Durrett 5th Chapter1 Solutions

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1 Probability Spaces

1.1.1. Let $\Omega = \mathbb{R}$, $\mathcal{F}$ =all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Solution. We firstly prove $\mathcal{F}$ is a σ-field. Let $\{A_i\} \subseteq \mathcal{F}$. It is obvious $A_i^c \in \mathcal{F}$. If A_i or A_j is countable, then $A_i \cap A_j$ is countable and hence contained in $\mathcal{F}$. Otherwise A_i^c and A_j^c is countable, and then $(A_i \cap A_j)^c = A_i^c \cup A_j^c$ is countable, which concludes $A_i \cap A_j$ is always in $\mathcal{F}$. Finally, $\bigcup_{i\in\mathbb{N}} A_i$ is countable if every A_i is countable, Otherwise $(\bigcup_{i\in\mathbb{N}} A_i)^c = \bigcap_{i\in\mathbb{N}} A_i^c \subseteq A_0^c$ is countable since A_0^c is countable, which concludes $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$. Hence $\mathcal{F}$ is a σ-field.

P is well-defined since $A \cup A^c = \Omega = \mathbb{R}$ is uncountable, whence A and A^c can not be countable simutaneously. Finally we prove P is a probability. It is trivial $P(\Omega) = 1$. For disjoint sets, $A_i \in \mathcal{F}$, if A_i are all countable, $P(A_i) = 0$ and then $+_{i=1}^{\infty} A_i$ is countable, $P(+_{i=1}^{\infty} A_i) = 0 = \sum_{i=1}^{\infty} P(A_i)$. Otherwise A_k^c is countable, and then $A_i \subseteq A_k^c, i \neq k$ is countable since they are disjoint. And $(+_{i\in\mathbb{N}} A_i)^c = \bigcap_{i\in\mathbb{N}} A_i^c \subseteq A_k^c$ is countable. So $P(+_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) = 1 = P(+_{i\in\mathbb{N}} A_i)$, when P is a probability. □

1.1.2. Recall the definition of S_d from Example 1.1.5. Show that $\sigma(S_d) = \mathcal{R}^d$, the Borel subsets of $\mathbb{R}^d$.

Solution. The definition of S_d is

$$S_d = \left\{ \prod_{i=1}^{d} (a_i, b_i] \;\middle|\; -\infty \leq a, b \leq +\infty \right\}$$

It is obvious $S_d \subseteq \mathcal{R}^d$ whence $\sigma(S_d) \subseteq \mathcal{R}^d$.Then for any $\prod_{i=1}^{d} (a_i, b_i)$,

$$\bigcup_{n=1}^{\infty} (\prod_{i=1}^{d} (a_i, b_i - \frac{1}{n}]) = \prod_{i=1}^{d} (a_i, b_i)$$

The desired result follows from $\mathcal{R}^d$ is the σ-field of the right. □

1.1.3. A σ-field $\mathcal{F}$ is said to be countably generated if there is a countable collection $\mathcal{C} \subseteq \mathcal{F}$ so that $\sigma(C) = F$. Show that $\mathcal{R}^d$ is countably generated.

Solution. Let

$$\mathcal{U} = \left\{ \prod_{i=1}^{d} (a_i, b_i) \;\middle|\; a_i, b_i \in \mathbb{Q} \right\}$$

Then $\mathcal{U}$ is countable since $|\mathcal{U}| \leq \left|\mathbb{Q}^{2d}\right|$. And $\sigma(\mathcal{U}) = \mathcal{R}^d$ since for each $\prod_{i=1}^{d} (a_i, b_i)$, there are $a_{i,k}, b_{i,k} \in \mathbb{Q}$ such that $a_{i,k} \to a_i + 0$ and $b_{i,k} \to b_i - 0$ as $k \to \infty$ and then

$$\bigcup_{k=1}^{\infty} (\prod_{i=1}^{d} (a_{i,k}, b_{i,k})) = \prod_{i=1}^{d} (a_i, b_i)$$

□

1.1.4.

