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Clifford Algebras and Lie Theory

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Dedicated to my family Nozomi and Emma

Preface

Given a symmetric bilinear form B on a vector space V, one defines the Clifford algebra $\mathrm{Cl}(V; B)$ to be the associative algebra generated by the elements of V, with relations

$$v_1 v_2 + v_2 v_1 = 2B(v_1, v_2), \quad v_1, v_2 \in V.$$

If $B = 0$ this is just the exterior algebra $\wedge(V)$, and for general B there is an isomorphism (the *quantization map*)

$$q \colon \wedge(V) \to \mathrm{Cl}(V; B).$$

Hence, the Clifford algebra may be regarded as $\wedge(V)$ with a new, "deformed" product.

Clifford algebras enter the world of Lie groups and Lie algebras from multiple directions. For example, they are used to give constructions of the spin groups $\mathrm{Spin}(n)$, the simply connected double coverings of $\mathrm{SO}(n)$ for $n \geq 3$. Going a little further, one then obtains the spin representations of $\mathrm{Spin}(n)$, which is an irreducible representation if n is odd and breaks up into two inequivalent irreducible representations if n is even. The "accidental" isomorphisms of Lie groups in low dimensions, such as $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$, all find natural explanations using the spin representation. Furthermore, there are explicit constructions of the exceptional groups E_6, E_7, E_8, F_4, G_2, using special features of the spin groups and the spin representation. There are many remarkable aspects of these constructions, see e.g., Adams' book [1] or the survey article by Baez [20].

Further relationships between Lie groups and Clifford algebras come from the theory of Dirac operators on homogeneous spaces. Recall that by the Borel–Weil Theorem, any irreducible representation of a compact Lie group G is realized as a space of holomorphic sections of a line bundle over an appropriate coadjoint orbit. These sections may be regarded as solutions of the Dolbeault–Dirac operator for the standard complex structure on the coadjoint orbit. For non-compact Lie groups, there are many constructions of representations using more general types of Dirac operators, beginning with the work of Atiyah–Schmid [16] and Parthasarathy [106]. See the book by Huang–Pandzic [66] for further references and more recent developments.

Given a Lie algebra $\mathfrak{g}$ with a non-degenerate invariant symmetric bilinear form B (e.g., $\mathfrak{g}$ semisimple, with B the Killing form), it is also of interest to consider the Clifford algebra of $\mathfrak{g}$ itself. Let $\phi \in \wedge^3\mathfrak{g}$ be the structure constant tensor of $\mathfrak{g}$, defined using the metric. By a beautiful observation of Kostant and Sternberg [90], the quantized element $q(\phi) \in \mathrm{Cl}(\mathfrak{g})$ squares to a scalar. If $\mathfrak{g}$ is complex semisimple, the cubic element is one of the basis elements

$$\phi_i \in \wedge^{2m_i+1}(\mathfrak{g}), \quad i = 1, \ldots, \mathrm{rank}(\mathfrak{g}),$$

of the *primitive subspace* $P(\mathfrak{g}) \subseteq \wedge(\mathfrak{g})^{\mathfrak{g}}$ of the invariant subalgebra of the exterior algebra. According to the Hopf–Koszul–Samelson Theorem, $\wedge(\mathfrak{g})^{\mathfrak{g}}$ is itself an exterior algebra $\wedge P(\mathfrak{g})$ with generators $\phi_1, \ldots, \phi_l$. In [86], Kostant proved the marvelous result that, similarly, the invariant subalgebra of the Clifford algebra $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ is itself a Clifford algebra $\mathrm{Cl}(P(\mathfrak{g}))$ over the quantized generators $q(\phi_i)$. In particular, the elements $q(\phi_i)$ all square to scalars—a surprising fact given that the ϕ_i can have very high degree.

My own involvement with this subject goes back to work with Anton Alekseev on group-valued moment maps [11]. We had developed a non-standard non-commutative equivariant de Rham theory tailored towards these applications, and were looking for a natural framework in order to understand its relation to the standard equivariant de Rham theory. Inspired by [86, 90], we were led to consider a *quantum Weil algebra*

$$\mathscr{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}),$$

where $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$, and equipped with a differential similar to the usual Weil differential on the standard Weil algebra $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$. The differential was explicitly given as a graded commutation with a cubic element

$$\mathscr{D}_{\mathfrak{g}} = \sum_i e^i \otimes e_i + 1 \otimes q(\phi) \in \mathscr{W}(\mathfrak{g}),$$

where e_i is a basis of $\mathfrak{g}$, e^i the dual basis, and $\phi \in \wedge^3\mathfrak{g}$ as above. The fact that $[\mathscr{D}_{\mathfrak{g}}, \cdot]$ squares to zero is due to the fact that $\mathscr{D}_{\mathfrak{g}}^2$ lies in the center of $\mathscr{W}(\mathfrak{g})$—in fact one finds that

$$\mathscr{D}_{\mathfrak{g}}^2 = \frac{1}{2}\mathrm{Cas}_g \otimes 1 + \mathrm{const.},$$

where $\mathrm{Cas}_{\mathfrak{g}} \in U(\mathfrak{g})$ is the quadratic Casimir element. In [4] and its sequel [7], we used this theory to give a new proof for the case of quadratic Lie algebras of *Duflo's Theorem*, giving an algebra isomorphism

$$\mathrm{sym} \circ \widehat{J^{1/2}} \colon S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$$

(cf. Section 5.6 for notation) between invariants in the symmetric and enveloping algebras of $\mathfrak{g}$.

Independently, Kostant [87] had introduced a more general cubic element

$$\mathscr{D}_{\mathfrak{g},\mathfrak{k}} \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}),$$

associated to a quadratic Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ with orthogonal complement $\mathfrak{p} = \mathfrak{k}^{\perp}$, which he called the *cubic Dirac operator*. Taking $\mathfrak{k} = 0$, one recovers $\mathscr{D}_{\mathfrak{g},0} = \mathscr{D}_{\mathfrak{g}}$. For the case where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of compact Lie groups G and K, with K a maximal rank subgroup of G, he used the cubic Dirac operator to obtain an algebraic version of the Borel–Weil construction [87, 88], not requiring invariant complex structures on G/K. (In the complex case, an algebraic version of the Borel–Weil method had been accomplished several decades earlier in Kostant's work [84].) Other beautiful applications include a generalization of the Weyl character formula, and the discovery of multiplets of representations due to Gross–Kostant–Ramond–Sternberg [57].

It is possible to go in the opposite direction and interpret $\mathscr{D}_{\mathfrak{g},\mathfrak{k}}$ as a geometric Dirac operator on a homogeneous space. As such, it corresponds to a particular affine connection with a non-zero torsion. Such geometric Dirac operators had been studied by Slebarski [114, 115] in the 1980s; the precise relation with Kostant's cubic Dirac operator was clarified in [2]. There are also precursors in conformal field theory and Kac–Moody theory; see in particular Kac–Todorov [73], Kazami–Suzuki [77] and Wassermann [119].

This book had its beginnings as lecture notes for a graduate course at the University of Toronto, taught in the fall of 2005. The plan for the lectures was to give an introduction to the cubic Dirac operator, covering some of the applications to Lie theory described above. To set the stage, it was necessary to develop some foundational material on Clifford algebras. In a similar graduate course in the fall of 2009, we included other aspects such as the theory of pure spinors and Petracci's new proof [107] of the Poincaré–Birkhoff–Witt Theorem. The book itself contains further topics not covered in the lectures; in particular it describes Kostant's structure theory of $\mathrm{Cl}(\mathfrak{g})$ for a complex reductive Lie algebra and surveys the recent developments concerning the properties of the Harish-Chandra projection for Clifford algebras, $\mathsf{hc}_{\mathrm{Cl}}\colon \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{t})$.

A number of interesting topics related to the theme of this book had to be omitted. For example, we did not include applications of the cubic Dirac operator to Kac–Moody algebras [53, 73, 79, 94, 103, 109, 119]. We also decided to omit a discussion of Dirac geometry [13, 43], group-valued moment maps [11] or generalized complex geometry [59, 60, 63], even though some of the material presented here is motivated by those topics. Furthermore, the book does not cover the applications to the classical dynamical Yang–Baxter equation [6, 51] or to the Kashiwara–Vergne conjecture [5, 8, 75].

Many of the topics in this book play a role in theoretical physics. We have already indicated some relationships with conformal field theory. The prototype of a Dirac operator (with respect to the Minkowski metric) appeared in Dirac's 1928 article on his quantum theory of the electron [46]. More general Dirac operators on pseudo-Riemannian manifolds were studied later, and became a standard tool in differential geometry with the advent of Atiyah–Singer's index theory [18, 97]. The role of representation theory in quantum mechanics was explored in Weyl's 1929 book [120]. The "multi-valued" representations of the rotation group, i.e., representations of the spin group, are of basic importance in quantum electrodynamics, corresponding to particles with possibly half-integer spin. A systematic study, introducing the group Spin(n), was made in 1935 in the work of Brauer–Weyl [29]. Detailed information on the numerous applications of Clifford algebras, spinors and Dirac operators in physics may be found in the books [23, 64].

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Conventions

Throughout this book $\mathbb{K}$ will denote a ground field. The precise convention will usually be stated at the beginning of a chapter—initially $\mathbb{K}$ is any field of characteristic $\neq 2$; towards the end of the book we take $\mathbb{K}$ to be the real or complex numbers. We will make frequent use of super-conventions, as outlined in Appendix A. For instance, given a $\mathbb{Z}_2$-graded algebra, we will refer to "commutators" where some authors prefer the term "super commutator", and we will call "derivations" what some authors would call "super derivations" or even "anti-derivations". Unless specified otherwise or clear from the context, vector spaces or Lie algebras are taken to be finite dimensional, and algebras are taken to be associative algebras with unit. Some background material and conventions about reductive Lie algebras may be found in Appendix B.

Contents

List of Symbols

$\mathbb{R}, \mathbb{C}, \mathbb{H}$	real numbers, complex numbers, quaternions
$\mathrm{Der}(\cdot)$	derivations
$\mathrm{Aut}(\cdot)$	automorphisms
$\mathrm{End}(\cdot)$	endomorphisms
id_X	identity map from X to itself
ev_x	evaluation at element x
$\mathrm{Mat}_n(\mathbb{K})$	$n \times n$-matrices with coefficients in $\mathbb{K}$
$\Omega(M)$	differential forms
$C^\infty(M)$	smooth functions
$\mathfrak{X}(M)$	vector fields
$B^\flat$	map $V \to V^*$ defined by a bilinear form B, p. 1
$B^\sharp$	inverse to $B^\flat$, for B non-degenerate
$\wedge(V)$, $S(V)$, $T(V)$	exterior algebra (p. 23), symmetric algebra, tensor algebra
$\mathrm{Cl}(V)$, $\mathrm{Cl}(V, B)$	Clifford algebra, p. 27
$\mathbb{C}l(V)$	complex Clifford algebra, p. 74
σ, q	symbol map, quantization map for the Clifford algebra, p. 32
$\mathrm{Pin}(V)$, $\mathrm{Spin}(V)$	Pin group, Spin group, p. 51
$\Gamma(V)$	Clifford group, p. 49
N	norm homomorphism, p. 51
$V^{\mathfrak{g}}$, V^G	invariant subspace of a $\mathfrak{g}$-representation, resp. G-representation
S_F, S^F, S	spinor modules, pp. 56, 58, 60
ρ, ρ_E	action on a Clifford module, p. 54
$U(\mathfrak{g})$	universal enveloping algebra, p. 109
$W(\mathfrak{g})$	Weil algebra, p. 149
$\mathcal{W}(\mathfrak{g})$	quantum Weil algebra, p. 169
$\widetilde{W}(\mathfrak{g})$	non-commutative Weil algebra, p. 154
$\mathcal{D}$, $\mathcal{D}_{\mathfrak{g}}$, $\mathcal{D}(\mathfrak{g})$	cubic Dirac operator, p. 171
hc_U	Harish-Chandra projection for $U(\mathfrak{g})$, pp. 182, 192
$\mathsf{hc}_{\mathrm{Cl}}$	Harish-Chandra projection for $\mathrm{Cl}(\mathfrak{g})$, p. 184
$\mathsf{hc}_{\mathcal{W}}$	Harish-Chandra projection for $\mathcal{W}(\mathfrak{g})$, p. 188

hc_{KNV}	generalized Harish-Chandra projection, p. 273
t	transgression map, p. 158
$P(\mathfrak{g})$	primitive elements in $(\wedge\mathfrak{g})^{\mathfrak{g}}$, p. 241
$P_S(\mathfrak{g})$	a generating subspace of $S(\mathfrak{g})^{\mathfrak{g}}$, p. 264
$\mathrm{Har}_S(\mathfrak{g})$	harmonic polynomials, p. 263
$\underline{n}$	n modulo 2, p. 275
$\mathrm{gr}(\cdot)$	associated graded space of a filtered vector space, p. 280
$\mathsf{ch}(V)$	formal character of a representation, p. 293
$\mathfrak{D}(M, E)$	differential operators on a vector bundle $E \to M$, p. 42
$\mathrm{Pol}(\cdot)$	algebra of polynomials
$U(\mathfrak{g})$	universal enveloping algebra, p. 109
$J^{1/2}$	Duflo factor, p. 133
$\iota(\cdot)$	contraction operator
$\varepsilon(\cdot)$	exterior multiplication
$L(\cdot)$	Lie derivative
E_{bas}	basic subspace, p. 144
$\mathrm{d}_{\wedge}$	Lie algebra differential on $\wedge\mathfrak{g}^*$ p. 147
d_{CE}	Chevalley–Eilenberg differential, p. 147
d_K	Koszul differential, p. 140
∂	differential on $\wedge\mathfrak{g}$, p. 233
h	homotopy operator, p. 139
Δ, ε	comultiplication, counit, p. 118
$\mathfrak{R}, \mathfrak{R}_+$	roots, positive roots, p. 285
$(\cdot)[n]$	degree shift, p. 278
W	Weyl group, p. 288
$l(w)$	length of a Weyl group element, p. 290
w_α	Weyl reflection defined by a root, p. 288
ht	height, p. 290
ρ	half-sum of positive roots, p. 291
$\mathfrak{t}^*_+$	positive Weyl chamber, p. 291
P, Q	weight lattice, root lattice, p. 287
$P^\vee$, $Q^\vee$	coweight lattice, corrot lattice, p. 287
ξ^L, ξ^R	left-, right-invariant vector field defined by $\xi \in \mathfrak{g}$, p. 104
ξ_M	generating vector field for a Lie algebra action, p. 302
θ^L, θ^R	left-, right-invariant Maurer–Cartan forms, p. 104

Chapter 1
Symmetric bilinear forms

In this chapter we will describe the foundations of the theory of non-degenerate symmetric bilinear forms on finite-dimensional vector spaces and their orthogonal groups. Among the highlights of this discussion are the *Cartan–Dieudonné Theorem*, which states that any orthogonal transformation is a finite product of reflections, and *Witt's Theorem* giving a partial normal form for quadratic forms. The theory of *split* symmetric bilinear forms is found to have many parallels to the theory of symplectic forms, and we will give a discussion of the Lagrangian Grassmannian for this case. Throughout $\mathbb{K}$ will denote a ground field of characteristic $\neq 2$. We are mainly interested in the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and sometimes we specialize to those two cases.

1.1 Quadratic vector spaces

Suppose V is a finite-dimensional vector space over $\mathbb{K}$. For any bilinear form $B: V \times V \to \mathbb{K}$, define a linear map

$$B^\flat:\ V \to V^*,\ v \mapsto B(v, \cdot).$$

The bilinear form B is called *symmetric* if it satisfies $B(v_1, v_2) = B(v_2, v_1)$ for all $v_1, v_2 \in V$. Since $\dim V < \infty$ this is equivalent to $(B^\flat)^* = B^\flat$. The symmetric bilinear form B is uniquely determined by the associated quadratic form $Q_B(v) = B(v, v)$, using the *polarization identity*

$$B(v, w) = \frac{1}{2}\big(Q_B(v + w) - Q_B(v) - Q_B(w)\big). \tag{1.1}$$

The *kernel* (also called *radical*) of B is the subspace

$$\ker(B) = \{v \in V \mid B(v, w) = 0 \text{ for all } w \in V\},$$

i.e., the kernel of the linear map $B^\flat$. The bilinear form B is called *non-degenerate* if $\ker(B) = 0$, i.e., if and only if $B^\flat$ is an isomorphism. A vector space V together with a non-degenerate symmetric bilinear form B will be referred to as a *quadratic*

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vector space. Assume for the rest of this chapter that (V, B) is a quadratic vector space.

Definition 1.1 A vector $v \in V$ is called *isotropic* if $B(v, v) = 0$.

For instance, if $V = \mathbb{C}^n$ over $\mathbb{K} = \mathbb{C}$, with the standard bilinear form $B(z, w) = \sum_{i=1}^n z_i w_i$, then $v = (1, i, 0, \ldots, 0)$ is an isotropic vector. If $V = \mathbb{R}^2$ over $\mathbb{K} = \mathbb{R}$, with bilinear form $B(x, y) = x_1 y_1 - x_2 y_2$, then the set of isotropic vectors $x = (x_1, x_2)$ is given by the "light cone" $x_1 = \pm x_2$.

The *orthogonal group* $\mathrm{O}(V)$ is the group

$$\mathrm{O}(V) = \{A \in \mathrm{GL}(V) |\ \ B(Av, Aw) = B(v, w) \text{ for all } v, w \in V\}. \tag{1.2}$$

The subgroup of orthogonal transformations of determinant 1 is denoted by $\mathrm{SO}(V)$, and is called the *special orthogonal group*.

For any subspace $F \subseteq V$, the *orthogonal* or *perpendicular* subspace is defined as

$$F^\perp = \{v \in V |\ B(v, v_1) = 0 \text{ for all } v_1 \in F\}.$$

The image of $B^\flat(F^\perp) \subseteq V^*$ is the annihilator of F. From this one deduces the dimension formula

$$\dim F + \dim F^\perp = \dim V \tag{1.3}$$

and the identities

$$(F^\perp)^\perp = F, \quad (F_1 \cap F_2)^\perp = F_1^\perp + F_2^\perp, \quad (F_1 + F_2)^\perp = F_1^\perp \cap F_2^\perp$$

for all $F, F_1, F_2 \subseteq V$. For any subspace $F \subseteq V$ the restriction of B to F has kernel $\ker(B|_{F \times F}) = F \cap F^\perp$.

Definition 1.2 A subspace $F \subseteq V$ is called a *quadratic subspace* if the restriction of B to F is non-degenerate, that is $F \cap F^\perp = 0$.

Using $(F^\perp)^\perp = F$ we see that F is quadratic $\Leftrightarrow$ $F^\perp$ is quadratic $\Leftrightarrow$ $F \oplus F^\perp = V$. As a simple application, one finds that any non-degenerate symmetric bilinear form B on V can be "diagonalized". Let us call a basis $E_1, \ldots, E_n$ of V an *orthogonal basis* if $B(E_i, E_j) = 0$ for all $i \neq j$.

Proposition 1.1 *Any quadratic vector space (V, B) admits an orthogonal basis $E_1, \ldots, E_n$. If $\mathbb{K} = \mathbb{C}$, one can arrange that $B(E_i, E_i) = 1$ for all i. If $\mathbb{K} = \mathbb{R}$ or $K = \mathbb{Q}$, one can arrange that $B(E_i, E_i) = \pm 1$ for all i.*

Proof The proof is by induction on $n = \dim V$, the case $\dim V = 1$ being obvious. If $n > 1$, choose any non-isotropic vector $E_1 \in V$. The 1-dimensional subspace $\mathrm{span}(E_1)$ is a quadratic subspace, hence so is the orthogonal subspace $\mathrm{span}(E_1)^\perp$. By induction, there is an orthogonal basis $E_2, \ldots, E_n$ of $\mathrm{span}(E_1)^\perp$.

If $\mathbb{K}=\mathbb{C}$ (resp. $\mathbb{K}=\mathbb{R},\mathbb{Q}$), one can rescale the E_i such that $B(E_i,E_i)=1$ (resp. $B(E_i,E_i)=\pm 1$). $\square$

We will denote by $\mathbb{K}^{n,m}$ the vector space $\mathbb{K}^{n+m}$ with bilinear form given by

$$B(E_i,E_j)=\begin{cases}\delta_{ij} & : i=1,\ldots,n,\\ -\delta_{ij} & : i=n+1,\ldots,n+m.\end{cases}$$

If $m=0$ we simply write $\mathbb{K}^n=\mathbb{K}^{n,0}$, and refer to the bilinear form as *standard*. The proposition above shows that for $\mathbb{K}=\mathbb{C}$, any quadratic vector space (V,B) is isomorphic to $\mathbb{C}^n$ with the standard bilinear form, while for $\mathbb{K}=\mathbb{R}$, it is isomorphic to some $\mathbb{R}^{n,m}$. (Here n,m are uniquely determined, although it is not entirely obvious at this point.)

1.2 Isotropic subspaces

Let (V,B) be a quadratic vector space.

Definition 1.3 A subspace $F\subseteq V$ is called *isotropic*[1] if $B|_{F\times F}=0$, that is $F\subseteq F^\perp$.

The polarization identity (1.1) shows that a subspace $F\subseteq V$ is isotropic if and only if all of its vectors are isotropic. If $F\subseteq V$ is isotropic, then

$$\dim F\le \dim V/2 \tag{1.4}$$

since $\dim V=\dim F+\dim F^\perp\ge 2\dim F$.

Proposition 1.2 *For isotropic subspaces F, F' the following three conditions*

(a) *$F+F'$ is quadratic,*
(b) *$V=F\oplus (F')^\perp$,*
(c) *$V=F'\oplus F^\perp$*

are equivalent and imply that $\dim F=\dim F'$. *Given an isotropic subspace $F\subseteq V$ one can always find an isotropic subspace F' satisfying these conditions.*

Proof We have

$$\begin{aligned}(F+F')\cap(F+F')^\perp &= (F+F')\cap F^\perp\cap(F')^\perp\\ &= (F+(F'\cap F^\perp))\cap(F')^\perp\\ &= (F\cap(F')^\perp)+(F'\cap F^\perp).\end{aligned}$$

[1] In some of the literature (e.g., C. Chevalley [40] or L. Grove [58]), a subspace is called isotropic if it contains at least one non-zero isotropic vector, and totally isotropic if all of its vectors are isotropic.

Thus

$$(F+F')\cap(F+F')^{\perp}=0 \Leftrightarrow F\cap(F')^{\perp}=0 \text{ and } F'\cap F^{\perp}=0$$
$$\Leftrightarrow F\cap(F')^{\perp}=0 \text{ and } F+(F')^{\perp}=V. \quad (1.5)$$

This shows that (a)$\Leftrightarrow$(b), and similarly (a)$\Leftrightarrow$(c). Property (b) shows that $\dim V = \dim F + (\dim F')^{\perp} = \dim F + \dim V - \dim F'$, hence $\dim F = \dim F'$. Given an isotropic subspace F, we find an isotropic subspace F' satisfying (c) as follows. Choose any complement W to $F^{\perp}$, so that

$$V = F^{\perp} \oplus W.$$

Thus $V = F^{\perp} + W$ and $0 = F^{\perp} \cap W$. Taking orthogonals, this is equivalent to $0 = F \cap W^{\perp}$ and $V = F + W^{\perp}$, that is,

$$V = F \oplus W^{\perp}.$$

Let $S:\ W \to F \subseteq F^{\perp}$ be the projection along $W^{\perp}$. Then $w - S(w) \in W^{\perp}$ for all $w \in W$. The subspace

$$F' = \{w - \frac{1}{2}S(w) |\ w \in W\},$$

being the graph of a map $W \to F^{\perp}$, is again a complement to $F^{\perp}$. For $w \in W$, we have

$$B(w - \frac{1}{2}S(w), w - \frac{1}{2}S(w)) = B(w, w - S(w)) + \frac{1}{4}B(S(w), S(w)) = 0,$$

where the first term vanishes since $w - S(w) \in W^{\perp}$ and the second term vanishes since $S(w) \in F$ is isotropic. This shows that F' is isotropic. □

An isotropic subspace is called *maximal isotropic* if it is not properly contained in another isotropic subspace. Put differently, an isotropic subspace F is maximal isotropic if and only if it contains all isotropic elements of $F^{\perp}$.

Proposition 1.3 *Suppose F, F' are maximal isotropic. Then*

(a) *the kernel of the restriction of B to $F + F'$ equals $F \cap F'$. (In particular, $F + F'$ is quadratic if and only if $F \cap F' = 0$.)*
(b) *The images of F, F' in the quadratic vector space $(F + F')/(F \cap F')$ are maximal isotropic.*
(c) $\dim F = \dim F'$.

Proof Since F is maximal isotropic, it contains all isotropic vectors of $F^{\perp}$, and in particular it contains $F^{\perp} \cap F'$. Thus

$$F^{\perp} \cap F' = F \cap F'.$$

Similarly, $F \cap (F')^{\perp} = F \cap F'$ since F' is maximal isotropic. The calculation (1.5) hence shows that

$$(F + F') \cap (F + F')^{\perp} = F \cap F',$$

proving (a). Let $W=(F+F')/(F\cap F')$ with the bilinear form B_W induced from B, and let $\pi:\ F+F'\to W$ be the quotient map. Clearly, B_W is non-degenerate, and $\pi(F),\pi(F')$ are isotropic. Hence the sum $W=\pi(F)+\pi(F')$ is a direct sum, and the two subspaces are maximal isotropic of dimension $\frac{1}{2}\dim W$. It follows that $\dim F=\dim\pi(F)+\dim(F\cap F')=\dim\pi(F')+\dim(F\cap F')=\dim F'$. □

Definition 1.4 The *Witt index* of a non-degenerate symmetric bilinear form B is the dimension of a maximal isotropic subspace.

By Eq. (1.4), the maximal possible Witt index is $\frac{1}{2}\dim V$ if $\dim V$ is even, and $\frac{1}{2}(\dim V-1)$ if $\dim V$ is odd. Clearly, $\mathbb{R}^n$ has Witt index zero, while $\mathbb{R}^{n,n}$ has Witt index n.

1.3 Split bilinear forms

Definition 1.5 The non-degenerate symmetric bilinear form B on an even-dimensional vector space V is called *split* if its Witt index is $\frac{1}{2}\dim V$. In this case maximal isotropic subspaces are also called *Lagrangian subspaces*.

Equivalently, the Lagrangian subspaces are characterized by the property

$$F=F^{\perp},$$

and B is split if and only if such subspaces exist.

Definition 1.6 The set $\mathrm{Lag}(V)$ of Lagrangian subspaces of V is called the *Lagrangian Grassmannian*.

Proposition 1.4 *Suppose V is a vector space with split bilinear form and $F\in$* $\mathrm{Lag}(V)$ *is a Lagrangian subspace.*

1. *The set of subspaces R complementary to F has a canonical affine structure, with $F\otimes F$ as its space of translations.*
2. *The set of Lagrangian subspaces complementary to F has a canonical affine structure, with $\wedge^2 F$ as its space of translations.*
3. *If R is complementary to F, then so is $R^{\perp}$. The map $R\mapsto R^{\perp}$ is an affine-linear involution on the set of complements. The fixed point set of this involution is the affine subspace consisting of the Lagrangian complements to F in V.*
4. *For any complement R, the midpoint of the line segment between R and $R^{\perp}$ is a Lagrangian complement.*

Proof For *any* subspace F of any vector space V, consider the additive group $F\otimes\mathrm{ann}(F)\subseteq V\otimes V^*=\mathrm{End}(V)$ of linear maps whose kernel and range are in F. Elements f in this subspace satisfy $f^2=0$, hence

$$F\otimes\mathrm{ann}(F)\hookrightarrow\mathrm{GL}(V),\ \ f\mapsto A_f,\quad A_f(v)=v+f(v)$$

is an inclusion as a subgroup. The resulting action on V preserves the set of complements to F, and is free and transitive on the set of complements. (Given two complements R, R', one can think of $R' \subseteq R \oplus F$ as the graph of a linear map $V/F \cong R \to F$.) If V comes endowed with a split bilinear form and F is Lagrangian, the isomorphism $B^\flat$ identifies $F = F^\perp$ with $\mathrm{ann}(F)$. This proves (1).

For (2), we similarly consider an inclusion

$$\wedge^2(F) \to \mathrm{O}(V),\ \phi \mapsto A_\phi, \quad A_\phi(v) = v + \iota(\pi(v))\phi,$$

where $\pi : V \to V/F = F^*$ is the projection. This subgroup of $\mathrm{O}(V)$ acts freely and transitively on the set of Lagrangian complements to F: One checks that if F' is any Lagrangian complement to F, then all other Lagrangian complements are obtained as graphs of skew-symmetric linear maps $F^* = F' \to F$. This proves (2).

For (3), observe first that if R is a complement to F, then $R^\perp$ is a complement to $F^\perp = F$. Hence, $R \mapsto R^\perp$ defines an involution on the set of complements. The fixed point set of this involution consists of complements with $R = R^\perp$, that is, Lagrangian complements. From

$$\langle A_f(w), v\rangle = \langle w + f(w), v\rangle = \langle w, v + f^*(w)\rangle = \langle w, A_{f^*}(v)\rangle,$$

we see that $A_f(R)^\perp = (A_{f^*})^{-1}(R^\perp) = A_{-f^*}(R^\perp)$, proving that the involution is affine-linear. For (4), we use that the involution preserves the line through R and $R^\perp$ and exchanges R and $R^\perp$. Hence it fixes the midpoint, which is therefore Lagrangian. □

Remark 1.1 The construction in (4) can also be phrased more directly, as follows. Let R be a complement to F. Let $S : R \to F$ be the map defined by $\langle v, S(w)\rangle = \langle v, w\rangle$ for all $v, w \in R$. Then $R^\perp = \{v - S(v) |\ v \in R\}$, and the midpoint between R and $R^\perp$ is $\{v - \frac{1}{2}S(v) |\ v \in R\}$. Note that this is the construction used in the proof of Proposition 1.2.

Proposition 1.5 *Let $F \subseteq V$ be a Lagrangian subspace. The group of orthogonal transformations of V fixing all points of F is the additive group $\wedge^2(F)$, embedded into $\mathrm{O}(V)$ by the map*

$$\phi \mapsto A_\phi, \quad A_\phi(v) = v + \iota(\pi(v))\phi.$$

(Here $\pi : V \to V/F \cong F^*$ is the projection.) The group $\mathrm{O}(V)_F$ of orthogonal transformations $A \in \mathrm{O}(V)$ taking F to itself is an extension

$$1 \to \wedge^2(F) \to \mathrm{O}(V)_F \to \mathrm{GL}(F) \to 1.$$

One has $\mathrm{O}(V)_F \subseteq \mathrm{SO}(V)$.

Proof It is clear that the subgroup $\wedge^2(F)$ fixes all points of F. Conversely, suppose $A \in \mathrm{O}(V)$ fixes all points of F. Pick a Lagrangian complement F' to F. Then $A(F')$ is again a complement to F, hence is related to F' by some $\phi \in \wedge^2(F)$. The transformation $\tilde{A} = A_\phi^{-1} \circ A$ preserves F'. For all $v \in F$ and $w \in F'$, we have $\langle v, \tilde{A}w\rangle = \langle \tilde{A}^{-1}v, w\rangle = \langle v, w\rangle$, hence $\tilde{A}w = w$. This shows that $\tilde{A}$ fixes all points of F, F', and is hence equal to the identity. That is, $A = A_\phi$. For any $A \in \mathrm{O}(V)_F$, the restriction to F defines an element of $\mathrm{GL}(F)$, which is trivial if and only if A fixes F pointwise, i.e., if it lies in the subgroup $\wedge^2(F)$. The map to $\mathrm{GL}(F)$ is surjective: Given any $g \in \mathrm{GL}(F)$, the transformation $A = (g^{-1})^* \oplus g$ of $F^* \oplus F \cong F' \oplus F = V$ is orthogonal, and restricts to g on F. Note that this transformation has determinant 1, as do all transformations in $\wedge^2(F) \subseteq \mathrm{O}(V)$. $\square$

If F is a Lagrangian subspace, the choice of a Lagrangian complement $F' \cong F^*$ identifies V with $F^* \oplus F$, with the quadratic form given by the pairing

$$B((\mu, v), (\mu, v)) = \langle \mu, v \rangle.$$

That is, $B((\mu_1, v_1), (\mu_2, v_2)) = \frac{1}{2}(\langle \mu_1, v_2 \rangle + \langle \mu_2, v_1 \rangle)$. Given such a Lagrangian splitting of V, one can construct an adapted basis.

Proposition 1.6 *Let (V, B) be a quadratic vector space with a split bilinear form. Then there exists a basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ of V in which the bilinear form is given as follows*:

$$B(e_i, e_j) = 0, \quad B(e_i, f_j) = \frac{1}{2}\delta_{ij}, \quad B(f_i, f_j) = 0. \tag{1.6}$$

Proof Choose a pair of complementary Lagrangian subspaces F and F'. Since B defines a non-degenerate pairing between F and F', it defines an isomorphism $F' \cong F^*$. Choose a basis $e_1, \ldots, e_k$, and define $f_1, \ldots, f_k \in F'$ by $B(e_i, f_j) = \frac{1}{2}\delta_{ij}$. It is automatic that $B(e_i, e_j) = B(f_i, f_j) = 0$ since F, F' are Lagrangian. $\square$

The basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ is not orthogonal. However, it may be replaced by an orthogonal basis

$$E_i = e_i + f_i, \quad \tilde{E}_i = e_i - f_i.$$

In the new basis, the bilinear form reads as

$$B(E_i, E_j) = \delta_{ij}, \quad B(E_i, \tilde{E}_j) = 0, \quad B(\tilde{E}_i, \tilde{E}_j) = -\delta_{ij}. \tag{1.7}$$

The orthogonal group of $F^* \oplus F$ will be discussed in detail in Section 4.2.2 below. At this point, let us rephrase Proposition 1.5 in terms of the splitting:

Lemma 1.1 *The subgroup of orthogonal transformations fixing all points of $F \subseteq F^* \oplus F$ consists of all transformations of the form*

$$A_D : (\mu, v) \mapsto (\mu, v + D\mu),$$

where $D : F^* \to F$ is skew-adjoint: $D^* = -D$.

1.4 E. Cartan–Dieudonné Theorem

Throughout this section we assume that (V, B) is a quadratic vector space. The following simple result will be frequently used.

Lemma 1.2 *For any* $A \in \mathrm{O}(V)$, *the orthogonal of the space of* A*-fixed vectors equals the range of* $A - I$:

$$\mathrm{ran}(A - I) = \ker(A - I)^{\perp}.$$

Proof For any $L \in \mathrm{End}(V)$, the transpose $L^{\top}$ relative to B satisfies $\mathrm{ran}(L) = \ker(L^{\top})^{\perp}$. We apply this to $L = A - I$, and observe that $\ker(A^{\top} - I) = \ker(A - I)$ since a vector is fixed under A if and only if it is fixed under $A^{\top} = A^{-1}$. □

Definition 1.7 An orthogonal transformation $R \in \mathrm{O}(V)$ is called a *reflection* if its fixed point set $\ker(R - I)$ has codimension 1.

Equivalently, $\mathrm{ran}(R - I) = \ker(R - I)^{\perp}$ is 1-dimensional. If $v \in V$ is a non-isotropic vector, then the formula

$$R_v(w) = w - 2\frac{B(v, w)}{B(v, v)} v$$

defines a reflection, since $\mathrm{ran}(R_v - I) = \mathrm{span}(v)$ is 1-dimensional.

Proposition 1.7 *Any reflection* R *is of the form* R_v, *where the non-isotropic vector* v *is unique up to a non-zero scalar.*

Proof Suppose R is a reflection, and consider the 1-dimensional subspace $F = \mathrm{ran}(R - I)$. We claim that F is a quadratic subspace of V. Once this is established, we obtain $R = R_v$ for any non-zero $v \in F$, since R_v then acts as -1 on F and as $+1$ on $F^{\perp}$. To prove the claim, suppose on the contrary that F is not quadratic. Since $\dim F = 1$, it is thus isotropic. Let F' be an isotropic complement to $F^{\perp}$, so that $F + F'$ is quadratic. Since R fixes $(F + F')^{\perp} \subseteq F^{\perp} = \ker(R - I)$, it preserves $F + F'$, and restricts to a reflection of $F + F'$. This reduces the problem to the case $\dim V = 2$, with $F \subseteq V$ maximal isotropic, and such that R fixes F pointwise. By Proposition 1.5, the group of such transformations is identified with $\wedge^2(F)$, but for $\dim F = 1$ this group is $\{0\}$. Hence R is the identity, contradicting $\dim \mathrm{ran}(R - I) = 1$. □

Some easy properties of reflections are

1. $\det(R) = -1$,
2. $R^2 = I$,
3. $\mathrm{ran}(R - I) = \ker(R + I)$,
4. $A R_v A^{-1} = R_{Av}$ for all non-isotropic $v \in V$ and all $A \in \mathrm{O}(V)$.

Lemma 1.3 *If* $R_1 \neq R_2$ *are distinct reflections defined by non-isotropic vectors* $v_i \in \mathrm{ran}(R_i - I)$, *then*

$$R_1 R_2 = R_2 R_1 \Leftrightarrow B(v_1, v_2) = 0.$$

Proof By direct calculation, using the formula for reflections,

$$(R_1R_2 - R_2R_1)(w) = \frac{4\,B(v_1, v_2)\,(B(w, v_1)v_2 - B(w, v_2)v_1)}{B(v_1, v_1)\,B(v_2, v_2)}$$

for all $w \in V$. Hence $R_1R_2 - R_2R_1$ if and only if $B(v_1, v_2) = 0$, or $B(w, v_1)v_2 = B(w, v_2)v_1$ for all w. The second case occurs if and only if v_1 is a multiple of v_2, i.e., $R_1 = R_2$. □

For any $A \in \mathrm{O}(V)$, let $l(A)$ denote the smallest number l such that $A = R_1 \cdots R_l$, where $R_i \in \mathrm{O}(V)$ are reflections. We put $l(I) = 0$, and for the time being we put $l(A) = \infty$ if A cannot be written as such a product. (The Cartan–Dieudonné Theorem below states that $l(A) < \infty$ always.) The following properties are easily obtained from the definition. For all $A, g, A_1, A_2 \in \mathrm{O}(V)$:

$$\begin{aligned} l(A^{-1}) &= l(A), \\ l(gAg^{-1}) &= l(A), \\ |l(A_1) - l(A_2)| \le l(A_1A_2) &\le l(A_1) + l(A_2), \\ \det(A) &= (-1)^{l(A)}. \end{aligned}$$

A little less obvious is the following estimate.

Proposition 1.8 *For any $A \in \mathrm{O}(V)$, the number $l(A)$ is bounded below by the codimension of the fixed point set*:

$$\dim(\mathrm{ran}(A - I)) \le l(A).$$

Proof Let $n(A) = \dim(\mathrm{ran}(A - I))$. If $A_1, A_2 \in \mathrm{O}(V)$, we have

$$\ker(A_1A_2 - I) \supseteq \ker(A_1A_2 - I) \cap \ker(A_1 - I) = \ker(A_2 - I) \cap \ker(A_1 - I).$$

Taking orthogonals, we obtain

$$\mathrm{ran}(A_1A_2 - I) \subseteq \mathrm{ran}(A_2 - I) + \mathrm{ran}(A_1 - I)$$

which shows that

$$n(A_1A_2) \le n(A_1) + n(A_2).$$

Thus, if $A = R_1 \cdots R_l$ is a product of $l = l(A)$ reflections, we have

$$n(A) \le n(R_1) + \ldots + n(R_l) = l(A),$$

proving the proposition. □

The following upper bound for $l(A)$ is much more tricky.

Theorem 1.1 (E. Cartan–Dieudonné) *Any orthogonal transformation $A \in \mathrm{O}(V)$ can be written as a product of $l(A) \le \dim V$ reflections.*

Proof By induction, we may assume that the theorem is true for quadratic vector spaces of dimension $\leq \dim V - 1$. We will consider three cases.

Case 1: Let $v \in \ker(A - I)$ be a non-isotropic vector. Pick any non-isotropic vector $v \in \ker(A - I)$. Then A fixes the span of v and restricts to an orthogonal transformation A_1 of $V_1 = \mathrm{span}(v)^\perp$. Using the induction hypothesis, we obtain

$$l(A) = l(A_1) \leq \dim V - 1. \tag{1.8}$$

Case 2: $\mathrm{ran}(A - I)$ is non-isotropic. We claim:

(C) There exists a non-isotropic element $w \in V$ such that $v = (A - I)w$ is non-isotropic.

Given v, w as in (C), we may argue as follows. Since $v = (A - I)w$ and hence $(A + I)w \in \mathrm{span}(v)^\perp$, we have

$$R_v(A - I)w = -(A - I)w, \quad R_v(A + I)w = (A + I)w.$$

Adding the two equations, and dividing by 2 we find $R_v A w = w$. Since w is non-isotropic, this shows that the kernel of $R_v A - I$ is non-isotropic. Equation (1.8) applied to the orthogonal transformation $R_v A$ shows that $l(R_v A) \leq \dim V - 1$. Hence $l(A) \leq \dim V$. It remains to prove the claim (C). Suppose it is false, so that we have

$(\neg C)$ The transformation $A - I$ takes the set of non-isotropic elements into the set of isotropic elements.

Let $v = (A - I)w$ be a non-isotropic element in $\mathrm{ran}(A - I)$. By $(\neg C)$ the element w is isotropic. The orthogonal space $\mathrm{span}(w)^\perp$ is non-isotropic for dimensional reasons, hence there exists a non-isotropic element w_1 with $B(w, w_1) = 0$. Then $w_1,\ w + w_1,\ w - w_1$ are all non-isotropic, and by $(\neg C)$ their images

$$v_1 = (A - I)w_1,\ v + v_1 = (A - I)(w + w_1),\ v - v_1 = (A - I)(w - w_1)$$

are isotropic. But then the polarization identity

$$Q_B(v) = \frac{1}{2}(Q_B(v + v_1) + Q_B(v - v_1)) - Q_B(v_1) = 0$$

shows that v is isotropic, a contradiction. This proves (C).

Case 3: Both $\ker(A - I)$ and $\mathrm{ran}(A - I)$ are isotropic. Since these two subspaces are orthogonal, it follows that they are equal and are both Lagrangian. This reduces the problem to the case $V = F^* \oplus F$, where $F = \ker(A - I)$, that is, A fixes F pointwise. By Lemma 1.1 this implies that $\det(A) = 1$. Let R_v be *any* reflection. Then $A_1 = R_v A \in \mathrm{O}(V)$ has $\det(A_1) = -1$. Hence $\ker(A_1 - I)$ and $\mathrm{ran}(A_1 - I)$ cannot both be isotropic, and by the first two cases $l(A_1) \leq \dim V = 2 \dim F$. But since $\det(A_1) = -1$, $l(A_1)$ must be *odd*, hence $l(A_1) < \dim V$ and therefore $l(A) \leq \dim V$. □

Remark 1.2 Our proof of the Cartan–Dieudonné Theorem is a small modification of Artin's proof in [14]. If $\mathrm{char}(\mathbb{K}) = 2$, there exist counterexamples to the statement of the theorem. See Chevalley [41, p. 83].

1.5 Witt's Theorem

The following result is of fundamental importance in the theory of quadratic forms.

Theorem 1.2 (Witt's Theorem) *Suppose $F, \tilde{F}$ are subspaces of a quadratic vector space (V, B), such that there exists an isometric isomorphism $\phi: F \to \tilde{F}$, i.e., $B(\phi(v), \phi(w)) = B(v, w)$ for all $v, w \in F$. Then ϕ extends to an orthogonal transformation $A \in \mathrm{O}(V)$.*

Proof By induction, we may assume that the theorem is true for quadratic vector spaces of dimension $\le \dim V - 1$. We will consider two cases.

Case 1: F is non-isotropic. Let $v \in F$ be a non-isotropic vector, and let $\tilde{v} = \phi(v)$. Then $Q_B(v) = Q_B(\tilde{v}) \neq 0$, and $v + \tilde{v}$ and $v - \tilde{v}$ are orthogonal. The polarization identity $Q_B(v) + Q_B(\tilde{v}) = \frac{1}{2}(Q_B(v + \tilde{v}) + Q_B(v - \tilde{v}))$ shows that $v + \tilde{v}$ and $v - \tilde{v}$ are not both isotropic; say $w = v + \tilde{v}$ is non-isotropic. The reflection R_w satisfies

$$R_w(v + \tilde{v}) = -(v + \tilde{v}), \quad R_w(v - \tilde{v}) = v - \tilde{v}.$$

Adding and dividing by 2, we find that $R_w(v) = -\tilde{v}$. Let $Q = R_w R_v$. Then Q is an orthogonal transformation with $Q(v) = \tilde{v} = \phi(v)$.

Replacing F with $F' = Q(F)$, v with $v' = Q(v)$ and ϕ with $\phi' = \phi \circ Q^{-1}$, we may thus assume that $F \cap \tilde{F}$ contains a non-isotropic vector v such that $\phi(v) = v$. Let

$$V_1 = \mathrm{span}(v)^{\perp}, \quad F_1 = F \cap V_1, \quad \tilde{F}_1 = \tilde{F} \cap V_1,$$

and let $\phi_1 : F_1 \to \tilde{F}_1$ be the restriction of ϕ. By induction, there exists an orthogonal transformation $A_1 \in \mathrm{O}(V_1)$ extending ϕ_1. Let $A \in \mathrm{O}(V)$ with $A(v) = v$ and $A|_{V_1} = A_1$; then A extends ϕ.

Case 2: F is isotropic. Let F' be an isotropic complement to $F^{\perp}$, and let $\tilde{F}'$ be an isotropic complement to $\tilde{F}^{\perp}$. The pairing given by B identifies $F' \cong F^*$ and $\tilde{F}' \cong \tilde{F}^*$. The isomorphism $\phi: F \to \tilde{F}$ extends to an isometry $\psi: F \oplus F' \to \tilde{F} \oplus \tilde{F}'$, given by $(\phi^{-1})^*$ on $F' \cong F^*$. By Case 1 above, ψ extends further to an orthogonal transformation of V. □

Some direct consequences are:

1. $\mathrm{O}(V)$ acts transitively on the set of isotropic subspaces of any given dimension.
2. If $F, \tilde{F}$ are isometric, then so are $F^{\perp}, \tilde{F}^{\perp}$. Indeed, any orthogonal extension of an isometry $\phi: F \to \tilde{F}$ restricts to an isometry of their orthogonals.
3. Suppose $F \subseteq V$ is a subspace isometric to $\mathbb{K}^n$, with standard bilinear form $B(E_i, E_j) = \delta_{ij}$, and F is maximal relative to this property. If $F' \subseteq V$ is isometric to $\mathbb{K}^{n'}$, then there exists an orthogonal transformation $A \in \mathrm{O}(V)$ with $F' \subseteq A(F)$. In particular, the dimension of such a subspace F is an invariant of (V, B).

A subspace $W \subseteq V$ of a quadratic vector space is called *anisotropic* if it does not contain isotropic vectors other than 0. In particular, W is a quadratic subspace.

Proposition 1.9 (Witt decomposition) *Any quadratic vector space* (V, B) *admits a decomposition*

$$V = F \oplus F' \oplus W,$$

where F, F' *are maximal isotropic*, W *is anisotropic*, *and* $W^\perp = F \oplus F'$. *If* $V = F_1 \oplus F_1' \oplus W_1$ *is another such decomposition*, *then there exists* $A \in \mathrm{O}(V)$ *with*

$$A(F) = F_1,\ A(F') = F_1',\ A(W) = W_1.$$

Proof To construct such a decomposition, let F be a maximal isotropic subspace, and let F' be an isotropic complement to $F^\perp$. Then $F \oplus F'$ is quadratic, hence so is $W = (F \oplus F')^\perp$. Since F is maximal isotropic, the subspace W cannot contain isotropic vectors other than 0. Hence W is anisotropic. Given another such decomposition $V = F_1 \oplus F_1' \oplus W_1$, choose an isomorphism $F \cong F_1$. As we saw (e.g., in the proof of Witt's Theorem), this extends canonically to an isometry $\phi :\ F \oplus F' \to F_1 \oplus F_1'$. Witt's Theorem gives an extension of ϕ to an orthogonal transformation $A \in \mathrm{O}(V)$. It is automatic that A takes $W = (F \oplus F')^\perp$ to $W = (F_1 \oplus F_1')^\perp$. □

Example 1.1 If $\mathbb{K} = \mathbb{R}$, the bilinear form on the anisotropic part of the Witt decomposition is either positive definite (i.e., $Q_B(v) > 0$ for non-zero $v \in W$) or negative definite (i.e., $Q_B(v) < 0$ for non-zero $v \in W$). By Proposition 1.1, any quadratic vector space (V, B) over $\mathbb{R}$ is isometric to $\mathbb{R}^{n,m}$ for some n, m. The Witt decomposition shows that n, m are uniquely determined by B. Indeed $\min(n, m)$ is the Witt index of B, while the sign of $n - m$ is given by the sign of Q_B on the anisotropic part.

1.6 Orthogonal groups for $\mathbb{K} = \mathbb{R}, \mathbb{C}$

In this section we discuss the structure of the orthogonal group $\mathrm{O}(V)$ for quadratic vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Being a closed subgroup of $\mathrm{GL}(V)$, the orthogonal group $\mathrm{O}(V)$ is a Lie group. Recall that for a Lie subgroup $G \subseteq \mathrm{GL}(V)$, the corresponding Lie algebra $\mathfrak{g}$ is the subspace of all $\xi \in \mathrm{End}(V)$ with the property $\exp(t\xi) \in G$ for all $t \in \mathbb{K}$ (using the exponential map of matrices). We have

Proposition 1.10 *The Lie algebra of* $\mathrm{O}(V)$ *is given by*

$$\mathfrak{o}(V) = \{A \in \mathrm{End}(V) |\ B(Av, w) + B(v, Aw) = 0 \textit{ for all } v, w \in V\},$$

with the bracket given by commutation.

Proof Suppose $A \in \mathfrak{o}(V)$, so that $\exp(tA) \in \mathrm{O}(V)$ for all t. Taking the t-derivative of $B(\exp(tA)v, \exp(tA)w) = B(v, w)$ we obtain $B(Av, w) + B(v, Aw) = 0$ for all $v, w \in V$. Conversely, given $A \in \mathfrak{gl}(V)$ with $B(Av, w) + B(v, Aw) = 0$ for all $v, w \in V$ we have

$$\begin{aligned}
B(\exp(tA)v, \exp(tA)w) &= \sum_{k,l=0}^{\infty} \frac{t^{k+l}}{k!l!} B(A^k v, A^l w) \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{t^k}{i!(k-i)!} B(A^i v, A^{k-i} w) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i=0}^{k} \binom{k}{i} B(A^i v, A^{k-i} w) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} B(v, A^k w) \sum_{i=0}^{k} (-1)^i \binom{k}{i} \\
&= B(v, w)
\end{aligned}$$

since $\sum_{i=0}^{k} (-1)^i \binom{k}{i} = \delta_{k,0}$. □

Thus $A \in \mathfrak{o}(V)$ if and only if $B^\flat \circ A : V \to V^*$ is a skew-adjoint map. In particular,

$$\dim_{\mathbb{K}} \mathfrak{o}(V) = N(N-1)/2,$$

where $N = \dim V$.

Let us now first discuss the case $\mathbb{K} = \mathbb{R}$. We have shown that any quadratic vector space (V, B) over $\mathbb{R}$ is isometric to $\mathbb{R}^{n,m}$, for unique n, m. The corresponding orthogonal group will be denoted by $\mathrm{O}(n, m)$, the special orthogonal group $\mathrm{SO}(n, m)$, and its identity component $\mathrm{SO}_0(n, m)$. The dimension of $\mathrm{O}(n, m)$ coincides with the dimension of its Lie algebra $\mathfrak{o}(n, m)$, $N(N-1)/2$, where $N = n + m$. If $m = 0$, we will write $\mathrm{O}(n) = \mathrm{O}(n, 0)$ and $\mathrm{SO}(n) = \mathrm{SO}(n, 0)$. These groups are compact, since they are closed subsets of the unit ball in $\mathrm{Mat}(n, \mathbb{R})$.

Lemma 1.4 *The groups* $\mathrm{SO}(n)$ *are connected for all* $n \geq 1$, *and have the fundamental group* $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ *for* $n \geq 3$.

Proof The defining action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ restricts to a transitive action on the unit sphere S^{n-1}, with the stabilizer at $(0, \ldots, 0, 1)$ equal to $\mathrm{SO}(n-1)$. Hence, for $n \geq 2$ the Lie group $\mathrm{SO}(n)$ is the total space of a principal fiber bundle over S^{n-1}, with fiber $\mathrm{SO}(n-1)$. This shows by induction that $\mathrm{SO}(n)$ is connected. The long exact sequence of homotopy groups

$$\cdots \to \pi_2(S^{n-1}) \to \pi_1(\mathrm{SO}(n-1)) \to \pi_1(\mathrm{SO}(n)) \to \pi_1(S^{n-1})$$

shows furthermore that the map $\pi_1(\mathrm{SO}(n-1)) \to \pi_1(\mathrm{SO}(n))$ is an isomorphism for $n > 3$ (since $\pi_2(S^{n-1}) = 0$ in that case). But $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$, since $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R}P(3) = S^3/\mathbb{Z}_2$ (see below). □

The groups SO(3) and SO(4) have a well-known relation with the group SU(2) of complex 2×2-matrices X satisfying $X^\dagger = X^{-1}$ and $\det X = 1$. Recall that the center of SU(2) is $\mathbb{Z}_2 = \{+I, -I\}$.

Proposition 1.11 *There are isomorphisms of Lie groups,*

$$\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2, \quad \mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2,$$

where in the second equality the quotient is by the diagonal subgroup $\mathbb{Z}_2 \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof Consider the algebra of quaternions $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$,

$$\mathbb{H} = \left\{ X = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}, \; z, w \in \mathbb{C} \right\}.$$

For any $X \in \mathbb{H}$ let $\|X\| = (|z|^2 + |w|^2)^{\frac{1}{2}}$. Note that $X^\dagger X = X X^\dagger = \|X\|^2 I$ for all $X \in \mathbb{H}$. Define a symmetric $\mathbb{R}$-bilinear form on $\mathbb{H}$ by

$$B(X_1, X_2) = \frac{1}{2} \mathrm{tr}(X_1^\dagger X_2).$$

The identification $\mathbb{H} \cong \mathbb{R}^4$ takes this to the standard bilinear form on $\mathbb{R}^4$ since $B(X, X) = \frac{1}{2}\|X\|^2 \mathrm{tr}(I) = \|X\|^2$. The unit sphere $S^3 \subseteq \mathbb{H}$, characterized by $\|X\|^2 = 1$, is the group $\mathrm{SU}(2) = \{X \mid X^\dagger = X^{-1}, \; \det(X) = 1\}$. Define an action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{H}$ by

$$(X_1, X_2) \cdot X = X_1 X X_2^{-1}.$$

This action preserves the bilinear form on $\mathbb{H} \cong \mathbb{R}^4$, and hence defines a homomorphism $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$. The kernel of this homomorphism is the finite subgroup $\{\pm(I, I)\} \cong \mathbb{Z}_2$. (Indeed, $X_1 X X_2^{-1} = X$ for all X implies in particular that $X_1 = X X_2 X^{-1}$ for all invertible X. But this is only possible if $X_1 = X_2 = \pm I$.) Since $\dim \mathrm{SO}(4) = 6 = 2 \dim \mathrm{SU}(2)$, and since SO(4) is connected, this homomorphism must be onto. Thus $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(I, I)\}$.

Similarly, identify $\mathbb{R}^3 \cong \{X \in \mathbb{H} \mid \mathrm{tr}(X) = 0\} = \mathrm{span}(I)^\perp$. The conjugation action of SU(2) on $\mathbb{H}$ preserves this subspace; hence we obtain a group homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. The kernel of this homomorphism is $\mathbb{Z}_2 \cong \{\pm I\} \subseteq \mathrm{SU}(2)$. Since SO(3) is connected and $\dim \mathrm{SO}(3) = 3 = \dim \mathrm{SU}(2)$, it follows that $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm I\}$. □

To study the more general groups $\mathrm{SO}(n, m)$ and $\mathrm{O}(n, m)$, we recall the polar decomposition of matrices. Let

$$\mathrm{Sym}(k) = \{A \mid A^\top = A\} \subseteq \mathfrak{gl}(k, \mathbb{R})$$

be the space of real symmetric $k \times k$-matrices, and $\mathrm{Sym}^+(k)$ its subspace of positive definite matrices. As is well known, the exponential map for matrices restricts to a diffeomorphism,

$$\exp : \; \mathrm{Sym}(k) \to \mathrm{Sym}^+(k),$$

with inverse $\log: \mathrm{Sym}^+(k) \to \mathrm{Sym}(k)$. Furthermore, the map

$$\mathrm{O}(k) \times \mathrm{Sym}(k) \to \mathrm{GL}(k, \mathbb{R}),\ (O, X) \mapsto Oe^X$$

is a diffeomorphism. The inverse map

$$\mathrm{GL}(k, \mathbb{R}) \to \mathrm{O}(k) \times \mathrm{Sym}(k),\ \mapsto (A|A|^{-1}, \log |A|),$$

where $|A| = (A^\top A)^{1/2}$, is called the *polar decomposition* for $\mathrm{GL}(k, \mathbb{R})$. We will need the following simple observation.

Lemma 1.5 *Let $X \in \mathrm{Sym}(k)$ be non-zero. Then the closed subgroup of $\mathrm{GL}(k, \mathbb{R})$ generated by e^X is non-compact.*

Proof Replacing X with $-X$ if necessary, we may assume $\|e^X\| > 1$. But then $\|e^{nX}\| = \|e^X\|^n$ goes to ∞ for $n \to \infty$. $\square$

This shows that $\mathrm{O}(k)$ is a maximal compact subgroup of $\mathrm{GL}(k, \mathbb{R})$. The polar decomposition for $\mathrm{GL}(k, \mathbb{R})$ restricts to a polar decomposition for any closed subgroup G that is invariant under the involution $A \mapsto A^\top$. Let

$$K = G \cap \mathrm{O}(k, \mathbb{R}),\ P = G \cap \mathrm{Sym}^+(k),\ \mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}(k).$$

The diffeomorphism $\exp: \mathrm{Sym}(k) \to \mathrm{Sym}^+(k)$ restricts to a diffeomorphism $\exp: \mathfrak{p} \to P$, with inverse the restriction of log. Hence the polar decomposition for $\mathrm{GL}(k, \mathbb{R})$ restricts to a diffeomorphism

$$K \times \mathfrak{p} \to G$$

whose inverse is called the polar decomposition of G. (It is a special case of a *Cartan decomposition*.) Using Lemma 1.5, we see that K is a maximal compact subgroup of G. Since $\mathfrak{p}$ is just a vector space, K is a deformation retract of G.

We will now apply these considerations to $G = \mathrm{O}(n, m)$. Let B_0 be the standard bilinear form on $\mathbb{R}^{n+m}$, and define the endomorphism J by

$$B(v, w) = B_0(Jv, w).$$

Thus J acts as the identity on $\mathbb{R}^n \oplus 0$ and as minus the identity on $0 \oplus \mathbb{R}^m$. An endomorphism of $\mathbb{R}^{n+m}$ commutes with J if and only if it preserves the direct sum decomposition $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$. A matrix $A \in \mathrm{Mat}(n+m, \mathbb{R})$ lies in $\mathrm{O}(n, m)$ if and only if $A^\top J A = J$, where $\top$ denotes as before the usual transpose of matrices, i.e., the transpose relative to B_0 (not relative to B). Similarly, $X \in \mathfrak{o}(n, m)$ if and only if $X^\top J + JX = 0$.

Remark 1.3 In block form we have

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}.$$

For $A \in \mathrm{Mat}(n+m, \mathbb{R})$ in block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1.9}$$

we have $A \in \mathrm{O}(n,m)$ if and only if

$$a^\top a = I + c^\top c, \ \ d^\top d = I + b^\top b, \ \ a^\top b = c^\top d. \tag{1.10}$$

Similarly, for $X \in \mathrm{Mat}(n+m,\mathbb{R})$, written in block form

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{1.11}$$

we have $X \in \mathfrak{o}(n,m)$ if and only if

$$\alpha^\top = -\alpha, \ \beta^\top = \gamma, \ \delta^\top = -\delta. \tag{1.12}$$

Since $\mathrm{O}(n,m)$ is invariant under $A \mapsto A^\top$ (and likewise for the special orthogonal group and its identity component) the polar decomposition applies. We find the following.

Proposition 1.12 *Relative to the polar decomposition of* $\mathrm{GL}(n+m,\mathbb{R})$, *the maximal subgroups of*

$$G = \mathrm{O}(n,m), \ \mathrm{SO}(n,m), \ \mathrm{SO}_0(n,m),$$

are, respectively,

$$K = \mathrm{O}(n) \times \mathrm{O}(m), \ \mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m)), \ \mathrm{SO}(n) \times \mathrm{SO}(m).$$

(*Here* $\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m))$ *are elements of* $(\mathrm{O}(n) \times \mathrm{O}(m))$ *of determinant* 1.) *In all of these cases the space* $\mathfrak{p}$ *in the Cartan decomposition is given by matrices of the form*

$$\begin{pmatrix} 0 & x \\ x^\top & 0 \end{pmatrix},$$

where x is an arbitrary $n \times m$-matrix.

Proof We start with $G = \mathrm{O}(n,m)$. Elements in $K = G \cap \mathrm{O}(n+m)$ are characterized by $A^\top JA = J$ and $A^\top A = I$. The two conditions give $AJ = JA$, so that A is a block diagonal element of $\mathrm{O}(n+m)$. Hence $A \in \mathrm{O}(n) \times \mathrm{O}(m) \subseteq \mathrm{O}(n,m)$. This shows that $K = \mathrm{O}(n) \times \mathrm{O}(m)$. The elements

$$X \in \mathfrak{p} = \mathfrak{o}(n,m) \cap \mathrm{Sym}(n+m)$$

satisfy $X^\top J + JX = 0$ and $X^\top = X$, hence they are symmetric block off-diagonal matrices. This proves our characterization of $\mathfrak{p}$ and proves the polar decomposition for $\mathrm{O}(n,m)$. The polar decompositions for $\mathrm{SO}(n,m)$ is an immediate consequence, and the polar decomposition for $\mathrm{SO}_0(n,m)$ follows since $\mathrm{SO}(n) \times \mathrm{SO}(m)$ is the identity component of $\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m))$. $\square$

Corollary 1.1 *Unless $n = 0$ or $m = 0$ the group* $\mathrm{O}(n,m)$ *has four connected components and* $\mathrm{SO}(n,m)$ *has two connected components.*

We next describe the space $P = \exp(\mathfrak{p})$.

Proposition 1.13 *The space $P=\exp(\mathfrak{p})\subseteq G$ consists of matrices*

$$P=\left\{\begin{pmatrix}(I+bb^\top)^{1/2} & b\\ b^\top & (I+b^\top b)^{1/2}\end{pmatrix}\right\},$$

where b ranges over all $n\times m$-matrices. In fact,

$$\log\begin{pmatrix}(I+bb^\top)^{1/2} & b\\ b^\top & (I+b^\top b)^{1/2}\end{pmatrix}=\begin{pmatrix}0 & x\\ x^\top & 0\end{pmatrix},$$

where x and b are related as follows:

$$b=\frac{\sinh(xx^\top)}{xx^\top}x,\quad x=\frac{\operatorname{arsinh}((bb^\top)^{1/2})}{(bb^\top)^{1/2}}b. \tag{1.13}$$

Note that $xx^\top$ and $bb^\top$ need not be invertible. The fraction in the expression (1.13) for b is to be interpreted as $f(xx^\top)$, where $f(z)$ is the entire holomorphic function $\frac{\sinh z}{z}$, and $f(xx^\top)$ is given in terms of the spectral theorem, or equivalently in terms of the power series expansion of f. The fraction in the expression (1.13) for x is to be interpreted similarly in terms of the holomorphic function $\frac{\operatorname{arsinh}(z)}{z}$.

Proof Let $X=\left(\begin{smallmatrix}0 & x\\ x^\top & 0\end{smallmatrix}\right)$. By induction on k,

$$X^{2k}=\begin{pmatrix}(xx^\top)^k & 0\\ 0 & (x^\top x)^k\end{pmatrix},\quad X^{2k+1}=\begin{pmatrix}0 & (xx^\top)^k x\\ x(x^\top x)^k & 0\end{pmatrix}.$$

This gives

$$\exp(X)=\begin{pmatrix}\cosh(xx^\top) & \frac{\sinh(xx^\top)}{xx^\top}x\\ x\frac{\sinh(x^\top x)}{x^\top x} & \cosh(x^\top x)\end{pmatrix},$$

which is exactly the form of elements in P with $b=\frac{\sinh(xx^\top)}{xx^\top}x$. The equation $\cosh(xx^\top)=(1+bb^\top)^{1/2}$ gives $\sinh(xx^\top)=(bb^\top)^{1/2}$. Plugging this into the formula for b, we obtain the second equation in (1.13). □

For later reference, we mention one more simple fact about the orthogonal and special orthogonal groups. Let $\mathbb{Z}_2$ be the center of $\mathrm{GL}(n+m,\mathbb{R})$ consisting of $\pm I$.

Proposition 1.14 *For all n,m, the center of the group $\mathrm{O}(n,m)$ is $\mathbb{Z}_2$. Except in the cases $(n,m)=(0,2),(2,0)$, the center of $\mathrm{SO}(n,m)$ is $\mathbb{Z}_2$ if $-I$ lies in $\mathrm{SO}(n,m)$, and is trivial otherwise.*

The proof is left as an exercise. (Note that the elements of the center of G commute in particular with the diagonal elements of G. In the case at hand, one uses this fact to argue that the central elements are themselves diagonal, and finally that they are multiples of the identity.)

A similar discussion applies to $\mathbb{K}=\mathbb{C}$. It is enough to consider the case $V=\mathbb{C}^n$, equipped with the standard symmetric bilinear form. Again, our starting point is the polar decomposition, but now for complex matrices. Let $\mathrm{Herm}(n) = \{A \mid A^\dagger = A\}$ be the space of Hermitian $n \times n$ matrices and $\mathrm{Herm}^+(n)$ the subset of positive definite matrices. The exponential map gives a diffeomorphism

$$\mathrm{Herm}(n) \to \mathrm{Herm}^+(n),\ \ X \mapsto e^X.$$

This is used to show that the map

$$\mathrm{U}(n) \times \mathrm{Herm}(n) \to \mathrm{GL}(n, \mathbb{C}),\quad (U, X) \mapsto Ue^X$$

is a diffeomorphism; the inverse map takes A to (Ae^{-X}, X) with $X = \frac{1}{2}\log(A^\dagger A)$. The polar decomposition of $\mathrm{GL}(n, \mathbb{C})$ gives rise to polar decompositions of any closed subgroup $G \subseteq \mathrm{GL}(n, \mathbb{C})$ that is invariant under the involution $\dagger$. In particular, this applies to $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$. Indeed, if $A \in \mathrm{O}(n, \mathbb{C})$, the matrix $A^\dagger A$ lies in $\mathrm{O}(n, \mathbb{C}) \cap \mathrm{Herm}(n)$, and hence its logarithm $X = \frac{1}{2}\log(A^\dagger A)$ lies in $\mathfrak{o}(n, \mathbb{C}) \cap \mathrm{Herm}(n)$. But clearly,

$$\mathrm{O}(n, \mathbb{C}) \cap \mathrm{U}(n) = \mathrm{O}(n, \mathbb{R}),$$
$$\mathrm{SO}(n, \mathbb{C}) \cap \mathrm{U}(n) = \mathrm{SO}(n, \mathbb{R}),$$

while

$$\mathfrak{o}(n, \mathbb{C}) \cap \mathrm{Herm}(n) = \sqrt{-1}\,\mathfrak{o}(n, \mathbb{R}).$$

Hence the maps $(U, X) \mapsto Ue^X$ restrict to polar decompositions

$$\mathrm{O}(n, \mathbb{R}) \times \sqrt{-1}\,\mathfrak{o}(n, \mathbb{R}) \to \mathrm{O}(n, \mathbb{C}),$$
$$\mathrm{SO}(n, \mathbb{R}) \times \sqrt{-1}\,\mathfrak{o}(n, \mathbb{R}) \to \mathrm{SO}(n, \mathbb{C}),$$

which shows that the algebraic topology of the complex orthogonal and special orthogonal group coincides with that of its real counterparts. Arguing as in the real case, the center of $\mathrm{O}(n, \mathbb{C})$ is given by $\{+I, -I\}$, while the center of $\mathrm{SO}(n, \mathbb{C})$ is trivial for n odd and $\{+I, -I\}$ for n even, provided $n \geq 3$.

1.7 Lagrangian Grassmannians

In this section we discuss the structure of the Lagrangian Grassmannian $\mathrm{Lag}(V)$ for quadratic vector spaces V with split bilinear form B. Recall that any such V is isomorphic to $\mathbb{K}^{n,n}$, where $\dim V = 2n$. For $\mathbb{K} = \mathbb{R}$ we have the following result.

Theorem 1.3 *The Lagrangian Grassmannian of* $V = \mathbb{R}^{n,n}$ *is a homogeneous space*

$$\mathrm{Lag}(\mathbb{R}^{n,n}) \cong \mathrm{O}(n) \times \mathrm{O}(n)/\mathrm{O}(n)_\Delta \cong \mathrm{O}(n).$$

In particular, $\mathrm{Lag}(\mathbb{R}^{n,n})$ *is a compact space with two connected components.*

Proof Let B_0 be the standard positive definite bilinear form on the vector space $\mathbb{R}^{n,n} \cong \mathbb{R}^{2n}$. The orthogonal group corresponding to B_0 is $\mathrm{O}(2n)$. Introduce an involution $J \in \mathrm{O}(2n)$ by

$$B(v,w) = B_0(Jv,w).$$

Thus J acts as $+1$ on $\mathbb{R}^{n,0} = \mathbb{R}^n \oplus 0$ and as -1 on $\mathbb{R}^{0,n} = 0 \oplus \mathbb{R}^n$. The maximal compact subgroup $\mathrm{O}(n) \times \mathrm{O}(n) \subseteq \mathrm{O}(n,n)$ may be characterized as the set of all $A \in \mathrm{O}(2n)$ satisfying $JA = AJ$. If $L \in \mathrm{Lag}(V)$, then $L \cap J(L)$ is J-invariant and decomposes as the direct sum of its intersections with the ± 1 eigenspaces of J. Since $L \cap J(L)$ is isotropic, elements of these eigenspaces satisfy $B_0(v,v) = \pm B(v,v) = 0$. This shows that $L \cap J(L) = 0$.

Given L, we may choose a B_0-orthonormal basis $v_1, \ldots, v_n$ of L. Then

$$v_1, \ldots, v_n, J(v_1), \ldots, J(v_n)$$

is a B_0-orthonormal basis of $\mathbb{R}^{n,n}$. If L' is another Lagrangian subspace, with B_0-orthonormal basis $v_1', \ldots, v_n'$, then the orthogonal transformation $A \in \mathrm{O}(2n)$ given by

$$Av_i = v_i', \quad AJ(v_i) = J(v_i'), \ i = 1, \ldots, n,$$

commutes with J, hence $A \in \mathrm{O}(n) \times \mathrm{O}(n)$. This shows that $\mathrm{O}(n) \times \mathrm{O}(n)$ acts transitively on $\mathrm{Lag}(\mathbb{R}^{n,n})$. Taking $L' = L$, the construction also identifies $\mathrm{O}(L, B_0|_L) \subseteq \mathrm{O}(n) \times \mathrm{O}(n)$ as the stabilizer subgroup of L under this action. In particular, taking

$$L_0 = (\mathbb{R}^n)_\Delta$$

to be the image under the diagonal inclusion $\mathbb{R}^n \to \mathbb{R}^{n,n}$, the stabilizer subgroup is the diagonal subgroup $\mathrm{O}(n)_\Delta$. Finally, since the multiplication map

$$(\mathrm{O}(n) \times \{1\}) \times \mathrm{O}(n)_\Delta \to \mathrm{O}(n) \times \mathrm{O}(n)$$

is a bijection, the quotient is just $\mathrm{O}(n)$. □

For more general fields $\mathbb{K}$, the group $\mathrm{O}(n,\mathbb{K}) \times \mathrm{O}(n,\mathbb{K})$ acts on the Lagrangian Grassmannian for $V = \mathbb{K}^{n,n}$, and the orbit of L_0 is identified with $\mathrm{O}(n,\mathbb{K}) \times \mathrm{O}(n,\mathbb{K})/\mathrm{O}(n,\mathbb{K})_\Delta \cong \mathrm{O}(n,\mathbb{K})$, by the same argument as for the case $\mathbb{K} = \mathbb{R}$. The Lagrangian subspaces in this orbit are transverse to both V_+ and V_-. But for $\mathbb{K} \neq \mathbb{R}$, a Lagrangian subspace does not necessarily have this property. E.g., if $\mathbb{K} = \mathbb{C}$ and $n = 2$, the span of $E_1 + \sqrt{-1}E_2$ and $\tilde{E}_1 + \sqrt{-1}\tilde{E}_2$ is a Lagrangian subspace not transverse to V_+ (or V_-). Nonetheless, there is a good description of the space Lag in the complex case $\mathbb{K} = \mathbb{C}$.

Theorem 1.4 *The space of Lagrangian subspaces of $V = \mathbb{C}^{2m}$ is a homogeneous space*

$$\mathrm{Lag}(\mathbb{C}^{2m}) \cong \mathrm{O}(2m)/\mathrm{U}(m).$$

In particular, it is a compact space with two connected components.

Proof Let $C : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ be the conjugate linear involution given by complex conjugation. Then $\langle v, w\rangle = B(Cv, w)$ is the standard Hermitian inner product on $\mathbb{C}^{2m}$. Let $L_0 \subseteq \mathbb{C}^{2m}$ be the Lagrangian subspace spanned by

$$v_1 = \frac{1}{\sqrt{2}}(E_1 - \sqrt{-1}E_{m+1}), \ldots, v_m = \frac{1}{\sqrt{2}}(E_m - \sqrt{-1}E_{2m}),$$

where $E_1, \ldots, E_{2m}$ is the standard basis of $\mathbb{C}^{2m}$. Note that $v_1, \ldots, v_m$ is orthonormal for the Hermitian inner product, and $v_1, \ldots, v_m, Cv_1, \ldots, Cv_m$ is a $\langle\cdot,\cdot\rangle$-orthonormal basis of $\mathbb{C}^{2m}$. If L' is another Lagrangian subspace, with orthonormal basis $v_1', \ldots, v_m'$, then the unitary transformation $A \in \mathrm{U}(2m)$ taking v_i, Cv_i to v_i', Cv_i' commutes with complex conjugation C, hence it actually lies in $\mathrm{O}(2m)$. This shows that $\mathrm{O}(2m)$ acts transitively on $\mathrm{Lag}(\mathbb{C}^{2m})$. The transformations $A \in \mathrm{O}(2m) \subseteq \mathrm{U}(2m)$ preserving L_0 are those for which $v_i' = A(v_i)$ is again an orthonormal basis of L. Hence the stabilizer of L is $\mathrm{U}(m) \subseteq \mathrm{O}(2m)$. $\square$

Remark 1.4 The orbit of L_0 under $\mathrm{O}(m,\mathbb{C}) \times \mathrm{O}(m,\mathbb{C})$ is open and dense in $\mathrm{Lag}(\mathbb{C}^{2m})$, and as in the real case is identified with $\mathrm{O}(m,\mathbb{C})$. Thus, $\mathrm{Lag}(\mathbb{C}^{2m})$ is a smooth compactification of the complex Lie group $\mathrm{O}(m,\mathbb{C})$.

Theorem 1.4 has a well-known geometric interpretation. View $\mathbb{C}^{2m}$ as the complexification of $\mathbb{R}^{2m}$. Recall that an *orthogonal complex structure* on $\mathbb{R}^{2m}$ is an automorphism $J \in \mathrm{O}(2m)$ with $J^2 = -I$. Let $\mathscr{J}(2m)$ denote the space of all orthogonal complex structures; it has a base point J_0 given by the standard complex structure

$$J_0(E_i) = E_{i+m}, \quad J_0(E_{i+m}) = -E_i, \quad i = 1, \ldots, m,$$

on $\mathbb{R}^{2m} \cong \mathbb{C}^m$. The action of $\mathrm{O}(2m)$ on $\mathscr{J}(2m)$ by conjugation $J \mapsto AJA^{-1}$ is transitive, with stabilizer at J_0 equal to $\mathrm{U}(m)$. Hence the space of orthogonal complex structures is identified with the complex Lagrangian Grassmannian:

$$\mathscr{J}(2m) = \mathrm{O}(2m)/\mathrm{U}(m) = \mathrm{Lag}(\mathbb{C}^{2m}).$$

Explicitly, this correspondence takes $J \in \mathscr{J}(2m)$ to its $\sqrt{-1}$ eigenspace

$$L = \ker(J - \sqrt{-1}I).$$

This has complex dimension m since $\mathbb{C}^{2m} = L \oplus C(L)$, and it is isotropic since $v \in L$ implies that

$$B(v, v) = B(Jv, Jv) = B(\sqrt{-1}v, \sqrt{-1}v) = -B(v, v).$$

Conversely, any Lagrangian subspace $L \in \mathrm{Lag}(\mathbb{C}^{2m})$ determines J by the condition

$$w - \sqrt{-1}Jw \in L \tag{1.14}$$

for all $w \in \mathbb{R}^{2m}$. Multiplying (1.14) by $\sqrt{-1}$, we obtain $Jw + \sqrt{-1}w \in L$, hence $J(Jw) = -w$. Since L is Lagrangian we have that

$$0 = B(w - \sqrt{-1}Jw, w - \sqrt{-1}Jw) = B(w, w) - B(Jw, Jw) - 2\sqrt{-1}B(w, Jw),$$

which proves that $J \in \mathrm{O}(2m)$. Hence J is an orthogonal complex structure.

Remark 1.5 There are parallel results in symplectic geometry. Suppose V is a vector space with a non-degenerate *skew*-symmetric linear form ω. If $\mathbb{K} = \mathbb{R}$, any such V is isomorphic to $\mathbb{R}^{2n} = \mathbb{C}^n$ with the standard symplectic form. $L_0 = \mathbb{R}^n \subseteq \mathbb{C}^n$ is a Lagrangian subspace, and the action of $\mathrm{U}(n) \subseteq \mathrm{Sp}(V, \omega)$ on L_0 identifies

$$\mathrm{Lag}_\omega(\mathbb{R}^{2n}) \cong \mathrm{U}(n)/\mathrm{O}(n).$$

If $\mathbb{K} = \mathbb{C}$, then V is isomorphic to $\mathbb{C}^{2n}$ with the complexification of the standard symplectic form on $\mathbb{R}^{2n}$. The space $\mathrm{Lag}(V)$ of complex Lagrangian subspaces of the complex symplectic vector space $\mathbb{C}^{2n} \cong \mathbb{H}^n$ is a homogeneous space

$$\mathrm{Lag}_\omega(\mathbb{C}^{2n}) \cong \mathrm{Sp}(n)/U(n),$$

where $\mathrm{Sp}(n)$ is the *compact symplectic group* (i.e., the quaternionic unitary group). See e.g., [52, p. 67].

Chapter 2
Clifford algebras

Associated to any vector space V with a symmetric bilinear form B is a Clifford algebra $\mathrm{Cl}(V;B)$. In the special case $B=0$, the Clifford algebra is just the exterior algebra $\wedge(V)$, and in the general case the Clifford algebra can be regarded as a deformation of the exterior algebra. In this chapter after constructing the Clifford algebra and describing its basic properties, we study in detail the quantization map $q\colon\wedge(V)\to\mathrm{Cl}(V;B)$ and justify the term "quantization". Throughout we assume that V is a finite-dimensional vector space over a field $\mathbb{K}$ of characteristic 0.

2.1 Exterior algebras

2.1.1 Definition

For any vector space V over a field $\mathbb{K}$, let $T(V)=\bigoplus_{k\in\mathbb{Z}}T^k(V)$ be the tensor algebra, with $T^k(V)=V\otimes\cdots\otimes V$ the k-fold tensor product. Consider the two-sided ideal $\mathscr{I}(V)$ generated by elements of the form $v\otimes w+w\otimes v$, with $v,w\in V$. The quotient of $T(V)/\mathscr{I}(V)$ is the *exterior algebra*, denoted by $\wedge(V)$. The product in $\wedge(V)$ is usually denoted $\alpha_1\wedge\alpha_2$, although we will frequently omit the wedge symbol and just write $\alpha_1\alpha_2$. Since $\mathscr{I}(V)$ is a *graded* ideal, the exterior algebra inherits a grading

$$\wedge(V)=\bigoplus_{k\in\mathbb{Z}}\wedge^k(V),$$

where $\wedge^k(V)$ is the image of $T^k(V)$ under the quotient map. We will write $|\phi|=k$ if $\phi\in\wedge^k(V)$. Clearly, $\wedge^0(V)=\mathbb{K}$ and $\wedge^1(V)=V$ so that we can think of V as a subspace of $\wedge(V)$. We may thus think of $\wedge(V)$ as the associative algebra linearly generated by V, subject to the relations $v\wedge w+w\wedge v=0$.

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Throughout we will regard $\wedge(V)$ as a *graded super algebra*, where the $\mathbb{Z}_2$-grading is the mod 2 reduction of the $\mathbb{Z}$-grading.[1] Thus tensor products, derivations, and other constructions with $\wedge(V)$ are all understood in the super sense, often without further specification. Since

$$[\phi_1, \phi_2] \equiv \phi_1 \wedge \phi_2 + (-1)^{k_1 k_2} \phi_2 \wedge \phi_1 = 0$$

for $\phi_1 \in \wedge^{k_1}(V)$ and $\phi_2 \in \wedge^{k_2}(V)$, we see that $\wedge(V)$ is *commutative* (as a super algebra).

If V has dimension n, with basis $e_1, \ldots, e_n$, the space $\wedge^k(V)$ has basis

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \tag{2.1}$$

for all ordered subsets $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$. (If $k = 0$, we put $e_\emptyset = 1$.) In particular, we see that $\dim \wedge^k(V) = \binom{n}{k}$, and

$$\dim \wedge(V) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

2.1.2 Universal property, functoriality

The exterior algebra is characterized among graded super algebras by its *universal property*: If $\mathscr{A}$ is a graded super algebra and $f: V \to \mathscr{A}^1$ is a linear map with $f(v)f(w) + f(w)f(v) = 0$ for all $v, w \in V$, then f extends uniquely to a morphism of graded super algebras $f_\wedge: \wedge(V) \to \mathscr{A}$. Similar universal properties characterize $\wedge(V)$ in the categories of algebras, super algebras, graded algebras, filtered algebras, or filtered super algebras.

Any linear map $L: V \to W$ extends uniquely to a morphism of graded super algebras $\wedge(L): \wedge(V) \to \wedge(W)$. This follows, e.g., by the universal property, applied to L viewed as a map into $\wedge(W)$. One has

$$\wedge(L_1 \circ L_2) = \wedge(L_1) \circ \wedge(L_2), \quad \wedge(\mathrm{id}_V) = \mathrm{id}_{\wedge(V)}.$$

As a special case, taking L to be the zero map $0: V \to V$, the resulting algebra homomorphism $\wedge(0)$ is the *augmentation map*

$$\wedge(V) \to \mathbb{K}, \quad \phi \mapsto \phi_{[0]}$$

given by the projection to $\wedge^0(V) \cong \mathbb{K}$. Taking L to be the map $v \mapsto -v$, the map $\wedge(L)$ is the *parity homomorphism* $\Pi \in \mathrm{Aut}(\wedge(V))$, equal to $(-1)^k$ on $\wedge^k(V)$.

The functoriality gives an algebra homomorphism

$$\mathrm{End}(V) \to \mathrm{End}(\wedge(V)), \quad A \mapsto \wedge(A)$$

and, by restriction to invertible elements, a group homomorphism

$$\mathrm{GL}(V) \to \mathrm{Aut}(\wedge(V)), \quad g \mapsto \wedge(g)$$

[1] We refer to Appendix A for terminology and background regarding super vector spaces.

into the group of degree-preserving algebra automorphisms of $\wedge(V)$. We will often write g in place of $\wedge(g)$, but reserve this notation for invertible transformations since, e.g., $\wedge(0) \neq 0$.

Suppose V_1, V_2 are two vector spaces. Then $\wedge(V_1) \otimes \wedge(V_2)$ (tensor product of graded super algebras) with the natural inclusion of $V_1 \oplus V_2$ satisfies the universal property of the exterior algebra over $V_1 \oplus V_2$. Hence the morphism of graded super algebras

$$\wedge(V_1 \oplus V_2) \to \wedge(V_1) \otimes \wedge(V_2)$$

intertwining the two inclusions is an isomorphism. As a special case, $\wedge(\mathbb{K}^n) = \wedge(\mathbb{K}) \otimes \cdots \otimes \wedge(\mathbb{K})$.

2.1.3 Derivations

The space $\mathrm{Der}(\wedge(V))$ of derivations of the graded super algebra $\wedge(V)$ is a left module over $\wedge(V)$, since $\wedge(V)$ is commutative. Any such derivation is uniquely determined by its restriction to the space $V \subseteq \wedge(V)$ of generators, and conversely, any linear map $V \to \wedge(V)$ extends to a derivation. Thus

$$\mathrm{Der}(\wedge(V)) \cong \mathrm{Hom}(V, \wedge(V))$$

as graded super vector spaces, where the grading on the right-hand side is

$$\mathrm{Hom}^k(V, \wedge(V)) = \mathrm{Hom}(V, \wedge^{k+1}(V)).$$

In particular, $\mathrm{Der}^k(\wedge(V))$ vanishes if $k < -1$. Elements of the space $\mathrm{Der}^{-1}(\wedge(V)) = \mathrm{Hom}(V, \mathbb{K}) = V^*$ are called *contractions*. Explicitly, the derivation $\iota(\alpha)$ corresponding to $\alpha \in V^*$ is given by $\iota(\alpha)1 = 0$ and

$$\iota(\alpha)(v_1 \wedge \cdots v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, v_i \rangle \; v_1 \wedge \cdots \widehat{v_i} \cdots \wedge v_k \tag{2.2}$$

for $v_1, \ldots, v_k \in V$. The contraction operators satisfy $\iota(\alpha)\iota(\beta) + \iota(\beta)\iota(\alpha) = 0$ for $\alpha, \beta \in V^*$. Hence the map $\iota \colon V^* \to \mathrm{End}(\wedge(V))$ extends, by the universal property, to a morphism of super algebras

$$\iota \colon \wedge(V^*) \to \mathrm{End}(\wedge(V)).$$

This map takes $\wedge^k(V^*)$ to $\mathrm{End}^{-k}(\wedge(V))$, hence it becomes a morphism of graded super algebras if we use the opposite $\mathbb{Z}$-grading on $\wedge(V^*)$.

On the other hand, left multiplication defines a morphism of graded super algebras

$$\varepsilon \colon \wedge(V) \to \mathrm{End}(\wedge(V)),$$

called *exterior multiplication*. The operators $\varepsilon(v)$ for $v \in V$ and $\iota(\alpha)$ for $\alpha \in V^*$ satisfy commutation relations,

$$[\varepsilon(v), \varepsilon(w)] \equiv \varepsilon(v)\varepsilon(w) + \varepsilon(w)\varepsilon(v) = 0, \tag{2.3}$$

$$[\iota(\alpha), \iota(\beta)] \equiv \iota(\alpha)\iota(\beta) + \iota(\beta)\iota(\alpha) = 0, \tag{2.4}$$

$$[\iota(\alpha), \varepsilon(v)] \equiv \iota(\alpha)\varepsilon(v) + \varepsilon(v)\iota(\alpha) = \langle\alpha, v\rangle. \tag{2.5}$$

For later reference, observe that

$$\ker(\iota(\alpha)) = \mathrm{ran}(\iota(\alpha)) = \wedge(\ker(\alpha)) \subseteq \wedge(V)$$

for all $\alpha \in V^*$. (To see this, decompose V into $\ker(\alpha)$ and a complement V_1, and use that $\wedge(V) = \wedge(\ker(\alpha)) \otimes \wedge(V_1)$.) Similarly, $\ker(\varepsilon(v)) = \mathrm{ran}(\varepsilon(v))$ is the ideal generated by $\mathrm{span}(v)$.

2.1.4 Transposition

An *anti-automorphism* of an algebra $\mathscr{A}$ is an invertible linear map $f\colon \mathscr{A} \to \mathscr{A}$ with the property $f(ab) = f(b)f(a)$ for all $a, b \in \mathscr{A}$. Put differently, if $\mathscr{A}^{\mathrm{op}}$ is $\mathscr{A}$ with the opposite algebra structure $a \cdot_{\mathrm{op}} b := ba$, an anti-automorphism is an algebra isomorphism $\mathscr{A} \to \mathscr{A}^{\mathrm{op}}$.

The tensor algebra has a unique involutive anti-automorphism that is equal to the identity on $V \subseteq T(V)$. It is called the *canonical anti-automorphism* or *transposition*, and is given by

$$(v_1 \otimes \cdots \otimes v_k)^\top = v_k \otimes \cdots \otimes v_1.$$

Since the transposition preserves the ideal $\mathscr{I}(V)$ defining the exterior algebra $\wedge(V)$, it descends to an anti-automorphism of the exterior algebra, $\phi \mapsto \phi^\top$. Furthermore, since the transposition is given by a permutation of the set $\{v_1, \ldots, v_k\}$ of length $(k-1) + \cdots + 2 + 1 = k(k-1)/2$, we have

$$\phi^\top = (-1)^{k(k-1)/2}\phi, \quad \phi \in \wedge^k(V). \tag{2.6}$$

If $\mathbb{K} = \mathbb{C}$, and V is the complexification of a real vector space $V_\mathbb{R}$, then the complex conjugation mapping $v \mapsto v^c$ extends to a conjugate linear algebra morphism $\wedge(V) \to \wedge(V),\ \phi \mapsto \phi^c$. We will denote by

$$\phi \mapsto \phi^* = (\phi^c)^\top$$

the resulting conjugate linear anti-automorphism of $\wedge(V)$.

2.1.5 Duality pairings

For any element $\chi \in \wedge(V)$, let $\chi_{[0]} \in \wedge^0(V) = \mathbb{K}$ denote its degree zero part (i.e., the image under the augmentation map). Then

$$\wedge(V^*) \times \wedge(V) \to \mathbb{K}, \quad \phi \times \chi \mapsto \langle\phi, \chi\rangle := (\iota(\phi^\top)\chi)_{[0]} \tag{2.7}$$

is a non-degenerate bilinear pairing. For $\mu \in V^*$ one has

$$\langle \mu \wedge \phi, \chi \rangle = \langle \phi, \iota(\mu)\chi \rangle,$$

so that $\varepsilon(\mu) = \iota(\mu)^*$. More generally, for $\psi \in \wedge(V^*)$, $\varepsilon(\psi) = \iota(\psi^\top)^*$. The pairing of $\phi \in \wedge^k(V^*)$ with $\chi \in \wedge^l(V)$ is zero unless $k = l$. On the other hand, if $\mu^1, \dots, \mu^k \in V^*$ and $v_1, \dots, v_k \in V$, then

$$\langle \mu^1 \wedge \cdots \wedge \mu^k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle \mu^i, v_j \rangle_{i,j=1}^k). \tag{2.8}$$

The pairing (2.7) will be used to identify

$$\wedge(V^*) \cong \wedge(V)^*. \tag{2.9}$$

Suppose $e_1, \dots, e_n$ is a basis of V, with dual basis $e^1, \dots, e^n$ of V^*. The dual basis to the collection of $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \in \wedge(V)$ for ordered subsets $I = \{i_1, \dots, i_k\}$ (cf. (2.1)) is the collection of all

$$e^I := e^{i_1} \wedge \cdots \wedge e^{i_k} \in \wedge(V^*), \quad I = \{i_1, \dots, i_k\}.$$

That is,

$$\langle e^I, e_J \rangle = \delta^I_J, \quad I, J \subseteq \{1, \dots, n\}.$$

If $\mathbb{K} = \mathbb{C}$, and V is a Hermitian vector space with metric $\langle \cdot, \cdot \rangle$, then the conjugate linear isomorphism $V \to V^*$, $v \mapsto (w \mapsto \langle v, w \rangle)$ defined by the metric, followed by the pairing $\wedge(V^*) \otimes \wedge(V)$, defines a Hermitian metric on $\wedge(V)$. It is given by (2.8), where now $\mu^i \in V$. Similarly, positive definite Euclidean metrics on V define positive definite Euclidean metrics on $\wedge(V)$. If $e_1, \dots, e_n$ is an orthonormal basis of V relative to the Hermitian (resp. Euclidean) metric, then the e_I, $I \subseteq \{1, \dots, n\}$, form an orthonormal basis of $\wedge(V)$ relative to the induced Hermitian (resp. Euclidean) metric.

2.2 Clifford algebras

Clifford algebras are a generalization of exterior algebras, defined in the presence of a symmetric bilinear form.

2.2.1 Definition and first properties

Let V be a vector space over $\mathbb{K}$, with a symmetric bilinear form $B: V \times V \to \mathbb{K}$ (possibly degenerate).

Definition 2.1 The *Clifford algebra* $\mathrm{Cl}(V; B)$ is the quotient

$$\mathrm{Cl}(V; B) = T(V)/\mathscr{I}(V; B),$$

where $\mathscr{I}(V; B) \subseteq T(V)$ is the two-sided ideal generated by all elements of the form

$$v \otimes w + w \otimes v - 2B(v, w)1, \quad v, w \in V.$$

Clearly, $\mathrm{Cl}(V;0) = \wedge(V)$.

Remark 2.1 Clifford algebras for $V = \mathbb{R}^{0,n}$ were introduced by William Kingdon Clifford (1845–1879) under the name of *geometric algebras*. One of his aims was to explain the multiplication rules for Hamilton's quaternions $\mathbb{H}$ (cf. Proposition 2.3). His unfinished manuscript, titled *On the classification of geometric algebras*, was published posthumously in his collection of papers [42, pp. 397–401]. Further historical information may be found in [45] and [100].

Proposition 2.1 *The inclusion* $\mathbb{K} \to T(V)$ *descends to an inclusion* $\mathbb{K} \to \mathrm{Cl}(V;B)$. *The inclusion* $V \to T(V)$ *descends to an inclusion* $V \to \mathrm{Cl}(V;B)$.

Proof Consider the linear map

$$f\colon V \to \mathrm{End}(\wedge(V)),\ v \mapsto \varepsilon(v) + \iota(B^\flat(v)),$$

and its extension to an algebra homomorphism $f_T\colon T(V) \to \mathrm{End}(\wedge(V))$. The commutation relations (2.4) show that $f(v)f(w) + f(w)f(v) = 2B(v,w)1$. Hence f_T vanishes on the ideal $\mathscr{I}(V;B)$, and therefore descends to an algebra homomorphism

$$f_{\mathrm{Cl}}\colon \mathrm{Cl}(V;B) \to \mathrm{End}(\wedge(V)). \tag{2.10}$$

That is, $f_{\mathrm{Cl}} \circ \pi = f_T$ where $\pi\colon T(V) \to \mathrm{Cl}(V;B)$ is the projection. Since $f_T(1) = 1$, we see that $\pi(1) \neq 0$, i.e., the inclusion $\mathbb{K} \hookrightarrow T(V)$ descends to an inclusion $\mathbb{K} \hookrightarrow \mathrm{Cl}(V;B)$. Similarly, from $f_T(v).1 = v$ we see that the inclusion $V \hookrightarrow T(V)$ descends to an inclusion $V \hookrightarrow \mathrm{Cl}(V;B)$. $\square$

The proposition shows that V is a subspace of $\mathrm{Cl}(V;B)$. We may thus characterize $\mathrm{Cl}(V;B)$ as the unital associative algebra, with generators $v \in V$ and relations

$$vw + wv = 2B(v,w), \quad v,w \in V. \tag{2.11}$$

Let us view $T(V) = \bigoplus_k T^k(V)$ as a filtered super algebra (cf. Appendix A), with the $\mathbb{Z}_2$-grading and filtration inherited from the $\mathbb{Z}$-grading.

Since the elements $v \otimes w + w \otimes v - 2B(v,w)1$ are even, of filtration degree 2, the ideal $\mathscr{I}(V;B)$ is a filtered super subspace of $T(V)$, and hence $\mathrm{Cl}(V;B)$ inherits the structure of a filtered super algebra. Simply put, the $\mathbb{Z}_2$-grading and filtration on $\mathrm{Cl}(V;B)$ are defined by the condition that the generators $v \in V$ are odd, of filtration degree 1. In the decomposition

$$\mathrm{Cl}(V;B) = \mathrm{Cl}^{\bar{0}}(V;B) \oplus \mathrm{Cl}^{\bar{1}}(V;B),$$

the two summands are spanned by products $v_1 \cdots v_k$ with k even, respectively odd. From now on, we will always regard $\mathrm{Cl}(V;B)$ as a filtered super algebra (unless stated otherwise); in particular commutators $[\cdot,\cdot]$ will be in the $\mathbb{Z}_2$-graded sense. In this notation, the defining relations for the Clifford algebra become

$$[v,w] = 2B(v,w), \quad v,w \in V.$$

If $\dim V = n$, and $e_1, \ldots, e_n$ are an orthogonal basis of V, then (using the same notation as for the exterior algebra), the products

$$e_I = e_{i_1} \cdots e_{i_k}, \quad I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\},$$

with the convention $e_\emptyset = 1$, span $\mathrm{Cl}(V; B)$. We will see in Section 2.2.5 that the e_I are a basis.

2.2.2 Universal property, functoriality

The Clifford algebra is characterized by the following universal property:

Proposition 2.2 *Let $\mathscr{A}$ be a filtered super algebra, and $f: V \to \mathscr{A}^{(1)}$ a linear map satisfying*

$$f(v_1)f(v_2) + f(v_2)f(v_1) = 2B(v_1, v_2) \cdot 1, \quad v_1, v_2 \in V.$$

Then f extends uniquely to a morphism of filtered super algebras $\mathrm{Cl}(V; B) \to \mathscr{A}$.

Proof By the universal property of the tensor algebra, f extends to an algebra homomorphism $f_{T(V)}: T(V) \to \mathscr{A}$. The property $f(v_1)f(v_2) + f(v_2)f(v_1) = 2B(v_1, v_2) \cdot 1$ shows that f vanishes on the ideal $\mathscr{I}(V; B)$, and hence descends to the Clifford algebra. Uniqueness is clear, since the Clifford algebra is generated by elements of V. □

Remark 2.2 We can also view $\mathrm{Cl}(V; B)$ as a super algebra (forgetting the filtration), as a filtered algebra (forgetting the $\mathbb{Z}_2$-grading), or as an (ordinary) algebra, and formulate universal properties for each of these contexts.

Suppose B_1, B_2 are symmetric bilinear forms on V_1, V_2, and $f: V_1 \to V_2$ is a linear map such that

$$B_2(f(v), f(w)) = B_1(v, w), \quad v, w \in V_1.$$

Viewing f as a map into $\mathrm{Cl}(V_2; B_2)$, the universal property provides a unique extension to a morphism of filtered super algebras

$$\mathrm{Cl}(f): \mathrm{Cl}(V_1; B_1) \to \mathrm{Cl}(V_2; B_2).$$

Clearly,

$$\mathrm{Cl}(f_1 \circ f_2) = \mathrm{Cl}(f_1) \circ \mathrm{Cl}(f_2), \quad \mathrm{Cl}(\mathrm{id}_V) = \mathrm{id}_{\mathrm{Cl}(V)}.$$

The functoriality gives in particular a group homomorphism

$$\mathrm{O}(V; B) \to \mathrm{Aut}(\mathrm{Cl}(V; B)), \ g \mapsto \mathrm{Cl}(g)$$

into algebra automorphisms of $\mathrm{Cl}(V; B)$ (preserving $\mathbb{Z}_2$-grading and filtration). We will usually just write g in place of $\mathrm{Cl}(g)$. For example, the involution $v \mapsto -v$ lies in $\mathrm{O}(V;B)$; hence it defines an involutive algebra automorphism Π of $\mathrm{Cl}(V;B)$ called the *parity automorphism*. The ± 1 eigenspaces are the even and odd part of the Clifford algebra, respectively.

Suppose again that (V_1, B_1) and (V_2, B_2) are two vector spaces with symmetric bilinear forms, and consider the direct sum $(V_1 \oplus V_2,\ B_1 \oplus B_2)$. Then

$$\mathrm{Cl}(V_1 \oplus V_2; B_1 \oplus B_2) = \mathrm{Cl}(V_1; B_1) \otimes \mathrm{Cl}(V_2; B_2)$$

as filtered super algebras. This follows since $\mathrm{Cl}(V_1; B_1) \otimes \mathrm{Cl}(V_2; B_2)$ satisfies the universal property of the Clifford algebra over $(V_1 \oplus V_2; B_1 \oplus B_2)$. In particular, if $\mathrm{Cl}(n,m)$ denotes the Clifford algebra for $\mathbb{K}^{n,m}$, we have

$$\mathrm{Cl}(n,m) = \mathrm{Cl}(1,0) \otimes \cdots \otimes \mathrm{Cl}(1,0) \otimes \mathrm{Cl}(0,1) \otimes \cdots \otimes \mathrm{Cl}(0,1)$$

(using the $\mathbb{Z}_2$-graded tensor product).

2.2.3 The Clifford algebras $\mathrm{Cl}(n,m)$

Consider the case $\mathbb{K} = \mathbb{R}$. For n, m small, one can determine the algebras $\mathrm{Cl}(n,m) = \mathrm{Cl}(\mathbb{R}^{n,m})$ by hand.

Proposition 2.3 *For $\mathbb{K} = \mathbb{R}$, one has the following isomorphisms of the Clifford algebras $\mathrm{Cl}(n,m)$ with $n + m \leq 2$:*

$$\begin{aligned}
&\mathrm{Cl}(0,1) \cong \mathbb{C}, && \Pi(z) = z^c,\\
&\mathrm{Cl}(1,0) \cong \mathbb{R} \oplus \mathbb{R}, && \Pi(u,v) = (v,u),\\
&\mathrm{Cl}(0,2) \cong \mathbb{H}, && \Pi = \mathrm{Ad}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\\
&\mathrm{Cl}(1,1) \cong \mathrm{Mat}_2(\mathbb{R}), && \Pi = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\\
&\mathrm{Cl}(2,0) \cong \mathrm{Mat}_2(\mathbb{R}), && \Pi = \mathrm{Ad}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

Here $\mathbb{C}$ and $\mathbb{H}$ are viewed as algebras over $\mathbb{R}$, and $\mathrm{Mat}_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$ is the algebra of real 2×2-matrices.

Proof By the universal property, an algebra $\mathscr{A}$ of dimension 2^{n+m} is isomorphic to $\mathrm{Cl}(n,m)$ if there exists a linear map $f\colon \mathbb{R}^{n,m} \to \mathscr{A}$ satisfying $f(e_i)f(e_j) + f(e_j)f(e_i) = \pm 2\delta_{ij}$, with a plus sign for $i \leq n$ and a minus sign for $i > n$. We state these maps for $n + m \leq 2$: For the Clifford algebra $\mathrm{Cl}(0,1)$, we take $f(e_1) = \sqrt{-1} \in \mathbb{C}$. For the Clifford algebra $\mathrm{Cl}(1,0)$ we use the map $f(e_1) = (1,-1) \in \mathbb{R} \oplus \mathbb{R}$. Next, for $\mathrm{Cl}(0,2)$ we take

$$f(e_1) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

regarded as elements of $\mathbb{H}$. For Cl(1, 1), we take

$$f(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as elements of $\mathrm{Mat}_2(\mathbb{R})$. Finally, for Cl(2, 0) we use

$$f(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

again as elements of $\mathrm{Mat}_2(\mathbb{R})$. One easily checks that the indicated automorphism Π is $+1$ on the even part and -1 on the odd part. (For $n+m=2$ the parity automorphism is given as conjugation by $f(e_1)f(e_2)$, see Section 2.2.7 below.) □

The full classification of the Clifford algebras $\mathrm{Cl}(n, m)$ may be found in the book by Lawson–Michelsohn [97] or in the monograph by Budinich–Trautman [31]. A systematic treatment may also be found in the article [81] by Kobayashi–Yoshino. The Clifford algebras $\mathrm{Cl}(n, m)$ exhibit a remarkable mod 8 periodicity. For any algebra $\mathscr{A}$ over $\mathbb{K}$ (here $\mathbb{K} = \mathbb{R}$), let $\mathrm{Mat}_k(\mathscr{A}) = \mathscr{A} \otimes \mathrm{Mat}_k(\mathbb{K})$ be the algebra of $k \times k$ matrices with entries in $\mathscr{A}$. Then

$$\mathrm{Cl}(n+8, m) \cong \mathrm{Mat}_{16}(\mathrm{Cl}(n, m)) \cong \mathrm{Cl}(n, m+8).$$

These isomorphisms are related to the mod 8 periodicity in real K-theory [19, 97].

2.2.4 *The Clifford algebras* $\mathbb{C}l(n)$

For $\mathbb{K} = \mathbb{C}$ the pattern is simpler. Denote by $\mathbb{C}l(n)$ the Clifford algebra of $\mathbb{C}^n$.

Proposition 2.4 *One has the following isomorphisms of algebras over* $\mathbb{C}$:

$$\mathbb{C}l(2m) = \mathrm{Mat}_{2^m}(\mathbb{C}), \quad \mathbb{C}l(2m+1) = \mathrm{Mat}_{2^m}(\mathbb{C}) \oplus \mathrm{Mat}_{2^m}(\mathbb{C}).$$

More precisely, $\mathbb{C}l(2m) = \mathrm{End}(\wedge \mathbb{C}^m)$ *as a super algebra, while* $\mathbb{C}l(2m+1) = \mathrm{End}(\wedge \mathbb{C}^m) \otimes (\mathbb{C} \oplus \mathbb{C})$ *as a super algebra. Here the parity automorphism of* $\mathbb{C} \oplus \mathbb{C} = \mathbb{C}l(1)$ *is* $(u, v) \mapsto (v, u)$.

Proof Consider first the case $n = 2$. The map $f : \mathbb{C}^2 \to \mathrm{End}(\mathbb{C}^2)$,

$$f(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix},$$

extends by the universality property to an isomorphism $\mathbb{C}l(2) \to \mathrm{End}(\mathbb{C}^2)$. The resulting $\mathbb{Z}_2$-grading on $\mathrm{End}(\mathbb{C}^2)$ is induced by the $\mathbb{Z}_2$-grading on $\mathbb{C}^2$, where the first component is even and the second is odd. Equivalently, it corresponds to the identification $\mathbb{C}^2 \cong \wedge \mathbb{C}$. This shows that $\mathbb{C}l(2) \cong \mathrm{End}(\wedge \mathbb{C})$ as super algebras. For $\mathbb{C}^{2m} = \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2$ we hence obtain

$$\begin{aligned}\mathbb{C}l(2m) &= \mathbb{C}l(2)\otimes\cdots\otimes\mathbb{C}l(2)\\ &\cong \mathrm{End}(\wedge\mathbb{C})\otimes\cdots\otimes\mathrm{End}(\wedge\mathbb{C})\\ &= \mathrm{End}(\wedge\mathbb{C}\otimes\cdots\otimes\wedge\mathbb{C})\\ &= \mathrm{End}(\wedge\mathbb{C}^m)\end{aligned}$$

as super algebras (using the tensor product of super algebras).

For $n=1$ we have $\mathbb{C}l(1)=\mathbb{C}\oplus\mathbb{C}$ with parity automorphism $\Pi(u,v)=(v,u)$, by the same argument as for the real Clifford algebra $\mathrm{Cl}(1,0)$. Hence

$$\mathbb{C}l(2m+1)=\mathbb{C}l(2m)\otimes\mathbb{C}l(1)=\mathrm{End}(\wedge\mathbb{C}^m)\otimes(\mathbb{C}\oplus\mathbb{C})$$

as super algebras. □

The mod 2 periodicity

$$\mathbb{C}l(n+2)\cong\mathrm{Mat}_2(\mathbb{C}l(n)),$$

apparent in this classification result, is related to the mod 2 periodicity in complex K-theory [19].

Remark 2.3 The result shows in particular that there is an isomorphism of (ungraded) algebras,

$$\mathbb{C}l(2m-1)\cong\mathbb{C}l^{\bar{0}}(2m).$$

This can be directly seen as follows: By the universal property, the map

$$\mathbb{C}^{2m-1}\to\mathbb{C}l^{\bar{0}}(2m),\ e_i\mapsto\sqrt{-1}\,e_ie_{2m},\ \ i=1,\ldots,2m-1$$

extends to an algebra homomorphism $\mathbb{C}l(2m-1)\to\mathbb{C}l^{\bar{0}}(2m)$.

2.2.5 Symbol map and quantization map

We now return to the representation

$$f_{\mathrm{Cl}}\colon\mathrm{Cl}(V;B)\to\mathrm{End}(\wedge(V)),\quad f_{\mathrm{Cl}}(v)=\varepsilon(v)+\iota(B^\flat(v))$$

of the Clifford algebra, (see (2.10)).

Definition 2.2 The *symbol map* is the linear map

$$\sigma\colon\mathrm{Cl}(V;B)\to\wedge(V)$$

given by $\sigma(x)=f_{\mathrm{Cl}}(x).1$, where $1\in\mathbb{K}$ is regarded as an element of $\mathbb{K}=\wedge^0(V)$.

Proposition 2.5 *The symbol map is an isomorphism of filtered super vector spaces. In low degrees,*

$$\begin{aligned}
\sigma(1) &= 1,\\
\sigma(v) &= v,\\
\sigma(v_1 v_2) &= v_1 \wedge v_2 + B(v_1, v_2),\\
\sigma(v_1 v_2 v_3) &= v_1 \wedge v_2 \wedge v_3 + B(v_2, v_3)v_1 - B(v_1, v_3)v_2 + B(v_1, v_2)v_3.
\end{aligned}$$

Proof Let $e_i \in V$ be an orthogonal basis. Since the odd operators $f(e_i)$ commute, we find that

$$\sigma(e_{i_1} \cdots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k},$$

for $i_1 < \cdots < i_k$. This directly shows that the symbol map is an isomorphism: It takes the element $e_I \in \mathrm{Cl}(V; B)$ to the corresponding element $e_I \in \wedge(V)$. The formulas in low degrees are obtained by straightforward calculation. □

The inverse of the symbol map is called the *quantization map*

$$q \colon \wedge(V) \to \mathrm{Cl}(V; B).$$

In terms of the basis, $q(e_I) = e_I$. In low degrees,

$$\begin{aligned}
q(1) &= 1,\\
q(v) &= v,\\
q(v_1 \wedge v_2) &= v_1 v_2 - B(v_1, v_2),\\
q(v_1 \wedge v_2 \wedge v_3) &= v_1 v_2 v_3 - B(v_2, v_3)v_1 + B(v_1, v_3)v_2 - B(v_1, v_2)v_3.
\end{aligned}$$

Proposition 2.6 *The symbol map induces an isomorphism of graded super algebras,*

$$\mathrm{gr}(\mathrm{Cl}(V)) \to \wedge(V).$$

Proof Since the symbol map $\mathrm{Cl}(V) \to \wedge(V)$ is an isomorphism of filtered super spaces, the associated graded map $\mathrm{gr}(\mathrm{Cl}(V)) \to \mathrm{gr}(\wedge(V)) = \wedge(V)$ is an isomorphism of graded super spaces. To check that the induced map preserves products, we must show that the symbol map intertwines products up to lower order terms. That is, for $x \in \mathrm{Cl}(V)^{(k)}$ and $y \in \mathrm{Cl}(V)^{(l)}$ we have $\sigma(xy) - \sigma(x)\sigma(y) \in \wedge^{k+l-1}(V)$. But this is clear from

$$\begin{aligned}
\sigma(v_1 \cdots v_r) &= (\varepsilon(v_1) + \iota(B^\flat(v_1)) \cdots (\varepsilon(v_r) + \iota(B^\flat(v_r)).1\\
&= v_1 \wedge \cdots \wedge v_r \mod \wedge^{r-1}(V),
\end{aligned}$$

for $v_i \in V$. □

The quantization map has the following alternative description.

