Homework #5:
Brief Solutions

1. On 2-dimensional space with real coordinates (x, y), define position eigenstates $|x, y\rangle$ with $\hat{x}|x, y\rangle = x|x, y\rangle$ and $\hat{y}|x, y\rangle = y|x, y\rangle$ and normalization $\langle x, y|x', y'\rangle = \delta(x-x')\delta(y-y')$; and momentum eigenstates $|p_x, p_y\rangle$ with $\hat{p_x}|p_x, p_y\rangle = p_x|p_x, p_y\rangle$ and $\hat{p_y}|p_x, p_y\rangle = p_y|p_x, p_y\rangle$ and normalization $\langle p_x, p_y|p'_x, p'_y\rangle = \delta(p_x - p'_x)\delta(p_y - p'_y)$. And $\langle x, y|p_x, p_y\rangle = \frac{e^{\mathbb{i}(p_x\cdot x+p_y\cdot y)/\hbar}}{2\pi\hbar}$.

Here $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = \mathbb{i}\hbar$, and other commutators between them are zero.

The rotations around the origin point form the $SO(2)$ group (also denoted by C_∞). Denote the counter-clockwise rotation of angle θ by $g(\theta)$, which maps the point (x, y) to $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. It is easy to check that $g(\theta)\cdot g(\theta') = g(\theta + \theta' \mod 2\pi)$, so this is an Abelian group. All its irreducible representations(irreps) are 1-dimensional(1-dim'l), and are labeled by integer n, $\chi_n[g(\theta)] = R_n[g(\theta)] = (e^{\mathbb{i}n\theta})$. The orthogonality relation is $\int_0^{2\pi} \mathrm{d}\theta\, \chi_n[g(\theta)]^*\chi_{n'}[g(\theta)] = \int_0^{2\pi} \mathrm{d}\theta\, e^{-\mathbb{i}n\theta}e^{\mathbb{i}n'\theta} = 2\pi\cdot\delta_{n,n'}$.

(a). (2pts) The unitary operator for $g(\theta)$ is $\widehat{g(\theta)} = \int \mathrm{d}x \int \mathrm{d}y\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle\langle x, y|$. Compute the matrix element of $\widehat{g(\theta)}$ under the momentum eigenbasis, $\langle p'_x, p'_y|\widehat{g(\theta)}|p_x, p_y\rangle$.

(b). (2pts) Compute the generator of this "Lie group", $\widehat{L_z} \equiv \left[\mathbb{i}\frac{\partial}{\partial\theta}\widehat{g(\theta)}\right]_{\theta=0}$. Represent the result by the $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ operators. [Hint: consider $\widehat{L_z}\psi(x, y) = \left\{\mathbb{i}\frac{\partial}{\partial\theta}[\widehat{g(\theta)}\psi(x, y)]\right\}_{\theta=0}$]

(c). (2pts) Consider the 2D harmonic oscillator, $\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2)$. Here m, ω are positive constants. It can be viewed as the sum of two independent harmonic oscillators, $\hat{H} = (\frac{\hat{p}_x^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}) + (\frac{\hat{p}_y^2}{2m} + \frac{m\omega^2\hat{y}^2}{2})$. The ladder operators for the x- and y-components can be defined as $\hat{b}_x = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{\mathbb{i}}{m\omega}\hat{p}_x)$ and $\hat{b}_y = \sqrt{\frac{m\omega}{2\hbar}}(\hat{y} + \frac{\mathbb{i}}{m\omega}\hat{p}_y)$. They satisfy the commutation relation of boson annihilation operators, $[\hat{b}_x, \hat{b}_x^\dagger] = [\hat{b}_y, \hat{b}_y^\dagger] = 1$, $[\hat{b}_x, \hat{b}_y^\dagger] = [\hat{b}_x, \hat{b}_y] = 0$. Denote the unique normalized ground state of $\hat{H}$ by $|\text{vac}\rangle$, then $\hat{b}_x|\text{vac}\rangle = \hat{b}_y|\text{vac}\rangle = 0$. Write down all eigenvalues and normalized eigenstates of $\hat{H}$.

(d). (2pts) Rewrite the $\widehat{L_z}$ in (b) in terms of the ladder operators in (c). Show that $[\hat{H}, \widehat{L_z}] = 0$. {Hint: use $[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}]\hat{D} + [\hat{A}, \hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B}, \hat{D}] + \hat{C}[\hat{A}, \hat{D}]\hat{B}$. }

(e). (4pts) The "raising" operators $\hat{b}_x^\dagger$ and $\hat{b}_y^\dagger$ form basis of a 2-dimensional representation of the $SO(2)$ group. $g(\theta)$ transforms them to their linear combinations, $(\widehat{g(\theta)}\hat{b}_x^\dagger\widehat{g(\theta)}^\dagger,\ \widehat{g(\theta)}\hat{b}_y^\dagger\widehat{g(\theta)}^\dagger) = (\hat{b}_x^\dagger,\ \hat{b}_y^\dagger)\cdot R[g(\theta)]$. Solve this 2×2 representation matrix $R_{\hat{b}^\dagger}[g(\theta)]$. Check that $R_{\hat{b}^\dagger}[g(\theta)]\cdot R_{\hat{b}^\dagger}[g(\theta')] = R_{\hat{b}^\dagger}[g(\theta+\theta')]$. [Hint: $\widehat{g(\theta)} = \exp(-\mathbb{i}\theta\widehat{L_z})$.]

(f). (5pts) We can decompose the $R_{\hat{b}^\dagger}$ representation into irreps by the "projection operator". Compute $\int_0^{2\pi} R_n[g(\theta)]^*\cdot\widehat{g(\theta)}\hat{b}_a^\dagger\widehat{g(\theta)}^\dagger$, for $a = x, y$ and all integer n. Only two of them are linearly independent, after normalization denote them as $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$, satisfying $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$. $\hat{a}_i^\dagger$ forms an irrep of the $SO(2)$ group, $\widehat{g(\theta)}\hat{a}_i^\dagger\widehat{g(\theta)}^\dagger = \hat{a}_i^\dagger\cdot R_{\hat{a}_i^\dagger}[g(\theta)]$, Write down $\hat{a}_{1,2}^\dagger$ in terms of $\hat{b}_{x,y}^\dagger$, and their corresponding irreps $R_{\hat{a}_i^\dagger}[g(\theta)]$. Rewrite $\hat{H}$ and $\widehat{L_z}$ in terms of $\hat{a}_{1,2}^\dagger$ and $\hat{a}_{1,2}$.

