Vector Subspaces of $\mathbb{R}^n$, Linear dependence

Linear Span

Vectors in $\mathbb{R}^n$ are always Column vectors ($n\times 1$) matrices.

A subset V <u>of $\mathbb{R}^n$</u> is called a <u>Vector</u> <u>Subspace</u>

if (i) V is non empty.

(ii) $x \in V$ and $a \in \mathbb{R} \Rightarrow ax \in V$

(iii) $x, y \in V \Rightarrow x+y \in V$

<u>Example</u>: Let $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,/\, x+y-3z=0 \right\}$.

Then V is a vector subspace.

Evidently zero vector $\in V$ and so $V \neq \phi$.

It is readily checked that if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V$ then $\begin{bmatrix} ax \\ ay \\ az \end{bmatrix} \in V$ for $a \in \mathbb{R}$.

Check: if $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in V$

$$\left.\begin{array}{l} x_1+y_1-3z_1=0 \\ x_2+y_2-3z_2=0 \end{array}\right\} \Rightarrow (x_1+x_2)+(y_1+y_2)-3(z_1+z_2)=0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in V$$

Exercise: Explain why $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,/\, x+y-3z=1 \right\}$

is **NOT** a vector subsp of $\mathbb{R}^3$

Suppose V is a vect. subsp. of $\mathbb{R}^n$

Since $V \neq \phi$ select $x \in V$ and select $a = 0 \in \mathbb{R}$

then $ax \in V$ i.e $0 \in V$

Ex: Show that (i) a line passing through origin in $\mathbb{R}^3$ is a vector subspace in $\mathbb{R}^3$

(ii) A plane passing through origin is a vect. Sub sp.

(iii) $\mathbb{R}^3$ itself is a vect. subsp of $\mathbb{R}^3$

(iv) The singleton $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a vector subsp of $\mathbb{R}^3$.

Que: Are there any others? Justification ??

Linear Combinations: Suppose $v_1, v_2, \ldots v_k \in \mathbb{R}^n$

and $a_1, \ldots, a_k \in \mathbb{R}$ then $a_1 v_1 + a_2 v_2 + \ldots + a_k v_k$

is called ~~the~~ a lin. comb. of $v_1, \ldots, v_k$ with

coeff. $a_1, \ldots, a_k$

Let us now consider the set of <u>all</u> possible Linear Combinations of $v_1, \dots, v_k$ with arb. real coeff.

$$V = \{ a_1 v_1 + \dots + a_k v_k \;/\; a_1, \dots, a_k \in \mathbb{R} \} \quad (*)$$

<u>Check:</u> V is a vector subspace of $\mathbb{R}^n$

This is called <u>the linear span</u> of $\{v_1, \dots, v_k\}$.

<u>Que:</u> Suppose V is a vector subsp. of $\mathbb{R}^n$

Can you find a <u>finite</u> subset S of V such that V is exactly the linear span of S. We shall answer this question <u>soon</u>.

Going back to $(*)$ we say V is <u>generated</u> <u>by</u> $\{v_1, \dots, v_k\}$

Example: $x + y - 3z = 0$ or

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;/\; x + y - 3z = 0 \right\}.$$

Certainly $v_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ lie in V.

Prove that Linear span of $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ is Exactly V

Since we are in $\mathbb{R}^3$ you could begin with a geometrical reasoning (Parallelogram law of vect. add. with $a_1 v_1$ and $a_2 v_2$).

Can you back it up with algebraic justification?

So $V =$ lin. Span $\{v_1, v_2\}$.

Is it possible to find a single vector w_0 s.t

$V =$ lin. Span $\{w_0\}$. ? Why ??

What about finding a set S with 3 vectors.

s.t Lin. Span $S = V$? If you were asked to select a spanning set which one would you choose? Optimality ?

You may now begin to Suspect that for this Example a Spanning set

(i) Cannot be a Singleton.

(ii) Can have more than 2 elts but that would cause redundancy (more than what you need)

So the Optimal two element Spanning set must have to do with the fact that a plane is two Dimensional.

Dimension is the most fundamental idea in Mathematics and we are converging towards a rigorous and precise notion of dimension.

Before defining dimension we need one more basic notion:

Linear dependence / lin. indep. of vectors $v_1, \dots, v_k \in \mathbb{R}^n$

Recall the notion of Collinearity in $\mathbb{R}^3$.

Vectors $v_1, v_2 \in \mathbb{R}^3$ are Collinear iff one of them is a multiple of the other say $v_2 = \lambda v_1$

i.e $\lambda v_1 + (-1) v_2 = 0$.