Proposition 2.7 *The quantization map is given by graded symmetrization. That is, for* $v_1, \ldots, v_k \in V$,

$$q(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \operatorname{sign}(s) v_{s(1)} \cdots v_{s(k)}.$$

Here $\mathfrak{S}_k$ *is the group of permutations of* $1, \ldots, k$, *and* $\operatorname{sign}(s) = \pm 1$ *is the parity of a permutation* s.

Proof By linearity, it suffices to check for the case in which the v_j are elements of an orthonormal basis $e_1, \ldots, e_n$ of V, that is, $v_j = e_{i_j}$ (the indices i_j need not be ordered or distinct). If the i_j are all distinct, then the e_{i_j} Clifford-commute in the graded sense, and the right-hand side equals $e_{i_1} \cdots e_{i_k} \in \mathrm{Cl}(V; B)$, which coincides with the left-hand side. If any two e_{i_j} coincide, then both sides are zero. □

2.2.6 Transposition

Given a symmetric bilinear form B on V, the canonical anti-automorphism of the tensor algebra (see Section 2.1.4) also preserves the ideal $\mathscr{I}(V; B)$. Hence it descends to an anti-automorphism of $\mathrm{Cl}(V; B)$, still called the *canonical anti-automorphism* or *transposition*, with

$$(v_1 \cdots v_k)^\top = v_k \cdots v_1.$$

The quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V; B)$ intertwines the transposition maps for $\wedge(V)$ and $\mathrm{Cl}(V; B)$. This is sometimes useful for computations.

Example 2.1 Suppose $\phi \in \wedge^k(V)$ for some k. Since $q(\phi)$ has filtration degree k, its square has an even element of filtration degree $2k$. Using the transposition automorphism, one can be more precise. Indeed

$$(q(\phi)^2)^\top = (q(\phi)^\top)^2 = q(\phi)^2$$

since $q(\phi)^\top = q(\phi^\top) = \pm q(\phi)$. It follows that

$$q(\phi)^2 \in q\big(\wedge^0(V) \oplus \wedge^4(V) \oplus \cdots \oplus \wedge^{4r}(V)\big),$$

where $r = \frac{k}{2}$ if k is even and $r = \frac{k-1}{2}$ if k is odd.

If $\mathbb{K} = \mathbb{C}$, and if V, B arises as the complexification of a real vector space $V_\mathbb{R}$ with real bilinear form $B_\mathbb{R}$, we obtain a conjugate linear automorphism $x \mapsto x^c$ of $\mathrm{Cl}(V; B)$, given on generators by complex conjugation. Its fixed point set is $\mathrm{Cl}(V_\mathbb{R}; B_\mathbb{R})$. We also obtain a conjugate linear anti-automorphism

$$x \mapsto x^* = (x^c)^\top.$$

The quantization map intertwines the complex conjugation and the anti-automorphism $*$ for the exterior and Clifford algebras.

2.2.7 Chirality element

Let $\dim V = n$. Then any generator $\Gamma_\wedge \in \det(V) := \wedge^n(V)$ quantizes to give an element $\Gamma = q(\Gamma_\wedge)$. This element (or a suitable normalization of this element) is called the *chirality element* of the Clifford algebra. The square Γ^2 of the chirality element is always a scalar, as is immediate by choosing an orthogonal basis e_i, and letting $\Gamma = e_1 \cdots e_n$. In fact, since $\Gamma^\top = (-1)^{n(n-1)/2}\Gamma$ by (2.6), we have

$$\Gamma^2 = (-1)^{n(n-1)/2} \prod_{i=1}^{n} B(e_i, e_i).$$

Remarks 2.1

1. In the case $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^n$ we can always normalize Γ to satisfy $\Gamma^2 = 1$; this normalization determines Γ up to sign.
2. Another important case where Γ admits such a normalization is that of a vector space V with split bilinear form. Choose a pair of transverse Lagrangian subspaces to identify $V = F^* \oplus F$, and pick dual bases $e_1, \ldots, e_m$ of F and $f^1, \ldots, f^m$ of F^*. Then $B(e_i, f^j) = \frac{1}{2}\delta_i^j$, and the vectors $e_i - f^i,\ e_i + f^i,\ i = 1, \ldots, m$, form an orthogonal basis of V. Using $(e_i - f^i)^2 = -1$, $(e_i + f^i)^2 = 1$ we see that

$$\Gamma = (e_1 - f^1)(e_1 + f^1) \cdots (e_m - f^m)(e_m + f^m) \tag{2.12}$$

satisfies $\Gamma^2 = 1$.

Returning to the general case, we observe that

$$\Gamma v = (-1)^{n+1} v \Gamma \tag{2.13}$$

for all $v \in V$, e.g., by checking in an orthogonal basis. (If $v = e_i$, then v anti-commutes with all e_j for $j \neq i$ in the product $\Gamma = e_1 \cdots e_n$, and commutes with e_i. Hence we obtain $n-1$ sign changes.) Thus, if n is odd, then Γ lies in the center of $\mathrm{Cl}(V; B)$, *viewed as an ordinary algebra*. If n is even, the element Γ is even, and lies in the center of $\mathrm{Cl}^{\bar{0}}(V; B)$. Furthermore, in this case

$$\Pi(x) = \Gamma x \Gamma^{-1},$$

for all $x \in \mathrm{Cl}(V; B)$, i.e., the chirality element *implements* the parity automorphism. Let us also record the following useful fact, related to (2.13).

Proposition 2.8 [86, Proposition 84] *Let $e_1, \ldots, e_n$ be a basis of V, with B-dual basis $e^1, \ldots, e^n$, so that $B(e_i, e^j) = \delta_i^j$. For all $\phi \in \wedge^k(V)$,*

$$(-1)^k \sum_{i=1}^{n} e_i q(\phi) e^i = (n - 2k) q(\phi).$$

Note that the left-hand side of this equation does not depend on the choice of basis. It is the result of the action of the element $\sum_i e_i \otimes e^i \in \mathrm{Cl}(V; B) \otimes \mathrm{Cl}(V; B)^{\mathrm{op}}$, which clearly does not depend on the choice of basis, on $q(\phi) \in \mathrm{Cl}(V; B)$.

Proof Since the sum is independent of the choice of basis, we may assume that the e_i are an orthogonal basis consisting of non-isotropic vectors. In particular, $e_i e^i = 1$ for all i. It suffices to prove the formula for $\phi = e_I$, with $I \subseteq \{1, \ldots, n\}$. Thus $q(e_I) = e_I$ is the chirality element for the subspace spanned by the e_i, $i \in I$. We obtain

$$\begin{aligned} i \in I \;&\Rightarrow\; e_i q(\phi) e^i = (-1)^{k+1} e_i e^i q(\phi) = (-1)^{k+1} q(\phi), \\ i \notin I \;&\Rightarrow\; e_i q(\phi) e^i = (-1)^k e_i e^i q(\phi) = (-1)^k q(\phi). \end{aligned}$$

Summing over i, we hence obtain $q(\phi)$ with a scalar factor of

$$(-1)^{k+1} k + (-1)^k (n-k) = (-1)^k (n-2k).$$

This completes the proof. □

2.2.8 *The trace and the super-trace*

For any super algebra $\mathscr{A}$ and (super) vector space Y, a *Y-valued trace* on $\mathscr{A}$ is an even linear map $\mathrm{tr}_s\colon \mathscr{A} \to Y$ vanishing on the subspace $[\mathscr{A}, \mathscr{A}]$ spanned by super-commutators: That is, $\mathrm{tr}_s([x, y]) = 0$ for $x, y \in \mathscr{A}$.

Proposition 2.9 *Let $n = \dim V$. The linear map*

$$\mathrm{tr}_s\colon \mathrm{Cl}(V; B) \to \det(V),$$

given as the quotient map to $\mathrm{Cl}_{(n)}(V; B)/\mathrm{Cl}_{(n-1)}(V; B) \cong \wedge^n(V) = \det(V)$, *is a* $\det(V)$*-valued trace on the super algebra* $\mathrm{Cl}(V; B)$.

Proof Let e_i be an orthogonal basis and e_I the associated basis of $\mathrm{Cl}(V; B)$. Then $\mathrm{tr}_s(e_I) = 0$ unless $I = \{1, \ldots, n\}$. The product of e_I and e_J is of the form $e_I e_J = c e_K$ where $K = (I \cup J) - (I \cap J)$ and $c \in \mathbb{K}$. Hence $\mathrm{tr}_s(e_I e_J) = 0 = \mathrm{tr}_s(e_J e_I)$ unless $I \cap J = \emptyset$ and $I \cup J = \{1, \ldots, n\}$. But in case $I \cap J = \emptyset$, the elements e_I and e_J super-commute: $[e_I, e_J] = 0$. □

The Clifford algebra also has an *ordinary trace*, vanishing on ordinary commutators.

Proposition 2.10 *The formula*

$$\mathrm{tr}\colon \mathrm{Cl}(V; B) \to \mathbb{K}, \quad x \mapsto \sigma(x)_{[0]}$$

defines an (ordinary) trace on $\mathrm{Cl}(V;B)$, *that is* $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ *for all* $x, y \in \mathrm{Cl}(V;B)$. *The trace satisfies* $\mathrm{tr}(x^\top) = \mathrm{tr}(x)$ *and* $\mathrm{tr}(1) = 1$, *and is related to the super-trace by the formula*

$$\mathrm{tr}_s(\Gamma x) = \mathrm{tr}(x)\,\Gamma_\wedge,$$

where $\Gamma = q(\Gamma_\wedge)$ *is the chirality element in the Clifford algebra defined by a choice of generator of* $\det(V)$.

Proof Again, we use an orthogonal basis e_i of V. The definition gives $\mathrm{tr}(e_\emptyset) = 1$, while $\mathrm{tr}(e_I) = 0$ for $I \neq \emptyset$. We will show that $\mathrm{tr}(e_I e_J) = \mathrm{tr}(e_J e_I)$. We have $e_I e_J = c e_K$, where $K = (I \cup J) - (I \cap J)$ and $c \in \mathbb{K}$. If $I \neq J$, the set K is non-empty, hence $\mathrm{tr}(e_I e_J) = 0 = \mathrm{tr}(e_J e_I)$. If $I = J$, the trace property is trivial. To check the formula relating trace and super-trace we may assume $\Gamma_\wedge = e_1 \cdots e_n$. For $x = e_J$ we see that $\mathrm{tr}_s(\Gamma x)$ vanishes unless $J = \emptyset$, in which case we obtain $\mathrm{tr}_s(\Gamma) = \Gamma_\wedge$. □

2.2.9 *Lie derivatives and contractions*

For any vector space V, there is an isomorphism of graded super vector spaces

$$\mathrm{Der}(T(V)) \cong \mathrm{Hom}(V, T(V)) = T(V) \otimes V^*.$$

Here $V \cong T(V)^1$ is regarded as a graded super vector space concentrated in degree 1, and $T(V)$ is regarded as a graded super algebra. Indeed, any derivation of $T(V)$ is uniquely determined by its restriction to V; conversely, any linear map $D\colon V \to T(V)$ of degree $r \in \mathbb{Z}$ extends to an element of $\mathrm{Der}(T(V))^r$ by the derivation property,

$$D(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^k (-1)^{r(i-1)} v_1 \otimes \cdots \otimes Dv_i \otimes \cdots \otimes v_k,$$

for $v_1, \ldots, v_k \in V$. (For a detailed proof, see Chevalley [41, p. 27].)

The graded super Lie algebra $\mathrm{Der}(T(V))$ has non-vanishing components only in degrees $r \geq -1$. The component $\mathrm{Der}(T(V))^{-1}$ is the space $\mathrm{Hom}(V, \mathbb{K}) = V^*$, acting by *contractions* $\iota(\alpha)$, $\alpha \in V^*$:

$$\iota(\alpha)(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^k (-1)^{i-1} \langle \alpha, v_i \rangle\, v_1 \otimes \cdots \widehat{v_i} \cdots \otimes v_k.$$

The component $\mathrm{Der}(T(V))^0 = \mathrm{Hom}(V, V) = \mathfrak{gl}(V)$ is the space of *Lie derivatives*, given for $A \in \mathfrak{gl}(V)$ by

$$L_A(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^k v_1 \otimes \cdots \otimes L_A(v_i) \otimes \cdots \otimes v_k.$$

We have $\mathrm{Der}(T(V))^{-1} \oplus \mathrm{Der}(T(V))^0 \cong V^* \rtimes \mathfrak{gl}(V)$ as graded super Lie algebras.

Remark 2.4 A parallel discussion describes the derivations of $T(V)$ as an ordinary graded algebra; here one omits the sign $(-1)^{r(i-1)}$ in the formula for the extension of D.

Both contractions and Lie derivatives preserve the ideal $\mathscr{I}(V)$ defining the exterior algebra, and hence descend to derivations of $\wedge(V)$, still called contractions and Lie derivatives. This defines a morphism of graded super Lie algebras $V^* \rtimes \mathfrak{gl}(V) \to \mathrm{Der}(\wedge(V))$.

Given a symmetric bilinear form B on V, the contraction operators also preserve the ideal $\mathscr{I}(V; B)$ since

$$\iota(\alpha)(v_1 \otimes v_2 + v_2 \otimes v_1 - 2B(v_1, v_2)) = 0, \quad v_1, v_2 \in V.$$

Hence they descend to odd derivations $\iota(\alpha)$ of $\mathrm{Cl}(V; B)$ of filtration degree -1, given as

$$\iota(\alpha)(v_1 \cdots v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, v_i \rangle v_1 \cdots \widehat{v_i} \cdots v_k. \tag{2.14}$$

On the other hand, the Lie derivatives L_A on $T(V)$ preserve the ideal $\mathscr{I}(V; B)$ if and only if $A \in \mathfrak{o}(V; B)$, that is, $B(Av_1, v_2) + B(v_1, Av_2) = 0$ for all v_1, v_2. Under this condition, L_A descends to an even derivation of filtration degree 0

$$L_A(v_1 \cdots v_k) = \sum_{i=1}^{k} v_1 \cdots L_A(v_i) \cdots v_k$$

on the Clifford algebra. Together with the contractions, this gives a morphism of filtered super Lie algebras $V^* \rtimes \mathfrak{o}(V; B) \to \mathrm{Der}(\mathrm{Cl}(V; B))$:

$$[\iota(\alpha_1), \iota(\alpha_2)] = 0, \ [L_{A_1}, L_{A_2}] = L_{[A_1, A_2]}, \ [L_A, \iota(\alpha)] = \iota(A.\alpha),$$

where $A.\alpha = -A^*\alpha$ with A^* the dual map.

Proposition 2.11 *The symbol map intertwines the action of the group* $V^* \rtimes \mathfrak{o}(V; B)$ *by contractions and Lie derivatives on* $\mathrm{Cl}(V; B)$ *with the corresponding action on* $\wedge(V)$.

Proof It suffices to check on elements $\phi = v_1 \wedge \cdots \wedge v_k \in \wedge(V)$, where $v_1, \ldots, v_k$ are pairwise orthogonal. Then $q(\phi) = v_1 \cdots v_k$, and the quantization of $\iota(\alpha)\phi$ (given by (2.2)) coincides with $\iota(\alpha)(q(\phi))$ (given by (2.14)). The argument for the Lie derivatives is similar. □

Any element $v \in V$ defines a derivation of $\mathrm{Cl}(V; B)$ by the super commutator $x \mapsto [v, x]$. For generators $w \in V$, we have $[v, w] = 2B(v, w) = 2\langle B^\flat(v), w\rangle$. This shows that this derivation agrees with the contraction by $2B^\flat(v)$:

$$[v, \cdot] = 2\iota(B^\flat(v)). \tag{2.15}$$

As a simple application, we find:

Lemma 2.1 *The center of the filtered super algebra* $\mathrm{Cl}(V; B)$ *is the exterior algebra over* $\mathrm{rad}(B) = \ker(B^\flat)$. *Hence, if* B *is non-degenerate, the center consists of the scalars.*

We stress that the lemma refers to the "super-center", not the center of $\mathrm{Cl}(V; B)$ as an ordinary algebra.

Proof Indeed, suppose x lies in the center. Then

$$0 = [v, x] = 2\iota(B^\flat(v))x$$

for all $v \in V$. Hence $\sigma(x)$ is annihilated by all contractions $B^\flat(v)$, and is therefore an element of the exterior algebra over $\mathrm{ann}(\mathrm{ran}(B^\flat)) = \ker(B^\flat)$. Consequently $x = q(\sigma(x))$ is in $\mathrm{Cl}(\ker(B^\flat)) = \wedge(\ker(B^\flat))$. $\square$

2.2.10 The Lie algebra $q(\wedge^2(V))$

The following important fact relates the "quadratic elements" of the Clifford algebra to the Lie algebra $\mathfrak{o}(V; B)$.

Theorem 2.1 *The elements* $q(\lambda)$, $\lambda \in \wedge^2(V)$ *span a Lie subalgebra of* $\mathrm{Cl}(V; B)$. *Let* $\{\cdot, \cdot\}$ *be the induced Lie bracket on* $\wedge^2(V)$ *so that*

$$[q(\lambda), q(\lambda')] = q(\{\lambda, \lambda'\}).$$

The transformation $v \mapsto A_\lambda(v) = [q(\lambda), v]$ *defines an element* $A_\lambda \in \mathfrak{o}(V; B)$, *and the map*

$$\wedge^2(V) \to \mathfrak{o}(V; B), \quad \lambda \mapsto A_\lambda$$

is a Lie algebra homomorphism. One has $L_{A_\lambda} = [q(\lambda), \cdot]$ *as derivations of* $\mathrm{Cl}(V; B)$.

Proof By definition, $A_\lambda(v) = [q(\lambda), v] = -2\iota(B^\flat(v))q(\lambda)$. Hence

$$A_\lambda(v) = -2\iota(B^\flat(v))\lambda$$

since the quantization map intertwines the contractions of the exterior and Clifford algebras. We have $A_\lambda \in \mathfrak{o}(V; B)$ since

$$B(A_\lambda(v), w) = -2\iota(B^\flat(w))A_\lambda(v) = -2\iota(B^\flat(w))\iota(B^\flat(v))\lambda$$

is anti-symmetric in v, w. It follows that $L_{A_\lambda} = [q(\lambda), \cdot]$ since the two sides are derivations which agree on generators. Define a bracket $\{\cdot, \cdot\}$ on $\wedge^2(V)$ by

$$\{\lambda, \lambda'\} = L_{A_\lambda}\lambda' \tag{2.16}$$

(using the Lie derivatives on $\wedge(V)$). The calculation

$$[q(\lambda), q(\lambda')] = L_{A_\lambda} q(\lambda') = q(L_{A_\lambda}\lambda') = q(\{\lambda, \lambda'\})$$

shows that q intertwines $\{\cdot,\cdot\}$ with the Clifford commutator; in particular $\{\cdot,\cdot\}$ is a Lie bracket. Furthermore, from

$$[q(\lambda),[q(\lambda'),v]]-[q(\lambda'),[q(\lambda),v]]=[[q(\lambda),q(\lambda')],v]=[q(\{\lambda,\lambda'\}),v]$$

we see that $[A_\lambda,A_{\lambda'}]=A_{\{\lambda,\lambda'\}}$, that is, the map $\lambda\mapsto A_\lambda$ is a Lie algebra homomorphism. □

Corollary 2.1 *Relative to the bracket* $\{\cdot,\cdot\}$ *on* $\wedge^2(V)$, *the map*

$$V\rtimes\wedge^2(V)\to V^*\rtimes\mathfrak{o}(V;B),\quad (v,\lambda)\mapsto(B^\flat(v),A_\lambda)$$

is a homomorphism of graded super Lie algebras. We have a commutative diagram,

$$\begin{array}{ccc} V\rtimes\wedge^2(V) & \longrightarrow & V^*\rtimes\mathfrak{o}(V;B)\\ \big\downarrow{\scriptstyle q} & & \big\downarrow\\ \mathrm{Cl}(V;B) & \xrightarrow[\mathrm{ad}]{} & \mathrm{Der}(\mathrm{Cl}(V;B)). \end{array}$$

Note that we can think of $V\rtimes\wedge^2(V)$ as a graded super subspace of $\wedge(V)[2]$, using the standard grading on $\wedge(V)$ shifted down by 2. We will see in the following Section 2.3 that the graded Lie bracket on $V\rtimes\wedge^2(V)$ extends to a graded Lie bracket on all of $\wedge(V)[2]$.

Proposition 2.12 *If B is non-degenerate, then the map $\lambda\mapsto A_\lambda$ is an isomorphism* $\wedge^2(V)\to\mathfrak{o}(V;B)$.

Proof In a basis e_i of V, with B-dual basis e^i, the inverse map $\mathfrak{o}(V;B)\to\wedge^2(V)$ is given by $A\mapsto\frac14\sum_i A(e_i)\wedge e^i$. Indeed, since $B(A(e_i),e^i)=0$ the quantization of such an element is just $\frac14\sum_i A(e_i)e^i\in\mathrm{Cl}(V;B)$, and one directly checks that

$$\Big[\frac14\sum_i A(e_i)e^i,v\Big]=A(v),$$

as required. □

The inverse map will be denoted by

$$\lambda\colon\mathfrak{o}(V;B)\to\wedge^2(V),\tag{2.17}$$

and its quantization

$$\gamma=q\circ\lambda\colon\mathfrak{o}(V;B)\to\mathrm{Cl}(V).\tag{2.18}$$

In a basis e_i if V, with B-dual basis e^i, we have

$$\lambda(A)=\frac14\sum_i A(e_i)\wedge e^i,\tag{2.19}$$

hence $\gamma(A)=\frac14\sum_i A(e_i)e^i$.

2.2.11 A formula for the Clifford product

It is sometimes useful to express the Clifford multiplication

$$m_{\mathrm{Cl}}\colon \mathrm{Cl}(V \oplus V) = \mathrm{Cl}(V) \otimes \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

in terms of the exterior algebra multiplication,

$$m_\wedge\colon \wedge(V \oplus V) = \wedge(V) \otimes \wedge(V) \to \wedge(V).$$

Recall that the bilinear form B on V has a canonical extension to a symmetric bilinear form on $\wedge(V)$. The linear map $\wedge(V) \otimes \wedge(V) \to \mathbb{K},\ \chi' \otimes \chi \mapsto B(\chi', \chi)$ may be regarded as an element

$$\Psi \in \wedge(V^*) \otimes \wedge(V^*).$$

In terms of a basis $\chi_r \in \wedge(V),\ r = 1, \ldots, 2^n$ (where each χ_r has definite parity), with dual basis $\phi^r \in \wedge(V^*)$, we have

$$\Psi = \sum_{r=1}^{2^n} (-1)^{|\chi_r|} \phi^r \otimes B^\flat(\chi_r^\top). \tag{2.20}$$

(To verify this formula, evaluate on $\chi_s \otimes \chi_{s'}$.) The following is a reformulation of [86, Theorem 16].

Proposition 2.13 *Under the quantization map, the exterior algebra product and the Clifford product are related as follows*:

$$m_{\mathrm{Cl}} \circ (q \otimes q) = q \circ m_\wedge \circ \iota(\Psi).$$

Proof Let $e_1, \ldots, e_n$ be an orthogonal basis of V. It determines a basis of $\wedge(V)$, consisting of elements $e_I = e_{i_1} \ldots e_{i_k}$ indexed by ordered subsets $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. The dual basis of $\wedge(V^*)$ consists of elements $e^I = e^{i_1} \cdots e^{i_k}$, where $e^1, \ldots, e^n \in V^*$ are the dual basis to $e_1, \ldots, e_n$. In terms of these bases,

$$\Psi = \sum_I (-1)^{|I|} e^I \otimes B^\flat(e_I^\top).$$

Let V_i be the 1-dimensional subspace spanned by e_i. Then $\wedge(V)$ is the graded tensor product over all $\wedge(V_i)$, and similarly $\mathrm{Cl}(V)$ is the graded tensor product over all $\mathrm{Cl}(V_i)$. The formula for Ψ factorizes as

$$\Psi = \prod_{i=1}^n \Psi_i, \quad \Psi_i = 1 - e^i \otimes B^\flat(e_i). \tag{2.21}$$

It hence suffices to prove the formula for the case $V = V_1$. It is obvious that the composition $q \circ m_\wedge \circ \iota(\Psi_1)$ coincides with m_{Cl} on the basis elements $1 \otimes 1,\ 1 \otimes e_1$ and $e_1 \otimes 1$. On the basis element $e_1 \otimes e_1$ we have

$$q \circ m_\wedge \circ \iota(\Psi_1)(e_1 \otimes e_1) = q \circ m_\wedge(e_1 \otimes e_1 + B(e_1, e_1)) = B(e_1, e_1)$$

which coincides with $m_{\mathrm{Cl}}(e_1 \otimes e_1)$. □

If $\mathrm{char}(\mathbb{K}) = 0$, we may also write the element Ψ as an exponential:

$$\Psi = \exp\left(- \sum_{i=1}^{n} e^i \otimes B^\flat(e_i) \right).$$

Indeed, the right-hand side is $\prod_{i=1}^n \exp(-e^i \otimes B^\flat(e_i)) = \prod_{i=1}^n \Psi_i$.

2.3 The Clifford algebra as a quantization

Using the quantization map, the Clifford algebra $\mathrm{Cl}(V; B)$ may be thought of as $\wedge(V)$ with a new associative product. In this section we will explain in which sense the Clifford algebra is a quantization of the exterior algebra. Much of the material in this section is motivated by the paper [90] of Kostant and Sternberg.

2.3.1 Differential operators

The prototype of the notion of "quantization" to be considered here is the algebra of differential operators on a manifold M. For all k, we let $\mathfrak{D}^{(k)}(M) \subseteq \mathrm{End}(C^\infty(M))$ denote the space of differential operators of degree $\leq k$. Thus $\mathfrak{D}^{(-1)}(M) = 0$, and inductively for $k \geq 0$,

$$\mathfrak{D}^{(k)}(M) = \{D \in \mathrm{End}(C^\infty(M)) |\ \forall f \in C^\infty(M) \colon [D, f] \in \mathfrak{D}^{(k-1)}(M)\}.$$

Note $\mathfrak{D}^{(0)}(M) = C^\infty(M)$. One may show that this is indeed the familiar notion of differential operators: in local coordinates $q_1, \ldots, q_n$ on M, any $D \in \mathfrak{D}^{(k)}(M)$ has the form

$$D = \sum_{|I| \leq k} a_I(x) \Big(\frac{\partial}{\partial x}\Big)^I,$$

using multi-index notation $I = (i_1, \ldots, i_n)$ with

$$|I| = \sum_{j=1}^n i_j, \quad \Big(\frac{\partial}{\partial x}\Big)^I = \prod_{j=1}^n \Big(\frac{\partial}{\partial x_j}\Big)^{i_j}.$$

The composition of operators on $C^\infty(M)$ defines a product

$$\mathfrak{D}^{(k)}(M) \times \mathfrak{D}^{(l)}(M) \to \mathfrak{D}^{(k+l)}(M),$$

making

$$\mathfrak{D}(M) = \bigcup_{k=0}^\infty \mathfrak{D}^{(k)}(M)$$

into a filtered algebra. Now let T^*M be the cotangent bundle of M (dual of the tangent bundle), and let $\mathrm{Pol}^k(T^*M) \subseteq C^\infty(T^*M)$ be the functions whose restriction to each fiber is a polynomial of degree k. Note that $\mathrm{Pol}^k(T^*M)$ is isomorphic to the sections of $S^k(TM)$, the k-th symmetric power of the tangent bundle.

Proposition 2.14 *For every degree k differential operator $D \in \mathfrak{D}^{(k)}(M)$, there is a unique function $\sigma^k(D) \in \mathrm{Pol}^k(T^*M)$ such that for all functions f,*

$$\sigma^k(D) \circ \mathrm{d}f = \underbrace{[[\cdots[D, f], f]\cdots, f]}_{k \ \text{times}}.$$

Proof (Sketch) Writing D in local coordinates as above, the right-hand side is the function

$$\sum_{|I|=k} a_I(x)\Big(\frac{\partial f}{\partial x_1}\Big)^{i_1} \cdots \Big(\frac{\partial f}{\partial x_k}\Big)^{i_k}. \tag{2.22}$$

In particular, its value at any $x \in M$ depends only the differential $\mathrm{d}_x f$, and is a polynomial of degree k in $\mathrm{d}_x f$. □

The function

$$\sigma^k(D) \in \mathrm{Pol}^k(T^*M) \cong \Gamma^\infty(M, S^k(TM))$$

is called the degree k *principal symbol* of P. By (2.22), it is given in local coordinates as follows:

$$\sigma^k(D)(x, p) = \sum_{|I|=k} a_I(x) p^I.$$

We see in particular that $\sigma^k(D) = 0$ if and only if $D \in \mathfrak{D}^{(k-1)}(M)$.

We obtain an exact sequence,

$$0 \to \mathfrak{D}^{(k-1)}(M) \to \mathfrak{D}^{(k)}(M) \xrightarrow{\sigma^k} \Gamma^\infty(M; S^k(TM)) \to 0.$$

If D_1, D_2 are differential operators of degrees k_1, k_2, then $D_1 \circ D_2$ is a differential operator of degree $k_1 + k_2$ and

$$\sigma^{k_1+k_2}(D_1 \circ D_2) = \sigma^{k_1}(D_1)\sigma^{k_2}(D_2).$$

The symbol map descends to a morphism of graded algebras,

$$\sigma^\bullet : \mathrm{gr}^\bullet \mathfrak{D}(M) \to \Gamma^\infty(M, S^\bullet(TM)).$$

Using a partition of unity, it is not hard to see that this map is an isomorphism.

If D_1, D_2 have degree k_1, k_2, then the degree $k_1 + k_2$ principal symbol of the commutator $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is zero. Hence $[D_1, D_2]$ has degree $k_1 + k_2 - 1$. A calculation of the leading terms shows that

$$\sigma_{k_1+k_2-1}([D_1, D_2]) = \{\sigma_{k_1}(D_1),\ \sigma_{k_2}(D_2)\},$$

where $\{\cdot, \cdot\}$ is the *Poisson bracket* on $\mathrm{Pol}^\bullet(T^*M) = \Gamma^\infty(M, S^\bullet TM)$ given in local coordinates by

$$\{f, g\} = \sum_{i=1}^n \Big(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial x_i}\Big).$$

(We recall that a Poisson bracket on a manifold Q is a Lie bracket $\{\cdot,\cdot\}$ on the algebra of functions $C^\infty(Q)$, such that for all $F \in C^\infty(M)$, the linear map $\{F,\cdot\}$ is a derivation of the algebra structure. One calls $\{F,\cdot\}$ the *Hamiltonian vector field* associated to F.) In this sense the algebra $\mathfrak{D}^\bullet(M)$ of differential operators is regarded as a "quantization" of the Poisson algebra $\mathrm{Pol}^\bullet(T^*M) = \Gamma^\infty(M, S^\bullet TM)$.

2.3.2 Graded Poisson algebras

To formalize this construction, we define a *graded Poisson algebra* of degree n to be a commutative graded algebra $\mathscr{P} = \bigoplus_{k\in\mathbb{Z}} \mathscr{P}^k$, together with a bilinear map $\{\cdot,\cdot\}\colon \mathscr{P} \times \mathscr{P} \to \mathscr{P}$ (the *Poisson bracket*) such that

1. The space $\mathscr{P}[n]$ is a graded Lie algebra, with the bracket $\{\cdot,\cdot\}$.
2. The map $f \mapsto \{f,\cdot\}$ defines a morphism of graded Lie algebras

$$\mathscr{P}[n] \to \mathrm{Der}_{\mathrm{alg}}(\mathscr{P}).$$

Here $\mathrm{Der}_{\mathrm{alg}}(\mathscr{P})$ is the space of derivations for the algebra structure. That is, for any $f \in \mathscr{P}^k$, the map $\{f,\cdot\}$ is a degree $k-n$ derivation of the algebra structure. Note that the Poisson bracket is uniquely determined by its values on generators.

Example 2.2 For any manifold M, the space $\Gamma^\infty(M, S^\bullet(TM))$ of fiberwise polynomial functions on T^*M is a graded Poisson algebra of degree 1.

Example 2.3 Suppose $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ is any Lie algebra. Then the Lie bracket on $\mathfrak{g}$ extends to a Poisson bracket on the graded algebra $S(\mathfrak{g})$, making the latter into a graded Poisson algebra of degree 1. Viewing $S(\mathfrak{g})$ as polynomial functions on $\mathfrak{g}^*$, this is the *Kirillov–Poisson structure* on $\mathfrak{g}^*$. Conversely, if V is any vector space, then the structure of a graded Poisson algebra of degree 1 on $S(V)$ is *equivalent* to a Lie algebra structure on V.

Suppose now that $\mathscr{A}$ is a filtered algebra, with the property that the associated graded algebra $\mathrm{gr}(\mathscr{A})$ is commutative. This means that $[\mathscr{A}^{(k)}, \mathscr{A}^{(l)}] \subseteq \mathscr{A}^{(k+l-1)}$ for all k, l. Then the associated graded algebra $\mathscr{P} = \mathrm{gr}(\mathscr{A})$ becomes a graded Poisson algebra of degree 1, with the bracket determined by a commutative diagram:

$$\begin{tikzcd}
\mathscr{A}^{(k)} \otimes \mathscr{A}^{(l)} \arrow[r, "{[\cdot,\cdot]}"'] \arrow[d] & \mathscr{A}^{(k+l-1)} \arrow[d] \\
\mathscr{P}^k \otimes \mathscr{P}^l \arrow[r, "{\{\cdot,\cdot\}}"'] & \mathscr{P}^{k+l-1}.
\end{tikzcd}$$

We will think of $\mathscr{A}$ as a "quantization" of $\mathscr{P} = \mathrm{gr}(\mathscr{A})$, since the Poisson bracket on $\mathscr{P}$ is induced by the commutator bracket on $\mathscr{A}$. For instance, as discussed later, the enveloping algebra $U(\mathfrak{g})$ defines a quantization of the Poisson algebra $S(\mathfrak{g})$.

2.3.3 Graded super Poisson algebras

The symbol map for Clifford algebras may be put into a similar framework, but in a super context. See [90] and [35]. See Appendix A for background on graded and filtered super spaces.

Definition 2.3 A *graded super Poisson algebra of degree n* is a commutative graded super algebra $\mathscr{P} = \bigoplus_{k\in\mathbb{Z}} \mathscr{P}^k$, together with a bilinear map $\{\cdot,\cdot\}\colon \mathscr{P}\times\mathscr{P}\to\mathscr{P}$ such that

1. The space $\mathscr{P}[n]$ is a graded super Lie algebra, with bracket $\{\cdot,\cdot\}$.
2. The map $f\mapsto\{f,\cdot\}$ defines a morphism of graded super Lie algebras, $\mathscr{P}[n]\to\mathrm{Der}_{\mathrm{alg}}(\mathscr{P})$.

Here $\mathrm{Der}_{\mathrm{alg}}(\mathscr{P})$ signifies the derivations of $\mathscr{P}$ as a graded super algebra. Thus, the bracket $\{\cdot,\cdot\}$ is a map of degree $-n$, with the properties

$$\begin{aligned}
\{f_1,\{f_2,f_3\}\} &= \{\{f_1,f_2\},f_3\} + (-1)^{(|f_1|-n)(|f_2|-n)}\{f_2,\{f_1,f_3\}\},\\
\{f_1,f_2\} &= -(-1)^{(|f_1|-n)\,(|f_2|-n)}\{f_2,f_1\},\\
\{f_1,f_2f_3\} &= \{f_1,f_2\}f_3 + (-1)^{(|f_1|-n)|f_2|}f_2\{f_1,f_3\}.
\end{aligned}$$

For $f\in\mathscr{P}^k$, the bracket $\{f,\cdot\}$ is a derivation of degree $k-n$ of the algebra and Lie algebra structures. Letting $\mathrm{Der}_{\mathrm{Poi}}(\mathscr{P})$ be the derivations of $\mathscr{P}$ as a graded Poisson algebra, we hence have a morphism of graded super Lie algebras,

$$\mathscr{P}[n]\to\mathrm{Der}_{\mathrm{Poi}}(\mathscr{P}),\quad f\mapsto\{f,\cdot\}. \tag{2.23}$$

Note that $\mathscr{P}^n=\mathscr{P}[n]^0$ is an ordinary Lie algebra under the Poisson bracket.

Suppose now that $\mathscr{A}$ is a filtered super algebra such that the associated graded super algebra $\mathrm{gr}(\mathscr{A})$ is commutative. Thus $[\mathscr{A}^{(k)},\mathscr{A}^{(l)}]\subseteq\mathscr{A}^{(k+l-1)}$. Using the compatibility condition for the $\mathbb{Z}_2$-grading and filtration (cf. Appendix A)

$$(\mathscr{A}^{(2k)})^{\bar 0}=(\mathscr{A}^{(2k+1)})^{\bar 0},\quad (\mathscr{A}^{(2k+1)})^{\bar 1}=(\mathscr{A}^{(2k+2)})^{\bar 1}, \tag{2.24}$$

we see that in fact

$$[\mathscr{A}^{(k)},\mathscr{A}^{(l)}]\subseteq\mathscr{A}^{(k+l-2)}.$$

Hence the associated graded super algebra $\mathscr{P}=\mathrm{gr}(A)$ becomes a graded super Poisson algebra of degree 2, with Poisson bracket determined by the commutative diagram

$$\begin{tikzcd}
\mathscr{A}^{(k)}\otimes\mathscr{A}^{(l)} \arrow[r, "{[\cdot,\cdot]}"'] \arrow[d] & \mathscr{A}^{(k+l-2)} \arrow[d] \\
\mathscr{P}^k\otimes\mathscr{P}^l \arrow[r, "{\{\cdot,\cdot\}}"'] & \mathscr{P}^{k+l-2}.
\end{tikzcd}$$

2.3.4 Poisson structures on $\wedge(V)$

Any symmetric bilinear form B on a vector space V induces on $\mathscr{A} = \wedge(V)$ the structure of a graded super Poisson algebra of degree 2. The Poisson bracket is given on generators $v, w \in V = \wedge^1(V)$ by

$$\{v, w\} = 2B(v, w).$$

In this way one obtains a one-to-one correspondence between Poisson brackets of degree -2 on $\wedge(V)$ and symmetric bilinear forms B. Clearly, this Poisson bracket is induced from the commutator on the Clifford algebra under the identification $\wedge(V) = \mathrm{gr}(\mathrm{Cl}(V; B))$ from Proposition 2.6.

Taking the Poisson bracket with a given element of $\wedge^k(V)$ defines a derivation of degree $k-2$ of $\wedge(V)$. In particular, Poisson bracket with an element $v \in V$ is a derivation of degree -1, i.e., a contraction:

$$\{v, \cdot\} = 2\iota(B^\flat(v)).$$

Similarly, for $\lambda \in \wedge^2(V)$ we have

$$\{\lambda, \cdot\} = L_{A_\lambda},$$

since both sides are derivations given by $A_\lambda(w)$ on generators $w \in V$. In particular, the Lie bracket on $\wedge^2(V) = (\wedge(V)[2])^0$ defined by the Poisson bracket recovers our earlier definition as $\{\lambda, \lambda'\} = L_{A_\lambda}\lambda'$.

The graded Lie algebra $V \rtimes \wedge^2(V)$ from Section 2.2.10 is now interpreted as

$$(\wedge^1(V) \oplus \wedge^2(V))[2] \subseteq \wedge(V)[2].$$

As we saw, the quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V)$ restricts to a Lie algebra homomorphism on this Lie subalgebra. That is, on elements of degree ≤ 2, the quantization map takes Poisson brackets to commutators. This is no longer true, in general, for higher order elements.

Example 2.4 Let $\phi \in \wedge^3(V)$, so that $\{\phi, \phi\} \in \wedge^4(V)$. As we saw in Example 2.1,

$$[q(\phi), q(\phi)] = 2q(\phi)^2 \in q(\wedge^0(V) \oplus \wedge^4(V)).$$

The leading term is given by the Poisson bracket, that is,

$$[q(\phi), q(\phi)] - q(\{\phi, \phi\}) \in \wedge^0(V) = \mathbb{K}.$$

In general, this scalar is non-zero. For instance, if $V = \mathbb{R}^3$ with the standard bilinear form, and $\phi = e_1 \wedge e_2 \wedge e_3$ (the volume element), then

$$[q(\phi), q(\phi)] = 2q(\phi)^2 = 2(e_1 e_2 e_3)^2 = -2.$$

More generally, suppose V is a finite-dimensional vector space, with non-degenerate symmetric bilinear form B, and let e_a be a basis of V, with B-dual basis e^a. Given $\phi \in \wedge^3(V)$, define its components in the two bases by

$$\phi = \frac{1}{6}\sum_{abc} \phi^{abc} e_a \wedge e_b \wedge e_c = \frac{1}{6}\sum_{abc} \phi_{abc} e^a \wedge e^b \wedge e^c.$$

According to Proposition 2.13, the constant term in $[q(\phi), q(\phi)] = 2q(\phi)^2$ is obtained by applying the operator $\frac{1}{3!}(-\sum_a \iota(e^a) \otimes \iota(e_a))^3$ to $2\phi \otimes \phi \in \wedge(V) \otimes \wedge(V)$. This gives

$$[q(\phi), q(\phi)] - q(\{\phi, \phi\}) = \frac{1}{3}\Big(-\sum_a \iota(e^a) \otimes \iota(e_a)\Big)^3 (\phi \otimes \phi)$$
$$= -\frac{1}{3}\sum_{abc} \phi^{abc}\phi_{abc}.$$

It can be shown (cf. Proposition 6.8 below) that $\{\phi, \phi\} = 0$ if and only if the formula $[v, w] := \{\{\phi, v\}, w\}$ defines a Lie bracket on V.

Consider the map (cf. (2.23))

$$\wedge(V)[2] \to \mathrm{Der}_{\mathrm{Poi}}(\wedge(V)),\ \ f \mapsto \{f, \cdot\}. \tag{2.25}$$

Its composition with the inclusion

$$\mathrm{Der}_{\mathrm{Poi}}(\wedge(V)) \to \mathrm{Der}_{\mathrm{alg}}(\wedge(V)) \cong \wedge(V) \otimes V^*$$

is given by

$$f \mapsto (-1)^{|f|} 2\sum_a \iota(e^a) f \otimes B^\flat(e_a),$$

where e_a is a basis of V with dual basis $e^a \in V^*$. To verify this identity, it suffices to evaluate on generators $v \in V$: Indeed,

$$\{f, v\} = (-1)^{|f|}\{v, f\} = (-1)^{|f|} 2\iota(B^\flat(v)) f = (-1)^{|f|} 2\sum_a B(e_a, v)\,\iota(e^a) f.$$

If B is non-degenerate, then all derivations of the Poisson structure on $\wedge(V)$ are inner:

Proposition 2.15 *Suppose B is a non-degenerate symmetric bilinear form on V. Then the map (2.25) is surjective. Its kernel is the set of scalars $\mathbb{K}[2]$.*

Proof Suppose $D \in \mathrm{Der}_{\mathrm{Poi}}(\wedge(V))^m$ is a derivation of degree m of the Poisson structure. Let $e_1, \ldots, e_n$ be a basis of V, with e^i the B-dual basis. Then $De_i \in \wedge^{m+1}(V)$, and

$$\{De_i,\ e_j\} + \{De_j, e_i\} = \{De_i,\ e_j\} + (-1)^m\{e_i, De_j\} = D\{e_i, e_j\} = 0.$$

As a consequence,

$$\Big\{\sum_j De_j \wedge e^j, e_i\Big\} = \sum_j De_j \wedge \{e^j, e_i\} - \sum_j \{De_j,\ e_i\} \wedge e^j$$
$$= 2De_i + \sum_j \{De_i, e_j\} \wedge e^j$$
$$= 2(m+2)De_i,$$

where we used $\sum_j \{e_I, e_j\} \wedge e^j = 2|I|e_I$. It follows that $D = \{f, \cdot\}$

$$f = \frac{1}{2(m+2)} \sum_j De_j \wedge e^j,$$

proving the surjectivity. On the other hand, if $f \in \wedge(V)[2]$ defines the zero derivation, then in particular $\{f, e_i\} = 0$ for all i, hence $f \in \mathbb{K}[2]$. $\square$

Chapter 3
The spin representation

The Clifford algebra for a vector space V with split bilinear form B has an (essentially unique) irreducible module called the *spinor module* S. The Clifford action restricts to a representation of the *Spin group* $\mathrm{Spin}(V)$, known as the *spin representation*. After developing the basic properties of spinor modules and the spin representation, we give a discussion of *pure spinors* and their relation with Lagrangian subspaces. Throughout we will assume that V is a finite-dimensional vector space over a field $\mathbb{K}$ of characteristic zero, and that the bilinear form B on V is non-degenerate. We will write $\mathrm{Cl}(V)$ in place of $\mathrm{Cl}(V; B)$.

3.1 The Clifford group and the spin group

3.1.1 The Clifford group

Recall that $\Pi:\ \mathrm{Cl}(V) \to \mathrm{Cl}(V),\ x \mapsto (-1)^{|x|}x$ denotes the parity automorphism of the Clifford algebra. Let $\mathrm{Cl}(V)^\times$ be the group of invertible elements in $\mathrm{Cl}(V)$.

Definition 3.1 The *Clifford group* $\Gamma(V)$ is the subgroup of $\mathrm{Cl}(V)^\times$, consisting of all $x \in \mathrm{Cl}(V)^\times$ such that

$$A_x(v) := \Pi(x)vx^{-1} \in V$$

for all $v \in V \subseteq \mathrm{Cl}(V)$.

Hence, by definition the Clifford group comes with a natural representation,

$$\Gamma(V) \to \mathrm{GL}(V),\quad x \mapsto A_x.$$

Let $S\Gamma(V) = \Gamma(V) \cap \mathrm{Cl}^{\bar{0}}(V)^\times$ denote the *special Clifford group*. The Clifford group and the special Clifford group are extensions of the orthogonal and special orthogonal group, respectively. In fact we have:

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Theorem 3.1 *The natural representation of the Clifford group takes values in* $\mathrm{O}(V)$, *and defines an exact sequence*

$$1 \longrightarrow \mathbb{K}^\times \longrightarrow \Gamma(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1.$$

It restricts to a similar exact sequence for the special Clifford group

$$1 \longrightarrow \mathbb{K}^\times \longrightarrow S\Gamma(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

The elements of $\Gamma(V)$ *are all products*

$$x = v_1 \cdots v_k, \tag{3.1}$$

where $v_1, \ldots, v_k \in V$ *are non-isotropic. The corresponding element* A_x *is a product of reflections*:

$$A_x = R_{v_1} \cdots R_{v_k}. \tag{3.2}$$

In particular, every element $x \in \Gamma(V)$ *is either even or odd, depending on the parity of* k *in* (3.1). *In particular,* $S\Gamma(V)$ *is given by products* (3.1) *with* k *even.*

Proof Let $x \in \mathrm{Cl}(V)$. The transformation A_x is trivial if and only if $\Pi(x)v = vx$ for all $v \in V$, i.e., if and only if $[v, x] = 0$ for all $v \in V$. That is, it is the intersection of the center $\mathbb{K} \subseteq \mathrm{Cl}(V)$ with $\Gamma(V)$. (See Lemma 2.1.) This shows that the kernel of the homomorphism $\Gamma(V) \to \mathrm{GL}(V)$, $x \mapsto A_x$ is the group $\mathbb{K}^\times$ of invertible scalars.

Applying $-\Pi$ to the definition of A_x, we obtain $A_x(v) = xv\Pi(x)^{-1} = A_{\Pi(x)}(v)$. This shows that $A_{\Pi(x)} = A_x$ for $x \in \Gamma(V)$. Thus $\Pi(x)$ is a scalar multiple of x; in fact $\Pi(x) = \pm x$ since Π is the parity operator. This shows that elements of $\Gamma(V)$ have definite parity. For $x \in \Gamma(V)$ and $v, w \in V$ we have, using again $A_{\Pi(x)} = A_x$,

$$\begin{aligned}
2B(A_x(v), A_x(w)) &= A_x(v)A_x(w) + A_x(w)A_x(v) \\
&= A_x(v)A_{\Pi(x)}(w) + A_x(w)A_{\Pi(x)}(v) \\
&= \Pi(x)(vw + wv)\Pi(x^{-1}) \\
&= 2B(v, w)\Pi(x)\Pi(x^{-1}) \\
&= 2B(v, w).
\end{aligned}$$

This proves that $A_x \in \mathrm{O}(V)$ for all $x \in \Gamma(V)$. Suppose now that $v \in V$ is non-isotropic. Then it is invertible in the Clifford algebra, with $v^{-1} = v/B(v, v)$ and $\Pi(v) = -v$. For all $w \in V$,

$$A_v(w) = -vwv^{-1} = (wv - 2B(v, w))v^{-1} = w - 2\frac{B(v, w)}{B(v, v)}v = R_v(w).$$

Hence $v \in \Gamma(V)$, with $A_v = R_v$ the reflection defined by v. More generally, this proves (3.2) whenever x is of the form (3.1). By the E. Cartan–Dieudonné Theorem 1.1, any $A \in \mathrm{O}(V)$ is a product of reflections R_{v_i}. This shows that the map $x \mapsto A_x$ is onto $\mathrm{O}(V)$, and that $\Gamma(V)$ is generated by the non-isotropic vectors in V. The remaining statements are clear. □

Since every $x \in \Gamma(V)$ can be written in the form (3.1), it follows that the element $x^\top x$ lies in $\mathbb{K}^\times$. This defines the *norm homomorphism*

$$\mathsf{N}\colon\ \Gamma(V) \to \mathbb{K}^\times,\ x \mapsto x^\top x. \tag{3.3}$$

It is a group homomorphism and has the property

$$\mathsf{N}(\lambda x) = \lambda^2 \mathsf{N}(x)$$

for $\lambda \in \mathbb{K}^\times$.

Example 3.1 The chirality element $\Gamma \in \mathrm{Cl}(V)$, defined by choice of a generator $\Gamma_\wedge \in \det(V)$, is an element of the Clifford group $\Gamma(V)$, and is contained in $S\Gamma(V)$ if and only if $\dim V = 2m$ is even. In the special case of a vector space with split bilinear form, and Γ normalized so that $\Gamma^2 = 1$ (see Eq. (2.12)), one has

$$\mathsf{N}(\Gamma) = \Gamma^\top \Gamma = (-1)^m.$$

Example 3.2 Consider $V = F^* \oplus F$ with $\dim F = 1$. Choose dual generators $e \in F$, $f \in F^*$ so that $B(e, f) = \frac{1}{2}$. One checks that an even element $x = s + tfe \in \mathrm{Cl}^{\bar{0}}(V)$ with $s, t \in \mathbb{K}$ lies in $S\Gamma(V)$ if and only if $s, s+t$ are both invertible, and in that case

$$A_x(e) = \frac{s}{s+t}\, e, \quad A_x(f) = \frac{s+t}{s}\, e.$$

We have $\mathsf{N}(x) = x^\top x = s(s+t)$.

3.1.2 *The groups* Pin(V) *and* Spin(V)

We give the following definitions.

Definition 3.2 The *Pin group* $\mathrm{Pin}(V)$ is the kernel of the norm homomorphism $\mathsf{N}: \Gamma(V) \to \mathbb{K}^\times$. Its intersection with $S\Gamma(V)$ is called the *Spin group*, and is denoted $\mathrm{Spin}(V)$.

The normalization $\mathsf{N}(x) = 1$ specifies $x \in \Gamma(V)$ up to sign. Hence one obtains exact sequences,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V) \longrightarrow \mathrm{O}(V),$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V).$$

In general, the maps to $\mathrm{SO}(V)$, $\mathrm{O}(V)$ need not be surjective. A sufficient condition for surjectivity is that every element in $\mathbb{K}$ admits a square root, since one may then rescale any $x \in \Gamma(V)$ so that $\mathsf{N}(x) = 1$. Theorem 3.1 shows that in this case $\mathrm{Pin}(V)$ is the set of products $v_1 \cdots v_k$ of elements $v_i \in V$ with $B(v_i, v_i) = 1$, while $\mathrm{Spin}(V)$ consists of similar products with k even.

Remark 3.1 If V has non-zero isotropic vectors, then the condition that all elements in $\mathbb{K}$ have square roots is also necessary. Indeed, let $e \neq 0$ be isotropic, and let f be an isotropic vector with $B(e, f) = \frac{1}{2}$. Let $A \in \mathrm{SO}(V)$ be equal to the identity on $\mathrm{span}\{e, f\}^{\perp}$, and

$$A(e) = r\, e, \quad A(f) = r^{-1} f$$

with $r \in \mathbb{K}^{\times}$. As shown in Example 3.2, the lifts of A to $S\Gamma(V)$ are elements of the form $x = s + tfe$ with $r = s(s+t)^{-1}$. Since $\mathsf{N}(x) = s(s+t) = \frac{s^2}{r}$ we see that $x \in \mathrm{Spin}(V)$ if and only if $r = s^2$. The choice of square root of r specifies the lift x.

Remark 3.2 If $\mathbb{K} = \mathbb{R}$, the Pin and Spin groups are sometimes defined using a weaker condition $\mathsf{N}(x) = \pm 1$. This then guarantees that the maps to $\mathrm{O}(V)$, $\mathrm{SO}(V)$ are surjective.

For $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}^{n,m}$ we use the notation

$$\mathrm{Pin}(n,m) = \mathrm{Pin}(\mathbb{R}^{n,m}), \quad \mathrm{Spin}(n,m) = \mathrm{Spin}(\mathbb{R}^{n,m}).$$

If $m = 0$, we simply write $\mathrm{Pin}(n) = \mathrm{Pin}(n,0)$ and $\mathrm{Spin}(n) = \mathrm{Spin}(n,0)$.

Theorem 3.2 *Let* $\mathbb{K} = \mathbb{R}$. *Then* $\mathrm{Spin}(n,m)$ *is a double cover of the identity component* $\mathrm{SO}_0(n,m)$. *If* $n \geq 2$ *or* $m \geq 2$, *the group* $\mathrm{Spin}(n,m)$ *is connected.*

Proof The cases of $(n,m) = (0,1)$, $(1,0)$ are trivial. If $(n,m) = (1,1)$ one has $\mathrm{SO}_0(1,1) = \mathbb{R}_{>0}$, and $\mathrm{Spin}(1,1) = \mathbb{Z}_2 \times \mathbb{R}_{>0}$ (see Example 3.2). Suppose $n \geq 2$ or $m \geq 2$. To show that $\mathrm{Spin}(n,m)$ is connected, it suffices to show that the elements ± 1 (the pre-image of the group unit in $\mathrm{SO}(n,m)$) are in the same connected component. Let

$$v(\theta) \in \mathbb{R}^{n,m}, \quad 0 \leq \theta \leq \pi$$

be a continuous family of non-isotropic vectors with the property $v(\pi) = -v(0)$. Such a family exists, since V contains a 2-dimensional subspace isomorphic to $\mathbb{R}^{2,0}$ or $\mathbb{R}^{0,2}$. Rescale the vectors $v(\theta)$ to satisfy $B(v(\theta), v(\theta)) = \pm 1$. Then

$$[0,\pi] \to \mathrm{Spin}(n,m), \ \theta \mapsto x(\theta) = v(\theta)v(0)$$

is a path connecting $+1$ and -1. $\square$

The groups $\mathrm{Spin}(n,m)$ are usually not simply connected. Indeed since $\mathrm{SO}_0(n,m)$ has maximal compact subgroup $\mathrm{SO}(n) \times \mathrm{SO}(m)$, the fundamental group is

$$\pi_1(\mathrm{SO}_0(n,m)) = \pi_1(\mathrm{SO}(n)) \times \pi_1(\mathrm{SO}(m)).$$

In particular, if $n, m > 2$ the fundamental group of $\mathrm{SO}(n,m)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and hence that of its double cover $\mathrm{Spin}(n,m)$ is $\mathbb{Z}_2$. The spin group is simply connected only in the cases $n > 2$ and $m = 0, 1$, or $n = 0, 1$ and $m > 2$, and only in those cases is $\mathrm{Spin}(n,m)$ the universal cover of $\mathrm{SO}_0(n,m)$. Of particular interest is the case $m = 0$, where $\mathrm{Spin}(n)$ defines the universal cover of $\mathrm{SO}(n)$ for $n > 2$. In low dimensions, one has the exceptional isomorphisms

$$\begin{aligned}\mathrm{Spin}(2) &= \mathrm{SO}(2),\\ \mathrm{Spin}(3) &= \mathrm{SU}(2),\\ \mathrm{Spin}(4) &= \mathrm{SU}(2) \times \mathrm{SU}(2),\\ \mathrm{Spin}(5) &= \mathrm{Sp}(2),\\ \mathrm{Spin}(6) &= \mathrm{SU}(4).\end{aligned}$$

Here $\mathrm{Sp}(n)$ is the compact symplectic group, i.e., the group of norm-preserving automorphisms of the n-dimensional quaternionic vector space $\mathbb{H}^n$. The isomorphisms for $\mathrm{Spin}(3)$, $\mathrm{Spin}(4)$ follow from Proposition 1.11, while the isomorphisms for $\mathrm{Spin}(5)$, $\mathrm{Spin}(6)$ are obtained from a discussion of the spin representation of these groups; see Section 3.7.6 below. For $n \geq 7$, there are no further accidental isomorphisms of this type.

Let us now turn to the case $\mathbb{K} = \mathbb{C}$, so that $V \cong \mathbb{C}^n$ with the standard bilinear form. We write $\mathrm{Pin}(n, \mathbb{C}) = \mathrm{Pin}(\mathbb{C}^n)$ and $\mathrm{Spin}(n, \mathbb{C}) = \mathrm{Spin}(\mathbb{C}^n)$.

Proposition 3.1 *The Lie groups* $\mathrm{Pin}(n, \mathbb{C})$ *and* $\mathrm{Spin}(n, \mathbb{C})$ *are double covers of* $\mathrm{O}(n, \mathbb{C})$ *and* $\mathrm{SO}(n, \mathbb{C})$. *Furthermore,* $\mathrm{Spin}(n, \mathbb{C})$ *is connected and simply connected, i.e., it is the universal cover of* $\mathrm{SO}(n, \mathbb{C})$. *The group* $\mathrm{Spin}(n)$ *is the maximal compact subgroup of* $\mathrm{Spin}(n, \mathbb{C})$.

Proof The first part is clear, since for $x \in \Gamma(\mathbb{C}^n)$ the condition $N(\lambda x) = 1$ determines λ up to a sign. The second part follows by the same argument as in the real case, or alternatively by observing that ± 1 are in the same component of $\mathrm{Spin}(n, \mathbb{R}) \subseteq \mathrm{Spin}(n, \mathbb{C})$. Finally, since $\mathrm{SO}(n)$ is the maximal compact subgroup of $\mathrm{SO}(n, \mathbb{C})$, its pre-image $\mathrm{Spin}(n)$ is the maximal compact subgroup of $\mathrm{Spin}(n, \mathbb{C})$. □

Suppose V is a vector over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with non-degenerate symmetric bilinear form B. Since $\mathrm{Spin}(V)$ is a double cover of the identity component of $\mathrm{SO}(V)$, its Lie algebra is $\mathfrak{o}(V)$. The following result realizes the exponential map for $\mathrm{Spin}(V)$ directly in terms of the Clifford algebra. View $\mathrm{Cl}^{\bar{0}}(V)$ as a Lie algebra under Clifford commutation, where the corresponding Lie group is $\mathrm{Cl}^{\bar{0}}(V)^\times$. The exponential map $\exp :\ \mathrm{Cl}^{\bar{0}}(V) \to \mathrm{Cl}^{\bar{0}}(V)^\times$ for this Lie group is given by the usual power series. Recall the Lie algebra homomorphism $\gamma :\ \mathfrak{o}(V) \to \mathrm{Cl}^{\bar{0}}(V)$ from Section 2.2.10.

Proposition 3.2 *The following diagram commutes*:

$$\begin{array}{ccc}\mathrm{Spin}(V) & \longrightarrow & \mathrm{Cl}^{\bar{0}}(V)^\times \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{o}(V) & \xrightarrow[\gamma]{} & \mathrm{Cl}^{\bar{0}}(V).\end{array}$$

Proof For $A \in \mathfrak{o}(V)$ we have $A(v) = [\gamma(A), v]$ for $v \in V$, and accordingly

$$\exp(A)(v) = \exp(\mathrm{ad}(\gamma(A))v.$$

Using the identity $\exp(a)b\exp(-a) = \exp(\mathrm{ad}(a))b$ for elements a, b in a finite-dimensional (ordinary) algebra, we obtain

$$\exp(A)(v) = \mathrm{e}^{\gamma(A)}v\mathrm{e}^{-\gamma(A)}.$$

Since the left-hand side lies in V, this shows that $\mathrm{e}^{\gamma(A)} \in S\Gamma(V)$, by definition of the Clifford group. Furthermore, since $\gamma(A)^\top = -\gamma(A)$ we have

$$(\mathrm{e}^{\gamma(A)})^\top = \mathrm{e}^{\gamma(A)^\top} = \mathrm{e}^{-\gamma(A)},$$

and therefore $N(\mathrm{e}^{\gamma(A)}) = 1$. That is,

$$\mathrm{e}^{\gamma(A)} \in \mathrm{Spin}(V).$$

This shows that the group $\mathrm{Spin}(V) \subseteq \mathrm{Cl}(V)^\times$ has Lie algebra $\gamma(\mathfrak{o}(V)) \subseteq \mathrm{Cl}^{\bar{0}}(V)$. □

Example 3.3 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $V = \mathbb{K}^2$ with the standard bilinear form. Consider the element $A \in \mathfrak{o}(V)$ defined by $\lambda(A) = e_1 \wedge e_2$. Then $\gamma(A) = e_1e_2$. Since $(e_1e_2)^2 = -1$, the 1-parameter group of elements

$$x(\theta) = \exp(\theta/2e_1e_2) \in \mathrm{Spin}(V)$$

is given by the formula,

$$x(\theta) = \cos(\theta/2) + \sin(\theta/2)e_1e_2.$$

To find its action $A_{x(\theta)}$ on V, we compute

$$\begin{aligned} x(\theta)e_1x(-\theta) &= (\cos(\theta/2) + \sin(\theta/2)e_1e_2)e_1(\cos(\theta/2) - \sin(\theta/2)e_1e_2) \\ &= (\cos(\theta/2)e_1 - \sin(\theta/2)e_2)(\cos(\theta/2) - \sin(\theta/2)e_1e_2) \\ &= (\cos^2(\theta/2) - \sin^2(\theta/2))e_1 - 2\sin(\theta/2)\cos(\theta/2)e_2 \\ &= \cos(\theta)e_1 - \sin(\theta)e_2. \end{aligned}$$

This verifies that $A_{x(\theta)}$ is given as rotations by θ. We see explicitly that $A_{x(\theta+2\pi)} = A_{x(\theta)}$ while $x(\theta + 2\pi) = -x(\theta)$.

3.2 Clifford modules

3.2.1 Basic constructions

Let V be a vector space with symmetric bilinear form B, and $\mathrm{Cl}(V)$ the corresponding Clifford algebra. A module over the super algebra $\mathrm{Cl}(V)$ is called a *Clifford module*, or simply a $\mathrm{Cl}(V)$-module. That is, a Clifford module is a finite-dimensional super vector space E together with a morphism of super algebras,

$$\rho_E : \mathrm{Cl}(V) \to \mathrm{End}(E).$$

Equivalently, a Clifford module is given by a linear map $\rho_E : V \to \mathrm{End}^{\bar{1}}(E)$ such that

$$\rho_E(v)\rho_E(w) + \rho_E(w)\rho_E(v) = 2\,B(v,w)1$$

for all $v, w \in V$. A morphism of Clifford modules E, E' is a morphism of super vector spaces $f : E \to E'$ intertwining the Clifford actions.

Remark 3.3 If E has a filtration, compatible with the $\mathbb{Z}_2$-grading in the sense of Appendix A and such that the Clifford action is filtration-preserving, then we call E a *filtered* Clifford module. To construct a compatible filtration on a given Clifford module, choose any subspace $E' \subseteq E$ of definite parity, and pick $l \in \mathbb{Z}$, even or odd depending on the parity of E'. Then put $E^{(l+m)} = \mathrm{Cl}(V)^{(m)} E'$ for $m \in \mathbb{Z}$.

Remark 3.4 One can also consider modules over $\mathrm{Cl}(V)$, viewed as an *ordinary* (rather than super) algebra. These will be referred to as *ungraded Clifford modules*.

There are several standard constructions with Clifford modules:

1. *Submodules, quotient modules*. A submodule of a $\mathrm{Cl}(V)$-module E is a super subspace E_1 which is stable under the module action. In this case the quotient E/E_1 becomes a $\mathrm{Cl}(V)$-module in an obvious way. A $\mathrm{Cl}(V)$-module E is called *irreducible* if there are no submodules other than E and $\{0\}$.
2. *Direct sum*. The direct sum of two $\mathrm{Cl}(V)$-modules E_1, E_2 is again a $\mathrm{Cl}(V)$-module, with $\rho_{E_1 \oplus E_2} = \rho_{E_1} \oplus \rho_{E_2}$.
3. *Dual modules*. If E is any Clifford module, then the dual space $E^* = \mathrm{Hom}(E, \mathbb{K})$ becomes a Clifford module, with module structure defined in terms of the canonical anti-automorphism of $\mathrm{Cl}(V)$ by

$$\rho_{E^*}(x) = \rho_E(x^\top)^*,\ x \in \mathrm{Cl}(V).$$

   That is, $\langle \rho_{E^*}(x)\psi, \beta\rangle = \langle \psi, \rho_E(x^\top)\beta\rangle$ for $\psi \in E^*$ and $\beta \in E$. If E is a filtered $\mathrm{Cl}(V)$-module, then E^* with the dual filtration (see Appendix A) is again a filtered $\mathrm{Cl}(V)$-module.
4. *Tensor products*. Suppose V_1, V_2 are vector spaces with symmetric bilinear forms B_1, B_2. If E_1 is a $\mathrm{Cl}(V_1)$-module and E_2 is a $\mathrm{Cl}(V_2)$-module, the tensor product $E_1 \otimes E_2$ is a module over $\mathrm{Cl}(V_1) \otimes \mathrm{Cl}(V_2) = \mathrm{Cl}(V_1 \oplus V_2)$, with

$$\rho_{E_1 \otimes E_2}(x_1 \otimes x_2) = \rho_{E_1}(x_1) \otimes \rho_{E_2}(x_2).$$

   In particular, $\mathrm{Cl}(V)$-modules E can be tensored with super vector spaces, viewed as modules over the Clifford algebra for the trivial vector space $\{0\}$.
5. *Opposite grading*. If E is any $\mathrm{Cl}(V)$-module, then the same space E with the opposite $\mathbb{Z}_2$-grading is again a $\mathrm{Cl}(V)$-module, denoted by E^{op}.

Given a $\mathrm{Cl}(V)$-module E, one obtains a group representation of the Clifford group $\Gamma(V)$ by restriction, and $S\Gamma(V)$ acquires two representations $E^{\bar{0}}, E^{\bar{1}}$.

The first example of a Clifford module is the Clifford algebra $\mathrm{Cl}(V)$ itself where the module structure is given by multiplication from the left. The exterior algebra

$\wedge(V)$ is a Clifford module, with the action given on generators by (2.10). The symbol map $\sigma:\ \mathrm{Cl}(V)\cong\wedge(V)$ from Section 2.2.5 is characterized as the unique isomorphism of Clifford modules taking $1\in\mathrm{Cl}(V)$ to $1\in\wedge(V)$. The Clifford module $\mathrm{Cl}(V)\cong\wedge(V)$ is self-dual:

Proposition 3.3 *The* $\mathrm{Cl}(V)$*-module* $E=\mathrm{Cl}(V)$ (*with action by left-multiplication*) *is canonically isomorphic to its dual.*

Proof The map

$$\mathrm{Cl}(V)\to\mathrm{Cl}(V)^*,\ \ y\mapsto\phi_y,$$

where $\langle\phi_y,z\rangle=\mathrm{tr}(y^\top z)$, is a linear isomorphism of super spaces. For $x\in\mathrm{Cl}(V)$ we have

$$\langle\phi_{xy},\,z\rangle=\mathrm{tr}(y^\top x^\top z)=\langle\phi_y,\,x^\top z\rangle=\langle x.\phi_y,z\rangle;$$

hence ϕ is $\mathrm{Cl}(V)$-equivariant. □

Note however that the isomorphism described above does *not* preserve filtrations.

3.2.2 *The spinor module* S_F

Let V be a vector space of dimension $n=2m$, equipped with a split bilinear form B. View $\mathrm{Cl}(V)$ as a Clifford module under multiplication from the left, and let $F\subseteq V$ be a Lagrangian subspace. Then the left-ideal $\mathrm{Cl}(V)F$ is a submodule of $\mathrm{Cl}(V)$. The *spinor module associated to* F is the quotient $\mathrm{Cl}(V)$-module,

$$\mathsf{S}_F=\mathrm{Cl}(V)/\mathrm{Cl}(V)F.$$

The spinor module S_F may also be viewed as an *induced module*. View $\wedge(F)=\mathrm{Cl}(F)$ as a subalgebra of $\mathrm{Cl}(V)$, by the natural homomorphism extending the inclusion $F\subseteq V$.

Proposition 3.4 *Let* $\mathbb{K}$ *be the trivial* $\wedge(F)$*-module, that is,*

$$\phi.t=\phi_{[0]}t,\quad \phi\in\wedge(F),\ t\in\mathbb{K}.$$

Then S_F *is the corresponding induced module*

$$\mathsf{S}_F=\mathrm{Cl}(V)\otimes_{\wedge(F)}\mathbb{K}.$$

Proof By definition of the tensor product over $\wedge(F)$, the right-hand side is the quotient of $\mathrm{Cl}(V)\otimes\mathbb{K}$ by the subspace generated by all $x\otimes\phi.t-x\phi\otimes t$. But this is the same as the subspace of $\mathrm{Cl}(V)$ by the subspace generated by all $x(\phi-\phi_{[0]})$ for $\phi\in\wedge(F)$, which is exactly $\mathrm{Cl}(V)F$. □

Proposition 3.5 *The choice of a Lagrangian complement $F' \cong F^*$ to F identifies*

$$\mathsf{S}_F = \wedge(F^*),$$

where the Clifford action is given on generators by $\rho(\mu, v) = \varepsilon(\mu) + \iota(v)$ for $v \in F$ and $\mu \in F^*$.

Proof The choice of F' identifies $V = F^* \oplus F$, with the bilinear form

$$B((\mu_1, v_1), (\mu_2, v_2)) = \frac{1}{2}(\langle \mu_1, v_2 \rangle + \langle \mu_2, v_1 \rangle).$$

Both $\wedge(F)$ and $\wedge(F^*)$ are embedded as subalgebras of $\mathrm{Cl}(V)$, and the multiplication map defines a homomorphism of filtered super vector spaces

$$\wedge(F^*) \otimes \wedge(F) \to \mathrm{Cl}(F^* \oplus F). \tag{3.4}$$

The associated graded map is the isomorphism

$$\wedge(F^*) \otimes \wedge(F) \to \wedge(F^* \oplus F)$$

given by wedge product. Hence (3.4) is a linear isomorphism. Under this identification, $\mathrm{Cl}(V)F = \wedge(F^*) \otimes \bigoplus_{k \geq 1} \wedge^k(F)$, which has a natural complement $\wedge(F^*)$. Consequently

$$\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F \cong \wedge(F^*).$$

The Clifford action of $(\mu, v) \in F^* \oplus F$ on any $\psi \in \wedge(F^*)$ is given by Clifford multiplication by $\mu + v$ from the left, followed by projection to $\wedge(F^*)$ along $\mathrm{Cl}(V)F$. Since

$$(\mu + v)\psi = \mu\psi + [v, \psi] + (-1)^{|\psi|}\psi v = (\mu \wedge \psi + \iota(v)\psi) + (-1)^{|\psi|}\psi v,$$

and $\psi v \in \mathrm{Cl}(V)F$, this confirms our description of the action on $\wedge(F^*)$. $\square$

The spinor module S_F has a canonical filtration, compatible with the $\mathbb{Z}_2$-grading, given as the quotient of the filtration of the Clifford algebra:

$$\mathsf{S}_F^{(k)} = \mathrm{Cl}(V)^{(k)}/\mathrm{Cl}(V)^{(k-1)}F.$$

Proposition 3.6

1. *The associated graded space for the filtration on the spinor module is*

$$\mathrm{gr}(\mathsf{S}_F) = \wedge(F^*).$$

2. *For $v \in V$, the operator $\rho(v)$ on S_F has filtration degree 1, and the associated graded operator $\mathrm{gr}^1(\rho(v))$ on $\mathrm{gr}(\mathsf{S}_F) = \wedge(F^*)$ is the wedge product with the image of v in $F^* = V/F$.*
3. *For $v \in F \subseteq V$, the operator $\rho(v)$ has filtration degree -1, and the associated graded operator is contraction: $\mathrm{gr}^{-1}(\rho(v)) = \iota(v)$. That is, for all $\phi \in \mathsf{S}_F^{(k)}$ the leading term of $\rho(v)\phi \in \mathsf{S}_F^{(k-1)}$ is*

$$\mathrm{gr}^{k-1}(\rho(v)\phi) = \iota(v)\mathrm{gr}^k(\phi).$$

Proof Since $\mathrm{gr}(\mathrm{Cl}(V)) = \wedge(V)$, the associated graded space is

$$\mathrm{gr}(\mathsf{S}_F) = \wedge(V)/\wedge(V)F = \wedge(V/F) \cong \wedge(F^*).$$

Choose a Lagrangian complement $F' \cong F^*$ to F to identify $\mathsf{S}_F = \wedge(F^*)$. Then $\mathsf{S}_F^{(k)} = \bigoplus_{i \le k} \wedge^i F^*$, and the remaining claims are immediate from Proposition 3.5. □

3.2.3 The dual spinor module S^F

We define a *dual spinor module associated to* F as

$$\mathsf{S}^F = \mathrm{Cl}(V)\,\det(F),$$

where $\det(F) = \wedge^m(F)$ is the determinant line.

Proposition 3.7 *The choice of a Lagrangian complement $F' \cong F^*$ to F identifies*

$$\mathsf{S}^F = \wedge(F),$$

where the Clifford action is given on generators by $\rho(\mu, v) = \iota(\mu) + \varepsilon(v)$.

Proof Following the notation from the proof of Proposition 3.5, we have

$$\mathsf{S}^F = \mathrm{Cl}(V)\,\det(F) = \wedge(F^*) \otimes \det(F) \cong \wedge(F),$$

where the isomorphism is given by the contraction $\wedge(F^*) \to \mathrm{End}(\wedge(F))$. One readily checks that this identification takes the Clifford action to $\iota(\mu) + \varepsilon(v)$. □

Propositions 3.5 and 3.7 suggest that the spinor modules $\mathsf{S}_F, \mathsf{S}^F$ are in duality. In fact, this duality does not depend on the choice of complement.

Proposition 3.8 *There is a non-degenerate pairing*

$$\mathsf{S}^F \times \mathsf{S}_F \to \mathbb{K}, \quad y \times [x] \mapsto (y, [x]) = \mathrm{tr}(y^\top x)$$

for $y \in \mathsf{S}^F \subseteq \mathrm{Cl}(V)$ and $[x] \in \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ (represented by an element $x \in \mathrm{Cl}(V)$). The pairing satisfies

$$(y,\ \rho(z)[x]) = (\rho(z^T)y, [x]),$$

hence it defines an isomorphism of Clifford modules $\mathsf{S}^F \cong \mathsf{S}_F^*$. Choosing a Lagrangian complement F' to identify $\mathsf{S}_F = \wedge(F^*)$ and $\mathsf{S}^F = \wedge(F)$, the pairing is just the usual pairing between $\wedge(F^*)$ and $\wedge(F)$.

Proof The pairing is well defined, since

$$y \in \mathrm{Cl}(V)\,\det(F),\ x \in \mathrm{Cl}(V)F \ \Rightarrow\ xy^\top = 0 \Rightarrow\ \mathrm{tr}(y^\top x) = 0.$$

The pairing satisfies

$$(y,\ \rho(z)[x]) = \mathrm{tr}(y^\top zx) = \mathrm{tr}((z^\top y)^\top x) = (\rho(z^T)y, [x]).$$

Choose a Lagrangian complement F' to F, and view S_F as a subspace $\wedge(F^*) \subseteq \mathrm{Cl}(V)$ as in (3.4). The pairing between

$$[x] = \phi \in \mathsf{S}_F, \qquad y = \phi'\chi \in \mathsf{S}^F = \wedge(F^*)\det(F),$$

corresponding to $\psi = \iota(\phi')\chi \in \wedge(F)$, reads

$$(\psi, \phi) = \mathrm{tr}(y^\top x) = \mathrm{tr}(x^\top y) = \sigma(\phi^\top \phi'\chi)_{[0]} = (\iota(\phi^\top)\iota(\phi')\chi)_{[0]} = (\iota(\phi^\top)\psi)_{[0]}$$

which is just the standard pairing between $\wedge(F^*)$ and $\wedge(F)$. □

3.2.4 Irreducibility of the spinor module

We encountered special cases of the following result in our discussion of the Clifford algebras for $\mathbb{K} = \mathbb{C}$. See Proposition 2.4.

Theorem 3.3 *Let V be a vector space with split bilinear form B, and $F \subseteq V$ a Lagrangian subspace. The spinor module S_F is irreducible, and the module map*

$$\rho:\ \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S}_F)$$

is an isomorphism of super algebras. It restricts to an isomorphism

$$\mathrm{Cl}(V)^{\bar{0}} \to \mathrm{End}(\mathsf{S}_F^{\bar{0}}) \oplus \mathrm{End}(\mathsf{S}_F^{\bar{1}}).$$

Hence both $\mathsf{S}_F^{\bar{0}}, \mathsf{S}_F^{\bar{1}}$ are irreducible modules over $\mathrm{Cl}(V)^{\bar{0}}$. These two modules are non-isomorphic.

Proof We may use the model $V = F^* \oplus F$, $\mathsf{S}_F = \wedge(F^*)$. To prove $\mathrm{Cl}(V) \cong \mathrm{End}(\mathsf{S}_F)$, note that both spaces have the same dimension. Hence it suffices to show that ρ is surjective. That is, we have to show that $\mathrm{End}(\wedge F^*)$ is generated by exterior multiplications and contractions. Suppose first that $\dim F = 1$, and let $e \in F$, $f \in F^*$ be dual generators, so that $B(e, f) = \frac{1}{2}$. Then $\wedge F^*$ has basis $1, f$. In terms of this basis,

$$\varepsilon(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \iota(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon(f)\iota(e) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Together with the identity map of $\wedge F^*$, these form a basis of $\mathrm{End}(\wedge F^*) \cong \mathrm{Mat}_2(\mathbb{K})$, as claimed. The general case follows from the 1-dimensional case, using

$$\mathrm{End}(\wedge(F_1^* \oplus F_2^*)) = \mathrm{End}(\wedge F_1^*) \otimes \mathrm{End}(\wedge F_2^*).$$

This proves $\mathrm{Cl}(V) \cong \mathrm{End}(\mathsf{S}_F)$, which also implies that the spinor module is irreducible. It also yields

$$\mathrm{Cl}(V)^{\bar{0}} \cong \mathrm{End}^{\bar{0}}(\mathsf{S}_F) = \mathrm{End}(\mathsf{S}_F^{\bar{0}}) \oplus \mathrm{End}(\mathsf{S}_F^{\bar{1}}),$$

a direct sum of two irreducible modules. To see that the even and odd part of the spinor module are non-isomorphic modules over $\mathrm{Cl}^{\bar{0}}(V)$, choose bases e_i of F and f^i of F^* such that $B(e_i, f^j) = \frac{1}{2}\delta_i^j$, thus $e_i f^j = \delta_i^j - f^j e_i$. Consider the chirality element (2.12), written in the "normal-ordered" form

$$\Gamma = (1 - 2f^1 e_1) \cdots (1 - 2f^m e_m). \tag{3.5}$$

Since $\rho(1 - 2f^i e_i) f^I = \pm f^I$, with a $-$ sign if $i \in I$ and a minus sign if $i \notin I$, we find that $\rho(\Gamma)$ is the parity operator on S_F: it acts as $+1$ on $\mathsf{S}_F^{\bar{0}}$ and as -1 on $\mathsf{S}_F^{\bar{1}}$. In particular, these two representations are non-isomorphic. $\square$

Remark 3.5 Let $V = F^* \oplus F$ and $\mathsf{S}_F = \wedge(F^*)$, as in the proof above. On the subalgebra $\wedge(F) \subseteq \mathrm{Cl}(V)$, the homomorphism ρ coincides with the extension of the contraction operation $\iota : F \to \mathrm{End}(\wedge F^*)$ to an algebra morphism (still denoted by ι). Similarly, on $\wedge(F^*)$ it coincides with the morphism (still denoted by ι). Similarly, on $\wedge(F^*)$ it coincides with the extension of exterior multiplication $\varepsilon : F^* \to \mathrm{End}(\wedge F^*)$ to an algebra morphism (still denoted by ε). Using the proposition, we obtain that the linear map

$$\wedge(F^*) \otimes \wedge(F) \to \mathrm{End}(\wedge(F^*)), \quad \sum_i \phi_i \otimes \psi_i \mapsto \sum_i \varepsilon(\phi_i)\iota(\psi_i^\top) \tag{3.6}$$

is an isomorphism of super vector spaces. The operators on the right-hand side may be thought of as differential operators on the super algebra $\wedge(F^*)$.

3.2.5 Abstract spinor modules

Theorem 3.3 motivates the following definition.

Definition 3.3 Let V be a vector space with split bilinear form. A *spinor* module over $\mathrm{Cl}(V)$ is a Clifford module S for which the Clifford action

$$\rho : \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$$

is an isomorphism of super algebras. An *ungraded spinor module* is defined as an ungraded Clifford module such that ρ is an isomorphism of (ordinary) algebras.

We stress that we take Clifford modules, spinor modules, etc., to be $\mathbb{Z}_2$-graded unless specified otherwise. By Theorem 3.3 the standard spinor module S_F is an example of a spinor module, as is its dual S^F.

Theorem 3.4 *Let V be a vector space with split bilinear form.*

1. *There is a unique isomorphism class of ungraded spinor modules over* $\mathrm{Cl}(V)$.
2. *There are exactly two isomorphism classes of spinor modules over* $\mathrm{Cl}(V)$, *represented by* $\mathsf{S}_F, \mathsf{S}_F^{\mathrm{op}}$.

3. *A given ungraded spinor module admits exactly two compatible $\mathbb{Z}_2$-gradings. The corresponding parity operator is given by the Clifford action of the chirality element Γ, normalized (up to sign) by the condition $\Gamma^2 = 1$.*

Proof Theorem 3.3 shows that as an ungraded algebra, $\mathrm{Cl}(V)$ is isomorphic to a matrix algebra. Hence it admits, up to isomorphism, a *unique* ungraded spinor module, proving (1). The chirality element $\Gamma \in \mathrm{Cl}(V)$ (cf. (3.5)) satisfies $v\Gamma = -\Gamma v$ for all $v \in V$. Hence $\rho(v)$ exchanges the ± 1 eigenspaces of $\rho(\Gamma)$, showing that $\rho(\Gamma)$ defines a compatible $\mathbb{Z}_2$-grading. Now suppose $\mathsf{S} = \mathsf{S}^{\bar{0}} \oplus \mathsf{S}^{\bar{1}}$ is any compatible $\mathbb{Z}_2$-grading. Since $\rho(v)$ for $v \neq 0$ exchanges the odd and even summands, they both have dimension $\frac{1}{2} \dim \mathsf{S}$. Hence they are both irreducible under the action of $\mathrm{Cl}(V)^{\bar{0}}$. Since Γ is in the center of $\mathrm{Cl}(V)^{\bar{0}}$, it acts as a scalar on each summand. It follows that $\mathsf{S}^{\bar{0}}$ must be one of the two eigenspaces of $\rho(\Gamma)$, and $\mathsf{S}^{\bar{1}}$ is the other. This proves Part 3. Part 2 is immediate from Part 1 and Part 3. □

The theorem shows that if S, S' are two spinor modules, then the space

$$\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}, \mathsf{S}')$$

of intertwining operators is a 1-dimensional super vector space.

Remark 3.6 For $\mathbb{K} = \mathbb{R}$, the choice of $\mathbb{Z}_2$-grading on an ungraded spinor module over S is equivalent to a choice of orientation of V. Indeed, the definition (2.12) of the chirality element $\Gamma \in \mathrm{Spin}(V)$ shows that the choice of sign of Γ is equivalent to a choice of orientation on V.

As a special case, it follows that the spinor modules defined by two Lagrangian subspaces F, F' are isomorphic, possibly up to parity reversal, where the isomorphism is unique up to a scalar. Recall that $\mathrm{O}(V)$ acts transitively on the set $\mathrm{Lag}(V)$ of Lagrangian subspaces of V. Furthermore, the stabilizer group $\mathrm{O}(V)_F$ of any $F \in \mathrm{Lag}(V)$ is contained in $\mathrm{SO}(V)$.

Definition 3.4 We say that $F, F' \in \mathrm{Lag}(V)$ have *equal parity* if they are related by a transformation $g \in \mathrm{SO}(V)$ and *opposite parity* otherwise.

(For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the relative parity indicates if F, F' are in the same component of $\mathrm{Lag}(V)$.)

Proposition 3.9 *Let $g \in \mathrm{O}(V)$ with $g.F = F'$. Then any lift $x \in \Gamma(V)$ of g determines an isomorphism of Clifford modules $\mathsf{S}_F \to \mathsf{S}_{F'}$. This isomorphism preserves parity if and only if F, F' have the same parity.*

Proof Suppose $x \in \Gamma(V)$ lifts g, so that $A_x = g$. Then $F' = A_x(F) = \Pi(x)Fx^{-1}$ (as subsets of $\mathrm{Cl}(V)$). Hence

$$\mathrm{Cl}(V)Fx^{-1} = \mathrm{Cl}(V)F'.$$

Thus, right-multiplication by x^{-1} on $\mathrm{Cl}(V)$ descends to an isomorphism of Clifford modules $\mathsf{S}_F \to \mathsf{S}_{F'}$. Note that this isomorphism preserves parity if and only if x is even, i.e., $g \in \mathrm{SO}(V)$. □

Given a spinor module S over $\mathrm{Cl}(V)$, one obtains by restriction a group representation of the Clifford group $\Gamma(V)$ and its subgroup $\mathrm{Pin}(V)$. This is called the *spin representation* of $\Gamma(V)$. The action of $S\Gamma(V)$ preserves the splitting $\mathsf{S} = \mathsf{S}^{\bar{0}} \oplus \mathsf{S}^{\bar{1}}$; the two summands are called the *half-spin representations* of the special Clifford group $S\Gamma(V)$ and of its subgroup $\mathrm{Spin}(V)$.

Theorem 3.5 *The spin representation of $\Gamma(V)$ on S is irreducible. Similarly, each of the half-spin representations $\mathsf{S}^{\bar{0}}$ and $\mathsf{S}^{\bar{1}}$ is an irreducible representation of $S\Gamma(V)$. The two half-spin representations of $S\Gamma(V)$ are non-isomorphic. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we can replace $\Gamma(V)$ with* $\mathrm{Pin}(V)$ *and $S\Gamma(V)$ with* $\mathrm{Spin}(V)$.

Proof If a subspace of S is invariant under the action of $\Gamma(V)$, then it is also invariant under the subalgebra generated by $\Gamma(V)$. But $\Gamma(V)$ contains in particular all non-isotropic vectors, and linear combinations of non-isotropic vectors span all of V and hence generate all of $\mathrm{Cl}(V)$. Hence the subalgebra generated by $\Gamma(V)$ is all of $\mathrm{Cl}(V)$, and the irreducibility under $\Gamma(V)$ follows from that under $\mathrm{Cl}(V)$. Similarly, the subalgebra generated by $S\Gamma(V)$ equals $\mathrm{Cl}^{\bar{0}}(V)$, and the irreducibility of the half-spin representations under $S\Gamma(V)$ follows from that under $\mathrm{Cl}^{\bar{0}}(V)$. □

3.3 Pure spinors

Let $\rho : \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$ be a spinor module. If $\phi \in \mathsf{S}$ is a non-zero spinor, we can consider the space of vectors in V which annihilate ϕ under the Clifford action:

$$F(\phi) = \{v \in V \mid \rho(v)\phi = 0\}.$$

Lemma 3.1 *For all non-zero spinors $\phi \in \mathsf{S}$, the space $F(\phi)$ is an isotropic subspace of V.*

Proof If $v_1, v_2 \in F(\phi)$ we have

$$0 = (\rho(v_1)\rho(v_2) + \rho(v_2)\rho(v_1))\phi = 2B(v_1, v_2)\phi,$$

hence $B(v_1, v_2) = 0$. □

Definition 3.5 A non-zero spinor $\phi \in \mathsf{S}$ is called *pure* if the subspace $F(\phi)$ is Lagrangian.

Consider, for instance, the standard spinor module $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ defined by a Lagrangian subspace F. Let $\phi_0 \in \mathsf{S}_F$ be the image of $1 \in \mathrm{Cl}(V)$. Then ϕ_0 is a pure spinor, with $F(\phi_0) = F$.

Theorem 3.6 *The representation of $\Gamma(V)$ on a spinor module* S *restricts to a transitive action on the set of pure spinors. The map*

$$\left\{ \begin{array}{c} \textit{pure} \\ \textit{spinors} \end{array} \right\} \to \mathrm{Lag}(V),\ \phi \mapsto F(\phi) \tag{3.7}$$

is a $\Gamma(V)$-equivariant surjection, with fibers $\mathbb{K}^\times$. That is, if $F(\phi) = F(\phi')$, then ϕ, ϕ' coincide up to a non-zero scalar. All pure spinors ϕ are either even or odd. The relative parity of pure spinors ϕ, ϕ' is equal to the relative parity of the Lagrangian subspaces $F(\phi)$, $F(\phi')$.

Proof For any $x \in \Gamma(V)$, mapping to $g \in \mathrm{O}(V)$,

$$F(\rho(x)\phi) = xF(\phi)x^{-1} = \Pi(x)F(\phi)x^{-1} = A_x.F(\phi) = g.F(\phi).$$

It follows that for any pure spinor ϕ, the element $\rho(x)\phi$ is again a pure spinor.

To prove the remaining claims, we work with the standard spinor module S_F defined by a fixed Lagrangian subspace F. Let $\phi_0 \in \mathsf{S}_F$ be the image of $1 \in \mathrm{Cl}(V)$, so that $F(\phi_0) = F$. Suppose ϕ is a pure spinor with $F(\phi) = F$, and consider the standard filtration on S_F. By Proposition 3.6, $\rho(v)$ for $v \in F$ has filtration degree -1, and $\mathrm{gr}^{-1}(\rho(v))$ is the operator of contraction by $\mathrm{gr}(\mathsf{S}_F) = \wedge(F^*)$ given by contraction with v. Since $\bigcap_{v\in V} \ker(\iota(v)) = \wedge^0(F^*) = \mathbb{K}$, we conclude that $\bigcap_{v\in V} \ker(\rho(v)) = \mathsf{S}_F^{(0)} = \mathbb{K}\phi_0$. That is, $F \subseteq F(\phi)$ if and only if ϕ is a scalar multiple of ϕ_0.

Consider now a general pure spinor ϕ. Pick $g \in \mathrm{O}(V)$ with $g.F(\phi) = F$, and choose a lift $x \in \Gamma(V)$ of g. Then $F(\rho(x)\phi) = g.F(\phi) = F$, so that $\rho(x)\phi$ is a scalar multiple of ϕ_0. Since $\mathbb{K}^\times \subseteq \Gamma(V)$, this shows that $\Gamma(V)$ acts transitively on the set of pure spinors. The last statement follows since any element of the Clifford group is either even or odd; thus $\rho(x)^{-1}\phi_0$ is even or odd depending on the parity of x. $\square$

The following proposition shows how the choice of a pure spinor identifies S with a spinor module of the form S_F.

Proposition 3.10 *Let* S *be a spinor module over* $\mathrm{Cl}(V)$.

(i) *For any pure spinor $\phi \in \mathsf{S}$, one has*

$$\{x \in \mathrm{Cl}(V) |\ \rho(x)\phi = 0\} = \mathrm{Cl}(V)F(\phi).$$

Hence, there is a unique isomorphism of spinor modules $\mathsf{S} \to \mathsf{S}_{F(\phi)}$ taking ϕ to the image of 1 *in* $\mathrm{Cl}(V)/\mathrm{Cl}(V)F(\phi)$. *This identification preserves or reverses the $\mathbb{Z}_2$-grading depending on the parity of ϕ.*

(ii) *Suppose $F \subseteq V$ is a Lagrangian subspace. Then the pure spinors defining F are exactly the non-zero elements of the* pure spinor line

$$l_F = \{\phi \in \mathsf{S} |\ \rho(v)\phi = 0\ \ \forall v \in F\}.$$

There is a canonical isomorphism,

$$l_F \cong \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_F, \mathsf{S}).$$

(iii) *If* S' *is another spinor module, and* l'_F *the pure spinor line for* F*, then*

$$\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}, \mathsf{S}') = \mathrm{Hom}_{\mathbb{K}}(\mathsf{l}_F, \mathsf{l}'_F)$$

canonically.

Proof (i) The left ideal $\mathrm{Cl}(V)F(\phi)$ annihilates ϕ, defining a non-zero $\mathrm{Cl}(V)$-equivariant homomorphism $\mathsf{S}_{F(\phi)} = \mathrm{Cl}(V)/\mathrm{Cl}(V)F(\phi) \to \mathsf{S}$, $[x] \mapsto \rho(x)\phi$. Since $\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_{F(\phi)}, \mathsf{S})$ is 1-dimensional, this map is an isomorphism. In particular, $\rho(x)\phi = 0$ if and only if $[x] = 0$, i.e., $x \in \mathrm{Cl}(V)F(\phi)$. (ii) By definition, l_F consists of spinors ϕ with $F \subseteq F(\phi)$. If ϕ is non-zero, this must be an equality since $F(\phi)$ is isotropic. This shows that the non-zero elements of l_F are precisely the pure spinors defining F, and (using Theorem 3.6) that $\dim l_F = 1$. The isomorphism $l_F \cong \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_F, \mathsf{S})$ is defined by the map taking $\phi \in l_F$ to the homomorphism $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F \to \mathsf{S}$, $[x] \mapsto \rho(x)\phi$. (iii) A $\mathrm{Cl}(V)$-equivariant isomorphism $\mathsf{S} \to \mathsf{S}'$ must restrict to an isomorphism of the pure spinor lines for any Lagrangian subspace F. This defines a non-zero map $\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}, \mathsf{S}') \to \mathrm{Hom}_{\mathbb{K}}(\mathsf{l}_F, \mathsf{l}'_F)$. Since both sides are 1-dimensional, it is an isomorphism. □

Having established these general results, we proceed to give an explicit description of all pure spinors for the standard spinor module S_F, using a Lagrangian complement to F to identify $V = F^* \oplus F$ and $\mathsf{S}_F = \wedge(F^*)$.

Proposition 3.11 *Let* $V = F^* \oplus F$*. Then any triple* (N, χ, ω_N) *consisting of a subspace* $N \subseteq F$*, a volume form* $\kappa \in \det(\mathrm{ann}(N))^\times$ *on* V/N*, and a 2-form* $\omega_N \in \wedge^2 N^*$ *on* N*, defines a pure spinor*

$$\phi = \exp(-\tilde{\omega}_N)\kappa \in \wedge(F^*).$$

Here $\tilde{\omega}_N \in \wedge^2 F^*$ *is an arbitrary extension of* ω_N *to a 2-form on* F*. (Note that* ϕ *does not depend on the choice of this extension.) The corresponding Lagrangian subspace is*

$$F(\phi) = \{(\mu, v) \in F^* \oplus F |\ v \in N,\ \forall w \in N:\ \langle \mu, w\rangle = \omega_N(v, w)\}.$$

Every pure spinor in S_F *arises in this way from a unique triple* (N, κ, ω_N)*.*

Proof We first observe that Lagrangian subspaces $L \subseteq F^* \oplus F$ are in bijective correspondence with pairs (N, ω_N). Indeed, any such pair defines a subspace of dimension $\dim F$,

$$L = \{(\mu, v) \in F^* \oplus F |\ v \in N,\ \mu|_N = \omega_N(v, \cdot)\}.$$

If $(\mu, v), (\mu', v') \in L$, then

$$\langle \mu, v'\rangle + \langle \mu', v\rangle = \omega_N(v, v') + \omega_N(v', v) = 0,$$

hence L is Lagrangian. Conversely, given $L \subseteq F^* \oplus F$ let $N \subseteq F$ be its projection, and define ω_N by

$$\omega_N(v, v') = \langle \mu, v'\rangle = -\langle \mu', v\rangle,$$

where $(\mu, v), (\mu', v') \in L$ are pre-images of v, v'.

Suppose now that (N, ω_N, κ) are given and define ϕ as above. It is straightforward to check that elements (μ, v) with $v \in N$ and $\omega|_N = \omega_N(v, \cdot)$ annihilate ϕ. Hence $F(\phi)$ contains all such elements, and equality follows for dimension reasons. Conversely, suppose ϕ is a pure spinor. Let $N \subseteq F$ be the projection of $F(\phi) \subseteq F^* \oplus F$ to F. Then $\mathrm{ann}(N) \subseteq F(\phi)$. Pick $\kappa \in \det(\mathrm{ann}(N))^\times$. For $v, w \in N$, let $\mu, \nu \in F^*$ such that $(\mu, v), (\nu, w) \in F(\phi)$. Since $F(\phi)$ is isotropic, we have $\langle \mu, w\rangle + \langle \nu, v\rangle = 0$. Hence

$$\omega_N(v, w) = \langle \mu, w\rangle$$

is a well-defined skew-symmetric 2-form on N. Let $\tilde{\omega}_N$ be an arbitrary extension to a 2-form on V. Then $F(\phi)$ has the description given in the proposition, and hence coincides with $F(\mathrm{e}^{-\tilde{\omega}_N}\kappa)$. It follows that ϕ is a non-zero scalar multiple of $\mathrm{e}^{-\tilde{\omega}_N}\kappa$, where the scalar can be absorbed in the choice of κ. □

In low dimensions it is easy to be pure:

Proposition 3.12 *Suppose V is a vector space with split bilinear form, with* $\dim V \le 6$, *and* S *a spinor module. Then all non-zero even or odd elements in* S *are pure spinors.*

Proof Consider the case $\dim V = 6$ (the case $\dim V < 6$ is even easier). We may use the model $V = F^* \oplus F$, $\mathsf{S} = \wedge(F^*)$. Suppose $\phi = \phi_{[0]} + \phi_{[2]} \in \mathsf{S}^{\bar{0}}$ is non-zero. Put $t = \phi_{[0]}$. If $t \neq 0$, we have $\phi = t\exp(\phi_{[2]}/t)$, which is a pure spinor by Proposition 3.11. If $\phi_{[0]} = 0$, then $\chi := \phi_{[2]}$ is a non-zero element of $\wedge^2 F^*$. Since $\dim F = 3$, it has a 1-dimensional kernel $N \subseteq F$, with χ a generator of $\det(\mathrm{ann}(N))$. But this again is a pure spinor by Proposition 3.11.

For non-zero odd spinors $\phi \in \mathsf{S}^{\bar{1}}$, choose a non-isotropic $v \in V$ with $\rho(v)\phi \neq 0$. Since $\phi' = \rho(v)\phi \in \mathsf{S}^{\bar{0}}$ is pure, the same is true of $\phi = B(v, v)^{-1}\rho(v)\phi'$. □

3.4 The canonical bilinear pairing on spinors

Given a spinor module S, the dual S^* is again a spinor module. The 1-dimensional super vector space

$$K_{\mathsf{S}} := \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}^*, \mathsf{S})$$

is called the *canonical line* for the spinor module. Its parity is even or odd depending on the parity of $\frac{1}{2}\dim V$. The evaluation map defines an isomorphism of Clifford modules,

$$\mathsf{S}^* \otimes K_{\mathsf{S}} \to \mathsf{S}.$$

Note also that $K_{\mathsf{S}^*} = K_{\mathsf{S}}^*$.

By Proposition 3.10, if $F \subseteq V$ is a Lagrangian subspace and $\mathsf{l}_{\mathsf{S}^*,F}$, $\mathsf{l}_{\mathsf{S},F}$ are the corresponding pure spinor lines, we have

$$K_{\mathsf{S}} = \mathrm{Hom}_{\mathbb{K}}(\mathsf{l}_{\mathsf{S}^*,F}, \mathsf{l}_{\mathsf{S},F}) = \mathsf{l}_{\mathsf{S},F} \otimes (\mathsf{l}_{\mathsf{S}^*,F})^*.$$

Example 3.4 Let $F \subseteq V$ be a Lagrangian subspace. We saw above that the dual of $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ is canonically isomorphic to $\mathsf{S}^F = \mathrm{Cl}(V)\det(F)$. The pure spinor lines l_F for these two spinor modules are

$$\mathsf{l}_{\mathsf{S}^F,F} = \det(F), \qquad \mathsf{l}_{\mathsf{S}_F,F} = \mathbb{K}.$$

Hence

$$K_{\mathsf{S}^F} = \det(F), \quad K_{\mathsf{S}_F} = \det(F^*).$$

In terms of the identifications $\mathsf{S}_F = \wedge(F^*)$, $\mathsf{S}^F = \wedge(F)$ given by the choice of a complementary Lagrangian subspace, the isomorphism

$$K_{\mathsf{S}^F} = \mathrm{Hom}_{\mathrm{Cl}(V)}(\wedge(F), \wedge(F^*)) = \det(F^*)$$

is given by the contraction $\wedge(F) \otimes \det(F^*) \to \wedge(F^*)$. Indeed, given a generator of $\det(F^*)$, the resulting map $\wedge(F) \to \wedge(F^*)$ intertwines $\iota(\mu), \varepsilon(v)$ with $\varepsilon(\mu), \iota(v)$.

Definition 3.6 The canonical bilinear pairing

$$(\cdot,\cdot)_{\mathsf{S}}: \quad \mathsf{S} \otimes \mathsf{S} \to K_{\mathsf{S}}, \quad \phi \otimes \psi \mapsto (\phi,\psi)_{\mathsf{S}}$$

is the isomorphism $\mathsf{S} \otimes \mathsf{S} \to \mathsf{S}^* \otimes \mathsf{S} \otimes K_{\mathsf{S}}$, followed by the duality pairing $\mathsf{S}^* \otimes \mathsf{S} \to \mathbb{K}$.

The pairing $(\cdot,\cdot)_{\mathsf{S}}$ satisfies

$$(\rho(x)\phi, \psi)_{\mathsf{S}} = (\phi, \rho(x^\top)\psi)_{\mathsf{S}}, \quad x \in \mathrm{Cl}(V), \tag{3.8}$$

by a similar property of the pairing between S^* and S (defining the Clifford action on S^*). Restricting to the Clifford group and replacing ψ with $\rho(x)\psi$ it follows that

$$(\rho(x)\phi,\ \rho(x)\psi)_{\mathsf{S}} = \mathsf{N}(x)\ (\phi,\psi)_{\mathsf{S}}, \quad x \in \Gamma(V), \tag{3.9}$$

where $\mathsf{N}:\ \Gamma(V) \to \mathbb{K}^\times$ is the norm homomorphism (3.3). The bilinear form is uniquely determined, up to a non-zero scalar, by its invariance property:

Proposition 3.13 *Let* S *be a spinor module over* $\mathrm{Cl}(V)$, *and* $(\cdot,\cdot):\ \mathsf{S} \times \mathsf{S} \to \mathbb{K}$ *is a bilinear pairing with the property*

$$(\rho(x)\phi,\ \rho(x)\psi)_{\mathsf{S}} = \mathsf{N}(x)\ (\phi,\psi)_{\mathsf{S}}, \quad x \in \Gamma(V),$$

where $\rho:\ \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$ *is the module action. Then* $(\cdot,\cdot)$ *coincides with the canonical pairing* $(\cdot,\cdot)_{\mathsf{S}}$, *for some trivialization* $K_{\mathsf{S}} \cong \mathbb{K}$.

Proof The invariance property implies that $(\rho(x)\phi,\psi) = (\phi, \rho(x^\top)\psi)$ for all $x \in \Gamma(V)$, hence (by linearity) for all $x \in \mathrm{Cl}(V)$. This shows that the bilinear pairing gives an isomorphism of Clifford modules $\mathsf{S} \to \mathsf{S}^*$. It hence provides a trivialization of $K_{\mathsf{S}} = \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}^*, \mathsf{S})$, identifying $(\cdot,\cdot)$ with the pairing $(\cdot,\cdot)_{\mathsf{S}}$. $\square$

Example 3.5 For the Clifford module $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$, defined by a Lagrangian subspace F, one has $K_{\mathsf{S}_F} = \det(F^*)$. To explicitly write the pairing, choose a Lagrangian complement in order to identify $\mathsf{S}_F = \wedge(F^*)$. Then the pairing is given by

$$(\phi, \psi)_{\mathsf{S}} = (\phi^\top \wedge \psi)_{[\mathrm{top}]}. \tag{3.10}$$

We next turn to the symmetry properties of the bilinear form.

Proposition 3.14 *Let* $\dim V = 2m$. *The canonical pairing* $(\cdot, \cdot)_{\mathsf{S}}$ *is*

- *symmetric if* $m = 0, 1 \bmod 4$,
- *skew-symmetric if* $m = 2, 3 \bmod 4$.

Furthermore, if $m = 0 \bmod 4$ (*resp.* $m = 2 \bmod 4$) *it restricts to a non-degenerate symmetric* (*resp. skew-symmetric*) *form on both* $\mathsf{S}^{\bar{0}}$ *and* $\mathsf{S}^{\bar{1}}$. *If m is odd, then the bilinear form vanishes on both* $\mathsf{S}^{\bar{0}}$, $\mathsf{S}^{\bar{1}}$, *and hence gives a non-degenerate pairing between them.*

Proof We may use the model $V = F^* \oplus F$, $\mathsf{S} = \wedge(F^*)$. Let $\phi \in \wedge^k(F^*)$ and $\psi \in \wedge^{m-k}(F^*)$. Then

$$\begin{aligned}
(\psi, \phi)_{\mathsf{S}} &= \psi^\top \wedge \phi \\
&= (-1)^{(m-k)(m-k-1)/2} \psi \wedge \phi \\
&= (-1)^{(m-k)(m-k-1)/2 + k(m-k)} \phi \wedge \psi \\
&= (-1)^{(m-k)(m-k-1)/2 + k(m-k) + k(k-1)/2} \phi^\top \wedge \psi \\
&= (-1)^{m(m-1)/2} (\phi, \psi)_{\mathsf{S}}.
\end{aligned}$$

This gives the symmetry property of the bilinear form. If m is even, then $\mathsf{S}^{\bar{0}}$ and $\mathsf{S}^{\bar{1}}$ are orthogonal under this bilinear form, and hence the bilinear form is non-degenerate on both. $\square$

Remark 3.7 Suppose $m = 0, 1 \bmod 4$, so that $(\cdot, \cdot)_{\mathsf{S}}$ is symmetric. Then

$$(\rho(v)\phi, \rho(w)\phi)_{\mathsf{S}} = B(v, w)(\phi, \phi)_{\mathsf{S}}. \tag{3.11}$$

If $v = w$, this follows from the invariance property since $\mathsf{N}(v) = v^\top v = B(v, v)$, and in general by polarization. The identity shows that if $(\phi, \phi)_{\mathsf{S}} \neq 0$, then the null space $F(\phi)$ is trivial. Indeed, for all $v \in F(\phi)$ the identity implies $B(v, w) = 0$ for all w, hence $v = 0$.

Theorem 3.7 (E. Cartan, Chevalley) *Let* S *be a spinor module. Let* $\phi, \psi \in \mathsf{S}$ *be pure spinors. Then the pairing* $(\phi, \psi)_{\mathsf{S}}$ *is non-zero if and only if the Lagrangian subspaces* $F(\phi)$ *and* $F(\psi)$ *are transverse.*

Proof We use the model $V = F^* \oplus F$ and $\mathsf{S} = \wedge F^*$, using the formula (3.10) for the pairing.

'$\Leftarrow$'. Suppose $F(\phi) \cap F(\psi) = 0$. Choose $A \in \mathrm{O}(V)$ such that A^{-1} takes $F(\phi), F(\psi)$ to F^*, F, respectively. Let $x \in \Gamma(V)$ be a lift, i.e., $A_x = A$. Then $\rho(x)^{-1}\phi$ and $\rho(x)^{-1}\psi$ are pure spinors representing F^* and F, hence they are elements of $\det(F^*)^\times$ and $\mathbb{K}^\times$ respectively. By (3.10) their pairing is non-zero, hence also

$$(\phi, \psi)_{\mathsf{S}} = \mathsf{N}(x)\ (\rho(x)^{-1}\phi, \rho(x)^{-1}\psi)_{\mathsf{S}} \neq 0.$$

'$\Rightarrow$'. Suppose $(\phi, \psi)_{\mathsf{S}} \neq 0$. Choose $x \in \Gamma(V)$ with $\psi = \rho(x).1$. Then

$$0 \neq \mathsf{N}(x)(\phi, \psi)_{\mathsf{S}} = (\rho(x)^{-1}\phi, 1)_{\mathsf{S}} = (\rho(x)^{-1}\phi)_{[\mathrm{top}]}.$$

In particular, $\rho(x)^{-1}\phi$ is not annihilated by any non-zero $\rho(v)$, $v \in F$. Hence $F(\rho(x)^{-1}\phi) \cap F = 0$, and consequently $F(\phi) \cap F(\psi) = F(\phi) \cap A_x(F) = 0$. □

Remark 3.8 More generally, we could also consider two different spinor modules $\mathscr{S}, \mathscr{S}'$. One obtains a pairing

$$(\cdot, \cdot) : \mathscr{S}' \otimes \mathscr{S} \cong \mathscr{S}^* \otimes \mathrm{Hom}(\mathscr{S}^*, \mathscr{S}') \otimes \mathscr{S} \to \mathrm{Hom}(\mathscr{S}^*, \mathscr{S}').$$

As before, the Lagrangian subspaces defined by $\phi \in \mathscr{S}, \phi' \in \mathscr{S}'$ are transverse if and only if $(\phi, \phi') \neq 0$.

In particular, Theorem 3.7 shows that pure spinors satisfy $(\phi, \phi)_{\mathsf{S}} = 0$. In dimension 8, the converse is true.

