Advanced Quantum Mechanics: Fall 2019
Final Exam: Brief Solutions

NOTE: Sentences in italic fonts are questions to be answered.

Possibly Useful facts:

- $\epsilon_{abc} \equiv \begin{cases} +1, & abc = xyz, \text{ or } yzx, \text{ or } zxy; \\ -1, & abc = zyx, \text{ or } xzy, \text{ or } yxz; \\ 0, & \text{otherwise.} \end{cases}$ $\epsilon_{abc} = \epsilon_{bca} = -\epsilon_{acb}$. $\delta_{ab} \equiv \begin{cases} 1, & a = b; \\ 0, & a \neq b. \end{cases}$
- Some Taylor expansions: $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)$, $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)$, $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + O(x^4)$.
- Baker-Hausdorff formula: $\exp(\hat{A}) \cdot \hat{B} \cdot \exp(-\hat{A}) = \hat{B} + \sum_{n=1}^{+\infty} \frac{1}{n!} \underbrace{[\hat{A}, [\hat{A}, \dots [\hat{A},}_{n\text{-fold commutator}} \hat{B}] \dots]]$.
- Spin (angular momentum) operators satisfy $[\hat{S}_a, \hat{S}_b] = \mathbb{i} \sum_c \epsilon_{abc} \hat{S}_c$. $(a, b, c = x, y, z)$
  - $\hat{\boldsymbol{S}}^2 \equiv \sum_a \hat{S}_a^2$, $\hat{S}_\pm \equiv \hat{S}_x \pm \mathbb{i} \hat{S}_y$. $[\hat{\boldsymbol{S}}^2, \hat{S}_{x,y,z}] = 0$, $[\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm$. Basis $|S, m\rangle$ satisfy, $\hat{S}_z |S, m\rangle = m |S, m\rangle$, $\hat{S}_\pm |S, m\rangle = \sqrt{(S \mp m)(S \pm m + 1)} |S, m \pm 1\rangle$, $\hat{\boldsymbol{S}}^2 |S, m\rangle = S(S+1) |S, m\rangle$. $2S$ is non-negative integer, $m = -S, -S+1, \dots, S$.
  - $e^{-\mathbb{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}} \cdot \hat{S}_a \cdot e^{\mathbb{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}} = \sum_b \hat{S}_b \cdot [R_{\boldsymbol{n}}(\theta)]_{ba}$. $SO(3)$ matrix for rotation around axis $\boldsymbol{n}$ by angle θ is $[R_{\boldsymbol{n}}(\theta)]_{ab} = n_a n_b + \cos\theta (\delta_{ab} - n_a n_b) - \sin\theta \sum_c \epsilon_{abc} n_c$, here $\boldsymbol{n}$ is 3D unit-length real vector, $\boldsymbol{n} \cdot \hat{\boldsymbol{S}} \equiv n_x \hat{S}_x + n_y \hat{S}_y + n_z \hat{S}_z$.
  - $\hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j \equiv \hat{S}_{iz} \hat{S}_{jz} + \hat{S}_{ix} \hat{S}_{jx} + \hat{S}_{iy} \hat{S}_{jy} = \hat{S}_{iz} \hat{S}_{jz} + \frac{1}{2}(\hat{S}_{i+} \hat{S}_{j-} + \hat{S}_{i-} \hat{S}_{j+})$.
- Spin-1/2: $\hat{S}_a = \sigma_a / 2$ under the $\hat{S}_z$ eigenbasis $(a = x, y, z)$.
  Pauli matrices σ_a are $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $\{\sigma_a, \sigma_b\} = 2\delta_{ab} \mathbb{1}$.
  $\exp(-\mathbb{i}\theta \boldsymbol{n} \cdot \boldsymbol{\sigma}) = \cos(\theta) \mathbb{1} - \mathbb{i} \sin(\theta) (\boldsymbol{n} \cdot \boldsymbol{\sigma})$. $|\hat{S}_z = \pm \frac{1}{2}\rangle$ are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$.
- Spin-1: $\hat{S}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\hat{S}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathbb{i} & 0 \\ \mathbb{i} & 0 & -\mathbb{i} \\ 0 & \mathbb{i} & 0 \end{pmatrix}$, $\hat{S}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, under the $\hat{S}_z$ eigenbasis.
- The D_4 group: $\{(C_4)^{(n \mod 4)} (\sigma_s)^{(m \mod 2)} | C_4^4 = \sigma_s^2 = C_4 \sigma_s C_4 \sigma_s = \mathbb{1}\}$.
  8 elements, 5 conjugacy classes: $\{\mathbb{1}\}$, $\{C_4, C_4^3\}$, $\{C_4^2\}$, $\{\sigma_s, C_4^2 \sigma_s\}$, and $\{C_4 \sigma_s \equiv \sigma_d, C_4^3 \sigma_s\}$.
  Character table for irreducible representations (irrep) $\Gamma_{1,2,3,4,5}$ is given on the right,

	$\mathbf{1}$	$2C_4$	C_4^2	$2\sigma_s$	$2\sigma_d$
Γ_1	1	1	1	1	1
Γ_2	1	1	1	-1	-1
Γ_3	1	-1	1	1	-1
Γ_4	1	-1	1	-1	1
Γ_5	2	0	-2	0	0

Problem 1. (15 points) For a single non-relativistic particle in harmonic potential, $\hat{H}_{\text{1-body}} = \frac{\hat{p}}{2m} + \frac{m\omega^2}{2}\hat{x}^2$, define ladder operators $\hat{a}_{\text{1-body}} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \mathbb{i}\frac{1}{m\omega}\hat{p})$, then $[\hat{a}_{\text{1-body}}, \hat{a}^\dagger_{\text{1-body}}] = 1$, and $\hat{H}_{0,\text{1-body}} = \hbar\omega \cdot (\hat{a}^\dagger_{\text{1-body}}\hat{a}_{\text{1-body}} + \frac{1}{2})$. It has a unique ground state $|\psi_{0,\text{1-body}}\rangle$ with $\hat{a}_{\text{1-body}}|\psi_{0,\text{1-body}}\rangle = 0$, $\psi_{0,\text{1-body}}(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\exp(-\frac{m\omega}{2\hbar}x^2)$, and excited states $|\psi_{n,\text{1-body}}\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}^\dagger_{\text{1-body}})^n|\psi_{0,\text{1-body}}\rangle$ with single particle energy $E_{n,\text{1-body}} = (n+\frac{1}{2})\hbar\omega$.

Consider two identical fermions, the "first quantized" form of the unperturbed Hamiltonian is $\hat{H}_0 = \frac{\hat{p}_1^2}{2m} + \frac{m\omega^2\hat{x}_1^2}{2} + \frac{\hat{p}_2^2}{2m} + \frac{m\omega^2\hat{x}_2^2}{2}$, here subscripts $_1$ and $_2$ label the two particles, $[\hat{x}_i, \hat{p}_j] = \mathbb{i}\hbar\delta_{ij}$. Consider a time-dependent perturbation $\hat{V}(t) = -f \cdot \cos(\Omega t) \cdot (\hat{x}_1 + \hat{x}_2)$, here f, Ω are positive constants, f is "small". The full Hamiltonian is $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$. The time evolution operator $\hat{U}(t)$ satisfy $\hat{U}(t=0) = \mathbb{1}$ and $\mathbb{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t) = \hat{H}(t)\cdot\hat{U}(t)$. Note that the 2-fermion wavefunctions should satisfy $\psi(x_1, x_2, t) = -\psi(x_2, x_1, t)$.

(a) (5pts) *Write down the orthonormal ground state(s)* $|\psi_{0,\textit{2-fermion}}\rangle$*, first excited state(s)* $|\psi_{1,\textit{2-fermion}}\rangle$*, second excited state(s)* $|\psi_{2,\textit{2-fermion}}\rangle$ *of unperturbed 2-fermion Hamiltonian* $\hat{H}_0$ *in terms of single particle basis* $\psi_{n,\textit{1-body}}$*, and the corresponding energy eigenvalues of* $\hat{H}_0$*.* [Hint: may have degeneracy; you don't need to write down the explicit functional form of 2-fermion wavefunctions; "second quantization" can be used but is not necessary]

(b) (5pts) *Compute the transition probability from the ground state(s) to the first excited state(s) over time t, namely* $|\langle\psi_{1,\textit{2-fermion}}|\hat{U}(t)|\psi_{0,\textit{2-fermion}}\rangle|^2$*, to lowest nontrivial order of f.* [Hint: use interaction picture]

(c) (5pts*) *Compute the transition probability from the ground state(s) to the second excited state(s) over time t, namely* $|\langle\psi_{2,\textit{2-fermion}}|\hat{U}(t)|\psi_{0,\textit{2-fermion}}\rangle|^2$*, to lowest nontrivial order of f.* [Hint: previous results may help]

Solution: this is related to Homework #3 Problem 4

(a) Method #1: "first quantized" form,

$E_{0,\text{2-fermion}} = E_{0,\text{1-body}} + E_{1,\text{1-body}} = 2\hbar\omega$ (non-degenerate),

$\psi_{0,\text{2-fermion}}(x_1, x_2) = \frac{1}{\sqrt{2}}(\psi_{0,\text{1-body}}(x_1)\psi_{1,\text{1-body}}(x_2) - \psi_{1,\text{1-body}}(x_1)\psi_{0,\text{1-body}}(x_2))$.

