Homotopy, Homotopy Groups, and Covering Spaces

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Abstract Homotopy groups are examples of topological invariants: topologically equivalent spaces have the same homotopy groups. Roughly, the nth **homotopy group** of a topological space M expresses the inequivalent ways an n-sphere can be continuously mapped into M, regarding two such maps as equivalent if one can be continuously morphed into the other. The homotopy group with $n = 1$ is called the **fundamental group**. This article introduces homotopy groups and the related concept of a **covering space**. A covering space E of M is like M but "unwrapped" so that E's fundamental group is only part of M's fundamental group.

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1 Outline and conventions

This article has three parts. The first part (sections 2-10) is about homotopy and homotopy equivalence. *Homotopy* is a kind of equivalence between continuous maps from one topological space to another, and *homotopy equivalence* is a kind of equivalence relation between topological spaces. The second part (sections 11-20) is about homotopy groups, which use a modified version of homotopy (homotopy with a fixed basepoint) to explore the topological properties of a space. The third part (sections 21-29) is about covering spaces, which have smaller first homotopy groups than the spaces they cover.

Some of the definitions and results reviewed here apply to topological spaces that are not necessarily manifolds, but most of the applications in this series of articles involve topological (or smooth) manifolds. Article 93875 reviews the concept of a manifold without boundary, and article 44113 reviews the generalization to manifolds with boundary. For the rest of this article, the unqualified word *manifold* means a finite-dimensional topological manifold with boundary.[1] The boundary may be empty, in which case it's a manifold without boundary. In this article, the unqualified word *map* always means *continuous map*.

This article uses the symbol π for two different things. When written with a subscript, π_k denotes the kth homotopy group (section 16). When written without a subscript, π denotes a covering map (section 21).

In this article, the statement $A \subset B$ is synonymous with $A \subseteq B$. (The case $A = B$ is not automatically excluded.) If G and H are groups, then the notation $G \simeq H$ means that G and H are isomorphic to each other.[2,3]

Some references to Lee (2011) are paired with references to the earlier edition Lee (2000), because the earlier edition is freely accessible online.

[1] Many math texts – including some of the sources cited in this article – use a different convention in which the word *manifold* by itself implies *without boundary*.

[2] Article 29682 defines *isomorphism* of groups.

[3] Sometimes, distinguishing between isomorphism (equality as abstract groups) and other forms of equality is important. Example: $SO(3)$ has infinitely many different subgroups that are all isomorphic to $SO(2)$. When this distinction is not important, isomorphism is sometimes written $G = H$.

2 Homotopy

Let M be a topological space. *Homotopy* is one way to formalize the idea that one given object inside M can or cannot be continuously morphed to another given object inside M. To define homotopy, both of the given objects are described as images of maps from another topological space X to M. Instead of only morphing one of the maps' images to the other one, we morph one of the maps to the other one.

To formalize this, let f and g be two maps from X to M, and let I be the closed interval $[0,1] \subset \mathbb{R}$. If a map $h : X \times I \to M$ exists with $h(x,0) = f(x)$ and $h(x,1) = g(x)$, then f and g are called **homotopic** to each other, and h is called a **homotopy** from f to g.[4,5] Changing the parameter s in $h(x,s)$ continuously from 0 to 1 corresponds to continuously morphing the map $f : X \to M$ to the map $g : X \to M$. Two maps f, g that are homotopic to each other are said to be in the same **homotopy class**.[6] Homotopy is an equivalence relation on the set of all maps from X to M.

A map is called **constant** if it sends its whole domain to a single point. A map that is homotopic to a constant map is called **nullhomotopic**[7] or **inessential**.[8,9]

[4] Eschrig (2011), pages 40-41

[5] Section 27.1 in Tu (2011) advocates using a map of the form $h : X \times \mathbb{R} \to M$ instead of $h : X \times I \to M$ (because $\mathbb{R}$ is a manifold without boundary) but still requires $h(x,0) = f(x)$ and $h(x,1) = g(x)$.

[6] Eschrig (2011), page 41

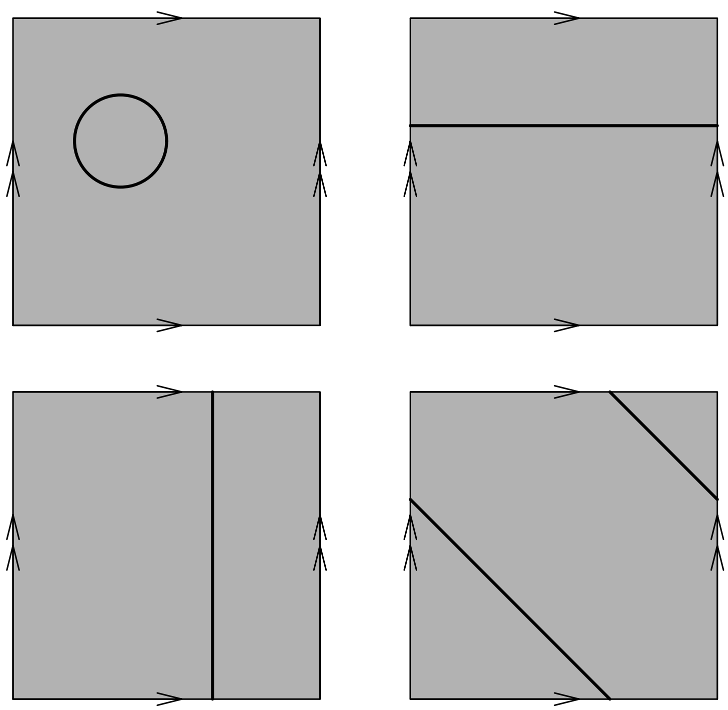
[7] Hatcher (2001), chapter 0; Eschrig (2011), section 2.5, page 41

[8] Hocking and Young (1961), beginning of section 4-3

[9] https://encyclopediaofmath.org/wiki/Inessential_mapping

3 Examples

A two-dimensional torus, $S^1 \times S^1$, may be represented as a square with opposite sides identified with each other. Using this representation, each of the four pictures below shows a single closed loop embedded in a torus. In each picture, the arrows on the edges of the square indicate how those edges should be identified with each other to define the torus,[10] and the loop is drawn as a thicker black line.



No two of these loops are homotopic to each other: none of them can be deformed continuously into any of the others without breaking the loop somewhere. The upper-left picture shows a loop that is nullhomotopic (homotopic to a point).

[10]This is a standard graphical notation. Edges with one arrow are identified with each other respecting the orientations indicated by the arrows, edges with two arrows are identified with each other respecting the orientations indicated by the arrows, and so on.

4 Smooth homotopy

Let X, M be smooth manifolds that may have boundaries. If the maps in the definition of *homotopy* are also required to be smooth (including the map $h : X \times I \to M$), then h is called a **smooth homotopy**.[11,12] Some basic facts:

- Smooth homotopy is an equivalence relation on the set of all smooth maps $X \to M$.[13]
- If two smooth maps f, g from X to M are homotopic to each other, then they are smoothly homotopic to each other.[14]
- If f is a continuous map $X \to M$, then f is homotopic to a smooth map.[15] Even better: f may be approximated by a smooth map homotopic to f.[16] This is the **Whitney approximation theorem**.