Proposition 3.15 (Chevalley [41, IV.1.1]) *Suppose V is a vector space with split bilinear form, with* $\dim V = 8$, *and* S *a spinor module. Then a non-zero even or odd spinor $\phi \in \mathsf{S}$ is pure if and only if $(\phi, \phi)_{\mathsf{S}} = 0$. Furthermore, the spinors ϕ with $(\phi, \phi)_{\mathsf{S}} \neq 0$ satisfy $F(\phi) = 0$.*

Proof We work in the model $V = F^* \oplus F$, with the spinor module $\mathsf{S}_F = \wedge F^*$. Suppose $\phi \in \mathsf{S}_F$ is an even or odd non-zero spinor with $(\phi, \phi)_{\mathsf{S}} = 0$. We will show that ϕ is pure. Suppose first that ϕ is even. Then

$$0 = (\phi, \phi)_{\mathsf{S}} = 2\phi_{[0]}\phi_{[4]} - \phi_{[2]} \wedge \phi_{[2]}.$$

If $\phi_{[0]} \neq 0$, we may rescale ϕ to arrange $\phi_{[0]} = 1$. The property $\phi_{[4]} = \frac{1}{2}\phi_{[2]} \wedge \phi_{[2]}$ then means $\phi = \exp(\phi_{[2]})$, which is a pure spinor. If $\phi_{[0]} = 0$, the property $\phi_{[2]} \wedge \phi_{[2]} = 0$ tells us that $\phi_{[2]} = \mu^1 \wedge \mu^2$ for suitable $\mu^1, \mu^2 \in F^*$. Let ω be a 2-form such that $\phi_{[2]} \wedge \omega = \phi_{[4]}$; then $\phi = \mu_1 \wedge \mu_2 \wedge \exp(\omega)$ which is again a pure spinor. If ϕ is odd, choose any non-isotropic v. Then $\rho(v)\phi$ is an even spinor, with $(\rho(v)\phi, \rho(v)\phi)_{\mathsf{S}} = B(v, v)(\phi, \phi)_{\mathsf{S}} = 0$. Hence $\rho(v)\phi$ is pure, and consequently ϕ is pure.

On the other hand, if $(\phi, \phi)_{\mathsf{S}} \neq 0$, and $v \in F(\phi)$, then (3.11) shows that

$$0 = (\rho(v)\phi, \rho(w)\phi)_{\mathsf{S}} = B(v, w)(\phi, \phi)_{\mathsf{S}}$$

for all $w \in V$. Hence $B(v, w) = 0$ for all w and therefore $v = 0$. □

3.5 The character $\chi : \Gamma(V)_F \to \mathbb{K}^\times$

Let $F \subseteq V$ be a Lagrangian subspace. By Proposition 1.5 the group $\mathrm{O}(V)_F$ of orthogonal transformations preserving F is contained in $\mathrm{SO}(V)$, and fits into an exact sequence

$$1 \to \wedge^2(F) \to \mathrm{O}(V)_F \to \mathrm{GL}(F) \to 1.$$

Let $\Gamma(V)_F \subseteq S\Gamma(V)$ be the pre-image of $\mathrm{O}(V)_F$ in the Clifford group, and

$$\mathrm{Spin}(V)_F = \Gamma(V)_F \cap \mathrm{Pin}(V).$$

Thus $x \in \Gamma(V)_F$ if and only if A_x preserves F. The action $\rho(x)$ of such an element on the spinor module S must preserve the pure spinor line l_F. This defines a group homomorphism

$$\chi : \Gamma(V)_F \to \mathbb{K}^\times,$$

with $\rho(x)\phi = \chi(x)\phi$ for all $x \in \Gamma(V)_F$, $\phi \in l_F$. Clearly, this character is independent of the choice of S.

Proposition 3.16 *The character χ satisfies*

$$\chi(x)^2 = \mathsf{N}(x)\ \det(A_x|_F)$$

for all $x \in \Gamma(V)_F$. Hence, the restriction of χ to the group $\mathrm{Spin}(V)_F$ defines a square root of the function $x \mapsto \det(A_x|_F)$.

Proof We use the model $V = F^* \oplus F$, $\mathsf{S} = \wedge(F^*)$, so that $\chi(x) = \rho(x).1$. It suffices to check the following two cases: (i) A_x fixes F pointwise, and (ii) A_x preserves both F and F^*.

In case (i), A_x is given by an element $\lambda \in \wedge^2(F)$, and hence $x = t\exp(-\lambda)$ for some non-zero $t \in \mathbb{K}$. We have $\det(A_x|_F) = 1$. The action of x on $1 \in \wedge(F^*)$ is multiplication by t, hence $\chi(x) = t$, while $\mathsf{N}(x) = t^2$. This verifies the formula in case (i).

In case (ii), let $Q = A_x|_F \in \mathrm{GL}(F)$. Then $A_x(\mu, v) = ((Q^{-1})^*\mu, Qv)$. For all $\nu \in F^*$ we have $x\nu x^{-1} = A_x(\nu) = (Q^{-1})^*\nu$, where we used $\Pi(x) = x$. It follows that

$$x\psi x^{-1} = (Q^{-1})^*\psi$$

for all $\psi \in \wedge(F^*) \subseteq \mathrm{Cl}(V)$. Take $\psi \in \det(F^*)^\times$, so that $(Q^{-1})^*\psi = \frac{1}{\det Q}\psi$. We obtain

$$\rho(x)\psi = \rho(x\psi)1 = \rho(x\psi x^{-1})\rho(x)1 = \frac{\chi(x)}{\det Q}\psi.$$

Pairing with $\rho(x)1 = \chi(x)$, and using the invariance property of the bilinear form, we find that

$$\mathsf{N}(x)\psi = (\rho(x).1, \rho(x)\psi)_{\mathsf{S}} = \frac{\chi(x)^2}{\det Q}\psi,$$

hence $\chi(x)^2 = \mathsf{N}(x)\det(Q)$. ☐

Let $\mathbb{K}_\chi$ denote the $\Gamma(V)_F$-representation on $\mathbb{K}$, with $x \in \Gamma(V)_F$ acting as multiplication by $\chi(x)$.

Proposition 3.17 *The fiber of the associated line bundle*

$$\Gamma(V) \times_{\Gamma(V)_F} \mathbb{K}_\chi \to \mathrm{Lag}(V)$$

at $L \in \mathrm{Lag}(V)$ *is the pure spinor line* $l_L \subseteq \mathsf{S}_F$.

Proof Since $x \in \Gamma(V)_F$ acts as $\chi(x)$ on $\mathsf{l}_F \subseteq \mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$, we have $x = \chi(x) \mod \mathrm{Cl}(V)F$ for all $x \in \Gamma(V)_F$. Thus

$$\chi(x)x^{-1} = 1 \mod \mathrm{Cl}(V)F.$$

Now let $(z, t) \in \Gamma(V) \times \mathbb{K}_\chi$, and put $L = A_z(F)$. The map

$$\mathrm{Cl}(V) \to \mathrm{Cl}(V), \ y \mapsto tyz$$

takes $\mathrm{Cl}(V)L$ to $\mathrm{Cl}(V)F$, hence it descends to an element of $\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_L, \mathsf{S}_F) = \mathsf{l}_L$. If $(z', t') = (zx^{-1}, \chi(x)t)$ with $x \in \Gamma(V)_F$, then

$$t'yz' = \chi(x)tyzx^{-1} = tyz \mod \mathrm{Cl}(V)F,$$

thus (z', t') defines the same homomorphism as the element (z, t). □

If the map $\mathrm{Pin}(V) \to \mathrm{O}(V)$ is onto (e.g., if $\mathbb{K} = \mathbb{C}$), the line bundle has the description

$$\mathrm{Pin}(V) \times_{\mathrm{Pin}(V)_F} \mathbb{K}_\chi,$$

where $\mathrm{Pin}(V)_F = \mathrm{Spin}(V)_F$ acts via the character $x \mapsto \chi(x) = \det^{1/2}(A_x|_F)$.

3.6 Cartan's triality principle

If $\dim V = 8$, one has the remarkable phenomenon of *triality*, stated (in essentially the form given below) by E. Cartan [34]. (A more geometric triality principle had been described earlier by E. Study [116]. See Porteus [108, Chapter 24] for historical background, and a discussion of Study's triality principle.)

The following discussion is based on Chevalley's exposition in [41]. As before, we assume that the bilinear form B on V is split (which is automatic if $\mathbb{K} = \mathbb{C}$). Let $\rho : \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$ be a spinor module, and let $\Gamma \in \mathrm{Spin}(V)$ be the chirality element, with the unique normalization for which $\rho(\Gamma)$ is the parity operator of S. Since $\Gamma, 1$ span the center of the algebra $\mathrm{Cl}^{\bar{0}}(V)$, and since the linear span of $\mathrm{Spin}(V)$ is all of $\mathrm{Cl}^{\bar{0}}(V)$, the center of the group $\mathrm{Spin}(V)$ consists of four elements

$$\mathrm{Cent}(\mathrm{Spin}(V)) = \{1, -1, \Gamma, -\Gamma\}. \tag{3.12}$$

The two half-spin representations $\mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$ of $\mathrm{Spin}(V)$ are irreducible representations of dimension 8. In addition, one has the 8-dimensional representation on V via $\pi : \mathrm{Spin}(V) \to \mathrm{SO}(V)$, $x \mapsto A_x$. These three representations are all non-isomorphic, and are distinguished by the action of the center (3.12). Indeed, the central element $-1 \in \mathrm{Spin}(V)$ acts trivially on V since $\pi(-1) = I$, while it acts as $-I$ in the half-spin representations. The triality principle, Theorem 3.8 below, shows that there is a degree 3 automorphism of $\mathrm{Spin}(V)$ relating the three representations.

Consider the direct sum

$$A = V \oplus \mathsf{S}^{\bar{0}} \oplus \mathsf{S}^{\bar{1}}.$$

Since $\dim V = 8$, Proposition 3.14 shows that the canonical bilinear form $(\cdot,\cdot)_{\mathsf{S}}$ of S is symmetric, and restricts to non-degenerate bilinear forms on both $\mathsf{S}^{\bar{0}}$ and $\mathsf{S}^{\bar{1}}$. After choice of a trivialization $K_{\mathsf{S}} \cong \mathbb{K}$, it becomes a scalar-valued symmetric bilinear form; its direct sum with the given bilinear form B on V is a non-degenerate symmetric bilinear form B_A on A. The corresponding orthogonal group is denoted $\mathrm{O}(A)$, as usual. Let $\rho_A : S\Gamma(V) \to \mathrm{Aut}(A)$ be the triagonal action on A. Then $\rho_A(\mathrm{Spin}(V)) \subseteq \mathrm{O}(A)$.

Theorem 3.8 (Triality) *There exists an orthogonal automorphism $J \in \mathrm{O}(A)$ of order 3 and a group automorphism $j \in \mathrm{Aut}(\mathrm{Spin}(V))$ of order 3 such that*

$$J(V) = \mathsf{S}^{\bar{1}}, \quad J(\mathsf{S}^{\bar{1}}) = \mathsf{S}^{\bar{0}}, \quad J(\mathsf{S}^{\bar{0}}) = V, \tag{3.13}$$

and

$$J \circ \rho_A(x) = \rho_A(j(x)) \circ J, \quad x \in \mathrm{Spin}(V). \tag{3.14}$$

One hence obtains a commutative diagram, for $x \in \Gamma(V)$,

$$\begin{tikzcd} V \arrow[r, "J"'] \arrow[d, "\rho(x)"] & \mathsf{S}^{\bar{1}} \arrow[r, "J"'] \arrow[d, "\rho(j(x))"] & \mathsf{S}^{\bar{0}} \arrow[d, "\rho(j(j(x)))"] \\ V \arrow[r, "J"'] & \mathsf{S}^{\bar{1}} \arrow[r, "J"'] & \mathsf{S}^{\bar{0}}. \end{tikzcd}$$

A key ingredient in the proof is the following cubic form on A:

$$C_A : A \to \mathbb{K}, \ \xi = (v, \phi^{\bar{0}}, \phi^{\bar{1}}) \mapsto (\rho(v)\phi^{\bar{0}}, \phi^{\bar{1}})_{\mathsf{S}}. \tag{3.15}$$

Lemma 3.2 *The cubic form C_A satisfies $C_A(\rho_A(x)\xi) = \mathsf{N}(x)C_A(\xi)$ for all $x \in S\Gamma(V)$. Hence, $\rho_A(x)$, $x \in S\Gamma(V)$ preserves C_A precisely if $x \in \mathrm{Spin}(V)$.*

Proof For $\xi = (v, \phi^{\bar{0}}, \phi^{\bar{1}})$,

$$\begin{aligned} C_A(\rho_A(x)\xi) &= (\rho(A_x(v))\rho(x)\phi^{\bar{0}}, \rho(x)\phi^{\bar{1}})_{\mathsf{S}} \\ &= (\rho(x)\rho(v)\phi^{\bar{0}}, \rho(x)\phi^{\bar{1}})_{\mathsf{S}} \\ &= \mathsf{N}(x)(\rho(v)\phi^{\bar{0}}, \phi^{\bar{1}})_{\mathsf{S}} = \mathsf{N}(x)C_A(\xi) \end{aligned}$$

as claimed. □

We will construct the triality automorphism J in such way that it also preserves C_A.

Lemma 3.3 *Any $f \in \mathrm{O}(A)$ preserving each of the subspaces $V, \mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$ and preserving the cubic form C_A is of the form $\rho_A(x)$, for a unique $x \in \mathrm{Spin}(V)$.*

Proof For all $\xi = (v, \phi^{\bar{0}}, \phi^{\bar{1}})$ we find

$$(\rho(f(v))f(\phi^{\bar{0}}), f(\phi^{\bar{1}}))_{\mathsf{S}} = (\rho(v)\phi^{\bar{0}}, \phi^{\bar{1}})_{\mathsf{S}} = (f(\rho(v)\phi^{\bar{0}}), f(\phi^{\bar{1}}))_{\mathsf{S}},$$

where the first equality used invariance of C_A and the second equality used the invariance of B_A. Consequently,

$$f(\rho(v)\phi) = \rho(f(v))f(\phi)$$

for all even spinors ϕ. On the other hand, any odd spinor can be written $\psi = \rho(v)\phi$, hence the identity gives

$$\begin{aligned} f(\psi) &= \rho(f(v))f(\rho(v)^{-1}\psi) \\ &= \rho(f(v))B(v,v)^{-1}f(\rho(v)\psi) \\ &= \rho(f(v))^{-1}f(\rho(v)\psi). \end{aligned}$$

That is, $f(\rho(v)\psi) = \rho(f(v))f(\psi)$ for all odd spinors. This shows that

$$\rho(f(v)) = (f|_{\mathsf{S}}) \circ \rho(v) \circ (f|_{\mathsf{S}})^{-1}. \tag{3.16}$$

Since $f|_{\mathsf{S}}$ is an even endomorphism of S, it is of the form $\rho(x)$ for some $x \in \mathrm{Cl}^{\bar{0}}(V)$. Equation (3.16) shows that $\rho(f(v)) = \rho(xvx^{-1}) = \rho(A_x(v))$, hence $f(v) = A_x(v)$ and in particular $x \in S\Gamma(V)$. We have shown that $f = \rho_A(x)$. Using again the invariance of C_A and the previous lemma, we obtain $\mathsf{N}(x) = 1$, so that $x \in \mathrm{Spin}(V)$. □

We are now in a position to construct the triality isomorphism.

Proof of Theorem 3.8 Pick $n \in V$ and $q \in \mathsf{S}^{\bar{0}}$ with $B(n,n) = 1$, $(q,q)_{\mathsf{S}} = 1$. Let R_n and R_q be the corresponding reflections in $V, \mathsf{S}^{\bar{0}}$. The map

$$V \to \mathsf{S}^{\bar{1}} : \ v \mapsto \rho(v)q$$

is an isometry (see Remark 3.7); let $T_q : \ \mathsf{S}^{\bar{1}} \to V$ be the inverse map. Define orthogonal involutions $\mu, \tau \in \mathrm{O}(A)$ by

$$\begin{aligned} \mu(v, \phi^{\bar{0}}, \phi^{\bar{1}}) &= (R_n(v), \ \rho(n)\phi^{\bar{1}}, \ \rho(n)\phi^{\bar{0}}), \\ \tau(v, \phi^{\bar{0}}, \phi^{\bar{1}}) &= (T_q(\phi^{\bar{1}}), R_q(\phi^{\bar{0}}), \rho(v)q). \end{aligned}$$

Note that μ preserves V and exchanges the spaces $\mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$, while τ preserves $\mathsf{S}^{\bar{0}}$ and exchanges the spaces $V, \mathsf{S}^{\bar{1}}$. Hence the composition

$$J = \tau \circ \mu \in \mathrm{O}(A)$$

satisfies (3.13). Let us verify that $J^3 = \mathrm{id}_A$. It suffices to check on elements $v \in V \subseteq A$. We have:

$$\begin{aligned}
\mu(v) &= R_n(v) = -nvn,\\
\tau\mu(v) &= -\rho(nvn)q,\\
\mu\tau\mu(v) &= -\rho(vn)q,\\
\tau\mu\tau\mu(v) &= -\rho(vn)q + 2(\rho(vn)q, q)_{\mathsf{S}}\, q\\
&= -\rho(vn)q + 2B(v,n)q\\
&= \rho(nv)q,\\
\mu\tau\mu\tau\mu(v) &= \rho(v)q,\\
\tau\mu\tau\mu\tau\mu(v) &= v.
\end{aligned}$$

Hence $J^3 v = v$ as claimed. We next show that the cubic form C_A changes sign under μ, τ, and hence is invariant under J. For $\xi = (v, \phi^{\bar{0}}, \phi^{\bar{1}})$, we have

$$C_A(\mu(\xi)) = (\rho(R_n(v)n)\phi^{\bar{1}}, \rho(n)\phi^{\bar{0}})_{\mathsf{S}} = -(\rho(v)\phi^{\bar{1}}, \phi^{\bar{0}})_{\mathsf{S}} = -C_A(\xi).$$

The computation for τ is a bit more involved. Let $w = T_q(\phi^{\bar{1}})$, so that $\rho(w)q = \phi^{\bar{1}}$. Then

$$\begin{aligned}
C_A(\tau(\xi)) &= (\rho(w)R_q(\phi^{\bar{0}}), \rho(v)q)_{\mathsf{S}}\\
&= (R_q(\phi^{\bar{0}}), \rho(wv)q)_{\mathsf{S}}\\
&= (\phi^{\bar{0}}, \rho(wv)q)_{\mathsf{S}} - 2(\phi^{\bar{0}}, q)_{\mathsf{S}}(q, \rho(wv)q)_{\mathsf{S}}\\
&= 2B(v,w)(\phi^{\bar{0}}, q)_{\mathsf{S}} - (\phi^{\bar{0}}, \rho(v)\phi^{\bar{1}})_{\mathsf{S}} - 2(\phi^{\bar{0}}, q)_{\mathsf{S}}(\rho(w)q, \rho(v)q)_{\mathsf{S}}\\
&= -(\phi^{\bar{0}}, \rho(v)\phi^{\bar{1}})_{\mathsf{S}}\\
&= -(\rho(v)\phi^{\bar{0}}, \phi^{\bar{1}})_{\mathsf{S}} = -C_A(\xi).
\end{aligned}$$

Hence $C_A(J(\xi)) = C_A(\xi)$. Suppose now that $x \in \mathrm{Spin}(V)$. Then

$$J \circ \rho_A(x) \circ J^{-1}$$

preserves B_A, C_A and the three subspaces $V, \mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$. By Lemma 3.3 we may write this composition as $\rho_A(j(x))$ for a unique element $j(x) \in \mathrm{Spin}(V)$. Using the uniqueness part of Lemma 3.3, we find $j(x_1)j(x_2) = j(x_1x_2)$ and $j(j(j(x))) = x$. □

The theory described here goes much further. Using polarization, the cubic form C_A defines a symmetric trilinear form $T_A \in S^3(A^*)$, with $T_A(\xi,\xi,\xi) = C_A(\xi)$. This form defines a "triality": I.e., $T_A(\xi_1, \xi_2, \xi_3)$ is a non-degenerate bilinear form in ξ_1, ξ_2 for arbitrary fixed non-zero ξ_3. In turn, this triality can be used to construct an interesting non-associative product on V, making V into the algebra of *octonions*. A beautiful discussion of this theory may be found in the paper [20] by Baez.

If $K = \mathbb{C}$, it is possible to arrange that the triality automorphism of $\mathrm{Spin}(8,\mathbb{C})$ preserves the compact group $\mathrm{Spin}(8) = \mathrm{Spin}(8,\mathbb{R})$. See the discussion in Section 3.7.6. Using a computation of roots, one finds that the fixed point set of this automorphism is the exceptional compact Lie group G_2.

3.7 The Clifford algebra $\mathbb{C}l(V)$

Throughout this section we denote by V a vector space over $\mathbb{K} = \mathbb{R}$, with a *positive definite* symmetric bilinear form B. The complexification of the Clifford algebra of V coincides with the Clifford algebra of the complexification of V, and will be denoted by $\mathbb{C}l(V)$. It has the additional structure of an involution, coming from the complex conjugation operation on $V^{\mathbb{C}}$, and one can consider *unitary* Clifford modules compatible with this involution. In this section, we will develop the theory of such unitary Clifford modules, and present a number of applications to the theory of compact Lie groups.

3.7.1 The Clifford algebra $\mathbb{C}l(V)$

Let $V^{\mathbb{C}}$ be the complexification of V. For $v \in V^{\mathbb{C}}$ we denote by v^c its complex conjugate. The Hermitian inner product of $V^{\mathbb{C}}$ will be denoted by $\langle\cdot,\cdot\rangle$, while the extension of B to a complex bilinear form will still be denoted by B. Thus $\langle v, w\rangle = B(v^c, w)$ for $v, w \in V^{\mathbb{C}}$. We put

$$\mathbb{C}l(V) := \mathrm{Cl}(V^{\mathbb{C}}) = \mathrm{Cl}(V)^{\mathbb{C}}.$$

The complex conjugation mapping $v \mapsto v^c$ on $V^{\mathbb{C}}$ extends to a conjugate linear algebra automorphism $x \mapsto x^c$ of the complex Clifford algebra $\mathbb{C}l(V)$. As in 2.2.6, define a conjugate linear anti-automorphism

$$x \mapsto x^* = (x^c)^\top.$$

Thus $(xy)^* = y^*x^*$ and $(ux)^* = u^c x^*$ for $u \in \mathbb{C}$.

Definition 3.7 A unitary Clifford module over $\mathbb{C}l(V)$ is a Hermitian super vector space E together with a morphism of super $*$-algebras $\mathbb{C}l(V) \to \mathrm{End}(E)$.

Thus, for a unitary Clifford module the action map ρ satisfies $\rho(x^*) = \rho(x)^*$ for all $x \in \mathbb{C}l(V)$. Equivalently, the elements of $v \in V \subseteq \mathbb{C}l(V)$ act as self-adjoint operators. Note that for a unitary Clifford module, the representations of $\mathrm{Spin}(V)$, $\mathrm{Pin}(V)$ preserve the Hermitian inner product. They are thus unitary representations.

Example 3.6 The Clifford algebra $\mathbb{C}l(V)$ itself has a Hermitian inner product, $\langle x, y\rangle = \mathrm{tr}(x^*y)$. Let $\rho : \mathbb{C}l(V) \to \mathrm{End}(\mathbb{C}l(V))$ be the action by left-multiplication. For $v \in V \subseteq V^{\mathbb{C}}$ we have $v^* = v$, hence $\langle x, vy\rangle = \langle vx, y\rangle$ for all $x \in \mathbb{C}l(V)$. This shows that $\mathbb{C}l(V)$ is a unitary $\mathbb{C}l(V)$-module. The quantization map intertwines the Hermitian inner product on $\wedge V^{\mathbb{C}}$, given by $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle = \det\langle v_i, w_j\rangle$, with the Hermitian inner product $\langle x, y\rangle = \mathrm{tr}(x^*y)$ on $\mathbb{C}l(V)$.

Since $\mathbb{C}l(V)$ has a faithful unitary representation, such as in the previous example, it follows that $\mathbb{C}l(V)$ is a C^*-algebra. That is, it admits a unique norm $\|\cdot\|$ relative to which it is a Banach algebra, and such that the C^*-identity $\|x^*x\| = \|x\|^2$ is satisfied. This norm is equal to the operator norm in any such presentation, and is explicitly given in terms of the trace by the formula

$$\|a\| = \lim_{n\to\infty} \left(\mathrm{tr}(a^*a)^n\right)^{\frac{1}{2n}}.$$

Note that this C^*-norm is different from the Hilbert space norm $\mathrm{tr}(a^*a)^{1/2}$.

3.7.2 *The groups* $\mathrm{Spin}_c(V)$ *and* $\mathrm{Pin}_c(V)$

Suppose $x \in \Gamma(V^{\mathbb{C}})$, defining a complex transformation $A_x(v) = (-1)^{|x|}xvx^{-1} \in \mathrm{O}(V^{\mathbb{C}})$ as before.

Lemma 3.4 *The element $x \in \Gamma(V^{\mathbb{C}})$ satisfies $A_x(v)^* = A_x(v^*)$ for all $v \in V^{\mathbb{C}}$, if and only if x^*x is a positive real number.*

Proof For all $x \in \Gamma(V^{\mathbb{C}})$ and all $v \in V^{\mathbb{C}}$, we have

$$A_x(v)^* = (-1)^{|x|}(x^{-1})^*v^*x^* = A_{(x^{-1})^*}(v^*).$$

This coincides with $A_x(v^*)$ for all v if and only if $x = \lambda(x^{-1})^*$ for some $\lambda \in \mathbb{C}^\times$, i.e., if and only if $x^*x \in \mathbb{C}^\times$. Since x^*x is a positive element, this condition is equivalent to $x^*x \in \mathbb{R}_{>0}$. □

Definition 3.8 We define

$$\begin{aligned} \Gamma_c(V) &= \{x \in \Gamma(V^{\mathbb{C}}) |\ x^*x \in \mathbb{R}_{>0}\}, \\ \mathrm{Pin}_c(V) &= \{x \in \Gamma(V^{\mathbb{C}}) |\ x^*x = 1\}, \\ \mathrm{Spin}_c(V) &= \mathrm{Pin}_c(V) \cap S\Gamma(V^{\mathbb{C}}). \end{aligned}$$

If $V = \mathbb{R}^n$ with the standard bilinear form, we write $\Gamma_c(n)$, $\mathrm{Pin}_c(n)$, $\mathrm{Spin}_c(n)$.

By definition, an element x of the Clifford group lies in $\Gamma_c(V)$ if and only if the automorphism $A_x \in \mathrm{O}(V^{\mathbb{C}})$ preserves the real subspace V. That is, $\Gamma_c(V) \subseteq \Gamma(V^{\mathbb{C}})$ is the inverse image of $\mathrm{O}(V) \subseteq \mathrm{O}(V^{\mathbb{C}})$. The exact sequence for $\Gamma(V^{\mathbb{C}})$ restricts to exact sequences,

$$\begin{aligned} &1 \to \mathbb{C}^\times \to \Gamma_c(V) \to \mathrm{O}(V) \to 1, \\ &1 \to \mathrm{U}(1) \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \to 1, \\ &1 \to \mathrm{U}(1) \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \to 1, \end{aligned}$$

where we have used $\mathbb{C}^\times \cap \mathrm{Pin}_c(V) = \mathbb{C}^\times \cap \mathrm{Spin}_c(V) = \mathrm{U}(1)$.

One can also directly define $\mathrm{Pin}_c(V)$, $\mathrm{Spin}_c(V)$ as the subgroups of $\Gamma(V^{\mathbb{C}})$ generated by $\mathrm{Pin}(V)$, $\mathrm{Spin}(V)$ together with $\mathrm{U}(1)$. That is, $\mathrm{Spin}_c(V)$ is the quotient of $\mathrm{Spin}(V) \times \mathrm{U}(1)$ by the relation

$$(x, \mathrm{e}^{\sqrt{-1}\psi}) \sim (-x, -\mathrm{e}^{\sqrt{-1}\psi}),$$

and similarly for $\mathrm{Pin}_c(V)$. A third viewpoint towards these groups, using the spinor module, is described in Section 3.7.3 below. The norm homomorphism for $\Gamma(V^{\mathbb{C}})$ restricts to a group homomorphism,

$$\mathsf{N}\colon\ \Gamma_c(V) \to \mathbb{C}^\times,\ x \mapsto x^\top x.$$

On the subgroup $\mathrm{Pin}_c(V)$ this may be written $\mathsf{N}(x) = (x^c)^{-1}x$, which evidently takes values in $\mathrm{U}(1)$.

Together with the map to $\mathrm{O}(V)$ this defines exact sequences,

$$1 \to \mathbb{Z}_2 \to \Gamma_c(V) \to \mathrm{O}(V) \times \mathbb{C}^\times \to 1,$$
$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \times \mathrm{U}(1) \to 1,$$
$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \times \mathrm{U}(1) \to 1.$$

One of the motivations for introducing the group $\mathrm{Spin}_c(V)$ is the following lifting property. Suppose J is an orthogonal complex structure on V, that is, $J \in \mathrm{O}(V)$ and $J^2 = -I$. Such a J exists if and only if $n = \dim V$ is even, and turns V into a vector space over $\mathbb{C}$, with scalar multiplication

$$(a + \sqrt{-1}b)x = ax + bJx.$$

Let $U_J(V) \subseteq \mathrm{SO}(V)$ be the corresponding unitary group (i.e., the elements of $\mathrm{SO}(V)$ commuting with J).

Theorem 3.9 *The inclusion* $\mathrm{U}_J(V) \hookrightarrow \mathrm{SO}(V)$ *admits a unique lift to a group homomorphism* $U_J(V) \hookrightarrow \mathrm{Spin}_c(V)$, *in such a way that its composite with the map* $\mathsf{N}\colon\ \mathrm{Spin}_c(V) \to \mathrm{U}(1)$ *is the complex determinant* $\mathrm{U}_J(V) \to \mathrm{U}(1)$, $A \mapsto \det_J(A)$.

Proof We have to show that the map

$$\mathrm{U}_J(V) \to \mathrm{SO}(V) \times \mathrm{U}(1),\ A \mapsto (A,\ \det_J(A))$$

lifts to the double cover. Since $U_J(V)$ is connected, if such a lift exists then it is unique. To prove existence, it suffices to check that any loop representing the generator of $\pi_1(\mathrm{U}_J(V)) \cong \mathbb{Z}$ lifts to a loop in $\mathrm{Spin}_c(V)$. The inclusion of any non-zero J-invariant subspace $V' \subseteq V$ induces an isomorphism of the fundamental groups of the unitary groups. It is hence sufficient to check for the case $V = \mathbb{R}^2$, with J the standard complex structure $Je_1 = e_2$, $Je_2 = -e_1$. Our task is to lift the map

$$\mathrm{U}(1) \to \mathrm{SO}(2) \times \mathrm{U}(1),\ \mathrm{e}^{\sqrt{-1}\theta} \mapsto (R(\theta), \mathrm{e}^{\sqrt{-1}\theta})$$

to the double cover, $\mathrm{Spin}_c(\mathbb{R}^2)$. This lift is explicitly given by the following modification of Example 3.3,

$$x(\theta) = \mathrm{e}^{\sqrt{-1}\theta/2}(\cos(\theta/2) + \sin(\theta/2)\, e_1e_2) \in \mathrm{Spin}_c(\mathbb{R}^2).$$

Indeed, $x(\theta + 2\pi) = x(\theta)$ and $\mathsf{N}(x(\theta)) = \mathrm{e}^{\sqrt{-1}\theta}$. □

Remark 3.9 The two possible square roots of $\det_J(A)$ for $A \in \mathrm{U}_J(V)$ define a double cover,

$$\tilde{\mathrm{U}}_J(V) = \{(A, z) \in \mathrm{U}_J(V) \times \mathbb{C}^\times \mid z^2 = \det_J(A)\}.$$

While the inclusion $\mathrm{U}_J(V) \hookrightarrow \mathrm{SO}(V)$ does not lift to the Spin group, the above proof shows that there exists a lift $\tilde{\mathrm{U}}_J(V) \to \mathrm{Spin}(V)$ for this double cover. Equivalently, $\tilde{\mathrm{U}}_J(V)$ is identified with the pull-back of the spin double cover.

3.7.3 Spinor modules over $\mathbb{C}l(V)$

We will now discuss special features of spinor modules over the complex Clifford algebra $\mathbb{C}l(V)$ for an even-dimensional real Euclidean vector space V.

The first point we wish to stress is that, similar to Remark 3.6, *the choice of a compatible $\mathbb{Z}_2$-grading on a spinor module S is equivalent to the choice of orientation on V*. Indeed, let $e_1, \ldots, e_{2m}$ be an oriented orthonormal basis of V, where $\dim V = 2m$. Then the chirality element is

$$\Gamma = (\sqrt{-1})^m e_1 \cdots e_{2m} \in \mathrm{Spin}_c(V).$$

The normalization of Γ is such that $\Gamma^2 = 1$. Changing the orientation changes the sign of Γ, and hence changes the parity operator $\rho(\Gamma)$.

If the spinor module is of the form S_F for a Lagrangian subspace $F \in \mathrm{Lag}(V^{\mathbb{C}})$, we also have the orientation defined by the complex structure J corresponding to F. (See Section 1.7.) These two orientations agree:

Proposition 3.18 *Let F be a Lagrangian subspace of $V^{\mathbb{C}}$, and J the corresponding orthogonal complex structure having F as its $+\sqrt{-1}$ eigenspace. Then the orientation on V defined by J coincides with that defined by the $\mathbb{Z}_2$-grading on S_F.*

Proof The orientation defined by J is given by the volume element $e_1 \wedge \cdots \wedge e_n$, where e_i is an orthonormal basis such that $Je_{2j-1} = e_{2j}$. The Lagrangian subspace F is spanned by the orthonormal (for the Hermitian metric) vectors

$$E_j = \frac{1}{\sqrt{2}}(e_{2j-1} - \sqrt{-1}e_{2j}).$$

We claim that the chirality element given by the basis e_i acts as $+1$ on $\mathsf{S}_F^{\bar{0}}$ and as -1 on $\mathsf{S}_F^{\bar{1}}$. We have

$$E_j E_j^c = \frac{1}{2}(e_{2j-1} - \sqrt{-1}e_{2j})(e_{2j-1} + \sqrt{-1}e_{2j}) = \sqrt{-1}e_{2j-1}e_{2j} + 1,$$

hence

$$\Gamma = (E_1 E_1^c - 1) \cdots (E_m E_m^c - 1).$$

For $I = \{i_1, \ldots, i_k\}$ let $E_I^c = E_{i_1}^c \wedge \cdots \wedge E_{i_k}^c \in \wedge F^c \cong \mathsf{S}_F$. The operator $\rho(E_i E_i^c - 1)$ acts as 0 on E_I^c if $i \notin I$, and as -1 if $i \in I$. Hence $\rho(\Gamma)$ acts on E_I^c as $(-1)^k$, proving the claim. □

As a special case of a unitary Clifford module, we have unitary spinor modules. These are Clifford modules S with the property that $\rho : \mathbb{C}l(V) \to \mathrm{End}(\mathsf{S})$ is an isomorphism of super $*$-algebras. Equivalently, the $\mathbb{C}l(V)$-action on S is irreducible.

Proposition 3.19 *Any spinor module* S *admits a Hermitian metric, unique up to positive scalar, for which it becomes a unitary spinor module.*

Proof Let $F \in \mathrm{Lag}(V^{\mathbb{C}})$. Then S_F is a unitary spinor module, and the choice of an isomorphism $\mathsf{S} \cong \mathsf{S}_F$ determines a Hermitian metric on S. Conversely, this Hermitian metric is uniquely determined by its restriction to the pure spinor line l_F, since $\rho(\mathrm{Cl}(V))l_F = \mathsf{S}$. □

For any two unitary spinor modules S, S', the space $\mathrm{Hom}_{\mathbb{C}l}(\mathsf{S}, \mathsf{S}')$ of intertwining operators inherits a Hermitian metric from the full space of homomorphisms $\mathrm{Hom}(\mathsf{S}, \mathsf{S}')$, and the map

$$\mathsf{S} \otimes \mathrm{Hom}_{\mathbb{C}l}(\mathsf{S}, \mathsf{S}') \to \mathsf{S}'$$

is an isomorphism of unitary Clifford modules. In the special case $\mathsf{S}' = \mathsf{S}$, we have $\mathrm{Hom}_{\mathbb{C}l}(\mathsf{S}, \mathsf{S}) = \mathbb{C}$ and the group of $\mathbb{C}l(V)$-equivariant unitary automorphisms of S is the group $\mathrm{U}(1) \subseteq \mathbb{C}$.

Using unitary spinor modules, one obtains another characterization of the groups $\mathrm{Pin}_c(V)$ and $\mathrm{Spin}_c(V)$. Suppose $\dim V$ is even, and pick a unitary spinor module S. An even or odd element $U \in \mathrm{U}(\mathsf{S})$ *implements* $A \in \mathrm{O}(V)$ if

$$\rho(A(v)) = \det(A)\ U \circ \rho(v) \circ U^{-1}$$

for all $v \in V$, with $\det(A) = \pm 1$ depending on the parity of U.

Proposition 3.20 *Suppose* $\dim V$ *is even. For any unitary spinor module* S, *the map* $\mathrm{Pin}_c(V) \to \mathrm{U}(\mathsf{S})$ *is injective. Its image is of implementers of orthogonal transformations of* V. *Similarly* $\mathrm{Spin}_c(V)$ *is isomorphic to the group of implementers of special orthogonal transformations.*

Proof Let $A \in \mathrm{O}(V)$ be given. If $x \in \mathrm{Pin}_c(V)$ is a lift A, then $\det(A)\, x v x^{-1} = A(v)$ for all $v \in V$, and consequently $U = \rho(x)$ satisfies $U\rho(v)U^{-1} = \det(A)\rho(A(v))$. Conversely, suppose U is a unitary element of parity $\det(A) = \pm 1$, implementing A. Let x be the unique element in $\mathrm{Cl}(V)$ with $\rho(x) = U$. Thus $\det(A) = (-1)^{|x|}$, and $\rho((-1)^{|x|} x v x^{-1}) = \rho(A(v))$, hence $(-1)^{|x|} x v x^{-1} = A(v)$ since ρ is faithful. Similarly, $N(x) = x^* x \in \mathrm{U}(1)$. □

Thus $\mathrm{Pin}_c(V)$ and $\mathrm{Spin}_c(V)$ are realized as unitary implementers in S, of orthogonal and special orthogonal transformations, respectively.

3.7.4 Classification of irreducible $\mathbb{C}l(V)$-modules

Let V be a Euclidean vector space. We have seen that if $\dim V$ is even, there are two isomorphism classes of irreducible $\mathbb{C}l(V)$-modules, related by parity reversal. We will now extend this discussion to include the case of $\dim V$ odd. Recall once again that by default, we take Clifford modules to be equipped with a $\mathbb{Z}_2$-grading. Such a module is irreducible if there is no non-trivial invariant $\mathbb{Z}_2$-*graded* subspace. As we will see, the classification of such $\mathbb{Z}_2$-graded Clifford modules is in a sense "opposite" to the classification of irreducible ungraded Clifford modules.

The orientation on V determines the chirality operator

$$\Gamma = (\sqrt{-1})^{n(n-1)/2} e_1 \cdots e_n \in \mathrm{Pin}_c(V),$$

where $n = \dim V$ and $e_1, \ldots, e_n$ is an oriented orthonormal basis; it satisfies $\Gamma^2 = 1$. For n odd, the element Γ is odd, and it is an element of the (ordinary) center of $\mathbb{C}l(V)$. For n even, the element Γ is even, and it lies in the super-center of $\mathbb{C}l(V)$. There is a canonical isomorphism of (ordinary) algebras

$$\mathbb{C}l(V) \to \mathbb{C}l^{\bar{0}}(V \oplus \mathbb{R}), \tag{3.17}$$

determined (using the universal property) by the map on generators $v \mapsto \sqrt{-1}ve$, where e is the standard basis vector for the $\mathbb{R}$ summand. Since $(\sqrt{-1}ve)^* = -\sqrt{-1}e\,v^* = \sqrt{-1}v^*e$ this map is a $*$-isomorphism. If n is odd, the isomorphism (3.17) takes the chirality element of $\mathbb{C}l(V)$ to the chirality element for $\mathbb{C}l(V \oplus \mathbb{R})$, up to a sign.

Theorem 3.10 *Let V be a Euclidean vector space of dimension n, and $\mathbb{C}l(V)$ its complexified Clifford algebra.*

(i) *Suppose n is even. Then there are*:
- *two isomorphism classes of irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(V)$-modules*,
- *a unique isomorphism class of irreducible ungraded $\mathbb{C}l(V)$-modules*,
- *two isomorphism classes of irreducible $\mathbb{C}l^{\bar{0}}(V)$-modules.*

(ii) *Suppose n is odd. Then there are*:
- *a unique isomorphism class of irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(V)$-modules*,
- *two isomorphism classes of irreducible ungraded $\mathbb{C}l(V)$-modules*,
- *a unique isomorphism class of irreducible $\mathbb{C}l^{\bar{0}}(V)$-modules.*

Note that an irreducible $\mathbb{Z}_2$-graded module may be reducible as an ungraded module: There may be invariant subspaces which are not $\mathbb{Z}_2$-graded subspaces.

Proof We may assume $V = \mathbb{R}^n$, and let $\Gamma_n \in \mathbb{C}l(n)$ be the chirality element for the standard orientation. Note also that the third item in (i), resp. (ii), is equivalent to the second item in (ii), resp. (i), since $\mathbb{C}l(n-1) \cong \mathbb{C}l^0(n)$.

(i) Suppose n is even. We denote by S_n a spinor module of $\mathbb{C}l(n)$, with $\mathbb{Z}_2$-grading given by the orientation of $\mathbb{R}^n$. By the results of Section 3.2.5, S_n represents the unique isomorphism class of ungraded irreducible $\mathbb{C}l(n)$-modules, while S_n and $\mathsf{S}_n^{\mathrm{op}}$ represent the two isomorphism classes of irreducible $\mathbb{Z}_2$-graded spinor modules. The latter are distinguished by the action of Γ_n.

(ii) Suppose n is odd. Then

$$\mathbb{C}l(n) \cong \mathbb{C}l^{\bar{0}}(n+1) \cong \mathrm{End}^{\bar{0}}(\mathsf{S}_{n+1}) = \mathrm{End}(\mathsf{S}_{n+1}^{\bar{0}}) \oplus \mathrm{End}(\mathsf{S}_{n+1}^{\bar{1}})$$

identifies $\mathbb{C}l(n)$ as a direct sum of two matrix algebras. Hence there are two classes of irreducible ungraded $\mathbb{C}l(n)$-modules (given by $\mathsf{S}_{n+1}^{\bar{0}}$ and $\mathsf{S}_{n+1}^{\bar{1}}$). These are distinguished by the action of the chirality element Γ_n (note that the map to $\mathbb{C}l^{\bar{0}}(n+1)$ takes Γ_n to Γ_{n+1}, up to sign).

It remains to classify irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(n)$-modules $E = E^{\bar{0}} \oplus E^{\bar{1}}$, for n odd. If $n = 1$, since $\dim \mathbb{C}l(1) = 2$, the Clifford algebra $\mathbb{C}l(1)$ itself is an example. Conversely, if E is an irreducible $\mathbb{C}l(1)$-module, the choice of any non-zero element $\phi \in E^{\bar{0}}$ defines an isomorphism $\mathbb{C}l(1) \to E$, $x \mapsto \rho(x)\phi$. For general odd n, write $\mathbb{C}l(n) = \mathbb{C}l(n-1) \otimes \mathbb{C}l(1)$. If E is an irreducible $\mathbb{Z}_2$-*graded* $\mathbb{C}l(n)$-module, then $E_1 = \mathrm{Hom}_{\mathbb{C}l(n-1)}(\mathsf{S}_{n-1}, E)$ is a $\mathbb{Z}_2$-graded $\mathbb{C}l(1)$-module. This gives a decomposition

$$E \cong \mathsf{S}_{n-1} \otimes \mathrm{Hom}_{\mathbb{C}l(n-1)}(\mathsf{S}_{n-1}, E)$$

as $\mathbb{Z}_2$-graded $\mathbb{C}l(n-1) \otimes \mathbb{C}l(1) = \mathbb{C}l(n)$-modules (using graded tensor products). Since E is irreducible, the $\mathbb{Z}_2$-graded $\mathbb{C}l(1)$-module E_1 must be irreducible, hence it is isomorphic to $\mathbb{C}l(1)$. This proves that $E \cong \mathsf{S}_{n-1} \otimes \mathbb{C}l(1)$ as a $\mathbb{Z}_2$-graded module over $\mathbb{C}l(n-1) \otimes \mathbb{C}l(1) = \mathbb{C}l(n)$. □

Remark 3.10 (Restrictions) Any $\mathbb{C}l(n)$-module can be regarded as a $\mathbb{C}l(n-1)$-module by restriction. By dimension count, one verifies:

1. If n is even, then the ungraded module S_n restricts to a direct sum of the two non-isomorphic ungraded $\mathbb{C}l(n-1)$-modules (given by the even and odd part of S_n). The two $\mathbb{Z}_2$-graded modules S_n and $\mathsf{S}_n^{\mathrm{op}}$ both become isomorphic to the unique $\mathbb{Z}_2$-graded module over $\mathbb{C}l(n-1)$.
2. If n is odd, then the restrictions of the two irreducible ungraded $\mathbb{C}l(n)$-modules to $\mathbb{C}l(n-1)$ are both isomorphic to S_{n-1}, while the restriction of the irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(n)$-module is isomorphic to a direct sum $\mathsf{S}_{n-1} \oplus \mathsf{S}_{n-1}^{\mathrm{op}}$.

3.7.5 Spin representation

We saw that up to isomorphism, the algebra $\mathbb{C}l^{\bar{0}}(V)$ has two irreducible modules if $n = \dim V$ is even, and a unique module if n is odd. These restrict to representations of the group $\mathrm{Spin}(V) \subseteq \mathbb{C}l^{\bar{0}}(V)$, called the two *half-spin representations* if n is even, respectively the *spin representation* if n is odd. If $V = \mathbb{R}^n$, it is customary to denote the two half-spin representations (for n even) by $\Delta_n^{\pm}$, and the spin representation (for n odd) by Δ_n. Here Δ_n^+ (resp. Δ_n^-) is the half-spin representation where Γ_n acts as $+1$ (resp. as -1).

More concretely, taking $V = \mathbb{R}^{2m}$ with the spin representation defined by its standard complex structure $Je_{2j-1} = e_{2j}$, we may take $\Delta_{2m}^{\pm}$ to be the even and odd part of $\mathsf{S}_{2m} = \wedge\mathbb{C}^m$, and $\Delta_{2m-1} = \mathsf{S}_{2m-2} = \wedge\mathbb{C}^{m-1}$ (the spinor module over $\mathrm{Cl}(2m-2) \cong \mathrm{Cl}^{\bar{0}}(2m-1)$).

Proposition 3.21

(i) *If n is even, the two half-spin representations $\Delta_n^{\pm}$ of* $\mathrm{Spin}(n)$ *are irreducible, and are non-isomorphic. Their restrictions to* $\mathrm{Spin}(n-1)$ *are both isomorphic to Δ_{n-1}.*
(ii) *If n is odd, the spin representation Δ_n is irreducible. Its restriction to* $\mathrm{Spin}(n-1)$ *is isomorphic to $\Delta_{n-1} = \Delta_{n-1}^+ \oplus \Delta_{n-1}^-$.*

Proof This is immediate from the classification of irreducible $\mathrm{Cl}^{\bar{0}}(n)$-modules, since $\mathrm{Spin}(n)$ generates $\mathrm{Cl}^{\bar{0}}(n)$ as an algebra. (Note e.g., that $\mathrm{Spin}(n)$ contains the basis e_I of $\mathrm{Cl}^{\bar{0}}(n)$, where I ranges over subsets of $\{1, \ldots, n\}$ with an even number of elements.) $\square$

We recall some terminology from the representation theory of compact Lie groups G (see e.g., [30, 48]). Let H be a Hermitian vector space with a unitary G-representation. The inner product on H will be denoted $\langle \cdot, \cdot \rangle$.

(i) H is of *real type* if it admits a G-equivariant conjugate linear endomorphism C with $C^2 = I$. In this case, H is the complexification of the real G-representation $\mathsf{H}_{\mathbb{R}}$ given as the fixed point set of C. Representations of real type admit a non-degenerate symmetric bilinear form $(\phi, \psi) = \langle C\phi, \psi \rangle$. Conversely, given a G-invariant non-degenerate skew-symmetric bilinear form, define a conjugate linear endomorphism T by $(\phi, \psi) = \langle T\phi, \psi \rangle$. The square $T^2 \in \mathrm{End}(\mathsf{H})$ is $\mathbb{C}$-linear and is positive definite. Hence $|T| = (T^2)^{1/2}$ commutes with T, and $C = T|T|^{-1}$ defines a real structure. We will call the bilinear form *compatible* with the Hermitian structure if $C = T$.
(ii) H is of *quaternionic type* if it admits a G-equivariant conjugate linear endomorphism C with $C^2 = -I$. In this case, C gives H the structure of a quaternionic G-representation, where scalar multiplication by the quaternions i, j, k is given by $i = \sqrt{-1}$, $j = C$, $k = ij$. From C one obtains a non-degenerate skew-symmetric bilinear form $(\phi, \psi) = \langle C\phi, \psi \rangle$. Conversely, given a G-invariant non-degenerate symmetric bilinear form, define a conjugate linear endomorphism T by $(\phi, \psi) = \langle T\phi, \psi \rangle$. Again, $|T| = (-T^2)^{1/2}$ commutes with T, and $C = T|T|^{-1}$ defines a quaternionic structure. We will call the bilinear form *compatible* with the Hermitian structure if $C = T$.

(iii) For G-representations H of real or quaternionic type, the structure map C gives an isomorphism with the dual G-representation H^*. That is, such representations are self-dual. We will call a unitary G-representation of *complex type* if it is not self-dual.

For a real or quaternionic representation, the corresponding bilinear form defines an element of $\mathrm{Hom}_G(\mathsf{H}, \mathsf{H}^*)$. If H is irreducible, then this space is 1-dimensional. Hence for irreducible representations the real and quaternionic cases are exclusive. This proves part of the following result (see [30, Chapter II.6]).

Theorem 3.11 *An irreducible unitary representation of a compact Lie group G is either of real, complex or quaternionic type.*

We now specialize to the spin representations. The canonical bilinear form on spinor modules S can be viewed as scalar-valued, after choice of a trivialization of the canonical line K_{S}.

Proposition 3.22 *Let V be a Euclidean vector space of even dimension $n = 2m$, and S a unitary spinor module over $\mathbb{C}l(V)$. Then the Hermitian metric on S and the canonical bilinear form are compatible.*

Proof We use the model $V^{\mathbb{C}} = F^* \oplus F$, $\mathsf{S} = \wedge F^*$. Let $f^1, \ldots, f^m$ be a basis of F^*, orthonormal relative to the Hermitian inner product. Then the f^I for subsets $I \subseteq \{1, \ldots, m\}$ define an orthonormal basis of $\wedge F^*$. For any subset I let I^c be the complementary subset. Use $f^1 \wedge \cdots \wedge f^m$ to trivialize $\det(F^*)$, and define signs $\varepsilon_I = \pm 1$ by

$$(f^I)^\top \wedge f^{I^c} = \varepsilon_I \, f^1 \wedge \cdots \wedge f^m.$$

Thus $(f^I, f^{I^c})_{\mathsf{S}} = \varepsilon_I$. Notice that $\varepsilon_I = \varepsilon_{I^c}$ in the symmetric case and $\varepsilon_I = -\varepsilon_{I^c}$ in the skew-symmetric case. Define C by

$$(f^I, f^J)_{\mathsf{S}} = \langle C f^I, f^J \rangle.$$

Then $Cf^I = \varepsilon_I f^{I_c}$. We note that $C^2 = I$ in the symmetric case and $C^2 = -I$ in the skew-symmetric case. □

Theorem 3.12 *The types of the spin representations of* $\mathrm{Spin}(n)$ *are as follows.*

$$\begin{aligned}
n = 0 \mod 8 &: \quad \Delta_n^\pm \textit{ real type},\\
n = 1, 7 \mod 8 &: \quad \Delta_n \textit{ real type},\\
n = 2, 6 \mod 8 &: \quad \Delta_n^\pm \textit{ complex type},\\
n = 3, 5 \mod 8 &: \quad \Delta_n \textit{ quaternionic type},\\
n = 4 \mod 8 &: \quad \Delta_n^\pm \textit{ quaternionic type}.
\end{aligned}$$

If $n = 2, 6 \mod 8$, *one has* $\Delta_n^- \cong (\Delta_n^+)^*$.

Proof We consider various subcases.

- Case 1a: $n = 2m$ with m even. The canonical bilinear form $(\cdot,\cdot)_S$ is still non-degenerate on the even and odd part of the spinor module; hence it defines a real structure if $m = 0$ mod 4 and a quaternionic structure for $m = 2$ mod 4.
- Case 1b: $n = 2m$ with m odd. Then $(\cdot,\cdot)_S$ defines a non-degenerate pairing between $\mathsf{S}_n^{\bar{0}} = \Delta_n^+$ and $\mathsf{S}_n^{\bar{1}} = \Delta_n^-$. Hence $\Delta_n^\pm \ncong \Delta_n^\mp = (\Delta_n^\pm)^*$, so that $\Delta_n^\pm$ are of complex type.
- Case 2a: $n = 2m - 1$ is odd, with m even. Recall that $\Delta_n = \mathsf{S}_{n+1}^{\bar{0}}$. The restriction of $(\cdot,\cdot)_\mathsf{S}$ gives the desired non-degenerate symmetric (if $m = 0$ mod 4) or skew-symmetric (if $m = 2$ mod 4) bilinear form.
- Case 2b: $n = 2m - 1$ with m is odd. Here the restriction of $(\cdot,\cdot)_\mathsf{S}$ to $\mathsf{S}_{n+1}^{\bar{0}}$ is zero. We instead use the form

$$(\phi, \psi)' := (\phi,\ \rho(e_{n+1})\psi)_\mathsf{S} \equiv (\rho(e_{n+1})\phi, \psi)_\mathsf{S},$$

for $\phi, \psi \in \Delta_{2m-1} = \Delta_{2m}$. This is no longer Spin$(n+1)$-invariant, but is still Spin(n)-invariant. Since

$$\begin{aligned}(\psi, \phi)' &= (\rho(e_{n+1})\psi,\ \phi)_\mathsf{S}\\ &= (-1)^{m(m-1)/2}(\phi,\ \rho(e_{n+1})\psi)_\mathsf{S}\\ &= (-1)^{m(m-1)/2}(\phi, \psi)'.\end{aligned}$$

We find that the bilinear form is symmetric for $m = 1$ mod 4 and skew-symmetric for $m = 3$ mod 4. □

3.7.6 Applications to compact Lie groups

Theorem 3.12 has several important Lie-theoretic implications.

Spin(3) The spin representation Δ_3 has dimension 2, and after choice of basis defines a homomorphism Spin(3) → U(2). Since Spin(3) is semisimple, the image lies in SU(2), and by dimension count, the resulting map Spin(3) → SU(2) must be an isomorphism. Since Δ_3 is of quaternionic type, one similarly has a homomorphism Spin(3) → Aut($\mathbb{H}$) = Sp(1), which is an isomorphism by dimension count. That is, we recover Spin(3) ≅ SU(2) ≅ Sp(1).

Spin(4) The two half-spin representations $\Delta_4^\pm$ are both 2-dimensional. By an argument similar to that for $n = 3$, we see that the representation on $\Delta_4^+ \oplus \Delta_4^-$ defines an isomorphism Spin(4) → SU(2) × SU(2).

Spin(5) Δ_5 is a 4-dimensional representation of quaternionic type. After choice of an orthonormal quaternionic basis, this gives a homomorphism to Sp(2) =

$\mathrm{Aut}(\mathbb{H}^2)$, which by dimension count must be an isomorphism. This realizes the isomorphism

$$\mathrm{Spin}(5) \cong \mathrm{Sp}(2).$$

In particular, we see that Spin(5) acts transitively on the unit sphere $S^7 \subseteq \Delta_5 \cong \mathbb{C}^4$, since Sp(2) acts transitively on the quaternions of unit norm. The stabilizer of the base point $(0, 1) \in \mathbb{H}^2$ is $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ (embedded in Sp(2) as the upper left block). We hence see that

$$S^7 = \mathrm{Spin}(5)/\mathrm{SU}(2).$$

This checks with dimensions, since Spin(5) has dimension 10, while SU(2) has dimension 3.

Spin(6) The half-spin representations $\Delta_6^\pm$ are 4-dimensional, and define homomorphisms $\mathrm{Spin}(6) \to \mathrm{U}(4)$. Since Spin(6) is semisimple, this homomorphism must take values in SU(4), realizing the isomorphism

$$\mathrm{Spin}(6) \cong \mathrm{SU}(4).$$

In particular Spin(6) acts transitively on $S^7 \subseteq \Delta_6^\pm \cong \Delta_5$, extending the action of Spin(5), with stabilizers SU(3).

Spin(7) The 8-dimensional spin representation Δ_7 is of real type, hence it can be regarded as the complexification of an 8-dimensional real representation $\Delta_7^{\mathbb{R}} \cong \mathbb{R}^8$. Restricting to Spin(6), we have $\Delta_7 = \Delta_6^+ \oplus \Delta_6^-$. Under the symmetric bilinear form on Δ_7, both $\Delta_6^\pm$ are Lagrangian.[1] This implies that $\Delta_7^{\mathbb{R}} \cong \Delta_6^\pm$ as real $\mathrm{Spin}(6) \subseteq \mathrm{Spin}(7)$-representations. Since Spin(6) acts transitively on the unit sphere $S^7 \subseteq \Delta_6^\pm$, this shows that Spin(7) acts transitively on the unit sphere $S^7 \subseteq \Delta_7^{\mathbb{R}}$. Let H be the isotropy group at some given base point on S^7. It is a compact Lie group of dimension

$$\dim H = \dim \mathrm{Spin}(7) - \dim S^7 = 21 - 7 = 14.$$

More information is obtained using some homotopy theory. For a compact, simply connected simple Lie group G, one knows that $\pi_1(G) = \pi_2(G) = 0$, while $\pi_3(G) = \mathbb{Z}$. By the long exact sequence of homotopy groups of a fibration,

$$\cdots \to \pi_{k+1}(S^7) \to \pi_k(H) \to \pi_k(\mathrm{Spin}(7)) \to \pi_k(S^7) \to \cdots,$$

and using that $\pi_k(S^7) = 0$ for $1 < k < 7$ (Hurewicz' Theorem), we find that $\pi_1(H) = \pi_2(H) = 0$ and $\pi_3(H) = \mathbb{Z}$. It follows that H is simply connected and simple (otherwise $\pi_3(H)$ would have more summands). But in dimension 14 there is a unique such group: the exceptional Lie group G_2. This proves the following remarkable result.

[1] In more detail, recall that the bilinear form on $\Delta_7 \cong \mathsf{S}_8^{\bar{0}}$ is $\mathbb{C}l(7) \cong \mathbb{C}l^0(8)$-invariant. Restricting to $\mathbb{C}l(6) \subseteq \mathbb{C}l(7)$, we obtain a $\mathbb{C}l(6)$-invariant bilinear form on $\Delta_6^+ \oplus \Delta_6^- \cong \mathsf{S}_6$, which must agree with the canonical bilinear form up to scalar multiple. But the latter vanishes on the even and odd part of S_6.

Theorem 3.13 *There is a transitive action of* Spin(7) *on* S^7. *The stabilizer subgroups for this action are isomorphic to the exceptional Lie group* G_2. *That is,*

$$S^7 = \mathrm{Spin}(7)/G_2.$$

We remark that one can also directly identify the root system for H, avoiding the use of algebraic topology or appealing to the classification of Lie groups. This is carried out in Adams' book [1, Chapter 5].

Spin(8) The triality principle from Section 3.6 specializes to give a degree 3 automorphism j of the group $\mathrm{Spin}(8,\mathbb{C})$, along with a degree 3 automorphism J of $\mathbb{C}^8 \oplus \Delta_8^+ \oplus \Delta_8^-$ interchanging the three summands, such that the induced maps $J:\ \Delta_8^- \to \mathbb{C}^8$, etc., are equivariant relative to the automorphism j. Since $\Delta_8^\pm$ are of real type, one may hope for J to preserve the real subspace $\mathbb{R}^8 \oplus \Delta_8^{+,\mathbb{R}} \oplus \Delta_8^{-,\mathbb{R}}$, and for j to preserve Spin(8). This is accomplished by taking the vectors n, q in the construction of J, j (see Section 3.6) to lie in $\mathbb{R}^8$ and $\Delta_8^{+,\mathbb{R}}$, respectively. The trilinear form on $\mathbb{C}^8 \oplus \Delta_8^+ \oplus \Delta_8^-$ (cf. (3.15)) restricts to the real part, and can be used [34, 41] to define on $\mathbb{R}^8$ an octonion multiplication, $\mathbb{R}^8 \cong \mathbb{O}$. The exceptional group G_2 is now realized as the fixed point group for the automorphism j of the compact group Spin(8), and also as the automorphism group $\mathrm{Aut}(\mathbb{O})$ of the octonions. A beautiful survey of this theory is given in Baez's article [20]. For the history and other constructions of G_2, see Agricola [3].

The other exceptional groups E_6, E_7, E_8, F_4 are related to spin groups as well. For example, F_4 contains a copy of Spin(9), and the action of Spin(9) on $\mathfrak{f}_4/\mathfrak{o}(9)$ is isomorphic to the (real) spin representation $\Delta_9^{\mathbb{R}}$. In a similar fashion, E_8 contains a copy of $\mathrm{Spin}(16)/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ is generated by the chirality element Γ_{16}), and the action of $\mathrm{Spin}(16)/\mathbb{Z}_2$ on the quotient $\mathfrak{e}_8/\mathfrak{o}(16)$ is isomorphic to the (real) spin representation of Spin(16) on $\Delta_{16}^{+,\mathbb{R}}$. (This checks with dimensions: E_8 is 248-dimensional, Spin(16) is 120-dimensional, and $\Delta_{16}^{+,\mathbb{R}}$ is $2^7 = 128$-dimensional.) Proofs, and a wealth of related results, can be found in Adams' book [1].

Chapter 4
Covariant and contravariant spinors

Suppose V is a quadratic vector space. In Section 2.2.10 we defined a map $\lambda : \mathfrak{o}(V) \to \wedge^2(V)$, which is a Lie algebra homomorphism relative to the Poisson bracket on $\wedge(V)$. We also considered its quantization $\gamma = q \circ \lambda : \mathfrak{o}(V) \to \mathrm{Cl}(V)$, which is a Lie algebra homomorphism relative to the Clifford commutator. One of the problems addressed in this chapter is to give explicit formulas for the Clifford exponential $\exp(\gamma(A)) \in \mathrm{Cl}(V)$. We will compute its image under the symbol map, and express its relation to the exterior algebra exponential $\exp(\lambda(A))$. These questions will be studied using the spin representation for $W = V^* \oplus V$, with bilinear form given by the pairing. (In particular, the bilinear form on V itself need not be split, and may even be degenerate).

4.1 Pull-backs and push-forwards of spinors

Let V be any vector space, and let $W = V^* \oplus V$, equipped with the bilinear form

$$B_W((\mu_1, v_1), (\mu_2, v_2)) = \frac{1}{2}(\langle \mu_1, v_2 \rangle + \langle \mu_2, v_1 \rangle). \tag{4.1}$$

We will occasionally use a basis $e_1, \ldots, e_m$ of V, with dual basis $f^1, \ldots, f^m$ of V^*. Thus $B_W(e_i, f^j) = \frac{1}{2}\delta_i^j$, and the Clifford relations in $\mathrm{Cl}(W)$ read, in terms of super commutators, as

$$[f^i, f^j] = 0, \quad [e_i, f^j] = \delta_i^j, \quad [e_i, e_j] = 0.$$

We define the standard or *contravariant* spinor module to be $\wedge(V^*)$, with generators $\mu \in V^*$ acting by exterior multiplication and $v \in V$ acting by contraction. We will also consider the dual or *covariant* spinor module $\wedge(V)$, with generators $v \in V$ acting by exterior multiplication and $\mu \in V^*$ acting by contraction. Recall (cf. Section 3.4) that there is a canonical isomorphism of Clifford modules,

$$\wedge(V^*) \cong \wedge(V) \otimes \det(V^*),$$

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defined by contraction. The choice of a generator $\Gamma_\wedge \in \det(V^*)$ gives an isomorphism, called the "star operator" for the volume form $\Gamma_\wedge$

$$*_{\Gamma_\wedge}: \wedge(V) \to \wedge(V^*),\ \chi \mapsto \iota(\chi)\Gamma_\wedge.$$

Thus

$$*_{\Gamma_\wedge} \circ \varepsilon(v) = \iota(v) \circ *_{\Gamma_\wedge},$$
$$*_{\Gamma_\wedge} \circ \iota(\mu) = \varepsilon(\mu) \circ *_{\Gamma_\wedge}.$$

In Proposition 3.11, we saw that the most general contravariant pure spinor is of the form

$$\phi = \exp(-\omega)\kappa, \tag{4.2}$$

where $\omega \in \wedge^2 V^*$ is a 2-form, and $\kappa \in \det(\mathrm{ann}(N))^\times$ is a volume form on V/N, for some subspace $N \subseteq V$. Given κ, the form ϕ only depends on the restriction $\omega_N \in \wedge^2(N^*)$ of ω to N.

By reversing the roles of V and V^*, the most general covariant spinor is of the form

$$\chi = \exp(-\pi)\nu, \tag{4.3}$$

where $\pi \in \wedge^2(V)$ and $\nu \in \det(S)^\times$, for some subspace $S \subseteq V$. Given ν, the spinor χ only depends on the image $\pi_S \in \wedge^2(V/S)$ of π. The corresponding Lagrangian subspace is

$$F(\mathrm{e}^{-\pi}\nu) = \{(\mu, v) \in V^* \oplus V \mid \mu \in \mathrm{ann}(S),\ \pi(\mu,\cdot) - v \in S\}. \tag{4.4}$$

Note that $S = F(\mathrm{e}^{-\pi}\nu) \cap V$, while $\mathrm{ann}(S)$ is characterized as projection of $F(\mathrm{e}^{-\pi}\nu)$ to V^*.

For a linear map $\Phi: V_1 \to V_2$ we denote by $\Phi_* = \wedge(\Phi): \wedge V_1 \to \wedge V_2$ the "push-forward" map, and by $\Phi^* = \wedge(\Phi^*): \wedge V_2^* \to \wedge V_1^*$ the "pull-back" map.

Proposition 4.1 (Push-forwards) *Suppose $\Phi: V_1 \to V_2$ is a linear map, and $\chi \in \wedge V_1$ is a pure spinor. Then the following are equivalent:*

1. $\Phi_*\chi \neq 0$,
2. $\ker(\Phi) \cap \{v_1 \in V_1 \mid v_1 \wedge \chi = 0\} = \{0\}$,
3. $\Phi_*\chi$ *is a pure spinor.*

In this case, the Lagrangian subspace defined by the pure spinor $\Phi_*\chi$ is

$$F(\Phi_*\chi) = \{(\mu_2, \Phi_* v_1) \mid (\Phi^*\mu_2, v_1) \in F(\chi)\}.$$

Proof Write $\chi = \mathrm{e}^{-\pi}\nu$ as in (4.3). Since ν is a generator of $\det(S)$, the subspace $S \subseteq V_1$ may be characterized as the set of all $v_1 \in V_1$ with $v_1 \wedge \nu = 0$, that is $v_1 \wedge \chi = 0$. Thus

$$\Phi_*\chi \neq 0 \Leftrightarrow \Phi_*\nu \neq 0 \Leftrightarrow \ker(\Phi) \cap S = \{0\}.$$

Furthermore, in this case $\Phi_*\nu$ is a generator of $\det(\Phi(S))$, and hence $\Phi_*\chi = e^{-\Phi_*\pi}\Phi_*\nu$ is a pure spinor. We have

$$(\mu_2, v_2) \in F(\Phi_*\chi) \;\Leftrightarrow\; \mu_2 \in \mathrm{ann}(\Phi(S)) \text{ and } \Phi_*\pi(\mu_2, \cdot) - v_2 \in \Phi(S). \qquad (4.5)$$

The first condition in (4.5) means that $\Phi^*\mu_2 \in \mathrm{ann}(S)$. Choose $w_1 \in V_1$ with $(\Phi^*\mu_2, w_1) \in F(\chi)$. Note that w_1 is unique modulo $F(\chi) \cap V_1 = S$. The difference $\pi(\Phi^*\mu_2, \cdot) - w_1$ lies in S, hence

$$\Phi_*\pi(\mu_2, \cdot) - \Phi(w_1) \in \Phi(S).$$

The second condition in (4.5) now shows that $v_2 - \Phi(w_1) = \Phi(u_1)$ for a unique element $u_1 \in S$. Putting $v_1 = w_1 + u_1$, we obtain $v_2 = \Phi(v_1)$, giving the desired description of $F(\Phi_*\chi)$. □

By a similar argument, one shows:

Proposition 4.2 (Pull-backs) *Suppose $\Phi : V_1 \to V_2$ is a linear map, and $\phi \in \wedge V_2^*$ is a pure spinor. Then the following are equivalent*:

1. $\Phi^*\phi \neq 0$,
2. $\ker(\Phi^*) \cap \{\mu_2 \in V_2^* |\ \mu_2 \wedge \phi = 0\} = \{0\}$,
3. *$\Phi^*\phi$ is a pure spinor.*

In this case, the Lagrangian subspace defined by $\Phi^*\phi$ is

$$F(\Phi^*\phi) = \{(\Phi^*\mu_2, v_1) |\ (\mu_2, \Phi_* v_1) \in F(\phi)\}.$$

The two propositions suggest notions of push-forwards and pull-backs of Lagrangian subspaces. Write

$$(\mu_1, v_1) \sim_\Phi (\mu_2, v_2) \;\Leftrightarrow\; v_2 = \Phi(v_1),\ \mu_1 = \Phi^*(\mu_2).$$

If $E \in \mathrm{Lag}(V_1^* \oplus V_1)$, we define the *forward image of E*:

$$\Phi_! E = \{(\mu_2, v_2) \in V_2^* \oplus V_2 |\ \exists(\mu_1, v_1) \in E :\ (\mu_1, v_1) \sim_\Phi (\mu_2, v_2)\}.$$

If $F \in \mathrm{Lag}(V_2^* \oplus V_2)$, we define the *backward image of F*:

$$\Phi^! F = \{(\mu_1, v_1) \in V_1^* \oplus V_1 |\ \exists(\mu_2, v_2) \in F :\ (\mu_1, v_1) \sim_\Phi (\mu_2, v_2)\}.$$

The two propositions above show that

$$\ker(\Phi) \cap (E \cap V_1) = \{0\} \Rightarrow \Phi_! E \in \mathrm{Lag}(V_2^* \oplus V_2),$$
$$\ker(\Phi^*) \cap (F \cap V_1^*) = \{0\} \Rightarrow \Phi^! F \in \mathrm{Lag}(V_1^* \oplus V_1).$$

Remark 4.1 In fact, $\Phi_! E$ and $\Phi^! F$ are Lagrangian even without these transversality assumptions. However, assuming that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the forward image $\Phi_! E$ (resp. backward image $\Phi^! F$) depends continuously on E (resp. on F) only on the open subset of $\mathrm{Lag}(V^* \oplus V)$ where the transversality condition is satisfied.

Proposition 4.3 *Let $\Phi: V_1 \to V_2$ be a linear map. Suppose E_1 is a Lagrangian subspace of $V_1^* \oplus V_1$, and F_2 is a Lagrangian subspace of $V_2^* \oplus V_2$. Let $E_2 = \Phi_! E_1$ be the forward image of E_1 and $F_1 = \Phi^! F_2$ the backward image of F_2. Then*

$$E_1 \cap F_1 = \{0\} \quad \Leftrightarrow \quad E_2 \cap F_2 = \{0\}.$$

Furthermore, in this case

$$\ker(\Phi) \cap (E_1 \cap V_1) = \{0\}, \quad \ker(\Phi^*) \cap (F_2 \cap V_2^*) = \{0\}.$$

Proof Consider the covariant spinor module over $\mathrm{Cl}(V_1^* \oplus V_1)$ and the contravariant one over $\mathrm{Cl}(V_2^* \oplus V_2)$. Let $\chi \in \wedge V_1$ be a pure spinor defining E_1, and $\phi \in \wedge V_2^*$ a pure spinor defining F_2. Then $\langle \phi, \Phi_* \chi \rangle = \langle \Phi^* \phi, \chi \rangle$, hence one pairing vanishes if and only if the other pairing vanishes. But by Theorem 3.7, the non-vanishing of the pairing is equivalent to transversality of the corresponding Lagrangian subspaces. Furthermore, the non-vanishing implies that $\Phi_* \chi$ and $\Phi^* \phi$ are both non-zero, which by Propositions 4.1 and 4.2 gives the transversality conditions. □

4.2 Factorizations

4.2.1 The Lie algebra $\mathfrak{o}(V^* \oplus V)$

Let $W = V^* \oplus V$, and recall the isomorphism (cf. Section 2.2.10)

$$\lambda: \mathfrak{o}(W) \to \wedge^2(W),$$

given in terms of the Poisson bracket on $\wedge(W)$ by $S(w) = \{\lambda(S), w\}$. We are interested in the action of $\gamma(S) = q(\lambda(S)) \in \mathrm{Cl}(W)$ in the contravariant spinor module $\wedge(V^*)$. To compute this action, decompose

$$\wedge^2(W) = \wedge^2(V) \oplus \wedge^2(V^*) \oplus (V^* \wedge V).$$

In $\mathfrak{o}(W)$, the three summands correspond to:

1. The commutative Lie algebra $\wedge^2(V)$ of skew-adjoint maps $E_1: V^* \to V$, acting as $(\mu, v) \mapsto (0, E_1(\mu))$. Equivalently, this is the subalgebra of $\mathfrak{o}(W)$ fixing V pointwise. We have
$$\lambda(E_1) = \frac{1}{2} \sum_i E_1(f^i) \wedge e_i. \tag{4.6}$$
The element $\gamma(E_1) \in \wedge^2(V)$ is given by the same formula (viewing $\wedge(V)$ as a subalgebra of $\mathrm{Cl}(W)$), and its action in the spinor module is contraction with $\lambda(E_1)$.
2. The commutative Lie algebra $\wedge^2(V^*)$ of skew-adjoint maps $E_2: V \to V^*$, acting as $(\mu, v) \mapsto (E_2(v), 0)$. Equivalently, this is the subalgebra of $\mathfrak{o}(W)$ fixing V^* pointwise. We have
$$\lambda(E_2) = \frac{1}{2} \sum_i E_2(e_i) \wedge f^i.$$

The element $\gamma(E_2) \in \wedge^2(V^*)$ is given by the same formula (viewing $\wedge(V^*)$ as a subalgebra of $\mathrm{Cl}(W)$), and its action in the spinor module is exterior multiplication by $\lambda(E_2)$.

3. The Lie algebra $\mathfrak{gl}(V)$, where $A: V \to V$ acts as $(\mu, v) \mapsto (-A^*\mu, Av)$. Equivalently, this is the subalgebra of $\mathfrak{o}(W)$ preserving the direct sum decomposition $V^* \oplus V$. We have

$$\lambda(A) = -\sum_i f^i \wedge A(e_i).$$

This quantizes to

$$\begin{aligned}\gamma(A) &= q(\lambda(A))\\ &= -\frac{1}{2}\sum_i (f^i A(e_i) - A(e_i) f^i)\\ &= -\sum_i f^i A(e_i) + \frac{1}{2}\mathrm{tr}(A),\end{aligned}$$

where we used $\sum_i \langle f^i,\ A(e_i)\rangle = \mathrm{tr}(A)$. Letting L_A denote the canonical action of $\mathfrak{gl}(V)$ on $\wedge(V^*)$, given as the derivation extensions of the action as $-A^*$ on V^*, we find that $\gamma(A)$ acts as

$$\rho(\gamma(A)) = L_A + \frac{1}{2}\mathrm{tr}(A).$$

That is, the action of $\mathfrak{gl}(V)$ on the spinor module $\wedge(V^*)$ differs from the "standard" action by the 1-dimensional character $A \mapsto \frac{1}{2}\mathrm{tr}(A)$.

Writing elements of $W = V^* \oplus V$ as column vectors, we see that

$$\mathfrak{o}(W) \cong \wedge^2(V^*) \oplus \mathfrak{gl}(V) \oplus \wedge^2(V)$$

consists of block matrices of the form

$$S = \begin{pmatrix} -A^* & E_2 \\ E_1 & A \end{pmatrix},$$

and our discussion shows that

$$\rho(\gamma(S)) = \iota(\lambda(E_1)) + \varepsilon(\lambda(E_2)) + L_A + \frac{1}{2}\mathrm{tr}(A).$$

4.2.2 The group SO($V^* \oplus V$)

Corresponding to the three Lie subalgebras of $\mathfrak{o}(W)$ there are three subgroups of $\mathrm{SO}(W)$:

1. $\wedge^2(V^*)$, given as matrices in block form

$$\begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix},$$

where $E_2: V \to V^*$ is a skew-adjoint linear map.

2. $\wedge^2(V)$, given as matrices in block form

$$\begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix},$$

where $E_1 : V^* \to V$ is a skew-adjoint linear map.

3. $\mathrm{GL}(V)$, embedded as matrices in block form

$$\begin{pmatrix} (Q^{-1})^* & 0 \\ 0 & Q \end{pmatrix},$$

where $Q : V \to V$ is invertible.

If $\mathbb{K} = \mathbb{R}, \mathbb{C}$, these are Lie subgroups of $\mathrm{SO}(W)$, with respective Lie subalgebras $\wedge^2(V^*)$, $\mathfrak{gl}(V)$, $\wedge^2(V)$ of $\mathfrak{o}(W)$. Now consider arbitrary orthogonal transformations. An endomorphism $g \in \mathrm{End}(W)$, written in block form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.7}$$

is orthogonal if and only if

$$a^*c + c^*a = 0, \quad b^*d + d^*b = 0, \quad a^*d + c^*b = I.$$

Proposition 4.4 (Factorization formulas)

(i) *The map* $\wedge^2(V) \times \mathrm{GL}(V) \times \wedge^2(V^*) \to \mathrm{SO}(W)$ *taking* (g_1, g_2, g_3) *to the product* $g = g_1g_2g_3$ *is injective. Its image is the set of all orthogonal transformations* (4.7), *for which the block* $a \in \mathrm{End}(V^*)$ *is invertible.*

(ii) *The map* $\wedge^2(V^*) \times \mathrm{GL}(V) \times \wedge^2(V) \to \mathrm{SO}(W)$ *taking* (g_1, g_2, g_3) *to* $g_1g_2g_3$ *is injective. Its image is the set of all orthogonal transformations* (4.7) *for which the block* $d \in \mathrm{End}(V)$ *is invertible.*

In particular, the orthogonal transformations (4.7) for which the block a or the block d are invertible, are contained in $\mathrm{SO}(V^* \oplus V)$.

Proof For (i) we try to write (4.7) as a product

$$\begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix} \begin{pmatrix} (Q^{-1})^* & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} (Q^{-1})^* & (Q^{-1})^* E_2 \\ E_1(Q^{-1})^* & Q + E_1(Q^{-1})^* E_2 \end{pmatrix}.$$

If a is invertible, we can solve for E_1, E_2, Q in terms of the blocks a, b, c:

$$Q = (a^{-1})^*, \; E_1 = ca^{-1}, \; E_2 = a^{-1}b.$$

(Note that $d = (a^{-1})^*(I - c^*b)$ if a is invertible.) The proof of (ii) is similar. □

4.2.3 *The group* Spin$(V^* \oplus V)$

The factorizations of $\mathrm{SO}(W)$ give rise to factorizations of the special Clifford group $S\Gamma(W)$ and of the spin group $\mathrm{Spin}(W)$.

1. The inclusion $\wedge^2(V) \hookrightarrow \mathrm{SO}(W)$ lifts to an inclusion as a subgroup of $\mathrm{Spin}(W) \subseteq S\Gamma(W)$, by the map
$$\begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix} \mapsto \tilde{E}_1 := \exp(\lambda(E_1)),$$
where the right-hand side is an element of $\wedge(V^*) \subseteq \mathrm{Cl}(W)$. Indeed, for all $(\mu, v) \in W$ one has
$$\tilde{E}_1(\mu, v)\tilde{E}_1^{-1} = (\mu, v - \iota_\mu \lambda(E_1)),$$
showing that $\tilde{E}_1$ lies in $S\Gamma(W)$ and that it lifts the orthogonal transformation defined by E_1. Since $\mathsf{N}(\tilde{E}_1) = 1$ it is an element of the spin group. The action of this factor in the spinor module is
$$\rho(\tilde{E}_1)\phi = \iota(\mathrm{e}^{\lambda(E_1)})\phi.$$
2. Similarly, the group homomorphism $\wedge^2(V^*) \hookrightarrow \mathrm{SO}(W)$ lifts to an inclusion into $\mathrm{Spin}(W) \subseteq S\Gamma(W)$ by the map
$$\begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix} \mapsto \tilde{E}_2 := \exp(\lambda(E_2)),$$
where the right-hand side is viewed as an element of $\wedge(V^*) \subseteq \mathrm{Cl}(W)$. The action of this factor in the spinor module is
$$\rho(\tilde{E}_2)\phi = \mathrm{e}^{\lambda(E_2)} \wedge \phi.$$
3. The inclusion $\mathrm{GL}(V) \subseteq \mathrm{SO}(W)$ does not have a natural lift. Let $\mathrm{GL}_\Gamma(V) \subseteq S\Gamma(W)$ denote the pre-image of $\mathrm{GL}(V)$, so that there is an exact sequence
$$1 \to \mathbb{K}^\times \to \mathrm{GL}_\Gamma(V) \to \mathrm{GL}(V) \to 1.$$
If $\tilde{Q} \in \mathrm{GL}_\Gamma(V)$ is a lift of $Q \in \mathrm{GL}(V)$, then its action in the spinor module reads as
$$\rho(\tilde{Q})\phi = \chi(\tilde{Q})\ Q.\phi.$$
Here $Q.\phi$ denotes the "standard" action of $\mathrm{GL}(V)$ on $\wedge V^*$, given by the extension of $Q \mapsto (Q^{-1})^* \in \mathrm{End}(V^*)$ to an algebra homomorphism of $\wedge(V^*)$, while $\chi :\ \mathrm{GL}_\Gamma(V) \to \mathbb{K}^\times$ is the restriction of the character $\chi :\ S\Gamma(W)_V \to \mathbb{K}^\times$ from Section 3.5. Recall its property $\chi(\tilde{Q})^2 = \mathsf{N}(\tilde{Q}) \det(Q)$. If $\tilde{Q}$ can be normalized to lie in $\mathrm{Spin}(W) \subseteq \Gamma(W)_V$, we have
$$\chi(\tilde{Q}) = \det{}^{1/2}(Q),$$
where the sign of the square root depends on the choice of the lift.

Example 4.1 Suppose V is 1-dimensional. Let $e \in V$ be a generator, and let $f \in V^*$ be the dual generator so that $B_W(e, f) = \frac{1}{2}$. The spinor module $\mathsf{S} = \wedge V^*$ has basis $\{1, f\}$. Elements $Q \in \mathrm{GL}(V)$ are non-zero scalars $r \in \mathbb{K}^\times \cong \mathrm{GL}(V)$. The possible lifts of
$$\begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix} \in \mathrm{SO}(W)$$

are given by

$$\tilde{Q} = t\big(1 - (1 - r^{-1}) f e\big) \in S\Gamma(W),$$

where $t \in \mathbb{K}^\times$. One checks $\mathsf{N}(\tilde{Q}) = r^{-1}t^2$. If r admits a square root (e.g., if $\mathbb{K} = \mathbb{C}$), one obtains two lifts $\tilde{Q} \in \mathrm{Spin}(W)$ (one for each choice of square root $t = r^{1/2}$). The action of $\tilde{Q}$ in the spin representation is given by

$$\rho(\tilde{Q})1 = t, \quad \rho(\tilde{Q}) f = \frac{t}{r} f$$

which is consistent with the formula given above.

Remark 4.2 One can also consider the spin representation of $S\Gamma(W)$ on the *dual* spinor module $\mathsf{S}^V = \wedge(V)$. Here the formulas for the action of the three factors read as

$$\rho(\tilde{E}_1)\psi = \mathrm{e}^{\lambda(E_1)} \wedge \psi, \quad \rho(\tilde{E}_2)\psi = \iota(\mathrm{e}^{\lambda(E_2)})\, \psi, \quad \rho(\tilde{Q})\psi = \frac{\chi(\tilde{Q})}{\det Q}\, Q_* \,\psi.$$

(Due to the factor $\det(Q)^{-1}$, the action of $\tilde{Q}$ on the pure spinor line $l_V = \det(V)$ is multiplication by $\chi(\tilde{Q})$.)

4.3 The quantization map revisited

Up to this point we discussed spinors only for quadratic vector spaces whose bilinear form B is *split*. By the following construction, the spinor module may be used for Clifford algebras of arbitrary symmetric bilinear forms, even degenerate ones. In particular, we can interpret the symbol map in terms of the spinor module.

4.3.1 The symbol map in terms of the spinor module

Suppose V is a vector space with a symmetric bilinear form B. Then the map

$$j : V \mapsto W := V^* \oplus V, \; v \mapsto B^\flat(v) \oplus v,$$

using the bilinear form B_W on W given by (4.1), is a partial isometry:

$$B_W(j(v_1), j(v_2)) = \frac{1}{2}\langle B^\flat(v_1), v_2\rangle + \frac{1}{2}\langle B^\flat(v_2), v_1\rangle = B(v_1, v_2).$$

Hence it extends to an injective algebra homomorphism,

$$j : \mathrm{Cl}(V) \to \mathrm{Cl}(W).$$

Using the covariant spinor module, $\mathrm{Cl}(W)$ is identified with $\mathrm{End}(\wedge(V))$.

Proposition 4.5 *The composition*

$$\mathrm{Cl}(V) \xrightarrow{j} \mathrm{Cl}(W) \xrightarrow{\rho} \mathrm{End}(\wedge(V))$$

is equal to the standard representation of $\mathrm{Cl}(V)$ *on* $\wedge(V)$. *In particular, the symbol map can be written in terms of the covariant spinor module as*

$$\sigma(x) = \rho(j(x)).1.$$

Proof The elements $j(v) = B^\flat(v) \oplus v$ act on $\wedge(V)$ as $\varepsilon(v) + \iota(B^\flat(v))$, as required. □

If B is non-degenerate, we may interpret the symbol map also in terms of the contravariant spinor module $\wedge(V^*)$. The Clifford action of $j(v)$ is now $\iota(v)+\varepsilon(B^\flat(v))$, and we again have $\sigma(x) = \rho(j(x)).1$. In fact, the isomorphism $V^* \cong V$ given by B defines an isomorphism of the two spinor modules.

4.3.2 The symbol of elements in the spin group

We will assume that the bilinear form B on V is non-degenerate. In this section we will give formulas for the image of elements in $\mathrm{Spin}(V) \subseteq \mathrm{Cl}(V)$, or more generally of elements in the special Clifford group $S\Gamma(V)$ under the symbol map. Denote by $B^\sharp :\ V^* \to V$ the inverse of $B^\flat$. Let V^- denote the vector space V with bilinear form $-B$. Then the map

$$\kappa :\ V \oplus V^- \to W,\ v_1 \oplus v_2 \mapsto B^\flat(v_1 + v_2) \oplus (v_1 - v_2)$$

is an isomorphism of quadratic vector spaces. Indeed,

$$Q_{B_W}(B^\flat(v_1 + v_2) \oplus (v_1 - v_2)) = B(v_1 + v_2, v_1 - v_2) = Q_B(v_1) - Q_B(v_2).$$

The inverse map reads as

$$\kappa^{-1}(\mu \oplus v) = \frac{B^\sharp(\mu) + v}{2} \oplus \frac{B^\sharp(\mu) - v}{2}.$$

In matrix form,

$$\kappa = \begin{pmatrix} B^\flat & B^\flat \\ I & -I \end{pmatrix}, \qquad \kappa^{-1} = \frac{1}{2}\begin{pmatrix} B^\sharp & I \\ B^\sharp & -I \end{pmatrix}.$$

Conjugation by κ gives a group isomorphism between $\mathrm{O}(V \oplus V^-)$ and $\mathrm{O}(W)$. In particular, $\mathrm{O}(V) \subseteq \mathrm{O}(V \oplus V^-)$ acts by orthogonal transformations on W. We find, for all $C \in \mathrm{O}(V)$,

$$\kappa \circ \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \circ \kappa^{-1} = \frac{1}{2}\begin{pmatrix} B^\flat(C + I)B^\sharp & B^\flat(C - I) \\ (C - I)B^\sharp & C + I \end{pmatrix}. \tag{4.8}$$

If $C+I$ is invertible, we may apply our factorization formula to this expression. We obtain

$$\kappa \circ \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \circ \kappa^{-1} = \begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} (T^{-1})^* & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix},$$

where

$$E_1 = \frac{C-I}{C+I} \circ B^\sharp, \quad E_2 = B^\flat \circ \frac{C-I}{C+I}, \quad T = \frac{C+I}{2}.$$

Remark 4.3 Note that since the right-hand side of this product is in $\mathrm{SO}(W)$, so is the left-hand side. We hence see that

$$C \in \mathrm{O}(V), \ \det(C+I) \neq 0 \ \Rightarrow \ C \in \mathrm{SO}(V)$$

for any field $\mathbb{K}$ of characteristic 0. (If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, this fact easily follows from eigenvalue considerations.)

We will now consider lifts of $C \in \mathrm{SO}(V)$ to the Clifford group $S\Gamma(V)$, and consider the action of such a lift on the spinor module. This formula will involve the elements $\lambda(E_1)$ and $\lambda(E_2)$. For any $X \in \mathfrak{o}(V)$, the composition $E_1 = X \circ B^\sharp : V^* \to V$ is skew-adjoint; hence it defines an element

$$\lambda(X \circ B^\sharp) \in \wedge^2(V) \subseteq \wedge^2(W),$$

as in Section 4.2.1. It is related to the element $\lambda(X) \in \wedge^2(V)$ (defined using B) by

$$\lambda(X \circ B^\sharp) = 2\lambda(X),$$

as follows from the explicit formula (4.6). Similarly, we have

$$\lambda(B^\flat \circ X) = 2\, B^\flat(\lambda(X)).$$

A lift $\widetilde{C} \in S\Gamma(V)$ to the spin group determines a lift

$$\widetilde{T} = \frac{\widetilde{C+I}}{2} \in \mathrm{GL}_\Gamma(V) \subseteq S\Gamma(W).$$

From the known action of the factors in the spinor module, we therefore deduce, using Proposition 4.5:

Theorem 4.1 *Let $\tilde{C} \in S\Gamma(V)$ be a lift of $C \in \mathrm{SO}(V)$. Then the action of $\tilde{C}$ on the Clifford module $\wedge V$ is given by the formula*

$$\rho(\tilde{C}) = \chi\Big(\frac{\widetilde{C+I}}{2}\Big)\, \varepsilon\big(e^{2\lambda(\frac{C-I}{C+I})}\big) \circ \Big(\frac{C+I}{2}\Big)_* \circ \iota\big(e^{2\lambda(\frac{C-I}{C+I})}\big).$$

Here $T_* = (T^{-1})^*$ denotes the action of $T \in \mathrm{End}(V)$ as an algebra homomorphism of $\wedge(V^*) \cong \wedge(V)$. If $\tilde{C} \in \mathrm{Spin}(V)$, the scalar factor may be written as

$$\chi\Big(\frac{\widetilde{C+I}}{2}\Big) = \det{}^{1/2}\Big(\frac{C+I}{2}\Big), \tag{4.9}$$

where the sign of the square root depends on the choice of the lift.

Applying this formula to $1 \in \wedge(V)$, and using that $\sigma(\tilde{C}) = \rho(\tilde{C}).1$ we find:

Proposition 4.6 *Suppose $\tilde{C} \in S\Gamma(V) \subseteq \mathrm{Cl}(V)$ is a lift of $C \in \mathrm{SO}(V)$. Then the symbol $\sigma(\tilde{C})$ is given by the formula*

$$\sigma(\tilde{C}) = \chi\Big(\widetilde{\frac{C+I}{2}}\Big)\, \mathrm{e}^{2\lambda(\frac{C-I}{C+I})}.$$

If $\tilde{C} \in \mathrm{Spin}(V)$, the scalar factor may be written as $\det^{1/2}\big(\frac{C+I}{2}\big)$.

Proposition 4.6 has the following immediate consequence:

Corollary 4.1 *Suppose $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$). The pull-back of the function*

$$\mathrm{SO}(V) \to \mathbb{K}, \quad C \mapsto \det\Big(\frac{C+I}{2}\Big)$$

to $\mathrm{Spin}(V)$ *has a unique smooth (resp. holomorphic) square root, equal to* 1 *at the group unit.*

Proof The form degree 0 part $\sigma(\tilde{C})_{[0]}$ of the symbol of $\tilde{C} \in \mathrm{Spin}(V)$ provides such a square root. □

The element

$$\psi_C = \sigma(\tilde{C}) \in \wedge(V^*)$$

is a pure spinor, since it is obtained from the pure spinor $1 \in \wedge(V^*)$ by the action of an element of the spin group. The Lagrangian subspace of W defined by $1 \in \wedge(V^*)$ is V; hence the Lagrangian subspace defined by $\psi_C = \rho(\tilde{C}).1$ is the image of V under $(\kappa \circ (C \oplus I) \circ \kappa^{-1})V$. That is,

$$F_C = \{((C-I)v,\ (C+I)v) \in W, \quad v \in V\}.$$

A Lagrangian subspace transverse to F_C is given as the image of V^* under the map $\kappa \circ (C \oplus I) \circ \kappa^{-1}$:

$$E_C = \{((C+I)v,\ (C-I)v) \in W, \quad v \in V\}.$$

Any volume element $\Gamma_\wedge \in \det(V^*)$ is a pure spinor defining V^*. Hence a pure spinor defining E_C is

$$\phi_C = \rho(\tilde{C})\Gamma_\wedge.$$

Note that $(\phi_C, \psi_C) = \Gamma_\wedge$ is non-zero, as required by Theorem 3.7.

4.3.3 Another factorization

Other types of factorizations of the matrix (4.8) defined by C lead to different formulas for symbols. We will use the following expression in which the block-diagonal part is moved all the way to the right:

Proposition 4.7 *Suppose $C \in \mathrm{SO}(V)$ with $\det(C - I) \neq 0$, and let $D: V \to V^*$ be skew-adjoint and invertible. Then*

$$\kappa \circ \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \circ \kappa^{-1}$$
$$= \begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix} \begin{pmatrix} I & D \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ E_2 & I \end{pmatrix} \begin{pmatrix} (T^{-1})^* & 0 \\ 0 & T \end{pmatrix},$$

where

$$E_1 = \frac{C+I}{C-I}\, B^\sharp - D^{-1},$$
$$E_2 = D^{-1}\Big(B^\flat\, \frac{C - C^{-1}}{4} D^{-1} - I\Big),$$
$$T = D^{-1} B^\flat\, \frac{C-I}{2}.$$

Proof The matrix product on the left-hand side of the desired equality is given by (4.8), while the right-hand side is, by direct computation,

$$\text{r.h.s.} = \begin{pmatrix} (I + DE_2)(T^{-1})^* & DT \\ (E_1 + E_1 D E_2 + E_2)(T^{-1})^* & (I + DE_1)T \end{pmatrix}.$$

The two expressions coincide if and only if E_1, E_2, T are as stated in the proposition. For instance, a comparison of the upper right corners gives $T = D^{-1}\frac{C-I}{2}$. Similarly, one finds E_1, E_2 by comparing the upper left and lower right corners. (One may verify that, with the resulting choices of E_1, E_2, T, the lower left corners match as well.) $\square$

Using the known action of the factors in the spinor module (cf. Section 4.2.3), we obtain:

Corollary 4.2 *Let $\tilde{C} \in S\Gamma(V)$ be a lift of $C \in \mathrm{SO}(V)$. For any choice of D as above, the action of $\tilde{C}$ on $\psi \in \wedge V$ is given by the formula*

$$\rho(\tilde{C})\psi = \chi(\tilde{T})\, \iota\big(\mathrm{e}^{\lambda(E_1)}\big)\, \mathrm{e}^{\lambda(D)}\, \iota\big(\mathrm{e}^{\lambda(E_2)}\big)\, T_*\psi.$$

Here $\tilde{T}$ is the lift of T determined by the lift $\tilde{C}$. In particular, taking $\psi = 1$ we obtain the following formula for the symbol of elements in the special Clifford group:

$$\sigma(\tilde{C}) = \chi(\tilde{T})\, \iota\big(\mathrm{e}^{\lambda(E_1)}\big)\, \mathrm{e}^{\lambda(D)}.$$

The choice $D = B^\flat\, \frac{C-I}{C+I}$ gives $E_1 = 0$, and we recover our first formula (Proposition 4.6) for $\sigma(\tilde{C})$. In the following section we will instead consider the case $C = \exp(A)$ with the choice $D = B^\flat\, A/2$.

4.3.4 *The symbol of elements* $\exp(\gamma(A))$

Suppose $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, and consider exponentials $C=\exp(A)$ for $A\in\mathfrak{o}(V)$. Thus

$$\det\Big(\frac{C+I}{2}\Big)=\det\big(\cosh\big(A/2\big)\big),\quad \frac{C-I}{C+I}=\tanh\big(A/2\big).$$

By Corollary 4.1 we obtain a smooth square root of the function

$$\mathfrak{o}(V)\to\mathbb{C},\ A\mapsto\det(\cosh(A/2)),$$

equal to 1 at $A=0$. There is a distinguished lift

$$\tilde{C}=\exp(\gamma(A))\in\mathrm{Spin}(V),$$

and the formula from Proposition 4.6 now reads:

$$\sigma(\mathrm{e}^{\gamma(A)})=\det{}^{1/2}(\cosh(A/2))\,\mathrm{e}^{2\lambda(\tanh(A/2))}. \tag{4.10}$$

Example 4.2 Let us verify that (4.10) is consistent with the computation from Example 3.3, in the case $V=\mathbb{C}^2$ with $\lambda(A)=-\theta/2\,e_1\wedge e_2$. We found that

$$\exp(\gamma(A))=\cos(\theta/2)-\sin(\theta/2)e_1e_2. \tag{4.11}$$

On the other hand, $A=\theta J$ where $Je_1=e_2,\ Je_2=-e_1$, hence $\exp(A/2)=\cos(\theta/2)+\sin(\theta/2)J$. It follows that

$$\cosh(A/2)=\cos(\theta/2)\mathrm{id},$$
$$\tanh(A/2)=\tan(\theta/2)J.$$

This yields $\lambda(\tanh(A/2))=-\frac{1}{2}\tan(\theta/2)e_1\wedge e_2$ and

$$\det{}^{1/2}(\cosh(A/2))=\cos(\theta/2),\quad \mathrm{e}^{2\lambda(\tanh(A/2))}=1-\tan(\theta/2)e_1\wedge e_2.$$

Their product is indeed the symbol of (4.11).

4.3.5 *Clifford exponentials versus exterior algebra exponentials*

We first assume $\mathbb{K}=\mathbb{C}$. Let $A\in\mathfrak{o}(V)$, and consider the formula from Section 4.3.3 for $C=\exp A,\ D=B^\flat\circ\frac{A}{2}$. We obtain

$$E_1=2f(A)\circ B^\sharp,\quad E_2=2B^\flat\circ g(A),\quad T=j^R(A)$$

with the following functions of $z\in\mathbb{C}$,

$$f(z)=\frac{1}{2}\coth\Big(\frac{z}{2}\Big)-\frac{1}{z},\quad g(z)=\frac{\sinh(z)-z}{z^2},\quad j^R(z)=\frac{\mathrm{e}^z-1}{z}.$$

For later use we also define

$$j(z)=\frac{\sinh(z/2)}{z/2},\quad j^L(z)=\frac{1-\mathrm{e}^{-z}}{z}.$$

Note that $g,\ j^L,\ j^R$, and j are entire holomorphic functions on $\mathbb{C}$, while f is meromorphic with poles at $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. Since f and g are odd functions, the elements $f(A)$ and $g(A)$ are again in $\mathfrak{o}(V)$, while $j(A)^\top = j(A)$ and $j^L(A)^\top = j^R(A)$. Furthermore, $j^L(A), j^R(A)$, and $j(A)$ are invertible if and only if A has no eigenvalues of the form $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. The resulting formula for the symbol gives:

Theorem 4.2 *Suppose* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *For all* $A \in \mathfrak{o}(V)$ *with the property that* A *has no eigenvalue* $2\pi\sqrt{-1}k$ *with* $k \neq 0$, *the symbol of* $\exp(\gamma(A)) \in \mathrm{Cl}(V)$ *is given by the formula*

$$\sigma(\exp(\gamma(A)) = \iota(\mathscr{S}(A)) \exp(\lambda(A)),$$

where $\mathscr{S}: \mathfrak{o}(V) \to \wedge(V)$ *is the map*

$$\mathscr{S}(A) = \det{}^{1/2}(j(A))\ \exp\big(4\lambda(f(A))\big).$$

Once again, while Proposition 4.7 requires that A be invertible (which never happens if $\dim V$ is odd), the resulting formula holds without this assumption. In case A is invertible we can directly write

$$\mathscr{S}(A) = \det{}^{1/2}\Big(\frac{\sinh(A/2)}{A/2}\Big)\ \mathrm{e}^{4\lambda(\frac{1}{2}\coth(\frac{A}{2}) - \frac{1}{A})}\ .$$

As it stands, $\mathscr{S}: \mathfrak{o}(V) \to \wedge(V)$ is a meromorphic function, holomorphic on the set whose elements A do not have eigenvalues of the form $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. We will see below that it is in fact holomorphic everywhere.

Example 4.3 We continue the calculations from Example 4.2, where

$$\exp(\gamma(A)) = \cos(\theta/2) - \sin(\theta/2) e_1 e_2.$$

We have $\frac{\sinh(A/2)}{A/2} = \frac{\sin(\theta/2)}{\theta/2} I$, hence

$$\det{}^{1/2}\Big(\frac{\sinh(A/2)}{A/2}\Big) = \frac{\sin(\theta/2)}{\theta/2}.$$

On the other hand,

$$f(A) = \frac{1}{2}\coth\frac{A}{2} - \frac{1}{A} = -\Big(\frac{1}{2}\cot\Big(\frac{\theta}{2}\Big) - \frac{1}{\theta}\Big) J,$$

so

$$4\lambda(f(A)) = \Big(\cot\Big(\frac{\theta}{2}\Big) - \frac{2}{\theta}\Big) e_1 \wedge e_2.$$

Hence

$$\mathscr{S}(A) = \frac{\sin(\theta/2)}{\theta/2}\Big(1 + \Big(\cot\Big(\frac{\theta}{2}\Big) - \frac{2}{\theta}\Big) e_1 \wedge e_2\Big).$$

Note that $\mathscr{S}(A)$ has no poles. Contracting with $\exp(\lambda(A)) = 1 - \frac{\theta}{2} e_1 \wedge e_2$, we find

$$\iota(\mathscr{S}(A)) \exp(\lambda(A)) = \cos\Big(\frac{\theta}{2}\Big) - \sin\Big(\frac{\theta}{2}\Big) e_1 \wedge e_2 = \sigma(\exp(\gamma(A))),$$

as desired.

4.3.6 The symbol of elements $\exp(\gamma(A) - \sum_i e_i \tau^i)$

Theorem 4.2 has a useful generalization, allowing for linear terms. Let P be a vector space of odd parameters. Let e_i be a basis of V, and consider expressions $e_i \otimes \tau^i \in V \otimes P$ with $\tau^i \in P$. Note that we can view the parameters τ^i as the components $\tau(e^i)$ of a linear map $\tau: V \to P$. The following theorem shows that the same element $\mathscr{S}(A) \in \wedge(V)$ as before relates the exponentials of elements

$$\lambda(A) - \sum_i e_i \tau^i \in \wedge(V) \otimes \wedge(P),$$

$$\gamma(A) - \sum_i e_i \tau^i \in \mathrm{Cl}(V) \otimes \wedge(P).$$

Quite remarkably, it is not necessary to introduce a τ-dependence into the definition of $\mathscr{S}(A)$.

Theorem 4.3 *Let* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *With* $\mathscr{S}(A)$ *as above,*

$$\mathrm{e}^{\gamma(A) - \sum_i e_i \tau^i} = q\left(\iota(\mathscr{S}(A))\mathrm{e}^{\lambda(A) - \sum_i e_i \tau^i}\right).$$

Proof Think of $\mathrm{Cl}(V) \otimes \wedge(P) = \mathrm{Cl}(V \oplus P)$ as the Clifford algebra for the degenerate bilinear form $B \oplus 0$. Pick an arbitrary non-degenerate symmetric bilinear form B_P on P, and consider the bilinear form $B \oplus \varepsilon B_P$ on $V \oplus P$. Then $\lambda(A) - \sum_i e_i \tau^i = \lambda(\tilde{A}_\varepsilon)$ with

$$\tilde{A}_\varepsilon = \begin{pmatrix} A & -2\varepsilon\tau^\top \\ 2\tau & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 2\tau & 0 \end{pmatrix} + O(\varepsilon),$$

where $O(\varepsilon)$ denotes a term that goes to 0 for $\varepsilon \to 0$. By induction, it follows that the powers of $\tilde{A}_\varepsilon$ have the form

$$\tilde{A}_\varepsilon^m = \begin{pmatrix} A^m & 0 \\ 2\tau A^{m-1} & 0 \end{pmatrix} + O(\varepsilon).$$

Hence $f(\tilde{A}_\varepsilon) \in \mathrm{End}(V \oplus P)$ is given by

$$f(\tilde{A}_\varepsilon) = \begin{pmatrix} f(A) & 0 \\ Q & 0 \end{pmatrix} + O(\varepsilon),$$

with $Q = \tau f(A) A^{-1}$. The skew-adjoint map $V \oplus P \to V^* \oplus P^*$ defined by $\lambda(f(\tilde{A}_\varepsilon)) \in \wedge^2(V^* \oplus P^*)$ is the composition

$$\begin{pmatrix} B_V^\flat & 0 \\ 0 & \varepsilon B_P^\flat \end{pmatrix} \circ f(\tilde{A}_\varepsilon) = \begin{pmatrix} B_V^\flat \circ f(A) & 0 \\ 0 & 0 \end{pmatrix} + O(\varepsilon).$$

This shows that

$$\lambda(f(\tilde{A}_\varepsilon)) = \lambda(f(A)) + O(\varepsilon).$$

Similarly,

$$\det(j(\tilde{A}_\varepsilon)) = \det(j(A)) + O(\varepsilon),$$

since only the block diagonal term contributes. Letting $\varepsilon \to 0$ in our general formula

$$\exp(\gamma(\tilde{A}_\varepsilon)) = \iota(\mathscr{S}(\tilde{A}_\varepsilon))\exp(\lambda(\tilde{A}_\varepsilon))$$

proves the theorem. □

Example 4.4 In the 2-dimensional setting of Examples 4.2 and 4.3, consider exponentials of the form

$$\begin{aligned}\exp\left(\gamma(A) - \sum_{i=1}^{2} e_i\tau^i\right) &= \exp\left(-\frac{\theta}{2}e_1e_2 - \sum_{i=1}^{2} e_i\tau^i\right)\\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{\theta}{2}e_1e_2 + \sum_{i=1}^{2} e_i\tau^i\right)^m.\end{aligned}$$

Using that e_1e_2 anti-commutes with $\sum_i e_i\tau^i$, the sum becomes

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{\theta}{2}\right)^m(e_1e_2)^m + \left(\sum_{i=1}^{2} e_i\tau^i\right)\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}a_m\left(\frac{\theta}{2}e_1e_2\right)^{m-1}\\ &+\left(\sum_{i=1}^{2} e_i\tau^i\right)^2\sum_{m=2}^{\infty}\frac{(-1)^m}{m!}b_m\left(\frac{\theta}{2}e_1e_2\right)^{m-2},\end{aligned}$$

where $a_m = \sum_{j=0}^{m-1}(-1)^j = \frac{1}{2}(1+(-1)^m)$ is 1 if m is odd, 0 otherwise, while

$$b_m = \sum_{j,j'\geq 0,\, j+j'\leq m-2}(-1)^{j+j'} = \frac{1}{2}\Big((-1)^m m + \frac{1}{2}(1-(-1)^m)\Big).$$

With $(e_1e_2)^{2k} = (-1)^k$, the result can as before be expressed in terms of trigonometric functions:

$$\begin{aligned}\exp\left(\gamma(A) - \sum_{i=1}^{2} e_i\tau^i\right) &= \cos(\theta/2) - \sin(\theta/2)e_1e_2 - \frac{\sin(\theta/2)}{\theta/2}\sum_{i=1}^{2} e_i\tau^i\\ &+\left(\frac{\sin(\theta/2)}{(\theta/2)^2} - \frac{\cos(\theta/2)}{\theta/2}\right)\tau_1\tau_2 - \frac{\sin(\theta/2)}{\theta/2}e_1e_2\tau_1\tau_2.\end{aligned}$$

It is straightforward to verify that this formula coincides with $q\circ\iota(\mathscr{S}(A))$ applied to

$$\exp\left(\lambda(A) - \sum_{i=1}^{2} e_i\wedge\tau^i\right) = 1 - \frac{\theta}{2}e_1\wedge e_2 - \sum_{i=1}^{2} e_i\tau^i + e_1\wedge e_2\wedge\tau_1\wedge\tau_2.$$

4.3.7 The function $A \mapsto \mathscr{S}(A)$

Until now the function $\mathscr{S}$ was defined on the set of $A \in \mathfrak{o}(V)$ such that A has no non-zero eigenvalues in $2\pi\sqrt{-1}\mathbb{Z}$. We now show that the function $\mathscr{S}$ extends to an analytic function on all of $\mathfrak{o}(V)$. In particular, the formulas established above hold on all of $\mathfrak{o}(V)$.

Theorem 4.4 *Let $\mathbb{K} = R$ or $\mathbb{C}$. The function $A \mapsto \mathscr{S}(A)$ extends to an analytic function $\mathfrak{o}(V) \to \wedge(V)$. In particular, its degree zero part*

$$A \mapsto \det{}^{1/2}(j(A)) = \det{}^{1/2}\left(\frac{\sinh(A/2)}{A/2}\right)$$

is a well-defined analytic function $\mathfrak{o}(V) \to \mathbb{K}$.