(i) Show that if $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ are σ-algebras, then $\bigcup_i \mathcal{F}_i$ is an algebra.

(ii) Give an example to show that $\bigcup_i \mathcal{F}_i$ need not be a σ-algebra.

Solution.

(i) For $A, B \in \bigcup_i \mathcal{F}_i$, there is F_k such that $A, B \in \mathcal{F}_k$, and then $A \cup B, A^c \in \mathcal{F}_k \subseteq \bigcup_i \mathcal{F}_i$, whence $\bigcup_i \mathcal{F}_i$ is an algebra.

(ii) Let $\Omega = \mathbb{R}^{\mathbb{N}}$ and $F_i = \left\{U \times \mathbb{R}^{\mathbb{N}} \;\middle|\; U \in \mathcal{R}^i\right\}$. While $[0,1]^i \times \mathbb{R}^{\mathbb{N}} \in \mathcal{F}_i$, but

$$[0,1]^{\mathbb{N}} = \bigcap_{i=1}^{\infty} [0,1]^i \times \mathbb{R}^{\mathbb{N}} \notin \bigcup_i \mathcal{F}_i$$

□

1.1.5. A set $A \subseteq \{1, 2, \dots\}$ is said to have asymptotic density θ if

$$\lim_{n\to\infty} \frac{|A \cap \{1, 2, ..., n\}|}{n} = \theta$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a σ-algebra? an algebra?

Solution. $\mathcal{A}$ is not an algebra. The counterexample is shown in the following.

Let $A = \{2n\}$ and $B = \{b_n\}, b_n \in \{2n-1, 2n\}$. Then A, B have asymptotic densities $1/2$. And then let

$$S_n = |A \cap B \cap \{1, 2, \dots, 2n\}| = \sum_{k=1}^{n} |A \cap B \cap \{2k-1, 2k\}| = \sum_{k=1}^{n} c_k$$

where

$$c_k = \begin{cases} 0 & b_k = 2k-1 \\ 1 & b_k = 2k \end{cases}$$

It means for any series $\{x_i\}, x_i \in \{0, 1\}$, there exists B with asymptotic density $1/2$ such that $c_i = x_i$. Hence let

$$c_i = \begin{cases} 0 & i = 1 \\ 1 & i = [3^{2k}+1, 3^{2k+1}] \\ 0 & i = [3^{2k+1}+1, 3^{2k+2}] \end{cases} \qquad k \in \mathbb{N}$$

Then for $n = 3^{2k+1}$, $S_n \geq \sum_{3^{2k}+1}^{3^{2k+1}} c_i = 2 \times 3^{2k} = \frac{2}{3}n$ and for $n = 3^{2k+2}$, $S_n = S_{3^{2k+1}} \leq 3^{2k+1} = \frac{1}{3}n$. So $L_n := |A \cap B \cap \{1, 2, \dots, n\}|/n \geq 1/3$ when $n = 2 \times 3^{2k+1}$ and $L_n \leq \frac{1}{6}$ when $n = 2 \times 3^{2k+2}$, whence L_n diverges and then $\mathcal{A}$ is not an algebra.

(The motivation here is for $0 < S_n/n$ <1 we could add a lot of 1's to making $S_{n+m}/n+m > 1-\epsilon$ for any epsilon; and a lot of 0's to making $S_{n+m}/n+m < \epsilon$ for any $\epsilon > 0$. Thus the quotinet could fluctuates and then not converge.) □

2 Distributions

1.2.1. Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

Solution. $Z = X \cdot 1_A + Y \cdot 1_{A^c}$ is a random variable. □

1.2.2. Let χ have the standard normal distribution. Use Theorem 1.2.6 to get upper and lower bounds on $P(\chi \leq 4)$.

Solution. Theorem 1.2.6 is listed at Theorem 1.2.2.

$$P(\chi \leq 4) = \int_{-\infty}^{4} e^{x^2/2}\,\mathrm{d}x = 1 - \int_{x}^{+\infty} e^{x^2/2}\,\mathrm{d}x$$

