Homework #2:
Brief solutions

***** (about lecture #1) *****

1. (5pts) The definition of unitary operator is that a linear operator $\hat{U}$ is unitary if the inner product $(\hat{U}\phi, \hat{U}\psi) = (\phi, \psi)$ for any states ϕ and ψ. *Prove that this condition is equivalent to:* $(\hat{U}\psi, \hat{U}\psi) = (\psi, \psi)$ *for any state* ψ. [Hint: the former condition obviously imply the latter one, try to derive the former condition from the latter one, by assuming an arbitrary linear combination of states]

Solutions:

Consider the inner product $(c_1\phi+c_2\psi, c_1\phi+c_2\psi) = (c_1^*, c_2^*)\begin{pmatrix}\langle\phi|\phi\rangle, & \langle\phi|\psi\rangle \\ \langle\psi|\phi\rangle, & \langle\psi|\psi\rangle\end{pmatrix}\begin{pmatrix}c_1 \\ c_2\end{pmatrix}$, where $c_{1,2}$ are two complex numbers, ϕ, ψ are two quantum states.

According to the latter condition, this equals to $(\hat{U}(c_1\phi + c_2\psi), \hat{U}(c_1\phi + c_2\psi))$
$= (c_1^*, c_2^*)\begin{pmatrix}\langle\hat{U}\phi|\hat{U}\phi\rangle, & \langle\hat{U}\phi|\hat{U}\psi\rangle \\ \langle\hat{U}\psi|\hat{U}\phi\rangle, & \langle\hat{U}\psi|\hat{U}\psi\rangle\end{pmatrix}\begin{pmatrix}c_1 \\ c_2\end{pmatrix} = (c_1^*, c_2^*)\begin{pmatrix}\langle\phi|\phi\rangle, & \langle\hat{U}\phi|\hat{U}\psi\rangle \\ \langle\hat{U}\psi|\hat{U}\phi\rangle, & \langle\psi|\psi\rangle\end{pmatrix}\begin{pmatrix}c_1 \\ c_2\end{pmatrix}$.

Then we have, $c_1^*c_2\langle\phi|\psi\rangle + c_2^*c_1\langle\psi|\phi\rangle = c_1^*c_2\langle\hat{U}\phi|\hat{U}\psi\rangle + c_2^*c_1\langle\hat{U}\psi|\hat{U}\phi\rangle$.
Choose $c_1 = c_2 = 1$, this condition is, $2 \cdot \mathrm{Re}(\langle\phi|\psi\rangle) = 2 \cdot \mathrm{Re}(\langle\hat{U}\phi|\hat{U}\psi\rangle)$;
choose $c_1 = \mathbb{i}, c_2 = 1$, this is, $2 \cdot \mathrm{Im}(\langle\phi|\psi\rangle) = 2 \cdot \mathrm{Im}(\langle\hat{U}\phi|\hat{U}\psi\rangle)$.
Here Re and Im denote the real and imaginary parts of a complex number, respectively.

Therefore $\langle\phi|\psi\rangle = \langle\hat{U}\phi|\hat{U}\psi\rangle$.

2. $\mathcal{H}_1$ and $\hat{H}_2$ are both 2-dimensional Hilbert spaces. $\mathcal{H}_1$ has complete orthonormal basis $|e_1\rangle$ and $|e_2\rangle$, $\mathcal{H}_2$ has complete orthonormal basis $|e_1'\rangle$ and $|e_2'\rangle$. In the following we will represent operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ as matrices under these basis. Define three nontrivial hermitian operators $\hat{\sigma}_1 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$, $\hat{\sigma}_2 = \begin{pmatrix}0 & -\mathbb{i} \\ \mathbb{i} & 0\end{pmatrix}$, $\hat{\sigma}_3 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$, in $\mathcal{H}_1$; and $\hat{\sigma}_1' = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$, $\hat{\sigma}_2' = \begin{pmatrix}0 & -\mathbb{i} \\ \mathbb{i} & 0\end{pmatrix}$, $\hat{\sigma}_3' = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$, in $\mathcal{H}_2$.

