Homework #4:
Brief Solutions

***** (about lecture #3) *****

Problem 1. Consider the 1D harmonic oscillator $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$. Here $\hat{x}$ is position operator, $\hat{p}$ is momentum operator, $[\hat{x}, \hat{p}] = \mathbb{i}\hbar$, and in position representation $\hat{p} = -\mathbb{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{b} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \mathbb{i}\frac{1}{m\omega}\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{b}, \hat{b}^\dagger] = 1$ and $\hat{H}_0 = \hbar\omega\,(\hat{b}^\dagger\hat{b} + \frac{1}{2})$. It has a unique ground state $|0\rangle$ with $\hat{b}|0\rangle = 0$, and excited states $|n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{b}^\dagger)^n|0\rangle$ with energy $E_n = (n + \frac{1}{2})\hbar\omega$.

(a) (5pts) Let $\hat{H}' = \hat{H}_0 - f \cdot \hat{x}$, where f is a real constant. $\hat{H}'$ is related to $\hat{H}_0$ by $\hat{U} \cdot \hat{H}' \cdot \hat{U}^\dagger = \hat{H}_0 + c$. Here c is a real constant, $\hat{U} = \exp(-\mathbb{i}X\hat{p} - \mathbb{i}P\hat{x})$ is a unitary operator with real parameters X and P. *Solve X and P and c in terms of $f, m, \omega, \hbar$.*

(b) (5pts) Denote the normalized ground state of $\hat{H}'$ by $|0'\rangle$. *Evaluate $\langle 0'|\hat{x}|0'\rangle$ and $\langle 0'|\hat{p}|0'\rangle$.* [Hint: result of (a) may help.]

(c) (5pts) At $t = 0$, let the state $|\psi(t=0)\rangle = |0'\rangle$, evolve this state under $\hat{H}_0$, namely $|\psi(t)\rangle = \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t)|\psi(t=0)\rangle$. *Evaluate $\langle\psi(t)|\hat{x}|\psi(t)\rangle$ and $\langle\psi(t)|\hat{p}|\psi(t)\rangle$.* [Hint: you can use either Schrödinger or Heisenberg picture, you can directly use the Heisenberg equations of motion for $\hat{x}$ and $\hat{p}$ and their solutions for harmonic oscillator]

(d) (5pts) Define two Hermitian operators: $\hat{O}_1 = m^2\omega^2\hat{x}^2 - \hat{p}^2$, $\hat{O}_2 = m\omega(\hat{x}\hat{p} + \hat{p}\hat{x})$. Their Heisenberg picture under $\hat{H}_0$ are $\hat{O}_{i,\mathrm{H}}(t) = \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t) \cdot \hat{O}_i \cdot \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t)$. *Write down the Heisenberg equations of motion, $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_{i,H}(t) = \ldots$ for $i = 1, 2$.* The right-hand side of these equations should be expressed in terms of $\hat{O}_{j,\mathrm{H}}(t)$ with $j = 1, 2$.

(e) (5pts) *Solve the equations in (d). Namely solve $\hat{O}_{i,H}(t)$ in terms of $\hat{O}_{j,H}(t = 0)$.*

Solution:

(a) By the Baker-Hausdorff formula,
$\hat{U}^\dagger\hat{x}\hat{U} = \hat{x} + [\mathbb{i}X\hat{p} + \mathbb{i}P\hat{x}, \hat{x}] + \cdots = \hat{x} + (\mathbb{i}X)(-\mathbb{i}\hbar) + 0 + \cdots = \hat{x} + X\hbar$, and

$\hat{U}^\dagger \hat{p} \hat{U} = \hat{p} + [\mathbb{i}X\hat{p} + \mathbb{i}P\hat{x}, \hat{p}] + \cdots = \hat{p} + (\mathbb{i}P)(\mathbb{i}\hbar) + 0 + \cdots = \hat{p} - P\hbar.$

Then $\hat{U}^\dagger \cdot \hat{H}_0 \cdot \hat{U} = \frac{1}{2m}(\hat{U}^\dagger \hat{p} \hat{U})^2 + \frac{m\omega^2}{2}(\hat{U}^\dagger \hat{x} \hat{U})^2 = \frac{1}{2m}(\hat{p} - P\hbar)^2 + \frac{m\omega^2}{2}(\hat{x} + X\hbar)^2$.

Compare this with $\hat{H}' = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2 - f \cdot \hat{x} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}(\hat{x} - \frac{f}{m\omega^2})^2 - \frac{f^2}{2m\omega^2}$, we get $X = -\frac{f}{m\omega^2\hbar}$, $P = 0$, $c = -\frac{f^2}{2m\omega^2}$.

(b) According to (a), $\hat{U}^\dagger \hat{H}_0 \hat{U} = \hat{H}' - c$, there is one-to-one correspondence between the eigenstates of $\hat{H}_0$ and $\hat{H}'$:

if $\hat{H}_0|n\rangle = E_n|n\rangle$, then $\hat{H}' \cdot \hat{U}^\dagger|n\rangle = (\hat{U}^\dagger \hat{H}_0 \hat{U} + c) \cdot \hat{U}^\dagger|n\rangle = \hat{U}^\dagger \hat{H}_0|n\rangle + c\hat{U}^\dagger|n\rangle = (E_n + c) \cdot \hat{U}^\dagger|n\rangle$;

conversely, if $\hat{H}'|n'\rangle = E'_n|n'\rangle$, then $\hat{H}_0 \cdot \hat{U}|n'\rangle = (E'_n - c) \cdot \hat{U}|n'\rangle$.