Applying the theorem gives

$$(\frac{1}{4} - \frac{1}{64})e^{-8} \leq \int_{x}^{+\infty} e^{x^2/2}\,\mathrm{d}x \leq \frac{1}{4}e^{-8}$$

So

$$1 - \frac{1}{4}e^{-8} \leq P(\chi \leq 4) \leq 1 - \frac{15e^{-8}}{64}$$

□

1.2.3. Show that a distribution function has at most countably many discontinuities.

Solution. This is a common conlusion in real analysis. Let f be a distribution function, then f is monotonic increasing. Let P be the set of discontinuities. For each $x_0 \in P$, $f_-(x_0)$, $f_+(x_0)$ exist and $f_-(x_0) < f_-(x_0)$. Then letting

$$\mathcal{I} := \{(f_-(a), f_+(a)) \mid a \in P\}$$

and

$$\begin{aligned} \varphi :& P \to \mathcal{I} \\ & x_0 \mapsto (f_-(x_0), f_-(x_0)) \end{aligned}$$

intervals in $\mathcal{I}$ are disjoint and φ is bijective and in since for any $x < y$,

$$f_-(x) < f_+(x) \leq f_-(y) < f_+(y)$$

The desired result follows from $\mathcal{I}$, a collection of disjoint open intervals in $\mathbb{R}$, is at most countable since $\exists q_{a,b} \in \mathbb{Q}$, $q_{a,b} \in (a,b) \in \mathcal{I}$, and then $(a,b) \mapsto q_{a,b}$ is an injection to $\mathbb{Q}$. □

1.2.4. Show that if $F(x) = P(X \leq x)$ is continuous then $Y = F(X)$ has a uniform distribution on $(0,1)$, that is, if $y \in [0,1]$, $P(Y \leq y) = y$.

Solution. We have $0 \geq F(x) \leq 1$, so $P(Y \leq y) = 0$ for $y < 0$ and $P(Y \leq y) = 1$ for $y > 1$. For $y \in [0,1]$,

$$P(Y \leq y) = P(F(X) \leq y) = P(X \in \{h \mid F(h) \leq y\})$$

Let $h_0 = \sup\{h \mid F(h) \leq y\}$. Since F is increasing and continuous, $F(h_0) = y$. Therefore

$$P(Y \leq y) = P(X \leq h_0) = F(h_0) = y$$

□

1.2.5. Suppose X has continuous density f, $P(\alpha \leq X \leq \beta) = 1$ and g is a function that is strictly increasing and differentiable on (α, β). Then $g(X)$ has density

$$\rho(y) = \begin{cases} \frac{f(g^{-1}(y))}{g'(g^{-1}(y))} & y \in (g(\alpha), g(\beta)) \\ 0 & \textit{otherwise} \end{cases}$$

When $g(x) = ax + b$ with $a > 0$, $g^{-1}(y) = \frac{y-b}{a}$ so the answer is $\frac{1}{a}f(\frac{y-b}{a})$.

Solution. Since g is strictly increasing, g is invertible and therefore

$$P(g(X) \le y) = P(X \le g^{-1}(y)) = \begin{cases} 0 & y \in (-\infty, g(\alpha)] \\ \int_{\alpha}^{g^{-1}(y)} f(x)\,\mathrm{d}x & y \in (g(\alpha), g(\beta)) \\ 1 & y \in [g(\beta), +\infty) \end{cases}$$

We could direct differentiate the right or replace $x = g^{-1}(u)$, and then

$$\int_{\alpha}^{g^{-1}(y)} f(x)\,\mathrm{d}x = \int_{g(\alpha)}^{y} f(g^{-1}(u))\,\mathrm{d}g^{-1}(u) = \int_{g(\alpha)}^{y} \frac{f(g^{-1}(u))}{g'(g^{-1}(u))}\,\mathrm{d}u$$

□

1.2.6. Suppose X has a normal distribution. Use the previous exercise to compute the density of $\exp(X)$. (The answer is called the lognormal distribution.)