(a) (5pts) Consider a state in the 4-dimensional Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ described by the density matrix $\hat{\rho} = \frac{1}{4}\mathbb{1}_{4\times4} + \frac{1}{8}\hat{\sigma}_3 \otimes \hat{\sigma}_3' + \frac{1}{8}\hat{\sigma}_1 \otimes \hat{\sigma}_1'$, where $\mathbb{1}_{4\times4}$ is the 4×4 identity matrix(identity operator in $\mathcal{H}_1 \otimes \mathcal{H}_2$). *Compute the eigenvalues and orthonormal eigenstates* *of* ρ. [Hint: facts about Pauli matrices in Homework#1 might help]

(b) (5pts) *Check that* $\hat{\rho}$ *defined in (a) is a legitimate density matrix, namely that it is hermitian, positive semi-definite, and has unity trace. Check that whether* $\hat{\rho}$ *represents a pure state or not. Compute the von Neumann entropy* $S[\hat{\rho}] \equiv -\mathrm{Tr}[\hat{\rho} \log \hat{\rho}]$. [Hint: result of (a) is of course useful]

(c) (5pts) Consider an observable $\hat{O} = \hat{\sigma}_2 \otimes \hat{\sigma}'_2$ defined on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Measure $\hat{O}$ under the state $\hat{\rho}$ defined in (a). *What are the possible measurement results, and their corresponding probabilities? For each possible measurement result, what is the density matrix for the collapsed state after measurement?* [Hint: check that $[\hat{\rho}, \hat{O}] = 0$, this fact might help]

(d) (5pts) Consider an observable $\hat{Q} = \hat{\sigma}_2 \otimes \mathbb{1}_{2\times 2}$ defined on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Here $\mathbb{1}_{2\times 2}$ is the 2×2 identity matrix. Measure $\hat{Q}$ under the state $\hat{\rho}$ defined in (a). *What are the possible measurement results, and their corresponding probabilities? For each possible measurement result, what is the density matrix for the collapsed state after measurement?*

Solutions:

(a) Under the basis $(|e_1 \otimes e'_1\rangle, |e_1 \otimes e'_2\rangle, |e_2 \otimes e'_1\rangle, |e_2 \otimes e'_2\rangle$, $\hat{\rho}$ is $\begin{pmatrix} \frac{3}{8} & 0 & 0 & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & 0 & 0 & \frac{3}{8} \end{pmatrix}$.

This is block-diagonalized, similar to Homework #1 Problem 7(a). Rearrange the basis into $(|e_1 \otimes e'_1\rangle, |e_2 \otimes e'_2\rangle, |e_1 \otimes e'_2\rangle, |e_2 \otimes e'_1\rangle)$, this matrix becomes $\begin{pmatrix} \frac{3}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{8} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$.

The top-left 2×2 block has eigenvalue $\frac{3}{8} + \frac{1}{8} = \frac{1}{2}$ for eigenvector $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and eigenvalue $\frac{3}{8} - \frac{1}{8} = \frac{1}{4}$ for eigenvector $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The bottom-right 2×2 block has eigenvalue $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$ for eigenvector $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and eigenvalue $\frac{1}{8} - \frac{1}{8} = 0$ for eigenvector $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The final results are

eigenvalue of $\hat{\rho}$	eigenstate of $\hat{\rho}$
$\frac{1}{2}$	$\frac{1}{\sqrt{2}}(\lvert e_1 \otimes e_1'\rangle + \lvert e_2 \otimes e_2'\rangle)$
$\frac{1}{4}$	$\frac{1}{\sqrt{2}}(\lvert e_1 \otimes e_1'\rangle - \lvert e_2 \otimes e_2'\rangle)$
$\frac{1}{4}$	$\frac{1}{\sqrt{2}}(\lvert e_1 \otimes e_2'\rangle + \lvert e_2 \otimes e_1'\rangle)$
0	$\frac{1}{\sqrt{2}}(\lvert e_1 \otimes e_2'\rangle - \lvert e_2 \otimes e_1'\rangle)$

Method #2 for eigenvalues: not rigorous.

The three terms in $\hat{\rho}$ mutually commute, in particular $(\hat{\sigma}_3 \otimes \hat{\sigma}_3') \cdot (\hat{\sigma}_1 \otimes \hat{\sigma}_1') = -\hat{\sigma}_2 \otimes \hat{\sigma}_2' = (\hat{\sigma}_1 \otimes \hat{\sigma}_1') \cdot (\hat{\sigma}_3 \otimes \hat{\sigma}_3')$. So they can have simultaneous eigenstates.

$(\hat{\sigma}_3 \otimes \hat{\sigma}_3')$ has eigenvalues ± 1, and $(\hat{\sigma}_1 \otimes \hat{\sigma}_1')$ has eigenvalue ± 1. The 4 possible combinations of eigenvalues are $\frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$, $\frac{1}{4} + \frac{1}{8} - \frac{1}{8} = \frac{1}{4}$, $\frac{1}{4} - \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, $\frac{1}{4} - \frac{1}{8} - \frac{1}{8} = 0$.