Proof Let P, τ^i be as in Theorem 4.3. We assume that the τ^i are a basis of the parameter space P (e.g., we may take $P = V^*$, with τ^i the dual basis of e_i). Then $\exp(\lambda(A) - \sum_i e_i \tau^i)$ has a non-vanishing part of top degree $2\dim V$. By the lemma below, there exists an analytic function $\mathscr{S}' : \mathfrak{o}(V) \to \wedge^2(V \oplus P^*)$ satisfying

$$\iota\big(\mathscr{S}'(A)\big)\exp\Big(\lambda(A) - \sum_i e_i \tau^i\Big) = q^{-1}\Big(\exp\Big(\gamma(A) - \sum_i e_i \tau^i\Big)\Big).$$

By uniqueness, this function coincides with the function $\mathscr{S}(A)$ defined above. (Thus, it actually takes values in $\wedge(V)$.) $\square$

Lemma 4.1 *Let E be a vector space, $\chi, \psi \in \wedge(E)$, and suppose that the top degree part $\chi_{[\mathrm{top}]} \in \det(E)$ is non-zero. Then there is a unique solution $\phi \in \wedge(E^*)$ of the equation*

$$\psi = \iota(\phi)\chi.$$

If χ and ψ depend analytically on parameters, then so does the solution ϕ.

Proof Fix a generator $\Gamma_\wedge \in \det(E^*)$. Then the desired equation $\psi = \iota(\phi)\chi$ is equivalent to

$$\iota(\psi)\Gamma_\wedge = \phi \wedge \iota(\chi)\Gamma_\wedge.$$

Since $\chi_{[\dim E]} \neq 0$, we have $(\iota(\chi)\Gamma_\wedge)_{[0]} \neq 0$, i.e., $\iota(\chi)\Gamma_\wedge$ is invertible. Thus

$$\phi = (\iota(\psi)\Gamma_\wedge) \wedge (\iota(\chi)\Gamma_\wedge)^{-1}.$$

This shows existence and uniqueness, and also implies the statements regarding dependence on parameters. $\square$

4.4 Volume forms on conjugacy classes

As an application of some of the techniques developed here, we will prove the following fact:

the conjugacy classes of any connected, simply connected semisimple (real or complex) Lie group have distinguished volume forms.

In fact, it suffices to assume that the Lie algebra of G has an invariant non-degenerate symmetric bilinear form B—in the semisimple case this can be taken to be the Killing form. Also, the assumption that G is connected and simply connected can be relaxed (see below).

Let G be a real or complex Lie group, with a non-degenerate G-invariant symmetric bilinear form B on its Lie algebra $\mathfrak{g}$. Let $\theta^L,\ \theta^R \in \Omega^1(G,\mathfrak{g})$ be the (left-invariant, right-invariant) Maurer–Cartan forms. That is, if $\xi^L,\ \xi^R \in \mathfrak{X}(G)$ are the left-, right-invariant vector fields on G, equal to ξ at the group unit, then $\iota(\xi^L)\theta^L = \xi = \iota(\xi^R)\theta^R$. For $\xi \in \mathfrak{g}$ define two sections of the bundle $T^*G \oplus TG$:

$$\begin{aligned} e(\xi) &= B(\theta^L + \theta^R, \xi) \oplus (\xi^L - \xi^R),\\ f(\xi) &= B(\theta^L - \theta^R, \xi) \oplus (\xi^L + \xi^R). \end{aligned}$$

Proposition 4.8 *The sections $e(\xi)$ and $f(\xi)$ for $\xi \in \mathfrak{g}$ span transverse Lagrangian subbundles $E, F \subseteq T^*G \oplus TG$, respectively. One has*

$$\langle e(\xi_1), e(\xi_2)\rangle = 0, \quad \langle f(\xi_1), f(\xi_2)\rangle = 0, \quad \langle e(\xi_1), f(\xi_2)\rangle = 2B(\xi_1, \xi_2).$$

Proof Use left-trivialization of the tangent bundle to identify $TG = G \times \mathfrak{g}$ and $T^*G = G \times \mathfrak{g}^*$. The left-trivialization takes $B(\theta^L, \xi),\ \xi^L$ to the constant sections $g \mapsto B^\flat(\xi),\ \xi$ and $B(\theta^R, \xi), \xi^R$ to the sections $g \mapsto B^\flat(\mathrm{Ad}(g^{-1})\xi),\ \mathrm{Ad}(g^{-1})\xi$. Thus, in terms of left-trivialization,

$$\begin{aligned} e(\xi)|_g &= B^\flat((1 + \mathrm{Ad}(g^{-1}))\xi) \oplus (1 - \mathrm{Ad}(g^{-1})\xi),\\ f(\xi)|_g &= B^\flat((1 - \mathrm{Ad}(g^{-1}))\xi) \oplus (1 + \mathrm{Ad}(g^{-1})\xi). \end{aligned}$$

Let $\overline{TG}$ denote TG with the opposite bilinear form, and define the bundle map

$$\kappa:\ TG \oplus \overline{TG} \to T^*G \oplus TG,\ v_1 \oplus v_2 \mapsto B^\flat(v_1 + v_2) \oplus (v_1 - v_2)$$

as in Section 4.3.2. This is an isometry, and we observe that

$$\kappa^{-1}(e(\xi)) = \xi \oplus \mathrm{Ad}(g^{-1})\xi,\ \ \kappa^{-1}(f(\xi)) = \xi \oplus (-\mathrm{Ad}(g^{-1})\xi). \tag{4.12}$$

That is, $\kappa^{-1}(E)|_g$ is the graph of the isometry $\mathrm{Ad}(g^{-1})$, while $\kappa^{-1}(F)|_g$ is the graph of the isometry $-\mathrm{Ad}(g^{-1})$. These are transverse Lagrangian subspaces, hence so is their image under κ. The formulas for the inner products are immediate from (4.12). $\square$

The TG-components of the sections $e(\xi) \in \Gamma(T^*G \oplus TG)$ are the generating vector fields $\xi_G = \xi^L - \xi^R \in \mathfrak{X}(G)$ for the conjugation action on G. Suppose $\mathscr{C} \subseteq G$ is a conjugacy class, i.e., an orbit of the conjugation action, and define the subbundle $E_{\mathscr{C}} \subseteq T\mathscr{C} \oplus T^*\mathscr{C}$ to be the span of the sections

$$e_{\mathscr{C}}(\xi) = \iota_{\mathscr{C}}^* B(\theta^L + \theta^R, \xi) \oplus \xi_{\mathscr{C}}, \quad \xi \in \mathfrak{g},$$

where $\iota_{\mathscr{C}} : \mathscr{C} \hookrightarrow G$ is the inclusion. Since $\langle e_{\mathscr{C}}(\xi_1), e_{\mathscr{C}}(\xi_2)\rangle = \langle e(\xi_1), e(\xi_2)\rangle|_{\mathscr{C}} = 0$ for all $\xi_1, \xi_2 \in \mathfrak{g}$, the subbundle $E_{\mathscr{C}}$ is isotropic. For dimension reasons, it is in fact Lagrangian. Since the projection of $E_{\mathscr{C}}$ to $T\mathscr{C}$ is a bijection, $E_{\mathscr{C}}$ is the graph of a 2-form $-\omega_{\mathscr{C}}$, where

$$\iota(\xi_{\mathscr{C}})\omega_{\mathscr{C}} = -i_{\mathscr{C}}^* B(\theta^L + \theta^R, \xi).$$

Explicitly,

$$\omega_{\mathscr{C}}(\xi_{\mathscr{C}}|_g, \xi'_{\mathscr{C}}|_g) = B(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}\xi, \xi').$$

In terms of the contravariant spinor module $\wedge T^*\mathscr{C}$ over $\mathrm{Cl}(T^*\mathscr{C} \oplus T\mathscr{C})$, it follows that $E_{\mathscr{C}}$ is the Lagrangian subbundle spanned by the pure spinor $\phi_{\mathscr{C}} = \exp(-\omega_{\mathscr{C}}) \in \Omega(\mathscr{C})$.

By definition of $E_{\mathscr{C}}$, we have $E|_{\iota(g)} = (T_g\iota_{\mathscr{C}})_! E_{\mathscr{C}}|_g$ for all $g \in \mathscr{C}$, in the notation of Section 4.1. By Proposition 4.3, the Lagrangian subbundle $F_{\mathscr{C}} \subseteq T^*\mathscr{C} \oplus T\mathscr{C}$ given by $F_{\mathscr{C}}|_g = (T_g\iota_{\mathscr{C}})^! F|_{\iota(g)}$ is transverse to $E_{\mathscr{C}}$. Suppose we are given a pure spinor $\psi \in \Omega(G)$ defining $F_{\mathscr{C}}$. Then the pull-back $\psi_{\mathscr{C}} = \iota_{\mathscr{C}}^*\psi$ is a pure spinor defining $F_{\mathscr{C}}$. By Theorem 3.7, the transversality of $E_{\mathscr{C}}$, $F_{\mathscr{C}}$ is equivalent to the non-degeneracy of the pairing between $\phi_{\mathscr{C}}$ and $\psi_{\mathscr{C}}$. That is, $(\phi_{\mathscr{C}}, \psi_{\mathscr{C}}) = \exp(\omega_{\mathscr{C}}) \wedge \iota_{\mathscr{C}}^*\psi$ is a volume form on $\mathscr{C}$.

We will give a construction of ψ using the spin representation. This will require an additional ingredient: We assume that we are given a lift

$$\widetilde{\mathrm{Ad}}: \ G \to \mathrm{Pin}(\mathfrak{g})$$

of the adjoint action $\mathrm{Ad}: \ G \to \mathrm{O}(\mathfrak{g})$. If G is connected and simply connected, the lift is automatic.

Recall the description of F given in the proof of Proposition 4.8, presenting F as the image of $\{(\mathrm{Ad}(g)\xi, -\xi)|\ \xi \in \mathfrak{g}\} \subseteq T\mathscr{C} \oplus \overline{T\mathscr{C}}$ under the isometry κ. The image of the anti-diagonal $\{(\xi, -\xi)|\ \xi \in \mathfrak{g}\}$ under κ is T^*G, which is defined by the pure spinor $1 \in \Gamma(\wedge T^*G) = \Omega(G)$. Thus

$$\psi_g = \rho(\widetilde{\mathrm{Ad}}(g)).1$$

is a pure spinor representing $F|_g$. By Proposition 4.5, this is just the symbol of the element $\widetilde{\mathrm{Ad}}(g) \in \mathrm{Pin}(\mathfrak{g}) \subseteq \mathrm{Cl}(\mathfrak{g})$:

$$\psi_g = \sigma(\widetilde{\mathrm{Ad}}(g)).$$

Note that ψ is invariant under the adjoint action of G on itself. To summarize:

Theorem 4.5 (Volume forms on conjugacy classes [13, 101]) *Let G be a real or complex Lie group, whose Lie algebra comes equipped with an invariant non-degenerate symmetric bilinear form B. Assume that the adjoint action lifts to a Lie group morphism $G \to \mathrm{Pin}(\mathfrak{g})$, and let $\psi \in \Omega(G)$ be the resulting pure spinor. For any conjugacy class $\mathscr{C} \subseteq G$, let $\omega_{\mathscr{C}}$ be the 2-form defined above. Then the top degree part of the wedge product*

$$\exp(\omega_{\mathscr{C}}) \wedge i_{\mathscr{C}}^*\psi$$

defines an $\mathrm{Ad}(G)$-invariant volume form on $\mathscr{C}$.

Proposition 4.6 (applied to $C = \mathrm{Ad}(g)$) gives an explicit formula for ψ. Left-trivialization of TG identifies the map $g \mapsto \lambda(\frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}) \in \wedge^2 \mathfrak{g}^*$ with a 2-form on G, defined over the subset of G where Ad_g has no eigenvalue equal to -1. In terms of Maurer–Cartan forms the 2-form reads as

$$-\frac{1}{4} B\left(\frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}\theta^L, \theta^L\right).$$

We obtain:

Proposition 4.9 *Over the set of $g \in G$ where Ad_g has no eigenvalue equal to -1, the pure spinor $\psi \in \Omega(G)$ is given by the formula*

$$\psi = \det{}^{1/2}\left(\frac{\mathrm{Ad}_g + 1}{2}\right) \exp\left(-\frac{1}{4} B\left(\frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}\theta^L, \theta^L\right)\right). \tag{4.13}$$

Here the sign of the square root depends on the choice of lift $\widetilde{\mathrm{Ad}}$. If G is connected, the set of $g \in G$ such that $\det(\mathrm{Ad}_g + I) = 0$ is open and dense. On the other hand, if G is disconnected, then there may be connected components for which $\det(\mathrm{Ad}_g) = -1$. For any such component, Ad_g always has -1 as an eigenvalue.

Let us make a few comments on the volume form on $\mathscr{C}$.

Remarks 4.4

1. Theorem 4.5 shows in particular that, under the given assumptions, all conjugacy classes in G have a natural orientation. The simplest example of a non-orientable conjugacy class of a non-simply connected group is $\mathscr{C} = \mathbb{R}P(2) \subseteq \mathrm{SO}(3)$ (the conjugacy class of rotations by π).
2. If G is connected, the map to $\mathrm{Pin}(\mathfrak{g})$ necessarily takes values in $\mathrm{Spin}(\mathfrak{g})$, and hence the resulting form ψ is *even*. According to the theorem, the conjugacy classes in G must all be even-dimensional. The simplest example of an odd-dimensional conjugacy class of a disconnected Lie group is $\mathscr{C} = S^1 \subseteq \mathrm{O}(2)$, the conjugacy class of 2-dimensional reflections.
3. The adjoint action always lifts after passage to a double cover $\tilde{G}$. The volume forms on the conjugacy classes in $\tilde{G}$ determine in particular invariant measures, and these descend to the conjugacy classes in G. Thus, given the invariant metric B on $\mathfrak{g}$, there are distinguished invariant measures on all the conjugacy classes of G. Note that conjugacy classes in a general Lie group G need not admit invariant measures. An example is the group G generated by translations and dilations of the real line $\mathbb{R}$. The generic conjugacy classes are diffeomorphic to $\mathbb{R}$ with this action, and hence do not admit invariant measures. In this case, $\mathfrak{g}$ does not admit an invariant metric.
4. If G is semisimple, there is a distinguished B given by the Killing form. Hence in that case the volume forms on the conjugacy classes of G (assuming, e.g., that G is simply connected) are completely canonical.
5. The volume forms on conjugacy classes are analogous to the Liouville volume forms on coadjoint orbits $\mathscr{O} \subseteq \mathfrak{g}^*$. The latter are given by $(\exp \omega_{\mathscr{O}})_{[\mathrm{top}]}$, using

the Kirillov–Kostant–Souriau symplectic structure $\omega_{\mathscr{O}}$. Suppose G is a compact, real Lie group, with maximal torus T and positive Weyl chamber $\mathfrak{t}^*_+$. Then the integral of the Liouville volume form of a coadjoint orbit $\mathscr{O} = G.\mu = G/G_\mu$ for $\mu \in \mathfrak{t}^*_+$ is given by

$$\operatorname{vol}(G.\mu) = c \prod_\alpha \pi \langle \alpha, B^\sharp \mu \rangle,$$

where the constant c depends only on the face of the Weyl chamber $\mathfrak{t}^*_+$ which contains μ, and the product is over all positive roots α such that $\langle \alpha, B^\sharp \mu \rangle \neq 0$. Similarly, the volume of a conjugacy class $\mathscr{C} = G.\exp\xi = G/G_{\exp\xi}$, for ξ in the alcove, is given by an expression

$$\operatorname{vol}(G.\exp\xi) = c' \prod_\alpha \sin \pi \langle \alpha, \xi \rangle,$$

where c' depends only on the open face of the alcove which contains ξ, and the product is over all positive roots α such that $\langle \alpha, \xi \rangle \notin \mathbb{Z}$. See [12] for more details.

For further developments of the theory outlined here, see [13].

Chapter 5
Enveloping algebras

Enveloping algebras define a functor $\mathfrak{g} \mapsto U(\mathfrak{g})$ from the category of Lie algebras to the category of associative unital algebras in such a way that representations of $\mathfrak{g}$ on vector spaces V are equivalent to algebra representations of $U(\mathfrak{g})$ on V. A fundamental result in the theory of enveloping algebras is the Poincaré–Birkhoff–Witt Theorem, which (in one of its incarnations) states that a natural "quantization map" from the symmetric algebra $S(\mathfrak{g})$ into $U(\mathfrak{g})$ is an isomorphism of vector spaces. One of the goals of this chapter is to present a proof of this result, due to E. Petracci, which is similar in spirit to the proof that the quantization map for Clifford algebras is an isomorphism. Throughout this chapter $\mathbb{K}$ denotes a field of characteristic 0. The vector spaces E and Lie algebras $\mathfrak{g}$ considered in this chapter may be infinite-dimensional unless stated otherwise.

5.1 The universal enveloping algebra

5.1.1 Construction

For any Lie algebra $\mathfrak{g}$, one defines the *universal enveloping algebra* $U(\mathfrak{g}) = T(\mathfrak{g})/\mathscr{I}$ as the quotient of the tensor algebra by the two-sided ideal $\mathscr{I}$ generated by elements of the form

$$\xi \otimes \zeta - \zeta \otimes \xi - [\xi, \zeta].$$

Equivalently, the universal enveloping algebra is generated by elements $\xi \in \mathfrak{g}$ subject to relations $\xi\zeta - \zeta\xi = [\xi, \zeta]$. Since $\mathscr{I}(\mathfrak{g})$ is a filtered ideal in $T(\mathfrak{g})$, with $\mathscr{I} \cap \mathbb{K} = 0$, it follows that $U(\mathfrak{g})$ is a filtered algebra. The construction of the enveloping algebra $U(\mathfrak{g})$ from a Lie algebra $\mathfrak{g}$ is functorial: Any Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$ induces a morphisms of filtered algebras $U(\mathfrak{g}_1) \to U(\mathfrak{g}_2)$, with the appropriate property under composition.

The inclusion $\mathfrak{g} \to T(\mathfrak{g})$ descends to a Lie algebra homomorphism

$$j\colon\ \mathfrak{g} \to U(\mathfrak{g}),$$

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where the bracket on $U(\mathfrak{g})$ is the commutator. As a consequence of the Poincaré–Birkhoff–Witt Theorem, to be discussed below, this map is injective. We will usually denote the image $j(\xi)$ in the enveloping algebra simply by ξ, although strictly speaking this is only justified once the Poincaré–Birkhoff–Witt Theorem is proved.

5.1.2 Universal property

Theorem 5.1 (Universal property) *If $\mathscr{A}$ is an associative algebra, and $f:\ \mathfrak{g} \to \mathscr{A}$ is a homomorphism of Lie algebras, then there is a unique morphism of algebras $f_U:\ U(\mathfrak{g}) \to \mathscr{A}$ such that $f = f_U \circ j$.*

Proof The map f extends to an algebra homomorphism $T(\mathfrak{g}) \to \mathscr{A}$. This algebra homomorphism vanishes on the ideal $\mathscr{I}$, and hence descends to an algebra homomorphism $f_U:\ U(\mathfrak{g}) \to \mathscr{A}$ with the desired property. This extension is unique, since $j(\mathfrak{g})$ generates $U(\mathfrak{g})$ as an algebra. □

By the universal property, any module over the Lie algebra $\mathfrak{g}$ becomes a module over the algebra $U(\mathfrak{g})$.

Remark 5.1 If $\dim \mathfrak{g} < \infty$, the injectivity of the map $j:\ \mathfrak{g} \to U(\mathfrak{g})$ can also be obtained as a consequence of *Ado's Theorem*, stating that any such Lie algebra has a faithful finite-dimensional representation $f:\ \mathfrak{g} \to \mathrm{End}(V)$. The faithfulness means that f is injective, and since $f = f_U \circ j$ it follows that j is injective too.

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two Lie algebras one has an isomorphism of filtered algebras

$$U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2);$$

indeed the tensor product on the right satisfies the universal property of the enveloping algebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. A detailed proof may be found, e.g., in [76, Chapter V.2].

5.1.3 Augmentation map, anti-automorphism

The projection $\mathfrak{g} \to 0$ is a Lie algebra morphism. By the universal property it induces an algebra morphism $U(\mathfrak{g}) \to \mathbb{K}$. This morphism is referred to as *augmentation*. Its kernel is called the augmentation ideal, and is denoted by $U^+(\mathfrak{g})$. By contrast, Clifford algebras do not, in general, admit augmentation maps that are (super) algebra homomorphisms.

The map $\xi \mapsto -\xi$ is an anti-automorphism of the Lie algebra $\mathfrak{g}$, i.e., it preserves the bracket up to a sign. Define an algebra anti-automorphism of $T(\mathfrak{g})$ by

$$\xi_1 \otimes \cdots \otimes \xi_r \mapsto (-1)^r \xi_r \otimes \cdots \otimes \xi_1.$$

This preserves the ideal $\mathscr{I}$, and therefore descends to an anti-automorphism of $U(\mathfrak{g})$, denoted by s. That is,

$$\mathsf{s}(\xi_1 \cdots \xi_k) = (-1)^k \xi_k \cdots \xi_1.$$

Given a Lie algebra representation $\pi : \mathfrak{g} \to \mathrm{End}(V)$, with dual representation $\pi^* : \mathfrak{g} \to \mathrm{End}(V^*)$, $\pi^*(\xi) = -\pi(\xi)^*$, the corresponding algebra representations of $a \in U(\mathfrak{g})$ are related by

$$\pi^*(a) = \pi(\mathsf{s}(a))^*.$$

5.1.4 Derivations

The functoriality of the construction of $U(\mathfrak{g})$ shows in particular that any Lie algebra automorphism of $\mathfrak{g}$ extends uniquely to an algebra automorphism of $U(\mathfrak{g})$. Similarly, if D is a Lie algebra derivation of $\mathfrak{g}$, then its extension to a derivation of the tensor algebra $T(\mathfrak{g})$ preserves the ideal $\mathscr{I}$. Hence it descends to an algebra derivation of $U(\mathfrak{g})$, given by

$$D(\xi_1 \cdots \xi_k) = \sum_{i=1}^{k} \xi_1 \cdots \xi_{i-1}(D\xi_i)\xi_{i+1} \cdots \xi_k.$$

5.1.5 Modules over $U(\mathfrak{g})$

A module over $U(\mathfrak{g})$ is a vector space E together with an algebra homomorphism $U(\mathfrak{g}) \to \mathrm{End}(E)$. The universal property of the enveloping algebra shows that such a module structure is equivalent to a Lie algebra representation of $\mathfrak{g}$ on E. The left-action of the enveloping algebra on itself corresponds to the *left-regular representation* $\rho^L(\xi)x = \xi x$. There is also a *right-regular representation* $\rho^R(\xi)x = -x\xi$. The two actions commute, and the diagonal action is the *adjoint representation* $\mathrm{ad}(\xi)x = \xi x - x\xi = [\xi, x]$. An element x lies in the center of $U(\mathfrak{g})$ if and only if it commutes with all generators ξ. That is, it consists exactly of the invariants for the adjoint action:

$$\mathrm{Cent}(U(\mathfrak{g})) = U(\mathfrak{g})^{\mathfrak{g}}.$$

5.1.6 Unitary representations

Suppose $\mathfrak{g}$ is a real Lie algebra, and let $\mathfrak{g}^{\mathbb{C}}$ be its complexification. The enveloping algebra $U(\mathfrak{g}^{\mathbb{C}})$ admits a unique conjugate linear automorphism $x \mapsto x^c$ extending the complex conjugation map on $\mathfrak{g}^{\mathbb{C}}$. Define a conjugate linear anti-automorphism

$$* : U(\mathfrak{g}^{\mathbb{C}}) \to U(\mathfrak{g}^{\mathbb{C}}), \quad a \mapsto a^* = \mathsf{s}(a^c);$$

thus $\xi^* = -\xi^c$ for generators $\xi \in \mathfrak{g}$. A unitary representation of $\mathfrak{g}^{\mathbb{C}}$ on a Hermitian vector space E is a Lie algebra homomorphism $\pi : \mathfrak{g}^{\mathbb{C}} \to \mathrm{End}(E)$ such that the elements of $\mathfrak{g} \subseteq \mathfrak{g}^{\mathbb{C}}$ are represented as skew-adjoint operators. Equivalently, it is a $*$-homomorphism $\pi : U(\mathfrak{g}^{\mathbb{C}}) \to \mathrm{End}(E)$, i.e., $\pi(a)^* = \pi(a^*)$ for all $a \in U(\mathfrak{g}^{\mathbb{C}})$.

5.1.7 Graded or filtered Lie algebras and super Lie algebras

If $\mathfrak{g}$ is a graded (resp. filtered) Lie algebra, then the tensor algebra $T(\mathfrak{g})$ is a graded (resp. filtered) algebra, in such a way that the inclusion of $\mathfrak{g}$ is a morphism. Furthermore, the ideal $\mathscr{I}$ defining the enveloping algebra is a graded (resp. filtered) subspace, and hence the enveloping algebra $U(\mathfrak{g})$ inherits a grading (resp. filtration). By construction, this *internal grading* (resp. *internal filtration*) has the property that the inclusion $j : \mathfrak{g} \to U(\mathfrak{g})$ preserves degrees. The filtration from the construction in Section 5.1.1 will be called the *external filtration*. The *total filtration* degree is the sum of the internal and external filtration degrees. The total filtration degree is such that the map $j : \mathfrak{g} \to U(\mathfrak{g})$ defines a morphism of filtered spaces, $\mathfrak{g}[-1] \to U(\mathfrak{g})$. Given a filtered Lie algebra $\mathfrak{g}$, the same Lie algebra with shifted filtration $\mathfrak{g}[-1]$ is again a filtered Lie algebra, and the total filtration for $U(\mathfrak{g})$ agrees with the internal filtration for $U(\mathfrak{g}[-1])$.

If $\mathfrak{g}$ is a Lie super algebra, one defines the enveloping algebra $U(\mathfrak{g})$ as a quotient of the tensor algebra by the ideal generated by elements

$$\xi \otimes \zeta - (-1)^{|\zeta||\xi|} \zeta \otimes \xi - [\xi, \zeta].$$

Then $U(\mathfrak{g})$ becomes a super algebra in such a way that $j : \mathfrak{g} \to U(\mathfrak{g})$ is a morphism of super spaces. If $\mathfrak{g}$ is a graded (resp. filtered) Lie super algebra, then $U(\mathfrak{g})$ becomes a graded (resp. filtered) super algebra, relative to the *internal* grading (resp. filtration) defined by the condition that j is a morphism of graded (resp. filtered) super vector spaces. It is also a filtered super algebra relative to the *total* filtration, defined by the condition that j defines a morphism of filtered super spaces $\mathfrak{g}[-2] \to U(\mathfrak{g})$. (The degree shift by 2 is dictated by the super-sign convention: Recall that to view graded, filtered vector spaces as graded, filtered super spaces, one doubles the degree.) Given a filtered Lie super algebra $\mathfrak{g}$, the same Lie algebra with shifted filtration $\mathfrak{g}[-2]$ is again a filtered Lie super algebra, and the total filtration for $U(\mathfrak{g})$ agrees with the internal filtration for $U(\mathfrak{g}[-2])$.

5.1.8 Further remarks

1. Given a central extension

$$0 \to \mathbb{K}\mathsf{c} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of a Lie algebra $\mathfrak{g}$, one can define *level r enveloping algebras* as

$$U_r(\widehat{\mathfrak{g}}) := U(\widehat{\mathfrak{g}})/ < \mathsf{c} - r >, \quad r \in \mathbb{K},$$

specializing to $U(\mathfrak{g})$ for $r = 0$. A module over $U_r(\widehat{\mathfrak{g}})$ is given by a $\widehat{\mathfrak{g}}$-representation such that the central element c acts as multiplication by r. Again, this construction generalizes to central extensions of graded or filtered super Lie algebras.

2. Suppose V is a vector space, equipped with a symmetric bilinear form B. Define a graded super Lie algebra

$$\mathbb{K}[2] \oplus V[1],$$

where $\mathbb{K}[2]$ is the 1-dimensional space spanned by a central element c of degree -2, and where $[v, w] = 2B(v, w)\mathsf{c}$ for $v, w \in V[1]$. Shifting degree by 2, this becomes a filtered super Lie algebra

$$\mathbb{K} \oplus V[-1],$$

for which c now has degree 0. Its level 1 enveloping algebra is the Clifford algebra

$$\mathrm{Cl}(V; B) = U_1(\mathbb{K} \oplus V[-1]);$$

here the filtration on the Clifford algebra comes from the internal filtration of the enveloping algebra.

5.2 The Poincaré–Birkhoff–Witt Theorem

The Poincaré–Birkhoff–Witt Theorem appears in several equivalent versions. The first version is based on the following observation.

Lemma 5.1 *For any permutation σ of $\{1, \ldots, k\}$ and any $\xi_j \in \mathfrak{g}$,*

$$\xi_1 \cdots \xi_k - \xi_{\sigma(1)} \cdots \xi_{\sigma(k)} \in U^{(k-1)}(\mathfrak{g}).$$

Proof For transpositions of two adjacent elements, this is clear from the definition of the enveloping algebra. The general case follows since such transpositions generate the symmetric group. □

It follows that the commutator or two elements of filtration degree k, l has filtration degree $k + l - 1$. Hence, the associated graded algebra $\mathrm{gr}(U(\mathfrak{g}))$ is commutative (in the usual, ungraded sense) and the inclusion of $\mathfrak{g}$ extends to an algebra homomorphism

$$S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g})). \tag{5.1}$$

Theorem 5.2 (Poincaré–Birkhoff–Witt, version I) *The homomorphism $S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is an algebra isomorphism.*

For the second version, let $\{e_i,\ i \in P\}$ be a basis of $\mathfrak{g}$, with a totally ordered index set P. Using the lemma, one shows that $U(\mathfrak{g})$ is already spanned by elements of the form $e_{i_1} \cdots e_{i_k}$ where $i_1 \le \cdots \le i_k$. Since the similar products in the symmetric algebra are clearly a basis of $S(\mathfrak{g})$ we obtain a surjective linear map

$$S(\mathfrak{g}) \to U(\mathfrak{g}),\ e_{i_1} \cdots e_{i_k} \mapsto e_{i_1} \cdots e_{i_k}. \tag{5.2}$$

Theorem 5.3 (Poincaré–Birkhoff–Witt, version II) *The elements*

$$\{e_{i_1} \cdots e_{i_k} \,|\, i_1 \le \cdots \le i_k\}$$

form a basis of $U(\mathfrak{g})$.

Equivalently, the map (5.2) is an isomorphism. Since a map of $\mathbb{Z}_{\ge 0}$-filtered vector spaces is an isomorphism if and only if the associated graded map is an isomorphism, and since the associated graded map to (5.2) is (5.1), the equivalence of versions I and II is clear. A different lift of (5.1) is given by *symmetrization*

$$\mathrm{sym}:\ S(\mathfrak{g}) \to U(\mathfrak{g}),\quad \xi_1 \cdots \xi_k \mapsto \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \xi_{s(1)} \cdots \xi_{s(k)}.$$

It may be characterized as the unique linear map such that

$$\mathrm{sym}(\xi^k) = \xi^k$$

for all $\xi \in \mathfrak{g}$ and all k, where on the left-hand side the kth power $\xi^k = \xi \cdots \xi$ is a product in the symmetric algebra, while on the right-hand side it is taken in the enveloping algebra. (Note that the elements ξ^k with $\xi \in \mathfrak{g}$ span $S^k(\mathfrak{g})$, by polarization.) The symmetrization map is the direct analogue of the quantization map $q:\ \wedge(V) \to \mathrm{Cl}(V)$ for Clifford algebras, which was given by symmetrization in the graded sense.

Theorem 5.4 (Poincaré–Birkhoff–Witt, version III) *The symmetrization map*

$$\mathrm{sym}:\ S(\mathfrak{g}) \to U(\mathfrak{g})$$

is an isomorphism of filtered vector spaces.

Remark 5.2 It is rather easy to see that $\mathrm{sym}:\ S(\mathfrak{g}) \to U(\mathfrak{g})$ is surjective: For this, it suffices to show that the associated graded map (5.1) is surjective. But this follows, e.g., since the elements $e_{i_1} \cdots e_{i_k}$ for weakly increasing sequences $i_1 \le \cdots \le i_k$ span $U(\mathfrak{g})$. The difficult part of the Poincaré–Birkhoff–Witt Theorem is to show that the map sym is injective.

Let ρ^L be the left-regular representation of $\mathfrak{g}$ on $U(\mathfrak{g})$,

$$\rho^L(\xi)x = \xi x.$$

The isomorphism $\mathrm{sym}:\ S(\mathfrak{g}) \to U(\mathfrak{g})$ from version III takes ρ^L to a $\mathfrak{g}$-representation on the symmetric algebra $S(\mathfrak{g})$.

Theorem 5.5 (Poincaré–Birkhoff–Witt, version IV) *There exists a* $\mathfrak{g}$*-representation* $\rho: \mathfrak{g} \to \mathrm{End}(S(\mathfrak{g}))$, *with the property*

$$\rho(\zeta)(\zeta^n) = \zeta^{n+1} \tag{5.3}$$

for all $\zeta \in \mathfrak{g},\ n \geq 0$.

To see the equivalence with version III, suppose first that $\mathrm{sym}: S(\mathfrak{g}) \to U(\mathfrak{g})$ is known to be an isomorphism. Let ρ be the representation on $S(\mathfrak{g})$ obtained from ρ^L under this isomorphism. Equation (5.3) follows from $\rho^L(\zeta)\mathrm{sym}(\zeta^n) = \zeta\,\zeta^n = \zeta^{n+1} = \mathrm{sym}(\zeta^{n+1})$. Conversely, if $\rho: \mathfrak{g} \to \mathrm{End}(S(\mathfrak{g}))$ is a Lie algebra representation satisfying (5.3), extend to an algebra morphism $\rho: U(\mathfrak{g}) \to \mathrm{End}(S(\mathfrak{g}))$ and define a *symbol map*

$$\sigma: U(\mathfrak{g}) \to S(\mathfrak{g}),\ x \mapsto \rho(x).1.$$

Then

$$\sigma(\mathrm{sym}(\zeta^n)) = \rho(\zeta^n).1 = \rho(\zeta)^n.1 = \zeta^n,$$

proving $\sigma \circ \mathrm{sym} = \mathrm{id}_{S(\mathfrak{g})}$. Since sym is surjective, it follows that sym is an isomorphism.

A beautiful direct proof of version IV of the Poincaré–Birkhoff–Witt Theorem was obtained by Emanuela Petracci [107] in 2003. We will present this proof in Section 5.5 below. In fact, Petracci's argument yields the following explicit formula for the representation

$$\rho(\zeta)(\xi^n) = \sum_{k=0}^{n} \binom{n}{k} B_k\, \xi^{n-k}\, \mathrm{ad}_\xi^k(\zeta), \tag{5.4}$$

where B_n are the Bernoulli numbers, defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Note that if $\xi = \zeta$, only the term $k = 0$ contributes to (5.4) and we get (5.3), as required. The verification that (5.4) defines a Lie algebra representation on $S(\mathfrak{g})$ is the main task in this approach, and will be carried out in Section 5.5).

Recall that the Bernoulli numbers for odd $n \geq 3$ are all zero, while $B_0 = 1,\ B_1 = -\frac{1}{2}$, and

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66}, \ldots.$$

One deduces the following expressions in low degrees:

$$\begin{aligned}
\rho(\zeta)(1) &= \zeta, \\
\rho(\zeta)(\xi) &= \xi\zeta - \frac{1}{2}[\xi,\zeta], \\
\rho(\zeta)(\xi^2) &= \xi^2\zeta - \xi[\xi,\zeta] + \frac{1}{6}[\xi,[\xi,\zeta]], \\
&\cdots.
\end{aligned}$$

Once the Poincaré–Birkhoff–Witt Theorem is in place, we may use the symmetrization map $\mathrm{sym}: S(\mathfrak{g}) \to U(\mathfrak{g})$ to transfer the non-commutative product on $U(\mathfrak{g})$ to a product $*$ on $S(\mathfrak{g})$. By definition of symmetrization and of the enveloping algebra, we have

$$\xi_1\xi_2 = \frac{1}{2}(\xi_1 * \xi_2 + \xi_2 * \xi_1), \quad [\xi_1, \xi_2] = \xi_1 * \xi_2 - \xi_2 * \xi_1.$$

Hence

$$\xi_1 * \xi_2 = \xi_1\xi_2 + \frac{1}{2}[\xi_1, \xi_2].$$

The triple product is already much more complicated. One finds, after cumbersome computation, that

$$\begin{aligned}\xi_1 * \xi_2 * \xi_3 = \xi_1\xi_2\xi_3 &+ \frac{\xi_3[\xi_1, \xi_2] + \xi_1[\xi_2, \xi_3] + \xi_2[\xi_1, \xi_3]}{2} \\ &+ \frac{[\xi_1, [\xi_2, \xi_3]] - [\xi_3, [\xi_1, \xi_2]]}{6}.\end{aligned}$$

Remark 5.3 Similar to the discussion for Clifford algebras, the isomorphism

$$\mathrm{gr}(U(\mathfrak{g})) \cong S(\mathfrak{g})$$

induces the structure of graded Poisson algebra on $S(\mathfrak{g})$. The Poisson structure is determined by $\{\xi_1, \xi_2\} = [\xi_1, \xi_2]$ for generators $\xi_1, \xi_2 \in \mathfrak{g}$, hence it coincides with the Kirillov–Poisson structure on the space of polynomial functions on $\mathfrak{g}^*$; see Example 2.3.

5.3 $U(\mathfrak{g})$ as left-invariant differential operators

For any manifold M, let $\mathfrak{D}(M)$ denote the algebra of differential operators on M (cf. Section 2.3.1). Given an action of a Lie group G on M, one can consider the subalgebra $\mathfrak{D}(M)^G$ of differential operators that commute with the G-action. The isomorphism (defined by the principal symbol)

$$\sigma^\bullet : \mathrm{gr}^\bullet\mathfrak{D}(M) \to \Gamma^\infty(M, S^\bullet(TM))$$

is G-equivariant, and restricts to an isomorphism on G-invariants,

$$(\mathrm{gr}^\bullet\mathfrak{D}(M))^G \to \Gamma^\infty(M, S^\bullet(TM))^G.$$

We also have an injection $\mathrm{gr}^\bullet(\mathfrak{D}(M)^G) \to (\mathrm{gr}^\bullet\mathfrak{D}(M))^G$, but for non-compact Lie groups and ill-behaved actions this need not be an isomorphism.

Consider now the special case of the left-action of G on itself. Let $\mathfrak{D}^L(G)$ denote the differential operators on G that commute with left-translation. The Lie algebra isomorphism

$$\mathfrak{g} \mapsto \mathfrak{X}^L(G), \ \xi \mapsto \xi^L,$$

where $\mathfrak{g}$ is the Lie algebra of G, extends to an algebra homomorphism $T(\mathfrak{g}) \to \mathfrak{D}^L(G)$, which vanishes on the ideal $\mathscr{I}$. Hence we get an induced algebra homomorphism

$$U(\mathfrak{g}) \to \mathfrak{D}^L(G),$$

taking the image of $\xi \in \mathfrak{g} = T^1(\mathfrak{g})$ to ξ^L. The Poincaré–Birkhoff–Witt Theorem now has a differential-geometric interpretation.

Theorem 5.6 (Poincaré–Birkhoff–Witt, version V) *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra over* $\mathfrak{g}$, *and* G *a Lie group integrating* $\mathfrak{g}$. *The canonical map* $U(\mathfrak{g}) \to \mathfrak{D}^L(G)$ *is an isomorphism of algebras.*

Proof We have

$$\Gamma^\infty(G, S^\bullet(TG))^L = S(T_eG) = S(\mathfrak{g}),$$

where the superscript G indicates invariants under the action by left-multiplication. Consider the composition of maps

$$S^k\mathfrak{g} \to U^{(k)}\mathfrak{g} \to \mathfrak{D}^{(k),L}(G) \xrightarrow{\sigma} S^k\mathfrak{g},$$

where the first map is any choice of a left inverse to $U^{(k)}\mathfrak{g} \to S^k\mathfrak{g}$, for instance symmetrization. As noted earlier, the associated graded map $S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is independent of this choice, and we obtain a sequence of morphisms of graded algebras,

$$S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g})) \to \mathrm{gr}(\mathfrak{D}^L(G)) \to S(\mathfrak{g}). \tag{5.5}$$

The composition of these maps is the identity on $S(\mathfrak{g})$, since it is clearly the identity on $\mathfrak{g}$. The last map is injective (since it comes from the inclusion of $\mathrm{gr}(\mathfrak{D}^L(G))$ into $\mathrm{gr}(\mathfrak{D}(G))^L \cong S(\mathfrak{g})$), hence it must be an isomorphism. The first map is surjective (since the e_I span $U(\mathfrak{g})$), hence it too must be an isomorphism. But then the middle map must be an isomorphism as well. □

Note that the proof gives version III (and hence versions I, II and IV) of the Poincaré–Birkhoff–Witt Theorem for the case of finite-dimensional real Lie algebras. However, this argument depends on Lie's Third Theorem since it requires the Lie group G integrating $\mathfrak{g}$. By contrast, the argument in Section 5.5 of the Poincaré–Birkhoff–Witt Theorem is purely algebraic.

Remark 5.4 The Poincaré–Birkhoff–Witt Theorem for finite-dimensional Lie algebras, and the symbol isomorphism $\mathrm{gr}(\mathfrak{D}(M)) = \Gamma(S(TM))$ for the algebra of differential operators, may be regarded as special cases of a Poincaré–Birkhoff–Witt Theorem for Lie algebroids (see [111] and [104, Theorem 3]).

5.4 The enveloping algebra as a Hopf algebra

5.4.1 Hopf algebras

An algebra may be viewed as a triple $(\mathscr{A}, m, i)$ consisting of a vector space $\mathscr{A}$, together with linear maps $m: \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ (the *multiplication*) and $i: \mathbb{K} \to \mathscr{A}$ (the *unit*), such that

$$\begin{aligned} m \circ (m \otimes 1) &= m \circ (1 \otimes m) \quad \text{(Associativity)}, \\ m \circ (i \otimes 1) &= m \circ (1 \otimes i) = 1 \quad \text{(Unit property)}. \end{aligned} \tag{5.6}$$

It is called *commutative* if $m \circ \mathscr{T} = m$, where

$$\mathscr{T}: \mathscr{A} \otimes \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}, \; x \otimes x' \mapsto x' \otimes x$$

exchanges the two factors. A coalgebra is defined similar to an algebra, but with "arrows reversed":

Definition 5.1 A *coalgebra* is a vector space $\mathscr{A}$, together with linear maps

$$\Delta: \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}, \quad \varepsilon: \mathscr{A} \to \mathbb{K}$$

called *comultiplication* and *counit*, such that

$$\begin{aligned} (\Delta \otimes 1) \circ \Delta &= (1 \otimes \Delta) \circ \Delta \quad \text{(Coassociativity)}, \\ (\varepsilon \otimes 1) \circ \Delta &= (1 \otimes \varepsilon) \circ \Delta = 1 \quad \text{(Counit property)}. \end{aligned}$$

It is called *cocommutative* if $\mathscr{T} \circ \Delta = \Delta$.

It is fairly obvious from the definition that the dual of any coalgebra is an algebra. By contrast, the dual of an algebra $\mathscr{A}$ is not a coalgebra, in general, since the dual map $m^*: \mathscr{A}^* \to (\mathscr{A} \otimes \mathscr{A})^*$ does not take values in $\mathscr{A}^* \otimes \mathscr{A}^*$ unless $\dim \mathscr{A} < \infty$. There is an obvious notion of morphism of coalgebras; for example the counit provides such a morphism.

A Hopf algebra is a vector space with compatible algebra and coalgebra structures, as follows:

Definition 5.2 A *Hopf algebra* is a vector space $\mathscr{A}$, together with maps

$$\begin{aligned} m: \mathscr{A} \otimes \mathscr{A} &\to \mathscr{A} \quad \text{(multiplication)}, \\ i: \mathbb{K} &\to \mathscr{A} \quad \text{(unit)}, \\ \Delta: \mathscr{A} &\to \mathscr{A} \otimes \mathscr{A} \quad \text{(comultiplication)}, \\ \varepsilon: \mathscr{A} &\to \mathbb{K} \quad \text{(counit)}, \\ \mathsf{s}: \mathscr{A} &\to A \quad \text{(antipode)}, \end{aligned}$$

such that

1. $(\mathscr{A}, m, i)$ is an algebra,
2. $(\mathscr{A}, \Delta, \varepsilon)$ is a coalgebra,

3. Δ and ε are algebra morphisms,
4. s is a linear isomorphism and has the property,

$$m \circ (1 \otimes \mathsf{s}) \circ \Delta = m \circ (\mathsf{s} \otimes 1) \circ \Delta = i \circ \varepsilon.$$

Remarks 5.5 The condition 3 that Δ and ε be algebra morphisms is equivalent to m and i being coalgebra morphisms. Indeed, both properties are expressed by the formulas

$$\begin{aligned}
&\Delta \circ m = (m \otimes m) \circ (1 \otimes \mathscr{T} \otimes 1)(\Delta \otimes \Delta),\\
&\varepsilon \otimes \varepsilon = \varepsilon \circ m,\\
&\Delta \circ i = i \otimes i,\\
&\varepsilon \circ i = 1.
\end{aligned}$$

Furthermore, it is automatic [76, Theorem III.3.4] that s is an algebra anti-homomorphism as well as a coalgebra anti-homomorphism, and that $\mathsf{s} \circ i = i, \quad \varepsilon \circ \mathsf{s} = \varepsilon$.

Hopf algebras may be viewed as algebraic counterparts, or rather generalizations, of groups. Indeed any group defines a Hopf algebra:

Example 5.1 Let Γ be any group, and $\mathbb{K}[\Gamma]$ its *group algebra*, consisting of linear combinations $\sum_{g \in \Gamma} a_g\, g$ with $a_g \in \mathbb{K}$. The unit i and multiplication m are given on basis elements by

$$m(g \otimes g') = gg', \quad i(1) = e.$$

The algebra structure extends to a Hopf algebra structure by putting

$$\Delta(g) = g \otimes g, \quad \varepsilon(\sum_{g \in \Gamma} a_g\, g) = a_e, \quad \mathsf{s}(g) = g^{-1}.$$

In this example, the group Γ can be recovered from the Hopf algebra $\mathbb{K}[\Gamma]$ as the set of elements satisfying $\Delta(x) = x \otimes x$. This motivates the following notion.

Definition 5.3 An element x of a Hopf algebra $(\mathscr{A}, m, i, \Delta, \varepsilon, s)$ is called *group-like* if $\Delta(x) = x \otimes x$.

Proposition 5.1 *The set of group-like elements is a group, with multiplication m, inverse $x^{-1} = \mathsf{s}(x)$, and group unit $e = i(1)$.*

Proof Since Δ is an algebra morphism, the product of two group-like elements is again group-like. Since s is a coalgebra anti-homomorphism one has, for any group-like element x,

$$\Delta(\mathsf{s}(x)) = (\mathsf{s} \otimes \mathsf{s})(\mathscr{T}(\Delta(x))) = (\mathsf{s} \otimes \mathsf{s})(\Delta(x)) = \mathsf{s}(x) \otimes \mathsf{s}(x),$$

so that $\mathsf{s}(x)$ is group-like. By applying $\varepsilon \otimes 1$ to the definition, one verifies that group-like elements satisfy $\varepsilon(x) = 1$. This then shows that

$$m(x \otimes s(x)) = m(1 \otimes \mathsf{s})(\Delta(x)) = i(\varepsilon(x)) = i(1) = e,$$

and similarly $m(\mathsf{s}(x) \otimes x) = e$. Hence $\mathsf{s}(x) = x^{-1}$. □

Example 5.2 (Finite groups) Let $\mathscr{A} = C(\Gamma, \mathbb{K})$ be the algebra of functions on a finite group Γ. Here m is the pointwise multiplication, and i is given by the inclusion of $\mathbb{K}$ as constant functions. Define a comultiplication

$$\Delta : \; C(\Gamma, \mathbb{K}) \to C(\Gamma, \mathbb{K}) \otimes C(\Gamma, \mathbb{K}) = C(\Gamma \times \Gamma, \mathbb{K}),$$

a counit ε, and an antipode s by

$$\Delta(f)(g_1, g_2) = f(g_1 g_2), \quad \varepsilon(f) = f(e), \quad \mathsf{s}(f)(g) = f(g^{-1}).$$

Then $(\mathscr{A}, m, i, \Delta, \varepsilon, \mathsf{s})$ is a finite-dimensional Hopf algebra. Its group-like elements are given by $\widehat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{K}^\times) \subseteq \mathscr{A}$.

Remark 5.6 One can show [76, Proposition III.3.3] that the dual of any *finite-dimensional* Hopf algebra $(\mathscr{A}, m, i, \Delta, \varepsilon, \mathsf{s})$ is a Hopf algebra

$$(\mathscr{A}^*, \Delta^*, \varepsilon^*, m^*, i^*, \mathsf{s}^*).$$

For instance, if Γ is a finite group, the Hopf algebras $\mathbb{K}[\Gamma]$ and $C(\Gamma, \mathbb{K})$ are dual.

Remark 5.7 Any Hopf algebra $\mathscr{A}$ gives rise to a group

$$\Gamma_{\mathscr{A}} = \mathrm{Hom}_{\mathrm{alg}}(\mathscr{A}, \mathbb{K})$$

(algebra homomorphisms) with product

$$\phi_1 \phi_2 = (\phi_1 \otimes \phi_2) \circ \Delta,$$

inverse $\phi^{-1} = \phi \circ s$, and group unit $e = \varepsilon$. If $\mathscr{A} = C(\Gamma, \mathbb{K})$ for a finite group Γ, then the map

$$\Gamma \to \Gamma_{\mathscr{A}}, \; g \mapsto \left[\mathrm{ev}_g : \; f \mapsto f(g)\right]$$

is an isomorphism. (Tannaka–Krein duality, e.g., [30, Chapter III.7].)

5.4.2 Hopf algebra structure on $S(E)$

Let E be a vector space (possible $\dim E = \infty$), and $(S(E), m, i)$ the symmetric algebra over E. Any morphism of vector spaces induces an algebra morphism of their symmetric algebras; in particular the diagonal inclusion $E \to E \oplus E$ defines an algebra morphism

$$\Delta : \; S(E) \to S(E) \otimes S(E) = S(E \oplus E).$$

Let $\mathsf{s} : \; S(E) \to S(E)$ be the canonical anti-automorphism (equal to $v \mapsto -v$ on $E \subseteq S(E)$), and let $\varepsilon : \; S(E) \to \mathbb{K}$ be the augmentation map. Then $(S(E), m, i, \Delta, \varepsilon, \mathsf{s})$ is a Hopf algebra. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$, we have

$$\Delta(v^k) = (v \otimes 1 + 1 \otimes v)^k = \sum_{j=0}^{k} \binom{k}{j} v^{k-j} \otimes v^j.$$

By polarization, elements of the form v^k span all of $S^k(E)$, hence these formulas determine Δ.

Since all the structure maps preserve gradings, they extend to the degree completion

$$\overline{S}(E) = \prod_{k=0}^{\infty} S^k(E)$$

given by the direct product. Thus $\overline{S}(E)$ is again a Hopf algebra. For any $v \in V$, the exponential

$$\mathrm{e}^v = \sum_{k=0}^{\infty} \frac{1}{k!} v^k$$

is a group-like element of $\overline{S}(E)$. Indeed

$$\Delta(\mathrm{e}^v) = \mathrm{e}^v \otimes \mathrm{e}^v$$

by the formulas for $\Delta(v^k)$. For the multiplication map, counit and antipode, we similarly have

$$m(\mathrm{e}^v \otimes \mathrm{e}^{v'}) = \mathrm{e}^v \mathrm{e}^{v'}, \quad \varepsilon(\mathrm{e}^v) = 1, \quad \mathsf{s}(\mathrm{e}^v) = \mathrm{e}^{-v}.$$

5.4.3 Hopf algebra structure on $U(\mathfrak{g})$

Let $\mathfrak{g}$ be a Lie algebra, and $(U(\mathfrak{g}), m, i)$ its enveloping algebra $U(\mathfrak{g})$. The diagonal inclusion $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, $\xi \mapsto \xi \oplus \xi$ is a Lie algebra morphism; hence it extends to an algebra morphism

$$\Delta : \ U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) = U(\mathfrak{g} \oplus \mathfrak{g}).$$

Together with the counit $\varepsilon : \ U(\mathfrak{g}) \to \mathbb{K}$ given as the augmentation map, and the antipode $\mathsf{s} : \ U(\mathfrak{g}) \to U(\mathfrak{g})$ given by the canonical anti-automorphism of $U(\mathfrak{g})$, we find:

Theorem 5.7 $(U(\mathfrak{g}), m, i, \Delta, \varepsilon, \mathsf{s})$ *is a cocommutative Hopf algebra.*

The example $S(E)$ from the last section is a special case, thinking of E as a Lie algebra with zero bracket.

Proof By definition, Δ and ε are algebra homomorphisms. The coassociativity of Δ follows because both $(\Delta \otimes 1) \circ \Delta$ and $(1 \otimes \Delta) \circ \Delta$ are the maps

$$U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g}) = U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g})$$

induced by triagonal inclusion. The counital properties of ε are equally clear. It remains to check the properties of the antipode. The property $m \circ (1 \otimes \mathsf{s}) \circ \Delta = i \circ \varepsilon$

is clearly true on scalars $i(\mathbb{K}) \subseteq U(\mathfrak{g})$, while the general case follows by induction on the filtration degree: if the property holds on $x \in U^{(k)}(\mathfrak{g})$, and if $\xi \in \mathfrak{g}$, then $i(\varepsilon(\xi x)) = 0$ since ξx is in the augmentation ideal, and also

$$\begin{aligned} m((1 \otimes \mathsf{s})(\Delta(\xi x))) &= m((1 \otimes \mathsf{s})((\xi \otimes 1 + 1 \otimes \xi)\Delta(x)) \\ &= m((\xi \otimes 1)(1 \otimes \mathsf{s})(\Delta(x)) - m((1 \otimes \mathsf{s})(\Delta(x))(1 \otimes \xi)) \\ &= \xi m((1 \otimes \mathsf{s})(\Delta(x)) - m((1 \otimes \mathsf{s})(\Delta(x)))\xi \\ &= \xi i(\varepsilon(x)) - i(\varepsilon(x))\xi \\ &= 0, \end{aligned}$$

where we used that s is an anti-homomorphism. The property $m \circ (\mathsf{s} \otimes 1) \circ \Delta = i \circ \varepsilon$ is verified similarly. Cocommutativity $\mathscr{T} \circ \Delta = \Delta$ follows since it holds on generators $v \in \mathfrak{g}$, and since $\mathscr{T}: U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is an algebra homomorphism (induced by the Lie algebra isomorphism $(\xi_1, \xi_2) \mapsto (\xi_2, \xi_1)$). $\square$

Remark 5.8 To summarize, we can think of $U(\mathfrak{g})$ as an algebraic analogue or substitute for the Lie group G integrating $\mathfrak{g}$. The cocommutativity of $U(\mathfrak{g})$ is parallel to the fact that $C(G, \mathbb{K})$ is a commutative algebra. This point of view is taken in the definition of quantum groups [36, 76]; these are not actually groups but are defined as suitable Hopf algebras.

It is obvious that for $\mathfrak{g}$ non-Abelian, the symmetrization map $\mathrm{sym}: S(\mathfrak{g}) \to U(\mathfrak{g})$ does not intertwine the multiplications m. On the other hand, it intertwines all the other Hopf algebra structure maps:

Proposition 5.2 *The symmetrization map* $\mathrm{sym}: S(\mathfrak{g}) \to U(\mathfrak{g})$ *intertwines the comultiplications* Δ, *counits* ε, *antipodes* s, *and units* i *of the Hopf algebras* $S(\mathfrak{g})$ *and* $U(\mathfrak{g})$. *In particular,* sym *is a coalgebra homomorphism.*

Proof It is clear that $\mathrm{sym} \circ i = i$. The symmetrization map is functorial with respect to Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$. Functoriality for the diagonal inclusion $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ shows that sym intertwines Δ, while functoriality relative to the projection $\mathfrak{g} \mapsto \{0\}$ implies that sym intertwines ε. Finally, let $\xi_1, \ldots, \xi_k \in \mathfrak{g}$, and let $\xi_1 \cdots \xi_k \in S(\mathfrak{g})$ be their product in the enveloping algebra. Then

$$\begin{aligned} \mathrm{sym}(\mathsf{s}(\xi_1, \ldots, \xi_k)) &= (-1)^k \mathrm{sym}(\xi_1, \ldots, \xi_k) \\ &= \frac{(-1)^k}{k!} \sum_{s \in \mathfrak{S}_k} \xi_{s(1)} \ldots \xi_{s(k)} \\ &= \frac{1}{k!} \mathsf{s}\Big(\sum_{s \in \mathfrak{S}_k} \xi_{s(k)} \ldots \xi_{s(1)} \Big) \\ &= \mathsf{s}(\mathrm{sym}(\xi_1, \ldots, \xi_k)), \end{aligned}$$

which shows that sym also intertwines the antipodes s. $\square$

5.4.4 Primitive elements

It is in fact possible to recover $\mathfrak{g}$ from $U(\mathfrak{g})$. For this we need the following.

Definition 5.4 An element x of a Hopf algebra $(\mathscr{A}, m, i, \Delta, \varepsilon, \mathsf{s})$ is called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. Let $P(\mathscr{A})$ denote the space of primitive elements.

Lemma 5.2 *For any Hopf algebra $\mathscr{A}$, the space of primitives $P(\mathscr{A})$ is a Lie subalgebra under commutation.*

Proof Suppose x and y are primitive. Since Δ is an algebra homomorphism,

$$\begin{aligned}\Delta(xy - yx) &= \Delta(x)\Delta(y) - \Delta(y)\Delta(x)\\ &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x)\\ &= (xy - yx) \otimes 1 + 1 \otimes (xy - yx),\end{aligned}$$

which shows that $xy - yx$ is primitive. □

For any vector space E, we have $E \subseteq P(S(E))$ by definition of the coproduct. More generally, for any Lie algebra we have $\mathfrak{g} \subseteq P(U(\mathfrak{g}))$.

Lemma 5.3 *For any vector space E over $\mathbb{K}$, the set of primitive elements in the symmetric algebra is $P(S(E)) = E$.*

Proof It is clear that $P(S(E))$ is a graded subspace of $S(E)$ containing E. Since elements of degree 0 cannot be primitive, it remains to show that there are no primitive elements of degree $k > 1$. Given $\mu \in E^*$, let

$$(\mathrm{id}_{S(E)} \otimes \mu) : \ S(E) \otimes S(E) \to S(E) \otimes \mathbb{K} = S(E)$$

be the identity map on the first factor and the pairing with $\mu \in E^* \subseteq S(E)^*$ on the second factor. Since this map vanishes on $S(E) \otimes S^k(E)$ for $k \neq 0$, we have

$$\begin{aligned}(\mathrm{id}_{S(E)} \otimes \mu)\Delta(v^n) &= (\cdot \otimes \mu) \sum_{k=0}^{n} \binom{n}{k} v^{n-k} \otimes v^k\\ &= n v^{n-1} \langle \mu, v \rangle.\end{aligned}$$

We see hence that $(\cdot \otimes \mu) \circ \Delta = i_S(\mu)$. If $x \in S^k(E)$ is primitive, it follows that

$$\iota_S(\mu)x = (\mathrm{id}_{S(E)} \otimes \mu)(1 \otimes x + x \otimes 1) = \langle \mu,\, x \rangle.$$

If $k > 1$, the right-hand side vanishes. This shows that $x = 0$. □

According to the Poincaré–Birkhoff–Witt Theorem, the symmetrization map is an isomorphism of coalgebras $S(\mathfrak{g}) \to U(\mathfrak{g})$. Since the definition of primitive elements only involves comultiplication, we may conclude that

$$P(U(\mathfrak{g})) = \mathfrak{g},$$

an isomorphism of Lie algebras. In summary, $\mathfrak{g}$ can be recovered from the Hopf algebra structure of $U(\mathfrak{g})$.

5.4.5 Coderivations

A derivation of an algebra $(\mathscr{A}, m, i)$ is a linear map $D : \mathscr{A} \to \mathscr{A}$ satisfying $D \circ m = m \circ (D \otimes 1 + 1 \otimes D)$. Similarly, one defines:

Definition 5.5 A *coderivation* of a coalgebra $(\mathscr{A}, \Delta, \varepsilon)$ is a linear map $C : \mathscr{A} \to \mathscr{A}$ satisfying

$$\Delta \circ C = (C \otimes 1 + 1 \otimes C) \circ \Delta.$$

The space of coderivations is denoted by $\mathrm{Coder}(\mathscr{A})$.

The space of derivations of an algebra is a Lie algebra under commutation. The same is true for coderivations of a coalgebra:

Lemma 5.4 *The space* $\mathrm{Coder}(\mathscr{A})$ *of coderivations of a coalgebra* $\mathscr{A}$ *is a Lie algebra under commutation.*

Proof If C_1, C_2 are two coderivations, then

$$\begin{aligned}\Delta \circ C_1 \circ C_2 &= (C_1 \otimes 1 + 1 \otimes C_1) \circ (C_2 \otimes 1 + 1 \otimes C_2) \circ \Delta \\ &= (C_1 \circ C_2 \otimes 1 + 1 \otimes C_1 \circ C_2 + C_1 \otimes C_2 + C_2 \otimes C_1) \circ \Delta.\end{aligned}$$

Subtracting a similar equation for $\Delta \circ C_2 \circ C_1$, one obtains the derivation property of $[C_1, C_2] = C_1 \circ C_2 - C_2 \circ C_1$. □

Proposition 5.3 *For any Hopf algebra* $(\mathscr{A}, m, i, \Delta, \varepsilon)$, *the map*

$$P(\mathscr{A}) \times \mathscr{A} \to \mathscr{A},$$

given by the restriction of m, *is an action of the Lie algebra of primitive elements by coderivations* $P(\mathscr{A}) \to \mathrm{Coder}(\mathscr{A})$.

Proof For $\xi \in P(\mathscr{A})$ and $x \in \mathscr{A}$,

$$\Delta(\xi x) = \Delta(\xi)\Delta(x) = (\xi \otimes 1 + 1 \otimes \xi)\Delta(x),$$

proving that $P(\mathscr{A})$ acts by coderivations. □

Corollary 5.1 *The left-regular representation of* $\mathfrak{g}$ *on* $U(\mathfrak{g})$

$$\rho^L : \mathfrak{g} \to \mathrm{End}(U(\mathfrak{g})),$$

given by $\rho^L(\xi)x = \xi x$, *is an action by coderivations of* $U(\mathfrak{g})$.

Proof Immediate from the proposition, since the elements of $\mathfrak{g} \subseteq U(\mathfrak{g})$ are primitive. □

5.4.6 Coderivations of $S(E)$

Let E be a vector space. Recall that the space of derivations of $S(E)$ is isomorphic to the space of linear maps $E \to S(E)$, since any such map extends uniquely as a derivation. Thus

$$\mathrm{Der}(S(E)) \cong \mathrm{Hom}(E, S(E))$$

as graded vector spaces. Dually, one expects that the space of coderivations of the co-algebra $S(E)$ is isomorphic to the space $\mathrm{Hom}(S(E), E)$. In more geometric terms, we may think of the elements of

$$S(E)^* = \mathrm{Hom}(S(E), \mathbb{K}) = \prod_{k=0}^{\infty} S^k(E)^*$$

as *formal functions* on E, i.e., Taylor expansions at 0 of smooth functions on E. Accordingly we think of elements $X \in \mathrm{Hom}(S(E), E)$ as *formal vector fields*. Any formal vector field is determined by its action on elements $v^n \in S^n(E)$, for $n = 0, 1, \ldots$. It is convenient to introduce the "generating function" $\mathrm{e}^{tv} = \sum_{k=0}^{\infty} \frac{t^k}{k!} v^k \in S(E)[\![t]\!]$; then X is determined by $X(\mathrm{e}^{tv}) \in E[\![t]\!]$. A coderivation C of $S(E)$ defines a derivation C^* of the algebra $S(E)^*$, and hence should correspond to a formal vector field.

Theorem 5.8 *There is a canonical isomorphism*

$$\mathrm{Coder}(S(E)) \cong \mathrm{Hom}(S(E), E)$$

between the space of coderivations of $S(E)$ and the space of formal vector fields on E. The isomorphism takes $X \in \mathrm{Hom}(S(E), E)$ to the coderivation

$$C = m \circ (1 \otimes X) \circ \Delta.$$

Proof It is convenient to work with the generating function e^{tv}. Since $\Delta(\mathrm{e}^{tv}) = \mathrm{e}^{tv} \otimes \mathrm{e}^{tv} \in S(V \oplus V)[\![t]\!]$, the formula relating X and C reads as

$$C(\mathrm{e}^{tv}) = \mathrm{e}^{tv} X(\mathrm{e}^{tv}).$$

Explicitly, we have

$$C(v^n) = \sum_{k=0}^{n} \binom{n}{k} v^k X(v^{n-k})$$

for all $n = 0, 1, 2, \ldots$. We first show that if C is a coderivation, then

$$X(\mathrm{e}^{tv}) := \mathrm{e}^{-tv} C(\mathrm{e}^{tv})$$

lies in $E[\![t]\!] \subseteq S(E)[\![t]\!]$. Equivalently, we show that $X(\mathrm{e}^{tv})$ is primitive:

$$\begin{aligned}
\Delta(X(\mathrm{e}^{tv})) &= \Delta(\mathrm{e}^{-tv})\Delta(C(\mathrm{e}^{tv}))\\
&= (\mathrm{e}^{-tv}\otimes\mathrm{e}^{-tv})\cdot(C\otimes 1+1\otimes C)\Delta(\mathrm{e}^{tv})\\
&= (\mathrm{e}^{-tv}\otimes\mathrm{e}^{-tv})\cdot(C(\mathrm{e}^{tv})\otimes\mathrm{e}^{tv}+\mathrm{e}^{tv}\otimes C(\mathrm{e}^{tv}))\\
&= \mathrm{e}^{-tv}C(\mathrm{e}^{tv})\otimes 1+1\otimes\mathrm{e}^{-tv}C(\mathrm{e}^{tv})\\
&= X(\mathrm{e}^{tv})\otimes 1+1\otimes X(\mathrm{e}^{tv}).
\end{aligned}$$

Conversely, if $X\in\mathrm{Hom}(S(E),E)$, let $C(\mathrm{e}^{tv}):=\mathrm{e}^{tv}X(\mathrm{e}^{tv})$. The calculation

$$\begin{aligned}
\Delta(C(\mathrm{e}^{tv})) &= (\mathrm{e}^{tv}\otimes\mathrm{e}^{tv})\Delta(X(\mathrm{e}^{tv}))\\
&= (\mathrm{e}^{tv}\otimes\mathrm{e}^{tv})(X(\mathrm{e}^{tv})\otimes 1+1\otimes X(\mathrm{e}^{tv}))\\
&= C(\mathrm{e}^{tv})\otimes\mathrm{e}^{tv}+\mathrm{e}^{tv}\otimes C(\mathrm{e}^{tv})\\
&= (C\otimes 1+1\otimes C)\circ\Delta(\mathrm{e}^{tv})
\end{aligned}$$

shows that C defines a coderivation. $\square$

Proposition 5.4 *The Lie bracket on* $\mathrm{Hom}(S(E),E)$ *induced by the isomorphism with the Lie algebra* $\mathrm{Coder}(S(E))$ *of coderivations reads as*

$$[X_1,X_2](\mathrm{e}^{tv}) = X_1(\mathrm{e}^{tv}X_2(\mathrm{e}^{tv})) - X_2(\mathrm{e}^{tv}X_1(\mathrm{e}^{tv})). \tag{5.7}$$

Proof For any $Y\in E$ we have

$$\begin{aligned}
C_1(\mathrm{e}^{tv}Y) &= \frac{\partial}{\partial s}\Big|_{s=0} C_1(\mathrm{e}^{tv+sY})\\
&= \frac{\partial}{\partial s}\Big|_{s=0} \mathrm{e}^{tv+sY}X_1(\mathrm{e}^{tv+sY})\\
&= \mathrm{e}^{tv}X_1(\mathrm{e}^{tv})Y+\mathrm{e}^{tv}X_1(\mathrm{e}^{tv}Y).
\end{aligned}$$

Putting $Y=X_2(\mathrm{e}^{tv})\in E[\![t]\!]$ we find that

$$C_1(C_2(\mathrm{e}^{tv})) = \mathrm{e}^{tv}X_1(\mathrm{e}^{tv})X_2(\mathrm{e}^{tv})+\mathrm{e}^{tv}X_1(\mathrm{e}^{tv}X_2(\mathrm{e}^{tv})).$$

Hence $[C_1,C_2](\mathrm{e}^{tv})=\mathrm{e}^{tv}X_1(\mathrm{e}^{tv}X_2(\mathrm{e}^{tv}))-\mathrm{e}^{tv}X_2(\mathrm{e}^{tv}X_2(\mathrm{e}^{tv}))$. $\square$

5.5 Petracci's proof of the Poincaré–Birkhoff–Witt Theorem

Our goal in this section is to prove version IV of the Poincaré–Birkhoff–Witt Theorem. We will indeed prove a more precise version, Petracci's Theorem 5.9 below, explicitly describing the $\mathfrak{g}$-representation on $S(\mathfrak{g})$ corresponding to the left-regular representation on $U(\mathfrak{g})$.

To simplify notation, we will omit the parameter t from the "generating functions" e^{tv}, and simply write, e.g.,

$$C(\mathrm{e}^{v}) = \mathrm{e}^{v}X(\mathrm{e}^{v}).$$

This is a well-defined equality in $\overline{S}(E)$ if X (hence C) is homogeneous of some fixed degree; otherwise we view this identity as an equality of formal power series in t, obtained by replacing v with tv.

5.5.1 A $\mathfrak{g}$-representation by coderivations

The main idea in Petracci's approach to the Poincaré–Birkhoff–Witt Theorem is to define a $\mathfrak{g}$-representation on $S(\mathfrak{g})$, which under symmetrization $\mathrm{sym}: S(\mathfrak{g}) \to U(\mathfrak{g})$ goes to the left-regular representation

$$\rho^L: \mathfrak{g} \to \mathrm{End}(U(\mathfrak{g})), \quad \rho^L(\zeta).x = \zeta x$$

of $\mathfrak{g}$ on $U(\mathfrak{g})$. The representation should be by coderivations of $S(\mathfrak{g})$, since $\rho^L(\zeta)$ is a coderivation (cf. Corollary 5.1) and sym is a coalgebra homomorphism (cf. Proposition 5.2). Equivalently, we are interested in Lie algebra homomorphisms

$$\rho: \mathfrak{g} \to \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}), \quad \zeta \mapsto X^\zeta.$$

Using differential geometry, one can make a guess for X^ζ, as follows. Suppose $\mathbb{K} = \mathbb{R}$, and let G be a Lie group integrating $\mathfrak{g}$. The left-regular representation of $\mathfrak{g}$ on $U(\mathfrak{g})$ can be thought of as a counterpart to the left-action of G on itself, and the symmetrization map is an algebraic counterpart[1] to the exponential map $\exp: \mathfrak{g} \to G$. The exponential map is a local diffeomorphism on an open dense subset $\mathfrak{g}_\sharp \subseteq \mathfrak{g}$. We may view the pull-back $\exp^*(\zeta^R) \in \mathfrak{X}(\mathfrak{g}_\sharp)$ as a $\mathfrak{g}$-valued function on $\mathfrak{g}_\natural$. Explicit calculation (see Section C.4 in Appendix C) gives that this function is

$$\xi \mapsto \phi(\mathrm{ad}_\xi)\zeta,$$

with $\phi(z) = \frac{z}{e^z - 1}$. This suggests defining $X^\zeta \in \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g})$ by $X^\zeta(e^\xi) = \phi(\mathrm{ad}_\xi)\zeta$, and this formula makes sense for any field $\mathbb{K}$. Petracci's Theorem below shows that this is indeed the correct choice.

Theorem 5.9 (Petracci) *Let $X^\zeta \in \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g})$, $\zeta \in \mathfrak{g}$, be the formal vector fields defined by*

$$X^\zeta(e^\xi) = \phi(\mathrm{ad}_\xi)\zeta,$$

where $\phi(z) = \frac{z}{e^z - 1}$. Then

$$[X^{\zeta_1}, X^{\zeta_2}] = X^{[\zeta_1, \zeta_2]},$$

[1] More concretely, the symmetric algebra $S(\mathfrak{g})$ may be identified with the convolution algebra of distributions (generalized measures) on $\mathfrak{g}$ supported at 0, while $U(\mathfrak{g})$ is identified with the convolution algebra of distributions on G supported at the group unit e. The infinitesimal $\mathfrak{g}$-action generating the left-multiplication is given by the right-invariant vector fields,

$$\zeta \mapsto -\zeta^R.$$

Push-forward $\exp_*$ of distributions gives an isomorphism of distributions supported at 0 with those supported at e, and this isomorphism is exactly the symmetrization map.

and ϕ is the unique formal power series with $\phi(0) = 1$ having this property. Hence, the formal vector fields X^ζ define a representation by coderivations.

The proof will be given in Section 5.5.3, after some preparations.

5.5.2 The formal vector fields $X^\zeta(\phi)$

We have to develop a technique for calculating the commutator of formal vector fields of the form $e^\xi \mapsto \phi(\mathrm{ad}_\xi)\zeta$, for formal power series $\phi \in \mathbb{K}[\![z]\!]$. To this end, we introduce the following notations. For $\zeta \in \mathfrak{g}$, we define a linear map

$$\mathbb{K}[\![z]\!] \to \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}), \quad \phi \mapsto X^\zeta(\phi)$$

by $X^\zeta(\phi)(e^\xi) = \phi(\mathrm{ad}_\xi)(\zeta)$. We also define linear maps

$$\mathbb{K}[\![z_1, z_2]\!] \to \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}), \ \psi \mapsto X^{\zeta_1,\zeta_2}(\psi)$$

for $\zeta_1, \zeta_2 \in \mathfrak{g}$, taking a monomial $z_1^{k_1} z_2^{k_2}$ to the formal vector field

$$X^{\zeta_1,\zeta_2}(z_1^{k_1} z_2^{k_2})(e^\xi) = [\mathrm{ad}_\xi^{k_1}(\zeta_1), \mathrm{ad}_\xi^{k_2}(\zeta_2)].$$

Proposition 5.5 *For all $\phi \in \mathbb{K}[\![z]\!]$ and $\zeta_1, \zeta_2 \in \mathfrak{g}$,*

$$X^{[\zeta_1,\zeta_2]}(\phi) = X^{\zeta_1,\zeta_2}(\Delta\phi),$$

where $\Delta\phi \in \mathbb{K}[\![z_1, z_2]\!]$ is the formal power series $(\Delta\phi)(z_1, z_2) = \phi(z_1 + z_2)$.

Proof It is enough to check on monomials $\phi(z) = z^n$. By induction on n,

$$\mathrm{ad}_\xi^n[\zeta_1, \zeta_2] = \sum_{i=0}^{n} \binom{n}{i} [\mathrm{ad}_\xi^i \zeta_1, \mathrm{ad}_\xi^{n-i} \zeta_2];$$

hence $X^{[\zeta_1,\zeta_2]}(z^n) = X^{\zeta_1,\zeta_2}\big((z_1 + z_2)^n\big)$. □

Given $\phi \in \mathbb{K}[\![z]\!]$ we define power series in two variables by

$$\delta^{(1)}\phi(z_1, z_2) = \frac{\phi(z_1 + z_2) - \phi(z_2)}{z_1},$$
$$\delta^{(2)}\phi(z_1, z_2) = \frac{\phi(z_1 + z_2) - \phi(z_1)}{z_2}.$$

Proposition 5.6 *For any $\phi \in \mathbb{K}[\![z]\!]$, and any $\zeta, Y \in \mathfrak{g}$,*

$$X^\zeta(\phi) \circ Y = X^{Y,\zeta}(\delta^{(1)}\phi), \quad \zeta, Y \in \mathfrak{g}.$$

On the left-hand side Y is identified with the operator $S(\mathfrak{g}) \to S(\mathfrak{g})$ of multiplication by Y. Put differently, we think of Y as a "constant" element of $\mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}) \cong \mathrm{Coder}(S(\mathfrak{g}))$.

Proof On monomials z^n, the formula says that

$$X^{\zeta}(z^n)(Y\mathrm{e}^{\xi}) = X^{Y,\zeta}\Big(\frac{(z_1+z_2)^n - z_2^n}{z_1}\Big)(\mathrm{e}^{\xi}).$$

The proof is an induction on n: The cases $n=1$ is clear, while

$$\begin{aligned} X^{\zeta}(z^{n+1})(Y\mathrm{e}^{\xi}) &= \frac{\partial}{\partial s}\Big|_{s=0} \mathrm{ad}^{n+1}(\xi + sY)\zeta \\ &= [Y, \mathrm{ad}^n(\xi)\zeta] + \mathrm{ad}(\xi)\frac{\partial}{\partial s}\Big|_{s=0} \mathrm{ad}^n(\xi+sY)\zeta \\ &= [Y, \mathrm{ad}^n(\xi)\zeta] + \mathrm{ad}(\xi)X^{\zeta}(z^n)(Y\mathrm{e}^{\xi}) \\ &= X^{Y,\zeta}(z_2^n)(\mathrm{e}^{\xi}) + \mathrm{ad}(\xi)X^{Y,\zeta}\Big(\frac{(z_1+z_2)^n - z_2^n}{z_1}\Big)(\mathrm{e}^{\xi}) \\ &= X^{Y,\zeta}\Big(z_2^n + (z_1+z_2)\frac{(z_1+z_2)^n - z_2^n}{z_1}\Big)(\mathrm{e}^{\xi}) \\ &= X^{Y,\zeta}\Big(\frac{(z_1+z_2)^{n+1} - z_2^{n+1}}{z_1}\Big)(\mathrm{e}^{\xi}), \end{aligned}$$

using the induction hypothesis for the fourth equality sign. □

Proposition 5.7 *The Lie bracket of vector fields $X^{\zeta_1}(\phi_1)$ and $X^{\zeta_2}(\phi_2)$ is given by*

$$[X^{\zeta_1}(\phi_1), X^{\zeta_2}(\phi_2)] = -X^{\zeta_1,\zeta_2}\big(\delta^{(2)}\phi_1\, \pi_2^*\phi_2 + \delta^{(1)}\phi_2\, \pi_1^*\phi_1\big),$$

where $(\pi_j^*\phi)(z_1, z_2) = \phi(z_j)$ for $j = 1, 2$.

Proof By definition of the Lie bracket (5.7),

$$[X^{\zeta_1}(\phi_1), X^{\zeta_2}(\phi_2)](\mathrm{e}^{\xi}) = X^{\zeta_1}(\phi_1)(\mathrm{e}^{\xi} X^{\zeta_2}(\phi_2)(\mathrm{e}^{\xi})) - X^{\zeta_2}(\phi_2)(\mathrm{e}^{\xi} X^{\zeta_1}(\phi_1)(\mathrm{e}^{\xi})).$$

To compute the first term we put $Y = X^{\zeta_2}(\phi_2)(\mathrm{e}^{\xi}) = \phi_2(\mathrm{ad}_{\xi})\zeta_2$, and use the previous proposition:

$$X^{\zeta_1}(\phi_1)(\mathrm{e}^{\xi} X^{\zeta_2}(\phi_2)(\mathrm{e}^{\xi})) = (X^{Y,\zeta_1}(\delta^{(1)}\phi_1))(\mathrm{e}^{\xi}).$$

Writing $\delta^{(1)}\phi_1$ as a linear combination of monomials $z_1^{k_1} z_2^{k_2}$, and using the computation,

$$\begin{aligned} (X^{Y,\zeta_1}(z_1^{k_1} z_2^{k_2}))(\mathrm{e}^{\xi}) &= [\mathrm{ad}_{\xi}^{k_1} Y, \mathrm{ad}_{\xi}^{k_2}\zeta_1] \\ &= [\mathrm{ad}_{\xi}^{k_1}\phi_2(\mathrm{ad}_{\xi})\zeta_2, \mathrm{ad}_{\xi}^{k_2}\zeta_1] \\ &= X^{\zeta_2,\zeta_1}(z_1^{k_1} z_2^{k_2}\phi_2(z_1))(\mathrm{e}^{\xi}), \end{aligned}$$

we find that

$$X^{\zeta_1}(\phi_1)(\mathrm{e}^\xi X^{\zeta_2}(\phi_2)(\mathrm{e}^\xi)) = X^{\zeta_2,\zeta_1}\Big(\delta^{(1)}\phi_1\ \pi_1^*\phi_2\Big)(\mathrm{e}^\xi)$$
$$= -X^{\zeta_1,\zeta_2}\Big(\delta^{(2)}\phi_1\ \pi_2^*\phi_2\Big)(\mathrm{e}^\xi).$$

Similarly,

$$X^{\zeta_2}(\phi_2)(\mathrm{e}^\xi X^{\zeta_1}(\phi_1)(\mathrm{e}^\xi)) = X^{\zeta_1,\zeta_2}\Big(\delta^{(1)}\phi_2\ \pi_1^*\phi_1\Big)(\mathrm{e}^\xi).$$

Hence the Lie bracket is $-X^{\zeta_1,\zeta_2}\big(\delta^{(2)}\phi_1\ \pi_2^*\phi_2 + \delta^{(1)}\phi_2\ \pi_1^*\phi_1\big)$, as claimed. □

As a special case $\phi = \phi_1 = \phi_2$, we see that $[X^{\zeta_1}(\phi), X^{\zeta_2}(\phi)] = X^{\zeta_1,\zeta_2}(\psi)$ where

$$\psi(z_1,z_2) = -\frac{\phi(z_1+z_2)-\phi(z_2)}{z_1}\phi(z_1) - \frac{\phi(z_1+z_2)-\phi(z_1)}{z_2}\phi(z_2).$$

5.5.3 *Proof of Petracci's Theorem*

Since $X^{[\zeta_1,\zeta_2]}(\phi) = X^{\zeta_1,\zeta_2}(\Delta\phi)$ we see that $\zeta \mapsto X^\zeta(\phi)$ is a Lie algebra homomorphism if and only if ϕ satisfies the functional equation

$$\phi(z_1+z_2) + \frac{\phi(z_1+z_2)-\phi(z_2)}{z_1}\phi(z_1) + \frac{\phi(z_1+z_2)-\phi(z_1)}{z_2}\phi(z_2) = 0.$$

The equation may be re-written

$$\frac{\phi(z_1+z_2)}{z_1+z_2} = \Big(\frac{\phi(z_1)}{z_1}\frac{\phi(z_2)}{z_2}\Big)\Big(1+\frac{\phi(z_1)}{z_1}+\frac{\phi(z_2)}{z_2}\Big)^{-1}.$$

Suppose $\phi \in \mathbb{K}[\![z]\!]$ is a non-zero solution of this equation. Putting $z_1 = z_2 = z$, we see that the leading term (for $z \to 0$) in the expansion of ϕ cannot be of order z^k with $k > 1$, or the left-hand side would be of order z^{k-1} while the right-hand side would be of order z^{2k-2}. That is, if ϕ is a non-zero solution, then

$$\psi(z) = 1 + \frac{z}{\phi(z)}$$

is a well-defined element $\psi \in \mathbb{K}[\![z]\!]$. A short calculation shows that in terms of ψ, the functional equation is simply $\psi(z_1+z_2) = \psi(z_1)\psi(z_2)$. The solutions are $\psi(z) = \mathrm{e}^{cz}$ with $c \in \mathbb{K}$, together with the trivial solution $\psi(z) = 0$. We conclude that the solutions of the functional equation for ϕ are

$$\phi(z) = \frac{z}{\mathrm{e}^{cz}-1},\ \ c \neq 0,$$

together with the solutions $\phi(z) = -z$ and $\phi(z) = 0$. In particular, there is a unique solution with $\phi(0) = 1$, given by $\phi(z) = \frac{z}{\mathrm{e}^z-1}$. This proves Petracci's Theorem 5.9.

Remark 5.9 The $\mathfrak{g}$-representation on $S(\mathfrak{g})$ defined by the exceptional solution $\phi(z) = -z$ is just the adjoint representation:

$$C^\zeta(-z)(\mathrm{e}^\xi) \equiv \mathrm{e}^\xi X^\zeta(-z)(\mathrm{e}^\xi) = [\zeta,\xi]\,\mathrm{e}^\xi \ \ \Rightarrow\ \ C^\zeta(-z)(\xi^n) = n[\zeta,\xi]\,\xi^{n-1}.$$

5.6 The center of the enveloping algebra

We have already observed that the center of the enveloping algebra $U(\mathfrak{g})$ is just the ad-invariant subspace,

$$\mathrm{Cent}(U(\mathfrak{g})) = (U(\mathfrak{g}))^{\mathfrak{g}}.$$

Elements of the center are also called *Casimir elements*. The symmetrization map is $\mathfrak{g}$-equivariant, hence it defines an isomorphism of vector spaces sym: $(S\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cent}(U(\mathfrak{g}))$.

Example 5.3 Suppose that $\mathfrak{g}$ comes equipped with an invariant non-degenerate symmetric bilinear form B. If e_i is a basis of $\mathfrak{g}$, and e^i the B-dual basis (so that $B(e_i, e^j) = \delta_i^j$), the element $p = \sum_i e_i e^i \in S^2(\mathfrak{g})$ is invariant. Its image $\mathrm{sym}(p) \in U(\mathfrak{g})$ under symmetrization is called the *quadratic Casimir* of the bilinear form.

Remark 5.10 Suppose $\mathbb{K} = \mathbb{R}$, and that $\mathfrak{g}$ is the Lie algebra of Lie group G. In terms of the identification $U(\mathfrak{g}) \cong \mathfrak{D}^L(G)$, the center corresponds to the space $\mathfrak{D}^{L\times R}(G)$ of bi-invariant differential operators. For instance, if G is a quadratic Lie group (i.e., if $\mathfrak{g}$ comes with a non-degenerate invariant symmetric bilinear form B), then the operator

$$D = \sum_i e_i^L (e^i)^L$$

(where $e_i \in \mathfrak{g}$ is a basis of $\mathfrak{g}$, and $e^i \in \mathfrak{g}$ is the B-dual basis) is the bi-invariant differential operator corresponding to the quadratic Casimir element.

Suppose for the rest of this section that $\mathbb{K} = \mathbb{C}$ and $\dim \mathfrak{g} < \infty$. Suppose $\rho : \mathfrak{g} \to \mathrm{End}(E)$ is a $\mathfrak{g}$-representation on a finite-dimensional complex vector space E. Extend to a representation $\rho : U(\mathfrak{g}) \to \mathrm{End}(E)$. If $x \in \mathrm{Cent}(U(\mathfrak{g}))$, the operator $\rho(x)$ commutes with all $\rho(y)$, $y \in U(\mathfrak{g})$. If ρ is irreducible, this implies by Schur's lemma that $\rho(x)$ is a multiple of the identity. That is, any irreducible representation determines an algebra homomorphism

$$\mathrm{Cent}(U(\mathfrak{g})) \to \mathbb{K},\ x \mapsto \rho(x).$$

For semisimple Lie algebras, it is known that this algebra homomorphism characterizes ρ up to isomorphism. In fact, it suffices to know this map on a set of generators for $\mathrm{Cent}(U(\mathfrak{g}))$. For example, if $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, any irreducible representation is determined by the value of the quadratic Casimir in this representation. It is therefore of interest to understand the structure of $\mathrm{Cent}(U(\mathfrak{g}))$ as an algebra.

The symmetrization map $\mathrm{sym} : S(\mathfrak{g}) \to U(\mathfrak{g})$ restricts to an isomorphism on invariants, $(S(\mathfrak{g}))^{\mathfrak{g}} \to (U(\mathfrak{g}))^{\mathfrak{g}} = \mathrm{Cent}(U(\mathfrak{g}))$. Unfortunately this restricted map is not an algebra homomorphism.

Example 5.4 Suppose $\mathfrak{g}$ is a quadratic Lie algebra, with non-degenerate symmetric bilinear form B. Let e_i be a basis of $\mathfrak{g}$, let e^i be the B-dual basis, and consider

$p = \sum_i e_i e^i$. Let $f_{ijk} = B([e_i, e_j], e_k)$ be the structure constants. We use B to raise or lower indices, e.g., $f_{ij}{}^k = \sum_l f_{ijl} B(e^l, e^m)$. We have

$$\begin{aligned}\operatorname{sym}(p^2) &= \sum_{ij} \operatorname{sym}(e_i e^i e_j e^j)\\ &= \frac{1}{3} \sum_{ij} (e_i e^i e_j e^j + e^i e_j e_i e^j + e_i e_j e^j e^i),\end{aligned}$$

where we used basis independence to identify some of the expressions coming from symmetrization. (For instance, $\sum_i e_i e_j e^i e^j = \sum_i e^i e_j e_i e^j$.) Using the defining relations in the enveloping algebra, this becomes

$$\begin{aligned}\operatorname{sym}(p^2) &= \frac{1}{3} \sum_{ij} (2 e_i e^i e_j e^j - e^i [e_i, e_j] e^j + e_i e_j e^i e^j - e^i e^j [e_i, e_j])\\ &= \frac{1}{3} \sum_{ij} (3 e_i e^i e_j e^j - 2 e^i [e_i, e_j] e^j - e^i e^j [e_i, e_j])\\ &= (\operatorname{sym}(p))^2 + \frac{1}{3} \sum_{ijk} f_{ijk} e^i e^j e^k\\ &= (\operatorname{sym}(p))^2 + \frac{1}{6} \sum_{ijkl} f_{ijk} f^{ijl} e_l e^k.\end{aligned}$$

We hence see that $\operatorname{sym}(p^2) \neq (\operatorname{sym}(p))^2$ in general. Note that $\sum_{ij} f_{ijk} f^{ijl}$ are the coefficients of the Killing form on $\mathfrak{g}$. If $\mathfrak{g}$ is simple, the Killing form is a multiple of B, and the correction term is a multiple of $\operatorname{sym}(p)$. For instance, if $\mathfrak{g} = \mathfrak{so}(3)$, with B-orthonormal basis e_1, e_2, e_3 satisfying $[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2$, we obtain

$$\operatorname{sym}(p^2) = \operatorname{sym}(p)^2 + \frac{1}{3} \operatorname{sym}(p).$$

Duflo's Theorem says that this failure of $\operatorname{sym}: S(\mathfrak{g}) \to U(\mathfrak{g})$ to restrict to an algebra isomorphism on invariants can be repaired by pre-composing sym with a suitable infinite-order differential operator on $S(\mathfrak{g})$.

Let us introduce such infinite-order differential operators on $S(V)$ for any finite-dimensional vector space V. The symmetric algebra $S(V)$ is identified with the algebra of polynomials on V^*. Any $\mu \in V^*$ defines a derivation $\iota_S(\mu)$ of this algebra, given by

$$\iota_S(\mu) v = \langle \mu, v \rangle$$

on generators $v \in V$. The map $\mu \mapsto \iota_S(\mu)$ extends to an algebra homomorphism $S(V^*) \to \operatorname{End}(S(V))$, and even further to an algebra homomorphism $S(V)^* \to \operatorname{End}(S(V))$, where $S(V)^* = \overline{S}(V^*)$ is the algebraic dual

$$S(V)^* = \prod_{k=0}^{\infty} S^k(V^*).$$

One may think of $S(V)^*$ as an algebra of "infinite-order differential operators" acting on polynomials $S(V) = \mathrm{Pol}(V^*)$.

For $p \in S(V)^*$, the corresponding operator $\widetilde{p} = \iota_S(p) \in \mathrm{End}(S(V))$ is characterized by the equation

$$\widetilde{p}(\mathrm{e}^{tv}) = p(tv)\mathrm{e}^{tv},$$

an equality of formal power series in t with coefficients in $S(V)$.

More generally, if f is a smooth function defined on some open neighborhood of $0 \in V$, its Taylor series expansion is an element of $S(V)^*$, and hence the operator $\widetilde{f} \in \mathrm{End}(S(V))$ is well defined.

Returning to the setting for the Duflo Theorem, consider the function

$$J^{1/2}(\xi) = \det{}^{1/2}(j(\mathrm{ad}_\xi)) = \exp\Big(\frac{1}{2}\mathrm{tr}\,\log\big(j(\mathrm{ad}_\xi)\big)\Big),$$

with $j(z) = \frac{\sinh(z/2)}{z/2}$. As discussed in Section 4.3, $J^{1/2}$ is a well-defined holomorphic function of $\xi \in \mathfrak{g}$ if $\mathbb{K} = \mathbb{C}$. Modulo terms of order ≥ 4 in ξ, one finds that

$$J^{1/2}(\xi) = 1 + \frac{1}{48}\mathrm{tr}_\mathfrak{g}(\mathrm{ad}_\xi^2) + \cdots .$$

Note that the first correction term is a multiple of the quadratic form associated to the Killing form on $\mathfrak{g}$. If $\mathbb{K}$ is an arbitrary field of characteristic zero, $J^{1/2}$ is well defined (via its Taylor series) as an element

$$J^{1/2} \in \mathrm{Hom}(S(\mathfrak{g}), \mathbb{K}) = S(\mathfrak{g})^*.$$

The function $J^{1/2}$ defines an operator $\widetilde{J^{1/2}} \in \mathrm{End}(S(\mathfrak{g}))$.

Theorem 5.10 (Duflo [47]) *The composition*

$$\mathrm{sym} \circ \widetilde{J^{1/2}} :\; S(\mathfrak{g}) \to U(\mathfrak{g})$$

restricts to an algebra isomorphism $(S(\mathfrak{g}))^\mathfrak{g} \to \mathrm{Cent}(U(\mathfrak{g}))$.

In Section 7.3, we present a proof of the Duflo Theorem for the case that $\mathfrak{g}$ is a quadratic Lie algebra. This proof will relate the appearance of the factor $J^{1/2}$ in the Duflo Theorem with that in the theory of Clifford algebras.

Chapter 6
Weil algebras

For any Lie algebra $\mathfrak{g}$, the exterior and symmetric algebras over $\mathfrak{g}^*$ combine into a differential algebra $W(\mathfrak{g})$, the *Weil algebra*. This chapter will develop some aspects of the theory of differential spaces and $\mathfrak{g}$-differential spaces, as needed for our purposes. (We refer to Appendix A for background material on graded and filtered super vector spaces.) We will then introduce the Weil algebra as a universal object among commutative $\mathfrak{g}$-differential algebras with connection. By considering *non-commutative* $\mathfrak{g}$-differential algebras with connection, we are led to introduce also a *non-commutative Weil algebra*. We will discuss applications of the two Weil algebras to Chern–Weil theory and to transgression.

6.1 Differential spaces

Definition 6.1 A *differential space* is a super vector space $E = E^{\bar{0}} \oplus E^{\bar{1}}$, equipped with an odd operator $\mathrm{d}\colon\ E \to E$ such that $\mathrm{d} \circ \mathrm{d} = 0$, i.e., $\mathrm{im}(\mathrm{d}) \subseteq \ker(\mathrm{d})$. One calls

$$H(E, \mathrm{d}) = \frac{\ker(\mathrm{d})}{\mathrm{im}(\mathrm{d})}$$

the *cohomology* of the differential space (E, d). It is again a super vector space, with $\mathbb{Z}_2$-grading inherited from E. A *morphism of differential spaces* (also called a *cochain map*) $(E_1, \mathrm{d}_1) \to (E_2, \mathrm{d}_2)$ is a morphism of super vector spaces intertwining the differentials.

The category of differential spaces, with cochain maps as morphisms, has direct sums

$$(E_1, \mathrm{d}_1) \oplus (E_2, \mathrm{d}_2) = (E_1 \oplus E_2,\ \mathrm{d}_1 \oplus \mathrm{d}_2)$$

and tensor products

$$(E_1, \mathrm{d}_1) \otimes (E_2, \mathrm{d}_2) = (E_1 \otimes E_2,\ \mathrm{d}_1 \otimes 1 + 1 \otimes \mathrm{d}_2),$$

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with the usual compatibility properties. (Here tensor products are in the super sense, e.g., $(1 \otimes \mathrm{d}_2)(v_1 \otimes v_2) = (-1)^{|v_1|} v_1 \otimes \mathrm{d}_2 v_2$.) It is thus a *tensor category*, and one can consider its algebra objects, Lie algebra objects, and so on. For example, a *differential algebra* $(\mathscr{A}, \mathrm{d})$ is a differential space with a multiplication morphism

$$m: \ (\mathscr{A}, \mathrm{d}) \otimes (\mathscr{A}, \mathrm{d}) \to (\mathscr{A}, \mathrm{d})$$

and a unit morphism

$$i: \ (\mathbb{K}, 0) \to (\mathscr{A}, \mathrm{d})$$

satisfying the algebra axioms (Eq. (5.6)) in Section 5.4.1). One finds that this is equivalent to $\mathscr{A}$ being a super algebra, with a differential d that is a *derivation* of the product. The cohomology $H(\mathscr{A}, \mathrm{d})$ of a differential algebra is again a super algebra. One similarly defines differential Lie algebras, differential coalgebras, differential Hopf algebras, and so on. We will not spell out all of these definitions.

There are analogous definitions of categories of *graded differential spaces* (also known as *cochain complexes*) or *filtered differential spaces*. Thus, a graded (resp. filtered) differential space (E, d) is a graded (resp. filtered) super vector space E with a differential d of degree (resp. filtration degree) 1. Its cohomology $H(E, \mathrm{d})$ is again a graded (resp. filtered) super vector space. Morphisms of graded (resp. filtered) differential spaces are morphisms of graded (resp. filtered) super vector spaces intertwining the differentials.

Remark 6.1 If (E, d) is a (graded, filtered) differential space, and $n \in \mathbb{Z}$, then the same space with degree shift $(E[n], \mathrm{d})$ is a (graded, filtered) differential space. Indeed, we can regard $E[n]$ as a tensor product of differential spaces $E \otimes \mathbb{K}[n]$ where the differential on $\mathbb{K}[n]$ is trivial.

Example 6.1 For any manifold M, the algebra of differential forms $\Omega(M)$ is a graded differential algebra.

Example 6.2 Let $\mathbb{K}[\iota]$ be the commutative graded super algebra,[1] generated by an element ι of degree -1 satisfying $\iota^2 = 0$. (Thus $\mathbb{K}[\iota] = \mathbb{K} \cdot \iota \oplus \mathbb{K}$ as a graded super vector space.) Then $\mathbb{K}[\iota]$ is a graded differential algebra for the differential

$$\mathrm{d}(b\iota + a) = b.$$

Example 6.3 Let $\mathfrak{g}$ be a Lie algebra, and consider the graded super Lie algebra $\mathfrak{g}[1] \rtimes \mathfrak{g}$ with degree -1 generators $I_\xi \in \mathfrak{g}[1]$ and degree 0 generators $L_\xi \in \mathfrak{g}$, for $\xi \in \mathfrak{g}$. It is a graded differential Lie algebra, with differential given by $\mathrm{d}(I_\xi) = L_\xi$, $\mathrm{d}(L_\xi) = 0$. It may also be viewed as follows. Regard $\mathfrak{g}$ as a graded differential Lie algebra with trivial grading and zero differential. Then

$$\mathfrak{g}[1] \rtimes \mathfrak{g} = \mathfrak{g} \otimes \mathbb{K}[\iota]$$

is a tensor product with the commutative graded differential algebra $\mathbb{K}[\iota]$ from Example 6.2.

[1] Here $[\cdot]$ does not indicate a degree shift, but signifies a polynomial ring.

6.2 Symmetric and tensor algebra over differential spaces

Suppose E is a super vector space. Then the tensor algebra $T(E)$ and the symmetric algebra $S(E)$ are super algebras, in such a way that the inclusion of E is a morphism of super vector spaces. Recall that the definition of the symmetric algebra $S(E)$ uses the super-sign convention (cf. Appendix A): $S(E)$ is an algebra with generators $v \in E$, and relations

$$vw - (-1)^{|v||w|} wv = 0,$$

for homogeneous elements $v, w \in E$.

If E is a graded (resp. filtered) super space, the algebras $S(E)$ and $T(E)$ inherit an *internal grading* (resp. *internal filtration*), with the property that the inclusion of E preserves degrees. We can also consider the *total grading* (resp. *total filtration*), obtained by adding twice the external degree, i.e., such that the inclusion defines morphisms of graded (resp. filtered) super spaces $E[-2] \to S(E)$ and $E[-2] \to T(E)$. Note that $S(E)$ with the total grading is isomorphic to $S(E[-2])$ with the internal grading, and similarly for the tensor algebra.

From now on, unless specified otherwise, we will always work with the internal grading or filtration.

If (E, d) is a (graded, filtered) differential space, then $S(E)$ and $T(E)$ are (graded, filtered) differential algebras in such a way that the inclusion of E is a morphism. The differential on these super algebras is the derivation extension of the differential on E; the property $[\mathrm{d}, \mathrm{d}] = 0$ is immediate since it holds on generators. Similarly, if $(\mathfrak{g}, \mathrm{d})$ is a (graded, filtered) differential Lie algebra, then the enveloping algebra $U(\mathfrak{g})$ is a (graded, filtered) differential algebra.

6.3 Homotopies

Let $E_{\mathbb{K}} = \mathbb{K}[0] \oplus \mathbb{K}[-1]$, where $\mathbb{K}[0]$ is spanned by a generator t and $\mathbb{K}[-1]$ by a generator $\bar{t}$. Then $E_{\mathbb{K}}$ is a graded differential space, with differential

$$\mathrm{d}t = \bar{t}, \quad \mathrm{d}\bar{t} = 0.$$

The symmetric algebra over $E_{\mathbb{K}}$ is the commutative graded differential algebra $S(E_{\mathbb{K}}) = \mathbb{K}[t, \mathrm{d}t]$, with generators t of degree 0 and $\mathrm{d}t$ of degree 1, and with the single relation $(\mathrm{d}t)^2 = 0$. A general element of this algebra is a finite linear combination

$$y = \sum_k a_k t^k + \sum_l b_l t^l \mathrm{d}t \tag{6.1}$$

with $a_k, b_l \in \mathbb{K}$. Let $\pi_0, \pi_1 : \mathbb{K}[t, dt] \to \mathbb{K}$ be the morphisms of differential algebras, given on the element (6.1) by

$$\pi_0(y) = a_0, \quad \pi_1(y) = \sum_k a_k.$$

One can think of $\mathbb{K}[t, \mathrm{d}t]$ as an algebraic counterpart to differential forms on a unit interval, with π_0 and π_1 the evaluations at the end points. In the same spirit, we can define an "integration operator" $J:\ \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$,

$$J\Big(\sum_k a_k t^k + \sum_l b_l t^l \mathrm{d}t\Big) = \sum_l \frac{b_l}{l+1}.$$

This satisfies

Lemma 6.1 (Stokes' formula) *The integration operator $J:\ \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$ has the property $J \circ \mathrm{d} = \pi_1 - \pi_0$.*

Proof For $y = \sum_{k\geq 0} a_k t^k + \sum_{l\geq 0} b_l t^l \mathrm{d}t$, we have

$$J(\mathrm{d}y) = J\Big(\sum_{k>0} k a_k t^{k-1} \mathrm{d}t\Big) = \sum_{k>0} a_k = (\pi_1 - \pi_0)(y)$$

as claimed. □

Definition 6.2 A *homotopy* between two morphisms $\phi_0, \phi_1:\ E \to E'$ of (graded, filtered) differential spaces $(E, \mathrm{d}), (E', \mathrm{d}')$ is a morphism

$$\phi:\ E \to \mathbb{K}[t, \mathrm{d}t] \otimes E'$$

such that $\phi_0 = (\pi_0 \otimes 1) \circ \phi$ and $\phi_1 = (\pi_1 \otimes 1) \circ \phi$. In this case, ϕ_0 and ϕ_1 are called *homotopic*.

For instance, the two projections $\pi_0, \pi_1:\ \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$ are homotopic: the identity morphism of $\mathbb{K}[t, \mathrm{d}t]$ provides a homotopy. Homotopies can be composed: If

$$\phi:\ E \to \mathbb{K}[t, \mathrm{d}t] \otimes E', \quad \psi:\ E' \to \mathbb{K}[t, \mathrm{d}t] \otimes E''$$

are homotopies between $\phi_0, \phi_1:\ E \to E'$ and $\psi_0, \psi_1:\ E' \to E''$, respectively, then

$$(1 \otimes \psi) \circ \phi:\ E \to \mathbb{K}[t, \mathrm{d}t] \otimes \mathbb{K}[t, \mathrm{d}t] \otimes E'',$$

followed by multiplication in $\mathbb{K}[t, \mathrm{d}t]$, is a homotopy $\psi * \phi$ between $\psi_0 \circ \phi_0$ and $\psi_1 \circ \phi_1$. Note that the composition $*$ is associative. Let us write $\phi_0 \sim \phi_1$ for the relation of homotopy of morphisms of differential spaces.

Proposition 6.1 *The relation $\sim$ of homotopy of cochain maps between differential spaces E, E' is an equivalence relation.*

Proof If $\phi_0:\ E \to E'$ is a cochain map, then $\phi_0 \sim \phi_0$ by the homotopy

$$\phi = i \otimes \phi_0:\ E = \mathbb{K} \otimes E \to \mathbb{K}[t, \mathrm{d}t] \otimes E',$$

where $i:\ \mathbb{K} \to \mathbb{K}[t, \mathrm{d}t]$ is the inclusion of scalars. Hence $\sim$ is reflexive. To see that it is symmetric, consider the involution of the graded differential algebra $\mathbb{K}[t, \mathrm{d}t]$, given on generators by $t \mapsto 1 - t$, $\mathrm{d}t \mapsto -\mathrm{d}t$. This involution intertwines the two morphisms $\pi_0, \pi_1:\ \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$.

Hence, if $\phi_0 \sim \phi_1 :\ E \to E'$ by the homotopy $\phi :\ E \to \mathbb{K}[t, \mathrm{d}t] \otimes E'$, then $\phi_1 \sim \phi_0$ by the composition of ϕ with this involution. Finally, since one can add and subtract homotopies, $\phi_0 \sim \phi_1 :\ E \to E'$ and $\phi_1 \sim \phi_2 :\ E \to E'$ imply that

$$\phi_0 = \phi_0 + \phi_1 - \phi_1 \sim \phi_1 + \phi_2 - \phi_1 = \phi_2.$$

This shows that $\sim$ is transitive. □

Homotopies are often expressed in terms of homotopy operators.

Definition 6.3 A *homotopy operator* between $\phi_0, \phi_1 :\ E \to E'$ of differential spaces (E, d), (E', d') is an odd linear map $h :\ E \to E'$ with

$$h \circ \mathrm{d} + \mathrm{d}' \circ h = \phi_1 - \phi_0.$$

In the graded (resp. filtered) case, one requires that h is of degree (resp. filtration degree) -1.

For instance, the integration operator $J :\ \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$ is a homotopy operator between π_0, π_1.

Proposition 6.2 *Two morphisms $\phi_0, \phi_1 :\ E \to E'$ of differential spaces are homotopic if and only if there exists a homotopy operator. In this case the induced maps in cohomology are the same*: $H(\phi_0) = H(\phi_1)$.

Proof A homotopy operator h defines a homotopy $\phi :\ E \to \mathbb{K}[t, \mathrm{d}t] \otimes E'$ by

$$\phi(x) = \mathrm{d}t\ h(x) + \ t\phi_1(x) + (1 - t)\phi_0(x).$$

Conversely, given a homotopy ϕ between ϕ_0 and ϕ_1, the formula

$$h = (J \otimes 1) \circ \phi$$

defines a homotopy operator. The equation for the homotopy operator shows in particular that $\phi_0 - \phi_1$ takes $\ker(\mathrm{d})$ to $\mathrm{im}(\mathrm{d}')$. That is, $H(\phi_0) - H(\phi_1) = H(\phi_0 - \phi_1)$ $= 0$. □

Definition 6.4 Two morphisms $\phi :\ E \to E'$ and $\psi :\ E' \to E$ are called *homotopy inverses* if

$$\phi \circ \psi \sim \mathrm{id}_{E'}, \quad \psi \circ \phi \sim \mathrm{id}_E.$$

A morphism ϕ admitting a homotopy inverse ψ is also called a *homotopy equivalence*.

In this case $H(\phi)$ induces an isomorphism in cohomology, with inverse $H(\psi)$.

Example 6.4 Let $i :\ \mathbb{K} \to \mathbb{K}[t, \mathrm{d}t]$ be the inclusion of scalars, and $\pi :\ \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$ the augmentation map $\varepsilon(\sum_k a_k t^k + \sum_l b_l t^l \mathrm{d}t) = a_0$. Then $\varepsilon \circ i = \mathrm{id}_{\mathbb{K}}$. Let h be "integration from 0 to t",

$$h\Big(\sum_k a_k t^k + \sum_l b_l t^l \mathrm{d}t\Big) = \sum_l \frac{b_l}{l+1} t^{l+1}.$$

Then $h\mathrm{d}+\mathrm{d}h = \mathrm{id} - i\circ\varepsilon$, showing that i is a homotopy equivalence, with ε its homotopy inverse. That is, the differential algebra $\mathbb{K}[t,\mathrm{d}t]$ is acyclic.

6.4 Koszul algebras

Given a (graded, filtered) super vector space V, we obtain a (graded, filtered) differential space $E_V = V\otimes E_{\mathbb{K}}$. For $v\in V$, we will write $v\otimes t =: v$ and $v\otimes\overline{t} =: \overline{v}$; hence $E_V = V\oplus V[-1]$ with differential $\mathrm{d}v = \overline{v}$, $\mathrm{d}\overline{v} = 0$. It is characterized by the universal property that if E is a (graded filtered) differential space, then any morphism $V\to E$ of (graded, filtered) super spaces extends uniquely to a morphism $E_V\to E$ of (graded, filtered) differential spaces. Note that E_V has zero cohomology, since $\mathrm{im}(\mathrm{d}) = V[-1] = \ker(\mathrm{d})$.

Definition 6.5 The differential algebra $S(E_V)$ will be called the *Koszul algebra* for the (graded, filtered) super vector space V.

It is characterized by a universal property:

Proposition 6.3 (Universal property of Koszul algebra) *For any commutative* (*graded, filtered*) *differential algebra* $(\mathscr{A},\mathrm{d})$, *and any morphism of* (*graded, filtered*) *super vector spaces* $V\to\mathscr{A}$, *there is a unique extension to a homomorphism of* (*graded, filtered*) *differential algebras* $S(E_V)\to\mathscr{A}$.

Proof The morphism $V\to\mathscr{A}$ of super vector spaces extends uniquely to a morphism $E_V = V\oplus V[-1]\to\mathscr{A}$ of differential spaces. In turn, by the universal property of symmetric algebras (Appendix A, Proposition A.2) it extends further to a morphism $S(E_V)\to\mathscr{A}$ of super algebras; it is clear that this morphism intertwines differentials. □

One can also consider a *non-commutative version* of the Koszul algebra, using the tensor algebra $T(E_V)$ rather than the symmetric algebra. Using the same argument as for $S(E_V)$, we find:

Proposition 6.4 (Universal property of non-commutative Koszul algebra) *For any* (*graded, filtered*) *differential algebra* $(\mathscr{A},\mathrm{d})$ *and any morphism of* (*graded, filtered*) *super vector spaces* $V\to\mathscr{A}$, *there is a unique extension to a morphism of* (*graded, filtered*) *differential algebras* $T(E_V)\to\mathscr{A}$.

