ECE 311 - Assignment 1 - Models of Control Systems

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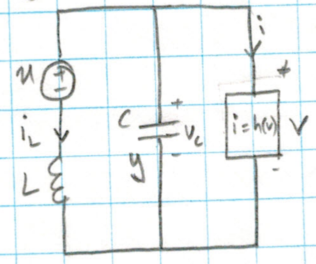
1. i)



$h(v) = -v + \frac{1}{3}v^3 = i$

KVL: $v = V_c = u + L\frac{di_L}{dt}$, $-V_c + u + L\frac{di_L}{dt} = 0$, $V_c = v$

KCL: $i_L + C\frac{dV_c}{dt} + i = 0$, $i = h(v)$

$$i_L + C\frac{dV_c}{dt} + \left(-v + \frac{1}{3}v^3\right) = 0$$

$$i_L + C\frac{dV_c}{dt} - V_c + \frac{1}{3}V_c^3 = 0$$

$$\therefore \begin{cases} -V_c + u + L\frac{di_L}{dt} = 0 \\ i_L + C\frac{dV_c}{dt} - V_c + \frac{1}{3}V_c^3 = 0 \end{cases}$$

ii) $x_1 = i_L$

$x_2 = V_c$, $y = x_2$ $\qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$-x_2 + u + L x_1' = 0$$

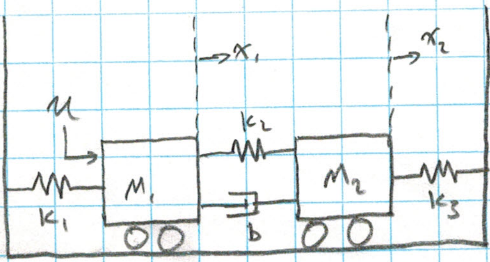
$$x_1' = \frac{1}{L}x_2 + \frac{1}{L}u$$

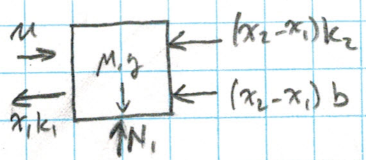
$$x_1 + C x_2' - x_2 + \frac{1}{3}x_2^3 = 0$$

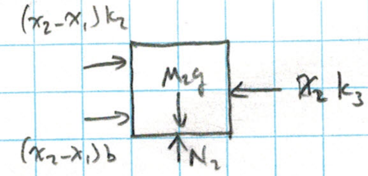
$$x_2' = \frac{1}{C}x_2 - \frac{1}{3C}x_2^3 - \frac{1}{C}x_1$$

$$\therefore \begin{cases} x' = \begin{bmatrix} \frac{1}{L}x_2 + \frac{1}{L}u \\ -\frac{1}{C}x_1 + \frac{1}{C}x_2 - \frac{1}{3C}x_2^3 \end{bmatrix} \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{cases}$$

2.



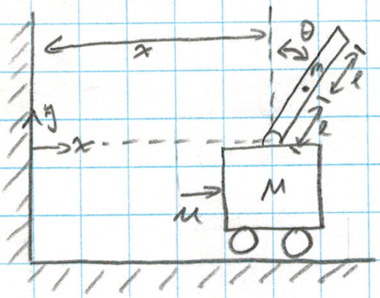


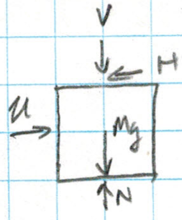


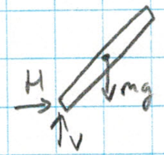
$$u - x_1 k_1 - (x_2 - x_1) k_2 - (x_2 - x_1) b = M_1 x_1''$$

$$(x_2 - x_1) k_2 + (x_2 - x_1) b - x_2 k_3 = M_2 x_2''$$

3.





