TENSOR CATEGORIES AND TOPOLOGICAL ORDER

KARLSTADS UNIVERSITET
Jürgen Fuchs

TENSOR CATEGORIES,

TOPOLOGICAL FIELD THEORY

AND

TOPOLOGICAL ORDER

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Topics

- ☞ mathematical tools useful for studying topological orders / phases
- ☞ algebraic structures :
  - ⇝ categories and functors
  - ⇝ fusion categories
  - ⇝ modular tensor categories
  - ⇝ higher categories
  - ⇝ algebras in tensor categories
  - ⇝ module and bimodule categories ......
- ☞ geometric structures : cobordisms
- ☞ from geometry / topology to algebraic structures : topological field theory

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- algebraic structures :
  - categories and functors
  - fusion categories
  - modular tensor categories
  - higher categories
  - algebras in tensor categories
  - module and bimodule categories ......
- geometric structures : cobordisms
- from geometry / topology to algebraic structures : topological field theory
- relevance to topological orders in thermodynamic limit
- in particular gapped boundaries and gapped interfaces

Part I / II

- categories
- $\cdots$
- $\cdots$
- $\cdots$
- modular tensor categories

Part I / II

- ☞ categories
- ☞ · · ·
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- ☞ modular tensor categories

- ☞ algebras
- ☞ · · ·
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- ☞ module and bimodule categories

Part I / II

- categories
- $\cdots$
- $\cdots$
- $\cdots$
- modular tensor categories

- algebras
- $\cdots$
- $\cdots$
- module and bimodule categories

in gory detail

Part III

- topological field theory for gapped boundaries and interfaces

Why categories ?

2-dimensional topological order

☞ gapped 2-d bulk

☞ **Question** : can one "put" the system on a disk without closing the gap **?**

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☞ **Question** : assuming vanishing obstruction , how many independent “ boundary conditions " on the disk such that system remains gapped **?**

Answer : . . . .

↯ suitable language and tools needed

↯ needed e.g. for telling precisely

what a boundary condition is and

when two boundary conditions independent

Why categories ?

Miscellaneous general motivations :

- ⇝ see what is really going on :

  recognize distinct situations as different realizations of the same structure

- ⇝ prove stuff only once

- ⇝ be able to tell precisely when two things are “the same” $=$ “isomorphic”

- ⇝ and even to keep track of isomorphisms

Why categories ?

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  recognize distinct situations as different realizations of the same structure

- prove stuff only once
- be able to tell precisely when two things are "the same" $=$ "isomorphic"
- and even to keep track of isomorphisms
- generalized notions of symmetry
- categories $\rightsquigarrow$ functors $\rightsquigarrow$ natural transformations
- quantization
- quantum filed theories and the dimensional ladder
- ....

Categories

Categories

DESIRABLE — Quasi-particles

- quasi-particle excitations
- may interact / be manipulated

INFORMAL DEFINITION — Category

- collection of things
- notion of relating two things
- sensible properties of the latter

DEFINITION — **Category $\mathcal{C}$**

Data :

- class Obj$(\mathcal{C})$ members called objects
- set of arrows $x \xrightarrow{f} y$ for any two objects x and y

  call x source and y target of $\xrightarrow{f}$

  write $x = \mathsf{s}(\xrightarrow{f}) \equiv \mathsf{s}(f)$ and $y = \mathsf{t}(\xrightarrow{f}) \equiv \mathsf{t}(f)$
- distinguished arrow $\mathrm{id}_x : x \longrightarrow x$ for each object x
- composed arrow $g \circ f : \mathsf{s}(f) \rightarrow \mathsf{t}(g)$

  for any two arrows f and g with $\mathsf{t}(f) = \mathsf{s}(g)$

Axioms :. . . . . .

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Axioms :

- associativity of composition :

  $h \circ (g \circ f) = (h \circ g) \circ f$ whenever either side defined
- unit properties of id :

  $f \circ \mathrm{id}_x = f$ for $\mathsf{s}(f) = x$ and $\mathrm{id}_y \circ f = f$ for $\mathsf{t}(f) = y$

Categories

COMMENTS

- for x an object of $\mathcal{C}$ write
  $x \in \mathrm{Obj}(\mathcal{C})$ or just $x \in \mathcal{C}$ though $\mathrm{Obj}(\mathcal{C})$ not necessarily a *set*
- call arrow $x \xrightarrow{f} y$ a morphism from x to y
- id_x called identity morphism
- call morphisms f and g composable iff $\mathtt{t}(f) = \mathtt{s}(g)$
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- denote set of all morphisms $x \xrightarrow{f} y$ by $\mathrm{Hom}_{\mathcal{C}}(x,y)$
- write $\mathrm{End}_{\mathcal{C}}(x) := \mathrm{Hom}_{\mathcal{C}}(x,x)$

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☞ write $\mathrm{End}_{\mathcal{C}}(x) := \mathrm{Hom}_{\mathcal{C}}(x, x)$

☞ alternative formulation :

- start from class of objects and class $\mathrm{Hom}(\mathcal{C})$ of all morphisms
- formulate rest of structure and axioms in terms of maps

$\mathrm{id}\colon\ \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Hom}(\mathcal{C})$ $\qquad$ $\mathtt{s}, \mathtt{t}\colon\ \mathrm{Hom}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{C})$

$\circ\colon\ \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Hom}(\mathcal{C}) \longrightarrow \mathrm{Hom}(\mathcal{C})$

EXAMPLES

- $\mathcal{S}et$
  - objects $=$ sets
  - morphisms $=$ maps between sets
  - identity morphism $=$ identity map
  - composition $=$ composition of maps
- $\mathcal{V}ect$
  - objects $=$ vector spaces   over some field $\Bbbk$
  - morphisms $=$ linear maps

- *for us* : $\Bbbk = \mathbb{C}$

Categories

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☞ $\mathcal{v}ect$

- objects $=$ finite-dimensional vector spaces   over some field $\Bbbk$
- morphisms $=$ linear maps

☞ *for us*: $\Bbbk = \mathbb{C}$

Categories

EXAMPLE

☞ $\boxed{X\text{-mod}}$ ($X = G$ group / $= R$ ring / $= A$ algebra)

- objects $=$ X-modules
- morphisms $=$ intertwiners

EXAMPLE

- $\boxed{X\text{-mod}}$ ($X = G$ group / $= R$ ring / $= A$ algebra)
  - objects $=$ X-modules
  - morphisms $=$ intertwiners

- module $=$ vector space V together with X-action (representation)

  $\rho\colon\; X \longrightarrow \mathrm{End}_{\mathcal{V}ect}(V)$ compatible with the structure of X

- intertwiner $\varphi\colon\, (V,\rho_V) \longrightarrow (W.\rho_W)$

  $=$ linear map $V \longrightarrow W$

  such that
$$\begin{tikzcd} V \arrow[r, "\rho_V(x)"] \arrow[d, "\varphi"'] & V \arrow[d, "\varphi"] \\ W \arrow[r, "\rho_W(x)"] & W \end{tikzcd}$$
  for all $x \in X$

EXAMPLE

- a single object $*$ and only isomorphisms
  - composition of morphisms gives associative product with inverse
  - identity morphism gives unit element
  - is nothing but group G with morphisms $=$ group elements
  - notation $\boxed{*/\!\!/G}$

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- isomorphism $=$ morphism having an inverse

Categories

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EXAMPLES

- generalization: groupoid : only *iso*morphisms
- path groupoid of manifold M
  - objects $=$ points of M
  - morphisms $=$ paths
  - composition $=$ concatenation of paths

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EXAMPLES

- generalization: groupoid : only isomorphisms
- fundamental groupoid of manifold M
  - objects $=$ points of M
  - morphisms $=$ homotopy classes of paths
  - composition inherited from concatenation

EXAMPLE

- a single object $*$ and only isomorphisms
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EXAMPLES

- generalization: groupoid : only isomorphisms
- action groupoid $\boxed{X/\!/G}$ of set X endowed with G-action (G group)
  - objects $=$ elements of X
  - morphisms $=$ group elements $x \xrightarrow{g} g.x$
  - composition inherited from group product

COMMENTS

- rarely sensible to require two objects x and y of a category to be *equal*
- sensible to require two objects to be isomorphic : $\boxed{x \cong y}$
  i.e. existence of an isomorphism $f\colon\; x \xrightarrow{\cong} y$
- Example : any finite-dimensional vector space V
  isomorphic to its dual vector space V^*
  but no distinguished isomorphism
  (need basis or non-degenerate bilinear form)
- Example : distinction between real and pseudoreal / quaternionic
  representations of a group (e.g. of SU(2))

EXAMPLES

☞ $\boxed{\mathcal{M}_d}$

- objects $=$ smooth manifolds
- morphisms $=$ smooth maps of manifolds

<u>*for us*</u>: manifold $=$ smooth manifold

Categories

EXAMPLES

- $\boxed{\mathcal{M}_d}$
  - objects $=$ smooth manifolds
  - morphisms $=$ smooth maps of manifolds
- $\boxed{\mathcal{C}obord_{d,d-1}}$
  - objects $=$ oriented $d-1$-manifolds
  - morphisms $=$ oriented d-manifolds with boundary up to diffeom.
  - source / target $=$ incoming / outgoing boundary

EXAMPLES

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Details :

- cobordism $B\colon M \to N$ $=$ oriented d-dimensional manifold B

  plus orientation preserving diffeomorphism $\phi_B\colon \overline{M} \sqcup N \xrightarrow{\cong} \partial B$

  (overline = orientation reveral)

- morphism $=$ equivalence class of cobordisms

Categories

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Details :

- equivalence of cobordisms (B,ϕ_B) and $(B'.\phi_{B'})$:

have orientation-preserving diffeomorphism φ such that

$$\begin{tikzcd} B \arrow[rr, "\varphi"] & & B' \\ & \overline{M} \sqcup N \arrow[ul, "\phi_B"] \arrow[ur, "\phi_{B'}"'] & \end{tikzcd}$$

EXAMPLES

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Details :

- identity morphism id_M : represented by cylinder $M \times [0,1]$
- composition of morphisms

  $=$ gluing : identify $\partial_- B$ with $\partial_+ B'$ (using collars)

EXAMPLES

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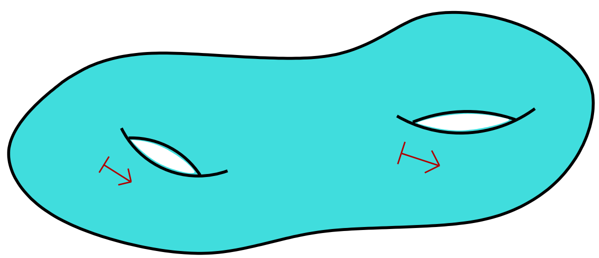
Details :

illustration : $d=3$



Categories

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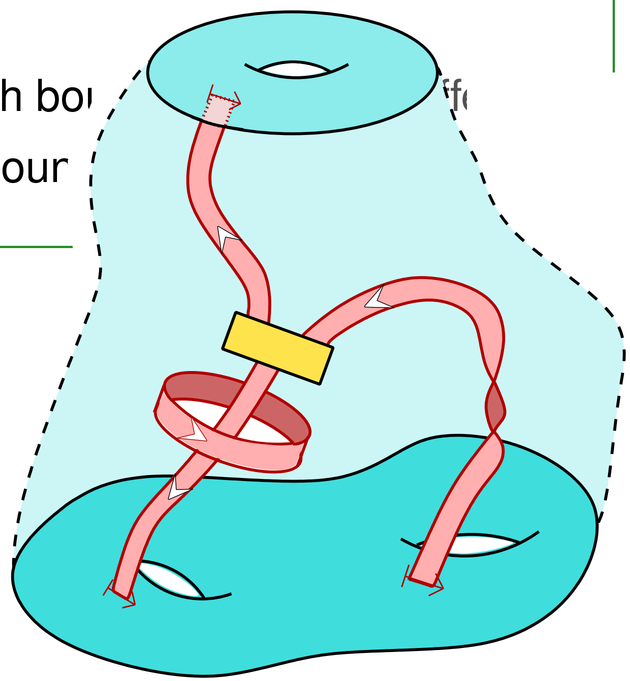
illustration : $d = 3$ (decorated version)



EXAMPLES

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EXAMPLE

☞ $\boxed{\mathit{Cobord}_{1,0}}$

- objects $=$ finite disjoint unions of oriented points $(\bullet, \pm)$
- morphisms $=$ finite disjoint unions of oriented intervals and circles

EXAMPLES

☞ $\mathcal{C}obord_{d,d-1}$

- objects $=$ oriented $d-1$-manifolds
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EXAMPLE

☞ $\mathcal{C}obord_{\,2,1}$

- objects $=$ finite disjoint unions of oriented circles
- morphisms generated via disjoint unions and gluings from 6 elementary morphisms

EXAMPLES

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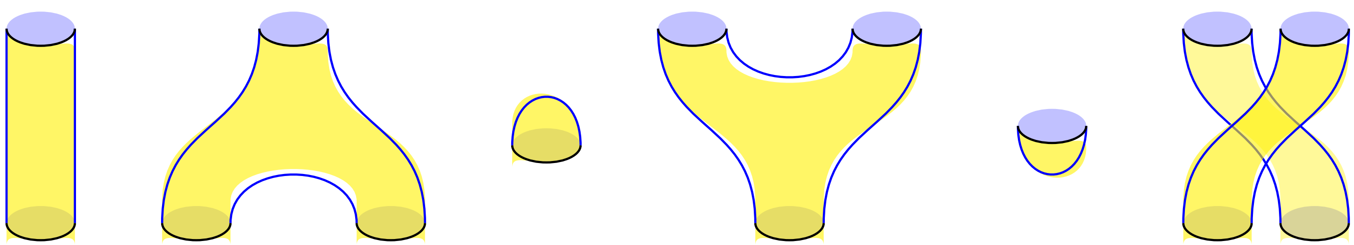

Categories

EXAMPLES

- $\boxed{\mathcal{C}obord_{d,d-1}}$
  - objects = oriented $d{-}1$-manifolds
  - morphisms = oriented d-manifolds with boundary up to diffeom.
  - source / target = incoming / outgoing boundary
- other versions also of interest
  - framed
  - combed
  - unoriented

Graphical calculus

Graphical calculus

- ☞ convenient to visualize morphisms graphically

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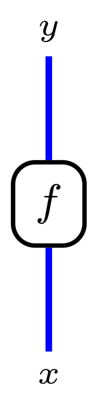
  first step :

GRAPHICAL CALCULUS

- general morphism $\boxed{x \xrightarrow{f} y}$
  - box / coupon
  - lines connecting coupon to domain (input) and codomain (output)
  - vertical straight lines – up to ambient isotopy
  - *optimistic* convention : read from bottom to top

$f \;=$ 

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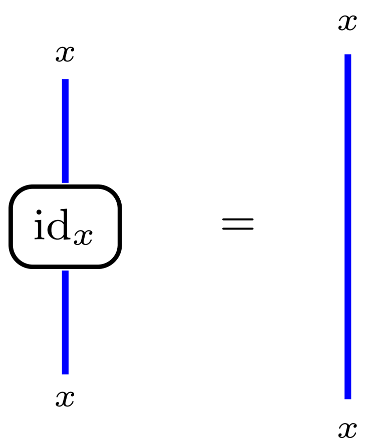

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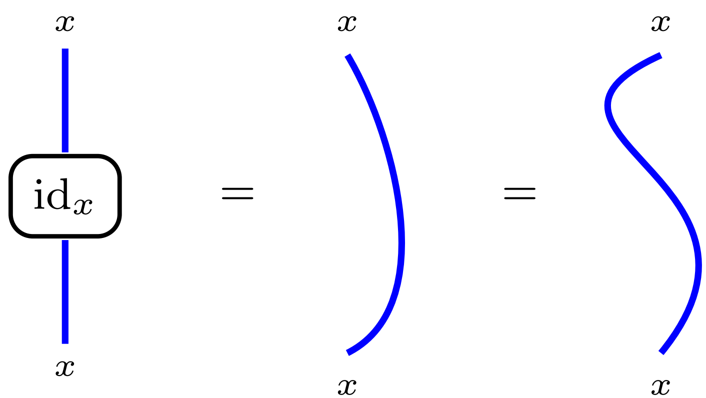

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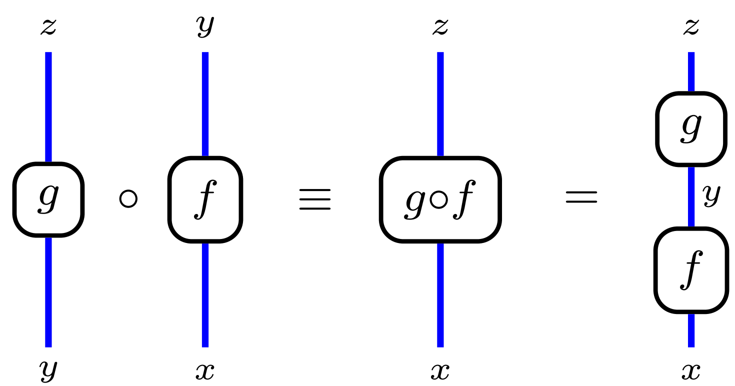
- identity morphism
- composition $=$ concatenation



Categories

DESIRABLE **State spaces**

- state spaces
- linear operators

- morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$ just a set in general

DESIRABLE — **State spaces**

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DEFINITION — **Linear category**

- Linear category
  - $\mathrm{Hom}_{\mathcal{C}}(x, y)$ structure of a vector space (over $\mathbb{C}$ / over $\Bbbk$)
  - composition of morphisms bilinear

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EXAMPLES

- $\mathcal{V}ect$, νect , A-mod
- linearization of a category $\mathcal{C}$: *functors* from $\mathcal{C}$ to $\mathcal{V}ect$

DESIRABLE — **"Symmetries"**

- ☞ symmetries in physics realized via representations
- ☞ require other familiar aspects of categories of vector spaces / of modules

Categories

DESIRABLE — **"Symmetries"**

- symmetries in physics realized via representations
- require other familiar aspects of categories of vector spaces / of modules

INFORMAL DEFINITION — **Abelian category**

- $\boxed{\text{abelian category}}$ $=$ equivalent to A-mod for some algebra A
  - can add morphisms
  - can add objects : biproduct / direct sum $x \oplus y$
  - kernels and cokernels
  - zero object
  - exact sequances $0 \rightarrow x \rightarrow z \rightarrow y \rightarrow 0$
  - projective and injective objects
  - ......