The Koszul algebras $S(E)$, resp. $T(E)$ play the role of "contractible spaces" in the category of commutative, resp. non-commutative differential algebras. In fact one has:

Theorem 6.1 *Let $\mathscr{A}$ be a (graded, filtered) differential algebra. Then any two morphisms of (graded, filtered) differential algebras $\phi_0, \phi_1 : T(E_V) \to \mathscr{A}$ are homotopic. If $\mathscr{A}$ is commutative, then any two homomorphisms of (graded, filtered) differential algebras $\phi_0, \phi_1 : S(E_V) \to \mathscr{A}$ are homotopic.*

Proof We present the proof for $T(E_V)$. (The proof for $S(E_V)$ is analogous.) Define a linear map

$$\phi : V \to \mathbb{K}[t, \mathrm{d}t] \otimes \mathscr{A}, \quad \phi(v) = (1-t)\phi_0(v) + t\phi_1(v),$$

and extend to a morphism of differential algebras

$$\phi : T(E_V) \to \mathbb{K}[t, \mathrm{d}t] \otimes \mathscr{A}$$

by the universal property of the Koszul algebra. Then $\phi_i = (\pi_i \otimes 1) \circ \phi$, since both sides are morphisms of differential algebras $T(E_V) \to \mathscr{A}$ that agree on V. □

Corollary 6.1 *The non-commutative Koszul algebra $T(E_V)$ is* acyclic. *That is, the augmentation map $\varepsilon : T(E_V) \to \mathbb{K}$ and the unit $i : \mathbb{K} \to T(E_V)$ are homotopy inverses. Similarly, the commutative Koszul algebra $S(E_V)$ is acyclic.*

Proof We have to show that $i \circ \varepsilon : T(E_V) \to T(E_V)$ is homotopic to $\mathrm{id} : T(E_V) \to T(E_V)$. But according to Theorem 6.1, *any* two morphisms of differential algebras $T(E_V) \to T(E_V)$ are homotopic. The proof for $S(E_V)$ is analogous. □

In particular, the unit maps $i : \mathbb{K} \to S(E_V), T(E_V)$ define isomorphisms of the cohomology algebras of $S(E_V)$ and $T(E_V)$ with $\mathbb{K}$.

Remark 6.2 Suppose that the (graded, filtered) super vector space V is a module over a Lie algebra $\mathfrak{g}$. Then the action of $\mathfrak{g}$ on E_V commutes with the differential, and so does its derivation extension to $S(E_V)$ and $T(E_V)$. The homotopy equivalences considered above are all $\mathfrak{g}$-equivariant. In particular, it follows that the $\mathfrak{g}$-invariant parts $S(E_V)^{\mathfrak{g}}$ and $T(E_V)^{\mathfrak{g}}$ are acyclic.

6.5 Symmetrization

Suppose E is a (graded, filtered) super vector space, $\mathscr{A}$ is a (graded filtered) super algebra (not necessarily commutative), and $\phi : E \to \mathscr{A}$ is a morphism of (graded, filtered) super vector spaces. Then ϕ extends canonically to a morphism of (graded, filtered) super vector spaces

$$\mathrm{sym}(\phi) : S(E) \to \mathscr{A},$$

by super symmetrization: For homogeneous elements $v_i \in E$,

$$v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} (-1)^{N_s(v_1, \ldots, v_k)} \phi(v_{s^{-1}(1)}) \cdots \phi(v_{s^{-1}(k)}).$$

Here $N_s(v_1, \ldots, v_k)$ is the number of pairs $i < j$ such that v_i, v_j are odd elements and $s^{-1}(i) > s^{-1}(j)$. (The sign is dictated by the super sign convention.) If the super algebra $\mathscr{A}$ is commutative, this is the unique extension as a morphism of super algebras. For the special case $\mathscr{A} = T(E)$, the symmetrization map is the inclusion as "symmetric tensors", and the general case may be viewed as this inclusion followed by the algebra homomorphism $T(E) \to \mathscr{A}$.

The symmetrization map can also be characterized as follows. Let e_i be a homogeneous basis of E, and let τ^i be formal parameters, with τ^i even or odd depending on whether e_i is even or odd. Then

$$\mathrm{sym}(\phi)\Big(\sum_i \tau^i e_i\Big)^k = \Big(\sum_i \tau^i \phi(e_i)\Big)^k$$

for all k. Equivalently,

$$\mathrm{sym}(\phi)\big(\mathrm{e}^{\sum_i \tau^i e_i}\big) = \mathrm{e}^{\sum_i \tau^i \phi(e_i)},$$

as an equality of formal power series in the τ_i.

Lemma 6.2 *Let E be a super vector space, and let $S(E)$ and $T(E)$ be the symmetric and tensor algebras respectively. Let D_S and D_T be the derivation extensions of a given (even or odd) endomorphism of $D \in \mathrm{End}(E)$. Then the inclusion $S(E) \to T(E)$ as symmetric tensors intertwines D_S, D_T.*

Proof This follows since the action of D_T on $T^k(E)$ commutes with the action of the symmetric group $\mathfrak{S}_k$, and in particular preserves the invariant subspace. For $k = 2$, with $\mathscr{T} \in \mathfrak{S}_2$ the transposition, this is checked by the computation

$$\begin{aligned}
\mathscr{T} D_T(v_1 \otimes v_2) &= \mathscr{T}\big(Dv_1 \otimes v_2 + (-1)^{|D||v_1|} v_1 \otimes Dv_2\big)\\
&= (-1)^{|v_1||v_2| + |D||v_2|} v_2 \otimes Dv_1 + (-1)^{|v_1||v_2|} Dv_2 \otimes v_1\\
&= (-1)^{|v_1||v_2|} D_T(v_2 \otimes v_1)\\
&= D_T \mathscr{T}(v_1 \otimes v_2).
\end{aligned}$$

The general case is reduced to the case $k = 2$, by writing a general element of $\mathfrak{S}_k$ as a product of transpositions. □

Proposition 6.5 *Suppose $\mathscr{A}$ is a differential algebra, E a differential space, and $\phi:\ E \to \mathscr{A}$ is a morphism of differential spaces. Then the symmetrized map*

$$\mathrm{sym}(\phi):\ S(E) \to \mathscr{A}$$

is a morphism of differential spaces.

Proof By Lemma 6.2, the inclusion $S(E) \to T(E)$ is a morphism of differential spaces, hence so is its composition with $T(E) \to \mathscr{A}$. □

As a special case if V is a super vector space, the symmetrization map for Koszul algebras $S(E_V) \to T(E_V)$ is a morphism of differential spaces.

Proposition 6.6 *The quotient map* $\pi : T(E_V) \to S(E_V)$ *is a homotopy equivalence of graded differential spaces, with homotopy inverse given by symmetrization*

$$\mathrm{sym} : S(E_V) \to T(E_V).$$

Proof Since $\pi \circ \mathrm{sym}$ is the identity, it suffices to show that $\mathrm{sym} \circ \pi$ is homotopic to the identity of $T(E_V)$. Let $\phi_0, \phi_1 : T(E_V) \to T(E_V) \otimes S(E_V)$ be the two morphisms of differential algebras $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$. Let

$$\psi : T(E_V) \otimes S(E_V) \to T(E_V),\; x \otimes y \mapsto x\mathrm{sym}(y).$$

Then ψ is a morphism of graded differential spaces. We have

$$\psi \circ \phi_0 = \mathrm{id}_{T(E_V)}, \quad \psi \circ \phi_1 = \mathrm{sym} \circ \pi.$$

By Theorem 6.1, ϕ_0 and ϕ_1 are homotopic in the category of graded differential spaces; hence so are $\psi \circ \phi_0$ and $\psi \circ \phi_1$. □

6.6 $\mathfrak{g}$-differential spaces

Differential algebras may be thought of as a generalization of differential forms on manifolds. Viewed in this way, the $\mathfrak{g}$-differential algebras discussed below are a generalization of differential forms on manifolds with $\mathfrak{g}$-actions. The concept of a $\mathfrak{g}$-differential algebra was introduced by H. Cartan in [32, 33].

Let $\mathfrak{g}$ be a Lie algebra. Recall the graded differential Lie algebra $\mathfrak{g}[1] \rtimes \mathfrak{g}$ introduced in Example 6.3.

Definition 6.6 A (graded, filtered) $\mathfrak{g}$-*differential space* is a (graded, filtered) differential space (E, d), together with a representation of the (graded, filtered) differential Lie algebra $\mathfrak{g}[1] \rtimes \mathfrak{g}$.

Here, "representation" is understood in the category of (graded, filtered) differential spaces: In particular, the action map $(\mathfrak{g}[1] \rtimes \mathfrak{g}) \otimes E \to E$ intertwines the differentials. Letting $\iota(\xi) \in \mathrm{End}^{\bar{1}}(E)$ and $L(\xi) \in \mathrm{End}^{\bar{0}}(E)$ be the images of $I_\xi, L_\xi \in \mathfrak{g}[1] \rtimes \mathfrak{g}$ under the representation, we find that the axioms of a $\mathfrak{g}$-differential space are equivalent to the following super commutation relations:

$$\begin{aligned}
[\mathrm{d}, \mathrm{d}] &= 0,\\
[\iota(\xi), \mathrm{d}] &= L(\xi),\\
[L(\xi), \mathrm{d}] &= 0,\\
[L(\xi), L(\zeta)] &= L([\xi, \zeta]),\\
[L(\xi), \iota(\zeta)] &= \iota([\xi, \zeta]),\\
[\iota(\xi), \iota(\zeta)] &= 0.
\end{aligned}$$

For a *graded (resp. filtered)* $\mathfrak{g}$-*differential space*, the operators d, $L(\xi)$, and $\iota(\xi)$ have degree (resp. filtration degree) $1, 0$, and -1.

Example 6.5 If V is a (graded, filtered) vector space with a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$, then the differential space structure of $E_V = V \oplus V[-1]$ extends uniquely to that of a (graded, filtered) $\mathfrak{g}$-differential space in such a way that $L(\xi)v$ for $v \in V$ is the given representation and $\iota(\xi)v = 0$. (The formulas on $\overline{v} \in V[-1]$ are determined by the commutation relations.)

Remark 6.3 View $\mathbb{K}[-1]$ as a commutative graded super Lie algebra, with generator D of degree 1. A (graded, filtered) differential space may be regarded as a representation of $\mathbb{K}[-1]$, in the category of (graded, filtered) super spaces, with D represented as the differential. Since the action of D on $\mathfrak{g}[1] \rtimes \mathfrak{g}$ is by super Lie derivations, we can form the semidirect product $(\mathfrak{g}[1] \rtimes \mathfrak{g}) \rtimes \mathbb{K}[-1]$. It is a graded Lie super algebra, where the bracket relations between generators D, I_ξ, L_ξ are similar to the commutator relations between d, $\iota(\xi)$, $L(\xi)$, as listed above. Hence, a (graded, filtered) $\mathfrak{g}$-differential space may also be viewed as a representation of $(\mathfrak{g}[1] \rtimes \mathfrak{g}) \rtimes \mathbb{K}[-1]$ in the category of (graded, filtered) super spaces.

Remark 6.4 The condition $[L(\xi), \mathrm{d}] = 0$ says that each $L(\xi)$ is a cochain map. The condition $[\iota(\xi), \mathrm{d}] = L(\xi)$ shows that these cochain maps are all homotopic to 0, with $\iota(\xi)$ as homotopy operators. In particular, $L(\xi)$ induces the 0 action on cohomology.

Direct sums and tensor products of $\mathfrak{g}$-differential spaces are defined in an obvious way, making the category of $\mathfrak{g}$-differential spaces into a tensor category. We may hence consider algebra objects, coalgebra objects, Lie algebra objects, and so on. For instance, a $\mathfrak{g}$-*differential (Lie) algebra* is a $\mathfrak{g}$-differential space, which is also a super (Lie) algebra, such that d, $L(\xi)$, $\iota(\xi)$ are super (Lie) algebra derivations.

Example 6.6 The differential Lie algebra $\mathfrak{g}[1] \rtimes \mathfrak{g}$ is an example of a $\mathfrak{g}$-differential Lie algebra. (The action of $\mathfrak{g}[1] \rtimes \mathfrak{g}$ is just the adjoint representation.)

Example 6.7 Suppose $\mathbb{K} = \mathbb{R}$, and let M be a manifold. An action of the Lie algebra $\mathfrak{g}$ on M is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ into the Lie algebra of vector fields on M. Given such an action, the algebra $\mathscr{A} = \Omega(M)$ of differential forms becomes a $\mathfrak{g}$-differential algebra, with d the de Rham differential, and $\iota(\xi)$ and $L(\xi)$ the contractions and Lie derivatives with respect to the vector fields $\rho(\xi)$.

Below we will encounter many other examples of $\mathfrak{g}$-differential algebras.

Definition 6.7 Let E be a (graded, filtered) $\mathfrak{g}$-differential space. One defines the *basic subspace* E_{bas} to be the (graded, filtered) differential subspace consisting of all $x \in E$ with $\iota(\xi)x = 0$ and $L(\xi)x = 0$ for all ξ. One calls

$$H_{\mathrm{bas}}(E) := H(E_{\mathrm{bas}}, \mathrm{d})$$

the *basic cohomology* of E.

Equivalently, E_{bas} is the subspace fixed by the action of $\mathfrak{g}[1] \rtimes \mathfrak{g}$. A morphism of $\mathfrak{g}$-differential spaces $E_1 \to E_2$ induces a morphism of differential spaces $(E_1)_{\mathrm{bas}} \to (E_2)_{\mathrm{bas}}$, hence of the basic cohomology.

We also define the *invariant subspace* $E_{\mathrm{inv}} = E^{\mathfrak{g}}$ to be the subspace annihilated by all $L(\xi)$, and the *horizontal subspace* E_{hor} to be the subspace annihilated by all $\iota(\xi)$. Thus $E_{\mathrm{bas}} = E_{\mathrm{hor}} \cap E^{\mathfrak{g}}$. Note however that the horizontal subspace is not in general a differential subspace.

Definition 6.8 A *connection* on a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathscr{A}$ is an odd linear map (of degree 1, filtration degree 1)

$$\theta:\ \mathfrak{g}^* \to \mathscr{A}$$

with the properties,

1. θ is $\mathfrak{g}$-equivariant: $\theta(L(\xi)\mu) = L(\xi)\theta(\mu)$,
2. $\iota(\xi)\theta(\mu) = \langle \mu, \xi\rangle$.

A $\mathfrak{g}$-differential algebra admitting a connection is called *locally free*.

Example 6.8 The $\mathfrak{g}$-differential algebra $\Omega(M)$ of differential forms on a $\mathfrak{g}$-manifold M is locally free if and only if the $\mathfrak{g}$ action on M is locally free, that is, $\xi \neq 0$ implies that the vector field $\rho(\xi) \in \mathfrak{X}(M)$ has no zeros.

The following is clear from the definition:

Proposition 6.7 *Let $\mathscr{A}$ be a (graded, filtered) $\mathfrak{g}$-differential algebra. If $\mathscr{A}$ is locally free, then the space of connections on $\mathscr{A}$ is an affine space with* $\mathrm{Hom}(\mathfrak{g}^*[-1], \mathscr{A}_{\mathrm{hor}})^{\mathfrak{g}}$ *as its space of motions.*

Here Hom denotes the space of morphisms of (graded, filtered) super spaces. The *curvature* of a connection is an even map $\mathfrak{g}^* \to \mathscr{A}$ of degree 2, defined by

$$F^{\theta} = \mathrm{d}\theta + \frac{1}{2}[\theta, \theta],$$

where we view θ as an element of $\mathscr{A} \otimes \mathfrak{g}$. Equivalently, if e_a is a basis of $\mathfrak{g}$, and writing $\theta = \sum_a \theta^a e_a$ with coefficients $\theta^a \in \mathscr{A}$, we have $F^{\theta} = \sum_a (F^{\theta})^a e_a$ where

$$(F^{\theta})^a = \mathrm{d}\theta^a + \frac{1}{2} f^a_{bc}\theta^b\theta^c.$$

The curvature map $F^{\theta}:\ \mathfrak{g}^* \to \mathscr{A}$ is $\mathfrak{g}$-equivariant, and it takes values in $\mathscr{A}_{\mathrm{hor}}$. That is, it defines an element of $\mathrm{Hom}(\mathfrak{g}^*[-2], \mathscr{A}_{\mathrm{hor}})^{\mathfrak{g}}$.

6.7 The $\mathfrak{g}$-differential algebra $\wedge\mathfrak{g}^*$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and $\wedge\mathfrak{g}^*$ the exterior algebra over the dual space. As explained in Section 2.1.2, the space of graded derivations is

$$\mathrm{Der}(\wedge\mathfrak{g}^*) \cong \mathrm{Hom}(\mathfrak{g}^*, \wedge\mathfrak{g}^*),$$

since any derivation is determined by its values on generators. Let

$$\mathrm{d} \in \mathrm{Hom}(\mathfrak{g}^*, \wedge^2\mathfrak{g}^*) \cong \mathrm{Der}^1(\wedge\mathfrak{g}^*)$$

be the map dual to the Lie bracket $[\cdot,\cdot]\colon\ \wedge^2\mathfrak{g} \to \mathfrak{g}$. That is, for $\mu \in \mathfrak{g}^*$ and $\xi_1, \xi_2 \in \mathfrak{g}$,

$$\iota(\xi_1)\iota(\xi_2)\mathrm{d}\mu = \langle \mu, [\xi_1, \xi_2]\rangle. \tag{6.2}$$

Then $(\wedge\mathfrak{g}^*, \mathrm{d})$ is a graded differential algebra. To see that $\mathrm{d}^2 = 0$, it suffices to check on generators. It is actually interesting to consider the derivation d defined by an arbitrary linear map $[\cdot,\cdot]\colon\ \wedge^2\mathfrak{g} \to \mathfrak{g}$ (not necessarily a Lie bracket). Define the *Jacobiator* $\mathrm{Jac}\colon\ \mathfrak{g}^* \to \wedge^3\mathfrak{g}^*$ by

$$\iota(\xi_1)\iota(\xi_2)\iota(\xi_3)\mathrm{Jac}(\mu) = \langle \mu, [\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]]\rangle,$$

so that $[\cdot,\cdot]$ is a Lie bracket if and only if $\mathrm{Jac} = 0$. Extend to a derivation $\mathrm{Jac} \in \mathrm{Der}^2(\wedge\mathfrak{g}^*)$.

Proposition 6.8 *Let* $\mathfrak{g}$ *be a finite-dimensional vector space, and let* $\mathrm{d} \in \mathrm{Der}(\wedge\mathfrak{g}^*)$ *be defined by dualizing a linear map (bracket)* $[\cdot,\cdot]\colon\ \wedge^2\mathfrak{g} \to \mathfrak{g}$. *Then* d *is a differential if and only if* $[\cdot,\cdot]$ *is a Lie bracket. In fact,*

$$\mathrm{d}^2 = \mathrm{Jac},$$

where Jac *is the Jacobiator of the bracket.*

Proof Define $L(\xi) \in \mathrm{Der}^0(\mathfrak{g})$ by $L(\xi) = [\mathrm{d}, \iota(\xi)]$. Thus $\langle L(\xi)\mu, \zeta\rangle = -\langle \mu, [\xi, \zeta]\rangle$ for all $\mu \in \mathfrak{g}^*, \xi, \zeta \in \mathfrak{g}$. Checking on generators, we find that

$$[L(\xi), \iota(\zeta)] = \iota([\xi, \zeta]).$$

The square of d is a derivation, since it may be written $\mathrm{d}^2 = \frac{1}{2}[\mathrm{d}, \mathrm{d}]$. To find what it is, we compute

$$\begin{aligned}\iota(\xi_1)\iota(\xi_2)\mathrm{d}\mathrm{d}\mu &= \iota(\xi_1)(L(\xi_2) - \mathrm{d}\iota(\xi_2))\mathrm{d}\mu\\ &= \big(\iota([\xi_1, \xi_2])\mathrm{d} + L(\xi_2)\iota(\xi_1)\mathrm{d} - \iota(\xi_1)\mathrm{d}L(\xi_2)\big)\mu\\ &= \big(L([\xi_1, \xi_2]) + L(\xi_2)L(\xi_1) - L(\xi_1)L(\xi_2)\big)\mu.\end{aligned}$$

Hence

$$\begin{aligned}\iota(\xi_1)\iota(\xi_2)\iota(\xi_3)\mathrm{d}\mathrm{d}\mu &= \iota(\xi_3)\iota(\xi_1)\iota(\xi_2)\mathrm{d}\mathrm{d}\mu\\ &= \langle \mu, [\xi_3, [\xi_1, \xi_2]] + [[\xi_3, \xi_2], \xi_1] - [[\xi_3, \xi_1], \xi_2]\rangle\\ &= \iota(\xi_1)\iota(\xi_2)\iota(\xi_3)\mathrm{Jac}(\mu),\end{aligned}$$

proving $\mathrm{d}^2 = \mathrm{Jac}$. □

Suppose for the remainder of this section that $\mathfrak{g}$ is a Lie algebra, so that $\mathrm{Jac} = 0$. The cohomology algebra $H(\wedge\mathfrak{g}^*, \mathrm{d})$ is called the *Lie algebra cohomology* of $\mathfrak{g}$, and is denoted by $H(\mathfrak{g})$. We will frequently denote the differential on $\wedge\mathfrak{g}^*$ by $\mathrm{d}_\wedge$. Using dual bases $e_a \in \mathfrak{g}$ and $e^a \in \mathfrak{g}^*$, the Lie algebra differential may be written as

$$\mathrm{d}_\wedge = \frac{1}{2}\sum_a e^a \circ L(e_a), \tag{6.3}$$

(with e^a acting by exterior multiplication), as is checked on generators. In particular, the $\mathfrak{g}$-invariants $(\wedge\mathfrak{g}^*)^{\mathfrak{g}}$ are all cocycles, defining a morphism of graded algebras

$$(\wedge\mathfrak{g}^*)^{\mathfrak{g}} \to H(\mathfrak{g}).$$

In Section 10.1 we will show that for $\mathfrak{g}$ complex reductive, this map is an isomorphism.

Remark 6.5 Suppose $\mathbb{K} = \mathbb{R}$, and $\mathfrak{g}$ is the Lie algebra of a Lie group G. The identification of $\mathfrak{g}$ with left-invariant vector fields dualizes to an identification of $\mathfrak{g}^*$ with left-invariant 1-forms. This extends to an isomorphism of $\wedge\mathfrak{g}^*$ with the algebra of left-invariant differential forms on G, and the Lie algebra differential corresponds to the de Rham differential under this identification. If G is connected, then $(\wedge\mathfrak{g}^*)^{\mathfrak{g}} = (\wedge\mathfrak{g}^*)^G$ is identified with the bi-invariant forms on G.

Remark 6.6 Let $f^a_{bc} = \langle e^a, [e_b, e_c]\rangle$ be the structure constants of $\mathfrak{g}$ relative to the given basis. Then the differential on $\wedge\mathfrak{g}^*$ reads as

$$\mathrm{d}_\wedge = -\frac{1}{2}\sum_{abc} f^a_{bc} e^b e^c \iota(e_a). \tag{6.4}$$

To summarize, the exterior algebra $\wedge\mathfrak{g}^*$, with the Lie algebra differential $\mathrm{d}_\wedge$, and with the usual Lie derivatives and contraction operators, is a graded $\mathfrak{g}$-differential algebra. The horizontal and basic subspace of $\wedge\mathfrak{g}^*$ consists of the scalars, hence the basic cohomology is just $\mathbb{K}$. The map $\theta: \mathfrak{g}^* \to \wedge\mathfrak{g}^*$, given as the inclusion of $\wedge^1\mathfrak{g}^*$, is the unique connection on $\wedge\mathfrak{g}^*$; the curvature is zero (since $(\wedge^2\mathfrak{g}^*)_{\mathrm{hor}} = 0$).

More generally, suppose $L_V: \mathfrak{g} \to \mathrm{End}(V)$ is a $\mathfrak{g}$-representation on a vector space V. Let

$$C^\bullet(\mathfrak{g}, V) = V \otimes \wedge^\bullet\mathfrak{g}^*$$

with grading induced from the grading on the exterior algebra, and define the (Chevalley–Eilenberg) differential

$$\mathrm{d}_{CE} = \sum_a L_V(e_a) \otimes e^a + 1 \otimes \mathrm{d}_\wedge. \tag{6.5}$$

Then $(C^\bullet(\mathfrak{g}, V), \mathrm{d}_{CE})$ is a graded differential space. The cohomology groups of the complex $C^\bullet(\mathfrak{g}, V)$ are denoted by $H^\bullet(\mathfrak{g}, V)$. Note that

$$H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}.$$

With the Lie derivatives $L(\xi) = L_V(\xi) \otimes 1 + 1 \otimes L_\wedge(\xi)$ and contractions $\iota(\xi) = 1 \otimes \iota_\wedge(\xi)$, the complex $C^\bullet(\mathfrak{g}, V)$ becomes a graded $\mathfrak{g}$-differential space. The basic subcomplex is $V^{\mathfrak{g}}$ with the zero differential.

6.8 $\mathfrak{g}$-homotopies

We next generalize the definition of homotopies:

Definition 6.9 Let $\phi_0, \phi_1 : E \to E'$ be morphisms of (graded, filtered) $\mathfrak{g}$-differential spaces.

1. A $\mathfrak{g}$-*homotopy* between ϕ_0 and ϕ_1 is a morphism of (graded, filtered) $\mathfrak{g}$-differential spaces $\phi : E \to \mathbb{K}[t, \mathrm{d}t] \otimes E'$ with
$$\phi_0 = (\pi_0 \otimes 1) \circ \phi, \ \phi_1 = (\pi_1 \otimes 1) \circ \phi.$$
2. A $\mathfrak{g}$-*homotopy operator* between ϕ_0 and ϕ_1 is a morphism of (graded, filtered) super spaces $h : E[1] \to E'$ such that
$$h \circ \iota(\xi) + \iota'(\xi) \circ h = 0,$$
$$h \circ \mathrm{d} + \mathrm{d}' \circ h = \phi_1 - \phi_0.$$

Thus h is odd (of degree -1 in the graded or filtered cases). The definition of h implies that it intertwines Lie derivatives as well:

$$\begin{aligned}[h, L(\xi)] &= [h, [\mathrm{d}, \iota(\xi)]] \\ &= [[h, \mathrm{d}], \iota(\xi)] - [\mathrm{d}, [h, \iota(\xi)]] \\ &= [\phi_1 - \phi_0, \iota(\xi)] = 0.\end{aligned}$$

(By a small abuse of notation, we wrote $[h, L(\xi)]$ for $h \circ L(\xi) - L'(\xi) \circ h$, etc.) In other words, a $\mathfrak{g}$-homotopy operator h is a homotopy operator such that the map $h : E[1] \to E'$ is $\mathfrak{g}[1] \rtimes \mathfrak{g}$-equivariant.

The discussion of homotopies in Section 6.3 extends to the case of $\mathfrak{g}$-homotopies, with the obvious changes. In particular, the same argument given in Proposition 6.2 shows that ϕ_0 and ϕ_1 are $\mathfrak{g}$-homotopic if and only if there is a $\mathfrak{g}$-homotopy operator. In this case the maps in basic cohomology are the same:

$$H_{\mathrm{bas}}(\phi_0) = H_{\mathrm{bas}}(\phi_1) : \ H_{\mathrm{bas}}(E) \to H_{\mathrm{bas}}(E').$$

6.9 The Weil algebra

Consider the Koszul algebra for $\mathfrak{g}^*[-1]$,

$$S(E_{\mathfrak{g}^*}[-1]) = S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*).$$

As before, we associate to each $\mu \in \mathfrak{g}^*$ the degree 1 generators $\mu \in \mathfrak{g}^*[-1] = \wedge^1 \mathfrak{g}^*$ and the degree 2 generators $\overline{\mu} \in \mathfrak{g}^*[-2] = S^1 \mathfrak{g}^*$, so that $\mathrm{d}\mu = \overline{\mu}$, $\mathrm{d}\overline{\mu} = 0$. The coadjoint $\mathfrak{g}$-representation on $\mathfrak{g}^*$ defines a representation on $E_{\mathfrak{g}^*}$, commuting with the differential. The resulting $\mathfrak{g}$-representation by derivations of $S(E_{\mathfrak{g}^*}[-1])$ is the tensor product of the coadjoint representations on $S(\mathfrak{g}^*)$ and $\wedge(\mathfrak{g}^*)$; the generators for the action will be denoted by $L(\xi)$. To turn $S(E_{\mathfrak{g}^*}[-1])$ into a $\mathfrak{g}$-differential algebra, we need to define the contraction operators. The action of $\iota(\xi)$ on $\overline{\mu}$ is determined:

$$\iota(\xi)\overline{\mu} = \iota(\xi)\mathrm{d}\mu = L(\xi)\mu - \mathrm{d}\iota(\xi)\mu = L(\xi)\mu.$$

On the degree 1 generators, it is natural to take $\iota(\xi)\mu = \langle \mu, \xi \rangle$. It is straightforward to check the relations involving $\iota(\xi)$ on generators, so that we have turned $S(E_{\mathfrak{g}^*}[-1])$ into a $\mathbb{Z}$-graded $\mathfrak{g}$-differential algebra.[2]

Definition 6.10 The graded $\mathfrak{g}$-differential algebra

$$W\mathfrak{g} = S(E_{\mathfrak{g}^*}[-1])$$

is called the *Weil algebra* for $\mathfrak{g}$.

In terms of a basis e_a of $\mathfrak{g}$, with dual basis e^a and structure constants $f^a_{bc} = \langle e^a, [e_b, e_c] \rangle$, the Weil differential, contractions, and Lie derivatives are given on generators by

$$\begin{aligned} \mathrm{d}e^a &= \overline{e}^a, & \mathrm{d}\overline{e}^a &= 0, \\ L(e_b)e^a &= -\sum_c f^a_{bc} e^c, & L(e_b)\overline{e}^a &= -\sum_c f^a_{bc} \overline{e}^c, \\ \iota(e_b)e^a &= \delta^a_b, & \iota(e_b)\overline{e}^a &= -\sum_c f^a_{bc} e^c. \end{aligned}$$

As a special case of Corollary 6.1 (and the subsequent remark) we have,

Proposition 6.9 *The Weil algebra $W\mathfrak{g}$, as well as its $\mathfrak{g}$-invariant part $(W\mathfrak{g})^{\mathfrak{g}}$, are acyclic differential algebras.*

Proposition 6.10 *The Weil algebra is locally free, with connection*

$$\theta_W : \mathfrak{g}^* \to W\mathfrak{g},\ \mu \mapsto \mu.$$

The curvature of the connection on $W\mathfrak{g}$ is given by

$$F^{\theta_W} : \mathfrak{g}^* \to W^2\mathfrak{g},\ \mu \mapsto \overline{\mu} - \mathrm{d}_\wedge \mu.$$

Proof It is immediate that $\theta_W(\mu) = \mu$ is a connection. In terms of a basis of $\mathfrak{g}$, the components of the connection 1-form on $W\mathfrak{g}$ are $\theta^a_W = e^a$; hence those of the curvature form are

$$(F^{\theta_W})^a = \mathrm{d}e^a + \frac{1}{2}\sum_{bc} f^a_{bc} e^b e^c.$$

The first term is $\overline{e^a}$, while the second term is $-\mathrm{d}_\wedge e^a$ (see Eq. (6.4)). $\square$

[2]More generally, for any $\kappa \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})^{\mathfrak{g}}$, one obtains a $\mathfrak{g}$-differential algebra by putting $\iota(\xi)\mu = \langle \mu, \kappa(\xi) \rangle$.

Remark 6.7 We may verify directly that $\overline{\mu} - \mathrm{d}_\wedge(\mu)$ is horizontal:

$$\iota(\xi)\overline{\mu} - \mathrm{d}_\wedge(\mu) = L(\xi)\mu - L_\wedge(\mu) + \mathrm{d}_\wedge(\iota(\xi)\mu) = 0,$$

since $L(\xi)$ agrees with $L_\wedge(\xi)$ on $\wedge\mathfrak{g}^* \subseteq W\mathfrak{g}$, and since $\mathrm{d}_\wedge$ vanishes on scalars. (Thus $\overline{\mu} - \mathrm{d}_\wedge\mu$ is the horizontal projection of $\overline{\mu}$, which gives an alternative proof of the curvature formula.)

By definition, the Weil algebra $W\mathfrak{g}$ is a tensor product $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$, where $S\mathfrak{g}^*$ is the symmetric algebra generated by the variables $\overline{\mu}$ and $\wedge\mathfrak{g}^*$ is the exterior algebra generated by the variables μ. It is often convenient to replace these generators μ and $\overline{\mu}$ with generators μ and $\widehat{\mu}$, where

$$\widehat{\mu} := \overline{\mu} - \mathrm{d}_\wedge\mu \in W^2\mathfrak{g}$$

are the curvature variables. Thus we obtain a second identification of the Weil algebra $W\mathfrak{g}$ with $S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$, where now $S\mathfrak{g}^*$ is the symmetric algebra generated by the variables $\widehat{\mu}$. The main advantage of this change of variables is that the contraction operators simplify to

$$\iota(\xi)\mu = \langle \mu, \xi\rangle, \quad \iota(\xi)\widehat{\mu} = 0.$$

Theorem 6.2 *The horizontal and basic subspaces of the Weil algebra $W\mathfrak{g}$ are*

$$(W\mathfrak{g})_{\mathrm{hor}} = S\mathfrak{g}^*, \quad (W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g}^*)_{\mathfrak{g}},$$

where $S\mathfrak{g}^*$ is the symmetric algebra generated by the variables $\widehat{\mu}$. The differential on the basic subcomplex is just 0, *so*

$$H_{\mathrm{bas}}(W\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}.$$

Proof The description of the horizontal and basic subspaces is immediate. Since the basic subcomplex $(W\mathfrak{g})_{\mathrm{bas}}$ only contains elements of even degree, its differential is zero and hence the complex coincides with its cohomology. □

We next describe the differential of $W\mathfrak{g}$ in the new variables.

Theorem 6.3 *Identify $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$, where $\wedge\mathfrak{g}^*$ is the exterior algebra generated by the variables μ, and $S\mathfrak{g}^*$ is the symmetric algebra generated by the variables $\widehat{\mu}$. Let d_{CE} be the Chevalley–Eilenberg differential* (6.5) *for the $\mathfrak{g}$-module $S\mathfrak{g}^*$, and let d_K be the Koszul differential relative to the generators $\mu, \widehat{\mu}$, that is, $\mathrm{d}_K\mu = \widehat{\mu}$, $\mathrm{d}_K\widehat{\mu} = 0$. Then the super derivations d_{CE} and d_K commute, and the Weil differential is their sum*:

$$\mathrm{d} = \mathrm{d}_K + \mathrm{d}_{CE}.$$

Proof For all $\mu \in \mathfrak{g}^*$, we have

$$\begin{aligned}\mathrm{d}\mu &= \overline{\mu} = \widehat{\mu} + \mathrm{d}_\wedge(\mu) = \mathrm{d}_K\mu + \mathrm{d}_{CE}\mu\\ \mathrm{d}\widehat{\mu} &= -\mathrm{d}(\mathrm{d}_\wedge\mu) = \sum_i \widehat{L(e_i)\mu} \otimes e^i = \mathrm{d}_{CE}\widehat{\mu} = (\mathrm{d}_K + \mathrm{d}_{CE})\widehat{\mu}.\end{aligned}$$

This gives the equality of derivations $\mathrm{d} = \mathrm{d}_K + \mathrm{d}_{CE}$ on generators, and hence everywhere. The fact that $\mathrm{d}_K, \mathrm{d}_{CE}$ commute follows since $0 = \mathrm{d}^2 - \mathrm{d}_K^2 - \mathrm{d}_{CE}^2 = \mathrm{d}_K \mathrm{d}_{CE} + \mathrm{d}_{CE} \mathrm{d}_K$. $\square$

In terms of the basis e_a, the Weil differential, Lie derivatives, and contractions are given on the new generators by

$$\begin{aligned} \mathrm{d}e^a &= \widehat{e}^a - \frac{1}{2}\sum_{bc} f^a_{bc} e^b e^c, \qquad & \mathrm{d}\widehat{e}^a &= \sum\nolimits_{bc} f^a_{bc} \widehat{e}^b e^c, \\ L(e_b)e^a &= -\sum\nolimits_c f^a_{bc} e^c, \qquad & L(e_b)\widehat{e}^a &= -\sum\nolimits_c f^a_{bc} \widehat{e}^c, \\ \iota(e_b)e^a &= \delta^a_b, \qquad & \iota(e_b)\widehat{e}^a &= 0. \end{aligned}$$

Remark 6.8 The automorphism of $S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ given on generators by $\mu \mapsto \mu$, $\widehat{\mu} \mapsto \frac{1}{r}\widehat{\mu}$, for $r \neq 0$, intertwines Lie derivatives and contractions, and takes d to the differential

$$\mathrm{d}^{(r)} = r\mathrm{d}_K + \mathrm{d}_{CE}.$$

Let $(W\mathfrak{g})^{(r)} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ be the family of $\mathfrak{g}$-differential algebras with the derivations $\mathrm{d}^{(r)}$, $\iota(\xi)$, $L(\xi)$. For $r \neq 0$ these are all isomorphic, but for $r = 0$ the family degenerates to the Chevalley–Eilenberg complex for the $\mathfrak{g}$-module $S\mathfrak{g}^*$. Later, we will find it convenient to work with $r = 2$.

6.10 Chern–Weil homomorphisms

The Weil algebra is universal among commutative $\mathfrak{g}$-differential algebras with connection.

Proposition 6.11 (Universal property of the Weil algebra) *For any commutative* (*graded, filtered*) $\mathfrak{g}$-*differential algebra* $\mathscr{A}$ *with connection* $\theta_{\mathscr{A}}$, *there is a unique morphism of* (*graded, filtered*) $\mathfrak{g}$-*differential algebras*

$$c:\ W\mathfrak{g} \to \mathscr{A} \tag{6.6}$$

such that $c \circ \theta_W = \theta_{\mathscr{A}}$.

Proof Suppose $\mathscr{A}$ is a commutative $\mathfrak{g}$-differential algebra with connection. By the universal property of Koszul algebras, the map $\theta:\ \mathfrak{g}^* \to \mathscr{A}^{\bar{1}}$ extends uniquely to a homomorphism of differential algebras $c:\ W\mathfrak{g} \to \mathscr{A}$. The calculation

$$\begin{aligned} \iota(\xi)c(\overline{\mu}) &= \iota(\xi)\mathrm{d}\theta(\mu) \\ &= L(\xi)\theta(\mu) - \mathrm{d}\iota(\xi)\theta(\mu) \\ &= \theta(L(\xi)\mu) - \mathrm{d}\langle \mu, \xi \rangle \\ &= c(L(\xi)\mu) = c(\iota(\xi)\overline{\mu}), \end{aligned}$$

together with $\iota(\xi)c(\mu) = \iota(\xi)\theta(\mu) = \langle\mu, \xi\rangle = c(\iota(\xi)\mu)$, shows that c intertwines contractions. Since $L(\xi) = [\iota(\xi), \mathrm{d}]$, it intertwines Lie derivatives as well. $\square$

The homomorphism c is called the *characteristic homomorphism* for the connection $\theta_{\mathscr{A}}$. Note that conversely if $\mathscr{A}$ is a commutative $\mathfrak{g}$-differential algebra, then a morphism of $\mathfrak{g}$-differential algebras $c: W\mathfrak{g} \to \mathscr{A}$ determines a connection by $\theta_{\mathscr{A}} = c \circ \theta_W$.

Examples 6.9

1. The connection $\theta_\wedge$ on the commutative $\mathfrak{g}$-differential algebra $\mathscr{A} = \wedge\mathfrak{g}^*$ defines a morphism of $\mathfrak{g}$-differential algebras
$$W\mathfrak{g} \to \wedge\mathfrak{g}^*. \tag{6.7}$$
This is the map taking the generators $\widehat{\mu}$ to zero.
2. For any commutative $\mathfrak{g}$-differential algebra $\mathscr{A}$, the tensor product $W\mathfrak{g} \otimes \mathscr{A}$ has a connection $\theta_W \otimes 1$. This defines a morphism $W\mathfrak{g} \to W\mathfrak{g} \otimes \mathscr{A}$, $w \mapsto w \otimes 1$, and hence a map in basic cohomology
$$(S\mathfrak{g}^*)^{\mathfrak{g}} = H_{\mathrm{bas}}(W\mathfrak{g}) \to H_{\mathrm{bas}}(W\mathfrak{g} \otimes \mathscr{A}).$$

The argument from the proof of Theorem 6.1 extends to the case of $\mathfrak{g}$-differential algebras.

Theorem 6.4 *Suppose $\mathscr{A}$ is a commutative (graded, filtered) $\mathfrak{g}$-differential algebra. Then any two morphisms of (graded, filtered) $\mathfrak{g}$-differential algebras $c_0, c_1: W\mathfrak{g} \to \mathscr{A}$ are $\mathfrak{g}$-homotopic.*

Proof Let $\theta_0, \theta_1: \mathfrak{g}^* \to \mathscr{A}$ be the connections defined by the restrictions of c_0, c_1. Then
$$\theta = (1-t)\theta_0 + t\theta_1$$
is a connection on $\mathbb{K}[t, \mathrm{d}t] \otimes \mathscr{A}$, and its characteristic homomorphism
$$c: W\mathfrak{g} \to \mathbb{K}[t, \mathrm{d}t] \otimes \mathscr{A}$$
is the desired $\mathfrak{g}$-homotopy. $\square$

As an immediate consequence, if $\mathscr{A}$ is a commutative $\mathfrak{g}$-differential algebra admitting a connection, then the resulting algebra morphism in basic cohomology
$$(S\mathfrak{g}^*)^{\mathfrak{g}} = H_{\mathrm{bas}}(W\mathfrak{g}) \to H_{\mathrm{bas}}(\mathscr{A}) \tag{6.8}$$
is independent of the connection.

Definition 6.11 Let $\mathscr{A}$ be a $\mathfrak{g}$-differential algebra with connection. The resulting algebra homomorphism
$$(S\mathfrak{g}^*)^{\mathfrak{g}} \to H_{\mathrm{bas}}(\mathscr{A})$$
is called the *Chern–Weil homomorphism*.

Remark 6.9 This construction goes back to the differential geometry of principal bundles. Let G be a Lie group, and $P \to B$ a principal G-bundle. Then there exists a continuous map $f: B \to BG$ into the classifying space of G, unique up to homotopy, such that $P \cong f^*(EG)$ is the pull-back of the classifying bundle. The resulting map in cohomology $f^*: H(BG) \to H(B)$ (defined for any choice of coefficients) is thus independent of the choice of f. Elements in its image are called the *characteristic classes* of P. Working in the smooth category and with real coefficients, the Chern–Weil theory realizes the characteristic classes as de Rham cohomology classes of differential forms, constructed using connections and curvature. The theory of characteristic classes was pioneered by S.S. Chern [37]; A. Weil introduced $W\mathfrak{g}$ as an algebraic model for differential forms on the classifying bundle EG, see H. Cartan's lectures [32].

Proposition 6.12 *Let $\mathscr{A}$ be a commutative $\mathfrak{g}$-differential algebra with connection, and denote by $c: W\mathfrak{g} \to \mathscr{A}$ the characteristic homomorphism. Then the map*

$$\phi: W\mathfrak{g} \otimes \mathscr{A} \to \mathscr{A}, \ w \otimes x \mapsto c(w)x$$

is a $\mathfrak{g}$-homotopy equivalence, with $\mathfrak{g}$-homotopy inverse the inclusion,

$$\psi: \mathscr{A} \to W\mathfrak{g} \otimes \mathscr{A}, \ x \mapsto 1 \otimes x.$$

Proof Clearly, $\phi \circ \psi = \mathrm{id}_{\mathscr{A}}$. On the other hand, $(\psi \circ \phi)(w \otimes x) = 1 \otimes c(w)x$. Let $\tau_0, \tau_1: W\mathfrak{g} \to W\mathfrak{g} \otimes \mathscr{A}$ be the characteristic homomorphisms for the connections $\theta_0 = 1 \otimes \theta_{\mathscr{A}}$ and $\theta_1 = \theta_W \otimes 1$ on $W\mathfrak{g} \otimes \mathscr{A}$. Thus $\tau_0(w) = 1 \otimes c(w)$, $\tau_1(w) = w \otimes 1$. We have

$$\psi \circ \phi = (\mathrm{id}_{W\mathfrak{g}} \otimes m_{\mathscr{A}}) \circ (\tau_0 \otimes \mathrm{id}_{\mathscr{A}}),$$
$$\mathrm{id}_{W\mathfrak{g}\otimes\mathscr{A}} = (\mathrm{id}_{W\mathfrak{g}} \otimes m_{\mathscr{A}}) \circ (\tau_1 \otimes \mathrm{id}_{\mathscr{A}}),$$

where $m_{\mathscr{A}}$ is the multiplication in $\mathscr{A}$. By Theorem 6.4, τ_0 is $\mathfrak{g}$-homotopic to τ_1. Since $\mathfrak{g}$-homotopies can be composed, it follows that $\psi \circ \phi$ is $\mathfrak{g}$-homotopic to $\mathrm{id}_{W\mathfrak{g}\otimes\mathscr{A}}$. $\square$

6.11 The non-commutative Weil algebra $\widetilde{W}\mathfrak{g}$

The Weil algebra $W\mathfrak{g} = S(E_{\mathfrak{g}^*}[-1])$ is characterized by its universal property among the commutative $\mathfrak{g}$-differential algebras with connection. To obtain a similar universal object among non-commutative $\mathfrak{g}$-differential algebras with connection, we only have to replace the symmetric algebra with the tensor algebra. As a differential algebra, $\widetilde{W}\mathfrak{g} = T(E_{\mathfrak{g}^*}[-1])$ is the non-commutative Koszul algebra (cf. Section 6.4), freely generated by degree 1 generators μ and degree 2 generators $\overline{\mu}$. The formulas for the contractions are given on generators by just the same formulas as for $W\mathfrak{g}$:

$$\iota(\xi)\mu = \langle \mu, \xi \rangle, \quad \iota(\xi)\overline{\mu} = L(\xi)\mu.$$

Definition 6.12 (Alekseev–Meinrenken [7]) The graded $\mathfrak{g}$-differential algebra

$$\widetilde{W}\mathfrak{g} = T(E_{\mathfrak{g}^*}[-1])$$

is called the *non-commutative Weil algebra*.[3]

Most of the results for the commutative Weil algebra carry over to the non-commutative case, with essentially the same proofs (simply replace W with $\widetilde{W}$). We will not repeat the proofs, but just state the results.

1. The non-commutative Weil algebra $\widetilde{W}\mathfrak{g}$ is a locally free $\mathfrak{g}$-differential algebra, with connection
$$\theta_{\widetilde{W}} : \mathfrak{g}^* \to \widetilde{W}\mathfrak{g}, \; \mu \mapsto \mu.$$
2. Given a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathscr{A}$ with connection $\theta_{\mathscr{A}}$, there is a unique morphism of (graded, filtered) $\mathfrak{g}$-differential algebras
$$c: \widetilde{W}\mathfrak{g} \to \mathscr{A}$$
such that $c \circ \theta_W = \theta_{\mathscr{A}}$ (cf. Proposition 6.11).
3. Given a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathscr{A}$, any two morphisms of (graded, filtered) $\mathfrak{g}$-differential algebras $c_0, c_1 : \widetilde{W}\mathfrak{g} \to \mathscr{A}$ are $\mathfrak{g}$-homotopic (cf. Theorem 6.4). We hence see that for a non-commutative $\mathfrak{g}$-differential algebra $\mathscr{A}$ with connection, the algebra morphism
$$H_{\mathrm{bas}}(\widetilde{W}\mathfrak{g}) \to H_{\mathrm{bas}}(\mathscr{A}),$$
induced by the characteristic homomorphism $c: \widetilde{W}\mathfrak{g} \to \mathscr{A}$ does not depend on the connection. Theorem 6.5 below shows that $H_{\mathrm{bas}}(\widetilde{W}\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$; hence we have generalized the Chern–Weil homomorphism to the non-commutative setting.
4. Let $\mathscr{A}$ be a $\mathfrak{g}$-differential algebra with connection, and let $c: \widetilde{W}\mathfrak{g} \to \mathscr{A}$ be the characteristic homomorphism. Then the map
$$\phi: \widetilde{W}\mathfrak{g} \otimes \mathscr{A} \to \mathscr{A}, \; w \otimes x \mapsto c(w)x$$
is a $\mathfrak{g}$-homotopy equivalence, with $\mathfrak{g}$-homotopy inverse the inclusion, $\psi : \mathscr{A} \to \widetilde{W}\mathfrak{g} \otimes \mathscr{A}$, $x \mapsto 1 \otimes x$ (cf. Proposition 6.12).

In the commutative case, we found that the horizontal subalgebra of $W\mathfrak{g}$ is the symmetric algebra generated by the curvature variables $\hat{\mu}$. As a consequence we deduced that $H_{\mathrm{bas}}(W\mathfrak{g}) = (W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g}^*)^{\mathfrak{g}}$. This aspect of $W\mathfrak{g}$ does *not* carry over to $\widetilde{W}\mathfrak{g}$, and changing the variables to $\mu, \hat{\mu}$ does not appear useful. We will show however that the quotient map

$$\widetilde{W}\mathfrak{g} \to W\mathfrak{g}$$

(the characteristic homomorphism for $W\mathfrak{g}$, regarded as a non-commutative $\mathfrak{g}$-differential algebra) is a $\mathfrak{g}$-homotopy equivalence. In particular, this will establish that $H_{\mathrm{bas}}(\widetilde{W}\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$.

[3] The algebra $\widetilde{W}\mathfrak{g}$ is different from the non-commutative Weil algebra $\mathscr{W}\mathfrak{g}$ of [4], which we will discuss below under the name of *quantum Weil algebra*.

Proposition 6.13 *The map*

$$W\mathfrak{g} = S(E_{\mathfrak{g}^*}[-1]) \to \widetilde{W}\mathfrak{g} = T(E_{\mathfrak{g}^*}[-1]) \tag{6.9}$$

given by the inclusion of symmetric tensors (i.e., the extension of $E_{\mathfrak{g}^*}[-1] \hookrightarrow T(E_{\mathfrak{g}^*}[-1])$ *by symmetrization) is a morphism of* $\mathfrak{g}$*-differential spaces.*

Proof The proof is similar to that of Proposition 6.5. However, in this case Lemma 6.2 does not immediately apply, since the differential space $E_{\mathfrak{g}^*}[-1]$ is not a $\mathfrak{g}$-differential space. Instead, consider the graded vector space

$$E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c},$$

where c is a generator of degree 0. It has the structure of a graded $\mathfrak{g}$-differential space, with the given differential and Lie derivative on $E_{\mathfrak{g}^*}[-1]$, with $\iota(\xi)\mu = \langle \mu, \xi \rangle \mathsf{c}$, $\iota(\xi)\overline{\mu} = L(\xi)\mu$, and with the trivial action of $\iota(\xi)$, $L(\xi)$, and d on c. We have

$$W\mathfrak{g} = S(E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c})/\langle \mathsf{c} - 1 \rangle. \tag{6.10}$$

Let $E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c} \to \widetilde{W}\mathfrak{g}$ be the map given by the inclusion of $E_{\mathfrak{g}^*}[-1]$ on the first summand, and by the map $\mathsf{c} \mapsto 1$ on the second summand. This map is a morphism of $\mathfrak{g}$-differential spaces; hence Lemma 6.2 shows that it extends to a map of $\mathfrak{g}$-differential spaces

$$S(E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c}) \to \widetilde{W}\mathfrak{g}.$$

The ideal in $S(E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c})$ generated by $\mathsf{c} - 1$ is a $\mathfrak{g}$-differential subspace, contained in the kernel of this map. Since "symmetrizing" and "setting c equal to 1" commute, this is the same as the map (6.9). □

Theorem 6.5 [7] *The quotient map* $\phi : \widetilde{W}\mathfrak{g} \to W\mathfrak{g}$ *is a* $\mathfrak{g}$*-homotopy equivalence, with homotopy inverse* $\psi : W\mathfrak{g} \to \widetilde{W}\mathfrak{g}$ *given by symmetrization,* $S(E_{\mathfrak{g}^*}[-1]) \to T(E_{\mathfrak{g}^*}[-1])$.

Proof The proof is analogous to that of Proposition 6.6. The main point is that the two morphisms $c_0, c_1 : \widetilde{W}\mathfrak{g} \to \widetilde{W}\mathfrak{g} \otimes W\mathfrak{g}$ given by

$$c_0(w) = w \otimes 1, \;\; c_1(w) = 1 \otimes \phi(w)$$

are the characteristic homomorphisms for the two natural connections on $\widetilde{W}\mathfrak{g} \otimes W\mathfrak{g}$. Hence they are $\mathfrak{g}$-homotopic (Property 3 above), and their compositions with

$$\widetilde{W}\mathfrak{g} \otimes W\mathfrak{g} \to \widetilde{W}\mathfrak{g}, \;\; w \otimes w' \mapsto w\,\mathrm{sym}(w')$$

are again $\mathfrak{g}$-homotopic. □

Corollary 6.2 *The quotient map* $\widetilde{W}\mathfrak{g} \to W\mathfrak{g}$ *induces an isomorphism* $H_{\mathrm{bas}}(\widetilde{W}\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$.

6.12 Equivariant cohomology of $\mathfrak{g}$-differential spaces

Definition 6.13 The *equivariant cohomology* of a $\mathfrak{g}$-differential space E is the cohomology of the basic subcomplex of $W\mathfrak{g} \otimes E$:

$$H_{\mathfrak{g}}(E) = H_{\mathrm{bas}}(W\mathfrak{g} \otimes E). \tag{6.11}$$

The left-multiplication of $(W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g}^*)^{\mathfrak{g}}$ on $(W\mathfrak{g} \otimes E)_{\mathrm{bas}}$ gives $H_{\mathfrak{g}}(E)$ the structure of a module over $(S\mathfrak{g}^*)^{\mathfrak{g}}$.

If $\mathscr{A}$ is a $\mathfrak{g}$-differential algebra, then $W\mathfrak{g} \otimes \mathscr{A}$ is a $\mathfrak{g}$-differential algebra; hence $H_{\mathfrak{g}}(\mathscr{A})$ inherits a super algebra structure. Similarly, if a $\mathfrak{g}$-differential space E is a module over the $\mathfrak{g}$-differential algebra $\mathscr{A}$, then $H_{\mathfrak{g}}(E)$ becomes a module over $H_{\mathfrak{g}}(\mathscr{A})$.

Remark 6.10 By Theorem 6.5, one can replace $W\mathfrak{g}$ with $\widetilde{W}\mathfrak{g}$ in the definition of $H_{\mathfrak{g}}(E)$. If $\mathscr{A}$ is a $\mathfrak{g}$-differential algebra, the resulting product on $H_{\mathfrak{g}}(\mathscr{A})$ does not depend on the use of $W\mathfrak{g}$ or $\mathscr{W}\mathfrak{g}$ in its definition. Indeed, since the quotient map $\widetilde{W}\mathfrak{g} \otimes \mathscr{A} \to W\mathfrak{g} \otimes \mathscr{A}$ is a super algebra morphism, the induced isomorphism in basic cohomology is a super algebra morphism.

Example 6.10 If $\mathfrak{g}$ is a real Lie algebra, and M is a $\mathfrak{g}$-manifold, the cohomology group $H_{\mathfrak{g}}(M) = H_{\mathfrak{g}}(\Omega(M))$ is called the *equivariant de Rham cohomology* of M. Suppose that $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, M is compact and the action of $\mathfrak{g}$ integrates to an action of G on M. According to a theorem of H. Cartan [33], $H_{\mathfrak{g}}(M)$ coincides in this case with the equivariant cohomology $H_G(M, \mathbb{R}) = H(EG \times_G M, \mathbb{R})$. Thus, $W\mathfrak{g}$, $\widetilde{W}\mathfrak{g}$ may be thought of as algebraic counterparts to the classifying bundle EG.

Proposition 6.14 *If $\mathscr{A}$ is a $\mathfrak{g}$-differential algebra admitting a connection, then its equivariant cohomology* (*cf.* (6.11)) *is canonically isomorphic to its basic cohomology*: *Thus*

$$H_{\mathfrak{g}}(\mathscr{A}) \cong H_{\mathrm{bas}}(\mathscr{A})$$

as super algebras.

Proof The $\mathfrak{g}$-homotopy equivalence $\mathscr{A} \to \widetilde{W}\mathfrak{g} \otimes \mathscr{A}$, $x \mapsto 1 \otimes x$ (cf. Section 6.11 (4)) restricts to a homotopy equivalence of the basic subcomplexes, hence to an isomorphism $H_{\mathrm{bas}}(\mathscr{A}) \to H_{\mathfrak{g}}(\mathscr{A})$. □

This result admits a generalization to (graded, filtered) $\mathfrak{g}$-differential spaces (E, d) having the structure of a module over $\widetilde{W}\mathfrak{g}$.[4]

[4] The following results will not be needed elsewhere in this book.

Definition 6.14 A module over a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathscr{A}$ is a (graded, filtered) $\mathfrak{g}$-differential space E, which is also a module over $\mathscr{A}$, in such a way that the module action $\phi: \mathscr{A} \otimes E \to E$ is a morphism of (graded, filtered) $\mathfrak{g}$-differential spaces.

Note that if $\mathscr{A}$ is a $\mathfrak{g}$-differential algebra with connection θ, then the characteristic homomorphism $c: \widetilde{W}\mathfrak{g} \to \mathscr{A}$ makes $\mathscr{A}$ into a module over $\widetilde{W}\mathfrak{g}$ via $\phi(w \otimes x) = c(w)x$. If the super algebra $\mathscr{A}$ is commutative, then $\mathscr{A}$ becomes a module over $W\mathfrak{g}$.

Let us make the convention that in the category of super spaces, $\mathrm{End}(E)$ denotes the super algebra of *all* linear maps $E \to E$, but in the category of graded (resp. filtered) super spaces, we take $\mathrm{End}(E)$ to be with the algebra of linear maps $E \to E$ of finite degree (resp. finite filtration degree). With this convention, if E is a (graded, filtered) $\mathfrak{g}$-differential space, the space $\mathrm{End}(E)$ becomes a (graded, filtered) $\mathfrak{g}$-differential algebra. The condition for a module E over a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathscr{A}$ is equivalent to the condition that the map $\mathscr{A} \to \mathrm{End}(E)$ given by the module action is a morphism of (graded, filtered) $\mathfrak{g}$-differential algebras.

Proposition 6.15 *Any two (graded, filtered) module structures*

$$\phi_0, \phi_1 : \widetilde{W}\mathfrak{g} \otimes E \to E$$

over the $\mathfrak{g}$-differential algebra $\widetilde{W}\mathfrak{g}$ are $\mathfrak{g}$-homotopic.

Proof The corresponding morphisms of $\mathfrak{g}$-differential algebras $\widetilde{W}\mathfrak{g} \to \mathrm{End}(E)$ are $\mathfrak{g}$-homotopic, by Section 6.11 (3). ☐

Proposition 6.16 *Let E be a (graded, filtered) module over the (graded, filtered) $\mathfrak{g}$-differential algebra $\widetilde{W}\mathfrak{g}$. Then the module map is a $\mathfrak{g}$-homotopy equivalence, with homotopy inverse the inclusion*

$$E \to \widetilde{W}\mathfrak{g} \otimes E, \ v \mapsto 1 \otimes v.$$

Proof The tensor product $E' = \widetilde{W}\mathfrak{g} \otimes E$ is a $\widetilde{W}\mathfrak{g}$-module in two ways, with $w \in \widetilde{W}\mathfrak{g}$ acting on $w_1 \otimes v$ by $ww_1 \otimes v$ or by $(-1)^{|w||w_1|} w_1 \otimes wv$. By the previous proposition, these two module structures are $\mathfrak{g}$-homotopic. Now argue as in the proof of Proposition 6.12. ☐

As a consequence, the map on basic subcomplexes $(\widetilde{W}\mathfrak{g} \otimes E)_{\mathrm{bas}} \to E_{\mathrm{bas}}$ induces an isomorphism in cohomology,

$$H_{\mathfrak{g}}(E) \cong H_{\mathrm{bas}}(E).$$

The book of Guillemin–Sternberg [61] gives a detailed discussion of modules over the *commutative* Weil algebra $W\mathfrak{g}$ (under the name "$W^\star$-modules").

6.13 Transgression in the Weil algebra

We next discuss *transgression* in the Weil algebra $W\mathfrak{g}$. Recall that $W\mathfrak{g}$, and also its invariant part $(W\mathfrak{g})^{\mathfrak{g}}$, are acyclic differential algebras. That is, the augmentation map $\varepsilon:\ (W\mathfrak{g})^{\mathfrak{g}} \to \mathbb{K}$ and the unit map $i:\ \mathbb{K} \to (W\mathfrak{g})^{\mathfrak{g}}$ are homotopy inverses. Since the Weil differential d vanishes on $(W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g}^*)^{\mathfrak{g}}$, any invariant polynomial $p \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ of positive degree is a cocycle, and hence is the coboundary of an element in $(W\mathfrak{g})^{\mathfrak{g}}$.

Definition 6.15 A *cochain of transgression* for $p \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ is an odd element $C \in (W\mathfrak{g})^{\mathfrak{g}}$ with $\mathrm{d}(C) = p$.

The connection on $\wedge\mathfrak{g}^*$ gives a morphism of graded $\mathfrak{g}$-differential algebras $\pi:\ W\mathfrak{g} \to \wedge\mathfrak{g}^*$ (cf. Example 6.9). It restricts to a morphism of graded differential algebras

$$\pi:\ (W\mathfrak{g})^{\mathfrak{g}} \to (\wedge\mathfrak{g}^*)^{\mathfrak{g}},$$

with the trivial differential on $(\wedge\mathfrak{g}^*)^{\mathfrak{g}}$. Note that π vanishes on the space

$$(W^+\mathfrak{g})^{\mathfrak{g}} \cap \ker(\mathrm{d}) = (W\mathfrak{g})^{\mathfrak{g}} \cap \mathrm{ran}(\mathrm{d})$$

since $(\wedge\mathfrak{g}^*)^{\mathfrak{g}} \cap \mathrm{ran}(\mathrm{d}_\wedge) = 0$.

Proposition 6.17 *There is a well-defined linear map*

$$\mathsf{t}:\ (S^+\mathfrak{g}^*)^{\mathfrak{g}} \to (\wedge\mathfrak{g}^*)^{\mathfrak{g}}, \quad p \mapsto \mathsf{t}(p) \tag{6.12}$$

such that $\mathsf{t}(p) = \pi(C)$ *for any cochain of transgression* C *with* $p = \mathrm{d}(C)$. *If* p *has degree* $r > 0$, *then* $\mathsf{t}(p)$ *has degree* $2r-1$. *The map* (6.12) *vanishes on the subspace* $((S^+\mathfrak{g}^*)^{\mathfrak{g}})^2$ *spanned by products of invariant polynomials with zero constant term.*

Proof Since $(W\mathfrak{g})^{\mathfrak{g}}$ is acyclic, and since the Weil differential vanishes on $(S\mathfrak{g}^*)^{\mathfrak{g}}$, any invariant polynomial $p \in (S^r\mathfrak{g}^*)^{\mathfrak{g}}$ of degree $r > 0$ is a coboundary. Hence we may write $p = \mathrm{d}(C)$ for some cochain of transgression $C \in (W^{2r-1}\mathfrak{g})^{\mathfrak{g}}$. If C' is another cochain of transgression for p, then $C' - C$ is closed, hence $\pi(C' - C) = 0$ as remarked above. Suppose $p, q \in (S^+\mathfrak{g}^*)^{\mathfrak{g}}$, and let $C \in \mathscr{T}$ be a cochain of transgression for p. Then qC^p is a cochain of transgression for qp. Since π is an algebra morphism, $\pi(qC) = \pi(q)\pi(C) = 0$. □

We will refer to the map t as *transgression*.

Remarks 6.11

1. The transgression construction was described in H. Cartan's article [32]. Cartan, however, refers not to the map t but to any right-inverse of the map $(S\mathfrak{g}^*)^{\mathfrak{g}} \to \mathrm{ran}(\mathsf{t})$ as a transgression. Calling t the transgression has become common since the work of Chern and Simons [38] in the theory of principal G-bundles.

2. For $\mathfrak{g}$ a complex reductive Lie algebra, it turns out that the kernel of t is *exactly* the space $((S^+\mathfrak{g}^*)^{\mathfrak{g}})^2$, while the image is the space of primitive elements of $\mathfrak{g}$. We will discuss this *Transgression Theorem* in Section 10.7.

Example 6.11 Suppose $B:\ \mathfrak{g}\times\mathfrak{g}\to\mathbb{K}$ is an invariant symmetric bilinear form on $\mathfrak{g}$, and let $p\in(S^2\mathfrak{g}^*)^{\mathfrak{g}}$ be the element defined by $\iota_S(\xi)p=2\,B(\xi,\cdot)$. In a basis $e_1,\ldots,e_n$ of $\mathfrak{g}$, with dual basis $e^1,\ldots,e^n$ of $\mathfrak{g}^*$, we have $p=\sum_{a=1}^n e^a B^\flat(e_a)$. Consider p as a degree 4 element of $W\mathfrak{g}$, using the inclusion $S\mathfrak{g}^*\hookrightarrow W\mathfrak{g}$ generated by the curvature variables $\widehat{\mu}=\overline{\mu}-\lambda(\mu)$. Then

$$p=\sum_{a=1}^n \overline{e^a}\,\overline{B^\flat(e_a)}-2\sum_{a=1}^n\overline{e^a}\lambda(B^\flat(e_a)).$$

(Here we used that the algebra morphism $\lambda:\ S\mathfrak{g}^*\to\wedge\mathfrak{g}^*$ takes $S^+\mathfrak{g}^*$ to $\operatorname{ran}(\mathrm{d}^\wedge)$, hence $\lambda((S^+\mathfrak{g}^*)^{\mathfrak{g}})=0$.) Thus $\sum_a\lambda(e^a)\lambda(B^\flat(e_a))=0$. A cochain of transgression $C\in(W^3\mathfrak{g})^{\mathfrak{g}}$ for this element is given by

$$\begin{aligned}C&=\sum_{a=1}^n\overline{e^a}\wedge B^\flat(e_a)-\frac{2}{3}\sum_{a=1}^n e^a\wedge\lambda(B^\flat(e_a))\\&=\sum_{a=1}^n\widehat{e^a}\wedge B^\flat(e_a)+\frac{1}{3}\sum_{a=1}^n e^a\wedge\lambda(B^\flat(e_a)).\end{aligned}$$

This shows $\mathsf{t}(p)=\phi\in(\wedge^3\mathfrak{g}^*)^{\mathfrak{g}}$, where

$$\phi=\frac{1}{3}\sum_{a=1}^n e^a\wedge\lambda(B^\flat(e_a))$$

is the cubic element determined by B.

If $h\in\operatorname{End}^{-1}(W\mathfrak{g})$ is a $\mathfrak{g}$-equivariant homotopy operator for the Weil algebra, then $C=h(p)$ is a cochain of transgression for $p\in(S^+\mathfrak{g}^*)^{\mathfrak{g}}$. Thus

$$\mathsf{t}=\pi\circ h.$$

In particular, we may take h to be the standard homotopy operator, obtained in Section 6.4 as a special case of the homotopy operator for the Koszul algebra. Recall that h is a composition of the morphism of differential algebras

$$W\mathfrak{g}\to\mathbb{K}[t,\mathrm{d}t]\otimes W\mathfrak{g},\tag{6.13}$$

given on generators $\xi,\overline{\xi}$ by

$$\xi\mapsto t\xi,\ \ \overline{\xi}\mapsto\mathrm{d}(t\xi)=\xi\mathrm{d}t+t\overline{\xi},$$

followed by

$$J\otimes 1:\ \mathbb{K}[t,\mathrm{d}t]\otimes W\mathfrak{g}\to W\mathfrak{g},$$

where $J:\ \mathbb{K}[t,\mathrm{d}t]\to\mathbb{K}$ is “integration over the unit interval”, see Section 6.3. Let us describe the restriction of $\pi\circ h$ to the symmetric algebra $S\mathfrak{g}\subseteq W\mathfrak{g}$ (generated by the curvature variables $\widehat{\xi} = \overline{\xi} - \lambda(\xi)$, $\xi \in \mathfrak{g}$). We will use the following notation. As before, let

$$\lambda: \ S(\mathfrak{g}^*) \to \wedge(\mathfrak{g}^*)$$

be the algebra morphism extending the map on generators, $\mu \mapsto \lambda(\mu)$. Also, for $\mu \in \mathfrak{g}$ let $\iota_S(\mu) \in \mathrm{End}(S\mathfrak{g}^*)$ be the derivation given on generators by $\iota_S(\mu)\mu = \langle \mu, \mu \rangle$. For $p \in S\mathfrak{g}^*$, viewed as a polynomial on $\mathfrak{g}$, $\iota_S(\mu)p$ is just the derivative in the direction of μ. Let e_a be a basis of $\mathfrak{g}$, with dual basis e^a.

Proposition 6.18 *The transgression of a homogeneous element $p \in (S^{m+1}\mathfrak{g}^*)^{\mathfrak{g}}$ is given by*

$$\mathsf{t}(p) = \frac{(m!)^2}{(2m+1)!} \sum_a e^a \wedge \lambda(\iota_S(e_a)p).$$

In fact, the right-hand side gives $\pi(h(p))$ for any $p \in S^{m+1}(\mathfrak{g}^*) = (W^{2m+2}\mathfrak{g})_{\mathrm{hor}}$ (not necessarily invariant).

Proof It suffices to prove the formula for $p = \widehat{\mu}_1 \cdots \widehat{\mu}_{m+1}$ with $\mu_i \in \mathfrak{g}^*$. The first map (6.13) takes p to

$$\begin{aligned}
&(\mathrm{d}t\mu_1 + t\overline{\mu}_1 - t^2\lambda(\mu_1)) \cdots (\mathrm{d}t\mu_{m+1} + t\overline{\mu}_{m+1} - t^2\lambda(\mu_{m+1})) \\
&= (\mathrm{d}t\mu_1 + t\widehat{\mu}_1 + (t - t^2\lambda(\mu_1)) \cdots (\mathrm{d}t\mu_{m+1} + t\widehat{\mu}_{m+1} + (t - t^2)\lambda(\mu_{m+1})).
\end{aligned}$$

The algebra morphism π is given on generators by $\pi(\widehat{\mu}) = 0$, $\pi(\mu) = \mu$ for $\mu \in \mathfrak{g}$; hence it takes the above to

$$\begin{aligned}
&\big((\mathrm{d}t\mu_1 + (t - t^2)\lambda(\mu_1)) \cdots (\mathrm{d}t\mu_{m+1} + (t - t^2)\lambda(\mu_{m+1})\big) \\
&= (t - t^2)^{m+1} \prod_{i=1}^{m+1} \lambda(\mu_i) + (t - t^2)^m \mathrm{d}t \sum_{i=1}^{m+1} \mu_i \wedge \prod_{j \neq i} \lambda(\mu_j) \\
&= (t - t^2)^{m+1} \lambda(p) + (t - t^2)^m \mathrm{d}t \sum_a e^a \wedge \lambda(\iota_S(e_a)p).
\end{aligned}$$

The "integral" J of the first term is zero (since it does not contain $\mathrm{d}t$), while

$$J((t - t^2)^m \mathrm{d}t) = \frac{(m!)^2}{(2m+1)!}$$

by direct calculation. □

Proposition 6.18 has an interesting refinement, due to Kostant.

Proposition 6.19 [86, Theorem 73] *For any homogeneous element $p \in S^{m+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, where $m \geq 0$, and any $\xi \in \mathfrak{g}$, the contraction $\iota(\xi)\mathsf{t}(p)$ lies in $\lambda(S(\mathfrak{g}^*)) \subseteq \wedge(\mathfrak{g}^*)$. It is given by*

$$\iota(\xi)\mathsf{t}(p) = \frac{(m!)^2}{(2m)!} \lambda(\iota_S(\xi)p).$$

Note that Proposition 6.19 implies Proposition 6.18, since $\sum_a e^a \iota(e_a)$ acts as the degree operator on $\wedge\mathfrak{g}^*$.

Proof Let $\eta: W(\mathfrak{g}) \to W(\mathfrak{g})$ be the derivation of degree 0, given on generators by $\eta(\mu) = 0$, $\eta(\widehat{\mu}) = \mu$. The derivation η is locally nilpotent (i.e., for any $x \in W(\mathfrak{g})$, there exists $N > 0$ with $\eta^N(x) = 0$), hence its exponential e^η is a well-defined automorphism of $W(\mathfrak{g})$. On generators, this automorphism is given by

$$e^\eta(\mu) = \mu, \quad e^\eta(\widehat{\mu}) = \widehat{\mu} + \lambda(\mu) = \overline{\mu}.$$

Thus, if $q \in S^m\mathfrak{g}^* = (W^{2m}\mathfrak{g})_{\mathrm{hor}}$, then

$$\lambda(q) = \pi(e^\eta(q)) = \frac{\eta^m}{m!}(q). \tag{6.14}$$

The derivation η commutes with the derivation $\iota_S(\xi)$, as one verifies on generators. We apply this result to $q = \iota_S(\xi)p$ for a given element $p \in S^{m+1}\mathfrak{g}^*$, $m \geq 0$. Since the derivation η commutes with the derivation $\iota_S(\xi)$, as one checks by evaluating on generators, we obtain

$$\lambda(\iota_S(\xi)p) = \frac{1}{m!}\iota_S(\xi)\eta^m(p). \tag{6.15}$$

Note $\eta^m(p) \in S^1(\mathfrak{g}^*) \otimes \wedge^{2m}(\mathfrak{g}^*)$. Decompose the Weyl differential as a sum $\mathrm{d}_W = \mathrm{d}_K + \mathrm{d}_{CE}$, where d_K is the Koszul differential and d_{CE} the Chevalley–Eilenberg differential. Both are derivations, and checking on generators one finds that

$$[\eta, \mathrm{d}_K] = \mathrm{d}_{CE}, \quad [\eta, \mathrm{d}_{CE}] = 0.$$

This shows that η preserves the subspace $\ker(\mathrm{d}_K) \cap \ker(\mathrm{d}_{CE})$. If p is $\mathfrak{g}$-invariant, then $p \in \ker(\mathrm{d}_K) \cap \ker(\mathrm{d}_{CE})$; hence also $\eta^m(p) \in \ker(\mathrm{d}_K) \cap \ker(\mathrm{d}_{CE})$. In particular,

$$\eta^m(p) \in \ker(\mathrm{d}_K).$$

Now let $g: W(\mathfrak{g}) \to W(\mathfrak{g})$ be the derivation of degree -1 given on generators by $g(\widehat{\mu}) = \mu$, $g(\mu) = 0$. Then $[\mathrm{d}_K, g]$ is the derivation equal to the identity on generators $\mu, \widehat{\mu}$. Let $\phi \in \mathrm{End}^0(W\mathfrak{g})$ be its inverse on $W^+\mathfrak{g}$, and equal to zero on $W^0\mathfrak{g}$. On $S^i(\mathfrak{g}^*) \otimes \wedge^j(\mathfrak{g}^*)$, g acts as multiplication by $i + j$, hence ϕ divides by $i + j$ if $i + j > 0$. Then $[\mathrm{d}_K, \phi] = 0$, $[g, \phi] = 0$, and $g \circ \phi$ is the standard homotopy operator for the Koszul differential:

$$[\mathrm{d}_K,\ g \circ \phi] = 1 - \varepsilon,$$

where ε is the augmentation map for $W\mathfrak{g}$. The calculation

$$\begin{aligned}
[\mathrm{d}_K, \iota_S(\xi) - \iota(\xi) \circ g \circ \phi] &= -\iota(\xi) + \iota(\xi) \circ [\mathrm{d}_K, g \circ \phi] \\
&= -\iota(\xi) \circ \varepsilon \\
&= 0
\end{aligned}$$

shows in particular that $\iota_S(\xi) - \iota(\xi) \circ g \circ \phi$ preserves $\ker(\mathrm{d}_K)$. Applying this to $\eta^m(p)$, the result lies in $\ker(\mathrm{d}_K) \cap \wedge(\mathfrak{g}^*) = 0$. Hence (6.15) becomes

$$
\begin{aligned}
\lambda(\iota_S(\xi)p) &= \frac{1}{m!}(\iota(\xi)\circ g\circ\phi)(\eta^m(p))\\
&= \frac{1}{m!}\frac{1}{(2m+1)}(\iota(\xi)\circ g)(\eta^m(p))\\
&= \frac{1}{m!}\frac{1}{(2m+1)}\iota(\xi)\sum_a e^a\wedge\eta^m(\iota_S(e_a)p)\\
&= \frac{1}{(2m+1)}\iota(\xi)\sum_a e^a\wedge\lambda(\iota_S(e_a)p).
\end{aligned}
$$

Now use Proposition 6.18. □

Chapter 7
Quantum Weil algebras

As explained in Chapter 2.3 the Clifford algebra $\mathrm{Cl}(V)$ of a Euclidean vector space can be regarded as a quantization of the exterior algebra $\wedge(V)$, and similarly the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra is a quantization of the symmetric algebra $S(\mathfrak{g})$. In this chapter we will consider a similar quantization of the Weil algebra $W(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$ with a non-degenerate invariant inner product B. Identifying $W(\mathfrak{g}) = S(\mathfrak{g}) \otimes \wedge(\mathfrak{g})$, where $S(\mathfrak{g})$ is the symmetric algebra generated by the "curvature variables", this quantum Weil algebra is a $\mathfrak{g}$-differential algebra $\mathscr{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$. The main result of this chapter (due to [4, 7]) is the existence of an isomorphism of $\mathfrak{g}$-differential spaces $q: W(\mathfrak{g}) \to \mathscr{W}(\mathfrak{g})$. On basic subcomplexes, this quantization map restricts to a linear isomorphism between the center of the enveloping algebra and invariants in $S\mathfrak{g}$. We will give a proof of *Duflo's Theorem* for the case of quadratic Lie algebras, stating that this isomorphism respects product structures.

7.1 The $\mathfrak{g}$-differential algebra $\mathrm{Cl}(\mathfrak{g})$

Suppose $\mathfrak{g}$ is a quadratic Lie algebra with bilinear form B. We will use B to identify $\mathfrak{g}^* \cong \mathfrak{g}$. The Lie bracket on $\mathfrak{g}$ will be denoted by $[\cdot,\cdot]_\mathfrak{g}$, to avoid confusion with commutation in the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$. Recall (cf. Section 2.3) that B defines a Poisson bracket on the graded super algebra $\wedge(\mathfrak{g})$, given on generators by $\{\xi,\zeta\} = 2B(\xi,\zeta)$, and $\iota(\xi) = \frac{1}{2}\{\xi,\cdot\}$. Define $\lambda(\xi) \in \wedge^2\mathfrak{g}$ and $\phi \in \wedge^3\mathfrak{g}$ by

$$\{\xi,\lambda(\zeta)\} = [\xi,\zeta]_\mathfrak{g}, \quad \{\xi,\phi\} = 2\lambda(\xi).$$

Thus $\lambda(\xi) = \lambda(\mathrm{ad}_\xi)$ in the notation of (2.17), while ϕ is the *structure constant tensor*.

As discussed in Section 6.7, the exterior algebra $\wedge(\mathfrak{g}) \cong \wedge(\mathfrak{g}^*)$ is a $\mathfrak{g}$-differential algebra. Observe that $\mathrm{d}\xi = 2\lambda(\xi)$, since

$$\iota(\zeta)\mathrm{d}\xi = L(\zeta)\xi = [\zeta,\xi]_\mathfrak{g} = \{\zeta,\lambda(\xi)\} = 2\iota(\zeta)\lambda(\xi).$$

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Proposition 7.1 *The differential, Lie derivatives, and contractions on $\wedge\mathfrak{g}$ are Poisson brackets*:

$$\mathrm{d} = \{\phi, \cdot\}, \quad \iota(\xi) = \frac{1}{2}\{\xi, \cdot\}, \quad L(\xi) = \{\lambda(\xi), \cdot\}.$$

The elements $\xi, \lambda(\xi)$, and ϕ satisfy the following Poisson bracket relations:

$$\begin{aligned} \{\phi, \phi\} &= 0, \\ \{\phi, \xi\} &= 2\lambda(\xi), \\ \{\phi, \lambda(\xi)\} &= 0, \\ \{\lambda(\xi), \lambda(\zeta)\} &= \lambda([\xi, \zeta]_{\mathfrak{g}}), \\ \{\lambda(\xi), \zeta\} &= [\xi, \zeta]_{\mathfrak{g}}, \\ \{\xi, \zeta\} &= 2\,B(\xi, \zeta). \end{aligned}$$

Proof For the first part it suffices to check on generators $\zeta \in \mathfrak{g} = \wedge^1\mathfrak{g}$, using that both sides are derivations of $\wedge\mathfrak{g}$. But $L(\xi)\zeta = \{\lambda(\xi), \zeta\}$, $\iota(\xi)\zeta = \frac{1}{2}\{\xi, \zeta\}$ and $\mathrm{d}\zeta = 2\lambda(\zeta) = \{\phi, \zeta\}$ all follow from the definition.

We have $\{\xi, \zeta\} = 2B(\xi, \zeta)$ by definition of the Poisson bracket. The identity $L(\xi) = \{\lambda(\xi), \cdot\}$ determines all Poisson brackets with elements $\lambda(\xi)$, while $\{\phi, \xi\} = \lambda(\xi)$ is the definition of ϕ. The remaining bracket $\{\phi, \phi\} = \mathrm{d}\phi$ vanishes since $\phi \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$ is invariant, and hence is a cocycle for the Lie algebra differential. □

Let us spell out the formulas in a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$, with B-dual basis $e^1, \ldots, e^n$. We have (cf. (2.19))

$$\lambda(\xi) = \frac{1}{4}\sum_a [\xi, e_a]_{\mathfrak{g}} \wedge e^a = -\frac{1}{4}\sum_{ab} B(\xi, [e_a, e_b]_{\mathfrak{g}})\, e^a \wedge e^b, \tag{7.1}$$

and

$$\phi = \frac{1}{3}\sum_a \lambda(e_a) \wedge e^a = -\frac{1}{12}\sum_{abc} B([e_a, e_b]_{\mathfrak{g}}, e_c)\, e^a \wedge e^b \wedge e^c. \tag{7.2}$$

The Poisson brackets from Proposition 7.1 quantize to commutators $[\cdot, \cdot]_{\mathrm{Cl}}$ in the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$. Let $\gamma(\xi) = q(\lambda(\xi))$. In the basis,

$$\gamma(\xi) = -\frac{1}{4}\sum_{ab} B(\xi, [e_a, e_b]_{\mathfrak{g}})\, e^a e^b,$$

where the product on the right-hand side is taken in the Clifford algebra.

Proposition 7.2 *The elements $q(\phi), \gamma(\xi), \xi$ in* $\mathrm{Cl}(\mathfrak{g})$ *satisfy the following commutation relations*:

$$\begin{aligned} [q(\phi), q(\phi)]_{\mathrm{Cl}} &= \frac{1}{12}\mathrm{tr}\big(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}})\big), \\ [q(\phi), \xi]_{\mathrm{Cl}} &= 2\gamma(\xi), \end{aligned}$$

$$
\begin{aligned}
[q(\phi), \gamma(\xi)]_{\mathrm{Cl}} &= 0, \\
[\gamma(\xi), \gamma(\zeta)]_{\mathrm{Cl}} &= \gamma([\xi, \zeta]_{\mathfrak{g}}), \\
[\gamma(\xi), \zeta]_{\mathrm{Cl}} &= [\xi, \zeta]_{\mathfrak{g}}, \\
[\xi, \zeta]_{\mathrm{Cl}} &= 2B(\xi, \zeta).
\end{aligned}
$$

Here

$$\mathrm{Cas}_{\mathfrak{g}} = \sum_a e_a e^a \in U(\mathfrak{g})$$

is the quadratic Casimir element, and $\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}))$ *is its trace in the adjoint representation on* $\mathfrak{g}$.

Proof The commutators with $\gamma(\xi)$ all follow from $L(\xi) = [\gamma(\xi), \cdot]_{\mathrm{Cl}}$, while the commutators with ξ are obtained from $\iota(\xi) = \frac{1}{2}[\xi, \cdot]_{\mathrm{Cl}}$. It remains to compute $[q(\phi), q(\phi)]_{\mathrm{Cl}} = 2q(\phi)^2$. Observe that it is a scalar since

$$[\xi, [q(\phi), q(\phi)]_{\mathrm{Cl}}]_{\mathrm{Cl}} = 2[[\xi, q(\phi)]_{\mathrm{Cl}}, q(\phi)]_{\mathrm{Cl}} = 4[\gamma(\xi), q(\phi)]_{\mathrm{Cl}} = 0.$$

To find this scalar, recall that by Proposition 2.13 the square of $q(\phi)$ may be computed by applying the operator $\exp(-1/2 \sum_a \iota(e^a) \otimes \iota(e_a))$ to $\phi \otimes \phi \in \wedge\mathfrak{g} \otimes \wedge\mathfrak{g}$, followed by the wedge product $\wedge\mathfrak{g} \otimes \wedge\mathfrak{g} \to \wedge\mathfrak{g}$, followed by q. Since we already know that only the scalar term survives, we obtain

$$
\begin{aligned}
q(\phi)q(\phi) &= \frac{-1}{6}\Big(\sum_a \iota(e^a) \otimes \iota(e_a)\Big)^3 (\phi \otimes \phi) \\
&= -\frac{1}{6}\sum_{abc} (\iota(e_a)\iota(e_b)\iota(e_c)\phi)(\iota(e^a)\iota(e^b)\iota(e^c)\phi) \\
&= -\frac{1}{24}\sum_{abc} B(e_a, [e_b, e_c]_{\mathfrak{g}}) B(e^a, [e^b, e^c]_{\mathfrak{g}}) \\
&= -\frac{1}{24}\sum_{bc} B([e_b, e_c]_{\mathfrak{g}}, [e^b, e^c]_{\mathfrak{g}}) \\
&= \frac{1}{24}\sum_{bc} B(e_c, [e_b, [e^b, e^c]_{\mathfrak{g}}]_{\mathfrak{g}}) \\
&= \frac{1}{24}\sum_{c} B(e_c, \mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}) e^c) \\
&= \frac{1}{24}\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}})).
\end{aligned}
$$

(See also the computation in Example 2.4.) □

Remark 7.1 The observation that $q(\phi)$ squares to the scalar goes back to Kostant and Sternberg [90].

As a consequence, we have

Corollary 7.1 *The Clifford algebra* $\mathrm{Cl}(\mathfrak{g})$ *is a filtered* $\mathfrak{g}$*-differential algebra with differential, Lie derivatives and contractions given as*

$$\mathrm{d}_{\mathrm{Cl}} = [q(\phi), \cdot]_{\mathrm{Cl}}, \quad L_{\mathrm{Cl}}(\xi) = [\gamma(\xi), \cdot]_{\mathrm{Cl}}, \quad \iota_{\mathrm{Cl}}(\xi) = \frac{1}{2}[\xi, \cdot]_{\mathrm{Cl}}.$$

The quantization map $q : \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ intertwines the Lie derivatives and contractions, but does *not* intertwine the differentials:

Proposition 7.3 *The Lie algebra differential* $\mathrm{d}_\wedge$ *on* $\wedge\mathfrak{g} \cong \wedge\mathfrak{g}^*$ *and the Clifford differential differ by contraction with twice the cubic element* $\phi \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$:

$$q^{-1} \circ \mathrm{d}_{\mathrm{Cl}} \circ q = \mathrm{d}_\wedge + 2\iota(\phi).$$

Proof For $x \in \mathrm{Cl}(\mathfrak{g})$, let x^L be the operator of left-multiplication by x, and x^R the operator of ($\mathbb{Z}_2$-graded) right-multiplication. Thus

$$x^L(y) = xy, \quad x^R(y) = (-1)^{|y||x|}yx$$

for homogeneous elements $x, y \in \mathrm{Cl}(\mathfrak{g})$, and $x^L - x^R = [x, \cdot]_{\mathrm{Cl}}$. If $\xi \in \mathfrak{g}$, we have

$$q^{-1} \circ \xi^L \circ q = \varepsilon(\xi) + \iota(\xi), \quad q^{-1} \circ \xi^R \circ q = \varepsilon(\xi) - \iota(\xi).$$

(The first formula follows since both sides define representations of $\mathrm{Cl}(\mathfrak{g})$ on $\wedge\mathfrak{g}$, and these two representations agree on $1 \in \wedge\mathfrak{g}$. The second formula is obtained similarly, or using that $q^{-1} \circ (\xi^L - \xi^R) \circ q = [\xi, \cdot]_{\mathrm{Cl}} = 2\iota(\xi)$.) Let e_a be a basis of $\mathfrak{g}$ with B-dual basis e^a, and write $\phi = \frac{1}{6}\sum_{abc}\phi^{abc}e_a \wedge e_b \wedge e_c$ where $\phi^{abc} = -\frac{1}{2}B([e^a, e^b]_{\mathfrak{g}}, e^c)$. Then

$$\mathrm{d}_\wedge = \sum_{abc}\phi^{abc}\varepsilon(e_a)\varepsilon(e_b)\iota(e_c).$$

Since $\iota(\xi)$ and $\varepsilon(\xi)$ are dual relative to the metric on $\wedge\mathfrak{g}$, the negative dual $\partial_\wedge = -\mathrm{d}_\wedge^*$ is

$$\partial_\wedge = \sum_{abc}\phi^{abc}\varepsilon(e_a)\iota(e_b)\iota(e_c).$$

The quantization map intertwines the operators $q(\phi)^L$ and $q(\phi)^R$ on $\mathrm{Cl}(\mathfrak{g})$ with the following operators on $\wedge(\mathfrak{g})$:

$$\begin{aligned} q^{-1} \circ q(\phi)^L \circ q &= \frac{1}{6}\sum_{abc}\phi^{abc}(\varepsilon(e_a) + \iota(e_a))(\varepsilon(e_b) + \iota(e_b))(\varepsilon(e_c) + \iota(e_c)) \\ &= \varepsilon(\phi) + \frac{1}{2}\sum_{abc}\phi^{abc}\big(\varepsilon(e_a)\iota(e_b)\iota(e_c) + \varepsilon(e_a)\varepsilon(e_b)\iota(e_c)\big) + \iota(\phi) \\ &= \varepsilon(\phi) + \iota(\phi) + \frac{1}{2}(\mathrm{d}_\wedge + \partial_\wedge), \end{aligned}$$

$$q^{-1}\circ q(\phi)^R\circ q=\varepsilon(\phi)+\frac{1}{2}\sum_{abc}\phi^{abc}\big(\varepsilon(e_a)\iota(e_b)\iota(e_c)-\varepsilon(e_a)\varepsilon(e_b)\iota(e_c)\big)$$
$$=\varepsilon(\phi)-\iota(\phi)-\frac{1}{2}(\mathrm{d}_\wedge-\partial_\wedge).$$

Subtracting the two results, we obtain

$$q^{-1}\circ \mathrm{d}_{\mathrm{Cl}}\circ q=\mathrm{d}_\wedge+2\iota(\phi),$$

as desired. □

Proposition 7.4 *If* $\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_\mathfrak{g}))\neq 0$, *the cohomology of* $\mathrm{Cl}(\mathfrak{g})$ *is equal to zero.*

Proof Let $q(\phi)$ act on $\mathrm{Cl}(\mathfrak{g})$ by multiplication from the left. If $x\in\mathrm{Cl}(\mathfrak{g})$, then

$$\mathrm{d}(q(\phi)x)+q(\phi)\mathrm{d}x=(\mathrm{d}q(\phi))x=[q(\phi),q(\phi)]x=\frac{1}{12}\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_\mathfrak{g}))x.$$

Hence $h:=12(\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_\mathfrak{g})))^{-1}q(\phi)$ is a homotopy operator between the identity map and the zero map. □

7.2 The quantum Weil algebra

7.2.1 Poisson structure on the Weil algebra

As in the last section, we use the bilinear form B to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and hence $W(\mathfrak{g})=S(\mathfrak{g})\otimes\wedge(\mathfrak{g})$. In Remark 6.8, we introduced a family of $\mathfrak{g}$-differential algebras $(W(\mathfrak{g}))^{(r)}$ with contractions, Lie derivatives, and differential given by

$$\iota(\xi)\zeta=B(\xi,\zeta),\quad \iota(\xi)\widehat{\zeta}=0,$$
$$L(\xi)\zeta=[\xi,\zeta]_\mathfrak{g},\quad L(\xi)\widehat{\zeta}=\widehat{[\xi,\zeta]_\mathfrak{g}},$$
$$\mathrm{d}^{(r)}\zeta=r\widehat{\zeta}+2\lambda(\zeta),\quad \mathrm{d}^{(r)}\widehat{\zeta}=\sum_i\widehat{L(e_i)\zeta}\,e^i,$$

(recall $\mathrm{d}_\wedge\xi=2\lambda(\xi)$). We observed that for $r\neq 0$, these are all isomorphic by a simple rescaling of the $\hat{\zeta}$-variable. Both $S\mathfrak{g}$ and $\wedge\mathfrak{g}$ are graded Poisson algebras, where the bracket on $S\mathfrak{g}$ is defined by the Lie bracket and that on $\wedge\mathfrak{g}$ is determined by B. Hence $W(\mathfrak{g})$ becomes a Poisson algebra. Explicitly, the Poisson brackets of the generators are

$$\{\xi,\zeta\}=2B(\xi,\zeta),\quad \{\widehat{\xi},\widehat{\zeta}\}=\widehat{[\xi,\zeta]_\mathfrak{g}},\quad \{\widehat{\xi},\zeta\}=0.$$

For the rest of this section we will make the choice $r=2$, since this is the unique choice for which the differential becomes a Poisson bracket.

Lemma 7.1 *The derivation* $\mathrm{d}^{(r)}$ *can be written as a Poisson bracket* $\{D,\cdot\}$ *if and only if* $r=2$. *In fact,* $\mathrm{d}^{(2)}=\{D,\cdot\}$, *where* $D\in(W(\mathfrak{g}))^3$ *is the cubic element*

$$D=\sum_i \widehat{e^i}e_i+\phi. \tag{7.3}$$

Proof Suppose $\mathrm{d}^{(r)}=\{D,\cdot\}$ for some element $D\in W(\mathfrak{g})$. Since d raises degree by 1, while $\{\cdot,\cdot\}$ has degree -2, we can take D of degree 2. Since

$$\mathrm{d}^{(r)}e_j=r\hat{e}_j+2\lambda(e_j),$$

and recalling $\{\phi,\xi\}=2\lambda(\xi)$, we see that we must have

$$D=\frac{r}{2}\sum_i \hat{e}_i e^i+\phi.$$

But then $\{D,\hat{e}_j\}$ equals $\mathrm{d}^{(r)}\hat{e}_j=\widehat{[e_i,e_j]_{\mathfrak{g}}}e^i$ if and only if $r=2$. □

For $r=2$, the variables

$$\xi,\quad \overline{\xi}=\mathrm{d}^{(2)}\xi=2(\widehat{\xi}+\lambda(\xi))$$

satisfy the bracket relations

$$\{\xi,\zeta\}=2B(\xi,\zeta),\quad \{\overline{\xi},\overline{\zeta}\}=2\overline{[\xi,\zeta]_{\mathfrak{g}}},\quad \{\overline{\xi},\zeta\}=2[\xi,\zeta]_{\mathfrak{g}}.$$

In particular

$$\iota(\xi)=\frac{1}{2}\{\xi,\cdot\},\quad L(\xi)=\frac{1}{2}\{\overline{\xi},\cdot\}.$$

Remark 7.2

1. For general $r\neq 0$, the bracket relation among the variables $\overline{\xi}=\mathrm{d}^{(r)}\xi=r\widehat{\xi}+2\lambda(\xi)$ takes on a more complicated form.
2. Rather than working with the modified differential $\mathrm{d}^{(2)}$, one can keep the original differential $\mathrm{d}^{(1)}$ but work with the Poisson bracket on $\wedge\mathfrak{g}$ and Clifford algebra structure defined by $\frac{1}{2}B$. This is the approach taken in [4].

For the rest of this chapter, we put $r=2$. The element $D\in(W(\mathfrak{g}))^3$ (cf. (7.3)) reads, in terms the variables $\xi,\overline{\xi}$, as

$$D=\frac{1}{2}\sum_i \overline{e}_i e^i-2\phi.$$

Proposition 7.5 *We have the following Poisson bracket relations in* $W(\mathfrak{g})$:

$$\begin{aligned}\{D,D\}&=2\sum_i \widehat{e_i}\,\widehat{e^i},\\ \{D,\xi\}&=\overline{\xi},\\ \{D,\overline{\xi}\}&=0,\end{aligned}$$

$$\{\overline{\xi}, \zeta\} = 2[\xi, \zeta]_{\mathfrak{g}},$$
$$\{\overline{\xi}, \overline{\zeta}\} = 2\overline{[\xi, \zeta]_{\mathfrak{g}}},$$
$$\{\xi, \zeta\} = 2B(\xi, \zeta).$$

In particular, $\{D, \cdot\} = \mathrm{d}^{(2)}$.

Proof The bracket relations involving $\xi, \overline{\xi}$ are all simple consequences of $\{\xi, \cdot\} = 2\iota(\xi)$ and $\{\overline{\xi}, \cdot\} = 2L(\xi)$. It remains to check the formula for $\{D, D\}$. We have

$$\{D, D\} = 2\sum_{ij} \widehat{e^i}\widehat{e^j} B(e_i, e_j) + \cdots = 2\sum_i \widehat{e^i}\widehat{e_i} + \cdots,$$

where the dots indicate terms in $S\mathfrak{g} \otimes \wedge^+\mathfrak{g}$. But $\{D, D\}$ lies in the symmetric algebra generated by the $\widehat{\xi}$, since

$$\iota(\xi)\{D, D\} = \frac{1}{2}\{\xi, \{D, D\}\} = \{\{\xi, D\}, D\} = \{\overline{\xi}, D\} = 0.$$

Hence the terms in $S\mathfrak{g} \otimes \wedge^+\mathfrak{g}$ all cancel. $\square$

Remark 7.3 Note that the quadratic Casimir element $\sum_i \widehat{e_i}\widehat{e^i}$ lies in the Poisson center of $W(\mathfrak{g})$. Hence the formula for $\{D, D\}$ is consistent with $[\mathrm{d}^{(2)}, \mathrm{d}^{(2)}] = 0$.

7.2.2 *Definition of the quantum Weil algebra*

Quantizing the Poisson bracket relations of the Weil algebra $W(\mathfrak{g})$, we arrive at the following definition.

Definition 7.1 The *quantum Weil algebra* $\mathcal{W}(\mathfrak{g})$ is the filtered super algebra generated by odd elements ξ of filtration degree 1 and even elements $\overline{\xi}$ of filtration degree 2, with commutation relations

$$[\xi, \zeta]_{\mathcal{W}} = 2B(\xi, \zeta), \quad [\overline{\xi}, \zeta]_{\mathcal{W}} = 2[\xi, \zeta]_{\mathfrak{g}}, \quad [\overline{\xi}, \overline{\zeta}]_{\mathcal{W}} = 2\overline{[\xi, \zeta]_{\mathfrak{g}}}.$$

That is, $\mathcal{W}(\mathfrak{g})$ is the *semidirect product*

$$\mathcal{W}(\mathfrak{g}) = U\mathfrak{g} \otimes_s \mathrm{Cl}(\mathfrak{g}),$$

where $U\mathfrak{g}$ is the enveloping algebra generated by the elements $\frac{1}{2}\overline{\xi}$, acting on $\mathrm{Cl}(\mathfrak{g})$ by the extension of the adjoint representation.

Put differently, $\mathcal{W}(\mathfrak{g})$ is the quotient of $\tilde{W}\mathfrak{g} = T(E_{\mathfrak{g}}[1])$ by the two-sided ideal generated by elements of the form

$$\xi \otimes \zeta + \zeta \otimes \xi - 2B(\xi, \zeta),$$
$$\overline{\xi} \otimes \overline{\zeta} + \overline{\zeta} \otimes \overline{\xi} - 2\overline{[\xi, \zeta]_{\mathfrak{g}}},$$
$$\overline{\xi} \otimes \zeta - \zeta \otimes \overline{\xi} - [\xi, \zeta]_{\mathfrak{g}}.$$

It is straightforward to see that this ideal is invariant under the differential, contractions and Lie derivatives on $\tilde{W}\mathfrak{g}$. Hence $\mathscr{W}(\mathfrak{g})$ becomes uniquely a filtered $\mathfrak{g}$-differential algebra, in such a way that the quotient map $\tilde{W}\mathfrak{g} \to \mathscr{W}(\mathfrak{g})$ is a morphism of $\mathfrak{g}$-differential algebras. The formulas for differential, contractions and Lie derivatives are induced from those on $\tilde{W}\mathfrak{g}$, and are given on generators by

$$\mathrm{d}\xi = \overline{\xi}, \quad \mathrm{d}\overline{\xi} = 0,$$

$$\iota(\xi)\zeta = B(\xi,\zeta), \quad \iota(\xi)\overline{\zeta} = [\xi,\zeta]_{\mathfrak{g}},$$

$$L(\xi)\zeta = [\xi,\zeta]_{\mathfrak{g}}, \quad L(\xi)\overline{\zeta} = \overline{[\xi,\zeta]_{\mathfrak{g}}}.$$

As for the commutative Weil algebra $W(\mathfrak{g})$, it is often convenient to change the variables to ξ, $\widehat{\xi} = \frac{1}{2}\overline{\xi} - \gamma(\xi)$ (where $\gamma(\xi) = q(\lambda(\xi))$). In terms of the new generators the commutation relations read as

$$[\xi,\zeta]_{\mathscr{W}} = 2B(\xi,\zeta), \quad [\widehat{\xi},\zeta]_{\mathscr{W}} = 0 \quad [\widehat{\xi},\widehat{\zeta}]_{\mathscr{W}} = \widehat{[\xi,\zeta]_{\mathfrak{g}}}.$$

Hence

$$\mathscr{W}(\mathfrak{g}) = U\mathfrak{g} \otimes \mathrm{Cl}\mathfrak{g},$$

where $U\mathfrak{g}$ denotes the enveloping algebra generated by the elements $\widehat{\xi}$. We see that the basic subcomplex is

$$(\mathscr{W}(\mathfrak{g}))_{\mathrm{bas}} = (U\mathfrak{g})^{\mathfrak{g}},$$

using the enveloping algebra generated by the $\widehat{\xi}$'s. Since it has no odd component, the differential on the basic subcomplex is zero. Thus

$$H_{\mathrm{bas}}(\mathscr{W}(\mathfrak{g})) = (U\mathfrak{g})^{\mathfrak{g}}.$$

In terms of the new variables, it is also clear that the associated graded algebra to the filtered algebra $\mathscr{W}(\mathfrak{g})$ is

$$\mathrm{gr}(\mathscr{W}(\mathfrak{g})) = W(\mathfrak{g}),$$

because $\mathrm{gr}(U\mathfrak{g}) = S\mathfrak{g}$ and $\mathrm{gr}(\mathrm{Cl}\mathfrak{g}) = \wedge\mathfrak{g}$. From the formulas on generators, we see that the Poisson bracket on $W(\mathfrak{g})$ is induced from the commutation on $\mathscr{W}(\mathfrak{g})$. Thus $\mathscr{W}(\mathfrak{g})$ is a quantization of $W(\mathfrak{g})$ in the sense of Section 2.3.

The *canonical anti-involution* $^\top$ on the Weil algebra $W(\mathfrak{g})$ is given on generators by

$$\overline{\xi}^{\top} = -\overline{\xi}, \quad \xi^{\top} = \xi, \quad \xi \in \mathfrak{g}. \tag{7.4}$$

On each graded component $W^k(\mathfrak{g})$, this anti-involution acts as -1 if $k = 2 \bmod 4$ or $k = 3 \bmod 4$, and as $+1$ if $k = 1 \bmod 4$ or $k = 0 \bmod 4$. The same formulas, Eq. (7.4), also define a canonical anti-involution $^\top$ on the quantum Weil algebra $\mathscr{W}(\mathfrak{g})$, and the quantization map q intertwines the two anti-involutions.

7.2.3 The cubic Dirac operator

Let $\mathscr{D} \in \mathscr{W}^{(3)}\mathfrak{g}$ be the "quantized version" of the element $D \in W^3\mathfrak{g}$,

$$\mathscr{D} = \frac{1}{2}\sum_i \overline{e^i} e_i - 2q(\phi).$$

Note that $\mathscr{D}$ is the unique odd element of filtration degree 3 with $\mathrm{gr}^3(\mathscr{D}) = D$, and such that $\mathscr{D}^\top = -\mathscr{D}$. In terms of the variables $\xi, \widehat{\xi}$, the cubic Dirac operator takes on the form

$$\mathscr{D} = \sum_i \widehat{e}^i e_i + q(\phi).$$

Similar to Proposition 7.5, we have:

Theorem 7.1 *We have the following commutation relations in $\mathscr{W}(\mathfrak{g})$:*

$$\begin{aligned}
[\mathscr{D}, \mathscr{D}]_{\mathscr{W}} &= 2\mathrm{Cas}_{\mathfrak{g}} + \frac{1}{12}\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}}),\\
[\overline{\xi}, \mathscr{D}]_{\mathscr{W}} &= 0,\\
[\xi, \mathscr{D}]_{\mathscr{W}} &= \overline{\xi},\\
[\overline{\xi}, \zeta]_{\mathscr{W}} &= 2[\xi, \zeta]_{\mathfrak{g}},\\
[\overline{\xi}, \overline{\zeta}]_{\mathscr{W}} &= 2\overline{[\xi, \zeta]_{\mathfrak{g}}},\\
[\xi, \zeta]_{\mathscr{W}} &= 2B(\xi, \zeta).
\end{aligned}$$

Here $\mathrm{Cas}_{\mathfrak{g}} = \sum_i \widehat{e_i}\widehat{e^i} \in U\mathfrak{g}$ *is the Casimir element, and* $\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}})$ *its trace in the adjoint representation. The contractions, Lie derivatives, and differential on* $\mathscr{W}(\mathfrak{g})$ *are inner derivations*:

$$\iota(\xi) = \frac{1}{2}[\xi, \cdot]_{\mathscr{W}}, \quad L(\xi) = \frac{1}{2}[\overline{\xi}, \cdot]_{\mathscr{W}}, \quad \mathrm{d} = [\mathscr{D}, \cdot]_{\mathscr{W}}. \tag{7.5}$$

Proof The identities $\iota(\xi) = \frac{1}{2}[\xi, \cdot]_{\mathscr{W}}$, $L(\xi) = \frac{1}{2}[\overline{\xi}, \cdot]_{\mathscr{W}}$ are clear from the definition of $\mathscr{W}(\mathfrak{g})$, and they imply the commutation relations involving $\xi, \overline{\xi}$. From $[\xi, [\mathscr{D}, \mathscr{D}]] = 2[\overline{\xi}, \mathscr{D}] = 0$, we see that $[\mathscr{D}, \mathscr{D}] \in U\mathfrak{g}$ (the enveloping algebra generated by the variables $\widehat{\xi}$). Denoting terms in $U\mathfrak{g} \otimes q(\wedge^+\mathfrak{g})$ by dots, and using our earlier computation of $[q(\phi), q(\phi)]$, we find:

$$\begin{aligned}
[\mathscr{D}, \mathscr{D}] &= \sum_{ij}[\widehat{e_i} e^i + q(\phi), \widehat{e_j} e^j + q(\phi)]\\
&= \sum_{ij} \widehat{e_i}\widehat{e_j}[e^i, e^j]_{\mathrm{Cl}} + [q(\phi), q(\phi)] + \cdots\\
&= 2\mathrm{Cas}_{\mathfrak{g}} + \frac{1}{12}\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}}),
\end{aligned}$$

since the terms in $U\mathfrak{g} \otimes q(\wedge^+\mathfrak{g})$ must cancel. The identity $\mathrm{d} = [\mathscr{D}, \cdot]_{\mathscr{W}}$ follows since both sides are derivations that agree on generators. □

The element $\mathscr{D}$ is called the *cubic Dirac operator* after Kostant [87]. It is an algebraic Dirac operator in the sense that its square

$$\mathscr{D}^2 = \mathrm{Cas}_{\mathfrak{g}} + \frac{1}{24}\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}})$$

is the quadratic Casimir (viewed as the algebraic counterpart to a Laplacian), up to lower order terms. In Chapter 9 we will discuss interpretations of $\mathscr{D}$ as a differential-geometric Dirac operator.

7.2.4 $\mathscr{W}(\mathfrak{g})$ as a level 1 enveloping algebra

We remarked that the Clifford algebra can be viewed as a "level 1 enveloping algebra" of a super Lie algebra, see Section 5.1.8. The quantum Weil algebra $\mathscr{W}(\mathfrak{g})$ can be viewed similarly, using the graded differential Lie algebra $E_{\mathfrak{g}}[1] = \mathfrak{g}[1] \rtimes \mathfrak{g}$. For $\xi \in \mathfrak{g}$ let I_ξ and L_ξ denote the corresponding generators of degree $-1, 0$. The bilinear form B defines a central extension

$$\mathbb{K}[2] \oplus \mathfrak{g}[1] \rtimes \mathfrak{g}, \tag{7.6}$$

where $\mathbb{K}[2]$ is spanned by a central generator c of degree -2, and with the new bracket relations

$$[I_\xi, I_\zeta] = 2B(\xi,\zeta)\mathsf{c}, \quad [L_\xi, I_\zeta] = I_{[\xi,\zeta]_{\mathfrak{g}}}, \quad [L_\xi, L_\zeta] = L_{[\xi,\zeta]_{\mathfrak{g}}}.$$

It is a graded $\mathfrak{g}$-differential Lie algebra, where the action of contractions and Lie derivatives is just the adjoint action of $\mathfrak{g}[1] \rtimes \mathfrak{g}$, and where $\mathrm{d}I_\xi = L_\xi$, $\mathrm{d}L_\xi = 0$, $\mathrm{d}\mathsf{c} = 0$. After degree shift by 2, (7.6) is a *filtered* differential Lie algebra

$$\mathbb{K} \oplus \mathfrak{g}[-1] \rtimes \mathfrak{g}[-2],$$

where c now has filtration degree 0. We may regard $\mathscr{W}(\mathfrak{g})$ as the level 1 enveloping algebra,

$$\mathscr{W}(\mathfrak{g}) = U(\mathbb{K} \oplus \mathfrak{g}[-1] \rtimes \mathfrak{g}[-2])/\langle \mathsf{c} - 1 \rangle,$$

(with the internal filtration). The generators ξ and $\overline{\xi}$ correspond to I_ξ and $2L_\xi$ with the shifted filtration degrees.

One can also go one step further and consider the graded differential Lie algebra

$$\mathbb{K}[2] \oplus \mathfrak{g}[1] \rtimes \mathfrak{g} \oplus \mathbb{K}[-1], \tag{7.7}$$

containing (7.6) as a subalgebra, and with the generator $D \in \mathbb{K}[-1]$ acting as the differential:

$$[D,D] = 0, \quad [D,\mathsf{c}] = 0, \quad [D, I_\xi] = L_\xi, \quad [D, L_\xi] = 0.$$

It is a graded super Lie algebra, with an odd invariant bilinear form given by $\langle I_\xi,\, L_\zeta \rangle = B(\xi,\zeta)$, $\langle \mathsf{c},\, D \rangle = 1$ (all other inner products among generators are 0).