Solution. By Exercise 1.2.5, letting $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, the distribution is

$$g(x) = \begin{cases} \frac{f(\ln x)}{\exp(\ln(x))} = \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2}} & x \in (0, +\infty) \\ 0 & x \in (-\infty, 0] \end{cases}$$

□

1.2.7.

(i) Suppose X has density function f. Compute the distribution function of X^2 and then differentiate to find its density function.

(ii) Work out the answer when X has a standard normal distribution to find the density of the chi-square distribution.

Solution.

(i) Since

$$P(X^2 \le y) = \begin{cases} 0 & y \in (-\infty, 0] \\ P\left(-\sqrt{y} \le X \le \sqrt{y}\right) & y \in (0, +\infty) \end{cases}$$

and replacing $x = \sqrt{u}$,

$$\begin{aligned} P\left(-\sqrt{y} \le X \le \sqrt{y}\right) &= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,\mathrm{d}x = \int_{0}^{\sqrt{y}} (f(x) + f(-x))\,\mathrm{d}x \\ &= \int_0^y (f(\sqrt{u}) + f(-\sqrt{u}))\,\mathrm{d}\sqrt{u} = \int_0^y \frac{f(\sqrt{u}) + f(-\sqrt{u})}{2\sqrt{u}}\,\mathrm{d}u \end{aligned}$$

X^2 has a density function,

$$g(y) = \begin{cases} 0 & y \in (-\infty, 0] \\ \frac{f(\sqrt{y}) + f(-\sqrt{y})}{2\sqrt{y}} & y \in (0, +\infty) \end{cases}$$

(*We can not directly differentiate the definite intergrals with variable limits, since it may be indifferentiable when f is discontinuous*)

(ii) Applying above conclusion, the density is

$$g(y) = \begin{cases} 0 & y \in (-\infty, 0] \\ \frac{e^{-y/2}}{\sqrt{2\pi y}} & y \in (0, +\infty) \end{cases}$$

□

3 Random Variables

1.3.1. Show that if $\mathcal{A}$ generates $\mathcal{S}$, then $X^{-1}(\mathcal{A}) \equiv \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.

Solution. It is equivalent to prove $\sigma(X^{-1}(\mathcal{A})) = \sigma(X)$. Since $X^{-1}(\mathcal{A}) \subseteq \sigma(X)$ is obvious and hence $\sigma(X^{-1}(\mathcal{A})) \subseteq \sigma(X)$, it suffices to prove $\sigma(X^{-1}(\mathcal{A})) \subseteq \sigma(X)$. Considering

$$\mathcal{E} := \left\{B \in \mathcal{S} \;\middle|\; X^{-1}(B) \in \sigma(X^{-1}(\mathcal{A}))\right\}$$

it is not diffcult to see $\mathcal{E}$ is σ-algebra containing $\mathcal{A}$, whence $\mathcal{S} \subseteq \mathcal{E}$ and then $\sigma(X^{-1}(\mathcal{A})) \subseteq \sigma(X)$.

A more common statement is

$$\sigma(X^{-1}(\mathcal{A})) = X^{-1}(\sigma(\mathcal{A}))$$

□

1.3.2. Prove Theorem 1.3.6 when $n = 2$ by checking $\{X_1 + X_2 < x\} \in \mathcal{F}$.

Solution. Theorem 1.3.6 is listed at Theorem 1.3.1. We observe that

$$\{X_1 + X_2 < x\} = \bigcup_{q \in \mathbb{Q}} \{X_1 < q\} \cap \{X_2 < x - q\}$$

, since the left set contains the right set is obvious and for each $(x_1, x_2) \in \{X_1 + X_2 < x\}$, there is $x_1 < x - x_2$ and $q_0 \in \mathbb{Q}$, $x_1 < q_0 < x - x_2$, whence $(x_1, x_2) \in \{X_1 < q_0\} \cap \{X_2 < x - q_0\}$ and the left set is contained in the right set.

For the complete proof of $n = k$, we could use the induction on n and the fact that $X_1 + \cdots + X_k = (X_1 + \cdots + X_{k-1}) + X_k$.

