\kappa\big[1 - \cos(2\pi/n)\big]\right), \tag{35}$$

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[31]The off-diagonal components are those with $\theta'(\mathbf{x}) \neq \theta(\mathbf{x})$ for at least one $\mathbf{x}$.

which occurs whenever the difference $\theta'(\mathbf{x}) - \theta(\mathbf{x})$ modulo $2\pi$ is equal to $\pm 2\pi/n$ for one point $\mathbf{x}$ and is equal to zero for all other points. If we choose $\kappa$ so that the quantity (35) is proportional to $dt$,[32,33] then we can use

$$T = 1 - dt\,H + O(dt^2)$$

to define the hamiltonian $H$. This will be done in section 17.

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[32]For the $Z_2$ case, other accounts of this way of taking $dt \to 0$ include Fradkin and Susskind (1978) and section IV-A in Kogut (1979).

[33]With this choice, the components with the third-largest magnitude go to zero faster than (35) does. To check this, use the fact that every component of $T_0$ is equal to $e^{-2\kappa b}$ with a $\kappa$-independent coefficient $b \geq 0$. If $b' > b$, then $e^{-2\kappa b'}$ goes to zero faster than $e^{-2\kappa b}$ does.

# 17   The $Z_n$ model: hamiltonian

This section derives the hamiltonian, the operator that generates time-translations in the special continuous-time limit that was described in the previous section. This is the limit $dt \to 0$ with the $dt$-dependence of $\kappa$ given by the relationship

$$\exp\left(-2\kappa\big[1 - \cos(2\pi/n)\big]\right) = \alpha\, dt \tag{36}$$

for some $dt$-independent constant $\alpha$. To derive the hamiltonian, expand the transfer matrix to first order in $dt$. According to equation (31), expanding the factor $e^{-s_1[\theta]}$ in equation (33) to first order in $dt$ gives

$$e^{-s_1[\theta]} = 1 + dt\,\lambda \sum_{\{\mathbf{x},\mathbf{y}\}} \cos\big(\theta(\mathbf{x}) - \theta(\mathbf{y})\big) + O(dt^2) \tag{37}$$

with $\lambda \equiv \epsilon^{D-2} R^2$. According to equations (30) and (36), expanding the factor $T_0$ in equation (33) to first order in $dt$ gives

$$T_0 = 1 + dt\,\alpha \sum_{\mathbf{x}} \frac{X(\mathbf{x}) + X^\dagger(\mathbf{x})}{2} + O(dt^2) \tag{38}$$

where $X(\mathbf{x})$ is the matrix whose component $X(\mathbf{x})[\theta', \theta]$ is equal to 1 whenever $e^{i(\theta'(\mathbf{x}) - \theta(\mathbf{x}))} = e^{2\pi i/n}$ and is equal to zero otherwise. For $dt \to 0$, the hamiltonian $H$ is related to the transfer matrix by $T = e^{-dt\,H} = 1 - dt\,H + O(dt^2)$, so equations (37)-(38) imply

$$H = -\alpha \sum_{\mathbf{x}} \frac{X(\mathbf{x}) + X^\dagger(\mathbf{x})}{2} - \lambda H_1 \tag{39}$$

where $H_1$ is the diagonal matrix defined by

$$(H_1 \Psi)[\theta] = \left(\sum_{\{\mathbf{x},\mathbf{y}\}} \cos\big(\theta(\mathbf{x}) - \theta(\mathbf{y})\big)\right) \Psi[\theta]. \tag{40}$$

23

# 18    Another way to write the hamiltonian

As in section 11, the matrix $H_1$ may also be written

$$(H_1\Psi)[\theta] = \left( \sum_{\{\mathbf{x},\mathbf{y}\}} \frac{Z(\mathbf{y})Z^\dagger(\mathbf{x}) + Z^\dagger(\mathbf{x})Z(\mathbf{y})}{2} \right) \Psi[\theta] \qquad (41)$$

with $Z(\mathbf{x})$ defined by (25), as before. The difference is that now each of the variables $\theta(\mathbf{x})$ takes only a finite number of values, so $Z(\mathbf{x})$ can be viewed as a (diagonal) matrix. The matrix $X(\mathbf{x})$ that was defined in section 17 commutes with $Z(\mathbf{y})$ and $Z^\dagger(\mathbf{y})$ when $\mathbf{y} \neq \mathbf{x}$, but not when $\mathbf{y} = \mathbf{x}$.