But you still have to prove that each of these 4 combinations appear once.

(b) $\hat{\rho}$ is obviously hermitian, has non-negative eigenvalues (positive semi-definite), and unity trace (sum of eigenvalues equal unity).

$S[\hat{\rho}] = -\sum \lambda \log(\lambda) = -\frac{1}{2}\log(\frac{1}{2}) - \frac{1}{4}\log(\frac{1}{4}) - \frac{1}{4}\log(\frac{1}{4}) - 0\log(0) = \frac{3}{2}\log(2)$.

$\hat{\rho}$ is not a pure state (non-zero entropy, more-than-one nonzero eigenvalues).

(c) Under the basis $(\lvert e_1 \otimes e_1'\rangle, \lvert e_1 \otimes e_2'\rangle, \lvert e_2 \otimes e_1'\rangle, \lvert e_2 \otimes e_2'\rangle)$, $\hat{O}$ is $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$. It is easy to check that $[\hat{\rho}, \hat{O}] = 0$. In fact the eigenstates in (a) are also eigenstates of $\hat{O}$,

eigenvalue of $\hat{\rho}$ (probability)	eigenstate of $\hat{\rho}$	eigenvalue of $\hat{O}$
$\frac{1}{2}$	$\lvert\psi_1\rangle \equiv \frac{1}{\sqrt{2}}(\lvert e_1 \otimes e_1'\rangle + \lvert e_2 \otimes e_2'\rangle)$	-1
$\frac{1}{4}$	$\lvert\psi_2\rangle \equiv \frac{1}{\sqrt{2}}(\lvert e_1 \otimes e_1'\rangle - \lvert e_2 \otimes e_2'\rangle)$	$+1$
$\frac{1}{4}$	$\lvert\psi_3\rangle \equiv \frac{1}{\sqrt{2}}(\lvert e_1 \otimes e_2'\rangle + \lvert e_2 \otimes e_1'\rangle)$	$+1$
0	$\lvert\psi_4\rangle \equiv \frac{1}{\sqrt{2}}(\lvert e_1 \otimes e_2'\rangle - \lvert e_2 \otimes e_1'\rangle)$	-1

Therefore the measurement result can be

-1 with probability $\frac{1}{2}+0=\frac{1}{2}$; and

$+1$ with probability $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

If the result is -1, the collapsed state density matrix is $\frac{1}{\frac{1}{2}+0}(\frac{1}{2}|\psi_1\rangle\langle\psi_1| + 0|\psi_4\rangle\langle\psi_4|)$, $\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$ under the above basis, which becomes a pure state after measurement!

If the result is $+1$, the collapsed state density matrix is $\frac{1}{\frac{1}{4}+\frac{1}{4}}(\frac{1}{4}|\psi_2\rangle\langle\psi_2| + \frac{1}{4}|\psi_3\rangle\langle\psi_3|)$, $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ under the above basis, which is still a mixed state.

(d) Under the basis $(|e_1 \otimes e'_1\rangle, |e_1 \otimes e'_2\rangle, |e_2 \otimes e'_1\rangle, |e_2 \otimes e'_2\rangle)$, $\hat{Q}$ is $\begin{pmatrix} 0 & 0 & -\mathbb{i} & 0 \\ 0 & 0 & 0 & -\mathbb{i} \\ \mathbb{i} & 0 & 0 & 0 \\ 0 & \mathbb{i} & 0 & 0 \end{pmatrix}$.

It has eigenvalues ± 1 (each is 2-fold degenerate). For eigenvalue $+1$, the two eigenstates can be chosen as $\frac{1}{\sqrt{2}}(|e_1\rangle + \mathbb{i}|e_2\rangle) \otimes |e'_{1,2}\rangle$. For eigenvalue -1, the two eigenstates can be chose as $\frac{1}{\sqrt{2}}(|e_1\rangle - \mathbb{i}|e_2\rangle) \otimes |e'_{1,2}\rangle$.

The projection operator onto the eigenvalue $+1$ subspace is

$\hat{P}_{\hat{Q}=+1} = \frac{1}{2}(|e_1\rangle + \mathbb{i}|e_2\rangle)(\langle e_1| - \mathbb{i}\langle e_2|) \otimes \mathbb{1}_{2\times 2}$, or $\frac{1}{2}\begin{pmatrix} 1 & 0 & -\mathbb{i} & 0 \\ 0 & 1 & 0 & -\mathbb{i} \\ \mathbb{i} & 0 & 1 & 0 \\ 0 & \mathbb{i} & 0 & 1 \end{pmatrix}$ under the above basis.