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for us : use some of these concepts freely

DESIRABLE — **"Finiteness"**

- ☞ finite number of elementary quasi-particle excitations
- ☞ composite excitations separable into 'sums' of elementary excitations

INFORMAL DEFINITION — **Finite category**

- ☞ $\boxed{\text{finite category}}$ $\equiv$ finite abelian linear category :

  $:=$ equivalent to A-mod for some finite-dimensional algebra A

  ($\Bbbk$ algebraically closed field)

- ☞ in particular :
  - finitely many isomorphism classes of simple objects
  - every object has finite length
  - every object has a projective cover

DESIRABLE — **"Finiteness"**

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DEFINITION — **Semisimple category**

- $\boxed{\text{semisimple category}}$:
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  - all such direct sums exist

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DEFINITION — **Semisimple category**

- semisimple category :
  - each object is direct sum of finitely many simple objects
  - all such direct sums exist
- finitely semisimple category :
  - finite abelian
  - semisimple

Functors

Functors

☞ recall : linearization

☞ can be regarded as an instance of "mappig one category to another one"

☞ should generalize the idea that

instead of group G can study G-representations on vector spaces

Functors

INFORMAL DEFINITION — Functor

- functor from $\mathcal{C}$ to $\mathcal{D}$:
  - map any object of $\mathcal{C}$ to an object of $\mathcal{D}$
  - map any morphism of $\mathcal{C}$ to a morphism of $\mathcal{D}$
  - in a way compatible with their structure and properties

Functors

DEFINITION — Functor

☞ $\boxed{\text{functor } F\colon\ \mathcal{C} \to \mathcal{D}}$

Data :

- map $F\colon\ \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D})$
- map $F \equiv F_{x,y}\colon\ \mathrm{Hom}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$

for each pair of objects x, y of $\mathcal{C}$

Axioms :

- $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$
- $F(g \circ f) = F(g) \circ F(f)$ (when defined)

Functors

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COMMENTS

☞ linear functor $F\colon\ \mathcal{C} \to \mathcal{D}$ between linear categories :

linear on morphism spaces

☞ composition of functors $F\colon\ \mathcal{C} \longrightarrow \mathcal{C}'$ and $F'\colon\ \mathcal{C}' \longrightarrow \mathcal{C}''$

to $F' \circ F\colon\ \mathcal{C} \longrightarrow \mathcal{C}''$ via composition of maps

Functors

DEFINITION — **Functor**

- $\boxed{\text{functor } F\colon\ \mathcal{C} \to \mathcal{D}}$
  - map $F\colon \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D})$
  - maps $F \equiv F_{x,y}\colon\ \mathrm{Hom}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$
  - $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$
  - $F(g\circ f) = F(g)\circ F(f)$

EXAMPLES

- identity functor $\mathrm{Id}_{\mathcal{C}}\colon\ \mathcal{C} \longrightarrow \mathcal{C}$: identity maps
- for any group G (and field $\Bbbk$)
  functor $*/\!/G \longrightarrow \mathcal{V}ect$ amounts to a linear representation of G
- associating to any vector space V its dual space $V^* = \mathrm{Hom}_{\mathcal{V}ect}(V,\Bbbk)$
  provides functor $\mathcal{V}ect \longrightarrow \mathcal{V}ect^{\mathrm{op}}$
- associating to any vector space V its bidual space V^{**}
  provides endofunctor $\mathcal{V}ect \longrightarrow \mathcal{V}ect$

Functors

DEFINITION — Functor

- ☞ $\boxed{\text{functor } F\colon\ \mathcal{C} \to \mathcal{D}}$
  - map $F\colon \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D})$
  - maps $F \equiv F_{x,y}\colon\ \mathrm{Hom}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$
  - $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$
  - $F(g \circ f) = F(g) \circ F(f)$

DEFINITION — Opposite category

- ☞ opposite category $\boxed{\mathcal{C}^{\mathrm{op}}}$
  - objects $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C})$
  - morphisms $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(y,x)$
  - composition $g \circ^{\mathrm{op}} f = f \circ g$
- ☞ contravariant functor $\mathcal{C} \longrightarrow \mathcal{D}$ $:=$ functor $\mathcal{C} \longrightarrow \mathcal{D}^{\mathrm{op}}$

Functors

DEFINITION — Functor

☞ functor $F\colon\ \mathcal{C} \to \mathcal{D}$

- map $F\colon \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D})$
- maps $F \equiv F_{x,y}\colon\ \mathrm{Hom}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$
- $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$
- $F(g\circ f) = F(g)\circ F(f)$

EXAMPLE

☞ d-dimensional topological field theory

$=$ functor $\mathbf{tft}\colon\ \mathcal{C}obord_{d,d-1} \longrightarrow \mathcal{V}ect$

satisfying compatibility conditions (to be spelt out later)

Thus :

- assignment $(d-1)$-manifold $M \longmapsto$ vector space $\mathbf{tft}(M)$
- assignment d-dimensional cobordism $M \xrightarrow{B} N$

  $\longmapsto$ linear map $\mathbf{tft}(B)\colon\ \mathbf{tft}(M) \longrightarrow \mathbf{tft}(N)$

- ☞ linear representation of group G $=$ "functor $*/\!/G \longrightarrow \mathcal{V}ect$
- ☞ thus $\boxed{\text{category } G\text{-mod of } G\text{-modules}}$ has functors as objects
- ☞ thus morphisms of G-mod (intertwiners)

  have functors as their domain and codomain

- ☞ linear representation of group G $=$ "functor $*/\!/G \longrightarrow \mathcal{V}ect$
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INFORMAL DEFINITION — **Natural transformation**

☞ $\boxed{\text{natural transformation } F \longrightarrow G}$

$=$ collection of morphisms relating objects $F(x)$ and $G(x)$ for all objects x in a manner compatible with morphisms

DEFINITION — **Natural transformation**

☞ natural transformation $\boxed{\psi\colon\ F \longrightarrow G}$

between functors $F,\, G\colon\ \mathcal{C} \longrightarrow \mathcal{C}'$

$:=$ family of morphisms $\psi_x : F(x) \longrightarrow G(x)$ in $\mathcal{C}'$ for $x \in \mathcal{C}$

such that
$$\begin{tikzcd} F(x) \arrow[r, "\psi_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\psi_y"] & G(y) \end{tikzcd}$$
for all $x \xrightarrow{f} y$ in $\mathcal{C}$

Functors

DEFINITION — **Natural transformation**

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for all $x \xrightarrow{f} y$ in $\mathcal{C}$

DEFINITION — **Natural isomorphism**

☞ natural isomorphism $\boxed{\psi\colon\; F \longrightarrow G}$

$:=$ natural transformation such that each ψ_x is an isomorphism in $\mathcal{C}'$

Functors

INFORMAL DEFINITION — **Equivalence of categories**

☞ existence of mutually "inverse" functors $F\colon\ \mathcal{C} \leftrightarrows \mathcal{D}\colon G$
up to ...

DEFINITION — **Equivalence of categories**

- functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ called equivalence of categories
  $:=$ existence of functor $G\colon \mathcal{D} \longrightarrow \mathcal{C}$
  and natural isomorphisms $\mathrm{Id}_{\mathcal{D}} \longrightarrow F\,G$ and $G\,F \longrightarrow \mathrm{Id}_{\mathcal{C}}$
- $\mathcal{C}$ and $\mathcal{D}$ called equivalent $:=$ existence of equivalence functor
  ⇝ notation $\boxed{F\colon \mathcal{C} \xrightarrow{\simeq} \mathcal{D}}$

Functors

DEFINITION — **Equivalence of categories**

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- $\mathcal{C}$ and $\mathcal{D}$ called equivalent $:=$ existence of equivalence functor
  - notation $\boxed{F\colon\ \mathcal{C} \xrightarrow{\simeq} \mathcal{D}}$

DEFINITION — $[\,\mathcal{C},\mathcal{D}\,]$

- small category $:=$ category whose class of objects is a set
- $\boxed{[\,\mathcal{C},\mathcal{D}\,]}$ for small category $\mathcal{C}$ $:=$ the category with
  - objects $=$ all functors from $\mathcal{C}$ to $\mathcal{D}$
  - morphisms $=$ natural transformations

DEFINITION — **Equivalence of categories**

- functor $F\colon\ \mathcal{C} \longrightarrow \mathcal{D}$ called equivalence of categories

  $:=$ existence of functor $G\colon\ \mathcal{D} \longrightarrow \mathcal{C}$

  and natural isomorphisms $\mathrm{Id}_{\mathcal{D}} \longrightarrow F\,G$ and $G\,F \longrightarrow \mathrm{Id}_{\mathcal{C}}$

- $\mathcal{C}$ and $\mathcal{D}$ called equivalent $:=$ existence of equivalence functor

  ⇝ notation $\boxed{F\colon\ \mathcal{C} \xrightarrow{\simeq} \mathcal{D}}$

EXAMPLES

- indeed: $[\,*/\!/G, \mathcal{V}ect\,] \simeq G$-mod

- family of linear maps $\psi_V\colon\ V \longrightarrow V^{**}$ for all $V \in \mathcal{V}ect$

  with $\psi_V(v)(\varphi) := \varphi(v)$ for $v \in V$, $\varphi \in V^*$

  furnishes natural transformation $\mathrm{Id}_{\mathcal{V}ect} \longrightarrow (-)^{**}$

- restricting to $\mathcal{v}ect$ gives a natural isomorphism

Criterion for recognizing an equivalence :

PROPOSITION — **Equivalence of categories**

$F\colon\ \mathcal{C} \longrightarrow \mathcal{D}$ equivalence

$\Longleftrightarrow$

(1) essentially surjective

i.e. for any $y \in \mathcal{D}$ there exists $x \in \mathcal{C}$ with $y \cong F(x)$ in $\mathcal{D}$

(2) fully faithful

i.e. $F_{x,x'}:$ "$\mathrm{Hom}_{\mathcal{C}}(x,x') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(x),F(x'))$ bijective map

for any two $x,x' \in \mathcal{C}$

Monoidal categories

Monoidal categories

DESIRABLE — **Several systems**

☞ standard situation : system consisting of several isolated subsystems

DEFINITION — **Cartesian product**

☞ Cartesian product $\mathcal{C} \times \mathcal{D}$ of categories $\mathcal{C}$ and $\mathcal{D}$:

category with

- objects $=$ pairs (x, y) with $x \in \mathcal{C}$ and $y \in \mathcal{D}$
- morphisms $=$ pairs of morphisms $(x \xrightarrow{f} x', y \xrightarrow{g} y')$

DESIRABLE — **Several systems**

- standard situation : system consisting of several isolated systems
- combine observables of subsystems

DEFINITION — **Tensor product**

- tensor product on a category $\mathcal{C}$: functor $\boxed{\otimes : \ \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}}$

  mapping $(x, y) \longmapsto x \otimes y$

  $$(x \xrightarrow{f} x' \,, y \xrightarrow{g} y') \longmapsto x{\otimes}y \xrightarrow{f \otimes g} x'{\otimes}y'$$

Note : $\mathrm{id}_x \otimes \mathrm{id}_y = \mathrm{id}_{x \otimes y}$

DEFINITION — **Monoidal category**

☞ monoidal category $\boxed{\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}, a, r, l)}$

$:=$ category $\mathcal{C}$ endowed with

- tensor product functor $\otimes$
- distinguished object $\mathbf{1} \in \mathcal{C}$
- natural isomorphism $a\colon\ \otimes \circ (\otimes \times \mathrm{Id}) \to \otimes \circ (\mathrm{Id} \times \otimes$
  satisfying pentagon identity
- natural isomorphisms $l\colon\ \mathbf{1} \otimes \mathrm{Id} \longrightarrow \mathrm{Id}$ and $r\colon\ \mathrm{Id} \otimes \mathbf{1} \longrightarrow \mathrm{Id}$
  satisfying triangle identity

Monoidal categories

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- monoidal category $\boxed{\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}, a, r, l)}$

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  - natural isomorphisms $l\colon\ \mathbf{1} \otimes \mathrm{Id} \longrightarrow \mathrm{Id}$ and $r\colon\ \mathrm{Id} \otimes \mathbf{1} \longrightarrow \mathrm{Id}$

    satisfying triangle identity

- Triangle identity :

$$\begin{tikzcd}
(x \otimes \mathbf{1}) \otimes y \arrow[rr, "a_{x,\mathbf{1},y}"] \arrow[dr, "r_x \otimes \mathrm{id}_y"'] & & x \otimes (\mathbf{1} \otimes y) \arrow[dl, "\mathrm{id}_x \otimes l_y"] \\
& x \otimes y &
\end{tikzcd}$$

Monoidal categories

☞ Pentagon identity :

$$\begin{tikzcd}
 & ((u\otimes v)\otimes x)\otimes y \arrow[dl, "a_{u,v,x}\otimes \mathrm{id}_y"'] \arrow[dr, "a_{u\otimes v,x,y}"] & \\
(u\otimes(v\otimes x))\otimes y \arrow[d, "a_{u,v\otimes x,y}"'] & & (u\otimes v)\otimes(x\otimes y) \arrow[d, "a_{u,v,x\otimes y}"] \\
u\otimes((v\otimes x)\otimes y) \arrow[rr, "\mathrm{id}_u\otimes a_{v,x,y}"] & & u\otimes(v\otimes(x\otimes y))
\end{tikzcd}$$

☞ Triangle identity :

$$\begin{tikzcd}
(x\otimes \mathbf{1})\otimes y \arrow[rr, "a_{x,\mathbf{1},y}"] \arrow[dr, "r_x\otimes \mathrm{id}_y"'] & & x\otimes(\mathbf{1}\otimes y) \arrow[dl, "\mathrm{id}_x\otimes l_y"] \\
 & x\otimes y &
\end{tikzcd}$$

COMMENTS

- terminology : $\mathbf{1}$ = tensor unit / monoidal unit / unit object
  a = associator / associativity constraint
  $l\,/\,r$ = left / right unit constraint
- pentagon $\Longrightarrow$ any two possibilities of relating any two bracketings of multiple tensor product are equal (Mac Lane's coherence theorem)

COMMENTS

- terminology : $\mathbf{1}$ $=$ tensor unit / monoidal unit / unit object

  a $=$ associator / associativity constraint

  $l\,/\,r$ $=$ left / right unit constraint

- pentagon $\Longrightarrow$ any two possibilities of relating any two bracketings of multiple tensor product are equal (Mac Lane's coherence theorem)

- origin of terminology *monoidal* :

  monoidal category $=$ categorified version of associative unital monoid $(R, \cdot, e)$

- but : structure (associator) instead of property (associativity) :

  given $\mathcal{C}$ with given tensor product $\otimes$ may admit inequivalent associators

- alternative terminology : tensor category (disfavored)

COMMENTS

- terminology : $\mathbf{1}$ $=$ tensor unit / monoidal unit / unit object

  a $=$ associator / associativity constraint

  $l\,/\,r$ $=$ left / right unit constraint

- pentagon $\Longrightarrow$ any two possibilities of relating any two bracketings of multiple tensor product are equal (Mac Lane's coherence theorem)

- explicit form of associativity constraint :

$$\begin{tikzcd}
(x\otimes y)\otimes z \arrow[r, "a_{x,y,z}"] \arrow[d, "(f\otimes g)\otimes h"'] & x\otimes(y\otimes z) \arrow[d, "f\otimes(g\otimes h)"] \\
(x'\otimes y')\otimes z' \arrow[r, "a_{x',y',z'}"] & x'\otimes(y'\otimes z')
\end{tikzcd}$$

for any triple $x \xrightarrow{f} x'$, $y \xrightarrow{g} y'$, $z \xrightarrow{h} z'$

Graphical calculus

Graphical calculus

☞ presence of associator in tensor products of morphisms

$\rightsquigarrow$ graphical calculus clumsy

Graphical calculus

☞ presence of associator in tensor products of morphisms

$\rightsquigarrow$ graphical calculus clumsy

DEFINITION — **Strict monoidal category**

☞ strict monoidal category:

$:=$ monoidal category with a, l and r identities

☞ presence of associator in tensor products of morphisms

$\rightsquigarrow$ graphical calculus clumsy

DEFINITION — **Strict monoidal category**

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$:=$ monoidal category with a, l and r identities

THEOREM — **Coherence**

Every monoidal category is equivalent to a strict monoidal category

☞ presence of associator in tensor products of morphisms

$\rightsquigarrow$ graphical calculus clumsy

DEFINITION — **Strict monoidal category**

☞ strict monoidal category:

$:=$ monoidal category with a , l and r identities

THEOREM — **Coherence**

Every monoidal category is equivalent to a strict monoidal category

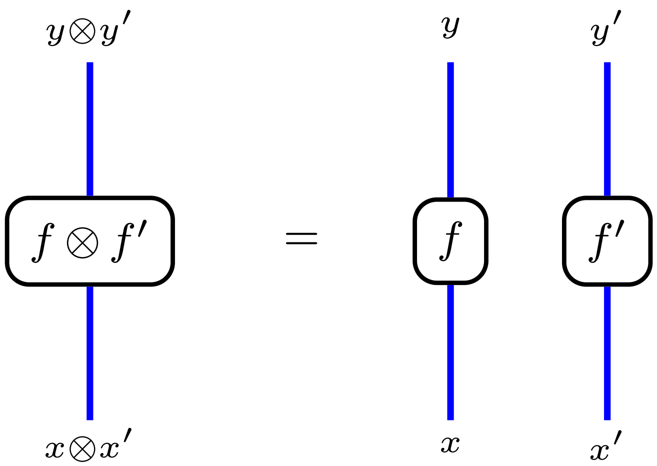
$\Longrightarrow$ strictification : replace monoidal category by a strict one

