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Statistical Inference

Inductive Inference | Deductive Inference

Inductive Inference

Eg:- * 10 million seeds population

white   red

→ take a sample of 10 seeds sample

white = $\frac{5}{10}$ red = $\frac{5}{10}$

conclusion : 5 million seeds are white and 5 million seeds are red.

Deductive Inference

Eg:- One of the interior ~~triangle A equals to 90°~~ angles of each right triangle equals 90°.

* Triangle A is a right triangle

conclusion :
One of the ~~equ~~ angles of triangle A equals to 90°

~~Has~~ Random sample = Independent and identically distributed sample

Methods of Estimation

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \leftarrow \text{estimator of } \mu \text{ (a function)}$$

$$= \bar{x}_n \leftarrow \text{estimate}$$

① Methods of moments

$$\mu_r' = M_r' \qquad \mu_r' = E[X^r]$$

μ_r' ↑ population moment M_r' ↑ sample moment

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$$M_r' = \frac{1}{n}\sum_{i=1}^{n} x_i^r$$

point estimate

$$\mu_r' = \cancel{M_r} \; M_r'$$

$$\mu_r' = \frac{1}{n}\sum_{i=1}^{n} x_i^r$$

Example 1

* $f(X;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(1/2\sigma^2)(x-\mu)^2}$; $x = 0, 1, 2..$

Let $(\theta_1, \theta_2) = (\mu, \sigma)$

1st population moment

$$\mu_1' = E[X^1] = \cancel{\mu}\, E[X] = \mu$$

1st sample moment

$$M_1' = \frac{1}{n}\sum_{i=1}^{n} x_i^1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

By method of moments

$$M_1' = \mu_1'$$

$$\bar{x} = \mu$$

$\mu = \bar{X}$ (estimator of μ is $\bar{x}$)

$\mu = \bar{x}$ (estimate of μ is $\bar{x}$)

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2nd population moment

$\mu_2' = E[X^2]$

$\cancel{\mu_2' = M_2' = \frac{1}{n}\sum_{i=1}^{n} x_i^2}$

$\mu_2' = (\sigma^2 + \mu^2)$

2nd sample moment

$M_2' = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

By method of moments

$M_2' = \mu_2'$

$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \mu^2 + \sigma^2$

$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \bar{x}^2 + \sigma^2$

$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$

$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$

$Var(X) = E[X^2] - (E[X])^2$

$\sigma^2 = E[X^2] - \mu^2$

$E[X^2] = (\sigma^2 + \mu^2)$

$\frac{1}{n}\sum x_i^2 - \bar{x}^2 =$

$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$

$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n^2}\sum_{i=1}^{n} x_i^2$

$\frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2 =$

$= \frac{1}{n}\sum_{i=1}^{n} (x_i^2 - 2x_i\bar{x} + \bar{x}^2)$

$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{2\bar{x}}{n}\sum_{i=1}^{n} x_i + \frac{1}{n} \times \sum_{i=1}^{n} \bar{x}^2$

$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}.\bar{x} + \frac{1}{n} n\bar{x}^2$

$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}^2 + x^2$

$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2\right] - \bar{x}^2$

use capital letters for the variable

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2} \quad \text{(estimator of } \sigma\text{)}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2} \quad \text{(estimate of } \sigma\text{)}$$