□

1.3.3. Show that if f is continuous and $X_n \to X$ almost surely then $f(X_n) \to f(X)$ almost surely.

Solution. For $\omega \in \{X_n \to X\}$, $f((X_n)(\omega)) \to f(X(\omega))$. Therefore, $\{X_n \to X\} \subseteq \{f(X_n) \to f(X)\}$ and hence

$$P\left(\{f(X_n) \to f(X)\}\right) \geq P\left(\{X_n \to X\}\right) = 1$$

□

1.3.4.

(i) Show that a continuous function from $\mathbb{R}^d \to \mathbb{R}$ is a measurable map from $(\mathbb{R}^d, \mathcal{R}^d)$ to $(\mathbb{R}, \mathcal{R})$.

(ii) Show that $\mathcal{R}^d$ is the smallest σ-field that makes all the continuous functions measurable.

Solution. Let f be a continuous function from $\mathbb{R}^d \to \mathbb{R}$.

(i) It suffices to prove for all open set $V \subseteq \mathbb{R}$, $f^{-1}(V) \in \mathcal{R}^d$. Actually $f^{-1}(V)$ is an open set in $\mathbb{R}^n$ and hence in $\mathcal{R}^d$, which is a directly result from general topology and can be proven by the following.

For each $x \in f^{-1}(V)$,there is $\epsilon_x > 0$ such that the open ball $U(f(x), \epsilon_x) \subseteq V$ and hence there is $\delta_x > 0$,

$$f(U(x, \delta_x)) \subseteq U(f(x), \epsilon_x) \subseteq V$$

So

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U(x, \delta_x)$$

is an open set contiained in $\mathcal{R}^n$.

(ii) Let $\mathcal{F}$ be the smallest σ-field that makes all the continuous functions measurable. For the continuous function,

$$f_{x_0,d_0}(x) = d_0 - d(x_0, x)$$

there is

$$\{f_{x_0,d_0} > 0\} = \{d(x_0, x) > d_0\} = U(x_0, d_0) \in \mathcal{F}$$

So $\mathcal{F}$ contains all open balls and hence contains all open set, which gives $\mathcal{R}^n \subseteq \mathcal{F}$. The desired result follows.

$\square$

1.3.5. A function f is said to be lower semicontinuous or l.s.c. if

$$\liminf_{y \to x} f(y) \geq f(x)$$

and upper semicontinuous (u.s.c.) if $-f$ is l.s.c. Show that f is l.s.c. if and only if $\{x : f(x) \leq a\}$ is closed for each $a \in \mathbb{R}$ and conclude that semicontinuous functions are measurable.

Solution. Let $S = \{x : f(x) \leq a\}$. To prove the forward direction, for $\{x_n\} \subseteq S$, $x_n \to x$ implies $f(x) \leq \liminf x_n \leq a$, whence $x \in S$ and S is closed. For the contrary direction, suppose $a = \liminf_{y \to x} f(y)$ and there is a sequence $x_n \to x$, $f(x_n) \to a$. Since $S_\epsilon = \{x \mid f(x) \leq a + \epsilon\}$ is closed and $\{x_n\}_{n \geq m} \subseteq S_\epsilon$ for a large m, $x \in S_\epsilon$. So

$$x \in \bigcap_{n=1}^{\infty} S_{\frac{1}{n}} = \{x \mid f(x) \leq a\}$$

Therefore $f(x) \leq a = \liminf_{y \to x} f(y)$ and f is l.s.c. Then f is measurable follows from $\{x \mid f(x) > a\}$ is an open set and then a measurable set. $\square$

1.3.6. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary function and let $f^\delta(x) = \sup\{f(y) : |y - x| < \delta\}$ and $f_\delta(x) = \inf\{f(y) : |y - x| < \delta\}$ where $|z| = (z_1^2 + \cdots + z_d^2)^{1/2}$. Show that f^δ is l.s.c. and f_δ is u.s.c. Let $f^0 = \lim_{\delta \to 0} f^\delta$, $f_0 = \lim \delta \to 0 f_\delta$, and conclude that the set of points at which f is discontinuous $= \{f^0 \neq f_0\}$ is measurable.