[11] Eschrig (2011), pages 40-41; Lee (2013), text above lemma 6.28

[12] When working with smooth manifolds, homotopies are usually required to be smooth, as in section 27.1 in Tu (2011) and the text above lemma 6.28 in Lee (2013).

[13] Lee (2013), lemma 6.28

[14] Lee (2013), theorem 9.28

[15] Lee (2013), theorem 9.27

[16] Hirsch (1976), chapter 5, lemma 1.5

5 Homotopy equivalence

Given a map $f : X \to M$, a **homotopy inverse** of f is a map in the opposite direction, $g : M \to X$, such that the compositions

$$X \xrightarrow{f} M \xrightarrow{g} X \qquad\qquad M \xrightarrow{g} X \xrightarrow{f} M$$

are homotopic to the identity maps on X and M, respectively. A map that has a homotopy inverse is called a **homotopy equivalence**,[17] and the topological spaces X and M are called **homotopy equivalent** to each other if such a map $f : X \to M$ exists.[18] Topological spaces that are homotopy equivalent to each other are said to have the same **homotopy type**.

Homotopy is an equivalence relation on the set of maps from one given topological space to another given topological space,[19] so it also defines an equivalence relation among the images of those maps, but these are both different than the thing called *homotopy equivalence*. The thing called *homotopy equivalence* is an equivalence relation between two topological spaces, not between two maps from one topological space to another, and not (just) between two subspaces of a given topological space.[20]

If two topological spaces are homeomorphic to each other, then they are also homotopy equivalent to each other, but two topological spaces may be homotopy equivalent to each other without being homeomorphic to each other. Example: $\mathbb{R}^n$ is homotopy equivalent to a point,[21] but it's not homeomorphic to a point.[22]

[17]Cutler (2021), definition 4

[18]Tu (2011), definition 27.3; Eschrig (2011), page 136

[19]Lee (2013), lemma 6.28

[20]Images of homotopic maps can fail to be homotopy equivalent. Example: if S^n is an n-dimensional sphere, then any map $S^n \to \mathbb{R}^n$ is homotopic to one that sends all of S^n to a single point in $\mathbb{R}^n$, but S^n is not homotopy equivalent to a single point – in other words, S^n is not *contractible* (section 6).

[21]In other words, $\mathbb{R}^n$ is *contractible* (section 6).

[22]Manifolds with different numbers of dimensions cannot be homeomorphic to each other (Lee (2011), problem 13-3; also Lee (2000), theorem 13.22).

6 Contractible topological spaces

A topological space M is called **contractible** if it is homotopy equivalent to a point.[23]

This definition can be reduced to something more intuitive. Saying that M is homotopy equivalent to a point p means that maps $f : M \to p$ and $g : p \to M$ exist for which

$$M \xrightarrow{f} p \xrightarrow{g} M \qquad\qquad p \xrightarrow{g} M \xrightarrow{f} p$$

are both homotopic to the identity map. Only one map of the form $f : M \to p$ exists: this is the constant map that sends every point of M to the same point p. We can think of a map of the form $g : p \to M$ as a way of selecting just one point of M. The composition

$$M \xrightarrow{f} p \xrightarrow{g} M$$

sends all of M to a single point of M, and the composition

$$p \xrightarrow{g} M \xrightarrow{f} p$$

is the identity map on p, so a topological space M is contractible if and only if a point $m \in M$ exists for which the constant map $M \to m$ is homotopic to the identity map on M.[24] We can use this as a simpler way to define *contractible*.[25]

All three of these conditions on M are equivalent to each other:[26]

- M is contractible.
- For every space X, every map $f : M \to X$ is nullhomotopic.
- For every space X, every map $f : X \to M$ is nullhomotopic.

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[23] Eschrig (2011), section 2.5, page 41; Hatcher (2001), chapter 0; Tu (2011), definition 27.5 (for topological manifolds, which are a special class of topological spaces)

[24] Møller (2015), proposition 2.7

[25] Compared to the definition that uses homotopy equivalence, this one is simpler in the sense that it doesn't refer to any auxiliary space N, and it only involves one homotopic-to-the-identity-map condition instead of two.

[26] Hatcher (2001), chapter 0, exercise 10

7 Example

A manifold with a boundary may be contractible even if the boundary is not contractible as a manifold by itself. To illustrate this, let M be the set of points in n-dimensional euclidean space $\mathbb{R}^n$ satisfying the condition

$$x_1^2 + \cdots + x_n^2 \leq 1.$$

This is an n-dimensional compact manifold with boundary. Its boundary ∂M is a sphere S^{n-1}, which is an $(n-1)$-dimensional compact manifold without boundary. The interior of M is an n-dimensional **open ball**, which is a non-compact manifold without boundary. This example has these properties:

- The manifold M is contractible.[27]
- The interior M, as manifold by itself, is contractible.[27]
- The boundary ∂M, as a manifold by itself, is not contractible.[28,29]
- Even though the boundary ∂M is not contractible as a manifold by itself, the inclusion map $\partial M \to M$ is nullhomotopic.[30]

The important message here is that the even though the inclusion map $\partial M \to M$ is homotopic to a constant map among homotopies that are allowed to explore all of M, it's not homotopic to a constant map among homotopies that are required to stay within ∂M itself. That's why the boundary of a contractible manifold can be a non-contractible manifold.

[27] To deduce this, consider the homotopy $h : M \times I \to M$ given by $h\big((x_1, ..., x_n), s\big) = (sx_1, ..., sx_n)$. This shows that the constant map $h(x, 0)$ is homotopic to the identity map $h(x, 1)$.

[28] This can be deduced using results that will be quoted in section 17: the homotopy group $\pi_k(S^k)$ is not trivial, but the homotopy groups of any contractible space are trivial.

[29] Finite-dimensional spheres are not contractible, but the infinite-dimensional sphere S^∞ is contractible (Hatcher (2001), chapter 0, exercise 16; and Freed (2012), theorem 6.56).

[30] This is a special case of the theorem that was quoted at the end of section 6, because M is contractible.

8 Retraction and deformation retraction

Let M be a topological space, and consider a map $r : M \to M$ with $r(M) = X$. If $r(x) = x$ for all $x \in X$, then r is called a **retraction** of M onto X. This is analogous to the linear-algebra concept of a projection,[31] because applying a retraction twice gives the same result as applying it once: $r(r(m)) = r(m)$ for all $m \in M$.

Let $i : X \to M$ denote the inclusion map, which is defined by $i(x) = x$ for all $x \in X$. If the map r-followed-by-i is homotopic to the identity map on M, then X is called a **deformation retract** of M, and the retraction r is called a **deformation retraction**.[32,33] This implies that X and M are homotopy equivalent to each other,[34] because the map i-followed-by-r is homotopic to the identity map on X (because it *is* the identity map on X).[35]

A few slightly different versions of *deformation retraction* are used in the math literature:[36,37]

- **weak deformation retraction**, which is less restrictive than the version defined above,
- **deformation retraction**, which is the version defined above,
- **strong deformation retraction**, which is more restrictive than the version defined above.

