Density Functional Theory and the Fractional Quantum Hall Effect

Based on the work of Jain el.

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Table of Contents

The integer quantum hall Effect

The Hamiltonian is

$$H = \frac{1}{2m}(p + eA)^2. \qquad (1)$$

A is chose such that $\nabla \times A = B_z$.
The energy eigenvalues are

$$E_n = \hbar\omega_B(n + \frac{1}{2}), \quad \omega_B = \frac{eB}{m}. \qquad (2)$$

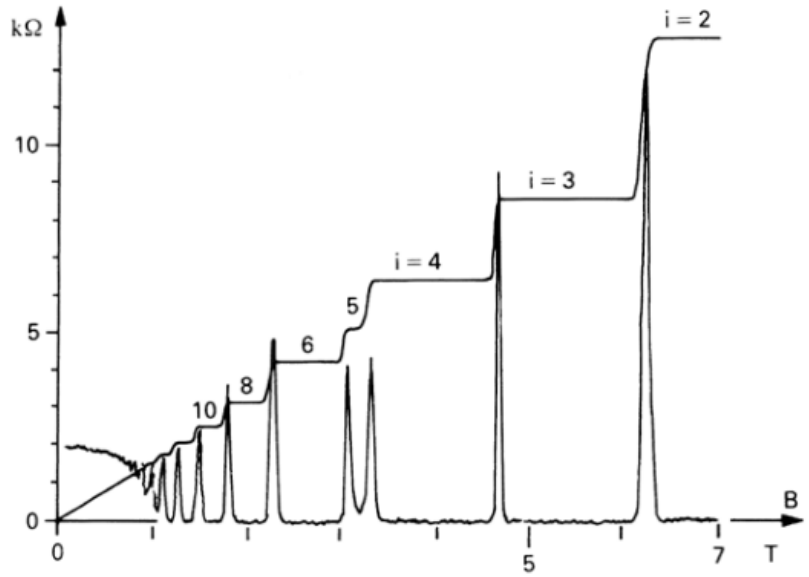

Figure: Integer quantum Hall effect. Figures, if not specified, are taken from David Tong's lecture [2].

The degeneracy and ground state

The number of states for each E_n is

$$N = \frac{eBA}{2\pi\hbar} = \frac{A}{2\pi l_B^2}, \quad l_B = \sqrt{\frac{\hbar}{eB}}. \tag{3}$$

