1 → Eigen vectors are orthonormal [Find why?]

Review

Field

~~Set~~ $(\mathbb{F}, +, \cdot)$ multiplication, addition

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$+ : F \times F \to F$

$\cdot : F \times F \to F$

$x, y, z \in \mathbb{F}$

1) $x + y = y + x$ (2) $(x+y)+z = x+(y+z)$

$\boxed{3)}$ $\exists!\, 0$, such that $x + 0 = x$ $\boxed{4)}$ $\forall\, x \in F$, $\exists!\, -x \in F$ such that $x + (-x) = 0$

5) $x \cdot y = y \cdot x$ 6) $(xy)z = x(yz)$

$\boxed{7)}$ $\exists!\, 1$, st $x \cdot 1 = x$ $\boxed{8)}$ $\forall\, x \in F$, $x \neq 0$, $\exists!\, x^{-1} \in \mathbb{F}$, s.t $x \cdot x^{-1} = 1$

9) $x(y+z) = x \cdot y + x \cdot z$

Characteristic of a field:

If it is possible to add 1, Finite number of times to get 0 in $\mathbb{F}$, the smallest number 'n' such that $1 + 1 + \dots + 1 = 0$, is called characteristic of $\mathbb{F}$.
If it is not possible, we say $\text{char}(\mathbb{F}) = 0$

$\mathbb{F}(t) = \dfrac{P(t)}{Q(t)}$ $P(t)$ and $Q(t)$ are polynomials in the variable t with coeffs in $\mathbb{F}$, and $Q(t)$ is not zero poly.

Vector space

- consists of
  (1) a Field $\mathbb{F}$, elements of $\mathbb{F}$ are all scalars
  (2) a set V whose objects are called vectors

and two operations

(A) vector addition: $v_1, v_2 \in V$, assign to it $v_1 + v_2 \in V$

(B) scalar addition: $\alpha \in \mathbb{F}$, $v \in V$, assign to it $\alpha v \in V$

such that :-

(i) $v_1 + v_2 = v_2 + v_1$ (ii) $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

$\boxed{(iii)}$ $\exists!\, 0 \in V$, such that $v + 0 = v$

$\boxed{(iv)}$ given $v \in V$, $\exists!\, -v \in V$, s.t $v + (-v) = 0$

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$\boxed{(v)}$ $1 \cdot v = v$ (vi) $(\alpha\beta)v = \alpha(\beta v)$

(vii) $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$

(viii) $(\alpha+\beta)v = \alpha v + \beta v$

Eg.1: $\mathbb{C} \supseteq \mathbb{R} \supseteq \mathbb{Q}$

$\mathbb{C}$ ↓ complex nos, $\mathbb{R}$ ↓ real nos., $\mathbb{Q}$ ↓ rational nos

$\mathbb{Q}(i) = \{a + ib \mid a, b \in \mathbb{Q}\}$

$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$

are vector spaces over $\mathbb{Q}$

$\mathbb{C}$, $\mathbb{R}$ · V.S over $\mathbb{R}$

$\mathbb{C}$ is VS over $\mathbb{C}$

Eg-2: $\mathbb{F}^n = \{(\alpha_1, \ldots, \alpha_n) \mid \alpha_i \in \mathbb{F}\}$

coordinate wise addition & coordinate wise scalar multiplication makes $\mathbb{F}^n$ a V.S over $\mathbb{F}$

Subspace: V VS / F (V is vector space over F)

V f.d.V.S /F → finite dimension vector space

W is subspace of V means $W \subseteq V$

and W is a vector space over $\mathbb{F}$ with same operations coming from V.

Eg: (1) 0, V are called trivial subspaces of V

(2) · $W_1 = \{(\alpha_1, \ldots, \alpha_{n-1}, 0) \mid \alpha_i \in \mathbb{F}\} \subseteq \mathbb{F}^n$

$W_2 = \{(\alpha_1, \ldots, \alpha_n) \mid \alpha_1 + \alpha_3 = 0\}$

Def: V vs/F $S \subseteq V$ (subset)

A vector $w \in V$ is linear combination of vectors in S if we can write $w = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$ with $c_i \in F$, $v_i \in S$

Def: V vs/F $S \subseteq V$ subset

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The subspace of V spanned by S is the intersection of all subspaces of V which contains S)?

$\iff$ (Equivalently) The subspace spanned by a non-empty set S is the set of all lin. comb of vectors in S

The subspace spanned by empty set is the 0 subspace.

$\rightarrow$ V vs/F $W_1, W_2 \subseteq V$ (subspaces)

$W_1 + W_2 = \{ v_1 + v_2 \mid v_i \in W_i \}$

$W_1 \cap W_2$ is subspace

Def: $S \subseteq V$ set S is lin dep if $\exists$ distinct vectors $v_1, v_2, \dots, v_n$ in S & scalars $a_1, \dots, a_n \in \mathbb{F}$ not all zero s.t $a_1 v_1 + \dots + a_n v_n = 0$.

A set S is called lin indep if it is not lin dep.

Def: V vs/$\mathbb{F}$

A basis of V is a lin. indep. set of vectors in V which spans V.

If V has a finite basis, we say V is finite dimensional otherwise we say V is infinite dimensional.

Theorem: If V is a f.d. v.s /F then any two basis have same no. of elements

Say a basis of V contains n vectors, then

$\dim(V) = n$

$\dim(0) = 0$.

Theorem: (Every vector space have a basis)?

$\dim_{\mathbb{R}}(\mathbb{C}) = 2$ $\dim_{\mathbb{Q}}(\mathbb{C}) = \infty$

$\mathbb{P}(\mathbb{R})$ - {set of real poly in 1 variable}

$\dim_{\mathbb{R}}[\mathbb{P}(\mathbb{R})] = \infty \rightarrow \{1, x, x^2, \dots\}$

$\mathbb{P}_n(\mathbb{R})$ = {set of real poly. in one var. of deg $\leq n$}

$\dim_{\mathbb{R}}(\mathbb{P}_n(\mathbb{R})) = n+1$ basis $\rightarrow \{1, x, \dots, x^n\}$

$\rightarrow$ V F.d.v.s / F

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$W \subseteq V$ (subspace), $\dim(W) < \dim(V)$

Def: Direct sum of subspaces

V v.s / $\mathbb{F}$ W_1, W_2 subspaces of V

we say $V = W_1 \oplus W_2$ if

① $V = W_1 + W_2$ and

② $W_1 \cap W_2 = \{0\}$

$\rightarrow \dim(V) = \dim(W_1) + \dim(W_2)$

Ordered basis (V is f.d.v.s / $\mathbb{F}$)

Def: A ordered basis of V is a finite sequence of vectors in V which are lin. independent & spans V.

[$V_1, V_2, \dots V_n$ are fixed, the order is fixed]

Eg: $\mathbb{P}_2(\mathbb{R}) \Rightarrow \beta = \{1, x, x^2\}, \beta' = \{1, x^2, x\}$

$\beta'' = \{x^2, x, 1\}$

$\rightarrow$ Given an ordered basis $\beta = \{v_1, \dots, v_n\}$ and a vector $v \in V$,

$$v = \sum_{i=1}^{n} x_i v_i$$

Defn: The n-tuple $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n$ is called the coordinate vector of v w.r.t ordered basis β.

$$[v]_\beta = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Eg $\mathbb{P}_2(\mathbb{R})$
$\beta = \{1, x, x^2\}$

$[3x^2+5x+7]_\beta = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$

$\beta' = \{1, x^2, x\}$
$[3x^2+5x+7]_{\beta'} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$

$\mathbb{P}^2(\mathbb{R}) \longrightarrow \mathbb{R}^3$
$(\beta) \qquad (e_1, e_2, e_3)$

$ax^2+bx+c \longmapsto \begin{bmatrix} c \\ b \\ a \end{bmatrix} := [ax^2+bx+c]_\beta$

(More generally, $\dim_{\mathbb{F}}(V)=n$, then a choice β of ordered bases gives a bijection b/w V & $\mathbb{F}^n$ sending v to $[v]_\beta$.) ?

Thm: V is n-dim V.S/F
$B = \{v_1, \ldots, v_n\}$ & $B' = \{v_1', \ldots, v_n'\}$ be two ordered bases of V
Then $\exists!$ invertible $(n\times n)$-matrix with entries in F s.t
$[v]_B = P[v]_{B'} \qquad \forall v \in V$

The columns of P are given by $P_j = [v_j']_B$ $j=1, \ldots, n$

Eg: $V = \mathbb{R}^2$ $\quad B = \{(1,0), (0,2)\}$
$B' = \{(1,1), (1,-2)\}$

$P = \begin{bmatrix} 1 & 1 \\ 1/2 & -1 \end{bmatrix}$

$v = (a,b) \in \mathbb{R}^2 \qquad [(a,b)]_B = \begin{bmatrix} a \\ b/2 \end{bmatrix} = P\begin{bmatrix} \frac{2a+b}{3} \\ \frac{a-b}{3} \end{bmatrix}$

$[(a,b)]_{B'} = \begin{bmatrix} \frac{2a+b}{3} \\ \frac{a-b}{3} \end{bmatrix}$

Linear transformation
Def: (T, V, W)
V, W are vector spaces over a field F and T is a map from V to W s.t
$T(av_1+v_2) = aT(v_1)+T(v_2) \quad \forall v_1, v_2 \in V$
$\forall a \in F$

Ex: Check: if T is l.t, $T(0)=0$

Eg: $T: V \to V$ (i) $Tv = v \quad \forall v$
(ii) $Tv = 0 \quad \forall v$

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Eg: $T: \mathbb{P}_2(\mathbb{R}) \longrightarrow \mathbb{P}_2(\mathbb{R})$
$a+bx+cx^2 \longmapsto b+2cx$

$T': \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_1(\mathbb{R})$
$a+bx+cx^2 \longmapsto b+2cx$

both are diff. l.t as vector space is not the same.

Eg: $T_1: \mathbb{P}_2(\mathbb{R}) \longrightarrow \mathbb{P}_3(\mathbb{R})$
$a+bx+cx^2 \longmapsto ax + \frac{b}{2}x^2 + \frac{c}{3}x^3$

Eg. $T_4: \mathbb{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}$
$a+bx+cx^2 \longmapsto a + b/2 + c/3$

EX: V f.d v.s$/F$, $B = \{v_1, \dots, v_n\}$ is the ordered basis of V
W v.s$/F$ $\quad w_1, \dots, w_n$ are vectors in W
Then $\exists!$ l.t $T: V \to W$ s.t $T(v_i) = w_i$ [if V is f.d.vs then W should also be v.s la?]
Proof: $v = \sum_{i=1}^{n} x_i v_i$ $\quad$ Define $Tv = \sum_{i=1}^{n} x_i w_i$
Check: This is l.t

$\to T: V \to W$ is a l.t

~~Def: Range$(T) = \{w \in W \mid \exists v \in V$ s.t $Tv = w\}$~~

Def: A lin. trans T is called surjective if Range$(T) = W$
Def: Suppose W is f.d.vs
Rank$(T) = \dim_F(\text{Range}(T))$

Ex: Check range(T) is a subspace of W.
Defn: $\text{Ker}(T) = \{v \in V \mid Tv = 0\}$
We say T is injective if $\text{Ker}(T) = 0$

Ex: Check $\text{Ker}(T)$ is a subspace of V
Defn: Suppose V is f.d.vs$/F$
Nullity$(T) = \dim_F(\text{Ker}(T))$

Theorem: Rank-Nullity Thm → full doubt.

V, W f.d.v.s/F $T: V \to W$ is a l.t

$$\text{rank}(T) + \text{nullity}(T) = \dim_F(V)$$

Ex: There is no surjective linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Defn: $T: V \to W$ is a l.t which is inj & surj (bijective)

Then $T^{-1}: W \to V$ by $T^{-1}(w) = v$ if $Tv = w$

Ex: Check T^{-1} is a lin. trans from W to V

Def: Two vector spaces V and W are called isomorphism if $\exists$ a bijective linear trans from V to W.

In that case, we say, T is an isomorphism b/w V and W.

Eg: $V = P^2(\mathbb{R})$, $W = \mathbb{R}^3$

$T = P^2(\mathbb{R}) \to \mathbb{R}^3$ $\quad T_1 : P^2(\mathbb{R}) \to \mathbb{R}^3$

$$a + bx + cx^2 \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thm: Any two finite dimensional vector spaces over F of same dimension are isomorphic

P.f: V has a basis $\{v_1, \ldots, v_n\}$ over F

W has a basis $\{w_1, \ldots, w_n\}$ over F.

Define $T: V \to W$ by $Tv_i = w_i$ for $i = 1, \ldots, n$

Matrix representation of linear transformation

$T: V \to W$ a l.t

$\beta = \{v_1, \ldots, v_n\}$ ordered basis of V

$\beta' = \{w_1, \ldots, w_m\}$ ordered basis of W

$Tv_1, Tv_2, \ldots, Tv_n$ uniquely determines T.

Write $T v_j = \sum_{i=1}^{m} \alpha_{ij} w_i \qquad 1 \le j \le n$

$$[T v_j]_{\beta'} = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & & | \\ [T v_1]_{\beta'} & [T v_2]_{\beta'} & \cdots & [T v_n]_{\beta'} \\ | & | & & | \end{bmatrix}_{m \times n} = \alpha_{ij}$$

→ We call A the matrix of linear transformation T w.r.t ordered basis β and β'.

$$T v = T\left(\sum_{j=1}^{n} x_j v_j\right) = \sum_{j=1}^{n} x_j T v_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} \alpha_{ij} w_i$$

$v \in V$

$$[v]_{\beta} = X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} x_j\right) w_i$$

$$[T v]_{\beta'} = A X = A [v]_{\beta}$$

<u>Theorem:</u> Let V be an 'n' dim V.S/F with ordered basis β. Let W be an 'm' dim V.S/F, with ordered basis β'. Then for each linear transformation T from V to W, there exists an $m \times n$ matrix A with entries in F s.t $[T v]_{\beta'} = A [v]_{\beta}$ for all $v \in V$

However the assignment T to A is a one-one correspondence b/w the set of linear transformat

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from V to W and set of (m×n) matrices with entries in F.

$\beta = \{v_1, \ldots, v_n\}$ ordered basis of V
$\beta' = \{w_1, \ldots, w_m\}$ ordered basis of W

The j^{th} column of A is given by $[Tv_j]_{\beta'}$

$$\underset{2\times 3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} \begin{matrix} 1 \\ x \\ x^2 \end{matrix} = \begin{bmatrix} b \\ 2c \end{bmatrix} \begin{matrix} 1 \\ x \end{matrix}$$

Recall: V, $\beta = \{v_1, \ldots, v_n\}$; W, $\beta' = \{w_1, \ldots, w_m\}$
$T: V \to W$

- The l.t T determines a matrix A (m×n) over F s.t the j^{th} column of A, $A_j = [Tv_j]$
We call the matrix A to be the matrix of T w.r.t ordered basis β and β'.