(g). (5pts) Write down the simultaneous eigenstates of $\hat{H}$ and $\widehat{L_z}$, $|\hat{H} = E, \widehat{L_z} = \ell\rangle$, in terms of $|\text{vac}\rangle$ and ladder operators $\hat{a}_i^\dagger$. What are the possible eigenvalues E and ℓ?

(h). (3pts) Each state $|\hat{H} = E, \widehat{L_z} = \ell\rangle$ in (g) forms an irrep of $SO(2)$. $\widehat{g(\theta)}|\hat{H} = E, \widehat{L_z} = \ell\rangle = |\hat{H} = E, \widehat{L_z} = \ell\rangle\cdot R_{E,\ell}[g(\theta)]$. Compute this 1×1 representation 'matrix' $R_{E,\ell}[g(\theta)]$.

(i). (5pts) Compute the "matrix elements", $\langle\hat{H} = E, \widehat{L_z} = \ell|\hat{a}_i^\dagger|\hat{H} = E', \widehat{L_z} = \ell'\rangle$, for $i = 1, 2$. Check that the results satisfy the "selection rule" for $SO(2)$ group, namely that if $R_{E,\ell}^*\otimes R_{\hat{a}_i^\dagger}\otimes R_{E',\ell'}$ is not the trivial irrep., then this matrix element vanishes.

(j). (5pts) Define two hermitian operators $\widehat{L_x} = \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1$ and $\widehat{L_y} = -\mathbb{i}\hat{a}_1^\dagger\hat{a}_2 + \mathbb{i}\hat{a}_2^\dagger\hat{a}_1$. Check that $[\hat{H}, \widehat{L_x}] = [\hat{H}, \widehat{L_y}] = 0$. Compute the commutators $[\widehat{L_x}, \widehat{L_y}]$, $[\widehat{L_y}, \widehat{L_z}]$, $[\widehat{L_z}, \widehat{L_x}]$, represent the results in terms of linear combinations of $\widehat{L_{x,y,z}}$. [Side remark: $SO(2)$ has only 1-dim'l irrep., but $\hat{H}$ has degenerate eigenvalues. In fact $\hat{H}$ has a larger non-Abelian symmetry. The $\widehat{L_{x,y,z}}$ are generators of this symmetry group and commute with $\hat{H}$.]

Solution:

(a) $\langle x, y|p_x, p_y\rangle = \frac{1}{2\pi\hbar}e^{\mathbb{i}(p_x x + p_y y)/\hbar}$.

Then $\widehat{g(\theta)}|p_x, p_y\rangle = \int \mathrm{d}x \int \mathrm{d}y\, \widehat{g(\theta)}|x, y\rangle\langle x, y|p_x, p_y\rangle$

$= \int \mathrm{d}x \int \mathrm{d}y\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle \cdot \frac{1}{2\pi\hbar}e^{\mathbb{i}(p_x x + p_y y)/\hbar}$.

Change the dummy variables to $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$, or equivalently $x = x'\cos\theta + y'\sin\theta$ and $y = -x'\sin\theta + y'\cos\theta$, the Jacobian of this variable change is unity, $|\frac{\partial(x,y)}{\partial(x',y')}| = |\det\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}| = 1$.

$\widehat{g(\theta)}|p_x, p_y\rangle = \int \mathrm{d}x' \int \mathrm{d}y' \, |x', y'\rangle \cdot \frac{1}{2\pi\hbar} e^{\mathbb{i}([(p_x\cos\theta - p_y\sin\theta)x' + (p_y\cos\theta + p_x\sin\theta)y']/\hbar}$

$= |p_x\cos\theta - p_y\sin\theta, p_x\sin\theta + p_y\cos\theta\rangle$ is a momentum eigenstate.

$\langle p_x', p_y'|\widehat{g(\theta)}|p_x, p_y\rangle = \delta(p_x' - (p_x\cos\theta - p_y\sin\theta)) \cdot \delta(p_y' - (p_x\sin\theta + p_y\sin\theta))$

$= \delta(p_x - (p_x'\cos\theta + p_y'\sin\theta)) \cdot \delta(p_y - (-p_x'\sin\theta + p_y'\cos\theta))$.

(b) Method #1:

It'll be most clear to consider the action of $\widehat{g(\theta)}$ on a generic state $|\psi\rangle$ with wavefunction $\psi(x,y) = \langle x, y|\psi\rangle$.

$\widehat{g(\theta)}|\psi\rangle = \int \mathrm{d}x \int \mathrm{d}y \, \widehat{g(\theta)}|x, y\rangle\langle x, y|\psi\rangle = \int \mathrm{d}x \int \mathrm{d}y \, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle \, \psi(x,y)$

$= \int \mathrm{d}x' \int \mathrm{d}y' \, |x', y'\rangle \, \psi(x'\cos\theta + y'\sin\theta, -x'\sin\theta + y'\cos\theta)$.

Therefore the action of $\widehat{g(\theta)}$ on wavefunctions $\psi(x,y)$ is

$\widehat{g(\theta)}: \quad \psi(x,y) \mapsto \psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$.

So the action of $\widehat{L}_z \equiv \left[\mathbb{i}\frac{\partial}{\partial\theta}\widehat{g(\theta)}\right]_{\theta=0}$ is

$\left[\mathbb{i}\frac{\partial}{\partial\theta}\widehat{g(\theta)}\right]_{\theta=0}: \; \psi(x,y) \mapsto \mathbb{i}\frac{\partial}{\partial\theta}\psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)\big|_{\theta=0} = \mathbb{i}[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}]\psi(x,y)$.