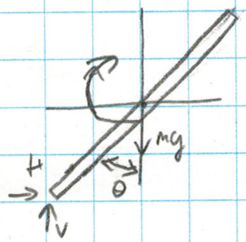


TRANSLATION OF CART (x-DIR) :

$F = ma, \quad u - H = M x''$ ①

ROTATION OF PENDULUM :

$\tau = I \theta''$



$V l \sin\theta - H l \cos\theta = I\theta''$ ②

TRANSLATION OF PENDULUM :

x :

$H = m a_{p,x} \; ; \; x_p = x + l \sin\theta$

$= m \frac{d^2}{dt^2}(x + l\sin\theta)$

$H = m \frac{d^2}{dt^2}(l \sin\theta) + m x'' = m x'' + m l \cos\theta\, \theta'' - m l \sin\theta\, \theta'^2$ ③

y :

$V - mg = m a_{p,y} \; ; \; y_p = l \cos\theta$

$V = mg + m \frac{d^2}{dt^2}(l\cos\theta)$

$V = mg - ml\cos\theta\,\theta'^2 - ml\sin\theta\,\theta''$ ④

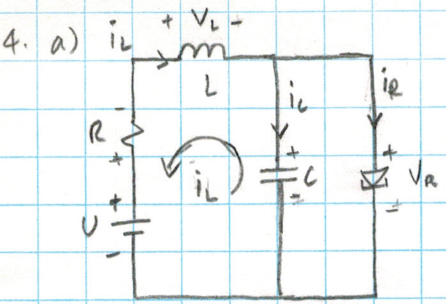
③ → ① : $u - m x'' - m l \cos\theta\,\theta'' + m l \sin\theta\,\theta'^2 = M x''$

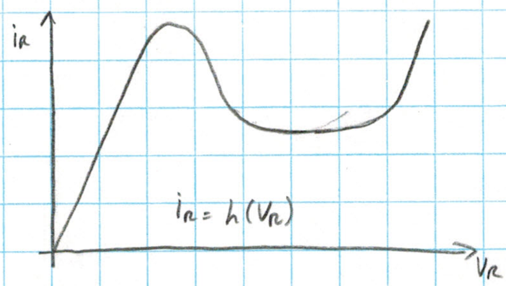
③,④ → ② : $(-ml\cos\theta\,\theta'^2 - ml\sin\theta\,\theta'' + mg)\, l \sin\theta - (m x'' + m l\cos\theta\,\theta'' + m l \sin\theta\,\theta'')\, l\cos\theta = I\theta''$

$-ml^2\theta'' - ml\cos\theta\, x'' + mgl\sin\theta = I\theta''$

$\therefore$ EQUATIONS OF MOTION
$$\begin{cases} u - m x'' - ml\cos\theta\,\theta'' + ml\sin\theta\,\theta'^2 = M x'' \\ -ml^2\theta'' - ml\cos\theta\, x'' + mgl\sin\theta = I\theta'' \end{cases}$$

4. a)





KVL: $U = V_c + V_L + i_L R$

$U = V_c + L\frac{di_L}{dt} + i_L R$

KCL: $i_L = i_R + i_c$

$i_L = h(V_c) + C\frac{dV_c}{dt}$

$V_R = V_c$

$$\therefore \begin{cases} U = V_c + L\frac{di_L}{dt} + i_L R \\ i_L = h(V_R) + C\frac{dV_c}{dt} \end{cases}$$

b) LET $\begin{cases} x_1 = V_c \\ x_2 = i_L \end{cases}$, $y = x_1$.

$$x_1' = \frac{1}{C}x_2 - \frac{1}{C}h(V_R)$$

$$x_2' = \frac{1}{L}U - \frac{1}{L}x_1 - \frac{R}{L}x_2$$

$$\therefore \begin{cases} x' = \begin{bmatrix} -\frac{1}{C}h(V_R) + \frac{1}{C}x_2 \\ -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}U \end{bmatrix} \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

c) U IS CONSTANT, $u^* = u$

$f(x^*, u^*) = 0$

$0 = -\frac{1}{C}h(V_R) + \frac{1}{C}x_2$ $\qquad x_2 = h(V_R)$

$0 = -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u$ $\qquad x_1 = u - Rx_2$

$$\therefore (x^*, u^*) = \left(\begin{bmatrix} u - Rh(V_R) \\ h(V_R) \end{bmatrix}, u \right)$$