The projection operator onto the eigenvalue -1 subspace is

$\hat{P}_{\hat{Q}=-1} = \frac{1}{2}(|e_1\rangle - \mathbb{i}|e_2\rangle)(\langle e_1| + \mathbb{i}\langle e_2|) \otimes \mathbb{1}_{2\times 2}$, or $\frac{1}{2}\begin{pmatrix} 1 & 0 & \mathbb{i} & 0 \\ 0 & 1 & 0 & \mathbb{i} \\ -\mathbb{i} & 0 & 1 & 0 \\ 0 & -\mathbb{i} & 0 & 1 \end{pmatrix}$ under the above basis.

The probability for getting measurement result $+1$ is $\mathrm{Tr}(\hat{\rho}\hat{P}_{\hat{Q}=+1}) = \frac{1}{2}$, the collapsed state is $\hat{P}_{\hat{Q}=+1}\hat{\rho}\hat{P}_{\hat{Q}=+1}/\text{Tr}(\hat{\rho}\hat{P}_{\hat{Q}=+1}) = \frac{1}{4}\begin{pmatrix} 1 & 0 & -\mathbb{i} & 0 \\ 0 & 1 & 0 & -\mathbb{i} \\ \mathbb{i} & 0 & 1 & 0 \\ 0 & \mathbb{i} & 0 & 1 \end{pmatrix}$, which is a mixed state.

The probability for getting measurement result -1 is $\text{Tr}(\hat{\rho}\hat{P}_{\hat{Q}=-1}) = \frac{1}{2}$, the collapsed state is $\hat{P}_{\hat{Q}=-1}\hat{\rho}\hat{P}_{\hat{Q}=-1}/\text{Tr}(\hat{\rho}\hat{P}_{\hat{Q}=-1}) = \frac{1}{4}\begin{pmatrix} 1 & 0 & \mathbb{i} & 0 \\ 0 & 1 & 0 & \mathbb{i} \\ -\mathbb{i} & 0 & 1 & 0 \\ 0 & -\mathbb{i} & 0 & 1 \end{pmatrix}$, which is a mixed state.

Method #2: brief descriptions only.

Once we know that $\hat{Q}$ has only eigenvalues ± 1, we can directly write down the projection operators, $\hat{P}_{\hat{Q}=+1} = \frac{\mathbb{1}_{4\times 4}+\hat{Q}}{2}$, $\hat{P}_{\hat{Q}=-1} = \frac{\mathbb{1}_{4\times 4}-\hat{Q}}{2}$.

We can then use $(\hat{A}\otimes\hat{B})\cdot(\hat{C}\otimes\hat{D}) = (\hat{A}\hat{C})\otimes(\hat{B}\hat{D})$, and $\text{Tr}_{\mathcal{H}_1\otimes\mathcal{H}_2}(\hat{A}\otimes\hat{B}) = \text{Tr}_{\mathcal{H}_1}(\hat{A})\cdot\text{Tr}_{\mathcal{H}_2}(\hat{B})$, and the multiplication rules of Pauli matrices, to do the remaining calculations.

***** (about lecture #2) *****

3. Consider a single-boson Hilbert space with two complete orthonormal basis states, $|1\rangle$ & $|2\rangle$. Denote the corresponding creation,annihilation operators by $\hat{b}_1^\dagger$,$\hat{b}_1$ (for $|1\rangle$) and $\hat{b}_2^\dagger$,$\hat{b}_2$ (for $|2\rangle$), then $|1\rangle = \hat{b}_1^\dagger|\text{vac}\rangle$, $|2\rangle = \hat{b}_2^\dagger|\text{vac}\rangle$, where $|\text{vac}\rangle$ is the normalized 'vacuum' state, and $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{i,j}$, $[\hat{b}_i, \hat{b}_j] = [\hat{b}_i^\dagger, \hat{b}_j^\dagger] = 0$.

(a). (3pts) *Write down a complete orthonormal basis for the Hilbert space of two bosons*, in terms of tensor product states $|i\rangle\otimes|j\rangle$, $i, j = 1, 2$.