Compare this with the actions of $\hat{x}$,$\hat{y}$,$\hat{p}_x$,$\hat{p}_y$, we have $\widehat{L}_z = \frac{1}{\hbar}[\hat{x}\hat{p}_y - \hat{y}\hat{p}_x]$.

Method #2:

$\widehat{g(\theta)} = \int \mathrm{d}x \int \mathrm{d}y \, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle\langle x, y|$

$= \int \mathrm{d}p_x \int \mathrm{d}p_y \, \int \mathrm{d}x \int \mathrm{d}y \, |p_x, p_y\rangle\langle p_x, p_y|x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle$

$= \int \mathrm{d}p_x \int \mathrm{d}p_y \, \int \mathrm{d}x \int \mathrm{d}y \, |p_x, p_y\rangle \frac{e^{-\mathbb{i}p_x\cdot(x\cos\theta - y\sin\theta) - \mathbb{i}p_y\cdot(x\sin\theta + y\cos\theta)}}{2\pi\hbar}\langle x, y|$.

Here we have used $\langle p_x, p_y|x, y\rangle = \frac{e^{(-\mathbb{i}p_x x - \mathbb{i}p_y y)/\hbar}}{2\pi\hbar}$.

Now only the numerical factor involves θ, take derivative with respect to θ,

$\widehat{L}_z \equiv \left[\mathbb{i}\frac{\partial}{\partial\theta}\widehat{g(\theta)}\right]_{\theta=0} = \int \mathrm{d}p_x \int \mathrm{d}p_y \, \int \mathrm{d}x \int \mathrm{d}y \, |p_x, p_y\rangle \left(\frac{1}{\hbar}(-p_x y + p_y x) \cdot \frac{e^{(-\mathbb{i}p_x x - \mathbb{i}p_y y)/\hbar}}{2\pi\hbar}\right) \langle x, y|$

$= \int \mathrm{d}p_x \int \mathrm{d}p_y \, \int \mathrm{d}x \int \mathrm{d}y \, |p_x, p_y\rangle \left(\frac{1}{\hbar}(-p_x y + p_y x)\langle p_x, p_y|x, y\rangle\right) \langle x, y|$.

Compare this with the diagonal form of $\hat{p}_{x,y}$ and $\hat{x}$,$\hat{y}$ operators,

$\hat{p}_x = \int \mathrm{d}p_x \int \mathrm{d}p_y \, |p_x, p_y\rangle p_x \langle p_x, p_y|$, $\hat{p}_y = \int \mathrm{d}p_x \int \mathrm{d}p_y \, |p_x, p_y\rangle p_y \langle p_x, p_y|$, and

$\hat{x} = \int \mathrm{d}x \int \mathrm{d}y \, |x, y\rangle x \langle x, y|$, $\hat{y} = \int \mathrm{d}x \int \mathrm{d}y \, |x, y\rangle y \langle x, y|$.

We have $\widehat{L}_z = \frac{1}{\hbar}(-\hat{p}_x\hat{y} + \hat{p}_y\hat{x})$.

(c) $\hat{H} = \hbar\omega \cdot (\hat{b}_x^\dagger \hat{b}_x + \hat{b}_y^\dagger \hat{b}_y + 1) = \hbar\omega \cdot (\hat{n}_x + \hat{n}_y + 1)$. Here $\hat{n}_x = \hat{b}_x^\dagger \hat{b}_x$ and $\hat{n}_y = \hat{b}_y^\dagger \hat{b}_y$.

The eigenvalues are $E_{n_x,n_y} = \hbar\omega \cdot (n_x + n_y + 1)$, with normalized eigenstate $|\hat{n}_x = n_x, \hat{n}_y = n_y\rangle = \frac{1}{\sqrt{n_x! n_y!}} (\hat{b}_x^\dagger)^{n_x} (\hat{b}_y^\dagger)^{n_y} |\text{vac}\rangle$. Here n_x, n_y are non-negative integers.

(d)

Use $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b}_x + \hat{b}_x^\dagger)$, $\hat{y} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b}_y + \hat{b}_y^\dagger)$, $\hat{p}_x = -\mathbb{i}\sqrt{\frac{m\omega\hbar}{2}}(\hat{b}_x - \hat{b}_x^\dagger)$, $\hat{p}_y = -\mathbb{i}\sqrt{\frac{m\omega\hbar}{2}}(\hat{b}_y - \hat{b}_y^\dagger)$.

$\widehat{L}_z = -\mathbb{i}\hat{b}_x^\dagger \hat{b}_y + \mathbb{i}\hat{b}_y^\dagger \hat{b}_x$.

Use $[\hat{H}, \hat{b}_x] = -\hbar\omega \hat{b}_x$, $[\hat{H}, \hat{b}_y] = -\hbar\omega \hat{b}_y$, $[\hat{H}, \hat{b}_x^\dagger] = +\hbar\omega \hat{b}_x^\dagger$, $[\hat{H}, \hat{b}_y^\dagger] = +\hbar\omega \hat{b}_y^\dagger$,

$[\hat{H}, \hat{b}_x^\dagger \hat{b}_y] = [\hat{H}, \hat{b}_x^\dagger]\hat{b}_y + \hat{b}_x^\dagger[\hat{H}, \hat{b}_y] = \hbar\omega \hat{b}_x^\dagger \cdot \hat{b}_y - \hat{b}_x^\dagger \cdot \hbar\omega \hat{b}_y = 0$, and similarly $[\hat{H}, \hat{b}_y^\dagger \hat{b}_x] = 0$.

Therefore $[\hat{H}, \widehat{L}_z] = 0$.