COMMENT

☞ no loss of generality – can still detect shadow of associator

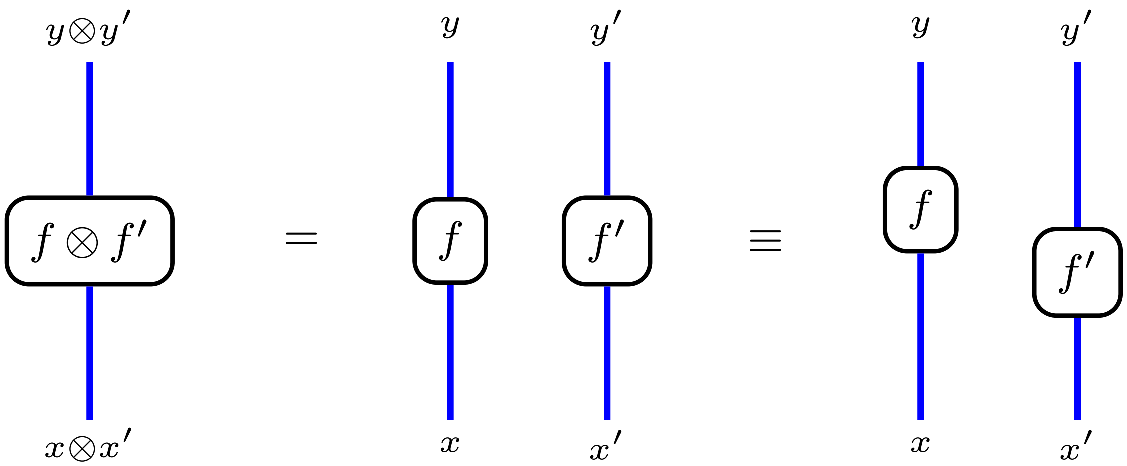
GRAPHICAL CALCULUS

☞ tensor product = juxtaposition



GRAPHICAL CALCULUS

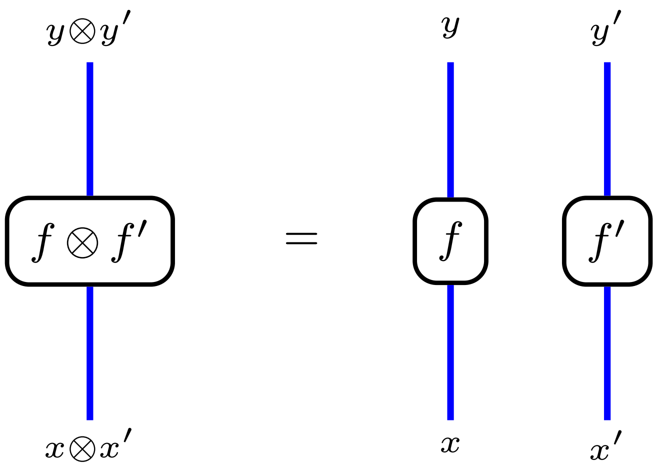
☞ tensor product = juxtaposition



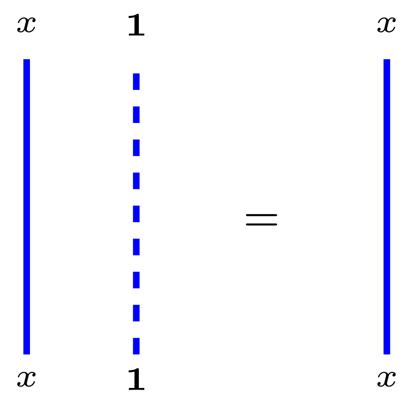
Graphical calculus

GRAPHICAL CALCULUS

☞ tensor product $=$ juxtaposition



☞ tensor unit $\mathbf{1}$ invisible



Fusion

DEFINITION — **Grothendieck group**

☞ Grothendieck group $\boxed{K_0(\mathcal{C})}$ of abelian category $\mathcal{C}$

$:=$ abelian group presented by generator and relations :

- one generator $[x]$ for each isomorphism class of objects of $\mathcal{C}$
- one relation $[z]-[x]-[y]=0$ for each exact sequence $0 \to x \to z \to y \to 0$

COMMENTS

☞ can be generalized to larger class of categories

☞ in particular $[z]=[x]+[y]$ for $z=x\oplus y$

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DEFINITION — **Exact tensor product**

☞ $\otimes$ exact

$:=$ for every exact sequence $0\to x\to z\to y\to 0$

also $0\to u\otimes x\to u\otimes z\to u\otimes y\to 0$

and $0\to x\otimes u\to z\otimes u\to y\otimes u\to 0$ exact for any $u\in\mathcal{C}$

DEFINITION — **Grothendieck group**

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$:=$ abelian group presented by generator and relations :

- one generator $[x]$ for each isomorphism class of objects of $\mathcal{C}$
- one relation $[z] - [x] - [y] = 0$ for each exact sequence $0 \to x \to z \to y \to 0$

PROPOSITION — **Grothendieck ring**

For $\mathcal{C}$ monoidal with exact tensor product , setting

$$\boxed{[x] * [y] := [x \otimes y]}$$

endows $K_0(\mathcal{C})$ with a ring structure

with unit element $[\mathbf{1}]$

Fusion

☞ assume $\mathcal{C}$ monoidal and semisimple

$\Longrightarrow$

↯ $x \otimes y$ $=$ direct sum of simple objects for any $x, y \in \mathcal{C}$

☞ assume $\mathcal{C}$ monoidal and semisimple and with exact tensor product

$\Longrightarrow$

- $x \otimes y$ $=$ direct sum of simple objects
- all tensor products $x \otimes y$ determined up to iso by those of simple objects
- select family $\boxed{\{ X_i \,|\, i \in \mathcal{I} \}}$

  of representatives for the isomorphism classes of simple objects with $X_0 = \mathbf{1}$

Fusion

☞ assume $\mathcal{C}$ monoidal and semisimple and with exact tensor product

$\Longrightarrow$

- $x \otimes y$ $=$ direct sum of simple objects
- all tensor products $x \otimes y$ determined up to iso by those of simple objects
- select family $\boxed{\{ X_i \,|\, i \in \mathcal{I} \}}$

  of representatives for the isomorphism classes of simple objects with $X_0 = \mathbf{1}$

PROPOSITION — **Fusion rules**

For $\mathcal{C}$ semisimple monoidal with exact tensor product :

$$\boxed{[X_i] * [X_j] = \sum_{k \in \mathcal{I}} \mathrm{N}_{ij}{}^{k} \, [X_k]}$$

- $\mathrm{N}_{ij}{}^{k} \in \mathbb{Z}_{\geq 0}$
- associative : $\sum_{l \in \mathcal{I}} \mathrm{N}_{ij}{}^{l} \, \mathrm{N}_{lk}{}^{m} = \sum_{l \in \mathcal{I}} \mathrm{N}_{jk}{}^{l} \, \mathrm{N}_{il}{}^{m}$
- unital : $\mathrm{N}_{i0}{}^{k} = \delta_j{}^{k} = \mathrm{N}_{0i}{}^{k}$

Fusion

☞ assume $\mathcal{C}$ monoidal and semisimple and with exact tensor product

$\Longrightarrow$

- $x \otimes y$ $=$ direct sum of simple objects
- all tensor products $x \otimes y$ determined up to iso by those of simple objects
- select family $\boxed{\{X_i \,|\, i \in \mathcal{I}\}}$

  of representatives for the isomorphism classes of simple objects with $X_0 = \mathbf{1}$

PROPOSITION — **Fusion rules**

For $\mathcal{C}$ semisimple monoidal with exact tensor product :

$$\boxed{[X_i] * [X_j] = \sum_{k\in\mathcal{I}} \mathrm{N}_{ij}{}^{k}\, [X_k]}$$

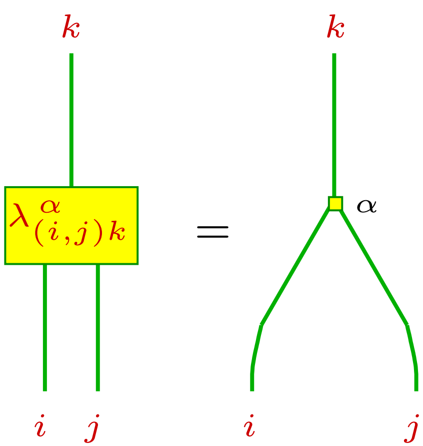
- $\mathrm{N}_{ij}{}^{k} \in \mathbb{Z}_{\geq 0}$ called fusion rules / fusion coefficients
- associative : $\sum_{l\in\mathcal{I}} \mathrm{N}_{ij}{}^{l}\, \mathrm{N}_{lk}{}^{m} = \sum_{l\in\mathcal{I}} \mathrm{N}_{jk}{}^{l}\, \mathrm{N}_{il}{}^{m}$
- unital : $\mathrm{N}_{i0}{}^{k} = \delta_j{}^{k} = \mathrm{N}_{0i}{}^{k}$

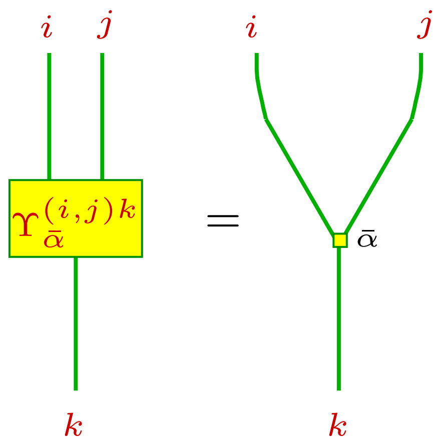
Graphical calculus

GRAPHICAL CALCULUS

for finitely semisimple monoidal categories

☞ bases for copuling spaces





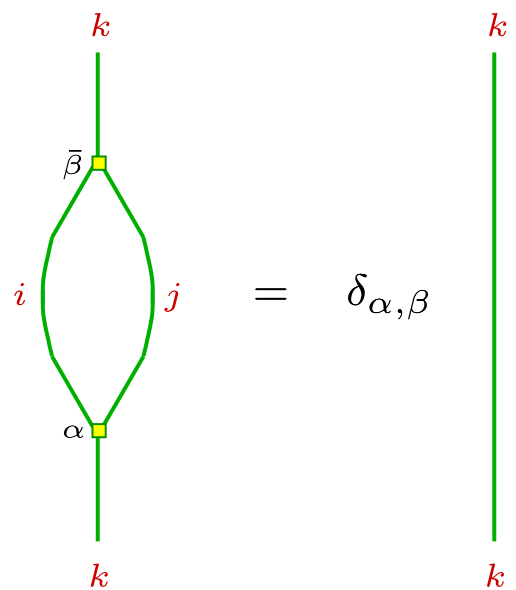
GRAPHICAL CALCULUS

for finitely semisimple monoidal categories

☞ bases for copuling spaces

$$\{ \lambda^{\alpha}_{(i,j)k} \} \subset \mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_k) \qquad \{ \Upsilon^{(i,j)k}_{\bar\alpha} \} \subset \mathrm{Hom}_{\mathcal{C}}(X_k, X_i \otimes X_j)$$

↯ can be chosen dual :

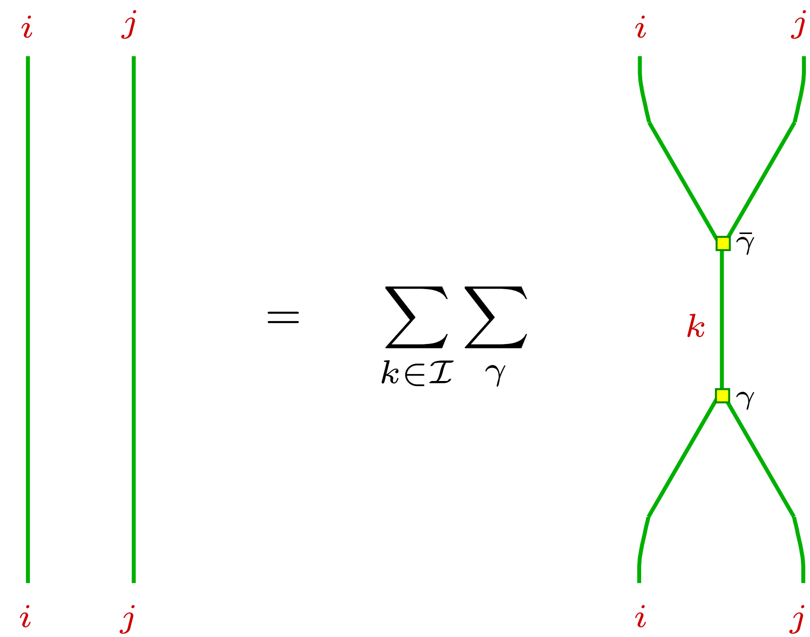


GRAPHICAL CALCULUS

for finitely semisimple monoidal categories .

☞ bases for copuling spaces

- can be chosen dual
- dominance / completeness :

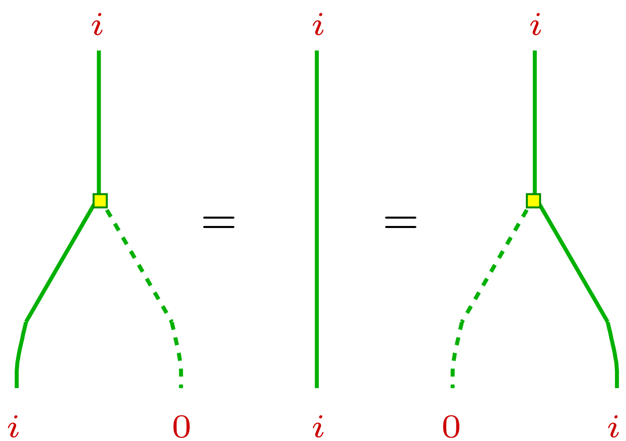


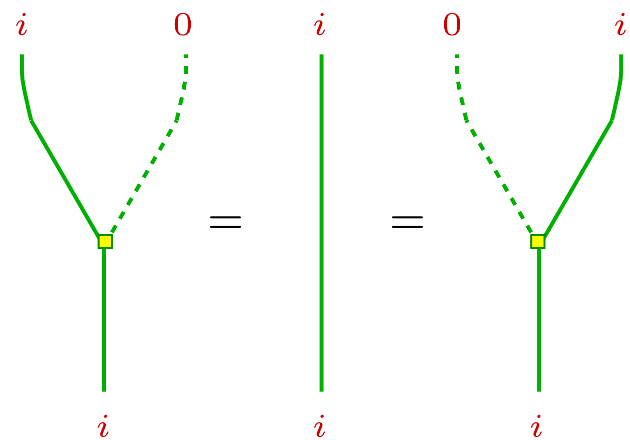
GRAPHICAL CALCULUS

for finitely semisimple monoidal categories .