The ground state of $\hat{H}'$ is $\hat{U}^\dagger|0\rangle$ where $|0\rangle$ is the ground state of $\hat{H}_0$.

$\langle 0'|\hat{x}|0'\rangle = \langle 0|\hat{U}\hat{x}\hat{U}^\dagger|0\rangle$, $\langle 0'|\hat{p}|0'\rangle = \langle 0|\hat{U}\hat{p}\hat{U}^\dagger|0\rangle$.

Similar to the calculations in (a), $\hat{U}\hat{x}\hat{U}^\dagger = \hat{x} - X\hbar = \hat{x} + \frac{f}{m\omega}$, $\hat{U}\hat{p}\hat{U}^\dagger = \hat{p} + P\hbar = \hat{p}$.

In the ground state $|0\rangle$ of $\hat{H}_0$, $\langle 0|\hat{x}|0\rangle = 0$ and $\langle 0|\hat{p}|0\rangle = 0$.

This can be seen from $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b} + \hat{b}^\dagger)$, and $\hat{p} = -\mathbb{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{b} - \hat{b}^\dagger)$, and $\langle 0|\hat{b}|0\rangle = \langle 0|\hat{b}^\dagger|0\rangle^* = 0$.

Therefore $\langle 0'|\hat{x}|0'\rangle = \langle 0|(\hat{x} + \frac{f}{m\omega^2})|0\rangle = \frac{f}{m\omega^2}$, $\langle 0'|\hat{p}|0'\rangle = \langle 0|\hat{p}|0\rangle = 0$.

(c) Method #1: Schrödinger picture.

$|0'\rangle = \exp(-\mathbb{i}\frac{f}{m\omega^2\hbar}\hat{p})|0\rangle = \exp[-\frac{f}{m\omega^2\hbar}\sqrt{\frac{\hbar m\omega}{2}}(\hat{b} - \hat{b}^\dagger)]|0\rangle$.

For notation simplicity, define $z = \frac{f}{m\omega^2\hbar}\sqrt{\frac{\hbar m\omega}{2}}$, then $|0'\rangle = \exp(-z^*\hat{b} + z\hat{b}^\dagger)|0\rangle$ $= e^{-|z|^2/2}\exp(z\hat{b}^\dagger)\exp(-z\hat{b})|0\rangle = e^{-|z|^2/2}\exp(z\hat{b}^\dagger)|0\rangle$ is a boson coherent state.

Denote boson coherent states $e^{-|z|^2/2}\exp(z\hat{b}^\dagger)|0\rangle$ by $|z\rangle$ hereafter.

$|\psi(t)\rangle = \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t)|0'\rangle = \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t) \cdot e^{-|z|^2/2}\exp(z\hat{b}^\dagger) \cdot \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t) \cdot \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t)|0\rangle$ $= e^{-|z|^2/2}\exp\left[z \cdot \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t) \cdot \hat{b}^\dagger \cdot \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t)\right] \cdot e^{-\frac{\mathbb{i}}{\hbar}E_0 \cdot t}|0\rangle$. Here E_0 is the ground state energy of $\hat{H}_0$.

From $\hat{H}_0 = \hbar\omega \cdot (\hat{b}^\dagger\hat{b} + \frac{1}{2})$, the commutator $[-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t, \hat{b}^\dagger] = -\mathbb{i}\omega t \cdot \hat{b}$, then by the Baker-Hausdorff formula, $\exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t) \cdot \hat{b}^\dagger \cdot \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t) = \sum_{n=0}^\infty \frac{(-\mathbb{i}\omega t)^n}{n!}\hat{b}^\dagger = e^{-\mathbb{i}\omega t}\hat{b}^\dagger$.

$|\psi(t)\rangle = e^{-|z|^2/2} \cdot \exp(ze^{-\mathbb{i}\omega t}\hat{b}^\dagger) \cdot e^{-\frac{\mathbb{i}}{\hbar}E_0 \cdot t}|0\rangle = e^{-\frac{\mathbb{i}}{\hbar}E_0 \cdot t}|ze^{-\mathbb{i}\omega t}\rangle$, is still a boson coherent state.

Then $\langle\psi(t)|\hat{b}|\psi(t)\rangle = ze^{-\mathbb{i}\omega t}$, $\langle\psi(t)|\hat{b}^\dagger|\psi(t)\rangle = z^*e^{\mathbb{i}\omega t}$.

Finally

$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \langle\psi(t)|\sqrt{\frac{\hbar}{2m\omega}}(\hat{b} + \hat{b}^\dagger)|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}(ze^{-\mathbb{i}\omega t} + z^*e^{\mathbb{i}\omega t}) = \frac{f}{m\omega^2}\cos(\omega t)$,

$\langle\psi(t)|\hat{p}|\psi(t)\rangle = \langle\psi(t)| - \mathbb{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{b}-\hat{b}^\dagger)|\psi(t)\rangle = \sqrt{\frac{\hbar m\omega}{2}}(-\mathbb{i}ze^{-\mathbb{i}\omega t} + \mathbb{i}z^* e^{\mathbb{i}\omega t}) = -\frac{f}{\omega}\sin(\omega t)$,

Method #2: Heisenberg picture.

