Durrett 5th Chapter1 Solutions

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1 Probability Spaces

1.1.1. Let $\Omega = \mathbb{R}$, $\mathcal{F}$ =all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Solution. We firstly prove $\mathcal{F}$ is a σ-field. Let $\{A_i\} \subseteq \mathcal{F}$. It is obvious $A_i^c \in \mathcal{F}$. If A_i or A_j is countable, then $A_i \cap A_j$ is countable and hence contained in $\mathcal{F}$. Otherwise A_i^c and A_j^c is countable, and then $(A_i \cap A_j)^c = A_i^c \cup A_j^c$ is countable, which concludes $A_i \cap A_j$ is always in $\mathcal{F}$. Finally, $\bigcup_{i\in\mathbb{N}} A_i$ is countable if every A_i is countable, Otherwise $(\bigcup_{i\in\mathbb{N}} A_i)^c = \bigcap_{i\in\mathbb{N}} A_i^c \subseteq A_0^c$ is countable since A_0^c is countable, which concludes $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$. Hence $\mathcal{F}$ is a σ-field.

P is well-defined since $A \cup A^c = \Omega = \mathbb{R}$ is uncountable, whence A and A^c can not be countable simutaneously. Finally we prove P is a probability. It is trivial $P(\Omega) = 1$. For disjoint sets, $A_i \in \mathcal{F}$, if A_i are all countable, $P(A_i) = 0$ and then $+_{i=1}^{\infty} A_i$ is countable, $P(+_{i=1}^{\infty} A_i) = 0 = \sum_{i=1}^{\infty} P(A_i)$. Otherwise A_k^c is countable, and then $A_i \subseteq A_k^c, i \neq k$ is countable since they are disjoint. And $(+_{i\in\mathbb{N}} A_i)^c = \bigcap_{i\in\mathbb{N}} A_i^c \subseteq A_k^c$ is countable. So $P(+_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) = 1 = P(+_{i\in\mathbb{N}} A_i)$, when P is a probability. □

1.1.2. Recall the definition of S_d from Example 1.1.5. Show that $\sigma(S_d) = \mathcal{R}^d$, the Borel subsets of $\mathbb{R}^d$.

Proof. The definition of S_d is

$$S_d = \left\{ \prod_{i=1}^{d} (a_i, b_i] \;\middle|\; -\infty \leq a, b \leq +\infty \right\}$$

It is obvious $S_d \subseteq \mathcal{R}^d$ whence $\sigma(S_d) \subseteq \mathcal{R}^d$.Then for any $\prod_{i=1}^{d} (a_i, b_i)$,

$$\bigcup_{n=1}^{\infty} (\prod_{i=1}^{d} (a_i, b_i - \frac{1}{n}]) = \prod_{i=1}^{d} (a_i, b_i)$$

The desired result follows from $\mathcal{R}^d$ is the σ-field of the right.

1.1.3. A σ-field $\mathcal{F}$ is said to be countably generated if there is a countable collection $\mathcal{C} \subseteq \mathcal{F}$ so that $\sigma(C) = F$. Show that $\mathcal{R}^d$ is countably generated.

Solution. Let

$$\mathcal{U} = \left\{ \prod_{i=1}^{d} (a_i, b_i) \;\middle|\; a_i, b_i \in \mathbb{Q} \right\}$$

Then $\mathcal{U}$ is countable since $|\mathcal{U}| \leq \left|\mathbb{Q}^{2d}\right|$. And $\sigma(\mathcal{U}) = \mathcal{R}^d$ since for each $\prod_{i=1}^{d} (a_i, b_i)$, there are $a_{i,k}, b_{i,k} \in \mathbb{Q}$ such that $a_{i,k} \to a_i + 0$ and $b_{i,k} \to b_i - 0$ as $k \to \infty$ and then

$$\bigcup_{k=1}^{\infty} (\prod_{i=1}^{d} (a_{i,k}, b_{i,k})) = \prod_{i=1}^{d} (a_i, b_i)$$

□

1.1.4.

(i) Show that if $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ are σ-algebras, then $\bigcup_i \mathcal{F}_i$ is an algebra.

(ii) Give an example to show that $\bigcup_i \mathcal{F}_i$ need not be a σ-algebra.

Solution.

