Homework #2:
Due: tentatively Oct. 8, 2019

***** (about lecture #1) *****

1. (5pts) The definition of unitary operator is that a linear operator $\hat{U}$ is unitary if the inner product $(\hat{U}\phi, \hat{U}\psi) = (\phi, \psi)$ for any states ϕ and ψ. *Prove that this condition is equivalent to: $(\hat{U}\psi, \hat{U}\psi) = (\psi, \psi)$ for any state ψ.* [Hint: the former condition obviously imply the latter one, try to derive the former condition from the latter one, by assuming an arbitrary linear combination of states]

2. $\mathcal{H}_1$ and $\hat{H}_2$ are both 2-dimensional Hilbert spaces. $\mathcal{H}_1$ has complete orthonormal basis $|e_1\rangle$ and $|e_2\rangle$, $\mathcal{H}_2$ has complete orthonormal basis $|e_1'\rangle$ and $|e_2'\rangle$. In the following we will represent operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ as matrices under these basis. Define three nontrivial hermitian operators $\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_2 = \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}$, $\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, in $\mathcal{H}_1$; and $\hat{\sigma}_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_2' = \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}$, $\hat{\sigma}_3' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, in $\mathcal{H}_2$.

(a) (5pts) Consider a state in the 4-dimensional Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ described by the density matrix $\hat{\rho} = \frac{1}{4}\mathbb{1}_{4\times4} + \frac{1}{8}\hat{\sigma}_3 \otimes \hat{\sigma}_3' + \frac{1}{8}\hat{\sigma}_1 \otimes \hat{\sigma}_1'$, where $\mathbb{1}_{4\times4}$ is the 4×4 identity matrix(identity operator in $\mathcal{H}_1 \otimes \mathcal{H}_2$). *Compute the eigenvalues and orthonormal eigenstates of ρ.* [Hint: facts about Pauli matrices in Homework#1 might help]

(b) (5pts) *Check that $\hat{\rho}$ defined in (a) is a legitimate density matrix, namely that it is hermitian, positive semi-definite, and has unity trace. Check that whether $\hat{\rho}$ represents a pure state or not. Compute the von Neumann entropy $S[\hat{\rho}] \equiv -\mathrm{Tr}[\hat{\rho}\log\hat{\rho}]$.* [Hint: result of (a) is of course useful]

(c) (5pts) Consider an observable $\hat{O} = \hat{\sigma}_2 \otimes \hat{\sigma}_2'$ defined on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Measure $\hat{O}$ under the state $\hat{\rho}$ defined in (a). *What are the possible measurement results, and their corresponding probabilities? For each possible measurement result, what is the density matrix for the collapsed state after measurement?* [Hint: check that $[\hat{\rho}, \hat{O}] = 0$, this fact might help]

(d) (5pts) Consider an observable $\hat{Q} = \hat{\sigma}_2 \otimes \mathbb{1}_{2\times 2}$ defined on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Here $\mathbb{1}_{2\times 2}$ is the 2×2 identity matrix. Measure $\hat{Q}$ under the state $\hat{\rho}$ defined in (a). *What are the possible measurement results, and their corresponding probabilities? For each possible measurement result, what is the density matrix for the collapsed state after measurement?* [Hint: the probabilities will be very different from those in (c)]

***** (about lecture #2) *****

3. Consider a single-boson Hilbert space with two complete orthonormal basis states, $|1\rangle$ & $|2\rangle$. Denote the corresponding creation,annihilation operators by $\hat{b}_1^\dagger$,$\hat{b}_1$ (for $|1\rangle$) and $\hat{b}_2^\dagger$,$\hat{b}_2$ (for $|2\rangle$), then $|1\rangle = \hat{b}_1^\dagger|\text{vac}\rangle$, $|2\rangle = \hat{b}_2^\dagger|\text{vac}\rangle$, where $|\text{vac}\rangle$ is the normalized 'vacuum' state, and $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{i,j}$, $[\hat{b}_i, \hat{b}_j] = [\hat{b}_i^\dagger, \hat{b}_j^\dagger] = 0$.

(a). (3pts) *Write down a complete orthonormal basis for the Hilbert space of two bosons,* in terms of tensor product states $|i\rangle \otimes |j\rangle$, $i, j = 1, 2$.