(b). (2pts) A unitary transformation $\hat{U}$ is defined by its action on single-boson basis as: $|1\rangle \mapsto \hat{U}|1\rangle = (u|1\rangle - v|2\rangle)$, $|2\rangle \mapsto \hat{U}|2\rangle = (v^*|1\rangle + u^*|2\rangle)$, where u, v are two complex numbers and $|u|^2 + |v|^2 = 1$. *Show that the above definition of $\hat{U}$ is indeed a unitary transformation in single-boson Hilbert space.*

(c). (5pts) The action of $\hat{U}$ on a tensor product state will be transforming each of the factors, for example $|1\rangle \otimes |2\rangle \mapsto \hat{U}|1\rangle \otimes \hat{U}|2\rangle$. *Write down the transformation results of all two-boson basis in (a) induced by $\hat{U}$, as linear combinations of the original two-boson basis states. Explicitly show that this transformation in the two-boson Hilbert space is unitary.*

(d). (5pts) $\hat{U}$ can be extended to the entire Fock space as follows: The transformation of an operator $\hat{O}$ by $\hat{U}$ is formally $\hat{U}\hat{O}\hat{U}^\dagger$. We demand that the transformation results of $\hat{b}_i^\dagger$ are: $\hat{U}\hat{b}_1^\dagger\hat{U}^\dagger = (u\hat{b}_1^\dagger - v\hat{b}_2^\dagger)$, and $\hat{U}\hat{b}_2^\dagger\hat{U}^\dagger = (v^*\hat{b}_1^\dagger + u^*\hat{b}_2^\dagger)$. Together with $\hat{U}|\text{vac}\rangle = |\text{vac}\rangle$, this can reproduce the definition of $\hat{U}$ in single-boson space, *e.g.* $\hat{U}|1\rangle = \hat{U}\hat{b}_1^\dagger|\text{vac}\rangle = \hat{U}\hat{b}_1^\dagger\hat{U}^\dagger \cdot \hat{U}|\text{vac}\rangle = (u\hat{b}_1^\dagger - v\hat{b}_2^\dagger)|\text{vac}\rangle = u|1\rangle - v|2\rangle$. *Use the creation operators to represent the two-boson basis in (a), then apply $\hat{U}$ on them, represent the results as linear combinations of the original two-boson basis.* The results should be consistent with (c).

(e). (5pts) Consider $\hat{H} = t \cdot (\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1)$, where t is a real number. You can do a unitary transformation(basis change) to “diagonalize” $\hat{H}$: find a new set of orthonormal creation (annihilation) operators $\hat{b}_i'^\dagger\,(\hat{b}_i')$ as linear combinations of $\hat{b}_j^\dagger\,(\hat{b}_j)$, so that $\hat{H} = \epsilon_1\hat{b}_1'^\dagger\hat{b}_1' + \epsilon_2\hat{b}_2'^\dagger\hat{b}_2'$, where $\epsilon_{1,2}$ are two c-numbers. These new operators should satisfy the same kind of commutation relations as the old ones, *e.g.* $[\hat{b}_i', \hat{b}_j'^\dagger] = \delta_{i,j}$. *Solve the new creation operators $\hat{b}_i'^\dagger$ in terms of $\hat{b}_j^\dagger$, and solve $\epsilon_{1,2}$. Then write down all the eigenvalues and eigenstates of $\hat{H}$ in the entire Fock space.*

(f). (5pts) (DIFFICULT) The explicit form of operator $\hat{U}$ in (d) in the entire Fock space is, $\hat{U} = \exp\left[\mathbb{i}\sum_{i,j=1}^{2}\hat{b}_i^\dagger\,(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)_{i,j}\,\hat{b}_j\right]$. Here $a_{1,2,3}$ are three real numbers, $\sigma_{1,2,3}$ are Pauli matrices defined in Homework #1 Problem 6. $(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)_{i,j}$ is the i^{th}-row-j^{th}-column element of the 2×2 matrix in the bracket. *Solve the real numbers $a_{1,2,3}$ in terms of the complex numbers u, v used to define $\hat{U}$ in (b).* [Hint: compute $\hat{U}\hat{b}_{1,2}^\dagger\hat{U}^\dagger$ by the Baker-Hausdorff formula, compare the results with those in (d), some results in Homework #1 will be useful]

Solutions:

(a) by definition, the orthonormal occupation basis are,

$|n_1 = 2, n_2 = 0\rangle \equiv \frac{1}{\sqrt{2}}|1,1\rangle \equiv \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(|1\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle) = |1\rangle \otimes |1\rangle$,

$|n_1 = 1, n_2 = 1\rangle \equiv |1,2\rangle \equiv \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle)$,

$|n_1 = 0, n_2 = 2\rangle \equiv \frac{1}{\sqrt{2}}|2,2\rangle \equiv \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(|2\rangle \otimes |2\rangle + |2\rangle \otimes |2\rangle) = |2\rangle \otimes |2\rangle$.

(b) one can check that $(\hat{U}|i\rangle, \hat{U}|j\rangle) = (|i\rangle, |j\rangle) = \delta_{i,j}$, for $i, j = 1, 2$.