Altogether, the hamiltonian (39) may be written[34,35]

$$H = -\alpha \sum_{\mathbf{x}} \frac{X(\mathbf{x}) + X^\dagger(\mathbf{x})}{2} - \lambda \sum_{\{\mathbf{x},\mathbf{y}\}} \frac{Z(\mathbf{y})Z^\dagger(\mathbf{x}) + Z^\dagger(\mathbf{x})Z(\mathbf{y})}{2}, \qquad (42)$$

with $Z(\mathbf{x})$ defined by equation (25) and with $X(\mathbf{x})$ defined as in section 17.

The matrices $X(\mathbf{x})$ and $Z(\mathbf{x})$ are unitary, and they satisfy $X^n(\mathbf{x}) = I$ and $Z^n(\mathbf{x}) = I$. Their commutation relations are

$$X(\mathbf{x})Z(\mathbf{x}) = e^{-2\pi i/n} Z(\mathbf{x})X(\mathbf{x}) \qquad\qquad X(\mathbf{x})X^\dagger(\mathbf{x}) = X^\dagger(\mathbf{x})X(\mathbf{x})$$
$$X(\mathbf{x})Z^\dagger(\mathbf{x}) = e^{2\pi i/n} Z^\dagger(\mathbf{x})X(\mathbf{x}) \qquad\qquad Z(\mathbf{x})Z^\dagger(\mathbf{x}) = Z^\dagger(\mathbf{x})Z(\mathbf{x})$$

and $A(\mathbf{x})B(\mathbf{y}) = B(\mathbf{y})A(\mathbf{x})$ whenever $\mathbf{x} \neq \mathbf{y}$, for all pairs $A \in \{X, X^\dagger, Z, Z^\dagger\}$ and $B \in \{X, X^\dagger, Z, Z^\dagger\}$.

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[34]Ortiz *et al* (2012), section 3.2

[35]Again, the sum over $\{\mathbf{x}, \mathbf{y}\}$ means the sum over all unordered nearest-neighbor pairs of points in the spatial lattice.

# 19  Example: $d = 1$ with arbitrary $n$

For $d = 1$, the spatial lattice has only one point, so a state $\Psi[\theta]$ is a function of only one discrete variable

$$\theta = 2\pi k/n \qquad k \in \{0, 1, 2, ..., n-1\}.$$

In this case, in the matrix representation defined in section 14, a state has only $n$ components:

$$\Psi[\theta] = \begin{bmatrix} \Psi[0] \\ \Psi[2\pi/n] \\ \Psi[2\pi 2/n] \\ \vdots \\ \Psi[2\pi(n-1)/n] \end{bmatrix}. \tag{43}$$

The sum in equation (40) is empty because nearest-neighbor pairs of points don't exist, so the matrix $H_1$ is zero. The matrix $X$ that was defined in section 17 is

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{44}$$

so the hamiltonian is

$$H \propto \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{45}$$

# 20   Example: $n = 2$ with arbitrary $d$

For $n = 2$, each variable $\theta(\mathbf{x})$ has only two possible values: $0$ and $\pi$. In this case, we can think of the system as a lattice of **qubits**,[36] with one qubit for each site $\mathbf{x}$. This means that with each site $\mathbf{x}$ is associated a pair of operators $Z(\mathbf{x})$ and $X(\mathbf{x})$ defined by

$$Z(\mathbf{x})\Psi[\theta] = \begin{cases} \Psi[\theta] & \text{if } e^{i\theta(\mathbf{x})} = 1 \\ -\Psi[\theta] & \text{if } e^{i\theta(\mathbf{x})} = -1 \end{cases} \qquad X(\mathbf{x})\Psi[\theta] = \Psi\left[\hat{\theta}\right]$$

with $\hat{\theta}$ defined by

$$e^{i\hat{\theta}(\mathbf{y})} = \begin{cases} -e^{i\theta(\mathbf{y})} & \text{if } \mathbf{y} = \mathbf{x} \\ e^{i\theta(\mathbf{y})} & \text{otherwise.} \end{cases}$$