5. i)
$$\begin{aligned} x_1' &= -x_1 + u \\ x_2' &= -2x_2 + x_3 \\ x_3' &= e^{x_1} x_2 + u \\ y &= x_1 + x_2 \end{aligned}$$

SET $u = 1$

$$f(x,u) = \begin{bmatrix} -x_1 + u \\ -2x_2 + x_3 \\ e^{x_1} x_2 + u \end{bmatrix} = 0 \quad \begin{aligned} &\Rightarrow x_1^* = u^* = 1 \\ &\Rightarrow x_2^* = -u^* e^{-x_1^*} = -1/e \end{aligned} \quad \Rightarrow x_3^* = 2x_2^* = -2/e$$

$$\therefore (x^*, u^*) = \left(\begin{bmatrix} 1 \\ -1/e \\ -2/e \end{bmatrix}, 1 \right)$$

ii)
$$\frac{\partial f}{\partial x} = \begin{array}{c|c} & \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \hline \begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ x_2 e^{x_1} & e^{x_1} & 0 \end{bmatrix} \end{array} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x^*, u^*)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ -\frac{1}{e} e^1 & e^1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ -1 & e & 0 \end{bmatrix}$$

$$\frac{\partial h}{\partial x} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \qquad \frac{\partial h}{\partial u} = 0$$

$$\delta x = \begin{bmatrix} x_1 - 1 \\ x_2 + 1/e \\ x_3 + 1/e \end{bmatrix}, \quad \delta u = u - 1, \quad \delta y = y - y^* = y - x_1^* - x_2^* = y - 1 + 1/e$$

$$\therefore \begin{cases} (\delta x)' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ -1 & e & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \delta u \\ \delta y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \delta x \end{cases}$$

1. i) $f(t) = 3t^2 e^{-t} \cdot \mathbb{1}(t)$

$$F(s) = 3 \frac{2!}{(s-1)^3}$$

$$\therefore F(s) = \frac{6}{(s-1)^3}$$

ii) $f(t) = \sin t \cos t \cdot \mathbb{1}(t)$

$$= \left(\frac{e^{it} - e^{-it}}{2i}\right)\left(\frac{e^{it} + e^{-it}}{2}\right)$$

$$= \frac{1}{4i}\left(e^{2it} - e^{-2it}\right)$$

$$F(s) = \frac{1}{4i}\left(\frac{1}{s-2i} - \frac{1}{s+2i}\right)$$

$$= \frac{1}{4i}\left(\frac{s+2i - s-2i}{s^2+4}\right)$$

$$\therefore F(s) = \frac{1}{s^2+4}$$

iii) $f(t) = \sin(t-3) \cdot \mathbb{1}(t)$

$$= \sin(t) * \delta(t-3) \cdot \mathbb{1}(t)$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

$$\therefore F(s) = \frac{1}{s^2+1} e^{-3s}$$

iv) $f(t) = \sin(t-3) \cdot \mathbb{1}(t-3)$

$$\therefore F(s) = \frac{1}{s^2+1} e^{-3s}$$

v) $f(t) = t \sin(t-3) \cdot \mathbb{1}(t)$

$$= t(\sin t * \delta(t-3))$$

$$F(s) = -\frac{d}{ds}\left(\frac{1}{s^2+1} e^{-3s}\right)$$

$$\therefore F(s) = \frac{3e^{-3s}(s^2+1) + 2se^{-3s}}{(s^2+1)^2}$$

vi) $f(t) = (t-3)\, e^{(t-3)} \cdot \mathbb{1}(t)$
$= (te^{t}) * \delta(t-3) \cdot \mathbb{1}(t)$

