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1 author:

Mehdi Jafari
Ankara University

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Homothetic motions in Euclidean 3-space

MEHDI JAFARI

Department of Mathematics, Faculty of Science, Urmia University, Urmia, IRAN
E-mail: mj_msc@yahoo.com

ABSTRACT

A one-parameter homothetic motion of a rigid body in three-dimensional Euclidean space is defined by means of the Hamilton operators. We investigate some properties of this motion and show that it has only one pole point at every instant t. Furthermore, the Darboux vector of the motion can be written as multiplication of two quaternions.

Keywords: Quaternion; Hamilton operator; homothetic motion; pole point.

INTRODUCTION

Bottema & Roth (1979) have analytically investigated the one-parameter homothetic motion of a rigid body in n-dimensional Euclidean space. After a review of some properties of homothetic motion, it is shown by Hacısaliholu (1971) that the motion is regular and has one pole point at every instant t. Yayı (1992) has considered the homothetic motions with the aid of the Hamilton operators in four-dimensional Euclidean space E^4. Subsequently, the Hamilton motion by means of Hamilton operator in semi-Euclidean space E_2^4 is expressed by Kula & Yaylı, (2005) and is shown that this motion is a homothetic motion. Recently, the homothetic motions in different spaces are investigated e.g. Tosun *et al*. (2006) and Jafari & Yaylı (2010).

In this paper, with the aid of the Hamilton operators, we define a Hamilton motion in three-dimensional Euclidean space E^3 and it is shown that this is a one-parameter homothetic motion. Furthermore, it is found that the Hamilton motion defined by regular curve of order r has only one pole point at every instant t. Furthermore, the Darboux vector of the motion is obtained and it is demonstrated that this vector can be written as multiplication of two quaternions. Finally, we give some examples for more clarification.

QUATERNIONS AND HAMILTON OPERATORS

In this section, we give a brief summary of the real quaternions and Hamilton operators. For detailed information about these concepts, we refer the reader to Agrawal (1987) and Ward (1996).

A real quaternion q is an expression of form

$$q = a_\circ + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

where $a_\circ$, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units which satisfy the equalities

$$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1,$$

$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = \vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

A quaternion q is a sum of a scalar part $S_q = a_\circ$, and a vector part $\vec{V}_q = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$. If $S_q = 0$, then q is called pure quaternion. The set of all the pure quaternions is denoted by K. The quaternion product of two quaternions q and p is defined as

$$q \otimes p = S_q S_p - \left\langle \vec{V}_q, \vec{V}_p \right\rangle + S_q \vec{V}_p + S_p \vec{V}_q + \vec{V}_p \wedge \vec{V}_q$$

where $''\langle,\rangle''$ and $''\wedge''$ are the inner and vector products in $\mathbb{R}^3$, respectively. The conjugate of the quaternion $q = S_q + \vec{V}_q$ is denoted by q^*, and defined as $q^* = S_q - \vec{V}_q$. The norm of a quaternion $q = (a_\circ, a_1, a_2, a_3)$ is defined by $q\,q^* = q^*\,q = a_\circ^2 + a_1^2 + a_2^2 + a_3^2$ and is denoted by N_q and say that $q_\circ = q/N_q$ is unit quaternion where $N_q \neq 0$. Unit quaternions provide a convenient mathematical notation for representing orientations and rotations of objects in three dimensions. The set of all quaternions, H, is an associative and non commutative algebra that form a 4-dimensional real space which contains the real axis R and a 3-dimensional real linear space $\mathbb{R}^3$, so that, $\mathrm{H} = \mathbb{R} \oplus \mathbb{R}^3$.

Theorem 1. Let q be a real quaternion, then $\overset{+}{h} : \mathrm{H} \to \mathrm{H}$ and $\overset{-}{h} : \mathrm{H} \to \mathrm{H}$ are defined as follows:

$$\underset{q}{\overset{+}{h}}(x) = q \otimes x, \qquad \underset{q}{\overset{-}{h}}(x) = x \otimes q \qquad x \in \mathrm{H}.$$

