Homework #6:
Brief Solutions

NOTE: Condon-Shortley convention should be used unless specified otherwise. Bold symbols denote three component vectors, for example $\boldsymbol{S}$ has three components S_x, S_y, S_z.

1. (5 points) The generators of $SO(3)$ group are $\overleftrightarrow{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathbb{i} \\ 0 & \mathbb{i} & 0 \end{pmatrix}$, $\overleftrightarrow{J}_y = \begin{pmatrix} 0 & 0 & \mathbb{i} \\ 0 & 0 & 0 \\ -\mathbb{i} & 0 & 0 \end{pmatrix}$, $\overleftrightarrow{J}_z = \begin{pmatrix} 0 & -\mathbb{i} & 0 \\ \mathbb{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Consider $\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}} \equiv n_x \overleftrightarrow{J}_x + n_y \overleftrightarrow{J}_y + n_z \overleftrightarrow{J}_z$, where n_x, n_y, n_z are real numbers and $\boldsymbol{n}^2 \equiv n_x^2 + n_y^2 + n_z^2 = 1$.

(a) (3pts) *Compute* $(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^2$ *and* $(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^3$ *explicitly, show that* $(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^3 = \boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}}$.

(b) (2pts) *Use the result of (a) to compute* $\exp(-\mathbb{i}\theta \boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})$ *explicitly.* [Note: this of course should be $\overleftrightarrow{R}_{\boldsymbol{n}}(\theta)$]

Solution:

(a) $\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}} = \begin{pmatrix} 0 & -\mathbb{i}n_z & \mathbb{i}n_y \\ \mathbb{i}n_z & 0 & -\mathbb{i}n_x \\ -\mathbb{i}n_y & \mathbb{i}n_x & 0 \end{pmatrix}$, is pure imaginary and anti-symmetric.

$(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^2 = \begin{pmatrix} n_z^2 + n_y^2 & -n_x n_y & -n_z n_x \\ -n_y n_x & n_z^2 + n_x^2 & -n_z n_y \\ -n_x n_z & -n_y n_z & n_y^2 + n_x^2 \end{pmatrix}$, is real symmetric.

$(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^3 = \begin{pmatrix} 0 & -\mathbb{i}n_z & \mathbb{i}n_y \\ \mathbb{i}n_z & 0 & -\mathbb{i}n_x \\ -\mathbb{i}n_y & \mathbb{i}n_x & 0 \end{pmatrix} \cdot (n_x^2 + n_y^2 + n_z^2) = \boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}}$.

(b) From (a), $(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^{2m+1} = (\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})$, and $(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^{2m+2} = (\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^2$, for $m = 0, 1, \ldots$.

$\exp(-\mathbb{i}\theta \boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}}) = \sum_{n=0}^{\infty} \frac{(-\mathbb{i}\theta)^n}{n!} (\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^n$

$= \mathbb{1}_{3\times 3} + \sum_{m=0}^{\infty} \frac{-\mathbb{i}(-1)^m \theta^{2m+1}}{(2m+1)!} (\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}}) + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\theta^{2m+2}}{(2m+2)!} (\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^2$

$= \mathbb{1}_{3\times 3} - \mathbb{i}\sin\theta \cdot (\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}}) + (\cos\theta - 1) \cdot (\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^2$

$$= \begin{pmatrix} n_x^2 + (n_y^2 + n_z^2)\cos\theta & -n_z \sin\theta + (1-\cos\theta) n_x n_y & n_y \sin\theta + (1-\cos\theta) n_x n_z \\ n_z \sin\theta + (1-\cos\theta) n_y n_x & n_y^2 + (n_z^2 + n_x^2)\cos\theta & -n_x \sin\theta + (1-\cos\theta) n_y n_z \\ -n_y \sin\theta + (1-\cos\theta) n_z n_x & n_x \sin\theta + (1-\cos\theta) n_z n_y & n_z^2 + (n_x^2 + n_y^2)\cos\theta \end{pmatrix}$$

2. (8 points) Schwinger boson. $\hat{b}_1^\dagger$ and $\hat{b}_2^\dagger$ are creation operators for orthonormal boson modes, $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{i,j}$. The occupation basis $|n_1, n_2\rangle \equiv \frac{1}{\sqrt{n_1! n_2!}}(\hat{b}_1^\dagger)^{n_1}(\hat{b}_2^\dagger)^{n_2}|\text{vac}\rangle$ are complete orthonormal basis of the Fock space. Here $|\text{vac}\rangle$ is the boson vacuum, $\hat{b}_i|\text{vac}\rangle = 0$. Denote $|n_1, n_2\rangle$ by $|j, m\rangle$ where $j = \frac{n_1+n_2}{2}$, $m = \frac{n_1-n_2}{2}$. Define three hermitian operators, $\hat{J}_z = \frac{1}{2}(\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2)$, $\hat{J}_x = \frac{1}{2}(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1)$, $\hat{J}_y = \frac{1}{2}(-\mathbb{i}\hat{b}_1^\dagger \hat{b}_2 + \mathbb{i}\hat{b}_2^\dagger \hat{b}_1)$.

(a) (3pts) *Compute the commutators,* $[\hat{J}_x, \hat{J}_y]$, $[\hat{J}_y, \hat{J}_z]$, $[\hat{J}_z, \hat{J}_x]$. The results should be linear combinations of $\hat{J}_{x,y,z}$.

(b) (5pts) *In the fixed total boson number subspace (fixed j quantum number), compute the matrix elements* $(J_x)_{mm'} \equiv \langle j, m|\hat{J}_x|j, m'\rangle$, $(J_y)_{mm'} \equiv \langle j, m|\hat{J}_y|j, m'\rangle$, $(J_z)_{mm'} \equiv \langle j, m|\hat{J}_z|j, m'\rangle$. *Check that these* $(2j+1) \times (2j+1)$ *matrices satisfy the commutation relations in (a).*

(a) This is essentially the same as Homework #4 Problem 1(j).