use simple letters for the variable

Example 2

$$f(x;\lambda) = \frac{\lambda^x . e^{-\lambda}}{x!} \quad ; x = 0,1,2\ldots$$

1st population moment

$$\mu_r' = E[X^r]$$

$$\mu_1' = E[X] = \cancel{\lambda}\ \cancel{\lambda}\ \lambda$$

1st sample moment

$$M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i^1$$

$$M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

By method of moment

$$M_1' = \mu_1'$$

$$\bar{X} = \lambda$$

$$\lambda = \bar{X} \quad \text{(estimator of } \lambda\text{)}$$

$$\lambda = \bar{x} \quad \text{(estimate of } \lambda\text{)}$$

Example 3

* $f(x) = \alpha^2 x^{\alpha-2}$; $0 < x < 1$, $\alpha > 0$

1^{st} population moment

$\mu_r' = E[X^r]$

$\mu_1' = E[X^1] = E[X]$

$\mu_1' = \alpha$

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx$$

$$= \int_0^1 x \cdot \alpha^2 x^{\alpha-2}\, dx$$

$$\cancel{= \alpha^2 \int_0^1 x^{2\alpha-4}\, dx}$$

$$\cancel{= \alpha^2 \frac{x^{2\alpha-3}}{}}$$

$$= \int_0^1 \alpha^2 \cdot x^{\alpha-1}\, dx$$

$$= \alpha^2 \int_0^1 x^{\alpha-1}\, dx$$

$$= \alpha^2 \left[\frac{x^\alpha}{\alpha}\right]_0^1$$

$$= \alpha^2 [1-0] = \alpha$$

1^{st} sample moment

$M_r' = \frac{1}{n}\sum_{i=1}^{n} x_i^r$

$M_1' = \frac{1}{n}\sum_{i=1}^{n} x_i^1$

$M_1' = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X}$

By method of moment

$M_r' = \mu_r'$

$M_1' = \mu_1'$

$\bar{X} = \alpha$

$\alpha = \bar{X}$ (estimator of α) //

Negative Exponential Distribution (Mean = θ)

Example 4

$$f(X;\theta) = \cancel{\theta e^{-\theta x} I_{(0,\infty)}(x)} \quad \theta e^{-\theta x} \Big]_{(0,\infty)} (x)$$

$$E[X] = \frac{1}{\theta}$$

1st population moment

$$\mu_1' = E[X^1] = E[X] = \frac{1}{\theta}$$

1st sample moment

$$M_1' = \frac{1}{n}\sum_{i=1}^{n} x_i^1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X}$$

By the method of moments

$$M_1' = \mu_1'$$

$$\frac{1}{\theta} = \bar{X}$$

$$\theta = \frac{1}{\bar{X}} \quad \text{(estimator of } \theta\text{)}$$

$$\theta = \frac{1}{\bar{x}} \quad \text{(estimate of } \theta\text{)}$$

* $f(x, p) = \binom{n}{x} p^x q^{n-x}$

x	0	1	2	3
$f(x, \frac{3}{4})$	$1/64$	$9/64$	$27/64$	$27/64$
$f(x, \frac{1}{4})$	$27/64$	$27/64$	$9/64$	$1/64$

$$\hat{p} = \hat{p}(x) = \begin{cases} 0.25 & \text{for } x = 0, 1 \\ 0.75 & \text{for } x = 2, 3 \end{cases}$$

* $x = 6$ and $n = 25$

$$f(6, p) = \binom{25}{6} p^6 (1-p)^{19}$$

$$\frac{d(f(6,p))}{dp} = \binom{25}{6} p^5 (1-p)^{18} [6(1-p) - 19p] = 0$$

$$p = 0, 1, \frac{6}{25}$$

~~$f(x_1, x_2, x_3 \ldots$~~ x_n

Joint density of n random variable

$$f_{x_1 \ldots x_n}(x_1, x_2, x_3, \ldots, x_n; \theta) = f(x_1; \theta) . f(x_2; \theta) \ldots f(x_n; \theta)$$

1st parameter

$$L(\theta; x_1 \ldots x_n) \,/\, L(\cdot\,; x_1, x_2, \ldots x_n)$$

$$L(\theta; x_1, \ldots \ldots, x_n) = L(\theta) = f(x_1, \theta)\, f(x_2, \theta) \ldots f(x_n, \theta)$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta)$$

$$\frac{\partial L(\theta)}{\partial \theta} = 0$$

Example :

$f(x;p) = p^x q^{1-x}$

$L(p) = (p^{x_1} \cdot q^{1-x_1})(p^{x_2} \cdot q^{1-x_2}) \ldots (p^{x_n} q^{1-x_n})$

