Homework #1:
Due: Sept. 24, 2019

1. $|1\rangle$, $|2\rangle$, $|3\rangle$ form a set of complete *non-orthogonal* basis of a Hilbert space, with the following overlaps(inner products), $\langle 1|1\rangle = \langle 2|2\rangle = \langle 3|3\rangle = 1$, $\langle 1|2\rangle = \langle 2|3\rangle = \langle 3|1\rangle = 1/3$.

(a)(2pts). *Show that* $|1\rangle$, $|2\rangle$, $|3\rangle$ *are linearly independent.*

(b)(5pts). Use the "Gram-Schmidt orthogonalization" procedure to find a set of complete orthonormal basis, $|\tilde{1}\rangle$, $|\tilde{2}\rangle$, $|\tilde{3}\rangle$, as follows:
find $|\tilde{1}\rangle = c_{1,1}|1\rangle$ is normalized; then
find $|\tilde{2}\rangle = c_{2,2}|2\rangle + c_{2,1}|\tilde{1}\rangle$ is normalized, and orthogonal to $|\tilde{1}\rangle$; then
find $|\tilde{3}\rangle = c_{3,3}|3\rangle + c_{3,1}|\tilde{1}\rangle + c_{3,2}|\tilde{2}\rangle$ is normalized, and orthogonal to both $|\tilde{1}\rangle$ and $|\tilde{2}\rangle$.
Solve these coefficients $c_{j,i}$. *And finally represent the original basis* $|1\rangle, |2\rangle, |3\rangle$ *as linear combinations of the new basis* $|\tilde{1}\rangle, |\tilde{2}\rangle, |\tilde{3}\rangle$.

(c)(3pts). *Find the* reciprocal *basis,* $|1'\rangle$, $|2'\rangle$, $|3'\rangle$, *in terms of* $|1\rangle$, $|2\rangle$, $|3\rangle$, *such that the inner products* $(|i'\rangle, |j\rangle) = \delta_{i,j}$. *Show that* $|1\rangle\langle 1'| + |2\rangle\langle 2'| + |3\rangle\langle 3'|$ *is the identity operator.*

(d)(5pts). A linear operator $\hat{A}$ is defined by its action on this basis as follows: $\hat{A}|1\rangle = (-|1\rangle + |2\rangle + |3\rangle)$, $\hat{A}|2\rangle = (-|2\rangle + |3\rangle + |1\rangle)$, $\hat{A}|3\rangle = (-|3\rangle + |1\rangle + |2\rangle)$. *Is* $\hat{A}$ *a hermitian operator? Is* $\hat{A}$ *a unitary operator? Solve the eigenvalues and normalized eigenvectors (in terms of* $|1\rangle, |2\rangle, |3\rangle$*) of* $\hat{A}$.

2. (5pts) If $[\hat{A}, \hat{B}] = 0$, namely $\hat{A}$ and $\hat{B}$ commute, then $\exp(\hat{A} + \hat{B}) = \exp(\hat{A}) \cdot \exp(\hat{B})$. *Prove this by brute-force expansion:* $\exp(\hat{A}) \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(\hat{A})^n$.

3. (5pts) *Prove the Baker-Hausdorff formula,* $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + [\hat{A}, [\hat{A}, \hat{B}]]/2! + \ldots$, *by brute-force*: expand both sides into sums of monomials $(\hat{A})^m\, \hat{B}\, (\hat{A})^n$, compare coefficients.

4. If the commutator $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ is a c-number (that commutes with everything), *prove the following* (these formulas may be useful later in this course).

(a)(2pts). $\exp(\hat{A}) \cdot \exp(\hat{B}) = \exp(\hat{B}) \cdot \exp(\hat{A}) \cdot \exp([\hat{A}, \hat{B}])$. [Hint: try to use the Baker-Hausdorff formula]

(b)(3pts). $\exp(\hat{A}) \cdot \exp(\hat{B}) = \exp(\hat{A} + \hat{B}) \cdot \exp(\frac{1}{2}[\hat{A}, \hat{B}])$. [Hint: check the heuristic proof of the Baker-Hausdorff formula, try to derive a differential equation.] [Side remark: this is a special case of Baker-Campbell-Hausdorff formula, $e^{\hat{A}}e^{\hat{B}} = \exp(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \dots)$.]