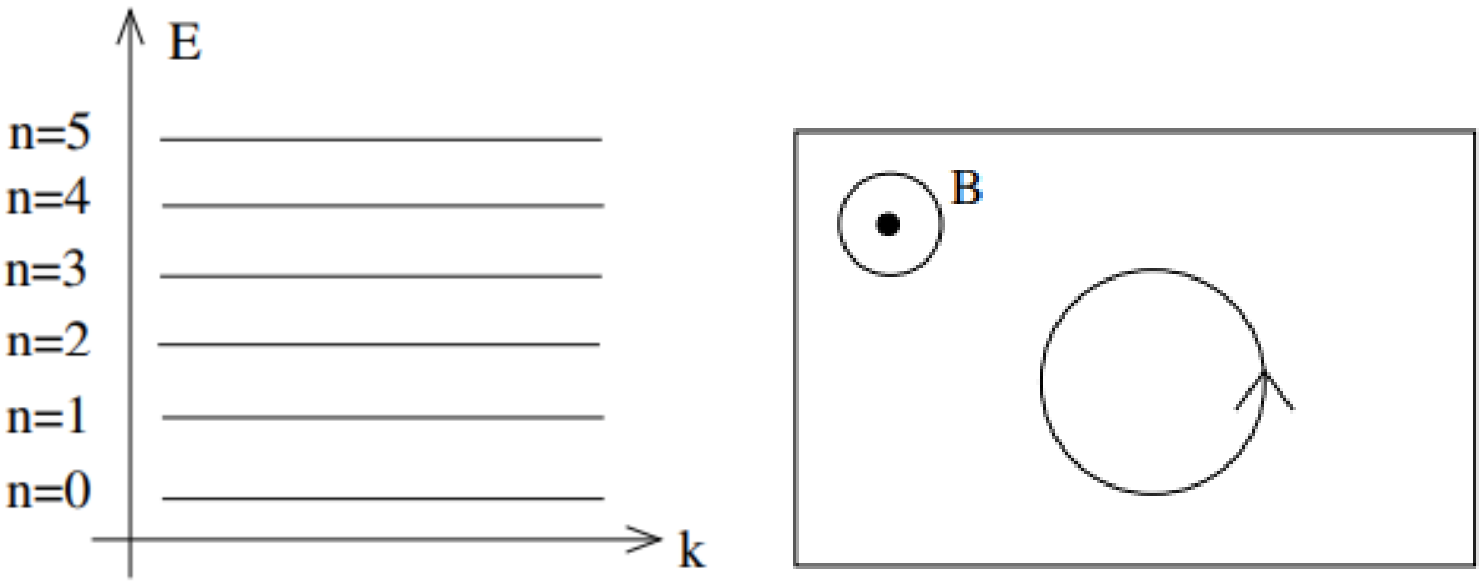

Figure: Landau levels and cyclotron behaviour

The fractional quantum Hall effect

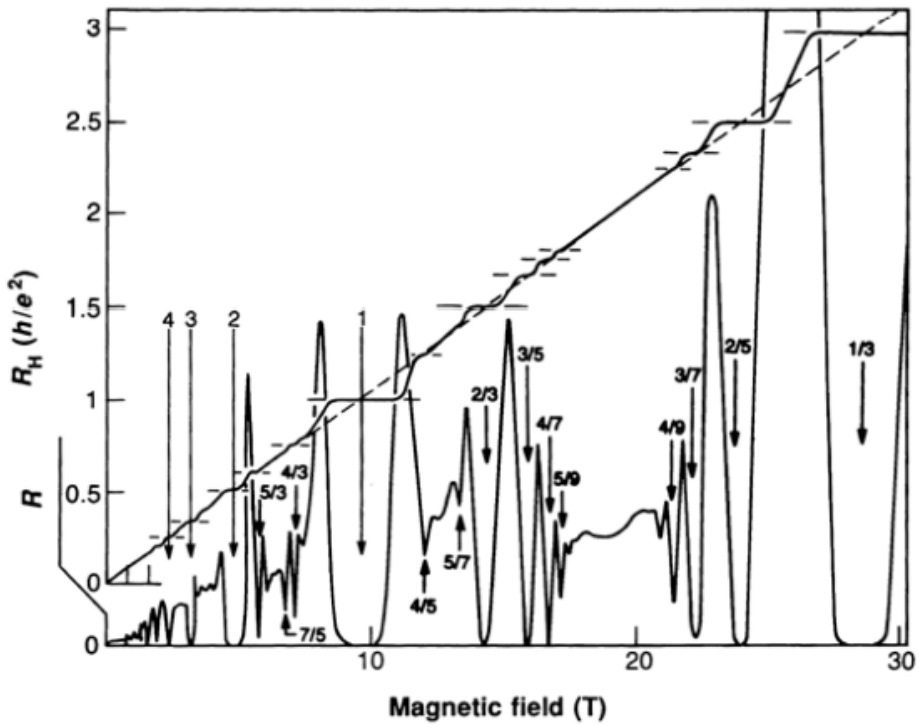

Figure: Fractional QHE

No longer free electron

When the filling fraction ν is not integer, the degeneracy of ground states is

$$C_N^{\nu N} \sim \left(\frac{1}{\nu}\right)^{\nu N} \left(\frac{1}{1-\nu}\right)^{(1-\nu)N} \tag{4}$$

which is HUGE.

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which is HUGE.
Electron-electron interaction has to be considered to pick out the true ground state. And this is a strongly coupled, non-perturbative system, because

$$\frac{E_{\mathrm{xc}}}{E_{gap}} = \infty. \tag{5}$$

The Laughlin state

Laughlin wrote down a wave function

$$\psi_m(z_i) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2} \tag{6}$$

where m is an odd integer. This wave function can explain FQHE at $\nu = 1/m$.

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Laughlin received one third of the Nobel Prize in Physics in 1998 for this discovery.

Composite fermions

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$$\prod_i (z_i - \eta). \tag{7}$$

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From this view point, Laughlin state describes free CFs.

