Homework #4:
Due: tentatively Oct. 24, 2019

***** (about lecture #3) *****

Problem 1. Consider the 1D harmonic oscillator $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$. Here $\hat{x}$ is position operator, $\hat{p}$ is momentum operator, $[\hat{x}, \hat{p}] = \mathbb{i}\hbar$, and in position representation $\hat{p} = -\mathbb{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{b} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \mathbb{i}\frac{1}{m\omega}\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{b}, \hat{b}^\dagger] = 1$ and $\hat{H}_0 = \hbar\omega\,(\hat{b}^\dagger\hat{b} + \frac{1}{2})$. It has a unique ground state $|0\rangle$ with $\hat{b}|0\rangle = 0$, and excited states $|n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{b}^\dagger)^n|0\rangle$ with energy $E_n = (n + \frac{1}{2})\hbar\omega$.

(a) (5pts) Let $\hat{H}' = \hat{H}_0 - f \cdot \hat{x}$, where f is a real constant. $\hat{H}'$ is related to $\hat{H}_0$ by $\hat{U} \cdot \hat{H}' \cdot \hat{U}^\dagger = \hat{H}_0 + c$. Here c is a real constant, $\hat{U} = \exp(-\mathbb{i}X\hat{p} - \mathbb{i}P\hat{x})$ is a unitary operator with real parameters X and P. *Solve X and P and c in terms of $f, m, \omega, \hbar$.*

(b) (5pts) Denote the normalized ground state of $\hat{H}'$ by $|0'\rangle$. *Evaluate $\langle 0'|\hat{x}|0'\rangle$ and $\langle 0'|\hat{p}|0'\rangle$.* [Hint: result of (a) may help.]

(c) (5pts) At $t = 0$, let the state $|\psi(t = 0)\rangle = |0'\rangle$, evolve this state under $\hat{H}_0$, namely $|\psi(t)\rangle = \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t)|\psi(t = 0)\rangle$. *Evaluate $\langle\psi(t)|\hat{x}|\psi(t)\rangle$ and $\langle\psi(t)|\hat{p}|\psi(t)\rangle$.* [Hint: you can use either Schrödinger or Heisenberg picture, you can directly use the Heisenberg equations of motion for $\hat{x}$ and $\hat{p}$ and their solutions for harmonic oscillator]

(d) (5pts) Define two Hermitian operators: $\hat{O}_1 = m^2\omega^2\hat{x}^2 - \hat{p}^2$, $\hat{O}_2 = m\omega(\hat{x}\hat{p} + \hat{p}\hat{x})$. Their Heisenberg picture under $\hat{H}_0$ are $\hat{O}_{i,\mathrm{H}}(t) = \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t) \cdot \hat{O}_i \cdot \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t)$. *Write down the Heisenberg equations of motion, $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_{i,H}(t) = \dots$ for $i = 1, 2$.* The right-hand side of these equations should be expressed in terms of $\hat{O}_{j,\mathrm{H}}(t)$ with $j = 1, 2$.

(e) (5pts) *Solve the equations in (d). Namely solve $\hat{O}_{i,H}(t)$ in terms of $\hat{O}_{j,H}(t = 0)$.*

2. Consider a spin-1/2 moment. Its state belongs to a two-dimensional Hilbert space, with complete orthonormal basis $|\uparrow\rangle$ and $|\downarrow\rangle$ (spin up and down). Consider a periodic magnetic

field $B(t)$ with period $T > 0$, $B(t) = \begin{cases} B, & \text{if } n < \frac{t}{T} < n + \frac{1}{2} \text{ for some integer } n; \\ -B, & \text{if } n + \frac{1}{2} < \frac{t}{T} < n + 1 \text{ for some integer } n. \end{cases}$.

Here B is a positive constant. The Hamiltonian is $\hat{H}(t) = -B(t) \cdot \sigma_3$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is Pauli matrix.

(a). (5pts) *Write down the explicit form of time-evolution operator $\hat{U}(t)$, in terms of Pauli matrices.* [Hint: although $\hat{H}$ is time-dependent, $\hat{H}$ at different time commute]

(b). (5pts) Given the state at $t = 0$ as $|\psi(t = 0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ under the above basis. *Compute the time-dependent expectation values $\langle\psi(t)|\sigma_1|\psi(t)\rangle$, $\langle\psi(t)|\sigma_2|\psi(t)\rangle$, $\langle\psi(t)|\sigma_3|\psi(t)\rangle$.*

(c). (5pts) *Compute the "retarded Green's function", the Fourier transform of $\hat{U}(t)$ over $t > 0$, $\hat{G}(\omega) = \mathbb{i} \int_0^\infty \mathrm{Tr}[\hat{U}(t)]\, e^{\mathbb{i}\omega t}\, \mathrm{d}t$. Find out the "energy spectrum" namely the poles of $\hat{G}(\omega)$.* Here Tr is the (matrix) trace. [Hint: to make this integral absolutely convergent, you can add an infinitesimal positive imaginary part to ω, namely compute $\tilde{G}(\omega + \mathbb{i}\delta)$ and eventually take $\delta \to +0$ limit]

(NOT REQUIRED) At any instant of time, $\hat{H}(t)$ has the same eigenvalues $\pm B$. However these are not the poles of $\tilde{G}(\omega)$ solved in (c). When the period $T \to +\infty$, will the spectrum in (c) goes back to the spectrum of a time-independent Hamiltonian with only two poles at $\omega = \pm B/\hbar$?

3. Consider the spin-1/2 moment defined in Problem 2. Under the $|\uparrow\rangle, |\downarrow\rangle$ basis, the Hamiltonian at time t is $\hat{H}(t) = -B\cos(\omega t)\sigma_1 - B\sin(\omega t)\sigma_2$. Here B, ω are positive constants, $\sigma_{1,2}$ are Pauli matrices.

(a). (5pts) The time evolution operator $\hat{U}(t)$ satisfies $\mathbb{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t) = \hat{H}(t) \cdot \hat{U}(t)$, and $\hat{U}(t = 0) = \hat{\mathbb{1}}$, and is a 2×2 matrix under the $|\uparrow\rangle, |\downarrow\rangle$ basis. Assume that B is a small parameter, *compute $\hat{U}(t)$ up to B^2 order by the Dyson series.*

(b). (DIFFICULT) (5pts) The time evolution can be solved exactly. Assume the solution to the Schrödinger equation, $\mathbb{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi,t\rangle = \hat{H}(t)|\psi,t\rangle$, is $|\psi,t\rangle = c_1(t)|\uparrow\rangle + c_2(t)e^{\mathbb{i}\omega\cdot t}|\downarrow\rangle$. *Solve $c_1(t)$ and $c_2(t)$ in terms of the initial values $c_1(t=0)$ and $c_2(t=0)$, and therefore solve the unitary time evolution operator $\hat{U}(t)$ as a 2×2 matrix under $|\uparrow\rangle, |\downarrow\rangle$ basis.* [note that $\begin{pmatrix} c_1(t) \\ c_2(t)e^{\mathbb{i}\omega\cdot t} \end{pmatrix} = \hat{U}(t)\cdot\begin{pmatrix} c_1(t=0) \\ c_2(t=0) \end{pmatrix}$ under the time-independent basis $|\uparrow\rangle, |\downarrow\rangle$]