$[\hat{J}_x, \hat{J}_y] = \mathbb{i}\hat{J}_z$, $[\hat{J}_y, \hat{J}_z] = \mathbb{i}\hat{J}_x$, $[\hat{J}_z, \hat{J}_x] = \mathbb{i}\hat{J}_y$.

(b) Use $\hat{b}_1^\dagger|n_1, n_2\rangle = \sqrt{n_1+1}|n_1+1, n_2\rangle$, $\hat{b}_2^\dagger|n_1, n_2\rangle = \sqrt{n_2+1}|n_1, n_2+1\rangle$, $\hat{b}_1|n_1, n_2\rangle = \sqrt{n_1}|n_1-1, n_2\rangle$, $\hat{b}_2|n_1, n_2\rangle = \sqrt{n_2}|n_1, n_2-1\rangle$. And $n_1 = j+m$, $n_2 = j-m$.

$(J_x)_{mm'} = \frac{1}{2}\big(\delta_{m,m'+1}\sqrt{(j+m'+1)(j-m')} + \delta_{m+1,m'}\sqrt{(j+m')(j-m'+1)}\big)$,

$(J_y)_{mm'} = \frac{1}{2}\big(-\mathbb{i}\delta_{m,m'+1}\sqrt{(j+m'+1)(j-m')} + \mathbb{i}\delta_{m+1,m'}\sqrt{(j+m')(j-m'+1)}\big)$,

$(J_z)_{mm'} = \delta_{m,m'} \cdot m$

Use Einstein convention hereafter for computing the commutators,

$$
\begin{aligned}
&([J_x, J_y])_{m,m'} \equiv (J_x)_{m,m''}(J_y)_{m'',m'} - (J_y)_{m,m''}(J_x)_{m'',m'} \\
&= \tfrac{1}{4}\Big(-\mathbb{i}\delta_{m,m'+2}\sqrt{(j+m)(j-m+1)(j+m-1)(j-m+2)} + \mathbb{i}\delta_{m,m'}(j+m)(j-m+1) \\
&\qquad -\mathbb{i}\delta_{m,m'}(j+m+1)(j-m) + \mathbb{i}\delta_{m+2,m'}\sqrt{(j+m'+2)(j-m'-1)(j+m')(j-m'+1)}\Big) \\
&\quad - \tfrac{1}{4}\Big(-\mathbb{i}\delta_{m,m'+2}\sqrt{(j+m)(j-m+1)(j+m-1)(j-m+2)} - \mathbb{i}\delta_{m,m'}(j+m)(j-m+1) \\
&\qquad +\mathbb{i}\delta_{m,m'}(j+m+1)(j-m) + \mathbb{i}\delta_{m+2,m'}\sqrt{(j+m'+2)(j-m'-1)(j+m')(j-m'+1)}\Big) \\
&= \tfrac{\mathbb{i}}{2}\delta_{m,m'}((j+m)(j-m+1) - (j+m+1)(j-m)) = \mathbb{i}\delta_{m,m'}m = (J_z)_{m,m'}
\end{aligned}
$$

$$
\begin{aligned}
&([J_y, J_z])_{m,m'} \equiv (J_y)_{m,m''}(J_z)_{m'',m'} - (J_z)_{m,m''}(J_y)_{m'',m'} = (J_y)_{m,m'} \cdot (m'-m) \\
&= \tfrac{1}{2}\Big(-\mathbb{i}\delta_{m,m'+1}\sqrt{(j+m'+1)(j-m')}\cdot(m'-m) + \mathbb{i}\delta_{m+1,m'}\sqrt{(j+m')(j-m'+1)}\cdot(m'-m)\Big) \\
&= \mathbb{i}\tfrac{1}{2}\Big(\delta_{m,m'+1}\sqrt{(j+m'+1)(j-m')} + \delta_{m+1,m'}\sqrt{(j+m')(j-m'+1)}\Big) = \mathbb{i}(J_x)_{m,m'}
\end{aligned}
$$

$$([J_z, J_x])_{m,m'} \equiv (J_z)_{m,m''}(J_x)_{m'',m'} - (J_x)_{m,m''}(J_z)_{m'',m'} = (J_x)_{m,m'} \cdot (m - m')$$
$$= \tfrac{1}{2}\Big(\delta_{m,m'+1}\sqrt{(j+m'+1)(j-m')}\cdot(m-m') + \delta_{m+1,m'}\sqrt{(j+m')(j-m'+1)}\cdot(m-m')\Big)$$
$$= \mathbb{i}\tfrac{1}{2}\Big(-\mathbb{i}\delta_{m,m'+1}\sqrt{(j+m'+1)(j-m')} + \mathbb{i}\delta_{m+1,m'}\sqrt{(j+m')(j-m'+1)}\Big) = \mathbb{i}(J_y)_{m,m'}$$

Note: these matrix elements are the same as the angular momentum operators' matrix elements under the Condon-Shortley convention in the $\hat{J}_z$ eigenbasis.

3. (17 points) Consider a spin-1 moment, denote the angular momentum operator by $\hat{\boldsymbol{S}}$. Then $[\hat{S}_a, \hat{S}_b] = \sum_c \mathbb{i}\epsilon_{abc}\hat{S}_c$, and $\hat{\boldsymbol{S}}^2 = 1 \cdot (1+1) = 2$ in this 3-dimensional Hilbert space. An obvious complete orthonormal basis is the S_z basis, $|S_z = +1, 0, -1\rangle$.