In fact, the commutator of creation/annihilation operator bilinears is still a bilinear, $[\sum_{i,j} \hat{b}_i^\dagger P_{ij} \hat{b}_j, \sum_{k,\ell} \hat{b}_k^\dagger Q_{k\ell} \hat{b}_\ell] = \sum_{i,j,k,\ell} (\hat{b}_i^\dagger P_{ij} \delta_{jk} Q_{k\ell} \hat{b}_\ell - \hat{b}_k^\dagger Q_{k\ell} \delta_{i\ell} P_{ij} \hat{b}_j) = \sum_{i,\ell} \hat{b}_i^\dagger ([P,Q])_{i\ell} \hat{b}_\ell$. Here $[P,Q] \equiv P \cdot Q - Q \cdot P$ is the commutator of the coefficient matrices. Then

$$[\hat{H}, \widehat{L}_z] = \hbar\omega \cdot [\hat{n}_x + \hat{n}_y, -\mathbb{i}\hat{b}_x^\dagger \hat{b}_y + \mathbb{i}\hat{b}_y^\dagger \hat{b}_x] = \hbar\omega \cdot (\hat{b}_x^\dagger, \hat{b}_y^\dagger) \cdot [\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}] \cdot \begin{pmatrix} \hat{b}_x \\ \hat{b}_y \end{pmatrix} = 0.$$

(e) Method #1:

Consider the action of $(\widehat{g(\theta)}\hat{x}\widehat{g(\theta)}^\dagger)$ on position basis $|x,y\rangle$, note that $\widehat{g(\theta)}^\dagger = \widehat{g(-\theta)}$,

$(\widehat{g(\theta)}\hat{x}\widehat{g(\theta)}^\dagger)|x,y\rangle = \widehat{g(\theta)}\hat{x}|x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta\rangle$

$= \widehat{g(\theta)}(x\cos\theta + y\sin\theta)|x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta\rangle = (x\cos\theta + y\sin\theta)|x,y\rangle$. Therefore $(\widehat{g(\theta)}\hat{x}\widehat{g(\theta)}^\dagger) = \hat{x}\cos\theta + \hat{y}\sin\theta$.

Similarly one can show that $(\widehat{g(\theta)}\hat{y}\widehat{g(\theta)}^\dagger) = -\hat{x}\sin\theta + \hat{y}\cos\theta$.

Consider the action of $(\widehat{g(\theta)}\hat{p}_{x,y}\widehat{g(\theta)}^\dagger)$ on momentum basis $|p_x, p_y\rangle$, and use the result of (a), one can show that $(\widehat{g(\theta)}\hat{p}_x\widehat{g(\theta)}^\dagger) = \hat{p}_x\cos\theta + \hat{p}_y\sin\theta$, $(\widehat{g(\theta)}\hat{p}_y\widehat{g(\theta)}^\dagger) = -\hat{p}_x\sin\theta + \hat{p}_y\cos\theta$.

Then by the definition of $\hat{b}_{x,y}^\dagger$, we have

$(\widehat{g(\theta)}\hat{b}_x^\dagger\widehat{g(\theta)}^\dagger, \quad \widehat{g(\theta)}\hat{b}_y^\dagger\widehat{g(\theta)}^\dagger) = (\hat{b}_x^\dagger, \quad \hat{b}_y^\dagger) \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Namely, $R_{\hat{b}^\dagger}[g(\theta)] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Method #2:

Use $\widehat{g(\theta)} = \exp(-\mathbb{i}\theta\widehat{L}_z)$, and the Baker-Hausdorff formula.

Use the result of (d), $[\widehat{L}_z, \hat{b}_x^\dagger] = \mathbb{i}\hat{b}_y^\dagger$, $[\widehat{L}_z, \hat{b}_y^\dagger] = -\mathbb{i}\hat{b}_x^\dagger$. By mathematical induction,

$$\underbrace{[\widehat{L}_z, [\widehat{L}_z, \dots [\widehat{L}_z}_{n\text{-fold commutator}}, \hat{b}_x^\dagger] \dots]] = \begin{cases} \hat{b}_x^\dagger, & n = 2m; \\ \mathbb{i}\hat{b}_y^\dagger, & n = 2m+1. \end{cases}$$

$$\underbrace{[\widehat{L}_z, [\widehat{L}_z, \dots [\widehat{L}_z}_{n\text{-fold commutator}}, \hat{b}_y^\dagger] \dots]] = \begin{cases} \hat{b}_y^\dagger, & n = 2m; \\ -\mathbb{i}\hat{b}_x^\dagger, & n = 2m+1. \end{cases}$$

$\widehat{g(\theta)}\hat{b}_x^\dagger\widehat{g(\theta)}^\dagger = \sum_{m=0}^\infty \frac{1}{(2m)!}(-\mathbb{i}\theta)^{2m}\hat{b}_x^\dagger + \sum_{m=0}^\infty \frac{1}{(2m+1)!}\mathbb{i}(-\mathbb{i}\theta)^{2m+1}\hat{b}_y^\dagger = \cos\theta \cdot \hat{b}_x^\dagger + \sin\theta \cdot \hat{b}_y^\dagger$, and $\widehat{g(\theta)}\hat{b}_y^\dagger\widehat{g(\theta)}^\dagger = \sum_{m=0}^\infty \frac{1}{(2m)!}(-\mathbb{i}\theta)^{2m}\hat{b}_y^\dagger + \sum_{m=0}^\infty \frac{1}{(2m+1)!}(-\mathbb{i})(-\mathbb{i}\theta)^{2m+1}\hat{b}_x^\dagger = \cos\theta \cdot \hat{b}_y^\dagger - \sin\theta \cdot \hat{b}_x^\dagger$.

$R_{\hat{b}^\dagger}[g(\theta)] \cdot R_{\hat{b}^\dagger}[g(\theta')] = R_{\hat{b}^\dagger}[g(\theta+\theta')]$ can be easily checked using the trigonometric identities, $\cos\theta\cos\theta' - \sin\theta\sin\theta' = \cos(\theta+\theta')$, $\cos\theta\sin\theta' + \sin\theta\cos\theta' = \sin(\theta+\theta')$.