Remark 7.4 The double extension (7.7) is similar to the standard double extension of the loop algebra of a quadratic Lie algebra. Indeed, both are obtained as double extensions for an orthogonal derivation [7]. See Ševera [113] for a more conceptual explanation of this relationship.

The Lie algebra (7.7) and the corresponding super Lie group appear in the work of Li-Bland and Ševera [99].

7.2.5 Conjugation

Suppose $\mathbb{K} = \mathbb{C}$. Recall that if V is a complex vector space with complex bilinear form B, given as the complexification of a real vector space $V_{\mathbb{R}}$ with a real bilinear form, the Clifford algebra $\mathrm{Cl}(V)$ admits a conjugate linear anti-involution $*$, given on generators by $v \in V$ by complex conjugation,

$$v^* = v^c.$$

Similarly, if a complex Lie algebra $\mathfrak{g}$ is given as the complexification of a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$, then the enveloping algebra $U(\mathfrak{g})$ has a conjugate linear anti-involution $x \mapsto x^*$, given on generators by $\xi \mapsto -\xi^c$. (Note the minus sign.) See Section 5.1.6.

Generalizing, there is a conjugate linear involution on the quantum Weil algebra $\mathscr{W}(\mathfrak{g})$, given on generators $\xi, \overline{\xi}$ by

$$\overline{\xi}^* = -\overline{\xi}^c, \quad \xi^* = \xi^c.$$

Indeed, the same formulas define a conjugate linear involution of $\widetilde{W}\mathfrak{g} = T(E_{\mathfrak{g}}[1])$, and the ideal defining $\mathscr{W}(\mathfrak{g})$ is invariant under $*$. For example,

$$\begin{aligned}
(\overline{\xi} \otimes \zeta - \zeta \otimes \overline{\xi} - [\xi, \zeta]_{\mathfrak{g}})^* &= \zeta^* \otimes \overline{\xi}^* - \overline{\xi}^* \otimes \zeta^* - [\xi, \zeta]_{\mathfrak{g}}^* \\
&= -\zeta^c \otimes \overline{\xi}^c + \overline{\xi}^c \otimes \zeta^c - [\xi, \zeta]_{\mathfrak{g}}^c \\
&= \overline{\xi}^c \otimes \zeta^c - \zeta^c \otimes \overline{\xi}^c - [\xi^c, \zeta^c]_{\mathfrak{g}}.
\end{aligned}$$

The element $q(\phi) \in \mathrm{Cl}(\mathfrak{g})$ and the cubic Dirac operator $\mathscr{D}$ are skew-adjoint element for the involution $*$

$$\mathscr{D}^* = -\mathscr{D}.$$

Let us also note that the Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g}),\ \xi \mapsto \gamma(\xi)$ is compatible with the involutions,

$$\gamma(\xi^*) = \gamma(\xi)^*.$$

Hence the algebra morphism $U(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ extending this Lie algebra morphism is again compatible with the involutions.

7.3 Application: Duflo's Theorem

Using the quantum Weil algebra, we will now prove Duflo's Theorem (cf. Theorem 5.10) for the case of quadratic Lie algebras $\mathfrak{g}$, following [4, 7]. Use the bilinear form on $\mathfrak{g}$ to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Think of $W(\mathfrak{g})$ (with differential $\mathrm{d}^{(2)}$) as $S(E_{\mathfrak{g}}[1])$, and let

$$q:\ W(\mathfrak{g}) \to \mathscr{W}(\mathfrak{g}) \tag{7.8}$$

be the map, extending $\xi \mapsto \xi$, $\overline{\xi} \mapsto \overline{\xi}$ on generators by super symmetrization. It is a vector space isomorphism since its associated graded map is the identity map.

Remark 7.5 The quantization map q can also be viewed as the map defined by super symmetrization

$$S(\mathbb{K} \oplus \mathfrak{g}[-1] \rtimes \mathfrak{g}[-2]) \to U(\mathbb{K} \oplus \mathfrak{g}[-1] \rtimes \mathfrak{g}[-2])$$

after taking a quotient of both sides by the respective ideals $\langle \mathsf{c} - 1 \rangle$.

By Proposition 6.5, the map (7.8) is an isomorphism of $\mathfrak{g}$-differential spaces. It restricts to vector space isomorphisms

$$(W(\mathfrak{g}))_{\mathrm{hor}} = S\mathfrak{g} \to (\mathscr{W}(\mathfrak{g}))_{\mathrm{hor}} = U\mathfrak{g} \tag{7.9}$$

and hence

$$(W(\mathfrak{g}))_{\mathrm{bas}} = (S\mathfrak{g})^{\mathfrak{g}} \to (\mathscr{W}(\mathfrak{g}))_{\mathrm{bas}} = (U\mathfrak{g})^{\mathfrak{g}}. \tag{7.10}$$

We may also think of (7.10) as the map in basic cohomology

$$H_{\mathrm{bas}}(W(\mathfrak{g})) = (S\mathfrak{g})^{\mathfrak{g}} \to H_{\mathrm{bas}}(\mathscr{W}(\mathfrak{g})) = (U\mathfrak{g})^{\mathfrak{g}},$$

since the differential on the basic subcomplexes is zero.

Theorem 7.2 *The map $(S\mathfrak{g})^{\mathfrak{g}} \to (U\mathfrak{g})^{\mathfrak{g}}$ given by (7.10) is an isomorphism of algebras.*

Proof The symmetrization map factors through the non-commutative Weil algebra $\tilde{W}\mathfrak{g}$:

$$W(\mathfrak{g}) \to \tilde{W}\mathfrak{g} \to \mathscr{W}(\mathfrak{g}),$$

hence (7.10) factors as

$$(S\mathfrak{g})^{\mathfrak{g}} \to H_{\mathrm{bas}}(\tilde{W}\mathfrak{g}) \to (U\mathfrak{g})^{\mathfrak{g}}.$$

The second map is an algebra homomorphism since it is the map in basic cohomology induced by the homomorphism of $\mathfrak{g}$-differential algebras $\tilde{W}\mathfrak{g} \to \mathscr{W}(\mathfrak{g})$. The first map is an algebra homomorphism, since it is inverse to the map in basic cohomology induced by the homomorphism of $\mathfrak{g}$-differential algebras $\tilde{W}\mathfrak{g} \to W(\mathfrak{g})$. □

We stress that the isomorphism $S\mathfrak{g} \to U\mathfrak{g}$ in (7.9) is *not* the symmetrization map for the enveloping algebra even though it comes from the symmetrization map q

of the Weil algebras $\mathscr{W}(\mathfrak{g})$. Indeed, q is defined using symmetrization with respect to $\xi, \overline{\xi}$, but $U\mathfrak{g}$ is the enveloping algebra generated by the variables $\widehat{\xi}$. To get an explicit formula for (7.9), we want to express q in terms of the symmetrization map

$$\mathrm{sym} = \mathrm{sym}_U \otimes q_{\mathrm{Cl}}: \; S\mathfrak{g} \otimes \wedge\mathfrak{g} \to U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g})$$

relative to $\xi, \widehat{\xi}$.

Recall that the formula relating exponentials in the exterior and in the Clifford algebra (cf. Theorem 4.2) involves a smooth function

$$\mathscr{S} \in C^\infty(\mathfrak{g}) \otimes \wedge\mathfrak{g}$$

of the form $\mathscr{S}(\xi) = J^{1/2}(\xi)\exp(\mathfrak{r}(\xi))$ where $J^{1/2}$ is the "Duflo factor" and $\mathfrak{r}$ is a certain meromorphic function with values in $\wedge^2\mathfrak{g}$. It gives rise to an element

$$\widetilde{\mathscr{S}} \in \overline{S}\mathfrak{g}^* \otimes \wedge\mathfrak{g}$$

which acts on $W(\mathfrak{g}) = S\mathfrak{g} \otimes \wedge\mathfrak{g}$ in a natural way: The $\overline{S}\mathfrak{g}^*$ factor acts as an infinite-order differential operator, while the second factor acts by contraction.

Theorem 7.3 (Alekseev–Meinrenken [4, 7]) *The isomorphism of $\mathfrak{g}$-differential spaces $q: W(\mathfrak{g}) \to \mathscr{W}(\mathfrak{g})$ is given in terms of the generators $\xi, \widehat{\xi}$ by*

$$q = \mathrm{sym} \circ \widetilde{\mathscr{S}}: \; S\mathfrak{g} \otimes \wedge\mathfrak{g} \to U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g}).$$

In particular, its restriction to $S\mathfrak{g} \subseteq W(\mathfrak{g})$ is the Duflo map,

$$\mathrm{Duf} = \mathrm{sym} \circ \widetilde{J^{1/2}}: \; S\mathfrak{g} \to U\mathfrak{g}.$$

Proof By definition, q is the symmetrization map relative to the variables $\xi, \overline{\xi}$. It may be characterized as follows: For all odd variables $\nu^i \in \mathfrak{g}^*$ and all even variables $\mu^j \in \mathfrak{g}^*$, and all $N = 0, 1, 2, \ldots$,

$$q\left(\left(\sum_i \nu^i e_i + \frac{1}{2}\sum_j \mu^j \overline{e}_j\right)^N\right) = \left(\sum_i \nu^i e_i + \frac{1}{2}\sum_j \mu^j \overline{e}_j\right)^N.$$

These conditions may be summarized in a single condition,

$$q\left(\exp_W\left(\sum_i \nu^i e_i + \frac{1}{2}\sum_j \mu^j \overline{e}_j\right)\right) = \exp_{\mathscr{W}}\left(\sum_i \nu^i e_i + \frac{1}{2}\sum_j \mu^j \overline{e}_j\right),$$

to be interpreted as an equality of formal power series in the variables ν and μ.

We want to express q in terms of the generators $e_i, \widehat{e_i} = \frac{1}{2}\overline{e}_i - \lambda(e_i)$ of $W(\mathfrak{g})$, respectively $e_i, \widehat{e_i} = \frac{1}{2}\overline{e}_i - \gamma(e_i)$ of $\mathscr{W}(\mathfrak{g})$. Using that $\widehat{e_i}$ and e_j commute in $\mathscr{W}(\mathfrak{g})$,

$$\exp_{\mathscr{W}}\left(\sum_i \nu^i e_i + \frac{1}{2}\sum_j \mu^j \overline{e}_j\right) = \exp_U\left(\sum_j \mu^j \widehat{e}_j\right)\exp_{\mathrm{Cl}}\left(\sum_i \nu^i e_i + \sum_j \mu^j \gamma(e_j)\right).$$

The first factor is $\mathrm{sym}(\exp_S(\sum_j \mu^j \widehat{e}_j))$ by definition of the symmetrization map $\mathrm{sym}: \; S\mathfrak{g} \to U\mathfrak{g}$. The second factor is the Clifford exponential of a quadratic element, and is related to the corresponding exponential in the exterior algebra,

$$\exp_{\mathrm{Cl}}\left(\sum_i \nu^i e_i + \sum_j \mu^j \gamma(e_j)\right) = q_{\mathrm{Cl}}\left(\iota(\mathscr{S}(\mu))\exp_\wedge\left(\sum_i \nu^i e_i + \sum_j \mu^j \lambda(e_j)\right)\right).$$

Hence

$$\begin{aligned}
&\exp_{\mathscr{W}}\Big(\sum_i \nu^i e_i + \frac{1}{2}\sum_j \mu^j \overline{e}_j\Big)\\
&= \mathrm{sym}_W\Big(\iota\big(\mathscr{S}(\mu)\big)\exp_S\Big(\sum_j \mu^j \widehat{e}_j\Big)\exp_\wedge\Big(\sum_i \nu^i e_i + \sum_j \mu^j \lambda(e_j)\Big)\Big)\\
&= \mathrm{sym}_W \circ \widetilde{S}\Big(\exp_W\Big(\sum_i \nu^i e_i + \sum_j \mu^j \lambda(e_j) + \sum_j \mu^j \widehat{e}_j\Big)\Big)\\
&= \mathrm{sym}_W \circ \widetilde{S}\Big(\exp_W\Big(\sum_i \nu^i e_i + \frac{1}{2}\sum_j \mu^j \overline{e}_j\Big)\Big),
\end{aligned}$$

which completes the proof. □

This finishes our proof of Duflo's Theorem in the case of quadratic Lie algebras. Note that in this proof, the "Duflo factor" $J^{1/2}$ is naturally interpreted in terms of Clifford algebra computations. On the other hand, Duflo's Theorem is valid for *arbitrary* Lie algebras (not only quadratic ones). In recent years, new proofs for the general case have been found using Kontsevich's theory of deformation quantization [82], and more recently by Alekseev–Torossian [10] in their approach to the Kashiwara–Vergne conjecture [75]. However, all of these proofs are a great deal more involved than the argument for the quadratic case.

7.4 Relative Dirac operators

In Kostant's paper [87], the cubic Dirac operator $\mathscr{D}$ appears as a special case of a *relative* cubic Dirac operator, associated to a pair of quadratic Lie algebras. Of particular interest is the case of equal rank reductive pairs. Independently, some of the ideas were also presented in lecture notes by A. Wassermann [119]. The discussion below uses further simplifications from [7].

Suppose $\mathfrak{g}$ is a quadratic Lie algebra, with bilinear form B, and $\mathfrak{k} \subseteq \mathfrak{g}$ is a quadratic subalgebra, i.e., the restriction of B to $\mathfrak{k}$ is non-degenerate. Let $\mathfrak{p} = \mathfrak{k}^\perp$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is a $\mathfrak{k}$-invariant orthogonal decomposition.

Definition 7.2 The *relative quantum Weil algebra* for the pair $\mathfrak{g}, \mathfrak{k}$ is the subalgebra

$$\mathscr{W}(\mathfrak{g},\mathfrak{k}) = (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{k}\text{-inv}}$$

of $\mathscr{W}(\mathfrak{g})$.

Equivalently, viewing $\mathscr{W}(\mathfrak{g})$ as a $\mathfrak{k}$-differential subalgebra by restriction, $\mathscr{W}(\mathfrak{g},\mathfrak{k})$ is the $\mathfrak{k}$-basic subalgebra. Define an injective algebra homomorphism

$$j:\ \mathscr{W}(\mathfrak{k})\to\mathscr{W}(\mathfrak{g}), \tag{7.11}$$

by sending the generators $\xi,\ \overline{\xi}\in\mathscr{W}(\mathfrak{k})$ for $\xi\in\mathfrak{k}$ to the corresponding generators in $\mathscr{W}(\mathfrak{g})$:

$$j(\xi)=\xi,\ \ j(\overline{\xi})=\overline{\xi}.$$

Checking on generators, it is immediate that j is a morphism of $\mathfrak{k}$-differential algebras.

Proposition 7.6 *The relative quantum Weil algebra* $\mathscr{W}(\mathfrak{g},\mathfrak{k})$ *is the commutant of the subalgebra* $j(\mathscr{W}(\mathfrak{k}))\subseteq\mathscr{W}(\mathfrak{g})$.

Proof Since $j(\mathscr{W}(\mathfrak{k}))$ has generators $j(\xi)=\xi$ and $j(\overline{\xi})=\overline{\xi}$, for $\xi\in\mathfrak{k}$, its commutant consists of elements annihilated by all $\iota(\xi)=\frac{1}{2}[\xi,\cdot]$ and $L(\xi)=\frac{1}{2}[\overline{\xi},\cdot]$ for $\xi\in\mathfrak{k}$. In other words, the commutant is the $\mathfrak{k}$-basic subalgebra $(\mathscr{W}(\mathfrak{g}))_{\mathfrak{k}\text{-bas}}=\mathscr{W}(\mathfrak{g},\mathfrak{k})$. □

Remark 7.6 The morphism j can also be viewed as follows: The inclusion $\mathfrak{k}\hookrightarrow\mathfrak{g}$ gives a morphism of graded $\mathfrak{k}$-differential Lie algebras,

$$\mathbb{K}[2]\oplus(\mathfrak{k}[1]\rtimes\mathfrak{k})\to\mathbb{K}[2]\oplus(\mathfrak{g}[1]\rtimes\mathfrak{g}).$$

Degree shift by 2 turns it into a morphism of filtered $\mathfrak{k}$-differential Lie algebras, and the enveloping functor gives a morphism of filtered $\mathfrak{k}$-differential algebras,

$$U(\mathbb{K}\oplus(\mathfrak{k}[-1]\rtimes\mathfrak{k}[-2]))\to U(\mathbb{K}\oplus(\mathfrak{g}[-1]\rtimes\mathfrak{g}[-2])).$$

(Here we are using the internal filtrations, as in Section 5.1.7.)

The map of quantum Weil algebras is obtained by taking the quotient by the ideal $\langle\mathsf{c}-1\rangle$ on both sides, where c is the generator of $\mathbb{K}$, and 1 is the unit of the enveloping algebra.

Definition 7.3 The difference

$$\mathscr{D}(\mathfrak{g},\mathfrak{k})=\mathscr{D}_{\mathfrak{g}}-j(\mathscr{D}_{\mathfrak{k}})$$

is called the *relative Dirac operator* for the pair $\mathfrak{g},\mathfrak{k}$.

Theorem 7.4 *The relative Dirac operator* $\mathscr{D}(\mathfrak{g},\mathfrak{k})$ *lies in the relative quantum Weil algebra* $\mathscr{W}(\mathfrak{g},\mathfrak{k})$ *and generates its differential*:

$$\mathrm{d}x=[\mathscr{D}(\mathfrak{g},\mathfrak{k}),x],\quad x\in\mathscr{W}(\mathfrak{g},\mathfrak{k}).$$

The commutator of $\mathscr{D}(\mathfrak{g},\mathfrak{k})$ *with itself is given by*

$$[\mathscr{D}(\mathfrak{g},\mathfrak{k}),\mathscr{D}(\mathfrak{g},\mathfrak{k})]=2\mathrm{Cas}_{\mathfrak{g}}-2j(\mathrm{Cas}_{\mathfrak{k}})+\frac{1}{12}\mathrm{tr}_{\mathfrak{g}}\mathrm{Cas}_{\mathfrak{g}}-\frac{1}{12}\mathrm{tr}_{\mathfrak{k}}\mathrm{Cas}_{\mathfrak{k}}, \tag{7.12}$$

where $\mathrm{Cas}_{\mathfrak{g}}\in U(\mathfrak{g})$ *and* $\mathrm{Cas}_{\mathfrak{k}}\in U(\mathfrak{k})$ *are the quadratic Casimir elements for the enveloping algebra generated by the "curvature variables".*

Proof The element $\mathscr{D}(\mathfrak{g},\mathfrak{k})$ is $\mathfrak{k}$-horizontal since

$$[\mathscr{D}(\mathfrak{g},\mathfrak{k}),\xi]=[\mathscr{D}_{\mathfrak{g}},\xi]-j([\mathscr{D}_{\mathfrak{k}},\xi])=\overline{\xi}-j(\overline{\xi})=0$$

for all $\xi\in\mathfrak{k}$. Similarly, it is $\mathfrak{k}$-invariant. This shows $\mathscr{D}(\mathfrak{g},\mathfrak{k})\in\mathscr{W}(\mathfrak{g},\mathfrak{k})$. Next, for $x\in\mathscr{W}(\mathfrak{g},\mathfrak{k})$ we have $[\mathscr{D}(\mathfrak{g},\mathfrak{k}),x]=[\mathscr{D}_{\mathfrak{g}},x]=\mathrm{d}x$, since elements of $\mathscr{W}(\mathfrak{g},\mathfrak{k})$ commute with elements of $\mathscr{W}(\mathfrak{k})$.

Write $\mathscr{D}_{\mathfrak{g}}=j(\mathscr{D}_{\mathfrak{k}})+\mathscr{D}(\mathfrak{g},\mathfrak{k})$. Since $j(\mathscr{D}_{\mathfrak{k}})\in j(\mathscr{W}(\mathfrak{k}))$ and $\mathscr{D}(\mathfrak{g},\mathfrak{k})\in\mathscr{W}(\mathfrak{g},\mathfrak{k})$ commute, we obtain

$$[\mathscr{D}_{\mathfrak{g}},\mathscr{D}_{\mathfrak{g}}]=j([\mathscr{D}_{\mathfrak{k}},\mathscr{D}_{\mathfrak{k}}])+[\mathscr{D}(\mathfrak{g},\mathfrak{k}),\mathscr{D}(\mathfrak{g},\mathfrak{k})].$$

The formula for $[\mathscr{D}(\mathfrak{g},\mathfrak{k}),\mathscr{D}(\mathfrak{g},\mathfrak{k})]$ now follows from the known formulas for $[\mathscr{D}_{\mathfrak{g}},\mathscr{D}_{\mathfrak{g}}]$ and $[\mathscr{D}_{\mathfrak{k}},\mathscr{D}_{\mathfrak{k}}]$, see Theorem 7.1. □

Let us now give a formula for $\mathscr{D}(\mathfrak{g},\mathfrak{k})$ in terms of the variables $\xi,\ \widehat{\xi}$. Invariance of the bilinear form implies that $[\mathfrak{k},\mathfrak{p}]\subseteq\mathfrak{p}$. Hence $\mathrm{ad}_{\xi},\ \xi\in\mathfrak{k}$, decomposes as a direct sum of its $\mathfrak{k}$ and $\mathfrak{p}$ components. Accordingly the element $\lambda_{\mathfrak{g}}(\xi)\in\wedge^2\mathfrak{g}$ decomposes as

$$\lambda_{\mathfrak{g}}(\xi)=\lambda_{\mathfrak{k}}(\xi)+\lambda_{\mathfrak{p}}(\xi), \tag{7.13}$$

for some $\lambda_{\mathfrak{p}}(\xi)=\lambda(\mathrm{ad}(\xi)|_{\mathfrak{p}})\in\wedge^2\mathfrak{p}$, and hence by quantization

$$\gamma_{\mathfrak{g}}(\xi)=\gamma_{\mathfrak{k}}(\xi)+\gamma_{\mathfrak{p}}(\xi).$$

Commutation with $\gamma_{\mathfrak{p}}(\xi)=q(\lambda_{\mathfrak{p}}(\xi)),\ \xi\in\mathfrak{k}$, generates the adjoint action of $\mathfrak{k}$ on $\mathrm{Cl}(\mathfrak{p})$.

Lemma 7.2 *The homomorphism $j:\mathscr{W}(\mathfrak{k})\to\mathscr{W}(\mathfrak{g})$ is given in terms of generators $\xi,\widehat{\xi}$ by*

$$j(\xi)=\xi,\quad j(\widehat{\xi})=\widehat{\xi}+\gamma_{\mathfrak{p}}(\xi),$$

for $\xi\in\mathfrak{k}$. In particular, the inclusion $U(\mathfrak{k})\to U(\mathfrak{g})\otimes\mathrm{Cl}(\mathfrak{p})$ is defined by the generators for the $\mathfrak{k}$-action.

Proof We have

$$j(\widehat{\xi})=j\left(\frac{1}{2}\overline{\xi}-\gamma_{\mathfrak{k}}(\xi)\right)=\frac{1}{2}\overline{\xi}-\gamma_{\mathfrak{k}}(\xi)=\widehat{\xi}+\gamma_{\mathfrak{p}}(\xi)$$

for all $\xi\in\mathfrak{k}$. □

Let $\phi_{\mathfrak{p}}\in\wedge^3\mathfrak{p}$ denote the cubic element, given as the projection of $\phi_{\mathfrak{g}}\in\wedge^3\mathfrak{g}$ relative to the splitting $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. Fix a basis $\{e_a\}$ of $\mathfrak{g}$, given by a basis of $\mathfrak{k}$ followed by a basis of $\mathfrak{p}$.

Lemma 7.3 *The decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ induces the following decomposition of the cubic element $\phi_{\mathfrak{g}}$,*

$$\phi_{\mathfrak{g}}=\phi_{\mathfrak{k}}+\phi_{\mathfrak{p}}+\sum\nolimits_a^{(\mathfrak{k})}\lambda_{\mathfrak{p}}(e_a)\wedge e^a.$$

Here $\sum_a^{(\mathfrak{k})}$ indicates summation over the basis of $\mathfrak{k}$.

Proof Since $\phi_{\mathfrak{p}}$ is the $\wedge^3\mathfrak{p}$-projection of $\phi_{\mathfrak{g}}$, we can find the difference $\phi_{\mathfrak{g}} - \phi_{\mathfrak{p}}$ by taking the contraction with elements $\xi \in \mathfrak{k}$. But

$$\iota(\xi)\left(\phi_{\mathfrak{k}} + \sum\nolimits_a^{(\mathfrak{k})} \lambda_{\mathfrak{p}}(e_a) \wedge e^a\right) = \lambda_{\mathfrak{k}}(\xi) + \lambda_{\mathfrak{p}}(\xi) = \lambda_{\mathfrak{g}}(\xi) = \iota(\xi)(\phi_{\mathfrak{g}} - \phi_{\mathfrak{p}}),$$

where we used (7.13). □

From the formula for $\phi_{\mathfrak{g}}$ (see (7.2)) we determine that

$$\phi_{\mathfrak{p}} = -\frac{1}{12}\sum\nolimits_{abc}^{(\mathfrak{p})} B([e_a, e_b]_{\mathfrak{g}}, e_c)\, e^a \wedge e^b \wedge e^c,$$

where $\sum_{abc}^{(\mathfrak{p})}$ indicates a triple summation over the basis of $\mathfrak{p}$.

Proposition 7.7 *The element $\mathscr{D}(\mathfrak{g}, \mathfrak{k}) \in \mathscr{W}(\mathfrak{g}, \mathfrak{k}) = (U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{k}\text{-inv}}$ is given by*

$$\mathscr{D}(\mathfrak{g}, \mathfrak{k}) = \sum\nolimits_a^{(\mathfrak{p})} \widehat{e}_a e^a + q(\phi_{\mathfrak{p}}), \tag{7.14}$$

where $\sum_a^{(\mathfrak{p})}$ indicates summation over the basis of $\mathfrak{p}$.

Proof Using the formulas for $\mathscr{D}_{\mathfrak{g}}, \mathscr{D}_{\mathfrak{k}}$, and the property $j(\widehat{\xi}) = \widehat{\xi} + \gamma_{\mathfrak{p}}(\xi)$ we have

$$\mathscr{D}(\mathfrak{g}, \mathfrak{k}) = \sum\nolimits_a^{(\mathfrak{p})} \widehat{e}_a e^a + q\left(\phi_{\mathfrak{g}} - \phi_{\mathfrak{k}} - \sum\nolimits_a^{(\mathfrak{k})} \lambda_{\mathfrak{p}}(e_a) \wedge e^a\right).$$

By the lemma above, the expression in parentheses is $\phi_{\mathfrak{p}}$. □

Remark 7.7 The pair $\mathfrak{g}, \mathfrak{k}$ of quadratic Lie algebras is called a *symmetric pair* if $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{g}} \subseteq \mathfrak{k}$. Equivalently, $\phi_{\mathfrak{p}} = 0$. For this case, the element $\mathscr{D}(\mathfrak{g}, \mathfrak{k}) = \sum_a^{(\mathfrak{p})} \widehat{e}_a e^a$ was studied by Parthasarathy [106]. (Specifically, the context was that of a real semisimple Lie group G with maximal compact subgroup K, with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition.) Kostant [87] discovered that for arbitrary pairs of quadratic Lie algebras, one obtains a Dirac operator with good properties if one adds the cubic term $q(\phi_{\mathfrak{p}})$. This motivated the terminology "cubic Dirac operator".

One can also define a morphism of the commutative Weil algebras

$$j:\ W(\mathfrak{k}) \to W(\mathfrak{g})$$

by the map $j(\xi) = \xi,\ j(\overline{\xi}) = \overline{\xi}$. The Poisson commutant of its image $j(W(\mathfrak{k}))$ is the $\mathfrak{k}$-basic subalgebra $W(\mathfrak{g}, \mathfrak{k}) = (W(\mathfrak{g}))_{\mathfrak{k}\text{-bas}}$. Clearly, $W(\mathfrak{g}, \mathfrak{k})$ is the associated graded algebra to $\mathscr{W}(\mathfrak{g}, \mathfrak{k})$.

Lemma 7.4 *We have a commutative diagram of $\mathfrak{k}$-differential spaces,*

$$\begin{array}{ccc} W(\mathfrak{k}) & \xrightarrow{\ j\ } & W(\mathfrak{g}) \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle q} \\ \mathscr{W}(\mathfrak{k}) & \xrightarrow[\ j\ ]{} & \mathscr{W}(\mathfrak{g}), \end{array} \tag{7.15}$$

where the vertical maps are quantization maps for quantum Weil algebras, as defined in Section 7.3.

Proof Recall that the quantization maps are given by symmetrization in the variables $\xi, \overline{\xi}$. Since $j: \mathscr{W}(\mathfrak{k}) \to \mathscr{W}(\mathfrak{g})$ and $j: W(\mathfrak{k}) \to W(\mathfrak{g})$ are algebra homomorphisms, the symmetrization map commutes with j. □

The morphism of $\mathfrak{k}$-differential algebras $j: \mathscr{W}(\mathfrak{k}) \to \mathscr{W}(\mathfrak{g})$ restricts to a morphism of $\mathfrak{k}$-basic subcomplexes. Since $(\mathscr{W}(\mathfrak{k}))_{\mathfrak{k}\text{-bas}} = (U\mathfrak{k})^{\mathfrak{k}}$, while $(\mathscr{W}(\mathfrak{g}))_{\mathfrak{k}\text{-bas}} = \mathscr{W}(\mathfrak{g}, \mathfrak{k})$, this gives a cochain map

$$j: (U\mathfrak{k})^{\mathfrak{k}} \to \mathscr{W}(\mathfrak{g}, \mathfrak{k}), \tag{7.16}$$

where the differential on $(U\mathfrak{k})^{\mathfrak{k}}$ is zero. Similarly, we obtain a cochain map $j: (S\mathfrak{k})^{\mathfrak{k}} \to W(\mathfrak{g}, \mathfrak{k})$.

Proposition 7.8 *The maps in the commutative diagram*

$$\begin{tikzcd} (S\mathfrak{k})^{\mathfrak{k}} \arrow[r] \arrow[d] & H(W(\mathfrak{g}, \mathfrak{k})) \arrow[d] \\ (U\mathfrak{k})^{\mathfrak{k}} \arrow[r] & H(\mathscr{W}(\mathfrak{g}, \mathfrak{k})) \end{tikzcd} \tag{7.17}$$

obtained by taking the $\mathfrak{k}$-basic cohomology in (7.15) are all isomorphisms of algebras. In particular,

$$H(\mathscr{W}(\mathfrak{g}, \mathfrak{k})) = (U\mathfrak{k})^{\mathfrak{k}}.$$

Proof We first show that the inclusion $j: W(\mathfrak{k}) \to W(\mathfrak{g})$ is a $\mathfrak{k}$-homotopy equivalence, with homotopy inverse the projection $W(\mathfrak{g}) \to W(\mathfrak{k})$. Let

$$W\mathfrak{p} = S(E_{\mathfrak{p}}[-1]) \subseteq W(\mathfrak{g})$$

be the subalgebra generated by $\xi \in \mathfrak{p}[-1]$, $\overline{\xi} \in \mathfrak{p}[-2]$ for $\xi \in \mathfrak{p}$. Note that $E_{\mathfrak{p}}[-1] \cong E_{\mathfrak{p}}[-1] \oplus 0$ is a $\mathfrak{k}$-differential subspace of $E_{\mathfrak{g}}[-1] \oplus \mathbb{K}\mathsf{c}$. Thus $W(\mathfrak{g}) = W(\mathfrak{k}) \otimes W\mathfrak{p}$ is an isomorphism of $\mathfrak{k}$-differential spaces. The homotopy equivalence between $S(E_{\mathfrak{p}}[-1])$ and $\mathbb{K}$ is compatible with the $\mathfrak{k}$-differential structure, i.e., it is a $\mathfrak{k}$-homotopy equivalence.

This gives the desired $\mathfrak{k}$-homotopy equivalence between $W(\mathfrak{g})$ and $W(\mathfrak{k})$. In particular, $j: W(\mathfrak{k}) \to W(\mathfrak{g})$ induces an isomorphism in basic cohomology, proving that the upper horizontal map in (7.15) is an algebra isomorphism. Consequently, the lower horizontal map is an algebra isomorphism as well. The left vertical map is an algebra isomorphism by Duflo's Theorem. We conclude that the right vertical map is an algebra isomorphism. □

The inclusion

$$(U\mathfrak{g})^{\mathfrak{g}} = (\mathscr{W}(\mathfrak{g}))_{\mathfrak{g}\text{-bas}} \hookrightarrow (\mathscr{W}(\mathfrak{g}))_{\mathfrak{k}\text{-bas}} = \mathscr{W}(\mathfrak{g}, \mathfrak{k})$$

of the $\mathfrak{g}$-basic subalgebra into $\mathfrak{k}$-basic subalgebra is a cochain map. Passing to cohomology, and using $H(\mathscr{W}(\mathfrak{g},\mathfrak{k}))\cong(U\mathfrak{k})^{\mathfrak{k}}$ it defines an *algebra* homomorphism

$$(U\mathfrak{g})^{\mathfrak{g}}\to(U\mathfrak{k})^{\mathfrak{k}}.$$

Theorem 7.5 *The algebra homomorphism $(U\mathfrak{g})^{\mathfrak{g}}\to(U\mathfrak{k})^{\mathfrak{k}}$ fits into a commutative diagram,*

$$\begin{tikzcd}
(S\mathfrak{g})^{\mathfrak{g}} \arrow[r] \arrow[d, "q"] & (S\mathfrak{k})^{\mathfrak{k}} \arrow[d, "q"] \\
(U\mathfrak{g})^{\mathfrak{g}} \arrow[r] & (U\mathfrak{k})^{\mathfrak{k}},
\end{tikzcd}$$

where the vertical maps are Duflo isomorphisms, and the upper horizontal map is induced by the orthogonal projection $\mathrm{pr}_{\mathfrak{k}}:\ \mathfrak{g}\to\mathfrak{k}$.

Proof The result follows from the commutative diagram

$$\begin{tikzcd}
(W(\mathfrak{g}))_{\mathfrak{g}\text{-bas}} \arrow[r] \arrow[d] & W(\mathfrak{g},\mathfrak{k}) \arrow[d] \\
(\mathscr{W}(\mathfrak{g}))_{\mathfrak{g}\text{-bas}} \arrow[r] & \mathscr{W}(\mathfrak{g},\mathfrak{k})
\end{tikzcd}$$

by passing to cohomology, using our results

$$H(W(\mathfrak{g},\mathfrak{k}))=(S\mathfrak{k})^{\mathfrak{k}},\ \ H(\mathscr{W}(\mathfrak{g},\mathfrak{k}))=(U\mathfrak{k})^{\mathfrak{k}}.$$

The upper horizontal map $(S\mathfrak{g})^{\mathfrak{g}}\to(S\mathfrak{k})^{\mathfrak{k}}$ can be viewed as a composition

$$H_{\mathfrak{g}\text{-bas}}(W(\mathfrak{g}))\to H_{\mathfrak{k}\text{-bas}}(W(\mathfrak{g}))\to H_{\mathfrak{k}\text{-bas}}(W(\mathfrak{k})),$$

where the second map is induced by the morphism $W(\mathfrak{g})\to W(\mathfrak{k})$ given by projection of $\xi,\overline{\xi}$ to their $\mathfrak{k}$-components. This map takes $\lambda_{\mathfrak{g}}(\xi)$ for $\xi\in\mathfrak{g}$ to $\lambda_{\mathfrak{k}}(\mathrm{pr}_{\mathfrak{k}}(\xi))$. Hence, in terms of the variables $\xi,\widehat{\xi}$ it is still induced by the projection $\xi\mapsto\mathrm{pr}_{\mathfrak{k}}(\xi)$, $\widehat{\xi}\mapsto\widehat{\mathrm{pr}_{\mathfrak{k}}(\xi)}$. As a consequence, the map $(S\mathfrak{g})^{\mathfrak{g}}\to(S\mathfrak{k})^{\mathfrak{k}}$ is simply the map induced by the orthogonal projection. □

Remark 7.8 Theorem 7.5 is a version of Vogan's conjecture (as formulated by Huang–Pandzic [65]) for quadratic Lie algebras. It was proved by Huang–Pandzic for symmetric pairs, and by Kostant [89] for reductive pairs. Kumar [93] interpreted Vogan's conjecture in terms of an induction map in the non-commutative equivariant cohomology from [4]. The simple proof given here, based on the quantization map for Weil algebras, is taken from [7].

7.5 Harish-Chandra projections

7.5.1 Enveloping algebras

A *triangular decomposition* of a Lie algebra $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+$$

as vector spaces, where $\mathfrak{n}_-, \mathfrak{k}, \mathfrak{n}_+$ are Lie subalgebras of $\mathfrak{g}$ and $[\mathfrak{k}, \mathfrak{n}_\pm] \subseteq \mathfrak{n}_\pm$. The triangular decomposition determines characters

$$\kappa_\pm \in \mathfrak{k}^*, \quad \kappa_\pm(\xi) = \frac{1}{2}\mathrm{tr}_{\mathfrak{n}_\pm}(\mathrm{ad}_\xi).$$

By the Poincaré–Birkhoff–Witt Theorem, the multiplication map

$$U(\mathfrak{n}_-) \otimes U(\mathfrak{k}) \otimes U(\mathfrak{n}_+) \to U(\mathfrak{g})$$

is an isomorphism of filtered vector spaces. By composing the inverse map with the projection to $U(\mathfrak{k})$ (using the augmentation maps $U(\mathfrak{n}_\pm) \to \mathbb{K}$), one obtains a map of filtered vector spaces

$$\mathsf{hc}_U : \; U(\mathfrak{g}) \to U(\mathfrak{k}), \tag{7.18}$$

left-inverse to the inclusion $U(\mathfrak{k}) \hookrightarrow U(\mathfrak{g})$. We will refer to hc_U as the *Harish-Chandra projection* for the given triangular decomposition of $\mathfrak{g}$. Equivalently, it is the projection to $U(\mathfrak{k})$ relative to the decomposition

$$U(\mathfrak{g}) = (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+) \oplus U(\mathfrak{k}).$$

Example 7.1 Let $\mathfrak{g}$ be a complex reductive Lie algebra, given as the complexification of a compact real form $\mathfrak{g}_\mathbb{R}$. Given $\xi_0 \in \mathfrak{g}_\mathbb{R}$, the eigenvalues of $\mathrm{ad}(\xi_0)$ on $\mathfrak{g}$ are all purely imaginary. Let $\mathfrak{n}_-, \mathfrak{k}, \mathfrak{n}_+$ be the sum of eigenspaces of $\mathrm{ad}(\xi_0)$ of eigenvalues $2\pi\sqrt{-1}s$ with $s < 0, s = 0, s > 0$ respectively. Then $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+$ is a triangular decomposition. If ξ_0 is a regular element so that $\mathfrak{k}$ is a Cartan subalgebra, the projection hc_U is the classical Harish-Chandra projection. In this case, ξ_0 determines a positive Weyl chamber, hence a system of simple roots, and the character κ_+ coincides with $\rho \in \mathfrak{t}^*$, the half-sum of positive roots.

The projection (7.18) does not preserve the product structure, in general. But it preserves all the other Hopf algebra structure maps:

Proposition 7.9 *The Harish-Chandra projection* (7.18) *intertwines the units*, *counits*, *comultiplications*, *and antipodes of the Hopf algebras* $U(\mathfrak{g})$ *and* $U(\mathfrak{k})$. *Suppose* $\mathfrak{s} \subseteq \mathfrak{k}$ *is a Lie subalgebra with the property*

$$(U(\mathfrak{k}) \otimes U^+(\mathfrak{n}_+))^\mathfrak{s} = 0, \quad (U^+(\mathfrak{n}_-) \otimes U(\mathfrak{k}))^\mathfrak{s} = 0.$$

Then hc_U *restricts to an algebra morphism*

$$U(\mathfrak{g})^\mathfrak{s} \to U(\mathfrak{k})^\mathfrak{s}$$

on $\mathfrak{s}$-invariants. In particular, if such a subalgebra $\mathfrak{s}$ exists, then hc_U is an algebra morphism on $\mathfrak{g}$-invariants.

Proof It is obvious that hc_U intertwines units and counits. The subspace $\ker(\mathsf{hc}_U) = (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)$ is invariant under the antipode s of $U\mathfrak{g}$, hence hc_U intertwines antipodes. Finally, since the coproduct Δ satisfies

$$\begin{aligned}\Delta(\ker(\mathsf{hc}_U)) &= \Delta(\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)\\ &\subseteq (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+) \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)\\ &= \ker(\mathsf{hc}_U \otimes \mathsf{hc}_U),\end{aligned}$$

we see that hc_U intertwines the comultiplications as well. The assumption on $\mathfrak{s}$ implies that

$$(\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{s}} = (\mathfrak{n}_- U(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{s}},$$

which is an ideal in $U(\mathfrak{g})^{\mathfrak{s}}$. Hence the quotient map to $U(\mathfrak{k})^{\mathfrak{s}}$ is an algebra homomorphism. □

Remark 7.9 The assumptions are satisfied in the setting of Example 7.1, by taking $\mathfrak{s} = \mathfrak{t}$ to be a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$. This follows by studying the decomposition into $\mathfrak{t}$-weight spaces.

Suppose that $\mathfrak{g}$ is a quadratic Lie algebra, with bilinear form B, and that the subalgebra $\mathfrak{k}$ is quadratic while $\mathfrak{n}_\pm$ are isotropic. We will assume furthermore that

$$\mathfrak{g}^{\mathfrak{k}} \subseteq \mathfrak{k}, \tag{7.19}$$

i.e., the $\mathfrak{k}$-fixed elements of $\mathfrak{g}$ are contained in the center of $\mathfrak{k}$. Since B defines a $\mathfrak{k}$-equivariant non-degenerate pairing between $\mathfrak{n}_\pm$, we have $\kappa_+ = -\kappa_-$. Put $\kappa = \kappa_+$.

Letting e_α be a basis of $\mathfrak{n}_+$, and e^α the B-dual basis of $\mathfrak{n}_- \cong (\mathfrak{n}_+)^*$, we have, for all $\xi \in \mathfrak{k}$,

$$\kappa(\xi) = \frac{1}{2}\mathrm{tr}_{\mathfrak{n}_+}(\mathrm{ad}_\xi) = \frac{1}{2}\sum_\alpha B(e^\alpha,\ [\xi e_\alpha]_{\mathfrak{g}}) = \frac{1}{2}\sum_\alpha B([e_\alpha, e^\alpha]_{\mathfrak{g}}, \xi) = B(\kappa^\sharp, \xi),$$

where

$$\kappa^\sharp = \frac{1}{2}\sum_\alpha [e_\alpha, e^\alpha]_{\mathfrak{g}}.$$

Note that $\kappa^\sharp$ lies in $\mathfrak{g}^{\mathfrak{k}}$, and hence, using the assumption (7.19), $\kappa^\sharp = B^\sharp(\kappa) \in \mathfrak{k}$.

Let $\mathrm{Cas}_{\mathfrak{g}} \in U(\mathfrak{g})$ be the quadratic Casimir defined by B, and $\mathrm{Cas}_{\mathfrak{k}}$ the Casimir defined by the restriction of B to $\mathfrak{k}$.

Proposition 7.10 *The Harish-Chandra projection of the quadratic Casimir element is given by*

$$\mathsf{hc}_U(\mathrm{Cas}_{\mathfrak{g}}) = \mathrm{Cas}_{\mathfrak{k}} + 2\kappa^\sharp.$$

Proof Let e_i be a basis of $\mathfrak{k}$, with B-dual basis e^i, and let e_α be a basis of $\mathfrak{n}_+$, with B-dual basis e^α of $\mathfrak{n}_-$. Then

$$\begin{aligned}\mathrm{Cas}_{\mathfrak{g}} &= \sum_i e_i e^i + \sum_\alpha (e_\alpha e^\alpha + e^\alpha e_\alpha) \\ &= \mathrm{Cas}_{\mathfrak{k}} + 2\sum_\alpha e^\alpha e_\alpha + 2\kappa^\sharp.\end{aligned}$$

The Harish-Chandra projection removes the second term. □

Remark 7.10 Given a central extension $0 \to \mathbb{K}\mathsf{c} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$, the triangular decomposition of $\mathfrak{g}$ induces a triangular decomposition of $\widehat{\mathfrak{g}}$. For any $r \in \mathbb{K}$ one may thus define Harish-Chandra projections

$$\mathsf{hc}_U : U_r(\widehat{\mathfrak{g}}) \to U_r(\widehat{\mathfrak{k}})$$

for the level r enveloping algebras.

Remark 7.11 The notion of triangular decomposition and the corresponding Harish-Chandra projection generalizes with obvious changes to the case of super Lie algebras.

7.5.2 Clifford algebras

Given a quadratic vector space (V, B) and a decomposition $V = V_- \oplus V_0 \oplus V_+$, where V_0 is a quadratic subspace and $V_0^\perp = V_+ \oplus V_-$ a splitting of the orthogonal subspace by two isotropic subspaces, one has an isomorphism

$$\mathrm{Cl}(V) \cong \wedge(V_-) \otimes \mathrm{Cl}(V_0) \otimes \wedge(V_+),$$

as algebras, and the augmentation maps for $\wedge V_\pm$ define a Harish-Chandra projection

$$\mathsf{hc}_{\mathrm{Cl}} : \mathrm{Cl}(V) \to \mathrm{Cl}(V_0).$$

Equivalently, this is the projection along $V_-\mathrm{Cl}(V) + \mathrm{Cl}(V)V_+$.

Remark 7.12 Thinking of $\mathrm{Cl}(V)$ as the level 1 enveloping algebra of the filtered super Lie algebra $\mathbb{K} \oplus V[-1]$ (cf. Section 5.1.8) this may be viewed as a Harish-Chandra projection

$$U_1(\mathbb{K} \oplus V[-1]) \to U_1(\mathbb{K} \oplus V_0[-1]),$$

as in Remarks 7.10, 7.11.

The Harish-Chandra projection is related to the orthogonal projection

$$\mathsf{p}_\wedge : \wedge(V) \to \wedge(V_0),$$

as follows. Let e_α be a basis of V_+, and e^α the dual basis of V_-, so that $B(e_\alpha, e^\beta) = \delta_\alpha^\beta$. Then $\mathfrak{r} := \sum_\alpha e^\alpha \wedge e_\alpha \in \wedge^2(V)$ is independent of the choice of basis, and defines a nilpotent operator $\iota(\mathfrak{r})$ on $\wedge(V)$. Let $q : \wedge(V) \to \mathrm{Cl}(V)$ be the quantization map, and $q_0 : \wedge(V_0) \to \mathrm{Cl}(V_0)$ its restriction to V_0.

Proposition 7.11 *The Harish-Chandra projection* $\mathsf{hc}_{\mathrm{Cl}}$ *and the orthogonal projection* $\mathsf{p}_\wedge$ *are related by*

$$\mathsf{hc}_{\mathrm{Cl}} \circ q = q_0 \circ \mathsf{p}_\wedge \circ \exp(-\iota(\mathfrak{r})).$$

Put differently, the automorphism $\exp(-\iota(\mathfrak{r}))$ *of* $\mathrm{Cl}(\mathfrak{g})$ *takes* $\mathrm{Cl}(\mathfrak{h})^\perp = q(\wedge(\mathfrak{h})^\perp)$ *to* $\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g}) + \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$.

Proof Write V as an orthogonal direct sum $V = V_0 \oplus \bigoplus_\alpha V(\alpha)$, where $V(\alpha)$ is the subspace spanned by e_α, e^α. Then

$$\mathrm{Cl}(V) = \mathrm{Cl}(V_0) \otimes \bigotimes_\alpha \mathrm{Cl}(V(\alpha)),$$

and $\mathsf{hc}_{\mathrm{Cl}}$ is the tensor product of the identity on $\mathrm{Cl}(V_0)$ and the Harish-Chandra projections $\mathrm{Cl}(V(\alpha)) \to \mathbb{C}$ for the remaining factors. Similarly, $\exp(-\iota(\mathfrak{r})) = \prod_\alpha \exp(-\iota(\mathfrak{r}(\alpha)))$, where $\mathfrak{r}(\alpha) = e^\alpha \wedge e_\alpha$.

The space $\wedge(V(\alpha))$ has basis $1,\ e_\alpha,\ e^\alpha,\ e^\alpha \wedge e_\alpha$. It is immediate that $\mathsf{hc}_{\mathrm{Cl}} \circ q$ and $\mathsf{p}_\wedge \circ \exp(-\iota(\mathfrak{r}(\alpha)))$ agree on the basis elements $1,\ e_\alpha,\ e^\alpha$. For the basis element $e^\alpha \wedge e_\alpha$ we have

$$\mathsf{p}_\wedge \circ \exp(-\iota(\mathfrak{r}(\alpha)))(e^\alpha \wedge e_\alpha) = \mathsf{p}_\wedge(e^\alpha \wedge e_\alpha - 1) = -1,$$

but also $\mathsf{hc}_{\mathrm{Cl}} \circ q(e^\alpha \wedge e_\alpha) = \mathsf{hc}_{\mathrm{Cl}}(e^\alpha e_\alpha - 1) = -1$. □

We are interested in the setting from Section 7.5.1, where $V = \mathfrak{g}$ is a quadratic Lie algebra, and $V_0 = \mathfrak{k}$ and $V_\pm = \mathfrak{n}_\pm$ are the summands of a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+$.

Lemma 7.5 *The Harish-Chandra projection* $\mathsf{hc}_{\mathrm{Cl}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{k})$ *intertwines contractions* $\iota(\xi)$ *and Lie derivatives* $L(\xi)$ *for* $\xi \in \mathfrak{k}$.

Proof The subspace $\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g}) + \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$ is invariant under $\iota(\xi),\ L(\xi)$. □

The projection γ_{Cl} does not intertwine the Clifford differentials, in general: See Proposition 7.15 below.

Let $\kappa \in \mathfrak{k}^*$ as above. Let $\phi_\mathfrak{g} \in \wedge^3\mathfrak{g},\quad \phi_\mathfrak{k} \in \wedge^3\mathfrak{k}$ be the structure constant tensors (cf. Section 7.1).

Proposition 7.12 *The Harish-Chandra projection of the quantized structure constant tensor is given by*

$$\mathsf{hc}_{\mathrm{Cl}}(q(\phi_\mathfrak{g})) = q(\phi_\mathfrak{k}) + \kappa^\sharp.$$

For any $\xi \in \mathfrak{k}$, the Harish-Chandra projection of $\gamma_{\mathfrak{g}}(\xi)$ is

$$\mathsf{hc}_{\mathrm{Cl}}(\gamma_{\mathfrak{g}}(\xi)) = \gamma_{\mathfrak{k}}(\xi) + \kappa(\xi).$$

Proof We use the basis from the proof of Proposition 7.10. We have the formula

$$\gamma_{\mathfrak{g}}(\xi) = \frac{1}{4}\sum_i [\xi, e_i]e^i + \frac{1}{4}\sum_\alpha [\xi, e^\alpha]_{\mathfrak{g}} e_\alpha + \frac{1}{4}\sum_\alpha [\xi, e_\alpha]_{\mathfrak{g}} e^\alpha. \tag{7.20}$$

The first term is $\gamma_{\mathfrak{k}}(\xi)$. The second term vanishes under $\mathsf{hc}_{\mathrm{Cl}}$. The last term has Harish-Chandra projection

$$\frac{1}{2}\sum_\alpha B([\xi, e_\alpha]_{\mathfrak{g}}, e^\alpha) = \kappa(\xi).$$

The difference $\mathsf{hc}_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})) - q(\phi_{\mathfrak{k}}) \in \mathrm{Cl}(\mathfrak{k})$ is an odd element of filtration degree 2, hence it lies in $\mathfrak{k}$. To compute this element we apply $\iota(\xi)$. We have

$$\begin{aligned}\iota(\xi)(\mathsf{hc}_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})) - q(\phi_{\mathfrak{k}})) &= \mathsf{hc}_{\mathrm{Cl}}(\iota(\xi)q(\phi_{\mathfrak{g}})) - \iota(\xi)q(\phi_{\mathfrak{k}})\\ &= \mathsf{hc}_{\mathrm{Cl}}(\gamma_{\mathfrak{g}}(\xi)) - \gamma_{\mathfrak{k}}(\xi)\\ &= \kappa(\xi).\end{aligned}$$

This shows $\mathsf{hc}_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})) - q(\phi_{\mathfrak{k}}) = \kappa^\sharp$. □

Proposition 7.13 *Let $\Gamma \in \mathrm{Cl}(\mathfrak{g})$ be a chirality element for $\mathfrak{g}$ (i.e., the quantization of a generator $\Gamma_\wedge$ of $\det(\mathfrak{g})$). Then $\mathsf{hc}_{\mathrm{Cl}}(\Gamma)$ is a chirality element of $\mathrm{Cl}(\mathfrak{k})$. If Γ is normalized so that $\Gamma^2 = 1$, then $\mathsf{hc}_{\mathrm{Cl}}(\Gamma)^2 = 1$.*

Proof Let e_i be a basis of $\mathfrak{k}$, and e_α, e^α the B-dual bases of $\mathfrak{n}_\pm$. We may take Γ to be the quantization of the wedge product of all these basis vectors.

$$\begin{aligned}\Gamma &= q\Big(\prod_\alpha e^\alpha \wedge e_\alpha \prod_i e_i\Big)\\ &= \prod_\alpha q\big(e^\alpha \wedge e_\alpha\big)\, q\Big(\prod_i e_i\Big)\\ &= \prod_\alpha \big(e^\alpha e_\alpha - 1\big)\, q\Big(\prod_i e_i\Big).\end{aligned}$$

We conclude that the Harish-Chandra projection is $\mathsf{hc}_{\mathrm{Cl}}(\Gamma) = \pm q(\prod_i e_i)$. The last claim follows since $\mathsf{hc}_{\mathrm{Cl}}$ is an algebra morphism on invariants. □

The Lie algebra morphism $\gamma_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ extends to an algebra morphism

$$\gamma_{\mathfrak{g}} : U(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g}),$$

and similarly for $\mathfrak{k}$. It turns out that they intertwine the Harish-Chandra projections, up to a small "twist". Let $\tau : U(\mathfrak{k}) \to U(\mathfrak{k})$ be the automorphism of $U(\mathfrak{k})$ extending the map $\mathfrak{k} \to U(\mathfrak{k}),\ \xi \mapsto \xi + \kappa(\xi)$.

Proposition 7.14 *The following diagram commutes*:

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[r, "\gamma_{\mathfrak{g}}"'] \arrow[d, "\tau\circ\mathsf{hc}_U"] & \mathrm{Cl}(\mathfrak{g}) \arrow[d, "\mathsf{hc}_{\mathrm{Cl}}"] \\ U(\mathfrak{k}) \arrow[r, "\gamma_{\mathfrak{k}}"'] & \mathrm{Cl}(\mathfrak{k}). \end{tikzcd}$$

Proof Since $\gamma(\mathfrak{n}_-) \subseteq \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})$ and $\gamma(\mathfrak{n}_+) \subseteq \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$, we have

$$\gamma(\ker(\mathsf{hc}_U)) \subseteq \ker(\mathsf{hc}_{\mathrm{Cl}}).$$

Thus, both $\mathsf{hc}_{\mathrm{Cl}} \circ \gamma_{\mathfrak{g}}$ and $\gamma_{\mathfrak{k}} \circ \mathsf{hc}_U$ vanish on $\ker(\mathsf{hc}_U)$. Hence it suffices to compare the two maps on $U(\mathfrak{k})$. From (7.20) we deduce that for all $\xi \in \mathfrak{k}$,

$$\begin{aligned} \gamma_{\mathfrak{g}}(\xi) &= \gamma_{\mathfrak{k}}(\xi) + \kappa(\xi) \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+ \\ &= \gamma_{\mathfrak{k}}(\tau(\xi)) \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+. \end{aligned}$$

This implies that

$$\gamma_{\mathfrak{g}}(\xi_1\cdots\xi_r) = \gamma_{\mathfrak{k}}(\tau(\xi_1))\cdots\gamma_{\mathfrak{k}}(\tau(\xi_r)) \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$$

for all $\xi_1,\ldots,\xi_r \in \mathfrak{k}$. □

Let $\mathrm{d}_{\mathfrak{g}}$ be the Clifford differential on $\mathrm{Cl}(\mathfrak{g})$, and $\mathrm{d}_{\mathfrak{k}}$ the Clifford differential on $\mathrm{Cl}(\mathfrak{k})$. (See Section 7.1.)

Proposition 7.15 *The Harish-Chandra projection* $\mathsf{hc}_{\mathrm{Cl}}$ *intertwines the Clifford differential* $\mathrm{d}_{\mathfrak{g}}$ *with the differential* $\mathrm{d}_{\mathfrak{k}} + 2\iota(\kappa^\sharp)$ *on* $\mathrm{Cl}(\mathfrak{k})$.

Proof Recall that $\mathrm{d}_{\mathfrak{g}}\xi = 2\gamma(\xi), \quad \xi \in \mathfrak{g}$. Since $\gamma_{\mathfrak{g}}(\mathfrak{n}_-) \in \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})$ and $\gamma_{\mathfrak{g}}(\mathfrak{n}_+) \in \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$. It follows that $\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})$ and $\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$ are differential subspaces of $\mathrm{Cl}(\mathfrak{g})$, hence so is their sum $\ker(\mathsf{hc}_{\mathrm{Cl}})$. On the other hand, for $\xi \in \mathfrak{k} \subseteq \mathrm{Cl}(\mathfrak{g})$ we have

$$\begin{aligned} \mathrm{d}_{\mathfrak{g}}\xi &= 2\gamma_{\mathfrak{g}}(\xi) \\ &= 2(\gamma_{\mathfrak{k}}(\xi) + \kappa(\xi)) \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+ \\ &= (\mathrm{d}_{\mathfrak{k}} + 2\iota(\kappa^\sharp))\xi \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+. \end{aligned}$$

Hence, if $x \in \mathrm{Cl}(\mathfrak{k}) \subseteq \mathrm{Cl}(\mathfrak{g})$, $\mathrm{d}_{\mathfrak{g}}x = (\mathrm{d}_{\mathfrak{k}} + 2\iota(\kappa^\sharp))x \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$. □

Proposition 7.16 *Suppose* $\mathfrak{s} \subseteq \mathfrak{g}$ *preserves* $\mathfrak{k}, \mathfrak{n}_+, \mathfrak{n}_-$ *and that*

$$(\wedge^+(\mathfrak{n}_-) \otimes \mathrm{Cl}(\mathfrak{k}))^{\mathfrak{s}} = 0,\ (\mathrm{Cl}(\mathfrak{k}) \otimes \wedge^+(\mathfrak{n}_+))^{\mathfrak{s}} = 0.$$

Then the Harish-Chandra projection γ_{Cl} *restricts to an algebra morphism on* $\mathfrak{s}$*-invariants. In particular, if such a subalgebra* $\mathfrak{s}$ *exists, then* γ_{Cl} *restricts to an algebra morphism* $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cl}(\mathfrak{k})$. *Furthermore, in this case*

$$\frac{1}{12}\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}})) - \frac{1}{12}\mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{k}})) = 2\langle\kappa,\kappa^\sharp\rangle. \tag{7.21}$$

Proof The conditions on $\mathfrak{s}$ imply

$$\mathrm{Cl}(\mathfrak{g})^{\mathfrak{s}} = (\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{s}} \oplus \mathrm{Cl}(\mathfrak{k})^{\mathfrak{s}}.$$

Since $(\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{s}}$ is a 2-sided ideal in $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{s}}$, the quotient map $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{s}} \to \mathrm{Cl}(\mathfrak{k})^{\mathfrak{s}}$ is an algebra homomorphism. We use this fact, together with $\mathsf{hc}_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})) = q(\phi_{\mathfrak{k}}) + \kappa^{\sharp}$, to compute $[\mathsf{hc}_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})), \mathsf{hc}_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}}))]$. We have

$$[q(\phi_{\mathfrak{k}}), \kappa^{\sharp}] = 2\gamma_{\mathfrak{k}}(\kappa^{\sharp}) = 0$$

since $\kappa^{\sharp}$ is central in $\mathfrak{k}$. Hence

$$[\mathsf{hc}_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})), \mathsf{hc}_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}}))] = [q(\phi_{\mathfrak{k}}), q(\phi_{\mathfrak{k}})] + 2B(\kappa^{\sharp}, \kappa^{\sharp}).$$

On the other hand, we have

$$[q(\phi_{\mathfrak{k}}), q(\phi_{\mathfrak{k}})] = \frac{1}{12}\mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{k}})), \quad [q(\phi_{\mathfrak{g}}), q(\phi_{\mathfrak{g}})] = \frac{1}{12}\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}})),$$

cf. Section 7.1. □

The assumptions of the proposition hold for the standard example 7.1, by taking $\mathfrak{s} = \mathfrak{t}$ to be a Cartan subalgebra in $\mathfrak{k}$. In this case, the proposition above is a version of the *Freudenthal–de Vries strange formula*. (See [87, Proposition 1.84], and Proposition 8.4 below.)

7.5.3 Quantum Weil algebras

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+$ be as in the last section. Let $\mathscr{W}(\mathfrak{g})$ be the corresponding quantum Weil algebra. Note that the filtered Lie algebra $\widetilde{\mathfrak{n}_+} = \mathfrak{n}_+[-1] \rtimes \mathfrak{n}_+[-2]$ (spanned by $\xi, \overline{\xi}$ for $\xi \in \mathfrak{n}_+$) is a $\mathfrak{k}$-differential subspace of $\mathscr{W}(\mathfrak{g})$, and similarly for $\widetilde{\mathfrak{n}_-} = \mathfrak{n}_-[-1] \rtimes \mathfrak{n}_-[-2]$. Hence $\widetilde{\mathfrak{n}_-}\mathscr{W}(\mathfrak{g}) + \mathscr{W}(\mathfrak{g})\widetilde{\mathfrak{n}_+}$ is a $\mathfrak{k}$-differential subspace. It defines a complement to $\mathscr{W}(\mathfrak{k})$. The projection onto the second summand in

$$\mathscr{W}(\mathfrak{g}) = (\widetilde{\mathfrak{n}_-}\mathscr{W}(\mathfrak{g}) + \mathscr{W}(\mathfrak{g})\widetilde{\mathfrak{n}_+}) \oplus \mathscr{W}(\mathfrak{k})$$

will be called the Harish-Chandra projection

$$\mathsf{hc}_{\mathscr{W}} : \mathscr{W}(\mathfrak{g}) \to \mathscr{W}(\mathfrak{k})$$

for the quantum Weil algebras. Put differently, $\mathsf{hc}_{\mathscr{W}}$ is obtained by composing the inverse of the map

$$U(\widetilde{\mathfrak{n}_-}) \otimes \mathscr{W}(\mathfrak{k}) \otimes U(\widetilde{\mathfrak{n}_+}) \to \mathscr{W}(\mathfrak{g}) \tag{7.22}$$

with the augmentation maps for $U(\widetilde{\mathfrak{n}_\pm})$.

Proposition 7.17 *The Harish-Chandra projection $\mathsf{hc}_{\mathscr{W}}$ is a morphism of $\mathfrak{k}$-differential spaces, left-inverse to the inclusion $j : \mathscr{W}(\mathfrak{k}) \hookrightarrow \mathscr{W}(\mathfrak{g})$. It restricts to a cochain map $\mathscr{W}(\mathfrak{g}, \mathfrak{k}) \to U(\mathfrak{k})^{\mathfrak{k}}$.*

Proof The first part is evident from the construction. The second part follows since $\mathsf{hc}_{\mathscr{W}}$ takes $\mathscr{W}(\mathfrak{g})_{\mathfrak{k}\text{-bas}} = \mathscr{W}(\mathfrak{g}, \mathfrak{k})$ to $\mathscr{W}(\mathfrak{k})_{\mathfrak{k}\text{-bas}} = (U\mathfrak{k})^{\mathfrak{k}}$. $\square$

Proposition 7.18 *The Harish-Chandra projection of the cubic Dirac operator for* $\mathfrak{g}$ *is the cubic Dirac operator for* $\mathfrak{k}$:

$$\mathsf{hc}_{\mathscr{W}}(\mathscr{D}_{\mathfrak{g}}) = \mathscr{D}_{\mathfrak{k}}.$$

Proof Recall that $\mathscr{D}_{\mathfrak{g}} = j(\mathscr{D}_{\mathfrak{k}}) + \mathscr{D}(\mathfrak{g}, \mathfrak{k})$. We have $\mathsf{hc}_{\mathscr{W}}(\mathscr{D}(\mathfrak{g}, \mathfrak{k})) = 0$ since the subalgebra $j(\mathscr{W}(\mathfrak{g}, \mathfrak{k}))$ consists of even elements. $\square$

Using the identifications $\mathscr{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$, $\mathscr{W}(\mathfrak{k}) = U(\mathfrak{k}) \otimes \mathrm{Cl}(\mathfrak{k})$, one can compare $\mathsf{hc}_{\mathscr{W}}$ with the tensor product $\mathsf{hc}_U \otimes \mathsf{hc}_{\mathrm{Cl}}$.

Theorem 7.6 *The following diagram commutes*:

$$\begin{tikzcd}
\mathscr{W}(\mathfrak{g}) \arrow[r] \arrow[d, "\mathsf{hc}_{\mathscr{W}}"] & U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}) \arrow[d, "(\tau^{-1}\circ \mathsf{hc}_U)\otimes \mathsf{hc}_{\mathrm{Cl}}"] \\
W(\mathfrak{k}) \arrow[r] & U(\mathfrak{k}) \otimes \mathrm{Cl}(\mathfrak{k}).
\end{tikzcd}$$

The map in basic cohomology $H_{\mathfrak{g}\text{-bas}}(\mathscr{W}(\mathfrak{g})) = U(\mathfrak{g})^{\mathfrak{g}} \to H_{\mathfrak{k}\text{-bas}}(\mathscr{W}(\mathfrak{k})) = U(\mathfrak{k})^{\mathfrak{k}}$ *is therefore the composition* $\tau^{-1} \circ \mathsf{hc}_U$.

Proof The proof extends that of Proposition 7.14. Both $\mathsf{hc}_U \otimes \mathsf{hc}_{\mathrm{Cl}}$ and $\mathsf{hc}_{\mathscr{W}}$ vanish on $\widetilde{\mathfrak{n}_-}\mathscr{W}(\mathfrak{g}) + \mathscr{W}(\mathfrak{g})\widetilde{\mathfrak{n}_+}$. It hence suffices to compare the two maps on $\mathscr{W}(\mathfrak{k}) \subseteq \mathscr{W}(\mathfrak{g})$. $\mathscr{W}(\mathfrak{k})$ is spanned by elements of the form

$$\overline{\xi_1} \cdots \overline{\xi_r}\, x, \tag{7.23}$$

where $\xi_i \in k$ and $x \in \mathrm{Cl}(\mathfrak{k})$. Arguing as in the proof of Proposition 7.14, we see that

$$\overline{\xi} = \widehat{\xi} + \gamma^{\mathfrak{k}}(\xi) + \langle \kappa, \xi \rangle \mod \mathfrak{n}_- \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+, \quad \xi \in \mathfrak{k}.$$

Hence the image of (7.23) under the isomorphism with $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$ is

$$(\widehat{\xi_1} + \gamma^{\mathfrak{k}}(\xi_1) + \langle \kappa, \xi_1 \rangle) \cdots (\widehat{\xi_r} + \gamma^{\mathfrak{k}}(\xi_r) + \langle \kappa, \xi_1 \rangle) x + \ldots, \tag{7.24}$$

where $\ldots$ lies in $U(\mathfrak{g}) \otimes \ker(\mathsf{hc}_{\mathrm{Cl}})$. The projection $\mathsf{hc}_U \otimes \mathsf{hc}_{\mathrm{Cl}}$, followed by the shift $\tau^{-1} \otimes 1$, takes (7.24) to

$$(\widehat{\xi_1} + \gamma^{\mathfrak{k}}(\xi_1)) \cdots (\widehat{\xi_r} + \gamma^{\mathfrak{k}}(\xi_r)) x \in U(\mathfrak{k}) \otimes \mathrm{Cl}(\mathfrak{k}).$$

But this is exactly the image of (7.23) under the isomorphism with $\mathscr{W}\mathfrak{k}$. $\square$

For the commutative Weil algebras, one has a much simpler projection

$$\mathsf{p}_W : W(\mathfrak{g}) \to W(\mathfrak{k}),$$

induced by the dual map of the inclusion $\mathfrak{k} \to \mathfrak{g}$ (i.e., the projection $\mathfrak{g} \to \mathfrak{k}$).

Theorem 7.7 *The following diagram commutes up to $\mathfrak{k}$-homotopy*:

$$\begin{tikzcd} W(\mathfrak{g}) \arrow[r, "q"'] \arrow[d, "\mathsf{p}_W"] & \mathscr{W}(\mathfrak{g}) \arrow[d, "\mathsf{hc}_{\mathscr{W}}"] \\ W(\mathfrak{k}) \arrow[r, "q"'] & \mathscr{W}(\mathfrak{k}). \end{tikzcd}$$

In fact, all maps in this diagram are $\mathfrak{k}$-homotopy equivalences. It gives rise to a commutative diagram

$$\begin{tikzcd} S(\mathfrak{g})^{\mathfrak{g}} \arrow[r, "q"'] \arrow[d, "\mathsf{p}_S"] & U(\mathfrak{g})^{\mathfrak{g}} \arrow[d, "\tau^{-1}\circ \mathsf{hc}_U"] \\ S(\mathfrak{k})^{\mathfrak{k}} \arrow[r, "q"'] & U(\mathfrak{k})^{\mathfrak{k}}, \end{tikzcd}$$

where the horizontal maps are Duflo maps.

Proof Interpret $\mathsf{hc}_{\mathscr{W}}$ as in (7.22). Since $\widetilde{\mathfrak{n}_{\pm}}$ are $\mathfrak{k}$-differential spaces, the augmentation maps

$$U(\widetilde{\mathfrak{n}_{\pm}}) \to \mathbb{K}$$

are $\mathfrak{k}$-homotopy equivalences. Hence $\mathsf{hc}_{\mathscr{W}}$ is a $\mathfrak{k}$-homotopy equivalence. Similarly, p_W is a $\mathfrak{k}$-homotopy equivalence, as are the quantization maps

$$q:\ W(\mathfrak{g}) \to \mathscr{W}(\mathfrak{g}), \quad q:\ W(\mathfrak{k}) \to \mathscr{W}(\mathfrak{k}).$$

Using Theorem 7.6, we see that the resulting commutative diagram in basic cohomology

$$\begin{tikzcd} H_{\mathfrak{g}\text{-bas}}(W(\mathfrak{g})) \arrow[r] \arrow[d] & H_{\mathfrak{g}\text{-bas}}(\mathscr{W}(\mathfrak{g})) \arrow[d] \\ H_{\mathfrak{k}\text{-bas}}(W(\mathfrak{k})) \arrow[r] & H_{\mathfrak{k}\text{-bas}}(\mathscr{W}(\mathfrak{k})) \end{tikzcd}$$

is the second diagram described in the theorem. □

Note that this Theorem 7.7 realizes the map $(U\mathfrak{g})^{\mathfrak{g}} \to (U\mathfrak{k})^{\mathfrak{k}}$ from Theorem 7.5 in terms of the Harish-Chandra projection.

Chapter 8
Applications to reductive Lie algebras

We will now apply the results from the previous chapter to the case where $\mathfrak{g}$ is a complex reductive Lie algebra. We will discuss the Harish-Chandra projection $\mathsf{hc}_U : U(\mathfrak{g}) \to S(\mathfrak{t})$ for the enveloping algebra, its counterpart $\mathsf{hc}_{\mathrm{Cl}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{t})$ for the Clifford algebra, and their interaction under the natural algebra morphism $\gamma : U(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$. A number of classical results, such as the Freudenthal–de Vries "strange formula", will be proved in this spirit. In Section 8.4 we discuss the theory of multiplets of representations for equal rank Lie subalgebras, due to Gross–Kostant–Ramond–Sternberg, and its explanation in terms of the cubic Dirac operator, due to Kostant. The final section is devoted to Dirac induction.

8.1 Notation

We refer to Appendix B for background information on reductive Lie algebras. Let us list some of the notation introduced there.

Fix a compact real form $\mathfrak{g}_{\mathbb{R}}$, and let $\xi \to \xi^c$ be the corresponding complex conjugation mapping for $\mathfrak{g}$. It determines a conjugate-linear anti-involution, $\xi \mapsto \xi^* = -\xi^c$. (Cf. Section 5.1.6.) We assume that B is an invariant non-degenerate symmetric $\mathbb{C}$-bilinear form on $\mathfrak{g}$, given as the complexification of an $\mathbb{R}$-bilinear form on $\mathfrak{g}_{\mathbb{R}}$. Since B is non-degenerate, it identifies $\mathfrak{g}$ and $\mathfrak{g}^*$; the resulting bilinear form on $\mathfrak{g}^*$ will be denoted by B^*.

We denote by $\mathfrak{t} \subseteq \mathfrak{g}$ a Cartan subalgebra, obtained by complexifying a maximal Abelian subalgebra $\mathfrak{t}_{\mathbb{R}} \subseteq \mathfrak{g}_{\mathbb{R}}$. Let $\mathfrak{R} \subseteq \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^*$ be the set of roots of $\mathfrak{g}$, and $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$ the root space corresponding to the root $\alpha \in \mathfrak{R}$. Fix a positive Weyl chamber $\mathfrak{t}_+$, corresponding to a decomposition $\mathfrak{R} = \mathfrak{R}_+ \cup \mathfrak{R}_-$ into positive and negative roots. We let $e_\alpha \in \mathfrak{g}$ for $\alpha \in \mathfrak{R}$ be root vectors, normalized in such a way that $e_\alpha^c = e_{-\alpha}$ (thus $e_\alpha^* = -e_{-\alpha}$) and $B(e_\alpha, e_{-\alpha}) = 1$.

The weight lattice will be denoted by P, and the set of dominant weights by P_+. Then P_+ labels the irreducible finite-dimensional representations of $\mathfrak{g}$. We let $V(\mu)$ denote the irreducible $\mathfrak{g}$-representation of highest weight $\mu \in P_+$.

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8.2 Harish-Chandra projections

8.2.1 Harish-Chandra projection for $U(\mathfrak{g})$

Consider the nilpotent Lie subalgebras

$$\mathfrak{n}_+ = \bigoplus_{\alpha\in\mathfrak{R}_+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \bigoplus_{\alpha\in\mathfrak{R}_-} \mathfrak{g}_\alpha$$

with basis the root vectors e_α for $\alpha \in \mathfrak{R}_+$ resp. $\alpha \in \mathfrak{R}_-$. Then

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$$

is the standard triangular decomposition of $\mathfrak{g}$. The character $\kappa \in \mathfrak{t}^*$ (cf. Section 7.5.1) is calculated as follows:

$$\langle \kappa, \xi\rangle = \frac{1}{2}\mathrm{tr}_{\mathfrak{n}_+}(\mathrm{ad}(\xi)) = \frac{1}{2}\sum_{\alpha\in\mathfrak{R}_+}\langle \alpha, \xi\rangle,$$

for $\xi \in \mathfrak{t}$. Thus κ coincides with

$$\rho = \frac{1}{2}\sum_{\alpha\in\mathfrak{R}_+}\alpha,$$

the half-sum of positive roots of $\mathfrak{g}$. The Harish-Chandra projection (Section 7.5.1)

$$\mathsf{hc}_U : U(\mathfrak{g}) \to U(\mathfrak{t}) = S(\mathfrak{t})$$

defined by the triangular decomposition restricts to an algebra homomorphism on $\mathfrak{g}$-invariants (cf. Proposition 7.9). We are interested in its image.

Let $a \in S(\mathfrak{g})^{\mathfrak{g}}$, and let $q(a) \in U(\mathfrak{g})^{\mathfrak{g}}$ be its image under the Duflo map. Theorem 7.7 shows that the images $\mathsf{p}_S(a) \in S(\mathfrak{t})$ and $\mathsf{hc}_U(q(a)) \in U(\mathfrak{t}) = S(\mathfrak{t})$ are related by the algebra automorphism τ of $S(\mathfrak{t})$, given on generators by

$$\xi \mapsto \xi + \langle \rho, \xi\rangle.$$

That is,

$$\mathsf{hc}_U(q(a)) = \tau(\mathsf{p}_S(a)), \quad a \in S(\mathfrak{g})^{\mathfrak{g}}. \tag{8.1}$$

Theorem 8.1

1. (Chevalley) *The projection* $\mathsf{p}_S : S(\mathfrak{g}) \to S(\mathfrak{t})$ *restricts to an isomorphism from* $S(\mathfrak{g})^{\mathfrak{g}}$ *to the space* $S(\mathfrak{t})^W$ *of Weyl-invariant polynomials.*
2. (Harish-Chandra) *The projection* $\mathsf{hc}_U : U(\mathfrak{g}) \to S(\mathfrak{t})$ *restricts to an isomorphism from* $U(\mathfrak{g})^{\mathfrak{g}}$ *to the space of polynomials that are invariant under the shifted Weyl group action,*

$$w : \mu \mapsto w(\mu+\rho) - \rho.$$

Proof We may assume that $\mathfrak{g}$ is semisimple.

1. It will be more convenient to work with the dual spaces. (The choice of an invariant quadratic form identifies $\mathfrak{g} \cong \mathfrak{g}^*$, $\mathfrak{t} \cong \mathfrak{t}^*$.) That is, we will show that the natural projection $S(\mathfrak{g}^*) \to S(\mathfrak{t}^*)$ induces an isomorphism from $\mathfrak{g}$-invariants to W-invariants. Identify $S(\mathfrak{g}^*)$ and $S(\mathfrak{t}^*)$ with polynomials on $\mathfrak{g}$ and $\mathfrak{t}$ respectively. Let $\mathfrak{g}_{\mathbb{R}} \supseteq \mathfrak{t}_{\mathbb{R}}$ be the compact real forms, and let $G_{\mathbb{R}}$ be a compact Lie group integrating $\mathfrak{g}_{\mathbb{R}}$. A polynomial on $\mathfrak{g}$ is $\mathrm{ad}(\mathfrak{g}_{\mathbb{R}})$-invariant if and only if it is $\mathrm{Ad}(G_{\mathbb{R}})$-invariant. Since every $\mathrm{Ad}(G_{\mathbb{R}})$-orbit in $\mathfrak{g}$ intersects $\mathfrak{t}$ in an orbit of W, this shows that p_S restricts to an *injective* map $(S\mathfrak{g}_{\mathbb{R}}^*)^{\mathfrak{g}_{\mathbb{R}}}$ into $(S\mathfrak{t}_{\mathbb{R}}^*)^W$. Complexifying, it follows that the restriction $S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{t}^*)^W$ is injective. To see that it is surjective, note that the space $S\mathfrak{t}^*$ is spanned by monomials $\xi \mapsto \langle \mu, \xi \rangle^m$ with $\mu \in P$ and $m = 0, 1, 2, \ldots$. Hence, the space $(S\mathfrak{t}^*)^W$ of Weyl invariants is spanned by the polynomials
$$a_{\mu,m}(\xi) = \sum_{\nu \in W.\mu} \langle \nu, \xi \rangle^m,$$
with $\mu \in P_+$ and $m = 0, 1, 2, \ldots$. Let $V(\mu)$ be the irreducible $\mathfrak{g}$-representation of highest weight μ, and define polynomials on $\mathfrak{g}$ by
$$b_{\mu,m}(\xi) = \mathrm{tr}_{V(\mu)}(\xi^m), \quad \xi \in \mathfrak{g},$$
where $\xi^m \in U(\mathfrak{g})$ is the m-th power of ξ in the enveloping algebra. For $\xi \in \mathfrak{t}$, the element $\xi^m \in U(\mathfrak{t})$ acts on the weight space $V(\mu)_\nu$ for $\nu \in P$ as the scalar $\langle \nu, \xi \rangle^m$. Hence, for $\xi \in \mathfrak{t}$,
$$\begin{aligned} b_{\mu,m}(\xi) &= \sum_{\nu \in P} \dim(V(\mu)_\nu) \langle \nu, \xi \rangle^m \\ &= \sum_{\nu \in P_+} \dim(V(\mu)_\nu) a_{\nu,m}(\xi). \end{aligned}$$
Since $\dim(V(\mu)_\mu) = 1$, while all other weights appearing in $V(\mu)$ satisfy $\nu \prec \mu$, it follows that the restrictions $b_{\mu,m}|_{\mathfrak{t}}$ with $\mu \in P_+$, $m = 0, 1, 2, \ldots$, also span $S(\mathfrak{t}^*)^W$. This shows that the map $S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{t}^*)^W$ is surjective.
2. The automorphism τ intertwines the shifted W-action with the usual W-action. Hence the result is immediate from (a) and the identity (8.1). □

Remark 8.1 The "standard" proof of the isomorphism between $U(\mathfrak{g})^{\mathfrak{g}}$ and the invariants in $S(\mathfrak{t})$ for the shifted Weyl action on $\mathfrak{t}$ (see e.g., Knapp [80]) uses the theory of Verma modules. Together with Chevalley's Theorem, this then gives an algebra isomorphism $U(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{g})^{\mathfrak{g}}$, called the *Harish-Chandra isomorphism*. By contrast, our approach views the Harish-Chandra isomorphism $U(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{g})^{\mathfrak{g}}$ as a special case of Duflo's isomorphism (Section 7.3), and then uses Chevalley's Theorem to identify the image of the Harish-Chandra projection $U(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{t})$.

Remark 8.2 Another famous result of Chevalley states that the space $S(\mathfrak{g})^{\mathfrak{g}}$ is a symmetric algebra in $l = \mathrm{rank}(\mathfrak{g}) = \dim(\mathfrak{t})$ homogeneous generators. His argument uses the isomorphism with $S(\mathfrak{t})^W$, and actually establishes a more general result for symmetric invariants of finite reflection groups (see e.g., Humphreys [68]). A different argument, due to Koszul [92], uses transgression in the Weil algebra (see [56, Section 6.12]).

8.2.2 Harish-Chandra projection of the quadratic Casimir

For the quadratic Casimir element $\mathrm{Cas}_{\mathfrak{g}} = \sum_a e_a e^a \in U(\mathfrak{g})^{\mathfrak{g}}$, we have (cf. Proposition 7.10)

$$\mathsf{hc}_U(\mathrm{Cas}_{\mathfrak{g}}) = \mathrm{Cas}_{\mathfrak{t}} + 2B^\sharp(\rho).$$

Note that the right-hand side can be written as $\tau(\mathrm{Cas}_{\mathfrak{t}} - B^*(\rho,\rho)) \in \tau(S(\mathfrak{t})^W)$.

Proposition 8.1 *The action of the quadratic Casimir element* $\mathrm{Cas}_{\mathfrak{g}}$ *in the irreducible unitary representation* $\pi:\ \mathfrak{g} \to \mathrm{End}(V)$ *of highest weight* $\mu \in P_+ \subseteq \mathfrak{t}^*$ *is given by the scalar*

$$\pi(\mathrm{Cas}_{\mathfrak{g}}) = B^*(\mu+\rho, \mu+\rho) - B^*(\rho,\rho).$$

Proof Since V is irreducible, $\mathrm{Cas}_{\mathfrak{g}}$ acts as a scalar. To find this scalar, evaluate on the highest weight vector $v \in V$. $\mathrm{Cas}_{\mathfrak{g}}$ decomposes into two summands according to $U(\mathfrak{g})^{\mathfrak{t}} = (\mathfrak{n}_- U(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{t}} \oplus U(\mathfrak{t})$. The first summand acts trivially on v. The second summand is the Harish-Chandra projection $\mathrm{Cas}_{\mathfrak{t}} + 2B^\sharp(\rho)$, and it acts on v as $B^*(\mu,\mu) + 2B^*(\rho,\mu) = B^*(\mu+\rho,\mu+\rho) - B^*(\rho,\rho)$. □

Remark 8.3 Note that if the bilinear form B is *positive definite* on $\mathfrak{g}_{\mathbb{R}}$, then B^* is negative definite on the real subspace space spanned by the weights. In this case the right-hand side can be written $-\|\mu+\rho\|^2 + \|\rho\|^2$.

As a special case, suppose $\mathfrak{g}$ is simple, and take V to be the adjoint representation. The highest weight of this representation is, by definition, the highest root $\alpha_{\max}$ of $\mathfrak{g}$. Let $\alpha_{\max}^\vee$ be the corresponding co-root (cf. Definition B.2 in Appendix B), and

$$h^\vee = 1 + \langle \rho,\ \alpha_{\max}^\vee \rangle$$

the *dual Coxeter number*. The *basic inner product* on a simple Lie algebra $\mathfrak{g}$ is the unique invariant inner product such that $B_{\mathrm{basic}}(\alpha_{\max}^\vee, \alpha_{\max}^\vee) = 2$. Note that B_{basic} is *negative definite* on $\mathfrak{g}_{\mathbb{R}}$.

Proposition 8.2 *If* $\mathfrak{g}$ *is simple, and* $B = B_{\mathrm{basic}}$ *is the basic inner product on* $\mathfrak{g}$, *the adjoint action of the quadratic Casimir is given by the scalar*

$$\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}) = 2h^\vee.$$

Proof By Proposition 8.1, $\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}})$ is the scalar,

$$B^*(\alpha_{\max}, \alpha_{\max}) + 2B^*(\alpha_{\max}, \rho) = B^*(\alpha_{\max}, \alpha_{\max})\ h^\vee.$$

But $B^*(\alpha_{\max}, \alpha_{\max}) = 2$ since B is the basic inner product. □

Proposition 8.3 *The basic inner product B_{basic} is related to the Killing form*

$$B_{\mathrm{Killing}}(\xi,\xi') = \mathrm{tr}(\mathrm{ad}_\xi \mathrm{ad}_{\xi'}), \quad \xi,\xi' \in \mathfrak{g}$$

by twice the dual Coxeter number:

$$B_{\mathrm{Killing}} = 2h^\vee B_{\mathrm{basic}}.$$

Proof Let $\mathrm{Cas}'_{\mathfrak{g}}$ be the Casimir operator relative to $B' = B_{\mathrm{Killing}}$. Since $\mathfrak{g}$ is simple, we have $B_{\mathrm{Killing}} = t B_{\mathrm{basic}}$ for some $t \neq 0$, and hence $\mathrm{Cas}'_{\mathfrak{g}} = \frac{1}{t}\mathrm{Cas}_{\mathfrak{g}}$. By definition of the Killing form, the trace $\mathrm{ad}(\mathrm{Cas}'_{\mathfrak{g}})$ equals $\dim\mathfrak{g}$. This shows that $\mathrm{ad}(\mathrm{Cas}'_{\mathfrak{g}})$ acts as 1 in the adjoint representation. Comparing with Proposition 8.2, it follows that $\frac{1}{t} = 2h^\vee$. □

8.2.3 *Harish-Chandra projection for* $\mathrm{Cl}(\mathfrak{g})$

The Harish-Chandra projection for the Clifford algebras $\mathsf{hc}_{\mathrm{Cl}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{t})$ (Section 7.5.2) has interesting consequences as well. By Proposition 7.12 we have

$$\mathsf{hc}_{Cl}(q(\phi)) = B^\sharp(\rho).$$

Proposition 7.16 specializes to

Proposition 8.4 (Freudenthal–de Vries) *The length squared of the element ρ is given by*

$$B^*(\rho,\rho) = \frac{1}{24}\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}})).$$

Remark 8.4 The above version of the Freudenthal–de Vries formula, valid for arbitrary bilinear forms, was formulated by Kostant [87, Proposition 1.84]. If $\mathfrak{g}$ is semisimple and $B = B_{\mathrm{Killing}}$, so that $\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}) = 1$, one obtains the more standard version of the formula:

$$B^*_{\mathrm{Killing}}(\rho,\rho) = \frac{\dim\mathfrak{g}}{24}.$$

The irreducible representation $\pi : \mathfrak{g} \to \mathrm{End}(V(\rho))$ of highest weight ρ is closely related to Clifford algebras, due to the following fact.

Proposition 8.5 *Let E be a finite-dimensional* $\mathrm{Cl}(\mathfrak{g})$*-module. Let $\mathfrak{g}$ act on E via the map $\gamma : \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$. Then E is a direct sum of ρ-representations.*

Proof By Proposition 7.12,

$$\xi \in \mathfrak{t} \;\Rightarrow\; \gamma(\xi) = \langle \rho, \xi\rangle \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+. \tag{8.2}$$

This shows that ξ acts on highest weight vectors as a scalar $\langle\rho,\xi\rangle$. □

Thus, if E is a finite-dimensional $\mathrm{Cl}(\mathfrak{g})$-module, then any highest weight vector $v \in E^{\mathfrak{n}_+}$ generates an irreducible representation $U(\mathfrak{g}).v$ isomorphic to $V(\rho)$. To obtain a canonical model, take $E = \mathrm{Cl}(\mathfrak{g})$ with the left-regular representation of $\mathrm{Cl}(\mathfrak{g})$ on itself. The line $\det(\mathfrak{n}_+)\det(\mathfrak{n}_-) \subseteq \mathrm{Cl}(\mathfrak{g})^{\mathfrak{t}}$ has a unique generator

$$\mathsf{R} \in \det(\mathfrak{n}_+)\det(\mathfrak{n}_-)$$

with the property $\mathsf{R}^2 = \mathsf{R} = \mathsf{R}^*$. Equivalently, R is normalized by the property $\mathsf{hc}_{\mathrm{Cl}}(\mathsf{R}) = 1$. In terms of (normalized) root vectors,

$$\mathsf{R} = \prod_{\alpha \in \mathfrak{R}_+} \frac{1}{2} e_\alpha e_{-\alpha} \in \mathrm{Cl}(\mathfrak{g}). \tag{8.3}$$

This may also be written as a Clifford exponential,

$$\mathsf{R} = \exp\Big(-\frac{1}{2} \sum_{\alpha \in \mathfrak{R}_+} e_{-\alpha} e_\alpha\Big).$$

Letting $\gamma: U(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ be the extension of the Lie algebra morphism $\gamma: \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$, we have:

Proposition 8.6 (Model of the ρ-representation) *There is an isomorphism of $\mathfrak{g}$-representations,*

$$V(\rho) \cong \gamma(U(\mathfrak{g}))\mathsf{R}. \tag{8.4}$$

Proof The properties (8.2) and $\gamma(\mathfrak{n}_+) \subseteq \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$ show that R is a highest weight vector. □

Remark 8.5 Clearly, any non-zero $v \in \det(\mathfrak{n}_+)\mathrm{Cl}(\mathfrak{g})$ generates a ρ-representation $\gamma(U(\mathfrak{g}))v$ for the same reason. The special choice made here is particularly convenient for our purposes later on (cf. Chapter 11).

The projection $\mathsf{hc}_{\mathrm{Cl}}$ does not respect products. However, we have:

Proposition 8.7 *The Harish-Chandra projection* $\mathsf{hc}_{\mathrm{Cl}}$ *restricts to an algebra morphism on the subalgebra* $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{t}}$ *of* $\mathfrak{t}$*-invariants. In fact,*

$$\mathsf{hc}_{\mathrm{Cl}}(xy) = \mathsf{hc}_{\mathrm{Cl}}(x)\mathsf{hc}_{\mathrm{Cl}}(y)$$

if one of x, y lies in $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{t}}$.

Proof Note that

$$(\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g}) + \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{t}} = (\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{t}}. \tag{8.5}$$

Let $x, y \in \mathrm{Cl}(\mathfrak{g})$, and put $x' = x - \mathsf{hc}_{\mathrm{Cl}}(x)$, and similarly for y. Then

$$xy = \mathsf{hc}_{\mathrm{Cl}}(x)\mathsf{hc}_{\mathrm{Cl}}(y) + \big(x'\mathsf{hc}_{\mathrm{Cl}}(y) + \mathsf{hc}_{\mathrm{Cl}}(x)y' + x'y'\big),$$

hence

$$\mathsf{hc}_{\mathrm{Cl}}(xy) = \mathsf{hc}_{\mathrm{Cl}}(x)\mathsf{hc}_{\mathrm{Cl}}(y) + \mathsf{hc}_{\mathrm{Cl}}(x'y').$$

If x' is $\mathfrak{t}$-invariant, then $x'y' \in \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})$ by (8.5), and hence $\mathsf{hc}_{\mathrm{Cl}}(x'y') = 0$. Similarly, the projection is zero if y' is invariant. $\square$

Remark 8.6 In the statement of the proposition, one can replace $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{t}}$ with the larger subalgebra $\mathrm{Cl}(\mathfrak{t}) + \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+ \subseteq \mathrm{Cl}(\mathfrak{g})$.

We conclude with the following result, relating the Harish-Chandra projections for the Clifford and enveloping algebras.

Proposition 8.8 [22, Lemma 4.4] *The map* $\mathsf{hc}_{\mathrm{Cl}}$ *takes the subalgebra* $\gamma(U\mathfrak{g}) \subseteq \mathrm{Cl}(\mathfrak{g})$ *to scalars. In fact,*

$$\mathsf{hc}_{\mathrm{Cl}}(\gamma(x)) = \mathrm{ev}_\rho(\mathsf{hc}_U(x)), \quad x \in U(\mathfrak{g}),$$

where $\mathrm{ev}_\rho : U\mathfrak{t} = S\mathfrak{t} \to \mathbb{C}$ *is the algebra morphism extending* $\mathfrak{t} \to \mathbb{C}$, $\xi \mapsto \rho(\xi)$.

Proof This is a special case of Proposition 7.14, since the map $\gamma_{\mathfrak{t}} : U(\mathfrak{t}) \to \mathrm{Cl}(\mathfrak{t})$ appearing there is just $\mathrm{ev}_0 : U(\mathfrak{t}) = S(\mathfrak{t}) \to \mathbb{C}$, and τ is a ρ-shift. Thus $\gamma_{\mathfrak{t}} \circ \tau = \mathrm{ev}_\rho$. $\square$

In Section 11.1 we will investigate the relationship of $\mathrm{Cl}(\mathfrak{g})$ with the ρ-representation in greater depth.

8.3 Equal rank subalgebras

We will consider a reductive Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ of equal rank. With no loss of generality, we may assume $\mathfrak{k}_{\mathbb{R}} \subseteq \mathfrak{g}_{\mathbb{R}}$. Let $\mathfrak{t} \subseteq \mathfrak{k} \subseteq \mathfrak{g}$ be a Cartan subalgebra, given as the complexification of a maximal Abelian subalgebra $\mathfrak{t}_{\mathbb{R}} \subseteq \mathfrak{k}_{\mathbb{R}}$. The orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ will be denoted by $\mathfrak{p}$.

Examples 8.1 For any $\xi \in \mathfrak{g}_{\mathbb{R}}$, the centralizer $\mathfrak{k} = \ker(\mathrm{ad}(\xi)) \subseteq \mathfrak{g}$ is an equal rank subalgebra. As extreme cases, one has $\mathfrak{k} = \mathfrak{t}$ and $\mathfrak{k} = \mathfrak{g}$. Other examples of equal rank subalgebras $\mathfrak{k} \subseteq \mathfrak{g}$ include:

1. $\mathfrak{g}$ of type C_2 (i.e., $\mathfrak{sp}(4)$), $\mathfrak{k}$ of type $A_1 \times A_1$ (i.e., $\mathfrak{su}(2) \times \mathfrak{su}(2)$),
2. $\mathfrak{g}$ of type G_2, $\mathfrak{k}$ of type A_2 (i.e., $\mathfrak{su}(3)$),
3. $\mathfrak{g}$ of type F_4, $\mathfrak{k}$ of type B_4 (i.e., $\mathfrak{spin}(9)$),
4. $\mathfrak{g}$ of type E_8, $\mathfrak{k}$ of type D_8 (i.e., $\mathfrak{spin}(16)$).

Remark 8.7 A classification of semisimple Lie subalgebras of a semisimple Lie algebra was obtained by Dynkin [49], following earlier work of A. Borel and J. de Siebenthal [25] who classified equal rank subgroups of compact Lie groups. Dynkin's result may be summarized as follows. Let $\alpha_1, \ldots, \alpha_l$ be a set of simple roots for $\mathfrak{g}$, identified with the vertices of the Dynkin diagram. For each simple root α_i, there is an equal rank Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ having as its set of simple roots the α_j, $j \neq i$, together with the lowest root $\alpha_0 = -\alpha_{\max}$ of the simple summand $\mathfrak{g}'$ of $\mathfrak{g}$ containing the root space $\mathfrak{g}_{\alpha_i}$. That is, the Dynkin diagram of $\mathfrak{k}$ is obtained from that of $\mathfrak{g}$ by first replacing the component containing α_i (i.e., the Dynkin diagram of $\mathfrak{g}'$) with the *extended Dynkin diagram*, and then removing the vertex α_i to obtain an ordinary Dynkin diagram.[1] The resulting semisimple equal rank subalgebra $\mathfrak{k}$ is a *maximal* Lie subalgebra of $\mathfrak{g}$ if and only if the coefficient of α_i in the expression $\alpha_{\max} = \sum_j k_j \alpha_j$ is a prime number. Dynkin proved that any semisimple equal rank $\mathfrak{k}$ is obtained by repeating this procedure a finite number of times. For details, see [117] or [105].

We denote by $\mathfrak{R}_\mathfrak{k} \subseteq \mathfrak{R}_\mathfrak{g} \subseteq \mathfrak{t}^*$ the set of roots of $\mathfrak{k} \subseteq \mathfrak{g}$, and let $\mathfrak{R}_\mathfrak{p}$ be its complement. The choice of a decomposition $\mathfrak{R}_\mathfrak{g} = \mathfrak{R}_{\mathfrak{g},+} \cup \mathfrak{R}_{\mathfrak{g},-}$ into positive and negative roots induces similar decompositions for $\mathfrak{R}_\mathfrak{k}$, $\mathfrak{R}_\mathfrak{p}$. We denote by

$$\rho_\mathfrak{g} = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_{\mathfrak{g},+}} \alpha$$

the half-sum of positive roots of $\mathfrak{g}$, and similarly define $\rho_\mathfrak{k}$ and $\rho_\mathfrak{p}$. Then

$$\rho_\mathfrak{g} = \rho_\mathfrak{k} + \rho_\mathfrak{p}.$$

Let $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ be the Lagrangian splitting of $\mathfrak{p}$ defined by the decomposition into positive and negative roots:

$$\mathfrak{p}_+ = \bigoplus_{\alpha \in \mathfrak{R}_{\mathfrak{p},+}} \mathfrak{g}_\alpha, \quad \mathfrak{p}_- = \bigoplus_{\alpha \in \mathfrak{R}_{\mathfrak{p},-}} \mathfrak{g}_\alpha.$$

We will use it to define a spinor module

$$\mathsf{S}_\mathfrak{p} = \mathrm{Cl}(\mathfrak{p})/\mathrm{Cl}(\mathfrak{p})\mathfrak{p}_+ \cong \wedge \mathfrak{p}_-$$

over $\mathrm{Cl}(\mathfrak{p})$. Let $\rho : \mathrm{Cl}(\mathfrak{p}) \to \mathrm{End}(\mathsf{S}_\mathfrak{p})$ be the Clifford action on the spinor module.

Remark 8.8 The decomposition $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ (or equivalently the corresponding complex structure on $\mathfrak{p}_\mathbb{R}$) is not $\mathfrak{k}$-invariant, in general. Hence $\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$ does not in general define a triangular decomposition in the sense of Section 7.5.1. It does define a triangular decomposition in case $\mathfrak{k}$ is the centralizer in $\mathfrak{g}$ of some element $\xi \in \mathfrak{t}_\mathbb{R}$.

There is a representation of the Lie algebra $\mathfrak{k}$ on the spinor module $\mathsf{S}_\mathfrak{p}$, defined by the Lie algebra homomorphism

$$\gamma_\mathfrak{p} : \mathfrak{k} \to \mathrm{Cl}(\mathfrak{p})$$

[1] Suppose $\mathfrak{g}$ is simple, and let $G_\mathbb{R}$ be the compact, simply connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{R}$. Then the Lie algebras $\mathfrak{k}_\mathbb{R}$ obtained by this procedure are the Lie algebras of centralizers $K_\mathbb{R}$ of elements $g = \exp(\xi)$, where ξ is a vertex of the Weyl alcove of G. Up to conjugacy, these are precisely the centralizers that are semisimple.

followed by the Clifford action ρ. We will refer to this action as the *spinor representation of* $\mathfrak{k}$ *on* $\mathsf{S}_{\mathfrak{p}}$.

On the other hand, the adjoint action of the Cartan subalgebra $\mathfrak{t}$ on $\mathfrak{p}_-$ extends to a representation by derivations of the super algebra $\wedge\mathfrak{p}_-$. We will denote this action by $\xi \mapsto \mathrm{ad}(\xi)$. The two representations are related as follows.

Proposition 8.9 *Under the identification* $\mathsf{S}_{\mathfrak{p}} \cong \wedge\mathfrak{p}_-$, *the adjoint action of* $\mathfrak{t}$ *on* $\wedge\mathfrak{p}_-$ *and the spinor representation of* $\mathfrak{t} \subseteq \mathfrak{k}$ *on* $\mathsf{S}_{\mathfrak{p}}$ *are related by a* $\rho_{\mathfrak{p}}$*-shift*:

$$\rho(\gamma_{\mathfrak{p}}(\xi)) = \mathrm{ad}(\xi) + \langle \rho_{\mathfrak{p}}, \xi \rangle, \quad \xi \in \mathfrak{t}.$$

Proof This is a special case of item 3 in Section 4.2.1, applied to the vector space $V = \mathfrak{p}_+$, with $A = \mathrm{ad}_\xi|_{\mathfrak{p}_+} \in \mathfrak{gl}(\mathfrak{p}_+) \subseteq \mathfrak{o}(\mathfrak{p})$. Here $\frac{1}{2}\mathrm{tr}(A) = \langle \rho_{\mathfrak{p}}, \xi \rangle$. $\square$

Fix an ordering of the set $\mathfrak{R}_{\mathfrak{p},+}$. Then $\wedge\mathfrak{p}_-$ has a basis consisting of elements

$$\wedge_{\alpha \in X} e_{-\alpha}, \tag{8.6}$$

where X ranges over subsets of $\mathfrak{R}_{\mathfrak{p},+}$, and the wedge product uses the induced ordering of X.

Proposition 8.10 *The weights for the* $\mathfrak{k}$*-action on* $\mathsf{S}_{\mathfrak{p}}$ *are the elements of the form*

$$\nu_X = \rho_{\mathfrak{p}} - \sum_{\alpha \in X} \alpha,$$

where X *ranges over subsets of* $\mathfrak{R}_{\mathfrak{p},+}$. *The multiplicity of a weight* ν *is equal to the number of subsets* X *such that* $\nu = \nu_X$.

Proof The basis vector (8.6) is a weight vector for the adjoint action of $\mathfrak{t}$, with corresponding weight $-\sum_{\alpha \in X} \alpha$. It is therefore also a weight vector for the Clifford action of $\mathfrak{t}$, with the weight shifted by $\rho_{\mathfrak{p}}$. $\square$

For any completely reducible $\mathfrak{t}$-representation on a finite-dimensional super space W, with weight spaces W_ν, define the formal character

$$\mathsf{ch}(W) = \mathsf{ch}(W^{\bar{0}}) - \mathsf{ch}(W^{\bar{1}}) = \sum_\nu (\dim(W_\nu^{\bar{0}}) - \dim(W_\nu^{\bar{1}}))\mathrm{e}^\nu.$$

The formal character has the properties

$$\mathsf{ch}(W \oplus W') = \mathsf{ch}(W) + \mathsf{ch}(W'),$$
$$\mathsf{ch}(W \otimes W') = \mathsf{ch}(W)\mathsf{ch}(W'),$$
$$\mathsf{ch}(W^*) = \mathsf{ch}(W)^*.$$

The character of the 1-dimensional $\mathfrak{t}$-representation $\mathbb{C}_\mu$ for $\mu \in \mathfrak{t}^*$ is e^μ, hence $\mathsf{ch}(\wedge\mathbb{C}_\mu) = 1 - \mathrm{e}^\mu$. Decomposing $\mathfrak{p}_-$ into root spaces, it follows that the character for the adjoint representation on the super space $\wedge\mathfrak{p}_- = \bigotimes_{\alpha \in \mathfrak{R}_{\mathfrak{p},+}} \wedge(\mathbb{C}e_{-\alpha})$ is

$$\mathsf{ch}(\wedge\mathfrak{p}_-) = \prod_{\alpha \in \mathfrak{R}_{\mathfrak{p},+}} (1 - \mathrm{e}^{-\alpha}).$$

The character for the spinor representation on $\mathsf{S}_\mathfrak{p}$ is obtained from this by a $\rho_\mathfrak{p}$-shift. Thus

$$\begin{aligned}\mathsf{ch}(\mathsf{S}_\mathfrak{p}) &= e^{\rho_\mathfrak{p}} \prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (1-e^{-\alpha}) \\ &= \prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (e^{\alpha/2}-e^{-\alpha/2}).\end{aligned}$$

Remark 8.9 We can also view $\mathsf{S}_\mathfrak{p}$ as an *ungraded* $\mathfrak{g}$-representation. Repeating the calculation above without the signs, we find that its character is

$$\prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (e^{\alpha/2}+e^{-\alpha/2}).$$

We claim that for $\mathfrak{k}=\mathfrak{t}$, this is in fact the character of the ρ-representation:

$$\mathsf{ch}(V(\rho)) = \prod_{\alpha\in\mathfrak{R}_+} (e^{\alpha/2}+e^{-\alpha/2}). \tag{8.7}$$

To see this, consider $\mathrm{Cl}(\mathfrak{g})\mathsf{R}$ as an *ungraded* $\mathfrak{g}$-representation, where $\mathfrak{g}$ acts by $\gamma:\ \mathfrak{g}\to\mathrm{Cl}(\mathfrak{g})$ followed by the left-regular representation of $\mathfrak{g}$ on itself. As a representation of $\mathfrak{t}\subseteq\mathfrak{g}$, it breaks up into 2^l copies of the spinor representation of $\mathfrak{t}$ on $\mathsf{S}_{\mathfrak{t}^\perp}$:

$$\mathrm{Cl}(\mathfrak{g})\mathsf{R} = \mathrm{Cl}(\mathfrak{n}_-)\mathsf{R}\otimes\mathrm{Cl}(\mathfrak{t}) \cong \mathsf{S}_{\mathfrak{t}^\perp}\otimes\mathrm{Cl}(\mathfrak{t}),$$

hence the character is $2^l\prod_{\alpha\in\mathfrak{R}_+}(e^{\alpha/2}+e^{-\alpha/2})$. On the other hand, we saw that the submodule $\mathrm{Cl}(\mathfrak{g})\mathsf{R}$ consists of 2^l copies of the ρ-representation, hence the character is $2^l\mathsf{ch}(V(\rho))$. Comparing, we obtain Eq. (8.7). We see in particular that $\mathsf{S}_{\mathfrak{t}^\perp}$ is isomorphic to $V(\rho)$ as a $\mathfrak{t}$-representation.

Let $W_\mathfrak{k}\subseteq W_\mathfrak{g}=W$ be the Weyl groups of $\mathfrak{k}\subseteq\mathfrak{g}$. Define a set

$$W_\mathfrak{p} = \{w\in W|\ \mathfrak{R}_{\mathfrak{k},+}\subseteq w\mathfrak{R}_+\}.$$

In terms of the Weyl chambers $\mathfrak{t}_+\subseteq\mathfrak{t}_{\mathfrak{k},+}$ determined by the positive roots, the definition reads

$$W_\mathfrak{p} = \{w\in W|\ w\mathfrak{t}_+\supseteq\mathfrak{t}_{\mathfrak{k},+}\}.$$

For any $w\in W$ denote by

$$\mathfrak{R}_{+,w} = \mathfrak{R}_+\cap w\mathfrak{R}_-$$

the set of positive roots that become negative under w^{-1}. See Appendix B.6 for some basic properties of the set $\mathfrak{R}_{+,w}$.

Lemma 8.1

1. *For all* $w \in W$,
$$\rho - w\rho = \sum_{\alpha \in \mathfrak{R}_{+,w}} \alpha.$$
If $X \subseteq \mathfrak{R}_+$ *is a subset with* $\rho - w\rho = \sum_{\alpha \in X} \alpha$, *then* $X = \mathfrak{R}_{+,w}$.
2. $w \in W_\mathfrak{p}$ *if and only if* $\mathfrak{R}_{+,w} \subseteq \mathfrak{R}_{\mathfrak{p},+}$.
3. *The map*
$$W_\mathfrak{k} \times W_\mathfrak{p} \to W, \quad (w_1, w_2) \to w_1 w_2$$
is a bijection; *thus* $W_\mathfrak{p}$ *labels the left cosets of* $W_\mathfrak{k}$ *in* W.
4. *If* $w \in W_\mathfrak{p}$, *the element* $w\rho - \rho_\mathfrak{k}$ *is weight of* $\mathsf{S}_\mathfrak{p}$ *of multiplicity* 1.

Proof

1. The first part is Lemma B.1 in Appendix B; the second part follows from Proposition 8.10, since $w\rho$ has multiplicity 1. (See also Remark 8.9.)
2. We have $\mathfrak{R}_{+,w} \subseteq \mathfrak{R}_{\mathfrak{p},+}$ if and only if the intersection
$$\mathfrak{R}_{+,w} \cap \mathfrak{R}_{\mathfrak{k},+} = w\mathfrak{R}_- \cap \mathfrak{R}_{\mathfrak{k},+}$$
is empty. But this means precisely that $\mathfrak{R}_{\mathfrak{k},+} \subseteq w\mathfrak{R}_+$, i.e., $w \in W_\mathfrak{p}$.
3. Let $w \in W$ be given. Since $\mathfrak{R}_\mathfrak{k} \cap w\mathfrak{R}_+$ is a system of positive roots for $\mathfrak{k}$, there is a unique $w_1 \in W_\mathfrak{k}$ with
$$w_1 \mathfrak{R}_{\mathfrak{k},+} = \mathfrak{R}_\mathfrak{k} \cap w\mathfrak{R}_+ \subseteq w\mathfrak{R}_+.$$
Thus $w_2 = w_1^{-1} w$ satisfies $\mathfrak{R}_{\mathfrak{k},+} \subseteq w_2 \mathfrak{R}_+$, i.e., $w_2 \in W_\mathfrak{p}$.
4. Suppose $w \in W_\mathfrak{p}$. By (1) and since $\rho = \rho_\mathfrak{k} + \rho_\mathfrak{p}$, we have
$$w\rho - \rho_\mathfrak{k} = \rho_\mathfrak{p} - \sum_{\alpha \in \mathfrak{R}_{+,w}} \alpha;$$
moreover $X = \mathfrak{R}_{+,w}$ is the unique subset for which this equation holds. Hence, by Proposition 8.10, $w\rho - \rho_\mathfrak{k}$ is a weight of multiplicity 1. □

We will now assume that B is positive definite on $\mathfrak{g}_\mathbb{R}$, defining a Hermitian metric on $\mathfrak{g}$ with norm $\|\cdot\|$. Since $P \subseteq \sqrt{-1}\mathfrak{g}_\mathbb{R}$, we have $\|\mu\|^2 = -B^*(\mu, \mu)$ for $\mu \in P \otimes_\mathbb{Z} \mathbb{R}$. We will find it convenient to introduce the notation $(\mu|\nu) = -B^*(\mu, \nu)$. A weight μ for a finite-dimensional completely reducible $\mathfrak{g}$-representation V will be called a $\mathfrak{g}$-*extremal weight* if $\|\mu + \rho\|$ is maximal among all weights of V. Note that this notion depends on the choice of positive Weyl chamber (or equivalently, of $\mathfrak{R}_+$). The following fact is proved in Appendix B, see Section B.10.