Notation: $T: V \to V$, we say T is an operator on V

$V = W$, $\beta = \beta'$, $T: V \to V$, β is ordered basis of V.
A matrix of T w.r.t β

Eg. ①: $T: P^2(\mathbb{R}) \longrightarrow P^2(\mathbb{R})$, $\beta = \{1, x, x^2\}$
$a + bx + cx^2 \longmapsto b + 2cx$

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$$

$T(1) = 0$

T

~~T(1) = 0~~

$\cancel{T(x)} = 2c$

$P^{-1}AP = B$

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Def: Two $n \times n$ matrix A and B are called similar over the field F if there exists an invertible $(n \times n)$ matrix P with co-efficients in F such that
$B = P^{-1}AP$

Theorem: Let V be an n dim V.S/F.
Let $\beta = \{v_1, \ldots, v_n\}$ an ordered basis of V
and $\beta' = \{v_1', \ldots, v_n'\}$ be another ordered basis of V

Let $T: V \to V$ be a linear transformation
Let $P = \begin{bmatrix} | & & | \\ P_1 & \cdots & P_n \\ | & & | \end{bmatrix}$ be change of basis matrix

$P_j = [v_j']$

Then $[T]_{\beta'} = P^{-1}[T]_\beta P$

Read more thoroughly

Proof: $\forall v \in V$, $\boxed{[v]_\beta = P[v]_{\beta'}}$
$[Tv]_{\beta'} = [T]_{\beta'}[v]_{\beta'}$
$P[Tv]_{\beta'} = [T]_\beta P[v]_{\beta'}$

Multiply by P^{-1},
$$[Tv]_{\beta'} = [P^{-1}[T]_\beta P][v]_{\beta'} \quad \forall v \in V$$

By defn,
$$[Tv]_{\beta'} = [T]_{\beta'}[v]_{\beta'}$$
$$\Rightarrow \boxed{[T]_{\beta'} = P^{-1}[T]_\beta P}$$

Remark: Let $T: V \to V$, a l.t over F & A & B are two matrices of T w.r.t basis β & β' respectively. Then A and B are similar over F.
Conversely, $T: V \to V$ a l.t, $\beta = \{v_1, \ldots, v_n\}$ ordered basis of V.
$A = [T]_\beta$

and $P = (a_{ij})$ be any invertible $n \times n$ matrix over F.

Suppose $B = P^{-1}AP$

<u>Define</u>: $v_j' = \sum_{i=1}^{n} a_{ij} v_i$

<u>Ex</u>: $[T]_{B'} = B$

<u>Space of linear transformation</u>:

V, W v.s/F

[$Hom_F(V,W)$ is different from $Hom_F(W,V)$]

$Hom_F(V,W) = \{$ set of linear transformation from V to W $\}$

$c \in F$, $cT_1 : V \to W$ $\quad$ $T_1 : V \to W$ $\quad$ $T_1 + T_2 : V \to W$

$(cT_1)(v) = cT_1 v$ $\quad$ $T_2 : V \to W$ $\quad$ $(T_1+T_2)(v) = T_1 v + T_2 v$

<u>Ex</u>: $Hom_F(V,W)$ with these two ops is an vector space over F

<u>Special case</u>: $W = F$

$V^* = Hom_F(V,F)$ is a v.s over F

<u>Def</u>: We say V^* is dual space of V.

$T_1 : V_1 \to V_2$ l.t

$T_2 : V_2 \to V_3$ l.t

$T_2 T_1(v_1) = T_2(T_1 v_1)$

$T_2 T_1 : V_1 \to V_3$ check if $T_2 T_1$ is l.t

$T_2 T_1(\alpha v_1 + v_2)$
$= T_2[T_1(\alpha v_1 + v_2)]$
$= T_2[\alpha T_1 v_1 + T_1 v_2]$
$= \alpha T_2 T_1 v_1 + T_2 T_1 v_2$
$= \alpha X v_1 + X v_2$

$\rightarrow Hom_F(V,V) = End_F(V)$

$T \in End_F(V)$

$T^2, T^3, \ldots$

<u>Define</u>: $T^{p,q} : V \to W$

$$T^{p,q}(v_i) = \begin{cases} 0 & \text{if } i \neq q \\ w_p & \text{if } i = q \end{cases}$$

[big ?]

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad = \delta_{iq} \cdot w_p$$

Claim: $\{T^{p,q}\}_{\substack{1\le p\le m \\ 1\le q\le n}}$ is a basis of $Hom_F(V,W)$ 03/09/19

Let $T: V\to W$ be a linear transformation from V to W.

$$Tv_j = \sum_{p=1}^{m} \alpha_{pj} w_p \qquad j=1,\dots,n$$

Consider: $U = \sum_{p=1}^{m}\sum_{q=1}^{n} \alpha_{pq} T^{p,q}$

$$U(v_j) = \sum_p \sum_q \alpha_{pq} T^{p,q}(v_j)$$

$$= \sum_p \sum_q \alpha_{pq} \delta_{jq} w_p$$

$$= \sum_p \alpha_{pj} w_p \equiv T(v_j)$$

Step-2: $\{T^{p,q}\}$ is linearly independent

$$U = \sum_p \sum_q \beta_{pq} T^{p,q} \equiv 0$$

$\Rightarrow Uv_j = 0 \quad \forall j$

$= \sum_p \beta_{pj} w_p \qquad \Rightarrow \beta_{pq} = 0 \ \forall p,q$

$\Rightarrow \{T^{p,q}\}$ are linearly independent

$\{w_1, \dots, w_m\}$ is a basis of W, hence $\beta_{1j} = \beta_{2j} = \dots = \beta_{mj}$ (true $\forall j$)

Theorem: Let V be an n dim $V.S/F$ & W be a m dim $V.S/F$

Then $Hom_F(V,W)$ is an mn dim $V.S/F$.
Moreover, if $\beta = \{v_1, \dots, v_n\}$ a basis of V
$\beta' = \{w_1, \dots, w_m\}$ a basis of W

Then $\{T^{p,q}\}_{\substack{1\le p\le m \\ 1\le q\le n}}$ defined by $T^{p,q}(v_i) = \delta_{iq} w_p$ is a basis of $Hom_F(V,W)$

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from

Recall: $V^* = Hom_F(V, F) = \{f : V \to F \mid f(cv_1 + v_2) = cf(v_1) + f(v_2)\}$
↳ called the dual space of V

Def: By an linear functional on V we mean linear transformation from V to F that is an element of V^*.

Consider: Let V be a f.d v.s/F & $\dim_F(V) = n$. Then $\dim_F(V^*) = n$. Hence $V \cong V^*$ as F vector space.
Let $\beta = \{v_1, \dots, v_n\}$ a basis of V.
Define $f_j(v_i) = \delta_{ij}$
then $\beta^* = \{f_1, \dots, f_n\}$ is a basis of V^*

Def^n: We call β^* the dual basis of β. Let V be a f.d.v.s/F & $\beta = \{v_1, \dots, v_n\}$ a basis of V

Remark: ① The dual basis of β, $\beta^* = \{f_1, \dots, f_n\}$ of V^* defined by $f_j(v_i) = \delta_{ij}$ is unique.

② $f \in V^*$ then $f = \sum_{i=1}^{n} f(v_i) f_i$

Pf: write $f = \sum c_i f_i$, $f(v_j) = \sum c_i f_i(v_j) = \sum c_i \delta_{ij} = c_j$

③ Let $v \in V$, then $v = \sum_{i=1}^{n} f_i(v) v_i$

Pf: Write $v = \sum c_i v_i$

$$f_j(v) = \sum c_i f_j(v_i) = \sum c_i \delta_{ij} = c_j$$

④ $T : V \to V^*$

$$T(v_i) = f_i$$

If V is not finite dimensional v.s then $V \ncong V.S$

Eg: $V = \{(a_n)_{n \geq 0} \mid a_n \neq 0 \text{ for only finitely many } n\}$

$$\{e_i\} = \begin{cases} e_{ji} = 0 & \text{for } j \neq i \\ e_{ii} = 1 \end{cases}$$

then $\{e_i\}$ is a basis of V

V fdvs$/F$, $V^* = \mathrm{Hom}_F(V,F)$; $\dim(V) = \dim(V^*)$

If $\beta = \{v_1, \dots, v_n\}$ basis of V, $\exists!\ \beta^* = \{f_1, \dots, f_n\}$ of V^*, $f_i(v_j) = \delta_{ij}$

Example: $V = P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$

$f_i : V \to \mathbb{R}$

$f_1(a+bx+cx^2) = a$

$f_2(a+bx+cx^2) = b$

$f_3(a+bx+cx^2) = c$

$f_1(1) = 1$, $f_1(x) = f_1(x^2) = 0$ | $f_2(1) = f_2(x^2) = 0$, $f_2(x) = 1$

$f_3(1) = f_3(x) = 0$ $f_3(x^2) = 1$

More generally, if $\beta = \{v_1, v_2, \dots, v_n\}$ $f_i(v) = i^{th}$ coordinate of $[v]_\beta$

b/c $v = \sum_{i=1}^{n} f_i(v) v_i$

If I give you an ordered basis B of V, then we can find β^* the dual basis of B of V^*.

Qn: Start with an ordered basis $\beta^* = \{f_1, \dots, f_n\}$ of V^*. Can you find an ordered basis B of V such that B^* is the dual basis of β

Eg: $V = P_2(\mathbb{R})$; x_1, x_2, x_3 distinct real numbers

Define: $L_i : V \to \mathbb{R}$

$L_i(p) = p(x_i)$

Check: L_1, L_2, L_3 are linearly independent

$c_1 L_1 + c_2 L_2 + c_3 L_3 = 0$

means for every $p(x) \in V$

$c_1 L_1(p) + c_2 L_2(p) + c_3 L_3(p) = 0$

Take $p = 1 \Rightarrow c_1 + c_2 + c_3 = 0$

Take $p = x \Rightarrow c_1 x + c_2 x + c_3 x = 0$

Take $p = x^2 \Rightarrow c_1 x^2 + c_2 x^2 + c_3 x^2 = 0$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow c_1 = c_2 = c_3 = 0$

Claim $\{L_1, L_2, L_3\}$ is a basis of V^*
because $\dim(V^*) = 3$
& $\{L_1, L_2, L_3\}$ are linear independent

05/09/19

→ Want a basis $\beta = \{p_1, p_2, p_3\}$ of V such that
$p_j(x_i) = L_i(p_j) = \delta_{ij}$

$$p_1(x_1) = 1 \; ; \; p_1(x_2) = p_1(x_3) = 0$$

$$p_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \qquad p_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \qquad p_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

$$v = \sum_{i=1} f_i(v)\, v_i$$

$$\Rightarrow p(x) = L_1(p)\, p_1 + L_2(p)\, p_2 + L_3(p)\, p_3$$
$$= p(x_1)\, p_1 + p(x_2)\, p_2 + p(x_3)\, p_3$$

Thus if $p(x_1) = c_1$, $p(x_2) = c_2$, $p(x_3) = c_3$
then $p = c_1 p_1 + c_2 p_2 + c_3 p_3$
This is the unique polynomial p that takes value
c_i at x_i for $i = 1, 2, 3$

<u>Double Dual</u>

$$V^{**} = (V^*)^* = \mathrm{Hom}_F(V^*, F)$$
$$= \{\text{set of linear transformation from } V^* \text{ to } F\}$$

Fix $v \in V$

<u>Define</u>: $L_v : V^* \to F$
$$L_v(f) = f(v)$$

$$L_v(cf+g) = (cf+g)v = cf(v) + g(v)$$
$$\parallel \qquad\qquad \parallel$$
$$cL_v(f) + L_v(g) = cf(v) + g(v)$$

Thus L_v is a l.t from V to F. Hence $L_v \in V^{**}$

Thm: Let V be a f.d. v.s $/F$

Then for $v \in V$, define $L_v \in V^{**}$ by $L_v(f) = f(v)$ $\forall f \in V^*$

The map $T: V \to V^{**}$

$v \mapsto L_v$

is an isomorphism of vector spaces

Rmk: This is canonical isomorphism from V to V^{**}

Pf: Check if T is a linear trans

$$T(cv_1 + v_2) = cTv_1 + Tv_2$$

$$L_{cv_1+v_2}(f) = f(cv_1+v_2) = cf(v_1) + f(v_2)$$
$$= cL_{v_1}(f) + L_{v_2}(f) \quad \forall f \in V^*$$
$$L_{cv_1+v_2} = cL_{v_1} + L_{v_2}$$

T is injective

$T(v) = 0 \Rightarrow v = 0$

$\parallel$

$L_v = 0 \Rightarrow L_v(f) = 0 \;\; \forall f \in V^* \Rightarrow v = 0$

Suppose $v \neq 0$. Yes $\exists$ a f s.t $f(v) \neq 0$

T is surj: V is f.d.v.s

$\dim(V) = n$ & $\dim(V^{**}) = n$

$\text{rank}(T) + \text{null}(T) = n$ $\quad$ $\text{null}(T) = 0$

$\text{rank}(T) = n \Rightarrow T$ is surjective

$\to V$ f.d.v.s$/F$, $L \in V^{**}$

Then $\exists!\, v \in V$, s.t

$L(f) = f(v) \quad \forall f \in V^*$

Pf: T is surj $\Rightarrow$ given any $L \in V^{**}$ $\exists\, v$ s.t $L = T(v) = L_v$
$\Rightarrow L(f) = L_v(f) = f(v)$
$\forall f \in V^*$

T is inj $\Rightarrow v$ is unique

$\rightarrow$ V f.d v.s/F. Each basis of V^* is the dual basis of some basis for V

Pf: $\beta^* = \{f_1, \ldots, f_n\}$ basis of V^*
$\Rightarrow \exists!$ dual basis β^{**} of β^* of V^{**}
$\beta^{**} = \{L_1, \ldots, L_n\}$
s.t $L_i(f_j) = \delta_{ij}$

given L_i $\exists\, v_i \in V$ s.t
$L_i(f) = f(v_i) = L_{v_i}(f)$
Consider $\beta = \{v_1, \ldots, v_n\}$ is a basis of V
(b/c $T: V \to V^{**}$ is an iso)
$\delta_{ij} = L_i(f_j) = f_j(v_i)$
$\Rightarrow \{f_1, \ldots, f_n\}$ is the dual bases of $\beta\{v_1, \ldots, v_n\}$

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Eg: X_t = # of dogs after t years
Y_t = # of deers after t years
$X_{t+1} = 0.4 X_t + 0.3 Y_t$
$Y_{t+1} = -0.5 X_t + 1.2 Y_t$
$X_0 = 100$, $Y_0 = 140$, Find X_{20} and Y_{20}

$A = \begin{bmatrix} 0.4 & 0.3 \\ -0.5 & 1.2 \end{bmatrix}$

$$\begin{bmatrix} X_{20} \\ Y_{20} \end{bmatrix} = A \begin{bmatrix} X_{19} \\ Y_{19} \end{bmatrix} = A\left(A \begin{bmatrix} X_{18} \\ Y_{18} \end{bmatrix}\right) = \cdots = A^{20} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} 0.4-\lambda & 0.3 \\ -0.5 & 1.2-\lambda \end{bmatrix} = (0.4-\lambda)(1.2-\lambda) + \frac{8 \times 3}{16 \times 10}$$
$= 0.48 + \lambda^2 - 1.6\lambda + 0.15$
$= \lambda^2 - 1.6\lambda + 0.63$
$= (\lambda - 0.9)(\lambda - 0.7)$
$\lambda = 0.9$ (or) 0.7

$$\begin{bmatrix} 0.4 & 0.3 \\ -0.5 & 1.2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0.9x \\ 0.9y \end{pmatrix}$$

__/__/__

$0.4x + 0.3y = 0.9x \Rightarrow 0.3y = 0.5x$

$-0.5x + 1.2y = 0.9y \Rightarrow 0.3y = 0.5x$

$x = 3$
$y = 5$

$0.4x + 0.3y = 0.7x \Rightarrow 0.3y = 0.3x$

$-0.5x + 1.2y = 0.7y \Rightarrow 0.5x = 0.5y$

$x = 1$
$y = 1$

$$P = \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.7 \end{bmatrix}$$

$$P^{-1}\begin{bmatrix} 1 & -5 \\ -1 & 3 \end{bmatrix}$$

$$A^{20} = P^{-1} D^{20} P = \begin{bmatrix} 1 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} (0.9)^{20} & 0 \\ 0 & (0.7)^{20} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (0.9)^{20} & -5(0.7)^{20} \\ -(0.9)^{20} & 3(0.7)^{20} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(0.9)^{20} - 25(0.7)^{20} & (0.9)^{20} - 5(0.7)^{20} \\ 15(0.7)^{20} - 3(0.9)^{20} & 3(0.7)^{20} - (0.9)^{20} \end{bmatrix}$$

$$= (0.9)^{20}\begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} + (0.7)^{20}\begin{bmatrix} -25 & -5 \\ 15 & 3 \end{bmatrix}$$

$$A^{20}\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} (0.9)^{20}(3X_0 + Y_0) + (0.7)^{20}(-25X_0 - 5Y_0) \\ (0.9)^{20}(-3X_0 - Y_0) + (0.7)^{20}(15X_0 + 3Y_0) \end{bmatrix}$$

(or)

$$A^{20}\begin{bmatrix} 30 \\ 50 \end{bmatrix} = (0.9)^{20}\begin{bmatrix} 30 \\ 50 \end{bmatrix}; \quad A^{20}\begin{bmatrix} 40 \\ 40 \end{bmatrix} = (0.7)^{20}\begin{bmatrix} 40 \\ 40 \end{bmatrix}$$

0.9, 0.7 = eigenvalues

→ eigenvectors corresponding to their eigenvalue

$$\begin{bmatrix} 100 \\ 140 \end{bmatrix} = 2\begin{bmatrix} 30 \\ 50 \end{bmatrix} + 1\begin{bmatrix} 40 \\ 40 \end{bmatrix}$$

$$A^{20}\begin{bmatrix} 100 \\ 140 \end{bmatrix} = 2A^{20}\begin{bmatrix} 30 \\ 50 \end{bmatrix} + A^{20}\begin{bmatrix} 40 \\ 40 \end{bmatrix}$$

$$= 2(0.9)^{20}\begin{bmatrix} 30 \\ 50 \end{bmatrix} + (0.7)^{20}\begin{bmatrix} 40 \\ 40 \end{bmatrix}$$

$$= \begin{bmatrix} (0.9)^{20}\,60 + (0.7)^{20}\,40 \\ (0.9)^{20}\,100 + (0.7)^{20}\,40 \end{bmatrix}$$

Def: Let A be a square $(n \times n)$ matrix
A <u>non-zero</u> vector v is called an eigenvector of A if $Av = \lambda v$ for some scalar λ. In that case, we say λ is an eigenvalue of A and v is an eigenvector of A corresponding to eigenvalue λ.