(i) For $A, B \in \bigcup_i \mathcal{F}_i$, there is F_k such that $A, B \in \mathcal{F}_k$, and then $A \cup B, A^c \in \mathcal{F}_k \subseteq \bigcup_i \mathcal{F}_i$, whence $\bigcup_i \mathcal{F}_i$ is an algebra.

(ii) Let $\Omega = \mathbb{R}^{\mathbb{N}}$ and $F_i = \left\{U \times \mathbb{R}^{\mathbb{N}} \;\middle|\; U \in \mathcal{R}^i\right\}$. While $[0,1]^i \times \mathbb{R}^{\mathbb{N}} \in \mathcal{F}_i$, but

$$[0,1]^{\mathbb{N}} = \bigcap_{i=1}^{\infty} [0,1]^i \times \mathbb{R}^{\mathbb{N}} \notin \bigcup_i \mathcal{F}_i$$

□

1.1.5. A set $A \subseteq \{1, 2, \dots\}$ is said to have asymptotic density θ if

$$\lim_{n\to\infty} \frac{|A \cap \{1, 2, ..., n\}|}{n} = \theta$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a σ-algebra? an algebra?

Solution. $\mathcal{A}$ is not an algebra. The counterexample is shown in the following.

Let $A = \{2n\}$ and $B = \{b_n\}, b_n \in \{2n-1, 2n\}$. Then A, B have asymptotic densities $1/2$. And then let

$$S_n = |A \cap B \cap \{1, 2, \dots, 2n\}| = \sum_{k=1}^{n} |A \cap B \cap \{2k-1, 2k\}| = \sum_{k=1}^{n} c_k$$

where

$$c_k = \begin{cases} 0 & b_k = 2k-1 \\ 1 & b_k = 2k \end{cases}$$

It means for any series $\{x_i\}, x_i \in \{0, 1\}$, there exists B with asymptotic density $1/2$ such that $c_i = x_i$. Hence let

$$c_i = \begin{cases} 0 & i = 1 \\ 1 & i = [3^{2k}+1, 3^{2k+1}] \\ 0 & i = [3^{2k+1}+1, 3^{2k+2}] \end{cases} \qquad k \in \mathbb{N}$$

Then for $n = 3^{2k+1}$, $S_n \geq \sum_{3^{2k}+1}^{3^{2k+1}} c_i = 2 \times 3^{2k} = \frac{2}{3}n$ and for $n = 3^{2k+2}$, $S_n = S_{3^{2k+1}} \leq 3^{2k+1} = \frac{1}{3}n$. So $L_n := |A \cap B \cap \{1, 2, \dots, n\}|/n \geq 1/3$ when $n = 2 \times 3^{2k+1}$ and $L_n \leq \frac{1}{6}$ when $n = 2 \times 3^{2k+2}$, whence L_n diverges and then $\mathcal{A}$ is not an algebra.

(The motivation here is for $0 < S_n/n <$1 we could add a lot of 1's to making $S_{n+m}/n+m > 1-\epsilon$ for any epsilon; and a lot of 0's to making $S_{n+m}/n+m < \epsilon$ for any $\epsilon > 0$. Thus the quotinet could fluctuates and then not converge.) □

2 Distributions

1.2.1. Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

Solution. $Z = X \cdot 1_A + Y \cdot 1_{A^c}$ is a random variable. □

1.2.2. Let χ have the standard normal distribution. Use Theorem 1.2.6 to get upper and lower bounds on $P(\chi \leq 4)$.

Solution. Theorem 1.2.6 is listed at Theorem 1.2.2.

$$P(\chi \leq 4) = \int_{-\infty}^{4} e^{x^2/2}\,\mathrm{d}x = 1 - \int_{x}^{+\infty} e^{x^2/2}\,\mathrm{d}x$$

Applying the theorem gives

$$(\frac{1}{4} - \frac{1}{64})e^{-8} \leq \int_{x}^{+\infty} e^{x^2/2}\,\mathrm{d}x \leq \frac{1}{4}e^{-8}$$

