№1

$$\varepsilon^{ijk}\varepsilon_{kjn} = -\varepsilon^{kji}\varepsilon_{kjn} = -\left(\delta^j_j\delta^i_n - \delta^i_j\delta^j_n\right) =$$

$$= -\left(3\delta^i_n - \delta^i_n\right) = -2\delta^i_n$$

$$a_{ij}g^{jk} = a_i^k$$

$$\left(a_{ij}a^{jk} + \delta_i^j a_{lj} g^{lk}\right) g_{ks} = 2a_{is}.$$

$$a_{li}\, g^{lk} g_{ks} = a_{li}\,\delta^l_s = a_{si} \overset{?}{=} a_{is}$$

$$\vec a = \frac{\vec h\times\vec r}{2} = \frac12 \varepsilon^{i\alpha\beta} h_\alpha r_\beta \vec e_i$$

$$\vec\nabla\times\vec a = \varepsilon^{j\xi\gamma}\frac{\partial}{\partial x^\xi} a_\gamma \vec e_j = \varepsilon^{j\xi\gamma}\frac{\partial}{\partial x^\xi}\left(\frac12\varepsilon^{\gamma\alpha\beta}h_\alpha r_\beta\right)\vec e_j =$$

$$= -\varepsilon^{\gamma\xi j}\varepsilon^{\gamma\alpha\beta}\frac{\partial}{\partial x^\xi}\left(\frac12 h_\alpha r_\beta\right)\vec e_j =$$

$$= -\left(\delta^\alpha_\xi\delta^\beta_j - \delta^\beta_\xi\delta^\alpha_j\right)\frac{\partial}{\partial x^\xi}\left(\frac12 h_\alpha r_\beta\right)\vec e_j =$$

$*$ $\frac{\partial}{\partial x^\alpha}\left(\frac12 h_\alpha r_\beta\right)\vec e_\beta = \frac12 h_\alpha \frac{\partial r_\beta}{\partial x^\alpha}\vec e_\beta = \frac12 h_\alpha\delta^\beta_\alpha \vec e_\beta = \frac12 h_\beta \vec e_\beta = \frac{\vec h}{2}$

$*$ $\frac{\partial}{\partial x^\beta}\left(\frac12 h_\alpha r_\beta\right)\vec e_\alpha = \frac12 h_\alpha\frac{\partial r_\beta}{\partial x^\beta}\vec e_\alpha = \frac12 h_\alpha \delta^\beta_\beta \vec e_\alpha = \frac32 h_\alpha \vec e_\alpha = \frac32\vec h$

$$= -\left(\frac{\vec h}{2} - \frac32\vec h\right) = \vec h$$

№3 a

$\nabla\cdot\vec h = 0$, действительно $\vec h = (h_x, h_y, h_z)$ константное векторное поле, оно не зависит от координаты точки и поэтому все частные производные $\frac{\partial h}{\partial x_i} = 0$.

Поэтому первое слагаемое 0.

$\nabla\cdot\vec r = 3$ (или размерности пространства).

$\vec r = (x, y, z)$

$\nabla\vec r = (1, 1, 1)$

$\nabla\cdot\vec r = 1+1+1 = 3$

Откуда $\vec B = \operatorname{rot}\vec a = \frac32\vec h$.

Я не знаю что с этим делать.

№3.а

$\vec r(x,y,z) = (x,y,z)$, $r = |\vec r| = \sqrt{x^2+y^2+z^2}$

$\frac{\vec r}{r}(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{}}, \frac{z}{\sqrt{}}\right)$

Посмотрим на производную первой коорд. по x:

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{1\cdot\sqrt{} - x\cdot\frac12\cdot 2x\cdot\frac{1}{\sqrt{}}}{x^2+y^2+z^2} = \left|\cdot\sqrt{}\right|$$

$$= \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)\sqrt{}} = \frac{r^2-x^2}{r^3}$$

Аналогично

$$\frac{\partial}{\partial y}\left(\left(\frac{\vec r}{r}\right)_y\right) = \frac{r^2-y^2}{r^3},\quad \frac{\partial}{\partial z}\left(\left(\frac{\vec r}{r}\right)_z\right) = \frac{r^2-z^2}{r^3}$$

$\operatorname{div}\frac{\vec r}{r}$ – это сумма всех трех, т.е.

$$\operatorname{div}\frac{\vec r}{r} = \frac{r^2-x^2}{r^3} + \frac{r^2-y^2}{r^3} + \frac{r^2-z^2}{r^3} = \frac{3r^2-r^2}{r^3} = \frac{2}{r}.$$

Можно заметить, что все работает если размерность >3.