(a). (3pts) Given unit vector $\boldsymbol{n} = (\sin\eta\cos\phi, \sin\eta\sin\phi, \cos\eta)$, where η, ϕ are real parameters, *compute the eigenvalues of* $\boldsymbol{n} \bullet \hat{\boldsymbol{S}}$. [Hint: eigenvalues can be obtained without calculation, consider $\exp(-\mathbb{i}\theta\boldsymbol{n'} \bullet \hat{\boldsymbol{S}}) \cdot (\hat{\boldsymbol{S}} \bullet \boldsymbol{n}) \cdot \exp(\mathbb{i}\theta\boldsymbol{n'} \bullet \hat{\boldsymbol{S}}) = \hat{\boldsymbol{S}} \bullet \overleftrightarrow{R}_{\boldsymbol{n'}}(\theta) \bullet \boldsymbol{n}$, where $\overleftrightarrow{R}_{\boldsymbol{n'}}(\theta)$ is the $SO(3)$ matrix for rotation around $\boldsymbol{n'}$ by angle θ.]

(b). (5pts) *Use the result of (a) to show that* $(\boldsymbol{n} \bullet \hat{\boldsymbol{S}})^3 = \boldsymbol{n} \bullet \hat{\boldsymbol{S}}$. *Use this fact to compute the* 3×3 *matrix* $\exp(-\mathbb{i}\theta\, \boldsymbol{n} \bullet \hat{\boldsymbol{S}})$ *in terms of real parameters* η, ϕ, θ, *under the* S_z *basis.* [Side remark: this is just $D^{(j=1)}(e^{-\mathbb{i}\theta\boldsymbol{n}\bullet\boldsymbol{\sigma}/2})$]

(c). (3pts) For the $\boldsymbol{n}$ in (a), *Compute the normalized eigenstates of* $\boldsymbol{n} \bullet \hat{\boldsymbol{S}}$. [Hint: can be done by brute-force, or using the result of (b) and the Hint of (a).]

(d). (3pts) The solution of (c) contains the "uniaxial spin nematic state", the eigenstate of $\boldsymbol{n} \bullet \hat{\boldsymbol{S}}$ with eigenvalue 0. Denote this state by $|\boldsymbol{n} \bullet \hat{\boldsymbol{S}} = 0\rangle$. *Compute for* $\boldsymbol{n}$ *along* x, y, z *directions the spin-nematic states, namely* $|S_x = 0\rangle$ *and* $|S_y = 0\rangle$ *and* $|S_z = 0\rangle$, *in terms of the* S_z *basis. Choose their overall complex phase factors carefully so that they are invariant under time-reversal symmetry. Check that they form complete orthonormal basis.* [Hint: time-reversal symmetry action on S_z basis is, $\hat{\mathcal{T}}|S_z\rangle = (-1)^{S_z}|-S_z\rangle$]

(e). (3pts) *Write down the matrix representation of spin operators* $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$, *in the basis of the three spin-nematic states* $|S_x = 0\rangle$ *and* $|S_y = 0\rangle$ *and* $|S_z = 0\rangle$ *solved in (d).*

Namely compute $(S_a)_{bc} \equiv \langle S_b = 0|\hat{S}_a|S_c = 0\rangle$. [Note: if you have solved these basis correctly, these three 3×3 hermitian matrices should be purely imaginary, according to time-reversal symmetry properties of spin operators]

Solution:

(a) Under the basis $(|S_z = 1\rangle, |S_z = 0\rangle, |S_z = -1\rangle)$,

$\hat{S}_z$ is diagonal $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and $\hat{S}_+ \equiv \hat{S}_x + \mathbb{i}\hat{S}_y$ is $\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$.

Then $(\boldsymbol{n}\cdot\hat{\boldsymbol{S}}) \equiv \sum_a n_a \hat{S}_a = \begin{pmatrix} \cos\eta & \frac{1}{\sqrt{2}}e^{-\mathbb{i}\phi}\sin\eta & 0 \\ \frac{1}{\sqrt{2}}e^{\mathbb{i}\phi}\sin\eta & 0 & \frac{1}{\sqrt{2}}e^{-\mathbb{i}\phi}\sin\eta \\ 0 & \frac{1}{\sqrt{2}}e^{\mathbb{i}\phi}\sin\eta & -\cos\eta \end{pmatrix}$,

From $\exp(-\mathbb{i}\theta\boldsymbol{n'}\cdot\hat{\boldsymbol{S}})\cdot(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})\cdot\exp(\mathbb{i}\theta\boldsymbol{n'}\cdot\hat{\boldsymbol{S}}) = \hat{\boldsymbol{S}}\cdot R_{\boldsymbol{n'}}(\theta)\cdot\boldsymbol{n}$, we see that $(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})$ has the same eigenvalues as $\hat{S}_z$, because we can always find a $SO(3)$ rotation matrix $R_{\boldsymbol{n'}}(\theta)$ so that $R_{\boldsymbol{n'}}(\theta)\cdot\boldsymbol{n}$ is the unit vector along $+z$ direction.

Therefore the eigenvalues of $(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})$ are $+1, 0, -1$, the same as those of $\hat{S}_z$.

(b). The eigenvalues λ of $(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})$ satisfy $\lambda^3 = \lambda$. The hermitian operator $(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})$ is a diagonal matrix in its eigenbasis, with diagonal entries being the eigenvalues. Therefore it satisfies $(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})^3 = (\hat{\boldsymbol{S}}\cdot\boldsymbol{n})$.

Therefore $(\boldsymbol{n}\cdot\hat{\boldsymbol{S}})^{2m+1} = (\boldsymbol{n}\cdot\hat{\boldsymbol{S}})$, and $(\boldsymbol{n}\cdot\hat{\boldsymbol{S}})^{2m+2} = (\boldsymbol{n}\cdot\hat{\boldsymbol{S}})^2$, for non-negative integer m.