Define the Heisenberg picture operators $\hat{x}_H(t) = \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)\cdot\hat{x}\cdot\exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)$, and $\hat{p}_H(t) = \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)\cdot\hat{p}\cdot\exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)$.

They satisfy $[\hat{x}_H(t),\hat{p}_H(t)] = \mathbb{i}\hbar$. And the Heisenberg picture of $\hat{H}_0$ is simply $\hat{H}_{0,H}(t) = \frac{1}{2m}[\hat{p}_H(t)]^2 + \frac{m\omega^2}{2}[\hat{x}_H(t)]^2$.

The Heisenberg equations of motion for $\hat{x}_H$ and $\hat{p}_H$ are

$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H(t) = \frac{\mathbb{i}}{\hbar}[\hat{H}_{0,H}(t),\hat{x}_H(t)] = \frac{1}{m}\hat{p}_H(t)$, and $\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_H(t) = \frac{\mathbb{i}}{\hbar}[\hat{H}_{0,H}(t),\hat{p}_H(t)] = -m\omega^2\hat{x}_H(t)$.

The solution to these equations is

$\hat{x}_H(t) = \hat{x}_H(t=0)\cos(\omega t) + \frac{1}{m\omega}\hat{p}_H(t=0)\sin(\omega t) = \hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)$,

$\hat{p}_H(t) = \hat{p}_H(t=0)\cos(\omega t) - m\omega\hat{x}_H(t=0)\sin(\omega t) = \hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)$.

Finally,

$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \langle\psi(t=0)\hat{x}_H(t)|\psi(t=0)\rangle = \langle 0'|[\hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)]|0'\rangle = \frac{f}{m\omega^2}\cos(\omega t)$,

and $\langle\psi(t)|\hat{p}|\psi(t)\rangle = \langle\psi(t=0)\hat{p}_H(t)|\psi(t=0)\rangle = \langle 0'|[\hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)]|0'\rangle$

$= -m\omega\frac{f}{m\omega^2}\sin(\omega t) = -\frac{f}{\omega}\sin(\omega t)$.

(d) According to the method #1 of (c), $|\psi(t)\rangle = e^{-\frac{\mathbb{i}}{\hbar}E_0\cdot t}|ze^{-\mathbb{i}\omega t}\rangle$ is a boson coherent state, $\hat{b}|\psi(t)\rangle = ze^{-\mathbb{i}\omega t}|\psi(t)\rangle$ with $z = \frac{f}{m\omega^2\hbar}\sqrt{\frac{\hbar m\omega}{2}}$.

$\hat{x}^2 = \frac{\hbar}{2m\omega}(\hat{b}+\hat{b}^\dagger)^2 = \frac{\hbar}{2m\omega}[\hat{b}^2 + (\hat{b}^\dagger)^2 + 2\hat{b}^\dagger\hat{b} + 1]$.

$\hat{p}^2 = -\frac{\hbar m\omega}{2}(\hat{b}-\hat{b}^\dagger)^2 = \frac{\hbar m\omega}{2}[-\hat{b}^2 - (\hat{b}^\dagger)^2 + 2\hat{b}^\dagger\hat{b} + 1]$.

Finally

$\langle\psi(t)|\hat{x}^2|\psi(t)\rangle = \frac{\hbar}{2m\omega}[z^2e^{-2\mathbb{i}\omega t} + (z^*)^2e^{2\mathbb{i}\omega t} + 2|z|^2 + 1] = \frac{\hbar}{2m\omega}[(ze^{-\mathbb{i}\omega t} + z^*e^{\mathbb{i}\omega t})^2 + 1]$

$= [\frac{f}{m\omega^2}\cos(\omega t)]^2 + \frac{\hbar}{2m\omega}$, and

$\langle\psi(t)|\hat{p}^2|\psi(t)\rangle = \frac{\hbar m\omega}{2}[-z^2e^{-2\mathbb{i}\omega t} - (z^*)^2e^{2\mathbb{i}\omega t} + 2|z|^2 + 1] = \frac{\hbar m\omega}{2}[-(ze^{-\mathbb{i}\omega t} - z^*e^{\mathbb{i}\omega t})^2 + 1]$

$= [\frac{f}{\omega}\sin(\omega t)]^2 + \frac{\hbar m\omega}{2}$.

Combine these with the result of (c), the variance of $\hat{x}$ and $\hat{p}$ under state $|\psi(t)\rangle$ are $\langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}$ and $\langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar m\omega}{2}$, independent of time, and satisfy the uncertainty relation $(\langle x^2\rangle - \langle x\rangle^2)(\langle p^2\rangle - \langle p\rangle^2) \geq \frac{\hbar^2}{4}$.

(e). $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_{1,H}(t) = 2\omega\hat{O}_{2,H}(t)$, and $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_{2,H}(t) = -2\omega\hat{O}_{1,H}(t)$.