Lemma 8.2 *If* $\mu \in P(V)$ *is a* $\mathfrak{g}$-*extremal weight*, *then the corresponding weight space is contained in the space of highest weight vectors*: $V_\mu \subseteq V^{\mathfrak{n}_+}$. *In particular*, $V(\mu)$ *appears in* V *with multiplicity equal to* $\dim V_\mu$.

In particular, for an irreducible $\mathfrak{g}$-representation the highest weight is the unique $\mathfrak{g}$-extremal weight. As another example, the weights $w\rho - \rho_{\mathfrak{k}}$, $w \in W_{\mathfrak{p}}$ for the $\mathfrak{k}$-representation on $S_{\mathfrak{p}}$ are exactly the $\mathfrak{k}$-extremal weights of $S_{\mathfrak{p}}$. Indeed, if ν is any weight of $S_{\mathfrak{p}}$, then $\nu + \rho_{\mathfrak{k}}$ is a weight of $V(\rho)$. Hence $\|\nu + \rho_{\mathfrak{k}}\| \le \|\rho\|$. Equality holds if $\nu + \rho_{\mathfrak{k}} = w\rho$ for some $w \in W$, but as we saw $w\rho - \rho_{\mathfrak{k}}$ is a weight of $S_{\mathfrak{p}}$ if and only if $w \in W_{\mathfrak{p}}$. More generally we have:

Proposition 8.11 *Let $\mathfrak{k} \subseteq \mathfrak{g}$ as above. For any $\mathfrak{g}$-dominant weight $\mu \in P_+$, the elements*

$$w(\mu + \rho) - \rho_{\mathfrak{k}}, \quad w \in W_{\mathfrak{p}}$$

are $\mathfrak{k}$-extremal weights for the $\mathfrak{k}$-representation $V(\mu) \otimes \mathsf{S}_{\mathfrak{p}}$, each appearing with multiplicity 1. *The irreducible $\mathfrak{k}$-representation with highest weight $w(\mu + \rho) - \rho_{\mathfrak{k}}$ appears in the even* (*resp. odd*) *component if $l(w)$ is even* (*resp. odd*).

Proof Write $V = V(\mu)$. The weights of $V \otimes \mathsf{S}_{\mathfrak{p}}$ are sums $\nu = \nu_1 + \nu_2$, where ν_1 is a weight of V and ν_2 is a weight of $\mathsf{S}_{\mathfrak{p}}$. Given such a weight, choose $w \in W$ such that $w^{-1}(\nu + \rho_{\mathfrak{k}})$ lies in the positive chamber for $\mathfrak{g}$. That is,

$$(w^{-1}(\nu + \rho_{\mathfrak{k}})|\alpha) \ge 0 \tag{8.8}$$

for all $\alpha \in \mathfrak{R}_+$. Since $\rho_{\mathfrak{k}}$ is a weight of $\mathsf{S}_{\mathfrak{k}\cap\mathfrak{t}^\perp}$, the sum $\nu_2 + \rho_{\mathfrak{k}}$ is among the $\mathfrak{t}$-weights of $\mathsf{S}_{\mathfrak{t}^\perp} = \mathsf{S}_{\mathfrak{p}} \otimes \mathsf{S}_{\mathfrak{k}\cap\mathfrak{t}^\perp}$, i.e., it lies in $P(V(\rho))$. Hence also $w^{-1}(\nu_2 + \rho_{\mathfrak{k}}) \in P(V(\rho))$. It follows that (cf. Proposition B.14)

$$\rho = w^{-1}(\nu_2 + \rho_{\mathfrak{k}}) + \sum_{\alpha \in \mathfrak{R}_+} k_\alpha \alpha$$

with $k_\alpha \ge 0$. Similarly, since $w^{-1}\nu_1$ is a weight of V,

$$\mu = w^{-1}\nu_1 + \sum_{\alpha \in \mathfrak{R}_+} l_\alpha \alpha,$$

where $l_\alpha \ge 0$. Adding, we obtain

$$\mu + \rho = w^{-1}(\nu + \rho_{\mathfrak{k}}) + \sum_{\alpha \in \mathfrak{R}_+} (k_\alpha + l_\alpha)\alpha.$$

Consider $\|\mu + \rho\| = (\mu + \rho|\mu + \rho)^{1/2}$. The inequality (8.8) gives

$$\|\mu + \rho\| \ge \|w^{-1}(\nu + \rho_{\mathfrak{k}})\| = \|\nu + \rho_{\mathfrak{k}}\|,$$

with equality if and only if all l_α, k_α are zero. The latter case is equivalent to $\nu_1 = w\mu$ and $\nu_2 = w\rho - \rho_{\mathfrak{k}}$. Since these are indeed weights for V resp. $\mathsf{S}_{\mathfrak{p}}$, it follows that their sum $\nu := w(\mu + \rho) - \rho_{\mathfrak{k}}$ is a $\mathfrak{k}$-extremal weight for $V \otimes \mathsf{S}_{\mathfrak{p}}$. Since $\nu + \rho_{\mathfrak{k}}$ lies in the interior of the positive chamber for $\mathfrak{k}$, while $\mu + \rho$ lies in the interior of the positive chamber for $\mathfrak{g}$, the equality $w(\mu + \rho) = \nu + \rho_{\mathfrak{k}}$ shows $w \in W_{\mathfrak{p}}$. Here w is uniquely determined by the equation $w(\mu + \rho) = \nu + \rho_{\mathfrak{k}}$. Suppose conversely that $w \in W_{\mathfrak{p}}$ is given, and that ν_1, ν_2 are weights for $V, \mathsf{S}_{\mathfrak{p}}$ with $\nu_1 + \nu_2 = w(\mu + \rho) - \rho_{\mathfrak{k}}$. The argument above shows that $\nu_1 = w'\mu$, $\nu_2 = w'\rho - \rho$ for some $w' \in W$. But then $w(\mu+\rho) = w'(\mu+\rho)$, thus $w' = w$. Hence ν_1, ν_2 are uniquely determined. It follows that the weight $w(\mu+\rho) - \rho_{\mathfrak{k}}$ of $V \otimes \mathsf{S}_{\mathfrak{p}}$ has multiplicity 1, and the corresponding weight space is just the tensor product $V_{w\mu} \otimes (\mathsf{S}_{\mathfrak{p}})_{w\rho-\rho_{\mathfrak{k}}}$. The weight space has even (resp. odd) parity if and only if $(\mathsf{S}_{\mathfrak{p}})_{w\rho-\rho_{\mathfrak{k}}}$ has even (resp. odd) parity, if and only if $l(w)$ is even (resp. odd). $\square$

8.4 The kernel of $\mathscr{D}_V$

We keep our assumption that B is the complexification of a *positive definite* invariant symmetric bilinear form on $\mathfrak{g}_{\mathbb{R}}$. Then $\mathsf{S}_{\mathfrak{p}}$ acquires a Hermitian structure, and $\mathrm{Cl}(\mathfrak{p})$ acts unitarily. Fix an irreducible unitary $\mathfrak{g}$-representation $V = V(\mu)$ of highest weight μ. Then $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ acts on $V \otimes \mathsf{S}_{\mathfrak{p}}$, and hence the relative Dirac operator $\mathscr{D}(\mathfrak{g}, \mathfrak{k}) \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{k}}$ is represented as a $\mathfrak{k}$-equivariant, skew-adjoint odd operator

$$\mathscr{D}_V \in \mathrm{End}(V \otimes \mathsf{S}_{\mathfrak{p}}).$$

We are interested in the kernel of $\mathscr{D}_V$. Denote by $M(\nu)$ the irreducible $\mathfrak{k}$-representation labeled by a dominant $\mathfrak{k}$-weight ν.

Theorem 8.2 (Kostant [87]) *As a $\mathfrak{k}$-representation, the kernel of the cubic Dirac operator on $V \otimes \mathsf{S}_{\mathfrak{p}}$ is a direct sum*

$$\ker(\mathscr{D}_V) \cong \bigoplus_{w \in W_{\mathfrak{p}}} M(w(\mu+\rho) - \rho_{\mathfrak{k}}).$$

The even part $\ker(\mathscr{D}_V)^{\bar{0}}$ *is the sum over $w \in W_{\mathfrak{p}}$ with $l(w)$ even, and the odd part* $\ker(\mathscr{D}_V)^{\bar{1}}$ *is the sum over $w \in W_{\mathfrak{p}}$ with $l(w)$ odd.*

Proof Since $\mathscr{D}_V$ is skew-adjoint, its kernel coincides with the kernel of its square. We will hence determine $\ker(\mathscr{D}_V^2)$. Consider the decomposition

$$V \otimes \mathsf{S}_{\mathfrak{p}} = \bigoplus_{\nu} (V \otimes \mathsf{S}_{\mathfrak{p}})_{[\nu]}$$

into the $\mathfrak{k}$-isotypical components, labeled by $\mathfrak{k}$-dominant weights. (We use a subscript $[\nu]$ to avoid confusion with the weight space.) Since $\mathscr{D}_V$ is $\mathfrak{k}$-invariant, it preserves each of these components. Using the formula

$$\mathscr{D}(\mathfrak{g}, \mathfrak{k})^2 = \mathrm{Cas}_{\mathfrak{g}} - j(\mathrm{Cas}_{\mathfrak{k}}) - \|\rho\|^2 + \|\rho_{\mathfrak{k}}\|^2$$

(cf. Eq. (7.12) and Proposition 8.4), together with the formula for the action of the Casimir element in an irreducible representation (cf. Proposition 8.1), it follows that the operator $\mathscr{D}_V^2$ acts on each $\mathfrak{k}$-isotypical subspace $(V \otimes \mathsf{S}_{\mathfrak{p}})_{[\nu]}$ as a scalar,

$$-\|\mu+\rho\|^2 + \|\nu+\rho_{\mathfrak{k}}\|^2.$$

In particular, $\ker(\mathscr{D}_V^2)$ is the sum over all $\mathfrak{k}$-isotypical components for which one has the equality $\|\mu+\rho\| = \|\nu+\rho_{\mathfrak{k}}\|$. Proposition 8.11 shows that the corresponding weights ν appear with multiplicity 1, and are exactly the weights $\nu = w(\mu+\rho)-\rho_{\mathfrak{k}}$ with $w \in W_{\mathfrak{p}}$. (We see once again that these weights are $\mathfrak{k}$-extremal, since $\mathscr{D}_V$ is skew-adjoint and hence $\mathscr{D}_V^2$ is non-positive.) $\square$

Following Gross–Kostant–Ramond–Sternberg [57] and Kostant [87], we will refer to the collection of irreducible representations

$$M(w(\mu+\rho)-\rho_{\mathfrak{k}}), \quad w \in W_{\mathfrak{p}}$$

as the *multiplet* indexed by μ. Note that an irreducible $\mathfrak{k}$-representation of highest weight ν belongs to some multiplet if and only if $\nu+\rho_{\mathfrak{k}}$ is a $\mathfrak{g}$-weight which furthermore is regular for the W-action. Theorem 8.2 has the following consequence,

Corollary 8.1 (Kostant [87]) *The dimensions of the irreducible representations in each multiplet satisfy*

$$\sum_{w\in W_{\mathfrak{p}}} (-1)^{l(w)} \dim M(w(\mu+\rho)-\rho_{\mathfrak{k}}) = 0.$$

Proof Since $\dim(V\otimes \mathsf{S}_{\mathfrak{p}})^{\bar{0}} = \dim(V\otimes \mathsf{S}_{\mathfrak{p}})^{\bar{1}}$, the exact sequence

$$0 \to \ker(\mathscr{D}_V)^{\bar{0}} \to (V\otimes \mathsf{S}_{\mathfrak{p}})^{\bar{0}} \xrightarrow{\mathscr{D}_V} (V\otimes \mathsf{S}_{\mathfrak{p}})^{\bar{1}} \to \operatorname{coker}(\mathscr{D}_V)^{\bar{1}} \to 0 \tag{8.9}$$

shows $\dim\ker(\mathscr{D}_V)^{\bar{0}} = \dim\operatorname{coker}(\mathscr{D}_V)^{\bar{1}}$. But

$$\operatorname{coker}(\mathscr{D}_V)^{\bar{1}} \cong V^{\bar{1}}/\mathscr{D}_V(V^{\bar{0}}) \cong \ker(\mathscr{D}_V)^{\bar{1}}.$$

Hence we obtain $\dim\ker(\mathscr{D}_V)^{\bar{0}} - \dim\ker(\mathscr{D}_V)^{\bar{1}} = 0$. Now use Theorem 8.2. $\square$

Using Proposition 8.1, it is immediate that $\mathrm{Cas}_{\mathfrak{k}}$ acts on the multiplet indexed by μ as a constant $-\|\mu+\rho\|^2 + \|\rho_{\mathfrak{k}}\|^2$. In fact, much more is true. Let $U(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{k})^{\mathfrak{k}}$ correspond to the inclusion $(S\mathfrak{g})^{\mathfrak{g}} \to (S\mathfrak{k})^{\mathfrak{k}}$ under the Duflo isomorphism. (Cf. Theorem 7.5.)

Theorem 8.3 [57, 87] *Let $y \in (U\mathfrak{k})^{\mathfrak{k}}$ be an element in the image of $U(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{k})^{\mathfrak{k}}$. Then the action of y on the members of any multiplet*

$$M(w(\mu+\rho)-\rho_{\mathfrak{k}}), \quad w \in W_{\mathfrak{p}}$$

is independent of $w \in W_{\mathfrak{p}}$.

Proof Let $x \in U(\mathfrak{g})^{\mathfrak{g}}$ be given, and $y \in U(\mathfrak{k})^{\mathfrak{k}}$ its image. By Theorem 7.5, there exists $z \in \mathscr{W}(\mathfrak{g},\mathfrak{k}) = (U(\mathfrak{g})\otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{k}}$ with

$$x - j(y) = [\mathscr{D}(\mathfrak{g},\mathfrak{k}), z],$$

where $j: \mathscr{W}\mathfrak{k} \to \mathscr{W}\mathfrak{g}$ is the morphism of $\mathfrak{k}$-differential algebras defined in that section. Under the action of $\mathscr{W}(\mathfrak{g},\mathfrak{k})$ on $\phi \in \ker(\mathscr{D}_V) \subseteq V \otimes \mathsf{S}_\mathfrak{p}$, this identity gives

$$x.\phi - j(y).\phi = \mathscr{D}_V(z.\phi).$$

Since $\mathscr{D}(\mathfrak{g},\mathfrak{k}) \in \mathscr{W}(\mathfrak{g},\mathfrak{k})$ commutes with elements of $j(\mathscr{W}\mathfrak{k})$, the action of $j(y)$ preserves $\ker(\mathscr{D}_V)$. On the other hand, $\mathscr{D}(\mathfrak{g},\mathfrak{k})$ also commutes with elements of $(U\mathfrak{g})^\mathfrak{g} \subseteq \mathscr{W}\mathfrak{g}$, hence also x preserves $\ker(\mathscr{D}_V)$.

We conclude that $x.\phi - j(y).\phi \in \ker(\mathscr{D}_V)$, hence $z.\phi \in \ker(\mathscr{D}_V^2) = \ker(\mathscr{D}_V)$. This proves $j(y).\phi = x.\phi$. But the action of $x \in (U\mathfrak{g})^\mathfrak{g}$ on $\ker(\mathscr{D}_V)$ is the scalar by which x acts on $V(\mu)$. □

Remark 8.10 As announced by Gross–Kostant–Ramond–Sternberg [57] and proved in Kostant's article [87], this property characterizes the triplets: Among the $\mathfrak{k}$-dominant weights ν such that $\nu + \rho_\mathfrak{k}$ is a regular weight for $\mathfrak{g}$, any triplet is determined by the values of elements in the image of $U(\mathfrak{g})^\mathfrak{g} \to U(\mathfrak{k})^\mathfrak{k}$.

Another application is the following generalized Weyl character formula.

Theorem 8.4 (Gross–Kostant–Ramond–Sternberg [57]) *Let $V(\mu)$ be the irreducible $\mathfrak{g}$-representation of highest weight μ. Then*

$$\mathsf{ch}(V(\mu)) = \frac{\sum_{w\in W_\mathfrak{p}} (-1)^{l(w)} \mathsf{ch}(M(w(\mu+\rho)-\rho_\mathfrak{k}))}{\prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (\mathrm{e}^{\alpha/2} - \mathrm{e}^{-\alpha/2})}.$$

Proof Write $V = V(\mu)$. The exact sequence (8.9) shows that

$$\mathsf{ch}(V \otimes \mathsf{S}_\mathfrak{p}) = \mathsf{ch}(\ker(\mathscr{D}_V)).$$

The left-hand side of this equality may be expressed as

$$\mathsf{ch}(V \otimes \mathsf{S}_\mathfrak{p}) = \mathsf{ch}(V)\mathsf{ch}(\mathsf{S}_\mathfrak{p}) = \mathsf{ch}(V) \prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (\mathrm{e}^{\alpha/2} - \mathrm{e}^{-\alpha/2}),$$

while the right-hand side equals $\sum_{w\in W_\mathfrak{p}} (-1)^{l(w)} \mathsf{ch}(M(w(\mu+\rho)-\rho_\mathfrak{k}))$. □

If $\mu = 0$, one obviously has $\mathsf{ch}(V(0)) = 1$. Hence

Corollary 8.2

$$\prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (\mathrm{e}^{\alpha/2} - \mathrm{e}^{-\alpha/2}) = \sum_{w\in W_\mathfrak{p}} (-1)^{l(w)} \mathrm{e}^{w\rho-\rho_\mathfrak{k}}.$$

For $\mathfrak{k} = \mathfrak{t}$, the GKRS character formula specializes to the usual Weyl character formula:

$$\mathsf{ch}(V(\mu)) = \frac{\sum_{w\in W} (-1)^{l(w)} \mathrm{e}^{w(\mu+\rho)}}{\prod_{\alpha\in\mathfrak{R}_+} (\mathrm{e}^{\alpha/2} - \mathrm{e}^{-\alpha/2})}. \tag{8.10}$$

8.5 q-dimensions

Kac, Möseneder-Frajria and Papi [74, Proposition 5.9] discovered that the dimension formula for multiplets generalizes to "q-dimensions". We recall the definition: Let $\mathfrak{g}$ be a semisimple Lie algebra with given choice of $\mathfrak{R}_+$, and $\alpha_1, \ldots, \alpha_l$ the corresponding simple roots. Let $\rho^\vee$ be the half-sum of positive co-roots $\alpha^\vee$. Alternatively, $\rho^\vee$ is characterized by its property

$$\langle \alpha, \rho^\vee \rangle = 1$$

for every simple root α of $\mathfrak{g}$. The q-dimension of a $\mathfrak{g}$-representation is defined as the polynomial in q,

$$\dim_{\mathsf{q}}(V) = \sum_\nu \dim(V_\nu)\mathsf{q}^{\langle \nu, \rho^\vee \rangle};$$

other normalizations exist in the literature. One has the following formula:

Proposition 8.12 *The q-dimension of the irreducible representation $V(\mu)$ of highest weight $\mu \in P_+$ is given by the formula,*

$$\dim_{\mathsf{q}}(V(\mu)) = \prod_{\alpha \in \mathfrak{R}_+} \frac{[\langle \mu + \rho, \alpha^\vee \rangle]_{\mathsf{q}}}{[\langle \rho, \alpha^\vee \rangle]_{\mathsf{q}}}$$

with the q-integers

$$[n]_{\mathsf{q}} = \frac{\mathsf{q}^{n/2} - \mathsf{q}^{-n/2}}{\mathsf{q}^{1/2} - \mathsf{q}^{-1/2}}.$$

Proof Write $q = e^s$. Then $\mathsf{q}^{\langle \nu, \rho^\vee \rangle}$ is the value of the character e^ν at the point $\exp(s\rho^\vee) \in T$. Hence, $\dim_{\mathsf{q}}(V(\mu))$ is the value of $\mathrm{ch}(V(\mu))$ at $\exp(s\rho^\vee)$. We may therefore calculate

$$\begin{aligned}
\dim_{\mathsf{q}}(V(\mu)) &= \frac{\sum_{w \in W} (-1)^{l(w)} \mathsf{q}^{\langle w(\mu+\rho), \rho^\vee \rangle}}{\sum_{w \in W} (-1)^{l(w)} \mathsf{q}^{\langle w\rho, \rho^\vee \rangle}} \\
&= \frac{\sum_{w \in W} (-1)^{l(w)} \mathsf{q}^{\langle \mu+\rho, w^{-1}\rho^\vee \rangle}}{\sum_{w \in W} (-1)^{l(w)} \mathsf{q}^{\langle \rho, w^{-1}\rho^\vee \rangle}} \\
&= \prod_{\alpha \in \mathfrak{R}_+} \frac{\mathsf{q}^{\langle \mu+\rho, \alpha^\vee \rangle/2} - \mathsf{q}^{-\langle \mu+\rho, \alpha^\vee \rangle/2}}{\mathsf{q}^{\langle \rho, \alpha^\vee \rangle/2} - \mathsf{q}^{-\langle \rho, \alpha^\vee \rangle/2}} \\
&= \prod_{\alpha \in \mathfrak{R}_+} \frac{[\langle \mu + \rho, \alpha^\vee \rangle]_{\mathsf{q}}}{[\langle \rho, \alpha^\vee \rangle]_{\mathsf{q}}},
\end{aligned}$$

where we used the Weyl character formula (8.10). □

For $\mathsf{q} \to 1$, one has $\lim_{\mathsf{q}\to 1}[n]_{\mathsf{q}} = n$, and one recovers the *Weyl dimension formula*,

$$\dim(V(\mu)) = \prod_{\alpha\in\mathfrak{R}_+} \frac{\langle \mu+\rho, \alpha^\vee\rangle}{\langle \rho, \alpha^\vee\rangle}.$$

(Indeed, the proof above is the standard proof of the dimension formula, cf. Duistermaat–Kolk [48, Chapter 4.9].) As a special case, note that

$$\dim(V(\rho)) = 2^{|\mathfrak{R}_+|}.$$

Lemma 8.3 [74, Lemma 5.8] *Suppose* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *as above, where* $\mathfrak{g}$ *is a semisimple Lie algebra, and* $\mathfrak{k} \subseteq \mathfrak{g}$ *is a semisimple subalgebra of equal rank. There exists a root* $\alpha \in \mathfrak{R}_\mathfrak{p}$ *with*

$$\langle \alpha, \rho_\mathfrak{k}^\vee\rangle = 0.$$

More generally, if $\mathfrak{k} \subseteq \mathfrak{g}$ *are equal rank reductive Lie algebras, there exists an element* $\tilde\rho_\mathfrak{k} \in \mathfrak{t}$ *(not unique) such that* $\langle \alpha, \tilde\rho_\mathfrak{k}\rangle = 1$ *for all simple roots of* $\mathfrak{k}$*, and* $\langle \alpha, \tilde\rho_\mathfrak{k}\rangle = 0$ *for at least one* $\alpha \in \mathfrak{R}_\mathfrak{p}$.

Proof We first assume $\mathfrak{k}$ semisimple. Consider a chain $\mathfrak{k} = \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_n = \mathfrak{g}$ of subalgebras with $\mathfrak{g}_i$ maximal in $\mathfrak{g}_{i+1}$. Then each $\mathfrak{g}_i$ is a semisimple subalgebra, and the set of roots of $\mathfrak{g}_i$ is a subset of those of $\mathfrak{g}$. We may thus assume that $\mathfrak{k}$ is maximal in $\mathfrak{g}$. Furthermore, by splitting $\mathfrak{g}$ into its simple components we may assume that $\mathfrak{g}$ is simple. Let $\alpha_1, \ldots, \alpha_l$ be the simple roots, and let $\alpha_{\max} = \sum_j k_j\alpha_j$ be the highest root. The coefficients k_j are called the Dynkin marks. As described in Remark 8.7, the maximal equal rank semisimple subalgebras are classified (up to conjugacy) by the set of all $i \in \{1, \ldots, l\}$ for which the Dynkin mark k_i is prime. More precisely, the set of simple roots of the subalgebra is

$$\{\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_l\}$$

with $\alpha_0 = -\alpha_{\max}$. Writing $\alpha_i = \frac{1}{k_i}(\alpha_{\max} - \sum_{j\neq i} k_j\alpha_j)$ we obtain

$$\langle \alpha_i, \rho_\mathfrak{k}^\vee\rangle = \frac{1}{k_i}(-1 - \sum_{j\neq i} k_j) = 1 - \frac{h}{k_i},$$

where $h = 1 + \langle \alpha_{\max}, \rho^\vee\rangle = 1 + \sum_j k_j$ is the Coxeter number. It is a standard fact (which may be verified, e.g., by consulting the tables for simple Lie algebras) that every prime Dynkin mark k_i divides the Coxeter number h. Hence $\langle \alpha_i, \rho_\mathfrak{k}^\vee\rangle \in \mathbb{Z}$. This proves the existence of a root $\alpha \in \mathfrak{R}_\mathfrak{p}$ with $\langle \alpha, \rho_\mathfrak{k}^\vee\rangle \in \mathbb{Z}_{\geq 0}$. If $\langle \alpha, \rho_\mathfrak{k}^\vee\rangle > 0$, then $\langle \alpha, \beta^\vee\rangle > 0$ for some simple root β of $\mathfrak{k}$. Thus $\alpha - \beta$ is a root. Since $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, one has $\alpha - \beta \in \mathfrak{R}_\mathfrak{p}$, and

$$\langle \alpha - \beta, \rho_\mathfrak{k}^\vee\rangle = \langle \alpha, \rho_\mathfrak{k}\rangle - 1.$$

By repeating this procedure, one eventually finds the desired root in $\mathfrak{R}_\mathfrak{p}$ whose pairing with $\rho_\mathfrak{k}^\vee$ is zero. This proves the lemma for $\mathfrak{g}, \mathfrak{k}$ semisimple. If $\mathfrak{k}$ is not semisimple, let $\mathfrak{z} \neq 0$ be its center. Choose any root $\alpha \in \mathfrak{R}_\mathfrak{p}$ that is not orthogonal to $\mathfrak{z}$. For suitable $\xi \in \mathfrak{z}$, we then have

$$\langle \alpha, \xi \rangle = \langle \alpha, \rho_\mathfrak{k}^\vee \rangle,$$

so that $\widetilde{\rho}_\mathfrak{k}^\vee = \rho_\mathfrak{k}^\vee - \xi$ has the desired properties. □

Theorem 8.5 (Kac, Möseneder-Frajria and Papi [74]) *Suppose $\mathfrak{k}$ is an equal rank semisimple Lie subalgebra of $\mathfrak{g}$. The q-dimensions of the irreducible $\mathfrak{k}$-representations in each multiplet satisfy*

$$\sum_{w \in W_\mathfrak{p}} (-1)^{l(w)} \dim_\mathsf{q} M(w(\mu+\rho) - \rho_\mathfrak{k}) = 0.$$

Proof By the generalized Weyl character formula, Theorem 8.4, the left-hand side of the displayed equation equals

$$\dim_\mathsf{q}(V) \prod_{\alpha \in \mathfrak{R}_{\mathfrak{p},+}} (\mathsf{q}^{\langle \alpha, \rho_\mathfrak{k}^\vee \rangle/2} - \mathsf{q}^{\langle \alpha, \rho_\mathfrak{k}^\vee \rangle/2}),$$

where V is viewed as a $\mathfrak{k}$-representation by restriction. By Lemma 8.3, at least one factor in the product over $\mathfrak{R}_{\mathfrak{p},+}$ is zero. □

As shown in [74], the results extends to arbitrary equal rank reductive Lie subalgebras $\mathfrak{k} \subseteq \mathfrak{g}$, provided the q-dimension is defined using $\widetilde{\rho}_\mathfrak{k}^\vee$ (cf. Lemma 8.3) instead of $\rho_\mathfrak{k}^\vee$.

8.6 The shifted Dirac operator

Return to the full Dirac operator $\mathscr{D} \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$ for a reductive Lie algebra $\mathfrak{g}$. (We still assume that B is positive definite on $\mathfrak{g}_\mathbb{R}$.) Fix a unitary module S over $\mathrm{Cl}(\mathfrak{g})$, and let $V = V(\mu)$ be an irreducible unitary $\mathfrak{g}$-representation of highest weight μ. Then $\mathscr{D}$ becomes a skew-adjoint odd operator $\mathscr{D}_V$ on $V \otimes \mathsf{S}$. Since $\mathscr{D}^2 = \mathrm{Cas}_\mathfrak{g} - \|\rho\|^2$, the action on $V \otimes \mathsf{S}$ is as a scalar, $-\|\mu+\rho\|^2$. In particular, D is invertible as an operator on $V \otimes \mathsf{S}$.

As noted by Freed–Hopkins–Teleman [53], one obtains interesting results by shifting the Dirac operator by elements $\tau \in \sqrt{-1}\mathfrak{g}$:

$$\mathscr{D}_\tau = \mathscr{D} - \tau \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}).$$

Note that $\mathscr{D}_\tau$ no longer squares to a central element, but instead satisfies

$$\mathscr{D}_\tau^2 = \mathrm{Cas} - \|\rho\|^2 + B(\tau,\tau) - 2(\widehat{\tau} + \gamma(\tau)).$$

We denote by $\tau^* = B^\flat(\tau) \in \sqrt{-1}\mathfrak{g}^*$ the image of τ under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ defined by B. Then $B(\tau,\tau) = B^*(\tau^*,\tau^*) = -\|\tau^*\|^2$. The element $\mathscr{D}_\tau$ is represented on $V \otimes \mathsf{S}$ as a skew-adjoint operator. The identity above becomes

$$\mathscr{D}_\tau^2\big|_{V\otimes \mathsf{S}} = -\|\mu+\rho\|^2 - \|\tau^*\|^2 - 2L(\tau),$$

where $L(\tau)$ indicates the diagonal action of $\tau \in \mathfrak{g}$ on $V \otimes \mathsf{S}$.

Proposition 8.13 (Freed–Hopkins–Teleman [53]) *Let $V = V(\mu)$ be the irreducible representation of highest weight $\mu \in P_+$. For $\tau \in \sqrt{-1}\mathfrak{g}$, the operator $\mathscr{D}_\tau$ on $V \otimes \mathsf{S}$ is invertible, unless τ^* lies in the coadjoint orbit of $\mu + \rho$. Moreover, for $\tau^* = \mu + \rho$ the kernel of $\mathscr{D}_\tau$ is on $V \otimes \mathsf{S}$ is the weight space $(V \otimes \mathsf{S})_{\mu+\rho} = V_\mu \otimes \mathsf{S}_\rho$.*

Proof Since the map $\tau \mapsto \mathscr{D}_\tau$ is $G_{\mathbb{R}}$-equivariant, we may assume that τ lies in the positive Weyl chamber. In particular, $\mathscr{D}_\tau$ is then $\mathfrak{t}$-equivariant, and hence preserves all weight spaces. On the weight space $(V \otimes \mathsf{S})_\nu \neq 0$ the operator $\mathscr{D}_\tau^2$ acts as

$$\begin{aligned}\mathscr{D}_\tau^2\big|_{(V\otimes\mathsf{S})_\nu} &= -\|\mu+\rho\|^2 - \|\tau^*\|^2 - 2\langle \nu, \tau\rangle \\ &= -\|\mu+\rho-\tau^*\|^2 + 2\langle \mu+\rho-\nu, \tau\rangle.\end{aligned}$$

The weight ν is a sum of weights ν_1 of V and ν_2 of S. Both $\mu - \nu_1$ and $\rho - \nu_2$ are linear combinations of positive roots with non-negative coefficients, hence so is $\mu + \rho - \nu$. Hence both terms in the formula for $\mathscr{D}_\tau^2|_{(V\otimes\mathsf{S})_\nu}$ are ≤ 0. Hence D_τ^2 is non-zero on the weight space $(V \otimes \mathsf{S})_\nu$ unless

$$\mu + \rho - \tau^* = 0, \quad \langle \mu + \rho - \nu, \tau\rangle = 0.$$

But $\tau^* = \mu + \rho$ implies that τ lies in the interior of the Weyl chamber, and hence the condition $\langle \mu + \rho - \nu, \tau\rangle = 0$ forces $\nu = \mu + \rho$. It follows that the kernel of D_τ^2 on $V \otimes \mathsf{S}$ is the highest weight space, $(V \otimes \mathsf{S})_{\mu+\rho} = V_\mu \otimes \mathsf{S}_\rho$. ☐

The proposition shows that the family of Dirac operators $\tau^* \mapsto \mathscr{D}_\tau$ on $\sqrt{-1}\mathfrak{g}_{\mathbb{R}}^*$, viewed as a representative for an equivariant K-theory class, is supported on the coadjoint orbit of $G_{\mathbb{R}}.(\mu + \rho)$. As shown in [53], the $G_{\mathbb{R}}$-equivariant K-theory class of this family of operators is identified with the class in $K_{G_{\mathbb{R}}}(\mathrm{pt}) = R(G_{\mathbb{R}})$ defined by the representation $[V]$.

8.7 Dirac induction

In contrast to the previous sections, we will denote by $\mathfrak{t}, \mathfrak{k}, \mathfrak{g}, \ldots$ compact real Lie algebras. We denote by $\mathfrak{t}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}, \ldots$ the reductive Lie algebras obtained by complexification; the corresponding Lie groups will be denoted by $T, K, G, \ldots$ and $T^{\mathbb{C}}, K^{\mathbb{C}}, G^{\mathbb{C}}, \ldots$ respectively. We will discuss *Dirac induction* from twisted representations of equal rank subgroups $K \subseteq G$ of a compact Lie group, using the cubic Dirac operator. It is similar to *holomorphic induction*, but works in more general settings since G/K need not have an invariant complex structure. References for this section include the papers by Kostant [87, 88] as well as [66, 95, 96, 119].

8.7.1 Central extensions of compact Lie groups

We will need some background material on central extensions of Lie groups G. In this section, *central extension* will always mean a central extension by the circle group,

$$1 \to \mathrm{U}(1) \to \widehat{G} \to G \to 1.$$

We will refer to the image of U(1) in this sequence as "the" central U(1) of the extension, even though there may be other U(1) subgroups of the center. A *morphism of central extensions* is given by a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \widehat{K} & \longrightarrow & K & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \widehat{G} & \longrightarrow & G & \longrightarrow & 1, \end{array}$$

where the vertical maps are Lie group morphisms. If a Lie group morphism $K \to G$ is given, one can use the diagram to define a *pull-back* of a central extension of G to a central extension of K.

An *automorphism* of a central extension $\widehat{G}$ is a morphism from the central extension to itself. That is, it is a Lie group automorphism of $\widehat{G}$ commuting with the action of the central $\mathrm{U}(1)$. The automorphism is called a *gauge transformation* of $\widehat{G}$ if the underlying group automorphism of G is the identity. It is easy to see that the group of gauge transformations of a central extension $\widehat{G}$ is the Abelian group

$$\mathrm{Hom}(G, \mathrm{U}(1))$$

of all Lie group morphisms $G \to \mathrm{U}(1)$. More generally, any two morphisms of central extensions $\widehat{K} \to \widehat{G}$ with a given underlying map $K \to G$ are related by a gauge transformation of $\widehat{K}$. In particular, if $\widehat{G}$ is isomorphic to a trivial central extension $G \times \mathrm{U}(1)$, then any two trivializations are related by an element of $\mathrm{Hom}(G, \mathrm{U}(1))$.

We will label central extensions by a notation $\widehat{G}^{(\kappa)}$; the *trivial central extension* is denoted by $\widehat{G}^{(0)} = G \times \mathrm{U}(1)$. The *exterior product* of two central extensions $\widehat{G_1}^{(\kappa_1)}$ and $\widehat{G_2}^{(\kappa_2)}$ is the quotient

$$\widehat{G_1 \times G_2}^{(\kappa_1+\kappa_2)} = \big((\widehat{G_1}^{(\kappa_1)} \times \widehat{G_2}^{(\kappa_2)}) \times \mathrm{U}(1)\big)\big/(\mathrm{U}(1) \times \mathrm{U}(1))$$

under the action

$$(w_1, w_2).(\widehat{g}_1, \widehat{g}_2, z) = (\widehat{g}_1 w_1^{-1}, \widehat{g}_2 w_2^{-1}, w_1 w_2 z);$$

it is a central extension

$$1 \to \mathrm{U}(1) \to \widehat{G_1 \times G_2}^{(\kappa_1+\kappa_2)} \to G_1 \times G_2 \to 1.$$

If $G_1 = G_2 = G$, we define an (interior) *product* $\widehat{G}^{(\kappa_1+\kappa_2)}$ as the pull-back of the exterior product under the diagonal inclusion $G \to G \times G$. Finally, the *opposite* $\widehat{G}^{(-\kappa)}$ of a central extension $\widehat{G}^{(\kappa)}$ is a quotient of the trivial central extension $\widehat{G}^{(\kappa)} \times \mathrm{U}(1)$ by the action $w.(\widehat{g}, z) = (\widehat{g}w^{-1}, zw^{-1})$. As the notation suggests, the product of $\widehat{G}^{(\kappa)}$ and $\widehat{G}^{(-\kappa)}$ is canonically isomorphic to the trivial extension.

For a compact, connected Lie group G, the group of isomorphism classes of central extensions of G by $\mathrm{U}(1)$ is canonically isomorphic to the cohomology group $H^3_G(\mathrm{pt}, \mathbb{Z}) = H^3(BG, \mathbb{Z})$. (For any space X, the group $H^3(X, \mathbb{Z})$ labels equivalence classes of principal bundles with structure group $PU(\mathscr{H})$, the projective unitary group of an infinite-dimensional separable Hilbert space $\mathscr{H}$. Given a central extension $\widehat{G}^{(\kappa)}$, there exists a representation of $\widehat{G}^{(\kappa)}$ on $\mathscr{H}$, where the central circle acts with weight 1. This defines a group morphism $G \to PU(\mathscr{H})$, and an associated bundle $(EG \times PU(\mathscr{H}))/G$. Cf. [17, 102] for further information.) Note that $H^3_G(\mathrm{pt}, \mathbb{Z})$ is a torsion group, since $H^3_G(\mathrm{pt}, \mathbb{R}) = 0$. Hence, if $\widehat{G}^{(\kappa)}$ is a given central extension of G, then there exists $m > 0$ such that $\widehat{G}^{(m\kappa)}$ is isomorphic to the trivial central extension. In particular, the central extension is isomorphic to the trivial central extension on the level of Lie algebras. Indeed, the choice of an invariant inner product on $\widehat{\mathfrak{g}}^{(\kappa)}$ defines a splitting of the extension by embedding $\mathfrak{g}$ as the orthogonal complement of $\mathfrak{u}(1) \subseteq \widehat{\mathfrak{g}}^{(\kappa)}$. The splitting is unique, up to a Lie algebra morphism $\mathfrak{g} \to \mathbb{R}$, i.e., up to an element of $(\mathfrak{g}^*)^{\mathfrak{g}}$.

8.7.2 Twisted representations

Suppose G is a compact, connected Lie group. Given a central extension $\widehat{G}^{(\kappa)}$, we define a *κ-twisted representation of G* to be a representation of $\widehat{G}^{(\kappa)} \to \mathrm{U}(V)$, where the central circle acts with weight 1. One may think of a κ-twisted representation as a projective representation $G \to \mathrm{PU}(V)$ together with an isomorphism of $\widehat{G}^{(\kappa)}$ with the pull-back of the central extension $1 \to \mathrm{U}(1) \to \mathrm{U}(V) \to \mathrm{PU}(V) \to 1$. The isomorphism classes of κ-twisted representations form a semigroup under direct sum; let $R^{(\kappa)}(G)$ denote the Grothendieck group. The tensor product of representations defines a product

$$R^{(\kappa_1)}(G) \times R^{(\kappa_2)}(G) \to R^{(\kappa_1+\kappa_2)}(G). \tag{8.11}$$

In particular, $R^{(\kappa)}(G)$ is a module over $R^{(0)}(G) = R(G)$.

Fix a maximal torus $T \subseteq G$, and let $P_G \equiv P_T \cong \mathrm{Hom}(T, \mathrm{U}(1))$ be the weight lattice of T. We will consider P_T as a subset of $\sqrt{-1}\mathfrak{t}^*$, consisting of all $\nu \in (\mathfrak{t}^*)^{\mathbb{C}}$ such that $\langle \nu, \xi \rangle \in 2\pi\sqrt{-1}\mathbb{Z}$ whenever $\xi \in \mathfrak{t}$ is in the kernel of $\exp_T : \mathfrak{t} \to T$. Let $\widehat{T}^{(\kappa)} \subseteq \widehat{G}^{(\kappa)}$ be the maximal torus given as the pre-image of T, and define the κ-twisted weights

$$P_G^{(\kappa)} \equiv P_T^{(\kappa)} \subseteq \mathrm{Hom}(\widehat{T}^{(\kappa)}, \mathrm{U}(1))$$

to be the homomorphisms whose restriction to the central $\mathrm{U}(1)$ is the identity. The affine lattice $P_T^{(\kappa)}$ labels the isomorphism classes of κ-twisted representations of T; in particular

$$R^{(\kappa)}(T) = \mathbb{Z}[P_T^{(\kappa)}].$$

$P_T^{(\kappa)}$ is the affine sublattice of $P_{\widehat{G}^{(\kappa)}} = P_{\widehat{T}^{(\kappa)}}$ given as the pre-image of the generator of $P_{\mathrm{U}(1)}$. The tensor product of twisted representations of T (cf. (8.11) for $G = T$) gives an "addition" map

$$P_T^{(\kappa_1)} \times P_T^{(\kappa_2)} \to P_T^{(\kappa_1+\kappa_2)}. \tag{8.12}$$

The image of the roots $\mathfrak{R} \subseteq \mathfrak{t}^*$ of G under the inclusion $\mathfrak{t}^* \to (\widehat{\mathfrak{t}}^{(\kappa)})^*$ are the roots of $\widehat{G}^{(\kappa)}$. We denote by $P^{(\kappa)}_{G,+}$ the corresponding dominant κ-twisted weights. Then $P^{(\kappa)}_{G,+}$ labels the irreducible κ-twisted representations of G, and

$$R^{(\kappa)}(G) = \mathbb{Z}[P^{(\kappa)}_{G,+}].$$

Remark 8.11 As mentioned above, any central extension $\widehat{G}^{(\kappa)}$ of a compact connected Lie group becomes trivial on the level of Lie algebras. The restriction of a given Lie algebra splitting $\mathfrak{g} \to \mathfrak{g}^{(\kappa)}$ to $\mathfrak{t} \to \mathfrak{t}^{(\kappa)}$ dualizes to give a projection, $(\mathfrak{t}^{(\kappa)})^* \to \mathfrak{t}^*$, and embeds $P^{(\kappa)}_G$ as a subset of $\mathfrak{t}^*$ of the form

$$P^{(\kappa)}_G = P_G + \delta^{(\kappa)}.$$

The "shift" $\delta^{(\kappa)} \in \sqrt{-1}\mathfrak{t}^*$ is defined modulo P_G; changing the trivialization by an element of $(\mathfrak{g}^*)^{\mathfrak{g}} \subseteq \mathfrak{t}^*$ modifies $\delta^{(\kappa)}$ accordingly.

8.7.3 The ρ-representation of $\mathfrak{g}$ as a twisted representation of G

Let G be compact and connected Lie group, with a given invariant inner product on its Lie algebra $\mathfrak{g}$. Define a central extension $\widehat{G}^{(\sigma)}$ as the pull-back of the central extension

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}_c(\mathfrak{g}) \to \mathrm{SO}(\mathfrak{g}) \to 1$$

under the adjoint representation $G \to \mathrm{SO}(\mathfrak{g})$. Note that $\widehat{G}^{(\sigma)}$ is associated to a central extension of G by $\mathbb{Z}_2$, obtained by pulling back the double cover

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(\mathfrak{g}) \to \mathrm{SO}(\mathfrak{g}) \to 1.$$

By the following lemma, this central extension is "2-torsion".

Lemma 8.4 *Suppose* $1 \to \mathbb{Z}_l \to \widetilde{G}^{(\kappa)} \to G \to 1$ *is a central extension of* G *by* $\mathbb{Z}_l$, *for some* $l \geq 2$, *and let*

$$\widehat{G}^{(\kappa)} = \widetilde{G}^{(\kappa)} \times_{\mathbb{Z}_l} \mathrm{U}(1)$$

for the natural action of $\mathbb{Z}_l$ *as a subgroup of* $\mathrm{U}(1)$. *Then the central extension* $\widehat{G}^{(l\kappa)}$ *is* canonically *isomorphic to the trivial central extension* $\widehat{G}^{(0)}$.

Proof The m-th power $\widehat{G}^{(m\kappa)}$ is an associated bundle to the m-th power

$$\widetilde{G}^{(m\kappa)} = \widetilde{G}^{(\kappa)} \times_{\mathbb{Z}_l} \mathbb{Z}_l,$$

where $\mathbb{Z}_l$ acts on $\mathbb{Z}_l$ by $z.w = z^m w$. If m is a multiple of l, this is the trivial action. In particular, $\widetilde{G}^{(l\kappa)}$ is canonically trivial, hence so is $\widehat{G}^{(l\kappa)}$. □

Realize the ρ-representation of $\mathfrak{g}$ as

$$V(\rho) \cong \gamma(U\mathfrak{g}).\mathsf{R} \subseteq \mathrm{Cl}(\mathfrak{g}),$$

as in Proposition 8.6. If G is not simply connected, this representation may not exponentiate to the group level, and indeed ρ need not lie in P_G. But it always integrates to a representation of $\widehat{G}^{(\sigma)}$, where the central circle acts with weight 1. (Since $\xi \in \mathfrak{g}$ acts on $\gamma(U\mathfrak{g}).\mathsf{R}$ as multiplication by $\gamma(\xi)$, its exponential in the double cover $\widetilde{G}^{(\sigma)}$ acts as multiplication by $\exp_{\mathrm{Cl}}(\gamma(\xi)) \in \mathrm{Spin}(\mathfrak{g})$.) Thus $V(\rho)$ is an irreducible σ-twisted representation of G. We denote by

$$\tilde{\rho} \in P^{(\sigma)}_{G,+} \tag{8.13}$$

its highest weight. The natural splitting of the Lie algebra of $\mathrm{Spin}_c(\mathfrak{g})$ determines a splitting $\widehat{\mathfrak{g}}^{(\sigma)} = \mathfrak{g} \times \mathfrak{u}(1)$. The resulting inclusion $P^{(\sigma)}_G \hookrightarrow \mathfrak{t}^*$ (cf. Remark 8.11) takes $\tilde{\rho}$ to the half-sum of positive roots, ρ.

Remark 8.12 Similarly, given any Clifford module S over $\mathrm{Cl}(\mathfrak{g})$, the resulting action of $\mathfrak{g}$ on S exponentiates to a σ-twisted representation of G on S, isomorphic to a direct sum of $V(\rho)$'s.

Suppose $K \subseteq G$ is a closed subgroup, with Lie algebra $\mathfrak{k} \subseteq \mathfrak{g}$. Put $\mathfrak{p} = \mathfrak{k}^{\perp}$, and let $\widehat{K}^{(\tau)}$ be the central extension of K defined as the pull-back of the central extension

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}_c(\mathfrak{p}) \to \mathrm{SO}(\mathfrak{p}) \to 1$$

under the adjoint representation $K \to \mathrm{SO}(\mathfrak{p})$. Again, this twisting is 2-torsion, and the Lie algebra extension $\widehat{\mathfrak{k}}^{(\tau)}$ has a canonical splitting.

Any Clifford module over $\mathrm{Cl}(\mathfrak{p})$ becomes a τ-twisted representation of K. Suppose K is a connected subgroup of maximal rank, and choose a maximal torus $T \subseteq K \subseteq G$. Let $\mathfrak{R}_{K,+} \subseteq \mathfrak{R}_+$ be positive roots of $K \subseteq G$ relative to some choice of Weyl chamber. Let $\mathsf{S}_{\mathfrak{p}}$ be the spinor module over $\mathrm{Cl}(\mathfrak{p})$, equipped with the τ-twisted representation of K as above. Every irreducible component for the $\mathfrak{k}$-action on $\mathsf{S}_{\mathfrak{p}}$ is also an irreducible component for the κ-twisted action of K. Let $\tilde{\rho}_{\mathfrak{p}}$ be the highest weight if the irreducible component whose underlying $\mathfrak{k}$-action has highest weight $\rho_{\mathfrak{p}}$. As for the case $K = G$, the affine lattice $P^{(\tau)}_K$ has a canonical embedding into $\mathfrak{t}^*$, under which $\tilde{\rho}_{\mathfrak{p}}$ goes to $\rho_{\mathfrak{p}}$. Letting $\tilde{\rho}_{\mathfrak{k}} \in P^{(\sigma_{\mathfrak{k}})}_{K,+}$ be defined similar to $\widetilde{\rho}$ (cf. (8.13)), we have

$$\tilde{\rho} = \tilde{\rho}_{\mathfrak{k}} + \tilde{\rho}_{\mathfrak{p}},$$

using the addition $P^{(\sigma_{\mathfrak{k}})}_T \times P^{(\tau)}_T \to P^{(\sigma)}_T$ (cf. (8.12)).

8.7.4 Definition of the induction map

Suppose now that G is compact and connected, K is a maximal rank subgroup, and T a maximal torus in K. Let $\widehat{K}^{(\tau)}$ be the central extension of K described in Section 8.7.3. We will use the cubic Dirac operator to define *induction maps*

$$R^{(\kappa-\tau)}(K) \to R^{(\kappa)}(G).$$

Let M be a $(\kappa - \tau)$-twisted representation of K. Then $\mathsf{S}_\mathfrak{p}^* \otimes M$ is a $\mathbb{Z}_2$-graded κ-twisted representation of K. Here $\mathsf{S}_\mathfrak{p}^*$ is the dual of the spinor module $\mathsf{S}_\mathfrak{p}$.

Remark 8.13 Recall from Section 3.4 that the super space $\mathsf{K} = \mathrm{Hom}_{\mathrm{Cl}}(\mathsf{S}_\mathfrak{p}, \mathsf{S}_{\mathfrak{p}^*})$ is 1-dimensional, with parity given by $(-1)^{\dim \mathfrak{p}/2} = (-1)^{|\mathfrak{R}_{\mathfrak{p},+}|}$.

The L^2-sections of the associated bundle

$$\mathsf{E} = \widehat{G}^{(\kappa)} \times_{\widehat{K}^{(\kappa)}} (\mathsf{S}_\mathfrak{p}^* \otimes M) \to G/K$$

are identified with the $\widehat{K}^{(\kappa)}$-invariant subspace

$$\Gamma_{L^2}(\mathsf{E}) = \left(L^2(\widehat{G}^{(\kappa)}) \otimes \mathsf{S}_\mathfrak{p}^* \otimes M\right)^{\widehat{K}^{(\kappa)}}, \tag{8.14}$$

where $\widehat{K}^{(\kappa)}$ acts on $L^2(\widehat{G}^{(\kappa)})$ by the right-regular representation $(\hat{k}.f)(\hat{g}) = f(\hat{g}\hat{k})$, and on $\mathsf{S}_\mathfrak{p}^* \otimes M$ by the κ-twisted representation as above. Since the central $\mathrm{U}(1) \subseteq \widehat{K}^{(\kappa)}$ acts on $\mathsf{S}_\mathfrak{p}^* \otimes M$ with weight 1, and on $L^2(\widehat{G}^{(\kappa)})$ with weight -1, it acts trivially on the tensor product. That is, the $\widehat{K}^{(\kappa)}$-action descends to an action of K, and the superscript in (8.14) may be replaced with K.

The group $\widehat{G}^{(\kappa)}$ acts on (8.14) via the left-regular representation on $L^2(\widehat{G}^{(\kappa)})$, $(\hat{g}_1.f)(\hat{g}) = f(\hat{g}_1^{-1}\hat{g})$. Since the left-regular representation of $z \in \mathrm{U}(1)$ on $L^2(\widehat{G}^{(\kappa)})$ coincides with the right-regular representation of $z^{-1} \in \mathrm{U}(1)$, we see that the central circle $\mathrm{U}(1) \subseteq \widehat{G}^{(\kappa)}$ acts with weight 1 on (8.14). In (8.14), we may replace $L^2(\widehat{G}^{(\kappa)})$ with the subspace $L^2(G)^{(\kappa)} \subseteq L^2(\widehat{G}^{(\kappa)})$ on which the left-regular representation of $\mathrm{U}(1)$ has weight 1. Thus

$$\Gamma_{L^2}(\mathsf{E}) = (L^2(G)^{(\kappa)} \otimes \mathsf{S}_\mathfrak{p}^* \otimes M)^K. \tag{8.15}$$

From the usual direct sum decomposition of the L^2-functions on a compact Lie group one obtains that

$$L^2(G)^{(\kappa)} = \bigoplus_\pi V_\pi \otimes V_\pi^*, \tag{8.16}$$

where the sum is now over all level κ representations $\pi : \widehat{G}^{(\kappa)} \to \mathrm{Aut}(V_\pi)$. Thus

$$\Gamma_{L^2}(\mathsf{E}) = \bigoplus_\pi V_\pi \otimes \left(V_\pi^* \otimes \mathsf{S}_\mathfrak{p}^* \otimes M\right)^K,$$

where V_π^* is regarded as a $\widehat{K}^{(\kappa)}$-representation by restriction.

To obtain a finite-dimensional κ-twisted representation from the infinite-dimensional space $\Gamma_{L^2}(\mathsf{E})$, we use the relative cubic Dirac operator (cf. (7.14))

$$\mathscr{D}(\mathfrak{g}, \mathfrak{k}) = \sum\nolimits_a^{(\mathfrak{p})} \widehat{e}_a e^a + q(\phi_\mathfrak{p}) \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}).$$

Recall that $\sum_a^{(\mathfrak{p})}$ indicates summation over a basis of $\mathfrak{p}$. Since K has maximal rank in G, we may identify $\mathfrak{p}$ with the unique K-invariant complement to $\widehat{\mathfrak{k}}^{(\kappa)}$ in $\widehat{\mathfrak{g}}^{(\kappa)}$. Thus we may identify $\mathscr{D}(\mathfrak{g}, \mathfrak{k})$ with the element $\mathscr{D}(\widehat{\mathfrak{g}}^{(\kappa)}, \widehat{\mathfrak{k}}^{(\kappa)}) \in U(\widehat{\mathfrak{g}}^{(\kappa)}) \otimes \mathrm{Cl}(\mathfrak{p})$. The algebra $U(\widehat{\mathfrak{g}}^{(\kappa)}) \otimes \mathrm{Cl}(\mathfrak{p})$ acts on $V_\pi^* \otimes \mathsf{S}_\mathfrak{p}^*$, hence $\mathscr{D}(\mathfrak{g}, \mathfrak{k})$ acts as an operator on

$V_\pi^* \otimes \mathsf{S}_\mathfrak{p}^* \otimes M$. Since $\mathscr{D}(\mathfrak{g}, \mathfrak{k})$ is K-invariant, it restricts to the K-invariant subspace, giving a collection of skew-adjoint operators

$$\mathscr{D}_M^\pi \in \mathrm{End}((V_\pi^* \otimes \mathsf{S}_\mathfrak{p}^* \otimes M)^K).$$

Tensoring each operator $\mathscr{D}_{M^\mathfrak{p}}$ with the identity operator on V_π, and summing over π, one obtains a skew-adjoint (unbounded) operator $\mathscr{D}_M$ on the Hilbert space $(L^2(G)^{(\kappa)} \otimes \mathsf{S}_\mathfrak{p}^* \otimes M)^K$. Since $\mathscr{D}_M$ is equivariant, its kernel $\ker(\mathscr{D}_M)$ is a κ-twisted representation of G. We will show below that the kernel is finite-dimensional.

Definition 8.1 The *Dirac induction* is the map from $(\kappa - \tau)$-twisted representations of K to $\mathbb{Z}_2$-graded κ-twisted representations of G, taking

$$M \mapsto \ker(\mathscr{D}_M).$$

It induces a map on isomorphism classes,

$$R^{(\kappa-\tau)}(K) \to R^{(\kappa)}(G), \quad [M] \mapsto [\ker(\mathscr{D}_M)^{\bar{0}}] - [\ker(\mathscr{D}_M)^{\bar{1}}],$$

taking $[M]$ to the equivariant index of $\mathscr{D}_M$.

8.7.5 The kernel of $\mathscr{D}_M$

The following theorem gives a characterization of the Dirac induction map in terms of weights. For $\nu \in P_{K,+}^{(\kappa-\tau)}$, let $M(\nu)$ denote the corresponding irreducible $(\kappa - \tau)$-twisted representation of K. Similarly, for $\mu \in P_{G,+}^{(\kappa)}$ let $N(\mu)$ denote the corresponding irreducible κ-twisted representation of G. Observe that

$$P_K^{(\kappa-\tau)} + \tilde{\rho}_\mathfrak{k} = P_G^{(\kappa)} + \tilde{\rho} = P_G^{(\kappa+\sigma)}.$$

The following is a version of Kostant's *generalized Borel–Weil Theorem* [87, 88]; see also Landweber [95] and Wassermann [119, Section 20].

Theorem 8.6 *Let $\nu \in P_{K,+}^{(\kappa-\tau)}$ be given. If there exists $w \in W$ (necessarily unique) such that*

$$\nu + \tilde{\rho}_\mathfrak{k} = w(\mu + \tilde{\rho})$$

for some $\mu \in P_{G,+}^{(\kappa)}$, then the Dirac induction takes $M(\nu)$ to $N(\mu)$, with parity change by $(-1)^{l(w)}$. The Dirac induction takes $M(\nu)$ to 0 if no such w exists.

Hence, on the level of Grothendieck groups of twisted representations, the induction $R^{(\kappa-\tau)}(K) \to R^{(\kappa)}(G)$ is given by

$$[M(\nu)] \mapsto (-1)^{l(w)}[N(\mu)]$$

if $\nu + \tilde{\rho}_\mathfrak{k} = w(\mu + \tilde{\rho})$ for some $w \in W$, while $[M(\nu)] \mapsto 0$ if no such W exists.

Proof Let $M = M(\nu)$ with $\nu \in R^{(\kappa-\tau)}_{K,+}$. Clearly,

$$\ker(\mathscr{D}_M) = \bigoplus_{\pi} V_\pi^* \otimes \ker(\mathscr{D}_M^\pi).$$

Identify

$$\begin{aligned}(V_\pi^* \otimes \mathsf{S}_\mathfrak{p} \otimes M(\nu))^K &\cong \mathrm{Hom}_{\widehat{K}^{(\kappa-\tau)}}(M(\nu)^*, V_\pi^* \otimes \mathsf{S}_\mathfrak{p}^*)\\ &= \mathrm{Hom}_{\widehat{\mathfrak{k}}^{(\kappa)}}(M(\nu)^*, V_\pi^* \otimes \mathsf{S}_\mathfrak{p}^*).\end{aligned}$$

Here we used that the equivariance condition relative to the action of the connected group $\widehat{K}^{(\kappa-\tau)}$ is the same as that relative to its Lie algebra, but $\widehat{\mathfrak{k}}^{(\kappa-\tau)} \equiv \widehat{\mathfrak{k}}^{(\kappa)}$ since $\widehat{\mathfrak{k}}^{(\tau)}$ has a canonical splitting. As indicated in Remark 8.13, we write $\mathsf{S}_\mathfrak{p}^* = \mathsf{S}_\mathfrak{p} \otimes \mathsf{K}$. Letting $\mathscr{D}_{V_\pi^*}$ denote the action of $\mathscr{D}(\mathfrak{g},\mathfrak{k})$ on $V_\pi^* \otimes \mathsf{S}_\mathfrak{p}$, it follows that

$$\ker(\mathscr{D}_M^\pi) \cong \mathrm{Hom}_{\widehat{\mathfrak{k}}^{(\kappa)}}(M(\nu)^*, \ker(\mathscr{D}_{V_\pi^*})) \otimes \mathsf{K}.$$

Let $\mu \in (\widehat{\mathfrak{t}}^{(\kappa)})^*$ be the highest weight of V_π. Then the dual $\widehat{\mathfrak{k}}^{(\kappa)}$-representation V_π^* has highest weight $*\mu = w_0(-\mu)$ with w_0 the longest Weyl group element of W. Similarly, $M(\nu)^*$ has highest weight $*_\mathfrak{k}\nu \equiv w_{0,\mathfrak{k}}(-\nu)$, where $w_{0,\mathfrak{k}}$ is the longest element in $W_\mathfrak{k}$. By Theorem 8.2, the space

$$\mathrm{Hom}_{\widehat{\mathfrak{k}}^{(\kappa)}}(M(\nu)^*, \ker(\mathscr{D}_{V_\pi^*}))$$

is zero unless there exists $w_1 \in W$ such that $*\mu + \rho = w_1(*_\mathfrak{k}\nu + \rho_\mathfrak{k})$, and in the latter case the multiplicity is 1. The parity of this isotypical component is given by the length $(-1)^{l(w_1)}$.

Since $w_0\mathfrak{R}_- = \mathfrak{R}_+$, one has $*\rho = \rho$, and hence

$$*\mu + \rho = *(\mu + \rho) = -w_0(\mu + \rho).$$

Likewise, $*_\mathfrak{k}\nu + \rho_\mathfrak{k} = -w_{0,\mathfrak{k}}(\nu + \rho)$. We may hence rewrite the condition as

$$w(\mu + \rho) = \nu + \rho_\mathfrak{k},$$

where $w = w_{0,\mathfrak{k}}^{-1} w_1 w_0$. Here the condition is written in terms of Lie algebra weights; if we are using Lie group weights for the central extensions of K resp. G, the same condition reads $w(\mu + \tilde{\rho}) = \nu + \tilde{\rho}_\mathfrak{k}$. Since $l(w_0) = |\mathfrak{R}_+|$ and $l(w_{0,\mathfrak{k}}) = |\mathfrak{R}_{\mathfrak{k},+}|$, we have

$$(-1)^{l(w)} = (-1)^{l(w_1) + |\mathfrak{R}_{\mathfrak{p},+}|}.$$

The line K has parity $(-1)^{|\mathfrak{R}_{\mathfrak{p},+}|}$. Hence $\ker(\mathscr{D}_M^\pi)$ has parity $(-1)^{l(w)}$. We conclude that $\ker(\mathscr{D}_M)$ is isomorphic to $N(\mu)$ if $w(\mu + \tilde{\rho}) = \nu + \tilde{\rho}_\mathfrak{k}$ for some $w \in W$, and the parity of $\ker(\mathscr{D}_M)$ is given by the parity of $l(w)$. □

Remark 8.14 Let us briefly compare with *holomorphic induction*. Let $\mathfrak{p} = \mathfrak{k}^\perp$ have the complex structure defined by the set $\mathfrak{R}_{\mathfrak{p},+}$ of positive roots. In general, this complex structure is only T-invariant. *Assuming* that it is actually K-invariant, we obtain a complex structure on $T(G/K) = G \times_K \mathfrak{p}$. It is well known that this complex structure on the tangent bundle is integrable, hence G/K is a complex manifold.

Given a finite-dimensional unitary K-representation M, the associated bundle $G \times_K M$ is a holomorphic vector bundle. One therefore has a *Dolbeault–Dirac operator* $\bar{\partial}_M$, acting on the sections of

$$G \times_K (M \times \wedge\mathfrak{p}^-).$$

(See e.g., [24, Chapter 3.6].) It is a G-equivariant elliptic operator, hence its kernel $\ker(\bar{\partial}_M)$ is a finite-dimensional $\mathbb{Z}_2$-graded G-representation. One defines the holomorphic induction map $R(K) \to R(G)$ by

$$[M] \mapsto [\ker(\bar{\partial}_M)] = [\ker(\bar{\partial}_M)^{\bar{0}}] - [\ker(\bar{\partial}_M)^{\bar{1}}].$$

In terms of weights, the induction map is given by

$$[M(\nu)] \mapsto (-1)^{l(w)}[N(\mu)]$$

provided $\nu = w(\mu + \rho) - \rho$ for some $w \in W$, and is zero otherwise. (Recall again that ρ need not be in the weight lattice P_G. But $\rho - w\rho$ is a linear combination of roots with integer coefficients (Lemma B.1), and hence is always in P_G.) Similar to the discussion of Dirac induction, the computation of $\ker(\bar{\partial}_M)$ may be reduced to an algebraic Dirac operator on spaces

$$V \otimes \wedge\mathfrak{p}_-,$$

where V is an irreducible $\mathfrak{g}$-representation (viewed as a $\mathfrak{k}$-representation by restriction). This program was carried out in Kostant's classical paper [84], over 35 years before [87]. A detailed comparison of the two induction procedures can be found in [88].

Chapter 9
$\mathscr{D}(\mathfrak{g}, \mathfrak{k})$ as a geometric Dirac operator

We had alluded to interpretations of the element $\mathscr{D}(\mathfrak{g}, \mathfrak{k})$ as a geometric Dirac operator over a homogeneous space G/K. We will now discuss this interpretation in more detail.

9.1 Differential operators on homogeneous spaces

We begin with a general discussion of differential operators on homogeneous spaces G/K, where G is a connected Lie group and K is a closed subgroup. We denote by $\mathfrak{g}, \mathfrak{k}$ the Lie algebras of G, K. As discussed in Section 5.3, the identification $\mathfrak{g} \cong \mathfrak{X}^L(G)$ of the Lie algebra $\mathfrak{g}$ with left-invariant vector fields on G extends to an algebra isomorphism

$$U(\mathfrak{g}) \cong \mathfrak{D}^L(G) \tag{9.1}$$

with left-invariant differential operators on G, intertwining the adjoint action of G on $U(\mathfrak{g})$ with the action on $\mathfrak{D}^L(G)$ given by right-translations. In particular, there is an isomorphism

$$U(\mathfrak{g})^K \cong \mathfrak{D}(G)^{G\times K}$$

of the K-invariant part of the enveloping algebra with differential operators on G that are invariant under the left G-action and under the right K-action. The action of such differential operators on $C^\infty(G)^{\{e\}\times K} \cong C^\infty(G/K)$ defines an algebra morphism

$$\mathfrak{D}(G)^{G\times K} \to \mathfrak{D}(G/K)^G.$$

The generators for the action of K by right-multiplication are the left-invariant vector fields ξ^L for $\xi \in \mathfrak{k}$. Thus $C^\infty(G)^{\{e\}\times K}$ is annihilated by such vector fields, and more generally by the ideal in $\mathfrak{D}^L(G)$ generated by such vector fields. Under the isomorphism (9.1), this left ideal corresponds to $U(\mathfrak{g})\mathfrak{k} \subseteq U(\mathfrak{g})$. Passing to K-invariants, this defines a morphism of filtered vector spaces

$$U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K \to \mathfrak{D}(G/K)^G. \tag{9.2}$$

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The subspace $(U(\mathfrak{g})\mathfrak{k})^K$ is clearly a left ideal in $U(\mathfrak{g})^K$, and since elements of $U(\mathfrak{g})^K$ commute with elements of $\mathfrak{k} \subseteq U(\mathfrak{g})$ it is also a right ideal. Hence $U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K$ acquires an algebra structure, and (9.2) is a morphism of filtered algebras.

Proposition 9.1 [62, p. 285] *Suppose $\mathfrak{k} \subseteq \mathfrak{g}$ admits a K-invariant complement. Then* (9.2) *is an isomorphism of filtered algebras.*

Proof It suffices to show that the associated graded map to (9.2) is an isomorphism. The principal symbol (Section 2.3.1) of differential operators on G/K defines a G-equivariant isomorphism of graded algebras

$$\mathrm{gr}(\mathfrak{D}(G/K)) \cong \Gamma^\infty(G/K, S(T(G/K))).$$

Passing to invariants, and using $T_{eK}(G/K) \cong \mathfrak{g}/\mathfrak{k}$, this becomes an isomorphism

$$\mathrm{gr}(\mathfrak{D}(G/K))^G \cong \Gamma^\infty(G/K, S(T(G/K)))^G \cong S(\mathfrak{g}/\mathfrak{k})^K.$$

The inclusion $\mathfrak{D}(G/K)^G \hookrightarrow \mathfrak{D}(G/K)$ descends to an injective map of the associated graded algebras, $\mathrm{gr}(\mathfrak{D}(G/K)^G) \hookrightarrow \mathrm{gr}(\mathfrak{D}(G/K))$. Hence, the symbol map on $\mathfrak{D}(G/K)^G$ defines an injective algebra morphism

$$\mathrm{gr}(\mathfrak{D}(G/K)^G) \to \mathrm{gr}(\mathfrak{D}(G/K))^G \cong S(\mathfrak{g}/\mathfrak{k})^K. \tag{9.3}$$

We claim that if $\mathfrak{k}$ admits a K-invariant complement $\mathfrak{p} \subseteq \mathfrak{g}$, then this last map is an isomorphism. Let $S(\mathfrak{p})^K \to U(\mathfrak{g})^K \cong \mathfrak{D}(G)^{G\times K}$ be the inclusion given by symmetrization. Its composition with the quotient map to $\mathfrak{D}(G/K)^G$ is a filtration-preserving map $S(\mathfrak{p})^K \to \mathfrak{D}(G/K)^G$, whose associated graded map is a right inverse to (9.3). This proves that (9.3) is an isomorphism.

The Poincaré–Birkhoff–Witt Theorem gives a K-equivariant isomorphism of filtered vector spaces, $U(\mathfrak{g}) = U(\mathfrak{g})\mathfrak{k} \oplus S(\mathfrak{p})$; hence

$$U(\mathfrak{g})^K = (U(\mathfrak{g})\mathfrak{k})^K \oplus S(\mathfrak{p})^K$$

and consequently

$$S(\mathfrak{p})^K = \mathrm{gr}\big(U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K\big). \tag{9.4}$$

We conclude that (9.2) is an isomorphism. □

Remark 9.1 For a simple example where $\mathfrak{k}$ does not admit a K-invariant complement, take $G = \mathrm{SL}(2,\mathbb{R})$, the real matrices of determinant 1, and $K = \mathbb{R}$, embedded in G as the upper triangular matrices with 1's on the diagonal. Then $\mathfrak{k}$ consists of the strictly upper triangular 2×2-matrices. In terms of the standard basis e, f, h of $\mathfrak{sl}(2,\mathbb{R})$ from Example B.1, we have $\mathfrak{k} = \mathrm{span}(e)$. Since ad_e is nilpotent, with kernel $\ker(\mathrm{ad}_e) = \mathfrak{k}$, it is clear that $\mathfrak{k}$ does not have a $\mathfrak{k}$-invariant complement. The map $S(\mathfrak{g})^K \to S(\mathfrak{g}/\mathfrak{k})^K$ is non-surjective already in degree 1. Indeed, the map takes $S^1(\mathfrak{g})^K = \mathfrak{k}$ to 0, but $S^1(\mathfrak{g}/\mathfrak{k})^K = (\mathfrak{g}/\mathfrak{k})^K$ is 1-dimensional, with generator the image $[h]$ of h in $\mathfrak{g}/\mathfrak{k}$. Accordingly, the map $U(\mathfrak{g})^K = \mathfrak{D}(G)^{G\times K} \to \mathfrak{D}(G/K)^G$ is not surjective: the G-invariant vector field on G/K defined by $[h] \in (\mathfrak{g}/\mathfrak{k})^K$ does not lift to a $G \times K$-invariant differential operator on G.

Proposition 9.1 generalizes to differential operators acting on vector bundles. For any vector bundle $E \to M$, let $\mathfrak{D}(M, E)$ denote the filtered algebra of differential operators on M acting on sections of E. The principal symbol of differential operators (Section 2.3.1) identifies

$$\mathrm{gr}(\mathfrak{D}(M, E)) = \Gamma^{\infty}(M, S(TM) \otimes \mathrm{End}(E))$$

as graded algebras. The G-equivariant vector bundles over G/K are of the form $E = G \times_K V$, where V is the K-module defined by the fiber of E at eK.

Proposition 9.2 *Suppose $\mathfrak{k}$ admits a K-invariant complement $\mathfrak{p}$ in $\mathfrak{g}$, and let $\mathscr{I} \subseteq U(\mathfrak{g}) \otimes \mathrm{End}(V)$ be the left ideal generated by the diagonal embedding*

$$\mathfrak{k} \hookrightarrow \mathfrak{g} \otimes \mathrm{End}(V),\ \xi \mapsto \xi \otimes 1 + 1 \otimes \pi(\xi).$$

Then there is an isomorphism of filtered algebras

$$(U(\mathfrak{g}) \otimes \mathrm{End}(V))^K / \mathscr{I}^K \to \mathfrak{D}(G/K, G \times_K V)^G. \tag{9.5}$$

Proof The algebra $U(\mathfrak{g}) \otimes \mathrm{End}(V)$ is identified with the G-invariant differential operators on the trivial bundle $G \times V \to G$, and its K-invariant part is identified with the $G \times K$-equivariant differential operators:

$$(U(\mathfrak{g}) \otimes \mathrm{End}(V))^K \cong \mathfrak{D}(G, G \times V)^{G \times K}.$$

The identification $\Gamma^{\infty}(G \times_K V) = C^{\infty}(G, V)^K$ defines an algebra morphism

$$\mathfrak{D}(G, G \times V)^{G \times K} \to \mathfrak{D}(G/K, G \times_K V)^G.$$

Elements in the image of the diagonal embedding $\mathfrak{k} \to \mathfrak{g} \otimes \mathrm{End}(V)$ annihilate all K-invariant sections; hence the same is true for all elements in the left ideal $\mathscr{I} \subseteq U(\mathfrak{g}) \otimes \mathrm{End}(V)$ generated by this image. This defines a morphism of filtered algebras (9.5).

We claim that there is a K-invariant direct sum decomposition

$$U(\mathfrak{g}) \otimes \mathrm{End}(V) = \mathscr{I} \oplus (S(\mathfrak{p}) \otimes \mathrm{End}(V)), \tag{9.6}$$

using the embedding $S(\mathfrak{p}) \to U\mathfrak{g}$ by symmetrization. Give $U(\mathfrak{g}) \otimes \mathrm{End}(V)$ the filtration defined by the filtration of $U(\mathfrak{g})$. This induces filtrations on $\mathscr{I}$ and on $\mathscr{F} = (S\mathfrak{p} \otimes \mathrm{End}(V))$. From $U(\mathfrak{g}) = U(\mathfrak{g})\mathfrak{k} \oplus S(\mathfrak{p})$ we have

$$U(\mathfrak{g})_{(r)} \otimes \mathrm{End}(V) = \mathscr{I}'_{(r)} \oplus \mathscr{F}_{(r)} \tag{9.7}$$

with $\mathscr{I}' = U(\mathfrak{g})\mathfrak{k} \otimes \mathrm{End}(V)$. But

$$\mathscr{I}_{(r)} = \mathscr{I}'_{(r)} \mod U(\mathfrak{g})_{(r-1)} \otimes \mathrm{End}(V);$$

hence an inductive argument deduces (9.6) from (9.7). The rest of the proof is parallel to that for $V = \mathbb{R}$. □

Suppose $\mathfrak{g}$ is a quadratic Lie algebra, $\mathfrak{k}$ a quadratic Lie subalgebra, and $\mathfrak{p} = \mathfrak{k}^{\perp}$. Suppose that the K-module V is also a $\mathbb{C}l(\mathfrak{p})$-module, and that the Clifford action is K-equivariant for the given action by automorphism of $\mathbb{C}l(\mathfrak{p})$. Then the relative Dirac operator $\mathscr{D}(\mathfrak{g}, \mathfrak{k}) \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^K$ defines an element of the algebra $(U(\mathfrak{g}) \otimes \mathrm{End}(V))^K$, and hence a G-invariant differential operator on sections of $G \times_K V$.

9.2 Dirac operators on manifolds

A (generalized) Dirac operator on a Riemannian manifold M is a first-order differential operator whose square is the Laplace operator for the metric (modulo lower order terms). Concrete examples of such operators are obtained from modules over the Clifford algebra bundle $\mathbb{C}l(T^*M)$.

9.2.1 Linear connections

We will give a review of connections on vector bundles $\mathscr{V} \to M$. Let

$$\Omega^\bullet(M,\mathscr{V}) = \Gamma^\infty(M, \wedge^\bullet T^*M \otimes \mathscr{V})$$

be the space of $\mathscr{V}$-valued differential forms on M. In particular, $\Omega^0(M,\mathscr{V}) = \Gamma^\infty(M,\mathscr{V})$ is the space of sections. A *connection* on $\mathscr{V} \to M$ is a linear map

$$\nabla:\ \Omega^0(M,\mathscr{V}) \to \Omega^1(M,\mathscr{V})$$

satisfying

$$\nabla(f\sigma) = f\nabla(\sigma) + \mathrm{d}f \wedge \sigma$$

for all $f \in C^\infty(M),\ \ \sigma \in \Gamma^\infty(M,\mathscr{V})$. The connections on $\mathscr{V}$ form an affine space, with $\Omega^1(M,\mathrm{End}(\mathscr{V}))$ as its space of motions.

A section $\sigma \in \Gamma^\infty(M,\mathscr{V})$ is *horizontal* for the connection ∇ if $\nabla\sigma = 0$. The connection ∇ extends uniquely to a linear operator of degree 1 on $\Omega(M,\mathscr{V})$, with the property

$$[\nabla,\alpha] = \mathrm{d}\alpha$$

for $\alpha \in \Omega^\bullet(M)$. Here elements of $\Omega(M)$ act on $\Omega(M,\mathscr{V})$ by multiplication, and the commutator is a graded commutator $\nabla\circ\alpha - (-1)^{|\alpha|}\alpha\circ\nabla$. One observes that the degree 2 operator $\nabla\circ\nabla$ commutes with multiplication by functions $f \in C^\infty(M)$, and is hence given by the action of an element $R \in \Omega^2(\mathscr{V},\mathrm{End}(\mathscr{V}))$ called the *curvature*. The curvature vanishes if and only if $\mathscr{V}$ admits local trivializations $\mathscr{V}|_U = U \times \mathbb{R}^N$ taking the connection ∇ to the exterior differential. A connection on $\mathscr{V}$ canonically induces connections on the bundles $\mathscr{V}^*,\ S^k(\mathscr{V}),\ T^k(\mathscr{V}),\ \wedge^k(\mathscr{V})$, and so on. The composition $\nabla_X = \iota_X \circ \nabla$ for $X \in \mathfrak{X}(M)$ is called the *covariant derivative in the direction of* X. In terms of ∇_X, the defining property of a connection reads as

$$\nabla_X(f\sigma) = f\nabla_X\sigma + X(f)\sigma,$$

for all $f \in C^\infty(M),\ X \in \mathfrak{X}(M),\ \ \sigma \in \Gamma^\infty(M,\mathscr{V})$, and the curvature is

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

for $X, Y \in \mathfrak{X}(M)$ (note that the right-hand side is tensorial in X, Y).

If $\mathscr{V}$ comes equipped with additional structure, one can require that the connection respects this structure. For example, if the bundle $\mathscr{V}$ has a pseudo-Euclidean metric B, a connection is called a *metric connection* if $\nabla B = 0$. That is,

$$(\nabla_X B)(\sigma, \sigma') \equiv X(B(\sigma, \sigma')) - B(\nabla_X \sigma, \sigma') - B(\sigma, \nabla_X \sigma') = 0$$

for all $X \in \mathfrak{X}(M)$ and $\sigma, \sigma' \in \Gamma^\infty(M, \mathscr{V})$. The metric connections form an affine space, with $\Gamma^\infty(M, \mathfrak{o}(\mathscr{V}))$ as its space of motions.

Let us now consider connections ∇ on $\mathscr{V} = TM$. The *torsion* of ∇ is a section of $T^*M \otimes T^*M \otimes TM$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for $X, Y \in \mathfrak{X}(M)$. It may be regarded as an element of $\Omega^1(M, \mathrm{End}(TM))$, taking $X \in \mathfrak{X}(M)$ to $T(X, \cdot)$. The *torsion-free part* of a connection ∇ on TM is the connection

$$\overline{\nabla}_X = \nabla_X - \frac{1}{2} T(X, \cdot);$$

as suggested by the name it has vanishing torsion.

Suppose B is a non-degenerate symmetric bilinear form on TM, so that M is a pseudo-Riemannian manifold. The *Levi-Civita Theorem* (see e.g., [55, p. 341 f.]) asserts that TM has a unique torsion-free metric connection ∇^{met}. Given any metric connection ∇ such that the 3-tensor $(X, Y, Z) \mapsto B(T(X, Y), Z)$ is skew-symmetric in X, Y, Z (i.e., $T(X, \cdot) \in \Gamma^\infty(M, \mathfrak{o}(\mathscr{V}))$), the torsion-free part is again a metric connection, hence it coincides with the Levi-Civita connection.

A metric connection on $(\mathscr{V}, B)$ induces a connection on the Clifford bundle $\mathbb{C}l(\mathscr{V})$, compatible with the Clifford multiplication in the sense that $\nabla(x \cdot x') = (\nabla x) \cdot x' + x \cdot (\nabla x')$ for sections $x, x' \in \Gamma^\infty(M, \mathbb{C}l(\mathscr{V}))$.

9.2.2 Principal connections

The compatibility of connections on $\mathscr{V}$ with additional structures (such as fiber metrics, complex structures, spin structures, etc.) is best expressed by viewing $\mathscr{V}$ as associated to a principal bundle with the appropriate structure group.

Let K be a Lie group with Lie algebra $\mathfrak{k}$. A principal K-bundle is a manifold $\mathscr{P}$ with a Lie group action $\mathscr{A} : K \to \mathrm{Diff}(\mathscr{P}),\ k \to \mathscr{A}_k$, such that the action is free and the map $\pi : \mathscr{P} \to M = \mathscr{P}/K$ is a locally trivial fibration. A (principal) connection on $\mathscr{P}$ is a Lie-algebra valued 1-form $\theta \in \Omega^1(\mathscr{P}, \mathfrak{k})$ that is K-equivariant,

$$\mathscr{A}^*_{k^{-1}} \theta = \mathrm{Ad}_k \theta$$

and satisfies

$$\iota(\mathscr{A}_\xi)\theta = \xi, \quad \xi \in \mathfrak{k},$$

where $\mathfrak{k} \to \mathfrak{X}(\mathscr{P}),\ \xi \mapsto \mathscr{A}_\xi$ is the infinitesimal action. The choice of a principal connection is equivalent to the choice of a K-equivariant *horizontal bundle*, complementary to the vertical bundle $\ker(T\pi) \subseteq T\mathscr{P}$. Given θ, the horizontal bundle is the kernel $\ker(\theta) \subseteq T\mathscr{P}$, while the choice of an invariant subbundle determines θ as the projection $T\mathscr{P} \to \ker(T\pi)$, followed by the trivialization $\ker(T\pi) \cong \mathscr{P} \times \mathfrak{k}$ given by the vector fields $\mathscr{A}_\xi$. For any $X \in \mathfrak{X}(M)$, we denote by $\mathrm{hor}(X) \in \mathfrak{X}(\mathscr{P})$ its *horizontal lift* with respect to θ, defined as the unique horizontal vector field on $\mathscr{P}$ projecting to X. The *curvature* of θ is the $\mathfrak{k}$-valued 2-form $F^\theta = \mathrm{d}\theta + \frac{1}{2}[\theta,\theta] \in \Omega^2(\mathscr{P},\mathfrak{k})$. It is *$K$-basic*, i.e., it is K-equivariant and $\iota(\mathscr{A}_\xi)F^\theta = 0$ for all $\xi \in \mathfrak{k}$.

For any K-representation on a vector space V, one can form the associated vector bundle

$$\mathscr{V} = \mathscr{P} \times_K V = (\mathscr{P} \times V)/K.$$

Sections of $\mathscr{V} \to M$ are identified with K-invariant sections of the trivial bundle $\mathscr{P} \times V \to \mathscr{P}$, or equivalently with K-equivariant functions $\mathscr{P} \to V$. This defines an isomorphism

$$\Gamma^\infty(M, \mathscr{P} \times_K V) \to C^\infty(\mathscr{P}, V)^K, \quad \sigma \mapsto \tilde{\sigma}.$$

It extends further to $\mathscr{V}$-valued differential forms,

$$\Omega^\bullet(M,\mathscr{V}) \to \Omega^\bullet(\mathscr{P}, V)^{K-\mathrm{bas}}, \quad \alpha \mapsto \tilde{\alpha}.$$

In terms of this correspondence, a principal connection θ induces a linear connection ∇ on $\mathscr{V} = \mathscr{P} \times_K V$ by the formula

$$\widetilde{\nabla\sigma} = (\mathrm{d} + \theta.)\tilde{\sigma}.$$

(Here the dot signifies the action of $\mathfrak{k}$ on V; thus $\theta.\tilde{\sigma} \in \Omega^1(\mathscr{P}, V)$.) That is,

$$\widetilde{\nabla_X\sigma} = \mathrm{hor}(X)\tilde{\sigma}$$

for all vector fields $X \in \mathfrak{X}(M)$; here $\mathrm{hor}(X) \in \mathfrak{X}(\mathscr{P})$ is the horizontal lift with respect to the connection. The curvature $R \in \Omega^2(M, \mathrm{End}(\mathscr{V}))$ of ∇ is the image of $F^\theta \in \Omega^2(\mathscr{P},\mathfrak{k})^{K-\mathrm{bas}} \cong \Omega^2(M, \mathscr{P} \times_K \mathfrak{k})$ under the map

$$\mathscr{P} \times_K \mathfrak{k} \to \mathscr{P} \times_K \mathrm{End}(V) \cong \mathrm{End}(\mathscr{V}).$$

Any linear connection on a vector bundle $\mathscr{V}$ corresponds in this way to a principal connection on the *frame bundle* $\mathrm{Fr}_{\mathrm{GL}}(\mathscr{V}) \to M$. (The fiber of this bundle at $m \in M$ is given by the linear isomorphisms $\mathscr{V}_m \to \mathbb{R}^N$, where $N = \mathrm{rank}(\mathscr{V})$. The structure group $\mathrm{GL}(N,\mathbb{R})$ acts on such isomorphisms by composition.) A change of structure group amounts to extra structure on $\mathscr{V}$: For example, the choice of a Euclidean fiber metric reduces the structure group to $\mathrm{O}(N)$, defined by the principal bundle $\mathrm{Fr}_{\mathrm{O}}(\mathscr{V}) \to M$ of orthonormal frames; the additional choice of an orientation reduces the structure group further to $\mathrm{SO}(N)$.

9.2.3 Dirac operators

Suppose now that M is a pseudo-Riemannian manifold, with metric B. Let $\mathbb{C}l(T^*M)$ be the complexified Clifford bundle relative to the dual metric B^* on T^*M.[1] Suppose the complex vector bundle $\mathscr{E} \to M$ is a bundle of Clifford modules. The module action defines a bundle map

$$\rho:\ \mathbb{C}l(T^*M) \to \mathrm{End}(\mathscr{E});$$

we will use similar notation for the corresponding sections.

Definition 9.1 Given a connection $\nabla^{\mathscr{E}}$ on $\mathscr{E}$, one defines the corresponding *Dirac operator*

$$\partial\!\!\!/^{\mathscr{E}}:\ \Gamma^\infty(M,\mathscr{E}) \to \Gamma^\infty(M,\mathscr{E})$$

as a composition

$$\Gamma^\infty(M,\mathscr{E}) \xrightarrow{\nabla^{\mathscr{E}}} \Gamma^\infty(M, T^*M \otimes \mathscr{E}) \xrightarrow{\rho} \Gamma^\infty(M,\mathscr{E}).$$

Recall that with our conventions, the Clifford module $\mathscr{E}$ is $\mathbb{Z}_2$-graded and the module action $\rho:\ \mathbb{C}l(T^*M) \to \mathrm{End}(\mathscr{E})$ preserves the $\mathbb{Z}_2$-grading. Hence, $\partial\!\!\!/^{\mathscr{E}}$ is an odd operator relative to the $\mathbb{Z}_2$-grading on $\Gamma^\infty(M,\mathscr{E})$.

Lemma 9.1 *The Dirac operator $\partial\!\!\!/^{\mathscr{E}}$ is a first-order differential operator, with principal symbol*

$$\sigma^1(\partial\!\!\!/^{\mathscr{E}}) \in \Gamma^\infty(M,\ TM \otimes \mathrm{End}(\mathscr{E}))$$

the section defined by the bundle map $\rho:\ T^*M \to \mathrm{End}(\mathscr{E})$.

Proof The principal symbol is determined by the commutator $[\partial\!\!\!/^{\mathscr{E}}, f]$ for $f \in C^\infty(M)$. By definition of a connection, $[\nabla^{\mathscr{E}}, f] = \mathrm{d}f$ (where $\mathrm{d}f$ acts as a multiplication operator on $\Omega^\bullet(M,\mathscr{E})$). Hence,

$$[\partial\!\!\!/^{\mathscr{E}}, f] = \rho \circ [\nabla^{\mathscr{E}}, f] = \rho \circ \mathrm{d}f$$

is the operator of Clifford multiplication $\rho(\mathrm{d}f)$. □

Remarks 9.1

1. Note that the principal symbol does not depend on the choice of connection on $\mathscr{E}$; indeed a change of connection changes $\partial\!\!\!/^{\mathscr{E}}$ by a zeroth-order differential operator.
2. In terms of a local frame u_a for TM, with B-dual frame u^a for T^*M,

$$\partial\!\!\!/^{\mathscr{E}} = \sum_a \rho(u^a)\,\nabla^{\mathscr{E}}_{u_a}, \tag{9.8}$$

and the principal symbol is $\sigma^1(\partial\!\!\!/^{\mathscr{E}}) = \sum_a u_a \otimes \rho(u^a)$.

[1] The definition of the Dirac operator given below works more generally for possibly degenerate metrics on T^*M.

3. The square of the Dirac operator is a second-order differential operator with principal symbol
$$\sigma^2((\partial\!\!\!/^{\mathscr{E}})^2) = (\sigma^1(\partial\!\!\!/^{\mathscr{E}}))^2 \in \Gamma^\infty(M,\ S^2(TM) \otimes \mathrm{End}(\mathscr{E}))$$
the quadratic function $\mu \mapsto \rho(\mu)^2 = \|\mu\|^2\mathrm{Id}$. Thus, $(\partial\!\!\!/^{\mathscr{E}})^2$ is a generalized Laplacian.
4. Suppose B is positive definite, so that M is a Riemannian manifold. Then the formula $\rho(\mu)^2 = \|\mu\|^2\mathrm{Id}$ for $\mu \in T^*M$ shows that the principal symbol of $\partial\!\!\!/^{\mathscr{E}}$ is invertible away from the zero section of T^*M. That is, Dirac operators $\partial\!\!\!/^{\mathscr{E}}$ for Riemannian manifolds are *elliptic operators*.
5. For analytic computations with Dirac operators, it is often useful to take $\nabla^{\mathscr{E}}$ to be a *Clifford connection* relative to the given metric connection ∇ on TM. That is,
$$\nabla^{\mathscr{E}}(\rho(x).\sigma) = \rho(\nabla x)\sigma + \rho(x).\nabla^{\mathscr{E}}\sigma$$
for all $x \in \Gamma^\infty(M, \mathbb{C}l(T^*M))$, $\sigma \in \Gamma^\infty(M, \mathscr{E})$. Any Clifford module admits a Clifford connection, see e.g., [24, Chapter 3.3].

Example 9.1 Let $\mathscr{E} = \wedge_{\mathbb{C}} T^*M$, with the action of $\mathbb{C}l(T^*M)$ defined by the isomorphism $\wedge_{\mathbb{C}} T^*M \cong \mathbb{C}l(T^*M)$ (using the symbol map) and the left-regular representation of the Clifford algebra on itself. The Levi-Civita connection on T^*M induces a Clifford connection on $\wedge_{\mathbb{C}} T^*M$. The resulting Dirac operator is the *de Rham Dirac operator*
$$\mathrm{d} + \mathrm{d}^* :\ \Gamma^\infty(M, \wedge_{\mathbb{C}} T^*M) \to \Gamma^\infty(M, \wedge_{\mathbb{C}} T^*M).$$
Here d^* is the dual of the de Rham differential relative to the metric. See e.g., [24, Proposition 3.53].

Example 9.2 Suppose M is an oriented Riemannian manifold, and let $\mathrm{Fr}_{\mathrm{SO}}(TM)$ be the principal $\mathrm{SO}(N)$-bundle of oriented orthonormal frames. A Spin-*structure on M* can be defined to be a lift of the structure group to the group $\mathrm{Spin}(N)$. That is, it is given by a principal $\mathrm{Spin}(N)$-bundle $\mathscr{P} \to M$ together with an isomorphism of principal bundles,
$$\mathscr{P} \times_{\mathrm{Spin}(N)} \mathrm{SO}(n) \to \mathrm{Fr}_{\mathrm{SO}}(TM).$$
If N is even, let S_0 be the standard spinor module over $\mathbb{C}l(\mathbb{R}^N)$. Then $\mathsf{S} = \mathscr{P} \times_{\mathrm{Spin}(N)} \mathsf{S}_0$ is a Clifford module over $\mathbb{C}l(T^*M)$. The Levi-Civita connection on TM defines a principal connection on $\mathrm{Fr}_{\mathrm{SO}}(TM)$, which pulls back to a principal connection on $\mathscr{P}$ and defines a Clifford connection on S. The corresponding Dirac operator, acting on sections of S, is called the Spin-*Dirac operator*.

A Spin_c-structure on M is defined similarly to a Spin-structure—just replace $\mathrm{Spin}(N)$ with $\mathrm{Spin}_c(N)$. The connection on the principal $\mathrm{Spin}_c(N)$-bundle $\mathscr{P}$ is not canonically determined by the Levi-Civita connection on $\mathrm{Fr}_{\mathrm{SO}}(TM)$, but one can choose a connection that *descends* to the Levi-Civita connection. The resulting Dirac operator is called the Spin_c-*Dirac operator*.

9.3 Dirac operators on homogeneous spaces

Let G be a Lie group, and K a closed subgroup. We will regard G as a G-equivariant principal K-bundle over G/K, with principal action $\mathscr{A}_k(g) = gk^{-1}$. The generating vector fields for the principal action are $\mathscr{A}_\xi = \xi^L,\ \xi \in \mathfrak{k}$.

The choice of a G-invariant principal connection $\theta \in \Omega^1(G, \mathfrak{k})^{G\times K}$ is equivalent to the choice of a K-equivariant splitting of the sequence

$$0 \to \mathfrak{k} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{k} \to 0,$$

i.e., to the choice of a K-invariant complement $\mathfrak{p} \subseteq \mathfrak{g}$. Given such a complement, the connection 1-form is

$$\theta = \mathrm{pr}_{\mathfrak{k}}\theta^L,$$

where $\mathrm{pr}_{\mathfrak{k}}:\ \mathfrak{g} \to \mathfrak{k}$ denotes the projection along $\mathfrak{p}$. For the curvature $F^\theta = \mathrm{d}\theta + \frac{1}{2}[\theta, \theta]$ one finds (using $\mathrm{d}\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0$, and decomposing into $\mathfrak{k}$-parts and $\mathfrak{p}$-parts) that

$$F^\theta = -\frac{1}{2}\mathrm{pr}_{\mathfrak{k}}[\mathrm{pr}_{\mathfrak{p}}\theta^L, \mathrm{pr}_{\mathfrak{p}}\theta^L].$$

As above, the G-invariant connection θ induces G-invariant linear connections ∇ on all associated vector bundles $G \times_K V$. As a special case, we may take $V = \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$. We will call the resulting connection on

$$T(G/K) = G \times_K \mathfrak{p}$$

the *natural connection*; it will be denoted ∇^{nat}. In the case $K = \{e\}$, this is the connection ∇^{nat} defined by left-trivialization of the tangent bundle.

The function $\widetilde{X} \in C^\infty(G, \mathfrak{p})$ defined by a vector field

$$X \in \mathfrak{X}(G/K) \cong \Gamma^\infty(G/K,\ G \times_K \mathfrak{p})$$

(cf. Section 9.2.2) is related to the horizontal lift $\mathrm{hor}(X) \in \mathfrak{X}(G)$ by

$$\widetilde{X} = \iota(\mathrm{hor}(X))\theta^L.$$

That is, $\mathrm{hor}(X)$ is the vector field corresponding to $\widetilde{X}$ under the identification $\mathfrak{X}(G) \cong C^\infty(G, \mathfrak{g})$ given by left-trivialization. The formula for the natural G-invariant connection on G/K reads as

$$\widetilde{\nabla^{\mathrm{nat}}_X Y} = \mathrm{hor}(X)\widetilde{Y}, \quad X, Y \in \mathfrak{X}(G/K).$$

Lemma 9.2 *The identification $\mathfrak{X}(G/K) \cong C^\infty(G, \mathfrak{p})^K,\ X \to \widetilde{X}$ takes the Lie bracket on vector fields to*

$$\widetilde{[X, Y]} = \mathrm{hor}(X)\tilde{Y} - \mathrm{hor}(Y)\tilde{X} + \mathrm{pr}_{\mathfrak{p}}[\tilde{X}, \tilde{Y}].$$

Proof Using the definition, we calculate

$$\begin{aligned}\widetilde{[X, Y]} &= \iota(\mathrm{hor}([X, Y]))\theta^L \\ &= \iota([\mathrm{hor}(X), \mathrm{hor}(Y)])\theta^L + F^\theta(\mathrm{hor}(X), \mathrm{hor}(Y)) \\ &= \mathrm{hor}(X)\tilde{Y} - \mathrm{hor}(Y)\tilde{X} + [\tilde{X}, \tilde{Y}] + F^\theta(\mathrm{hor}(X), \mathrm{hor}(Y)) \\ &= \mathrm{hor}(X)\tilde{Y} - \mathrm{hor}(Y)\tilde{X} + \mathrm{pr}_\mathfrak{p}[\tilde{X}, \tilde{Y}]\end{aligned}$$

for all $X, Y \in \mathfrak{X}(G/K)$. □

Lemma 9.3 *The torsion of the connection* ∇^{nat} *is given by* $\widetilde{T(X, Y)} = -\mathrm{pr}_\mathfrak{p}[\widetilde{X}, \widetilde{Y}]$.

Proof We have

$$\widetilde{\nabla^{\mathrm{nat}}_X Y} - \widetilde{\nabla^{\mathrm{nat}}_Y X} - \widetilde{[X, Y]} = \mathrm{hor}(X)\widetilde{Y} - \mathrm{hor}(Y)\widetilde{X} - \widetilde{[X, Y]} = -\mathrm{pr}_\mathfrak{p}[\widetilde{X}, \widetilde{Y}]$$

as claimed. □

Since the torsion tensor T for the G-invariant connection ∇^{nat} is G-invariant, we may use it to define a 1-parameter family of G-invariant connections,

$$\nabla^t_X = \nabla^{\mathrm{nat}}_X - tT(X, \cdot).$$

In particular, the value at $t = \frac{1}{2}$ is the torsion-free part of ∇^{nat}.

Suppose now that $\mathfrak{g}$ is a quadratic Lie algebra, $\mathfrak{k}$ a quadratic Lie subalgebra, and $\mathfrak{p} = \mathfrak{k}^\perp$. The pseudo-Riemannian metric on G descends to a left-invariant pseudo-Riemannian metric on G/K, and one observes that the connections ∇^t are all metric connections. In particular, $\nabla^{1/2}$ is the Levi-Civita connection ∇^{met}, and the family of connections may be written as

$$\nabla^t = (1 - 2t)\nabla^{\mathrm{nat}} + 2t\nabla^{\mathrm{met}}.$$

For $\xi \in \mathfrak{p}$, let $\mathrm{ad}^\mathfrak{p}_\xi : \mathfrak{p} \to \mathfrak{p}$ be the skew-adjoint map $\zeta \mapsto \mathrm{pr}_\mathfrak{p}[\xi, \zeta]$. Then

$$\widetilde{\nabla^t_X(Y)} = \mathrm{hor}(X)\widetilde{Y} + t\mathrm{ad}^\mathfrak{p}_{\tilde{X}}(\tilde{Y}).$$

Let $\gamma^\mathfrak{p}(\xi) \in \mathbb{C}l(\mathfrak{p})$ be the Clifford algebra element corresponding to $\mathrm{ad}^\mathfrak{p}_\xi$ (cf. Section 7.4).

If V is a K-equivariant $\mathbb{C}l(\mathfrak{p})$-module, define a connection on $\mathscr{E} = G \times_K V$ by

$$\widetilde{\nabla^{\mathscr{E},t}_X \sigma} = \mathrm{hor}(X)\widetilde{\sigma} + t\rho(\gamma^\mathfrak{p}(\tilde{X}))\widetilde{\sigma}.$$

It may be shown to be a Clifford connection (but we will not need this fact.) We compute the resulting family of Dirac operators in terms of the identification $\Gamma^\infty(G/K, \mathscr{E}) \cong C^\infty(G, V)^K$, $\sigma \mapsto \widetilde{\sigma}$. Let e_a be a basis of $\mathfrak{g}$, given by a basis of $\mathfrak{k}$ followed by a basis of $\mathfrak{p}$. Let e^a be the B-dual basis.

Proposition 9.3 *The family of Dirac operators* $\partial\!\!\!/^t$ *is given by the formula*

$$\widetilde{\partial\!\!\!/^t \sigma} = \Big(\sum\nolimits_a^{(\mathfrak{p})} \rho(e^a)e_a^L + 3t\rho(q(\phi_\mathfrak{p}))\Big)\widetilde{\sigma},$$

where the superscript indicates summation over the basis of $\mathfrak{p}$.

Proof Let u_a be a local frame for $T(G/K)$, with B-dual frame u^a. Then $\widetilde{u^a} \in C^\infty(G, \mathfrak{p})^K$. Equation (9.8) gives

$$\begin{aligned}\widetilde{\partial\!\!\!/^t\sigma} &= \sum_a \rho(\widetilde{u^a})\widetilde{\nabla^{\mathscr{E},t}_{u_a}\sigma} \\ &= \sum_a \rho(\widetilde{u^a})\big(\mathrm{hor}(u_a) + t\rho(\gamma^{\mathfrak{p}}(\widetilde{u_a}))\big)\widetilde{\sigma}.\end{aligned}$$

Here $\mathrm{hor}(u_a) \in \mathfrak{X}(G)$ is the horizontal lift relative to the natural connection, thus $\iota(\mathrm{hor}(u_a))\theta^L = \widetilde{u_a}$. The functions $\widetilde{u_a}$ form a local frame for the trivial vector bundle $G \times \mathfrak{p}$, with B-dual frame $\widetilde{u^a}$. The expression above is independent of the choice of such a frame. Thus, we may replace $\widetilde{u_a}$ with a basis e_a of $\mathfrak{p}$ (regarded as constant functions), and $\widetilde{u^a}$ with its B-dual basis e^a. The vector fields $\mathrm{hor}(u_a)$ are replaced by the left-invariant vector fields e_a^L. The formula becomes

$$\begin{aligned}\widetilde{\partial\!\!\!/^t\sigma} &= \sum\nolimits_a^{(\mathfrak{p})} \rho(e^a)\big(e_a^L + t\rho(\gamma^{\mathfrak{p}}(e_a))\big)\widetilde{\sigma} \\ &= \big(\sum\nolimits_a^{(\mathfrak{p})} \rho(e^a)e_a^L + 3t\rho(q(\phi_{\mathfrak{p}}))\big)\widetilde{\sigma},\end{aligned}$$

proving the proposition. □

Comparing with the expression for the relative Dirac operator

$$\mathscr{D}(\mathfrak{g}, \mathfrak{k}) = \sum\nolimits_a^{(\mathfrak{p})} \widehat{e_a}e^a + q(\phi_{\mathfrak{p}}) \in (U(\mathfrak{g}) \otimes \mathbb{C}l(\mathfrak{p}))^K,$$

(cf. Chapter 7, Eq. (7.14)), we note the following:

Theorem 9.1 *The image of* $\mathscr{D}(\mathfrak{g}, \mathfrak{k})$ *under the map*

$$(U(\mathfrak{g}) \otimes \mathbb{C}l(\mathfrak{p}))^K \to \mathfrak{D}(G/K,\ G \times_K V)$$

is the geometric Dirac operator for the connection $\nabla^{1/3} = \frac{1}{3}\nabla^{\mathrm{nat}} + \frac{2}{3}\nabla^{\mathrm{met}}$.

Thus, $\mathscr{D}(\mathfrak{g}, \mathfrak{k})$ is represented as the geometric Dirac operator $\partial\!\!\!/^{1/3}$.

Remark 9.2 Dirac operators on homogeneous spaces are discussed in a number of references, such as Ikeda [69]. The family of geometric Dirac operators $\partial\!\!\!/^t$ was first considered by Slebarski [114, 115], who also observed that the square of this operator is given by a simple formula if $t = 1/3$. The identification of $\partial\!\!\!/^{1/3}$ with the algebraic Dirac operator was observed by Goette [54] and Agricola [2].

Chapter 10
The Hopf–Koszul–Samelson Theorem

Suppose $\mathfrak{g}$ is a complex reductive Lie algebra. The Hopf–Koszul–Samelson Theorem (Theorem 10.2 below) identifies the space of invariants in the exterior algebra $(\wedge\mathfrak{g})^{\mathfrak{g}}$ with the exterior algebra over its subspace of primitive elements. The proof makes extensive use of Lie algebra homology and cohomology.

10.1 Lie algebra cohomology

Let $\mathfrak{g}$ be a complex Lie algebra of dimension $n = \dim\mathfrak{g}$. Fix a basis $e_a,\ a = 1,\ldots,n$ of $\mathfrak{g}$, with dual basis $e^a,\ a = 1,\ldots,n$ of $\mathfrak{g}^*$.

Recall that the Lie algebra cohomology $H^\bullet(\mathfrak{g})$ is the cohomology of the Chevalley–Eilenberg complex $(\wedge\mathfrak{g}^*, \mathrm{d})$ (cf. Section 6.7; we will write d in place of $\mathrm{d}_\wedge$ unless there is the risk of confusion). In terms of the basis

$$\mathrm{d} = \frac{1}{2}\sum_{a=1}^{n} e^a \circ L(e_a), \tag{10.1}$$

where $L(\xi),\ \xi \in \mathfrak{g}$ are the generators for the coadjoint action of $\mathfrak{g}$ on $\wedge\mathfrak{g}^*$. As a consequence, all invariant elements are cocycles. This defines a morphism of graded superalgebras

$$(\wedge^\bullet\mathfrak{g}^*)^{\mathfrak{g}} \to H^\bullet(\mathfrak{g}). \tag{10.2}$$

We will show that for reductive Lie algebras, this map is an isomorphism. The proof will use that reductive Lie algebras are unimodular.

Definition 10.1 The (infinitesimal) *modular character* of a Lie algebra $\mathfrak{g}$ is the Lie algebra morphism

$$\kappa:\ \mathfrak{g} \to \mathbb{C},\ \xi \mapsto \mathrm{tr}(\mathrm{ad}_\xi).$$

A Lie algebra is called *unimodular* if its modular character vanishes.

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Remarks 10.1

1. κ is a Lie algebra morphism by the calculation,

$$\kappa([\xi,\zeta]) = \mathrm{tr}(\mathrm{ad}_{[\xi,\zeta]}) = \mathrm{tr}([\mathrm{ad}_\xi, \mathrm{ad}_\zeta]) = 0.$$

2. If $\mathfrak{g}$ is a quadratic Lie algebra, with bilinear form B, then the operator ad_ξ, $\xi \in \mathfrak{g}$ is skew-adjoint relative to B. Hence, its trace is zero. It follows that quadratic Lie algebras, and in particular reductive Lie algebras, are unimodular.
3. By definition of the trace, $\kappa(\xi)$ is the scalar by which the derivation extension of ad_ξ acts on the top exterior power $\det(\mathfrak{g}) = \wedge^n \mathfrak{g}$. Dually,

$$L(\xi)\phi = -\kappa(\xi)\phi, \quad \phi \in \det(\mathfrak{g}^*).$$

4. In terms of the basis, $\kappa(\xi) = \sum_{a=1}^n \langle e^a, \mathrm{ad}_\xi e_a\rangle = \sum_{a=1}^n \langle L(e_a)e^a, \xi\rangle$. This gives the formula

$$\kappa = \sum_{a=1}^n L(e_a)e^a. \tag{10.3}$$

For any $\mathfrak{g}$-representation V, we will denote by $\mathfrak{g}.V$ the image of the action map $\mathfrak{g} \times V \to V$.

Lemma 10.1 *If $\mathfrak{g}$ is unimodular, the formula for the Lie algebra differential can be written*

$$\mathrm{d} = \frac{1}{2}\sum_{a=1}^n L(e_a) \circ e^a.$$

In particular, $\mathrm{ran}(\mathrm{d}) \subseteq \mathfrak{g}.\wedge\mathfrak{g}$.

Proof Permuting e^a and $L(e_a)$ in (10.1), and using (10.3), we obtain

$$\mathrm{d} = -\frac{1}{2}\kappa + \frac{1}{2}\sum_{a=1}^n L(e_a) \circ e^a.$$

For unimodular Lie algebras, the first term is zero. □

Suppose now that $\mathfrak{g}$ is a *reductive* Lie algebra. Then

$$\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}], \tag{10.4}$$

where $\mathfrak{z}$ is the center of $\mathfrak{g}$, and the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is the semisimple part of $\mathfrak{g}$. We will fix an invariant non-degenerate symmetric bilinear form B on $\mathfrak{g}$. (For instance, take the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ and any non-degenerate symmetric bilinear form on $\mathfrak{z}$.) One of the key properties of semisimple Lie algebras $\mathfrak{g}$ is *Weyl's Theorem* on the complete reducibility of finite-dimensional $\mathfrak{g}$-representations. (Cf. Appendix B, Section B.1.) For reductive Lie algebras $\mathfrak{g}$, the complete reducibility of

$\mathfrak{g}$-representations V holds, provided the representation of the center $\mathfrak{z}$ is diagonalizable. It then follows that every invariant subspace of V admits an invariant complement. In particular

$$V = V^{\mathfrak{g}} \oplus \mathfrak{g}.V. \tag{10.5}$$

The decomposition (10.4) is a special case, taking $V = \mathfrak{g}$ to be the adjoint representation.