$(\hat{U}|1\rangle, \hat{U}|1\rangle) = (u|1\rangle - v|2\rangle, u|1\rangle - v|2\rangle) = u^* u + (-v)^*(-v) = 1,$

$(\hat{U}|1\rangle, \hat{U}|2\rangle) = (u|1\rangle - v|2\rangle, v^*|1\rangle + u^*|2\rangle) = u^* v^* + (-v)^* u^* = 0,$

$(\hat{U}|2\rangle, \hat{U}|1\rangle) = (\hat{U}|1\rangle, \hat{U}|2\rangle)^* = 0,$

$(\hat{U}|2\rangle, \hat{U}|2\rangle) = (v^*|1\rangle + u^*|2\rangle, v^*|1\rangle + u^*|2\rangle) = (v^*)^*(v^*) + (u^*)^* u^* = 1.$

The action of $\hat{U}$ on the 1-particle basis is, $(|1\rangle, |2\rangle) \mapsto (|1\rangle, |2\rangle) \begin{pmatrix} u & v^* \\ -v & u^* \end{pmatrix}$, it is also easy to check that the 2×2 matrix is unitary, under the condition that $|u|^2 + |v|^2 = 1$, $\begin{pmatrix} u & v^* \\ -v & u^* \end{pmatrix}^\dagger \cdot \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix} = \begin{pmatrix} |u|^2 + |v|^2 & 0 \\ 0 & |u|^2 + |v|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) consider the basis in (a),

$|1\rangle \otimes |1\rangle \mapsto (u|1\rangle - v|2\rangle) \otimes (u|1\rangle - v|2\rangle)$
$= u^2 \cdot |1\rangle \otimes |1\rangle - \sqrt{2}uv \cdot \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle) + v^2 \cdot |2\rangle \otimes |2\rangle,$

$\frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle) \mapsto \frac{1}{\sqrt{2}}((u|1\rangle - v|2\rangle) \otimes (v^*|1\rangle + u^*|2\rangle) + (v^*|1\rangle + u^*|2\rangle) \otimes (u|1\rangle - v|2\rangle))$
$= \sqrt{2}uv^* \cdot |1\rangle \otimes |1\rangle + (u^* u - v^* v) \cdot \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle) - \sqrt{2}vu^* \cdot |2\rangle \otimes |2\rangle,$

$|2\rangle \otimes |2\rangle \mapsto (v^*|1\rangle + u^*|2\rangle) \otimes (v^*|1\rangle + u^*|2\rangle)$
$= (v^*)^2 \cdot |1\rangle \otimes |1\rangle + \sqrt{2}v^* u^* \cdot \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle) + (u^*)^2 \cdot |2\rangle \otimes |2\rangle.$

This can be written as, $(|n_1 = 2, n_2 = 0\rangle, |n_1 = 1, n_2 = 1\rangle |n_1 = 0, n_2 = 2\rangle)$
$\mapsto (|n_1 = 2, n_2 = 0\rangle, |n_1 = 1, n_2 = 1\rangle |n_1 = 0, n_2 = 2\rangle) \cdot \begin{pmatrix} u^2 & \sqrt{2}uv^* & (v^*)^2 \\ -\sqrt{2}uv & (u^* u - v^* v) & \sqrt{2}v^* u^* \\ v^2 & -\sqrt{2}vu^* & (u^*)^2 \end{pmatrix}$.

One can check that the 3×3 matrix $(\dots)$ is unitary by brute-force computation, $(\dots)^\dagger \cdot (\dots) = (|u|^2 + |v|^2)^2 \cdot \mathbb{1}_{3\times3} = \mathbb{1}_{3\times3}$.

(d) represent the basis in (a) by creation operators,

$|n_1 = 2, n_2 = 0\rangle = \frac{1}{\sqrt{2}}(\hat{b}_1^\dagger)^2 |\text{vac}\rangle,$

$|n_1 = 1, n_2 = 1\rangle = \hat{b}_1^\dagger \hat{b}_2^\dagger |\text{vac}\rangle,$

$|n_1 = 0, n_2 = 2\rangle = \frac{1}{\sqrt{2}}(\hat{b}_2^\dagger)^2 |\text{vac}\rangle.$

Then (note that $\hat{b}_1^\dagger \hat{b}_2^\dagger = \hat{b}_2^\dagger \hat{b}_1^\dagger$),