5.(5pts). Given the commutation relations $[\hat{A}, \hat{B}] = \mathbb{i}\hat{C}$, $[\hat{B}, \hat{C}] = \mathbb{i}\hat{A}$, and $[\hat{C}, \hat{A}] = \mathbb{i}\hat{B}$. *Compute* $\exp(\mathbb{i}\theta\hat{A}) \cdot (a\hat{A} + b\hat{B} + c\hat{C}) \cdot \exp(-\mathbb{i}\theta\hat{A})$ *where* θ, a, b, c *are c-numbers (result should be a finite-degree polynomial of* $\hat{A}$, $\hat{B}$, $\hat{C}$*).* Hereafter $\mathbb{i} \equiv \sqrt{-1}$ denotes the imaginary unit. [Hint: use the Baker-Hausdorff formula, try to write down several terms in the expansion, and find some pattern.]

6. Define the Pauli matrices, $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It is easy to check that $\{\sigma_1, \sigma_2\} = \{\sigma_2, \sigma_3\} = \{\sigma_3, \sigma_1\} = 0$, and $(\sigma_i)^2 = \mathbb{1}_{2\times 2}$, and $[\sigma_1, \sigma_2] = 2\mathbb{i}\sigma_3$, $[\sigma_2, \sigma_3] = 2\mathbb{i}\sigma_1$, $[\sigma_3, \sigma_1] = 2\mathbb{i}\sigma_2$.

(a)(5pts). Consider a 2×2 matrix $M = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$, where $a_{0,1,2,3}$ are complex numbers. *If* M *is a hermitian matrix, what is the condition on* $a_{0,1,2,3}$*? Show that* $M \cdot M = c_0\mathbb{1}_{2\times 2} + c_1 M$ *and solve the numbers* c_0 *and* c_1 *in terms of* $a_{0,1,2,3}$*. Then solve the eigenvalues of* M *in terms of* $a_{0,1,2,3}$*.* [Hint: you don't really need to diagonalize a 2×2 matrix. This result will be useful later. This result will be useful later in this course.]

(b)(5pts). *Compute* $\exp[\mathbb{i}(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)]$, *where* $a_{1,2,3}$ *are real numbers.* The result should be a linear combination of Pauli matrices. [Hint: compute the first few terms in the Taylor expansion and try to find some pattern. This result will be useful later in this course.]

7. $\mathcal{H}_1$ and $\hat{H}_2$ are both 2-dimensional Hilbert spaces. $\mathcal{H}_1$ has complete orthonormal basis $|e_1\rangle$ and $|e_2\rangle$, $\mathcal{H}_2$ has complete orthonormal basis $|e_1'\rangle$ and $|e_2'\rangle$.

(a)(4pts). Define operators $\hat{\sigma} = |e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|$ in $\mathcal{H}_1$, and $\hat{\sigma'} = |e_1'\rangle\langle e_2'| + |e_2'\rangle\langle e_1'|$ in $\mathcal{H}_2$. *Write down all the eigenvalues and normalized eigenstates of* $\hat{\sigma} \otimes \hat{\sigma'}$ *in* $\mathcal{H}_1 \otimes \mathcal{H}_2$*.*

(b)(1pt). Define a state $|\varphi\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |e_2'\rangle + |e_2\rangle \otimes |e_1'\rangle)$. *Show that this state CANNOT be written as a single tensor product,* $|\psi\rangle \otimes |\psi'\rangle$, *where* $|\psi\rangle$ *is a state in* $\mathcal{H}_1$, *and* $|\psi'\rangle$ *is a state in* $\mathcal{H}_2$*.* [Hint: assume this is $|\psi\rangle \otimes |\psi'\rangle$, try to solve $|\psi\rangle$ and $|\psi'\rangle$ in terms of basis]