Method #1: use the Heisenberg equations of motion, $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_H(t) = \frac{\mathbb{i}}{\hbar}[\hat{H}_{0,H}(t), \hat{O}_H(t)]$, and compute the commutators using $[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}]\hat{D} + [\hat{A}, \hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B}, \hat{D}] + \hat{C}[\hat{A}, \hat{D}]\hat{B}$ and $[\hat{x}_H(t), \hat{p}_H(t)] = \mathbb{i}\hbar$.

Method #2: use the Heisenberg equations of motion for $\hat{x}_H$ and $\hat{p}_H$ in method #2 of (c). $\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H = \frac{1}{m}\hat{p}_H$, and $\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_H = -m\omega^2\hat{x}_H$.

For notation simplicity, the argument t for Heisenberg picture operators are omitted here.

$\frac{\mathrm{d}}{\mathrm{d}t}(m^2\omega^2\hat{x}_H^2 - \hat{p}_H^2) = m^2\omega^2(\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H \cdot \hat{x}_H + \hat{x}_H \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H) - (\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_H \cdot \hat{p}_H + \hat{p}_H \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_H)$
$= m^2\omega^2 \cdot \frac{1}{m}(\hat{p}_H\hat{x}_H + \hat{x}_H\hat{p}_H) - (-m\omega^2)(\hat{x}_H\hat{p}_H + \hat{p}_H\hat{x}_H) = 2\omega \cdot m\omega(\hat{x}_H\hat{p}_H + \hat{p}_H\hat{x}_H)$

$\frac{\mathrm{d}}{\mathrm{d}t}[(m\omega(\hat{x}_H\hat{p}_H + \hat{p}_H\hat{x}_H)] = m\omega(\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H \cdot \hat{p}_H + \hat{x}_H \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_H + \frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_H \cdot \hat{x}_H + \hat{p}_H \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H)$
$= m\omega(\frac{1}{m}\hat{p}_H \cdot \hat{p}_H - m\omega^2\hat{x}_H \cdot \hat{x}_H - m\omega^2\hat{x}_H \cdot \hat{x}_H + \frac{1}{m}\hat{p}_H \cdot \hat{p}_H) = -2\omega \cdot (m^2\omega^2\hat{x}_H^2 - \hat{p}^2)$.

(f). The solution is

$\hat{O}_{1,H}(t) = \hat{O}_{1,H}(t=0)\cos(2\omega t) + \hat{O}_{2,H}(t=0)\sin(2\omega t)$,
$\hat{O}_{2,H}(t) = \hat{O}_{2,H}(t=0)\cos(2\omega t) - \hat{O}_{1,H}(t=0)\sin(2\omega t)$.

Method #1: write the equations in (e) as $\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{O}_{1,H}\\ \hat{O}_{2,H}\end{pmatrix} = \begin{pmatrix}0 & 2\omega\\ -2\omega & 0\end{pmatrix} \cdot \begin{pmatrix}\hat{O}_{1,H}\\ \hat{O}_{2,H}\end{pmatrix}$.

The solution is $\begin{pmatrix}\hat{O}_{1,H}\\ \hat{O}_{2,H}\end{pmatrix} = \exp\left[\begin{pmatrix}0 & 2\omega\\ -2\omega & 0\end{pmatrix} \cdot t\right] \cdot \begin{pmatrix}\hat{O}_{1,H}(t=0)\\ \hat{O}_{2,H}(t=0)\end{pmatrix}$.

$\exp[\begin{pmatrix}0 & 2\omega t\\ -2\omega t & 0\end{pmatrix}] = \exp[\mathbb{i}\cdot(2\omega t)\cdot\sigma_2] = \cos(2\omega t)\sigma_0 + \mathbb{i}\sin(2\omega t)\sigma_2 = \begin{pmatrix}\cos(2\omega t) & \sin(2\omega t)\\ -\sin(2\omega t) & \cos(2\omega t)\end{pmatrix}$.

One can also first diagonalize the 2×2 matrix $\begin{pmatrix}0 & 2\omega t\\ -2\omega t & 0\end{pmatrix}$. Or equivalently consider $\frac{\mathrm{d}}{\mathrm{d}t}(\hat{O}_{1,H} \pm \mathbb{i}\hat{O}_{2,H}) = \pm(2\omega\mathbb{i}) \cdot (\hat{O}_{1,H} \pm \mathbb{i}\hat{O}_{2,H})$, whose solution is $(\hat{O}_{1,H} \pm \mathbb{i}\hat{O}_{2,H}) = e^{\pm 2\omega t\mathbb{i}}[\hat{O}_{1,H}(t=0) \pm \mathbb{i}\hat{O}_{2,H}(t=0)]$.

Method #2: In fact these can be obtained without using the equations of motion in (e). Use $\hat{x}_H = \hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)$, and $\hat{p}_H = \hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)$. Then
$\hat{O}_{1,H} = m^2\omega^2\hat{x}_H^2 - \hat{p}_H^2 = m^2\omega^2[\hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)]^2 - [\hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)]^2$
$= (m^2\omega^2\hat{x}^2 - \hat{p}^2) \cdot [\cos(\omega t)^2 - \sin(\omega t)^2] + m\omega(\hat{x}\hat{p} + \hat{p}\hat{x}) \cdot 2\cos(\omega t)\sin(\omega t)$, and

$\hat{O}_{2,H} = m\omega(\hat{x}_H\hat{p}_H + \hat{p}_H\hat{x}_H) = m\omega \cdot \Big\{[\hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)] \cdot [\hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)]$
$+[\hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)] \cdot [\hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)]\Big\}$
$= (m^2\omega^2\hat{x}^2 - \hat{p}^2) \cdot [-2\cos(\omega t)\sin(\omega t)] + m\omega(\hat{x}\hat{p} + \hat{p}\hat{x}) \cdot [\cos(\omega t)^2 - \sin(\omega t)^2]$.