☞ bases for copuling spaces

- can be chosen dual
- dominance / completeness
- basis elements involving the tensor unit can be chosen trivial :





GRAPHICAL CALCULUS

for finitely semisimple strict monoidal categories

☞ two distinct distinguished bases for $\mathrm{Hom}_{\mathcal{C}}(U_i \otimes U_j \otimes U_k, U_l)$ corresponding to decompositions

$$\bigoplus_{q\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(U_i \otimes U_j, U_q) \otimes \mathrm{Hom}_{\mathcal{C}}(U_q \otimes U_k, U_l)$$

and

$$\bigoplus_{p\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(U_j \otimes U_k, U_p) \otimes \mathrm{Hom}_{\mathcal{C}}(U_i \otimes U_p, U_l)$$

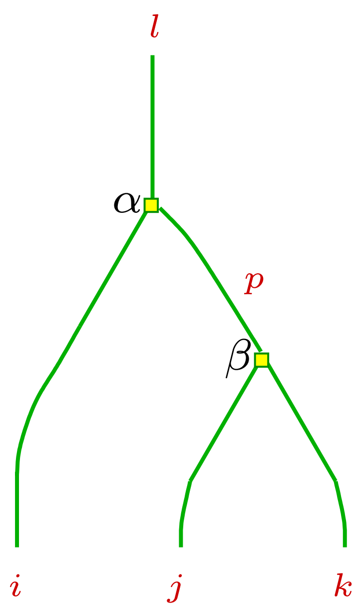
(shadow of the associator)

Graphical calculus

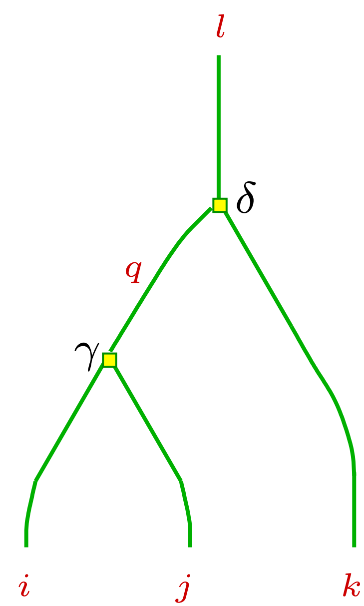
GRAPHICAL CALCULUS

for finitely semisimple strict monoidal categories

- ☞ two distinct distinguished bases for $\mathrm{Hom}_{\mathcal{C}}(U_i \otimes U_j \otimes U_k, U_l)$
- ☞ coefficients of basis transformation : fusing matrices / F-matrices / 6j-symbols



$$= \sum_{q\in\mathcal{I}} \sum_{\gamma,\delta} \mathsf{F}^{(i\,j\,k)\,l}_{\alpha p \beta, \gamma q \delta}$$

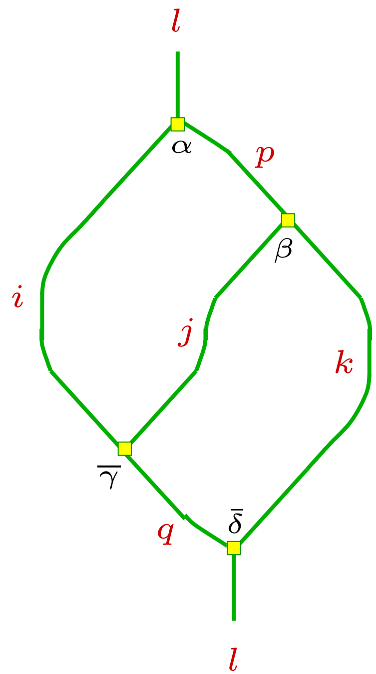


GRAPHICAL CALCULUS

for finitely semisimple strict monoidal categories

- ☞ two distinct distinguished bases for $\mathrm{Hom}_{\mathcal{C}}(U_i \otimes U_j \otimes U_k, U_l)$
- ☞ coefficients of basis transformation : fusing matrices

composing
with dual morphism
gives



$= \mathsf{F}^{(i\,j\,k)\,l}_{\alpha p \beta, \gamma q \delta}$

Functors

Monoidal functors

☞ adapt notions of functor and natural transformation to monoidal setting

INFORMAL DEFINITION — **Monoidal functor / natural transformation**

☞ monoidal functor $=$ functor
plus compatibility with tensor products

☞ monoidal natural transformation $=$ natural transformation
plus compatibility with tensor products

Monoidal functors

☞ adapt notions of functor and natural transformation to monoidal setting

INFORMAL DEFINITION — **Monoidal functor / natural transformation**

☞ monoidal functor $=$ functor
plus compatibility with tensor products (structure)

☞ monoidal natural transformation $=$ natural transformation
plus compatibility with tensor products (properties)

Monoidal functors

DEFINITION — **Monoidal functor**

☞ monoidal functor $\boxed{(F, \varphi_0, \varphi_2)}$ from $\mathcal{C}$ to $\mathcal{D}$:

Data:

- functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$
- isomorphism $\varphi_0\colon \mathbf{1}_\mathcal{D} \xrightarrow{\cong} F(\mathbf{1}_\mathcal{C})$ in $\mathcal{D}$
- natural isomorphism $\varphi_2\colon \otimes_\mathcal{D} \circ (F \times F) \xrightarrow{\cong} F \circ \otimes_\mathcal{C}$
  of functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{D}$

Monoidal functors

DEFINITION — **Monoidal functor**

☞ monoidal functor $\boxed{(F, \varphi_0, \varphi_2)}$ from $\mathcal{C}$ to $\mathcal{D}$:

Data :

- functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$
- isomorphism $\varphi_0\colon\ \mathbf{1}_\mathcal{D} \xrightarrow{\cong} F(\mathbf{1}_\mathcal{C})$ in $\mathcal{D}$
- natural isomorphism $\varphi_2\colon\ \otimes_\mathcal{D} \circ (F \times F) \xrightarrow{\cong} F \circ \otimes_\mathcal{C}$

Axioms :

- compatibility with associativity constraint

$$\begin{tikzcd}
\big(F(x) \otimes F(y)\big) \otimes F(z) \arrow[r, "a_{F(x),F(y),F(z)}"'] \arrow[d, "\varphi_2(x,y)\otimes\mathrm{id}_{F(z)}"'] & F(x) \otimes \big(F(y) \otimes F(z)\big) \arrow[d, "\mathrm{id}_{F(u)}\otimes\varphi_2(y,z)"] \\
F(z \otimes y) \otimes F(z) \arrow[d, "\varphi_2(x\otimes y,z)"'] & F(x) \otimes F(y \otimes z) \arrow[d, "\varphi_2(x\otimes y,z)"] \\
F((x \otimes y) \otimes z) \arrow[r, "F(a_{x,y,z})"] & F(x \otimes (y \otimes z))
\end{tikzcd}$$

DEFINITION — **Monoidal functor**

☞ monoidal functor $\boxed{(F, \varphi_0, \varphi_2)}$ from $\mathcal{C}$ to $\mathcal{D}$:

Data:

- functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$
- isomorphism $\varphi_0\colon\ \mathbf{1}_{\mathcal{D}} \xrightarrow{\cong} F(\mathbf{1}_{\mathcal{C}})$ in $\mathcal{D}$
- natural isomorphism $\varphi_2\colon\ \otimes_{\mathcal{D}} \circ (F \times F) \xrightarrow{\cong} F \circ \otimes_{\mathcal{C}}$

Axioms:

- compatibility with associativity constraint
- compatibility with left unit constraint

$$\begin{tikzcd} & \mathbf{1}_{\mathcal{D}} \otimes F(x) \arrow[dl, "\varphi_0 \otimes \mathrm{id}_{F(x)}"'] \arrow[dr, "l_{F(x)}"] & \\ F(\mathbf{1}_{\mathcal{C}}) \otimes F(x) \arrow[r, "\varphi_2(\mathrm{id}_{\mathcal{C}}, x)"] & F(\mathbf{1}_{\mathcal{C}} \otimes x) \arrow[r, "F(l_x)"] & F(x) \end{tikzcd}$$

- compatibility with right unit constraint

Monoidal functors

COMMENTS

- ☞ suppress labels $\mathcal{C}$, $\mathcal{D}$ when notation too clumsy otherwise
- ☞ φ_2 gives in particular isomorphisms

$$(\varphi_2)_{(x,y)}:\; F(x) \otimes_{\mathcal{D}} F(y) \xrightarrow{\cong} F(x \otimes_{\mathcal{C}} y) \quad \text{for } x\,, y \in \mathcal{C}$$

- ☞ weakened versions : lax monoidal functor / oplax monoidal functor : morphisms φ_0 , φ_2 only in one direction

Monoidal functors

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- weakened versions : lax monoidal functor / oplax monoidal functor : morphisms $\varphi_0\,,\varphi_2$ only in one direction

DEFINITION — **Monoidal natural transformation**

- monoidal natural transformation $\eta\colon\; (F,\varphi_0,\varphi_2) \longrightarrow (G,\psi_0,\psi_2)$

  between monoidal functors : $\eta\colon\; F\to G$

  s.t.

$$\begin{tikzcd} & \mathbf{1}_D \arrow[dl, "F(\mathbf{1}_\mathcal{C})"'] \arrow[dr, "G(\mathbf{1}_\mathcal{C})"] & \\ F(\mathbf{1}_\mathcal{C}) \arrow[rr, "\eta_\mathbf{1}"] & & G(\mathbf{1}_\mathcal{C}) \end{tikzcd}$$

and

$$\begin{tikzcd} F(x)\otimes F(y) \arrow[r, "\varphi_2(x,y)"] \arrow[d, "\eta_x\otimes\eta_y"] & F(x\otimes y) \arrow[d, "\eta_{x\otimes y}"] \\ G(x)\otimes G(y) \arrow[r, "\psi_2(x,y)"] & G(x\otimes y) \end{tikzcd}$$

for all $x,y\in\mathcal{C}$

Algebras

☞ recall : categories

EXAMPLES

☞ category $\boxed{\mathcal{V}ect}$

- objects = vector spaces
- morphisms = linear maps

☞ category $\boxed{\mathcal{v}ect}$

- objects = finite-dimensional vector spaces
- morphisms = linear maps

☞ indeed : monoidal categories

EXAMPLES

☞ monoidal category $\boxed{\mathcal{V}ect}$

- objects $=$ vector spaces
- morphisms $=$ linear maps
- tensor product $\otimes_{\Bbbk}$ of vector spaces
- tensor unit $\mathbf{1} = \Bbbk$

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- objects $=$ finite-dimensional vector spaces
- morphisms $=$ linear maps
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- indeed : monoidal categories

EXAMPLES

- monoidal category $\boxed{\mathcal{V}ect}$
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- monoidal category $\boxed{vect}$
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  - tensor product $\otimes_{\Bbbk}$ of vector spaces
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- unital associative algebra

  $=$ vector space with multiplication and unit element

☞ indeed : monoidal categories

EXAMPLES

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☞ unital associative algebra

$=$ object of $\mathcal{V}ect$ / $\mathcal{v}ect$ endowed with " multiplication and unit "

☞ indeed : monoidal categories

EXAMPLES

☞ monoidal category $\boxed{\mathcal{V}ect}$

- objects = vector spaces
- morphisms = linear maps
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☞ unital associative algebra

= object of $\mathcal{V}ect$ / vect endowed with multiplication and unit morphisms

Algebras

☞ interpret

- multiplication $\equiv$ bilinear map $A \times A \longrightarrow A$ $\rightsquigarrow$ linear map $m\colon A \otimes_{\Bbbk} A \longrightarrow A$
- unit element $1_A \in A$ $\rightsquigarrow$ linear map $\eta\colon \Bbbk \longrightarrow A$ $\qquad \eta(c) = c\,1$

☞ interpret

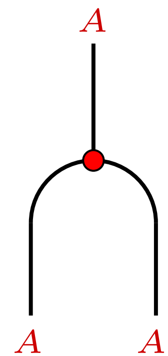
- multiplication $\equiv$ bilinear map $A \times A \longrightarrow A$ $\rightsquigarrow$ linear map $m\colon A \otimes_{\Bbbk} A \longrightarrow A$
- unit element $1_A \in A$ $\rightsquigarrow$ linear map $\eta\colon \Bbbk \longrightarrow A$ $\qquad \eta(c) = c\,1$

$\Longrightarrow$ $\Bbbk$-algebra $=$ algebra object $\boldsymbol{(A, m, \eta)} \in \boldsymbol{\mathcal{V}ect}$

Algebras

☞ associative algebra $=$ object A $+$ morphism

$$m: \quad A \otimes A \longrightarrow A$$



Algebras

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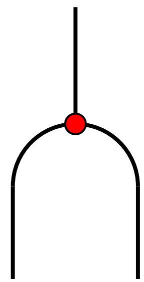
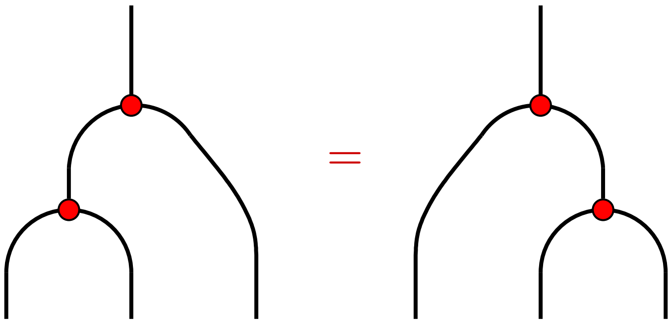


such that

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$$



- associative algebra $=$ object A $+$ morphism

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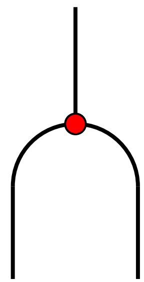
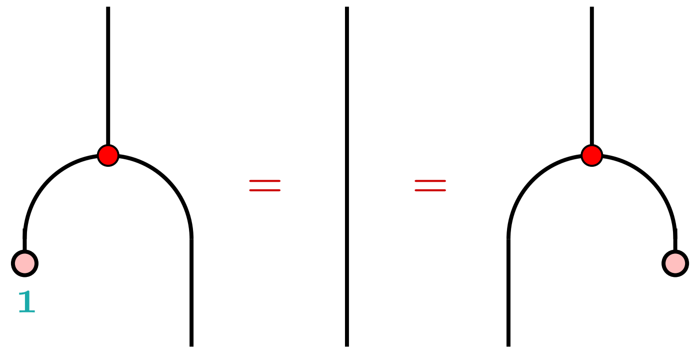


- unital algebra (A, m, η) :

$$\eta: \quad \mathbf{1} \to A$$

$$m \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes \eta)$$



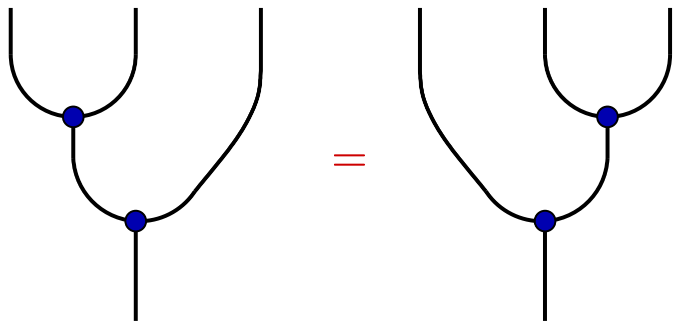
- natural setting : monoidal categories $(\mathcal{C}, \otimes, \mathbf{1})$

Coalgebras

☞ coassociative coalgebra (C,Δ) :

$$\Delta : \quad A \to A \otimes A$$

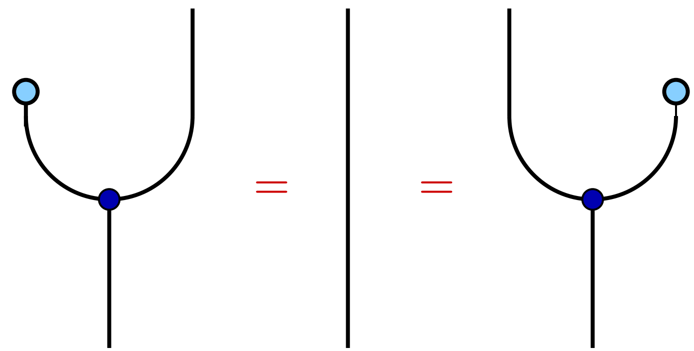
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$



☞ co-unital coalgebra (C,Δ,ε) :

$$\varepsilon : \quad A \to \mathbf{1}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$$



☞ natural setting : monoidal categories

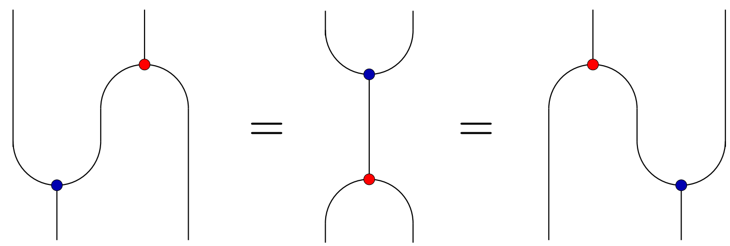
Frobenius algebras

☞ Algebra :

☞ Coalgebra :

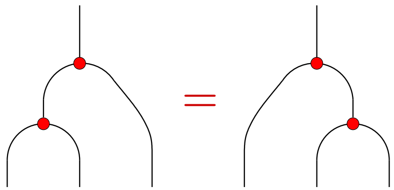
☞ Frobenius algebra : algebra and coalgebra

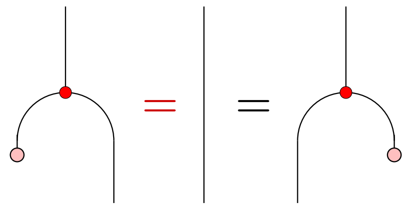
and coproduct a bimodule morphism :



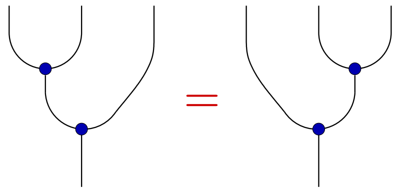
Frobenius algebras

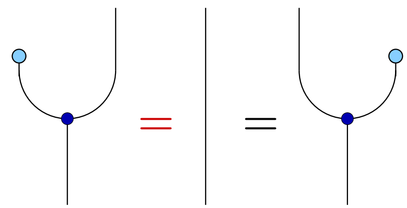
- ☞ Algebra :



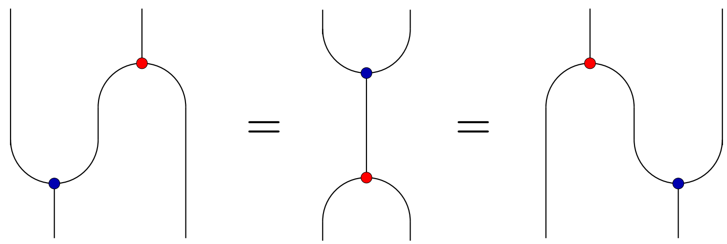


- ☞ Coalgebra :

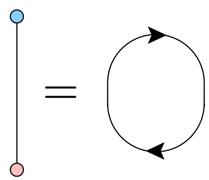




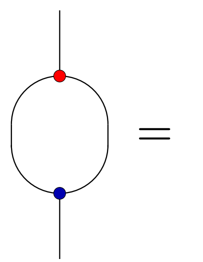
- ☞ Frobenius algebra : algebra and coalgebra
  and coproduct a bimodule morphism :



- ☞ special Frobenius algebra :



$\neq 0$



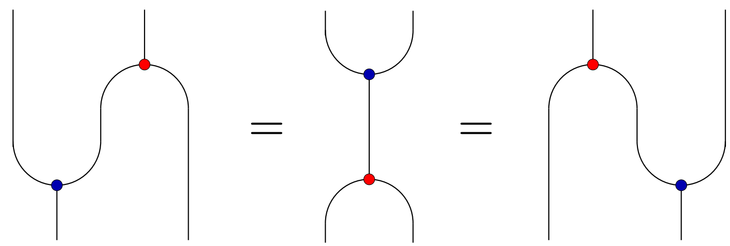
Frobenius algebras

- ☞ Algebra :

