ASSIGNMENT

CH5115 :

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CH18B118

(1) $v[k] = A\cos^2(2\pi f k + \phi)$

ϕ is constant.

A is a RV with $\mu_A = 0$, $\sigma_A^2 = 1$.

$v[k_1] = A\cos^2(2\pi f k_1 + \phi)$

$E(v[k_1]) = \int A\cos^2(2\pi f k_1 + \phi)\, dA = \cos^2(2\pi f k_1 + \phi)\mu_A$
$= 0$

$E(v[k_2]) = 0$

$\sigma_{vv}[k_1, k_2] = E\left((A\cos^2(2\pi f k_1 + \phi))(A\cos^2(2\pi f k_2 + \phi))\right)$
$= E(A^2\cos^2(2\pi f k_1 + \phi)\cos^2(2\pi f k_2 + \phi))$

$= E(A^2)\left[\cos^2(2\pi f k_1 + \phi)\cos^2(2\pi f k_2 + \phi)\right]$

$$\boxed{\begin{aligned}\sigma_A^2 = 1 &= E(A^2) - (E(A))^2 \\ &= E(A^2)\end{aligned}}$$
$[E(A) = 0]$

$= \cos^2(2\pi f k_1 + \phi)\cos^2(2\pi f k_2 + \phi)$

$\sigma_{vv}[k_1, k_2] = \frac{1}{4}\left[\cos(2\pi f(k_1 + k_2) + 2\phi) + \cos(2\pi f(k_1 - k_2))\right]^2$

Let $l = k_1 - k_2$

$= 1/4\left[\cos(2\pi f(k_1 + k_2) + 2\phi) + \cos(2\pi f l)\right]^2$

It depends on k value also. Hence is NOT covariance stationary.

(b) $v[k] = v[k-1] + e[k]$

$v[k-1] = v[k-2] + e[k-1]$

$\vdots$

$v[1] = v[0] + e[1]$

$v[K] = \sum_{i=1}^{K} e[i] + v[0]$ where $e[i]$ is WN with, say,

$\mu = 0$, σ_e^2.

$v[0] = 0$.

$\cancel{\Rightarrow E(v[K]) = 0}$

$\cancel{var(v[K]) = K\sigma_e^2}$ ~~(i.i.d's)~~

~~dependent on K. Also, note that as $K \to \infty$, variance explodes.~~

$$E(v[K]) = E\left(\sum_{i=1}^{K} e[i]\right)$$

$$= \sum_{i=1}^{K} E(e[i])$$

$$= 0$$

$$var[v[K]] = var(\sum e[i])$$

$$= var(e[0]) + var(e[1]) + \dots + var(e[K])$$

(since $e[K]$'s are i.i.d's)

$$= \sigma_e^2 + \sigma_e^2 + \dots + \sigma_e^2$$

$$= K\sigma_e^2$$

Notice the dependency on K. As $K \to \infty$, variance explodes to infinity.

Hence the Random walk process is variance non stationary.

(2) $y[k] = \dfrac{b_2 q^{-2}}{1+f_1 q^{-1}} u[k] + e[k]$

$\Rightarrow y[k] = -f_1 y[k-1] + b_2 u[k-2] + e[k] + f_1 e[k-1] \rightarrow (1)$

$\sigma_{yy}(l) = E(y[k]\, y[k-l])$ $\quad\because E(y[k]) = E(y[k-l]) = 0$

$= E\{(-f_1 y[k-1] + b_2 u[k-2] + e[k] + f_1 e[k-1])$
$(y[k-l])\}$

$$\boxed{\sigma_{yy}(l) = -f_1 \sigma_{yy}(l-1) + b_2 \sigma_{uy}(l-2) + \sigma_{ey}(l) + f_1 \sigma_{ey}(l-2) \rightarrow (*)}$$

$\sigma_{ye}(l) = -f_1 \sigma_{ye}(l-1) + \sigma_{ee}(l) + f_1 \sigma_{ee}(l-1) \rightarrow (A)$

$\sigma_{ye}(1) = -f_1 \sigma_{ye}(0) + f_1 \sigma^2_e$

$\sigma_{ye}(0) = -f_1 \sigma_{ye}(-1) + \sigma^2_e$

$= f_1 \sigma_{ye}(1) + \sigma^2_e$

On solving :- $\sigma_{ye}(0) = \sigma^2_e$

$\sigma_{ye}(1) = 0$

$= \sigma_{ye}(l) = 0 \quad |l| > 0$

$\sigma_{yu}[l] = E(y[k] u[k-l])$

$= -f_1 \sigma_{yu}(l-1) + b_2 \sigma_{uu}(l-2) + \sigma_{eu}(l)$

$+ f_1 \sigma_{eu}(l-1)$

$$\Rightarrow \boxed{\sigma_{yu}[l] = -f_1 \sigma_{yu}(l-1) + b_2 \sigma_{uu}(l-2)}$$

$(\sigma_{eu}(l) = 0)$

From (1), $\sigma_{yv}(0)$ is covariance btw $y[k]$ & $v[k]$ and should be 0.

