Homework #5:
Due: Nov. 14, 2019

1. On 2-dimensional space with real coordinates (x, y), define position eigenstates $|x, y\rangle$ with $\hat{x}|x, y\rangle = x|x, y\rangle$ and $\hat{y}|x, y\rangle = y|x, y\rangle$ and normalization $\langle x, y|x', y'\rangle = \delta(x-x')\delta(y-y')$; and momentum eigenstates $|p_x, p_y\rangle$ with $\hat{p_x}|p_x, p_y\rangle = p_x|p_x, p_y\rangle$ and $\hat{p_y}|p_x, p_y\rangle = p_y|p_x, p_y\rangle$ and normalization $\langle p_x, p_y|p'_x, p'_y\rangle = \delta(p_x - p'_x)\delta(p_y - p'_y)$. And $\langle x, y|p_x, p_y\rangle = \frac{e^{\mathbb{i}(p_x\cdot x+p_y\cdot y)/\hbar}}{2\pi\hbar}$.

Here $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = \mathbb{i}\hbar$, and other commutators between them are zero.

The rotations around the origin point form the $SO(2)$ group (also denoted by C_∞). Denote the counter-clockwise rotation of angle θ by $g(\theta)$, which maps the point (x, y) to $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. It is easy to check that $g(\theta)\cdot g(\theta') = g(\theta+\theta' \mod 2\pi)$, so this is an Abelian group. All its irreducible representations(irreps) are 1-dimensional(1-dim'l), and are labeled by integer n, $\chi_n[g(\theta)] = R_n[g(\theta)] = (e^{\mathbb{i}n\theta})$. The orthogonality relation is $\int_0^{2\pi} \mathrm{d}\theta\, \chi_n[g(\theta)]^*\chi_{n'}[g(\theta)] = \int_0^{2\pi} \mathrm{d}\theta\, e^{-\mathbb{i}n\theta}e^{\mathbb{i}n'\theta} = 2\pi\cdot\delta_{n,n'}$.

(a). (2pts) The unitary operator for $g(\theta)$ is $\widehat{g(\theta)} = \int \mathrm{d}x \int \mathrm{d}y\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle\langle x, y|$. Compute the matrix element of $\widehat{g(\theta)}$ under the momentum eigenbasis, $\langle p'_x, p'_y|\widehat{g(\theta)}|p_x, p_y\rangle$.

(b). (2pts) Compute the generator of this "Lie group", $\widehat{L_z} \equiv \left[\mathbb{i}\frac{\partial}{\partial\theta}\widehat{g(\theta)}\right]_{\theta=0}$. Represent the result by the $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ operators. [Hint: consider $\widehat{L_z}\psi(x, y) = \left\{\mathbb{i}\frac{\partial}{\partial\theta}[\widehat{g(\theta)}\psi(x, y)]\right\}_{\theta=0}$]

(c). (2pts) Consider the 2D harmonic oscillator, $\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2)$. Here m, ω are positive constants. It can be viewed as the sum of two independent harmonic oscillators, $\hat{H} = (\frac{\hat{p}_x^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}) + (\frac{\hat{p}_y^2}{2m} + \frac{m\omega^2\hat{y}^2}{2})$. The ladder operators for the x- and y-components can be defined as $\hat{b}_x = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{\mathbb{i}}{m\omega}\hat{p}_x)$ and $\hat{b}_y = \sqrt{\frac{m\omega}{2\hbar}}(\hat{y} + \frac{\mathbb{i}}{m\omega}\hat{p}_y)$. They satisfy the commutation relation of boson annihilation operators, $[\hat{b}_x, \hat{b}_x^\dagger] = [\hat{b}_y, \hat{b}_y^\dagger] = 1$, $[\hat{b}_x, \hat{b}_y^\dagger] = [\hat{b}_x, \hat{b}_y] = 0$. Denote the unique normalized ground state of $\hat{H}$ by $|\text{vac}\rangle$, then $\hat{b}_x|\text{vac}\rangle = \hat{b}_y|\text{vac}\rangle = 0$. Write down all eigenvalues and normalized eigenstates of $\hat{H}$.