[31] Hatcher (2001), chapter 0

[32] Lee (2011), chapter 7, page 200 (also Lee (2000), chapter 7, page 161)

[33] Fox (1943) says it this way: a retraction $r : M \to X \subset M$ is called a *deformation retraction* if it's homotopic to the identity map on M.

[34] *Homotopy equivalence* was defined in section 5.

[35] Fox (1943) shows that two topological spaces X and Y are homotopy equivalent to each other if and only if they are both homeomorphic to deformation retracts of a third topological space M. Given a homotopy equivalence $f : X \to Y$, such a M may be constructed as the **mapping cylinder** associated with f, but this M is not necessarily a manifold even if X and Y are. (This is clear from the description on page 206 in Lee (2011).)

[36] Cutler (2021), example 1.5

[37] This article uses the name *deformation retraction* the way it's used in Fox (1943), Lee (2011), and Cutler (2021). Chapter 0 in Hatcher (2001) and chapter 27 in Tu (2011) use the name *deformation retraction* for the more restrictive version that Cutler (2021) calls *strong deformation retraction*.

9 Examples

A deformation retraction is automatically a homotopy equivalence,[38] but the converse is false. The converse doesn't even make sense, because homotopy equivalence doesn't require either space to be a subset of the other, but deformation retraction does. Here's an example. Let M be the manifold obtained from $\mathbb{R}^{n+1}$ by deleting a single point p. This $(n+1)$-dimensional manifold M is homotopy equivalent to an n-dimensional sphere, S^n. In fact, if we treat the sphere as a submanifold of M that encloses the deleted point, then that submanifold is a (strong) deformation retract of M.[39] On the other hand, if we treat the sphere as a submanifold of M that doesn't enclose the deleted point, then that submanifold is not even a retract of M, much less a deformation retract.[40,41]

A retraction is not necessarily a deformation retraction. Consider an n-sphere S^n with $n \geq 1$, and let p be a point in S^n. Then the map $S^n \to p$ is a clearly a retraction, but it's not a deformation retraction.[42]

If a manifold has a boundary, then it may or may not admit a deformation retraction onto its boundary. A slight modification of the first example illustrates this. Let M be the set of points in $\mathbb{R}^{n+1}$ that satisfy $x_1^2 + \cdots + x_{n+1}^2 \leq 1$. This is an $(n+1)$-dimensional compact manifold whose boundary $X \equiv \partial M$ is an n-sphere, S^n. The boundary X cannot be obtained from M by a retraction,[40,41] much less a deformation retraction: a (continuous) map from M onto ∂M does not exist. Now let M' be the manifold obtained from M by deleting one point p from the interior of M, say the point $p = (0, ..., 0)$. This is a non-compact manifold with the same boundary as before, but now the boundary is a (strong) deformation retract of M'.[39]

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[38] Section 8

[39] Lee (2011), example 7.37 (also Lee (2000), example 7.23)

[40] Lee (2011), problem 13-6 (also Lee (2000), problem 13-3)

[41] Intuitively, a function from M to X would need to be discontinuous somewhere in the region bounded by X, because it would need to "rip a hole" somewhere in the interior in order to push all of the interior points to the boundary while leaving each boundary point where it was. References to various proofs are cited in Kannai (1981).

[42] A deformation retraction is automatically a homotopy equivalence (section 8), and a sphere is not homotopy equivalent to a point (sections 6-7).

10 Fiber bundles with contractible fiber

Article 70621 introduces the concept of a **fiber bundle**, a map $\pi : E \to B$ satisfying some special conditions that won't be repeated here. The map π is called the **bundle projection**, and for any point $b \in B$, the manifold $F \equiv \pi^{-1}(b) \subset E$ is called the **fiber**. The manifolds E and B are called the **total space** and the **base space**, respectively. A fiber bundle with base space B is often called a **fiber bundle over B**.

Given a fiber bundle in which E, B, and F are smooth manifolds, if the fiber F is contractible, then the total space E and the base space B are homotopy equivalent to each other.[43]

For an example, let E be the manifold obtained by deleting one point from $\mathbb{R}^{n+1}$, and let $B \subset E$ is an n-sphere S^n enclosing the deleted point. Then the deformation retraction from E to B that was noted in section 9 is the bundle projection of a trivial fiber bundle with contractible fiber $\mathbb{R}$.

Here's another example. The unbounded cylinder and unbounded Möbius band are both smooth fiber bundles with base space S^1 and with contractible fiber $\mathbb{R}$, so their total spaces are both homotopy equivalent to the circle S^1. This implies that they are also homotopy equivalent to each other, so this example shows that an orientable manifold can be homotopy equivalent to a non-orientable one, even if they both have the same number of dimensions.

[43] Freed (2012), proposition 6.42, using the fact that every smooth manifold has the homotopy type of a CW complex (article 93875)

11 Loops based at a point

Section 15 will define the *fundamental group* of a topological space M. This section introduces a basic ingredient in that definition.[44]

Let M be a topological space. Choose any point $p \in M$, and call it the **basepoint**. Imagine drawing a continuous curve in M that starts and ends at p. This can be described mathematically as a map $c : I \to M$, where I is the closed interval $[0, 1]$ and $c(0) = c(1) = p$. Call this a **loop based at** p. If c and c' are two loops based at p, then they will be called **loop-homotopic** if a map $h : I \times I \to M$ exists satisfying these conditions:

- $h(x, 0) = c(x)$ and $h(x, 1) = c'(x)$ for all $x \in I$.
- $h(0, s) = h(1, s) = p$ for all $s \in I$.

The first condition says that c and c' are homotopic to each other, as defined in section 2. The second condition requires the homotopy to preserve the basepoint while it's morphing one loop to the other. Loop-homotopy serves as an equivalence relation that is similar to homotopy but that keeps the basepoint fixed.

The constant map $I \to p$ will be called the **trivial loop based at** p.

[44]The language in this section is similar to the language used in Lee (2013), appendix A, pages 612-613.

12 Composing loops based at a point

In the context of a topological space M, if c and c' are two loops based at p, they can be composed with each other to obtain another loop based at p. Their composition $c \cdot c'$ is defined by connecting the end of c to the start of c'. More precisely: $c \cdot c' : I \to M$ is the map defined by

$$(c \cdot c')(s) = \begin{cases} c(2s) & \text{if } s \leq 1/2, \\ c'(2s-1) & \text{if } s \geq 1/2. \end{cases} \tag{1}$$

This satisfies $(c \cdot c')(0) = (c \cdot c')(1) = p$, so it is another loop based at p. It also passes through p at an intermediate value of s, namely $s = 1/2$, which is allowed. It's still a loop based at p even if we deform the loop slightly so that it doesn't pass through p at this intermediate value of s, as long as its endpoints at $s = 0$ and $s = 1$ are still at p.

If c is a loop based at p, then the set of all loops based at p that are loop-homotopic to c will be denoted $[c]$ and called the **loop-homotopy class** of c.