$E_{1,\text{2-fermion}} = E_{0,\text{1-body}} + E_{2,\text{1-body}} = 3\hbar\omega$ (non-degenerate),

$\psi_{1,\text{2-fermion}}(x_1, x_2) = \frac{1}{\sqrt{2}}(\psi_{0,\text{1-body}}(x_1)\psi_{2,\text{1-body}}(x_2) - \psi_{2,\text{1-body}}(x_1)\psi_{0,\text{1-body}}(x_2))$.

$E_{1,\text{2-fermion}} = E_{0,\text{1-body}} + E_{3,\text{1-body}} = E_{1,\text{1-body}} + E_{2,\text{1-body}} = 4\hbar\omega$ (2-fold degenerate),
$\psi_{2,\text{2-fermion},(1)}(x_1,x_2) = \frac{1}{\sqrt{2}}(\psi_{0,\text{1-body}}(x_1)\psi_{3,\text{1-body}}(x_2) - \psi_{3,\text{1-body}}(x_1)\psi_{0,\text{1-body}}(x_2))$,
$\psi_{2,\text{2-fermion},(2)}(x_1,x_2) = \frac{1}{\sqrt{2}}(\psi_{1,\text{1-body}}(x_1)\psi_{2,\text{1-body}}(x_2) - \psi_{2,\text{1-body}}(x_1)\psi_{1,\text{1-body}}(x_2))$.

Method #2: "second quantized" form,

define creation operators $\widehat{\psi_n}^\dagger$ for single particle state $\psi_{n,\text{1-body}}$, denote the fermion vacuum by $|\text{vac}\rangle$, then (check Problem 4 of Homework #3) $\hat{H}_0 = \sum_{n=0}^\infty E_{n,\text{1-body}}\widehat{\psi_n}^\dagger\widehat{\psi_n}$,
$\hat{V}(t) = -f\cdot\cos(\Omega t)\cdot\sqrt{\frac{\hbar}{2m\omega}}\sum_{n=0}^\infty\sqrt{n+1}(\widehat{\psi_n}^\dagger\widehat{\psi_{n+1}} + \widehat{\psi_{n+1}}^\dagger\widehat{\psi_n})$.
$|\psi_{0,\text{2-fermion}}\rangle = \widehat{\psi_0}^\dagger\widehat{\psi_1}^\dagger|\text{vac}\rangle$,
$|\psi_{1,\text{2-fermion}}\rangle = \widehat{\psi_0}^\dagger\widehat{\psi_2}^\dagger|\text{vac}\rangle$,
$|\psi_{2,\text{2-fermion},(1)}\rangle = \widehat{\psi_0}^\dagger\widehat{\psi_3}^\dagger|\text{vac}\rangle$, $|\psi_{2,\text{2-fermion},(2)}\rangle = \widehat{\psi_1}^\dagger\widehat{\psi_2}^\dagger|\text{vac}\rangle$.

(b) use interaction picture, $|\psi_I(t)\rangle \equiv e^{\mathbb{i}\hat{H}_0\cdot t/\hbar}|\psi(t)\rangle$, $\hat{O}_I(t) \equiv e^{\mathbb{i}\hat{H}_0\cdot t/\hbar}\hat{O}e^{-\mathbb{i}\hat{H}_0\cdot t/\hbar}$. Then $\mathbb{i}\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \hat{V}_I(t)\cdot|\psi_I(t)\rangle$, the evolution operator in interaction picture is $\hat{U}_I(t) \equiv e^{\mathbb{i}\hat{H}_0\cdot t/\hbar}\hat{U}(t) = \mathbb{1} + (\frac{-\mathbb{i}}{\hbar})\int_0^t \mathrm{d}t_1\,\hat{V}_I(t_1) + (\frac{-\mathbb{i}}{\hbar})^2\int_0^t\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,\hat{V}_I(t_1)\hat{V}_I(t_1) + \ldots$.

$|\langle\psi_f|\hat{U}(t)|\psi_i\rangle|^2 = |\langle\psi_f|\hat{U}_I(t)|\psi_i\rangle|^2$, as long as the final state $|\psi_f\rangle$ is an eigenstate of $\hat{H}_0$.

Note that $\hat{V}(t)$ and also $\hat{V}_I(t)$ can change energy of $\hat{H}_0$ by only$\pm\hbar\omega$.

The lowest non-trivial order of
$\langle\psi_{1,\text{2-fermion}}|\hat{U}_I(t)|\psi_{0,\text{2-fermion}}\rangle \approx (\frac{-\mathbb{i}}{\hbar})\int_0^t\mathrm{d}t_1\,\langle\psi_{1,\text{2-fermion}}|\hat{V}_I(t_1)|\psi_{0,\text{2-fermion}}\rangle$
$= (\frac{-\mathbb{i}}{\hbar})\int_0^t\mathrm{d}t_1\,\left[e^{\mathbb{i}E_{1,\text{2-fermion}}t/\hbar}\langle\psi_{1,\text{2-fermion}}|\hat{V}(t_1)|\psi_{0,\text{2-fermion}}\rangle e^{-\mathbb{i}E_{1,\text{1-fermion}}t/\hbar}\right]$
$= (\frac{-\mathbb{i}}{\hbar})\int_0^t\mathrm{d}t_1\,[e^{\mathbb{i}\omega t_1}\cdot(-f)\cdot\cos(\Omega t_1)\cdot\sqrt{\frac{\hbar}{2m\omega}}\cdot\sqrt{2}] = f\sqrt{\frac{1}{m\omega\hbar}}\cdot\frac{1}{2}(\frac{e^{\mathbb{i}(\omega+\Omega)t}-1}{\omega+\Omega} + \frac{e^{\mathbb{i}(\omega-\Omega)t}-1}{\omega-\Omega})$.

The matrix element $\langle\psi_{1,\text{2-fermion}}|\hat{V}(t_1)|\psi_{0,\text{2-fermion}}\rangle$ can be computed using either the "second quantized" formalism, or the "first quantized" formalism, $\langle\psi_{1,\text{2-fermion}}|\hat{V}(t_1)|\psi_{0,\text{2-fermion}}\rangle = \int\mathrm{d}x_1\int\mathrm{d}x_2\Big[$
$\frac{1}{\sqrt{2}}(\psi^*_{0,\text{1-body}}(x_1)\psi^*_{2,\text{1-body}}(x_2) - \psi^*_{2,\text{1-body}}(x_1)\psi^*_{0,\text{1-body}}(x_2))\times(-f\cos(\Omega t)(x_1+x_2))$
$\times\frac{1}{\sqrt{2}}(\psi_{0,\text{1-body}}(x_1)\psi_{1,\text{1-body}}(x_2) - \psi_{1,\text{1-body}}(x_1)\psi_{0,\text{1-body}}(x_2))\Big]$.

$|\langle\psi_{1,\text{2-fermion}}|\hat{U}(t)|\psi_{0,\text{2-fermion}}\rangle|^2 = \frac{f^2}{4m\omega\hbar}\cdot|\frac{e^{\mathbb{i}(\omega+\Omega)t}-1}{\omega+\Omega} + \frac{e^{\mathbb{i}(\omega-\Omega)t}-1}{\omega-\Omega}|^2$
$= \frac{f^2}{m\omega\hbar}\cdot[\frac{\sin^2((\omega+\Omega)t/2)}{(\omega+\Omega)^2} + 2\cos(\Omega t)\frac{\sin((\omega+\Omega)t/2)\sin((\omega-\Omega)t/2)}{(\omega+\Omega)(\omega-\Omega)} + \frac{\sin^2((\omega-\Omega)t/2)}{(\omega-\Omega)^2}]$.

(c) The second excited states are 2-fold degenerate $\psi_{2,\text{2-fermion},(i)}$, $i=1,2$.