2. Consider a spin-1/2 moment. Its state belongs to a two-dimensional Hilbert space, with complete orthonormal basis $|\uparrow\rangle$ and $|\downarrow\rangle$ (spin up and down). Consider a periodic magnetic field $B(t)$ with period $T > 0$, $B(t) = \begin{cases} B, & \text{if } n < \frac{t}{T} < n + \frac{1}{2} \text{ for some integer } n; \\ -B, & \text{if } n + \frac{1}{2} < \frac{t}{T} < n + 1 \text{ for some integer } n. \end{cases}$. Here B is a positive constant. The Hamiltonian is $\hat{H}(t) = -B(t) \cdot \sigma_3$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is Pauli matrix.

(a). (5pts) *Write down the explicit form of time-evolution operator $\hat{U}(t)$, in terms of Pauli matrices.* [Hint: although $\hat{H}$ is time-dependent, $\hat{H}$ at different time commute]

(b). (5pts) Given the state at $t = 0$ as $|\psi(t = 0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ under the above basis. *Compute the time-dependent expectation values $\langle\psi(t)|\sigma_1|\psi(t)\rangle$, $\langle\psi(t)|\sigma_2|\psi(t)\rangle$, $\langle\psi(t)|\sigma_3|\psi(t)\rangle$.*

(c). (5pts) *Compute the "retarded Green's function", the Fourier transform of $\hat{U}(t)$ over $t > 0$, $\hat{G}(\omega) = \mathbb{i}\int_0^\infty \mathrm{Tr}[\hat{U}(t)]\, e^{\mathbb{i}\omega t}\, \mathrm{d}t$. Find out the "energy spectrum" namely the poles of $\hat{G}(\omega)$.* Here Tr is the (matrix) trace. [Hint: to make this integral absolutely convergent, you can add an infinitesimal positive imaginary part to ω, namely compute $\tilde{G}(\omega + \mathbb{i}\delta)$ and eventually take $\delta \to +0$ limit]

(NOT REQUIRED) At any instant of time, $\hat{H}(t)$ has the same eigenvalues $\pm B$. However these are not the poles of $\tilde{G}(\omega)$ solved in (c). When the period $T \to +\infty$, will the spectrum in (c) goes back to the spectrum of a time-independent Hamiltonian with only two poles at $\omega = \pm B/\hbar$?

Solution:

(a) Because $\hat{H}(t)$ for different time commute,

$\hat{U}(t) = \exp(-\frac{\mathbb{i}}{\hbar}\int_0^t \hat{H}(t)\mathrm{d}t) = \exp(\frac{\mathbb{i}}{\hbar}\hat{\sigma_3}\int_0^t B(t)\mathrm{d}t)$

$= \cos(\frac{1}{\hbar}\int_0^t B(t)\mathrm{d}t)\sigma_0 + \mathbb{i}\sin(\frac{1}{\hbar}\int_0^t B(t)\mathrm{d}t)\sigma_3$ [check Homework #1 Problem 6(b)]

$$= \begin{pmatrix} \exp(\frac{\mathbb{i}}{\hbar}\int_0^t B(t)\mathrm{d}t) & 0 \\ 0 & \exp(-\frac{\mathbb{i}}{\hbar}\int_0^t B(t)\mathrm{d}t) \end{pmatrix}.$$

$$\int_0^t B(t)\mathrm{d}t = \begin{cases} B\cdot(t-nT), & \text{if } n < \frac{t}{T} < n+\frac{1}{2} \text{ for some integer } n; \\ B\cdot((n+1)T-t), & \text{if } n+\frac{1}{2} < \frac{t}{T} < n+1 \text{ for some integer } n. \end{cases}$$

$$\text{So } \hat{U}(t) = \begin{cases} \cos(\frac{B}{\hbar}(t-nT))\sigma_0 + \mathbb{i}\sigma_3\sin(\frac{B}{\hbar}(t-nT)), & \text{if } n < \frac{t}{T} < n+\frac{1}{2}; \\ \cos(\frac{B}{\hbar}((n+1)T-t))\sigma_0 + \mathbb{i}\sigma_3\sin(\frac{B}{\hbar}((n+1)T-t)), & \text{if } n+\frac{1}{2} < \frac{t}{T} < n+1. \end{cases}$$

(b) for notation simplicity, define

$$f(t) = \begin{cases} (t-nT), & \text{if } n < \frac{t}{T} < n+\frac{1}{2} \text{ for some integer } n; \\ ((n+1)T-t), & \text{if } n+\frac{1}{2} < \frac{t}{T} < n+1 \text{ for some integer } n. \end{cases}$$

which is a periodic function, $f(t) = f(t+T)$.