(f)

$\int_0^{2\pi} \mathrm{d}\theta\, (\chi_n[g(\theta)])^* \cdot \widehat{g(\theta)}\hat{b}_x^\dagger\widehat{g(\theta)}^\dagger = \int_0^{2\pi} \mathrm{d}\theta\, e^{-\mathbb{i}n\theta} \cdot (\hat{b}_x^\dagger \cos\theta + \hat{b}_y^\dagger \sin\theta)$

$$= \begin{cases} \pi \cdot (\hat{b}_x^\dagger + \mathbb{i}\hat{b}_y^\dagger), & n = -1; \\ \pi \cdot (\hat{b}_x^\dagger - \mathbb{i}\hat{b}_y^\dagger), & n = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$\int_0^{2\pi} \mathrm{d}\theta\, (\chi_n[g(\theta)])^* \cdot \widehat{g(\theta)}\hat{b}_y^\dagger\widehat{g(\theta)}^\dagger = \int_0^{2\pi} \mathrm{d}\theta\, e^{-\mathbb{i}n\theta} \cdot (\hat{b}_y^\dagger \cos\theta - \hat{b}_x^\dagger \sin\theta)$

$$= \begin{cases} \pi \cdot (\hat{b}_y^\dagger - \mathbb{i}\hat{b}_x^\dagger), & n = -1; \\ \pi \cdot (\hat{b}_y^\dagger + \mathbb{i}\hat{b}_x^\dagger), & n = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then one can see the basis for $n = -1$ irrep is proportional to $(\hat{b}_x^\dagger + \mathbb{i}\hat{b}_y^\dagger)$, for $n = 1$ irrep is proportional to $(\hat{b}_x^\dagger - \mathbb{i}\hat{b}_y^\dagger)$. And this 2-dimensional representation $R_{\hat{b}^\dagger}$ does not contain other irreps.

The normalized basis of creation operators can be chosen as $\hat{a}_1^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_x^\dagger + \mathbb{i}\hat{b}_y^\dagger)$, $\hat{a}_2^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_x^\dagger - \mathbb{i}\hat{b}_y^\dagger)$, then $R_{\hat{a}_1^\dagger}[g(\theta)] = R_{-1}[g(\theta)] = e^{-\mathbb{i}\theta}$, $R_{\hat{a}_2^\dagger}[g(\theta)] = R_1[g(\theta)] = e^{\mathbb{i}\theta}$.

Then $\hat{H} = \hbar\omega \cdot (\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1)$, $\hat{L}_z = \hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2$.

You may have switched the definitions of $\hat{a}_1$ and $\hat{a}_2$, then just exchange the subscripts $_1$ and $_2$ in the above results.

(g) Define $\hat{n}_1 = \hat{a}_1^\dagger \hat{a}_1$, $\hat{n}_2 = \hat{a}_2^\dagger \hat{a}_2$. Then $|\hat{n}_1 = n_1, \hat{n}_2 = n_2\rangle \equiv \frac{1}{\sqrt{n_1! n_2!}} (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} |\text{vac}\rangle$ is the simultaneous eigenstate of $\hat{H}$ and $\widehat{L_z}$, with eigenvalues $E = \hbar\omega \cdot (n_1 + n_2 + 1)$, and $\ell = (n_1 - n_2)$. Here $n_{1,2}$ are non-negative integers.

(h)
$\widehat{g(\theta)} |\hat{H} = E, \widehat{L_z} = \ell\rangle = \exp(-\mathbb{i}\theta \widehat{L_z}) |\hat{H} = E, \widehat{L_z} = \ell\rangle = |\hat{H} = E, \widehat{L_z} = \ell\rangle \cdot e^{-\mathbb{i}\theta\ell}$.

Therefore $R_{E,\ell}[g(\theta)] = R_{-\ell}[g(\theta)] = e^{-\mathbb{i}\theta\ell}$

(i)
$|\hat{H} = E', \widehat{L_z} = \ell'\rangle = |\hat{n}_1 = \frac{1}{2}(\frac{E'}{\hbar\omega} - 1 + \ell'), \hat{n}_2 = \frac{1}{2}(\frac{E'}{\hbar\omega} - 1 - \ell')\rangle$.

And $\hat{a}_1^\dagger |\hat{n}_1 = n_1, \hat{n}_2 = n_2\rangle = \sqrt{n_1 + 1} |\hat{n}_1 = n_1 + 1, \hat{n}_2 = n_2\rangle$,
$\hat{a}_2^\dagger |\hat{n}_1 = n_1, \hat{n}_2 = n_2\rangle = \sqrt{n_2 + 1} |\hat{n}_1 = n_1, \hat{n}_2 = n_2 + 1\rangle$.

Then $\hat{a}_1^\dagger |\hat{H} = E', \widehat{L_z} = \ell'\rangle = \sqrt{\frac{1}{2}(\frac{E'}{\hbar\omega} + 1 + \ell')} |\hat{H} = E' + \hbar\omega, \widehat{L_z} = \ell' + 1\rangle$,
$\hat{a}_2^\dagger |\hat{H} = E', \widehat{L_z} = \ell'\rangle = \sqrt{\frac{1}{2}(\frac{E'}{\hbar\omega} + 1 - \ell')} |\hat{H} = E' + \hbar\omega, \widehat{L_z} = \ell' - 1\rangle$.

Therefore $\langle \hat{H} = E, \widehat{L_z} = \ell | \hat{a}_1^\dagger | \hat{H} = E', \widehat{L_z} = \ell'\rangle = \sqrt{\frac{1}{2}(\frac{E'}{\hbar\omega} + 1 + \ell')} \delta_{\frac{E}{\hbar\omega}, \frac{E'}{\hbar\omega}+1} \delta_{\ell, \ell'+1}$,
$\langle \hat{H} = E, \widehat{L_z} = \ell | \hat{a}_2^\dagger | \hat{H} = E', \widehat{L_z} = \ell'\rangle = \sqrt{\frac{1}{2}(\frac{E'}{\hbar\omega} + 1 - \ell')} \delta_{\frac{E}{\hbar\omega}, \frac{E'}{\hbar\omega}+1} \delta_{\ell, \ell'-1}$.

Note that
$R^*_{E,\ell} \otimes R_{\hat{a}_1^\dagger} \otimes R_{E',\ell'} = R^*_{-\ell} \otimes R_{-1} \otimes R_{-\ell'} = R_\ell \otimes R_{-1} \otimes R_{-\ell'} = R_{\ell-1-\ell'}$;
$R^*_{E,\ell} \otimes R_{\hat{a}_2^\dagger} \otimes R_{E',\ell'} = R^*_{-\ell} \otimes R_{1} \otimes R_{-\ell'} = R_\ell \otimes R_{1} \otimes R_{-\ell'} = R_{\ell+1-\ell'}$.