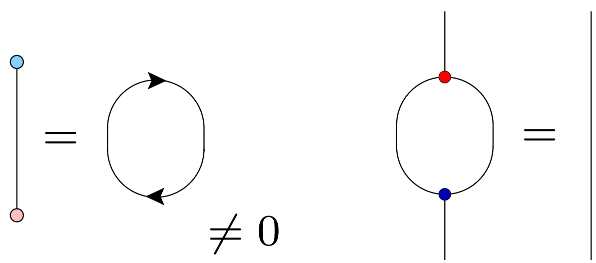
- ☞ Coalgebra :

- ☞ Frobenius algebra : algebra and coalgebra

  and coproduct a bimodule morphism :



- ☞ special Frobenius algebra :

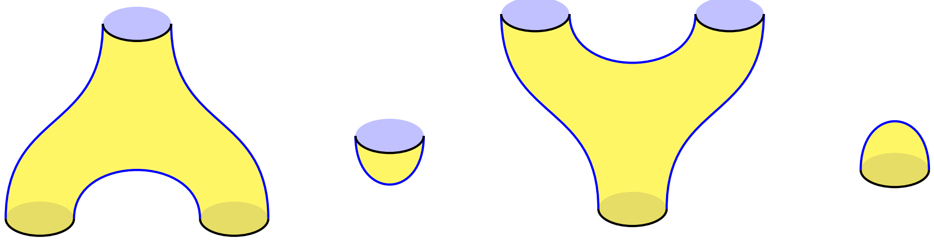


- ☞ natural setting : monoidal categories

Solution exercises

Solution exercise 1

☞ the circle as a symmetric Frobenius algebra in $\mathcal{C}obord_{2,1}$:



m η Δ ε

Solution exercise 2

☞ Lesson : graphical calculus allows one to

- memorize definitions and results
- visualize proofs

Lemma [I:5.2]: For any symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ the morphism

$$P := (\mathrm{id} \otimes d) \circ (c^{-1}_{A,A^\vee} \otimes \mathrm{id}) \circ (\mathrm{id}^\vee \otimes m \otimes \mathrm{id}) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent

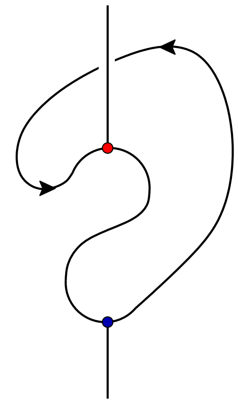
Proof:

$$\begin{aligned}
P \circ P &= (\mathrm{id}\otimes d) \circ (c^{-1}_{A,A^\vee} \otimes\mathrm{id}) \circ (\mathrm{id}^\vee \otimes m\otimes\mathrm{id}) \circ (\tilde{b}\otimes\Delta) \circ (\mathrm{id}\otimes d) \circ (c^{-1}_{A,A^\vee} \otimes\mathrm{id}) \circ (\mathrm{id}^\vee \otimes m\otimes\mathrm{id}) \circ (\tilde{b}\otimes\Delta) \\
&= \cdots \\
&= (\mathrm{id}\otimes d) \circ (c^{-1}_{A,A^\vee} \otimes\mathrm{id}) \circ (\mathrm{id}^\vee \otimes m\otimes\mathrm{id}) \circ (\mathrm{id}^\vee \otimes m\otimes\Delta) \circ (\tilde{b}\otimes\mathrm{id}\otimes\mathrm{id}\otimes d) \circ (c^{-1}_{A,A^\vee} \otimes c^{-1}_{A,A^\vee} \otimes\mathrm{id}) \circ (\tilde{b}\otimes\Delta) \\
&= \cdots \\
&= (\mathrm{id}\otimes\tilde{d}) \circ (c^{-1}_{A,A} \otimes d\otimes\mathrm{id}) \circ (\mathrm{id}\otimes c^{-1}_{A,A^\vee} \otimes\mathrm{id}\otimes\mathrm{id}^\vee) \circ (\mathrm{id}\otimes\mathrm{id}^\vee \otimes m\otimes m\otimes\mathrm{id}^\vee) \circ (\mathrm{id}\otimes\tilde{b}\otimes\Delta\otimes d\otimes b) \\
&\qquad \circ (c^{-1}_{A,A^\vee} \otimes c^{-1}_{A,A^\vee} \otimes\mathrm{id}) \circ (\tilde{b}\otimes\Delta) \\
&= \cdots \\
&= (\mathrm{id}\otimes d) \circ (c^{-1}_{A,A^\vee} \otimes\mathrm{id}) \circ (\mathrm{id}^\vee \otimes m\otimes\mathrm{id}) \circ (\tilde{b}\otimes\mathrm{id}\otimes m) \circ (\Delta\otimes\mathrm{id}) \circ \Delta \\
&= \cdots \\
&= \cdots \\
&= P
\end{aligned}$$

Solution exercise 2

Lemma [I:5.2]: For any symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ the morphism

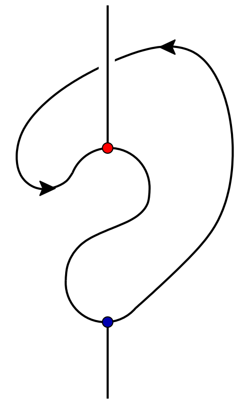
$$P := (\mathrm{id} \otimes d) \circ (c^{-1}_{A,A^\vee} \otimes \mathrm{id}) \circ (\mathrm{id}^\vee \otimes m \otimes \mathrm{id}) \circ (\tilde{b} \otimes \Delta) \quad =$$



is an idempotent

Lemma [I:5.2]: For any symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ the morphism

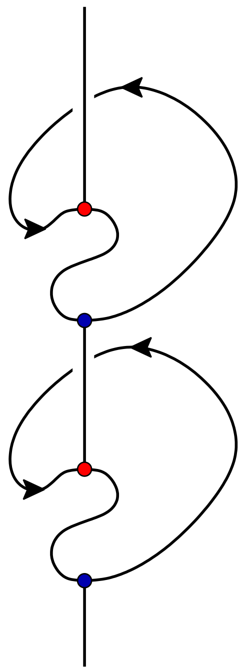
$$P := (\mathrm{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \mathrm{id}) \circ (\mathrm{id}^\vee \otimes m \otimes \mathrm{id}) \circ (\tilde{b} \otimes \Delta) \quad =$$



is an idempotent

Proof :

$$P \circ P \quad =$$



<u>Lemma</u> [I: 5.2] : For any symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ the morphism

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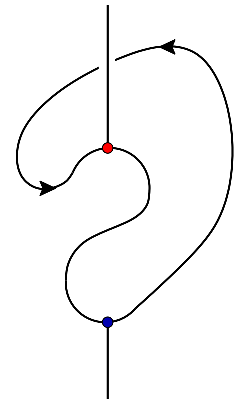

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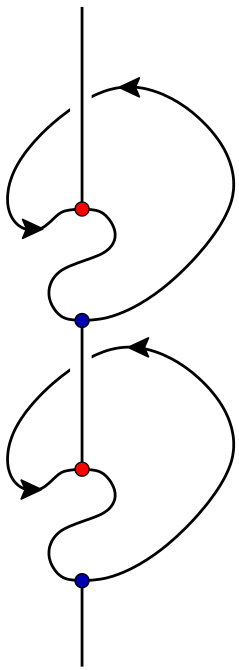
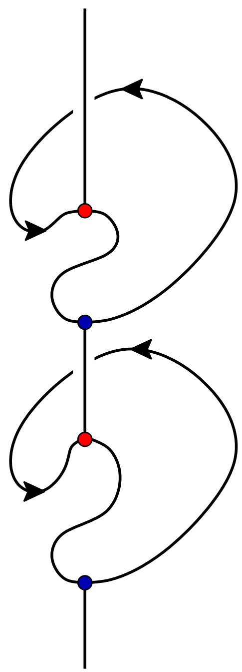
$P \circ P \quad =$  $=$ 

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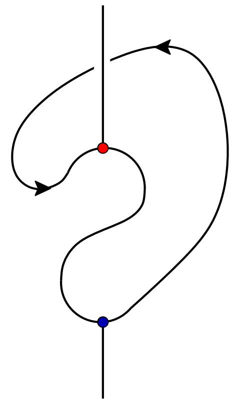
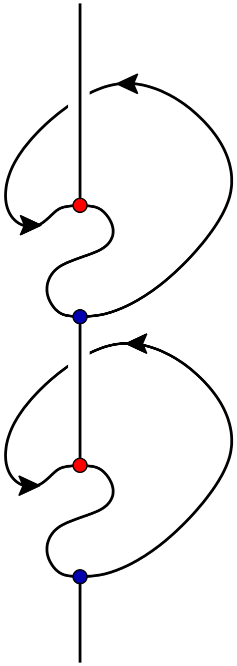
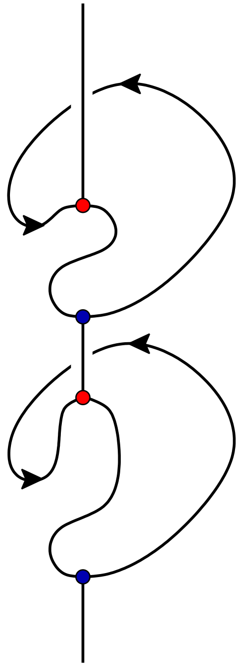
$P \;:=\; (\mathrm{id} \otimes d) \circ (c^{-1}_{A,A^\vee} \otimes \mathrm{id}) \circ (\mathrm{id}^\vee \otimes m \otimes \mathrm{id}) \circ (\tilde{b} \otimes \Delta) \;=$ 

is an idempotent

Proof :

$P \circ P \;=$  $=$ 

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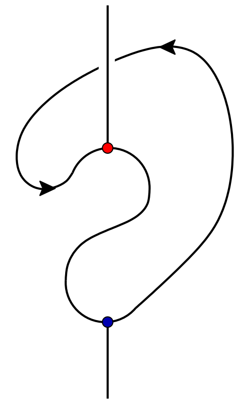

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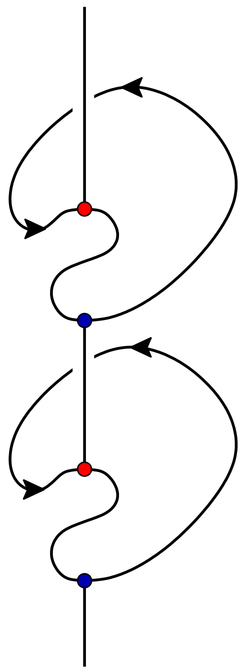
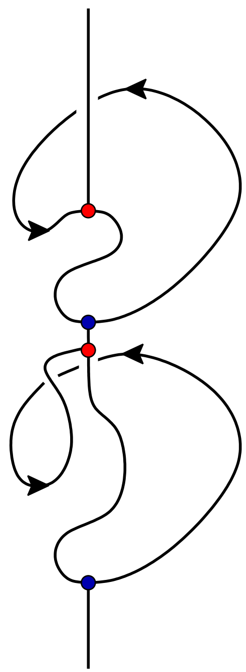


$$=$$



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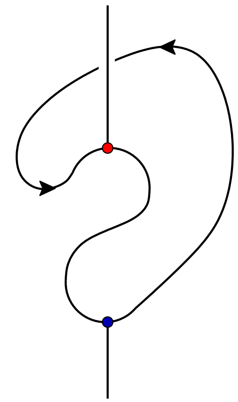

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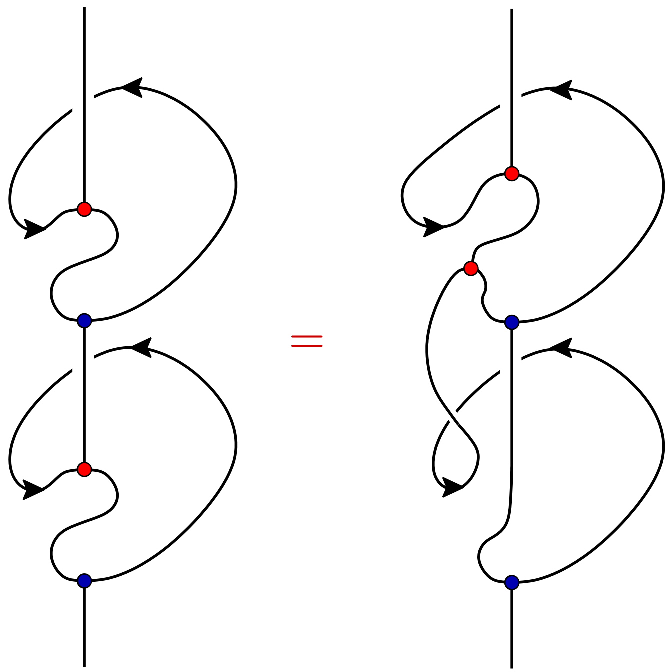

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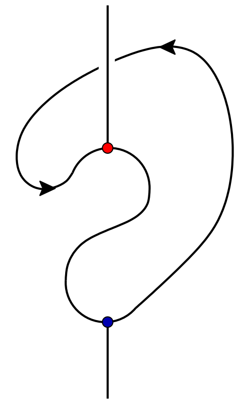

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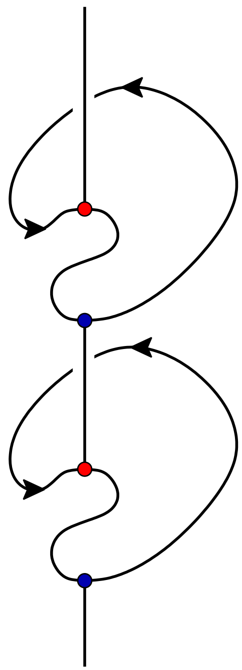
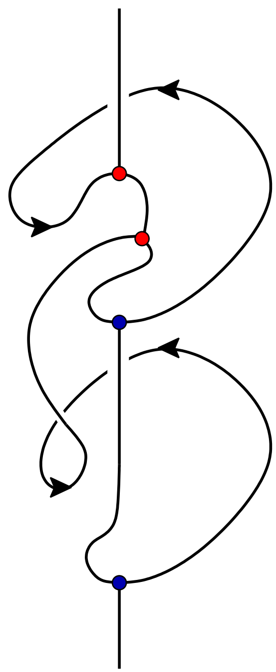


$=$



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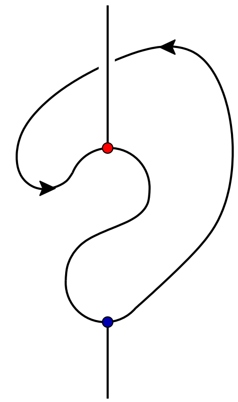

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Proof :

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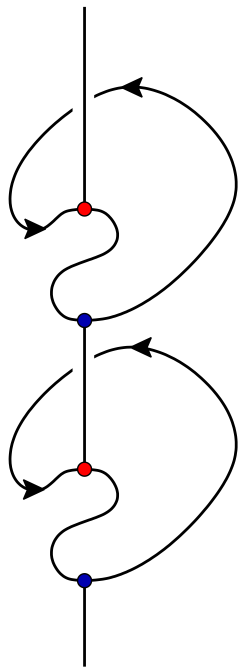
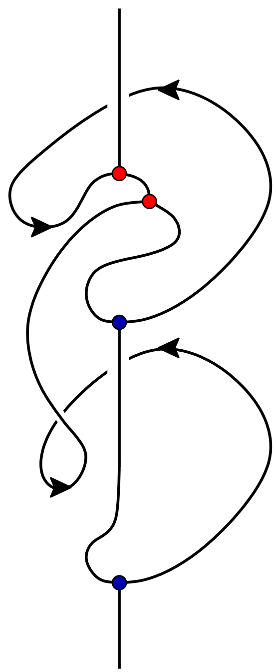
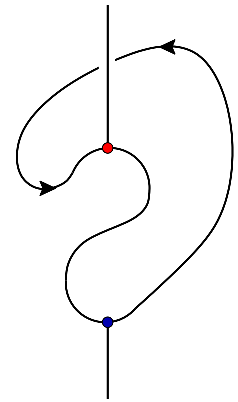


$$=$$



Lemma [I:5.2]: For any symmetric special Frobenius algebra A in a ribbon category $\mathcal{C}$ the morphism

$$P := (\mathrm{id} \otimes d) \circ (c^{-1}_{A,A^\vee} \otimes \mathrm{id}) \circ (\mathrm{id}^\vee \otimes m \otimes \mathrm{id}) \circ (\tilde{b} \otimes \Delta) \quad =$$



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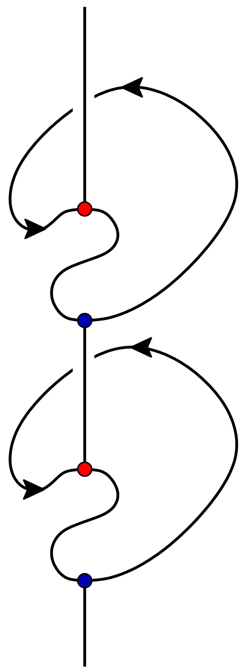

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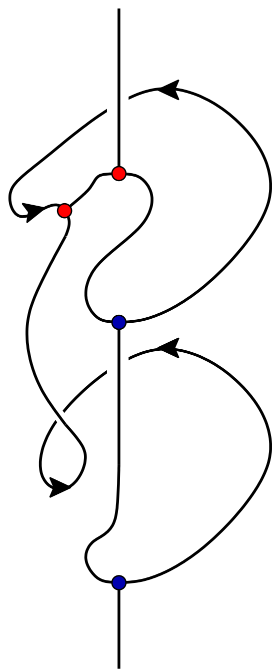

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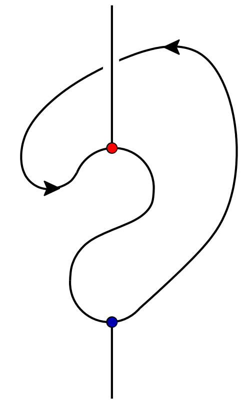