Then $\hat{U}(t) = \exp[\frac{\mathbb{i}B}{\hbar}f(t)\hat{\sigma}_3] = \cos[\frac{B}{\hbar}f(t)]\sigma_0 + \mathbb{i}\sin[\frac{B}{\hbar}f(t)]\sigma_3$. And

$\hat{U}^\dagger(t)\sigma_1\hat{U}(t) = \cos[2\frac{B}{\hbar}f(t)]\sigma_1 + \sin[2\frac{B}{\hbar}f(t)]\sigma_2$,

$\hat{U}^\dagger(t)\sigma_2\hat{U}(t) = \cos[2\frac{B}{\hbar}f(t)]\sigma_2 - \sin[2\frac{B}{\hbar}f(t)]\sigma_1$, and

$\hat{U}^\dagger(t)\sigma_3\hat{U}(t) = \sigma_3$.

Use the initial conditions $\langle\psi(t=0)|\sigma_1|\psi(t=0)\rangle = 1$, $\langle\psi(t=0)|\sigma_2|\psi(t=0)\rangle = 0$, $\langle\psi(t=0)|\sigma_3|\psi(t=0)\rangle = 0$.

The expectation values at time t are $\langle\psi(t)|\sigma_a|\psi(t)\rangle = \langle\psi(t=0)|\hat{U}^\dagger(t)\sigma_a\hat{U}(t)|\psi(t=0)\rangle$ for $a = 1,2,3$.

Finally, $\langle\psi(t)|\sigma_1|\psi(t)\rangle = \cos[2\frac{B}{\hbar}f(t)]$, $\langle\psi(t)|\sigma_2|\psi(t)\rangle = -\sin[2\frac{B}{\hbar}f(t)]$, $\langle\psi(t)|\sigma_3|\psi(t)\rangle = 0$.

The spin is actually oscillating back and forth in xy-plane.

(c) $\hat{G}(\omega) = \mathbb{i}\int_0^\infty \mathrm{Tr}[\hat{U}(t)]e^{\mathbb{i}\omega t}\mathrm{d}t = \mathbb{i}\int_0^\infty 2\cos[\frac{B}{\hbar}f(t)]e^{\mathbb{i}\omega t}\,\mathrm{d}t$.

Because of the periodicity of $f(t)$,

$2\mathbb{i}\int_0^\infty \cos[\frac{B}{\hbar}f(t)]e^{\mathbb{i}\omega t}\,\mathrm{d}t = 2\mathbb{i}\sum_{n=0}^\infty e^{\mathbb{i}\omega Tn}\int_0^T \cos[\frac{B}{\hbar}f(t)]e^{\mathbb{i}\omega t}\,\mathrm{d}t$.

By adding a small positive imaginary part to ω, this infinite series becomes absolutely convergent. Then formally this is $2\mathbb{i}\cdot(1-e^{\mathbb{i}\omega T})^{-1}\cdot\int_0^T \cos[\frac{B}{\hbar}f(t)]e^{\mathbb{i}\omega t}\,\mathrm{d}t$.

The last integral over a period is $\int_0^{T/2}\cos(\frac{B}{\hbar}t)e^{\mathbb{i}\omega t}\mathrm{d}t + \int_{T/2}^T \cos(\frac{B}{\hbar}\cdot(T-t))e^{\mathbb{i}\omega t}\mathrm{d}t$

$= \int_0^{T/2}\frac{1}{2}[e^{\mathbb{i}t\cdot(\omega+\frac{B}{\hbar})} + e^{\mathbb{i}t\cdot(\omega-\frac{B}{\hbar})}]\mathrm{d}t + \int_{T/2}^T \frac{1}{2}[e^{\mathbb{i}t\cdot(\omega-\frac{B}{\hbar})}e^{\mathbb{i}\frac{B}{\hbar}T} + e^{\mathbb{i}t\cdot(\omega+\frac{B}{\hbar})}e^{-\mathbb{i}\frac{B}{\hbar}T}]\mathrm{d}t$

$= \frac{1}{2\mathbb{i}}[\frac{e^{\mathbb{i}T\cdot(\omega+\frac{B}{\hbar})/2}-1}{\omega+B/\hbar} + \frac{e^{\mathbb{i}T\cdot(\omega-\frac{B}{\hbar})/2}-1}{\omega-B/\hbar} + \frac{e^{\mathbb{i}T\cdot(\omega-\frac{B}{\hbar})}-e^{\mathbb{i}T\cdot(\omega-\frac{B}{\hbar})/2}}{\omega-B/\hbar}e^{\mathbb{i}\frac{B}{\hbar}T} + \frac{e^{\mathbb{i}T\cdot(\omega+\frac{B}{\hbar})}-e^{\mathbb{i}T\cdot(\omega+\frac{B}{\hbar})/2}}{\omega+B/\hbar}e^{-\mathbb{i}\frac{B}{\hbar}T}]$

$= \frac{1}{2\mathbb{i}}[\frac{(e^{\mathbb{i}T\cdot(\omega+\frac{B}{\hbar})/2}-1)(e^{\mathbb{i}T\cdot(\omega-\frac{B}{\hbar})/2}+1)}{\omega+B/\hbar} + \frac{(e^{\mathbb{i}T\cdot(\omega-\frac{B}{\hbar})/2}-1)(e^{\mathbb{i}T\cdot(\omega+\frac{B}{\hbar})/2}+1)}{\omega-B/\hbar}]$