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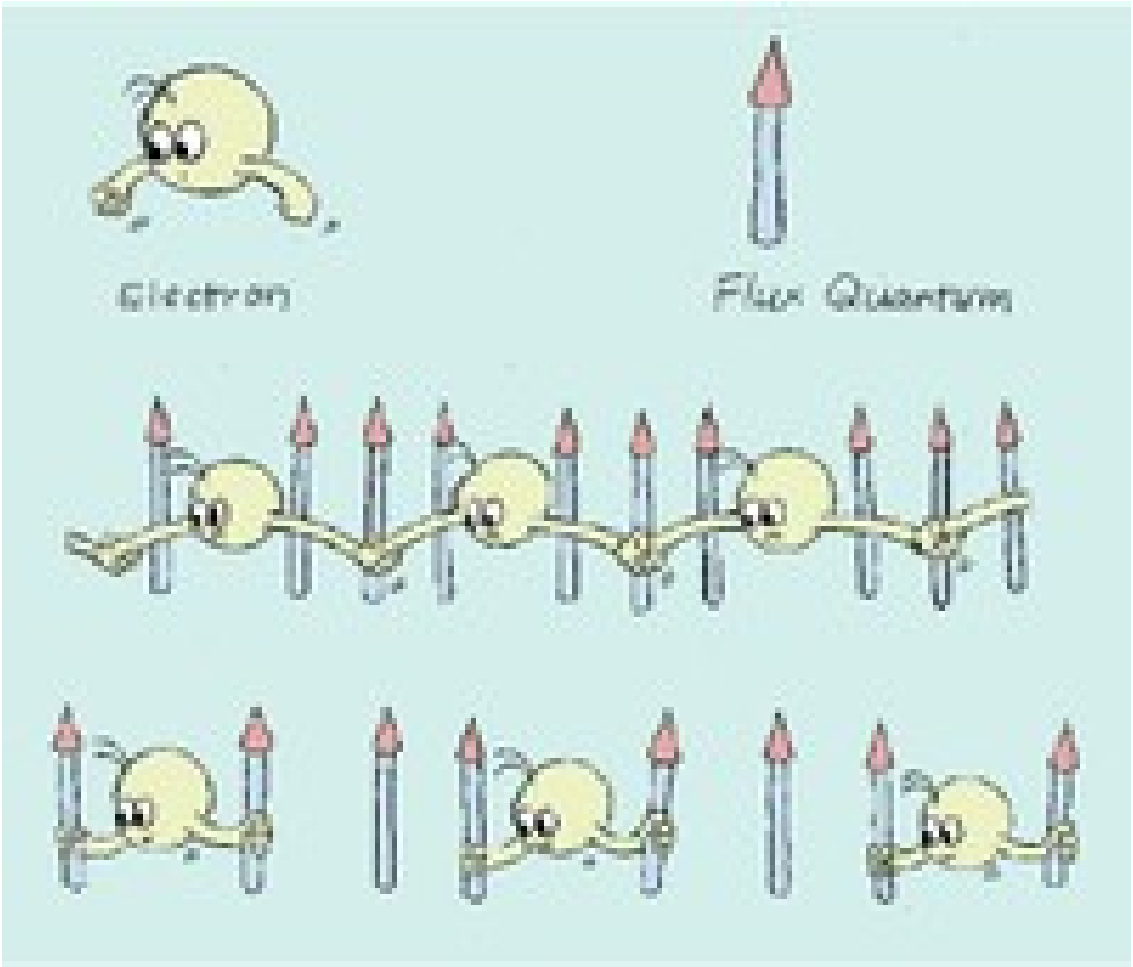

Figure: Composite fermion cartoon by Jain

Other filling fractions

From arguments of Berry phase, CFs experience a different magnetic field[1]

$$B^* = B - (m-1)n\Phi_0, \quad \Phi_0 = \frac{2\pi\hbar}{e}. \tag{9}$$

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$$\nu^* = \frac{\nu}{1-(m-1)\nu}, \quad \nu = \frac{\nu^*}{1+(m-1)\nu^*}. \tag{10}$$

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In other words, a full-filled ν^* gives us partially-filled ν

$$\nu^* = 1 \Rightarrow \nu = \frac{1}{m}; \quad \nu^* = 2 \Rightarrow \nu = \frac{2}{2m-1}. \tag{11}$$

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The KS Equation

The Kohn-Sahm Equation reads

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$$V_{\mathrm{KS}} = \underbrace{V_{\mathrm{H}}}_{\text{Hatree}} + \underbrace{V_{\mathrm{ext}}}_{\text{external potential}} + \underbrace{V_T^*}_{\text{nonstandard potential}} + \underbrace{V_{\mathrm{xc}}^*}_{\text{exchange-correlation}}. \tag{13}$$

The Hatree and external potential terms are normal Coulomb potential. While the exchange-correlation term V_{xc}^*, according to Jain [1], is

$$V_{\mathrm{xc}}^* = \frac{3}{2}a\nu^{\frac{1}{2}} + 2(b-\frac{f}{2})\nu + g, \tag{14}$$

where ν is local filling fraction, and a, b, f, g are some parameters.