These satisfy

$$Z^2(\mathbf{x}) = I \qquad X^2(\mathbf{x}) = I \qquad X(\mathbf{x})Z(\mathbf{y}) = \begin{cases} -Z(\mathbf{y})X(\mathbf{x}) & \text{if } \mathbf{y} = \mathbf{x} \\ Z(\mathbf{y})X(\mathbf{x}) & \text{otherwise,} \end{cases}$$

where $I$ is the identity matrix. These operators $Z(\mathbf{x})$ and $X(\mathbf{x})$ are the $n = 2$ version of the operators $Z(\mathbf{x})$ and $X(\mathbf{x})$ that were defined in sections 11 and 17, respectively, for arbitrary $n$. Both $X(\mathbf{x})$ and $Z(\mathbf{x})$ are self-adjoint when $n = 2$, so in this case the hamiltonian (42) may be written more simply as

$$H = -\alpha \sum_{\mathbf{x}} X(\mathbf{x}) - \lambda \sum_{\{\mathbf{x},\mathbf{y}\}} Z(\mathbf{x})Z(\mathbf{y}). \tag{46}$$

This is the hamiltonian for the **(quantum) Ising model**[37] Article 81040 uses this model in one-dimensional space (two-dimensional spacetime) to study spontaneous symmetry breaking.

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[36]Article 36176

[37]Section 25 clarifies some terminology.

# 21   From the $Z_n$ hamiltonian to the $O(2)$ hamiltonian

The hamiltonian of the $O(2)$ model (section 11) may be viewed as the $n \to \infty$ limit of the hamiltonian of the $Z_n$ model (section 18). This is obvious for the terms involving $Z(\mathbf{x})$. This section explains how it works for the other terms.

The hamiltonian for the $Z_n$ model, equation (42), involves a sum of $X(\mathbf{x})$ over all lattice sites $\mathbf{x}$, where $X(\mathbf{x})$ is the matrix that was defined in section 17. The corresponding term in the hamiltonian for the $O(2)$ model is the operator $H_0(\mathbf{x})$ defined by equation (24). The goal is to understand how $H_0(\mathbf{x})$ emerges from $X(\mathbf{x})$ when $n \to \infty$.

Consider a single point $\mathbf{x}$, and let $\psi(\theta)$ be a function of the single discrete variable $\theta \equiv \theta(\mathbf{x})$. This function can be represented as a column-matrix $\psi$ with $n$ components $\psi(0)$, $\psi(2\pi/n)$, ..., $\psi(2\pi(n-1)/n)$, as in equation (43). On $\psi$, the operator $X \equiv X(\mathbf{x})$ acts as the matrix (44), which implies

$$\left( \frac{X + X^\dagger - 2}{(2\pi/n)^2} \psi \right)(\theta) = \frac{\psi(\theta + 2\pi/n) + \psi(\theta - 2\pi/n) - 2\psi(\theta)}{(2\pi/n)^2}.$$

When $n \to \infty$, this becomes

$$\left( \frac{X + X^\dagger - 2}{(2\pi/n)^2} \psi \right)(\theta) \to -\left( \frac{d}{d\theta} \right)^2 \psi(\theta).$$

This shows that if we choose the coefficient $\alpha$ in (36) to be

$$\alpha = \frac{1}{R^2 \epsilon^D (2\pi/n)^2},$$

then the $n \to \infty$ limit of the hamiltonian (42) for the $Z_n$ model is the hamiltonian (23) for the $O(2)$ model.