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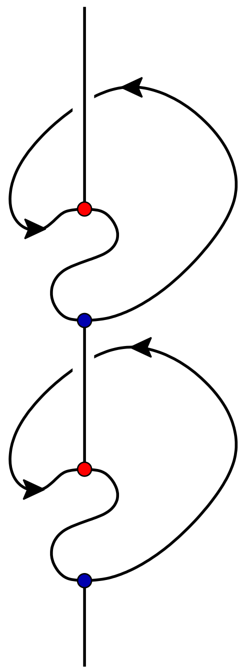

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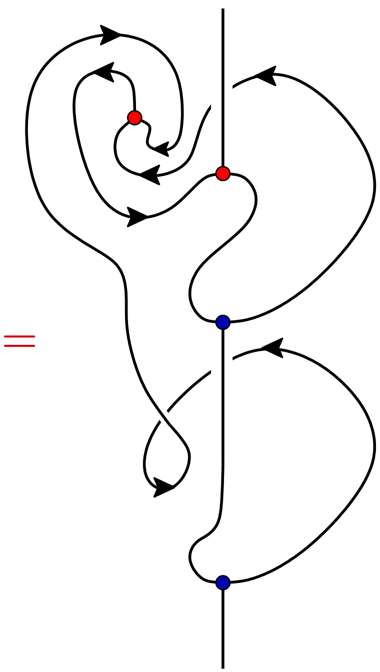

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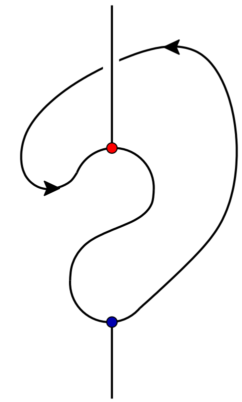

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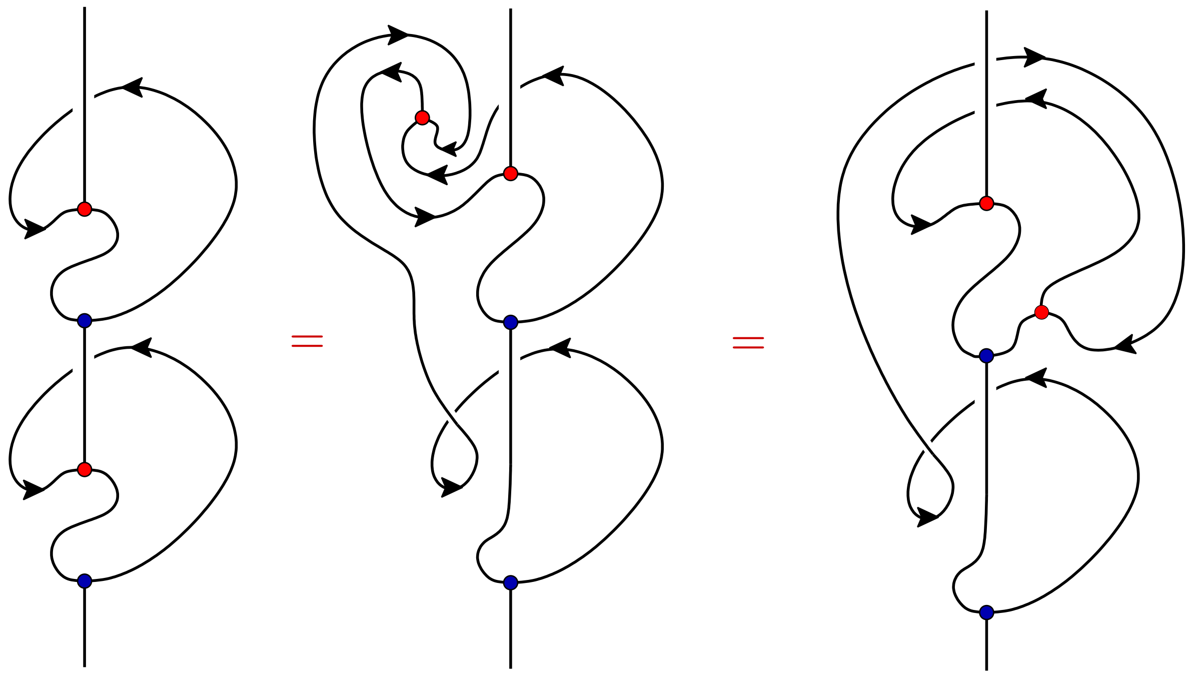

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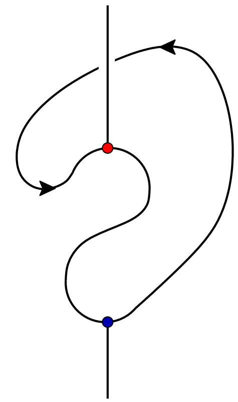

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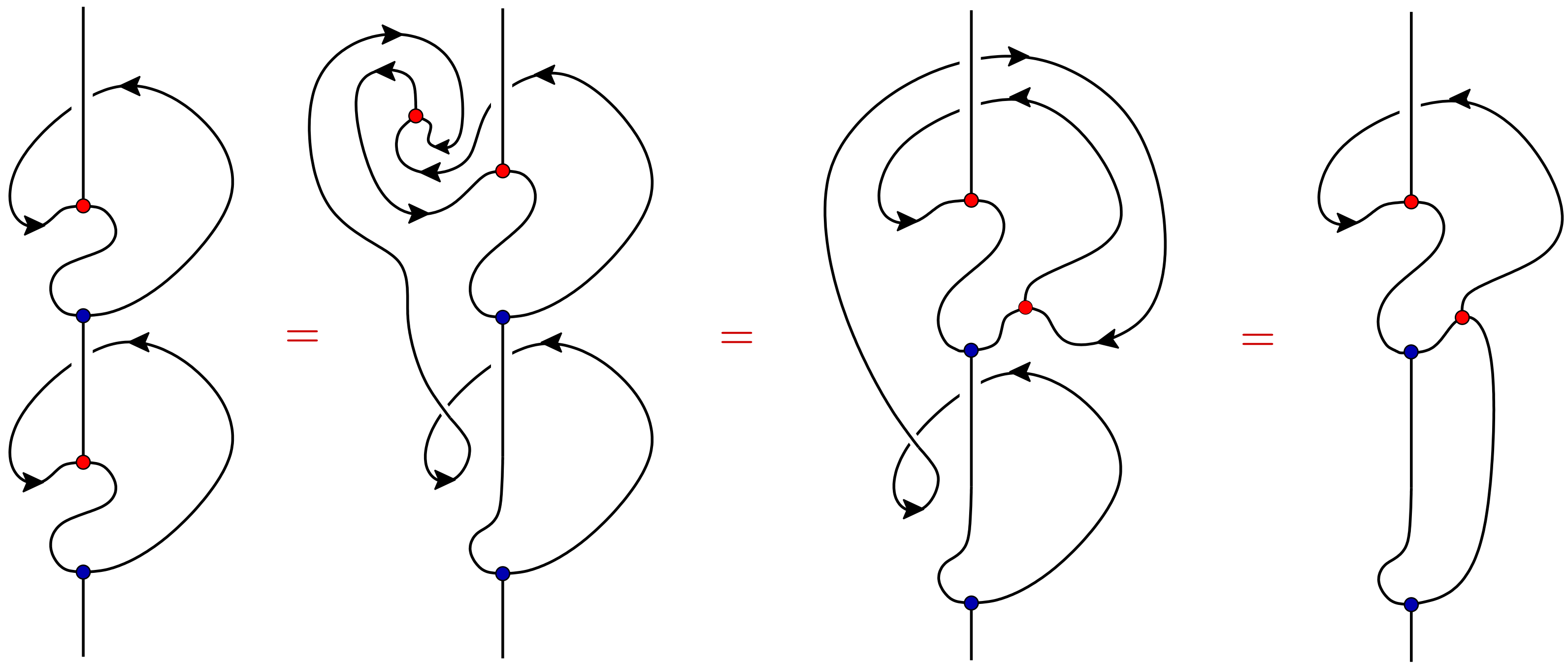

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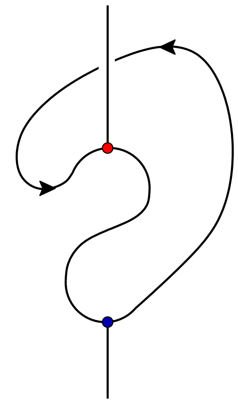

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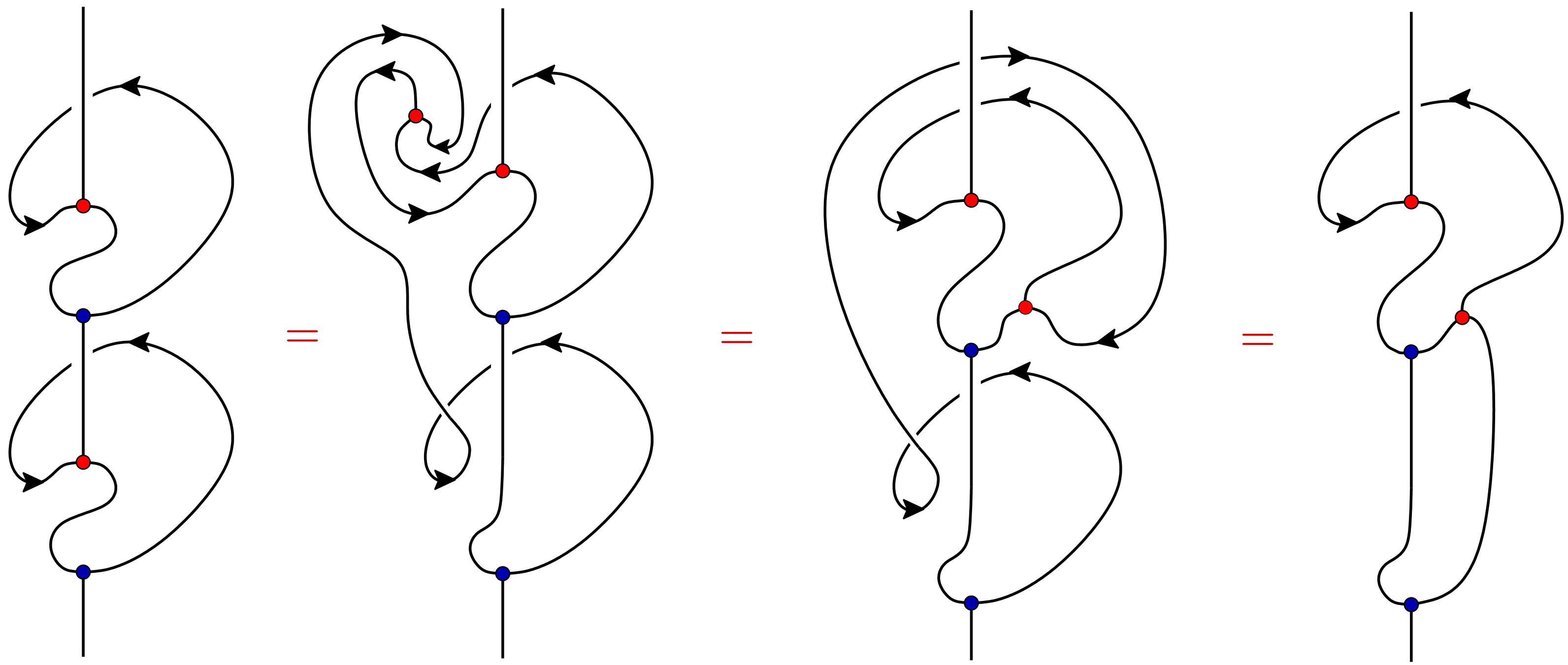

$= \cdots = P$

Extended exercise

Lemma [I: 5.2]: For any object U of $\mathcal{C}$ the morphism

$$P_U \;:=\; (\mathrm{id}_A \otimes d \otimes \mathrm{id}_U) \circ (c^{-1}_{A,A^\vee} \otimes c_{U,A}) \circ (\mathrm{id}^\vee \otimes m \otimes \mathrm{id}_A \otimes \mathrm{id}_U) \circ (\tilde{b} \otimes c_{A,U}) \circ (\Delta \otimes \mathrm{id}_U)$$

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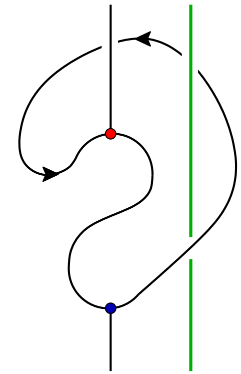

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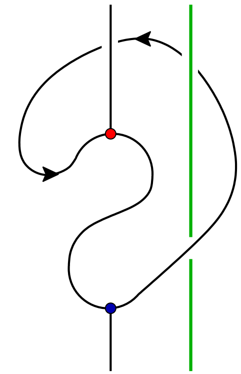

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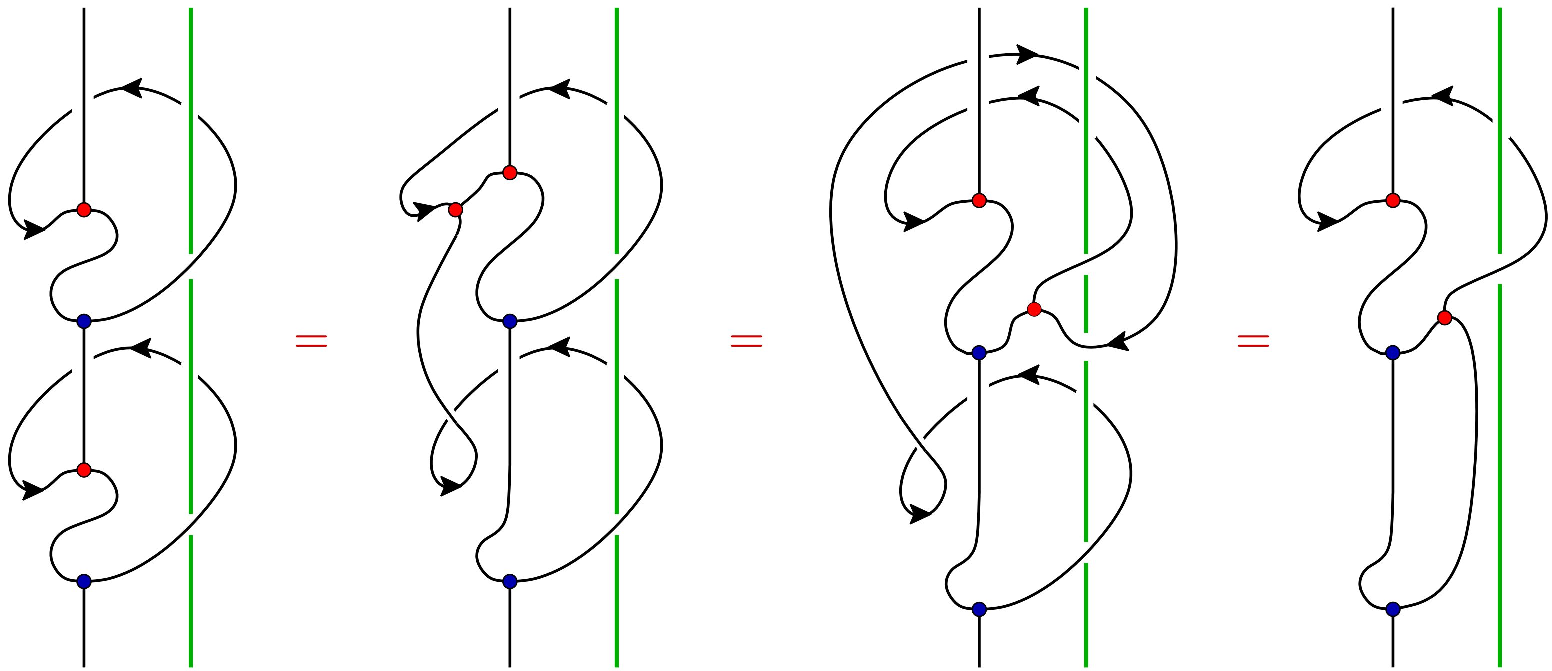
Proof:

$$P_U \circ P_U =$$



$$= \cdots = P_U$$

Braided categories

Rigid monoidal categories

Fusion categories

Ribbon categories

Ribbon categories

ribbons@hyatt

Modular tensor categories

Drinfeld center

Drinfeld center

DESIRABLE **Braiding**

- ☞ many interesting monoidal categories non-braided
- ☞ would like to promote them to braided categories

Drinfeld center

DESIRABLE — **Braiding**

- many interesting monoidal categories non-braided
- would like to promote them to braided categories

DEFINITION — **Drinfeld center**

- $\boxed{\text{Drinfeld center } \mathcal{Z}(\mathcal{C})}$ of monoidal category $\mathcal{C}$ $:=$ category with
  - objects $=$ pairs (x,γ) with $x \in \mathcal{C}$ and
    $\gamma = (\gamma_y)_{y\in\mathcal{C}}$ natural isomorphism $\gamma_y : x \otimes y \longrightarrow y \otimes x$
    satisfying one half of the properties of a braiding ("half-braiding")
  - morphisms $=$ compatible morphisms of $\mathcal{C}$

PROPOSITION — **Drinfeld center**

☞ $\mathcal{Z}(\mathcal{C})$ monoidal with same associativity constraint as $\mathcal{C}$

☞ forgetting the half-braiding furnishes monoidal functor $U: \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{C}$

Drinfeld center

PROPOSITION — **Drinfeld center**

- ☞ $\mathcal{Z}(\mathcal{C})$ monoidal with same associativity constraint as $\mathcal{C}$
- ☞ $\mathcal{Z}(\mathcal{C})$ braided with braiding $(\gamma^{\mathcal{Z}(\mathcal{C})})_{(a,\gamma),(a',\gamma')} = \gamma_{a'}$

