Exact solution, quantum liquid, quantum criticality and hydrodynamics

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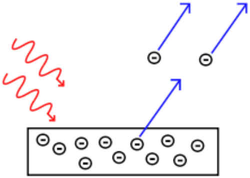
- **Elementary introduction to Exact solution, quantum statistics, thermodynamics**
- **Understanding fundamental many-body phenomena: Quantum liquid, quantum criticality, hydrodynamics, spin-charge separation**

My lectures

1. **Lieb-Liniger model: Exact solution and thermodynamics**
2. **Lieb-Liniger model: Quantum criticality and generalized hydrodynamics**
3. **Yang-Gaudin model: Spin charge separation and critical phenomenon**

M. Planck & A. Einstein

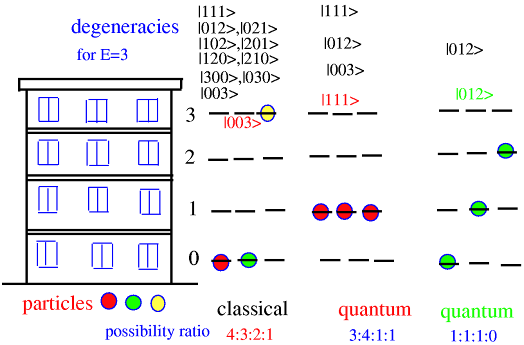


the photoelectric effect

Simple concepts may present a deep understanding of Nature: k, h, c

- Time and space: from Aristotle, to Newton's classical mechanics, Einstein's relativistic mechanics, quantum mechanics, evolution mechanics, and to relativistic mechanics
- Boltzmann's statistics : $S = k \log W$
- Nuclear energy: $E = m\, c^2$
- Max Planck's constant: $E = h\nu$
- de Broglie wave: $\lambda_d = h/mv$
- Emergent phenomena: superfluidity, superconductivity, fractional charge, polaron, magnon, spion, exciton, Skyrmion, Majorana fermion, $\cdots$

Ludwig Boltzmann (1844 – 1906) Austrian physicist was born in Vienna in 1944



- **For a given distribution, the complexion(degeneracy) reads:**

$$P = \frac{n!}{n_0!n_1!\cdots n_p!}$$

- For $n_1 = 3$, $P = \frac{3!}{3!} = 1$
  For $n_0 = 1, n_1 = 1, n_2 = 1$, $P = \frac{3!}{1!1!1!} = 6$
  For $n_0 = 2, n_3 = 1$, $P = \frac{3!}{2!1!} = 3$

- **Boltzmann's great perception(Using Stirling's formula $n! = n^n e^{-n} U(n)$):**
  $-\log P = n_0 \log n_0 + n_1 \log n_1 + n_2 \log n_2 + \cdots n_p \log n_p - n \log n + r_n$
- Applied Langrange's maximum method, i.e. define

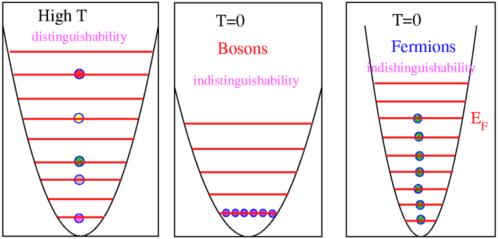
$$\begin{aligned} A &= n_0 + n_1 + n_2 \cdots + n_p \\ B &= n_1 + 2n_2 + \cdots pn_p \\ Y &\equiv -\log p + \alpha A + \beta B \end{aligned}$$

- let $dY/dn_k = 0 \longrightarrow 1 + \log n_k + \alpha + \beta k\epsilon = 0$
- the distribution $n_k = e^{-\alpha - 1 - \beta k\epsilon} = e^{-\alpha - 1 - \beta E_k}$
- the most possible probability(partition function $Q = \sum_k e^{-\beta E_k}$):

$$p_k = \frac{e^{-\beta E_k}}{\sum_k e^{-E_k/T}}, \qquad E = \frac{1}{Q} \sum_j E_j e^{-\beta E_j}, \qquad \beta = \frac{1}{k_B T}$$

- The most possible probability becomes the true distribution when n tends to infinity
- $\log P$ equals (up to a normalization) the entropy in equilibrium: $S = k \log W$
- The entropy reaches a maximum value in a equilibrium state that gives time's arrow (irreversibility), i.e. the second thermodynamic law, the law of Nature.
- Boltzmann equation: dynamic evolution equation

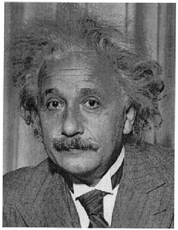
Quantum Statistics



quantum statistics

Bose-Einstein condensation *1924

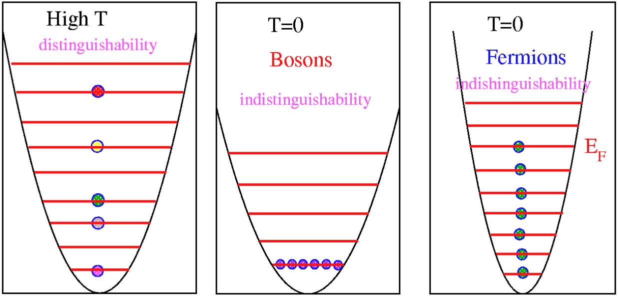
Satyendra Nath Bose



Albert Einstein

- **Particles either obey Bose-Einstein statistics or Fermi-Dirac statistics.**

  Strikingly different features of quantum statistical effects are revealed as matter is cooled down to a few nano-Kelvin above absolute zero—*bosons* with integer spin can form novel states of matter such as superconductors and superfluids, whereas *fermions* with half odd integer spin are not allowed to occupy a single quantum state due to the Pauli exclusion principle.

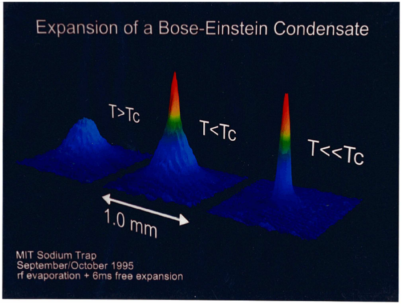


- Maxwell-Boltzmann distribution:
  $< n_j > \propto 1/\left(\exp\left(\left(\epsilon_j - \mu\right)/T\right)\right)$
- Bose-Einstein distribution:
  $< n_j >= \left(\exp\left[(\epsilon_j - \mu)/T\right] - 1\right)^{-1}$
- Fermi-Dirac distribution:
  $< n_j >= \left(\exp\left[(\epsilon_j - \mu)/T\right] + 1\right)^{-1}$

Quantum statistics:

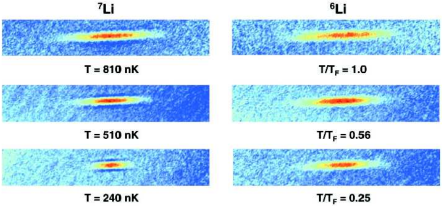
1. quantum man-body systems
2. microscopic state energy E_i
3. partition function $Z = \sum_{i=1}^{\infty} W_i e^{-E_i/(k_B T)}$
4. free energy $F = -k_B T \ln Z$

Bose-Einstein Condensation

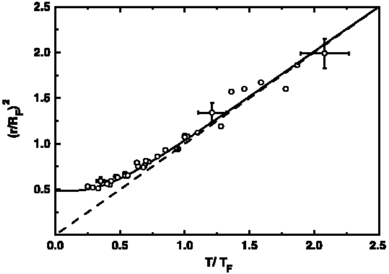


BEC of **alkali** atoms (Nobel Laureates: Cornell & Wieman, Ketterle, laser cooling & magnetic trapping). A sharp peak in the velocity distribution was observed below the critical temperature. This provided a clear signature of BEC.