Then $\exp(-\mathbb{i}\theta\boldsymbol{n}\cdot\hat{\boldsymbol{S}}) = \sum_{m=0}^{\infty}\frac{(-\mathbb{i}\theta)^m}{m!}(\boldsymbol{n}\cdot\hat{\boldsymbol{S}})^m$

$= \hat{\mathbb{1}} + \sum_{m=0}^{\infty}\frac{(-\mathbb{i}\theta)^{2m+1}}{(2m+1)!}(\boldsymbol{n}\cdot\hat{\boldsymbol{S}}) + \sum_{m=0}^{\infty}\frac{(-\mathbb{i}\theta)^{2m+2}}{(2m+2)!}(\boldsymbol{n}\cdot\hat{\boldsymbol{S}})^2$

$= \hat{\mathbb{1}} - \mathbb{i}\sin\theta(\boldsymbol{n}\cdot\hat{\boldsymbol{S}}) + (\cos\theta - 1)(\boldsymbol{n}\cdot\hat{\boldsymbol{S}})^2$

Here $(\boldsymbol{n}\cdot\hat{\boldsymbol{S}})$ has been given in (a), and

$$(\boldsymbol{n}\cdot\hat{\boldsymbol{S}})^2 = \begin{pmatrix} \cos^2\eta + \frac{\sin^2\eta}{2} & \frac{e^{-\mathbb{i}\phi}}{\sqrt{2}}\sin\eta\cos\eta & \frac{e^{-2\mathbb{i}\phi}}{2}\sin^2\eta \\ \frac{e^{\mathbb{i}\phi}}{\sqrt{2}}\sin\eta\cos\eta & \sin^2\eta & -\frac{e^{-\mathbb{i}\phi}}{\sqrt{2}}\sin\eta\cos\eta \\ \frac{e^{2\mathbb{i}\phi}}{2}\sin^2\eta & -\frac{e^{\mathbb{i}\phi}}{\sqrt{2}}\sin\eta\cos\eta & \frac{\cos^2\eta}{2} + \sin^2\eta \end{pmatrix}.$$

(c). the eigenvalue=1 eigenvector can be chosen as $\begin{pmatrix} e^{-\mathbb{i}\phi}\cos^2\frac{\eta}{2} \\ \sqrt{2}\cos\frac{\eta}{2}\sin\frac{\eta}{2} \\ e^{\mathbb{i}\phi}\sin^2\frac{\eta}{2} \end{pmatrix}$, namely the state is $|\boldsymbol{n}\cdot\hat{\boldsymbol{S}}=1\rangle = |S_z=1\rangle\cdot e^{-\mathbb{i}\phi}\cos^2\frac{\eta}{2} + |S_z=0\rangle\cdot\sqrt{2}\cos\frac{\eta}{2}\sin\frac{\eta}{2} + |S_z=-1\rangle\cdot e^{\mathbb{i}\phi}\sin^2\frac{\eta}{2}$.

This can also be obtained by combining two spin-1/2 spin polarized state $|\boldsymbol{n}\rangle_{S=\frac{1}{2}} = |\uparrow\rangle e^{-\mathbb{i}\phi/2}\cos\frac{\eta}{2} + |\downarrow\rangle e^{\mathbb{i}\phi/2}\sin\frac{\eta}{2}$, and $|\boldsymbol{n}\cdot\hat{\boldsymbol{S}}=+1\rangle \sim |\boldsymbol{n}\rangle_{S=\frac{1}{2}}\otimes|\boldsymbol{n}\rangle_{S=\frac{1}{2}}$ with $|\uparrow\rangle\otimes|\uparrow\rangle = |S_z=1\rangle$, $|\downarrow\rangle\otimes|\downarrow\rangle = |S_z=-1\rangle$, and $|\uparrow\rangle\otimes|\downarrow\rangle + |\downarrow\rangle\otimes|\uparrow\rangle = \sqrt{2}|S_z=0\rangle$.

the eigenvalue=-1 eigenvector can be chosen as $\begin{pmatrix} -e^{-\mathbb{i}\phi}\sin^2\frac{\eta}{2} \\ \sqrt{2}\cos\frac{\theta}{2}\sin\frac{\eta}{2} \\ -e^{\mathbb{i}\phi}\cos^2\frac{\eta}{2} \end{pmatrix}$, namely the state is $|\boldsymbol{n}\cdot\hat{\boldsymbol{S}}=-1\rangle = |S_z=1\rangle\cdot(-e^{-\mathbb{i}\phi}\sin^2\frac{\eta}{2}) + |S_z=0\rangle\cdot\sqrt{2}\cos\frac{\eta}{2}\sin\frac{\eta}{2} + |S_z=-1\rangle\cdot(-e^{\mathbb{i}\phi}\cos^2\frac{\eta}{2})$,

the eigenvalue=0 eigenvector can be chosen as $\begin{pmatrix} -\frac{1}{\sqrt{2}}e^{-\mathbb{i}\phi}\sin\eta \\ \cos\eta \\ \frac{1}{\sqrt{2}}e^{\mathbb{i}\phi}\sin\eta \end{pmatrix}$, namely the state is $|\boldsymbol{n}\cdot\hat{\boldsymbol{S}}=0\rangle = |S^z=1\rangle\cdot(-\frac{1}{\sqrt{2}}e^{-\mathbb{i}\phi}\sin\eta) + |S^z=0\rangle\cdot\cos\eta + |S^z=-1\rangle\cdot\frac{1}{\sqrt{2}}e^{\mathbb{i}\phi}\sin\eta$,

These eigenvectors can also be obtained by the relation, $\exp(-\mathbb{i}\theta'\boldsymbol{n'}\cdot\hat{\boldsymbol{S}})\cdot(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})\cdot\exp(\mathbb{i}\theta'\boldsymbol{n'}\cdot\hat{\boldsymbol{S}}) = \hat{\boldsymbol{S}}\cdot R_{\boldsymbol{n'}}(\theta')\cdot\boldsymbol{n}$. Choose $\boldsymbol{n'} = (\sin\phi, -\cos\phi, 0)$ and $\theta' = \eta$, then $R_{\boldsymbol{n'}}(\theta')\cdot\boldsymbol{n}$ is the unit vector along the $+z$ direction. Then the eigenstate of $(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})$ are $\exp(\mathbb{i}\theta'\boldsymbol{n'}\cdot\hat{\boldsymbol{S}})|S_z=\lambda\rangle$ for $\lambda=+1,0,-1$.

