Homework #3:
Due: tentatively Oct. 17, 2019

***** (about lecture #2) *****

1. Consider the boson coherent state $|z\rangle \equiv e^{-|z|^2/2} e^{z\,\hat{b}^\dagger}|\text{vac}\rangle$, where z is a complex number, $\hat{b}^\dagger$ is a boson creation operator ($[\hat{b}, \hat{b}^\dagger] = 1$), $|\text{vac}\rangle$ is the normalized boson vacuum state(so $\hat{b}|\text{vac}\rangle = 0$).

(a). (2pts) *Compute the overlap $\langle z'|z\rangle$, where z' and z are two complex numbers.* [Hint: you can expand $|z\rangle$ into occupation basis states, or use some results of Homework #1]

(b). (3pts) *Prove the resolution of identity in terms of these overcomplete basis,* $\mathbb{1} = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} |x + \mathbb{i}y\rangle\langle x + \mathbb{i}y|\, \frac{\mathrm{d}x\mathrm{d}y}{\pi}$, *where x, y are real numbers.* [Hint: represent $x + \mathbb{i}y$ by $r\,e^{\mathbb{i}\theta}$.]

2. Consider a "boson pairing state" $|\psi_\lambda\rangle \equiv \sqrt{1 - |\lambda|^2}\cdot \exp(\lambda \hat{b}_1^\dagger \hat{b}_2^\dagger)|\text{vac}\rangle$. Here $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{i,j}$, $[\hat{b}_i, \hat{b}_j] = 0$, λ is a complex number with $|\lambda| < 1$, $|\text{vac}\rangle$ is the normalized boson vacuum state (so $\hat{b}_i|\text{vac}\rangle = 0$).

(a). (2pts) *Compute $\langle\psi_\lambda|\psi_\lambda\rangle$ and show that this state is normalized.*

(b). (3pts) Define "Bogoliubov quasiparticle" annihilation operators, $\hat{\gamma}_1 = u\hat{b}_1 + v\hat{b}_2^\dagger$, $\hat{\gamma}_2 = u\hat{b}_2 + v\hat{b}_1^\dagger$, where $u = (1 - |\lambda|^2)^{-1/2}$ and $v = -\lambda\,(1 - |\lambda|^2)^{-1/2}$. *Check explicitly that $[\hat{\gamma}_i, \hat{\gamma}_j^\dagger] = \delta_{i,j}$, and $\hat{\gamma}_i|\psi_\lambda\rangle = 0$.*

3. Consider a single-fermion Hilbert space, with complete orthonormal basis $|e_1\rangle$,$|e_2\rangle$,$|e_3\rangle$. Denote the corresponding creation (annihilation) operators as $\hat{c}_i^\dagger$ ($\hat{c}_i$), for $i = 1, 2, 3$ respectively. Then $\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{i,j}$, $\{\hat{c}_i, \hat{c}_j\} = \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0$, and $|e_i\rangle = \hat{c}_i^\dagger|\text{vac}\rangle$, for $i, j = 1, 2, 3$, where $|\text{vac}\rangle$ is the normalized fermion 'vacuum' state.

(a). (5pts) Define three operators $\hat{S}_x = -\mathbb{i}(\hat{c}_2^\dagger\hat{c}_3 - \hat{c}_3^\dagger\hat{c}_2)$, $\hat{S}_y = -\mathbb{i}(\hat{c}_3^\dagger\hat{c}_1 - \hat{c}_1^\dagger\hat{c}_3)$, $\hat{S}_z = -\mathbb{i}(\hat{c}_1^\dagger\hat{c}_2 - \hat{c}_2^\dagger\hat{c}_1)$. *Compute the commutators, $[\hat{S}_x, \hat{S}_y]$, $[\hat{S}_y, \hat{S}_z]$, $[\hat{S}_z, \hat{S}_x]$.* Represent the results as linear combinations of $\hat{S}_{x,y,z}$. (Hint: check and use the identity $[\hat{A}\hat{B},\hat{C}\hat{D}] = \hat{A}\{\hat{B},\hat{C}\}\hat{D} - \{\hat{A},\hat{C}\}\hat{B}\hat{D} + \hat{C}\hat{A}\{\hat{B},\hat{D}\} - \hat{C}\{\hat{A},\hat{D}\}\hat{B}$.)