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where ν is local filling fraction, and a, b, f, g are some parameters. And V_T^* formally is

$$V_T^*(x,y)=\sum_\alpha\frac{\delta}{\delta n(x,y)}\langle\Psi_\alpha|T^*|\Psi_\alpha\rangle. \tag{15}$$

Self-consistency procedure

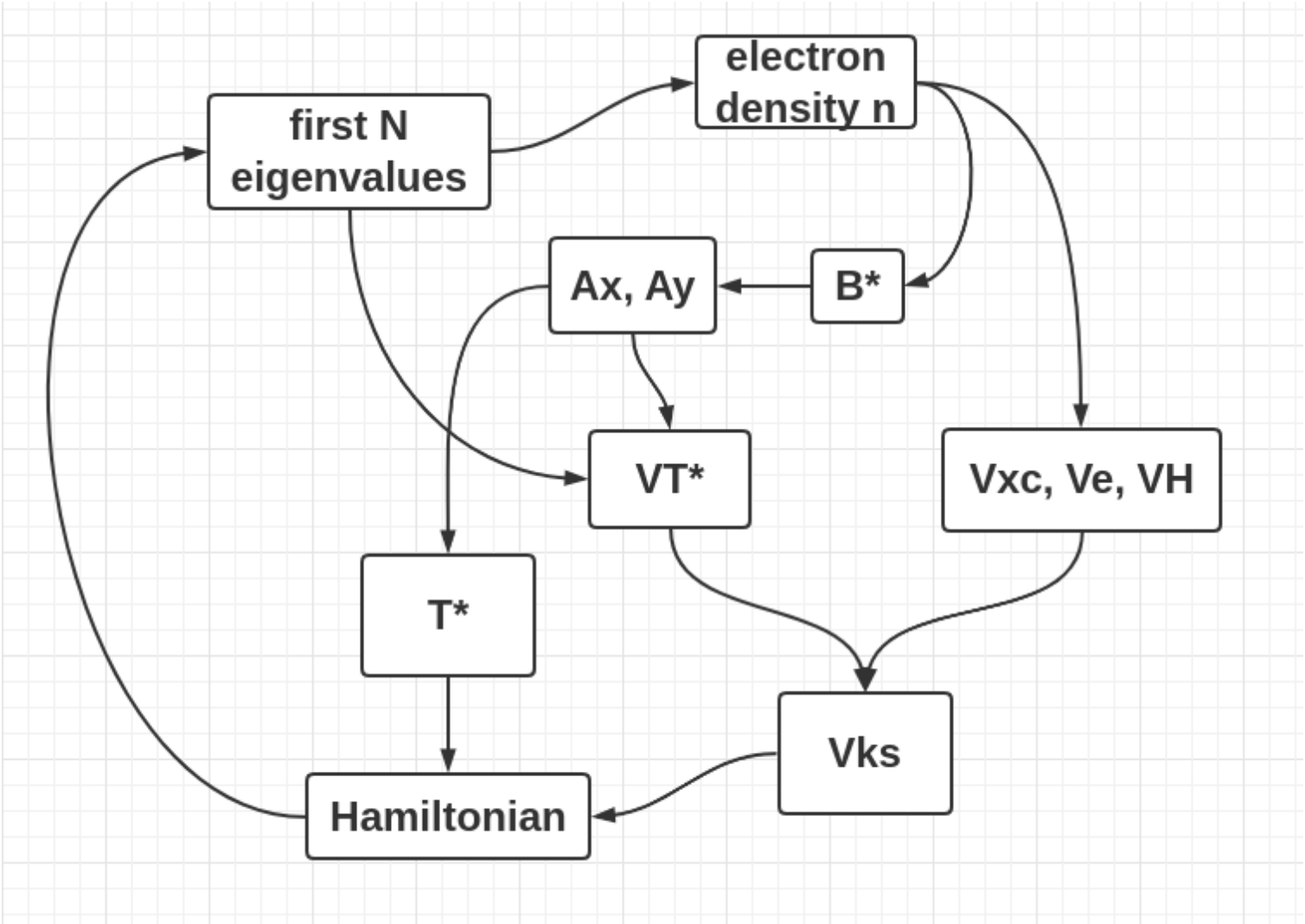

Figure: The self-consistency procedure

Density profile

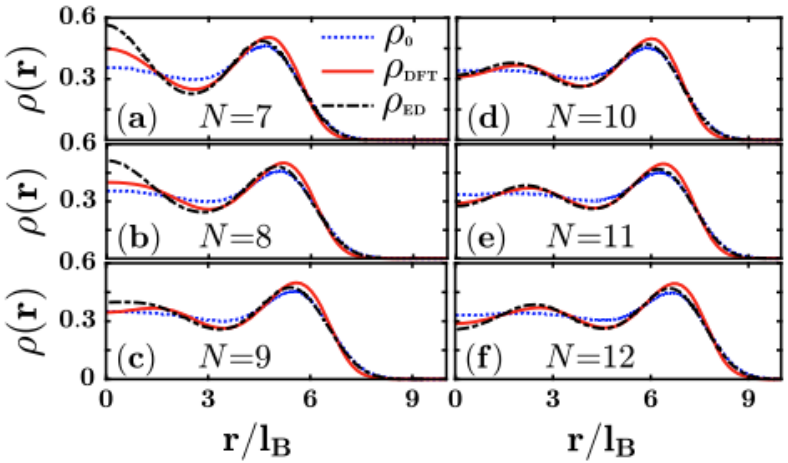

FIG. 1. Density profile for 1/3 droplets. This figure shows the density of a system of N composite fermions. ρ_0 is the density for Laughlin's 1/3 wave function [6], and ρ_{ED} is obtained from exact diagonalization (ED) of the Coulomb interaction at total angular momentum $L_{\mathrm{total}} = 3N(N-1)/2$ [28]. The density ρ_{DFT} is calculated from the solution of the KS equations for composite fermions in an external potential produced by a uniform positively charged disk of radius R so that $\pi R^2 \rho_b = N$. The total angular momentum of the CF state is L^*_{tot}, which is related to the total