Durrett 5th Chapter1 Solutions

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1 Probability Spaces

1.1.1. Let $\Omega = \mathbb{R}$, $\mathcal{F}$ =all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Solution. We firstly prove $\mathcal{F}$ is a σ-field. Let $\{A_i\} \subseteq \mathcal{F}$. It is obvious $A_i^c \in \mathcal{F}$. If A_i or A_j is countable, then $A_i \cap A_j$ is countable and hence contained in $\mathcal{F}$. Otherwise A_i^c and A_j^c is countable, and then $(A_i \cap A_j)^c = A_i^c \cup A_j^c$ is countable, which concludes $A_i \cap A_j$ is always in $\mathcal{F}$. Finally, $\bigcup_{i \in \mathbb{N}} A_i$ is countable if every A_i is countable, Otherwise $(\bigcup_{i \in \mathbb{N}} A_i)^c = \bigcap_{i \in \mathbb{N}} A_i^c \subseteq A_0^c$ is countable since A_0^c is countable, which concludes $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$. Hence $\mathcal{F}$ is a σ-field.

P is well-defined since $A \cup A^c = \Omega = \mathbb{R}$ is uncountable, whence A and A^c can not be countable simutaneously. Finally we prove P is a probability. It is trivial $P(\Omega) = 1$. For disjoint sets, $A_i \in \mathcal{F}$, if A_i are all countable, $P(A_i) = 0$ and then $+_{i=1}^{\infty} A_i$ is countable, $P(+_{i=1}^{\infty} A_i) = 0 = \sum_{i=1}^{\infty} P(A_i)$. Otherwise A_k^c is countable, and then $A_i \subseteq A_k^c, i \neq k$ is countable since they are disjoint. And $(+_{i \in \mathbb{N}} A_i)^c = \bigcap_{i \in \mathbb{N}} A_i^c \subseteq A_k^c$ is countable. So $P(+_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) = 1 = P(+_{i \in \mathbb{N}} A_i)$, when P is a probability. $\square$

1.1.2. Recall the definition of S_d from Example 1.1.5. Show that $\sigma(S_d) = \mathcal{R}^d$, the Borel subsets of $\mathbb{R}^d$.

Proof. The definition of S_d is

$$S_d = \left\{ \prod_{i=1}^{d} (a_i, b_i] \ \middle| \ -\infty \le a, b \le +\infty \right\}$$

It is obvious $S_d \subseteq \mathcal{R}^d$ whence $\sigma(S_d) \subseteq \mathcal{R}^d$. Then for any $\prod_{i=1}^{d} (a_i, b_i)$,

$$\bigcup_{n=1}^{\infty} \left(\prod_{i=1}^{d} (a_i, b_i - \frac{1}{n}] \right) = \prod_{i=1}^{d} (a_i, b_i)$$

The desired result follows from $\mathcal{R}^d$ is the σ-field of the right.

1.1.3. A σ-field $\mathcal{F}$ is said to be countably generated if there is a countable collection $\mathcal{C} \subseteq \mathcal{F}$ so that $\sigma(C) = F$. Show that $\mathcal{R}^d$ is countably generated.

Solution. Let

$$\mathcal{U} = \left\{ \prod_{i=1}^{d} (a_i, b_i) \ \middle| \ a_i, b_i \in \mathbb{Q} \right\}$$

Then $\mathcal{U}$ is countable since $|\mathcal{U}| \le \left| \mathbb{Q}^{2d} \right|$. And $\sigma(\mathcal{U}) = \mathcal{R}^d$ since for each $\prod_{i=1}^{d} (a_i, b_i)$, there are $a_{i,k}, b_{i,k} \in \mathbb{Q}$ such that $a_{i,k} \to a_i + 0$ and $b_{i,k} \to b_i - 0$ as $k \to \infty$ and then

$$\bigcup_{k=1}^{\infty} \left(\prod_{i=1}^{d} (a_{i,k}, b_{i,k}) \right) = \prod_{i=1}^{d} (a_i, b_i)$$

$\square$

1.1.4.

(i) Show that if $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ are σ-algebras, then $\bigcup_i \mathcal{F}_i$ is an algebra.