Solution. Use the conclusion in Exercise 1.3.5 to prove f^δ is a l.s.c. Let $S = \{x : f(x) \leq a\}$ and $\{x_n\} \subseteq S$, $x_n \to x$. Suppose $f^\delta(x) > a$ for the sake of contradiction. There is $f(y) > a + \epsilon$ where $|y - x| < \delta - \tau$ and $|x_k - x| < \tau$ for a large k. Thus $|x_k - y| < \delta$ and $f^\delta(x_k) \geq f(y) = a + \epsilon$ leads a contradiction with $x_k \in a$. f^δ is l.s.c follows from $x \in S$ and then S is closed. And f_δ is a u.s.c since

$$-f_\delta = -\inf\{f(y) : |y - x| < \delta\} = \sup\{-f(y) : |y - x| < \delta\} = (-f)^\delta$$

whence $-f_\delta$ is a l.s.c.

It suffices to prove $f_0(x) = f(x) \wedge \liminf_{y \to x} f(y)$ and $f^0(x) = f(x) \vee \limsup_{y \to x} f(y)$. And it follows from

$$f_\delta(x) = \inf\{f(y) : |y - x| < \delta\} = f(x) \wedge \inf\{f(y) : 0 < |y - x| < \delta\} \to f(x) + \liminf_{y \to x} f(y)$$

as $\delta \to 0$ and so does f^0. $\square$

1.3.7. A function $\varphi : \Omega \to \mathbb{R}$ is said to be simple if

$$\varphi(\omega) = \sum_{m=1}^{n} c_m 1_{A_m}(\omega)$$

where the c_m are real numbers and $A_m \in \mathcal{F}$. Show that the class of $\mathcal{F}$ measurable functions is the smallest class containing the simple functions and closed under pointwise limits.

Solution. By Theorem 1.3.6 in PTE5th(listed at Theorem 1.3.2), it is obvious simple functions and their pointwise limits are measurable functions. It is suffices to prove for each measurable function f, there is a sequence of simple functions $\{f_n\}$, $f_n \to f$. Define

$$f_n = \sum_{m=-n}^{n-1} \sum_{k=0}^{n-1} (m + \frac{k}{n}) 1_{A_{m,k}}$$

where $A_{m,k} = f^{-1}([m + \frac{k}{n}, m + \frac{k+1}{n}))$ and then $f_n \to f$. □

1.3.8. Use the previous exercise to conclude that Y is measurable with respect to $\sigma(X)$ if and only if $Y = f(X)$ where $f : R \to R$ is measurable.

Solution. It is easy to prove it's sufficient. For the contrary direction, we define $Y_n \to Y$ like $f_n \to f$ in Exercise 1.3.7. Thus

$$Y_n = \sum_{m=-n2^n}^{n2^n-1} \frac{m}{2^n} 1_{A_{m,n}}$$

where $A_{m,n} = Y^{-1}([\frac{m}{2^n}, \frac{m+1}{2^n})) \in \sigma(X)$ are disjoint. And since $\sigma(X) = X^{-1}(\mathcal{B})$, there must be $C \in \mathcal{B}$ such that $X^{-1}(C) = A_{m,n}$. Let

$$B_{m,n} = \bigcap_{C_i \in \mathcal{C}} C_i, \quad \mathcal{C} = \left\{ X \in \mathcal{B} \mid X^{-1}(C) = A_{m,n} \right\}$$

and then $B_{m,n} \in \mathcal{B}$, $X^{-1}(B_{m,n}) = A_{m,n}$ and $B_{m,n}$ are disjoint for a fixing n and different m. So we define

$$f_n = \sum_{m=-n2^n}^{n2^n-1} \frac{m}{2^n} 1_{B_{m,n}}$$

and hence $Y_n = f_n(X)$, f_n is measurable.