$\Rightarrow \begin{bmatrix} 30 \\ 50 \end{bmatrix}$ & $\begin{bmatrix} 40 \\ 40 \end{bmatrix}$ forms a basis of $\mathbb{R}^2$

Def: Let A be a $(n \times n)$ square matrix, with entries in F. A set of vectors $\{v_1, \ldots, v_n\}$ is called an eigenbasis of A
if (1) $\{v_1, \ldots, v_n\}$ form a basis of F^n
(2) For each $i = 1, 2, \ldots, n$, v_i is an eigenvector of A

$P = \begin{bmatrix} 30 & 40 \\ 50 & 40 \end{bmatrix}$ $\quad 1200 - 2000 = -800 \quad P^{-1}AP \quad 16-25 \quad 12-60 \quad -16-15 \quad -12+36$

$P^{-1}A = \frac{-1}{800}\begin{bmatrix} 40 & -50 \\ -40 & 30 \end{bmatrix}\begin{bmatrix} 0.4 & 0.3 \\ -0.5 & 1.2 \end{bmatrix} = \begin{bmatrix} 9 & -48 \\ -31 & 24 \end{bmatrix}$ $\quad 270 - 2400$

$P^{-1}AP = \frac{-1}{800}\begin{bmatrix} 9 & -48 \\ -31 & 24 \end{bmatrix}\begin{bmatrix} 30 & 40 \\ 50 & 40 \end{bmatrix} = \frac{-1}{800}\Big[-2130$

$P^{-1}AP = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.7 \end{bmatrix}$; $Q = \begin{bmatrix} 40 & 30 \\ 40 & 50 \end{bmatrix} \Rightarrow Q^{-1}AQ = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.9 \end{bmatrix}$

Def: A square matrix A is diagonalizable over a field F if there exists (entries in F) an invertible matrix P with entries in F such that $P^{-1}AP = D$ where D is a diagonal matrix with entries in F.

A = matrix with eigenvectors as the column

$\rightarrow \quad P^{-1}AP = D \Rightarrow A = PDP^{-1}$
$A^{2019} = (PDP^{-1})^{2019} = PD^{2019}P^{-1}$

Eigenvalue & eigenvector

Def: V is a vector space over F. $T: V \to V$ is a linear operator. λ is an eigenvalue of T if there exists $0 \neq v \in V$ such that $Tv = \lambda v$

• $v \neq 0 \in V$, s.t $Tv = \lambda v \iff \exists\ 0 \neq v \in V$ s.t $(T-\lambda I)\, v = 0$

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$$\Updownarrow$$

$T-\lambda I$ is not injective $\iff Ker(T-\lambda I) \neq 0$ (Suppose V is finite dimensional as F vector space)

$$\Updownarrow \qquad\qquad \Updownarrow$$

$T-\lambda I$ is not isom $\iff \det(T-\lambda I) = 0$

→ V f-d-vs/F, β be a basis | β' another basis

$A = [T]_\beta$ | $B = [T]_{\beta'}$

$\det(T) = \det(A)$ ⟵ $\det(T) = \det(B)$

$\exists$ inv P, $B = P^{-1}AP \Rightarrow \det(B) = \det(A) \Rightarrow$ this is true

$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$

→ For an eigenvalue λ of T, $Ker(T-\lambda I)$ is called λ-eigenspace of T, we denote by V_λ.
Moreover, $\dim_F(V) < \infty$, then $\dim_F(V_\lambda) < \infty$
We say geometric multiplicity of λ is the dimension of $V_\lambda := Ker(T-\lambda I)$

$(a-\lambda)(a-\lambda) = 0$

For a f-d-v.s V, λ eigenvalue T $\iff \det(T-\lambda I) = 0$

$$\Updownarrow$$

$$\det(\lambda I - T) = 0$$

<u>Def</u>: If A is a $n \times n$ square matrix over F, an eigenvalue of A in F is a scalar λ s.t $(A-\lambda I)$ is non-invertible;
$\det(A-\lambda I) = \det(\lambda I - A) = 0$.
We define characteristic poly of A

$$p_A(x) = \det(xI - A)$$

[Roots of $p_A(x)$ are the eigenvalues of A]

<u>Property</u>: If A & B are similar, then $p_A(x) = p_B(x)$.

<u>Proof</u>: $B = P^{-1}AP$

$$\begin{aligned} p_B(x) = \det(xI - B) &= \det(xI - P^{-1}AP) \\ &= \det[P^{-1}(xI-A)P] \\ &= \det(xI - A) \\ &= p_A(x) \end{aligned}$$

<u>Rmk</u>: $\dim_F(V) = n$

(1) (A) $p_T(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_{n-1}x + c_n$

[Char poly has deg $n = \dim_F(V)$]

(B) T has atmost n eigen values.

(2) $c_n = p_T(0) = \det(-A) = (-1)^n \det(A)$

(3) $c_1 = -(a_{11}+a_{22}+\dots+a_{nn}) = -\operatorname{tr}(A)$

$$p_T(x) = \det\begin{pmatrix} x-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & x-a_{22} & & \\ \vdots & & \ddots & \\ -a_{n1} & & & x-a_{nn} \end{pmatrix} =$$

$$= x-a_{11} \det\begin{pmatrix} x-a_{22} & & \\ & \ddots & \\ & & x-a_{nn} \end{pmatrix}$$

$$\vdots$$

$$= (x-a_{11})(x-a_{22})\cdots(x-a_{nn}) + \text{poly of deg } (n-2)$$

$$= x^n - (a_{11}+a_{22}+\dots+a_{nn}) + \text{poly of deg } (n-2)$$

$\dim_F(V)=2 \quad T: V\to V,\ A=[T]_\beta \Rightarrow p_A(x) = x^2 - \operatorname{tr}(A)x + \det(A)$

In general

$$p_T(x) = x^n - \operatorname{tr}(A)x^{n-1} + c_2 x^{n-2} + \dots + c_{n-1}x + (-1)^n\det(A)$$

Eg: (1) $F=\mathbb{R}$, $T:\mathbb{R}^2\to\mathbb{R}^2$, $T(x,y)=(-y,x)$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$xI - A = \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \quad \det(\lambda I - A) = p_A(x)$$

$$p_A(x) = x^2+1$$

No real eigenvalues

(2) $F=\mathbb{C}$, $T:\mathbb{C}^2\to\mathbb{C}^2$, $T(x,y)=(-y,x)$

$p_T(x) = x^2+1 \Rightarrow$ eigenvalues $= +i, -i$

Eigenvector corresponding to i

$0\neq v\in\mathbb{C}^2$, $Tv = iv \Leftrightarrow v \in \ker(T-iI)$

$(0,0)\neq v=(x,y) \Rightarrow (0,0) = (T-iI)(x,y) = (-y-ix, x-iy)$

$$-y-ix=0$$

$$x-iy=0 \qquad x=iy,\ x\neq 0$$

$v=(i,1)$ is an eigenvector of T with eigenvalue i

Eigenvector corres. to $-i$

$(0,0) \neq (x,y) \in \mathbb{C}^2$, $(0,0) = (T+iI)(x,y) = (-y+ix, x+iy)$

$\Updownarrow$

$y = ix,\ y \neq 0$

$v = (1,i)$ is an eigenvector of T with eigenvalue $-i$

Eg: $V = \{$sequences of complex numbers$\}$

$T: V \to V$

$T(\{a_1, a_2, \ldots\}) = \{0, a_1, a_2, \ldots\}$

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & \cdots & \cdots \\ & & & \end{bmatrix}$$

$\ker(T) = \{0, 0, \ldots\} \Leftrightarrow 0$ is not an eigenvalue of T

(λ cannot be $=0$)

$(T-\lambda I)(\{a_1, a_2, \ldots\})$

$= \{-\lambda a_1, a_1 - \lambda a_2, a_2 - \lambda a_3, \ldots\}$

$= \{0, 0, \ldots\}$ $\qquad \ker(T-\lambda I) = 0$

$-\lambda a_1 = 0 \Rightarrow a_1 = 0$

$a_{n-1} = \lambda a_n \Rightarrow a_n = \dfrac{a_{n-1}}{\lambda} = 0 \quad \forall\ \lambda \neq 0$

Thm: $\dim_{\mathbb{C}}(V) < \infty$, then every linear operator T on V has atleast one eigenvalue.

Pf: Since $\dim_{\mathbb{C}}(V) < \infty$, the eigenvalues are roots of the poly $p_T(x)$. Since all complex polynomial has atleast one complex root, T has atleast one eigenvalue.

From now: For $F = \mathbb{C}$ $\dim_{\mathbb{C}}(V) = n < \infty$

$T: V \to V$ linear transf

$p_T(x)$ has roots $\lambda_1, \lambda_2, \ldots, \lambda_n$

$p_T(x) = (x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n)$

$= x^n - Tr(A)x^{n-1} + \ldots + (-1)^n \det(A)$

In particular

$Tr(A) = \sum_{i=1}^{n} \lambda_i$

$\det(A) = \prod_{i=1}^{n} \lambda_i$

Set up: $F = \mathbb{C}$, $\dim_{\mathbb{C}}(V) < \infty$, $T: V \to V$ lin transf

Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of T

$V_\lambda = \ker(T - \lambda I) = \lambda$-eigenspace of T

$m_\lambda = \dim_{\mathbb{C}}(V_\lambda) =$ geometric multiplicity of λ

Lemma: $(X-\lambda)^{m_\lambda} \mid p_T(x)$, i.e, $p_T(x) = (X-\lambda)^{m_\lambda} f(x)$

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Pf: $V_\lambda \subseteq V$, choose a basis $\{v_1, \ldots, v_{m_\lambda}\}$ of V_λ
Choose $w_1, \ldots, w_{n-m_\lambda}$ vector of V s.t $\{v_1, \ldots, v_{m_\lambda}, w_1, \ldots w_{n-m_\lambda}\}$
is a basis of V, call it B.

Def: M_λ = highest power of $(X-\lambda)$ that divides $p_T(x)$
:= algebraic mult of λ

$\Rightarrow$ alg mult of $\lambda \geq$ geom mult of λ

Eg:
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$$a\begin{pmatrix}3\\1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\3\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\1\\3\\1\end{pmatrix} + d\begin{pmatrix}1\\1\\1\\3\end{pmatrix} = 0$$

$X^4 - 4X^3$

$3a+b+c+d=0$ $2a-2b=0;\ a=b$
$a+3b+c+d=0$ $2a-2c=0;\ a=c$
$a+b+3c+d=0$ $2a-2d=\ a=d$
$a+b+c+3d=0$ $6a=0 \Rightarrow a=0$

$rk(A)=1$
Nullity$(A)=3$
$\dim[\ker(A-0I)]=3$
$\Rightarrow$ alg mult of 0 is atleast 3
$tr(A)=4 \Rightarrow \lambda_4 = 4$
$p_A(x) = x^3(X-4)$
$0+\lambda_4 = 4 \Rightarrow$

Nullity $=0$

$B = A + 2I$
$\ker(B-2I) = \ker(A) = \dim$ of 3
$M_2 \geq 3$
$2+2+2+\lambda_4 = 12 \Rightarrow \lambda_4 = 6$
$p_B(x) = (x-2)^3(x-6)$

$\rightarrow$ $T: \mathbb{C}^2 \to \mathbb{C}^2$ $T(x,y) = (-y, x)$
$p_T(x) = x^2+1$ $\beta = \langle e_1, e_2 \rangle$
$$A = [T]_\beta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1\,I$$
$$A^2 + I = 0 = p_A(A)$$

Theorem (Cayley Hamilton)

Let A be a $(n \times n)$ matrix and $p_A(x) = x^n + c_1 x^{n-1} + \dots + c_n$ be the characteristic polynomial of A. Then,

$$p(A) = A^n + c_1 A^{n-1} + \dots + c_{n-1} A + c_n I = 0$$

Pf: $M \cdot \operatorname{adj}(M) = I \cdot \det(M)$

$M = xI - A \Rightarrow (xI - A)\operatorname{adj}(xI - A) = p_A(x) \cdot I$

$\Rightarrow \operatorname{adj}(xI - A) = B_0 + B_1 x + \dots + B_{n-1} x^{n-1}$

$$(xI - A)(B_0 + B_1 x + \dots + B_{n-1} x^{n-1}) = x^n I + c_1 x^{n-1} I + \dots + c_n I$$

Equating coeff: $-AB_0 = c_n I$

$B_{n-1} = I$

$B_{k-1} - AB_k = c_{n-k} I$

$$p_n(A) = A^n + c_1 A^{n-1} + \dots + c_{n-1} A + c_n I$$
$$= -AB_0 + (B_0 - AB_1)A + (B_1 - AB_2)A^2 + \dots + (B_{n-2} - AB_{n-1})A^{n-1} + B_{n-1}A^n$$
$$= 0$$

Thm. $S = \{h(x) \in \mathbb{C}[x] \mid h(A) = 0\}$

Then there exists a unique monic polynomial $p(x) \in S$ such that for any $h(x) \in S$, we have $h(x) = p(x)q(x)$ for some $q(x)$ poly.

Note: Monic polynomial: coeff. of term with highest power $= 1$

Proof: $S \neq 0$ because $p_A(x) \in S$

Suppose $p(x)$ is a monic polynomial of smallest degree in S.

Take any $h(x) \in S$; $h(x) = p(x)q(x) + r(x)$ such that $p(x) = 0$ or $\deg[p(x)] > \deg[r(x)]$

Note: $r(A) = h(A) - p(A)q(A) = 0 - 0 = 0$

$r(x) \in S \Rightarrow r(x) = 0$

C because $\deg(r) < \deg(p)$, p is the polynomial of smallest degree.

Let $g(x)$ be another monic poly in S of smallest degree

$$\deg(p(x)-g(x)) < \deg(p) = \deg(g)$$

$0 = p(A) - g(A) \Rightarrow p(x) - g(x) \notin S \Rightarrow p(x) - g(x) = 0 \Rightarrow p(x) = g(x)$

Def: Minimal polynomial of A is the monic polynomial $p(x)$ of smallest degree, such that $p(A) = 0$

Cor: Minimal polynomial divides the characteristic polynomial
↳ (Corollary)

Remark: $f(x) = x^k + a_1 x^{k-1} + \dots + a_k$

$$f(P^{-1}AP) = P^{-1}A^kP + a_1 P^{-1}A^{k-1}P + \dots + a_k$$
$$= P^{-1}[f(A)]P$$

So we can talk about minimal poly of an operator T on a f.d.v.s.