Here we have used $R^*_n = R_{-n}$, and $R_n \otimes R_m = R_{n+m}$ for the irreps of $SO(2)$. The trivial irrep is R_0, so the $\delta_{\ell,\ell'\pm 1}$ factors in the above results are consistent with the selection rule for this $SO(2)$ group.

NOTE: the $\delta_{\frac{E}{\hbar\omega}, \frac{E'}{\hbar\omega}+1}$ factor is the selection rule for "energy conservation symmetry".

(j) Use the commutator formula for bilinear operators in (d), these three operators satisfy the commutation relation of Pauli matrices.

$[\widehat{L}_x, \widehat{L}_y] = 2\mathbb{i}\widehat{L}_z$, $[\widehat{L}_y, \widehat{L}_z] = 2\mathbb{i}\widehat{L}_x$, $[\widehat{L}_z, \widehat{L}_x] = 2\mathbb{i}\widehat{L}_y$.

If you have switched the definitions of $\hat{a}_1$ and $\hat{a}_2$ in (f), this would be $[\widehat{L}_x, \widehat{L}_y] = -2\mathbb{i}\widehat{L}_z$, $[\widehat{L}_y, \widehat{L}_z] = -2\mathbb{i}\widehat{L}_x$, $[\widehat{L}_z, \widehat{L}_x] = -2\mathbb{i}\widehat{L}_y$.

2. Considered $\hat{H} = (\hat{f}_1^\dagger \hat{f}_2 + \hat{f}_2^\dagger \hat{f}_3 + \hat{f}_3^\dagger \hat{f}_4 + \hat{f}_4^\dagger \hat{f}_1 + \text{h.c.})$.

Here $\hat{f}_i(\hat{f}_i^\dagger)$ are annihilation(creation) operators for 4 fermion modes, satisfying $\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{ij}$ and $\{\hat{f}_i, \hat{f}_j\} = 0$, and h.c. means hermitian conjugate of the previous 4 terms.

The model conserves total particle number $\hat{n} = \sum_{i=1}^4 \hat{f}_i^\dagger \hat{f}_i$, namely $[\hat{H}, \hat{n}] = 0$.

$\hat{H}$ also has the D_4 point group symmetry, generated by

"4-fold rotation" $C_4: \ \hat{f}_1 \to \hat{f}_2 \to \hat{f}_3 \to \hat{f}_4 \to \hat{f}_1$, (this means $\widehat{C_4}\hat{f}_1\widehat{C_4}^\dagger = \hat{f}_2$, etc.), and

"principal axis reflection" $\sigma_s: \ \hat{f}_1 \to \hat{f}_1, \hat{f}_2 \to \hat{f}_4, \hat{f}_3 \to \hat{f}_3, \hat{f}_4 \to \hat{f}_2$.

This group has 8 elements, and 5 conjugacy classes: $\{\mathbf{1}\}, \{C_4, C_4^3\}, \{C_4^2\}, \{\sigma_s, C_4^2\sigma_s\}, \{\sigma_d \equiv C_4\sigma_s, C_4^3\sigma_s\}$. The character table for the five irreducible representations, $\Gamma_{1,2,3,4,5}$, is

	$\mathbf{1}$	$2C_4$	C_4^2	$2\sigma_s$	$2\sigma_d$
Γ_1	1	1	1	1	1
Γ_2	1	1	1	-1	-1
Γ_3	1	-1	1	1	-1
Γ_4	1	-1	1	-1	1
Γ_5	2	0	-2	0	0

(a) (5pts) A group element $g \in D_4$ will transform $\hat{f}_i^\dagger$ as $\hat{f}_i^\dagger \mapsto \sum_j \hat{f}_j^\dagger \cdot R[g]_{ji}$, where $R[g]$ is the 4×4 representation matrix. Decompose this into irreducible representations. Namely find $\hat{f'}_i^\dagger = \sum_j \hat{f}_j^\dagger \cdot U_{ji}$, where U_{ji} is a 4×4 unitary matrix, so that $\hat{f'}_i^\dagger$ transform under $g \in D_4$ as $\hat{f'}_i^\dagger \mapsto \sum_j \hat{f'}_j^\dagger \cdot R'[g]_{ji}$ with $R'[g]$ block-diagonalized, and each diagonal block is one of the irreducible representations. Solve the new basis $\hat{f'}_i^\dagger$ in terms of $\hat{f}_i^\dagger$ (or equivalently solve U), and the block-diagonalized representation $R'[g]$ for the generators $g = C_4$ and $g = \sigma_s$. [Hint: use the "projection operator" to find the new basis]

(b) (5pts) The Hilbert space with fixed total particle number $\hat{n}$ is a representation space of the D_4 group. Assume that the vacuum state $|\text{vac}\rangle$ is invariant under D_4 group. Then the transformation rules for $\hat{f}_i^\dagger$ completely determine the transformation rules for any states, for example C_4 transforms $\hat{f}_1^\dagger \hat{f}_2^\dagger |\text{vac}\rangle \mapsto \hat{f}_2^\dagger \hat{f}_3^\dagger |\text{vac}\rangle$. Decompose the 6-dimensional 2-particle Hilbert space, with occupation basis $\hat{f}_i^\dagger \hat{f}_j^\dagger |\text{vac}\rangle$ for $i < j$, into irreducible representations of D_4. [Hint: one can first work out the 6×6 representation and then change basis to block-diagonalize it; or use the result of (a) to construct the irreducible representation basis]

(c) (4pts) Rewrite $\hat{H}$ in terms of the $\hat{f'}_i^\dagger$ and $\hat{f'}_i$ solved in (a). Solve all the eigenvalues and eigenstates of $\hat{H}$ in the 2-particle Hilbert space.