So

$$1 - \frac{1}{4}e^{-8} \leq P(\chi \leq 4) \leq 1 - \frac{15e^{-8}}{64}$$

□

1.2.3. Show that a distribution function has at most countably many discontinuities.

Solution. This is a common conlusion in real analysis. Let f be a distribution function, then f is monotonic increasing. Let P be the set of discontinuities. For each $x_0 \in P$, $f_-(x_0)$, $f_+(x_0)$ exist and $f_-(x_0) < f_-(x_0)$. Then letting

$$\mathcal{I} := \{(f_-(a), f_+(a)) \mid a \in P\}$$

and

$$\begin{aligned} \varphi :& P \to \mathcal{I} \\ & x_0 \mapsto (f_-(x_0), f_-(x_0)) \end{aligned}$$

intervals in $\mathcal{I}$ are disjoint and φ is bijective and in since for any $x < y$,

$$f_-(x) < f_+(x) \leq f_-(y) < f_+(y)$$

The desired result follows from $\mathcal{I}$, a collection of disjoint open intervals in $\mathbb{R}$, is at most countable since $\exists q_{a,b} \in \mathbb{Q}$, $q_{a,b} \in (a,b) \in \mathcal{I}$, and then $(a,b) \mapsto q_{a,b}$ is an injection to $\mathbb{Q}$. □

1.2.4. Show that if $F(x) = P(X \leq x)$ is continuous then $Y = F(X)$ has a uniform distribution on $(0,1)$, that is, if $y \in [0,1]$, $P(Y \leq y) = y$.

Proof. We have $0 \geq F(x) \leq 1$, so $P(Y \leq y) = 0$ for $y < 0$ and $P(Y \leq y) = 1$ for $y > 1$. For $y \in [0,1]$,

$$P(Y \leq y) = P(F(X) \leq y) = P(X \in \{h \mid F(h) \leq y\})$$

Let $h_0 = \sup\{h \mid F(h) \leq y\}$. Since F is increasing and continuous, $F(h_0) = y$. Therefore

$$P(Y \leq y) = P(X \leq h_0) = F(h_0) = y$$

1.2.5. Suppose X has continuous density f, $P(\alpha \leq X \leq \beta) = 1$ and g is a function that is strictly increasing and differentiable on (α, β). Then $g(X)$ has density

$$\rho(y) = \begin{cases} \frac{f(g^{-1}(y))}{g'(g^{-1}(y))} & y \in (g(\alpha), g(\beta)) \\ 0 & otherwise \end{cases}$$

When $g(x) = ax + b$ with $a > 0$, $g^{-1}(y) = \frac{y-b}{a}$ so the answer is $\frac{1}{a}f(\frac{y-b}{a})$.

Proof. Since g is strictly increasing, g is invertible and therefore

$$P(g(X) \leq y) = P(X \leq g^{-1}(y)) = \begin{cases} 0 & y \in (-\infty, g(\alpha)] \\ \int_{\alpha}^{g^{-1}(y)} f(x)\,\mathrm{d}x & y \in (g(\alpha), g(\beta)) \\ 1 & y \in [g(\beta), +\infty) \end{cases}$$

We could direct differentiate the right or replace $x = g^{-1}(u)$, and then

$$\int_{\alpha}^{g^{-1}(y)} f(x)\,\mathrm{d}x = \int_{g(\alpha)}^{y} f(g^{-1}(u))\,\mathrm{d}g^{-1}(u) = \int_{g(\alpha)}^{y} \frac{f(g^{-1}(u))}{g'(g^{-1}(u))}\,\mathrm{d}u$$

1.2.6. Suppose X has a normal distribution. Use the previous exercise to compute the density of $\exp(X)$. (The answer is called the lognormal distribution.)

Proof. By Exercise 1.2.5, letting $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, the distribution is

$$g(x) = \begin{cases} \frac{f(\ln x)}{\exp(\ln(x))} = \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2}} & x \in (0, +\infty) \\ 0 & x \in (-\infty, 0] \end{cases}$$

1.2.7.

(i) Suppose X has density function f. Compute the distribution function of X^2 and then differentiate to find its density function.

(ii) Work out the answer when X has a standard normal distribution to find the density of the chi-square distribution.

Proof.