Fermi Pressure



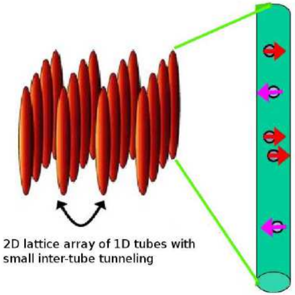
Size of Clouds

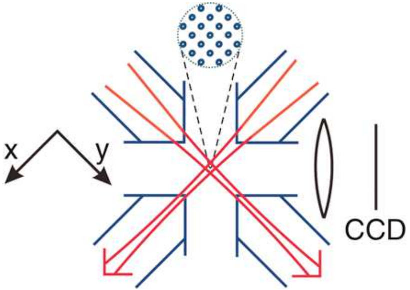


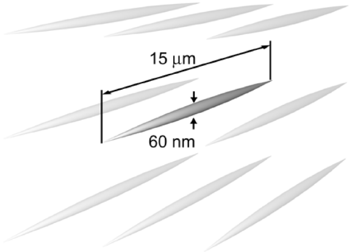
Fermi Pressure

The Fermi gas has been cooled to a temperature less than the Fermi temperature by thermal collisions with the evaporatively cooled bosons. At this temperature, the spatial size of the gas is strongly affected by the Fermi pressure resulting from the Pauli exclusion principle and gives clear experimental evidence for quantum degeneracy. Truscott et al, Science **291**, 2570 (2001)

From toy models into the lab





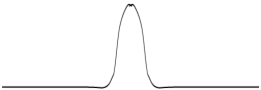


Use the tools and precision of atomic physics to realize integrable many-body problems: 1D Bose gases and Fermi gases, which provide fundamental understanding the nature of quantum mechanics and quantum statistics. They continue to inspire significant developments in mathematics, theories and experiments.

Integrability: classical and quantum

Classical Integrability

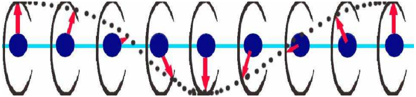
1834 Scott Russel
Discovered soliton



Soliton solutions of Exactly solvable models

Quantum Integrability

1931 Hans Bethe
Heisenberg chain



$$H = \frac{J}{2}\sum_{i=1}^{L}(\sigma_i \cdot \sigma_{i+1} + 1) - \frac{1}{2}h$$

Many conserved quantities

Quantum Integrability

Hans Bethe

1. Many-body Hamiltonian

$$H = \frac{J}{2} \sum_{i=1}^{L} (\sigma_i \cdot \sigma_{i+1} + 1) - \frac{1}{2} h$$

2. Bethe hypothesis: Wave function

$$a(x_1, \ldots, x_N) = \sum_{\{P\}} A_{\{P\}} e^{\mathrm{i}(k_{P_1} x_1 + \ldots + k_{P_N} x_N)}$$

3. Energy spectrum: Bethe equations

$$\exp(ik_j L) = (-1)^{n-1} \prod_{\ell \neq j} \frac{s_{\ell j}}{s_{j \ell}}$$

$$E = JL + J \sum_{j=1}^{n} (2 \cos k_j - 2).$$

4. Physics: Solving BA equations

Exactly solvable models

- 1931, Bethe: Heisenberg spin chain
- 1963, Lieb, Liniger: 1D Bose gas
- 1967, Yang, Gaudin: spin-1/2 Fermi gas
- 1968, Lieb, Wu: 1D Hubbard model
- 1968, Sutherland, SU(2s+1) quantum gases
- 1969, Yang, Yang: thermodynamics of 1D Bose gas
- 1972, Baxter, 2D XYZ vertex model
- 1979, Faddeev, et al., quantum inverse scattering method
- Kondo problems, Gaudin megnets, BCS model⋯
- New frontier with integrability: universal many-body phenomena

Yang-Baxter equation $Y_{12}Y_{23}Y_{12}=Y_{23}Y_{12}Y_{23}$

See a review: Guan, Batchelor, Lee, Review of Modern Physics, 85, 1633 (2013)

Group of Quantum Integralable System
量子可积系统研究组

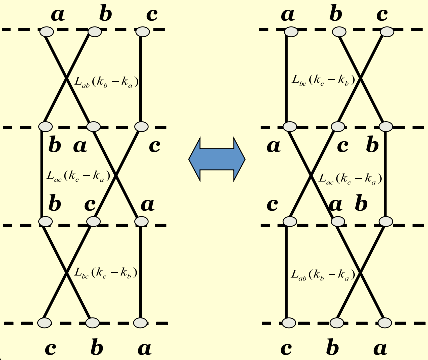
CN Yang's 1967

- **The many-body scattering matrix reduces to a product of many two-body scattering matrices.** Outgoing waves only consist of reflected waves, namely, no diffracted waves.

$$A_{21} = S_{12}(k_1 - k_2)A_{12}$$

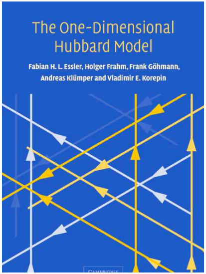
$$S_{bc}(k_c - k_b)S_{ac}(k_c - k_a)S_{ab}(k_b - k_a)$$

$$= S_{ab}(k_b - k_a)S_{ac}(k_c - k_a)S_{bc}(k_c - k_b)$$



Schrodinger Eigenvalueproblems → **Bethe ansatz eqs**

Yang-Baxter solvable models in physics



1D quantum physics

Exact Solution of the BCS Model

$$H_P = \sum_k \varepsilon_k n_k + g \sum_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

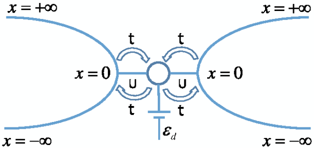
Eigenvalue equation:

$$H_P |\Psi\rangle = E |\Psi\rangle$$

Ansatz for the eigenstates (generalized Cooper ansatz)

$$|\Psi\rangle = \prod_{\alpha=1}^{M} \Gamma^\dagger_\alpha |0\rangle, \quad \Gamma^\dagger_\alpha = \sum_k \frac{1}{2\varepsilon_k - E_\alpha} c^+_{k\uparrow} c^+_{-k\downarrow}$$

Bethe ansatz for ultrasmall metallic grains



Kondo physics

Cold atoms in 1D

Exactly solvable models–Paradigm of physics

- Fractional excitations (anyon, spinon, magnon, ...)
- Quasi-long range order (correlation function, M-body reduced density matrices)
- Quantum liquid (Luttinger liquid, spin charge separation)
- Quantum criticality (scaling laws, field theory, quantum refrigeration)
- Quantum dynamics (transport properties, hydrodynamic, thermalization)
- Strongly correlated matter (topological matters, disorder, dynamical phase transition)
- Entanglement entropy (quantum information, quantum metrology)
- Conformal Field Theory, gauge field, Yang-Mills theory . . . . . .