(ii) Give an example to show that $\bigcup_i \mathcal{F}_i$ need not be a σ-algebra.

Solution.

(i) For $A, B \in \bigcup_i \mathcal{F}_i$, there is F_k such that $A, B \in \mathcal{F}_k$, and then $A \cup B, A^c \in \mathcal{F}_k \subseteq \bigcup_i \mathcal{F}_i$, whence $\bigcup_i \mathcal{F}_i$ is an algebra.

(ii) Let $\Omega = \mathbb{R}^{\mathbb{N}}$ and $F_i = \left\{ U \times \mathbb{R}^{\mathbb{N}} \mid U \in \mathcal{R}^i \right\}$. While $[0,1]^i \times \mathbb{R}^{\mathbb{N}} \in \mathcal{F}_i$, but

$$[0,1]^{\mathbb{N}} = \bigcap_{i=1}^{\infty} [0,1]^i \times \mathbb{R}^{\mathbb{N}} \notin \bigcup_i \mathcal{F}_i$$

$\square$

1.1.5. A set $A \subseteq \{1, 2, \dots\}$ is said to have asymptotic density θ if

$$\lim_{n \to \infty} \frac{|A \cap \{1, 2, ..., n\}|}{n} = \theta$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a σ-algebra? an algebra?

Solution. $\mathcal{A}$ is not an algebra. The counterexample is shown in the following.

Let $A = \{2n\}$ and $B = \{b_n\}, b_n \in \{2n - 1, 2n\}$. Then A, B have asymptotic densities $1/2$. And then let

$$S_n = |A \cap B \cap \{1, 2, \dots, 2n\}| = \sum_{k=1}^{n} |A \cap B \cap \{2k-1, 2k\}| = \sum_{k=1}^{n} c_k$$

where

$$c_k = \begin{cases} 0 & b_k = 2k - 1 \\ 1 & b_k = 2k \end{cases}$$

It means for any series $\{x_i\}, x_i \in \{0, 1\}$, there exists B with asymptotic density $1/2$ such that $c_i = x_i$. Hence let

$$c_i = \begin{cases} 0 & i = 1 \\ 1 & i = [3^{2k} + 1, 3^{2k+1}] \qquad k \in \mathbb{N} \\ 0 & i = [3^{2k+1} + 1, 3^{2k+2}] \end{cases}$$

Then for $n = 3^{2k+1}$, $S_n \geq \sum_{3^{2k}+1}^{3^{2k+1}} c_i = 2 \times 3^{2k} = \frac{2}{3}n$ and for $n = 3^{2k+2}$, $S_n = S_{3^{2k+1}} \leq 3^{2k+1} = \frac{1}{3}n$. So $L_n := |A \cap B \cap \{1, 2, \dots, n\}|/n \geq 1/3$ when $n = 2 \times 3^{2k+1}$ and $L_n \leq \frac{1}{6}$ when $n = 2 \times 3^{2k+2}$, whence L_n diverges and then $\mathcal{A}$ is not an algebra.

(The motivation here is for $0 < S_n/n < 1$ we could add a lot of 1's to making $S_{n+m}/n+m > 1 - \epsilon$ for any epsilon; and a lot of 0's to making $S_{n+m}/n+m < \epsilon$ for any $\epsilon > 0$. Thus the quotinet could fluctuates and then not converge.) $\square$

2 Distributions

1.2.1. Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

Solution. $Z = X \cdot 1_A + Y \cdot 1_{A^c}$ is a random variable. $\square$

1.2.2. Let χ have the standard normal distribution. Use Theorem 1.2.6 to get upper and lower bounds on $P(\chi \leq 4)$.

Solution. Theorem 1.2.6 is listed at Theorem 1.3.2.

$$P(\chi \le 4) = \int_{-\infty}^{4} e^{x^2/2} \, \mathrm{d}x = 1 - \int_{x}^{+\infty} e^{x^2/2} \, \mathrm{d}x$$

Applying the theorem gives

$$(\frac{1}{4} - \frac{1}{64})e^{-8} \le \int_{x}^{+\infty} e^{x^2/2} \, \mathrm{d}x \le \frac{1}{4}e^{-8}$$