$\hat{U}|n_1=2,n_2=0\rangle = \frac{1}{\sqrt{2}}(\hat{U}\hat{b}_1^\dagger\hat{U}^\dagger)^2\hat{U}|\text{vac}\rangle = \frac{1}{\sqrt{2}}(u\hat{b}_1^\dagger - v\hat{b}_2^\dagger)^2|\text{vac}\rangle$
$= [u^2\cdot\frac{1}{\sqrt{2}}(\hat{b}_1^\dagger)^2 - \sqrt{2}uv\cdot\hat{b}_1^\dagger\hat{b}_2^\dagger + v^2\cdot\frac{1}{\sqrt{2}}(\hat{b}_2^\dagger)^2]|\text{vac}\rangle$,

$\hat{U}|n_1=1,n_2=1\rangle = (\hat{U}\hat{b}_1^\dagger\hat{U}^\dagger)(\hat{U}\hat{b}_2^\dagger\hat{U}^\dagger)\hat{U}|\text{vac}\rangle = (u\hat{b}_1^\dagger - v\hat{b}_2^\dagger)(v^*\hat{b}_1^\dagger + u^*\hat{b}_2^\dagger)|\text{vac}\rangle$
$= [\sqrt{2}uv^*\cdot\frac{1}{\sqrt{2}}(\hat{b}_1^\dagger)^2 + (uu^* - vv^*)\cdot\hat{b}_1^\dagger\hat{b}_2^\dagger - \sqrt{2}vu^*\cdot\frac{1}{\sqrt{2}}(\hat{b}_2^\dagger)^2]|\text{vac}\rangle$,

$\hat{U}|n_1=0,n_2=2\rangle = \frac{1}{\sqrt{2}}(\hat{U}\hat{b}_2^\dagger\hat{U}^\dagger)^2\hat{U}|\text{vac}\rangle = \frac{1}{\sqrt{2}}(v^*\hat{b}_1^\dagger + u^*\hat{b}_2^\dagger)^2|\text{vac}\rangle$
$= [(v^*)^2\cdot\frac{1}{\sqrt{2}}(\hat{b}_1^\dagger)^2 + \sqrt{2}v^*u^*\cdot\hat{b}_1^\dagger\hat{b}_2^\dagger + (u^*)^2\cdot\frac{1}{\sqrt{2}}(\hat{b}_2^\dagger)^2]|\text{vac}\rangle$.

This is consistent with the result of (c).

(e) $\hat{H} = (\hat{b}_1^\dagger, \hat{b}_2^\dagger)\begin{pmatrix}0 & t\\ t & 0\end{pmatrix}\begin{pmatrix}\hat{b}_1\\ \hat{b}_2\end{pmatrix}$.

The 2×2 matrix in the middle is $t\sigma_1$, with eigenvalues t and $-t$ [c.f. Homework #1 Problem 6.(a)]. $\begin{pmatrix}0 & t\\ t & 0\end{pmatrix} = U\cdot\begin{pmatrix}t & 0\\ 0 & -t\end{pmatrix}\cdot U^\dagger$, where $U = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}$ is a unitary matrix (choice of U is not unique).

Define $(\hat{b}_1'^\dagger, \hat{b}_2'^\dagger) = (\hat{b}_1^\dagger, \hat{b}_2^\dagger)\cdot U$, namely, $\hat{b}_1'^\dagger = \frac{1}{\sqrt{2}}\hat{b}_1^\dagger + \frac{1}{\sqrt{2}}\hat{b}_2^\dagger$, $\hat{b}_2'^\dagger = \frac{1}{\sqrt{2}}\hat{b}_1^\dagger - \frac{1}{\sqrt{2}}\hat{b}_2^\dagger$, then $[\hat{b}_i', \hat{b}_j'^\dagger] = \delta_{i,j}$, and $\hat{H} = t\hat{b}_1'^\dagger\hat{b}_1' - t\hat{b}_2'^\dagger\hat{b}_2' = t\hat{n}_1' - t\hat{n}_2'$. Here $\hat{n}_i' \equiv \hat{b}_i'^\dagger\hat{b}_i'$ is the boson occupation number operator under the new basis. This basis change does not change the boson vacuum, $\hat{b}_i'|\text{vac}\rangle = 0$.

$\epsilon_1 = t$, $\epsilon_2 = -t$, $\hat{b}_1'^\dagger = \frac{1}{\sqrt{2}}\hat{b}_1^\dagger + \frac{1}{\sqrt{2}}\hat{b}_2^\dagger$, $\hat{b}_2'^\dagger = \frac{1}{\sqrt{2}}\hat{b}_1^\dagger - \frac{1}{\sqrt{2}}\hat{b}_2^\dagger$. [You may swap $\epsilon_{1,2}$, then $\hat{b}_{1,2}'^\dagger$ should also be swapped. $\hat{b}_i'^\dagger$ can be multiplied by independent complex phases.]