Thacker, *Rev. Mod. Phys.* 53, 253 (1981)

F. Y. Wu, *Rev. Mod. Phys.* 54, 235 (1982)

Andrei, Furuya and Lowenstein, *Rev. Mod. Phys.* 55, 331 (1983)

Dukelsky, Pittel and Sierra, *Rev. Mod. Phys.* 76, 643 (2004)

Cazalilla, Citro, Giamarchi, Orignac, & Rigol, *Rev. Mod. Phys.* 83, 1405 (2011)

Guan, Batchelor, and Lee, *Rev. Mod. Phys.* 85, 1633 (2013)

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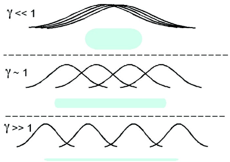
I. Lieb-Liniger model: Exact solution and thermodynamics



Lieb & Liniger in 1963 solved the 1D δ-function interacting Bose gas (Interaction range is much less than the mean distance between atoms)

$$H = \int_0^L dx \left[\partial_x \Psi^\dagger(x)\partial_x \Psi(x) + c\,\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)\right]$$

$$c = -4\hbar^2/ma_{1\mathrm{D}}, \qquad \gamma = c/n$$

- Commutation relations:

$$\left[\hat\psi(x), \hat\psi^\dagger(y)\right] = \delta(x-y), \quad \left[\hat\psi(x), \hat\psi(y)\right] = \left[\hat\psi^\dagger(x), \hat\psi^\dagger(y)\right] = 0$$

- The eigenvalue problem of the Schrödinger equation $\hat H|\Psi\rangle = E|\Psi\rangle$ with

$$|\Psi\rangle = \frac{1}{\sqrt{N!}}\int_0^L \mathrm{d}^N x\, \Psi(x)|x\rangle, \quad x = \{x_1, x_2, \cdots, x_N\}, \quad |x\rangle = \hat\psi^\dagger(x_1)\cdots\hat\psi^\dagger(x_N)|0\rangle$$

in N-particle sector reduces to the quantum mechanic man-body problem which is described by the Schrödinger equation $H\Psi(x) = E\Psi(x)$ with

$$H = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c\sum_{i<j}^N \delta(x_i - x_j)$$

- Two-particles

$$\begin{aligned}\Psi(x_1,x_2) &= \theta(x_2-x_1)\left[A_{P_1P_2}e^{\mathrm{i}(k_1x_1+k_2x_2)}+A_{P_2P_1}e^{\mathrm{i}(k_2x_1+k_1x_2)}\right]\\ &\quad+\theta(x_1-x_2)\left[A_{P_1P_2}e^{\mathrm{i}(k_1x_2+k_2x_1)}+A_{P_2P_1}e^{\mathrm{i}(k_2x_2+k_1x_1)}\right]\end{aligned}$$

- Discontinuity of the derivative

$$X=\frac{1}{2}(x_1+x_2),\ Y=x_2-x_1$$
$$\left[-\frac{1}{2}\frac{\partial^2}{\partial X^2}-2\frac{\partial^2}{\partial Y^2}\right]\Psi+2c\delta(Y)\Psi=E\Psi$$
$$\frac{\partial\Psi}{\partial Y}\Big|_{Y=0^+}-\frac{\partial\Psi}{\partial Y}\Big|_{Y=0^-}=c\Psi|_{Y=0}$$

- Two-body scattering relation

$$\frac{A_{P_2P_1}}{A_{P_1P_2}}=-\frac{c-\mathrm{i}(k_2-k_1)}{c+\mathrm{i}(k_2-k_1)}=-e^{\mathrm{i}\theta(k_2-k_1)}=Y_{12}(k_2-k_1)$$
$$\theta(x)=-2tan^{-1}\frac{x}{c},\ -\pi<\theta(x)<\pi$$
$$A_{P_2P_1}(k_2,k_1)=Y_{12}(k_2-k_1)A_{P_1P_2}(k_1,k_2)$$

Periodic boundary conditions for three particles

$x_1 < x_2 < x_3$

$$\begin{aligned}\Psi(x_1, x_2, x_3) \quad = \quad & A_{123} e^{\mathrm{i}(k_1 x_1 + k_2 x_2 + k_3 x_3)} + A_{213} e^{\mathrm{i}(k_2 x_1 + k_1 x_2 + k_3 x_3)} \\ & + A_{132} e^{\mathrm{i}(k_1 x_1 + k_3 x_2 + k_2 x_3)} + A_{312} e^{\mathrm{i}(k_3 x_1 + k_1 x_2 + k_2 x_3)} \\ & + A_{231} e^{\mathrm{i}(k_2 x_1 + k_3 x_2 + k_3 x_3)} + A_{321} e^{\mathrm{i}(k_3 x_1 + k_2 x_2 + k_1 x_3)}\end{aligned}$$

$x_2 < x_3 < x_1$

$$\begin{aligned}\Psi(x_2, x_3, x_1) \quad = \quad & A_{123} e^{\mathrm{i}(k_1 x_2 + k_2 x_3 + k_3 x_1)} + A_{213} e^{\mathrm{i}(k_2 x_2 + k_1 x_3 + k_3 x_1)} \\ & A_{132} e^{\mathrm{i}(k_1 x_2 + k_3 x_3 + k_2 x_1)} + A_{312} e^{\mathrm{i}(k_3 x_2 + k_1 x_3 + k_2 x_1)} \\ & A_{231} e^{\mathrm{i}(k_2 x_2 + k_3 x_3 + k_1 x_1)} + A_{321} e^{\mathrm{i}(k_3 x_2 + k_2 x_3 + k_1 x_1)}\end{aligned}$$

Bethe ansatz equations :

$$\Psi(0, x_2, x_3) = \Psi(x_2, x_3, L)$$

$$A_{123} = A_{231} e^{\mathrm{i}k_1 L}, \; A_{213} = A_{132} e^{\mathrm{i}k_2 L}, \; A_{312} = A_{123} e^{\mathrm{i}k_3 L}$$