②

№3.б

Посмотрим на производную первой коорд. по y:

$$\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = x\cdot(-1/2)\cdot 2y\cdot\frac{1}{(\sqrt{x^2+y^2+z^2})^3} = -\frac{xy}{r^3}$$

Это выражение симметрично по x и y.

Если обозначить поле за $\vec F = \frac{\vec r}{r}$, то

$$\frac{\partial F^\alpha}{\partial x^\beta} = -\frac{x^\alpha x_\beta}{r^3} = \frac{\partial F^\beta}{\partial x^\alpha}$$

Перейдем к ротору:

$$\operatorname{rot}\vec F = \nabla\times\vec F = \varepsilon^{i\alpha\beta}\frac{\partial}{\partial x^\alpha}F_\beta\vec e_i = \varepsilon^{i\alpha\beta}\frac{\partial F_\beta}{\partial x^\alpha}\vec e_i =$$

$$= \varepsilon^{i\alpha\beta}\frac{\partial F_\alpha}{\partial x^\beta}\vec e_i = -\varepsilon^{i\cancel{\alpha\beta}\,\beta\alpha}\frac{\partial}{\partial x^\beta}F_\alpha\vec e_i = -\nabla\times\vec F,\ \text{т.е.}$$

$\operatorname{rot}\vec F = -\operatorname{rot}\vec F \Rightarrow \operatorname{rot}\vec F = 0$

№4.

$\vec r = r^\alpha\vec e_\alpha = (r^1, r^2, r^3, \ldots)$

$\frac{\partial r^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$, т.е. например $\frac{\partial r^1}{\partial x_2} = 0$, $\frac{\partial r^1}{\partial x_1} = 1$.

Обозначим $\vec c = (\vec a\cdot\vec r)\vec b = (a_\alpha r^\alpha) b_\beta \vec e^\beta$

$$\operatorname{div}\vec c = \nabla\cdot\vec c = \frac{\partial}{\partial x_\gamma}\cdot c_\gamma = \frac{\partial}{\partial x_\gamma}\cancel{\left(a_\alpha r^\alpha b_\beta\vec e^\beta\right) = \frac{\partial r^\alpha}{\partial x_\gamma}a_\alpha b_\beta\vec e^\beta =}$$

$$\cancel{= \delta^\alpha_\gamma a_\alpha b_\beta\vec e^\beta} = \frac{\partial}{\partial x_\gamma}\left(a_\alpha r^\alpha b_\gamma\right) = \frac{\partial r^\alpha}{\partial x^\gamma}\cdot a_\alpha b_\gamma =$$

$$= \delta^\alpha_\gamma\cdot a_\alpha b_\gamma = a_\alpha b_\alpha = \vec a\cdot\vec b$$

$$\operatorname{rot}\vec c = \varepsilon^{i\xi\gamma}\frac{\partial}{\partial x^\xi}c_\gamma\vec e_i = \varepsilon^{i\xi\gamma}\frac{\partial}{\partial x^\xi}\left(a_\alpha r^\alpha b_\gamma\right)\vec e_i =$$

$$= \varepsilon^{i\xi\gamma}\boxed{\frac{\partial r^\alpha}{\partial x^\xi}}a_\alpha b_\gamma\vec e_i = \varepsilon^{i\xi\gamma}\delta^\alpha_\xi a_\alpha b_\gamma\vec e_i = \varepsilon^{i\xi\gamma}a_\xi b_\gamma\vec e_i = \vec a\times\vec b$$

$\underbrace{\alpha\ \beta\ \gamma\ \xi}_{\alpha}\ \sigma\ \rho\ \cancel{ijkl}\ \mu\ \tau\ \psi$

③

$f^\alpha = x^i a^\alpha$

$$а)\ \oint_S \boldsymbol r(\boldsymbol a\cdot d\boldsymbol S) = \oint_S x^i\boldsymbol e_i a^\alpha dS_\alpha = \boldsymbol e_i\oint_S a^\alpha\frac{\partial x^i}{\partial x^\alpha}dV = \oint_S a^i\boldsymbol e_i dV = \boldsymbol a V$$

$$б)\ \oint_S(\boldsymbol a\cdot\boldsymbol r)d\boldsymbol S = \oint_V grad\left(x^\alpha a_\alpha\right)dV = \oint_S \boldsymbol a dV = \boldsymbol a V$$

$a_\alpha\, \mathrm{grad}\, x^\alpha = a_\alpha\delta^\beta_\alpha\vec e_\beta$

$$\int_S f\, dS = \int_V \nabla f\, dV.$$

$$\int_V a_\beta\vec e_\beta\, dV = \int_V \vec a\, dV = \vec a V$$

$$\int_S f^\alpha dS_\alpha = \int_V \frac{\partial f}{\partial x^\alpha}dV$$