The Hamilton's operators $\overset{+}{H}$ and $\overset{-}{H}$, could be represented as the matrices;

$$\overset{+}{H}(q) = \begin{bmatrix} a_\circ & -a_1 & -a_2 & -a_3 \\ a_1 & a_\circ & -a_3 & a_2 \\ a_2 & a_3 & a_\circ & -a_1 \\ a_3 & -a_2 & a_1 & a_\circ \end{bmatrix} \tag{1}$$

and

$$\overset{-}{H}(q) = \begin{bmatrix} a_\circ & -a_1 & -a_2 & -a_3 \\ a_1 & a_\circ & a_3 & -a_2 \\ a_2 & -a_3 & a_\circ & a_1 \\ a_3 & a_2 & -a_1 & a_\circ \end{bmatrix} \tag{2}$$

Theorem 2. If q and p are two real quaternions, λ is a real number and $\overset{+}{H}$ and $\overset{-}{H}$ are operators as defined in the equations (1) and (2), respectively, then the following identities hold:

i. $q = p \Leftrightarrow \overset{+}{H}(q) = \overset{+}{H}(p) \Leftrightarrow \overset{-}{H}(q) = \overset{-}{H}(p)$.

ii. $\overset{+}{H}(q+p) = \overset{+}{H}(q) + \overset{+}{H}(p), \quad \overset{-}{H}(q+p) = \overset{-}{H}(q) + \overset{-}{H}(p)$.

iii. $\overset{+}{H}(\lambda q) = \lambda \overset{+}{H}(q), \quad \overset{-}{H}(\lambda q) = \lambda \overset{-}{H}(q)$.

iv. $\overset{+}{H}(qp) = \overset{+}{H}(q)\,\overset{+}{H}(p), \quad \overset{-}{H}(qp) = \overset{-}{H}(p)\,\overset{-}{H}(q)$.

v. $\overset{+}{H}(q^{-1}) = \left[\overset{+}{H}(q)\right]^{-1}, \quad \overset{-}{H}(q^{-1}) = \left[\overset{-}{H}(q)\right]^{-1}, \quad (N_q)^2 \neq 0.$

vi. $\overset{+}{H}(\bar{q}) = \left[\overset{+}{H}(q)\right]^{T}, \quad \overset{-}{H}(\bar{q}) = \left[\overset{-}{H}(q)\right]^{T}.$

vii. $\det\left[\overset{+}{H}(q)\right] = (N_q)^2, \quad \det\left[\overset{-}{H}(q)\right] = (N_q)^2.$

Proof: The proof can be found in Agrawal (1987) and GroB *et al.* (2001).

The Euclidean motions in E^3 are represented by 3×3 orthogonal matrices $A = \left[a_{ij}\right]$, where $A^t A = AA^t = I_3$. The Lie algebra $SO(3)$ of the group $GL(3)$ of 3×3 positive orthogonal matrices A is the algebra of skew-symmetric matrices

$$\Omega = \dot{A}A^t = \begin{bmatrix} 0 & \Omega_z & -\Omega_y \\ -\Omega_z & 0 & \Omega_x \\ \Omega_y & -\Omega_x & 0 \end{bmatrix}$$

where $\dot{A}$ indicates the differentiation of A with respect to the real parameter t. Ω is called the instantaneous rotation vector (Darboux vector) of the motion.

HOMOTHETIC MOTIONS AT E^3

The 1-parameter homothetic motions of a body in three-dimensional space E^3 are generated by transformation

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix},$$

where A is a 3×3 orthogonal matrix and h is homothetic scalar. The matrix $B = hA$ is called a homothetic matrix and Y, X and C are 3×1 real matrices. The homothetic scalar h and the elements of A and C are continuously differentiable functions of a real parameter t.

Y and X correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space $R_\circ$ and the fixed space R, respectively. At the initial time $t = t_\circ$, we consider the coordinate systems of $R_\circ$ and R are coincident.

To avoid the case of affine transformation we assume that

$$h(t) \neq cons.$$

and to avoid the case of a pure translation or a pure rotation, we also assume that

$$\frac{d}{dt}(hA) \neq 0, \quad \frac{d}{dt}(C) \neq 0.$$

Theorem 3. The homothetic motions of Euclidean space E^3 are regular motions.

For detailed information about the homothetic motions, we refer the reader to Hacisalihoglu (1971).

HAMILTON MOTIONS IN EUCLIDEAN 3-SAPCE

Let us consider the curve $\alpha : I \subset \mathrm{R} \to E^4$ defined by

$$\alpha(t) = (a_\circ(t), a_1(t), a_2(t), a_3(t)), \tag{3}$$

for every $t \in I$. We suppose that $\alpha(t)$ is a differentiable curve of order r which does not pass through the origin.