$\exp(\mathbb{i}\theta'\boldsymbol{n'}\cdot\hat{\boldsymbol{S}})$ has been computed in (b). Make the following replacement in the result of (b), $\theta\to-\eta$, $\eta\to\frac{\pi}{2}$, $\phi\to\phi-\frac{\pi}{2}$. We obtain $\exp(\mathbb{i}\theta'\boldsymbol{n'}\cdot\hat{\boldsymbol{S}})$ as the 3×3 unitary matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathbb{i}\sin\eta\cdot\begin{pmatrix} 0 & \frac{\mathbb{i}}{\sqrt{2}}e^{-\mathbb{i}\phi} & 0 \\ \frac{-\mathbb{i}}{\sqrt{2}}e^{\mathbb{i}\phi} & 0 & \frac{\mathbb{i}}{\sqrt{2}}e^{-\mathbb{i}\phi} \\ 0 & \frac{-\mathbb{i}}{\sqrt{2}}e^{\mathbb{i}\phi} & 0 \end{pmatrix} + (\cos\eta-1)\cdot\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2}e^{-2\mathbb{i}\phi} \\ 0 & 1 & 0 \\ -\frac{1}{2}e^{2\mathbb{i}\phi} & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\cos\eta+1}{2} & \frac{-1}{\sqrt{2}}e^{-\mathbb{i}\phi}\sin\eta & -\frac{\cos\eta-1}{2}e^{-2\mathbb{i}\phi} \\ \frac{1}{\sqrt{2}}e^{\mathbb{i}\phi}\sin\eta & \cos\eta & \frac{-1}{\sqrt{2}}e^{-\mathbb{i}\phi}\sin\eta \\ -\frac{\cos\eta-1}{2}e^{2\mathbb{i}\phi} & \frac{1}{\sqrt{2}}e^{\mathbb{i}\phi}\sin\eta & \frac{\cos\eta+1}{2} \end{pmatrix}.$$

The three columns of this unitary matrix are the eigenvectors of $(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})$.

(d). Use the result of (c), up to plus/minus signs for each state,
$|S_x=0\rangle = -\frac{1}{\sqrt{2}}(|S_z=1\rangle - |S_z=-1\rangle)$,

$|S_y = 0\rangle = \frac{\mathbb{i}}{\sqrt{2}}(|S_z = 1\rangle + |S_z = -1\rangle)$,
$|S_z = 0\rangle = |S_z = 0\rangle$.

Note that the choices of the above phases satisfy that under time-reversal the $|S_{x,y,z} = 0\rangle$ states are "real" (invariant), under the convention $\mathcal{T}|S_z = m\rangle = (-1)^m |S_z = -m\rangle$.

It should be easy to check that the $|S_{x,y,z} = 0\rangle$ states form a complete orthonormal basis of spin-1 Hilbert space, $\langle S_a = 0|S_b = 0\rangle = \delta_{ab}$.

(e). Under the phase convention of (d) of the basis $(|S_x = 0\rangle, |S_y = 0\rangle, |S_z = 0\rangle)$,

$$\hat{S}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathbb{i} \\ 0 & \mathbb{i} & 0 \end{pmatrix}, \hat{S}_y = \begin{pmatrix} 0 & 0 & \mathbb{i} \\ 0 & 0 & 0 \\ -\mathbb{i} & 0 & 0 \end{pmatrix}, \hat{S}_z = \begin{pmatrix} 0 & -\mathbb{i} & 0 \\ \mathbb{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that under the "real" basis of (d), the matrix elements of spin operators are all pure imaginary.

4. (20 points) Consider three spin-1/2 moments (labeled by subscripts $i = 1, 2, 3$). Each spin-1/2 has a 2-dimensional Hilbert space with complete orthonormal basis $|s_i = \pm\frac{1}{2}\rangle$, and spin operators $\hat{S}_{i,a} = \frac{1}{2}\sigma_a$, for $a = x, y, z$, under the above basis in the 2-dim'l Hilbert space.

The entire 8-dimensional Hilbert space is the tensor product of the three spin-1/2 Hilbert spaces. The S_z tensor product basis are denoted by $|s_1, s_2, s_3\rangle$ with $s_i = \pm\frac{1}{2}$. Then $\hat{S}_{i,z}|s_1, s_2, s_3\rangle = s_i|s_1, s_2, s_3\rangle$.

The commutation relations between spin operators are $[\hat{S}_{i,a}, \hat{S}_{j,b}] = \delta_{i,j} \sum_c \mathbb{i}\epsilon_{abc}\hat{S}_{i,c}$.