- Many-particle wave function($x_1 < x_2 < \cdots < x_N$)

$$\Psi = \sum_P A_{P_1,\ldots,P_N} \exp \mathrm{i}(k_{P_1} x_1 + \ldots + k_{P_N} x_N)$$

- Periodic conditions

$$\Psi(x_1, x_2, \ldots, x_N) = \Psi(x_2, \ldots, x_N, x_1)$$

$$\sum_P A_{P_1,P_2,\cdots,P_N} e^{\mathrm{i}(k_{P_1} x_1 + k_{P_2} x_2 + \cdots k_{P_N} x_N)} = \sum_{P'} A_{P_2,\cdots,P_N,P_1} e^{\mathrm{i}(k_{P_2} x_2 + \cdots k_{P_N} x_N + k_{P_1} x_1)}$$

$$P = \begin{pmatrix} k_1 & k_2 & \cdots & k_N \\ P_1 & P_2 & \cdots & P_N \end{pmatrix}, \quad P' = \begin{pmatrix} k_1 & k_2 & \cdots & k_{N-1} & k_N \\ P_2 & P_3 & \cdots & P_N & P_1 \end{pmatrix}$$

$$A_P = A_{P'} e^{\mathrm{i}k_{P1} L}, \;\; A_P = \left[-e^{\mathrm{i}\theta(k_{P_1} - k_{P_2})} \right] A_{P_2 P_1 \ldots P_N} = (-1)^{N-1} e^{\mathrm{i}\sum_j \theta(k_{P1} - k_j)} A_{P'}$$

$$e^{\mathrm{i}k_{P1} L} = (-1)^{N-1} e^{\mathrm{i}\sum_j \theta(k_{P1} - k_j)}$$

- Lieb-Liniger equation

$$E = \frac{\hbar^2}{2m} \sum_{j=1}^{N} k_j^2, \;\; \exp(\mathrm{i}k_j L) = -\prod_{\ell=1}^{N} \frac{k_j - k_\ell + \mathrm{i}\,c}{k_j - k_\ell - \mathrm{i}\,c}$$

Lieb and Liniger, 1963 *Phys. Rev.* **130** 1605

- Repulsive interaction

$$K_i L = 2\pi I_i + \sum_{\ell=1}^{N} \theta(k_i - k_\ell), \quad I = -\frac{N-1}{2}, -\frac{N-1}{2} + 1, \ldots \frac{N-1}{2}$$

- Thermodynamic limit

$$\begin{aligned} \frac{dI(k)}{Ldk} &= \frac{1}{2\pi} + \frac{1}{L}\sum_{\ell=1}^{N} \frac{1}{2\pi} \frac{2c}{c^2 + (k-k_\ell)^2} \\ \rho(k) &= \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-Q}^{Q} dq \, \frac{2c\rho(q)}{c^2 + (k-q)^2} \end{aligned}$$

- Love equation

$$\begin{aligned} k &= Qx, \; c = Q\lambda, \; \rho(Qx) = g(x) \\ \frac{E}{N} &= \frac{\hbar^2 n^2}{2m} e_0(\gamma), \; e_0(\gamma) = \frac{\gamma^3}{\lambda^3}\int_{-1}^{1} x^2 g_0(x) dx \\ g_0(x) &= \frac{1}{2\pi} + \frac{\lambda}{\pi}\int_{-1}^{1} \frac{g_0(y)}{\lambda^2 + (x-y)^2} \mathrm{d}y \end{aligned}$$

- Weak coupling $\gamma \ll 1$ (Prolhac, JPA 50, 1751(2017))

$$e_0 = \gamma - \frac{4}{3\pi}\gamma^{3/2} + \left(\frac{1}{6} - \frac{1}{\pi^2}\right)\gamma^2 + O(\gamma^2)$$

- Semi-circle law

$$q_j = \sum_{l \neq j}^{N} \frac{1}{q_j - q_l}, \quad q_j = k_j \sqrt{L/2c}, \ \mathrm{H}_N(q) = \prod_{i=1}^{N} (q - q_j)$$

$$\lim_{q \to q_j} \frac{\mathrm{H}_N''(q)}{\mathrm{H}_N'(q)} = 2q_j, \ F(q) \equiv \mathrm{H}_N''(q) - 2q\mathrm{H}_N'(q)$$

$$\mathrm{H}_N''(q) - 2q\mathrm{H}_N'(q) + 2N\mathrm{H}_N(q) = 0$$

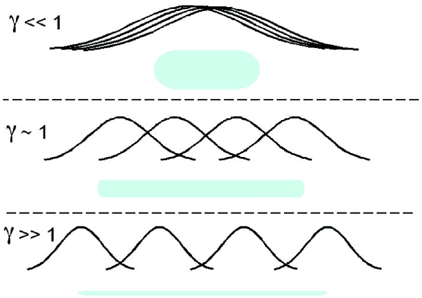
- Fermionization

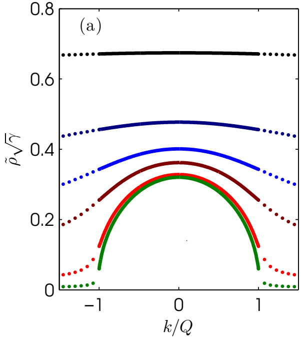
$$k_j L = 2\pi I_j - 2\sum_{\ell=1}^{N} tan^{-1} \frac{k_j - k_\ell}{c}, \ I_i = (N+1)/2 - j$$

$$k_j L = 2\pi I_j - 2\frac{Nk_j}{c} + \frac{2}{3c^2} \sum_{\ell=1}^{N} (k_j - k_\ell)^3 + O(c^{-5})$$

$$k_j L = 2\pi I_j - 2\frac{Nk_j}{c} + \frac{16\pi^3}{3c^3 L^3} \sum_{\ell=1}^{N} (\ell - j)^3 + O(c^{-5})$$

$$k_j = \frac{2\pi I_j}{L} \left(1 - \frac{2}{\gamma} + 4\gamma^4 - \frac{8}{\gamma^3}\right) + \frac{16\pi^3}{3c^3 L^4} \left[(N + \frac{1}{2} - j)^4 - (\frac{1}{2} - j)^4\right] + O(c^{-5})$$





Hermite polynomials

$$H_N''(q) - 2qH_N'(q) + 2NH_N(q) = 0$$

semicircle-law

$$\rho(\mathbf{k}) = \frac{\mathbf{1}}{\pi\sqrt{\gamma}}\sqrt{\mathbf{1} - \frac{\mathbf{k^2}}{\mathbf{4}|\gamma|\mathbf{n^2}}}$$

Gaudin 1971; Batchelor, Guan & McGuire 2004

Jiang, Chen, Guan, Chinese Physics B, 24, 5 (2015)