Theorem: Let A be a $n\times n$ matrix. Then the roots of minimal polynomial of A and characteristic polynomial of A are same, except multiplicity.

Pf: $p(x)$ minimal poly of A, $p_A(x)$ is characteristic poly of A

$p(\lambda) = 0 \Leftrightarrow p_A(\lambda) = 0$ (follows from corollary)

$\rightarrow \lambda$ is a root of $p_A(x) \Rightarrow \lambda$ is an eigenvalue of A

$\Rightarrow$ there exists $0 \neq v \in \mathbb{R}^n$ s.t. $Av = \lambda v$

Say, $p(x) = x^k + a_1 x^{k-1} + \dots + a_k$

$$p(A)v = (\lambda^k + a_1 \lambda^{k-1} + \dots + a_k)v$$
$$= p(\lambda)v$$

$\Rightarrow p(\lambda) = 0 \Rightarrow \lambda$ is a root of min poly of A.

Eg: $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$; $A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$; $A^3 = 4A$

Find minimal polynomial of A.

A) Let $p(x) = x^3 - 4x \Rightarrow p(A) = 0$

Factors of $p(x) \Rightarrow 1, x, x-2, x+2$

$q(x) = x(x-2)$, $x(x+2)$, $x^2 - 4$

None of the satisfy $q(A) = 0 \Rightarrow p(x)$ is the minimal polynomial.

$p_A(x) = x^2(x^2 - 4)$

Eg: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $p_A(x) = (x-1)(x-2)^2$

$p(x) = (x-1)(x-2)$

Eg: $A = \begin{bmatrix} 3 & -1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

$$\begin{matrix} 2 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & -1 & -1 \\ 0 & 0 & 1 \end{matrix}$$

$$a\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + c\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 0$$

$2a + 2b + 3c = 0$

$2a + 2b - c = 0$ $2a + 2b + b = 0$

$-b - c = 0$ $2a + 3b = 0$

$\Rightarrow -c = b$ $a = -\frac{3b}{2}$

$c = -b$

$2\left(-\frac{3b}{2}\right) + 2b + 3(-b) \Rightarrow b = 0$

$$\det \begin{bmatrix} 3-\lambda & -1 & -1 \\ 2 & 2-\lambda & -1 \\ 2 & 2 & -\lambda \end{bmatrix} = 0$$

$(\lambda-2)\lambda$

$(3-\lambda)\left[(2-\lambda)(-\lambda) - [-2]\right] + 1\left[-2\lambda - (-2)\right]$

$-1\left[4 - 2(2-\lambda)\right]$

$= (3-\lambda)(\lambda^2 - 2\lambda + 2) + [2 - 2\lambda] - [\cancel{4} - \cancel{4} + 2\lambda]$

$= 3\lambda^2 - 6\lambda + 6 - \lambda^3 + 2\lambda^2 - 2\lambda + 2 - 2\lambda - 2\lambda$

$= -\lambda^3 + 5\lambda^2 - 12\lambda + 8$

$= -[\lambda^3 - 5\lambda^2 + 12\lambda - 8]$

Def: A $(n\times n)$ is called diagonalizable if there exists an invertible matrix P and a diagonal matrix D s.t. $D = P^{-1}AP$.

Equivalently: A linear operator $T: V \to V$ ($\dim(V) = n < \infty$) is called diagonalizable if there exists a basis $\beta = \{v_1, v_2, \ldots v_n\}$ of V such that $[T]_\beta$ is diagonalized such a basis of V is called an eigenbasis of T.

$\beta = \{v_1, \dots, v_n\}$ $\quad [T]_\beta = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$

$$Tv_i = a_{ii} v_i$$

__/__/__

diag $\Rightarrow p_T(\lambda)$ product of distinct linear factors

$\Rightarrow$ alg mult(λ) = geom mult(λ) $\forall \lambda$

$\Rightarrow$ Σ geom mult = dim V

<u>Thm</u>: Let T be a linear on V a f.d.v.s $/k$ $\quad$ 26/09/19

k ($\mathbb{R}$ or $\mathbb{C}$)

Let $\lambda_1, \lambda_2, \dots, \lambda_k \in k$ are distinct eigenvalues of T

$V_{\lambda_i} = \ker(T - \lambda_i I)$ $\quad$ λ_i - eigenspace of T

Then the following are equivalent:-

(1) T is diagonalizable

(2) $p_T(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$ where $r_i = \dim V_{\lambda_i}$ for $i = 1, \dots, k$

(3) $\sum_{i=1}^{k} \dim(V_{\lambda_i}) = \dim(V)$

<u>Proof</u>: (1) $\Rightarrow$ (2)

T is diagonalizable $\Rightarrow$ $[T]_\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$

$$p_T(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}, \quad r_i = \dim(V_{\lambda_i})$$

(2) $\Rightarrow$ (3) $\quad \deg p_T(x) = \dim V$

$$\Downarrow$$

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} \dim(V_{\lambda_i}) = \dim V$$

(3) $\Rightarrow$ (1) $\quad w_i \in V_{\lambda_i} \quad i = 1, \dots, k$

<u>Claim</u>: $w_1 + \dots + w_k = 0 \Rightarrow w_1 = w_2 = \dots = w_k = 0$

f polynomial

$$\begin{aligned} 0 &= f(T)(0) \\ &= f(T)(w_1 + \dots + w_k) \\ &= f(T)w_1 + \dots + f(T)w_k \\ &= f(\lambda_1)w_1 + \dots + f(\lambda_k)w_k \\ &= \sum_{j=1}^{k} f(\lambda_j) w_j \end{aligned}$$

<u>Choose</u> $\rightarrow$ $f_i(\lambda_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$0 = f_i(T)(0) = \sum_{j=1}^{k} f_i(\lambda_j) w_j = \sum_{j=1}^{k} \delta_{ij} w_j = w_i$

Choose β_i a basis of V_{λ_i}. Let $W = V_{\lambda_1} + \dots + V_{\lambda_k}$

Claim: $\beta = \{\beta_1, \dots, \beta_k\}$ is a basis of W

$w \in W \Rightarrow w = w_1 + \dots + w_k$ with $w_i \in V_{\lambda_i}$

$W \subseteq V$

$\dim W = |\beta_1| + \dots + |\beta_k| = \dim V_{\lambda_1} + \dots + \dim V_{\lambda_k} = \dim V$

$\Rightarrow W = V \Rightarrow \beta = \{\beta_1, \dots, \beta_k\}$ is a basis of V.

$$[T]_\beta = \begin{bmatrix} \lambda_1 & & & & & & 0 \\ & \lambda_1 & & & & & \\ & & \lambda_2 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_k & \\ 0 & & & & & & \lambda_k \end{bmatrix}$$

Cor: $\dim_k(V) = n$. $T: V \to V$ is a linear operator on V which has 'n' distinct eigenvalues. Then T is diagonalizable

Pf: $\lambda_1, \dots, \lambda_n$ are distinct eigenvalues of T

$\dim(V_{\lambda_i}) \geq 1 \Rightarrow \sum_{i=1}^{n} \dim(V_{\lambda_i}) \geq n = \dim(V) \Rightarrow \sum_{i=1}^{n} \dim(V_{\lambda_i}) = \dim(V)$

$\Downarrow$ T is diag

Eg. $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ Determine if A is diagonalizable and if yes, find P and D

$(5-\lambda)$

$\begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{bmatrix}$

$\lambda^2 - 3\lambda + 2$

$(\lambda-1)(\lambda-2)$

$6 - 12 + 3\lambda$

$(5-\lambda)[(\lambda-4)(+4+\lambda)+12] + 6[4+\lambda-6] - 6[6-3(4-\lambda)]$

$(5-\lambda)[\lambda^2 - 16 + 12] + 6[\lambda - 2] - 6[3\lambda - 6]$

$= (5-\lambda)[\lambda^2 - 4] + 6(\lambda-2) - 6(3)(\lambda-2)$

$(\lambda-2)[(5-\lambda)(\lambda+2) + 6 - 18] = (\lambda-2)[5\lambda - \lambda^2 + 10 - 2\lambda - 12]$

$= (\lambda-2)(-\lambda^2 + 3\lambda - 2) = (\lambda-2)^2(\lambda-1)$

$\dim(N(A-2I)) = 2$

Nullity$(A-2I) = 2 \Rightarrow \dim V_2 = 2$; 1 is an eigenvalue $\Rightarrow \dim V_1 \geq 1$

$\Rightarrow \dim(V_1) + \dim(V_2) \geq 3 = \dim(V) \Rightarrow$ equality $\Rightarrow T$ is diagonalizable

$\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \end{bmatrix} \Rightarrow$ $3v_1 - 6v_2 - 6v_3 = 2v_1$; $-v_1 + 2v_2 + 2v_3 = 2v_2 \Rightarrow 2v_3 = v_1$; $3v_1 - 6v_2 - 6v_3 = 2v_3$; $v_1 = v_3$

Defn: $T: V \to V$ is a linear operator. $W \subseteq V$, W is invariant subspace of T if $TW \subseteq W$

Rmk: f poly, $TW \subseteq W \Rightarrow f(T)W \subseteq W$

Eg: $V = \mathbb{R}^3$, $T: \mathbb{R}^3 \to \mathbb{R}^3$, matrix of T w.r.t std basis $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$W = \langle e_1, e_2 \rangle$, $Te_1 = e_1 + e_2$, $Te_2 = e_2 - e_1 \Rightarrow TW \subseteq W \Leftarrow$

Def^n: $T: V \to V$ is a linear op. $W \subseteq V$

W is invariant subspace of T if $TW \subseteq W$

Remark: f polynomial, $TW \subseteq W \Rightarrow f(T) W \subseteq W$

(mean any power of x is also invariant)

$V = \mathbb{R}^2$; $T: \mathbb{R}^2 \to \mathbb{R}^3$; matrix of T w.r.t std. basis

$$A = \begin{bmatrix} \boxed{\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}} & 2 \\ & 1 \\ 0 \quad 0 & -1 \end{bmatrix}$$ invariant subspace

if $W = \langle e_1, e_2 \rangle$

$$\left. \begin{aligned} Te_1 &= e_1 + e_2 \\ Te_2 &= e_2 - e_1 \end{aligned} \right\} \text{ So } TW \subseteq W$$

Eg: $\beta = \begin{bmatrix} & & \\ 0 & & \end{bmatrix} \Rightarrow A = \begin{bmatrix} 5 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

$W_0 = \{0\}$
$W_1 = \langle e_1 \rangle$
$W_2 = \langle e_1, e_2 \rangle$
$W_3 = \langle e_1, e_2, e_3 \rangle = \mathbb{R}^3$

→ Every linear operator has atleast 2 invariant subspaces, $\{0\}$ and V itself
as $T0 = 0$ & TV takes elements in V to V.

→ If a matrix is upper Δ^{le}

$$TV_1 = a_{11} V_1$$
$$TV_2 = a_{12} V_1 + a_{22} V_2$$
$$TV_k = a_{1k} V_1 + a_{2k} V_2 + \cdots + a_{kk} V_k$$
$$TV_n = a_{1n} V_1 + a_{2n} V_n + \cdots + a_{kn} V_n$$

Proposition: V f.d.v.s$/F$. $T: V \to V$ a linear op.
$W \neq V$, a T invariant subspace of V. Suppose the minimal polynomial of T

$$P(T) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_k)^{e_k}$$

Then there exists a $v \in V$ such that $v \notin W$
$\exists$ some λ_i, such that $(T - \lambda_i I) v \in W$

Def: $\dim_F(V) < \infty$, T lin. op on V
We say V is triangulable if $\exists$ a basis B of V such that $[T]_\beta$ is upper triangular.

Thm: V f.d. v.s/F, T lin. op on V
T is triangulable $\Leftrightarrow$ minimal poly of T factors as product of linear terms

Pf: $[T]_\beta = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ 0 & & \ddots & \\ & & & a_{nn} \end{bmatrix}$

$p_T(x) = (x - a_{11})(x - a_{22}) \cdots (x - a_{nn})$

$p(x) \mid p_T(x) \Rightarrow p(x)$ is product of linear factors

$\Leftarrow$ Let $p(x) = (x-\lambda_1)^{e_1} \cdots (x-\lambda_k)^{e_k}$
Apply prop. with $w=0$ $\exists\, 0 \neq v_1$, $Tv_1 = \lambda_1 v_1$ for some λ_i
call $W_1 = \langle v_1 \rangle$

Apply prop with $W = W_1$
$\exists\, v_2 \notin W_1$, s.t $(T - \lambda_{i_2} I)v_2 \in W$
$\Downarrow$
$Tv_2 = c_1 v_1 + \lambda_{i_2} v_2$

$W_2 = \langle v_1, v_2 \rangle$ W_2 is invariant under T.

Apply prop with $W = W_2$ $\exists\, w_3 \notin W_2$
$Tv_3 = a_1 v_1 + a_2 v_2 + \lambda_{i_3} v_3$
$W_3 = \langle v_1, v_2, v_3 \rangle$

Continue the process with $W_n = V$, $\langle v_1, v_2, \ldots, v_n \rangle$
$Tv_1 = \lambda_{i_1} v_1$
$Tv_2 = a_{12} v_1 + \lambda_{i_2} v_2$
$\vdots$
$T v_n = a_{1n} v_1 + a.$

$\Rightarrow [T]_B = \begin{bmatrix} \lambda_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & & \lambda_n \end{bmatrix}$

Thm: $\dim_F(V) < \infty$, $T: V \to V$ lin op.
T is diagonalizable over $F \Leftrightarrow$ minimal poly. of T is a product of distinct linear factors

Eg: $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ min poly is x^2, which has repeated roots.
$\Rightarrow$ Not diagonalizable

Proof: ($\Rightarrow$) T is diagonalizable)
$\exists$ a basis β of V, s.t $Tv_i = \lambda_i v_i$ for $v_i \in \beta$
Assume V has $\{\lambda_1, \dots, \lambda_k\}$ distinct eigenvalues
$v_i \in \beta$
One of $(T-\lambda_1)v_i$, $(T-\lambda_2 I)v_i, \dots (T-\lambda_k I)v_i$ is zero.
$\Rightarrow (T-\lambda_1 I)(T-\lambda_2 I) \dots (T-\lambda_k I)v_i = 0$ for all i
$\Rightarrow (T-\lambda_1 I)\dots(T-\lambda_k I)v = 0 \quad \forall v \in V$
If $f(x) = (x-\lambda_1)\dots(x-\lambda_k)$ then $f(T) = 0$
Minimal polynomial $p(x)$ of T divides $f(x)$, hence have distinct roots

($\Leftarrow$) $p(x) = (x-\lambda_1)\dots(x-\lambda_k)$ λ_i's are distinct
W = subspace of V spanned by the eigenvalue of T.
We need to prove $W = V$
Assume $W \subsetneq V$ (Note W is invariant under T)
$\exists v \in V$, $v \notin W$ s.t $(T-\lambda_j I)v \in W$ for some λ_j
$w \in W$
$w = w_1 + \dots w_k$ with $Tw_i = \lambda_i w_i$
Any poly $h(x)$
$h(T)w = h(\lambda_1)w_1 + \dots + h(\lambda_k)w_k \in W$
with
$p(x) = (x-\lambda_j)q(x)$ & $q(x_j) \neq 0$

$q(x) - q(\lambda_j) = (x - \lambda_j)\, q(x)$

$q(T)v - q(\lambda_j)v = h(T)\,(T - \lambda_j I)v = h(T)w \in W$

$p(T)v = (T - \lambda_i I)\, q(T)v$

$\Rightarrow q(T)v$ is an eigenvector of T with eigenvalue λ

$\Rightarrow q(T)v \in V$

$\Rightarrow q(\lambda_j)v \in W \Rightarrow q(\lambda_j) = 0$

Inner Product Space

($F = \mathbb{R}$ or $\mathbb{C}$)

Def: V is a v.s/F. An inner product on V a function $\langle\ ,\ \rangle : V \times V \to F$ such that $(a,b) \mapsto \langle a, b\rangle$.