Solution:

(a)

The basis can be chosen as

irrep. R'	basis	$R'[C_4]$	$R'[\sigma_s]$
Γ_1	$\hat{f'}_1^\dagger \equiv \hat{\Gamma}_1^\dagger = \frac{1}{2}(\hat{f}_1^\dagger + \hat{f}_2^\dagger + \hat{f}_3^\dagger + \hat{f}_4^\dagger)$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
Γ_3	$\hat{f'}_2^\dagger \equiv \hat{\Gamma}_3^\dagger = \frac{1}{2}(\hat{f}_1^\dagger - \hat{f}_2^\dagger + \hat{f}_3^\dagger - \hat{f}_4^\dagger)$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
Γ_5	$(\hat{f'}_3^\dagger \equiv \hat{\Gamma}_{5,x}^\dagger = \frac{1}{2}(\hat{f}_1^\dagger - \hat{f}_3^\dagger), \hat{f'}_4^\dagger \equiv \hat{\Gamma}_{5,y}^\dagger = \frac{1}{2}(\hat{f}_2^\dagger - \hat{f}_4^\dagger))$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

The procedures of using "projection operator" are summarized in the following tables,

g	$\mathbf{1}$	C_4	C_4^3	C_4^2	σ_s	$C_4^2\sigma_s$	$C_4\sigma_s$	$C_4^3\sigma_s$
$\widehat{gf_1^\dagger}$	$\hat{f}_1^\dagger$	$\hat{f}_2^\dagger$	$\hat{f}_4^\dagger$	$\hat{f}_3^\dagger$	$\hat{f}_1^\dagger$	$\hat{f}_3^\dagger$	$\hat{f}_2^\dagger$	$\hat{f}_4^\dagger$
$\widehat{gf_2^\dagger}$	$\hat{f}_2^\dagger$	$\hat{f}_3^\dagger$	$\hat{f}_1^\dagger$	$\hat{f}_4^\dagger$	$\hat{f}_4^\dagger$	$\hat{f}_2^\dagger$	$\hat{f}_1^\dagger$	$\hat{f}_3^\dagger$
$\widehat{gf_3^\dagger}$	$\hat{f}_3^\dagger$	$\hat{f}_4^\dagger$	$\hat{f}_2^\dagger$	$\hat{f}_1^\dagger$	$\hat{f}_3^\dagger$	$\hat{f}_1^\dagger$	$\hat{f}_4^\dagger$	$\hat{f}_2^\dagger$
$\widehat{gf_4^\dagger}$	$\hat{f}_4^\dagger$	$\hat{f}_1^\dagger$	$\hat{f}_3^\dagger$	$\hat{f}_2^\dagger$	$\hat{f}_2^\dagger$	$\hat{f}_4^\dagger$	$\hat{f}_3^\dagger$	$\hat{f}_1^\dagger$
$\chi_{\Gamma_1}(g)$	1	1	1	1	1	1	1	1
$\chi_{\Gamma_2}(g)$	1	1	1	1	-1	-1	-1	-1
$\chi_{\Gamma_3}(g)$	1	-1	-1	1	1	1	-1	-1
$\chi_{\Gamma_4}(g)$	1	-1	-1	1	-1	-1	1	1
$\chi_{\Gamma_5}(g)$	2	0	0	-2	0	0	0	0

$\sum_g \chi_R^*(g)\widehat{gf_i^\dagger}$	$R=\Gamma_1$	$R=\Gamma_2$	$R=\Gamma_3$	$R=\Gamma_4$	$R=\Gamma_5$
$i=1$	$2(\hat{f}_1^\dagger + \hat{f}_2^\dagger + \hat{f}_3^\dagger + \hat{f}_4^\dagger)$	0	$2(\hat{f}_1^\dagger - \hat{f}_2^\dagger + \hat{f}_3^\dagger - \hat{f}_4^\dagger)$	0	$2(\hat{f}_1^\dagger - \hat{f}_3^\dagger)$
$i=2$	$2(\hat{f}_2^\dagger + \hat{f}_3^\dagger + \hat{f}_4^\dagger + \hat{f}_1^\dagger)$	0	$2(\hat{f}_2^\dagger - \hat{f}_3^\dagger + \hat{f}_4^\dagger - \hat{f}_1^\dagger)$	0	$2(\hat{f}_2^\dagger - \hat{f}_4^\dagger)$
$i=3$	$2(\hat{f}_3^\dagger + \hat{f}_4^\dagger + \hat{f}_1^\dagger + \hat{f}_2^\dagger)$	0	$2(\hat{f}_3^\dagger - \hat{f}_4^\dagger + \hat{f}_1^\dagger - \hat{f}_2^\dagger)$	0	$2(\hat{f}_3^\dagger - \hat{f}_1^\dagger)$
$i=4$	$2(\hat{f}_4^\dagger + \hat{f}_1^\dagger + \hat{f}_2^\dagger + \hat{f}_3^\dagger)$	0	$2(\hat{f}_4^\dagger - \hat{f}_1^\dagger + \hat{f}_2^\dagger - \hat{f}_3^\dagger)$	0	$2(\hat{f}_4^\dagger - \hat{f}_2^\dagger)$

(b) Use $\hat{f'}_i^\dagger$ to construct the basis, $\hat{f'}_i\hat{f'}_j|\text{vac}\rangle$, the transformation rules of these states can be deduced from the transformation rules of $\hat{f'}_i^\dagger$ given in (a). The results are summarized in the following table,

irrep. R'	basis	$R'[C_4]$	$R'[\sigma_s]$
$\Gamma_3 = \Gamma_1 \otimes \Gamma_3$	$\hat{f'}_1^\dagger \hat{f'}_2^\dagger \lvert\text{vac}\rangle$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$\Gamma_5 = \Gamma_1 \otimes \Gamma_5$	$(\hat{f'}_1^\dagger \hat{f'}_3^\dagger \lvert\text{vac}\rangle, \hat{f'}_1^\dagger \hat{f'}_4^\dagger \lvert\text{vac}\rangle)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
$\Gamma_5 = \Gamma_3 \otimes \Gamma_5$	$(\hat{f'}_2^\dagger \hat{f'}_3^\dagger \lvert\text{vac}\rangle, -\hat{f'}_2^\dagger \hat{f'}_4^\dagger \lvert\text{vac}\rangle)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
Γ_2, from $\Gamma_5 \otimes \Gamma_5 = \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4$	$\hat{f'}_3^\dagger \hat{f'}_4^\dagger \lvert\text{vac}\rangle$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$