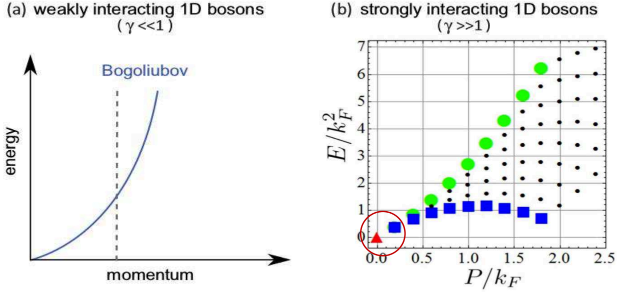
Table 1. Experiments of Lieb–Liniger gas.

quantum dynamics	^{87}Rb[61,71,74–76]
thermalization	^{87}Rb[61,70,74,75]
solitons	^{87}Rb[68,77]
fermionization	^{39}K[57,72]
YY thermodynamics	^{87}Rb[56,61,62,65–67,78]
strong coupling	^{87}Rb[24,25]
phase diagram	Cs[79]
3-body correlations	^{87}Rb,[58,63] Cs[60]
excited state	Cs[64]

Jiang, Chen, Guan, Chin. Phys. B 24, 050311 (2015)

However, the observation of the **Luttinger liquid and quantum criticality** of a 1D quantum system is a long-standing challenge.

Cazalilla, Citro, Giamarchi, Orignac, & Rigol, *Rev. Mod. Phys.* 83, 1405 (2011)
Guan, Batchelor, and Lee, *Rev. Mod. Phys.* 85, 1633 (2013)



particle-hole exitations

$\gamma \ll 1$, Bogoliubov dispersion: $\epsilon_p = v_s P\sqrt{1+\frac{p^2}{4m^2v^2}}$

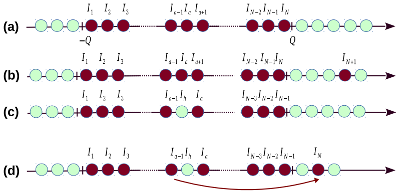
$\gamma \gg 1$ create a Fermi surface with $k_F = \pi\rho$: $\epsilon_p = v_s p + \frac{P^2}{2m^*} + O(p^3)$

One particle excitation

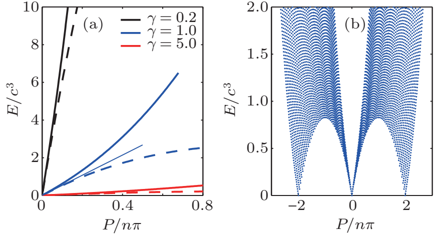
$$\bar{\rho}(k) = \frac{1}{2\pi} + \int_{-Q}^{Q} a(k-k')\bar{\rho}(k')dk' + \frac{1}{L}a(k-k_e)$$

$$\Delta\rho(k) = \frac{1}{L}a(k-k_e) + \int_{-Q}^{Q} a(k-k')\Delta\rho(k'), \quad \Delta\rho(k) = \bar{\rho}(k) - \rho_0(k)$$

$$\Delta E(k_c) = L\int_{-Q}^{Q} \Delta\rho(k)(k^2-\mu)\mathrm{d}k + (k_e^2-\mu)$$

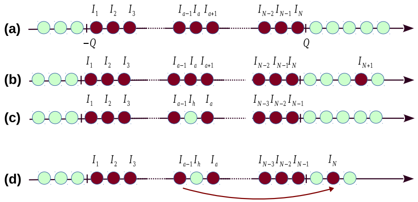


Elementary excitations



Particle and Hole excitations

- (a) the quantum numbers for the ground state
- (b) adding a particle near right Fermi point
- (c) a hole excitation. The total number of particles is $N - 1$
- (d) a single particle-hole excitation.



Elementary excitations

- (a) the quantum numbers for the ground state
- (b) adding a particle near right Fermi point (ΔN or backward scattering $2\Delta D$).
- (c) a hole excitation. The total number of particles is $N-1$.
- (d) a single particle-hole excitation ($N^{\pm}$).

Total momentum and excitation energy

$$
\begin{aligned}
\Delta P &= \frac{2\pi}{L}\left[\Delta N \Delta D + N^{+} - N^{-}\right] + 2\Delta D k_{\mathrm{F}}, \\
\Delta E &= \frac{2\pi v_s}{L}\left[\frac{1}{4}(\Delta N/Z)^2 + (\Delta D Z)^2 + N^{+} + N^{-}\right] \\
&= \frac{\pi}{2L}\left[v_s Z^{-2}(\Delta N)^2 + v_s Z^2 J^2 + 4v_s N^{+} + 4v_s N^{-}\right], \quad J = 2\Delta D \\
V_J &= v_s Z^2, \quad V_N = \frac{V_s}{Z^2}, \quad K = \frac{v_s}{v_z} = Z^2, \quad Z = 2\pi\rho(Q)
\end{aligned}
$$

Origin of Quantum liquid

$$\text{Bethe Ansatz result} \quad \Delta E = \frac{2\pi v_s}{L}(N^- + N^+) + \frac{\pi}{2L}\left(\frac{v_s}{Z^2}\Delta N^2 + v_s Z^2 (2D)^2\right)$$

$$\text{Bosonization Hamiltonian} \quad H = v_s \sum_{q\neq 0} |q| \hat{b}_q^\dagger \hat{b}_q + \frac{\pi}{2L}\left(v_N \, \Delta N^2 + v_J \, J^2 \right)$$

$$H = \int dx \left(\frac{\pi v_s K}{2} \Pi^2 + \frac{v_s}{2\pi K} (\partial_x \phi)^2 \right)$$

Luttinger parameter $\quad K = \sqrt{v_J / v_N} = Z^2$

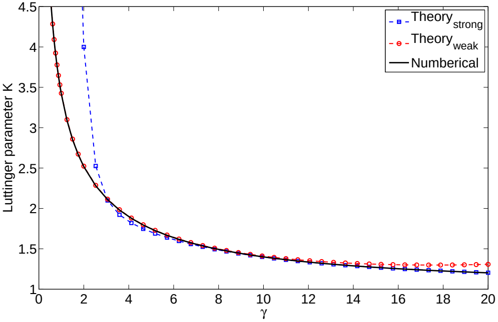
Sound velocity $\quad v_s = \sqrt{v_N v_J} = \sqrt{\frac{L^2}{mN}\frac{\partial^2 E}{\partial L^2}}$

Density stiffness $\quad v_N = \frac{v_s}{K} = \frac{L}{\pi\hbar}\left(\frac{\partial^2 E}{\partial N^2}\right)_{N=N_0}$

Phase stiffness $\quad v_J = v_s K = \pi L \frac{\partial^2}{\partial \alpha} E$

e.g. K determines the leading order correlation $\langle \psi^\dagger(x)\psi(0)\rangle \sim 1/x^{1/2K}$