Finally $\hat{G}(\omega) = \frac{1}{1-e^{\mathbb{i}\omega T}}[\frac{(e^{\mathbb{i}T\cdot(\omega+\frac{B}{\hbar})/2}-1)(e^{\mathbb{i}T\cdot(\omega-\frac{B}{\hbar})/2}+1)}{\omega+B/\hbar} + \frac{(e^{\mathbb{i}T\cdot(\omega-\frac{B}{\hbar})/2}-1)(e^{\mathbb{i}T\cdot(\omega+\frac{B}{\hbar})/2}+1)}{\omega-B/\hbar}]$

Only the factor $\frac{1}{1-e^{\mathbb{i}\omega T}}$ has simple poles at $\omega = \frac{2\pi}{T}n$ for integer n . The last factor in $[\dots]$ has no singularity for finite ω, even for $\omega \to \pm\frac{B}{\hbar}$.

Side remarks: the residue at $\omega = \frac{2\pi}{T}n$ is $\frac{\sin(\frac{BT}{2\hbar})}{\frac{BT}{2\hbar}} \cdot \frac{2\cdot(-1)^n}{(\frac{2\pi n}{T}\cdot\frac{\hbar}{B})^2-1}$.

Use $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n/x)^2-1} = -\frac{\pi x}{\sin(\pi x)}$, we have the "sum rule", the sum the these residues is a constant, $\sum_n \frac{\sin(\frac{BT}{2\hbar})}{\frac{BT}{2\hbar}} \cdot \frac{2\cdot(-1)^n}{(\frac{2\pi n}{T}\cdot\frac{\hbar}{B})^2-1} = \frac{\sin(\frac{BT}{2\hbar})}{\frac{BT}{2\hbar}} \cdot 2 \cdot (-\frac{BT/2\hbar}{\sin(BT/2\hbar)}) = -2$.

Note that these "energy levels" depends only on the period T, but is independent of the field strength B. The distribution of spectral weights (the residues) does depend on B.

3. Consider the spin-1/2 moment defined in Problem 2. Under the $|\uparrow\rangle, |\downarrow\rangle$ basis, the Hamiltonian at time t is $\hat{H}(t) = -B\cos(\omega t)\sigma_1 - B\sin(\omega t)\sigma_2$. Here B, ω are positive constants, $\sigma_{1,2}$ are Pauli matrices.

(a). (5pts) The time evolution operator $\hat{U}(t)$ satisfies $\mathbb{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t) = \hat{H}(t)\cdot\hat{U}(t)$, and $\hat{U}(t=0) = \hat{\mathbb{1}}$, and is a 2×2 matrix under the $|\uparrow\rangle, |\downarrow\rangle$ basis. Assume that B is a small parameter, *compute* $\hat{U}(t)$ *up to* B^2 *order by the Dyson series.*

(b). (DIFFICULT) (5pts) The time evolution can be solved exactly. Assume the solution to the Schrödinger equation, $\mathbb{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi,t\rangle = \hat{H}(t)|\psi,t\rangle$, is $|\psi,t\rangle = c_1(t)|\uparrow\rangle + c_2(t)e^{\mathbb{i}\omega\cdot t}|\downarrow\rangle$. *Solve* $c_1(t)$ *and* $c_2(t)$ *in terms of the initial values* $c_1(t=0)$ *and* $c_2(t=0)$*, and therefore solve the unitary time evolution operator* $\hat{U}(t)$ *as a* 2×2 *matrix under* $|\uparrow\rangle, |\downarrow\rangle$ *basis.* [note that $\begin{pmatrix} c_1(t) \\ c_2(t)e^{\mathbb{i}\omega\cdot t}\end{pmatrix} = \hat{U}(t)\cdot\begin{pmatrix} c_1(t=0) \\ c_2(t=0)\end{pmatrix}$ under the time-independent basis $|\uparrow\rangle, |\downarrow\rangle$]

Solution:

(a). $\hat{U}(t) = \hat{\mathbb{1}} + \frac{-\mathbb{i}}{\hbar}\int_0^t \mathrm{d}t_1\, \hat{H}(t_1) + (\frac{-\mathbb{i}}{\hbar})^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, \hat{H}(t_1)\hat{H}(t_2) + \dots$.

$\int_0^t \mathrm{d}t_1\, \hat{H}(t_1) = -\frac{B}{\omega}[\sin(\omega t)\sigma_1 + (1-\cos(\omega t))\sigma_2]$

$\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, \hat{H}(t_1)\hat{H}(t_2) = \int_0^t \mathrm{d}t_1\, \hat{H}(t_1)\cdot(-\frac{B}{\omega})[\sin(\omega t_1)\sigma_1 + (1-\cos(\omega t_1))\sigma_2]$

$$= \frac{B^2}{\omega}\int_0^t \mathrm{d}t_1 \{(\cos(\omega t)\sin(\omega t) + \sin(\omega t)(1-\cos(\omega t)) \\ + [\cos(\omega t)(1-\cos(\omega t)) - \sin(\omega t)\sin(\omega t)](\mathbb{i}\sigma_3)\}$$

$$= \frac{B^2}{\omega^2}[(1-\cos(\omega t)) + (\sin(\omega t) - \omega t)(\mathbb{i}\sigma_3)]$$