Note: if you use "projection operator" directly on the $\hat{f}_i^\dagger \hat{f}_j^\dagger |\text{vac}\rangle$ basis, you will get the following Γ_5 irreps,

irrep. R'	basis	$R'[C_4]$	$R'[\sigma_s]$
Γ_5	$(\frac{\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_3^\dagger \hat{f}_4^\dagger}{\sqrt{2}} \lvert\text{vac}\rangle, \frac{\hat{f}_2^\dagger \hat{f}_3^\dagger - \hat{f}_4^\dagger \hat{f}_1^\dagger}{\sqrt{2}} \lvert\text{vac}\rangle)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$
Γ_5	$(\hat{f}_1^\dagger \hat{f}_3^\dagger \lvert\text{vac}\rangle, \hat{f}_2^\dagger \hat{f}_4^\dagger \lvert\text{vac}\rangle)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

They are just linear superpositions of the two copies of Γ_5 irrep. given by $\hat{f}'$ basis. To see this, change the first Γ_5 basis above so that the representation matrices are exactly the same as before,

irrep. R'	basis	$R'[C_4]$	$R'[\sigma_s]$
Γ_5	$(\frac{(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_3^\dagger \hat{f}_4^\dagger)+(\hat{f}_2^\dagger \hat{f}_3^\dagger - \hat{f}_4^\dagger \hat{f}_1^\dagger)}{2} \lvert\text{vac}\rangle, \frac{-(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_3^\dagger \hat{f}_4^\dagger)+(\hat{f}_2^\dagger \hat{f}_3^\dagger - \hat{f}_4^\dagger \hat{f}_1^\dagger)}{2} \lvert\text{vac}\rangle)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Then $\frac{(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_3^\dagger \hat{f}_4^\dagger)+(\hat{f}_2^\dagger \hat{f}_3^\dagger - \hat{f}_4^\dagger \hat{f}_1^\dagger)}{2} = \frac{-(\hat{f}_2^\dagger + \hat{f}_4^\dagger)(\hat{f}_1^\dagger - \hat{f}_3^\dagger)}{2} = \frac{1}{\sqrt{2}}(-\hat{f'}_1 \hat{f'}_3 + \hat{f'}_2 \hat{f'}_3)$, $\frac{-(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_3^\dagger \hat{f}_4^\dagger)+(\hat{f}_2^\dagger \hat{f}_3^\dagger - \hat{f}_4^\dagger \hat{f}_1^\dagger)}{2} = \frac{-(\hat{f}_1^\dagger + \hat{f}_3^\dagger)(\hat{f}_2^\dagger - \hat{f}_4^\dagger)}{2} = \frac{1}{\sqrt{2}}(-\hat{f'}_1 \hat{f'}_4 + (-\hat{f'}_2 \hat{f'}_4))$. And $\hat{f}_1 \hat{f}_3 = \frac{-(\hat{f}_1 + \hat{f}_3)(\hat{f}_1 - \hat{f}_3)}{2} = \frac{1}{\sqrt{2}}(-\hat{f'}_1 \hat{f'}_3 - \hat{f'}_2 \hat{f'}_3)$, $\hat{f}_2 \hat{f}_4 = \frac{-(\hat{f}_2 + \hat{f}_4)(\hat{f}_2 - \hat{f}_4)}{2} = \frac{1}{\sqrt{2}}(-\hat{f'}_1 \hat{f'}_4 - (-\hat{f'}_2 \hat{f'}_4))$.

(c) $\hat{H} = 2\hat{f'}_1^\dagger \hat{f'}_1 - 2\hat{f'}_2^\dagger \hat{f'}_2 \equiv 2\hat{\Gamma}_1^\dagger \hat{\Gamma}_1 - 2\hat{\Gamma}_3^\dagger \hat{\Gamma}_3$.

Note that because $\hat{H}$ is invariant under the D_4 group, namely that $\hat{H}$ is a trivial representation, the bilinear operators in $\hat{H}$ must contain creation and annihilation operators of the same irrep. (in fact the creation operator should be in the irrep conjugate to the annihilation operator, but all these irreps here are self-conjugate).

The occupation basis of $\hat{f}'$, $|n_1', n_2', n_3', n_4'\rangle = (\hat{f'}_1^\dagger)^{n_1'} (\hat{f'}_2^\dagger)^{n_2'} (\hat{f'}_3^\dagger)^{n_3'} (\hat{f'}_4^\dagger)^{n_4'} |\text{vac}\rangle$, are eigenstates with eigenvalues $2(n'_1 - n'_2)$. Here $n'_{1,2,3,4}$ can only be 0 or 1, so the normalization factor $\frac{1}{\sqrt{n'_1!n'_2!n'_3!n'_4!}}$ is omitted.

For the 2-particle states in (b),

irrep. R'	basis	$\hat{H}$ eigenvalue
$\Gamma_3 = \Gamma_1 \otimes \Gamma_3$	$\hat{f'}_1^\dagger \hat{f'}_2^\dagger \lvert\text{vac}\rangle$	$2 + (-2) = 0$
$\Gamma_5 = \Gamma_1 \otimes \Gamma_5$	$(\hat{f'}_1^\dagger \hat{f'}_3^\dagger \lvert\text{vac}\rangle, \hat{f'}_1^\dagger \hat{f'}_4^\dagger \lvert\text{vac}\rangle)$	$2 + 0 = 2$ (2-fold degenerate)
$\Gamma_5 = \Gamma_3 \otimes \Gamma_5$	$(\hat{f'}_2^\dagger \hat{f'}_3^\dagger \lvert\text{vac}\rangle, -\hat{f'}_2^\dagger \hat{f'}_4^\dagger \lvert\text{vac}\rangle)$	$-2 + 0 = -2$ (2-fold degenerate)
Γ_2	$\hat{f'}_3^\dagger \hat{f'}_4^\dagger \lvert\text{vac}\rangle$	$0 + 0 = 0$