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- $\mathcal{C}$ endowed with (right / left) duality

  $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ endowed with duality with

  $(x,\gamma)^\vee = (x^\vee, \gamma^-)$ and same evaluation and coevaluation

  with $(\gamma^-)_b$

  $$:= (\mathrm{ev}_a \otimes \mathrm{id}_b \otimes \mathrm{id}_{a^\vee}) \circ (\mathrm{id}_{a^\vee} \otimes \gamma_b^{-1} \otimes \mathrm{id}_{a^\vee}) \circ (\mathrm{id}_{a^\vee} \otimes \mathrm{id}_b \otimes \mathrm{coev}_a)$$

Drinfeld center

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- $\mathcal{C}$ ribbon $\Rightarrow$ natural ribbon structure on $\mathcal{Z}(\mathcal{C})$

Drinfeld center

DEFINITION — **Revese category**

☞ $\boxed{\text{Reverse category } \overline{\mathcal{C}}}$ of braided / ribbon category $\mathcal{C}$

$:=$ same monoidal category with opposite braiding / and inverse twist

INFORMAL DEFINITION — **Deligne product**

☞ $\boxed{\text{Deligne product } \mathcal{C} \boxtimes \mathcal{D}}$ of finite abelian categories $\mathcal{C}$ and $\mathcal{D}$

- sensible generalization of tensor product
- defined through a universal property

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right exact functors $\mathcal{C} \boxtimes \mathcal{D} \longrightarrow \mathcal{E}$

equivalent to functors $\mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ right exact in each argument

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DEFINITION — **Enveloping category**

☞ enveloping category of $\mathcal{C}$ $:=$ $\boxed{\overline{\mathcal{C}} \boxtimes \mathcal{C}}$

PROPOSITION — **Enveloping category vs center**

For any finite ribbon category $\mathcal{C}$ have canonical braided monoidal functor

$$\boxed{G_{\mathcal{C}}:\ \overline{\mathcal{C}}\boxtimes\mathcal{C}\longrightarrow\mathcal{Z}(\mathcal{C})}$$

☞ as a functor : $\overline{\mathcal{C}}\boxtimes\mathcal{C}\ni\ \overline{u}\boxtimes v\ \longmapsto\ \big(\overline{u}\otimes v\,,(\gamma^{\mathcal{Z}(\mathcal{C})})_{\overline{u}\otimes v}\big)$

- $\overline{u}\boxtimes v\ \in\mathcal{C}$
- $(\gamma^{\mathcal{Z}(\mathcal{C})})_{\overline{u}\otimes v;c}=\big((\gamma^{\mathcal{C}})^{-1}_{c,\overline{u}}\otimes\mathrm{id}_v\big)\circ\big(\mathrm{id}_{\overline{u}}\otimes(\gamma^{\mathcal{C}})_{v,c}\big)$

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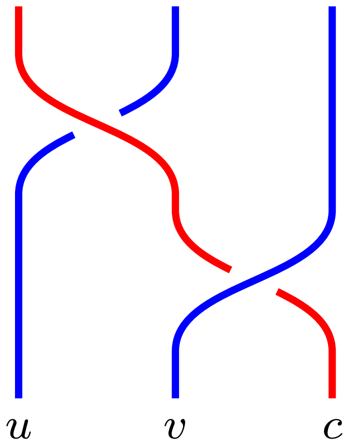
- pictorially :

$(\gamma^{\mathcal{Z}(\mathcal{C})})_{\overline{u}\otimes v;c} \ =$



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THEOREM — **Modularity**

Braided fusion category $\mathcal{C}$ modular $\Longleftrightarrow$ $G_{\mathcal{C}}$ braided equivalence

Bicategories

Bicategories

☞ recall : category $\mathcal{C}obord_{d,d-1}$

with $d{-}1$-manifolds as objects and d-manifolds as morphisms

☞ can cut a d-manifold along a $d{-}1$-submanifold $\rightsquigarrow$ locality property

Bicategories

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DESIRABLE **Locality**

☞ realize locality in strong sense :

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DESIRABLE — **Locality**

☞ realize locality in strong sense :

allow for cutting also $d-1$-submanifold along $d-2$-submanifold etc.

☞ this way get multi-layered structure :

- d-manifolds
- $d-1$-submanifolds
- $d-2$-submanifolds
- ......

☞ other example of three-layered structure : collection of all categories

- categories
- functors
- natural transformations

Bicategories

DEFINITION — **Bicategory**

☞ $\boxed{\text{bicategory } \mathcal{B}}$:

Data :

- class $\mathrm{Obj}(\mathcal{B})$ of objects
- category $\mathcal{H}om(A,B)$ for each pair $A,B \in \mathcal{B}$
- functor $c_{A,B,C}\colon \mathcal{H}om(B,C) \times \mathcal{H}om(A,B) \to \mathcal{H}om(A,C)$

  for each triple $A,B,C \in \mathcal{B}$

  mapping pairs of objects $(g,f) \mapsto g \circ f \quad \in \mathcal{H}om(A,C)$

  and pairs of morphisms $(\beta,\alpha) \mapsto \beta \circ \alpha \quad \in \mathrm{Hom}_{\mathcal{H}om(A,C)}(-,-)$
- functor $I_A\colon */\!/\,\mathrm{id}_* \longrightarrow \mathcal{H}om(A,A)$ for each $A \in \mathcal{B}$
- $\ldots$

Axioms :

- $\ldots$

Bicategories

DEFINITION — Bicategory

☞ $\boxed{\text{bicategory}}$:

Data : ⇝ class $\mathrm{Obj}()$ / categories $\mathcal{H}om(A,B)$

⇝ composition functors $c_{A,B,C}$ / identity functors I_A

⇝ natural isomorphisms a, r, l of functors expressing

– associativity :

$$\begin{tikzcd}
& \mathcal{H}om(C,D)\times\mathcal{H}om(B,C)\times\mathcal{H}om(A,B) \arrow[dl, "c_{B,C,D}\times\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}\times c_{A,B,C}"] & \\
\mathcal{H}om(B,D)\times\mathcal{H}om(A,B) \arrow[rr, Rightarrow, "a_{A,B,C,D}"] \arrow[dr, "c_{A,B,D}"'] & & \mathcal{H}om(C,D)\times\mathcal{H}om(A,C) \arrow[dl, "c_{A,C,D}"] \\
& \mathcal{H}om(A,D) &
\end{tikzcd}$$

– unitality :

$$\begin{tikzcd}
\mathcal{H}om(A,B)\times *//\mathrm{id}_* \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{Id}\times I_A"'] & & \mathcal{H}om(A,B) \\
& \mathcal{H}om(A,B)\times\mathcal{H}om(A,A) \arrow[ur, "a_{A,A.B}"'] \arrow[u, phantom, "\Uparrow r_{A,B}"] &
\end{tikzcd}$$

and ...

Bicategories

DEFINITION — Bicategory

☞ $\boxed{\text{bicategory } \mathcal{B}}$:

Data : class $\mathrm{Obj}()$ / categories $\mathcal{H}om(A,B)$

- composition functors $c_{A,B,C}$ / identity functors I_A
- natural isomorphisms a, r, l

Axioms :

- pentagon :

$$\begin{tikzcd}[column sep=small]
& ((kh)g)f \arrow[dl, "\alpha_{k,h,g}\otimes\mathrm{id}"'] \arrow[dr, "\alpha_{kh,g,f}"] & \\
(k(hg)f) \arrow[d, "\alpha_{k,hg,f}"'] & & (kh)\otimes(gf) \arrow[d, "\alpha_{k,h,gf}"] \\
k(hg)f \arrow[rr, "\mathrm{id}\otimes\alpha_{h,g,f}"] & & k(h(gf))
\end{tikzcd}$$

- triangle :

$$\begin{tikzcd}
(g\,\mathrm{id})\,f \arrow[rr, "\alpha_{g,\mathrm{id},f}"] \arrow[dr, "r_g*\mathrm{id}"'] & & g\,(\mathrm{id}\,f) \arrow[dl, "\mathrm{id}*l_f"] \\
& g\,f &
\end{tikzcd}$$

COMMENTS

- terminology : objects / 1-morphisms / 2-morphisms
  or 0-cells / 1-cells / 2-cells
- common notation : $*/\!\!/\mathrm{id}_* =: \mathbf{1}$
- common notation : "$\Longrightarrow$" for 2-cells
- natural isomorphisms give specific 2-cell components

$$a_{h,g,f} \colon (hg)f \xrightarrow{\cong} h(gf) \qquad l_f \colon \mathbf{1}_B \circ f \xrightarrow{\cong} f \qquad r_f \colon f \circ \mathbf{1}_A \xrightarrow{\cong} f$$

Bicategories

EXAMPLES

- $\boxed{\mathcal{C}at}$
  - objects $=$ small categories
  - 1-morphisms $=$ functors
  - 2-morphisms $=$ natural transformations
- bicategory with a single object $*$
  - specified by the category $\mathcal{E}nd(*)$ (monoidal)
  - composition of 1-morphisms $=$ tensor product of $\mathcal{E}nd(*)$
- $\boxed{\mathcal{A}lg}$
  - objects $=$ $\mathbb{k}$- algebras
  - $\mathcal{H}om(A, A')$ $=$ category A-Bimod-A' of A-A'-bimodukes
  - composition of 1-morphisms $=$ tensor product of bimodules

special case :

DEFINITION — **2-category**

☞ 2-category $:=$ strict bicategory

i.e. $a\,,\,l\,,\,r$ identities

EXAMPLE

☞ $\mathcal{C}at$

☞ two different compositions of 2-cells :

- vertical composition $=$ composition of morphisms in $\mathcal{H}om(-,-)$:

$$\begin{tikzcd} C \arrow[rr, bend left=60, "F"{name=U}] \arrow[rr, "F'"{name=M, description}] \arrow[rr, bend right=60, "F''"'{name=L}] & & D \arrow[from=U, to=M, Rightarrow, "\varphi"] \arrow[from=M, to=L, Rightarrow, "\psi"] \end{tikzcd} \quad\longmapsto\quad \begin{tikzcd} C \arrow[rr, bend left=60, "F"{name=U}] \arrow[rr, bend right=60, "F''"'{name=L}] & & D \arrow[from=U, to=L, Rightarrow, "\psi\circ\varphi"] \end{tikzcd}$$

- horizontal composition :

$$\begin{tikzcd} C \arrow[r, bend left=60, "F"{name=U1}] \arrow[r, bend right=60, "F'"'{name=L1}] & D \arrow[r, bend left=60, "G"{name=U2}] \arrow[r, bend right=60, "G'"'{name=L2}] & E \arrow[from=U1, to=L1, Rightarrow, "\varphi"] \arrow[from=U2, to=L2, Rightarrow, "\psi"] \end{tikzcd} \quad\longmapsto\quad \begin{tikzcd} C \arrow[rr, bend left=60, "G\circ F"{name=U}] \arrow[rr, bend right=60, "G'\circ F'"'{name=L}] & & E \arrow[from=U, to=L, Rightarrow, "\psi*\varphi"] \end{tikzcd}$$

☞ two different compositions of 2-cells :

- vertical composition $=$ composition of morphisms in $\mathcal{H}om(-,-)$:

$$\begin{tikzcd} C \arrow[rr, bend left=60, "F"] \arrow[rr, "F'"] \arrow[rr, bend right=60, "F''"'] \arrow[rr, phantom, shift left=3, "\Downarrow\varphi"] \arrow[rr, phantom, shift right=3, "\Downarrow\psi"] & & D \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} C \arrow[rr, bend left=60, "F"] \arrow[rr, bend right=60, "F''"'] \arrow[rr, phantom, "\Downarrow\psi\circ\varphi"] & & D \end{tikzcd}$$

- horizontal composition :

$$\begin{tikzcd} C \arrow[rr, bend left=60, "F"] \arrow[rr, bend right=60, "F'"'] \arrow[rr, phantom, "\Downarrow\varphi"] & & D \arrow[rr, bend left=60, "G"] \arrow[rr, bend right=60, "G'"'] \arrow[rr, phantom, "\Downarrow\psi"] & & E \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} C \arrow[rr, bend left=60, "G\circ F"] \arrow[rr, bend right=60, "G'\circ F'"'] \arrow[rr, phantom, "\Downarrow\psi*\varphi"] & & E \end{tikzcd}$$

- horizontal composition associative up to natural isomorphism

- interchange law : $\boxed{(\alpha*\beta)\circ(\gamma*\delta) = (\alpha\circ\gamma)\ *\ (\beta\circ\delta)}$

Defects

Defects

- ☞ Goal : 2-d defects in 3-d TFT as models for line defects in topological orders
- ☞ Warmup : lne defects in 2-d RCFT
- ☞ Then : topological defects in 3-d TFT of Reshetikhin-Turaev type

Defects in QFT

☞ Codimension-1 defect $\mathbf{QFT}_1 \,\Big|\, \mathbf{QFT}_2$

$=$ interface separating region supporting QFT_1 from region supporting QFT_2

Defects in QFT

☞ Codimension-1 defect $\quad \mathbf{QFT}_1 \,\big|\, \mathbf{QFT}_2$

$=$ interface separating region supporting QFT$_1$ from region supporting QFT$_2$

- ubiquitous in nature
- natural part of the structure of quantum field theory
- *physical boundaries* as special case

$\mathbf{QFT}_1 \,\big|$

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☞ *Topological* defect : correlators do not change when deforming the defect

without crossing other substructures

☞ Example : 2-d Ising model

- ferromagnetic nearest-neighbour interaction
- change coupling to *anti*-ferromagnetic on all bonds crossed by some line

$\rightsquigarrow$ topological defect line

☞ Codimension-1 defect $\text{QFT}_1 \mid \text{QFT}_2$

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☞ Some general features of topological defects :

- codimension-2 defects $\text{def}_1 \mid \text{def}_2$ etc
- transparent defect
- invert orientation $\rightsquigarrow$ dual defect
- move two topological defects to coincidence $\rightsquigarrow$ fusion product of defects

Defects in QFT

- Codimension-1 defect $\textbf{QFT}_1 \,|\, \textbf{QFT}_2$
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  - move two topological defects to coincidence $\rightsquigarrow$ fusion product of defects
- Mathematical formulation : $\rightsquigarrow$ higher categories

Invertible defects and symmetries

assume : defects form a rigid monoidal category
(proven for 2-d RCFT)

☞ Subclass : *invertible* topological defects : $\boxed{D \otimes D^\vee \;\cong\; \mathbf{1} \;\cong\; D^\vee \otimes D}$

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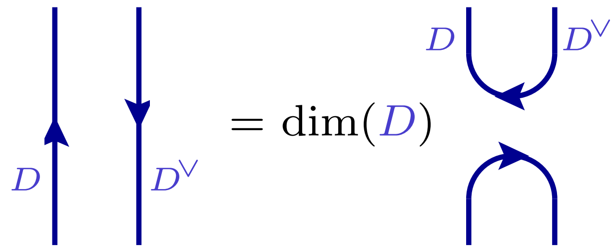

drawn for $d = 2$

$\dim(D) = \pm 1$

$\rightsquigarrow$ identity of correlators when applied locally in any configuration of fields & defects

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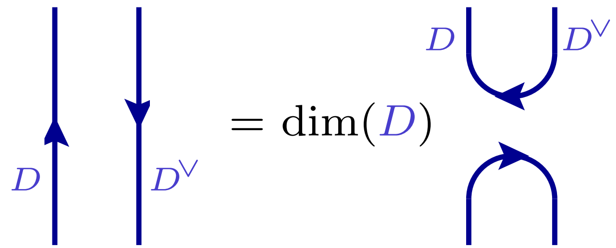

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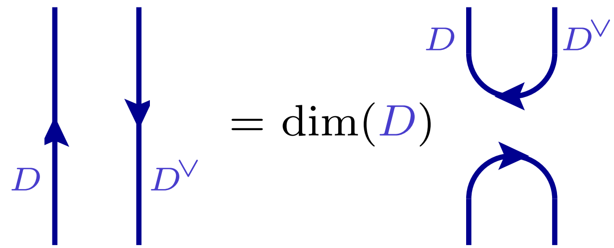

$\rightsquigarrow$ identity of correlators when applied locally in any configuration of fields & defects

- invertible defects form a group under fusion
- act on *all* data of the theory as a symmetry group
- e.g. critical 2-d Ising model : $\mathbb{Z}_2$
  critical three-state Potts model : $\mathfrak{S}_3$

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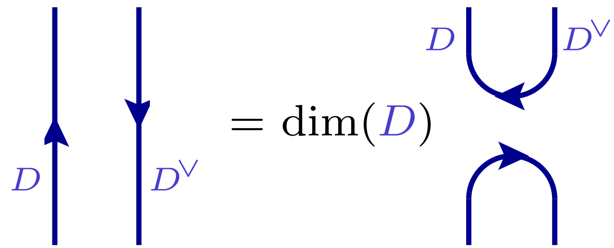

$\rightsquigarrow$ identity of correlators when applied locally in any configuration of fields & defects