angular momentum of the electron state by $L_{\mathrm{tot}} = L^*_{\mathrm{tot}} + N(N-1)$ [29]. The CF-DFT solution produces $L^*_{\mathrm{tot}} = N(N-1)/2$, which is consistent with $L_{\mathrm{tot}} = 3N(N-1)/2$. All densities are quoted in units of $(2\pi l_B^2)^{-1}$, the density at $\nu = 1$. We take $\rho_b = 1/3$.

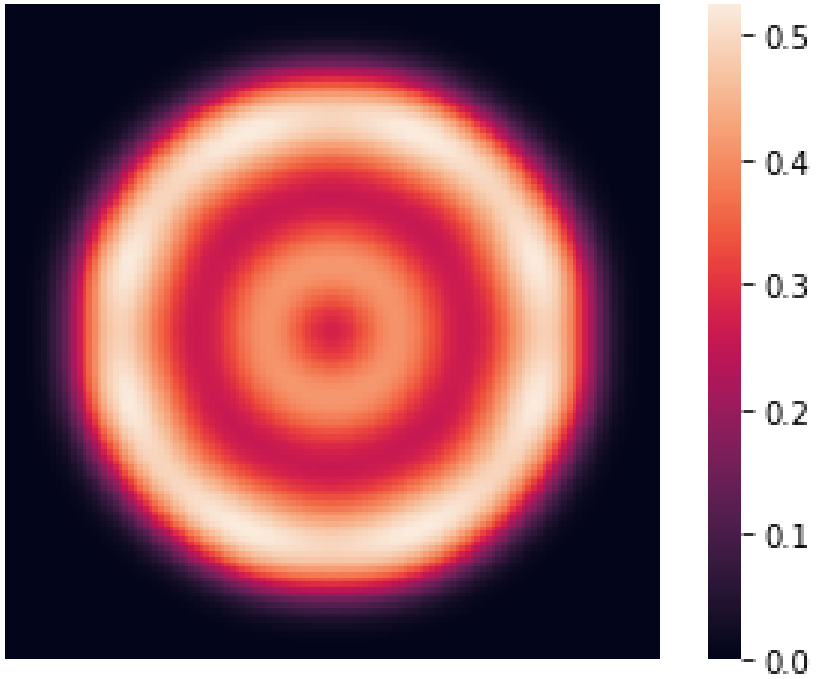

Figure: Jain's result [1] (left) and my result (right). The right figure is for $N = 10$.

Gauge choice in computation

We know clearly that the quantum theory is gauge invarient

$$(P + eA)e^{-ief} = e^{-ief}(p + eA - e\nabla f). \tag{16}$$

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However, the magnitude of changes in A will influence the magnitude of wave function's momentum. We want wave function's momentum to be as small as possible because the discretization can only be finite and always small. So we should choose the gauge such that $|A|$ is minimized.

Thanks for listening!

References


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