So

$$1 - \frac{1}{4}e^{-8} \le P(\chi \le 4) \le 1 - \frac{15e^{-8}}{64}$$

$\square$

1.2.3. Show that a distribution function has at most countably many discontinuities.

Solution. This is a common conlusion in real analysis. Let f be a distribution function, then f is monotonic increasing. Let P be the set of discontinuities. For each $x_0 \in P$, $f_-(x_0)$, $f_+(x_0)$ exist and $f_-(x_0) < f_-(x_0)$. Then letting

$$\mathcal{I} := \{(f_-(a), f_+(a)) \mid a \in P\}$$

and

$$\varphi : P \to \mathcal{I}$$
$$x_0 \mapsto (f_-(x_0), f_-(x_0))$$

intervals in $\mathcal{I}$ are disjoint and φ is bijective and in since for any $x < y$,

$$f_-(x) < f_+(x) \le f_-(y) < f_+(y)$$

The desired result follows from $\mathcal{I}$, a collection of disjoint open intervals in $\mathbb{R}$, is at most countable since $\exists q_{a,b} \in \mathbb{Q}$, $q_{a,b} \in (a, b) \in \mathcal{I}$, and then $(a, b) \mapsto q_{a,b}$ is an injection to $\mathbb{Q}$. $\square$

A   Related Theorem Details

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Theorem 1.3.1. *For $x > 0$, leting*

$$I(x) := \int_{x}^{+\infty} e^{-y^2/2}$$

then for $n \in \mathbb{N}$,

$$I(x) = I_n(x) + R_n(x)$$

where

$$I_n(x) := e^{-x^2/2} \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \cdots + (-1)^{n-1} \frac{(2n-3)!!}{x^{2n-1}} \right)$$

$$= e^{-x^2/2} \sum_{k=1}^{n} (-1)^{k-1} \frac{(2k-3)!!}{x^{2k-1}}$$

and

$$R_n(x) := (-1)^n (2n-1)!! \int_{x}^{+\infty} \frac{e^{-y^2/2}}{y^{2n}} \, \mathrm{d}y$$

by setting $(-1)!! = 1$. Then $I_{2k}(x) < I(x) < I_{2k+1}(x)$ for any $k \in \mathbb{N}$.

Proof. Use induction on n. It's trivial for $n = 0$ and assume it holds for $n = k$. Since

$$\frac{R_k(x)}{(-1)^k(2k-1)!!} = \int_x^\infty \frac{e^{-y^2/2}}{y^{2k}}\,\mathrm{d}y = \int_x^{+\infty} -\frac{1}{y^{2k+1}}\,\mathrm{d}e^{-y^2/2}$$

$$= \left.\frac{-e^{-y^2/2}}{y^{2k+1}}\right|_x^{+\infty} - \int_x^{+\infty} \frac{(2k+1)e^{-y^2/2}}{y^{2k+2}}\,\mathrm{d}y$$

$$= \frac{-e^{-x^2/2}}{x^{2k+1}} - (2k+1)\int_x^{+\infty} \frac{e^{-y^2/2}}{y^{2k+2}}\,\mathrm{d}y$$

$$= \frac{-e^{-x^2/2}}{x^{2k+1}} + \frac{R_{k+1}(x)}{(-1)^k(2k-1)!!}$$

then

$$I = I_k + R_k = I_k + (-1)^k(2k-1)!!\frac{-e^{-x^2/2}}{x^{2k+1}} + R_{k+1}(x)$$

$$= I_{k+1}(x) + R_{k+1}(x)$$

The last conclusion is obvious since $R_{2k} > 0$ and $R_{2k+1} < 0$ for any $k \in \mathbb{N}$.

Proof.

Theorem 1.3.2 (Theorem 1.2.6 in PTE5th). *For $x > 0$,*

$$(x^{-1} - x^{-3})e^{-x^2/2} \leq \int_x^{+\infty} e^{-y^2/2}\,\mathrm{d}y \leq x^{-1}e^{-x^2/2}$$

Proof. We give a stronger conclusion and a more general proof in Theorem 1.3.1. The desired result directly follows from the cases of $n = 1$ and $n = 2$.