To lowest non-trivial order,

$\langle\psi_{2,2\text{-fermion},(i)}|\hat{U}_I(t)|\psi_{0,2\text{-fermion}}\rangle \approx (\frac{-\mathbb{i}}{\hbar})^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, \langle\psi_{2,2\text{-fermion},(i)}|\hat{V}_I(t_1)\hat{V}_I(t_1)|\psi_{0,2\text{-fermion}}\rangle$
$= (\frac{-\mathbb{i}}{\hbar}) \int_0^t \mathrm{d}t_1 \left[\langle\psi_{2,2\text{-fermion},(i)}|\hat{V}_I(t_1)|\psi_{1,2\text{-fermion}}\rangle \times (\frac{-\mathbb{i}}{\hbar}) \int_0^{t_1} \mathrm{d}t_2\, \langle\psi_{1,2\text{-fermion}}|\hat{V}_I(t_1)|\psi_{0,2\text{-fermion}}\rangle\right]$
$= (\frac{-\mathbb{i}}{\hbar}) \int_0^t \mathrm{d}t_1 \left[e^{\mathbb{i}\omega t_1}\langle\psi_{2,2\text{-fermion},(i)}|\hat{V}(t_1)|\psi_{1,2\text{-fermion}}\rangle \times f\sqrt{\frac{1}{m\omega\hbar}} \cdot \frac{1}{2}(\frac{e^{\mathbb{i}(\omega+\Omega)t_1}-1}{\omega+\Omega} + \frac{e^{\mathbb{i}(\omega-\Omega)t_1}-1}{\omega-\Omega})\right]$

$\langle\psi_{2,2\text{-fermion},(1)}|\hat{V}(t_1)|\psi_{1,2\text{-fermion}}\rangle = (-f)\cdot\cos(\Omega t_1)\cdot\sqrt{\frac{\hbar}{2m\omega}}\cdot\sqrt{3}.$

$\langle\psi_{2,2\text{-fermion},(2)}|\hat{V}(t_1)|\psi_{1,2\text{-fermion}}\rangle = (-f)\cdot\cos(\Omega t_1)\cdot\sqrt{\frac{\hbar}{2m\omega}}.$

Then $\langle\psi_{2,2\text{-fermion},(i)}|\hat{U}_I(t)|\psi_{0,2\text{-fermion}}\rangle$

$$\approx \frac{f^2}{4\sqrt{2}m\omega\hbar}\left(\frac{e^{2\mathbb{i}(\omega+\Omega)t}-2e^{\mathbb{i}(\omega+\Omega)t}+1}{2(\omega+\Omega)^2} + \frac{1-e^{\mathbb{i}(\omega+\Omega)t}-e^{\mathbb{i}(\omega-\Omega)t}+e^{2\mathbb{i}\omega t}}{(\omega+\Omega)(\omega-\Omega)} + \frac{e^{2\mathbb{i}(\omega-\Omega)t}-2e^{\mathbb{i}(\omega-\Omega)t}+1}{2(\omega-\Omega)^2}\right)\cdot\begin{cases}\sqrt{3}, & i=1;\\ 1, & i=2.\end{cases}$$

$$= \frac{f^2}{8\sqrt{2}m\omega\hbar}\left(\frac{e^{\mathbb{i}(\omega+\Omega)t_1}-1}{\omega+\Omega} + \frac{e^{\mathbb{i}(\omega-\Omega)t_1}-1}{\omega-\Omega}\right)^2\cdot\begin{cases}\sqrt{3}, & i=1;\\ 1, & i=2.\end{cases}$$

Finally, $|\langle\psi_{2,2\text{-fermion},(i)}|\hat{U}(t)|\psi_{0,2\text{-fermion}}\rangle|^2 = \frac{f^4}{128m^2\omega^2\hbar^2}\left|\frac{e^{\mathbb{i}(\omega+\Omega)t_1}-1}{\omega+\Omega} + \frac{e^{\mathbb{i}(\omega-\Omega)t_1}-1}{\omega-\Omega}\right|^4\cdot\begin{cases}3, & i=1;\\ 1, & i=2.\end{cases}$

$$= \frac{f^4}{8m^2\omega^2\hbar^2}\cdot\left[\frac{\sin^2((\omega+\Omega)t/2)}{(\omega+\Omega)^2} + 2\cos(\Omega t)\frac{\sin((\omega+\Omega)t/2)\sin((\omega-\Omega)t/2)}{(\omega+\Omega)(\omega-\Omega)} + \frac{\sin^2((\omega-\Omega)t/2)}{(\omega-\Omega)^2}\right]^2\cdot\begin{cases}3, & i=1;\\ 1, & i=2.\end{cases}$$

Problem 2. (10 points)(*) *Solve the nonzero C.-G. coefficients $\langle j_1, m_1; j_2, m_2|j,m\rangle$ for $j_2 = \frac{1}{2}$ and $m_2 = \pm\frac{1}{2}$ and generic j_1, m_1, j, m.* [Hint: consider an "addition of angular momentum" problem, define angular momentum operators $\hat{\boldsymbol{J}}_1$ and $\hat{\boldsymbol{J}}_2$ for $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ Hilbert spaces respectively, define $\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_2$, then solve $|j,m\rangle$ satisfying $\hat{\boldsymbol{J}}^2|j,m\rangle = j(j+1)|j,m\rangle$ and $\hat{J}_z|j,m\rangle = m|j,m\rangle$, in terms of tensor product basis $|j_1, m_1\rangle|j_2, m_2\rangle$; you might need to use mathematical induction]

Solution:

Use $|\uparrow\rangle$ and $|\downarrow\rangle$ to denote $|j_2 = \frac{1}{2}, m_2 = \pm\frac{1}{2}\rangle$ respectively. Nonzero C.-G. coefficients must have $m_1 + m_2 = m$.

If $j_1 = 0$, then j must be $\frac{1}{2}$, and m_2 must equal to m, then (up to phase factor) $\langle j_1 = 0, m_1 = 0; j_2 = \frac{1}{2}, m_2 = m|j = \frac{1}{2}, m\rangle = 1$, $m = \pm\frac{1}{2}$.

If $j_1 > 0$, then j can be $j_1 - \frac{1}{2}$ or $j_1 + \frac{1}{2}$.

Method #1: use ladder operators,

For $j = j_1 + \frac{1}{2}$, $|j = j_1 + \frac{1}{2}, m = j\rangle = |j_1, m_1 = j_1\rangle|\uparrow\rangle$.

Apply lowering operators $\hat{J}_- = \hat{J}_{1-} + \hat{J}_{2-}$ repeatedly, we have
$|j = j_1 + \frac{1}{2}, m\rangle = \sqrt{\frac{j+m}{2j}}|j_1, m_1 = m - \frac{1}{2}; \uparrow\rangle + \sqrt{\frac{j-m}{2j}}|j_1, m_1 = m + \frac{1}{2}; \downarrow\rangle$.

This can be proved by mathematical induction: the $m = j$ case is correct, and
$|j = j_1 + \frac{1}{2}, m-1\rangle = \frac{1}{\sqrt{(j+m)(j-m+1)}}\hat{J}_-|j = j_1 + \frac{1}{2}, m\rangle$
$= \frac{1}{\sqrt{(j+m)(j-m+1)}}(\sqrt{\frac{j+m}{2j}}\sqrt{(j+m-1)(j-m+1)}|j_1, m_1 = m - \frac{3}{2}; \uparrow\rangle$
$+ \sqrt{\frac{j+m}{2j}}|j_1, m_1 = m - \frac{1}{2}; \downarrow\rangle + \sqrt{\frac{j-m}{2j}}\sqrt{(j+m)(j-m)}|j_1, m_1 = m - \frac{1}{2}; \downarrow\rangle)$
$= \sqrt{\frac{j+(m-1)}{2j}}|j_1, m_1 = (m-1) - \frac{1}{2}; \uparrow\rangle + \sqrt{\frac{j-(m-1)}{2j}}|j_1, m_1 = (m-1) + \frac{1}{2}; \downarrow\rangle$.
Therefore this formula is correct for all m. Then (up to overall phase factor)
$\langle j_1, m_1 = m - \frac{1}{2}; \uparrow |j = j_1 + \frac{1}{2}, m\rangle = \sqrt{\frac{j+m}{2j}} = \sqrt{\frac{j_1+m_1+1}{2j_1+1}}$,
$\langle j_1, m_1 = m + \frac{1}{2}; \downarrow |j = j_1 + \frac{1}{2}, m\rangle = \sqrt{\frac{j-m}{2j}} = \sqrt{\frac{j_1-m_1+1}{2j_1+1}}$. Here $m = -j, -j+1, \ldots, j$.

For $j = j_1 - \frac{1}{2}$, $|j = j_1 - \frac{1}{2}, m = j\rangle$ is a linear combination of $|j_1, m_1 = m - \frac{1}{2}\rangle| \uparrow\rangle$ and $|j_1, m_1 = m + \frac{1}{2}\rangle| \downarrow\rangle$, and is orthogonal to
$|j = j_1 + \frac{1}{2}, m = j_1 - \frac{1}{2}\rangle = \sqrt{\frac{2j_1}{2j_1+1}}|j_1, m_1 = m - \frac{1}{2}\rangle| \uparrow\rangle + \sqrt{\frac{1}{2j_1+1}}|j_1, m_1 = m + \frac{1}{2}\rangle| \downarrow\rangle$. So
$|j = j_1 - \frac{1}{2}, m = j\rangle = \sqrt{\frac{1}{2j_1+1}}|j_1, m_1 = m - \frac{1}{2}\rangle| \uparrow\rangle - \sqrt{\frac{2j_1}{2j_1+1}}|j_1, m_1 = m + \frac{1}{2}\rangle| \downarrow\rangle$.