(d). (2pts) Rewrite the $\widehat{L_z}$ in (b) in terms of the ladder operators in (c). Show that $[\hat{H}, \widehat{L_z}] = 0$. {Hint: use $[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}]\hat{D} + [\hat{A}, \hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B}, \hat{D}] + \hat{C}[\hat{A}, \hat{D}]\hat{B}$. }

(e). (4pts) The "raising" operators $\hat{b}_x^\dagger$ and $\hat{b}_y^\dagger$ form basis of a 2-dimensional representation of the $SO(2)$ group. $g(\theta)$ transforms them to their linear combinations, $(\widehat{g(\theta)}\hat{b}_x^\dagger\widehat{g(\theta)}^\dagger,\ \widehat{g(\theta)}\hat{b}_y^\dagger\widehat{g(\theta)}^\dagger) = (\hat{b}_x^\dagger,\ \hat{b}_y^\dagger)\cdot R[g(\theta)]$. Solve this 2×2 representation matrix $R_{\hat{b}^\dagger}[g(\theta)]$. Check that $R_{\hat{b}^\dagger}[g(\theta)]\cdot R_{\hat{b}^\dagger}[g(\theta')] = R_{\hat{b}^\dagger}[g(\theta+\theta')]$. [Hint: $\widehat{g(\theta)} = \exp(-\mathbb{i}\theta\widehat{L_z})$.]

(f). (5pts) We can decompose the $R_{\hat{b}^\dagger}$ representation into irreps by the "projection operator". Compute $\int_0^{2\pi} R_n[g(\theta)]^*\cdot\widehat{g(\theta)}\hat{b}_a^\dagger\widehat{g(\theta)}^\dagger$, for $a = x, y$ and all integer n. Only two of them are linearly independent, after normalization denote them as $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$, satisfying $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$. $\hat{a}_i^\dagger$ forms an irrep of the $SO(2)$ group, $\widehat{g(\theta)}\hat{a}_i^\dagger\widehat{g(\theta)}^\dagger = \hat{a}_i^\dagger\cdot R_{\hat{a}_i^\dagger}[g(\theta)]$, Write down $\hat{a}_{1,2}^\dagger$ in terms of $\hat{b}_{x,y}^\dagger$, and their corresponding irreps $R_{\hat{a}_i^\dagger}[g(\theta)]$. Rewrite $\hat{H}$ and $\widehat{L_z}$ in terms of $\hat{a}_{1,2}^\dagger$ and $\hat{a}_{1,2}$.

(g). (5pts) Write down the simultaneous eigenstates of $\hat{H}$ and $\widehat{L_z}$, $|\hat{H} = E, \widehat{L_z} = \ell\rangle$, in terms of $|\text{vac}\rangle$ and ladder operators $\hat{a}_i^\dagger$. What are the possible eigenvalues E and ℓ?

(h). (3pts) Each state $|\hat{H} = E, \widehat{L_z} = \ell\rangle$ in (g) forms an irrep of $SO(2)$. $\widehat{g(\theta)}|\hat{H} = E, \widehat{L_z} = \ell\rangle = |\hat{H} = E, \widehat{L_z} = \ell\rangle\cdot R_{E,\ell}[g(\theta)]$. Compute this 1×1 representation 'matrix' $R_{E,\ell}[g(\theta)]$.

(i). (5pts) Compute the "matrix elements", $\langle\hat{H} = E, \widehat{L_z} = \ell|\hat{a}_i^\dagger|\hat{H} = E', \widehat{L_z} = \ell'\rangle$, for $i = 1, 2$. Check that the results satisfy the "selection rule" for $SO(2)$ group, namely that if $R_{E,\ell}^*\otimes R_{\hat{a}_i^\dagger}\otimes R_{E',\ell'}$ is not the trivial irrep., then this matrix element vanishes.

(j). (5pts) Define two hermitian operators $\widehat{L_x} = \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1$ and $\widehat{L_y} = -\mathbb{i}\hat{a}_1^\dagger\hat{a}_2 + \mathbb{i}\hat{a}_2^\dagger\hat{a}_1$. Check that $[\hat{H}, \widehat{L_x}] = [\hat{H}, \widehat{L_y}] = 0$. Compute the commutators $[\widehat{L_x}, \widehat{L_y}]$, $[\widehat{L_y}, \widehat{L_z}]$, $[\widehat{L_z}, \widehat{L_x}]$, represent the results in terms of linear combinations of $\widehat{L_{x,y,z}}$. [Side remark: $SO(2)$ has only 1-dim'l irrep., but $\hat{H}$ has degenerate eigenvalues. In fact $\hat{H}$ has a larger non-Abelian symmetry. The $\widehat{L_{x,y,z}}$ are generators of this symmetry group and commute with $\hat{H}$.]