The occupation basis states $|n_1', n_2'\rangle \equiv \frac{1}{\sqrt{(n_1')!(n_2')!}}(\hat{b}_1'^\dagger)^{n_1'}(\hat{b}_2'^\dagger)^{n_2'}|\text{vac}\rangle$ are normalized eigenstates of $\hat{H}$, with eigenvalue $\epsilon_1 n_1' + \epsilon_2 n_2'$.

(f) For notation simplicity, define 2×2 matrix $A = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$, and define $\hat{A} = \sum_{i,j}\hat{b}_i^\dagger A_{i,j}\hat{b}_j$, Then $\hat{U} = \exp(\mathbb{i}\hat{A})$. Consider $\hat{U}\cdot\hat{b}_k^\dagger\cdot\hat{U}^\dagger = \exp(\mathbb{i}\hat{A})\cdot\hat{b}_k^\dagger\cdot\exp(-\mathbb{i}\hat{A})$. By the Baker-Hausdorff formula, this is, $\hat{b}_k^\dagger + \sum_{n=1}^\infty\frac{1}{n!}\underbrace{[\mathbb{i}\hat{A},[\mathbb{i}\hat{A},\ldots[\mathbb{i}\hat{A},\hat{b}_k^\dagger]}_{n\text{-fold commutator}}\ldots]]$.

Use the identity $[\hat{b}_i^\dagger\hat{b}_j, \hat{b}_k^\dagger] = \delta_{j,k}\hat{b}_i^\dagger$, we have $[\mathbb{i}\hat{A}, \hat{b}_k^\dagger] = \mathbb{i}\sum_i\hat{b}_i^\dagger A_{i,k}$.

By mathematical induction (steps omitted), $\underbrace{[\mathbb{i}\hat{A},[\mathbb{i}\hat{A},\ldots[\mathbb{i}\hat{A},\hat{b}_k^\dagger]}_{n\text{-fold commutator}}\ldots]] = \mathbb{i}^n\sum_i\hat{b}_i^\dagger(A^n)_{i,k}$. Here A^n is the n-th power of the matrix A, which is also a 2×2 matrix.

Finally, $\hat{U}\cdot\hat{b}_k^\dagger\cdot\hat{U}^\dagger = \hat{b}_k^\dagger + \sum_{n=1}^\infty\frac{1}{n!}\mathbb{i}^n\sum_i\hat{b}_i^\dagger(A^n)_{i,k} = \sum_i\hat{b}_i^\dagger\,[\exp(\mathbb{i}A)]_{i,k}$.

Comparing to (d), we should have $\exp(\mathbb{i}A) = \begin{pmatrix} u & v^* \\ -v & u^* \end{pmatrix}$.

Use the result of Homework #1 Problem 6(b),
$\exp(\mathbb{i}A) = \cos(\sqrt{a_1^2 + a_2^2 + a_3^2})\mathbb{1}_{2\times 2} + \mathbb{i}\frac{\sin(\sqrt{a_1^2+a_2^2+a_3^2})}{\sqrt{a_1^2+a_2^2+a_3^2}} \cdot (a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)$.

Then $u = \cos(\sqrt{a_1^2 + a_2^2 + a_3^2}) + \mathbb{i}\frac{\sin(\sqrt{a_1^2+a_2^2+a_3^2})}{\sqrt{a_1^2+a_2^2+a_3^2}} \cdot a_3$,
$v = \frac{\sin(\sqrt{a_1^2+a_2^2+a_3^2})}{\sqrt{a_1^2+a_2^2+a_3^2}} \cdot a_2 - \mathbb{i}\frac{\sin(\sqrt{a_1^2+a_2^2+a_3^2})}{\sqrt{a_1^2+a_2^2+a_3^2}} \cdot a_1$.

Conversely,
$a_1 = -\mathrm{Im}(v) \cdot \frac{\arccos[\mathrm{Re}(u)]}{\sin(\arccos[\mathrm{Re}(u)])}$, $a_2 = \mathrm{Re}(v) \cdot \frac{\arccos[\mathrm{Re}(u)]}{\sin(\arccos[\mathrm{Re}(u)])}$, $a_3 = \mathrm{Im}(u) \cdot \frac{\arccos[\mathrm{Re}(u)]}{\sin(\arccos[\mathrm{Re}(u)])}$. [Note: solution of $a_{1,2,3}$ is not unique, 'arccos' function is not single-valued.]