Homework #7:
Due: tentatively Dec. 12, 2019

***** (about lecture #5) *****

1. (9points) For $j_1 = 2$, $j_2 = 1$, $j = 1$, *compute all the nonzero Clebsch-Gordon coefficients,* $\langle j, m | j_1, m_1; j_2, m_2 \rangle$. [Hint: think of this as an "addition of angular momentum" problem, define angular momentum operators $\hat{\boldsymbol{J}}_1$ and $\hat{\boldsymbol{J}}_2$ for $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ Hilbert spaces respectively, $m_i = -j_i, -j_i + 1, \ldots, j_i$, define total angular momentum operators $\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_2$, then solve the total angular momentum eigenbasis $|j, m\rangle$ in terms of tensor product basis $|j_1, m_1\rangle|j_2, m_2\rangle$]

2. (21points) Consider a spin-1 moment, denote the spin angular momentum operators by $\hat{\boldsymbol{S}}$. Then $[\hat{S}_a, \hat{S}_b] = \sum_c \mathbb{i}\epsilon_{abc}\hat{S}_c$. A complete orthonormal basis for the 3-dimensional Hilbert space is the S_z-eigenbasis $|S_z = +1, 0, -1\rangle$. Define "magnetic quadrupole" operators, $\hat{Q}_1 = \hat{S}_y\hat{S}_z + \hat{S}_z\hat{S}_y$, $\hat{Q}_2 = \hat{S}_z\hat{S}_x + \hat{S}_x\hat{S}_z$, $\hat{Q}_3 = \hat{S}_x\hat{S}_y + \hat{S}_y\hat{S}_x$, $\hat{Q}_4 = \hat{S}_x\hat{S}_x - \hat{S}_y\hat{S}_y$, $\hat{Q}_5 = \frac{1}{\sqrt{3}}(\hat{S}_x\hat{S}_x + \hat{S}_y\hat{S}_y - 2\hat{S}_z\hat{S}_z)$. They are obviously hermitian.

(a) (5pts) *Check that* $\sum_{a=x,y,z}[\hat{S}_a, [\hat{S}_a, \hat{Q}_i]] = 6 \cdot \hat{Q}_i$, $i = 1, \ldots, 5$. [Therefore the $\hat{Q}_i$ operators "angular momentum" quantum number is $k = 2$, because $6 = 2 \cdot (2 + 1)$] [The commutators $[\hat{S}_a, \hat{Q}_i]$ will be useful later]

(b) (7pts) By making linear combinations of $\hat{Q}_i$, we can form "irreducible tensor operators" $\hat{T}_q^{(k=2)}$, $q = -2, -1, 0, 1, 2$. And $[\hat{S}_z, \hat{T}_q^{(k=2)}] = q \cdot \hat{T}_q^{(k=2)}$, $[\hat{S}_\pm, \hat{T}_q^{(k=2)}] = \sqrt{(k \mp q)(k \pm q + 1)} \cdot \hat{T}_{q\pm 1}^{(k=2)}$. *Solve* $\hat{T}_q^{(k=2)}$ *as linear combinations of* $\hat{Q}_i$. [Hint: find $\hat{T}_{q=0}^{(k=2)}$ first, then generate others]

(c) (9pts) *Compute the matrix elements* $\langle S_z = m | \hat{T}_{q=m_1}^{(k=2)} | S_z = m_2 \rangle$, *for* $m = -1, 0, 1$, $m_1 = -2, -1, 0, 1, 2$, $m_2 = -1, 0, 1$. *Show that this is proportional to the C-G coefficients* $\langle j = 1, m | j_1 = 2, m_1; j_2 = 1, m_2 \rangle$ *solved in Problem 1.*

***** (about lecture #5 & #6) *****

3. (20points) Consider two spin-1 moments, $\hat{\boldsymbol{S}}_1$ and $\hat{\boldsymbol{S}}_2$. They satisfy $[\hat{S}_{i,a}, \hat{S}_{j,b}] = \delta_{i,j}\sum_c \mathbb{i}\epsilon_{abc}\hat{S}_{i,c}$ (here a, b, c label x, y, z components), and $\hat{\boldsymbol{S}}_1^2 = \hat{\boldsymbol{S}}_2^2 = 1 \cdot (1 + 1) = 2$. A

complete orthonormal basis for the 9-dimensional Hilbert space is the S_z-basis, $|s_1, s_2\rangle$, with $s_{1,2} = -1, 0, 1$ and $\hat{S}_{1,z}|s_1, s_2\rangle = s_1|s_1, s_2\rangle$ and $\hat{S}_{2,z}|s_1, s_2\rangle = s_2|s_1, s_2\rangle$. The matrix elements of $\hat{S}_{i,x}$ and $\hat{S}_{i,y}$ under this basis follow the Condon-Shortley convention.

(1) (9pts) Consider $\hat{H}_0 = -J \cdot \hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 \equiv -J \sum_a \hat{S}_{1,a}\hat{S}_{2,a}$. Here $J > 0$ is a positive real constant. *Solve all the eigenvalues and eigenstates (in terms of S_z-basis) of $\hat{H}_0$.* [Hint: $\hat{H}_0 = -\frac{J}{2}(\hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2)^2 + \frac{J}{2}\hat{\boldsymbol{S}}_1^2 + \frac{J}{2}\hat{\boldsymbol{S}}_2^2$.]

(2) (6pts) Define vector spin chirality $\hat{\boldsymbol{\chi}} = \hat{\boldsymbol{S}}_1 \times \hat{\boldsymbol{S}}_2$ (namely $\hat{\chi}_x = \hat{S}_{1,y}\hat{S}_{2,z} - \hat{S}_{1,z}\hat{S}_{2,y}$, $\ldots$). Define total spin operator (spin rotation generator) $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$. *Check that $\hat{\boldsymbol{\chi}}$ transforms like a vector under spin rotation, namely $[\hat{S}_a, \hat{\chi}_b] = \mathbb{i}\epsilon_{abc}\hat{\chi}_c$. Evaluate the matrix elements of $\hat{\chi}_a$ ($a = x, y, z$) between the degenerate ground states of $\hat{H}_0$ solved in (1).* [Hint: certain symmetry may help, and you can use the “projection theorem”]

(3) (5pts) Add a small staggered magnetic field term to the Hamiltonian as perturbation, $\hat{H} = \hat{H}_0 - B \cdot (\hat{S}_{1,z} - \hat{S}_{2,z})$. Treat the real constant B as a small parameter. *Solve the second order perturbation results for the energies of the original ground states of $\hat{H}_0$.* [NOTE: the unperturbed ground states of $\hat{H}_0$ are degenerate, but degenerate perturbation theory can be avoided by dividing the Hilbert spaces by symmetry (conserved quantity)]