By mathematical induction (steps omitted), or orthogonality to $|j = j_1 + \frac{1}{2}, m\rangle$,
$|j = j_1 - \frac{1}{2}, m = j\rangle = \sqrt{\frac{j-m+1}{2j+2}}|j_1, m_1 = m - \frac{1}{2}\rangle| \uparrow\rangle - \sqrt{\frac{j+m+1}{2j+2}}|j_1, m_1 = m + \frac{1}{2}\rangle| \downarrow\rangle$.
Then (up to overall phase factor)
$\langle j_1, m_1 = m - \frac{1}{2}; \uparrow |j = j_1 - \frac{1}{2}, m\rangle = \sqrt{\frac{j-m+1}{2j+2}} = \sqrt{\frac{j_1-m_1}{2j_1+1}}$,
$\langle j_1, m_1 = m + \frac{1}{2}; \downarrow |j = j_1 - \frac{1}{2}, m\rangle = -\sqrt{\frac{j+m+1}{2j+2}} = -\sqrt{\frac{j_1+m_1}{2j_1+1}}$. Here $m = -j, -j+1, \ldots, j$.

Method #2: solve eigenvalue problem $\hat{\boldsymbol{J}}^2|j, m\rangle = j(j+1)|j, m\rangle$,
$\hat{\boldsymbol{J}}^2 = \hat{\boldsymbol{J}}_1^2 + \hat{\boldsymbol{J}}_2^2 + 2\hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}}_2 = j_1(j_1+1) + \frac{3}{4} + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$.
assume $|j, m\rangle = c_1|j_1, m_1 = m - \frac{1}{2}\rangle| \uparrow\rangle + c_2|j_1, m_1 = m + \frac{1}{2}\rangle| \downarrow\rangle$, then $\hat{\boldsymbol{J}}^2|j, m\rangle$
$= [(j_1(j_1+1) + \frac{3}{4} + 2 \cdot (m - \frac{1}{2}) \cdot \frac{1}{2})c_1 + \sqrt{(j_1 + m + \frac{1}{2})(j_1 - m + \frac{1}{2})c_2}]|j_1, m_1 = m - \frac{1}{2}\rangle| \uparrow\rangle$
$+ [(j_1(j_1+1) + \frac{3}{4} + 2 \cdot (m + \frac{1}{2}) \cdot (-\frac{1}{2}))c_2 + \sqrt{(j_1 - m + \frac{1}{2})(j_1 + m + \frac{1}{2})c_1}]|j_1, m_1 = m + \frac{1}{2}\rangle| \downarrow\rangle$
$= (|j_1, m_1 = m - \frac{1}{2}\rangle| \uparrow\rangle, |j_1, m_1 = m + \frac{1}{2}\rangle| \downarrow\rangle)\begin{pmatrix} (j_1+\frac{1}{2})^2 + m & \sqrt{(j_1+\frac{1}{2})^2 - m^2} \\ \sqrt{(j_1+\frac{1}{2})^2 - m^2} & (j_1+\frac{1}{2})^2 - m \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

The components of the normalized eigenvectors of this 2×2 matrix for eigenvalues $(j_1 + \frac{1}{2})(j_1 + \frac{3}{2})$ and $(j_1 - \frac{1}{2})(j_1 + \frac{1}{2})$ are the wanted C.-G. coefficients. But you need to choose overall phase factors of the eigenvectors in order to satisfy the Condon-Shortley convention.

Problem 3 (30 points) Consider four spin-1/2 moments (labeled by subscripts $i = 1, 2, 3, 4$). The spin operators satisfy $[\hat{S}_{i,a}, \hat{S}_{j,b}] = \delta_{ij} \sum_c \mathbb{i}\epsilon_{abc}\hat{S}_{i,c}$ for $i, j = 1, 2, 3, 4$ and $a, b, c = x, y, z$. A complete orthonormal basis is the tensor product of S_z-eigenbasis, $|S_{1z}, S_{2z}, S_{3z}, S_{4z}\rangle$ (see page 1). For notation simplicity, use $\uparrow$ ($\downarrow$) to denote $S_{iz} = +\frac{1}{2}$ $(-\frac{1}{2})$. For example $|\uparrow\downarrow\downarrow\uparrow\rangle$ means $|S_{1z} = +\frac{1}{2}, S_{2z} = -\frac{1}{2}, S_{3z} = -\frac{1}{2}, S_{4z} = +\frac{1}{2}\rangle$.

(a) (5pts) Define $\hat{S}_{1+2,a} = \hat{S}_{1,a} + \hat{S}_{2,a}$, $\hat{S}_{1+2+3,a} = \hat{S}_{1+2,a} + \hat{S}_{3,a}$, and $\hat{S}_{1+2+3+4,a} = \hat{S}_{1+2+3,a} + \hat{S}_{4,a}$, for $a = x, y, z$. *Show that* $\hat{\boldsymbol{S}}^2_{1+2+3+4}$, $\hat{S}_{1+2+3+4,z}$, $\hat{\boldsymbol{S}}^2_{1+2+3}$, $\hat{\boldsymbol{S}}^2_{1+2}$, *mutually commute, namely all commutators between them vanish.*

(b) (15pts) According to (a), we can find eigenbasis $|S_{1+2+3+4}, S_{1+2+3+4,z}, S_{1+2+3}, S_{1+2}\rangle$ for $\hat{\boldsymbol{S}}^2_{1+2+3+4}$ eigenvalue $S_{1+2+3+4}(S_{1+2+3+4} + 1)$, $\hat{S}_{1+2+3+4,z}$ eigenvalue $S_{1+2+3+4,z}$, $\hat{\boldsymbol{S}}^2_{1+2+3}$ eigenvalue $S_{1+2+3}(S_{1+2+3} + 1)$, and $\hat{\boldsymbol{S}}^2_{1+2}$ eigenvalue $S_{1+2}(S_{1+2} + 1)$. *Solve all* $|S_{1+2+3+4}, S_{1+2+3+4,z}, S_{1+2+3}, S_{1+2}\rangle$ *states in terms of tensor product S_z-basis.* [Hint: this can be viewed as three steps of "addition of angular momentum" problem, first add $\hat{\boldsymbol{S}}_1$ and $\hat{\boldsymbol{S}}_2$ to get $|S_{1+2}, S_{1+2,z}\rangle$, then further add $\hat{\boldsymbol{S}}_3$ to get $|S_{1+2+3}, S_{1+2+3,z}, S_{1+2}\rangle$, and finally add $\hat{\boldsymbol{S}}_4$, some previous results may help]

(c) (10pts*) Consider $\hat{H} = \hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 + \hat{\boldsymbol{S}}_2 \cdot \hat{\boldsymbol{S}}_3 + \hat{\boldsymbol{S}}_3 \cdot \hat{\boldsymbol{S}}_4 + \hat{\boldsymbol{S}}_4 \cdot \hat{\boldsymbol{S}}_1$. It satisfies $[\hat{H}, \hat{\boldsymbol{S}}^2_{1+2+3+4}] = 0$ and $[\hat{H}, \hat{S}_{1+2+3+4,a}] = 0$ for $a = x, y, z$. *Compute all the nonzero matrix elements of $\hat{H}$ under the basis $|S_{1+2+3+4}, S_{1+2+3+4,z}, S_{1+2+3}, S_{1+2}\rangle$ solved in (b).* [Hint: rewrite $\hat{H}$ using ladder operators (see page 1), by symmetry a lot of matrix elements will be zero, and some nonzero matrix elements will be the same]

Solution: this is similar to Homework #6 Problem 4(b)

(a) Fact #1: if $\hat{J}_{x,y,z}$ satisfy $[\hat{J}_a, \hat{J}_b] = \sum_c \mathbb{i}\epsilon_{abc}\hat{J}_c$, then $[\hat{\boldsymbol{J}}^2, \hat{J}_a] = 0$.

This fact can be used directly without proof. $\hat{S}_{1+2,a}$, $\hat{S}_{1+2+3,a}$, $\hat{S}_{1+2+3+4,a}$ all satisfy the above form of commutation relations. Then $[\hat{\boldsymbol{S}}^2_{1+2+3+4}, \hat{S}_{1+2+3+4,z}] = 0$.

Fact #2: if $\hat{J}_{1,a}$ and $\hat{J}_{2,a}$ satisfy $[\hat{J}_{i,a}, \hat{J}_{j,b}] = \delta_{ij} \sum_c \mathbb{i}\epsilon_{abc}\hat{J}_{i,c}$, then $[\hat{\boldsymbol{J}}^2_1, \hat{J}_{i,a}] = 0$ for $i = 1, 2$ and $a = x, y, z$ ($i = 1$ case is fact #1, $i = 2$ case is trivial), and therefore $[\hat{\boldsymbol{J}}^2_1, \hat{J}_{1,a} + \hat{J}_{2,a}] = 0$.

Then $[\hat{\boldsymbol{S}}^2_{1+2+3}, \hat{S}_{1+2+3+4,z} = \hat{S}_{1+2+3,z} + \hat{S}_{4,z}] = 0$, $[\hat{\boldsymbol{S}}^2_{1+2}, \hat{S}_{1+2+3+4,z} = \hat{S}_{1+2,z} + \hat{S}_{3+4,z}] = 0$.