When the composition $c \cdot c'$ is deformed as described above, it remains in the same loop-homotopy class. This allows us to define the composition of two loop-homotopy classes by $[c] \cdot [c'] \equiv [c \cdot c']$. This composition is clearly associative, but it can be noncommutative: $[c \cdot c']$ is not necessarily the same as $[c' \cdot c]$. The set of loop-homotopy classes, equipped with the rule (1) for composing them, defines a group.[45] This group is called the **fundamental group of M with basepoint p**, denoted $\pi_1(M, p)$.[46] Each element of $\pi_1(M, p)$ is a loop-homotopy class $[c]$. The identity element is the loop-homotopy class of the trivial loop $I \to p$, and the inverse $[c]^{-1}$ of $[c]$ is defined by switching the roles of the "start" and "end" of c.

[45] Article 29682 introduces the concept of a *group*.

[46] Lee (2013), appendix A, page 613; and Hatcher (2001), section 1.1

13 Composing loops: another perspective

Section 12 reviewed the standard way of describing the composition of two loops based at p. That description is constructive: it tells us how to *make* the composition $c \cdot c'$ from c and c'.

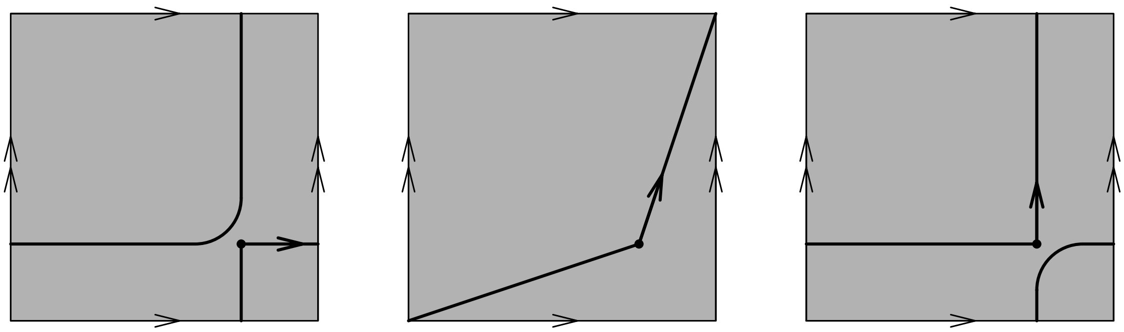
We can also describe the composition of loop-homotopy classes in a different way: instead of giving a recipe for how to *make* the composition $c \cdot c'$, we can give a criterion for how to *recognize* that a loop is in the same loop-homotopy class as $c \cdot c'$, which is ultimately what really matters. Start with any loop based at p and call it c''. Any continuous deformation of c'' that keeps its starting and ending points fixed at p is a loop-homotopy: it remains in the same loop-homotopy class. We can think of c'' as a stretchable but unbreakable rubber band that can be continuously morphed, keeping its endpoints fixed at p and keeping the whole thing within M. Now, suppose that we take some intermediate point on this rubber band and pull it back to the point p, without moving the endpoints of c'' and without letting any part of it leave M, so that it remains in the same loop-homotopy class. The result is a pair of consecutive loops based at p, which we can call c and c', and now c'' is just the composition $c \cdot c'$ that was defined in section 12.

Altogether, we can think of the composition $[c] \cdot [c']$ as the class $[c'']$ for any loop c'' that can be deformed into $c \cdot c'$ as described above.

14 The importance of the basepoint

Loop-homotopy is required to leave the basepoint fixed throughout the homotopy.[47] This section illustrates why that requirement matters.

First consider a two-dimensional torus, $T^2 \equiv S^1 \times S^1$. The fundamental group of T^2 with any given basepoint is abelian:[48,49] the order in which loop-homotopy classes are composed doesn't matter. This is illustrated by the three pictures shown below. The basepoint is shown as a dot. The picture on the left shows the a loop in the class $[c \cdot c']$, where c is a loop that wraps once around the torus in the left-to-right direction, and c' is a loop that wraps once around the torus in the bottom-to-top direction. The picture on the right shows the a loop in the class $[c' \cdot c]$.



The loops shown in the left and right pictures are loop-homotopic to each other: they can both be continuously deformed to the loop in the middle picture. To deform the left picture to the middle one, move the upper-left section of the loop all the way into the upper-left corner of the square. To deform the right picture to the middle one, move the lower-right section of the loop all the way into the lower-right corner of the square. This shows that $[c \cdot c'] = [c' \cdot c]$, illustrating the fact that the the fundamental group of T^2 with the given basepoint is abelian.

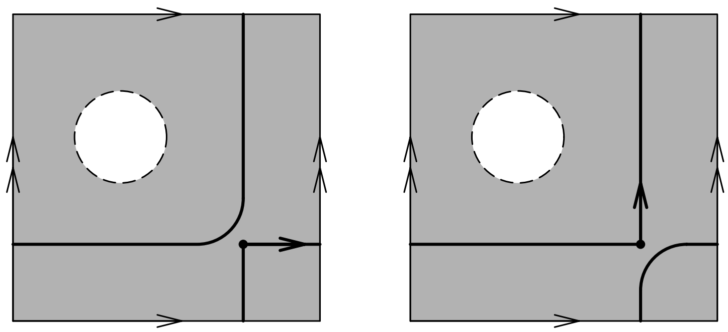
Now consider a different two-dimensional manifold M, obtained from the torus T^2 by deleting a single point. Topologically, this is the same as deleting a circu-

[47]Section 11

[48]This can be deduced from results that will be quoted in section 17.

[49]A group is called **abelian** if all of its elements commute with each other. Otherwise it is **nonabelian**.

lar disk. The result is a two-dimensional non-compact manifold, still without a boundary. This is depicted below, with a white hole where the deleted disk was. The dashed outline around the hole is a reminder that the disk's boundary was also deleted, so the remaining manifold does not have a boundary.[50] The left and right pictures below show curves in the classes $[c \cdot c']$ and $[c' \cdot c]$, respectively, with c and c' defined as before.



Unlike the situation in the unmodified torus, now these two curves are not loop-homotopic to each other: $[c \cdot c'] \neq [c' \cdot c]$. In the picture on the left, the hole prevents us from pulling the upper-left section of the loop all the way to the corner of the square, like we could do before when the hole was absent. Thanks to the hole, the fundamental group of this new manifold, with the given basepoint, is nonabelian.

Now we can understand why the keeping the basepoint fixed is important. If we think of the two closed curves shown above as the images of maps $S^1 \to M$, without giving the basepoint any special treatment, then they would clearly be homotopic to each other: we could deform one to the other just by rounding the sharp corner and sharpening the round corner. In contrast, the previous paragraph showed that they are not loop-homotopic to each other as loops based at p (the point represented by the dot). If we didn't require the basepoint to remain fixed during the homotopies, then this informative effect of the hole on the properties of the fundamental group would be lost.

[50] This two-dimensional manifold is called a **punctured torus** and is homeomorphic to (topologically equivalent to) an infinite plane with one **handle** attached. (The *handle* concept is reviewed in `https://en.wikipedia.org/wiki/Handle_decomposition`.) Approaching the dashed outline from within the shaded region corresponds to approaching infinity from within in the infinite plane.