Also, the map F_α acting on a pure quaternion ω:

$$F_\alpha : \mathrm{K} \to \mathrm{K}, \quad F_\alpha(\omega) = \alpha \otimes \omega \otimes \alpha^* \tag{4}$$

where α^* is conjugate of the α. We put $F_\alpha(\omega) = \omega'$. Using the definition of $\overset{+}{H}$ and $\overset{-}{H}$ the equation (4) is written as

$$\omega' = \overset{+}{H}(\alpha)\,\overset{-}{H}(\alpha^*)\,\omega\,.$$

From (1) and (2), we obtain

$$\overset{+}{H}(\alpha)\,\overset{-}{H}(\alpha^*) = \begin{bmatrix} a_\circ^2+a_1^2+a_2^2+a_3^2 & 0 & 0 & 0 \\ 0 & a_\circ^2+a_1^2-a_2^2-a_3^2 & 2(a_1a_2-a_\circ a_3) & 2(a_\circ a_2+a_1a_3) \\ 0 & 2(a_1a_2+a_\circ a_3) & a_\circ^2-a_1^2+a_2^2-a_3^2 & 2(a_2a_3-a_\circ a_1) \\ 0 & 2(a_1a_3-a_\circ a_2) & 2(a_\circ a_1+a_2a_3) & a_\circ^2-a_1^2-a_2^2+a_3^2 \end{bmatrix}$$

This simplifies to

$$\overset{+}{H}(\alpha)\,\overset{-}{H}(\alpha^*) = \begin{bmatrix} h' & 0 \\ 0 & B \end{bmatrix},$$

where $h' = a_\circ^2 + a_1^2 + a_2^2 + a_3^2$ and

$$B = \left[b_{ij}\right]_{3\times 3} = \begin{bmatrix} a_\circ^2+a_1^2-a_2^2-a_3^2 & 2(a_1a_2-a_\circ a_3) & 2(a_\circ a_2+a_1a_3) \\ 2(a_1a_2+a_\circ a_3) & a_\circ^2-a_1^2+a_2^2-a_3^2 & 2(a_2a_3-a_\circ a_1) \\ 2(a_1a_3-a_\circ a_2) & 2(a_\circ a_1+a_2a_3) & a_\circ^2-a_1^2-a_2^2+a_3^2 \end{bmatrix}.$$

For the matrix B, we have $BB^T = h'^2 I_3$ and $\det B = h'^3$.

The $1-$parameter Hamilton motions of a body in Euclidean $3-$space are generated by transformation

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_\circ \\ 1 \end{bmatrix} \tag{5}$$

where B is the above matrix. X, $X_\circ$ and C are 3×1 real matrices. B and C are continuously differentiable functions of a real parameter t; X and $X_\circ$ correspond to the position vectors of the same point P.

Theorem 4. The Hamilton motion determined by the equation (5) is a homothetic motion in E^3.

Proof: The matrix B can be represented as

$$B = h\begin{bmatrix} \dfrac{b_{11}}{h} & \dfrac{b_{12}}{h} & \dfrac{b_{13}}{h} \\ \dfrac{b_{21}}{h} & \dfrac{b_{22}}{h} & \dfrac{b_{23}}{h} \\ \dfrac{b_{31}}{h} & \dfrac{b_{32}}{h} & \dfrac{b_{33}}{h} \end{bmatrix} = hA,$$

where $h : I \subset \mathrm{R} \to \mathrm{R}$,

$$t \to h(t) = a_\circ^2(t) + a_1^2(t) + a_2^2(t) + a_3^2(t).$$

So, we find $A \in SO(3)$ and $h \in \mathrm{R}$. Thus B is a homothetic matrix and the equation (5) determines a homothetic motion.

POLE POINTS AND POLE CURVES OF THE MOTION

To find the pole points, we have to solve the equation

$$\dot{B}X + \dot{C} = 0. \tag{6}$$

Any solution of the equation (6) is a pole point of the motion at that instant in $R_\circ$. Since $\dot{B}$ is regular, the equation (6) has only one solution, *i.e.*, $X_\circ = (-\dot{B})^{-1}\dot{C} = 0$ at every instant t. This pole point in the fixed system is

$$X = B(-\dot{B})^{-1}\dot{C} + C.$$

Theorem 5. During the homothetic motion of Euclidean space of 3-dimensions, there is a unique instantaneous pole point at every time t.