Fact #3: from fact #2, it is obvious $(\hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_2)^2 = \sum_a (\hat{J}_{1,a} + \hat{J}_{2,a})^2$ commutes with $\hat{\boldsymbol{J}}_1^2$. Then $[\hat{\boldsymbol{S}}^2_{1+2+3+4}, \hat{\boldsymbol{S}}^2_{1+2+3}] = 0$, $[\hat{\boldsymbol{S}}^2_{1+2+3+4}, \hat{\boldsymbol{S}}^2_{1+2}] = 0$, $[\hat{\boldsymbol{S}}^2_{1+2+3}, \hat{\boldsymbol{S}}^2_{1+2}] = 0$.

(b) this is the direct application of the results of Problem 2,

Relevant C.-G. coefficients are $\langle j_1, m_1; j_2 = \frac{1}{2}, m_2 = \pm\frac{1}{2}|j, m\rangle$ for $j_1 = \frac{1}{2}$ or 1 or $\frac{3}{2}$.

Add spin "1" and "2" first,

$|S_{1+2} = 0, S_{1+2,z} = 0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$.

$|S_{1+2} = 1, S_{1+2,z} = 1\rangle = |\uparrow\uparrow\rangle$.

$|S_{1+2} = 1, S_{1+2,z} = 0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$.

$|S_{1+2} = 1, S_{1+2,z} = -1\rangle = |\downarrow\downarrow\rangle$.

Then add spin "3",

$|S_{1+2+3} = \frac{1}{2}, S_{1+2+3,z} = +\frac{1}{2}, S_{1+2} = 0\rangle = |S_{1+2} = 0, S_{1+2,z} = 0\rangle|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle)$,

$|S_{1+2+3} = \frac{1}{2}, S_{1+2+3,z} = -\frac{1}{2}, S_{1+2} = 0\rangle = |S_{1+2} = 0, S_{1+2,z} = 0\rangle|\downarrow\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle)$,

$|S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = +\frac{3}{2}, S_{1+2} = 1\rangle = |S_{1+2} = 1, S_{1+2,z} = 1\rangle|\uparrow\rangle = |\uparrow\uparrow\uparrow\rangle$,

$|S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = +\frac{1}{2}, S_{1+2} = 1\rangle$

$= \sqrt{\frac{2}{3}}|S_{1+2} = 1, S_{1+2,z} = 0\rangle|\uparrow\rangle + \sqrt{\frac{1}{3}}|S_{1+2} = 1, S_{1+2,z} = 1\rangle|\downarrow\rangle = \frac{1}{\sqrt{3}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)$,

$|S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = -\frac{1}{2}, S_{1+2} = 1\rangle$

$= \sqrt{\frac{1}{3}}|S_{1+2} = 1, S_{1+2,z} = -1\rangle|\uparrow\rangle + \sqrt{\frac{2}{3}}|S_{1+2} = 1, S_{1+2,z} = 0\rangle|\downarrow\rangle$

$= \frac{1}{\sqrt{3}}(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle)$,

$|S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = -\frac{3}{2}, S_{1+2} = 1\rangle = |S_{1+2} = 1, S_{1+2,z} = -1\rangle|\downarrow\rangle = |\downarrow\downarrow\downarrow\rangle$,

$|S_{1+2+3} = \frac{1}{2}, S_{1+2+3,z} = +\frac{1}{2}, S_{1+2} = 1\rangle$

$= \sqrt{\frac{1}{3}}|S_{1+2} = 1, S_{1+2,z} = 0\rangle|\uparrow\rangle - \sqrt{\frac{2}{3}}|S_{1+2} = 1, S_{1+2,z} = 1\rangle|\downarrow\rangle = \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle)$,

$|S_{1+2+3} = \frac{1}{2}, S_{1+2+3,z} = -\frac{1}{2}, S_{1+2} = 1\rangle$

$= \sqrt{\frac{2}{3}}|S_{1+2} = 1, S_{1+2,z} = -1\rangle|\uparrow\rangle - \sqrt{\frac{1}{3}}|S_{1+2} = 1, S_{1+2,z} = 0\rangle|\downarrow\rangle$

$= \frac{1}{\sqrt{6}}(2|\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle)$.

Finally add spin "4", the results are summarized in the following table,

$S_{1+2+3+4}$	$S_{1+2+3+4,z}$	S_{1+2+3}	S_{1+2}	state
0	0	$\frac{1}{2}$	0	$\frac{1}{2}(\lvert\downarrow\uparrow\downarrow\uparrow\rangle - \lvert\uparrow\downarrow\downarrow\uparrow\rangle - \lvert\downarrow\uparrow\uparrow\downarrow\rangle + \lvert\uparrow\downarrow\uparrow\downarrow\rangle)$
1	1	$\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}(\lvert\downarrow\uparrow\uparrow\uparrow\rangle - \lvert\uparrow\downarrow\uparrow\uparrow\rangle)$
1	0	$\frac{1}{2}$	0	$\frac{1}{2}(\lvert\downarrow\uparrow\downarrow\uparrow\rangle - \lvert\uparrow\downarrow\downarrow\uparrow\rangle + \lvert\downarrow\uparrow\uparrow\downarrow\rangle - \lvert\uparrow\downarrow\uparrow\downarrow\rangle)$
1	-1	$\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}(\lvert\downarrow\uparrow\downarrow\downarrow\rangle - \lvert\uparrow\downarrow\downarrow\downarrow\rangle)$
0	0	$\frac{1}{2}$	1	$\frac{1}{2\sqrt{3}}(2\lvert\downarrow\downarrow\uparrow\uparrow\rangle - \lvert\downarrow\uparrow\downarrow\uparrow\rangle - \lvert\uparrow\downarrow\downarrow\uparrow\rangle - \lvert\downarrow\uparrow\uparrow\downarrow\rangle - \lvert\uparrow\downarrow\uparrow\downarrow\rangle + 2\lvert\uparrow\uparrow\downarrow\downarrow\rangle)$
1	1	$\frac{1}{2}$	1	$\frac{1}{\sqrt{6}}(\lvert\downarrow\uparrow\uparrow\uparrow\rangle + \lvert\uparrow\downarrow\uparrow\uparrow\rangle - 2\lvert\uparrow\uparrow\downarrow\uparrow\rangle)$
1	0	$\frac{1}{2}$	1	$\frac{1}{2\sqrt{3}}(2\lvert\downarrow\downarrow\uparrow\uparrow\rangle - \lvert\downarrow\uparrow\downarrow\uparrow\rangle - \lvert\uparrow\downarrow\downarrow\uparrow\rangle + \lvert\downarrow\uparrow\uparrow\downarrow\rangle + \lvert\uparrow\downarrow\uparrow\downarrow\rangle - 2\lvert\uparrow\uparrow\downarrow\downarrow\rangle)$
1	-1	$\frac{1}{2}$	1	$\frac{1}{\sqrt{6}}(2\lvert\downarrow\downarrow\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\downarrow\downarrow\rangle - \lvert\uparrow\downarrow\uparrow\downarrow\rangle)$
1	1	$\frac{3}{2}$	1	$\frac{1}{2\sqrt{3}}(\lvert\downarrow\uparrow\uparrow\uparrow\rangle + \lvert\uparrow\downarrow\uparrow\uparrow\rangle + \lvert\uparrow\uparrow\downarrow\uparrow\rangle - 3\lvert\uparrow\uparrow\uparrow\downarrow\rangle)$
1	0	$\frac{3}{2}$	1	$\frac{1}{\sqrt{6}}(\lvert\downarrow\downarrow\uparrow\uparrow\rangle + \lvert\downarrow\uparrow\downarrow\uparrow\rangle + \lvert\uparrow\downarrow\downarrow\uparrow\rangle - \lvert\downarrow\uparrow\uparrow\downarrow\rangle - \lvert\uparrow\downarrow\uparrow\downarrow\rangle - \lvert\uparrow\uparrow\downarrow\downarrow\rangle)$
1	-1	$\frac{3}{2}$	1	$\frac{1}{2\sqrt{3}}(3\lvert\downarrow\downarrow\downarrow\uparrow\rangle - \lvert\downarrow\downarrow\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\downarrow\downarrow\rangle - \lvert\uparrow\downarrow\downarrow\downarrow\rangle)$
2	2	$\frac{3}{2}$	1	$\lvert\uparrow\uparrow\uparrow\uparrow\rangle$
2	1	$\frac{3}{2}$	1	$\frac{1}{2}(\lvert\downarrow\uparrow\uparrow\uparrow\rangle + \lvert\uparrow\downarrow\uparrow\uparrow\rangle + \lvert\uparrow\uparrow\downarrow\uparrow\rangle + \lvert\uparrow\uparrow\uparrow\downarrow\rangle)$
2	0	$\frac{3}{2}$	1	$\frac{1}{\sqrt{6}}(\lvert\downarrow\downarrow\uparrow\uparrow\rangle + \lvert\downarrow\uparrow\downarrow\uparrow\rangle + \lvert\uparrow\downarrow\downarrow\uparrow\rangle + \lvert\downarrow\uparrow\uparrow\downarrow\rangle + \lvert\uparrow\downarrow\uparrow\downarrow\rangle + \lvert\uparrow\uparrow\downarrow\downarrow\rangle)$
2	-1	$\frac{3}{2}$	1	$\frac{1}{2}(\lvert\downarrow\downarrow\downarrow\uparrow\rangle + \lvert\downarrow\downarrow\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\downarrow\downarrow\rangle + \lvert\uparrow\downarrow\downarrow\downarrow\rangle)$
2	-2	$\frac{3}{2}$	1	$\lvert\downarrow\downarrow\downarrow\downarrow\rangle$

(c) $\hat{H}$ commutes with $\hat{S}_{1+2+3+4,z}$ and $\hat{\boldsymbol{S}}^2_{1+2+3+4}$. Therefore $\langle S'_{1+2+3+4}, S'_{1+2+3+4,z}, S'_{1+2+3}, S'_{1+2}|\hat{H}|S_{1+2+3+4}, S_{1+2+3+4,z}, S_{1+2+3}, S_{1+2}\rangle$ will vanish if $S'_{1+2+3+4} \neq S_{1+2+3+4}$, or $S'_{1+2+3+4,z} \neq S_{1+2+3+4,z}$.