Let $f = \lim_{n\to\infty} f_n$ where f_n is convergent since if x belongs to a certain $B_{m,k}^{(n)}$, then $f_n(x)$ is increasing over n otherwise always 0. Then $Y = f(X)$ since $Y_n = f_n(X) \to Y = f(x)$ as $n \to \infty$ and f is measurable follows from f is a limit of simple functions. □

1.3.9. To get a constructive proof of the last result, note that $\{\omega : m2^{-n} \le Y < (m+1)2^n\} = X \in B_{m,n}$ for some $B_{m,n} \in R$ and set $f_n(x) = m2^{-n}$ for $x \in B_{m,n}$ and show that as $n \to \infty$, $fn(x) \to f(x)$. and $Y = f(X)$.

Solution. Just a hint for Exercise 1.3.8. The specific forms of simple functions approaching to Y is not critical. But it's curious that if the diction here implies there is a simpler and non-constructive proof. □

4 Integration

1.4.1. Show that if $f \ge 0$ and $\int f \,\mathrm{d}\mu = 0$ then $f = 0$ a.e.

Solution. Let $A_n = f \ge \frac{1}{n}$. Then

$$0 = \int f \,\mathrm{d}\mu \ge \int \frac{1_{A_n}}{n} \,\mathrm{d}\mu = \frac{1}{n}\mu(A_n) \ge 0$$

Hence $\mu(A_n) = 0$ and $\mu(A_n^c) = 1$. Thus $\mu(\{f = 0\}) = \mu(\bigcap A_n^c) = 1$ □

1.4.2. Let $f \ge 0$ and $E_{n,m} = \{x : m/2^n \le f(x) < (m+1)/2^n\}$. As $n \to \infty$,

$$\sum_{m=1}^{\infty} \frac{m}{2^n} \mu(E_{n,m}) \nearrow \int f \,\mathrm{d}\mu$$

Solution. Let $f_n = \sum_{m=1}^{\infty} \frac{m}{2^n} 1_{E_{n,m}}$. And then left hand is equal to $\int f_n \,\mathrm{d}\mu$ and $\int f_n \,\mathrm{d}\mu \le \int f \,\mathrm{d}\mu$. Recall

$$\int f \,\mathrm{d}\mu = \sup \left\{ \int h \,\mathrm{d}\mu : 0 \le h \le f, h \text{ is bounded and } \mu(\{x : h(x) > 0\}) < 1 \right\}$$

For such h and $C = \{x : h(x) > 0\}$, $h \le 1_C \sum_{m=1}^{\infty} \frac{m+1}{2^n} 1_{E_{n,m}} \le 1_C (f_n + \frac{1}{2^n})$. So

$$\int f_n \,\mathrm{d}\mu \ge \int h \,\mathrm{d}\mu - \frac{\mu(C)}{2^n} \to \int h \,\mathrm{d}\mu$$

The desired result follows from

$$\int f \,\mathrm{d}\mu \ge \limsup \int f \,\mathrm{d}\mu \ge \liminf \int f \,\mathrm{d}\mu \ge \sup_h \left\{ \int h \,\mathrm{d}\mu \right\} = \int f \,\mathrm{d}\mu$$

□

1.4.3. Let g be an integrable function on $\mathbb{R}$ and $\epsilon > 0$.

(i) Use the definition of the integral to conclude there is a simple function $\varphi = \sum_k b_k 1_{A_k}$ with $\int |g - \varphi| \,\mathrm{d}x < \epsilon$.

(ii) Use Lemma A.2.1 to approximate the A_k by finite unions of intervals to get a step function

$$q = \sum_{j=1}^{k} c_j 1(a_{j-1}, a_j)$$

with $a_0 < a_1 < \cdots < a_k$, so that $\int |\varphi - q| \,\mathrm{d}x < \epsilon$.

(iii) Round the corners of q to get a continuous function r so that $\int |q - r| \,\mathrm{d}x < \epsilon$.

(i)

Solution.