(a). (4pts) Define $\hat{S}_{2+3,a} = \hat{S}_{2,a} + \hat{S}_{3,a}$. *What are the possible values of the spin for the sum of spin 2 and 3, $\hat{\boldsymbol{S}}_{2+3} = \hat{\boldsymbol{S}}_2 + \hat{\boldsymbol{S}}_3$? Or equivalently what are the possible eigenvalues of $\hat{\boldsymbol{S}}^2_{2+3} \equiv \sum_a \hat{S}^2_{2+3,a}$? Write down the $|S_{2+3}, S_{2+3,z}\rangle$ basis in terms of the S_z tensor product basis $|s_2, s_3\rangle$.*

(b) (8pts) *What are the possible values of total spin for the sum of the three spins, $\hat{\boldsymbol{S}}_{1+2+3} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2 + \hat{\boldsymbol{S}}_3$? Write down the $|S_{1+2+3}, S_{1+2+3,z}\rangle$ basis in terms of the S_z tensor product basis $|s_1, s_2, s_3\rangle$.* [Hint: the result of (a) may be useful.]

(c). (8pts) Consider the "symmetries" generated by

$C_3: \quad |s_1, s_2, s_3\rangle \mapsto |s_2, s_3, s_1\rangle, \quad \hat{S}_{1,a} \mapsto \hat{S}_{3,a}, \ \hat{S}_{2,a} \mapsto \hat{S}_{1,a}, \ \hat{S}_{3,a} \mapsto \hat{S}_{2,a}$; and

$\sigma: \quad |s_1, s_2, s_3\rangle \mapsto |s_1, s_3, s_2\rangle, \quad \hat{S}_{1,a} \mapsto \hat{S}_{1,a}, \ \hat{S}_{2,a} \mapsto \hat{S}_{3,a}, \ \hat{S}_{3,a} \mapsto \hat{S}_{2,a}.$

This is the D_3 group. with 6 group elements $\{\mathbb{1}, C_3, C_3^2, \sigma, \sigma C_3, \sigma C_3^2\}$, classified into 3 conjugacy classes, $\{\mathbb{1}\}$, $\{C_3, C_3^2\}$, $\{\sigma, \sigma C_3, \sigma C_3^2\}$. The character table of its irreducible representations ($\Gamma_{1,2,3}$) is

	$\mathbb{1}$	$2C_3$	3σ
Γ_1	1	1	1
Γ_2	1	1	-1
Γ_3	2	-1	0

Note that $\hat{\boldsymbol{S}}_{1+2+3}$ is invariant under this D_3 group. Therefore we can label states with simultaneous eigenvalues of $\hat{\boldsymbol{S}}^2_{1+2+3}$ and $\hat{S}_{1+2+3,z}$, and D_3 irreducible representations.

Find new complete orthonormal basis of 8-dimensional Hilbert space (in terms of S_z tensor product basis), which form irreducible representations of D_3 group and are simultaneous eigenstates of $\hat{\boldsymbol{S}}^2_{1+2+3}$ and $\hat{S}_{1+2+3,z}$. [Hint: the result of (b) may be helpful.]

Solution:

For notation simplicity, use $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ to denote $S_{i,z} = +\frac{1}{2}$ and $-\frac{1}{2}$ states respectively.

(a). total spin of two spin-1/2 labeled by 2 & 3 can be $0 = \frac{1}{2} - \frac{1}{2}$ or $1 = \frac{1}{2} + \frac{1}{2}$: namely the tensor product of two spin-1/2 irreducible representations can be decomposed into the direct sum of a spin-0 irrep and a spin-1 irrep, $\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0} \oplus \mathbf{1}$.

define ladder operators $\hat{S}_{2+3,\pm} = \hat{S}_{2,\pm} + \hat{S}_{3,\pm} = \hat{S}_{2+3,x} \pm \mathbb{i}\hat{S}_{2+3,y}$, under Condon-Shortley convention $\hat{S}_{2+3,\pm}|S_{2+3} = j, S_{2+3,z} = m\rangle = \sqrt{j(j+1) - m(m\pm 1)}|S_{2+3} = j, S_{2+3,z} = m\pm 1\rangle$.

triplet (spin-1 states):

$|S_{2+3} = 1, S_{2+3,z} = 1\rangle = |\uparrow\rangle_2 |\uparrow\rangle_3,$

$|S_{2+3} = 1, S_{2+3,z} = 0\rangle = \frac{1}{\sqrt{2}}\hat{S}_{2+3,-}|S_{2+3} = 1, S_{2+3,z} = 1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_2 |\downarrow\rangle_3 + |\downarrow\rangle_2 |\uparrow\rangle_3),$

$|S_{2+3} = 1, S_{2+3,z} = -1\rangle = |\downarrow\rangle_2 |\downarrow\rangle_3.$

singlet (spin-0 states):

$|S_{2+3} = 0, S_{2+3,z} = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_2 |\downarrow\rangle_3 - |\downarrow\rangle_2 |\uparrow\rangle_3).$

This can be obtained by the fact that it is orthogonal to the $|S_{2+3} = 1, S_{2+3,z} = 0\rangle$ state, and must be a linear combination of $|\uparrow\rangle_2 |\downarrow\rangle_3$ and $|\downarrow\rangle_2 |\uparrow\rangle_3$, and should be vanished by the ladder operator $\hat{S}_{2+3,+}$. This determines the state $|S_{2+3} = 0, S_{2+3,z} = 0\rangle$ up to an overall phase factor.