$\hat{H}$ commutes with ladder operators $\hat{S}_{1+2+3+4,\pm}$. Therefore $\langle S_{1+2+3+4}, S_{1+2+3+4,z}, S'_{1+2+3}, S'_{1+2}|\hat{H}|S_{1+2+3+4}, S_{1+2+3+4,z}, S_{1+2+3}, S_{1+2}\rangle$ is independent of $S_{1+2+3+4,z}$.

These are the "selection rules" for the rotation group generated by $\hat{S}_{1+2+3+4,a}$. So we only need to compute $\langle S_{1+2+3+4}, S_{1+2+3+4,z}, S'_{1+2+3}, S'_{1+2}|\hat{H}|S_{1+2+3+4}, S_{1+2+3+4,z}, S_{1+2+3}, S_{1+2}\rangle$ for $S_{1+2+3+4,z} = S_{1+2+3+4}$.

Rewrite $\hat{H} = (\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{2z}\hat{S}_{3z} + \hat{S}_{3z}\hat{S}_{4z} + \hat{S}_{4z}\hat{S}_{1z}) + \frac{1}{2}(\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + \hat{S}_{2+}\hat{S}_{3-} + \hat{S}_{2-}\hat{S}_{3+} + \hat{S}_{3+}\hat{S}_{4-} + \hat{S}_{3-}\hat{S}_{4+} + \hat{S}_{4+}\hat{S}_{1-} + \hat{S}_{4-}\hat{S}_{1+})$.

$\hat{H}|2,m,\frac{3}{2},1\rangle = |\uparrow\uparrow\uparrow\uparrow\rangle$.

$\langle 2,m,\frac{3}{2},1|\hat{H}|2,m,\frac{3}{2},1\rangle = 1$, for $m=-2,-1,0,1,2$.

$\hat{H}|1,1,\frac{3}{2},1\rangle = \frac{1}{2\sqrt{3}}(-|\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\uparrow\downarrow\rangle)$.

$\hat{H}|1,1,\frac{1}{2},1\rangle = \frac{1}{2\sqrt{6}}(|\downarrow\uparrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\uparrow\rangle - |\uparrow\uparrow\uparrow\downarrow\rangle)$.

$\hat{H}|1,1,\frac{1}{2},0\rangle = \frac{1}{2\sqrt{2}}(-|\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\uparrow\downarrow\rangle)$.

$\langle 1,m,\frac{3}{2},1|\hat{H}|1,m,\frac{3}{2},1\rangle = -\frac{1}{3}$, for $m=-1,0,1$.

$\langle 1,m,\frac{1}{2},1|\hat{H}|1,m,\frac{3}{2},1\rangle = \langle 1,m,\frac{3}{2},1|\hat{H}|1,m,\frac{1}{2},1\rangle = \frac{1}{3\sqrt{2}}$, for $m=-1,0,1$.

$\langle 1,m,\frac{1}{2},0|\hat{H}|1,m,\frac{3}{2},1\rangle = \langle 1,m,\frac{3}{2},1|\hat{H}|1,m,\frac{1}{2},0\rangle = -\frac{1}{\sqrt{6}}$, for $m=-1,0,1$.

$\langle 1,m,\frac{1}{2},1|\hat{H}|1,m,\frac{1}{2},1\rangle = -\frac{1}{6}$, for $m=-1,0,1$.

$\langle 1,m,\frac{1}{2},0|\hat{H}|1,m,\frac{1}{2},1\rangle = \langle 1,m,\frac{1}{2},1|\hat{H}|1,m,\frac{1}{2},0\rangle = \frac{1}{2\sqrt{3}}$, for $m=-1,0,1$.

$\langle 1,m,\frac{1}{2},0|\hat{H}|1,m,\frac{1}{2},0\rangle = -\frac{1}{2}$, for $m=-1,0,1$.

(not required) $\hat{H}$ in $S_{1+2+3+4}=1$ subspace is 3 copies of 3×3 matrix,

$$\begin{pmatrix} -\frac{1}{3} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{3\sqrt{2}} & -\frac{1}{6} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2} \end{pmatrix} = (-1)\cdot \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$ which has eigenvalues $-1,0,0$.

$\hat{H}|0,0,\frac{1}{2},1\rangle = \frac{1}{2\sqrt{3}}(-|\downarrow\downarrow\uparrow\uparrow\rangle + 2|\downarrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle + 2|\uparrow\downarrow\uparrow\downarrow\rangle - \uparrow\uparrow\downarrow\downarrow\rangle)$.

$\hat{H}|0,0,\frac{1}{2},0\rangle = \frac{1}{2}(|\downarrow\downarrow\uparrow\uparrow\rangle - 2|\downarrow\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle - 2|\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle)$.

$\langle 0,0,\frac{1}{2},1|\hat{H}|0,0,\frac{1}{2},1\rangle = -\frac{1}{2}$.

$\langle 0,0,\frac{1}{2},0|\hat{H}|0,0,\frac{1}{2},1\rangle = \langle 0,0,\frac{1}{2},1|\hat{H}|0,0,\frac{1}{2},0\rangle = \frac{\sqrt{3}}{2}$.

$\langle 0,0,\frac{1}{2},0|\hat{H}|0,0,\frac{1}{2},0\rangle = -\frac{3}{2}$.

(not required) $\hat{H}$ in $S_{1+2+3+4}=0$ subspace is 2×2 matrix,

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{pmatrix} = (-2)\cdot \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix},$$ which has eigenvalues $-2,0$.

Problem 4. (35 points) Consider four fermion modes $\hat{f}_{1,2,3,4}$. They satisfy $\{\hat{f}_i,\hat{f}_j^\dagger\}=\delta_{ij}$. The occupation basis $|n_1,n_2,n_3,n_4\rangle = (\hat{f}_1^\dagger)^{n_1}(\hat{f}_2^\dagger)^{n_2}(\hat{f}_3^\dagger)^{n_3}(\hat{f}_4^\dagger)^{n_4}|\text{vac}\rangle$ are complete orthonormal basis of the Fock space, where $n_{1,2,3,4}=0$ or 1 are eigenvalues of $\hat{n}_i=\hat{f}_i^\dagger\hat{f}_i$. $|\text{vac}\rangle$ is the normalized "vacuum". Consider the D_4 point group (see page 1), generated by
"4-fold rotation" $C_4:\ \hat{f}_1^\dagger\to\hat{f}_2^\dagger\to\hat{f}_3^\dagger\to\hat{f}_4^\dagger\to\hat{f}_1^\dagger$, (this means $\widehat{C_4}\hat{f}_1^\dagger\widehat{C_4}^\dagger=\hat{f}_2^\dagger$, etc.), and
"principal axis reflection" $\sigma_s:\ \hat{f}_1^\dagger\to\hat{f}_1^\dagger, \hat{f}_2^\dagger\to\hat{f}_4^\dagger, \hat{f}_3^\dagger\to\hat{f}_3^\dagger, \hat{f}_4^\dagger\to\hat{f}_2^\dagger$.