# 22    A generalization: nonlinear sigma models

The $O(N)$ models with $N \geq 2$ are examples of **nonlinear sigma models**: their field variables can be viewed as coordinates on a smooth manifold called the **target space**.[38] In the $O(N)$ model, the target space is the sphere $S^{N-1}$ defined by (2).[39]

Section 23 will define a family of nonlinear sigma models in which the scalar field satisfies a different kind of constraint (equation (47)). In these models, the target space is a Lie group. Such models are called **principal chiral models**.[40,41] The name includes *principal* because the target space (a Lie group) is a special kind of homogeneous space[42] that mathematicians call a *principal homogeneous space*.[43] The name includes *chiral* because models of this type are often used to explore the consequences of *chiral symmetry breaking* in QCD.[44,45]

Section 23 constructs the simplest examples of principal chiral models. Much of the literature about principal chiral models includes another term called a **Wess-Zumino-Witten (WZW) term**, which has interesting effects. When $d = 2$, adding WZW term with the right coefficient turns the model into a conformal field theory (CFT), even though it has a nonzero mass gap without that term.[46] When $d = 4$, adding a WZW term together with a **Skyrme term** modifies the model's spectrum of particles to include baryon-like bosons and fermions.[47]

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[38]Fradkin (2021), section 16.5.3

[39]Using (2) to express one of the components of $\phi$ in terms of the others, as in section 6, gives a coordinate system that covers half of the sphere (only half because we must choose a sign for the square-root in the solution).

[40]Schwarz (1995), section 2

[41]The name *principle chiral model* is most commonly used in the context of two-dimensional spacetime, but I'm using it here for arbitrary $d$.

[42]The target space of the models defined in section 5 is a sphere $S^{N-1}$, which is a homogeneous space but (for $N \geq 3$) not a *principal* homogenous space. Kaplunovsky (2022) concisely explains why most of the nonlinear sigma models that people study have homogeneous target spaces, like spheres and Lie groups.

[43]Michiels (2013), section 2.3.1, definition 35

[44]Tong (2018), section 5.2

[45]The scalar field in a principal chiral model is sometimes called a **chiral field**. The same name is also used for a type of spinor field.

[46]Witten (1984), text below equation (11)

[47]Tong (2018), section 5.3.2

# 23    Principal chiral models: definition

In these models, the field variables at each spacetime point $x$ are $\mathbb{F}$-valued variables $U_{jk}(x)$ with $j, k \in \{1, 2, ..., N\}$, where $\mathbb{F}$ is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The field variables satisfy the constraints[48]

$$U^\dagger(x)U(x) = U(x)U^\dagger(x) = I \qquad\qquad \det U(x) = 1, \qquad\qquad (47)$$

where $U(x)$ is the matrix with components $U_{jk}(x)$, $U^\dagger(x)$ is its conjugate transpose,[49] and $I$ is the identity matrix. In other words, these are models in which the target space is a Lie group $G$, with $G = SO(N)$ if $\mathbb{F} = \mathbb{R}$, or $G = SU(N)$ if $\mathbb{F} = \mathbb{C}$.

A state is represented by a complex-valued function $\Psi[U]$ of the field variables $U_{ij}(x)$. The structure of inner product and the evolution equation described in section 3 still apply here, after replacing $\phi \to U$ and adopting a new definition of the measure. The measure is now $[dU] = \prod_x dU(x)$, where the measure $dU(x)$ at each spacetime point is the **Haar measure** over the group $G$, which is invariant under multiplication by an arbitrary element of $G$.[50]

The action is

$$S[U]_X = \epsilon^d \sum_{x \in X} \sum_a \frac{1}{2g^2} \operatorname{Trace}\Big( \big(\partial_a U^\dagger(x)\big)\big(\partial_a U(x)\big)\Big), \qquad\qquad (48)$$

with the lattice version of $\partial$ defined as usual. This defines the **principal chiral model** with target space $G = SO(N)$ or $G = SU(N)$.

The measure and the action are both invariant under the transformation

$$U(x) \to g_L U(x) g_R$$

where $g_L$ and $g_R$ are arbitrary elements of (this matrix representation of) the group $G$, so these transformations are symmetries of the model.

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[48]The new symbol $U$ is being used here for the scalar field as a reminder that the new set of constraints (47) has a different structure than (2).

[49]If the components of $M$ are $M_{jk}$, then the components of $M^\dagger$ are $M_{kj}^*$.