$$F(s) = -\frac{d}{ds}\left(\frac{1}{s-1}\right) e^{-3s}$$

$$\therefore F(s) = \frac{1}{(s-1)^2} e^{-3s}$$

vii) $f(t) = (t-3)\, e^{(t-3)} \cdot \mathbb{1}(t-3)$

$$F(s) = \mathcal{L}\{te^{t}\} e^{-3s}$$

$$\therefore F(s) = \frac{1}{(s-1)^2} e^{-3s}$$

viii) $f(t) = te^{-at} + 2t\cos t$

$$F(s) = \frac{1}{(s+1)^2} - 2\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{1}{(s+1)^2} - \frac{2(s^2+1) - 4s^2}{s^2+1}$$

$$\therefore F(s) = \frac{1}{(s+1)^2} - \frac{2-2s^2}{s^2+1}$$

ix) $f(t) = t^2 + e^{-at}\sin(bt)$

$$\therefore F(s) = \frac{2}{s^3} + \frac{1}{(s-a)^2+b^2}$$

2. i) $F(s) = \dfrac{s^2+s+1}{(s+1)(s+2)(s+3)}$

$$f(t) = \mathrm{Res}[F(s)e^{st}, s=-1] + \mathrm{Res}[F(s)e^{st}, s=-2] + \mathrm{Res}[F(s)e^{st}, s=-3]$$

$$= \left.\frac{(s^2+s+1)e^{st}}{(s+2)(s+3)}\right|_{s=-1} + \left.\frac{(s^2+s+1)e^{st}}{(s+1)(s+3)}\right|_{s=-2} + \left.\frac{(s^2+s+1)e^{st}}{(s+1)(s+2)}\right|_{s=-3}$$

$$\therefore f(t) = \left[\frac{1}{2}e^{-t} - 3e^{-2t} + \frac{7}{2}e^{-3t}\right]\cdot \mathbb{1}(t)$$

ii) $F(s) = \dfrac{s^2+s+1}{(s+1)(s+2)(s+3)}\left(e^{-s}\right)$

$$\mathcal{L}\{f(t-T)\,\mathbb{1}(t-T)\} = F(s)\,e^{-Ts}$$

$$\therefore f(t) = \left[\frac{1}{2}e^{-(t-T)} - 3e^{-2(t-1)} + \frac{7}{2}e^{-3(t-1)}\right]\cdot \mathbb{1}(t-1)$$

iii) $F(s) = \dfrac{e^{-2s}}{s-3}$

$$\therefore f(t) = e^{3(t-2)}\cdot \mathbb{1}(t-2)$$

iv) $F(s) = \dfrac{e^{1-s}}{s} = \dfrac{e\cdot e^{-s}}{s}$

$$\therefore f(t) = e\cdot \mathbb{1}(t-1)$$

v) $F(s) = \dfrac{1}{s^6} = \dfrac{1}{5!}\cdot\dfrac{5!}{s^6}$

$$\therefore f(t) = \frac{1}{240}t^5\cdot \mathbb{1}(t)$$

Vi)
$$F(s) = \frac{10}{s(s+1)(s+10)}$$

$$f(t) = \mathrm{Res}\left[F(s)e^{st}, s=0\right] + \mathrm{Res}\left[F(s)e^{st}, s=-1\right] + \mathrm{Res}\left[F(s)e^{st}, s=-10\right]$$
$$= \left.\frac{10e^{st}}{(s+1)(s+10)}\right|_{s=0} + \left.\frac{10e^{st}}{s\cdot(s+10)}\right|_{s=-1} + \left.\frac{10e^{st}}{s(s+1)}\right|_{s=-10}$$
$$\therefore f(t) = \left[1 - \frac{10}{9}e^{-t} + \frac{1}{9}e^{-10t}\right]\cdot \mathbb{1}(t)$$

