Homework #8:
Due: tentatively Dec. 19, 2019

***** (about lecture #6) *****

1. (20points) This problem is based on Problem 4 of Homework #3. Let the unperturbed Hamiltonian be the "second quantized" Hamiltonian for identical non-interacting particles in 1D harmonic potential, $\hat{H}_0 = \int \mathrm{d}x\, \widehat{\psi(x)}^\dagger \cdot (-\frac{\hbar^2\partial_x^2}{2m} + \frac{m\omega^2 x^2}{2}) \cdot \widehat{\psi(x)} = \sum_{n=0}^{\infty} E_{n,\text{1-body}} \widehat{\psi}_n^\dagger \widehat{\psi}_n$. Here $E_{n,\text{1-body}} = \hbar\omega \cdot (n + \frac{1}{2})$ is the single-particle eigenvalue, $\widehat{\psi}_n^\dagger$ is the creation operator for the nth single-particle eigenstate of harmonic oscillator, $\widehat{\psi(x)}^\dagger$ is the creation operator for the position basis $|x\rangle$. Let the perturbation term be the "second quantized" 2-body interaction, $\hat{V} = \frac{1}{2}\int \mathrm{d}x \int \mathrm{d}x'\, \widehat{\psi(x)}^\dagger \widehat{\psi(x')}^\dagger \cdot (x - x')^2 \cdot \widehat{\psi(x')}\widehat{\psi(x)}$. You can use the result of Problem 4(c) of Homework #3 to rewrite $\hat{V}$ in terms of $\widehat{\psi}_n^\dagger$ and $\widehat{\psi}_n$. The full Hamiltonian is $\hat{H} = \hat{H}_0 + \lambda\hat{V}$, where λ is a small real parameter.

(a) (7pts) *For N identical bosons, solve the ground state energy of $\hat{H}$ up to λ^2 order by perturbation theory.* Here integer $N \geq 2$.

(b) (7pts) *For N identical fermions, solve the ground state energy of $\hat{H}$ up to λ^2 order by perturbation theory.* Here integer $N \geq 2$.

(c) (6pts)(*) When particle number $N = 2$, eigenvalues and eigenstates of $\hat{H}$ can be solved exactly. Use the "first quantized" form, $\hat{H} = -\frac{\hbar^2}{2m}[(\frac{\partial}{\partial x_1})^2 + (\frac{\partial}{\partial x_2})^2] + \frac{m\omega^2}{2}(x_1^2 + x_2^2) + \lambda\,(x_1 - x_2)^2$. Then $\hat{H}\psi(x_1, x_2) = E\,\psi(x_1, x_2)$ can be solved by changing variables to the "center of mass" position $x_{\text{COM}} \equiv \frac{x_1+x_2}{2}$ and the relative position $X \equiv x_1 - x_2$. *Solve the exact ground state energy for two-boson and two-fermion cases respectively. Compare with the results of (a)(b) for $N = 2$.* [Note: $\psi(x_1, x_2)$ has different symmetry for boson and fermion cases.]

2. (15points) Consider three fermion modes, denote their annihilation operators as $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$. They satisfy $\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{i,j}$. The unperturbed Hamiltonian is $\hat{H}_0 = E_0 \cdot (\hat{n}_1 + \hat{n}_2) + E_1 \cdot \hat{n}_3$. Here $\hat{n}_i = \hat{f}_i^\dagger \hat{f}_i$ is the occupation number operator, $E_1 > E_0$ are real parameters. The occupation basis $|n_1, n_2, n_3\rangle$ are eigenstates of $\hat{H}_0$ with eigenvalue $E_0 \cdot (n_1 + n_2) + E_1 \cdot n_3$, where $n_{1,2,3} = 0$ or 1 are eigenvalues of $\hat{n}_{1,2,3}$ respectively.

Add a time-independent perturbation, $\hat{V} = -t \cdot (\hat{f}_1^\dagger \hat{f}_2 + \hat{f}_2^\dagger \hat{f}_1 + \hat{f}_1^\dagger \hat{f}_3 + \hat{f}_3^\dagger \hat{f}_1 - \hat{f}_2^\dagger \hat{f}_3 - \hat{f}_3^\dagger \hat{f}_2)$.

Here t is a real "small parameter". The perturbed Hamiltonian is $\hat{H} = \hat{H}_0 + \hat{V}$.

(a) (9pts) *Use one of the two approaches (formal series expansion, or unitary transformations) to compute all the energy eigenvalues of $\hat{H}$ in the 2-particle subspace to 3rd order of small parameter t.* [Hint: higher order degenerate perturbation theory can be avoided by changing to eigenbasis of 1st order secular equation; certain symmetry may help]

(b) (6pts) *Exactly diagonalize the Hamiltonian $\hat{H}_0 + \hat{V}$ in the 2-particle subspace, expand the exact energy formula to 3rd order of t, compare with the perturbation theory result in (a).*

3. (15points) Consider a 2-level system, $\hat{H}_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}$ under basis $(|\psi_0^{(0)}\rangle, |\psi_1^{(0)}\rangle)$ with $E_1 > E_0$. Add a time-dependent perturbation $\hat{V}(t) = V \begin{pmatrix} 0 & e^{-\mathbb{i}\omega t} \\ e^{\mathbb{i}\omega t} & 0 \end{pmatrix}$ under the above basis, where $V > 0$ is a "small parameter", ω is real. Denote the time-evolution operator in Schrödinger picture of perturbed Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$ as $\hat{U}_S(t)$, then $\mathbb{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} \hat{U}_S(t) = \hat{H}(t) \cdot \hat{U}_S(t)$.

(a) (5pts) *Compute the transition probability $|\langle \psi_1^{(0} | \hat{U}_S(t) | \psi_0^{(0)} \rangle|^2$ by perturbative expansion to lowest non-trivial order of V.* [Hint: use the interaction picture.]

(b) (5pts) *Compute $|\langle \psi_0^{(0} | \hat{U}_S(t) | \psi_0^{(0)} \rangle|^2$ to cubic order of V.* [Hint: you need to compute $\langle \psi_0^{(0} | \hat{U}_I(t) | \psi_0^{(0)} \rangle$ up to appropriate order of V]

(c) (5pts) An exact solution of $|\langle \psi_1^{(0} | \hat{U}_S(t) | \psi_0^{(0)} \rangle|^2$ is possible (Rabi oscillation). *Assume $|\psi(t)\rangle = c_0(t) e^{-\mathbb{i}E_0 t/\hbar} |\psi_0^{(0)}\rangle + c_1(t) e^{-\mathbb{i}E_1 t/\hbar} |\psi_0^{(1)}\rangle$ is the solution of $\mathbb{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$ with initial condition $|\psi(t=0)\rangle = |\psi_0^{(0)}\rangle$. Derive and solve differential equations for coefficients $c_0(t)$ and $c_1(1)$.* Then $|\langle \psi_1^{(0} | \hat{U}_S(t) | \psi_0^{(0)} \rangle|^2 = |c_1(t)|^2$. Check that the exact result and approximate result in (a) are consistent for small V.