(1) $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$

(2) $\langle \alpha v, w\rangle = \alpha \langle v, w\rangle$

(3) $\langle w, v\rangle = \overline{\langle v, w\rangle}$ [bar is complex conjugate)

(4) $\langle v, v\rangle \geq 0$ & $\langle v, v\rangle = 0$ iff $v = 0$

Def: An inner product space is a F vectorspace with a specified inner product.

[Linear in first coordi conjugate in secon

$F = \mathbb{R}$, $\dim(V) < \infty$, inner product space is called Euclidean space.

Complex innerproductspace is called unitary spa

Ex: $\langle v, \beta w\rangle = \bar{\beta}\langle v, w\rangle$

$\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$

Eg ③ More generally, $V = F^n$ ($F = \mathbb{R}$ or $\mathbb{C}$)

A is an invertible $(n\times n)$ matrix over F

$$A^* = \overline{A^t} = \overline{A}^t \qquad \langle A, B\rangle = tr(AB^*)$$

(consider F^n as $n\times 1$ column vectors)

$$\langle v, w\rangle = w^* A^* A v$$

this defines an inner product on V.

Eg ④ $C[a,b] = \{$continuous functions $f : [a,b] \to \mathbb{C}\}$

$$\langle f, g\rangle = \int_a^b f(t)\,\overline{g(t)}\,dt$$

is an inner product on $C[a,b]$

Eg: $V = F^{n\times 1}$, M is an $(n\times n)$ matrix over F

Such that: ① $M^* = M$

② $v^* M v > 0$ if $v \neq 0$

$\langle v, w\rangle = w^* M v$ is an inner product on $F^{n\times 1}$

[property ② $\Rightarrow$ M is invertible]

Conversely: $V = F^n$ with an inner product $\langle v, w\rangle$

Choose an ordered basis $\beta = \{v_1, \dots, v_n\}$ on V.

Define: $M_{ij} = \langle v_j, v_i\rangle = \overline{\langle v_i, v_j\rangle}$,

$M = (M_{ij})_{n\times n}$

(Check: $M_{ji} = \langle v_i, v_j\rangle = \overline{M_{ij}} \Rightarrow M = \overline{M}^t = M^*$)

$v = \sum_j x_j v_j$, $[v] = X$, $w = \sum_{i=1}^n y_i v_i$, $[w]_\beta = Y$

$$\langle v, w\rangle = \langle \sum_j x_j v_j, w\rangle$$
$$= \sum_j x_j \langle v_j, w\rangle = \sum_j x_j \langle w, v_j\rangle$$
$$= \sum_j x_j \langle \sum_i y_i v_i, v_j\rangle = \sum_j x_j \sum_i \bar{y}_i \langle v_i, v_j\rangle$$
$$= \sum_{j,i} \bar{y}_i M_{ij} X_j = Y^* M X$$

Since $\langle v, w\rangle$ is an inner product on V for $0 \neq v$,
$\langle v, v\rangle > 0$, $X^* M X > 0$

Conclusion: V is a f.d.v.s /F ($F = \mathbb{R}$ or $\mathbb{C}$), then any inner product on V is given by a $(n \times n)$ matrix M which satisfies (1) $M^* = M$ (2) $X^* M X > 0$ for all $X \neq 0$

Def: A matrix M is called Hermitian matrix if $M^* = M$
(A Hermitian matrix with real entries is a symmetric matrix)

Def: A Hermitian matrix M is called positive definite, if for every non-zero column vector X, we have $X^* M X > 0$
[Section 8.1 of Hoffman]

Def: V is an inner product space 04/10/19
$v \in V$, $\|v\| = \sqrt{\langle v, v\rangle}$ [Here $\sqrt{\ }$ is positive sq. rt.]

$$\langle \alpha v_1 + v_2, w\rangle = \alpha\langle v_1, w\rangle + \langle v_2, w\rangle$$
$$\langle v, w\rangle = \overline{\langle w, v\rangle} \Rightarrow \langle v, \alpha w_1 + w_2\rangle = \bar{\alpha}\langle v, w_1\rangle + \langle v, w_2\rangle$$

Thm: (1) $\|v\| \geq 0$ $\forall v$ & $\|v\| = 0$ iff $v = 0$

(2) $\|cv\| = |c| \, \|v\|$ $\forall v \in V, c \in F$

(3) $|\langle v, w\rangle| \leq \|v\| \|w\|$ [Cauchy-Schwartz inequality]

(4) $\|v + w\| \leq \|v\| + \|w\|$ [Triangle inequality]

Pf. ③ If $v=0$, inequality holds

~~Assume~~ $v \neq 0$, $w' = w - \frac{\langle w, v \rangle}{\|v\|^2} v$

$\langle w', v \rangle = 0$

$$0 \leq \|w'\|^2 = \langle w - \frac{\langle w, v \rangle}{\|v\|^2} v, w - \frac{\langle w, v \rangle}{\|v\|^2} v \rangle$$

$$= \langle w, w \rangle - \frac{\langle w, v \rangle \langle v, w \rangle}{\|v\|^2} \langle v, v \rangle$$

$$= \|w\|^2 - \frac{|\langle v, w \rangle|^2}{\|v\|^2}$$

$$\|w\|^2 - \frac{|\langle v, w \rangle|^2}{\|v\|^2} \geq 0 \Rightarrow |\langle v, w \rangle|^2 \leq \|v\|^2 \|w\|^2$$

→ Equality in C-S inequality holds if and only if $w = \lambda v$ for some $\lambda \in F$

Pf. ④ ⇒ $\|v+w\|^2 = \langle v+w, v+w \rangle = \langle v, v \rangle + \langle w + w \rangle + \langle v, w \rangle + \langle w, v \rangle$

$$= \|v\|^2 + \|w\|^2 + 2 \operatorname{Re} \langle v, w \rangle$$

$$\leq \|v\|^2 + \|w\|^2 + 2 |\langle v, w \rangle|$$

$$\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| = (\|w\| + \|v\|)^2$$

→ V inner product space, $v, w \in V$

$$0 \leq \frac{|\langle v, w \rangle|}{\|v\| \|w\|} \leq 1 \quad (\text{Assume } v \neq 0, w \neq 0)$$

Define angle b/w v and w as $\cos^{-1}\left(\frac{|\langle v, w \rangle|}{\|v\| \|w\|}\right)$

Def: We say two vectors v and w are orthogonal if $\langle v, w \rangle = 0$

Def: ~~Orthogonal and orthonormal set~~

$S \subseteq V$ is a set of vectors. We say

(1) S is orthogonal if $\langle x, y \rangle = 0$ for all $x \neq y$ in S

(2) S is orthonormal if S is orthogonal and $\|x\| = 1$ for all $x \in S$

Eg: ① Standard basis of $\mathbb{R}^n$ and $\mathbb{C}^n$ are orthonormal w.r.t std. inner product.

② $V = \mathbb{R}^2$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$\langle v, w \rangle = v_1 w_1 - v_2 w_1 - v_1 w_2 + 2 v_2 w_2$$

$$\langle v, w \rangle = 0 \Rightarrow v_1 w_1 - v_2 w_1 - v_1 w_2 + 2 v_2 w_2 = 0$$

$$w_1 (v_1 - v_2) + w_2 (2v_2 - v_1) = 0$$

$$w_1 = k(2v_2 - v_1) \ ; \ w_2 = -k(v_1 - v_2)$$

$$w_1 = 2v_2 - v_1 \qquad X = \begin{bmatrix} 2v_2 - v_1 \\ v_2 - v_1 \end{bmatrix}$$

$$w_2 = v_2 - v_1$$

Eg: ② $V = C[0,1]$, $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt$

$$f_n(x) = \sqrt{2} \cos(2\pi n x)$$

$$g_n(x) = \sqrt{2} \sin(2\pi n x)$$

Check: orthonormal set $\{1, f_1, g_1, f_2, g_2, \ldots\}$

$$h_n = \frac{1}{\sqrt{2}}(f_n + i g_n) = e^{2\pi i n x} \qquad \{1, h_1, h_2, \dots\}$$

[orthogonal set]

1/10/19

Theorem: An orthogonal set of non-zero vectors are linearly independent

Pf: Let S be a set of non-zero orthogonal vectors in V.
Let $v_1, \dots, v_m$ are distinct elements of S.
Suppose $w = c_1 v_1 + \dots + c_m v_m$

Want to show: $w = 0 \Rightarrow c_1 = c_2 = \dots = c_m = 0$

$$\langle w, v_k \rangle = \langle \sum_{i=1}^{n} c_i v_i, v_k \rangle = \sum_{i=1}^{m} c_i \langle v_i, v_k \rangle = c_k \|v_k\|^2$$

$$c_k = \frac{\langle w, v_k \rangle}{\|v_k\|^2}$$

$\Rightarrow w = 0 \Rightarrow c_k = 0 \ \forall K$

Corollary: Suppose $w = \mathrm{span}\{v_1, \dots, v_m\}$ where $\{v_1, \dots v_m\}$ is orthogonal set of non-zero vectors. Then

$$w = \sum_{K=1}^{m} \frac{\langle w, v_k \rangle}{\|v_k\|^2} \cdot v_k$$

$$\& \quad \|w\|^2 = \sum_{K=1}^{m} \frac{|\langle w, v_k \rangle|^2}{\|v_k\|^2}$$

Pf: $w = c_1 v_1 + \dots + c_m v_m$ with $c_k = \dfrac{\langle w, v_k \rangle}{\|v_k\|^2}$

$$\|w\|^2 = \langle w, w \rangle = \langle \sum_{i=1}^{m} c_i v_i, \sum_{j=1}^{m} c_j v_j \rangle$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} c_i \bar{c}_j \langle v_i, v_j \rangle = \sum_{K=1}^{m} |c_k|^2 \|v_k\|^2$$

Bessel's inequality

Cr: Let $S = \{v_1, \dots, v_m\}$ of nonzero orthogonal vectors in V. Then for any vector $v \in V$, we have

$$\sum_{k=1}^{m} \frac{|\langle v, v_k\rangle|^2}{\|v_k\|^2} \leq \|v\|^2$$

Moreover, the equality happens iff $v \in \operatorname{Span}(S)$

Pf: Let $w = \sum_{k=1}^{m} \frac{\langle v, v_k\rangle}{\|v_k\|^2} v_k \Rightarrow \|w\|^2 = \sum_{k=1}^{m} \frac{|\langle v, v_k\rangle|^2}{\|v_k\|^2}$

$\langle w, v_k\rangle = \langle v, v_k\rangle \Rightarrow \langle w - v, v_k\rangle = 0$

$\langle w - v, w\rangle = 0 \Rightarrow \langle v - w, w\rangle = 0$

$$\begin{aligned}
\|v\|^2 = \langle v, v\rangle &= \langle (v-w) + w, (v-w) + w\rangle \\
&= \langle v-w, v-w\rangle + \langle w, w\rangle + \langle v-w, w\rangle \\
&\quad + \langle w, v-w\rangle \\
&= \|w\|^2 + \|v-w\|^2 \\
&\geq \|w\|^2
\end{aligned}$$

Def: Let V be an inner product space. An orthonormal set in V which is also a basis of V is called an orthonormal basis of V.

$V \to$ inner product space 17/10/19

Thm: (Gram Schmidt orthogonalization)

V inner product space and $\{u_1, \dots, u_n\}$ are linearly independent vectors in V. Then there exists orthogonal vectors $\{v_1, \dots, v_n\}$ and orthonormal vectors $\{w_1, \dots, w_n\}$ in V st span $\{u_1, \dots, u_n\}$ = Span $\{v_1, \dots, v_n\}$ = Span $\{w_1, \dots, w_n\}$

Moreover, we can take

$$v_1 = u_1$$

$$v_{k+1} = u_{k+1} - \sum_{i=1}^{k} \frac{\langle u_{k+1}, v_i \rangle}{\langle v_i, v_i \rangle} v_i \quad , \quad w_{k+1} = \frac{v_{k+1}}{\|v_{k+1}\|} \quad \text{for } k=1, \dots, n-1$$

$$= u_{k+1} - \sum_{i=1}^{k} \langle u_{k+1}, w_i \rangle w_i$$

Pf: $\{u_1\}$ Span $\{u_1\}$ = Span $\{v_1\}$ = span $\{w_1\}$

Now $\{u_1, u_2\}$ lin indep.

$$u_2 = \alpha w_1 + v_2$$

We want to choose α s.t v_2 is orthogonal to w_1

$$0 = \langle v_2, w_1 \rangle = \langle u_2, w_1 \rangle - \alpha \langle w_1, w_1 \rangle \Rightarrow \alpha = \langle u_2, w_1 \rangle$$

$$v_2 = u_2 - \alpha w_1 = u_2 - \langle u_2, w_1 \rangle w_1$$

The theorem holds for $\{w_1, \dots, w_{m+1}\}$ for $m=1$

Assume theorem holds for $m=k$, now take $m=k+1$

Consider,

$$v_{k+2} = u_{k+2} - \sum_{i=1}^{k+1} \langle u_{k+2}, w_i \rangle w_i$$

$v_{k+2} \in$ span $\{u_{k+2}, w_1, \dots, w_{k+1}\}$ $\overset{\shortparallel}{}$ span $\{u_1, \dots, u_{k+1}, u_{k+2}\}$

$u_{k+2} \in$ span $\{v_{k+2}, w_1, \dots, w_{k+1}\}$ $\overset{\shortparallel}{}$ span $\{v_1, \dots, v_{k+2}\}$

$$\langle v_{k+2}, w_i \rangle \quad 1 \le i \le k+1$$

$$\shortparallel$$

$$\langle u_{k+2}, w_i \rangle - \sum_{j=1}^{k+1} \langle u_{k+2}, w_j \rangle \langle w_j, w_i \rangle = \langle u_{k+2}, w_i \rangle - \langle u_{k+2}, w_i \rangle = 0$$

Thm ①: If V is a finite dimensional inner product space, then V has an orthonormal basis.

Def: V inner product space, S subset of V
$x \in V$, x is orthogonal to S if $\langle x, y\rangle = 0$ for $y \in S$

$S^{\perp}$ = orthogonal to $S = \{x \in V \mid \langle x, y\rangle = 0 \ \forall y \in S\}$

Ex. $0^{\perp} = V$; $V^{\perp} = 0$

Def: V inner product space, $W \subseteq V$ (subspace). By orthogonal complment of W in V we mean a subspace $W' \subseteq V$ such that

① $W \cap W' = 0$ ② $W + W' = V$ ③ $W' = W^{\perp}$

In other words, W has an orthogonal complement if
$$V = W \oplus W^{\perp}$$

Cor②: V is a finite dimensional inner product space. Then any subspace W has an orthogonal complement.