[50]Montvay and Münster (1997), equation (3.90)

# 24   Principal chiral models: hamiltonian

This section derives the hamiltonian for principal chiral models with target space $G = SO(N)$ or $G = SU(N)$. When $G = SO(2)$, this derivation reduces to the one in sections 7-8 for the $O(2)$ model.

As in section 7, generalize the action (48) to use independent parameters $dt$ and $\epsilon$ for the step-sizes in the time and space directions, respectively. Let $[U^\dagger U']$ denote the collection of matrices $U^\dagger(\mathbf{x})U'(\mathbf{x})$, one for each $\mathbf{x}$. Then the action for a single time-step is

$$s[U, U'] = s_0[U^\dagger U'] + dt\, V[U] \tag{49}$$

with

$$s_0[U] \equiv -\beta \sum_{\mathbf{x}} \text{Trace}\big(U(\mathbf{x}) + U^\dagger(\mathbf{x})\big) \qquad \beta \equiv \frac{\epsilon^D}{2g^2\, dt}$$

$$V[U] \equiv -\epsilon^{D-2} \sum_{\{\mathbf{x},\mathbf{y}\}} \frac{1}{2g^2}\text{Trace}\big(U^\dagger(\mathbf{x})U(\mathbf{y}) + U^\dagger(\mathbf{y})U(\mathbf{x})\big) + \text{constant.} \tag{50}$$

If $\Psi[U]$ is the initial state, then the state after a single time-step is proportional to

$$\Psi'[U'] \equiv \int [dU]\ \exp\big(-s[U, U']\big)\Psi[U] = \int [dU]\ \exp\big(-s_0[U^\dagger U']\big)f[U]$$

with

$$f[U] \equiv e^{-dt\, V[U]}\Psi[U]. \tag{51}$$

Define $\tilde{U}(\mathbf{x}) \equiv \big(U'(\mathbf{x})\big)^\dagger U(\mathbf{x})$ and use the fact that the Haar measure is invariant under multiplication by arbitrary elements of the group to get

$$\Psi'[U'] = \int [d\tilde{U}]\ e^{-s_0[\tilde{U}]}f[U'\tilde{U}].$$

Now define $R[U]$ to be the operator whose effect on every function $f[U]$ is

$$R[U_1]f[U_2] = f[U_2 U_1].$$

Then

$$\Psi'[U'] = \int [d\tilde{U}] \; e^{-s_0[\tilde{U}]} R[\tilde{U}] f[U']. \tag{52}$$

This generalizes equation (17). So far, no approximations have been made.

The rest of the derivation will only be sketched here, enough to convey the basic idea.[51] Write

$$U(\mathbf{x}) = e^{i \sum_k \theta_k(\mathbf{x}) \tau_k}$$

where $\tau_k$ are generators for the Lie group $G$, normalized so that $\tau_k^\dagger = \tau_k$. Use this expression for $U(\mathbf{x})$ in equation (50) for $s_0$ to get

$$s_0[U] = \text{constant} + \beta \sum_{\mathbf{x}} \sum_{j,k} \theta_j(\mathbf{x}) \theta_k(\mathbf{x}) \text{Trace}(\tau_j \tau_k) + O(\theta^4). \tag{53}$$

Use the same parameters $\theta_k$ to write

$$R[U] = \prod_{\mathbf{x}} e^{i \sum_k \theta_k(\mathbf{x}) \ell_k(\mathbf{x})}$$

where $\ell_k(\mathbf{x})$ are generators for the operators $R[U]$. Now use the fact that $dt \to 0$ implies $\beta \to \infty$. For large enough $\beta$, the integral over $\tilde{U}$ in (52) is dominated by values of $U(\mathbf{x})$ close to the identity ($\theta$ close to zero), so we can use the approximation (53) to turn the integral over $\tilde{U}$ into a gaussian integral over $\theta$. Do this integral and write $\Psi'[U'] = (1 - H \, dt + O(dt^2)) f[U']$ to get the hamiltonian

$$\boxed{H = (\text{constant}) \times \sum_{\mathbf{x}} \sum_{k} \ell_k^2(\mathbf{x}) + V.}$$

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[51]A more detailed version is given on pages 103-105 in Creutz (1983). That version is for gauge fields instead of scalar fields, but this part of the derivation uses the same ideas. In the gauge-field case, the result is called the **Kogut-Susskind hamiltonian**.

