

# Durrett 5th Chapter2 Solutions

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## 1 Independence

**2.1.1.** Suppose  $(X_1, \dots, X_n)$  has density  $f(x_1, x_2, \dots, x_n)$ , that is

$$P((X_1, X_2, \dots, X_n) \in A) = \int_A f(x) dx \quad \text{for } A \in \mathcal{R}^n$$

If  $f(x)$  can be written as  $g_1(x_1) \cdots g_n(x_n)$  where the  $g_m \geq 0$  are measurable, then  $X_1, X_2, \dots, X_n$  are independent. Note that the  $g_m$  are not assumed to be probability densities.

**Solution.** Let  $A = B_1 \times \cdots \times B_n$  where  $B_i \in \mathcal{B}$ . Then by Fubini-Tonelli Theorem

$$\begin{aligned} P(X_1 \in B_1, \dots, X_n \in B_n) &= P((X_1, \dots, X_n) \in A) \\ &= \int_A f(x) dx = \int_{B_1 \times \cdots \times B_n} g_1 \cdots g_n dx \\ &= \prod_{1 \leq i \leq n} \int_{B_i} g_i dx \end{aligned}$$

Then let  $A_i = \mathbb{R} \times \cdots \times \mathbb{R} \times B_i \times \mathbb{R} \times \cdots \times \mathbb{R}$ , then since  $\prod_{1 \leq j \leq n} \int_{\mathbb{R}} g_j dx = \int_{\mathbb{R}^n} f dx = 1$ ,

$$P(X_i \in B_i) = P((X_1, \dots, X_n) \in A_i) = \frac{\int_{B_i} g_i dx}{\int_{\mathbb{R}} g_i dx} \prod_{1 \leq j \leq n} \int_{\mathbb{R}} g_j dx = \frac{\int_{B_i} g_i dx}{\int_{\mathbb{R}} g_i dx}$$

The desired result follows from

$$\begin{aligned} \prod_{1 \leq j \leq n} P(X_j \in B_j) &= \frac{\prod_{1 \leq j \leq n} \int_{B_j} g_j dx}{\prod_{1 \leq j \leq n} \int_{\mathbb{R}} g_j dx} \\ &= \prod_{1 \leq j \leq n} \int_{B_j} g_j dx = P(X_1 \in B_1, \dots, X_n \in B_n) \end{aligned}$$

□

**2.1.2.** Suppose  $X_1, \dots, X_n$  are random variables that take values in countable sets  $S_1, \dots, S_n$ . Then in order for  $X_1, \dots, X_n$  to be independent, it is sufficient that whenever  $x_i \in S_i$ ,

$$P(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i)$$

**Solution.** For any measurable set  $B_i \subset S_i$ ,  $B_i$  is countable and  $B_1 \times \cdots \times B_n$  is countable. Then

$$\begin{aligned} P(X_1 \in B_1, \dots, X_n \in B_n) &= \sum_{(x_1, \dots, x_n) \in B_1 \times \cdots \times B_n} P(X_1 = x_1, \dots, X_n = x_n) \\ &= \sum_{(x_1, \dots, x_n) \in B_1 \times \cdots \times B_n} \prod_{i=1}^n P(X_i = x_i) \\ &= \prod_{i=1}^n \left( \sum_{x_i \in B_i} P(X_i = x_i) \right) = \prod_{i=1}^n P(X_i \in B_i) \end{aligned}$$

is followed by the desired result.  $\square$

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### exercise

**2.7.1.** Consider  $\gamma(a)$  defined in (2.7.1). The following are equivalent:

- (a)  $\gamma(a) = -\infty$
- (b)  $P(X \geq a) = 0$
- (c)  $P(S_n \geq na) = 0$

**Solution.** According the context,

$$\gamma(a) := \lim_{n \rightarrow \infty} \frac{1}{n} \log P(S_n \geq na)$$

where  $S_n = X_1 + \cdots + X_n$  and  $X_i$ 's are i.i.d and  $a > EX_i$ . Its well-definition has been proven in DTE5th and is repeated in Theorem 2.7.2

- (a) $\Rightarrow$ (b). Assuming  $P(X_1 \geq a) = r > 0$ , then we have the contradiction that

$$\gamma(a) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln P(S_n \geq na) \geq \lim_{n \rightarrow \infty} \frac{1}{n} \ln (P(X_i \geq a))^n = \ln r > -\infty$$

- (b) $\Rightarrow$ (c) is trivial
- (c) $\Rightarrow$ (a) is weird but trivial.

$\square$

## A Related Theorem Details

### 2.7

**Lemma 2.7.1** (Lemma 2.7.1 in DTE5th). *If  $\gamma_{n+m} \geq \gamma_n + \gamma_m$  holds for any  $m$  and  $n$ , then  $\gamma(n)/n \rightarrow \sup_k \gamma(k)/k$  as  $n \rightarrow \infty$ .*

*Proof.* We have  $\limsup \frac{\gamma_n}{n} \leq \sup_k \frac{\gamma_k}{k}$  and let  $n = km + l$ , then

$$\frac{\gamma_n}{n} \geq \frac{km}{n} \frac{\gamma_m}{m} + \frac{\gamma_l}{n}$$

which conclude  $\liminf \gamma_n/n \geq \gamma_m/m$  for any  $m$  and hence the desired result.

**Theorem 2.7.2.** *Let  $X_i$ 's are i.i.d and  $Sn = X_1 + \dots + X_n$ . For any  $a > EX$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P(S_n \geq na)$$

*exists on  $\mathbb{R} \cup \{-\infty\}$ .*

*Proof.* We observe that

$$\ln P(S_{n+m} \geq (n+m)a) > \ln P(Sn \geq na) + \ln P(S_m \geq ma)$$

and hence by Lemma 2.7.1 as  $n \rightarrow \infty$ ,

$$\frac{\ln P(Sn \geq na)}{n} \rightarrow \sup_m \frac{\ln P(S_m \geq ma)}{m} < 0$$