(b). (5pts) *Write down a complete set of orthonormal basis of the Hilbert space of two fermions.* (Preferably in terms of creation operators) *Compute the matrix elements of* $\hat{S}_{x,y,z}$ *between these bases of two-fermion Hilbert space. Check that these matrix representations of* $\hat{S}_{x,y,z}$ *within the two-fermion Hilbert space do satisfy the commutation relations in (a).* [Hint: be careful about minus signs when computing matrix elements]

(c). (5pts) $\hat{S}_{x,y,z}$ in (a) are all hermitian operators. Then $\exp(\mathbb{i}\theta\hat{S}_x)$ is a unitary operator when θ is a real number. *Compute* $\exp(\mathbb{i}\theta\hat{S}_x)\cdot(a\hat{S}_x + b\hat{S}_y + c\hat{S}_z)\cdot\exp(-\mathbb{i}\theta\hat{S}_x)$ *and represent the result in terms of finite-degree polynomials of* $\hat{S}_{x,y,z}$. Here a,b,c are some complex numbers. [Hint: some results in Homework #1 will be useful]

(d). (5pts) *Solve the eigenvalues of the hermitian operator* $\hat{S}_y - \hat{S}_z$ *in the 8-dimensional Fock space.* [Hint: you can divide the Fock space into subspaces of fixed total fermion number, or try to 'diagonalize' a 'fermion bilinear' operator in similar way as Problem 2(d), some previous results may help]

(e). (5pts) (DIFFICULT) *Solve the eigenvalues and eigenvectors of* $\hat{H} = \hat{c}_1^\dagger\hat{c}_2 + \hat{c}_2^\dagger\hat{c}_1 + \hat{c}_3^\dagger\hat{c}_3 + \hat{c}_3(\hat{c}_1 + \hat{c}_2) + (\hat{c}_1^\dagger + \hat{c}_2^\dagger)\hat{c}_3^\dagger$, *in the 8-dimensional Fock space.* [Hint: use symmetry to divide the Fock space, certain particle-hole transformation and basis change may help]

4. Second quantization: identical non-interacting particles in 1D harmonic potential. Subscript $_{\text{1-body}}$ ($_{\text{Fock}}$) indicates operators for single-particle (in Fock space).

The single-particle Hamiltonian is $\hat{H}_{\text{1-body}} = \frac{\hat{p}^2_{\text{1-body}}}{2m} + \frac{m\omega^2}{2}\hat{x}^2_{\text{1-body}}$ [action on single-particle wavefunctions is $\hat{H}_{\text{1-body}}\,\psi(x) = (-\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2}{2}x^2)\,\psi(x)$], with normalized single-particle ground state $|\psi_0\rangle$ [wavefunction $\psi_0(x) \equiv \langle x|\psi_0\rangle = (\frac{m\omega}{\hbar\pi})^{1/4}\exp(-\frac{x^2}{2\hbar/m\omega})$], and normalized excited states $|\psi_n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger_{\text{1-body}})^n|\psi_0\rangle$. Here $\hat{a}^\dagger_{\text{1-body}} = \frac{m\omega\hat{x}-\mathbb{i}\hat{p}}{\sqrt{2m\omega\hbar}}$, $[\hat{a}_{\text{1-body}}, \hat{a}^\dagger_{\text{1-body}}] = 1$, and $\hat{H}_{\text{1-body}} = \hbar\omega\cdot(\hat{a}^\dagger_{\text{1-body}}\hat{a}_{\text{1-body}} + \frac{1}{2})$. The single-particle energy eigenvalues are $E_n = \hbar\omega\cdot(n+\frac{1}{2})$ for state $|\psi_n\rangle$, $n = 0, 1, \ldots$.