# 25   Terminology

Some of the models described in this article have other names, and some of those other names are also used for other models. This section sorts out some of the terminology, after reviewing a general mathematical relationship between quantum models and classical statistical models.

In the path integral formulation of a quantum model, the vacuum expectation value of a product of observables can be written as an integral equation with the schematic form[52]

$$\int [d\phi] \Psi^*[\phi]_{\text{final}} e^{iS[\phi]} F[\phi] \Psi[\phi]_{\text{init}}, \tag{54}$$

where $\phi$ are the field variables, $S[\phi]$ is the action, $F[\phi]$ represents the product of observables, $\Psi$ is the vacuum state, the factors $\Psi[\phi]_{\text{init}}$ and $\Psi^*[\phi]_{\text{final}}$ depend only on the field variables at the initial and final times, respectively. When treating spacetime as a lattice, the real-valued quantity $dt$ representing the lattice step-size in the time direction can be generalized to a complex-valued quantity, and its direction in the complex plane can be chosen (this is called **Wick rotation**) so that the factor $e^{iS[\phi]}$ becomes[52] $e^{-S_E[\phi]}$. The function $S_E[\phi]$ is called the **euclidean action**,[53] because (at least in some cases) it has the same form as the original action but with the original spacetime metric replaced by a new metric whose signature is euclidean instead of lorentzian. In this formulation, even if the explicit factors of the vacuum state $\Psi$ are excluded, the factor $e^{-S_E}$ automatically suppresses the contributions of all other states, so vacuum expectation values can be reconstructed from the simpler quantities[52]

$$\int [d\phi] e^{-S_E[\phi]} F[\phi]. \tag{55}$$

This is a **euclidean path integral** (section 3).

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[52]Article 63548

[53]In all of the previous sections, the euclidean action was denoted $S$ (without the subscript $E$) and was called the action (without the qualifier *euclidean*), as indicated in section 3.

Mathematically, the quantities (55) look just like[54] *statistical* expectation values for a classical model with hamiltonian $S_E[\phi]$, where the factor $e^{-S_E}$ acts as the Boltzmann distribution. Thanks to this mathematical correspondence, results about the thermodynamic phase structure of classical statistical models in $d$-dimensional euclidean space also apply to the vacuum-state phase structure of quantum models in $d$-dimensional lorentzian spacetime.[55] The correspondence is only mathematical, though: quantum and classical models that are related to each other through this correspondence live in different numbers of spatial dimensions and have different hamiltonians.[56]

Quantum and classical models that are related to each other through this correspondence often share the same name. The quantum $Z_2$ model, the $n = 2$ case of the $Z_n$ model that was defined in section 12, is also called the **(quantum) Ising model** in $d$-dimensional spacetime.[57] After re-interpreting $d$ as the number of spatial dimensions and re-interpreting $S_E$ as the hamiltonian, the same math describes a classical statistical model called the **(classical) Ising model**. Notice that the hamiltonian of the quantum Ising model (equation $(46)$)[58] is very different from the hamiltonian of the classical Ising model (which is mathematically the same as the euclidean action of the quantum Ising model, equations (28) with $n = 2$).

The quantum $O(2)$ model that was defined in section 4 is also called the **(quantum) XY model** in $d$-dimensional lorentzian spacetime,[59] at least when $d = 2$. After re-interpreting $S_E$ as the hamiltonian, the same math describes a classical statistical model called the **(classical) XY model** in two-dimensional euclidean space. Notice that the hamiltonian of the quantum XY model (equation $(22)$)[60] is

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[54]Their physical interpretations are different, but the math looks the same.

[55]Using this correspondence, results about the phase structure of corresponding classical and quantum models are interchangeable. Article 07246 exploits this interchangeability.