(b). total spin of three spin-1/2 labeled by 1 & 2 & 3 can be $\frac{1}{2}$ or $\frac{3}{2}$:

$\mathbf{\frac{1}{2}} \otimes (\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}}) = \mathbf{\frac{1}{2}} \otimes (\mathbf{0} \oplus \mathbf{1}) = (\mathbf{\frac{1}{2}} \otimes \mathbf{0}) \oplus (\mathbf{\frac{1}{2}} \otimes \mathbf{1}) = \mathbf{\frac{1}{2}} \oplus \mathbf{\frac{1}{2}} \oplus \mathbf{\frac{3}{2}}$.

define ladder operators $\hat{S}_{1+2+3,\pm} = \hat{S}_{1,\pm} + \hat{S}_{2+3,\pm} = \hat{S}_{1,\pm} + \hat{S}_{2,\pm} + \hat{S}_{3,\pm}$.

quartet (spin-3/2 states):

$|S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = \frac{3}{2}\rangle = |\uparrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = 1\rangle = |\uparrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3,$

$|S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = \frac{1}{2}\rangle = \frac{1}{\sqrt{3}} \hat{S}_{1+2+3,-} |S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = \frac{3}{2}\rangle$
$= \sqrt{\frac{1}{3}} |\downarrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = 1\rangle + \sqrt{\frac{2}{3}} |\uparrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = 0\rangle$
$= \sqrt{\frac{1}{3}} (|\downarrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3 + |\uparrow\rangle_1 |\downarrow\rangle_2 |\uparrow\rangle_3 + |\uparrow\rangle_1 |\uparrow\rangle_2 |\downarrow\rangle_3),$

$|S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = -\frac{1}{2}\rangle = \frac{1}{\sqrt{4}} \hat{S}_{1+2+3,-} |S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = \frac{1}{2}\rangle$
$= \sqrt{\frac{1}{3}} |\uparrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = -1\rangle + \sqrt{\frac{2}{3}} |\downarrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = 0\rangle$
$= \sqrt{\frac{1}{3}} (|\uparrow\rangle_1 |\downarrow\rangle_2 |\downarrow\rangle_3 + |\downarrow\rangle_1 |\uparrow\rangle_2 |\downarrow\rangle_3 + |\downarrow\rangle_1 |\downarrow\rangle_2 |\uparrow\rangle_3),$

$|S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = -\frac{3}{2}\rangle = |\downarrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = -1\rangle = |\downarrow\rangle_1 |\downarrow\rangle_2 |\downarrow\rangle_3.$

first doublet (spin-1/2 states):

$|S_{1+2+3} = \frac{1}{2}, S_{1+2+3,z} = \frac{1}{2}, S_{2+3} = 0\rangle = |\uparrow\rangle_1 |S_{2+3} = 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\uparrow\rangle_2 |\downarrow\rangle_3 - |\uparrow\rangle_1 |\downarrow\rangle_2 |\uparrow\rangle_3),$
$|S_{1+2+3} = \frac{1}{2}, S_{1+2+3,z} = -\frac{1}{2}, S_{2+3} = 0\rangle = |\downarrow\rangle_1 |S_{2+3} = 0\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle_1 |\uparrow\rangle_2 |\downarrow\rangle_3 - |\downarrow\rangle_1 |\downarrow\rangle_2 |\uparrow\rangle_3).$

second doublet (spin-1/2 states):

$|S_{1+2+3} = \frac{1}{2}, S_{1+2+3,z} = \frac{1}{2}, S_{2+3} = 1\rangle$
$= \sqrt{\frac{2}{3}} |\downarrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = 1\rangle - \sqrt{\frac{1}{3}} |\uparrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = 0\rangle$
$= \frac{1}{\sqrt{6}} (2|\downarrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3 - |\uparrow\rangle_1 |\uparrow\rangle_2 |\downarrow\rangle_3 - |\uparrow\rangle_1 |\downarrow\rangle_2 |\uparrow\rangle_3),$

$|S_{1+2+3} = \frac{1}{2}, S_{1+2+3,z} = -\frac{1}{2}, S_{2+3} = 1\rangle$
$= \sqrt{\frac{2}{3}} |\uparrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = -1\rangle - \sqrt{\frac{1}{3}} |\downarrow\rangle_1 |S_{2+3} = 1, S_{2+3,z} = 0\rangle$
$= \frac{1}{\sqrt{6}} (2|\uparrow\rangle_1 |\downarrow\rangle_2 |\downarrow\rangle_3 - |\downarrow\rangle_1 |\downarrow\rangle_2 |\uparrow\rangle_3 - |\downarrow\rangle_1 |\uparrow\rangle_2 |\downarrow\rangle_3).$

To construct the second doublet, one can use the C.-G. coefficients $\langle 1, m; \frac{1}{2}, m' | \frac{1}{2}, m+m' \rangle$ found in published tables, or the fact that $|S_{1+2+3} = \frac{1}{2}, S_{1+2+3,z} = \pm\frac{1}{2}, S_{2+3} = 1\rangle$ has to be orthogonal to $|S_{1+2+3} = \frac{3}{2}, S_{1+2+3,z} = \pm\frac{1}{2}\rangle$.

Note that here we have labeled the states by "quantum numbers" S_{1+2+3}, $S_{1+2+3,z}$, and S_{2+3}, because $\hat{\boldsymbol{S}}^2_{1+2+3}$ and $\hat{S}_{1+2+3,z}$ and $\hat{\boldsymbol{S}}^2_{2+3}$ mutually commute.