Denote the creation (annihilation) operators for single-particle states $|\psi_n\rangle$ by $\widehat{\psi_n}^\dagger$ ($\widehat{\psi_n}$). We will consider the case of fermions, then $\{\widehat{\psi_n}, \widehat{\psi_m}^\dagger\} = \delta_{n,m}$, $\{\widehat{\psi_n}, \widehat{\psi_m}\} = 0$.

The 'second quantized' Hamiltonian for identical particles is $\hat{H}_{\text{Fock}} = \sum_{n=0}^{\infty} E_n \widehat{\psi_n}^\dagger \widehat{\psi_n}$. This can be 'derived' from $\hat{H}_{\text{Fock}} = \int \mathrm{d}x\, \widehat{\psi(x)}^\dagger \cdot (-\frac{\hbar^2\partial_x^2}{2m} + \frac{m\omega^2 x^2}{2}) \cdot \widehat{\psi(x)}$. Here $\widehat{\psi(x)}^\dagger$ is the creation operator for position eigenbasis $|x\rangle$, and $\widehat{\psi(x)}^\dagger = \sum_{n=0}^{\infty} \widehat{\psi_n}^\dagger \langle\psi_n|x\rangle$. Then $\hat{H}_{\text{Fock}}$
$= \int \mathrm{d}x \sum_{n=0}^{\infty} \widehat{\psi_n}^\dagger \langle\psi_n|x\rangle \cdot (-\frac{\hbar^2\partial_x^2}{2m} + \frac{m\omega^2 x^2}{2}) \cdot \sum_{n'=0}^{\infty} \langle x|\psi_{n'}\rangle \widehat{\psi_{n'}}$
$= \sum_{n,n'=0}^{\infty} \int \mathrm{d}x\, \widehat{\psi_n}^\dagger \langle\psi_n|x\rangle \cdot E_{n'} \cdot \langle x|\psi_{n'}\rangle \widehat{\psi_{n'}} = \sum_{n,n'=0}^{\infty} \widehat{\psi_n}^\dagger \cdot E_{n'}\delta_{n,n'} \cdot \widehat{\psi_{n'}} = \sum_{n=0}^{\infty} E_n \widehat{\psi_n}^\dagger \widehat{\psi_n}$.

(a) (5pts) Consider the ground state for two fermions, $|\psi_{\text{GS}}^{(N=2)}\rangle$. *Write down this state (in terms of* $\widehat{\psi_n}^\dagger$ *and the fermion vacuum* $|vac\rangle$*) and its energy. Write down the explicit two particle wavefunction* $\psi_{GS}^{(N=2)}(x_1, x_2)$*, and compute the expectation value of* $(x_1 - x_2)^2$.

(b) (5pts) *Derive the 'second quantized' form of* $\hat{x}_{Fock} \equiv \int \mathrm{d}x\, \widehat{\psi(x)}^\dagger \cdot x \cdot \widehat{\psi(x)}$ *and* $\hat{p}_{Fock} \equiv \int \mathrm{d}x\, \widehat{\psi(x)}^\dagger \cdot (-\mathrm{i}\hbar\partial_x) \cdot \widehat{\psi(x)}$*, in terms of* $\widehat{\psi_n}^\dagger$ *and* $\widehat{\psi_n}$*. Compute the commutator* $[\hat{x}_{Fock}, \hat{p}_{Fock}]$. [Hint: represent x and $-\mathrm{i}\hbar\partial_x$ by ladder operators for 1-body wavefunctions]

(c) (5pts) *Derive the 'second quantized' form of the two-body term* $\hat{V}(x_1, x_2) = (x_1 - x_2)^2$, $\hat{V}_{Fock} = \frac{1}{2}\int \mathrm{d}x \int \mathrm{d}x'\, \widehat{\psi(x)}^\dagger \widehat{\psi(x')}^\dagger \cdot (x - x')^2 \cdot \widehat{\psi(x')}\widehat{\psi(x)}$*, in terms of* $\widehat{\psi_n}^\dagger$ *and* $\widehat{\psi_n}$*. Check that this produces the same expectation value on the state in (a) as the result of (a).* [Hint: use ladder operators for x and x'.]