$\sigma_{yv}(1)$ should also be 0

$\Rightarrow 0 = -f_1 \sigma_{yv}(-1) + b_2 \sigma_{vv}(-2)$

$\Rightarrow \sigma_{yv}(-1) = -\sigma_{yv}(1) = 0$

$\sigma_{yv}(2) = -f_1 \cancel{\sigma_{yv}(1)}^{0} + b_2 \sigma^2_v$

$$\boxed{\Rightarrow \sigma_{yv}(2) = b_2 \sigma^2_v}$$

$\sigma_{yv}(3) = -f_1 \sigma_y(2) + 0$

$= -f_1 b_2 \sigma^2_v$

$\sigma_{yv}(4) = f_1^2 b_2 \sigma^2_v$

$\sigma_{yv}(5) = -f_1^3 b_2 \sigma^2_v$

$\Rightarrow \sigma_{yv}(0) = 0$

$\boxed{\sigma_{yv}(1) = 0}$

$\sigma_{yv}(l) = (-f_1)^{l-2} b_2 \sigma^2_v. \qquad |l| > 1$

From (*)

$\sigma_{yy}(0) = \sigma^2_y = -f_1 \sigma_{yy}(1) + b_2 \sigma_{vy}(-2) + \sigma_{ey}(0) + f_1 \cancel{\sigma_{ey}(-1)}^{0}$

$\sigma_{yy}(0) = -f_1 \sigma_{yy}(1) + b_2^2 \sigma^2_v + \sigma^2_e \rightarrow (2)$

$\sigma_{yy}(1) = -f_1 \sigma_{yy}(0) + b_2 \cancel{\sigma_{vy}(-1)}^{0} + \cancel{\sigma_{ey}(1)}^{0} + f_1 \sigma_{ey}(0)$

$\sigma_{yy}(1) = -f_1 \sigma_{yy}(0) + f_1 \sigma^2_e \rightarrow (3)$

Put (3) in (2) :- $\sigma_{yy}(0) = f_1^2 \sigma_{yy}(0) - f_1^2 \sigma^2_e + b_2^2 \sigma^2_v + \sigma^2_e$

$$\Rightarrow \sigma_{yy}(0) = \frac{\sigma^2_e (1 - f_1^2) + b_2^2 \sigma^2_v}{1 - f_1^2} = \boxed{\sigma^2_e + \frac{b_2^2 \sigma^2_v}{1 - f_1^2}}$$

$$\sigma_{yy}(1) = -f_1\left(\sigma_e^2 + \frac{b_2^2\sigma_v^2}{1-f_1^2}\right) + f_1\sigma_e^2$$

$$\boxed{\sigma_{yy}(1) = \frac{-f_1 b_2^2 \sigma_v^2}{1-f_1^2}}$$

The values obtained (MATLAB) are consistent with theoretical results.

$$\boxed{\sigma_y^2 = \sigma_{yy}(0) = \sigma_e^2 + \frac{b_2^2\sigma_v^2}{1-f_1^2}}$$

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(4) (a) $y[k] \sim N(\mu, \sigma^2)$; $0 \le \mu < \infty$

↓ GWN.

$$L(\mu; y) = \ln f(y[k]|\mu) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\frac{(y[k]-\mu)^2}{\sigma^2}$$

$$s(\mu; y[k]) = \frac{\partial}{\partial\theta}L(\theta; y[k]) = \frac{y[k]-\mu}{\sigma^2}, \quad (\theta = \mu)$$

For N-observations.

$$L(\mu; y_N) = \ln f(y_N|\mu) = \ln \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y[k]-\mu)^2}{2\sigma^2}\right)$$

$$= c - \frac{N}{2}\ln\sigma - \frac{1}{2}\sum_{k=1}^{N}\frac{(y[k]-\mu)^2}{\sigma^2}$$

$$\Rightarrow s(\mu; y_N) = \frac{\partial}{\partial\mu}L(\mu; y_N) = \sum_{k=1}^{N}\frac{(y[k]-\mu)}{\sigma^2}$$

$$\Rightarrow \boxed{\hat{\mu}_{ML} = \frac{1}{N}\sum_{k=1}^{N} y[k]}$$

$$FI := -E\left(\frac{\partial S}{\partial \theta}\right) = -E\left(\sum_{K=1}^{N} \frac{y[K]-\mu}{\sigma^2}\right) = -E\left(\frac{\partial^2 L}{\partial \theta^2}\right)$$

$$\Rightarrow \frac{\partial L}{\partial \theta} = \sum_{K=1}^{N} \frac{y[K]-\mu}{\sigma^2}$$

$$\Rightarrow \frac{\partial^2 L}{\partial \theta^2} = \frac{-N}{\sigma^2} \Rightarrow \boxed{-E\left(\frac{\partial^2 L}{\partial \theta^2}\right) = \frac{N}{\sigma^2}}$$