Ex: Show that vectors $v_1, v_2 \in \mathbb{R}^3$ are collinear iff $\exists$ scalars a_1, a_2 $\boxed{\text{not both zero}}$ Symm formulation.

s.t $a_1 v_1 + a_2 v_2 = 0$

Vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ are said to be Coplanar if one of them lies in the plane of the other two say $v_3 = \lambda v_1 + \mu v_2$.

Ex: Prove: v_1, v_2, v_3 Coplanar iff $\exists$ Scalars a_1, a_2, a_3 not all zero such that $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$.

Def: Vectors $v_1, \ldots, v_K \in \mathbb{R}^n$ are said to be linearly dependent if $\exists$ scalars $a_1, \ldots, a_K$ $\boxed{\text{not all zero}}$ s.t. $a_1 v_1 + \ldots + a_K v_K = 0$.

If $v_1, \ldots, v_K$ are not linearly dependent we say they are linearly independent

That is $v_1, \ldots, v_K$ linearly independent if for scalars $a_1, \ldots, a_K$

$$a_1 v_1 + \ldots + a_K v_K = 0 \Longrightarrow a_1 = a_2 = \ldots = a_K = 0.$$

Convention: Empty set is linearly independent

If a set $\{v_1, \ldots, v_K\}$ contains zero vector it is linearly dependent.

Ex: Subset of a Lin Indep. Set is again lin. indep.

First Fundamental Lemma in Linear Algebra.

Suppose V is a vector subsp. of $\mathbb{R}^n$
and V is generated by k-vectors then
any set of $(k+1)$ or more vectors in V are
<u>linearly</u> <u>dependent</u>

<u>Proof:</u> By induction on k. <u>Case $k=1$:</u>
Let $V = \text{span}\{v_0\}$.
Let $v_1, v_2 \in V$ so $v_1 = a v_0$
$v_2 = b v_0$.
If $a=0$ or $b=0$ then $\{v_1, v_2\}$ is linearly dep.
If $a \neq 0$, $b \neq 0$ then
$b v_1 - a v_2 = 0$; Coeff. $b, -a$ <u>not</u> both zero.
$\therefore \{v_1, v_2\}$ lin. dep.
Assume the result for k - ~~That is wrong~~
We prove it for $k+1$

Well, let $\text{Span}\{v_1, \ldots, v_{k+1}\} = V$

Take $(k+2)$ vectors $w_1, w_2, \ldots, w_{k+1}, w_{k+2} \in V$

$$\therefore \quad w_1 = a_{11} v_1 + \cdots + a_{1k} v_k + a_{1\,k+1} v_{k+1}$$

$$w_2 = a_{21} v_1 + \cdots + a_{2k} v_k + a_{2,k+1} v_{k+1}$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$w_{k+1} = a_{k+1,1} v_1 + \cdots + a_{k+1,k} v_k + a_{k+1,k+1} v_{k+1}$$

$$w_{k+2} = a_{k+2,1} v_1 + \cdots + a_{k+2,k} v_k + a_{k+2,k+1} v_{k+1}$$

Now if $a_{1,k+1} = \cdots = a_{k+2,k+1} = 0$

then $w_1, w_2, \ldots, w_{k+1}, w_{k+2}$ <u>all</u> lie in the linear $\text{Span}\{v_1, \ldots, v_k\}$ = vect. Sp. gen. by k-vectors

By induction hypo. $w_1, \ldots, w_{k+1}, w_{k+2}$ <u>linearly</u> <u>dep</u>.

Next assume (one of) $a_{1,k+1}, \ldots, a_{k+2,k+1}$ is $\neq 0$.

By renaming $a_{k+2,k+1} \neq 0$.

Look at the $(k+1)$ vectors

$$w_1 - \frac{a_{1,k+1}}{a_{k+2,k+1}} w_{k+2}, \quad \cdots \quad w_{k+1} - \frac{a_{k+1,k+1}}{a_{k+2,k+1}} w_{k+2}$$

These lie in $\operatorname{span}\{v_1, \ldots, v_k\}$ (how?)

So by induction hypo. they are Lin. Dep.

$$\therefore \quad c_1 \left(w_1 - \frac{a_{1,k+1}}{a_{k+2,k+1}} w_{k+2} \right) + \cdots + c_{k+1} \left(w_{k+1} - \frac{a_{k+1,k+1}}{a_{k+2,k+1}} w_{k+2} \right) = 0$$

at least one of $c_1, \ldots, c_{k+1} \neq 0$.

$$\therefore \quad c_1 w_1 + \cdots + c_{k+1} w_{k+1} + \gamma w_{k+2} = 0$$

at least one of $c_1, \ldots, c_{k+1}, \gamma \neq 0$

$\therefore$ $w_1, \ldots, w_{k+2}$ Lin. Dep.

proof is complete.