(a) (5pts) A group element $g \in D_4$ transforms $\hat{f}_i^\dagger$ as $\hat{f}_i^\dagger \mapsto \sum_j \hat{f}_j^\dagger \cdot R[g]_{ji}$, where $R[g]$ is the 4×4 representation matrix. Decompose this into irreducible representations. Namely find $\hat{f'}_i^\dagger = \sum_j \hat{f}_j^\dagger \cdot U_{ji}$, where U_{ji} is a 4×4 unitary matrix, so that $\hat{f'}_i^\dagger$ transform under $g \in D_4$ as $\hat{f'}_i^\dagger \mapsto \sum_j \hat{f'}_j^\dagger \cdot R'[g]_{ji}$ with $R'[g]$ block-diagonalized, and each diagonal block is one of the irreducible representations. *Solve the new basis $\hat{f'}_i^\dagger$ in terms of $\hat{f}_i^\dagger$ (or equivalently solve U), and the block-diagonalized representation $R'[g]$ for the generators $g = C_4$ and $g = \sigma_s$.* [Hint: you may use the "projection operator" to find the new basis]

(b) (15pts) Assume that the vacuum state $|\text{vac}\rangle$ is invariant under D_4 group. Then the transformation rules for $\hat{f}_i^\dagger$ completely determine the transformation rules for any states, for example C_4 transforms $\hat{f}_1^\dagger \hat{f}_2^\dagger |\text{vac}\rangle \mapsto \hat{f}_2^\dagger \hat{f}_3^\dagger |\text{vac}\rangle$. *Decompose the 16-dimensional Fock space into irreducible representations(irrep) of D_4, namely find new basis of the Fock space so that they transform according to irreducible representations of the D_4 group.* [Hint: result of (a) may help; the D_4 group commutes with total particle number]

(c) (15pts*) Consider $\hat{H}_0 = E_0 \cdot \hat{n}$ where $\hat{n} = \sum_{i=1}^4 \hat{n}_i$, and $\hat{V} = \lambda \cdot (\hat{f}_1 \hat{f}_2 + \hat{f}_2 \hat{f}_3 + \hat{f}_3 \hat{f}_4 + \hat{f}_4 \hat{f}_1 + \hat{f}_2^\dagger \hat{f}_1^\dagger + \hat{f}_3^\dagger \hat{f}_2^\dagger + \hat{f}_4^\dagger \hat{f}_3^\dagger + \hat{f}_1^\dagger \hat{f}_4^\dagger)$, where $E_0 > 0$, λ is a ' 'small" real parameter. *Solve all eigenvalues of $\hat{H}_0 + \hat{V}$ in the entire Fock space up to λ^2 order.* [Hint: $\hat{V}$ is NOT invariant (trivial irrep) under the D_4 group, but results of (a)(b) will still help; you can use perturbation theory, or solve the exact eigenvalues and do Taylor expansion]

Solution: this is similar to Homework #5 Problem 2(a)(b).

(a) this is exactly the same as Homework #5 Problem 2(a).

The basis can be chosen as

irrep. R'	basis	$R'[C_4]$	$R'[\sigma_s]$
Γ_1	$\hat{f'}_1^\dagger \equiv \hat{\Gamma}_1^\dagger = \frac{1}{2}(\hat{f}_1^\dagger + \hat{f}_2^\dagger + \hat{f}_3^\dagger + \hat{f}_4^\dagger)$	$\begin{pmatrix} 1 \end{pmatrix}$	$\begin{pmatrix} 1 \end{pmatrix}$
Γ_3	$\hat{f'}_2^\dagger \equiv \hat{\Gamma}_3^\dagger = \frac{1}{2}(\hat{f}_1^\dagger - \hat{f}_2^\dagger + \hat{f}_3^\dagger - \hat{f}_4^\dagger)$	$\begin{pmatrix} -1 \end{pmatrix}$	$\begin{pmatrix} 1 \end{pmatrix}$
Γ_5	$(\hat{f'}_3^\dagger \equiv \hat{\Gamma}_{5,x}^\dagger = \frac{1}{\sqrt{2}}(\hat{f}_1^\dagger - \hat{f}_3^\dagger), \hat{f'}_4^\dagger \equiv \hat{\Gamma}_{5,y}^\dagger = \frac{1}{\sqrt{2}}(\hat{f}_2^\dagger - \hat{f}_4^\dagger))$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

They satisfy $\{\hat{f'}_i, \hat{f'}_j^\dagger\} = \delta_{ij}$.

(b) D_4 group preserves total particle number, so we can construct irreps. of D_4 group within each subspace of fixed total particle number. Irreps. in total particle number $n = 2$ subspace is the same as Homework #5 Problem 2(b).

n	irrep. R'	basis	$R'[C_4]$	$R'[\sigma_s]$
0	Γ_1	$\|\text{vac}\rangle$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
1	Γ_1	$\hat{f'}_1^\dagger\|\text{vac}\rangle$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
1	Γ_3	$\hat{f'}_2^\dagger\|\text{vac}\rangle$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
1	Γ_5	$(\hat{f'}_3^\dagger\|\text{vac}\rangle, \hat{f'}_4^\dagger\|\text{vac}\rangle)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
2	$\Gamma_3 = \Gamma_1 \otimes \Gamma_3$	$\hat{f'}_1^\dagger \hat{f'}_2^\dagger\|\text{vac}\rangle$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
2	$\Gamma_5 = \Gamma_1 \otimes \Gamma_5$	$(\hat{f'}_1^\dagger \hat{f'}_3^\dagger\|\text{vac}\rangle, \hat{f'}_1^\dagger \hat{f'}_4^\dagger\|\text{vac}\rangle)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
2	$\Gamma_5 = \Gamma_3 \otimes \Gamma_5$	$(\hat{f'}_2^\dagger \hat{f'}_3^\dagger\|\text{vac}\rangle, -\hat{f'}_2^\dagger \hat{f'}_4^\dagger\|\text{vac}\rangle)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
2	Γ_2, from $\Gamma_5 \otimes \Gamma_5 = \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4$	$\hat{f'}_3^\dagger \hat{f'}_4^\dagger\|\text{vac}\rangle$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
3	$\Gamma_5 = \Gamma_1 \otimes \Gamma_3 \otimes \Gamma_5$	$(\hat{f'}_1^\dagger \hat{f'}_2^\dagger \hat{f'}_3^\dagger\|\text{vac}\rangle, -\hat{f'}_1^\dagger \hat{f'}_2^\dagger \hat{f'}_4^\dagger\|\text{vac}\rangle)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
3	Γ_2, from $\Gamma_1 \otimes \Gamma_5 \otimes \Gamma_5$	$\hat{f'}_1^\dagger \hat{f'}_3^\dagger \hat{f'}_4^\dagger\|\text{vac}\rangle$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
3	Γ_4, from $\Gamma_3 \otimes \Gamma_5 \otimes \Gamma_5$	$\hat{f'}_2^\dagger \hat{f'}_3^\dagger \hat{f'}_4^\dagger\|\text{vac}\rangle$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
4	Γ_4, from $\Gamma_1 \otimes \Gamma_3 \otimes \Gamma_5 \otimes \Gamma_5$	$\hat{f'}_1^\dagger \hat{f'}_2^\dagger \hat{f'}_3^\dagger \hat{f'}_4^\dagger\|\text{vac}\rangle$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$

The representation matrices are not required.

(c) rewrite $\hat{H}$ in terms of $\hat{f'}_i^\dagger$ and $\hat{f'}_i$.
$\hat{H}_0 = E_0 \sum_{i=1}^{4} \hat{f'}_i^\dagger \hat{f'}_i$, forms Γ_1 irrep.
$\hat{V} = 2\lambda\,(\hat{f'}_3\hat{f'}_4 + \hat{f'}_4^\dagger \hat{f'}_3^\dagger)$, forms Γ_2 irrep.

Method #1: exact solution,
do particle-hole transformation on $\hat{f'}_3$, define $\hat{\tilde{f}}_1 = \hat{f'}_1$, $\hat{\tilde{f}}_2 = \hat{f'}_2$, $\hat{\tilde{f}}_3 = \hat{f'}_3^\dagger$, $\hat{\tilde{f}}_4 = \hat{f'}_4$,

then $\hat{H} = E_0 + \begin{pmatrix} \hat{f}_1^\dagger & \hat{f}_2^\dagger & \hat{f}_3^\dagger & \hat{f}_4^\dagger \end{pmatrix} \begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & -E_0 & 2\lambda \\ 0 & 0 & 2\lambda & E_0 \end{pmatrix} \begin{pmatrix} \hat{f}_1^\dagger \\ \hat{f}_2^\dagger \\ \hat{f}_3^\dagger \\ \hat{f}_4^\dagger \end{pmatrix}$.