(i) Since

$$P(X^2 \le y) = \begin{cases} 0 & y \in (-\infty, 0] \\ P\left(-\sqrt{y} \le X \le \sqrt{y}\right) & y \in (0, +\infty) \end{cases}$$

and replacing $x = \sqrt{u}$,

$$\begin{aligned} P\left(-\sqrt{y} \le X \le \sqrt{y}\right) &= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,\mathrm{d}x = \int_0^{\sqrt{y}} (f(x) + f(-x))\,\mathrm{d}x \\ &= \int_0^y (f(\sqrt{u}) + f(-\sqrt{u}))\,\mathrm{d}\sqrt{u} = \int_0^y \frac{f(\sqrt{u}) + f(-\sqrt{u})}{2\sqrt{u}}\,\mathrm{d}u \end{aligned}$$

X^2 has a density function,

$$g(y) = \begin{cases} 0 & y \in (-\infty, 0] \\ \frac{f(\sqrt{y}) + f(-\sqrt{y})}{2\sqrt{y}} & y \in (0, +\infty) \end{cases}$$

(*We can not directly differentiate the definite intergrals with variable limits, since it may be indifferentiable when f is discontinuous*)

(ii) Applying above conclusion, the density is

$$g(y) = \begin{cases} 0 & y \in (-\infty, 0] \\ \frac{e^{-y/2}}{\sqrt{2\pi y}} & y \in (0, +\infty) \end{cases}$$

3 Random Variables

1.3.1. Show that if $\mathcal{A}$ generates $\mathcal{S}$, then $X^{-1}(\mathcal{A}) \equiv \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.

Proof. It is equivalent to prove $\sigma(X^{-1}(\mathcal{A})) = \sigma(X)$. Since $X^{-1}(\mathcal{A}) \subseteq \sigma(X)$ is obvious and hence $\sigma(X^{-1}(\mathcal{A})) \subseteq \sigma(X)$, it suffices to prove $\sigma(X^{-1}(\mathcal{A})) \subseteq \sigma(X)$. Considering

$$\mathcal{E} := \left\{B \in \mathcal{S} \;\middle|\; X^{-1}(B) \in \sigma(X^{-1}(\mathcal{A}))\right\}$$

it is not diffcult to see $\mathcal{E}$ is σ-algebra containing $\mathcal{A}$, whence $\mathcal{S} \subseteq \mathcal{E}$ and then $\sigma(X^{-1}(\mathcal{A})) \subseteq \sigma(X)$.

A more common statement is

$$\sigma(X^{-1}(\mathcal{A})) = X^{-1}(\sigma(\mathcal{A}))$$

1.3.2. Prove Theorem 1.3.6 when $n = 2$ by checking $\{X_1 + X_2 < x\} \in \mathcal{F}$.

Proof. Theorem 1.3.6 is listed at Theorem 1.3.1. We observe that

$$\{X_1 + X_2 < x\} = \bigcup_{q \in \mathbb{Q}} \{X_1 < q\} \cap \{X_2 < x - q\}$$

, since the left set contains the right set is obvious and for each $(x_1, x_2) \in \{X_1 + X_2 < x\}$, there is $x_1 < x - x_2$ and $q_0 \in \mathbb{Q}$, $x_1 < q_0 < x - x_2$, whence $(x_1, x_2) \in \{X_1 < q_0\} \cap \{X_2 < x - q_0\}$ and the left set is contained in the right set.

For the complete proof of $n = k$, we could use the induction on n and the fact that $X_1 + \cdots + X_k = (X_1 + \cdots + X_{k-1}) + X_k$.

1.3.3. Show that if f is continuous and $X_n \to X$ almost surely then $f(X_n) \to f(X)$ almost surely.

Proof. For $\omega \in \{X_n \to X\}$, $f((X_n)(\omega)) \to f(X(\omega))$. Therefore, $\{X_n \to X\} \subseteq \{f(X_n) \to f(X)\}$ and hence

$$P\left(\{f(X_n) \to f(X)\}\right) \geq P\left(\{X_n \to X\}\right) = 1$$

1.3.4.

(i) Show that a continuous function from $\mathbb{R}^d \to \mathbb{R}$ is a measurable map from $(\mathbb{R}^d, \mathcal{R}^d)$ to $(\mathbb{R}, \mathcal{R})$.

(ii) Show that $\mathcal{R}^d$ is the smallest σ-field that makes all the continuous functions measurable.

Proof. Let f be a continuous function from $\mathbb{R}^d \to \mathbb{R}$.