Basis and Dimension

Two Conventions: Empty set ϕ is linearly independent.
Lin. Span of ϕ is $\{0\}$.

Def: Suppose V is a vector subspace of $\mathbb{R}^n$ then a subset S of V is called a Basis of V

if (i) S is linearly indep.
(ii) Linear Span of $S = V$

Ex: If $V = \{0\}$ set consisting of only the zero vector then ϕ is (the) basis of V

Ex: Let $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 / \; x+y-3z=0 \right\}$.

Check that $S = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis

We shall see that Every Vector subsp of $\mathbb{R}^n$ has a basis.

<u>Thm</u>: If V is a vector subsp. of $\mathbb{R}^n$ then

(i) V has a basis

(ii) Any two bases <u>have</u> the same # of Elements.

<u>proof</u>: Clear if V is the zero vector space.

So assume V contains non zero vectors. Pick $v_1 \in V$, $v_1 \neq 0$.

$V_1 = \{ a v_1 / a \in \mathbb{R} \}$ = Linear Span of $\{v_1\}$

If $V_1 = V$ we are through since $\{v_1\}$ spans V and is lin. indep. set.

Suppose $V_1 \subsetneqq V$. Pick $v_2 \in V - V_1$

<u>Claim</u>: $\{v_1, v_2\}$ is linearly indep.

Well, $a v_1 + b v_2 = 0 \Rightarrow v_2 = -\frac{a}{b} v_1 \in V_1$ (Contradiction) in case $b \neq 0$.

If $b = 0$ then we get $a v_1 = 0 \Rightarrow a = 0$ (Since $v_1 \neq 0$)

$\therefore a v_1 + b v_2 = 0 \Rightarrow a = b = 0$.

If $V_2 = \text{Lin. Span}\{v_1, v_2\} = V$ we are through

Since $\{v_1, v_2\}$ serves as a basis for V

Else, $V_2 \subsetneq V$ so Select $v_3 \in V - V_2$

<u>Claim:</u> $\{v_1, v_2, v_3\}$ lin. indep.

Well, Suppose $av_1 + bv_2 + cv_3 = 0$

Then $c = 0 \Rightarrow av_1 + bv_2 = 0 \Rightarrow a = b = 0$ ($\because \{v_1, v_2\}$ lin indep)

If $c \neq 0$ then $v_3 = -\frac{a}{c} v_1 - \frac{b}{c} v_2 \in V_2$ (Contradiction)

Proceeding thus we keep Constructing vectors $v_1, v_2, \ldots, v_k$

s.t $\{v_1, \ldots, v_k\}$ linearly indep.

$V_k = \text{lin. Span}\{v_1, \ldots, v_k\} = V$ in which case $\{v_1, \ldots, v_k\}$ is a basis. Process stops.

Or else the process goes to the next stage with $v_{k+1} \in V - V_k$ etc., Claim: The process must stop in at most n - steps!

Suppose not then we can construct linearly indep vectors $v_1, \ldots, v_{n+1}$ in $\mathbb{R}^n$ which contradicts the Fundamental Lemma ($\mathbb{R}^n$ is generated by n-vectors $\hat{e}_1, \ldots, \hat{e}_n$ so any set of $(n+1)$ vectors is linearly dependent). Claim is proved.

Thereby (i) is established.

To prove (ii) let S_1, S_2 be two bases $|S_1| = k$, $|S_2| = l$.

Linear Span $S_1 = V$ and $S_2 \subset V$

So V is generated by k vectors ; S_2 is lin. indep set of l vectors $\Rightarrow l \leq k$. Similarly $k \leq l$

$\therefore k = l$.

Definition: Let V be a vector subsp of $\mathbb{R}^n$ then $\operatorname{Dim} V$ is the Cardinality of any basis of V. (The Concept is well-defined)

Exercise: (i) Show that vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ are linearly indep iff $\det(v_1, v_2, v_3) \neq 0$.

(ii) Show that vectors $v_1, v_2 \in \mathbb{R}^3$ are linearly indep. iff $\det\begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_1 \cdot v_2 & v_2 \cdot v_2 \end{pmatrix} > 0$.