This bilinear form can be further diagonalized by a unitary transformation, $\begin{pmatrix} \hat{c}_1^\dagger & \hat{c}_2^\dagger & \hat{c}_3^\dagger & \hat{c}_4^\dagger \end{pmatrix} = \begin{pmatrix} \hat{f}_1^\dagger & \hat{f}_2^\dagger & \hat{f}_3^\dagger & \hat{f}_4^\dagger \end{pmatrix} \cdot U$, where U is a 4×4 unitary matrix, and $U^\dagger \cdot \begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & -E_0 & 2\lambda \\ 0 & 0 & 2\lambda & E_0 \end{pmatrix} \cdot U = \begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & -\sqrt{E_0^2+4\lambda^2} & 0 \\ 0 & 0 & 0 & \sqrt{E_0^2+4\lambda^2} \end{pmatrix}$, then $\hat{H} = E_0 + E_0\hat{c}_1^\dagger\hat{c}_1 + E_0\hat{c}_2^\dagger\hat{c}_2 - \sqrt{E_0^2+4\lambda^2}\hat{c}_3^\dagger\hat{c}_3 + \sqrt{E_0^2+4\lambda^2}\hat{c}_4^\dagger\hat{c}_4$, the occupation basis of $\hat{c}_i$, $(\hat{c}_1^\dagger)^{m_1}(\hat{c}_2^\dagger)^{m_2}(\hat{c}_3^\dagger)^{m_3}(\hat{c}_4^\dagger)^{m_4}|\text{vac of } c\rangle$ are eigenstates of $\hat{H}$. The eigenvalues are

$E_0 \cdot (m_1 + m_2 + 1) - \sqrt{E_0^2+4\lambda^2}m_3 + \sqrt{E_0^2+4\lambda^2}m_4$

$\approx E_0 \cdot (m_1 + m_2 + 1) - (E_0 + \frac{2\lambda^2}{E_0})m_3 + (E_0 + \frac{2\lambda^2}{E_0})m_4$, for $m_{1,2,3,4} = 0$ or 1.

Method #2: perturbation theory by unitary transformations,

Define $\hat{V}_+ = 2\lambda\hat{f'}_4^\dagger\hat{f'}_3^\dagger$, $\hat{V}_- = 2\lambda\hat{f'}_3\hat{f'}_4 = \hat{V}_+^\dagger$. Then $\hat{V} = \hat{V}_+ + \hat{V}_-$, $[\hat{H}_0, \hat{V}_\pm] = \pm(2E_0)\hat{V}_\pm$.

Let $\hat{H}^{(1)} = e^{\mathbb{i}\hat{S}}\hat{H}e^{-\mathbb{i}\hat{S}} = \hat{H}_0 + \hat{V} + [\mathbb{i}\hat{S}, \hat{H}_0] + [\mathbb{i}\hat{S}, \hat{V}] + \frac{1}{2}[\mathbb{i}\hat{S}, [\mathbb{i}\hat{S}, \hat{H}_0]] + \dots$.

Demand that $\hat{V}_+ + \hat{V}_- + [\mathbb{i}\hat{S}, \hat{H}_0] = 0$, then $\mathbb{i}\hat{S} = \frac{1}{2E_0}(\hat{V}_+ - \hat{V}_-)$.

Then up to t^2 order, $\hat{H}^{(1)} \approx \hat{H}_0 + (1 - \frac{1}{2})[\frac{1}{2E_0}(\hat{V}_+ - \hat{V}_-), \hat{V}_+ + \hat{V}_-] = \hat{H}_0 + \frac{1}{2E_0}[\hat{V}_+, \hat{V}_-]$

Use $[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}\{\hat{B}, \hat{C}\}\hat{D} - \{\hat{A}, \hat{C}\}\hat{B}\hat{D} + \hat{C}\hat{A}\{\hat{B}, \hat{D}\} - \hat{C}\{\hat{A}, \hat{D}\}\hat{B}$,

$[\hat{V}_+, \hat{V}_-] = 4\lambda^2(\hat{f'}_4^\dagger\hat{f'}_4 - \hat{f'}_3\hat{f'}_3^\dagger) = 4\lambda^2(\hat{f'}_4^\dagger\hat{f'}_4 + \hat{f'}_3^\dagger\hat{f'}_3 - 1)$.

Finally $\hat{H}^{(1)} \approx E_0\hat{f'}_1^\dagger\hat{f'}_1 + E_0\hat{f'}_2^\dagger\hat{f'}_2 + (E_0 + \frac{2\lambda^2}{E_0})\hat{f'}_3^\dagger\hat{f'}_3 + (E_0 + \frac{2\lambda^2}{E_0})\hat{f'}_4^\dagger\hat{f'}_4 - \frac{2t^2}{E_0}$. The eigenvalues are $E_0(n_1 + n_2) + (E_0 + \frac{2\lambda^2}{E_0})(n_3 + n_4) - \frac{2\lambda^2}{E_0}$, for $n_{1,2,3,4} = 0$ or 1.

Problem 5. (10 points) Consider three spin-1/2 moments $\hat{\boldsymbol{S}}_{1,2,3}$. The spin operators satisfy $[\hat{S}_{i,a}, \hat{S}_{j,b}] = \delta_{ij}\sum_c \mathbb{i}\epsilon_{abc}\hat{S}_{i,c}$ for $i,j = 1,2,3$ and $a,b,c = x,y,z$. A complete orthonormal basis is the tensor product of S_z-eigenbasis, $|S_{1z}, S_{2z}, S_{3z}\rangle$, with $S_{iz} = \pm\frac{1}{2}$.

(a) (8pts) *Solve all the eigenvalues of* $\hat{H} = \hat{S}_{1z}(\hat{S}_{2x}\hat{S}_{3y} - \hat{S}_{2y}\hat{S}_{3x})$.[Hint: rewriting $\hat{H}$ by ladder operators and using symmetries might help; or you can directly write down the 8×8 matrix and find its eigenvalues, note that $\hat{S}_{1a}\hat{S}_{2b}\hat{S}_{3c}$ means the tensor product $\hat{S}_{1a} \otimes \hat{S}_{2b} \otimes \hat{S}_{3c}$]

(b) (2pts*) *Explain the reason of eigenvalue degeneracy in (a).*

Solution:

(a) $\hat{H} = \frac{1}{2}\hat{S}_{1z}(\mathbb{i}\hat{S}_{2+}\hat{S}_{3-} - \mathbb{i}\hat{S}_{2-}\hat{S}_{3+})$ conserves $\hat{S}_{1z}$ and $\hat{S}_{2,z} + \hat{S}_{3,z}$.

$\hat{H}$ under S_z basis is $\begin{pmatrix} 0 &&&&&&& \\ & 0 & \frac{\mathbb{i}}{4} &&&&& \\ & -\frac{\mathbb{i}}{4} & 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 & -\frac{\mathbb{i}}{4} & \\ &&&&& \frac{\mathbb{i}}{4} & 0 & \\ &&&&&&& 0 \end{pmatrix}$, unspecified entries are zero.

Eigenvalues are $-\frac{1}{4}$ (2-fold degenerate), 0 (4-fold degenerate), $+\frac{1}{4}$ (2-fold degenerate)

(b) Answer #1: generalized Kramers theorem,

$\hat{H}$ is NOT time-reversal invariant, time-reversal changes sign of $\hat{H}$, because each term in $\hat{H}$ is a product of odd number of spin operators, so the original Kramers theorem cannot apply.

however $\hat{H}$ is invariant under an anti-unitary symmetry $\hat{U} = \hat{\sigma}_{2,3}\hat{\mathcal{T}}$, here $\hat{\mathcal{T}}$ is the anti-unitary time-reversal operator, $\hat{\sigma}_{2,3}$ is the unitary operator that exchanges spins "2" and "3", namely $\hat{\sigma}_{2,3}|s_1, s_2, s_3\rangle = |s_1, s_3, s_2\rangle$, $\hat{\sigma}_{2,3}\hat{S}_{2,a}\hat{\sigma}_{2,3}^\dagger = \hat{S}_{3,a}$, $\hat{\sigma}_{2,3}\hat{S}_{3,a}\hat{\sigma}_{2,3}^\dagger = \hat{S}_{2,a}$.

it is easy to check that $\hat{U}^2 = -\mathbb{1}$ in the Hilbert space,

then we have a generalized Kramers theorem, which says that all energy levels must be at least 2-fold degenerate: if $|\psi\rangle$ is eigenstate of $\hat{H}$, then $\hat{U}|\psi\rangle$ is also an eigenstate of $\hat{H}$ with the same eigenvalue, and $\langle\psi|\hat{U}\psi\rangle = 0$.

Answer #2: anti-commuting unitary symmetries,

$[\hat{H}, \hat{S}_{1,z}] = 0$, so we can find simultaneous eigenstates of $\hat{H}$ and $\hat{S}_{1,z}$, $|\hat{H} = E, \hat{S}_{1,z} = S_{1z}\rangle$.

Then we can find another unitary symmetry $\hat{U}$ such that $[\hat{H}, \hat{U}] = 0$, $\hat{U}\hat{S}_{1,z} = -\hat{S}_{1,z}\hat{U}$, namely that $\hat{U}$ does not change $\hat{H}$ eigenvalue, but changes sign of $\hat{S}_{1,z}$ eigenvalue, therefore $\hat{U}|\hat{H} = E, \hat{S}_{1,z} = S_{1z}\rangle = |\hat{H} = E, \hat{S}_{1,z} = -S_{1z}\rangle$. $\hat{S}_{1,z}$ eigenvalues are nonzero, so eigenstates of $\hat{H}$ have at least 2-fold degeneracy.

The choice of $\hat{U}$ is not unique, $\hat{U} = \exp(\mathbb{i}\pi(\hat{S}_{1,x} + \hat{S}_{2,x} + \hat{S}_{3,x}))$, or $\hat{U} = \exp(\mathbb{i}\pi\hat{S}_{1,x}) \cdot \hat{\sigma}_{2,3}$.