$$\hat{U}(t) = \mathbb{1}_{2\times 2} + \mathbb{i}\frac{B}{\hbar\omega}[\sin(\omega t)\sigma_1 + (1-\cos(\omega t))\sigma_2] - \frac{B^2}{\hbar^2\omega^2}[(1-\cos(\omega t)) + (\sin(\omega t)-\omega t)(\mathbb{i}\sigma_3)] + \dots$$

$$= \mathbb{1}_{2\times 2} + \frac{B}{\hbar\omega}\begin{pmatrix} 0 & (1-e^{-\mathbb{i}\omega t}) \\ (1-e^{\mathbb{i}\omega t}) & 0 \end{pmatrix} + \frac{B^2}{\hbar^2\omega^2}\begin{pmatrix} e^{-\mathbb{i}\omega t} - 1 + \mathbb{i}\omega t & 0 \\ 0 & e^{\mathbb{i}\omega t} - 1 - \mathbb{i}\omega t \end{pmatrix} + \dots$$

(b) From $\mathbb{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_1(t) \\ c_2(t)e^{\mathbb{i}\omega t}\end{pmatrix} = \begin{pmatrix} 0 & -Be^{-\mathbb{i}\omega t} \\ -Be^{\mathbb{i}\omega t} & 0 \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t)e^{\mathbb{i}\omega t}\end{pmatrix}$,

$$\mathbb{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_1(t) \\ c_2(t)\end{pmatrix} = \begin{pmatrix} 0 & -B \\ -B & \hbar\omega \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t)\end{pmatrix} = (-B\sigma_1 - \frac{\hbar\omega}{2}\sigma_3 + \frac{\hbar\omega}{2}\mathbb{1}_{2\times 2})\begin{pmatrix} c_1(t) \\ c_2(t)\end{pmatrix}.$$

Therefore $\begin{pmatrix} c_1(t) \\ c_2(t)\end{pmatrix} = \exp(-\frac{\mathbb{i}}{\hbar}\cdot[-B\sigma_1 - \frac{\hbar\omega}{2}\sigma_3 + \frac{\hbar\omega}{2}\mathbb{1}_{2\times 2}]\cdot t)\cdot\begin{pmatrix} c_1(t=0) \\ c_2(t=0)\end{pmatrix}$.

Use the result of Homework #1 Problem 6(b),

$\exp(-\frac{\mathbb{i}}{\hbar}\cdot[-B\sigma_1 - \frac{\hbar\omega}{2}\sigma_3 + \frac{\hbar\omega}{2}\mathbb{1}_{2\times 2}]\cdot t) = e^{-\mathbb{i}\omega t/2}[\cos(\Omega t)\mathbb{1}_{2\times 2} + \mathbb{i}\sin(\Omega t)(\frac{B/\hbar}{\Omega}\sigma_1 + \frac{\omega/2}{\Omega}\sigma_3)]$, where $\Omega = \sqrt{(\frac{B}{\hbar})^2 + (\frac{\omega}{2})^2}$.

$$\begin{pmatrix} c_1(t) \\ c_2(t)\end{pmatrix} = e^{-\mathbb{i}\omega t/2}\cdot\begin{pmatrix} \cos(\Omega t) + \mathbb{i}\frac{\omega}{2\Omega}\sin(\Omega t) & \mathbb{i}\frac{B}{\hbar\Omega}\sin(\Omega t) \\ \mathbb{i}\frac{B}{\hbar\Omega}\sin(\Omega t) & \cos(\Omega t) - \mathbb{i}\frac{\omega}{2\Omega}\sin(\Omega t)\end{pmatrix}\begin{pmatrix} c_1(t=0) \\ c_2(t=0)\end{pmatrix}.$$

Equivalently,

$$\begin{pmatrix} c_1(t) \\ c_2(t)e^{\mathbb{i}\omega t}\end{pmatrix} = \begin{pmatrix} e^{-\mathbb{i}\omega t/2}[\cos(\Omega t) + \mathbb{i}\frac{\omega}{2\Omega}\sin(\Omega t)] & e^{-\mathbb{i}\omega t/2}\mathbb{i}\frac{B}{\hbar\Omega}\sin(\Omega t) \\ e^{\mathbb{i}\omega t/2}\mathbb{i}\frac{B}{\hbar\Omega}\sin(\Omega t) & e^{\mathbb{i}\omega t/2}[\cos(\Omega t) - \mathbb{i}\frac{\omega}{2\Omega}\sin(\Omega t)]\end{pmatrix}\begin{pmatrix} c_1(t=0) \\ c_2(t=0)\end{pmatrix}$$

Therefore $U(t) = \begin{pmatrix} e^{-\mathbb{i}\omega t/2}[\cos(\Omega t) + \mathbb{i}\frac{\omega}{2\Omega}\sin(\Omega t)] & e^{-\mathbb{i}\omega t/2}\mathbb{i}\frac{B}{\hbar\Omega}\sin(\Omega t) \\ e^{\mathbb{i}\omega t/2}\mathbb{i}\frac{B}{\hbar\Omega}\sin(\Omega t) & e^{\mathbb{i}\omega t/2}[\cos(\Omega t) - \mathbb{i}\frac{\omega}{2\Omega}\sin(\Omega t)]\end{pmatrix}$.

One can check that expanding this formula as a series of B reproduces the result of (a).