$= \prod_{i=1}^{n} (p^{x_i} \cdot q^{1-x_i})$

$= p^{\Sigma x_i} \cdot q^{n-\Sigma x_i}$

$y = \Sigma x_i \Rightarrow$

$L(p) = p^y \cdot q^{n-y}$

$\log L(p) = \log p^y + \log q^{n-y}$

$\log L(p) = y \log p + (n-y) \log q$

$\frac{\partial [\log L(p)]}{\partial p} = \frac{y}{p} + \frac{n-y}{q}(-1)$

$\frac{\partial [\log L(p)]}{\partial p} = \frac{y}{p} + \frac{y-n}{1-p}$

$0 = \frac{y}{p} + \frac{y-n}{1-p}$

$0 = y(1-p) + p(y-n)$

$0 = \underbrace{y - yp + yp - pn}$

$\hat{p} = \frac{y}{n}$

Since $y = \Sigma x_i$,

$\hat{p} = \frac{\Sigma x_i}{n} = \bar{x}$ //

estimate

Example 2

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\left(\frac{1}{2\sigma^2}\right)(x-\mu)^2}$$

$$L(\mu) = \left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{(-1/2\sigma^2)(x_1-\mu)^2}\right] \cdot \left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{(-1/2\sigma^2)(x_2-\mu)^2}\right] \cdots$$

$$\left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{(-1/2\sigma^2)(x_n-\mu)^2}\right]$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{(-1/2\sigma^2)(x_i-\mu)^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{(-1/2\sigma^2)(\Sigma x_i-\mu)^2}$$

$$y = \Sigma x_i$$

$$\cancel{\log L(\mu)} = \log\left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(1/2\sigma^2)(y-\mu)^2}\right]$$

$$\cancel{L(\mu,\sigma^2;x)} = \log\frac{1}{\sqrt{2\pi\sigma^2}} + \log e^{(-1/2\sigma^2)} + \log e^{(y-\mu)^2}$$

$$L(\mu,\sigma^2;x) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{\cancel{n}} \cdot \exp\left[\frac{-1}{2\sigma^2}\cdot(x_i-\mu)^2\right]$$

$$L(\mu,\sigma^2;x) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}\,\Sigma(x_i-\mu)^2\right]$$

$$\log L(\mu,\sigma^2;x) = \frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\,\Sigma(x_i-\mu)^2$$

estimate of μ

$$\frac{\partial[\log L(\mu,\sigma^2;x)]}{\cancel{\log}\,\partial\mu} = \frac{1}{\sigma^2}\,\Sigma(x_i-\mu)$$

Here, log = ln

estimate of σ^2

$$\frac{\partial[\overset{\ln}{\log}\ L(\mu,\sigma^2;x)]}{\partial\sigma^2} = -\frac{n}{2}\times\frac{1}{\sigma^2} + \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

* $$\frac{\partial[\overset{\ln}{\log}\ L(\mu,\sigma^2,x)]}{\partial\mu} = 0$$

$$\frac{1}{\sigma^2}\sum(x_i-\mu) = 0$$

$$\hat{\mu} = \frac{1}{n}\sum x_i = \bar{x}$$ //

* $$\frac{\partial[\log L(\mu,\sigma^2;x)]}{\partial\sigma^2} = 0$$

$$-\frac{n}{2}\times\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i-\mu)^2 = 0$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\bar{x})^2$$ //

Interval Estimation

$$P[T_1 < \tau(\theta) < T_2] = \gamma$$

lower confidence limit (T_1)

upper confidence limit (T_2)

confidence co-efficient (γ)

Example 1

$X_i \sim N(\theta, \sigma^2)$

1.) $E(\bar{x}) = \theta$ —— $\left[E\left[\frac{\sum x_i}{n}\right] = \frac{1}{n} E\left[\sum x_i\right] = \frac{1}{n} \sum E[x_i] = \frac{1}{n} \times \sum \theta = \frac{1}{n} \times n\theta = \theta \right]$

$$\text{Var}[\bar{x}] = \text{Var}\left[\frac{\sum x_i}{n}\right]$$

$$= \frac{1}{n^2} \text{Var}\left[\sum x_i\right]$$

$$= \frac{1}{n^2} \sum \text{Var}[x_i]$$

$$= \frac{1}{n^2} \sum \sigma^2$$

$$= \frac{1}{n^2} \times n\sigma^2 = \frac{\sigma^2}{n}$$

2.) $E[\bar{x} - \theta] = E[\bar{x}] - E[\theta]$

$$= \theta - \theta$$

$$= 0$$

θ [illegible].