2. Considered $\hat{H} = (\hat{f}_1^\dagger\hat{f}_2 + \hat{f}_2^\dagger\hat{f}_3 + \hat{f}_3^\dagger\hat{f}_4 + \hat{f}_4^\dagger\hat{f}_1 + \text{h.c.})$.
Here $\hat{f}_i(\hat{f}_i^\dagger)$ are annihilation(creation) operators for 4 fermion modes, satisfying $\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{ij}$ and $\{\hat{f}_i, \hat{f}_j\} = 0$, and h.c. means hermitian conjugate of the previous 4 terms.

The model conserves total particle number $\hat{n} = \sum_{i=1}^4 \hat{f}_i^\dagger\hat{f}_i$, namely $[\hat{H}, \hat{n}] = 0$.

$\hat{H}$ also has the D_4 point group symmetry, generated by
"4-fold rotation" $C_4: \ \hat{f}_1 \to \hat{f}_2 \to \hat{f}_3 \to \hat{f}_4 \to \hat{f}_1$, (this means $\widehat{C_4}\hat{f}_1\widehat{C_4}^\dagger = \hat{f}_2$, etc.), and
"principal axis reflection" $\sigma_s: \ \hat{f}_1 \to \hat{f}_1, \hat{f}_2 \to \hat{f}_4, \hat{f}_3 \to \hat{f}_3, \hat{f}_4 \to \hat{f}_2$.
This group has 8 elements, and 5 conjugacy classes: $\{\mathbf{1}\}, \{C_4, C_4^3\}, \{C_4^2\}, \{\sigma_s, C_4^2\sigma_s\}, \{\sigma_d \equiv C_4\sigma_s, C_4^3\sigma_s\}$. The character table for the five irreducible representations, $\Gamma_{1,2,3,4,5}$, is

	$\mathbf{1}$	$2C_4$	C_4^2	$2\sigma_s$	$2\sigma_d$
Γ_1	1	1	1	1	1
Γ_2	1	1	1	-1	-1
Γ_3	1	-1	1	1	-1
Γ_4	1	-1	1	-1	1
Γ_5	2	0	-2	0	0

(a) (5pts) A group element $g \in D_4$ will transform $\hat{f}_i^\dagger$ as $\hat{f}_i^\dagger \mapsto \sum_j \hat{f}_j^\dagger \cdot R[g]_{ji}$, where $R[g]$ is the 4×4 representation matrix. Decompose this into irreducible representations. Namely find $\hat{f'}_i^\dagger = \sum_j \hat{f}_j^\dagger \cdot U_{ji}$, where U_{ji} is a 4×4 unitary matrix, so that $\hat{f'}_i^\dagger$ transform under $g \in D_4$ as $\hat{f'}_i^\dagger \mapsto \sum_j \hat{f'}_j^\dagger \cdot R'[g]_{ji}$ with $R'[g]$ block-diagonalized, and each diagonal block is one of the irreducible representations. Solve the new basis $\hat{f'}_i^\dagger$ in terms of $\hat{f}_i^\dagger$ (or equivalently solve U), and the block-diagonalized representation $R'[g]$ for the generators $g = C_4$ and $g = \sigma_s$. [Hint: use the "projection operator" to find the new basis]

(b) (5pts) The Hilbert space with fixed total particle number $\hat{n}$ is a representation space of the D_4 group. Assume that the vacuum state $|\text{vac}\rangle$ is invariant under D_4 group. Then the transformation rules for $\hat{f}_i^\dagger$ completely determine the transformation rules for any states, for example C_4 transforms $\hat{f}_1^\dagger\hat{f}_2^\dagger|\text{vac}\rangle \mapsto \hat{f}_2^\dagger\hat{f}_3^\dagger|\text{vac}\rangle$. Decompose the 6-dimensional 2-particle Hilbert space, with occupation basis $\hat{f}_i^\dagger\hat{f}_j^\dagger|\text{vac}\rangle$ for $i < j$, into irreducible representations of D_4. [Hint: one can first work out the 6×6 representation and then change basis to block-diagonalize it; or use the result of (a) to construct the irreducible representation basis]

(c) (5pts) Rewrite $\hat{H}$ in terms of the $\hat{f'}_i^\dagger$ and $\hat{f'}_i$ solved in (a). Solve all the eigenvalues and eigenstates of $\hat{H}$ in the 2-particle Hilbert space.