15 The fundamental group

Section 12 defined the fundamental group of a topological space M with respect to a chosen basepoint p. Now suppose that M is **path-connected**, which means that any two points of M may be connected to each other by a continuous path.[51,52] In this case, the fundamental groups of M with different basepoints p and p' are isomorphic to each other:[53] as abstract groups, they are the same. This abstract group is called the **fundamental group** of M, denoted $\pi_1(M)$. The fundamental group is an example of a topological invariant: it's the same for topological spaces that are homeomorphic to each other.

The fact that the fundamental group is independent of the basepoint is easy to understand intuitively, because we can think of a loop c with basepoint p as a stretchable but unbreakable rubber band that can be continuously morphed to a loop with basepoint p'. Keeping the basepoint fixed is important for defining the loop-homotopy classes (this was the message in section 14), but the resulting group is the same – as an abstract group – no matter which point we choose to serve as that one immovable basepoint.

As an example, suppose that M is a circle S^1. Then its fundamental group $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$, the additive group of integers. The integer $k \in \mathbb{Z}$ represents the class of loops that wind around the circle k times, and the sign of k corresponds to the direction of the winding (clockwise or counterclockwise).

Sections 17-18 will list more examples.

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[51] Lee (2000), chapter 4, page 69

[52] If M were not path-connected, then loops based at a given point p would only be able to explore one path-connected component of M, namely the one that contains the point p.

[53] Hatcher (2001), section 1.1

16 Higher homotopy groups

The fundamental group $\pi_1(M)$ that was defined in section 15 is the first in a series of groups $\pi_n(M)$ called *homotopy groups*, one for each integer $n \geq 1$.[54] Intuitively, each element of $\pi_1(M)$ is a circle S^1 in M – actually a map $c : S^1 \to M$ – together with a specific start-point and direction in which to travel around the circle, together with all such maps that are in the same loop-homotopy class as c. Similarly, each element of $\pi_n(M)$ is a map $c : S^n \to M$, together with all such maps that are in the same homotopy class as c. A map c from the n-sphere into M will be called an ***n*-loop**.[55]

To define the group structure of $\pi_n(M)$, we can use either of two approaches. The standard approach[56] is a constructive one: it tells us how to *make* the composition $c \cdot c'$ from c and c'. Here, I'll use a different approach, one that tells us how to *recognize* that a given n-loop $c'' : S^n \to M$ is in the same class as $[c] \cdot [c']$. We can use this as the definition of the composition $[c] \cdot [c']$, without needing to construct $c \cdot c'$ itself. This generalizes the approach that was described in section 13.

Start with a map $c'' : S^n \to M$ whose image includes a given point $p \in M$. Call this an ***n*-loop with basepoint *p***. For $n = 1$, this is a loop with basepoint p as defined in section 11. The class $[c'']$ is defined to include all such maps that are homotopic to c'' using homotopies that preserve the basepoint p. Now consider a submanifold $S^{n-1} \subset S^n$ that separates S^n into two parts, like the equator (a circle S^1) separates the surface of the earth (a 2-sphere S^2) into two parts. Choose this $(n-1)$-dimensional "equator" so that its image under c'' also includes the given basepoint p. If we continuously morph the map c'' so that the image of the whole equator is squeezed into the single point p, then the image of the resulting map is two spheres S^n in M that touch each other at p. These can be described as maps $c : S^n \to M$ and $c' : S^n \to M$, and the composition $[c] \cdot [c']$ of the classes $[c]$ and $[c']$ is given by $[c'']$. We can use this as the definition of the composition $[c] \cdot [c']$.

54 The general definition can be also applied for $n = 0$, but $\pi_0(M)$ is not a group (https://ncatlab.org/nlab/show/homotopy+group). It counts the number of connected components of M.

55 Nash and Sen (1983), section 5.2

56 Hatcher (2001), section 4.1; Eschrig (2011), pages 42-45

One deficiency of this definition is that it doesn't distinguish between $[c] \cdot [c']$ and $[c'] \cdot [c]$ when $n \geq 2$. When $n = 1$, we can distinguish between them by choosing an ordering for the points of the loop c'' (because it's the image of the interval $[0, 1] \subset \mathbb{R}$) and require that it be consistent with the ordering of the points in c and c' and of the factors in $c \cdot c'$, but that doesn't work when $n \geq 2$ because then points of an n-dimensional space cannot be ordered in any natural way. The standard constructive definition addresses that deficiency, but that turns out to be unnecessary, because even when the standard constructive definition is used, the homotopy groups $\pi_n(M)$ with $n \geq 2$ turn out to be commutative anyway.[57] So, in hindsight, the definition given in the preceding paragraph is sufficient. It doesn't distinguish between $[c] \cdot [c']$ and $[c'] \cdot [c]$ when $n \geq 2$, and it doesn't need to.

The set of homotopy classes of n-loops equipped with this rule for composing them defines a group, and this group is independent of the basepoint if M is path-connected.[58] This is the ***n*th homotopy group** of M, denoted $\pi_n(M)$. The homotopy groups are topological invariants: if two topological spaces are homeomorphic to each other, then they have the same homotopy groups.[59] The converse is false. Article 28539 describes a counterexample.

[57]Fushida-Hardy, "A non-visual proof that higher homotopy groups are abelian" (`https://stanford.edu/~sfh/homotopy.pdf`)

[58]More precisely, the groups defined using different basepoints are isomorphic to each other (Hatcher (2001), section 4.1, page 341).

[59]Hocking and Young (1961), text below corollary 4-29

17 Examples

This section lists a few examples of homotopy groups, using this notation: $\mathbb{Z}$ is the infinite cyclic group (isomorphic to the additive group of integers), $\mathbb{Z}_n$ is the cyclic group of order n, and 0 is the trivial group.[60]

- If M is contractible, then $\pi_k(M) = 0$ for all positive integers k.[61]
- If M is a topological manifold, then $\pi_1(M)$ is countable.[62]
- $\pi_n(S^n) \simeq \mathbb{Z}$ for all $n \geq 1$.[63]
- $\pi_k(S^n) = 0$ if $k < n$,[64] and $\pi_k(S^1) = 0$ if $k \geq 2$.[65]
- The beginning of section 4.1 in Hatcher (2001) shows a table of examples of $\pi_k(S^n)$. One of the entries with $k > n$ is $\pi_{10}(S^4) \simeq \mathbb{Z}_{24} \times \mathbb{Z}_3$.
- **Real projective space** $\mathbb{R}\mathrm{P}^n$ is an n-dimensional manifold defined by identifying opposite points of S^n. Its homotopy groups with $k \geq 2$ are the same as for S^n, but its fundamental group is $\pi_1(\mathbb{R}\mathrm{P}^n) \simeq \mathbb{Z}_2$ when $n \geq 2$.[66]
- If $M = X_1 \times \cdots \times X_n$ for any collection of connected manifolds $X_1, ..., X_n$ without boundaries, then $\pi_k(M)$ is isomorphic to $\pi_k(X_1) \times \cdots \times \pi_k(X_n)$.[67] Example: $\pi_k(S^1 \times S^1)$ is $\mathbb{Z} \times \mathbb{Z}$ if $k = 1$ and is 0 if $k \geq 2$.