Vii)
$$F(s) = \frac{1}{s(s+2)^2}$$

$$f(t) = \mathrm{Res}\left[F(s)e^{st}, s=0\right] + \mathrm{Res}\left[F(s)e^{st}, s=-2\right]$$
$$= \left.\frac{e^{st}}{(s+2)^2}\right|_{s=0} \quad \left.\frac{d}{ds}\left[\frac{e^{st}}{s}\right]\right|_{s=-2}$$
$$= \frac{1}{4} + \left.\frac{te^{st} - e^{st}}{s^2}\right|_{s=-2}$$
$$\therefore f(t) = \left[\frac{1}{4} + \frac{1}{4}te^{-2t} - \frac{1}{4}e^{-2t}\right]\cdot \mathbb{1}(t)$$

Viii)
$$F(s) = \frac{2(s+2)}{(s+1)(s^2+4)} = \frac{2s+4}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$As^2 + 4A + Bs^2 + Bs + Cs + C = 2s+4$$
$$A + B = 0$$
$$B + C = 2$$
$$4A + C = 4$$
$$4A - B = 2$$
$$5A = 2$$
$$A = 2/5$$
$$B = -2/5$$
$$C = 8/5$$

$$f(t) = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{8}{5}\left\{\frac{1}{s^2+4}\right\}$$
$$= \frac{2}{5}e^{-t} - \frac{2}{5}\cos(2t) + \frac{8}{5}\sin(2t)$$

3. $F(s) = \dfrac{2-s}{(s-1)(s+2)}$, FIND $\lim\limits_{t\to\infty} f(t)$

$$f(t) = \text{Res}\left[F(s)e^{st}, s=1\right] + \text{Res}\left[F(s)e^{st}, s=-2\right]$$

$$= \left.\frac{(2-s)e^{st}}{s+2}\right|_{s=1} + \left.\frac{(2-s)e^{st}}{s-1}\right|_{s=-2}$$

$$f(t) = \left[\frac{1}{3}e^{t} - \frac{4}{3}e^{-2t}\right]\cdot \mathbb{1}(t)$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{1}{3}e^{t} - \frac{4}{3}e^{-2t}$$

$$= \frac{1}{3}\lim_{t\to\infty} e^{t} - \frac{4}{3}\lim_{t\to\infty} e^{-2t}$$

$$= \frac{1}{3}(\infty) - \frac{4}{3}(0)$$

$$\therefore \lim_{t\to\infty} f(t) = \infty$$

4. i) $y'' + 2y' + 5y = e^{-t}\mathbb{1}(t)$, $y(0) = y'(0) = 0$

$$s^2Y + 2sY + 5Y = \frac{1}{s+1}$$

$$Y = \frac{1}{(s+1)(s^2+2s+5)}$$

$$Y = \frac{1}{(s+1)(s-1+4i)(s-1-4i)}$$

$$y(t) = \text{Res}\left[Y(s)e^{st}, s=-1\right] + \text{Res}\left[Y(s)e^{st}, s=1-4i\right] + \text{Res}\left[Y(s)e^{st}, s=1+4i\right]$$

$$= \frac{e^{st}}{(s-1+4i)(s-1-4i)}\bigg|_{s=-1} + \frac{e^{st}}{(s+1)(s-1-4i)}\bigg|_{s=1-4i} + \frac{e^{st}}{(s+1)(s-1+4i)}\bigg|_{s=1+4i}$$

$$= \frac{1}{20}e^{-t} + (-32-16i)e^{(1-4i)t} + (-32+16i)e^{(1+4i)t}$$

$$= \frac{1}{20}e^{-t} + (-32e^{t})(e^{-4it} + e^{4it}) + (16ie^{t})(e^{4it} - e^{-4it})$$

$$\therefore y(t) = \left[\frac{1}{20}e^{-t} - 64e^{t}\cos(4t) - 32e^{t}\sin(4t)\right]\cdot\mathbb{1}(t)$$

ii) $y'' - y = \mathbb{1}(t)$, $y(0) = y'(0) = 1$

$$s^2Y - sy(0) - y'(0) - Y = \frac{1}{s}$$

$$(s^2-1)Y = \frac{1}{s} + s + 1$$

$$Y = \frac{1}{s(s^2-1)} + \frac{s}{s^2-1} + \frac{1}{s^2-1}$$

$$y(t) = \text{Res}\left[Y(s)e^{st}, s=0\right] + \text{Res}\left[Y(s)e^{st}, s=1\right] + \cos(t) + \sin(t)$$