(b). (2pts) A unitary transformation $\hat{U}$ is defined by its action on single-boson basis as: $|1\rangle \mapsto \hat{U}|1\rangle = (u|1\rangle - v|2\rangle)$, $|2\rangle \mapsto \hat{U}|2\rangle = (v^*|1\rangle + u^*|2\rangle)$, where u, v are two complex numbers and $|u|^2 + |v|^2 = 1$. *Show that the above definition of $\hat{U}$ is indeed a unitary transformation in single-boson Hilbert space.*

(c). (5pts) The action of $\hat{U}$ on a tensor product state will be transforming each of the factors, for example $|1\rangle \otimes |2\rangle \mapsto \hat{U}|1\rangle \otimes \hat{U}|2\rangle$. *Write down the transformation results of all two-boson basis in (a) induced by $\hat{U}$, as linear combinations of the original two-boson basis states. Explicitly show that this transformation in the two-boson Hilbert space is unitary.*

(d). (5pts) $\hat{U}$ can be extended to the entire Fock space as follows: The transformation of an operator $\hat{O}$ by $\hat{U}$ is formally $\hat{U}\hat{O}\hat{U}^\dagger$. We demand that the transformation results of $\hat{b}_i^\dagger$ are: $\hat{U}\hat{b}_1^\dagger\hat{U}^\dagger = (u\hat{b}_1^\dagger - v\hat{b}_2^\dagger)$, and $\hat{U}\hat{b}_2^\dagger\hat{U}^\dagger = (v^*\hat{b}_1^\dagger + u^*\hat{b}_2^\dagger)$. Together with $\hat{U}|\text{vac}\rangle = |\text{vac}\rangle$, this can reproduce the definition of $\hat{U}$ in single-boson space, *e.g.* $\hat{U}|1\rangle = \hat{U}\hat{b}_1^\dagger|\text{vac}\rangle = \hat{U}\hat{b}_1^\dagger\hat{U}^\dagger \cdot \hat{U}|\text{vac}\rangle = (u\hat{b}_1^\dagger - v\hat{b}_2^\dagger)|\text{vac}\rangle = u|1\rangle - v|2\rangle$. *Use the creation operators to represent the two-boson basis in (a), then apply $\hat{U}$ on them, represent the results as linear combinations of the original two-boson basis.* The results should be consistent with (c).

(e). (5pts) Consider $\hat{H} = t \cdot (\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1)$, where t is a real number. You can do a unitary transformation(basis change) to "diagonalize" $\hat{H}$: find a new set of orthonormal creation (annihilation) operators $\hat{b}_i'^\dagger$ $(\hat{b}_i')$ as linear combinations of $\hat{b}_j^\dagger$ $(\hat{b}_j)$, so that $\hat{H} = \epsilon_1 \hat{b}_1'^\dagger \hat{b}_1' + \epsilon_2 \hat{b}_2'^\dagger \hat{b}_2'$, where $\epsilon_{1,2}$ are two c-numbers. These new operators should satisfy the same kind of commutation relations as the old ones, *e.g.* $[\hat{b}_i', \hat{b}_j'^\dagger] = \delta_{i,j}$. *Solve the new creation operators $\hat{b}_i'^\dagger$ in terms of $\hat{b}_j^\dagger$, and solve $\epsilon_{1,2}$. Then write down all the eigenvalues and eigenstates of $\hat{H}$ in the entire Fock space.*

(f). (5pts) (DIFFICULT) The explicit form of operator $\hat{U}$ in (d) in the entire Fock space is, $\hat{U} = \exp\left[\mathbb{i} \sum_{i,j=1}^{2} \hat{b}_i^\dagger \, (a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)_{i,j} \, \hat{b}_j\right]$. Here $a_{1,2,3}$ are three real numbers, $\sigma_{1,2,3}$ are Pauli matrices defined in Homework #1 Problem 6. $(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)_{i,j}$ is the i^{th}-row-j^{th}-column element of the 2×2 matrix in the bracket. *Solve the real numbers $a_{1,2,3}$ in terms of the complex numbers u, v used to define $\hat{U}$ in (b).* [Hint: compute $\hat{U}\hat{b}_{1,2}^\dagger\hat{U}^\dagger$ by the Baker-Hausdorff formula, compare the results with those in (d), some results in Homework #1 will be useful]