(c) Note that the D_3 group actions commute with total spin operators $\hat{\boldsymbol{S}}_{1+2+3}$. Therefore the operators corresponding to D_3 group elements can be simultaneously block-diagonalized with $\hat{\boldsymbol{S}}^2_{1+2+3}$ and $\hat{S}_{1+2+3,z}$. The basis in (b) can be used to construct the irreducible representations of D_3. The results are listed in the following table.

irrep.	$\hat{\boldsymbol{S}}^2_{1+2+3}$	$\hat{S}_{1+2+3,z}$	basis state(s)	$R(C_3)$	$R(\sigma)$
Γ_1	$\frac{3}{2}\cdot(\frac{3}{2}+1)$	$\frac{3}{2}$	$\lvert\uparrow\uparrow\uparrow\rangle$	(1)	(1)
Γ_1	$\frac{3}{2}\cdot(\frac{3}{2}+1)$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}(\lvert\downarrow\uparrow\uparrow\rangle + \lvert\uparrow\downarrow\uparrow\rangle + \lvert\uparrow\uparrow\downarrow\rangle)$	(1)	(1)
Γ_1	$\frac{3}{2}\cdot(\frac{3}{2}+1)$	$-\frac{1}{2}$	$\frac{1}{\sqrt{3}}(\lvert\uparrow\downarrow\downarrow\rangle + \lvert\downarrow\uparrow\downarrow\rangle + \lvert\downarrow\downarrow\uparrow\rangle)$	(1)	(1)
Γ_1	$\frac{3}{2}\cdot(\frac{3}{2}+1)$	$-\frac{3}{2}$	$\lvert\downarrow\downarrow\downarrow\rangle$	(1)	(1)
Γ_3	$\frac{1}{2}\cdot(\frac{1}{2}+1)$	$\frac{1}{2}$	$\left(\frac{1}{\sqrt{2}}(\lvert\uparrow\uparrow\downarrow\rangle - \lvert\uparrow\downarrow\uparrow\rangle),\ \frac{1}{\sqrt{6}}(2\lvert\downarrow\uparrow\uparrow\rangle - \lvert\uparrow\downarrow\uparrow\rangle - \lvert\uparrow\uparrow\downarrow\rangle)\right)$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Γ_3	$\frac{1}{2}\cdot(\frac{1}{2}+1)$	$-\frac{1}{2}$	$\left(\frac{1}{\sqrt{2}}(\lvert\downarrow\uparrow\downarrow\rangle - \lvert\downarrow\downarrow\uparrow\rangle),\ \frac{1}{\sqrt{6}}(-2\lvert\uparrow\downarrow\downarrow\rangle + \lvert\downarrow\uparrow\downarrow\rangle + \lvert\downarrow\downarrow\uparrow\rangle)\right)$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Here the choice for the basis of the 2-dimensional irrep Γ_3 are of course not unique. One can also choose the basis so that the three-fold rotation C_3 is diagonal.

irrep.	$\hat{\boldsymbol{S}}^2_{1+2+3}$	$\hat{S}_{1+2+3,z}$	basis state(s)	$R(C_3)$	$R(\sigma)$
Γ_1	$\frac{3}{2}\cdot(\frac{3}{2}+1)$	$\frac{3}{2}$	$\lvert\uparrow\uparrow\uparrow\rangle$	(1)	(1)
Γ_1	$\frac{3}{2}\cdot(\frac{3}{2}+1)$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}(\lvert\downarrow\uparrow\uparrow\rangle + \lvert\uparrow\downarrow\uparrow\rangle + \lvert\uparrow\uparrow\downarrow\rangle)$	(1)	(1)
Γ_1	$\frac{3}{2}\cdot(\frac{3}{2}+1)$	$-\frac{1}{2}$	$\frac{1}{\sqrt{3}}(\lvert\uparrow\downarrow\downarrow\rangle + \lvert\downarrow\uparrow\downarrow\rangle + \lvert\downarrow\downarrow\uparrow\rangle)$	(1)	(1)
Γ_1	$\frac{3}{2}\cdot(\frac{3}{2}+1)$	$-\frac{3}{2}$	$\lvert\downarrow\downarrow\downarrow\rangle$	(1)	(1)
Γ_3	$\frac{1}{2}\cdot(\frac{1}{2}+1)$	$\frac{1}{2}$	$\left(\frac{1}{\sqrt{3}}(\lvert\downarrow\uparrow\uparrow\rangle + e^{2\pi\mathrm{i}/3}\lvert\uparrow\downarrow\uparrow\rangle + e^{-2\pi\mathrm{i}/3}\lvert\uparrow\uparrow\downarrow\rangle),\ \frac{1}{\sqrt{3}}(\lvert\downarrow\uparrow\uparrow\rangle + e^{-2\pi\mathrm{i}/3}\lvert\uparrow\downarrow\uparrow\rangle + e^{2\pi\mathrm{i}/3}\lvert\uparrow\uparrow\downarrow\rangle)\right)$	$\begin{pmatrix} e^{2\pi\mathrm{i}/3} & 0 \\ 0 & e^{-2\pi\mathrm{i}/3} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Γ_3	$\frac{1}{2}\cdot(\frac{1}{2}+1)$	$-\frac{1}{2}$	$\left(\frac{1}{\sqrt{3}}(\lvert\uparrow\downarrow\downarrow\rangle + e^{2\pi\mathrm{i}/3}\lvert\downarrow\uparrow\downarrow\rangle + e^{-2\pi\mathrm{i}/3}\lvert\downarrow\downarrow\uparrow\rangle),\ \frac{1}{\sqrt{3}}(\lvert\uparrow\downarrow\downarrow\rangle + e^{-2\pi\mathrm{i}/3}\lvert\downarrow\uparrow\downarrow\rangle + e^{2\pi\mathrm{i}/3}\lvert\downarrow\downarrow\uparrow\rangle)\right)$	$\begin{pmatrix} e^{2\pi\mathrm{i}/3} & 0 \\ 0 & e^{-2\pi\mathrm{i}/3} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

One can also use the "projection operator" to generate the irreps. For example, the projection operator for the Γ_3 irrep maps the original basis

$|\uparrow\uparrow\downarrow\rangle$ to $(2|\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle)$, and

$|\uparrow\downarrow\uparrow\rangle$ to $(2|\uparrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle)$, and

$|\downarrow\uparrow\uparrow\rangle$ to $(2|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle)$.

These three final states are not linearly independent (sum to zero), and span the two-dimensional representation space of one Γ_3 irrep. One still needs to find an orthonormal basis for this sub-space.