[60] A **trivial group** G is a group that only has one element, namely the identity element.
[61] Hocking and Young (1961), bottom of page 185
[62] Lee (2013), proposition 1.16
[63] Sorensen (2017), proposition 5.1
[64] Sorensen (2017), corollary 3.2.1; Maxim (2018), proposition 6.1
[65] Sorensen (2017), proposition 2.1
[66] Section 29
[67] Hatcher (2001), proposition 4.2; Maxim (2018), proposition 1.18

18 Examples with nonabelian fundamental groups

The homotopy groups $\pi_n(M)$ with $n \geq 2$ are necessarily abelian, but the fundamental group $\pi_1(M)$ can be nonabelian. In fact, if G is any finitely generated group,[68] abelian or not, then a four-dimensional compact manifold M exists with $\pi_1(M) = G$.[69,70,71] Here are a few examples of lower-dimensional manifolds with nonabelian fundamental groups:

- One example was already described in section 14: start with a two-dimensional torus and delete a single point. The result is a non-compact manifold whose fundamental group is nonabelian with an infinite number of elements.[72]

- The **Klein bottle** is an example of a compact manifold whose fundamental group is nonabelian with an infinite number of elements.[73]

- The **Poincaré homology sphere** is an example of a compact manifold whose fundamental group is nonabelian with a finite number of elements.[74]

[68] A group G is **generated by** a subset $S \subset G$ if G itself is the smallest subgroup of G that contains S (Scott (1987), section 2.4). A group is **finitely generated** if it is generated by a subset with a finite number of elements (Scott (1987), section 5.4).

[69] Schwartz (2021)

[70] Calegari (2019) says each finitely *presented* group is the fundamental group of a closed orientable 4-manifold.

[71] Some insight about how to construct these manifolds is given in https://mathoverflow.net/questions/15411/ and https://math.stackexchange.com/questions/788097/.

[72] This is called the **once-punctured torus**. Its fundamental group is the **free group with two generators** (Chas and Phillips (2010), first paragraph). Section 8.1 in Scott (1987) defines **free group**.

[73] Rolfsen (2014), text surrounding theorem 5.2 (theorem 5.0.17 in the preprint version). The fundamental group of the Klein bottle is also called the **Klein bottle group**. Another description of this group is given in Hatcher (2001), example 1B.13 (using some notation from example 1B.12).

[74] Kirby and Scharlemann (1979) gives eight different descriptions of this manifold. The fact that its fundamental group is the *binary icosohedral group* is mentioned in descriptions 2 and 6. Description 5 is illustrated in figure 29 on page 125 in Montesinos (1987), which shows how to identify faces of a dodecahedron in pairs to construct the Poincaré homology sphere. The *binary icosahedral group* (also called the **icosian group**) is a subgroup of Spin(3) with 120 elements. Under the covering homomorphism Spin(3) $\to SO(3)$, it is mapped to the *icosohedral group*, the 60-element group of rotational symmetries of a regular dodecahedron (https://ncatlab.org/nlab/show/icosahedral+group and Dechant (2012)).

19 Homotopy equivalence and homotopy groups

If topological spaces X and Y are homotopy equivalent to each other, then they have the same homotopy groups: $\pi_n(X) = \pi_n(Y)$ for all n.[75]

The converse is false.[76,77] **Whitehead's theorem**[78] says that if X and Y have isomorphic homotopy groups *and* those isomorphisms are induced by a map $X \to Y$ *and* X and Y are both homeomorphic to CW complexes,[79] then they are homotopy equivalent to each other. Every smooth compact manifold is homeomorphic to a CW complex,[80] but smooth manifolds X, Y with isomorphic homotopy groups may still fail to be homotopy equivalent if the isomorphisms between their homotopy groups are not induced by a map $X \to Y$.

If M is a smooth manifold with boundary, then M is homotopy equivalent to its interior,[81,82] which is a manifold without boundary. Homotopy equivalent manifolds have the same homotopy groups, so the homotopy groups of a manifold with boundary the same as the homotopy groups of its interior.

If X is a deformation retract of Y, then they have the same homotopy groups: $\pi_n(X) = \pi_n(Y)$.[83] This is a special case of the fact that homotopy equivalent spaces have the same homotopy groups, because deformation retraction is a special case of homotopy equivalence.[84]

[75] Maxim (2018), corollary 1.11

[76] Article 28539 describes an example of two smooth manifolds that have the same homotopy groups but different *homology groups*, which implies that they are not homotopy equivalent to each other.

[77] Here's a case where the converse is true: If Lie groups G and H are both connected and both compact, then they are homotopy equivalent to each other (and isomorphic to each other as Lie groups) if and only if they have the same homotopy groups (Toda (1976)).

[78] Hatcher (2001), theorem 4.5

[79] A **CW complex** is a type of topological space.

[80] Article 93875

[81] Lee (2013), text above theorem 9.26

[82] This is true even though M is not homeomorphic to its interior if the boundary is not empty.

[83] Proposition 1.17 in Hatcher (2001) addresses the case $n = 1$.

[84] Section 8

20 n-connected manifolds

A topological manifold is a special kind of topological space.[85] A topological space is called **connected** if it is not the union of two (or more) disjoint nonempty open subsets.[86] Every path-connected[87] topological space is also connected,[88] but the converse is not always true. For a topological manifold, the converse is always true,[89] so we don't need to distinguish between *connected* and *path-connected* for topological manifolds.

A connected manifold M is called **simply-connected** if $\pi_1(M)$ is trivial.[90] Equivalently, M is simply-connected if and only if all maps $S^1 \to M$ are homotopic to each other. Examples: an n-sphere S^n with $n \geq 2$ is simply-connected, but the circle S^1 and torus $S^1 \times S^1$ are not.

More generally, a manifold M is called ***n*-connected** if $\pi_k(M)$ is trivial for all $k \leq n$.[91] Equivalently, a manifold is n-connected if and only if every map $S^k \to M$ is homotopic to a constant map (a map that sends S^k to a single point in M) for all $k \leq n$.[91] Notice that n-connected implies $(n-1)$-connected.[92,93] In particular, if a manifold is n-connected for some $n \geq 2$, then it is also simply-connected.

The results reviewed in section 17 show that if $n \geq 2$, then an n-sphere S^n is $(n-1)$-connected.[94] They also show that the projective space $\mathbb{RP}^n$ is not even simply connected (1-connected), much less $(n-1)$-connected.

[85] Article 93875

[86] Tu (2011), definition A.41

[87] Section 15 defined *path-connected*.

[88] Lee (2013), proposition A.41(b)

[89] Lee (2013), text below exercise A.42

[90] Hatcher (2001), text above proposition 1.6

[91] Hatcher (2001), section 4.1, page 346

[92] n-connected also implies connected: a manifold is (path-)connected if and only if $\pi_0(M)$ is trivial (footnote 91).

[93] Eschrig (2011) uses a different definition of n-connected. There, a manifold is called n-connected if every map $S^n \to M$ is homotopic to a constant map (section 2.5, page 47). That's different that the definition used here, because it only refers to S^n instead of to S^k for all $k \leq n$.

[94] Cohen (2023), proposition 7.4

21 Covering maps and covering spaces

If E and M are both n-dimensional connected topological manifolds, then a map $\pi : E \to M$ is called a **(topological) covering map** if it has these properties:[95,96]

- π is surjective.
- Every point $x \in M$ has a neighborhood U for which each connected component of $\pi^{-1}(U)$ is an open set (called a **sheet** over U) that is mapped homeomorphically onto U by π.