(i) Let $g = g^+ - g^-$ where $g^+ = g \vee 0$ and $g^- = -g \vee 0$. Thus there are simple functions h_1, h_2 such that $\int |g^+ - h_1| \,\mathrm{d}x < \epsilon/2$ and $\int |g^- - h_2| \,\mathrm{d}x < \epsilon/2$. Then $\varphi = h_1 - h_2$ is a simple function and

$$\int |g - \varphi| \,\mathrm{d}x = \int \big|(g^+ - h_1) - (g^- - h_2)\big| \,\mathrm{d}x \le \int \big|g^+ - h_1\big| \,\mathrm{d}x + \int \big|g^- - h_2\big| \,\mathrm{d}x < \epsilon$$

(ii) To make a continuous function replace each $c_j 1_{(aj-1,aj)}$ by a function that is 0 **on** $(a_{j-1}, a_j)^c$, c_j on $(a_{j-1+\delta}, a_j - \delta)$, and linear otherwise. If

□

A Some related theorem details

2 Distributions

Theorem 1.2.1. *For $x > 0$, leting*

$$I(x) := \int_x^{+\infty} e^{-y^2/2}$$

then for $n \in \mathbb{N}$,

$$I(x) = I_n(x) + R_n(x)$$

where

$$\begin{aligned} I_n(x) := e^{-x^2/2}\left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{x^{2n-1}}\right) \\ = e^{-x^2/2}\sum_{k=1}^{n}(-1)^{k-1}\frac{(2k-3)!!}{x^{2k-1}} \end{aligned}$$

and

$$R_n(x) := (-1)^n(2n-1)!!\int_x^{+\infty}\frac{e^{-y^2/2}}{y^{2n}}\,\mathrm{d}y$$

by setting $(-1)!! = 1$*. Then* $I_{2k}(x) < I(x) < I_{2k+1}(x)$ *for any* $k \in \mathbb{N}$*.*

Proof. Use induction on n. It's trivial for $n = 0$ and assume it holds for $n = k$. Since

$$\begin{aligned} \frac{R_k(x)}{(-1)^k(2k-1)!!} &= \int_x^{\infty}\frac{e^{-y^2/2}}{y^{2k}}\,\mathrm{d}y = \int_x^{+\infty} -\frac{1}{y^{2k+1}}\,\mathrm{d}e^{-y^2/2} \\ &= \frac{-e^{-y^2/2}}{y^{2k+1}}\bigg|_x^{+\infty} - \int_x^{+\infty}\frac{(2k+1)e^{-y^2/2}}{y^{2k+2}}\,\mathrm{d}y \\ &= \frac{-e^{-x^2/2}}{x^{2k+1}} - (2k+1)\int_x^{+\infty}\frac{e^{-y^2/2}}{y^{2k+2}}\,\mathrm{d}y \\ &= \frac{-e^{-x^2/2}}{x^{2k+1}} + \frac{R_{k+1}(x)}{(-1)^k(2k-1)!!} \end{aligned}$$

then

$$\begin{aligned} I = I_k + R_k = I_k + (-1)^k(2k-1)!!\frac{-e^{-x^2/2}}{x^{2k+1}} + R_{k+1}(x) \\ = I_{k+1}(x) + R_{k+1}(x) \end{aligned}$$

The last conclusion is obvious since $R_{2k} > 0$ and $R_{2k+1} < 0$ for any $k \in \mathbb{N}$.

Theorem 1.2.2 (Theorem 1.2.6 in PTE5th)**.** *For* $x > 0$*,*

$$(x^{-1} - x^{-3})e^{-x^2/2} \le \int_x^{+\infty} e^{-y^2/2}\,\mathrm{d}y \le x^{-1}e^{-x^2/2}$$

Proof. We give a stronger conclusion and a more general proof in Theorem 1.2.1. The desired result directly follows from the cases of $n = 1$ and $n = 2$.

3 Random Variables

Theorem 1.3.1 (Theorem 1.3.6 in PTE5th)**.** *If* $X_1, \ldots, X_n$ *are random variables then* $X_1 + \cdots + X_n$ *is a random variable.*

Theorem 1.3.2 (Theorem 1.3.7 in PTE5th)**.** *If* $X_1, X_2, \ldots$ *are random variables then so are*

$$\inf_n X_n, \qquad \sup_n Xn, \qquad \limsup_n Xn, \qquad \liminf_n Xn$$