Cazalilla, Citro, Giamarchi, Orignac, & Rigol, *Rev. Mod. Phys.* 83, 1405 (2011)
Haldane, Phys. Lett. A 81, 153 (1981)



Luttinger parameter

$$K = \frac{\pi}{\sqrt{3e - 2\gamma \frac{de}{d\gamma} + \frac{1}{2}\gamma^2 \frac{d^2 e}{d\gamma^2}}}$$

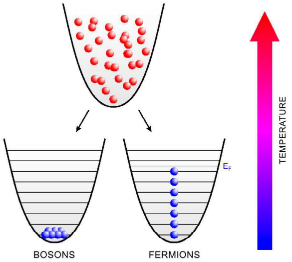
Strong coupling

$$K = 1 + \frac{4}{\gamma} + \frac{4}{\gamma^2} - \frac{16\pi^2}{3\gamma^3} + \frac{32\pi^2}{3\gamma^4} + O\left(\gamma^{-5}\right)$$

Weak coupling

$$K = \pi \left(\gamma - \frac{1}{2\pi}\gamma^{3/2}\right)^{-1/2}$$

Bosons and Fermions



Quantum statistics:

1. quantum man-body systems
2. microscopic state energy E_i
3. partition function $Z = \sum_{i=1}^{\infty} W_i e^{-E_i/(k_B T)}$
4. free energy $F = -k_B T \ln Z$
5. challenge: finding new physics

- Yang-Yang equation (Gibbs ensemble): a brilliant method(J. Math. Phys. 10, 1115 (1969))

$$\varepsilon(k) = k^2 - \mu - \frac{Tc}{\pi} \int \frac{dq}{c^2 + (k-q)^2} \ln\left(1 + e^{-\varepsilon(q)/T}\right)$$

- Equation of state: per length pressure

$$p(\mu, T) = \frac{T}{2\pi} \int_{-\infty}^{\infty} \ln\left(1 + e^{-\varepsilon(k)/T}\right) dk$$

- Bethe ansatz equations at finite temperatures

$$\rho(k) + \rho^h(k) = \frac{1}{2\pi} + \frac{c}{\pi} \int \frac{\rho(k')dk'}{c^2 + (k-k')^2}$$

- Entropy

$$\begin{aligned} W &= \frac{(L(\rho+\rho^h)dk)!}{(L\rho dk)!(L\rho^h dk)!}, \quad S = \int dS = \int \ln dW \\ s &= \frac{S}{L} = \int \left[(\rho+\rho^h)\ln(\rho+\rho^h) - \rho\ln\rho - \rho^h \ln\rho^h \right] dk \\ &= \int \left[(\rho+\rho^h)\ln\left(1+\frac{\rho}{\rho^h}\right) - \rho\ln\left(\frac{\rho}{\rho^h}\right) \right] dk \end{aligned}$$

- Partition function

$$\begin{aligned} \mathcal{Z} &= \mathrm{Tr}(e^{-H/T}) = \sum_{\rho,\rho^h} W(\rho,\rho^h) e^{-E(\rho,\rho^h)/T} \\ \mathcal{Z} &= \sum_{\rho,\rho^h} e^{-(E(\rho,\rho^h) - S(\rho,\rho^h)T)/T} \end{aligned}$$

- Gibbs ensemble

$$
\begin{aligned}
0 &= \frac{\delta G}{L} = \frac{\delta E}{L} - \mu\delta n - T\delta s, \ \frac{E}{L} = \int k^2 \rho(k)dk, \ n = \int \rho(k)dk \\
\delta s &= \int (\delta\rho + \delta\rho^h) \ln\left(1 + \frac{\rho}{\rho^h}\right) - \delta\rho \ln\left(\frac{\rho}{\rho^h}\right) dk \\
& \delta\rho(k) + \delta\rho^h(k) = \frac{c}{\pi}\int \frac{\delta\rho(k')dk'}{c^2 + (k-k')^2}
\end{aligned}
$$

- Yang-Yang method: dressed energy $\epsilon(k) = T \ln \frac{\rho_h(k)}{\rho(k)}$

$$
\begin{aligned}
0 &= \frac{\delta G}{L} \\
&= \int \left[k^2\delta\rho - \mu\delta\rho + T\delta\rho \ln\left(\frac{\rho}{\rho^h}\right) - \frac{c}{\pi}\int \frac{\delta\rho dq}{c^2 + (k-q)^2} \ln\left(1 + \frac{\rho}{\rho^h}\right)\right] dk \\
&= \int \left[k^2 - \mu + T\ln\left(\frac{\rho}{\rho^h}\right) - \frac{Tc}{\pi}\int \frac{dq}{c^2 + (k-q)^2} \ln\left(1 + \frac{\rho}{\rho^h}\right)\right] \delta\rho dk
\end{aligned}
$$

- Yang-Yang equation

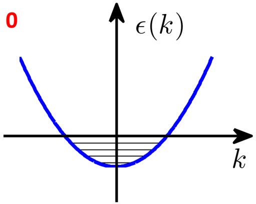
$$
\varepsilon(k) = k^2 - \mu - \frac{Tc}{\pi}\int \frac{dq}{c^2 + (k-q)^2} \ln\left(1 + e^{-\varepsilon(q)/T}\right)
$$

- Pressure

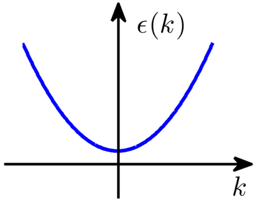
$$
\begin{aligned}
p &= \int_{-\infty}^{\infty} \left\{ \left(\mu - k^2\right) \rho(k) + T \left[\left(\rho(k) + \rho^h(k)\right) \right.\right. \\
&\quad \left.\left. \times \ln\left(\rho(k) + \rho^h(k)\right) - \rho(k) \ln \rho(k) - \rho^h(k) \ln \rho^h(k) \right] \right\} dk \\
&= \int_{-\infty}^{\infty} \left\{ \left[\left(\mu - k^2\right) - \ln \frac{\rho(k)}{\rho^h(k)} + T \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right] \rho(k) \right. \\
&\quad \left. + T\rho^h(k) \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right\} dk \\
&= \int_{-\infty}^{\infty} \left\{ T\rho(k) \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right. \\
&\quad \left. - \frac{Tc}{\pi} \int \frac{dq}{c^2 + (k-q)^2} \ln\left(1 + \frac{\rho(q)}{\rho^h(q)}\right) + T\rho^h(k) \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right\} dk \\
&= T \int_{-\infty}^{\infty} \left\{ \left[\rho(k) + \rho^h(k) - \frac{Tc}{\pi} \int \frac{\rho(q) dq}{c^2 + (k-q)^2} \right] \right. \\
&\quad \left. \times \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right\} dk \\
&= \frac{T}{2\pi} \int_{-\infty}^{\infty} \ln\left(1 + e^{-\varepsilon(k)/T}\right) dk
\end{aligned}
$$