Pf: $W = V \Rightarrow W^{\perp} = 0$ (statement holds)

Assume $\dim(W) = k < \dim(V) = n$

$\beta = \{w_1, \dots, w_k\}$ an orthonormal basis of W

$\Rightarrow \tilde{\beta}' = \{w_1, \dots, w_k, w_{k+1}, \dots, w_n\}$ of V

$\beta' = \{w_1, \dots, w_k, u_{k+1}, \dots, u_n\}$ a basis of V

Eg : $V = \mathbb{R}^3$, std inner product

$u_1 = (1,1,0)$

$u_2 = (0,1,1)$

$u_3 = (1,0,1)$

use Gram-Schmidt process on $\{u_1, u_2, u_3\}$ and calculate

$w_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$w_2 = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

$w_3 = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Eg: $V = \mathbb{P}(\mathbb{R}) = \{$poly in variable $x\}$

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)}\, dt$$

Hilbert space

LI set

$v_0 = 1$

$v_1 = x \quad \{1, x, \ldots, x^n\}$

$\vdots$

$v_n = x^n$ $\quad x^2$

$v_1 = x - \int_{-1}^{1} x\,dx =$

$v_2 =$

$v_3 = \int_{-1}^{1} x \cdot 1\, dt$

$V_2 =$

$\frac{x^2}{2}\Big|_{-1}^{1}$

$\frac{1}{2}(1-1) = 0$

$\int x^2 = \frac{x^3}{3}\Big|_{-1}^{1}$

$= \frac{1}{3} - \left(\frac{-1}{3}\right) = \frac{2}{3}$ $\quad \frac{x^3}{3} \quad \frac{1}{3}(1-$

$\int \frac{x\,dx}{3} \quad \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$ $\quad 1 \quad x \quad 1-(-1)$

$v_3 =$

$x^3 \cdot 1 \quad x^2$

$x^3 \cdot x$

$x^3\left(x^2 - \frac{1}{3}\right)x \cdot x$

x^2

$x^5 - \frac{1}{3}x^3 \quad \frac{x^3}{3}$

Apply Gram Schmidt process

$v_0 = p_0(x) = 1$

$v_1 = p_1(x) = x$

$u_0 = 1,\ u_1 = x,\ u_2 = x^2,\ u_3 = x^3$

$v_1 = u_1 = 1$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$= x - 0 = x$

$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$

$= x^2 - \frac{2}{3} \times \frac{1}{2} \times 1 = x^2 - \frac{1}{3}$

$v_4 = x^3 - 0 - \frac{2}{5} \times \frac{3}{2} \cdot x - 0$

$= x^3 - \frac{3}{5}x$

$\langle u_3, P_0\rangle = 0$, $\langle w_3, P_1\rangle = \frac{2}{5}$, $\langle u_3, P_2\rangle = 0$

$V_0 = P_0 = 1$; $V_1 = P_1 = x$; $V_2 = P_2 = x^2 - \frac{1}{3}$;

$V_3 = P_3 = x^3 - \frac{3}{5}x$

The polynomials $\underbrace{\{P_0, \dots, P_n\}}_{\text{forms an O.G basis of } \mathbb{P}_n(\mathbb{R})}$ are called Legendre polynomials

<u>Linear functionals and adjoints</u>

<u>Thm</u>: V fd IPS $f \in V^*$ then $\exists!\, y \in V$ s.t $f(v) = \langle v, y\rangle$ $\forall v \in V$
(Riesz representation)

<u>Pf</u>: Let $\{w_1, \dots, w_n\}$ be a basis of V (orthonormal)

Define $y = \sum \overline{f(w_j)}\, w_j$

check $\langle w_k, y\rangle = f(w_k)$

Hence proved.

<u>Thm</u>: T is lin op on a f.d. IPS V. Then $\exists!$ lin op T^* on V s.t $\langle Tv, w\rangle = \langle v, T^*w\rangle$ $\forall\, v, w \in V$

<u>Pf</u>: Consider $f_w : V \to \mathbb{F}$

$v \longrightarrow \langle Tv, w\rangle$

By Raiesz Representation, $\exists!\, y = v$ s.t

$\langle Tv, w\rangle = f_w(v) := \langle v, y\rangle$ We define $T^*(w) = y$

<u>Chk</u>. $\langle v, T^*(cw_1 + w_2)\rangle = \langle v, cT^*w_1 + T^*w_2\rangle$

Hence T^* is also a lin. operator

$= \bar{c}\langle v, T^*w_1\rangle + \langle v, T^*w_2\rangle$

$= \langle v, cT^*w_1\rangle + \langle v, T^*w_2\rangle$

$= \langle v, cT^*w_1 + T^*w_2\rangle$

Conclude: $T^*(cw_1 + w_2) = cT^*(w_1) + T^*(w_2)$

Hence proved that $\exists\, T^*$ s.t. $\langle Tv, w\rangle = \langle v, T^*w\rangle$

Thm: T is a lin op. on fd IPS V.

Let $\beta = \{w_1, \dots, w_n\}$ an O.N basis of V.

Let $A = [T]_\beta$ & $B = [T^*]_\beta$

Then $B = A^* = \bar{A}^T$

Pf: $$[T]_\beta = \begin{bmatrix} | & | & & | \\ [Tw_1]_\beta & [Tw_2]_\beta & \cdots & [Tw_n]_\beta \\ | & | & & | \end{bmatrix}$$

$$Tw_j = \langle Tw_j, w_1\rangle w_1 + \cdots + \langle Tw_j, w_n\rangle w_n$$

only true when we choose ON basis

$$[Tw_j]_\beta = \begin{bmatrix} \langle Tw_j, w_1\rangle \\ \vdots \\ \langle Tw_j, w_n\rangle \end{bmatrix}$$

$A_{ij} = \langle Tw_j, w_i\rangle$ where $A = (A_{ij})$

$B = (B_{ij}) = (\langle T^*w_j, w_i\rangle)$

$B_{ij} = \overline{\langle w_i, T^*w_j\rangle} = \overline{\langle Tw_i, w_j\rangle} = \bar{A}_{ji}$

Defn: T is lin op. on inner product space V (need not be fd)

We say T has an adjoint on V if $\exists$ a lnmap T^* on V s.t. $\langle Tv, w\rangle = \langle v, T^*w\rangle \;\forall\, v, w \in V$

[Note the diff b/w fdv & non-fd V if V is fd then T^* always exist if V is not fd then T^* may or may not exist if T^* exist then T has adj on V)

<u>Corollary</u>: V is fd IPS then T^* exists & is unique

<u>Def</u>: V is IPS. T is lin. op. on V

We say T is self adj if $\forall$ $v, w \in V$ we have

$$\langle Tv, w\rangle = \langle v, Tw\rangle$$

<u>Rmk</u>: if V is fd IPS then T is self adj. if $T = T^*$

Note
Matrix - only for fdVS V

<u>Corollary</u>: T is a self adj. op on a f.d. IPS V

$\beta = \{w_1, \dots, w_n\}$ ON basis of V

Then $[T]_\beta$ is Hermitian (symmetric if $\mathbb{F} = \mathbb{R}$)

i.e,

$[T]_\beta = A$, $[T^*]_\beta = A^*$

T is self-adj $\Rightarrow A = A^* \Leftrightarrow$ A is hermitian

next class we will try to prove hermitian & sym. matrices are diag. by unitary method.
$\hookrightarrow$ if $AA^* = I$

<u>Lemma</u>: T is a self adj. op on V

Then $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in V$

Pf:

$$\overline{\langle v, Tv \rangle} = \langle Tv, v \rangle = \langle v, Tv \rangle$$

$$\Rightarrow \langle Tv, v \rangle \in \mathbb{R}$$

→ Lemma: Eigenvalues of self adj. operators are real numbers.

Pf: Assume λ is an eigenvalue of T

$\Rightarrow \exists\ 0 \neq v \in V$, s.t $Tv = \lambda v$

$\langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \|v\|^2$

$\Rightarrow \lambda = \dfrac{\langle Tv, v \rangle}{\|v\|^2} \in \mathbb{R}$

Cor: All eigenvalues of a Hermitian matrices are real numbers.

→ Thm: T is a self adj. op on V.
The eigenvectors associated to distinct eigenvalues are orthogonal.

Pf: $0 \neq v \in V$ is an eigenvector of T with eigenvalue λ.
$0 \neq u \in V$ " " μ.

$\lambda \langle v, u \rangle = \langle \lambda v, u \rangle = \langle Tv, u \rangle$

$= \langle v, Tu \rangle = \langle v, \mu u \rangle = \mu \langle v, u \rangle$

$\Rightarrow (\lambda - \mu) \langle v, u \rangle = 0$

$\Rightarrow \boxed{\langle v, u \rangle = 0}$

Thm: V finite dimensional inner product space over $\mathbb{F}$
T is a self adj of V. Then T has an eigenvalue.

Pf: $F = \mathbb{C}$, we know the result
Assume $F = \mathbb{R}$, $T: V \to V$ self adj op.
$\dim(V) = n$, say $\beta = \{w_1, \dots, w_n\}$
orthonormal basis of V.

$A = (a_{ij}) \in M_n(\mathbb{R})$ matrix of T w.r.t β

$A: \mathbb{C}^n \to \mathbb{C}^n \Rightarrow A$ has an eigenvalue λ.

$\Rightarrow \quad x \in \mathbb{C}^n \mapsto Ax \Rightarrow \lambda \in \mathbb{R}$ (b/c A is symmetric)

$\Rightarrow 0 \neq \vec{x} \in \mathbb{C}^n$ s.t $Ax = \lambda x$

Write $\vec{x} = \vec{u} + i\vec{v}$ $\quad (\vec{u}, \vec{v} \in \mathbb{R}^n) \Rightarrow$ atleast one of $\vec{u}$ & $\vec{v}$ are non-zero

$$A(\vec{x}) = \lambda(\vec{x}) = \lambda(\vec{u} + i\vec{v}) = \lambda\vec{u} + i\lambda\vec{v}$$
$$= A\vec{u} + iA\vec{v}$$

$\Rightarrow A\vec{u} = \lambda\vec{u}$ & $A\vec{v} = \lambda\vec{v}$
atleast one of $\vec{u}, \vec{v}$ is non-zero

Assume $\vec{u} \neq 0$, $A\vec{u} = \lambda\vec{u}$ with $\vec{u} \neq 0$

$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$. check

$T(u_1 w_1 + \dots + u_n w_n)$
$= \lambda(u_1 w_1 + \dots + u_n w_n)$

Thm: $T: V \to V$ self adj op, $\dim(V) < \infty$

$V_0 \subseteq V$ such that $T(V_0) \subseteq V_0$

Then: $T(V_0^{\perp}) \subseteq V_0^{\perp}$

Recall: $V_0^{\perp} = \{ x \in V \mid \langle x, y \rangle = 0$ for all $y \in V_0 \}$

Moreover, $V + V_0^{\perp} = V$

Pf: $T(V_0) \subseteq V_0$

$x_0 \in V_0^{\perp}$, $y_0 \in V_0 \Rightarrow Ty_0 \in V_0$

Want to show: $Tx_0 \in V_0^{\perp}$ for all $y_0 \in V_0$

$\langle Tx_0, y_0 \rangle = \langle x_0, Ty_0 \rangle = 0$

Thm: $T: V \to V$ self adj op on a finite dimensional inner product space. Then there exists an eigenbasis of T consists of orthonormal vectors.

Pf: $V_0 = \ker(T - \lambda I) + \ldots + \ker(T - \lambda_k I)$

V_0 has an eigenbasis consisting of orthonormal vectors [$\ker(T - \lambda_i I)$ has an orthonormal basis by G-S for all i]

Let β_i is the orthonormal basis of $\ker(T - \lambda_i I)$

then $\beta = \beta_1 \cup \ldots \cup \beta_k$ is an orthonormal eigenbasis of V_0.

Claim: $V = V_0$

Suppose: $V_0 \neq V$, then $V_0^{\perp} \neq 0$

Let $v \in V_0$, write $v = v_1 + \cdots + v_k$ with $v_i \in Ker(T - \lambda_i I)$

$$Tv = \lambda_1 v_1 + \cdots + \lambda_k v_k \in V_0$$

$\Rightarrow T(V_0^{\perp}) \subseteq V_0^{\perp}$

Define: $T_1 : V_0^{\perp} \longrightarrow V_0^{\perp}$

$$x \longmapsto Tx$$

[In particular T_1 is self adj. (b/c T is self adj.)]

T_1 has an eigenvalue, say λ.

$\Rightarrow \lambda \in \{\lambda_1, \ldots, \lambda_k\}$

$\Rightarrow \exists!\ 0 \neq x \in V_0^{\perp}$ s.t $T_1 x = \lambda x$

$\Rightarrow x \in Ker(T - \lambda_i I)$ for some i

$\Rightarrow 0 \neq x \in V_0$

Contradiction b/c $V_0 \cap V_0^{\perp} = \{0\}$

Cor: Every Hermitian matrix is diagonalisable by a unitary matrix.

→ A matrix is called unitary if $AA^* = I$

Pf: $P = \begin{bmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_n \\ | & | & & | \end{bmatrix}$

where $\{w_1, \cdots, w_n\}$ is the eigenbasis of T consisting of orthonormal vectors

$$P^* P = I ; \quad A = P^{-1} D P$$

Recap

If A is hermitian of size $n \times n$

① Then all eigenvalues of A are real numbers.

② There exists a unitary matrix (orthogonal matrix) P and a diagonal matrix D such that

$$P^* A P = D$$

How to find P

Step ①: Compute all eigenvalues of A, say $\lambda_1, \dots, \lambda_k$

Step ②: For $1 \le i \le k$, find a basis β_i of $V_{\lambda_i} = \operatorname{Ker}(A - \lambda_i I)$. Use Gram-Schmidt to orthonormalize β_i to get an orthonormal basis

Step ③: $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$

P is the matrix whose columns are vectors from β.

Eg. ① $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Qn: Find an orthogonal matrix P and a diagonal matrix D s.t. $P^t A P = D$

$$\begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix} \Rightarrow \lambda[\lambda^2 - 1] + 1[-\lambda - 1] - 1[1 + \lambda]$$

$$= \lambda(\lambda^2 - 1) - 2(\lambda + 1)$$

$$= (\lambda + 1)(\lambda(\lambda - 1) - 2)$$

$$\lambda^2 - 2\lambda + \lambda - 2$$

$$= (\lambda + 1)(\lambda^2 - \lambda - 2) = (\lambda + 1)(\lambda - 2)(\lambda + 1)$$

$$= (\lambda + 1)^2 (\lambda - 2)$$

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

<u>QR decomposition of matrix</u>

<u>Gram-Schmidt process:</u> $\{u_1, \ldots, u_n\}$ linearly independent

$$v_1 = u_1, \quad w_1 = \frac{v_1}{\|v_1\|}$$

$$v_{k+1} = u_{k+1} - \sum_{i=1}^{k} \langle u_{k+1}, w_k \rangle w_k, \quad w_{k+1} = \frac{v_{k+1}}{\|v_{k+1}\|}$$

$$u_1 = \|u_1\| w_1 = \frac{\|u_1\|^2}{\|u_1\|} w_1 = \frac{\langle u_1, u_1 \rangle}{\|u_1\|} w_1$$

$$= \langle u_1, w_1 \rangle w_1$$

$$u_{k+1} = \sum_{i=1}^{k} \langle u_{k+1}, w_i \rangle w_i + v_{k+1}$$

$$= \sum_{i=1}^{k} \langle u_{k+1}, w_i \rangle w_i + \|v_{k+1}\| w_{k+1}$$

$$= \sum_{i=1}^{k} \langle u_{k+1}, w_i \rangle w_i + \langle v_{k+1}, w_{k+1} \rangle w_{k+1}$$

$$u_{k+1} = \sum_{i=1}^{k+1} \langle u_{k+1}, w_i \rangle w_i$$

$$A = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix} \begin{bmatrix} \langle u_1, w_1 \rangle & \langle u_2, w_1 \rangle & \langle u_n, w_1 \rangle \\ 0 & \langle u_2, w_2 \rangle & \langle u_n, w_2 \rangle \\ \vdots & \vdots & \vdots \\ 0 & 0 & \langle u_n, w_n \rangle \end{bmatrix}$$

<u>Special case:</u> V is a real inner product space of dimension n.

