

Durrett 5th Chapter2 Solutions

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1 Independence

2.1.1. Suppose (X_1, \dots, X_n) has density $f(x_1, x_2, \dots, x_n)$, that is

$$P((X_1, X_2, \dots, X_n) \in A) = \int_A f(x) dx \quad \text{for } A \in \mathcal{R}^n$$

If $f(x)$ can be written as $g_1(x_1) \cdots g_n(x_n)$ where the $g_m \geq 0$ are measurable, then X_1, X_2, \dots, X_n are independent. Note that the g_m are not assumed to be probability densities.

Solution. Let $A = B_1 \times \cdots \times B_n$ where $B_i \in \mathcal{B}$. Then by Fubini-Tonelli Theorem

$$\begin{aligned} P(X_1 \in B_1, \dots, X_n \in B_n) &= P((X_1, \dots, X_n) \in A) \\ &= \int_A f(x) dx = \int_{B_1 \times \cdots \times B_n} g_1 \cdots g_n dx \\ &= \prod_{1 \leq i \leq n} \int_{B_i} g_i dx \end{aligned}$$

Then let $A_i = \mathbb{R} \times \cdots \times \mathbb{R} \times B_i \times \mathbb{R} \times \cdots \times \mathbb{R}$, then since $\prod_{1 \leq j \leq n} \int_{\mathbb{R}} g_j dx = \int_{\mathbb{R}^n} f dx = 1$,

$$P(X_i \in B_i) = P((X_1, \dots, X_n) \in A_i) = \frac{\int_{B_i} g_i dx}{\int_{\mathbb{R}} g_i dx} \prod_{1 \leq j \leq n} \int_{\mathbb{R}} g_j dx = \frac{\int_{B_i} g_i dx}{\int_{\mathbb{R}} g_i dx}$$

The desired result follows from

$$\begin{aligned} \prod_{1 \leq j \leq n} P(X_j \in B_j) &= \frac{\prod_{1 \leq j \leq n} \int_{B_j} g_j dx}{\prod_{1 \leq j \leq n} \int_{\mathbb{R}} g_j dx} \\ &= \prod_{1 \leq j \leq n} \int_{B_j} g_j dx = P(X_1 \in B_1, \dots, X_n \in B_n) \end{aligned}$$

□

2.1.2. Suppose X_1, \dots, X_n are random variables that take values in countable sets S_1, \dots, S_n . Then in order for X_1, \dots, X_n to be independent, it is sufficient that whenever $x_i \in S_i$,

$$P(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i)$$

Solution. For any measurable set $B_i \subset S_i$, B_i is countable and $B_1 \times \cdots \times B_n$ is countable. Then

$$\begin{aligned} P(X_1 \in B_1, \dots, X_n \in B_n) &= \sum_{(x_1, \dots, x_n) \in B_1 \times \cdots \times B_n} P(X_1 = x_1, \dots, X_n = x_n) \\ &= \sum_{(x_1, \dots, x_n) \in B_1 \times \cdots \times B_n} \prod_{i=1}^n P(X_i = x_i) \\ &= \prod_{i=1}^n \left(\sum_{x_i \in B_i} P(X_i = x_i) \right) = \prod_{i=1}^n P(X_i \in B_i) \end{aligned}$$

is followed by the desired result.

□

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3 Borel-Cantelli Lemmas

2.3.1. Prove that $P(\limsup A_n) \geq \limsup P(A_n)$ and $P(\liminf A_n) \leq \liminf P(A_n)$.

Solution.

$$P(\limsup A_n) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) = \lim_{n \rightarrow \infty} P\left(\bigcup_{m=n}^{\infty} A_m\right) \geq \lim_{n \rightarrow \infty} \sup_{m \geq n} P(A_m) = \limsup P(A_n)$$

It can be proved by the similar method for \liminf or directly use Fatou's Lemma to 1_{A_n}

□

2.3.2. Prove the first result in Theorem 2.3.4 directly from the definition.

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exercise

2.7.1. Consider $\gamma(a)$ defined in (2.7.1). The following are equivalent:

- (a) $\gamma(a) = -\infty$
- (b) $P(X \geq a) = 0$
- (c) $P(S_n \geq na) = 0$

Solution. According the context,

$$\gamma(a) := \lim_{n \rightarrow \infty} \frac{1}{n} \log P(S_n \geq na)$$

where $S_n = X_1 + \dots + X_n$ and X_i 's are i.i.d and $a > EX_i$. Its well-definition has been proven in PTE5th and is repeated in Theorem 2.7.2

- (a) \Rightarrow (b). Assuming $P(X_1 \geq a) = r > 0$, then we have the contradiction that

$$\gamma(a) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln P(S_n \geq na) \geq \lim_{n \rightarrow \infty} \frac{1}{n} \ln (P(X_i \geq a))^n = \ln r > -\infty$$

- (b) \Rightarrow (c) is trivial
- (c) \Rightarrow (a) is weird but trivial.

□

A Related Theorem Details

2.7

Lemma 2.7.1 (Lemma 2.7.1 in PTE5th). *If $\gamma_{n+m} \geq \gamma_n + \gamma_m$ holds for any m and n , then $\gamma(n)/n \rightarrow \sup_k \gamma(k)/k$ as $n \rightarrow \infty$.*

Proof. We have $\limsup \frac{\gamma_n}{n} \leq \sup_k \frac{\gamma_k}{k}$ and let $n = km + l$, then

$$\frac{\gamma_n}{n} \geq \frac{km}{n} \frac{\gamma_m}{m} + \frac{\gamma_l}{n}$$

which conclude $\liminf \gamma_n/n \geq \gamma_m/m$ for any m and hence the desired result.

Theorem 2.7.2. *Let X_i 's are i.i.d and $S_n = X_1 + \dots + X_n$. For any $a > EX$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P(S_n \geq na)$$

exists on $\mathbb{R} \cup \{-\infty\}$.

Proof. We observe that

$$\ln P(S_{n+m} \geq (n+m)a) > \ln P(S_n \geq na) + \ln P(S_m \geq ma)$$

and hence by Lemma 2.7.1 as $n \rightarrow \infty$,

$$\frac{\ln P(S_n \geq na)}{n} \rightarrow \sup_m \frac{\ln P(S_m \geq ma)}{m} < 0$$