Quantum Phase Transition

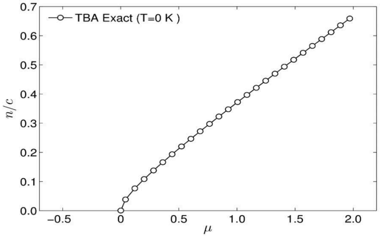
$T = 0$



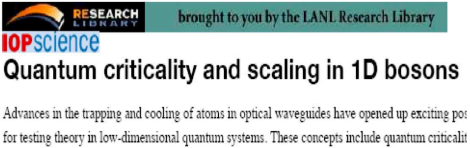
TLL , $\mu > 0$



Vacuum, $\mu \leq 0$



量子相变, $\mu = 0$



Quantum criticality and scaling in 1D bosons

Advances in the trapping and cooling of atoms in optical waveguides have opened up exciting pos
for testing theory in low-dimensional quantum systems. These concepts include quantum criticalit

Guan & Batchelor, J. Phys. A 44, 102001(2011)

$$n = \frac{\sqrt{\mu}}{\pi} + \frac{8}{3\pi^2}\mu\frac{1}{c} + \frac{70}{9\pi^3}\mu^{\frac{3}{2}}\frac{1}{c^2} + \cdots$$

- Weak coupling

$$\lim_{c\to 0}\frac{c}{\pi}\frac{1}{c^2+(k-q)^2}=\delta(k-q)\to\epsilon(k)=k^2-\mu-T\ln\left(1+e^{-\epsilon(k)/T}\right)$$

$$\frac{(k^2-\mu)}{T}=\ln\left(1+e^{-\epsilon(k)/T}\right),\ \ \epsilon(k)=T\ln\left(e^{-(k^2-\mu)/T}-1\right)$$

$$p=T\int\ln\left(1+e^{-\epsilon(k)/T}\right)dk=T\int\ln\left(1+\frac{1}{e^{(k^2-\mu)/T}-1}\right)dk$$

$$=-T\int\ln\left(1-e^{-(k^2-\mu)/T}\right)dk=\int\frac{\sqrt{\epsilon}d\epsilon}{e^{(\epsilon-\mu)/T}-1}$$

- High temperature

$$
\begin{aligned}
e^{-\epsilon(k)/T} &= ze^{-k^2/T} e^{\int dq a_2(k-q)\ln(1+ze^{-q^2/T})} \\
&= ze^{-k^2/T}\left(1 + z\int dq a_2(k-q) e^{-q^2/T}\right)
\end{aligned}
$$

- Equation of state

$$
\begin{aligned}
p &= \frac{T}{2\pi}\int_{-\infty}^{\infty} \ln\left(1 + e^{-\varepsilon(k)/T}\right) dk \\
&= \frac{T}{2\pi}\int dk \ln\left(1 + ze^{-k^2/T} + z^2 e^{-k^2/T}\int da a_2(k-q) e^{-q^2/T}\right) \\
&= -\frac{T}{2\pi}\int \ln\left(1 - ze^{-k^2/T}\right) dk \\
&+ \frac{T}{2\pi}\int dk \ln\left(1 + ze^{-k^2/T} + z^2 e^{-k^2/T}\int dk a_2(k-q) e^{-q^2/T}\right)\left(1 - ze^{-k^2/T}\right) \\
&= p_0 + \frac{T^{3/2}}{\sqrt{2\pi}} z^2 p_2, \quad p_2 = -\frac{1}{2} + \int dq' a_2(2q') e^{-2q'^2/T}
\end{aligned}
$$

- Strong coupling: $T = 0$

$$
\begin{aligned}
\epsilon(k) &= \epsilon^0(k) - \mu + \frac{c}{\pi}\frac{1}{c^2+k^2}\int_Q^Q dq\,\epsilon(q) \\
&\quad - \frac{c}{\pi}\int_{-Q}^{Q} dq\,\frac{\epsilon(q)}{(c^2+k^2)^2}[-2kq+q^2] \\
&\quad \frac{c}{\pi}\int_{-Q}^{Q} dq\,\frac{\epsilon(q)}{(c^2+k^2)^3}[-2kq+q^2]^2 + O\left(\frac{1}{c^6}\right)
\end{aligned}
$$

- solution

$$
\begin{aligned}
\epsilon(k) &= \epsilon^0(k) - \mu - \frac{2pc}{c^2+k^2} + \frac{4\mu^{5/2}}{15\pi|c|^3} + O\left(\frac{1}{c^4}\right) \\
E &\approx \frac{1}{3}n^3\pi^2\left(1 - \frac{4}{\gamma} + \frac{12}{\gamma^2} + \frac{\left(\frac{32}{15}\pi^2 - 32\right)}{\gamma^3}\right)
\end{aligned}
$$

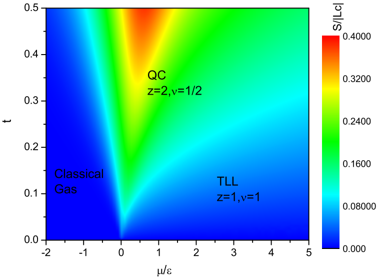
- Strong coupling: $T \neq 0$

$$\begin{aligned}\epsilon(k) &= \epsilon^0(k) - \mu - \frac{2c\,p(T)}{c^2+k^2} + \frac{T}{2\pi(c^2+k^2)^2}\int_{-\infty}^{\infty} dq\, q^2 \ln(1+\mathrm{e}^{-\epsilon(q)/T}) \\ &\approx \epsilon^0(k) - \mu - \frac{2c\,p(T)}{c^2+k^2} - \frac{1}{2\sqrt{\pi}c^3}\frac{T^{\frac{5}{2}}}{\left(\frac{\hbar^2}{2m}\right)^{\frac{3}{2}}}\mathrm{Li}_{\frac{5}{2}}(-\mathrm{e}^{A_0/T}) \\ A_0 &= \mu + \frac{2\,p(T)}{c} - \frac{4\mu^{5/2}}{15\pi|c|^3},\ \mathrm{Li}_n(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^n}\end{aligned}$$

