# Durrett 5th Chapter1 Solutions

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## 1 Probability Spaces

**1.1.1.** Let $\Omega = \mathbb{R}$, $\mathcal{F}$ =all subsets so that $A$ or $A^c$ is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

***Solution.*** We firstly prove $\mathcal{F}$ is a $\sigma$-field. Let $\{A_i\} \subseteq \mathcal{F}$. It is obvious $A_i^c \in \mathcal{F}$. If $A_i$ or $A_j$ is countable, then $A_i \cap A_j$ is countable and hence contained in $\mathcal{F}$. Otherwise $A_i^c$ and $A_j^c$ is countable, and then $(A_i \cap A_j)^c = A_i^c \cup A_j^c$ is countable, which concludes $A_i \cap A_j$ is always in $\mathcal{F}$. Finally, $\bigcup_{i\in\mathbb{N}} A_i$ is countable if every $A_i$ is countable, Otherwise $(\bigcup_{i\in\mathbb{N}} A_i)^c = \bigcap_{i\in\mathbb{N}} A_i^c \subseteq A_0^c$ is countable since $A_0^c$ is countable, which concludes $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$. Hence $\mathcal{F}$ is a $\sigma$-field.

$P$ is well-defined since $A \cup A^c = \Omega = \mathbb{R}$ is uncountable, whence $A$ and $A^c$ can not be countable simutaneously. Finally we prove $P$ is a probability. It is trivial $P(\Omega) = 1$. For disjoint sets, $A_i \in \mathcal{F}$, if $A_i$ are all countable, $P(A_i) = 0$ and then $+_{i=1}^{\infty} A_i$ is countable, $P(+_{i=1}^{\infty} A_i) = 0 = \sum_{i=1}^{\infty} P(A_i)$. Otherwise $A_k^c$ is countable, and then $A_i \subseteq A_k^c, i \neq k$ is countable since they are disjoint. And $(+_{i\in\mathbb{N}} A_i)^c = \bigcap_{i\in\mathbb{N}} A_i^c \subseteq A_k^c$ is countable. So $P(+_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) = 1 = P(+_{i\in\mathbb{N}} A_i)$, when $P$ is a probability. □

**1.1.2.** Recall the definition of $S_d$ from Example 1.1.5. Show that $\sigma(S_d) = \mathcal{R}^d$, the Borel subsets of $\mathbb{R}^d$.

*Proof.* The definition of $S_d$ is

$$S_d = \left\{ \prod_{i=1}^{d} (a_i, b_i] \;\middle|\; -\infty \le a, b \le +\infty \right\}$$

It is obvious $S_d \subseteq \mathcal{R}^d$ whence $\sigma(S_d) \subseteq \mathcal{R}^d$.Then for any $\prod_{i=1}^{d} (a_i, b_i)$,

$$\bigcup_{n=1}^{\infty} (\prod_{i=1}^{d} (a_i, b_i - \frac{1}{n}]) = \prod_{i=1}^{d} (a_i, b_i)$$

The desired result follows from $\mathcal{R}^d$ is the $\sigma$-field of the right.

**1.1.3.** A $\sigma$-field $\mathcal{F}$ is said to be countably generated if there is a countable collection $\mathcal{C} \subseteq \mathcal{F}$ so that $\sigma(C) = F$. Show that $\mathcal{R}^d$ is countably generated.

***Solution.*** Let

$$\mathcal{U} = \left\{ \prod_{i=1}^{d} (a_i, b_i) \;\middle|\; a_i, b_i \in \mathbb{Q} \right\}$$

Then $\mathcal{U}$ is countable since $|\mathcal{U}| \le \left|\mathbb{Q}^{2d}\right|$. And $\sigma(\mathcal{U}) = \mathcal{R}^d$ since for each $\prod_{i=1}^{d} (a_i, b_i)$, there are $a_{i,k}, b_{i,k} \in \mathbb{Q}$ such that $a_{i,k} \to a_i + 0$ and $b_{i,k} \to b_i - 0$ as $k \to \infty$ and then

$$\bigcup_{k=1}^{\infty} (\prod_{i=1}^{d} (a_{i,k}, b_{i,k})) = \prod_{i=1}^{d} (a_i, b_i)$$

□

**1.1.4.**

(i) Show that if $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ are $\sigma$-algebras, then $\bigcup_i \mathcal{F}_i$ is an algebra.

(ii) Give an example to show that $\bigcup_i \mathcal{F}_i$ need not be a $\sigma$-algebra.

***Solution.***

(i) For $A, B \in \bigcup_i \mathcal{F}_i$, there is $F_k$ such that $A, B \in \mathcal{F}_k$, and then $A \cup B, A^c \in \mathcal{F}_k \subseteq \bigcup_i \mathcal{F}_i$, whence $\bigcup_i \mathcal{F}_i$ is an algebra.

(ii) Let $\Omega = \mathbb{R}^{\mathbb{N}}$ and $F_i = \left\{U \times \mathbb{R}^{\mathbb{N}} \;\middle|\; U \in \mathcal{R}^i\right\}$. While $[0,1]^i \times \mathbb{R}^{\mathbb{N}} \in \mathcal{F}_i$, but

$$[0,1]^{\mathbb{N}} = \bigcap_{i=1}^{\infty} [0,1]^i \times \mathbb{R}^{\mathbb{N}} \notin \bigcup_i \mathcal{F}_i$$

□

**1.1.5.** A set $A \subseteq \{1, 2, \dots\}$ is said to have asymptotic density $\theta$ if

$$\lim_{n\to\infty} \frac{|A \cap \{1, 2, ..., n\}|}{n} = \theta$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a $\sigma$-algebra? an algebra?

***Solution.*** $\mathcal{A}$ is not an algebra. The counterexample is shown in the following.

Let $A = \{2n\}$ and $B = \{b_n\}, b_n \in \{2n-1, 2n\}$. Then $A, B$ have asymptotic densities $1/2$. And then let

$$S_n = |A \cap B \cap \{1, 2, \dots, 2n\}| = \sum_{k=1}^{n} |A \cap B \cap \{2k-1, 2k\}| = \sum_{k=1}^{n} c_k$$

where

$$c_k = \begin{cases} 0 & b_k = 2k-1 \\ 1 & b_k = 2k \end{cases}$$

It means for any series $\{x_i\}, x_i \in \{0, 1\}$, there exists $B$ with asymptotic density $1/2$ such that $c_i = x_i$. Hence let

$$c_i = \begin{cases} 0 & i = 1 \\ 1 & i = [3^{2k}+1, 3^{2k+1}] \\ 0 & i = [3^{2k+1}+1, 3^{2k+2}] \end{cases} \qquad k \in \mathbb{N}$$

Then for $n = 3^{2k+1}$, $S_n \geq \sum_{3^{2k}+1}^{3^{2k+1}} c_i = 2 \times 3^{2k} = \frac{2}{3}n$ and for $n = 3^{2k+2}$, $S_n = S_{3^{2k+1}} \leq 3^{2k+1} = \frac{1}{3}n$. So $L_n := |A \cap B \cap \{1, 2, \dots, n\}|/n \geq 1/3$ when $n = 2 \times 3^{2k+1}$ and $L_n \leq \frac{1}{6}$ when $n = 2 \times 3^{2k+2}$, whence $L_n$ diverges and then $\mathcal{A}$ is not an algebra.

□

# A Related Theorem Details