$\{u_1, \ldots, u_n\}$ a basis of V

$A = QR$ where Q is orthogonal
R is upper triangular

Special case: A real $(m\times n)$ matrix whose columns are linearly independent $(m \geq n)$

$$A_{m\times n} = Q_{m\times n} R_{n\times n}$$

where columns of Q are orthonormal

R is an upper triangular matrix $(Q^t Q = I)$

(R is invertible)

Eg: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find QR factorization of A

$$\cancel{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt2 & -1/\sqrt6 & 1/\sqrt3 \\ +1/\sqrt2 & -1/\sqrt6 & 1/\sqrt3 \\ 0 & 2/\sqrt6 & 1/\sqrt3 \end{bmatrix} \begin{bmatrix} \sqrt2 & 1/\sqrt2 & 0 \\ 0 & 3/\sqrt6 & 0 \\ 0 & 0 & \sqrt3 \end{bmatrix}}$$

$u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\sqrt{\left(\frac{2}{3}\right)^2 \times 3}$

$\frac{2}{3}\sqrt3$

$w_1 = \frac{1}{\sqrt2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\langle u_1, w_1\rangle = \sqrt2$

$$v_2 = u_2 - \underbrace{\langle u_2, w_1\rangle}_{=\,1/\sqrt2} w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt2}\,\frac{1}{\sqrt2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix} \Rightarrow w_2 = \frac{2}{\sqrt6}\begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 2/\sqrt6 \\ -1/\sqrt6 \\ 1/\sqrt6 \end{pmatrix}$$

$$v_3 = u_3 - \langle u_3, w_1\rangle w_1 - \langle u_3, w_2\rangle w_2$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt2}\begin{pmatrix} 0 \\ 1/\sqrt2 \\ 1/\sqrt2 \end{pmatrix} - \frac{1}{\sqrt6}\begin{pmatrix} 2/\sqrt6 \\ -1/\sqrt6 \\ 1/\sqrt6 \end{pmatrix}$$

$\frac{3}{6} + \frac{1}{6}$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 1/3 \\ -1/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$-\frac{3}{6} - \frac{1}{6}$

$$w_3 = \frac{1}{\sqrt3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$Q = \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}$$

$\frac{2}{\sqrt{6}} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \frac{3}{\sqrt{6}}$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$\frac{2}{2} + \frac{2}{6}$

$\frac{1}{2} - \frac{3}{6}$

$\frac{1}{2} + \frac{3}{6}$ $\quad \frac{1}{2} - \frac{1}{2} + \frac{4}{6}$

$\frac{1}{2} + \frac{3}{6}$

$\frac{3}{6} + \frac{1}{2} - \frac{2}{3}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

<u>Best approximation</u>

$p(x) = e^x$ on $[0,1]$

Say we want to approximate $f(x)$ by quadratic poly.

V = continuous functions (real valued) on $[0,1]$ $\quad e^x \in V$

$\mathbb{P}^2(\mathbb{R})$ = quadratic polynomial with real coefficients in one variable

V has a natural inner product $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$

$p(x) = a + bx + cx^2$ is best approx. of e^x on $[0,1]$

$\|e^x - p(x)\| \le \|e^x - h(x)\|$ for all $h(x) \in \mathbb{P}_2(\mathbb{R})$

Qns:

① Does best approximation exist?

② If it exists, is it unique?

③ If there is unique best approx., how do we find it?

Def: V is an inner product space.

V_0 is a subspace of V.

Let $v \in V$. We say $v_0 \in V_0$ is a best approximation of v inside V_0 if

$$\|v - v_0\| \leq \|v - w\| \text{ for all } w \in V_0$$

Prop: Let V be an inner product space

$V_0 \subseteq V$ and $v \in V$

If $v_0 \in V_0$, such that $(v - v_0)$ is orthogonal to V_0, then v_0 is the unique best approximation of v in V_0.

Conversely, if $v_0 \in V_0$ is the best approx of $v \in V$, then $v - v_0$ is orthogonal to V_0

Pf: $v_0 \in V_0$, s.t $\langle v - v_0, w \rangle = 0$ $\forall$ all $w \in V_0$

$$\|v - w\|^2 = \|(v - v_0) + (v_0 - w)\|^2$$

$$= \|v - v_0\|^2 + \|v_0 - w\|^2 \quad \left(\begin{array}{l} \text{b/c } v_0 - w \in V_0 \\ v - v_0 \text{ is orthogonal to } V_0 \end{array}\right)$$

$$\geq \|v - v_0\|^2$$

$$\Rightarrow \|v - w\| \geq \|v - v_0\|$$

$v_0 \in V_0$ is the best approx. of v in V_0.

$$\|v - v_0\| \leq \|v - w\| \quad \text{for all } w \in V_0$$

For all $\alpha \in F$, and $w \in V_0$

$$\|v - v_0\|^2 \leq \|v - (v_0 + \alpha w)\|^2$$

$$= \langle v - (v_0 + \alpha w), v - (v_0 + \alpha w) \rangle$$

$$\|v-v_0\|^2 \le \langle (v-v_0)-\alpha w,\ (v-v_0)-\alpha w\rangle$$
$$= \|v-v_0\|^2 + |\alpha|^2\|w\|^2 - 2\left(\mathrm{Re}\langle v-v_0, \alpha w\rangle\right)$$

Take: $\alpha = \dfrac{\langle v-v_0, w\rangle}{\|w\|^2}$

$$\langle v-v_0, \alpha w\rangle = \bar{\alpha}\langle v-v_0, w\rangle = \bar{\alpha}\alpha\|w\|^2$$
$$= |\alpha|^2\|w\|^2$$

For this choice of α,

$$\|v-v_0\|^2 \le \|v-(v_0+\alpha w)\|^2$$
$$= \|v-v_0\|^2 - |\alpha|^2\|w\|^2$$

$\Rightarrow \alpha = 0 \quad \Rightarrow \langle v-v_0, w\rangle = 0$ for all $w \in V_0$

Thm: V is an inner product space

V_0 is a finite dimensional subspace $v \in V$

Let $\{w_1, \dots, w_n\}$ an orthonormal basis of V_0.

Then $v_0 = \sum_{i=1}^{n} \langle v, w_i\rangle w_i$ is the unique best approximation of v in V_0

Pf: By previous proposition,

v_0 is the unique best approximation of v

if $\langle v-v_0, w_i\rangle = 0$ for $i=1,\dots,n$

$$\Updownarrow$$

$$\langle v, w_i\rangle = \langle v_0, w_i\rangle \text{ for } i=1,\dots,n$$

Check: Our v_0 has the property.

Eg: $V := \{f : [-1, 1] \to \mathbb{R}, \text{ continuous}\}$

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$$

Find the best approximation of e^x in $\mathbb{P}_2(\mathbb{R})$

Soln:

Recall Legendre polynomials

$\{1, t, t^2 - \frac{1}{3}\}$ are orthogonal basis of $\mathbb{P}_2(\mathbb{R})$

$$\underset{P_0}{1},\ \underset{P_1}{t},\ \underset{P_2}{t^2 - \tfrac{1}{3}}$$

The best approx is

$$f(x) = \left\langle e^x, \frac{P_0}{\|P_0\|} \right\rangle \frac{P_0}{\|P_0\|} + \left\langle e^x, \frac{P_1}{\|P_1\|} \right\rangle \frac{P_1}{\|P_1\|} + \left\langle e^x, \frac{P_2}{\|P_2\|} \right\rangle \frac{P_2}{\|P_2\|}$$

$$= \frac{\langle e^x, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0 + \frac{\langle e^x, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1 + \frac{\langle e^x, P_2 \rangle}{\langle P_2, P_2 \rangle} P_2$$

$\langle P_0, P_0 \rangle = \int_{-1}^{1} 1\, dt = 2$; $\langle P_1, P_1 \rangle = \int_{-1}^{1} t^2\, dt = \frac{2}{3}$

$\langle P_2, P_2 \rangle = \int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)^2 dt = 8/45$

$\langle e^x, P_0 \rangle = (e - e^{-1})$

$\langle e^x, P_1 \rangle = 2e^{-1}$

$\langle e^x, P_2 \rangle = \frac{2}{3}(e - 7e^{-1})$

Best approximate solution for system of linear eqn $AX = Y$ (Least square solution)

29/10/11

Def: V_1, V_2 are f.d.v.s's, V_2 inner product space
$T: V_1 \to V_2$ is a linear transformation.
Pick $y \in V_2$. Then a vector $x_0 \in V_1$ is called the best approximate solution for $Tx = y$, if
$$\|Tx_0 - y\| \leq \|Tx - y\| \quad \text{for } \forall\, x \in V_1$$

Note: x_0 is best approximate soln for $Tx = y$
$$\Updownarrow$$
Tx_0 is the best approximation of y from $\operatorname{im}(T)$

Thm: Let V_1, V_2 f.d.v.s. V_2 inner product
$T: V_1 \to V_2$ a linear transformation.
Then $Tx = y$ has a best approximate solution
Moreover, $x_0 \in V_1$ is a best approximate solution of $Tx = y$ iff $Tx_0 - y$ is orthogonal to $\operatorname{im}(T)$.
Finally, best approximate solution is unique if T is injective.

Cor: Let A be a $(m \times n)$ matrix with real entries.
Given $y \in \mathbb{R}^m$, the best approximate solution $x_0 \in \mathbb{R}^n$ of $Ax = y$ exists and is given by $A^t A x_0 = A^t y$
Moreover the solution is unique if the columns of A are linearly independent.

Proof:

$$\langle Ax_0 - y, u_i \rangle = 0$$
$$u_i^T (Ax_0 - y) = 0$$
$$A^T (Ax_0 - y) = 0$$
$$A^T A x_0 = A^T y$$

Recall: $A_{m\times n}$: matrix whose columns are linearly independent then $A = QR$, where Q is a matrix whose columns are orthonormal and R is an upper triangular matrix

[Note: $Q^t Q$ = identity matrix (I)] [Note: R is invertible]

Cor: Suppose A is an $(m\times n)$ matrix whose columns are linearly independent. Then for any $y \in \mathbb{R}^m$, best approximate solution $Ax = y$ exists and unique. Moreover it is given by $x_0 = R^{-1} Q^t y$, where $A = QR$

Eg: $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}$. Find the best approx. sol. for $y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$A_{3\times 2} = Q_{3\times 2} R_{2\times 2}$ $\quad 4+9+1$

$v_1 = u_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ $\quad w_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ $\quad -\frac{14}{14} + \frac{2}{14}$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \frac{(-1)}{14} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ $\quad 1 + \frac{1}{1}$

$= \frac{1}{14} \begin{pmatrix} -12 \\ 3 \\ 15 \end{pmatrix}$

$-24 + 9 + 15$

$\langle u_1, w_1 \rangle = \sqrt{14}$

$V_2 = u_2 - \langle u_2, w_1 \rangle w_1 = \frac{3}{14}\begin{bmatrix} -4 \\ 1 \\ 15 \end{bmatrix}$

$\langle u_2, w_1 \rangle = -\frac{1}{\sqrt{14}}$

$w_2 = \frac{1}{\sqrt{42}}\begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix}$

$\langle u_2, w_2 \rangle = \frac{9}{\sqrt{42}}$

$$Q = \begin{bmatrix} 2/\sqrt{14} & -4/\sqrt{42} \\ 3/\sqrt{14} & 1/\sqrt{42} \\ 1/\sqrt{14} & 5/\sqrt{42} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{14} & -1/\sqrt{14} \\ 0 & 9/\sqrt{42} \end{bmatrix}$$

Norms of vectors & matrices

→ Norm ($\| \ \|$) is a function from V to $\mathbb{R}$ such that

(1) $\|v\| > 0$ & $\|v\| = 0 \Leftrightarrow v = 0$

(2) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $\lambda \in F, v \in V$

(3) $\|v + w\| \le \|v\| + \|w\|$ for all $v, w \in V$

Eg: (1) V is a vector space with inner product $\langle \cdot, \cdot \rangle$

$\|v\| = \sqrt{\langle v, v \rangle}$ is a norm on V

(2) $V = \mathbb{R}^n$, $x = (x_1, \dots, x_n)$

$\|x\|_1 = |x_1| + |x_2| + |x_3| + \dots + |x_n|$

Check: $\|x\|_1$ is a norm $x = (x_1, \dots, x_n)$

(1) $\|x\|_1 \ge 0$

$\|x\|_1 = 0 \Leftrightarrow x_1 = x_2 = \dots = x_n = 0 \Leftrightarrow x = 0$

(2) $\|\lambda x\|_1 = |\lambda x_1| + |\lambda x_2| + |\lambda x_3| + \dots + |\lambda x_n|$

$= |\lambda| (|x_1| + \dots + |x_n|) = |\lambda| \, \|x\|_1$

③ $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$

$$\|x+y\|_1 = |x_1+y_1| + |x_2+y_2| + \cdots + |x_n+y_n|$$
$$\leq |x_1| + |y_1| + |x_2| + |y_2| \cdots + |x_n| + |y_n| = \|x\|_1 + \|y\|_1$$

Eg. ② $V = \mathbb{R}^n$ $\quad x = (x_1, \ldots, x_n)$

l^1 norm • $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

l^2 norm • $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$

⋮

l^p norm (for $p>1$) $\quad \|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$

l^∞ norm $\quad \|x\|_\infty = \max_{1\leq i\leq n} \{|x_i|\}$

For $x = (1, 1, \ldots, 1) \in \mathbb{R}^n$

$\|x\|_1 = n$, $\|x\|_2 = n^{1/2}$; $\ldots$, $\|x\|_p = n^{1/p}$, $\|x\|_\infty = 1$

Qn: V is a vector space with a given norm $\|v\|$

. Does there exist an inner product on V such that $\|v\| = \sqrt{\langle v, v\rangle}$?

Soln: V has a inner product $\langle x, y\rangle$

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$
$$= 2\,(\langle x,x\rangle + \langle y,y\rangle)$$
$$= 2\,(\|x\|^2 + \|y\|^2)$$

Take $x = (1,0)$ & $y = (0,2)$

$(\|x+y\|_1)^2 = (\|(1,2)\|_1)^2 = 3^2$

$(\|x-y\|_1)^2 = (\|(1,-2)\|_1)^2 = 3^2$

$\|x\|_1^2 = 1 \quad , \quad \|y_1\|^2 = 4$

$\Rightarrow \|x+y\|_1^2 + \|x-y\|_1^2 \neq 2(\|x\|_1^2 + \|y\|_1^2)$

$\Rightarrow \|\cdot\|_1$ does not come from any inner product on $\mathbb{R}^2$.

<u>Norms of matrices</u>

$A = (a_{ij})$ can define $\sum_i \sum_j |a_{ij}|$, $\max_{1 \le i,j \le n} |a_{ij}|$

A $(n \times n)$ matrix with real entries.

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$

<u>Define</u> $\|A\| := \sup_{\substack{x \\ \|x\|=1}} \|Ax\| = \sup_{\substack{x \\ x \neq 0}} \dfrac{\|Ax\|}{\|x\|}$

Sup $\equiv$ Supremum – the smallest number which is greater than all numbers in the set.