(b) $Y = aX + b + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2)$

$$l = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y[k]-b-ax[k])^2}$$

$$\Rightarrow L = \sum_{i=1}^{N}\left(\frac{-\log(2\pi\sigma^2)}{2} - \frac{1}{2\sigma^2}(y[k]-b-ax[k])^2\right)$$

$$\Rightarrow \frac{\partial L}{\partial a} = -\sum_{i=1}^{N} \frac{1}{\sigma^2}\left[y[k]-b-ax[k]\right]x[k] = 0$$

$$\Rightarrow \sum_{j=1}^{N} y[k]x[k] - b\sum_{j=1}^{N} x[k] - a\sum_{j=1}^{N} x[k]^2 = 0 \rightarrow (*)$$

$$\frac{\partial L}{\partial b} = -\sum_{j=1}^{N} \frac{1}{\sigma^2}(y[k]-b-ax[k]).1 = 0$$

$$= \sum_{j=1}^{N} y[k] - Nb - a\sum_{j=1}^{N} x[k] = 0 \rightarrow (\#)$$

Solving (*) & (#) for a and b:

$$\boxed{a = \frac{\sum_{j=1}^{N} y[k]x[k] - \frac{1}{N}\left(\sum_{j=1}^{N} y[k] \sum_{j=1}^{N} x[k]\right)}{\sum_{j=1}^{N} x[k]^2 - \frac{1}{N}\left(\sum_{j=1}^{N} x[k]\right)}}$$

$$\boxed{b = \frac{1}{N}\left[\sum_{j=1}^{N} y[k] - a\sum_{j=1}^{N} x[k]\right]}$$

$$\frac{\partial^2 L}{\partial \theta_1^2} = \frac{\partial^2 L}{\partial a^2} = \frac{-\sum_{k=1}^{N} (x[k])^2}{\sigma_e^2}$$

$$\frac{\partial^2 L}{\partial \theta_2^2} = \frac{\partial^2 L}{\partial b^2} = \frac{-N}{\sigma_e^2}$$

$$\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 L}{\partial a \partial b} = \frac{\partial^2 L}{\partial b \partial a} = \frac{-\sum_{k=1}^{N} x[k]}{\sigma_e^2}$$

$$I(\theta) = -E\left(\begin{bmatrix} \frac{\partial^2 L}{\partial a^2} & \frac{\partial^2 L}{\partial a \partial b} \\ \frac{\partial^2 L}{\partial b \partial a} & \frac{\partial^2 L}{\partial b^2} \end{bmatrix}\right)$$

$$= E\left(\begin{bmatrix} \sum_{k=1}^{N} \frac{(x[k])^2}{\sigma_e^2} & \sum_{k=1}^{N} \frac{x[k]}{\sigma_e^2} \\ \sum_{k=1}^{N} \frac{x[k]}{\sigma_e^2} & \frac{N}{\sigma_e^2} \end{bmatrix}\right)$$

There are no randomness in x,

$$\Rightarrow \boxed{I(\theta) = \begin{bmatrix} \sum_{k=1}^{N} \frac{(x[k])^2}{\sigma_e^2} & \sum_{k=1}^{N} \frac{x[k]}{\sigma_e^2} \\ \sum_{k=1}^{N} \frac{x[k]}{\sigma_e^2} & \frac{N}{\sigma_e^2} \end{bmatrix}}$$

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(3) (a) ACF shows slow decay. Hence, there is a integrating effect present.

(b) On differencing the data once and plotting the ACF, we would see sluggishness is removed. $\boxed{d=1}$

The ACF and PACF plot (of differenced data) shows that it is either MA(9) (or) AR(3). We choose AR(3) due to ease.

- For ARIMA, guess no. of parameters, say 3.

d = 1.

(P, d, M) could be (1, 1, 1) or (2, 1, 0) or (0, 1, 2)

These models result in underfit when tested with Ljung Box test.

- guess, no. of parameters = 4.

(P, d, M) could be (2, 1, 1), (1, 1, 2), (3, 1, 2), (0, 1, 3)

(1, 1, 2) and (0, 1, 3) results in underfit. ARIMA (2, 1, 1) and ARIMA (3, 1, 0) gives a proper fit.
The Akaike information criterion (AIC) for ARIMA (2, 1, 1) returns a smaller value than ARIMA (2, 1, 0)

Hence ARIMA (2, 1, 1) best fits.