Theorem 6. During the homothetic motion the pole curves slide and roll upon each others and the number of the sliding-rolling of the motion is h.

Example 1. Let $\alpha : I \subset \mathrm{R} \to E^4$ be a curve given by

$$t \to \alpha(t) = (\cos t,\ t,\ \sin t,\ -1),$$

for every $t \in I.\alpha(t)$ is a differentiable regular of order r. Since, $\alpha(t)$ does not pass though the origin, the matrix B can be represented as

$$B = \begin{bmatrix} \cos^2 t - \sin^2 t + t^2 - 1 & 2(t\sin t + \cos t) & 2(\cos t \sin t - t) \\ 2(t\sin t - \cos t) & -t^2 & -2(t\cos t + \sin t) \\ -2(\cos t\sin t + t) & 2(t\cos t + \sin t) & \cos^2 t - \sin^2 t - t^2 + 1 \end{bmatrix}$$

$$= (2 + t^2)A,$$

where $h(t) = (2 + t^2)$, $A \in SO(3)$. Thus, B is a homothetic matrix and it determines a homothetic motion in E^3.

DARBOUX VECTOR OF THE MOTION

Suppose that $\alpha(t)$ is a unit speed curve as defined in (3). In the homothetic motion defined by homothetic matrix B, Darboux matrix is $\Omega = \dot{A}A^{-1}$. So, we obtained

$$\Omega = \begin{bmatrix} 0 & \dot{a}_\circ a_3 - a_\circ \dot{a}_3 + \dot{a}_1 a_2 - a_1 \dot{a}_2 & \dot{a}_1 a_3 - a_1 \dot{a}_3 - \dot{a}_\circ a_2 + a_\circ \dot{a}_2 \\ -(\dot{a}_\circ a_3 - a_\circ \dot{a}_3 + \dot{a}_1 a_2 - a_1 \dot{a}_2) & 0 & \dot{a}_2 a_3 - a_2 \dot{a}_3 + \dot{a}_\circ a_1 + a_\circ \dot{a}_1 \\ -(\dot{a}_1 a_3 - a_1 \dot{a}_3 - \dot{a}_\circ a_2 + a_\circ \dot{a}_2) & -(\dot{a}_2 a_3 - a_2 \dot{a}_3 + \dot{a}_\circ a_1 + a_\circ \dot{a}_1) & 0 \end{bmatrix}.$$

The Darboux vector corresponds to skew-symmetric matrix Ω is defined by

$$\vec{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$$

Therefore, the Darboux vector of the motion

$$\vec{\Omega} = 2(-\dot{a}_2 a_3 + a_2 \dot{a}_3 - \dot{a}_\circ a_1 + a_\circ \dot{a}_1, \dot{a}_1 a_3 - a_1 \dot{a}_3 - \dot{a}_\circ a_2 + a_\circ \dot{a}_2, \dot{a}_1 a_3 - a_1 \dot{a}_3 - \dot{a}_\circ a_2 + a_\circ \dot{a}_2),$$

is obtained. This vector can be written as multiplication of two quaternions as

$$\vec{\Omega} = 2(\dot{\alpha} \otimes \alpha^*).$$

Example 2. Let $\alpha : I \subset \mathrm{R} \to E^4$ be a unit speed curve given by

$$t \to \alpha(t) = 12(\sqrt{2}\cos t, 1, \sqrt{2}\sin t, -1), \quad \text{for every } t \in I.$$

Since, $\alpha(t)$ does not pass though the origin, the matrix B can be represented as

$$B = \begin{bmatrix} \frac{1}{2}(\cos^2 t - \sin^2 t) & \frac{1}{\sqrt{2}}(\sin t + \cos t) & \cos t \sin t - \frac{1}{2} \\ \frac{1}{\sqrt{2}}(\sin t - \cos t) & 0 & -\frac{1}{\sqrt{2}}(\cos t + \sin t) \\ -\cos t \sin t - \frac{1}{2} & \frac{1}{\sqrt{2}}(\cos t - \sin t) & \frac{1}{2}(\cos^2 t - \sin^2 t) \end{bmatrix}.$$

B is a homothetic matrix and is defined a homothetic motion. The Darboux vector of this motion is

$$\vec{\Omega} = (1\sqrt{2}(\sin t + \cos t), 1, -1\sqrt{2}(\sin t + \cos t)).$$

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