(i) It suffices to prove for all open set $V \subseteq \mathbb{R}$, $f^{-1}(V) \in \mathcal{R}^d$. Actually $f^{-1}(V)$ is an open set in $\mathbb{R}^n$ and hence in $\mathcal{R}^d$, which is a directly result from general topology and can be proven by the following.

For each $x \in f^{-1}(V)$,there is $\epsilon_x > 0$ such that the open ball $U(f(x), \epsilon_x) \subseteq V$ and hence there is $\delta_x > 0$,

$$f(U(x, \delta_x)) \subseteq U(f(x), \epsilon_x) \subseteq V$$

So

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U(x, \delta_x)$$

is an open set contiained in $\mathcal{R}^n$.

(ii) Let $\mathcal{F}$ be the smallest σ-field that makes all the continuous functions measurable. For the continuous function,

$$f_{x_0, d_0}(x) = d_0 - d(x_0, x)$$

there is

$$\{f_{x_0, d_0} > 0\} = \{d(x_0, x) > d_0\} = U(x_0, d_0) \in \mathcal{F}$$

So $\mathcal{F}$ contains all open balls and hence contains all open set, which gives $\mathcal{R}^n \subseteq \mathcal{F}$. The desired result follows.

A Some related theorem details

2 Distributions

Theorem 1.2.1. *For $x > 0$, leting*

$$I(x) := \int_x^{+\infty} e^{-y^2/2}$$

then for $n \in \mathbb{N}$,

$$I(x) = I_n(x) + R_n(x)$$

where

$$\begin{aligned} I_n(x) :&= e^{-x^2/2}\left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{x^{2n-1}}\right) \\ &= e^{-x^2/2}\sum_{k=1}^{n}(-1)^{k-1}\frac{(2k-3)!!}{x^{2k-1}} \end{aligned}$$

and

$$R_n(x) := (-1)^n(2n-1)!! \int_x^{+\infty} \frac{e^{-y^2/2}}{y^{2n}}\,\mathrm{d}y$$

by setting $(-1)!! = 1$. Then $I_{2k}(x) < I(x) < I_{2k+1}(x)$ for any $k \in \mathbb{N}$.

Proof. Use induction on n. It's trivial for $n = 0$ and assume it holds for $n = k$. Since

$$\begin{aligned} \frac{R_k(x)}{(-1)^k(2k-1)!!} &= \int_x^{\infty} \frac{e^{-y^2/2}}{y^{2k}}\,\mathrm{d}y = \int_x^{+\infty} -\frac{1}{y^{2k+1}}\,\mathrm{d}e^{-y^2/2} \\ &= \left.\frac{-e^{-y^2/2}}{y^{2k+1}}\right|_x^{+\infty} - \int_x^{+\infty} \frac{(2k+1)e^{-y^2/2}}{y^{2k+2}}\,\mathrm{d}y \\ &= \frac{-e^{-x^2/2}}{x^{2k+1}} - (2k+1)\int_x^{+\infty} \frac{e^{-y^2/2}}{y^{2k+2}}\,\mathrm{d}y \\ &= \frac{-e^{-x^2/2}}{x^{2k+1}} + \frac{R_{k+1}(x)}{(-1)^k(2k-1)!!} \end{aligned}$$

then

$$\begin{aligned} I &= I_k + R_k = I_k + (-1)^k(2k-1)!!\frac{-e^{-x^2/2}}{x^{2k+1}} + R_{k+1}(x) \\ &= I_{k+1}(x) + R_{k+1}(x) \end{aligned}$$

The last conclusion is obvious since $R_{2k} > 0$ and $R_{2k+1} < 0$ for any $k \in \mathbb{N}$.

Proof.

Theorem 1.2.2 (Theorem 1.2.6 in PTE5th)**.** *For $x > 0$,*

$$(x^{-1} - x^{-3})e^{-x^2/2} \le \int_x^{+\infty} e^{-y^2/2}\,\mathrm{d}y \le x^{-1}e^{-x^2/2}$$

Proof. We give a stronger conclusion and a more general proof in Theorem 1.2.1. The desired result directly follows from the cases of $n = 1$ and $n = 2$.

3 Random Variables

Theorem 1.3.1 (Theorem 1.3.6 in PTE5th)**.** *If $X_1, \ldots, X_n$ are random variables then $X_1 + \cdots + X_n$ is a random variable.*