- Equation of State

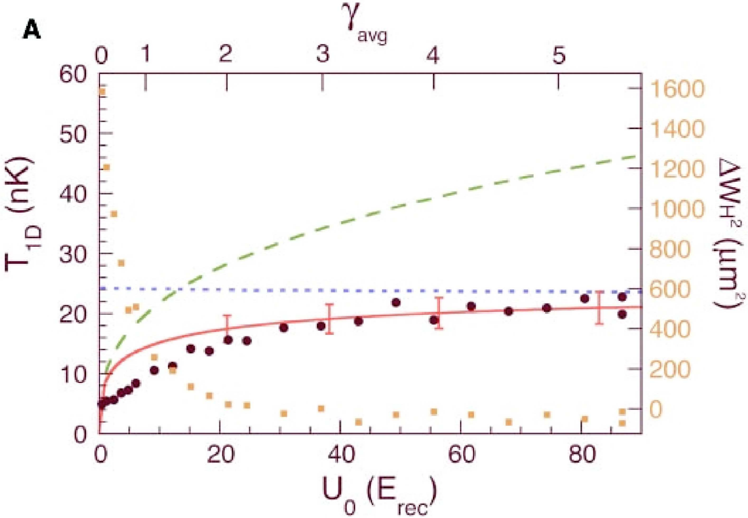
$$\begin{aligned}p(T) &\approx -\sqrt{\frac{m}{2\pi\hbar^2}}\,T^{\frac{3}{2}}\,\mathrm{Li}_{\frac{3}{2}}(-\mathrm{e}^{A/T})\left[1 + \frac{1}{2c^3\sqrt{\pi}}\left(\frac{T}{\frac{\hbar^2}{2m}}\right)^{\frac{3}{2}}\mathrm{Li}_{\frac{3}{2}}(-\mathrm{e}^{A/T})\right] \\ A &= \mu + \frac{2\,p(T)}{c} + \frac{1}{2\sqrt{\pi}c^3}\frac{T^{\frac{5}{2}}}{\left(\frac{\hbar^2}{2m}\right)^{\frac{3}{2}}}\mathrm{Li}_{\frac{5}{2}}(-\mathrm{e}^{A_0/T})\end{aligned}$$

Guan, Batchelor, JPA, 2011



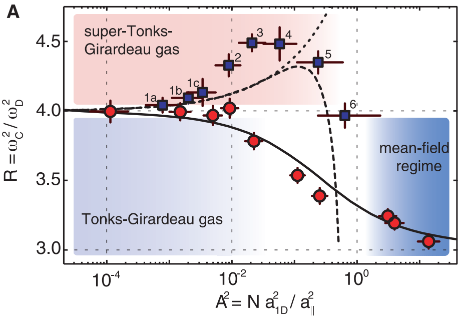
Boltzmann Statistics	$p_c = \sqrt{\frac{m}{2\pi\hbar^2}}\, T^{-\frac{3}{2}} \mathrm{e}^{\frac{\mu}{T}},\ p = p_0 + \frac{T^{\frac{3}{2}}}{\sqrt{2\pi}} \mathcal{Z}^2 p_2$	for $T \to \infty$
Fermi Statistics	$p = -\sqrt{\frac{m}{2\pi\hbar^2}}\, T^{\frac{3}{2}}\, \mathrm{Li}_{\frac{3}{2}}(-\mathrm{e}^{\frac{\mu}{T}}),$	for $c \to \infty$
Bose Statistics	$p = \sqrt{\frac{m}{2\pi\hbar^2}}\, T^{\frac{3}{2}}\, \mathrm{Li}_{\frac{3}{2}}(\mathrm{e}^{\frac{\mu}{T}}),$	for $c \to 0,$
Fractional Statistics	$(1+\omega_i)\prod_j \left(\frac{\omega_j}{1+\omega_j}\right)^{\alpha_{ji}} = e^{(\epsilon_i - \mu_i)/T}$	for $c,\ T \neq 0$

Haldane, Phys. Rev. Lett. (1991)
Wu, Phys. Rev. Lett. (1994)
Batchelor, Guan, Olkers, Rev. Rev. Lett. (2006)
Jiang, Chen, Guan, CPB (2015)



1D Tonks-Girardeau gas: $\gamma > 1$ for ^{87}Rb (Weiss's group, Nature **305** 2004 1125)

Super Tonks-Girardeau Gas



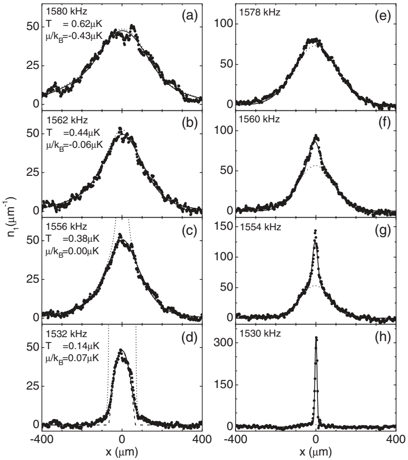
Using a tunable quantum gas of bosonic cesium atoms, Haller et al experimentally realized a 1D super Tonks-Girardeau gas, which is a highly excited quantum phase stabilized in the presence of strong attractive interaction.

Haller et al, Science **325**, 1224 (2009)

Astrakharchik, Boronat, Casulleras, Giorgini, Phys. Rev. Lett. **95**, 190407 (2005)

Batchelor, Bortz, Guan, Oelkers, J. Stat. Mech. **L10001** (2005)

Yang-Yang TBA equations (1969)



Druten'sgroup(2008)

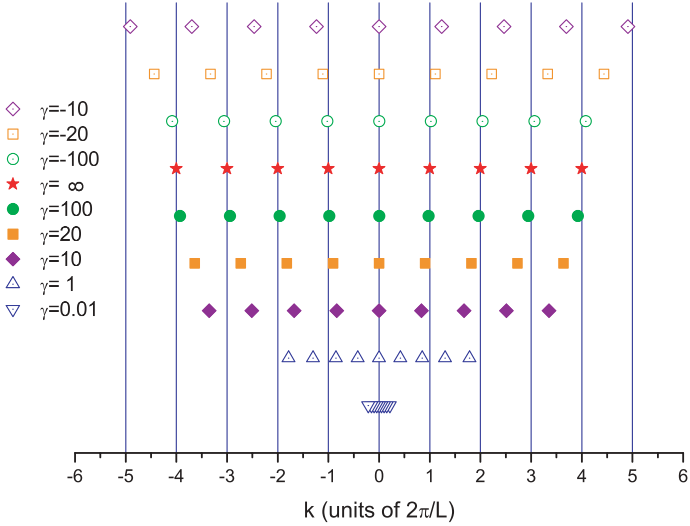
Exercises:

1. Prove that the eigenvalue problem of the Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$ of the 1D Bose gas in N-particle sector reduces to the quantum mechanic man-body problem which is described by the Schrödinger equation $H\Psi(x) = E\Psi(x)$ with the first quantized form of the Hamiltonian

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j}^{N} \delta(x_i - x_j).$$

2. Prove the Bethe ansatz equations of the Lieb-Liniger gas admit highly excited states with real values of $\{k_j\}$ in strongly attractive regime, i.e. $c \ll -1$. This state shows a novel existence of the super Tonks state.
Astrakharchik, Boronat, Casulleras, Giorgini, Phys. Rev. Lett. 95, 190407 (2005)
Batchelor, Bortz, Guan, Oelkers, J. Stat. Mech. L10001 (2005)

3. Prove the Yang-Yang thermodynamics Bethe ansatz equation and calculate the pressure in strong repulsion limit.



Batchelor, Bortz, Guan, Oelkers, J. Stat. Mech. **L10001** (2005)