If $\pi : E \to M$ is a covering map, then E is called a **covering space** for M.

The set $\pi^{-1}(x)$ is called the **fiber** over x. The concept of a covering map $\pi : E \to M$ is a special case of the concept of a (locally trivial) fiber bundle[97] with total space E, base space M, and bundle projection π. Covering maps are precisely the fiber bundles for which the fiber over each point is discrete (has a countable number of elements).[98] For a given covering map, every fiber has the same cardinality, called the **number of sheets**.[99]

A map $f : E \to M$ is called a **local homeomorphism** if each point $p \in E$ has a neighborhood U for which $f(U) \subset M$ is open and the restriction of f to U is a homeomorphism.[100] Any covering map is a local homeomorphism,[101] and any local homeomorphism between compact, connected manifolds is a covering map.[102] An injective covering map is a (global) homeomorphism.[103]

[95] Lee (2011), chapter 11, page 278 (also Lee (2000), chapter 11, page 234); Lee (2013), appendix A, page 615; Hatcher (2001), section 1.3

[96] The sources cited in the previous footnote give the definition for a more general class of topological spaces. The proof of proposition 4.40 in Lee (2013) shows that a covering of an n-dimensional topological manifold is an n-dimensional topological manifold.

[97] Article 70621

[98] Cohen (2023), last paragraph before section 2.1.1; Davis and Kirk (2015), section 4.3

[99] Lee (2011), proposition 11.11 (also Lee (2000), proposition 11.8); Lee (2013), exercise A.74

[100] Lee (2011), text before proposition 2.31

[101] Lee (2011), proposition 11.1; Lee (2013), exercise A.72

[102] Lee (2011), problem 11-9 combined with proposition 4.93

[103] Lee (2011), proposition 11.1; Lee (2013), exercise A.73

22 Covering transformations

Two covering maps $\pi : E \to M$ and $\pi' : E' \to M$ are called **isomorphic** to each other if

$$\pi'(\cdot) = \pi(f(\cdot)) \tag{2}$$

for some homeomorphism $f : E' \to E$.[104] Equation (2) may also be expressed by saying that the diagram

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[dr, "\pi'"'] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

commutes. If $E' = E$ and $\pi'(\cdot) = \pi(\cdot)$, then such a homeomorphism f is called a **covering transformation** or **deck transformation**.[105]

[104] Lee (2011), text above proposition 11.36 (also Lee (2000), text above lemma 12.1)
[105] Lee (2013), text above proposition 7.23

23 Smooth covering maps

In the context of smooth manifolds, a more refined definition of *covering map* is appropriate. If E and M are both n-dimensional connected smooth manifolds, then a map $\pi : E \to M$ is called a **smooth covering map** if it has the properties listed in section 21 for a topological covering map but with *mapped homeomorphically* replaced by *mapped diffeomorphically*.[106,107]

A map $f : E \to M$ is called a **local diffeomorphism** if each point $p \in E$ has a neighborhood U for which $f(U) \subset M$ is open and the restriction of f to U is a diffeomorphism.[108] Any smooth covering is a local diffeomorphism,[109] and any local diffeomorphism between compact, connected manifolds is a covering map.[110] An injective smooth covering is a diffeomorphism.[109]

If M is a connected smooth manifold and $\pi : E \to M$ is a topological covering map, then the topological manifold E admits a unique smooth structure that makes π a smooth covering map.[111] If M has a nonempty boundary, then the boundary of E is given by $\partial E = \pi^{-1}(\partial M)$.[112]

[106] Lee (2013), chapter 4, page 91

[107] A diffeomorphism is a smooth homeomorphism with a *smooth* inverse. A smooth homeomorphism might have an inverse without having a *smooth* inverse, so a smooth covering is not just a topological covering that happens to be smooth (Lee (2013), chapter 4, page 91).

[108] Lee (2013), text above theorem 4.5

[109] Lee (2013), proposition 4.33

[110] Lee (1997), exercise 11.2

[111] Lee (2013), proposition 4.40

[112] Lee (2013), proposition 4.41

24 Smooth covering transformations and orientations

Every nonorientable smooth manifold M has an orientable two-sheeted covering E called its **orientation covering**.[113] The orientation covering E of M has two sheets,[114] so it is also called the **oriented double covering** of M.[115] It is unique in the sense that any two orientation coverings of M are diffeomorphic to each other.[116] Examples:

- The n-sphere S^n is the orientation covering of $\mathbb{R}\mathrm{P}^n$. The covering map $S^n \to \mathbb{R}\mathrm{P}^n$ sends each pair of antipodal points in S^n to a single point in $\mathbb{R}\mathrm{P}^n$.

- The torus $S^1 \times S^1$ is the orientation covering of the Klein bottle. If the torus is described as $\mathbb{R}^2$ modulo $\mathbb{Z}^2$, then the Klein bottle is obtained by identifying $(x + 1/2, -y)$ with (x, y) for all $x, y \in \mathbb{R}^2$.

If M is a compact, connected, non-orientable smooth manifold and $\pi : E \to M$ is its orientation covering, then the map $E \to E$ that exchanges the two points in each fiber is a homeomorphism, and this homeomorphism is the only nontrivial covering transformation of π.[117]

If H is a subgroup of a group G, then the **index** of H in G is the number of distinct subsets of G that can be written as gH for some $g \in G$.[118] If M is a nonorientable connected smooth manifold, then its fundamental group $\pi_1(M)$ has a subgroup of index 2.[119] The fundamental group of a simply connected manifold is trivial, so every simply connected manifold is orientable.[119]

[113] Lee (2013), chapter 15, pages 393-394 and theorem 15.41(b)
[114] Lee (2013), theorem 15.41(b)
[115] Lee (2013), chapter 15, page 396
[116] Lee (2013), theorem 15.42
[117] Lee (2013), in the proof of theorem 15.43, and figure 15.9
[118] Scott (1987), section 1.7
[119] Lee (2013), theorem 15.43

25 Lifting a path to a covering space

If $I \equiv [0,1] \subset \mathbb{R}$, then a map $c : I \to M$ describes a continuous path (one-dimensional curve) in the base space M. If E is a covering space for M, then a map $\tilde{c} : I \to E$ for which $\pi(\tilde{c}(s)) = c(s)$ for all $s \in I$ is called a **lift** of the path from M to E. The lift is uniquely determined by its value at $s = 0$.[120] This is called the **path lifting property**.

If the fiber has more than one element, then the condition $c(1) = c(0)$ does not guarantee $\tilde{c}(1) = \tilde{c}(0)$, so the lift of a closed loop in M might not be a closed loop in E. Section 27 will use this to show that the fundamental group of E is only part of the fundamental group of M.