[56]Their hamiltonians are not even approximately equivalent to each other. This correspondence has nothing to do with the fact that a quantum model can sometimes be approximated by a classical model in the same number of spatial dimensions.

[57]Section IV-A in Kogut (1979)

[58]Also Fradkin (2022b), equation (18.18), published in Fradkin (2021)

[59]Example: equation (1) in Rana and Girvin (1993), which uses the name for arbitrary $d$.

[60]Also Fradkin (2022b), equation (18.19), published in Fradkin (2021)

very different from the hamiltonian of the classical XY model (which is mathematically the same as the euclidean action (6) of the quantum XY model).

Beware that the same name is also used for a different quantum model, one that is a special case of a family of quantum models called **XYZ models** or **Heisenberg models**.[61] In these models, the hamiltonian has the form

$$H = \sum_{(\mathbf{x},\mathbf{y})} \big( J_X \sigma_X(\mathbf{x})\sigma_X(\mathbf{y}) + J_Y \sigma_Y(\mathbf{x})\sigma_Y(\mathbf{y}) + J_Z \sigma_Z(\mathbf{x})\sigma_Z(\mathbf{y}) \big) \tag{56}$$

where the sum is over all pairs of neighboring lattice sites, the $J$s are real-valued coefficients, and the $\sigma$s are self-adjoint operators satisfying[62]

$$\sigma_X^2(\mathbf{x}) = \sigma_Y^2(\mathbf{x}) = \sigma_Z^2(\mathbf{x}) = 1$$
$$\sigma_Y(\mathbf{x}) \propto \sigma_X(\mathbf{x})\sigma_Z(\mathbf{x}) = -\sigma_Z(\mathbf{x})\sigma_X(\mathbf{x})$$
$$\big[\sigma_{\text{anything}}(\mathbf{x}),\ \sigma_{\text{anything}}(\mathbf{y})\big] = 0 \text{ if } \mathbf{x} \neq \mathbf{y}.$$

The name *XYZ model* alludes to the fact that the coefficients $J_X, J_Y, J_Z$ in (56) may all be different. A model with $J_X = J_Y = J_Z$ is called an **XXX model** to indicate that all three coefficients are equal. A model with $J_Z = 0$ is called an **XY model**, and an XY model with $J_X = J_Y$ is sometimes called an **XX model**. The important message here is that this quantum XY model is different than the one that was described in the previous paragraph, even though both models have the same name.

More generally, a common source of name-collisions is the existence of different types of correspondences between classical and quantum models. One type of correspondence is the one that was highlighted earlier in this section: the euclidean path integral formulation for vacuum expectation values of a quantum model has

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[61]Models like this are often called **spin chains**, at least in one-dimensional space. Even though I'm using them here as examples of different models having the same names, a useful relationship between some spin chains and some nonlinear sigma models does exist, at least in the large-spin limit. Equations (2), (5), (20), and (24) in Rao (2006) give a quick view of the relationship.

[62]In an XYZ model, a single qubit (article 36176) is associated with each site $\mathbf{x}$ of the spatial lattice, and the operators $\sigma_X(\mathbf{x}), \sigma_Y(\mathbf{x}), \sigma_Z(\mathbf{x})$ are observables associated with that qubit.

the same mathematical structure as statistical expectation values over a classical Boltzmann distribution. A different type of correspondence comes from prescriptively replacing the variables in a classical model with operators on a Hilbert space to get a "corresponding" quantum model.[63] Yet another type of correspondence comes from the fact that under some conditions, a quantum model may be well-approximated by a classical model. These three types of correspondence are all distinct. The name of a quantum model is sometimes inherited from a "corresponding" classical model (or conversely), so the existence of multiple correspondences is sometimes responsible for different quantum models having the same name.

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[63]Example: the quantum Heisenberg model with hamiltonian (56) "corresponds" to the classical Heisenberg model in which the $\sigma_{X,Y,Z}(\mathbf{x})$s are real numbers subject to the constraint $\sigma_X^2(\mathbf{x}) + \sigma_Y^2(\mathbf{x}) + \sigma_Z^2(\mathbf{x}) = 1$.

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