Fact: Vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly indep iff. $\det(v_i \cdot v_j)_{1 \leq i,j \leq k} \neq 0$. $\Big\}$ Proof: Needs More Theory

(this det is called the Gram-det.

Interpretation: Vol of the k-parallelpiped in $\mathbb{R}^n$ Spanned by $v_1, \ldots, v_k$).

Rank of a Matrix:

Let A be an $m \times n$ matrix.

Columns of A can be regarded as elements of $\mathbb{R}^m$ and their linear span is called the Column Space of A

Rows of A can be regarded as elements of $\mathbb{R}^n$ and their linear span is called the Row Space of A

Def: Row Rank A = Dim. (Row Sp. of A)

Column Rank A = Dim (Col. Space of A)

Exercise: Suppose S is a set of vectors and

Linear Span $S = V$

Show that (by deleting elements of S if necessary) S contains a subset S_0 which is a basis of V

In particular, we can select a subset of Rows of A to obtain a basis for Row Space of A. Ily for Columns.

Lemma: Let A be a $m \times n$ matrix

performing an elementary row operation does not alter the Column Rank of A.

Specifically if E is an elementary matrix or more generally any non sing matrix and $C_{i_1}, C_{i_2}, \dots, C_{i_\ell}$ are linearly indep cols of A then $EC_{i_1}, EC_{i_2}, \dots, EC_{i_\ell}$ ~~are~~ (the i_1th, i_2th, …, i_ℓth) Columns of EA are again linearly indep.

proof: i_1th col. of EA

$$= (EA)\hat{e}_{i_1} = E(A\hat{e}_{i_1}) = EC_{i_1}$$

Now Suppose $a_{i_1}(EC_{i_1}) + \dots + a_{i_\ell}(EC_{i_\ell}) = 0.$

$$\Rightarrow E(a_{i_1}C_{i_1} + \dots + a_{i_\ell}C_{i_\ell}) = 0.$$

$$\Rightarrow a_{i_1} c_{i_1} + \cdots + a_{i_\ell} c_{i_\ell} = 0 \quad (\because E \text{ is invertible})$$

$$\Rightarrow a_{i_1} = \cdots = a_{i_\ell} = 0 \quad (\because c_{i_1}, \ldots, c_{i_\ell} \text{ lin. Indep})$$

Thus the Columns of EA marked $i_1, \ldots, i_\ell$ are linearly independent.

Ex: Suppose $\{v_1, v_2, \ldots, v_k\}$ lin. indep vectors in $\mathbb{R}^n$

Show that $\{v_1, \ldots, v_{j-1}, \lambda v_j, v_{j+1}, \ldots, v_k\}$ $(\lambda \neq 0)$ is also lin. indep.

Show that $\{v_1, \ldots, v_{j-1}, v_j + c v_i, v_{j+1}, \ldots, v_k\}$ $(j \neq i)$ is also lin. indep.

Show that ~~if~~ ~~Rows~~ the Row Rank of A is not altered by performing Row Op. on A.

So, neither row rank <u>nor</u> Column Rank is affected by performing Row operations on A.

<u>Thm:</u> Row Rank A = Column Rank A.

<u>proof:</u> Let k = Row Rank A; l = Column Rank A.

By Row exchanges we may assume <u>first</u> k rows of A are linearly indep and last $m-k$ rows are linear Combinations of first k-rows $R_1, \ldots, R_k$.

Say $R_{k+1} = a_1 R_1 + \ldots + a_k R_k$.

Perform the row operations: $R_k' = R_k - a_1 R_1$
$R_k' = R_k - a_2 R_2, \ldots$
in Succession.

This Causes $(k+1)$st row to be zero.

$||^{ly}$ we can make $(k+2)$nd, ..., mth row of A as zero.

Resulting Matrix $\tilde{A}$ looks like

$$\tilde{A} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \cdots & b_{kn} \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

last $(m-k)$ rows are zero.

Row Rank $\tilde{A}$ = Row Rank $A = k$.

Col Rank $\tilde{A}$ = Col. Rank $A = l$.

Now look at the Columns of $\tilde{A}$ they may be regarded as Columns in $\mathbb{R}^k$ (gen. by k-vectors)

So $(k+1)$ or more Columns of $\tilde{A}$ are lin Dep.

$\therefore$ Column Rank $\tilde{A} \leq k$

$\therefore$ $l \leq k$.

Repeat the same with A^T and we get $k \leq l$

$\therefore$ $k = l$. Proof is Complete.