[120] Lee (2011), theorem 11.12 (also Lee (2000), proposition 11.10)

26 Universal covering spaces

If E is a simply connected covering space for M, then E is called a **universal covering space** for M.[121] A simply connected cover E is called *universal* because if E' is any other covering space for M, then the universal covering space E is a covering space for E', too.[122] If E' is already simply connected, then a covering map $E \to E'$ is a homeomorphism.[123] This implies that any two universal covering spaces for M are homeomorphic to each other. That statement about covering *spaces* are implied by this stronger statement about covering *maps*: the covering maps are isomorphic to each other.[124]

Every connected topological manifold M has a universal covering space,[125] and every connected smooth manifold M has a universal covering space with a smooth covering map.[126]

[121] Lee (2011), text after proposition 11.41 (also Lee (2000), text below proposition 12.6)
[122] Lee (2011), proposition 11.41(a) (also Lee (2000), proposition 12.6)
[123] Lee (2013), proposition A.79 (also Lee (2000), corollary 11.24)
[124] Lee (2011), proposition 11.41(b)
[125] Lee (2011), theorem 11.43 (also Lee (2000), theorem 12.8)
[126] Lee (2013), corollary 4.43

27 Covering spaces and homotopy groups

If M is a manifold and E is a covering of M, then they have mostly the same homotopy groups: $\pi_n(M) = \pi_n(E)$ for all $n \geq 2$.[127] The fundamental group $\pi_1(\cdot)$ is the only homotopy group that can distinguish between M and E. This section describes a correspondence between coverings of M and subgroups of $\pi_1(M)$.

Choose a basepoint $\tilde{m} \in E$ in the covering space, and let $m \equiv \pi(\tilde{m}) \in M$ be the corresponding basepoint in the base space. Use the abbreviations

$$F_E \equiv \pi_1(E, \tilde{m}) \qquad\qquad F_M \equiv \pi_1(M, m)$$

for the fundamental groups with the given basepoints. The group F_E is defined in terms of paths $\tilde{c} : I \to E$ with a given basepoint $\tilde{c}(0) = \tilde{c}(1) = \tilde{m}$,[128] and composing $\tilde{c}$ with the covering map π defines a path $c : I \to M$ in the base space M with basepoint $m \equiv \pi(\tilde{m})$:

$$\begin{tikzcd} I \arrow[r, "\tilde{c}"] \arrow[dr, "c"'] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

The group structure of F_E is defined by a rule for composing paths in E.[129] This rule carries over to a rule for composing paths in M, so this defines a group homomorphism $h : F_E \to F_M$.[130]

The homomorphism h is injective,[131] so we can think of F_E as a subgroup of F_M. This subgroup is said to be **induced** by the covering E.[132] The induced subgroup of F_M uses only those closed loops in M that remain closed when lifted to E.[133]

[127] Maxim (2018), corollary 1.13; Hocking and Young (1961), theorem 4-34

[128] Sections 15 and 25

[129] Section 12

[130] Lee (2000), text above lemma 7.15

[131] Lee (2000), theorem 11.13; Hatcher (2001), proposition 1.31

[132] Lee (2000), text below theorem 11.13

[133] Recall the simple fact that was highlighted in section 25: if the cover is nontrivial (has more than one sheet), then some closed paths in M don't come from any closed paths in E.

In this way, any given covering for M defines a subgroup F_M that is isomorphic to the fundamental group F_E of the covering space E.

The converse is also true: for any given subgroup H of F_M, a covering map $\pi : E \to M$ exists for which H is the induced subgroup,[134] so H is isomorphic to the fundamental group of E. The fact that every connected manifold M has a simply connected covering space[135] corresponds to the fact that F_M – like any group – has a subgroup H with only one element (the identity element).

Two subgroups H and H' of G are called **conjugate** to each other if $H' = g^{-1}Hg$ for some $g \in F_M$. Two coverings of M are isomorphic to each other if and only if the corresponding induced subgroups of F_M are conjugate to each other.[136] These results are the same no matter what basepoint we choose, so isomorphism classes of coverings of M are in one-to-one correspondence with conjugacy classes of subgroups of the fundamental group $\pi_1(M)$.[137]

[134]Hatcher (2001), proposition 1.36

[135]Section 26

[136]Lee (2000), theorem 12.5; also Hatcher (2001), section 1.3, text at the bottom of page 59

[137]Lee (2000), theorem 12.19; and Hatcher (2001), theorem 1.38

28 Example

Consider the base space $M = S^1$. Its fundamental group is $\pi_1(S^1) = \mathbb{Z}$. For every positive integer n, this has a subgroup consisting of integer multiples of n. The covering map corresponding to this subgroup is the map $S^1 \to S^1$, where the covering circle $E = S^1$ wraps n times around the base circle $M = S^1$. The universal covering map $\mathbb{R} \to S^1$ corresponds to the trivial subgroup of $\pi_1(S^1)$, consisting only of the identity element 0.[138]

This example illustrates the general fact that if M is a compact manifold and $\pi : E \to M$ is a covering map, then E is compact if and only if π has a finite number of sheets.[139]

[138] We can think of this as the subgroup consisting of integer multiples of n in the limit $n \to \infty$, because 0 is the only integer multiple of n that remains finite in this limit.

[139] Lee (2000), problem 11-4

29 More examples of covering spaces

- Take the base space M to be a Klein bottle. The Klein bottle can be constructed by identifying opposite edges of a square with each other in a particular way, so M has $\mathbb{R}^2$ as a covering space. The result quoted at the beginning of section 27 says $\pi_n(M) \simeq \pi_n(\mathbb{R}^2)$ for $n \geq 2$, and the homotopy groups of $\mathbb{R}^2$ are all trivial, so we can infer that $\pi_n(M) = 0$ for $n \geq 2$.

- The n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$ is defined by identifying opposite points of an n-sphere S^n with each other. This defines a covering map $S^n \to \mathbb{R}\mathrm{P}^n$ whose fiber consists of two points. When $n \geq 2$, the sphere S^n is simply connected, so S^n is the universal covering space for $\mathbb{R}\mathrm{P}^n$ for $n \geq 2$. The fact that the universal cover is a double cover (the fiber has only two points) shows that any covering space of $\mathbb{R}\mathrm{P}^n$ must be homeomorphic to either $\mathbb{R}\mathrm{P}^n$ or S^n, because the universal covering space for M is a covering space for every covering space for M (section 26). This shows that when $n \geq 2$, $\pi_1(\mathbb{R}\mathrm{P}^n)$ must have exactly two elements, so $\pi_1(\mathbb{R}\mathrm{P}^n) \simeq \mathbb{Z}_2$.[140]

- This example generalizes the previous one when $n = 3$. Let $E \equiv S^3$ be the unit sphere in 4d space, and let G be the group of symmetries generated by R, where R is the transformation that rotates through angle $2\pi/n$ in the 1-2 plane and through angle $2\pi m/n$ in the 3-4 plane, where m is relatively prime to n. Then $S^3 \to S^3/G$ is a covering map, the quotient space $M \equiv S^3/G$ is a manifold called a **lens space**, and its fundamental group is $\pi_1(S^3/G) \simeq G \simeq \mathbb{Z}_n$.[141,142]

[140] https://topospaces.subwiki.org/wiki/Homotopy_of_real_projective_space

[141] Lee (2011), example 12.28 (also Lee (2000), example 12.13)

[142] This is a special case of corollary 12.27 Lee (2011) (also corollary 12.12 Lee (2000)), which uses a broad class of quotient manifolds to generate examples of covering maps.

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