<u>Check:</u> $\|A\| = \sup_{\substack{x \\ \|x\|=1}} \|Ax\|$ is a norm

① $\|A\| \geq 0$ (by defn)

$A \neq 0 \Rightarrow \exists\, 0 \neq x_0$ s.t $Ax_0 \neq 0$ & $\|x_0\| = 1$

$\Rightarrow \|A\| = \sup_{\substack{x \\ \|x\|=1}} \|Ax\| \geq \|Ax_0\| > 0$

② $\|\lambda A\| = |\lambda| \, \|A\|$

$$\sup_{\substack{x \\ \|x\|=1}} \|\lambda A x\| = |\lambda| \sup_{\|x\|=1} \|Ax\| \quad \left(\sup_{\|x\|=1} |\lambda| \, \|Ax\| \right)$$

$$= |\lambda| \, \|A\|$$

③ $\|A+B\| \leq \|A\| + \|B\|$

$\|$

$$\sup_{\|x\|=1} \|(A+B)x\| = \sup_{\|x\|=1} \|Ax + Bx\|$$

$$\leq \sup_{\|x\|=1} \left(\|Ax\| + \|Bx\| \right)$$

$$\leq \left(\sup_{\|x\|=1} \|Ax\| \right) + \left(\sup_{\|x\|=1} \|Bx\| \right)$$

$$\leq \|A\| + \|B\|$$

Properties of matrix norm

① $\|I\| = 1$

Pf: $\|I\| = \sup_{\|x\|=1} \|Ix\| = \sup_{\|x\|=1} \|x\| = 1$

② $\|Ax\| \leq \|A\| \cdot \|x\|$ for $\forall \, x \in \mathbb{R}^n$

Case - 1 : $x = 0$, $LHS = RHS = 0$

Case -2 : $x_0 \neq 0$

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \geq \frac{\|Ax_0\|}{\|x_0\|}$$

③ $\|AB\| \leq \|A\| \, \|B\|$

Case-① : $A = 0 \Rightarrow LHS = RHS = 0$

Case-②: $A \neq 0$, $\|AB\| = \sup_{\|x\|=1} \|A(Bx)\| \leq \sup_{\|x\|=1} \left(\|A\| \cdot \|Bx\| \right)$

$$\leq \|A\| \left(\sup_{\|x\|=1} \|Bx\| \right) = \|A\| \cdot \|B\|$$

$\mathbb{R}^n$

$$\|(x_1,\dots,x_n)\|_p = \begin{cases} \sqrt[p]{|x_1|^p + \dots + |x_n|^p} & \text{for } 1 \le p \le \infty \\ \max\{|x_1|,\dots,|x_n|\} & p=\infty \end{cases}$$

$$\|A\|_p = \sup_{\|x\|_p = 1} \|Ax\|_p$$

Next time: We give formula for $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$

01/11/19

Thm: $A = (a_{ij})_{m\times n}$ matrix with real entries

$$\|A\|_1 = \max_{1\le j\le n} \left\{ \sum_{i=1}^n |a_{ij}| \right\}$$

$$\|A\|_\infty = \max_{1\le i\le n} \left\{ \sum_{j=1}^n |a_{ij}| \right\}$$

Pf: $\|Ax\|_1 = \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} x_j \right| \le \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|\,|x_j|$

$$= \sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}| \right) |x_j|$$

$$\le \max_j \left(\sum_i |a_{ij}| \right) \left(\sum_j |x_j| \right)$$

$$= \max_j \left(\sum_{i=1}^n |a_{ij}| \right) \|x\|_1$$

$\Rightarrow \|A\|_1 \le \max \{ \sum |a_{ij}| \}$ from prop & $\|Ax\|_1 \le k\|x\|_1$

$\|Ae_j\|_1 = \sum_{i=1}^n |a_{ij}| \le \|A\|_1$ (by defn) true for all j

$\Rightarrow \max_j \left\{ \sum_{i=1}^n |a_{ij}| \right\} \le \|A\|_1$

LHS $\le \|A\|_1 \le$ RHS

but LHS = RHS

$\Rightarrow \max_j \left\{ \sum_{i=1}^n |a_{ij}| \right\} = \|A\|_1$

Pf of $\|A\|_\infty$:

$$\|Ax\|_\infty \underset{(\text{defn})}{=} \max_i \left| \sum_j a_{ij} x_j \right| \le \max_i \left(\sum_{j=1}^n |a_{ij}|\, |x_j| \right)$$

$$\le \max_i \left(\left(\sum_{j=1}^n |a_{ij}| \right) \max |x_j| \right)$$

$$= \max_i \left(\sum |a_{ij}| \right) \|x\|_\infty$$

$$\|A\|_\infty \le \max_i \left(\sum_{j=1}^n |a_{ij}| \right)$$

Suppose $\max \left\{ \sum |a_{ij}| \right\}$ occurs at the row i_0.

$$\Rightarrow \sum_{i=1}^n |a_{i_0 j}| = \max_i \left\{ \sum_j |a_{ij}| \right\}$$

Define $x_0 = (\sigma_1, \dots, \sigma_n)$ where $\sigma_j = \begin{cases} \dfrac{|a_{i_0, j}|}{a_{i_0 j}} & \text{if } a_{i_0 j} \ne 0 \\ 0 & \text{if } a_{i_0 j} = 0 \end{cases}$

$= ($some order of $1, -1$ & $0)$

$\Rightarrow \|x_0\|_\infty = 1$

$$|a_{i_0 j}| = \sigma_j a_{i_0 j}$$

$$\max_i \left(\sum |a_{ij}| \right) = \sum_j |a_{i_0 j}| = \sum a_{i_0 j} \sigma_j$$

$$= (A x_0)_{i_0} \le \|A x_0\|_\infty \le \|A\|_\infty$$

$$\left(\text{Since } \|A x_0\|_\infty \le \|A\|_\infty \underbrace{\|x_0\|_\infty}_{1} \right)$$

Thm: ① $\|A\|_2 \leq \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$

② If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A^t A$

$$\|A\|_2 = \max_{1 \leq i \leq n} \sqrt{\lambda_i}$$

Pf. $\|Ax\|_2^2 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|^2$

$\leq \sum_{i=1}^{n} \left[\left(\sum_{j=1}^{n} |a_{ij}|^2 \right) \left(\sum_{j=1}^{n} |x_j|^2 \right) \right]$ (Cauchy-Schwartz)

$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \cdot \|x\|_2^2$

$\Rightarrow \|A\|_2 \leq \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2}$

② $B = A^t A$ is a real symmetric matrix

Eigenvalues of B are real & there exists an orthonormal basis of B say $u_1, u_2, \ldots, u_n$

$\lambda_j = \lambda_j \langle u_j, u_j \rangle = \langle \lambda_j u_j, u_j \rangle = \langle A^T A u_j, u_j \rangle$

$= \langle A u_j, A u_j \rangle = \|A u_j\|_2^2 \leq \|A\|_2^2$

$\Rightarrow \max_j \sqrt{\lambda_j} = \|A\|_2$

Now, to show that equality exists not just lesser than or equal

$$X = \sum_j \langle x, u_j \rangle u_j \quad , \quad \|X\|_2^2 = \sum_j |\langle x, u_j \rangle|^2$$

$$Bx = \sum_j \langle x, u_j \rangle B u_j = \sum_j \langle x, u_j \rangle \lambda_j u_j$$

$$\|Ax\|_2^2 = \langle Ax, Ax \rangle = \langle A^t A x, x \rangle = \langle Bx, x \rangle$$

$$= \sum_{i=1}^{n} |\langle x, u_i \rangle|^2 \lambda_i$$

$$\leq \left(\max_i \lambda_i\right) \sum_{i=1}^{n} |\langle x, u_i \rangle|^2$$

$$= \left(\max \lambda_i\right) \|x\|_2^2$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^t A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 6 & -1 \\ -1 & -1 & 1 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 4 + \sqrt{15} \\ \lambda_3 &= 4 - \sqrt{15} \end{aligned}$$

$$\Rightarrow \|A\|_2 = \sqrt{4 + \sqrt{15}}$$

Stability of solⁿ of sys of lin. eqⁿs

$Ax = b$; $A_{n \times n}$ = real matrix b is $(n \times 1)$ vector

Say $b + \Delta b$ is my i/p

Solⁿ is $X_0 + \Delta x_0$ (say actual soln is $Ax_0 + b$)

$$A(x_0 + \Delta x_0) = b + \Delta b$$

$$\Rightarrow A \Delta x_0 = \Delta b \Rightarrow \Delta x_0 = A^{-1} \Delta b$$

$$\Rightarrow \|\Delta x_0\| \leq \|A^{-1}\| \, \|\Delta b\| \quad -①$$

$$Ax_0 = b \Rightarrow \|b\| = \|Ax_0\| \leq \|A\| \, \|x_0\|$$

$$\Rightarrow \frac{1}{\|x_0\|} \leq \frac{\|A\|}{\|b\|} \quad -②$$

Multiply ① and ②

$$\frac{\|\Delta x_0\|}{\|x_0\|} \leq \|A\| \, \|A^{-1}\| \frac{\|\Delta b\|}{\|b\|}$$

Def^n

Condition number of a matrix A is given by

$$K(A) = \|A\|_2 \, \|A^{-1}\|_2$$

Def^n: We say a sys of lin. eq^n $Ax=b$ is numerically stable if $K(A)$ is small.

Note: $1 = \|I\| = \|A \cdot A^{-1}\| \leq \|A\| \cdot \|A^{-1}\| = K(A)$

So cond^n no is ≥ 1

So error in sol^n (x_0) will be greater than error in i/p (b)

Eg: where $K(A)$ is large

$$A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix} \qquad \text{Say } \epsilon = 0.01$$

$$\|A\|_2 = \sqrt{2+\epsilon^2+\sqrt{4+2\epsilon^2}}$$

$$\|A^{-1}\|_2 = \frac{1}{\epsilon^2}\sqrt{2+\epsilon^2+\sqrt{4+2\epsilon^2}}$$

$$K(A) = \frac{2+\epsilon^2+\sqrt{4+2\epsilon^2}}{\epsilon^2} \geq \frac{4}{\epsilon^2}$$

Even if we perturb ϵ by small value $K(A)$ ↑ a lot

If ϵ is large doesn't matter but as $\epsilon \to 0$, the matrix tends to $rk(A)=1$. That is the problem.

Solving system of linear eqns by iterations

Want to solve $Ax = b$. A is invertible. We start with some guess x_0

$$Q x_k = (Q-A) x_{k-1} + b$$

Pick an invertible matrix Q (Q^{-1} is easily computable)

Suppose

$$x_k \to x$$

$$Qx = (Q-A)x + b \iff Ax = b$$

→ Want $x_k \to x$ for no matter what x_0 is

→ Want the convergence to be fast

Jacobi method

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad Q = D = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & 0 \\ & 0 & \ddots & \\ & & & a_{nn} \end{pmatrix}$$

$$x_k = D^{-1}(D-A)x_{k-1} + D^{-1}b$$

Gauss-Siedel method

$$Q = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{pmatrix}$$

$$x_k = Q^{-1}(Q-A)x_{k-1} + Q^{-1}b$$

Eg: $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$, $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Jacobi method: $D = \begin{pmatrix} 2 & & 0 \\ & 3 & \\ 0 & & 2 \end{pmatrix}$

$$x_k = B x_{k-1} + b$$

$$B = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix} = (I - D^{-1}A)$$

$$b = \begin{pmatrix} 1/2 \\ 8/3 \\ -5/2 \end{pmatrix} = D^{-1} b \qquad x_{21} = [2, 3, -1] \text{ with precision } 10^{-4}$$

Gauss-Siedel method: $Q = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$

$$x_k = C x_{k-1} + p$$

$$C = I - Q^{-1}A = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/6 & 1/3 \\ 0 & 1/12 & 1/6 \end{pmatrix}$$

$$p = Q^{-1} b = \begin{pmatrix} 1/2 \\ 17/6 \\ -13/12 \end{pmatrix}$$

$X_9 = [2, 3, -1]$ precision 10^{-4}

Let e_k be the error after k iterations

$$e_k = \underset{\substack{\downarrow \\ \text{actual soln}}}{X} - X_k = X - Q^{-1}\left((Q-A)X_{k-1} + b\right)$$

$$= X - (I - Q^{-1}A) X_{k-1} - Q^{-1} b$$

$$= \underbrace{X - X_{k-1}}_{e_{k-1}} + Q^{-1}A X_{k-1} - Q^{-1}A x$$

$$= e_{k-1} - Q^{-1}A(e_{k-1})$$

$$= e_{k-1} (I - Q^{-1}A)$$

$$= (I - Q^{-1}A)^k e_0$$

$$\Rightarrow \|e_k\|_2 = \|(I - Q^{-1}A)^k e_0\|_2 \le \|(I - Q^{-1}A)^k\|_2 \cdot \|e_0\|_2$$

$$\|e_k\| \le \|(I - Q^{-1}A)^k\| \, \|e_0\|$$

Thm: The iterative method $x_k = (I - Q^{-1}A)x_{k-1} + Q^{-1}$
converges for all choice of x_0 if $\|I - Q^{-1}A\|_2 <$
Moreover the convergence is faster if $\|I - Q^{-1}A\|$
is small

Cor: $B = I - Q^{-1}A = b_{ij}$

$$\|B\|_2^2 \leq \sum_j \sum_i |b_{ij}|^2$$

So if $\sum_i \sum_j |b_{ij}|^2 < 1$ then method converges

Eg: $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Jacobi method: $B = I - D^{-1}A = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix}$

$$\|B\|_2^2 \leq \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 < 1$$

$\rho_B = \frac{1}{\sqrt{3}}$

Gauss-Siedel method:

$$C = I - Q^{-1}A = \begin{pmatrix} 0 & 11/20 & 0 \\ 0 & 1/6 & 1/3 \\ 0 & 1/12 & 1/6 \end{pmatrix}$$

$$\|C\|_2^2 \leq \left(\frac{11}{20}\right)^2 + \left(\frac{1}{6}\right)^2 2 + \left(\frac{1}{12}\right)^2 + \left(\frac{1}{3}\right)^2 < 1$$

$\|B\|_2 = \frac{1}{\sqrt{2}} \approx 0.707$

$\|C\|_2 = \frac{1}{60}\sqrt{857 + \sqrt{17 \times 31 \times 1137}} \approx 0.6731$

Spectral radius (ρ)

$B_{n\times n}$ complex eigenvalues are $\lambda_1, \dots, \lambda_n$

$$\rho(B) = \max\{|\lambda_1|, \dots, |\lambda_n|\}$$

Thm ①: $\rho(B) \le \|B\|$ (for [redacted] any matrix)

Thm ②: $\rho(B) < 1 \iff \lim_{k\to\infty} B^k = 0$

Let $B = I - Q^{-1}A$

$e_k = B^k e_0$

$\Rightarrow \lim e_k = 0 \iff \lim_{k\to\infty} B^k = 0 \iff \rho(B) < 1$

Thm: $x_k = (I - Q^{-1}A)x_{k-1} + Q^{-1}b$ converges for any x_0 iff $\rho(B) < 1$

If $\rho(B)$ is smaller, the x_k converges faster

$(+) = f$

$\|A\|_2 = \max \sqrt{\lambda_i}$

λ_j

~~$B = I - Q^{-1}A$~~

1×2

$V^* \to R$

$P_T(x) = x^2(x-1)$

$p(x) = x(x-1)$

Norms of vectors

Conditions for norms:

(1) $\|x\| = 0 \Rightarrow x = 0$

(2) $\|x+y\| \leq \|x\| + \|y\|$

(3) $\|\alpha x\| = |\alpha| \, \|x\|$

$\Rightarrow \|Ax\| \leq \|A\| \, \|x\|$

$\|AB\| \leq \|A\| \, \|B\|$

$\|I\| = 1$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty = \max_{1 \leq i \leq n} \{ |x_i| \}$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$\|x\|_n = \sqrt[n]{\sum_{i=1}^{n} |x_i|^n}$$

Norms of matrices

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \quad \left[\text{Max. of Column sum}\right]$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \quad \left[\text{Max of ROW sum}\right]$$

$$\|A\|_2 \leq \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

→ If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are (non-negative) eigenvalues of matrix $A^T A$, then

$$\|A\|_2 = \max_{1 \leq j \leq n} \sqrt{\lambda_j}$$

$\rightarrow \dfrac{\|\Delta x_0\|}{\|x_0\|} \le \|A\| \, \|A^{-1}\| \, \dfrac{\|\Delta b\|}{\|b\|}$

Condition number $K(A) = \|A\| \, \|A^{-1}\|$

- We say a sys. of lin eq^n $Ax=b$ is numerically stable if $K(A)$ is small.

Iterative methods for solving $Ax=b$

Jacobi method

A should be invertible.

$Q x^k = b - (A-Q) x^{k-1}$

Q = diagonal matrix

$A_1 x^{(k)} = b - A_2 x^{(k-1)}$

If $\|A_1^{-1} A_2\| < 1$, then $x^{(k)}$ converges to a unique solution $x = A_1^{-1} b - A_1^{-1} A_2 x$ which is $Ax=b$

Gauss-Siedel method

$A_1 x = b - A_2 x$

A_1 = lower triangular part of A including diagonal

$A_2 = A - A_1$

Convergence condition $\Rightarrow \|A_1^{-1} A_2\| < 1$

$\rightarrow \{w_1, \ldots, w_n\}$ is orthonormal basis of V_0

Then $v_0 = \sum_{i=1}^{n} \langle v, w_i \rangle w_i$ is the unique best approximation of v in V_0.