$$= \frac{e^{st}}{s^2-1}\bigg|_{s=0} + \frac{d}{ds}\left[\frac{e^{st}}{s}\right]\bigg|_{s=1}$$

$$= -1 + \frac{te^{st}s - e^{st}}{s^2}\bigg|_{s=1} + \cos(t) + \sin(t)$$

$$\therefore y(t) = \left[-1 + te^{t} + e^{t} + \cos(t) + \sin(t)\right]\cdot\mathbb{1}(t)$$

iii) $y^{(4)} + 4y^{(3)} + 8y^{(2)} + 8y^{(1)} + 4y = 3u' + 4u$, $y(0) = \ldots = y^{(4)}(0) = 0$, $u(t) = \mathbb{1}(t)$

$$s^4 Y + 4s^3 Y + 8s^2 Y + 8sY + 4Y = (3s+4)\left(\frac{1}{s}\right)$$

$$Y = \frac{3s+4}{s(s^4+4s^3+8s^2+8s+4)}$$

$$Y = \frac{3s+4}{s(s+1-i)^2(s+1+i)^2}$$

$$y(t) = \text{Res}\left[Y(s)e^{st}, s=0\right] + \text{Res}\left[Y(s)e^{st}, s=-1+i\right] + \text{Res}\left[Y(s)e^{st}, s=-1-i\right]$$

$$= \left.\frac{(3s+4)e^{st}}{(s+1-i)^2(s+1+i)^2}\right|_{s=0} + \frac{d}{ds}\left.\left[\frac{(3s+4)e^{st}}{s(s+1+i)^2}\right]\right|_{s=-1+i} + \frac{d}{ds}\left.\left[\frac{(3s+4)e^{st}}{s(s+1-i)^2}\right]\right|_{s=-1-i}$$

$$= 1 + \left(-\frac{1}{4}+\frac{1}{2}i\right)te^{(-1+i)t} + \left(\frac{1}{2}+\frac{1}{4}i\right)e^{(-1-i)t} + \left(-\frac{1}{4}-\frac{1}{2}i\right)te^{(-1-i)t} + \left(\frac{1}{2}-\frac{1}{4}i\right)e^{(-1-i)t}$$

$$\therefore y(t) = \left[1 - \frac{t+2}{2e^{t}}\cos t - \frac{2t+1}{2e^t}\sin t\right]\cdot \mathbb{1}(t)$$

iv) $y'' + y' + 3y = 0$, $y(0) = \alpha$, $y'(0) = \beta$

$$s^2 Y - sy(0) - y'(0) + sY - sy(0) + 3Y = 0$$

$$(s^2+s+3)Y = \alpha s + \alpha + \beta$$

$$Y = \frac{\alpha s + \alpha + \beta}{(s+\frac{1}{2})^2 + \frac{11}{4}}$$

$$Y = \alpha \frac{s}{(s+\frac{1}{2})^2 + \frac{11}{4}} + \frac{(\alpha+\beta)}{\frac{\sqrt{11}}{2}} \frac{\frac{\sqrt{11}}{2}}{(s+\frac{1}{2})^2+\frac{11}{4}}$$

$$y(t) = \alpha\mathcal{L}^{-1}\left\{\frac{s}{(s+\frac{1}{2})^2+\frac{11}{4}}\right\} + \frac{2(\alpha+\beta)}{\sqrt{11}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{11}}{2}}{(s+\frac{1}{2})^2+\frac{11}{4}}\right\}$$

$$\therefore y(t) = \left[\alpha e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{11}}{2}t\right) + \frac{2(\alpha+\beta)}{\sqrt{11}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{11}}{2}t\right)\right]\cdot\mathbb{1}(t)$$