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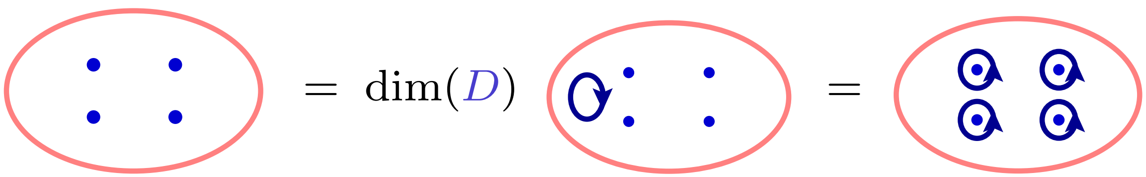
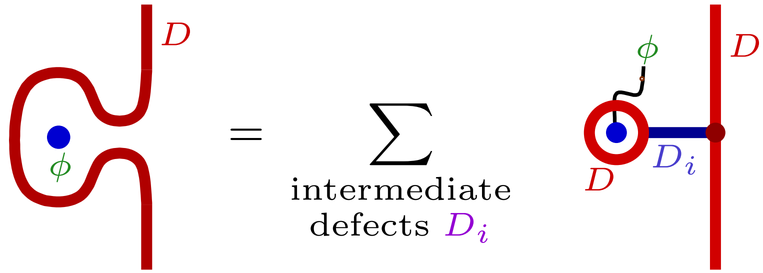
- Example : equalities for bulk field correlators on sphere :



Duality defects

(continuing in $d = 2$)

☞ Wrapping of general topological defect around a bulk field :



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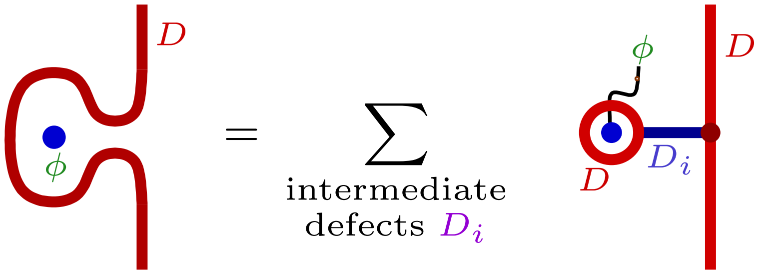

↯ bulk field turned into disorder field

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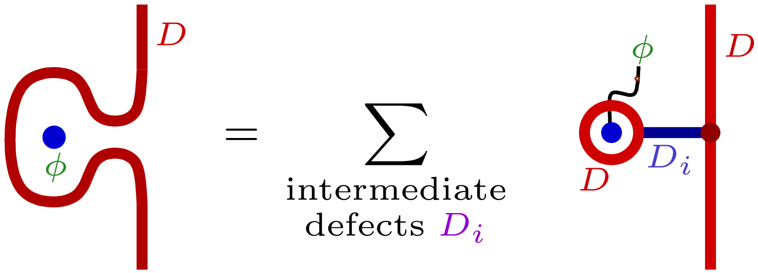

- bulk field turned into disorder field
- wrapping with dual defect turns disorder field back to bulk field if and only if $D \otimes D^{\vee}$ is direct sum of invertible defects
- in this case have an order-disorder duality

  e.g. critical 2-d Ising model : remnant of Kramers-Wannier duality
- again action on all field theoretic quantities

Duality defects

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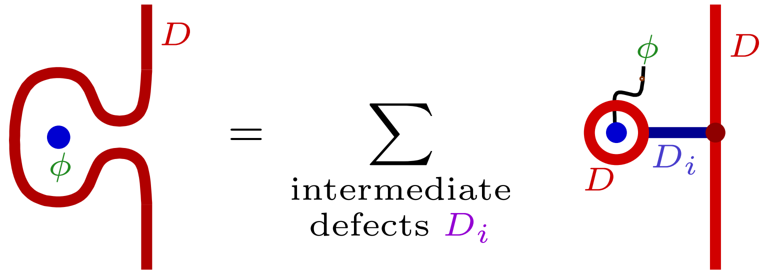

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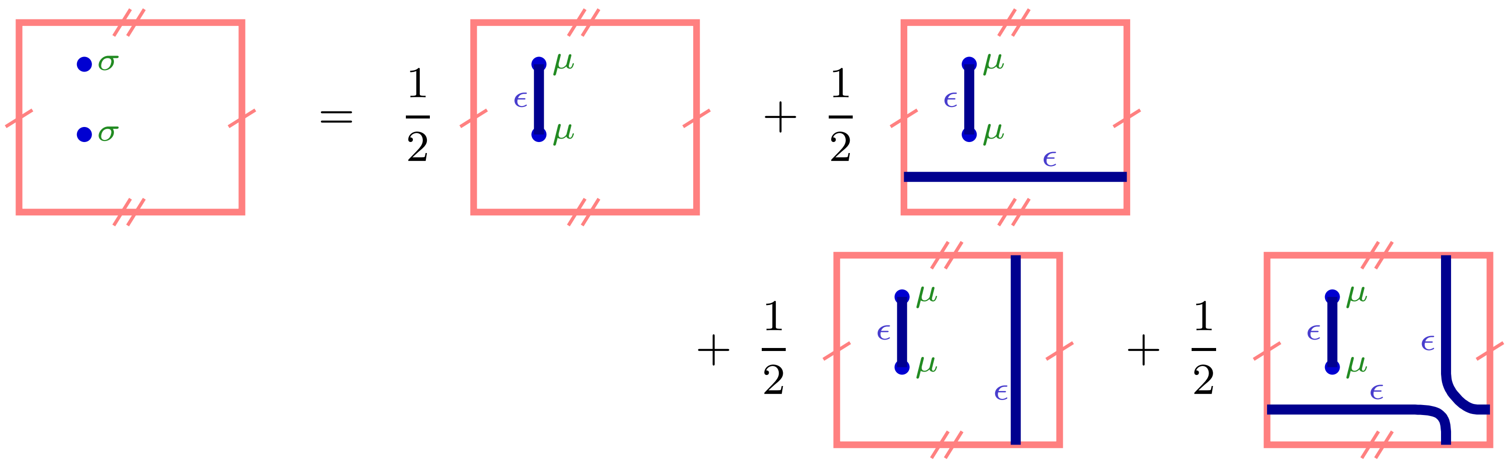
☞ Example : correlator of two Ising spin fields on a torus :



RT-type TFT with defects

☞ RT-type TFT : symmetric monoidal functor $\mathbf{tft}^{\mathcal{D}}_{3,2} : \mathcal{C}obord_{3,2} \longrightarrow \mathcal{V}ect$

resp. 2-functor $\mathbf{tft}^{\mathcal{D}}_{3,2,1} : \mathcal{C}obord_{3,2,1} \longrightarrow 2\text{-}\mathcal{V}ect$

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- insertions on Wilson lines / junctions labeled by morphisms of $\mathcal{D}$
- 2-d cut-and-paste boundaries on which Wilson lines can end
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☞ RT-type TFT with boundaries and defects :

- include in $\mathcal{C}obord$ three-manifolds with physical boundary
- include in $\mathcal{C}obord$ three-manifolds with surface defects

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- **RT-type TFT with boundaries and defects :**

  - include three-manifolds with physical boundary and / or surface defects
  - 3-d bulk regions labeled by modular tensor categories $\mathcal{D}_1,\, \mathcal{D}_2,\, \ldots$

    (bulk Wilson lines in such a region labeled by objects of $\mathcal{D}_i$)
  - boundary Wilson lines and defect Wilson lines
  - several layers of insertions and of junctions

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In particular :

- determine labels for physical boundaries / for surface defects
- determine labels for boundary and defect Wilson lines and for insertions

Conjecture : *Fit together to form bicategories of module categories*

Boundary Wilson lines

☞ Select boundary "a" to some bulk region labeled by a modular tensor cateory $\mathcal{C}$

- can contain boundary Wilson lines
- Wilson line can contain insertions
- insertions can be composed

$\rightsquigarrow$ category $\mathcal{W}_a$ of Wilson lines on boundary a

Boundary Wilson lines

- Select boundary "a" to some bulk region labeled by a modular tensor cateory $\mathcal{C}$
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  - boundary Wilson lines can be fused and can be deformed

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- can contain boundary Wilson lines
- Wilson line can contain insertions
- insertions can be composed
- boundary Wilson lines can be fused and can be deformed
- also impose : finitely semisimple etc

$\rightsquigarrow$ spherical fusion category $\mathcal{W}_a$ of Wilson lines on boundary a

Boundary Wilson lines

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- ☞ Postulate process of moving bulk Wilson lines to boundary

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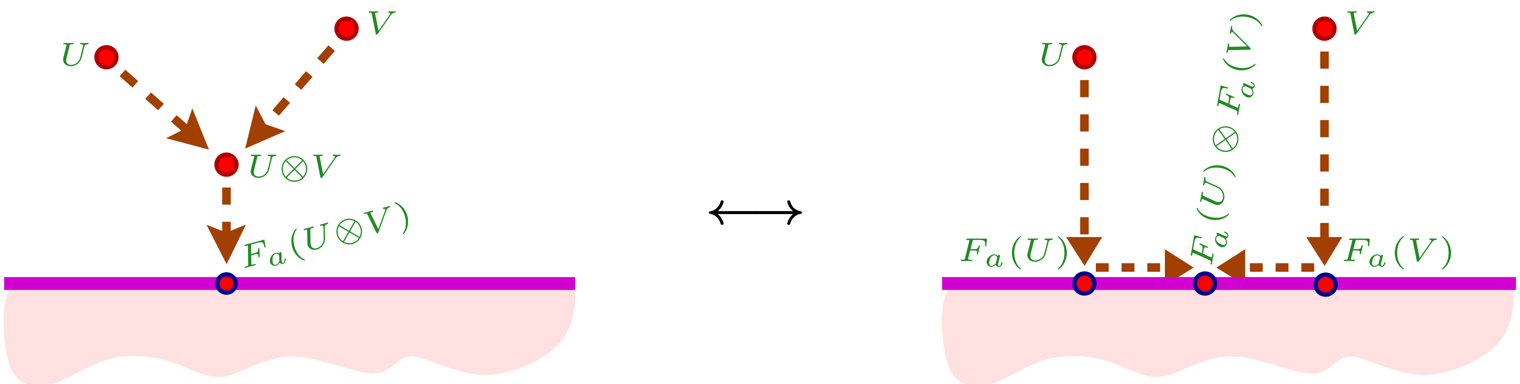

$\rightsquigarrow$ monoidal structure $F_a(U \otimes_{\mathcal{C}} V) \xrightarrow{\cong} F_a(U) \otimes_{\mathcal{W}_a} F_a(V)$

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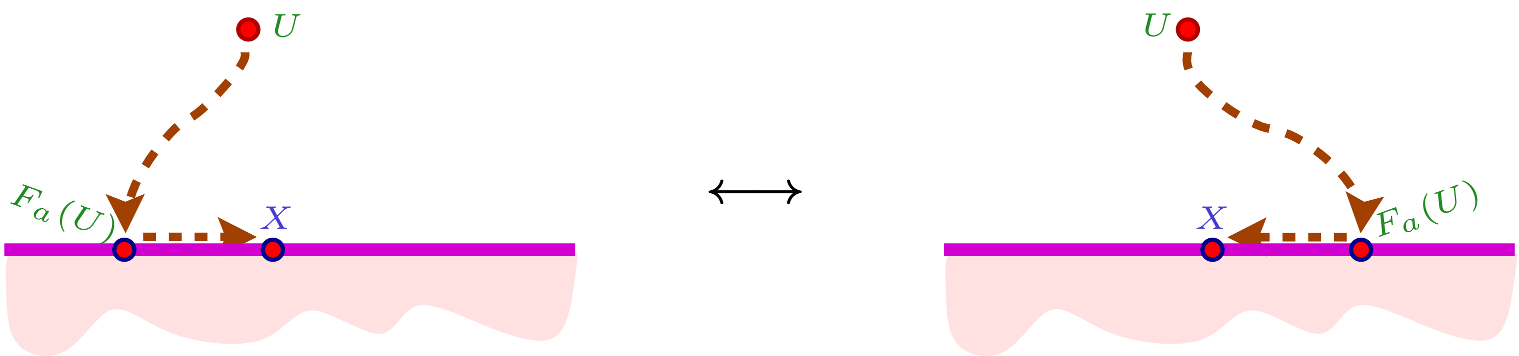

$\rightsquigarrow$ central structure $F_a(U) \otimes_{\mathcal{W}_a} X \xrightarrow{\cong} X \otimes_{\mathcal{W}_a} F_a(U)$

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equivalently : choice of lift

$$\begin{tikzcd} & \mathcal{Z}(\mathcal{W}_a) \arrow[d, "\mathrm{forget}"] \\ \mathcal{C} \arrow[ur, dashed, "\widetilde{F}_a"] \arrow[r, "F_a"'] & \mathcal{W}_a \end{tikzcd}$$

to Drinfeld center of $\mathcal{W}_a$

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  only reason for being able to consistently move boundary Wilson line $Y \in \mathcal{W}_a$ past any $X \in \mathcal{W}_a$ should be that $Y = F_a(U)$ for some $U \in \mathcal{C}$

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In short : Compatible boundary condition for bulk region $\mathcal{C}$

$=$ *Witt trivialization* $\widetilde{F}_a : \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)$ for some fusion category $\mathcal{W}_a$

Boundary conditions

☞ Thus for single boundary condition a : $\boxed{\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)}$

↯ in particular obstruction : no compatible boundary condition unless $[\mathcal{C}] = 0$ in *Witt group* of modular tensor categories

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- ☞ Other boundary condition b :

  other fusion category $\mathcal{W}_b$ of Wilson lines in region b

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  - ↯ in particular obstruction : no compatible boundary condition unless $[\mathcal{C}] = 0$ in *Witt group* of modular tensor categories
- ☞ Other boundary condition b :
  - ↯ category $\mathcal{W}_{a,b}$ of Wilson lines separating boundary region labeled a from region labeled b
  - ↯ fusion of Wilson lines in region a $\rightsquigarrow$ functor $\mathcal{W}_a \times \mathcal{W}_{a,b} \longrightarrow \mathcal{W}_{a,b}$
  - ↯ gives action of $\mathcal{W}_a$ on $\mathcal{W}_{a,b}$: $\mathcal{W}_{a,b}$ is left module category over $\mathcal{W}_a$
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module endofunctors

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- Impose naturality : $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b}) \simeq \mathcal{W}_b$

  Consistency check : $\mathcal{Z}(\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b})) \simeq \mathcal{Z}(\mathcal{W}_a)$ canonically

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  $\Longrightarrow$ can work with a single *reference boundary condition* a
- Conjecture : *Boundary conditions for $\mathcal{C}$ form the bicategory $\mathcal{W}_a$-$\mathcal{M}od$* of module categories over a fusion category $\mathcal{W}_a$ satisfying $\mathcal{Z}(\mathcal{W}_a) \simeq \mathcal{C}$

Boundary conditions

- ☞ Will assume : Boundary conditions given by $\mathcal{W}_a$-$\mathcal{M}od$
- ☞ Then $\mathcal{W}_{b,c} \simeq \mathcal{F}un_{\mathcal{W}_a}(\mathcal{W}_b, \mathcal{W}_c)$ for any pair of boundary conditions $b\,,\; c$

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- Warning :

  via $\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a) \xrightarrow{\text{forget}} \mathcal{W}_a$

  any $\mathcal{M} \in \mathcal{W}_a$-$\mathcal{M}od$ has natural structure of $\mathcal{C}$-module category

  But not every $\mathcal{C}$-module category of a Witt-trivial $\mathcal{C}$ gives a boundary condition

  Illustration : Toric code

  ↯ 2 elementary boundary conditions

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  Illustration : Toric code

  - 2 elementary boundary conditions
  - $\mathcal{C} = \mathcal{Z}(\mathcal{V}ect(\mathbb{Z}_2))$
  - 6 inequivalent indecomposable module categories over $\mathcal{C}$
  - 2 inequivalent indecomposable module categories over $\mathcal{W} = \mathcal{V}ect(\mathbb{Z}_2)$

Bicategories of surface defects

- Parallel analysis for surface defects :
  - defect d separating bulk regions labeled by $\mathcal{C}_1$ and $\mathcal{C}_2$
  - two monoidal functors $\mathcal{C}_1 \longrightarrow \mathcal{W}_d$ and $\mathcal{C}_2^{\mathrm{rev}} \to \mathcal{W}_d$ to fusion category $\mathcal{W}_d$

inverse braiding

Bicategories of surface defects

- Parallel analysis for surface defects :
  - defect d separating bulk regions labeled by $\mathcal{C}_1$ and $\mathcal{C}_2$
  - two monoidal functors $\mathcal{C}_1 \longrightarrow \mathcal{W}_d$ and $\mathcal{C}_2^{\mathrm{rev}} \to \mathcal{W}_d$ to fusion category $\mathcal{W}_d$
  - combine to central functor $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \to \mathcal{W}_d$

Deligne product

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- naturality $\rightsquigarrow$ braided equivalence $\boxed{\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)}$
- obstruction : no defects between $\mathcal{C}_1$ and $\mathcal{C}_2$ unless $[\mathcal{C}_1] = [\mathcal{C}_2]$ in Witt group

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☞ Example : Canonical Witt trivialization $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$ ($\mathcal{C}$ modular)

- defects separating $\mathcal{C}$ from itself $=$ $\mathcal{C}$-module catgeories

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  - *regular* $\mathcal{C}$-module category $(\mathcal{C}, \otimes)$ $\rightsquigarrow$ transparent defect $\mathcal{T}$
  - serves as monoidal unit for fusion of surface defects
  - Wilson lines separating transparent defect from itself $=$ ordinary Wilson lines

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- Example : *Turaev-Viro* TFT : $\mathcal{C}_1 \simeq \mathcal{Z}(\mathcal{A}_1)$ and $\mathcal{C}_2 \simeq \mathcal{Z}(\mathcal{A}_2)$

  $\rightsquigarrow$ $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{A}_1) \boxtimes \mathcal{Z}(\mathcal{A}_2^{\mathrm{op}}) \simeq \mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2^{\mathrm{op}})$

  $\rightsquigarrow$ defects separating $\mathcal{C}_1$ from $\mathcal{C}_2$ form bicategory $\mathcal{A}_1$-$\mathcal{A}_2$-$\mathcal{B}imod$