Exercise: Let $\tilde{A}$ be obtained from A via an Elementary row Op. (both Square Matrices)

(Tutorial)

$$\det A = \lambda \cdot (\det \tilde{A}) \; ; \; \lambda \neq 0.$$

Next, if $\tilde{A}$ is obtained from A through a Sequence of Elem. Row. Op.

$$\det A = \lambda (\det \tilde{A}) \; ; \; \lambda \neq 0.$$

Deduce that if A can be reduced to I_n through a Seq of Elem. Row Op. then

$$\det A \neq 0.$$

(Tutorial)

So if Columns of A linearly Indep then (A is a Sq. Matrix).

$$\det A \neq 0$$

Que Hint: If cols of A lin. Indep REF $= I_n$.

Exercises: Suppose $\text{Dim } V = k$ (Tutorial Problem)

(i) S is a linearly indep subset of V containing k vectors then S is a basis (*)

(ii) S is a set of k vectors such that (**)

Linear Span $S = V$

then S is a basis.

In particular if $C_1, \ldots, C_n$ are lin indep columns of a $n \times n$ matrix A

$$\text{Lin. Span}\{C_1, \ldots, C_n\} = \mathbb{R}^n$$

$$\therefore \quad \hat{e}_1 = b_{11} C_1 + b_{21} C_2 + \cdots + b_{n1} C_n$$

$$\hat{e}_2 = b_{12} C_1 + b_{22} C_2 + \cdots + b_{n2} C_n$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\hat{e}_n = b_{1n} C_1 + b_{2n} C_2 + \cdots + b_{nn} C_n$$

Deduce that $\text{Rank } A = n \Rightarrow A$ invertible
(nxn matrix)

Tutorial.

Prove the converse using (*) and/or (**)

Thm: Let A be a Square Matrix
The Following Are Equivalent
(i) Rows of A lin. Indep.
(ii) Cols of A lin Indep.
(iii) A is invertible
(iv) $\operatorname{Det} A \neq 0$.

Pf: (i) and (ii) are Equivalent since Row Rank = Col. Rank (= n in this case).

(ii) $\Leftrightarrow$ (iii) Exercise above

(ii) $\Rightarrow$ (iv) Exercise above

To Prove: $\neg$(ii) $\Rightarrow$ $\neg$(iv) : Cols of A lin. dep.
$\therefore$ Rows of A lin dep.
$\therefore$ $A \longrightarrow \begin{bmatrix} B \\ 0\,0 \cdots 0 \end{bmatrix}$ through a seq. of Row op.
B is $(n-1)\times n$

$$\det A = \lambda \det \begin{bmatrix} B \\ 0 \end{bmatrix} = 0 \quad (\lambda \neq 0).$$

$\therefore$ $\neg$(ii) $\Rightarrow$ $\neg$(iv) proof is Complete

Kronecker - Capelli Thm:

A System of lin. equations $Ax = b$

has a Solution <u>iff</u> $\text{Rank } A = \text{Rank } [A : b]$.

$\left(\begin{bmatrix} c_1 \\ \cdots \\ c_n \end{bmatrix} = c_1 \hat{e}_1 + \ldots + c_n \hat{e}_n \right)$

<u>proof:</u> Assume $\exists$ a Sol. $\begin{bmatrix} c_1 \\ \cdots \\ c_n \end{bmatrix}$

$\therefore A(c_1 \hat{e}_1 + \ldots + c_n \hat{e}_n) = b$

$\therefore c_1 (\text{Col } 1) + c_2 (\text{Col } 2) + \ldots + c_n (\text{Col } n) = b$

Recall $A\hat{e}_j = j\text{th Col } A$

$\therefore b \in$ Linear span Cols.

$\therefore b \in$ Column. Sp. of A or Col. Space A = Col. Space $[A : b]$

$\therefore \text{Rank } A = \text{Rank } [A : b]$

Conversely if $\text{Rank } A = \text{Rank } [A : b]$

$\Rightarrow b \in$ Col. Space of A

$\Rightarrow b = c_1 (\text{Col } 1) + \ldots + c_n (\text{Col } n)$ for some scalars $c_1, \ldots, c_n$

$\Rightarrow b = c_1 A\hat{e}_1 + \ldots + c_n A\hat{e}_n$

$\Rightarrow b = A(c_1 \hat{e}_1 + \ldots + c_n \hat{e}_n) \Rightarrow b = A \begin{bmatrix} c_1 \\ \cdots \\ c_n \end{bmatrix}$

and $\begin{bmatrix} c_1 \\ \cdots \\ c_n \end{bmatrix}$ is a Sol. of the System.