Proposition 10.1 *If $\mathfrak{g}$ is a reductive Lie algebra, the map $(\wedge\mathfrak{g}^*)^{\mathfrak{g}} \to H^\bullet(\mathfrak{g})$ (cf. (10.2)) is an isomorphism of graded super algebras.*

Proof Consider (10.5) for $V = \ker(\mathrm{d})$. Since d vanishes on invariants, we have $\ker(\mathrm{d})^{\mathfrak{g}} = (\wedge\mathfrak{g}^*)^{\mathfrak{g}}$. On the other hand, $\mathfrak{g}.\ker(\mathrm{d}) \subseteq \mathrm{ran}(\mathrm{d})$ by the formula $L(\xi) = [\mathrm{d}, \iota(\xi)]$, while Lemma 10.1 gives $\mathrm{ran}(\mathrm{d}) \subseteq \mathfrak{g}. \wedge \mathfrak{g}^* \cap \ker(\mathrm{d}) = \mathfrak{g}.\ker(\mathrm{d})$, since $\ker(\mathrm{d}) \subseteq \wedge\mathfrak{g}^*$ admits an invariant complement. Thus $\mathrm{ran}(\mathrm{d}) = \mathfrak{g}.\ker(\mathrm{d})$. The resulting decomposition

$$\ker(\mathrm{d}) = (\wedge\mathfrak{g}^*)^{\mathfrak{g}} \oplus \mathrm{ran}(\mathrm{d}) \tag{10.6}$$

proves $H^\bullet(\mathfrak{g}) = (\wedge\mathfrak{g}^*)^{\mathfrak{g}}$. □

10.2 Lie algebra homology

10.2.1 Definition and basic properties

Let $\mathfrak{g}$ be a complex Lie algebra. Recall (cf. Section 2.1.5) that the pairing between elements $\psi \in \wedge\mathfrak{g}^*$ and $\chi \in \wedge\mathfrak{g}$ is given by

$$\langle\psi, \chi\rangle = (\iota(\psi^\top)\chi)_{[0]}.$$

Dual to the differential d on $\wedge\mathfrak{g}^*$ there is a differential $\partial = -\mathrm{d}^*$ of degree -1 on $\wedge\mathfrak{g}$. That is,

$$\langle\psi, \partial\chi\rangle = -\langle \mathrm{d}\psi, \chi\rangle$$

for $\psi \in \wedge\mathfrak{g}^*,\ \chi \in \wedge\mathfrak{g}$. The homology $H_\bullet(\mathfrak{g}) := H(\wedge\mathfrak{g}, \partial)$ is called the *Lie algebra homology* of $\mathfrak{g}$. (The subscript indicates the grading, but is mainly used here to distinguish the homology from the cohomology.) By dualizing (10.1), and using that the operator $L(\xi) = -\mathrm{ad}_\xi^*$ on $\wedge\mathfrak{g}^*$ is dual to $L(\xi) = \mathrm{ad}_\xi$ on $\wedge\mathfrak{g}$, one obtains

$$\partial = \frac{1}{2}\sum_{a=1}^{n} L(e_a) \circ \iota(e^a). \tag{10.7}$$

Remark 10.2 Note that for $\xi, \zeta \in \mathfrak{g}$,

$$\partial\xi = 0, \quad \partial(\xi \wedge \zeta) = [\xi, \zeta].$$

Hence $\partial(\wedge^1\mathfrak{g})=0,\ \ \partial(\wedge^2\mathfrak{g})=[\mathfrak{g},\mathfrak{g}]$, and therefore

$$H_0(\mathfrak{g})=\mathbb{C},\quad H_1(\mathfrak{g})=\mathfrak{g}/[\mathfrak{g},\mathfrak{g}].$$

In particular, $H_1(\mathfrak{g})=0$ if and only if $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$.

Remark 10.3 If $\mathfrak{g}$ is *unimodular*, e.g., reductive, the formula for ∂ may be written

$$\partial=\frac{1}{2}\sum_{a=1}^{n}\iota(e^a)\circ L(e_a),$$

by dualizing the formula from Lemma 10.1. This shows that $(\wedge\mathfrak{g})^{\mathfrak{g}}\subseteq\ker(\partial)$, dual to the formula $\mathrm{ran}(\mathrm{d})\subseteq\mathfrak{g}.\wedge\mathfrak{g}^*$. Since $\det(\mathfrak{g})\subseteq(\wedge\mathfrak{g})^{\mathfrak{g}}$, by definition of a unimodular Lie algebra, it follows in particular that $\dim H_n(\mathfrak{g})=1$.

Since ∂ is defined by duality to d, there is a canonical isomorphism

$$H_\bullet(\mathfrak{g})=\frac{\ker(\partial)}{\mathrm{ran}(\partial)}=\frac{\mathrm{ann}(\mathrm{ran}(\mathrm{d}))}{\mathrm{ann}(\ker(\mathrm{d}))}=\left(\frac{\ker(\mathrm{d})}{\mathrm{ran}(\mathrm{d})}\right)^*=(H^\bullet(\mathfrak{g}))^*.$$

For all degrees k this defines a duality pairing between $H^k(\mathfrak{g})$ and $H_k(\mathfrak{g})$.

Lemma 10.2 *For $\phi\in\wedge^k\mathfrak{g}^*,\ \chi\in\wedge^l\mathfrak{g}$,*

$$\partial(\iota(\phi)\chi)=\iota(\mathrm{d}\phi)\chi+(-1)^k\iota(\phi)\partial\chi.$$

Proof By (10.7), we have

$$\begin{aligned}\partial\circ\iota(\phi)&=(-1)^k\iota(\phi)\circ\partial+\frac{1}{2}(-1)^k\sum_{a=1}^{n}\iota(L(e_a)\phi)\iota(e^a)\\&=(-1)^k\iota(\phi)\circ\partial+\iota(\mathrm{d}\phi),\end{aligned}$$

as desired. □

Proposition 10.2 *The contraction $\iota:\ \wedge^k\mathfrak{g}^*\times\wedge_l\mathfrak{g}\to\wedge_{l-k}\mathfrak{g}$ descends to a map in Lie algebra (co)homology*:

$$\cap:\ H^k(\mathfrak{g})\times H_l(\mathfrak{g})\to H_{l-k}(\mathfrak{g}).$$

Proof Lemma 10.2 shows that

$$\begin{aligned}\iota(\ker(\mathrm{d}))\ker(\partial)&\subseteq\ker(\partial),\\\iota(\mathrm{ran}(\mathrm{d}))\ker(\partial)&\subseteq\mathrm{ran}(\partial),\\\iota(\ker(\mathrm{d}))\mathrm{ran}(\partial)&\subseteq\mathrm{ran}(\partial).\end{aligned}$$

Consequently, the contraction descends to (co)homology, as claimed. □

Proposition 10.3 *Suppose* $\mathfrak{g}$ *is a unimodular of dimension* n, *so that* $\dim H_n(\mathfrak{g}) = 1$. *Then the contraction defines a* Poincaré duality isomorphism

$$H^k(\mathfrak{g}) \otimes H_n(\mathfrak{g}) \xrightarrow{\cong} H_{n-k}(\mathfrak{g}).$$

In particular, $\dim H^k(\mathfrak{g}) = \dim H^{n-k}(\mathfrak{g})$ *for all* $k = 0, \ldots, n$, *and similarly in homology.*

Proof Fix a generator $\Gamma \in \wedge^n \mathfrak{g}$. Since $\partial \Gamma = 0$, Lemma 10.2 gives the formula, for $\phi \in \wedge^k \mathfrak{g}^*$,

$$\partial(\iota(\phi)\Gamma) = \iota(\mathrm{d}\phi)\Gamma.$$

Hence the isomorphism $\wedge \mathfrak{g}^* \to \wedge \mathfrak{g}$, $\phi \mapsto \iota(\phi)\Gamma$ takes cocycles to cycles and coboundaries to boundaries. □

10.2.2 Schouten bracket

The differential ∂ is not a derivation of the wedge product on $\wedge \mathfrak{g}$. The discrepancy involves the Schouten bracket.

Proposition 10.4 *The Lie bracket on* $\mathfrak{g}$ *extends uniquely to a bilinear bracket* $[\cdot, \cdot]$ *on the exterior algebra* $\wedge \mathfrak{g}$, *in such a way that the following two properties are satisfied*:

(i) *For homogeneous elements* $\alpha, \beta \in \wedge \mathfrak{g}$,

$$[\alpha, \beta] = -(-1)^{|\alpha|-1}(-1)^{|\beta|-1}[\beta, \alpha].$$

(ii) *For homogeneous elements* $\alpha, \beta, \gamma \in \wedge \mathfrak{g}$,

$$[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(|\alpha|-1)|\beta|} \alpha \wedge [\beta, \gamma].$$

In the basis,

$$[\alpha, \beta] = \sum_{a,b=1}^{n} [e_a, e_b] \wedge \iota(e^a)\alpha \wedge \iota(e^b)\beta. \tag{10.8}$$

Proof The derivation property (ii) shows that $[\cdot, \cdot]$ is uniquely determined by its restriction to $\wedge \mathfrak{g} \times \mathfrak{g}$. Using property (i), it is uniquely determined by its restriction to $\mathfrak{g} \times \wedge \mathfrak{g}$, and using (ii) again it is uniquely determined by its restriction to $\mathfrak{g} \times \mathfrak{g}$. On the other hand, (10.8) clearly satisfies (i), (ii). □

Definition 10.2 The bracket $[\cdot, \cdot]$ described in Proposition 10.4 is called the *Schouten bracket* on $\wedge \mathfrak{g}$.

If there is risk of confusion (with other brackets, or commutators), we will denote the Schouten bracket by $[\cdot, \cdot]_{\wedge \mathfrak{g}}$. Recall that $[n]$ stands for a degree shift of a graded super vector space (cf. Appendix A).

Proposition 10.5 *The Schouten bracket satisfies the graded Jacobi identity*

$$[\alpha, [\beta, \gamma]] = [[\alpha, \beta], \gamma] + (-1)^{(|\alpha|-1)(|\beta|-1)}[\alpha, [\beta, \gamma]].$$

It therefore makes $\wedge\mathfrak{g}[1]$ *into a graded super Lie algebra.*

This may be proved, for example, by induction on degrees. We leave the details to the reader. The signs in Property (ii) of Proposition 10.4 are such that the Schouten bracket defines a morphism of graded super Lie algebras,

$$\wedge\mathfrak{g}[1] \to \mathrm{Der}_{\mathrm{alg}}(\wedge\mathfrak{g}).$$

Let us also note the following properties of the Schouten bracket on $\wedge\mathfrak{g}$.

Lemma 10.3

(i) *For all* $\alpha, \beta \in \wedge\mathfrak{g}$,

$$\begin{aligned}[\alpha, \beta] &= -\sum_{a=1}^{n} L(e_a)\alpha \wedge \iota(e^a)\beta \\ &= 2\alpha \wedge \partial\beta - \sum_{a=1}^{n} L(e_a)(\alpha \wedge \iota(e^a)\beta).\end{aligned}$$

(ii) *If* α *or* β *lies in* $\ker(\partial)$, *then* $[\alpha, \beta] \in \mathfrak{g}. \wedge \mathfrak{g}$.
(iii) *If* α *or* β *lies in* $(\wedge\mathfrak{g})^{\mathfrak{g}}$, *then* $[\alpha, \beta] = 0$.

Proof (i) The first formula is rewriting (10.8) using $\sum_{b=1}^{n}[e_a, e_b] \circ \iota(e^b) = L(e_a)$. In terms of the formula for ∂, the second formula reads as

$$[\alpha, \beta] = \sum_{a=1}^{n} \alpha \wedge L(e_a)\iota(e^a)\beta - \sum_{a=1}^{n} L(e_a)(\alpha \wedge \iota(e^a)\beta),$$

which is clearly equivalent to the first. (ii) The second formula in (i) shows that if $\partial\beta = 0$, then $[\alpha, \beta] \in \mathfrak{g}. \wedge \mathfrak{g}$. By the skew-symmetry property of the Schouten bracket, the same is true if $\partial\alpha = 0$. (iii) The first formula in (i) shows that if α is invariant, then $[\alpha, \beta] = 0$. By the skew-symmetry of the Schouten bracket, the same is true if β is invariant. □

Proposition 10.6 *The Schouten bracket and the differential* ∂ *on* $\wedge\mathfrak{g}$ *are related by the formula*

$$\partial(\alpha \wedge \beta) - (\partial\alpha) \wedge \beta - (-1)^{|\alpha|}\alpha \wedge \partial\beta = (-1)^{|\alpha|-1}[\alpha, \beta], \tag{10.9}$$

for homogeneous elements $\alpha, \beta \in \wedge\mathfrak{g}$.

Proof By direct computation, using $\partial = \frac{1}{2}\sum_{a=1}^{n} L(e_a) \circ \iota(e^a)$, the left-hand side in (10.9) equals

$$\frac{1}{2}(-1)^{|\alpha|} \sum_{a=1}^{n} L(e_a)\alpha \wedge \iota(e^a)\beta + \frac{1}{2}\sum_{a=1}^{n} \iota(e^a)\alpha \wedge L(e_a)\beta.$$

By Lemma 10.3, the first term is $\frac{1}{2}(-1)^{|\alpha|-1}[\alpha, \beta]$, while the second term is $-\frac{1}{2}(-1)^{(|\alpha|-1)|\beta|}[\beta, \alpha]$. Using the skew-symmetry of the Schouten bracket, this is again equal to $\frac{1}{2}(-1)^{|\alpha|-1}[\alpha, \beta]$. □

Proposition 10.7 *Given $\alpha \in \wedge\mathfrak{g}$, we have the following equality of operators on $\wedge\mathfrak{g}^*$,*

$$[\mathrm{d}, \iota(\alpha)] = \iota(\partial\alpha) + \sum_{a=1}^{n} \iota\big(\iota(e^a)\alpha\big) L(e_a). \tag{10.10}$$

The second term in this formula can be written $(-1)^{|\alpha|} \sum_{b=1}^{n} \iota(L(e_b)\alpha) \circ e^b$. In particular, if $\alpha \in (\wedge\mathfrak{g})^{\mathfrak{g}}$, then

$$[\mathrm{d}, \iota(\alpha)] = 0.$$

Proof The proof is by induction on $|\alpha|$, the case $|\alpha| = 1$ being Cartan's formula. Suppose $\alpha = \xi \wedge \beta$ with $\xi \in \mathfrak{g}$ and $|\beta| = |\alpha| - 1$. By induction, we may assume the formula holds for β. We have

$$\begin{aligned}
[\mathrm{d}, \iota(\alpha)] &= [\mathrm{d}, \iota(\xi)\iota(\beta)] \\
&= L(\xi)\iota(\beta) - \iota(\xi)[\mathrm{d}, \iota(\beta)] \\
&= \iota(L(\xi)\beta) + \iota(\beta)L(\xi) - \iota(\xi)\Big(\iota(\partial\beta) + \sum_{a=1}^{n} \iota\big(\iota(e^a)\beta\big)L(e_a)\Big) \\
&= \iota([\xi, \beta] - \xi \wedge \partial\beta) + \iota(\beta)L(\xi) - \iota(\xi)\sum_{a=1}^{n} \iota\big(\iota(e^a)\beta\big)L(e_a) \\
&= \iota(\partial(\xi \wedge \beta)) + \sum_{a=1}^{n} \iota\big(\iota(e^a)(\xi \wedge \beta)\big)L(e_a).
\end{aligned}$$

Here we used $\partial(\xi \wedge \beta) = [\xi, \beta] - \xi \wedge \partial\beta$, which is a special case of Proposition 10.6. The second term in (10.10) can be rewritten, using

$$L(e_a) = \sum_{b=1}^{n} e^b \circ \iota([e_b, e_a]) = \sum_{b=1}^{n} \iota([e_a, e_b]) \circ e^b$$

as

$$\begin{aligned}
\sum_{a=1}^{n} \iota\big(\iota(e^a)\alpha\big)L(e_a) &= (-1)^{|\alpha|-1} \sum_{a,b=1}^{n} \iota\big([e_a, e_b] \wedge \iota(e^a)\alpha\big) \circ e^b \\
&= (-1)^{|\alpha|} \sum_{b=1}^{n} \iota(L(e_b)\alpha) \circ e^b.
\end{aligned}$$

If α is invariant, this term vanishes. The first term in (10.10) vanishes as well, since invariant elements are ∂-closed. □

The following consequence of Proposition 10.6 will be needed below.

Lemma 10.4 [91] *The wedge product of a cycle with a boundary in* $\wedge\mathfrak{g}$ *lies in* $\mathfrak{g}.\wedge\mathfrak{g}$:

$$\ker(\partial)\wedge\operatorname{ran}(\partial)\subseteq\mathfrak{g}.\wedge\mathfrak{g}.$$

Proof Let $\alpha,\beta\in\wedge\mathfrak{g}$ with $\partial\alpha=0$. The proposition shows that

$$(-1)^{|\alpha|}\alpha\wedge\partial\beta=\partial(\alpha\wedge\beta)-(-1)^{|\alpha|-1}[\alpha,\beta].$$

By the definition of ∂, and using Lemma 10.3, both terms on the right are in $\mathfrak{g}.\wedge\mathfrak{g}$. □

10.3 Lie algebra homology for reductive Lie algebras

We will now return to the case that the Lie algebra $\mathfrak{g}$ is reductive. We will view $\mathfrak{g}$ as the complexification of a compact Lie subalgebra $\mathfrak{g}_{\mathbb{R}}$, and we will fix a positive definite symmetric bilinear form $B_{\mathbb{R}}:\ \mathfrak{g}_{\mathbb{R}}\times\mathfrak{g}_{\mathbb{R}}\to\mathfrak{g}_{\mathbb{R}}$. We will use this bilinear form to identify $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}}^*$, and we will use its complexification $B:\ \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Thus, d and ∂ will be regarded as operators on the same space $\wedge\mathfrak{g}\cong\wedge\mathfrak{g}^*$. The pairing $\langle\cdot,\cdot\rangle:\ \wedge\mathfrak{g}^*\otimes\wedge\mathfrak{g}\to\mathbb{C}$ becomes a non-degenerate symmetric bilinear form on $\wedge\mathfrak{g}$, extending B. Note that $\langle\cdot,\cdot\rangle$ is the complexification of a positive definite form on $\wedge\mathfrak{g}_{\mathbb{R}}$. It follows that $\langle\cdot,\cdot\rangle$ is non-degenerate on every conjugation invariant subspace of $\wedge\mathfrak{g}$, such as for instance the kernel and range of the operators d,∂. In what follows, *orthogonality* will be with respect to this bilinear form. We also obtain a positive definite *Hermitian form* on $\wedge\mathfrak{g}$, extending the Hermitian form on $\mathfrak{g}$, by $\phi\times\psi\mapsto\langle\phi^c,\psi\rangle$.

Remark 10.4 We refrain from denoting this bilinear form on $\wedge\mathfrak{g}$ by B, since this notation will be used in the next chapter for a slightly different extension.

Proposition 10.8 *Let* $\mathfrak{g}$ *be a complex reductive Lie algebra. Then there is an orthogonal "Hodge decomposition"*,

$$\wedge\mathfrak{g}=(\wedge\mathfrak{g})^{\mathfrak{g}}\oplus\operatorname{ran}(\partial)\oplus\operatorname{ran}(\mathrm{d}).\tag{10.11}$$

Here

$$\begin{aligned}\ker(\mathrm{d})&=(\wedge\mathfrak{g})^{\mathfrak{g}}\oplus\operatorname{ran}(\mathrm{d}),\\ \ker(\partial)&=(\wedge\mathfrak{g})^{\mathfrak{g}}\oplus\operatorname{ran}(\partial).\end{aligned}\tag{10.12}$$

The space $(\wedge\mathfrak{g})^{\mathfrak{g}}=\ker(\mathrm{d})\cap\ker(\partial)$ *is identified with both Lie algebra homology and cohomology*:

$$H_\bullet(\mathfrak{g})=(\wedge\mathfrak{g})^{\mathfrak{g}}=H^\bullet(\mathfrak{g}).$$

Proof Since ∂ is the negative transpose of d, we have decompositions

$$\wedge\mathfrak{g}=\ker(\mathrm{d})\oplus\operatorname{ran}(\partial)=\ker(\partial)\oplus\operatorname{ran}(\mathrm{d}).\tag{10.13}$$

The decomposition (10.12) for $\ker(\mathrm{d})$ was obtained in (10.6), and we used it to conclude $H^\bullet(\mathfrak{g}) = (\wedge\mathfrak{g})^\mathfrak{g}$. Together with (10.13) it implies the Hodge decomposition. The decomposition (10.12) for $\ker(\partial)$ follows since $\ker(\partial) = \mathrm{ran}(\mathrm{d})^\perp$, and it implies $H_\bullet(\mathfrak{g}) = (\wedge\mathfrak{g})^\mathfrak{g}$. □

Proposition 10.9 *Suppose* $\mathfrak{g}$ *is a complex reductive Lie algebra. There is a canonical* $\mathfrak{g}$*-equivariant homotopy operator* $h \in \mathrm{End}^{-1}(\wedge\mathfrak{g})$ *between the identity operator and the projection* $\pi : \wedge\mathfrak{g} \to (\wedge\mathfrak{g})^\mathfrak{g}$ *along* $\mathfrak{g}.\wedge\mathfrak{g}$.

Proof The "Laplacian" $[\mathrm{d}, \partial] = (\mathrm{d} + \partial)^2$ has kernel $\ker(\mathrm{d}) \cap \ker(\partial) = (\wedge\mathfrak{g})^\mathfrak{g}$, and it is invertible on $\mathfrak{g}.\wedge\mathfrak{g}$. Let $k \in \mathrm{End}^0(\wedge\mathfrak{g})$ be its inverse on $\mathfrak{g}.\wedge\mathfrak{g}$, extended by 0 on $(\wedge\mathfrak{g})^\mathfrak{g}$. Since $[\mathrm{d}, \partial]$ commutes with d, ∂, the same is true of k. Finally, $h = k \circ \partial$ satisfies $[\mathrm{d}, h] = [\mathrm{d}, \partial] \circ k = 1 - \pi$. It is clear that k is $\mathfrak{g}$-equivariant; hence so is $\mathfrak{g}$. □

For later reference, we also note the following result.

Lemma 10.5 [86, Proposition 33] *Let* $\mathfrak{g}$ *be a complex reductive Lie algebra. Suppose* $\chi \in (\wedge\mathfrak{g})^\mathfrak{g}$.

1. *If* $\phi \in \mathrm{ran}(\mathrm{d})$ *then* $\iota(\phi)\chi \in \mathrm{ran}(\partial)$.
2. *If* $\phi \in \mathrm{ran}(\partial)$ *then* $\iota(\phi)\chi \in \mathrm{ran}(\mathrm{d})$.

Proof

1. Since $\mathrm{ran}(\partial) = \ker(\mathrm{d})^\perp$, and since $\iota(\phi)$ is dual to exterior multiplication by $\phi^\top \in \mathrm{ran}(\mathrm{d})$, the statement is equivalent to $\chi \in (\mathrm{ran}(\mathrm{d}) \wedge \ker(\mathrm{d}))^\perp = \ker(\mathrm{d})^\perp$.
2. Similarly, since $\mathrm{ran}(\mathrm{d}) = \ker(\partial)^\perp$, and since $\iota(\phi)$ is dual to exterior multiplication by $\phi^\top \in \mathrm{ran}(\partial)$, the statement is equivalent to $\chi \in (\mathrm{ran}(\partial) \wedge \ker(\partial))^\perp$. But this follows from $\mathrm{ran}(\partial) \wedge \ker(\partial) \subseteq \mathfrak{g}.\wedge\mathfrak{g}$ (see Lemma 10.4). □

Since ∂ is not a derivation of the super algebra $\wedge\mathfrak{g}$, the Lie algebra homology does not "naturally" inherit a product structure. On the other hand, we can simply use the identification with $(\wedge\mathfrak{g})^\mathfrak{g}$ to define a product.

Definition 10.3 The product on $H_\bullet(\mathfrak{g})$ defined by its identification with $(\wedge\mathfrak{g})^\mathfrak{g}$ is called the *Pontryagin product*.

We will see that despite this somewhat "artificial" construction, the Pontryagin product has good properties. One reason is the following fact. Let

$$\pi : \wedge\mathfrak{g} \to (\wedge\mathfrak{g})^\mathfrak{g} \tag{10.14}$$

be the projection with kernel $\mathfrak{g}.\wedge\mathfrak{g}$.

Lemma 10.6 *For* $\alpha, \beta \in \ker(\partial)$,

$$\pi(\alpha \wedge \beta) = \pi(\alpha) \wedge \pi(\beta).$$

Proof Write $\ker(\partial) = (\wedge\mathfrak{g})^{\mathfrak{g}} \oplus \operatorname{ran}(\partial)$. The formula is obviously true if $\alpha, \beta \in (\wedge\mathfrak{g})^{\mathfrak{g}} \subseteq \ker(\partial)$. But Lemma 10.4 shows that $\pi(\ker(\partial) \wedge \operatorname{ran}(\partial)) \subseteq \pi(\mathfrak{g}. \wedge \mathfrak{g}) = 0$. □

Proposition 10.10 *For any morphism $f : \mathfrak{k} \to \mathfrak{g}$ of reductive Lie algebras, the induced map $H_\bullet(f) : H_\bullet(\mathfrak{k}) \to H_\bullet(\mathfrak{g})$ in Lie algebra homology is an algebra homomorphism.*

Proof We will use the same letter f to denote the algebra morphism $\wedge\mathfrak{k} \to \wedge\mathfrak{g}$ induced by $f : \mathfrak{k} \to \mathfrak{g}$. For any cycle, we use square brackets to denote the corresponding homology class. Let $\alpha, \beta \in (\wedge\mathfrak{k})^{\mathfrak{k}}$ be two invariant cycles, defining classes $[\alpha], [\beta]$. Their images under $H(f)$ are represented by cycles $f(\alpha),\ f(\beta)$, and the image of their product $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$ is represented by $f(\alpha \wedge \beta)$. Note that these elements are not $\mathfrak{g}$-invariant, in general. Using Lemma 10.6, we find that

$$\begin{aligned}
[f(\alpha \wedge \beta)] &= [f(\alpha) \wedge f(\beta)] \\
&= [\pi(f(\alpha) \wedge f(\beta))] \\
&= [\pi(f(\alpha)) \wedge \pi(f(\beta))] \\
&= [\pi(f(\alpha))] \wedge [\pi(f(\beta)] \\
&= [f(\alpha)] \wedge [f(\beta)].
\end{aligned}$$

That is, $H(f)([\alpha] \wedge [\beta]) = H(f)([\alpha]) \wedge H(f)([\beta])$. □

10.3.1 Hopf algebra structure on $(\wedge\mathfrak{g})^{\mathfrak{g}}$

Recall that the exterior algebra $\wedge V$ over any vector space V has the structure of a graded super coalgebra, with comultiplication dual to the product in $\wedge V^*$, and counit the augmentation map. In fact, it is a graded super Hopf algebra for the antipode given by the parity operator. Its space of primitive elements is $V \subseteq \wedge V$, cf. Section 5.4.4.

As a special case $V = \mathfrak{g}$ we have the Hopf algebra $(\wedge\mathfrak{g},\ m_0,\ i_0,\ \Delta_0,\ \varepsilon_0,\ \mathsf{s}_0)$. The structure maps $m_0, i_0, \varepsilon_0, \mathsf{s}_0$ restrict to the product m, unit i, counit (augmentation map) ε, and antipode (parity operator) s on the invariant subalgebra $(\wedge\mathfrak{g})^{\mathfrak{g}}$. However, the coproduct Δ_0 does not restrict, since $\Delta_0((\wedge\mathfrak{g})^{\mathfrak{g}})$ does not lie in $(\wedge\mathfrak{g})^{\mathfrak{g}} \otimes (\wedge\mathfrak{g})^{\mathfrak{g}}$, in general. Nonetheless, we can define a coproduct Δ on $(\wedge\mathfrak{g})^{\mathfrak{g}}$, as the dual of the product on $(\wedge\mathfrak{g}^*)^{\mathfrak{g}} \cong (\wedge\mathfrak{g})^{\mathfrak{g}}$.

Let $\pi : \wedge\mathfrak{g} \to (\wedge\mathfrak{g})^{\mathfrak{g}}$ be the projection along $\mathfrak{g}. \wedge \mathfrak{g}$; its dual is the inclusion $j : (\wedge\mathfrak{g})^{\mathfrak{g}} \to \wedge\mathfrak{g}$. Dualizing $m = \pi \circ m_0 \circ (j \otimes j)$, we find that

$$\Delta = (\pi \otimes \pi) \circ \Delta_0 \circ j.$$

Theorem 10.1 *The space $(\wedge\mathfrak{g})^{\mathfrak{g}}$ has the structure of a graded super Hopf algebra $((\wedge\mathfrak{g})^{\mathfrak{g}}, m, i, \Delta, \varepsilon, \mathsf{s})$.*

Proof Under the identification $(\wedge\mathfrak{g})^{\mathfrak{g}} = H_\bullet(\mathfrak{g})$, the coproduct is the map

$$\Delta_\bullet:\ H_\bullet(\mathfrak{g}) \to H_\bullet(\mathfrak{g}\oplus\mathfrak{g}) = H_\bullet(\mathfrak{g})\otimes H_\bullet(\mathfrak{g})$$

induced by the diagonal inclusion $\mathfrak{g}\to\mathfrak{g}\oplus\mathfrak{g}$. Hence, Proposition 10.10 shows that the coproduct $\Delta_\bullet$ is an algebra morphism. Clearly, the counit ε is an algebra morphism as well. It remains to prove the property of the antipode

$$m\circ(1\otimes \mathsf{s})\circ\Delta = m\circ(\mathsf{s}\otimes 1)\circ\Delta = i\circ\varepsilon.$$

Since $m\circ(\pi\otimes\pi)$ agrees with $\pi\circ m_0$ on cycles (cf. Lemma 10.6), we have

$$\begin{aligned}
m\circ(1\otimes \mathsf{s})\circ\Delta &= m\circ(1\otimes \mathsf{s})\circ(\pi\otimes\pi)\circ\Delta_0\circ j\\
&= m\circ(\pi\otimes\pi)\circ(1\otimes \mathsf{s}_0)\circ\Delta_0\circ j\\
&= \pi\circ m_0\circ(1\otimes \mathsf{s}_0)\circ\Delta_0\circ j\\
&= \pi\circ i_0\circ\varepsilon_0\circ j\\
&= i\circ\varepsilon.
\end{aligned}$$

Similarly for $m\circ(\mathsf{s}\otimes 1)\circ\Delta$. □

The Hopf algebra structure on $H_\bullet(\mathfrak{g})$ is functorial: For any Lie algebra morphism $f:\ \mathfrak{k}\to\mathfrak{g}$, the induced map $H_\bullet(f):\ H_\bullet(\mathfrak{k})\to H_\bullet(\mathfrak{g})$ is a morphism of graded super Hopf algebras. (The main issue is to show that $H_\bullet(f)$ intertwines the products, but this is Proposition 10.10.)

10.4 Primitive elements

Let us introduce the notation $I(\mathfrak{g})$ ("invariants") for the graded super Hopf algebra $(\wedge\mathfrak{g})^{\mathfrak{g}}$. Let $P(\mathfrak{g})\subseteq I(\mathfrak{g})$ denote its space of primitive elements, that is,

$$P(\mathfrak{g}) = \{x\in I(\mathfrak{g})|\ \Delta(x) = x\otimes 1 + 1\otimes x\}.$$

Clearly, $P(\mathfrak{g})$ is a graded subspace. It is invariant under complex conjugation, hence also under the involution $*$, and hence is a quadratic subspace of $I(\mathfrak{g})$. Its orthogonal can be characterized as follows. Let $I^+(\mathfrak{g})$ be the augmentation ideal.

Proposition 10.11 *There is an orthogonal direct sum decomposition,*

$$I^+(\mathfrak{g}) = P(\mathfrak{g})\oplus I^+(\mathfrak{g})^2. \tag{10.15}$$

Proof If $\alpha,\beta\in I^+(\mathfrak{g})$ and $x\in P(\mathfrak{g})$, then

$$\langle x,\alpha\beta\rangle = \langle\Delta(x),\alpha\otimes\beta\rangle = \langle x\otimes 1 + 1\otimes x,\alpha\otimes\beta\rangle = 0.$$

This shows that $P(\mathfrak{g})$ annihilates $I^+(\mathfrak{g})^2$. Conversely, suppose $x\in I^+(\mathfrak{g})$ annihilates $I^+(\mathfrak{g})^2$, so that $\langle\Delta(x),\alpha\otimes\beta\rangle = 0$ for all $\alpha,\beta\in I^+(\mathfrak{g})$. Equivalently, the projection of $\Delta(x)$ to $I^+(\mathfrak{g})\otimes I^+(\mathfrak{g})$ is zero. It follows that $\Delta(x) = x\otimes 1 + 1\otimes x$, so that x is primitive. □

Corollary 10.1 *Suppose $\phi, \chi \in P(\mathfrak{g})$ are primitive elements. Then $\iota(\phi)\chi \in I(\mathfrak{g})$ is a scalar, i.e., lies in $\mathbb{C} = I^0(\mathfrak{g})$.*

Proof Suppose $\phi \in P^k(\mathfrak{g})$ and $\chi \in P_l(\mathfrak{g})$ with $k \leq l$. If $k = l$ the claim is obvious, hence assume $k < l$. For all $\psi \in I^{l-k}(\mathfrak{g})$,

$$\langle \psi, \iota(\phi)\chi \rangle = \langle \phi^\top \wedge \psi, \chi \rangle = 0,$$

by Proposition 10.11. Hence $\iota(\phi)\chi = 0$. □

Proposition 10.12 *One has $P^k(\mathfrak{g}) = 0$ for k even. That is, all nonzero primitive elements in $I(\mathfrak{g})$ have odd degree.*

Proof Proposition 10.11 shows that there are no primitive elements of degree zero. Suppose x is a nonzero primitive element of even degree $r \geq 2$. Let k be the smallest natural number so that $x^k = 0$. Then

$$0 = \Delta(x^k) = \Delta(x)^k = \sum_i \binom{k}{i} x^i \otimes x^{k-i}.$$

The terms on the right-hand side all have different bidegrees, hence they must all vanish individually. In particular $x \otimes x^{k-1} = 0$. Since $x \neq 0$ by assumption, it follows that $x^{k-1} \neq 0$, contradicting our choice of k. □

10.5 Hopf–Koszul–Samelson Theorem

Let $\wedge P(\mathfrak{g})$ be the exterior algebra over the space of primitive elements. Define a grading on this algebra by requiring that the inclusion $P(\mathfrak{g}) \to \wedge P(\mathfrak{g})$ preserve degrees. Put differently, $\wedge P(\mathfrak{g})$ can be thought of as the symmetric algebra over the graded super space $P(\mathfrak{g})$.

Theorem 10.2 (Hopf–Koszul–Samelson) *The algebra morphism*

$$\wedge P(\mathfrak{g}) \to I(\mathfrak{g}) \equiv (\wedge \mathfrak{g})^{\mathfrak{g}},$$

extending the inclusion of $P(\mathfrak{g})$, is an isomorphism of graded super algebras.

Proof To show that the algebra homomorphism is injective, it suffices to show that its restriction to the top exterior power of $P(\mathfrak{g})$ is non-zero. Thus let $x_1, \ldots, x_l \in P(\mathfrak{g})$ be a basis, consisting of elements of homogeneous degrees $k_1 \leq k_2 \leq \cdots \leq k_l$. We have to show that their wedge product in $I(\mathfrak{g})$ is non-zero. Inductively, we show that the wedge product $x_1 \wedge \cdots \wedge x_r$ is non-zero, for all $r = 1, \ldots, l$. (Note that the wedge products in this proof are taken in $I(\mathfrak{g})$; as a consequence of the theorem, this will be equivalent to the wedge product in $\wedge P(\mathfrak{g})$.) In terms of the decomposition (10.15), the component of

$$\begin{aligned}\Delta(x_1 \wedge \cdots \wedge x_r) &= \Delta(x_1)\cdots\Delta(x_r)\\ &= (x_1 \otimes 1 + 1 \otimes x_1)\cdots(x_r \otimes 1 + 1 \otimes x_r)\end{aligned}$$

in $P(\mathfrak{g}) \otimes I(\mathfrak{g})$ is given by

$$\sum_{i=1}^{r}(-1)^{i-1} x_i \otimes (x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_r).$$

In particular, the component in $\mathrm{span}(x_r) \otimes I(\mathfrak{g})$ is

$$(-1)^{r-1} x_r \otimes x_1 \wedge \cdots \wedge x_{r-1}.$$

By induction $x_1 \wedge \cdots \wedge x_{r-1} \neq 0$. We hence see that $\Delta(x_1 \wedge \cdots \wedge x_r) \neq 0$, and thus $x_1 \wedge \cdots \wedge x_r \neq 0$.

We next show that the map $\wedge P(\mathfrak{g}) \to I(\mathfrak{g})$ is surjective. Let $J \subseteq I(\mathfrak{g})$ be the graded subalgebra given as its image, and write $I = I(\mathfrak{g})$. Let $I^+,\ J^+$ be the sums of components of positive degree. From (10.15) we have

$$I^+ = J^+ + (I^+)^2.$$

Squaring this identity, one obtains $(I^+)^2 = (J^+)^2 + (I^+)^3$, hence

$$I^+ = J^+ + (I^+)^3.$$

Proceeding in this manner, one obtains

$$I^+ = J^+ + (I^+)^i$$

for all $i \geq 2$. But $(I^+)^{n+1} = 0$. Hence $I^+ = J^+$, i.e., $J = I(\mathfrak{g})$. □

The operator of contraction by an element $\phi \in \wedge^k \mathfrak{g}$ with $k \geq 2$ is not a derivation of $\wedge\mathfrak{g}$. However, we have:

Proposition 10.13 *If $\phi \in P(\mathfrak{g})$ is a primitive element, then the operator of contraction $\iota(\phi)$ is a derivation of the graded super algebra $I(\mathfrak{g}) \subseteq \wedge\mathfrak{g}$. Put differently, it coincides with contraction by ϕ in the exterior algebra $\wedge P(\mathfrak{g})$.*

Proof Since $\Delta(\phi) = \phi \otimes 1 + 1 \otimes \phi$, the derivation property for $\iota(\phi)$ reads as

$$m \circ \iota(\Delta(\phi)) = \iota(\phi) \circ m.$$

Dualizing, this is equivalent to

$$\Delta(\phi^\top) \circ \Delta = \Delta \circ \phi^\top,$$

as operators $I(\mathfrak{g}) \to I(\mathfrak{g}) \otimes I(\mathfrak{g})$. (Here $\phi^\top \in I(\mathfrak{g})$ acts on $I(\mathfrak{g})$ by left-multiplication, and similar for $\Delta(\phi^\top) \in I(\mathfrak{g}) \otimes I(\mathfrak{g})$.) But this identity holds true since Δ is an algebra morphism. □

10.6 Consequences of the Hopf–Koszul–Samelson Theorem

As observed in Remark 11.1,

$$\dim I(\mathfrak{g}) = \dim(\wedge\mathfrak{g})^{\mathfrak{g}} = 2^{\mathrm{rank}(\mathfrak{g})}.$$

On the other hand, by the Hopf–Koszul–Samelson Theorem, this agrees with $\dim \wedge P(\mathfrak{g})$. This shows:

Corollary 10.2 *The space of primitive elements has dimension*

$$\dim P(\mathfrak{g}) = \mathrm{rank}(\mathfrak{g}).$$

From now on, we will write $l := \mathrm{rank}(\mathfrak{g})$.

Definition 10.4 An integer $m \geq 0$ is called an *exponent* of the reductive Lie algebra $\mathfrak{g}$ if $P^{2m+1}(\mathfrak{g}) \neq 0$.

Let $m_1 \leq \cdots \leq m_l$ be the exponents, where each m is listed with multiplicity $\dim P^{2m+1}(\mathfrak{g})$. Thus, $P(\mathfrak{g})$ has a homogeneous basis $x_1, \ldots, x_l$ with $x_i \in P^{2m_i+1}(\mathfrak{g})$. Recall that the *Poincaré polynomial* of a finite-dimensional non-negatively graded vector space $V^\bullet$ is the polynomial $p(t) = \sum_k t^k \dim V^k$.

Corollary 10.3 *The Poincaré polynomial of the graded algebra $I(\mathfrak{g}) = (\wedge\mathfrak{g})^{\mathfrak{g}}$ has the form*

$$p(t) = \prod_{i=1}^{l} (1 + t^{2m_i+1}),$$

where m_i are the exponents. One has

$$\sum_{i=1}^{l} (2m_i + 1) = n,$$

hence $p(t^{-1}) = t^{-n} p(t)$.

Proof The first assertion follows from the isomorphism $I(\mathfrak{g}) = \wedge P$. The equality $\sum_{i=1}^{l} (2m_i + 1) = n$ follows since the determinant line $\det(\mathfrak{g}) = I^n(\mathfrak{g})$ is 1-dimensional, and hence is spanned by $x_1 \wedge \cdots \wedge x_l \neq 0$. $\square$

Corollary 10.4 *If $\mathfrak{g}$ is semisimple, $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = H^4(\mathfrak{g}) = 0$, while $H^3(\mathfrak{g}) \neq 0$ is isomorphic to the space $P^3(\mathfrak{g})$ of primitive elements of degree 3.*

Proof For a semisimple Lie algebra, $H^1(\mathfrak{g}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* = 0$. Hence, 0 does not appear as an exponent of $\mathfrak{g}$, and the Poincaré polynomial of $I(\mathfrak{g})$ has the form

$$p(t) = 1 + \dim I^3(\mathfrak{g})\, t^3 + \cdots,$$

where the dots indicate terms of degree ≥ 5, and where $\dim I^3(\mathfrak{g})$ equals the multiplicity of the exponent 1. This multiplicity is non-zero, since $I^3(\mathfrak{g})$ contains the cubic element defined by the bilinear form B. □

Concerning the space $H^3(\mathfrak{g}) = (\wedge^3\mathfrak{g})^{\mathfrak{g}}$, one also has the following result. Let

$$(S^2\mathfrak{g})^{\mathfrak{g}} \to (\wedge^3\mathfrak{g})^{\mathfrak{g}} \tag{10.16}$$

be the map taking a bilinear form β to the cubic element $\phi \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$, with

$$\iota(\xi_1)\iota(\xi_2)\iota(\xi_3)\phi = \beta([\xi_1, \xi_2], \xi_3).$$

In Example 6.11, we saw that this is the transgression $\mathsf{t}\colon (S^2\mathfrak{g})^{\mathfrak{g}} \to (\wedge^3\mathfrak{g})^{\mathfrak{g}}$, up to a scalar multiple.

Proposition 10.14 *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the map* (10.16) *is an isomorphism.*

Proof An element $\beta \in (S^2\mathfrak{g})^{\mathfrak{g}}$ lies in the kernel of (10.16) if and only if

$$\beta([\xi_1, \xi_2], \xi_3) = 0, \quad \xi_1, \xi_2, \xi_3 \in \mathfrak{g},$$

i.e., $\beta([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$. Since $H_1(\mathfrak{g}) = 0$, we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. We hence obtain $\beta(\mathfrak{g}, \mathfrak{g}) = 0$, i.e., $\beta = 0$. Suppose conversely that $\phi \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$. Note that the Schouten bracket is symmetric on $\wedge^2\mathfrak{g}$,

$$[w, w'] = [w', w], \quad w \in \wedge^2\mathfrak{g}.$$

Recall also (Proposition 10.6) that $[w, w'] \in \mathfrak{g}.\wedge^3 \mathfrak{g}$ if w or w' lies in $\ker(\partial)$. Since ϕ annihilates $\mathfrak{g}.\wedge^3 \mathfrak{g}$ and since $\partial(\wedge^2\mathfrak{g}) = \mathfrak{g}$, this gives a well-defined symmetric bilinear form

$$\beta(\partial w, \partial w') = -\frac{1}{2}\iota([w, w'])\phi = \iota(w \wedge \partial w')\phi.$$

Taking $w = \xi_1 \wedge \xi_2$, and choosing w' with $\partial w' = \xi_3$, we see that ϕ is the image of β under (10.16). □

Remark 10.5 If $\mathfrak{g}$ is simple, then the space $(S^2\mathfrak{g})^{\mathfrak{g}}$ of symmetric bilinear forms β is easily seen to be 1-dimensional. (Indeed, let B be a fixed non-degenerate invariant symmetric bilinear form B on $\mathfrak{g}$. Then $(B^\flat)^{-1} \circ \beta^\flat \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \cong \mathbb{C}$, so that $\beta^\flat$ is a multiple of $B^\flat$.) For a semisimple Lie algebra, it then follows that $\dim(S^2\mathfrak{g})^{\mathfrak{g}} = \dim(\wedge^3\mathfrak{g})^{\mathfrak{g}}$ is the number of simple summands of $\mathfrak{g}$.

10.7 Transgression Theorem

The primitive subspace $P(\mathfrak{g})$ has an important interpretation in terms of transgression in the Weil algebra $W\mathfrak{g}$. Recall from Section 6.13 that for any Lie algebra $\mathfrak{g}$, the *transgression map*

$$\mathsf{t}\colon\ I_S^+(\mathfrak{g}) \to I^+(\mathfrak{g}), \tag{10.17}$$

where we write $I_S(\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$ and $I(\mathfrak{g}) = (\wedge\mathfrak{g}^*)^{\mathfrak{g}}$, is defined by a commutative diagram

$$\begin{tikzcd} & \mathscr{T} \arrow[dl, "\mathrm{d}_W"'] \arrow[d, "\pi"] \\ I_S^+(\mathfrak{g}) \arrow[r, "\mathsf{t}"'] & I^+(\mathfrak{g}) . \end{tikzcd}$$

Here the space $\mathscr{T} \subseteq (W\mathfrak{g})^{\mathfrak{g}}$ of *cochains of transgression* is the subspace of odd elements x satisfying $\mathrm{d}_W(x) \in I_S^+(\mathfrak{g})$, and $\pi: W\mathfrak{g} \to \wedge\mathfrak{g}^*$ is the projection. We saw that the kernel of t contains $(I_S^+(\mathfrak{g}))^2$. For complex reductive Lie algebras $\mathfrak{g}$ there is the following more precise statement. As before, we use B to identify $\mathfrak{g}^*$ with $\mathfrak{g}$.

Theorem 10.3 (Transgression Theorem) *Suppose $\mathfrak{g}$ is a complex reductive Lie algebra. Then*

$$\ker(\mathsf{t}) = (I_S^+(\mathfrak{g}))^2, \quad \mathrm{ran}(\mathsf{t}) = P(\mathfrak{g}).$$

Remark 10.6 The Transgression Theorem was explicitly stated in Chevalley's ICM address [39], and in H. Cartan's lectures [32]. A proof was also given in J. Leray's article [98].

The argument below follows the expositions in [56] and [86]. We will need several auxiliary results. Since $\mathfrak{g}$ is reductive, invariant elements $\psi \in I(\mathfrak{g})$ satisfy $\partial\psi = 0$. Hence Proposition 10.7 shows that

$$[\mathrm{d}_\wedge, \iota(\psi)] = 0, \quad \psi \in I(\mathfrak{g}),$$

as operators on $\wedge\mathfrak{g}$ (we write the Lie algebra differential as $\mathrm{d}_\wedge$, to avoid confusion with various other differentials encountered below.) The operators $\iota(\xi)$, $L(\xi)$ commute with $\iota(\psi)$ as well. It follows that the space

$$(\wedge\mathfrak{g})_{P\text{-hor}} = \{\phi \in \wedge\mathfrak{g} |\ \iota(\psi)\phi = 0 \text{ for all } \psi \in P(\mathfrak{g})\}$$

is a $\mathfrak{g}$-differential subspace of $\wedge\mathfrak{g}$. The formula for ∂ shows that

$$[\partial, \iota(\psi)] = 0, \quad \psi \in I(\mathfrak{g}).$$

Hence ∂ restricts to an operator on $(\wedge\mathfrak{g})_{P\text{-hor}}$.

Lemma 10.7 *The subcomplex $(\wedge\mathfrak{g})_{P\text{-hor}}$ is acyclic: The augmentation map*

$$\varepsilon: (\wedge\mathfrak{g})_{P\text{-hor}} \to \mathbb{C}$$

induces an isomorphism in cohomology. In fact, there is a $\mathfrak{g}$-equivariant homotopy operator $h \in \mathrm{End}^{-1}((\wedge\mathfrak{g})_{P\text{-hor}})$ such that $[\mathrm{d}_\wedge, h] = 1 - \varepsilon$.

Proof In Proposition 10.9, we constructed a homotopy operator $h \in \mathrm{End}^{-1}(\wedge\mathfrak{g})$ between the identity and projection to the subspace $I(\mathfrak{g}) = (\wedge\mathfrak{g})^{\mathfrak{g}}$. Reexamining the construction, we note that h preserves the subspace $(\wedge\mathfrak{g})_{P\text{-hor}}$ (since both $\mathrm{d}_\wedge$ and ∂ preserve that subspace). It hence defines a homotopy operator between the identity of $(\wedge\mathfrak{g})_{P\text{-hor}}$ and projection to $I(\mathfrak{g})\cap(\wedge\mathfrak{g})_{P\text{-hor}}$. By the Hopf–Koszul–Samelson Theorem 10.2 $I(\mathfrak{g})=\wedge P(\mathfrak{g})$, and since the contractions $\iota(\psi)$, $\psi\in P(\mathfrak{g})$ act as a derivation on $I(\mathfrak{g})$ (Proposition 10.13), this intersection consists of the scalars $\mathbb{C}=I^0(\mathfrak{g})$. □

Consider now the Weil algebra $W\mathfrak{g}=S\mathfrak{g}\otimes\wedge\mathfrak{g}$, where $S\mathfrak{g}$ is the symmetric algebra generated by the "curvature variables".

Proposition 10.15 *Let*

$$(W\mathfrak{g})_{P\text{-hor}}=\{x\in W\mathfrak{g}|\ \iota(\phi)x=0\ \textit{for all}\ \phi\in P(\mathfrak{g})\}.$$

1. *The subspace* $(W\mathfrak{g})_{P\text{-hor}}$ *is invariant under the action of* $\mathfrak{g}$ *on* $W\mathfrak{g}$, *and* $(W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$ *is a differential subspace.*
2. $I(\mathfrak{g})\cap(W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}=\mathbb{C}$.
3. *The Weil differential takes* $P(\mathfrak{g})$ *into cocycles in* $(W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$.
4. *The inclusion of* $I_S(\mathfrak{g})\hookrightarrow(W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$ *defines an isomorphism in cohomology,*

$$I_S(\mathfrak{g})\cong H((W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}},\mathrm{d}_W).$$

Proof

1. The Lie derivatives $L_W(\xi)$, $\xi\in\mathfrak{g}$ commute with contractions $\iota(\psi)$ for $\psi\in P(\mathfrak{g})$, hence they preserve $(W\mathfrak{g})_{P\text{-hor}}$. Recall that $\mathrm{d}_W=\mathrm{d}_K+\mathrm{d}_{CE}$, where $\mathrm{d}_K=\sum_{a=1}^n\widehat{e}^a\iota(e_a)$ is the Koszul differential and
$$\mathrm{d}_{CE}=\sum_a e^aL_S(e_a)+\mathrm{d}_\wedge=\sum_a e^aL_W(e_a)-\mathrm{d}_\wedge,$$
is the Chevalley–Eilenberg differential. Here $L_S(\xi)$ are the Lie derivatives on $S\mathfrak{g}$ (extended to $W\mathfrak{g}$), and $L_W(\xi)=L_S(\xi)+L_\wedge(\xi)$ is the Lie derivative on $W\mathfrak{g}$. On the space $(W\mathfrak{g})^{\mathfrak{g}}$ of *invariant* elements the Weil differential agrees with $\mathrm{d}_K-\mathrm{d}_\wedge$, which commutes with $\iota(\psi)$, $\psi\in P(\mathfrak{g})$. It follows that $(W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$ is a subcomplex of the Weil algebra.
2. As observed above, $(\wedge\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}=\mathbb{C}$.
3. For $\phi,\psi\in P(\mathfrak{g})$,
$$\iota(\psi)\mathrm{d}_W(\phi)=\iota(\psi)\mathrm{d}_K(\phi)=(-1)^{|\psi|}\mathrm{d}_K\iota(\psi)\phi=0,$$
where we used $\mathrm{d}_{CE}(\phi)=0$, and the fact that $\iota(\psi)\phi$ is a scalar (Proposition 10.13). Hence $\mathrm{d}_W(\phi)\in(W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$.
4. Suppose that $b\in(W^r\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$ is a cocycle, and write $b=b_0+\cdots+b_r$ where $b_i\in S\mathfrak{g}\otimes\wedge^i\mathfrak{g}$. Let k be the largest index with $b_k\neq 0$. The equations $\iota(\phi)b=0$ for all $\phi\in P(\mathfrak{g})$ and $\mathrm{d}_Wb=0$ imply $b_k\in(W^r\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$ and $\mathrm{d}_\wedge b_k=0$. If $k>0$, then $c=(1\otimes h)(b_k)$ (where h is the homotopy operator from Lemma 10.7) satisfies $\mathrm{d}_\wedge(c)=b_k$. Note that $c\in W^{r-1}(\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$, while
$$b_k+\mathrm{d}_W(c)=b_k-\mathrm{d}_\wedge(c)+\mathrm{d}_K(c)=\mathrm{d}_K(c)$$

lies in $S\mathfrak{g} \otimes \wedge^{k-1}\mathfrak{g}$. Then $b' = b + \mathrm{d}_W(c)$ has the form $b' = b'_0 + \cdots + b'_{k-1}$ with $b'_i \in S\mathfrak{g} \otimes \wedge^i \mathfrak{g}$. Proceeding in this manner, it follows that b is cohomologous to an element in $I_S(\mathfrak{g}) \subseteq (W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$. This shows that the inclusion $I_S(\mathfrak{g}) \hookrightarrow (W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$ induces a surjective map in cohomology. To see that the map in cohomology is also injective, suppose $p \in I_S(\mathfrak{g})$ satisfies $p = \mathrm{d}_W(b)$ for some $b \in (W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$. Write $b = b_0 + \cdots + b_k$ as above. If $k > 0$, then b_k is $P(\mathfrak{g})$-horizontal and $\mathrm{d}_\wedge$ closed. Hence, by the same inductive argument as above, we may change b by a coboundary to arrange $b \in I_S(\mathfrak{g})$. But d_W vanishes on $I_S(\mathfrak{g})$. It follows that $p = \mathrm{d}_W(b) = 0$. $\square$

We are now in a position to prove the Transgression Theorem.

Proof of Theorem 10.3 Since t vanishes on $(I^+_S(\mathfrak{g}))^2$, it descends to a linear map

$$I^+_S(\mathfrak{g})/(I^+_S(\mathfrak{g}))^2 \to I^+(\mathfrak{g}). \tag{10.18}$$

We have to show that the image of this map is $P(\mathfrak{g})$ and the kernel is zero. The space $I^+_S(\mathfrak{g})/(I^+_S(\mathfrak{g}))^2$ has dimension $l = \mathrm{rank}(\mathfrak{g})$, as a consequence of Chevalley's Theorem 8.1 and Remark 8.2. Since also $\dim P(\mathfrak{g}) = l$, it hence suffices to show that the image of (10.18) contains $P(\mathfrak{g})$. Let $\phi \in P(\mathfrak{g})$ be given. By part (3) of Proposition 10.15, $\mathrm{d}_W(\phi)$ is a cocycle in $(W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$, and by part (4) this cocycle is cohomologous to some $p \in I_S(\mathfrak{g})$. That is, there exists an odd cochain $b \in (W\mathfrak{g})^{\mathfrak{g}}_{P\text{-hor}}$ with

$$p = \mathrm{d}_W(b) + \mathrm{d}_W(\phi) = \mathrm{d}_W(b + \phi).$$

Let $C = b + \phi$. Then $\pi(C) = \phi$, $\mathrm{d}_W(C) = p$, proving $\phi = \mathsf{t}(p)$. $\square$

Remark 10.7 Note that this proof gives an explicit right inverse

$$P(\mathfrak{g}) \to I^+_S(\mathfrak{g})$$

to the transgression map t. Let $\mathscr{T}_{\mathrm{dist}} \subseteq \mathscr{T}$ be the *distinguished cochains of transgression*, consisting of $C \in \mathscr{T}$ with $\iota(\psi)C \in \mathbb{C}$ for all $\psi \in P(\mathfrak{g})$. Then the inverse map is uniquely described by the commutative diagram

$$\begin{tikzcd} & \mathscr{T}_{\mathrm{dist}} \arrow[dl, "\mathrm{d}_W"'] \arrow[d, "\pi"] \\ I^+_S(\mathfrak{g}) & P(\mathfrak{g}) \arrow[l] \end{tikzcd}$$

Chapter 11
The Clifford algebra of a reductive Lie algebra

In this final chapter we will review Kostant's theory [86] of the structure of $\mathrm{Cl}(\mathfrak{g})$ for a complex reductive Lie algebra $\mathfrak{g}$. One of his results (Theorem 11.6) is the "Clifford analogue" of the Hopf–Koszul–Samelson Theorem $(\wedge\mathfrak{g})^{\mathfrak{g}} = \wedge(P(\mathfrak{g}))$, stating that $(\mathrm{Cl}(\mathfrak{g}))^{\mathfrak{g}} = \mathrm{Cl}(P(\mathfrak{g}))$ for a suitably chosen bilinear form on $P(\mathfrak{g})$. Further results include the isomorphism $\gamma(U\mathfrak{g}) \cong \mathrm{End}(V(\rho))$ (Theorem 11.1), the ρ-decomposition $\mathrm{Cl}(\mathfrak{g}) = \mathrm{Cl}(P(\mathfrak{g})) \otimes \mathrm{End}(V(\rho))$ (Theorem 11.3), and the expansion of linear elements $\xi \in \mathfrak{g}$ in terms of the ρ-decomposition (Theorem 11.7 and subsequent corollary). It leads to the remarkable fact (Theorem 11.8) that the Harish-Chandra projection for Clifford algebras takes the quantization of primitive elements $q(P(\mathfrak{g}))$ to the linear subspace $\mathfrak{t} \subseteq \mathrm{Cl}(\mathfrak{t})$. Kostant conjectured a description of the resulting filtration of $\mathfrak{t}$ in terms of the "principal TDS" (Theorem 11.10); this conjecture was established by Joseph [72], in conjunction with work of Alekseev–Moreau [9].

11.1 $\mathrm{Cl}(\mathfrak{g})$ and the ρ-representation

Most of the results in this section are due to Kostant [86]. However, our proofs are rather different. We resume our discussion from Section 8.2.3. As in that section, we denote by $\mathsf{R}^2 = \mathsf{R} \in \mathrm{Cl}(\mathfrak{g})$ the unique projector in the 1-dimensional subspace $\det(\mathfrak{n}_+)\det(\mathfrak{n}_-)$. Its main properties are

$$\mathfrak{n}_+\mathsf{R} = 0, \quad \mathsf{R}\mathfrak{n}_- = 0, \quad \mathsf{hc}_{\mathrm{Cl}}(\mathsf{R} - 1) = 0.$$

We found that

$$\gamma(U(\mathfrak{g}))\mathsf{R} \cong V(\rho)$$

as $\mathfrak{g}$-representations, where $\xi \in \mathfrak{g}$ acts as left-multiplication by $\gamma(\xi)$. One of the goals of this section is to prove,

Theorem 11.1 [86, Section 5.3] *The left-multiplication of $\gamma(U(\mathfrak{g}))$ on $\gamma(U(\mathfrak{g}))\mathsf{R} \cong V(\rho)$ defines an algebra isomorphism*

$$\gamma(U(\mathfrak{g})) \cong \mathrm{End}(V(\rho)).$$

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To prove this result we will regard $\mathrm{Cl}(\mathfrak{g})$ as a $\mathfrak{g}\times\mathfrak{g}$-representation by composing the Lie algebra morphism $\gamma:\ \mathfrak{g}\to\mathrm{Cl}(\mathfrak{g})$ with the left- and right-regular representations of $\mathrm{Cl}(\mathfrak{g})$ on itself. That is,

$$(\xi_1,\xi_2).x=\gamma(\xi_1)x-x\gamma(\xi_2)$$

for $\xi_1,\xi_2\in\mathfrak{g}$ and $x\in\mathrm{Cl}(\mathfrak{g})$. Take the Weyl chamber for $\mathfrak{g}\times\mathfrak{g}$ to be the product $\mathfrak{t}_+\times(-\mathfrak{t}_+)$. The set of positive roots for $\mathfrak{g}\times\mathfrak{g}$ is then

$$\mathfrak{R}_+\times\{0\}\cup\{0\}\times\mathfrak{R}_-,$$

and the space of highest weight vectors for a $\mathfrak{g}\times\mathfrak{g}$-representation is the subspace fixed by $\mathfrak{n}_+\times\mathfrak{n}_-$. Proposition 8.5 shows that the highest weights for the $\mathfrak{g}\times\mathfrak{g}$-action on $\mathrm{Cl}(\mathfrak{g})$ are all equal to $(\rho,-\rho)$. The same is true for all $\mathfrak{g}\times\mathfrak{g}$-subrepresentations, in particular for $\gamma(U(\mathfrak{g}))\subseteq\mathrm{Cl}(\mathfrak{g})$.

Given a finite-dimensional completely reducible $\mathfrak{g}$-representation V, the space $\mathrm{End}(V)$ becomes a $\mathfrak{g}\times\mathfrak{g}$-module by composing $\pi:\ \mathfrak{g}\to\mathrm{End}(V)$ with the left- and right-regular representations of $\mathrm{End}(V)$ on itself:

$$(\xi_1,\xi_2).A=\pi(\xi_1)A-A\pi(\xi_2).$$

If $V(\mu)$ is an irreducible $\mathfrak{g}$-representation of highest weight $\mu\in P_+$, then

$$\mathrm{End}(V(\mu))\cong V(\mu)\otimes V(\mu)^*$$

is an irreducible $\mathfrak{g}\times\mathfrak{g}$-representation of highest weight $(\mu,-\mu)$, and with highest weight vector the orthogonal projection

$$\mathrm{pr}_{V(\mu)^{\mathfrak{n}_+}}\in\mathrm{End}(V(\mu))$$

onto $V(\mu)^{\mathfrak{n}_+}$. We will be interested in the case $\mu=\rho$. As a preliminary version of Theorem 11.1, we have the following.

Lemma 11.1 *The subspace $\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))\subseteq\mathrm{Cl}(\mathfrak{g})$ is a subalgebra. The left-regular representation of this subalgebra on $\gamma(U(\mathfrak{g}))\mathsf{R}\cong V(\rho)$ defines an algebra isomorphism*

$$\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))\cong\mathrm{End}(V(\rho)),$$

taking R *to* $\mathrm{pr}_{V(\rho)^{\mathfrak{n}_+}}$.

Proof Write $U(\mathfrak{g})=U(\mathfrak{n}_-)U(\mathfrak{t})U(\mathfrak{n}_+)$. Using

$$\gamma(U(\mathfrak{n}_-))\subseteq\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g}),\quad \gamma(U(\mathfrak{n}_+))\subseteq\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$$

together with the $\mathfrak{t}$-invariance of R, we have

$$\mathsf{R}\gamma(U(\mathfrak{g}))\mathsf{R}=\mathsf{R}\gamma(U(\mathfrak{t}))\mathsf{R}=\mathsf{R}\gamma(U(\mathfrak{t})).$$

This implies that

$$(\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g})))^2=\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g})),$$

proving that $\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))$ is a subalgebra. The element R is a highest weight vector for the $\mathfrak{g}\times\mathfrak{g}$-action on $\mathrm{Cl}(\mathfrak{g})$, of weight $(\rho,-\rho)$. Hence $\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))\subseteq$

Cl($\mathfrak{g}$) is an irreducible $\mathfrak{g}\times\mathfrak{g}$-representation of highest weight $(\rho,-\rho)$. The action on $\gamma(U(\mathfrak{g}))\mathsf{R}$ by left-multiplication defines a $\mathfrak{g}\times\mathfrak{g}$-equivariant algebra morphism $\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))\to\mathrm{End}(V(\rho))$. Since both sides are irreducible $\mathfrak{g}\times\mathfrak{g}$-representations, this map is an isomorphism. □

Lemma 11.2 *The subspace*

$$\mathrm{Cl}(\mathfrak{t})\mathsf{R}\subseteq\mathrm{Cl}(\mathfrak{g})$$

is the space of highest weight vectors for the $\mathfrak{g}\times\mathfrak{g}$*-action on* Cl($\mathfrak{g}$).

Proof Since $\gamma(\mathfrak{n}_-)\subseteq\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})$, the space $\mathrm{Cl}(\mathfrak{t})\mathsf{R}$ is annihilated by right multiplication by elements of $\gamma(\mathfrak{n}_-)$, and since $\mathrm{Cl}(\mathfrak{t})\mathsf{R}=\mathsf{R}\,\mathrm{Cl}(\mathfrak{t})$ and $\gamma(\mathfrak{n}_+)\subseteq\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$ it is also annihilated by left-multiplication by elements of $\gamma(\mathfrak{n}_+)$. It is thus contained in the space of highest weight vectors for the $\mathfrak{g}\times\mathfrak{g}$ action on Cl($\mathfrak{g}$). The $\mathfrak{g}\times\mathfrak{g}$-representation generated by this subspace has dimension

$$\dim(\mathrm{Cl}(\mathfrak{t}))\dim(\mathrm{End}(V(\rho)))=2^{\dim\mathfrak{t}}2^{2|\mathfrak{R}_+|}=2^{\dim\mathfrak{g}}=\dim(\mathrm{Cl}(\mathfrak{g})).$$

This establishes that $\mathrm{Cl}(\mathfrak{t})\mathsf{R}$ is the entire space of highest weight vectors. □

As a consequence there is a unique isomorphism of $\mathfrak{g}\times\mathfrak{g}$-representations

$$\mathrm{Cl}(\mathfrak{t})\otimes\mathrm{End}(V(\rho))\to\mathrm{Cl}(\mathfrak{g}),\tag{11.1}$$

taking $x\otimes\mathrm{pr}_{V(\mu)^{\mathfrak{n}_+}}$ to $x\mathsf{R}$. Consider the diagonal $\mathfrak{g}$-action. By Schur's Lemma, the space $\mathrm{End}(V(\rho))^{\mathfrak{g}}$ is 1-dimensional, and is spanned by $\mathrm{id}_{V(\rho)}$. The diagonal $\mathfrak{g}$-action on Cl($\mathfrak{g}$) is just the adjoint action. Hence, there is an isomorphism of vector spaces

$$\mathrm{Cl}(\mathfrak{t})\cong\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}},\tag{11.2}$$

taking x to the image of $x\otimes\mathrm{id}_{V(\rho)}$ under (11.1).

Remark 11.1 Using the symbol map (11.2), this also shows $(\wedge\mathfrak{g})^{\mathfrak{g}}\cong\wedge\mathfrak{t}$. In particular, $\dim(\wedge\mathfrak{g})^{\mathfrak{g}}=2^l$ where $l=\mathrm{rank}(\mathfrak{g})$.

Theorem 11.2 *The Harish-Chandra projection* $\mathsf{hc}_{\mathrm{Cl}}$ *restricts to an isomorphism of algebras*,

$$\mathsf{hc}_{\mathrm{Cl}}:\ \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}\to\mathrm{Cl}(\mathfrak{t}).\tag{11.3}$$

Proof By Proposition 8.7 the map $\mathsf{hc}_{\mathrm{Cl}}$ restricts to an algebra homomorphism on $\mathfrak{t}$-invariants, hence in particular on $\mathfrak{g}$-invariants. Since the two sides of (11.3) have equal dimensions, it is hence enough to show that (11.2) is a right inverse. Let $\pi:\ U(\mathfrak{g})\to\mathrm{End}(V(\rho))$ be the ρ-representation. Let $v_0\in V(\rho)$ be a normalized highest weight vector, with dual highest weight vector $v_0^*=\langle v_0,\cdot\rangle$. Thus $v_0\otimes v_0^*=\mathrm{pr}_{V(\rho)^{\mathfrak{n}_+}}\in V(\rho)\otimes V(\rho)^*=\mathrm{End}(V(\rho))$ is the highest weight vector in $\mathrm{End}(V(\rho))$. Extend to an orthonormal basis $v_0,\dots,v_N$ of $V(\rho)$ consisting of weight vectors, and choose $a_i\in U(\mathfrak{n}_-)$ with $v_i=\pi(a_i)v_0$, and such that $a_0=1$ and

$a_i \in U^+(\mathfrak{n}_-)$ for $i > 0$. The sum $\sum_{i=1}^N v_i \otimes v_i^*$ corresponds to the identity operator on $V(\rho)$. It follows that

$$\mathrm{id}_{V(\rho)} = \sum_{i=0}^{N} \pi(a_i) \circ \mathrm{pr}_{V(\rho)^{\mathfrak{n}_+}} \circ \pi(a_i^*).$$

Hence, (11.2) takes $x \in \mathrm{Cl}(\mathfrak{t})$ to

$$f(x) = \sum_{i=0}^{N} \gamma(a_i) x \mathsf{R} \gamma(a_i^*) \in \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}.$$

The image of $f(x)$ under $\mathsf{hc}_{\mathrm{Cl}}$ only involves the term $i = 0$. Since $\mathsf{hc}_{\mathrm{Cl}}(\mathsf{R}) = 1$, we obtain $\mathsf{hc}_{\mathrm{Cl}}(f(x)) = \mathsf{hc}_{\mathrm{Cl}}(x\mathsf{R}) = x$. $\square$

Remark 11.2 We learned Theorem 11.2 in 2003 from Bert Kostant, who attributed it to Dale Peterson (unpublished). Kostant explained how to obtain the result as a consequence of his theorem $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \cong \mathrm{Cl}(P(\mathfrak{g}))$ (Theorem 11.6 below), but remarked that Peterson had a direct proof. See Section 8.2 for Kostant's proof; a similar argument appeared in the 2008 preprint by Yuri Bazlov [22].

Having established this result, we can prove:

Proposition 11.1 *The projector* R *lies in the subalgebra* $\gamma(U(\mathfrak{g}))$. *In fact,*

$$\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g})) = \gamma(U(\mathfrak{g})).$$

Proof Consider $\gamma(U(\mathfrak{g}))$ as a subrepresentation for the $\mathfrak{g} \times \mathfrak{g}$-action on $\mathrm{Cl}(\mathfrak{g})$. By Lemma 11.2, its space of highest weight vectors is $\mathrm{Cl}(\mathfrak{t})\mathsf{R} \cap \gamma(U(\mathfrak{g}))$. Suppose $x\mathsf{R}$ with $x \in \mathrm{Cl}(\mathfrak{t})$ lies in $\gamma(U(\mathfrak{g}))$. In particular, this element commutes (in the super sense) with all elements of $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$. Since $\mathsf{hc}_{\mathrm{Cl}}$ is an algebra morphism on $\mathfrak{t}$-invariants, it follows that $\mathsf{hc}_{\mathrm{Cl}}(x\mathsf{R}) = x$ commutes with all elements of $\mathsf{hc}_{\mathrm{Cl}}(\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}) = \mathrm{Cl}(\mathfrak{t})$. Hence x is a scalar. This proves that $\mathrm{Cl}(\mathfrak{t})\mathsf{R} \cap \gamma(U(\mathfrak{g})) = \mathbb{C}\mathsf{R}$. In particular, we see that $\gamma(U(\mathfrak{g})) = \gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))$. $\square$

Theorem 11.1 is an immediate consequence of this result, together with Lemma 11.1.

Example 11.1 Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with standard basis e, f, h and bracket relations $[e, f]_{\mathfrak{g}} = h$, $[h, e]_{\mathfrak{g}} = 2e$, $[h, f]_{\mathfrak{g}} = -2f$. Take B to be the basic inner product; thus $B(e, f) = 1$, $B(h, h) = 2$, and all other inner products among basis elements equal to 0. Since

$$\gamma(h) = 1 - fe, \quad \gamma(e) = \frac{1}{2}he, \quad \gamma(f) = \frac{1}{2}fh,$$

we see that $\gamma(U\mathfrak{g})$ coincides with the even subalgebra of $\mathrm{Cl}(\mathfrak{g})$. The ρ-representation is the defining representation $\pi : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}(\mathbb{C}^2)$, given in terms of matrices as

$$\pi(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

One may verify directly that the map $\gamma(\xi) \mapsto \pi(\xi)$ extends to an isomorphism of algebras, $\gamma(U(\mathfrak{g})) \to \mathrm{End}(V(\rho))$, since the product relations among the matrices $\pi(\xi)$, $\xi \in \mathfrak{g}$ are parallel to those of the Clifford elements $\gamma(\xi)$, $\xi \in \mathfrak{g}$. For instance,

$$\gamma(e)\gamma(f) = \frac{1}{2}(1+\gamma(h)), \quad \pi(e)\pi(f) = \frac{1}{2}(1+\pi(h)).$$

The projection matrix to the highest weight vector in $\mathbb{C}^2$ is $\frac{1}{2}(1+\pi(h)) \in \mathrm{End}(\mathbb{C}^2)$. It corresponds to $\mathsf{R} = \frac{1}{2}(1+\gamma(h)) = \frac{1}{2}ef$, as required.

Since $\gamma(U\mathfrak{g}) \cong \mathrm{End}(V(\rho))$ is a matrix algebra, the algebra morphism

$$\gamma(U\mathfrak{g}) \otimes \gamma(U\mathfrak{g})^{\mathrm{op}} \to \mathrm{End}(\gamma(U\mathfrak{g})),\ c = \sum_i c_i' \otimes c_i'' \mapsto \Big(x \mapsto c(x) := \sum_i c_i' x c_i''\Big)$$

is an isomorphism.

Definition 11.1 We denote by

$$\mathsf{S} \in \gamma(U\mathfrak{g}) \otimes \gamma(U\mathfrak{g})^{\mathrm{op}} \cong \mathrm{End}(\gamma(U\mathfrak{g}))$$

the projection operator with range $\mathbb{C} \subseteq \gamma(U\mathfrak{g})$ and kernel $\ker(\mathsf{S}) = \ker(\mathsf{R} \otimes \mathsf{R})$.

Write $\mathsf{S} = \sum_i \mathsf{S}_i' \otimes \mathsf{S}_i''$. From $\mathsf{S}(\mathsf{R}x) = \mathsf{S}(x) = \mathsf{S}(x\mathsf{R})$ for all $x \in \gamma(U\mathfrak{g})$ and $\mathsf{S}(1) = 1$, we obtain

$$\sum_i \mathsf{S}_i'\mathsf{R} \otimes \mathsf{S}_i'' = \mathsf{S} = \mathsf{S}_i' \otimes \mathsf{R}\mathsf{S}_i'', \qquad \sum_i \mathsf{S}_i'\mathsf{S}_i'' = 1.$$

Furthermore, the obvious property $\gamma(\xi)\mathsf{S}(x) = \mathsf{S}(x)\gamma(\xi)$ for $x \in \gamma(U\mathfrak{g})$, $\xi \in \mathfrak{g}$ shows that

$$\sum_i \gamma(\xi)\mathsf{S}_i' \otimes \mathsf{S}_i'' = \mathsf{S}_i' \otimes \mathsf{S}_i''\gamma(\xi),\ \ \xi \in \mathfrak{g}. \tag{11.4}$$

Under the inclusion $\gamma(U\mathfrak{g}) \hookrightarrow \mathrm{Cl}(\mathfrak{g})$, the map $x \mapsto \mathsf{S}(x)$ extends to the full Clifford algebra.

Proposition 11.2 *The map*

$$\mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g}),\ x \mapsto \mathsf{S}(x) = \sum_i \mathsf{S}_i' x \mathsf{S}_i''$$

is a projection onto the subspace $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ *of* $\mathfrak{g}$*-invariant elements.*

Proof Equation (11.4) implies that for all $x \in \mathrm{Cl}(\mathfrak{g})$, the element $\mathsf{S}(x) = \sum_i \mathsf{S}_i' x \mathsf{S}_i''$ commutes with $\gamma(\xi)$ for all $\xi \in \mathfrak{g}$. That is, $\mathsf{S}(x) \in \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$. Since invariant elements of the Clifford algebra commute with the elements $\mathsf{S}_i' \in \gamma(U\mathfrak{g})$, we obtain

$$\mathsf{S}(\mathsf{S}(x)) = \mathsf{S}_i'\mathsf{S}(x)\mathsf{S}_i'' = \mathsf{S}(x)\sum_i \mathsf{S}_i'\mathsf{S}_i'' = \mathsf{S}(x).$$

This shows that S is a projection with range $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$. $\square$

By contrast, $x \mapsto (\mathsf{R} \otimes \mathsf{R})(x) = \mathsf{R}x\mathsf{R}$ is the projection onto the space of highest weight vectors for the $\mathfrak{g} \times \mathfrak{g}$-action on $\mathrm{Cl}(\mathfrak{g})$.

Lemma 11.3 *The difference of the two projections* S, $\mathsf{R} \otimes \mathsf{R}$ *satisfies*

$$\mathsf{S} - \mathsf{R} \otimes \mathsf{R} \in \gamma(\mathfrak{n}_- \, U\mathfrak{g}) \otimes \gamma(U\mathfrak{g}\,\mathfrak{n}_+).$$

Proof By definition $\mathsf{S}(x) \in \mathbb{C}$ for $x \in \gamma(U\mathfrak{g})$ is the scalar defined by $\mathsf{R}x\mathsf{R} = \mathsf{S}(x)\mathsf{R} = \mathsf{R}\mathsf{S}(x)$. This implies that

$$\mathsf{S} - \mathsf{R} \otimes \mathsf{R} = \sum_i (I - \mathsf{R})\mathsf{S}_i' \otimes \mathsf{S}_i''(I - \mathsf{R}),$$

since both sides give the same result if evaluated on x. The right-hand side lies in $(I - \mathsf{R})\gamma(U\mathfrak{g}) \otimes \gamma(U\mathfrak{g})(I - \mathsf{R})$. Recalling $\gamma(U\mathfrak{g}) = \mathrm{End}(V(\rho)) = V(\rho) \otimes V(\rho)^*$, and using that R projects to the highest weight vector, we see that $(I - \mathsf{R})\gamma(U\mathfrak{g}) = \gamma(\mathfrak{n}_- \, U\mathfrak{g})$. Similarly $\gamma(U\mathfrak{g})(I - \mathsf{R}) = \gamma(U\mathfrak{g}\,\mathfrak{n}_+)$. □

Proposition 11.3 *The inverse map to the algebra isomorphism* (11.3) *is given by*

$$\mathrm{Cl}(\mathfrak{t}) \to \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}, \quad x \mapsto \mathsf{S}(x). \tag{11.5}$$

Proof Lemma 11.3 shows that for all $x \in \mathrm{Cl}(\mathfrak{t})$, $\mathsf{S}(x) - \mathsf{R}x\mathsf{R} \in \ker(\mathsf{hc}_{\mathrm{Cl}})$. Using Proposition 8.7 together with $\mathsf{hc}_{\mathrm{Cl}}(\mathsf{R}) = 1$, we obtain

$$\mathsf{hc}_{\mathrm{Cl}}(\mathsf{S}(x)) = \mathsf{hc}_{\mathrm{Cl}}(\mathsf{R}x\mathsf{R}) = \mathsf{hc}_{\mathrm{Cl}}(\mathsf{R})\mathsf{hc}_{\mathrm{Cl}}(x)\mathsf{hc}_{\mathrm{Cl}}(\mathsf{R}) = x,$$

as required. □

Theorem 11.3 (Kostant) *The multiplication map*

$$\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \otimes \gamma(U(\mathfrak{g})) \to \mathrm{Cl}(\mathfrak{g})$$

is an isomorphism of algebras.

Proof Since the two factors $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ and $\gamma(U(\mathfrak{g}))$ commute, and since

$$\dim(\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}})\dim(\gamma(U(\mathfrak{g}))) = \dim \mathrm{Cl}(\mathfrak{t})\dim(\mathrm{End}(V(\rho))) = \dim \mathrm{Cl}(\mathfrak{g}),$$

it is enough to show that the product map is surjective. For $y \in \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$, we have $y - \mathsf{hc}_{\mathrm{Cl}}(y) \in \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$. By Theorem 11.2, it follows that

$$\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}\mathsf{R} = \mathrm{Cl}(\mathfrak{t})\mathsf{R}.$$

Using $\gamma(U(\mathfrak{g})) = \gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))$, this shows that

$$\begin{aligned}
\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}\gamma(U(\mathfrak{g})) &= \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))\\
&= \gamma(U(\mathfrak{g}))\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}\mathsf{R}\gamma(U(\mathfrak{g}))\\
&= \gamma(U(\mathfrak{g}))\mathrm{Cl}(\mathfrak{t})\mathsf{R}\gamma(U(\mathfrak{g}))\\
&= \mathrm{Cl}(\mathfrak{g}),
\end{aligned}$$

as claimed. □

Example 11.2 We continue the discussion of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, cf. Example 11.1. We found that $\gamma(U\mathfrak{g})$ coincides with the even subalgebra of $\mathrm{Cl}(\mathfrak{g})$. The invariant part $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ is the 2-dimensional subalgebra spanned by 1 together with the cubic element $q(\phi) = \frac{1}{2}(fhe + ehf) = fhe + h$. One checks that

$$h = q(\phi)\gamma(h), \quad e = q(\phi)\gamma(e), \quad f = q(\phi)\gamma(f).$$

Hence, the subalgebra $\gamma(U\mathfrak{g})\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ contains all generators, and hence coincides with all of $\mathrm{Cl}(\mathfrak{g})$. We have $\mathsf{R} = \frac{1}{2}(1 + \gamma(h)) = 1 - \frac{1}{2}fe$, while $\mathsf{S} \in \mathrm{Cl}(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$ is given by

$$\begin{aligned}\mathsf{S} &= \mathsf{R} \otimes \mathsf{R} + \gamma(f) \otimes \gamma(e)\\ &= \left(1 - \frac{1}{2}fe\right) \otimes \left(1 - \frac{1}{2}fe\right) + \left(\frac{1}{2}fh\right) \otimes \left(\frac{1}{2}he\right).\end{aligned}$$

Applying this to the element $h \in \mathfrak{t}$, one obtains

$$\mathsf{S}(h) = \left(1 - \frac{1}{2}fe\right)h\left(1 - \frac{1}{2}fe\right) + \left(\frac{1}{2}fh\right)h\left(\frac{1}{2}he\right) = h + fhe = q(\phi),$$

which is the unique invariant element mapping to h under $\mathsf{hc}_{\mathrm{Cl}}$.

11.2 Relation with extremal projectors

In the last section we considered projectors

$$\mathsf{R} \in \gamma(U\mathfrak{g}), \quad \mathsf{S} \in \gamma(U\mathfrak{g}) \otimes \gamma(U\mathfrak{g})^{\mathrm{op}}.$$

It is natural to ask about canonically defined pre-images in the enveloping algebras. While the enveloping algebra itself does not contain any non-trivial projectors, we will find natural pre-images in suitable extensions of the enveloping algebra. This involves the theory of *extremal projectors*, due to Asherova–Smirnov–Tolstoy [15] and Zhelobenko [121, 122]. I thank A. Alekseev for suggesting this connection. The results of this section will not be needed elsewhere.

Consider the decomposition of $U(\mathfrak{g})$ into weight spaces for the adjoint action of the Lie subalgebra $\mathfrak{t}$,

$$U(\mathfrak{g}) = \bigoplus_{\mu \in P} U(\mathfrak{g})_\mu, \quad U(\mathfrak{g})_\mu = \bigoplus_{\lambda \in P} U(\mathfrak{n}_-)_\lambda \otimes S(\mathfrak{t}) \otimes U(\mathfrak{n}_+)_{\mu-\lambda}.$$

Let $\widetilde{S}(\mathfrak{t})$ be the localization of the commutative algebra $S(\mathfrak{t})$ with respect to the set of denominators $\alpha^\vee + t$, for $\alpha \in \mathfrak{R}$ and $t \in \mathbb{C}$. (Thus, thinking of $S(\mathfrak{t})$ as polynomials on $\mathfrak{t}^*$, the elements of $\widetilde{S}(\mathfrak{t})$ are rational functions with denominators given as products of linear forms $\mu \mapsto \langle \mu, \alpha^\vee \rangle + t$.) Following [122] we define

$$\widetilde{U}(\mathfrak{g}) = \bigoplus_{\mu \in P} \widetilde{U}(\mathfrak{g})_\mu, \quad \widetilde{U}(\mathfrak{g})_\mu = \prod_{\lambda \in P} U(\mathfrak{n}_-)_\lambda \otimes \widetilde{S}(\mathfrak{t}) \otimes U(\mathfrak{n}_+)_{\mu-\lambda}.$$

The algebra $\widetilde{U}(\mathfrak{g})$ has a natural interpretation in terms of the *universal Verma module* $M = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}_+ \cong U(\mathfrak{n}_-) \otimes S(\mathfrak{t})$. The right $S(\mathfrak{t})$-module structure on $U(\mathfrak{g})$ preserves $U(\mathfrak{g})\mathfrak{n}_+$, and hence descends to a right module structure on M. Let

$$\widetilde{M} = M \otimes_{S(\mathfrak{t})} \widetilde{S}(\mathfrak{t}) \cong U(\mathfrak{n}_-) \otimes \widetilde{S}(\mathfrak{t}).$$

Note that M inherits a decomposition into weight spaces from $U(\mathfrak{g})$; hence $\widetilde{M}$ again comes with a decomposition $\widetilde{M} = \bigoplus_{\mu \in P} \widetilde{M}_\mu$. Let $\mathrm{End}(\widetilde{M})_\nu$, $\nu \in P$, be the space of endomorphisms taking $\widetilde{M}_\mu$ to $\widetilde{M}_{\mu+\nu}$ for all $\mu \in P$, and consider the algebra $\mathrm{End}(\widetilde{M})_{\mathrm{fin}} = \bigoplus_{\nu \in P} \mathrm{End}(\widetilde{M})_\nu$. The action of $U(\mathfrak{g})$ on $\widetilde{M}$ extends to $\widetilde{U}(\mathfrak{g})$, and defines a map

$$\widetilde{U}(\mathfrak{g}) \to \mathrm{End}(\widetilde{M})_{\mathrm{fin}}$$

which is easily seen to be injective. As shown in [122], the image of this map is the subalgebra of $\mathrm{End}(\widetilde{M})_{\mathrm{fin}}$ of endomorphisms which commute with the right $\widetilde{S}(\mathfrak{t})$ action. In particular, $\widetilde{U}(\mathfrak{g})$ is an algebra. Let us also note that the antipode s and the involution $*$ extend to $\widetilde{U}(\mathfrak{g})$, and the Harish-Chandra projection extends to a map $\mathsf{hc}_U : \widetilde{U}(\mathfrak{g}) \to \widetilde{S}(\mathfrak{t})$.

Theorem 11.4 [15, 121, 122] *The algebra $\widetilde{U}(\mathfrak{g})$ contains a unique projector $\mathsf{P} = \mathsf{P}^2$ with the properties*

$$\mathfrak{n}_+\mathsf{P} = 0, \;\; \mathsf{P}\mathfrak{n}_- = 0, \;\; \mathsf{hc}_U \circ (\mathsf{P} - 1) = 0.$$

The projector P is called the *extremal projector*. Realizing $\widetilde{U}(\mathfrak{g})$ as operators on $\widetilde{M}$, as above, the extremal projector is just the projection to the degree 0 component $\widetilde{M}^0 \cong \widetilde{S}(\mathfrak{t})$.

There is an explicit formula for P, see [122]. For any root α, let $e_\alpha, f_\alpha, h_\alpha$ be standard generators of the corresponding Lie subalgebra $\mathfrak{sl}(2,\mathbb{C})_\alpha$; see Proposition B.2. In particular $h_\alpha = \alpha^\vee$, and e_α, f_α are root vectors for the roots $\alpha, -\alpha$ with $[e_\alpha, f_\alpha] = h_\alpha$. Let $\mathsf{P}_\alpha \in \widetilde{U}(\mathfrak{sl}(2,\mathbb{C})_\alpha)$ be the element

$$\mathsf{P}_\alpha = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\prod_{j=1}^{m} \frac{1}{h_\alpha + \langle \rho, h_\alpha \rangle + j} \right) f_\alpha^m \, e_\alpha^m. \tag{11.6}$$

Definition 11.2 A *normal ordering* of the set of positive roots is an ordering $\alpha_1, \ldots, \alpha_r$ with the property that for any indices $i < j$ such that $\alpha_i + \alpha_j = \alpha_k$ is a root, one has $i < k < j$.

Remark 11.3 As shown in [121], the ordering defined by a reduced decomposition of the longest Weyl group element $w_0 = w_{i_1} \cdots w_{i_r}$ (see Proposition B.9) is a normal ordering, and all normal orderings arise in this way.

Theorem 11.5 [122, Theorem 4] *The extremal projector is given by the formula*

$$\mathsf{P} = \mathsf{P}_{\alpha_1} \cdots \mathsf{P}_{\alpha_r},$$

for any normal ordering $\alpha_1, \ldots, \alpha_r$ of the positive roots.

We are interested in the image of P in the Clifford algebra. Unfortunately, the algebra morphism $\gamma:\ U(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ does not extend to the full algebra $\widetilde{U}(\mathfrak{g})$. Nonetheless the images $\gamma(\mathsf{P}_\alpha) =: \mathsf{R}_\alpha$ are well defined, as we will now explain. For $\nu \in \mathfrak{t}^*$, let κ_ν be the automorphism of $S(\mathfrak{t})$ given on generators by $h \mapsto h + \langle \nu, h\rangle$. Thus, viewing elements of $S(\mathfrak{t})$ as polynomials, this automorphism shifts the argument by ν. The automorphism preserves the set of denominators $\alpha^\vee + t$, $\alpha \in \mathfrak{R}$, $t \in \mathbb{C}$, hence it extends to $\widetilde{S}(\mathfrak{t})$.

Given t, α, and a finite-dimensional completely reducible $\mathfrak{g}$-representation $\pi:\ \mathfrak{g} \to \mathrm{End}(V)$, the endomorphism $\pi(\kappa_{z\rho}(\alpha^\vee + t))$ is invertible for almost all $z \in \mathbb{C}$. Hence, for any $Y \in \widetilde{U}(\mathfrak{g})$, the function $Y(z) = \kappa_{z\rho}(Y)$ has a well-defined image $\pi(Y(z))$ as a rational function of z with values in $\mathrm{End}(V)$.

Proposition 11.4 *The image $\gamma(\mathsf{P}_\alpha)$ is a well-defined element of $\gamma(U\mathfrak{g})$, in the sense that $\gamma(\mathsf{P}_\alpha(z))$ has no poles at $z = 0$.*

Proof Since $\gamma(U\mathfrak{g})$ is a direct sum of ρ-representations, it suffices to prove the analogous statement for $\pi:\ \mathfrak{g} \to \mathrm{End}(V(\rho))$. Consider the restriction of π to $\mathfrak{sl}(2,\mathbb{C})_\alpha$. By the representation theory of $\mathfrak{sl}(2,\mathbb{C})$, the eigenvalues of $\pi(h_\alpha)$ are all integers. Since the largest eigenvalue equals $\langle\rho, h_\alpha\rangle$, the spectrum of $\pi(h_\alpha)$ lies in the set of integers between $-\langle\rho, h_\alpha\rangle$ and $\langle\rho, h_\alpha\rangle$. Only weights in this interval can be dominant weights for the $\mathfrak{sl}(2,\mathbb{C})_\alpha$ action. In particular, $\pi(e_\alpha)^m = 0$, $\pi(f_\alpha)^m = 0$ for all $m > \langle\rho, h_\alpha\rangle$. In the definition (11.6) of P_α we need therefore only consider terms with $m \leq \langle\rho, h_\alpha\rangle$. But for any such m, and $j = 1, \ldots, m$, the eigenvalues of $\pi(h_\alpha) + \langle\rho, h_\alpha\rangle + j$ are ≥ 1, hence the element is invertible. ☐

Proposition 11.5 *The element $\gamma(\mathsf{P})$ coincides with the projector* R.

Proof Let $\mathsf{R}' = \gamma(\mathsf{P})$. The properties $e_\alpha\mathsf{P} = 0 = \mathsf{P}f_\alpha$ imply $\gamma(e_\alpha)\mathsf{R}' = 0 = \mathsf{R}'\gamma(f_\alpha)$. They show that R' is a highest weight vector for the $\mathfrak{g} \times \mathfrak{g}$-action on $\gamma(U\mathfrak{g})$. It is hence a multiple of R. Since $\mathsf{P} - 1 \in \mathfrak{n}_-\widetilde{U}(\mathfrak{g})\mathfrak{n}_+$, we have $\mathsf{R}' - 1 \in \gamma(\mathfrak{n}_-)\mathrm{Cl}(\mathfrak{g})\gamma(\mathfrak{n}_+) \subseteq \ker \mathsf{hc}_{\mathrm{Cl}}$ and therefore $\mathsf{hc}_{\mathrm{Cl}}(\mathsf{R}') = 1 = \mathsf{hc}_{\mathrm{Cl}}(\mathsf{R})$. Hence $\mathsf{R}' = \mathsf{R}$. ☐

Let us next define the *Shapovalov pairing* $U(\mathfrak{n}_+) \times U(\mathfrak{n}_-) \to S\mathfrak{t}$ by

$$(x, y) \mapsto \mathsf{hc}_U(xy). \tag{11.7}$$

(It is standard to compose this definition with a Chevalley anti-involution in the first entry, in order to get a bilinear form on $U\mathfrak{n}_-$. But the convention above is more convenient for our purposes.) Let $x_0, x_1, \ldots \in U(\mathfrak{n}_+)$ be a basis consisting of weight vectors, i.e., $x_i \in U(\mathfrak{n}_+)_{\mu_i}$. Then $x_i^* \in U(\mathfrak{n}_-)_{-\mu_i}$ are a basis of $U(\mathfrak{n}_-)$. Using a Gram–Schmidt procedure we may assume that $\mathsf{hc}_U(x_i, x_j^*) = \delta_{ij}\phi_i$ for some $\phi_i \in \widetilde{S}(\mathfrak{t})$.

Proposition 11.6 [122, Proposition 7.1] *The elements*

$$\mathsf{P}_n = x_n^* \phi_n^{-1} \mathsf{P} x_n$$

form a partition of unity in $\widetilde{U}(\mathfrak{g})$. *That is, the* P_n *are pairwise commuting self-adjoint projections, with*

$$\sum_{n=0}^{\infty} \mathsf{P}_n = 1.$$

Proof Observe that for all $x \in \widetilde{U}(\mathfrak{g})$, $\mathsf{P}x\mathsf{P} = \mathsf{hc}_U(x)\mathsf{P}$. Hence, using the orthogonality of the bases,

$$\mathsf{P}_i\mathsf{P}_j = x_i^*\phi_i^{-1}\mathsf{P}x_i x_j^*\phi_j^{-1}\mathsf{P}x_j = x_i^*\phi_i^{-1}\mathsf{hc}_U(x_i x_j^*)\phi_j^{-1}\mathsf{P}x_j = \delta_{ij}\mathsf{P}_j.$$

Identify $\widetilde{M}$ with the submodule $\widetilde{U}(\mathfrak{g})\mathsf{P} \subseteq \widetilde{U}(\mathfrak{g})$. The elements $x_m^*\mathsf{P}$ form a basis of $\widetilde{M}$ as a module over $\widetilde{S}(\mathfrak{t})$. The calculation

$$\begin{aligned}\sum_n \mathsf{P}_n x_m^*\mathsf{P} &= \sum_n x_n^*\phi_n^{-1}\mathsf{P}x_n x_m^*\mathsf{P}\\ &= \sum_n x_n^*\phi_n^{-1}\mathsf{hc}_U(x_n x_m^*)\mathsf{P}\\ &= x_m^*\mathsf{P}\end{aligned}$$

shows that $\sum_n \mathsf{P}_n = 1$. □

Proposition 11.7 *The element*

$$\mathsf{Q} = \sum_n x_n^*\phi_n^{-1}\mathsf{P} \otimes \mathsf{P}x_n \in \prod_{\mu\in P} \widetilde{U}(\mathfrak{n}_- \oplus \mathfrak{t})_{-\mu} \otimes \widetilde{U}(\mathfrak{n}_+)_\mu$$

does not depend on the choice of basis. Under the action of the right-hand side as an endomorphism of $\widetilde{U}(\mathfrak{g})$, *it satisfies*

$$\mathsf{Q}(x) = \mathsf{Q}(\mathsf{P}x) = \mathsf{Q}(x\mathsf{P}), \tag{11.8}$$

and $\mathsf{Q}(1) = 1$.

Proof A basis-free description of Q is as follows: Since the pairing (11.7) is non-degenerate, there exists a unique element

$$T = \sum_i T_i' \otimes T_i'' \in \prod_{\mu\in P} \widetilde{U}(\mathfrak{n}_- \oplus \mathfrak{t})^{-\mu} \otimes \widetilde{U}(\mathfrak{n}_+)_\mu$$

with the property $y = \sum_i T_i'\mathsf{hc}_U(T_i'' y)$ for all $y \in U(\mathfrak{n}_-)$. Then

$$\mathsf{Q} = \sum_i T_i'\mathsf{P} \otimes \mathsf{P}T_i''.$$

Equations (11.8) are immediate from the definition, and $\mathsf{Q}(1) = 1$, is an immediate consequence of Proposition 11.6. □

Proposition 11.8 *The image of* Q *under* $\gamma \otimes \gamma$ *is well defined, and*

$$(\gamma \otimes \gamma)(\mathsf{Q}) = \mathsf{S}.$$

Here, "well defined" is understood similar to Proposition 11.4: The image of $\mathsf{Q}(z)$ has no pole at $z = 0$.

Proof (Sketch) Since $\mathrm{Cl}(\mathfrak{g})$ is a sum of copies of the ρ-representation $\pi : \mathfrak{g} \to \mathrm{End}(V(\rho))$, it is enough to show that $(\pi \otimes \pi)(\mathsf{Q})$ is well defined. For this, we make a convenient choice of basis. For each weight space $U(\mathfrak{n}_+)_\mu$, pick a basis x_i with the orthogonality property $\mathrm{hc}_U(x_i x_j^*) = 0$ for $i \neq j$, and such that a subset of the basis spans $\{x \in U(\mathfrak{n}_+)_\mu |\ \pi(x^*) = 0\}$. Rearranging, this gives a basis $x_0, x_1, \ldots$ of $U(\mathfrak{n}_+)$ with $\mathrm{hc}_U(x_i x_j^*) = 0$ for $i \neq j$, where $\pi(x_0^*)v_0, \ldots, \pi(x_N^*)v_0$ (with $N + 1 = \dim V(\rho)$) defines a basis of $V(\rho)$, while $\pi(x_i^*) = 0$ for $i > N$.

With this choice of basis, we need only keep terms $n \leq N$ in the expression for $(\pi \otimes \pi)(\mathsf{Q})$. Since $\pi(\mathsf{P})$ is projection to the highest weight space, we have

$$\pi(\phi_n^{-1}\mathsf{P}) = (\mathrm{ev}_\rho \mathsf{hc}_U(x_n x_n^*))^{-1}\pi(\mathsf{P}).$$

But

$$\mathrm{ev}_\rho \mathsf{hc}_U(x_n x_n^*) = \langle \pi(x_n)\pi(x_n)^* v_0, v_0\rangle = \langle \pi(x_n)^* v_0,\ \pi(x_n)^* v_0\rangle > 0.$$

The properties (11.8) with the normalization $\mathsf{Q}(1) = 1$ translate into similar properties of $(\gamma \times \gamma)(\mathsf{Q})$, identifying this element with S. $\square$

Example 11.3 Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, with standard basis e, f, h. Then $U(\mathfrak{n}_+)$ has basis $1, e, e^2, \ldots$, while $U(\mathfrak{n}_-)$ has basis $1, f, f^2, \ldots$. The Shapovalov pairing is

$$\mathsf{hc}_U(e^n f^m) = \delta_{n,m}\, m! \prod_{j=1}^{m}(h + 1 - j).$$

One finds that

$$\mathsf{Q} = \sum_{m=0}^{\infty} \frac{1}{m!}(f^m\mathsf{P})\left(\prod_{j=1}^{m}\frac{1}{h+1-j}\right) \otimes (\mathsf{P}\, e^m).$$

Consider the image under $\gamma \otimes \gamma$. Since $\gamma(f^2) = 0$, we need only consider the sum up to $m = 1$:

$$(\gamma \otimes \gamma)(\mathsf{Q}) = \mathsf{R} \otimes \mathsf{R} - \gamma(f)\gamma(h)^{-1}\mathsf{R} \otimes \mathsf{R}\gamma(e).$$

The element $\gamma(h) = 1 - fe$ has inverse $\gamma(h)^{-1} = 1 - 3fe$, hence $\gamma(h)^{-1}\mathsf{R} = \mathsf{R}$ (as is also clear since R projects onto highest weight vectors). Since $\mathsf{R} = \frac{1}{2}ef$ satisfies $\mathsf{R}\gamma(e) = \gamma(e)$, $\gamma(f)\mathsf{R} = \gamma(f)$, we find that

$$(\gamma \otimes \gamma)(\mathsf{Q}) = \mathsf{R} \otimes \mathsf{R} + \left(\frac{1}{2}fh\right) \otimes \left(\frac{1}{2}he\right),$$

which is the element S obtained earlier.

11.3 The isomorphism $(\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} \cong \mathrm{Cl}(P(\mathfrak{g}))$

We now turn to Kostant's "Clifford algebra analogue" of the Hopf–Koszul–Samelson Theorem. Recall that by the usual Hopf–Koszul–Samelson Theorem 10.2, the inclusion $P(\mathfrak{g}) \to I(\mathfrak{g}) = (\wedge\mathfrak{g})^{\mathfrak{g}}$ extends to an isomorphism of graded super algebras, $\wedge P(\mathfrak{g}) \to I(\mathfrak{g})$.

The Clifford analogue of this result uses the following extension of B to a non-degenerate symmetric bilinear form $B_\wedge$ on the exterior algebra,

$$B_\wedge(\phi, \psi) = (\iota(\phi)\psi)_{[0]},$$

where B is used to identify $\mathfrak{g} \cong \mathfrak{g}^*$, and hence $\wedge\mathfrak{g} \cong \wedge\mathfrak{g}^*$. We stress that this is different from the bilinear form $\langle\phi, \psi\rangle = (\iota(\phi^\top)\psi)_{[0]}$ used in Section 10.2. Both bilinear forms are non-degenerate, and the decomposition into graded components is orthogonal. On $\wedge^k\mathfrak{g}$, the two bilinear forms differ by a sign $(-1)^{k(k-1)/2}$. The form $\langle\cdot, \cdot\rangle$ is the complexification of a positive definite bilinear form on $\wedge\mathfrak{g}_{\mathbb{R}}$, and hence is non-degenerate on every conjugation invariant subspace. Hence $B_\wedge$ is non-degenerate on every conjugation invariant *graded* subspace. In particular, it is non-degenerate on the primitive subspace $P(\mathfrak{g})$. The form $B_\wedge$ is also non-degenerate on the summands of the decomposition

$$\wedge\mathfrak{g} = I(\mathfrak{g}) \oplus \mathrm{ran}(\mathrm{d}) \oplus \mathrm{ran}(\partial) = I(\mathfrak{g}) \oplus \mathfrak{g}. \wedge \mathfrak{g}, \tag{11.9}$$

and the decomposition is $B_\wedge$ orthogonal. Let $\mathrm{Cl}(P(\mathfrak{g}))$ be the Clifford algebra corresponding to the restriction of $B_\wedge$ to $P(\mathfrak{g})$.

Theorem 11.6 (Kostant [86]) *The inclusion $P(\mathfrak{g}) \hookrightarrow (\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}}$ extends to an isomorphism of filtered super algebras*

$$\mathrm{Cl}(P(\mathfrak{g})) \to (\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}}.$$

Proof By the universal property of the Clifford algebra, and since the two sides have equal dimension, it suffices to show that for all $\phi, \psi \in P(\mathfrak{g})$,

$$[q(\phi), q(\psi)] = 2B_\wedge(\phi, \psi).$$

Let τ_i, $i = 1, \ldots, 2^n$, be a homogeneous basis of $\wedge\mathfrak{g}$, given by a basis of $I(\mathfrak{g})$, followed by a basis of ran(d), followed by a basis of ran(∂). Let τ^i, $i = 1, \ldots, 2^n$, be the $B_\wedge$-dual basis, so that $B_\wedge(\tau^i, \tau_j) = \delta^i_j$. By the formula for the Clifford product given in Section 2.2.11, and using that ϕ is odd, we have

$$q^{-1}(q(\phi)\, q(\psi)) = \sum_{i=1}^{2^n} \iota(\tau_i)\phi \wedge \iota(\tau^i)\psi. \tag{11.10}$$

(Note that in contrast to the formula from Section 2.2.11, the formula does not involve the transposition $\top$. This is because τ^i is the $B_\wedge$-dual basis, rather than the $\langle\cdot, \cdot\rangle$-dual basis.) For any given i, the elements τ_i, τ^i have the same degree and lie in the same summand of the decomposition (11.9). Let us consider

$$\iota(\tau_i)\phi \wedge \iota(\tau^i)\psi \tag{11.11}$$

for these three cases.

(i) If $\tau_i, \tau^i \in I(\mathfrak{g})$, then clearly (11.11) lies in $I(\mathfrak{g})$.
(ii) For $\tau_i, \tau^i \in \mathrm{ran}(\mathrm{d})$, then Lemma 10.5 gives $\iota(\tau_i)\phi, \iota(\tau^i)\psi \in \mathrm{ran}(\partial)$. Hence Lemma 10.4 shows that (11.11) lies in $\mathfrak{g}. \wedge \mathfrak{g}$.
(iii) If $\tau_i, \tau^i \in \mathrm{ran}(\partial)$, then Lemma 10.5 gives $\iota(\tau_i)\phi, \iota(\tau^i)\psi \in \mathrm{ran}(\mathrm{d})$. Hence (11.11) lies in $\mathrm{ran}(\mathrm{d}) \subseteq \mathfrak{g}. \wedge \mathfrak{g}$.

Since the left-hand side of (11.10) lies in $I(\mathfrak{g})$, the terms from (ii), (iii) must cancel out. We can therefore replace the summation $\sum_{i=1}^{2^n}$ with $\sum_{i=1}^{2^l}$. Let $\nu_1, \ldots, \nu_l$ be a basis of $P(\mathfrak{g})$, and let $\nu^1, \ldots, \nu^l \in P(\mathfrak{g})$ be the $B_\wedge$-dual basis. We may take the basis of $I(\mathfrak{g})$ to consist of products $\nu_I = \nu_{i_1} \wedge \cdots \nu_{i_k}$ for ordered subsets $I = \{i_1, \ldots, i_k\}$. The formula becomes

$$q^{-1}(q(\phi)\,q(\psi)) = \sum_I \iota(\nu_I)\phi \wedge \iota(\nu^I)\psi. \tag{11.12}$$

But since $\iota(\nu_i)$, $i = 1, \ldots, l$ act as derivations on $I(\mathfrak{g})$ (Proposition 10.13), and since we are assuming that $\phi, \psi \in P(\mathfrak{g})$, only the terms with $|I| \leq 1$ contribute:

$$q^{-1}(q(\phi)\,q(\psi)) = \phi \wedge \psi + \sum_{i=1}^{l} \iota(\nu_i)\phi \wedge \iota(\nu^i)\psi = \phi \wedge \psi + B_\wedge(\phi, \psi).$$

That is,

$$q(\phi)q(\psi) = q(\phi \wedge \psi) + B_\wedge(\phi, \psi). \tag{11.13}$$

Adding a similar identity for $q(\psi)q(\phi)$, we obtain $[q(\phi), q(\psi)] = 2B_\wedge(\phi, \psi)$, as desired. □

Remark 11.4 According to [86, Remark 41], Kostant had obtained Theorem 11.6 already in the 1960s. His proof is contained in the 1979 master's thesis of his student Mohammed Najafi.

Remark 11.5

(a) From Example 2.1, we know that if $\phi \in \wedge^{2m+1}(\mathfrak{g})$, then

$$q^{-1}(q(\phi)^2) \in I^0(\mathfrak{g}) \oplus I^4(\mathfrak{g}) \oplus \cdots \oplus I^{4m}(\mathfrak{g}).$$

Kostant's Theorem 11.6 shows that if ϕ lies in the subspace $P^{2m+1}(\mathfrak{g})$, then the terms in $I^4(\mathfrak{g}), \ldots, I^{4m}(\mathfrak{g})$ are all equal to zero. Indeed, since $q(\phi)$ is among the generators of $(\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} = \mathrm{Cl}(P(\mathfrak{g}))$, it squares to a constant. In Chapter 7 we made extensive use of the fact that the quantized cubic generator squares to a scalar; we now see that the same is true for the higher generators as well.
(b) The identity (11.13) also shows that

$$q(\phi)q(\psi) - q(\psi)q(\phi) = 2q(\phi \wedge \psi).$$

The following result (where the key step is (11.13)) will be needed below. Let e_a be a basis of $\mathfrak{g}$, and e^a the B-dual basis.

Proposition 11.9 [86] *Let $\phi, \psi \in P(\mathfrak{g})$ be primitive elements of homogeneous degrees k, k'. Then*

$$\sum_{a=1}^{n} q(\iota(e_a)\phi)\, q(\iota(e^a)\psi) = k\, B_\wedge(\phi, \psi).$$

Proof Write $q(\iota(e_a)\phi) = \iota(e_a)q(\phi) = \frac{1}{2}[e_a, q(\phi)]_{\mathrm{Cl}}$, and similarly for the term involving ψ. Expanding, we find that the left-hand side is $\frac{1}{4}$ times

$$\sum_{a=1}^{n} e_a q(\phi) e^a q(\psi) + \sum_{a=1}^{n} q(\phi) e_a q(\psi) e^a + \sum_{a=1}^{n} q(\phi) e^a e_a q(\psi)$$
$$+ \sum_{a=1}^{n} e_a q(\phi) q(\psi) e^a.$$

In the last term we may write $q(\phi)q(\psi) = q(\phi \wedge \psi) + B_\wedge(\phi, \psi)$, cf. (11.13). Using Proposition 2.8 to evaluate the sums over a, the first three terms give

$$(-n + 2(k + k'))q(\phi)q(\psi) = (-n + 2(k + k'))\big(q(\phi \wedge \psi) + B_\wedge(\phi, \psi)\big),$$

while the last term gives

$$(n - 2(k + k'))q(\phi \wedge \psi) + n B_\wedge(\phi, \psi).$$

Adding, we obtain $2(k + k')B_\wedge(\phi, \psi) = 4kB_\wedge(\phi, \psi)$, where we used that $B_\wedge(\phi, \psi) = 0$ for $k \neq k'$. □

11.4 The ρ-decomposition of elements $\xi \in \mathfrak{g} \subseteq \mathrm{Cl}\mathfrak{g}$

Combining the results

$$\begin{array}{ll}
(\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} = \mathrm{Cl}(P(\mathfrak{g})) & \text{(Theorem 11.6)}, \\
\mathrm{Cl}\mathfrak{g} = (\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} \otimes \gamma(U\mathfrak{g}) & \text{(Theorem 11.3)}, \\
\mathrm{End}(V(\rho)) \cong \gamma(U\mathfrak{g}) & \text{(Theorem 11.1)},
\end{array}$$

we obtain the isomorphism of super algebras

$$\mathrm{Cl}\mathfrak{g} = \mathrm{Cl}(P(\mathfrak{g})) \otimes \mathrm{End}(V(\rho)), \tag{11.14}$$

called the *ρ-decomposition* in Kostant's paper [86]. Following [86], we will discuss how to expand a given element $\xi \in \mathfrak{g} \subseteq \mathrm{Cl}\mathfrak{g}$ in terms of this *ρ-decomposition*.

11.4.1 The space $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \lambda(S\mathfrak{g}))$

Recall that for $\xi \in \mathfrak{g}$, the derivation $\iota_S(\xi) \in \mathrm{End}(S\mathfrak{g})$ is given on generators $\zeta \in \mathfrak{g}$ by $\iota_S(\xi)(\zeta) = B(\xi, \zeta)$. Extend ι_S to an algebra morphism

$$\iota_S :\ S\mathfrak{g} \to \mathrm{End}(S\mathfrak{g}).$$

Define a symmetric bilinear form on $S\mathfrak{g}$, by putting

$$B_S(p,q) = (\iota_S(q)p)_{[0]},$$

where the subscript indicates the component in $S^0\mathfrak{g} = \mathbb{C}$. In particular, $S^k\mathfrak{g}$ and $S^{k'}\mathfrak{g}$ for $k \neq k'$ are orthogonal. Clearly, this bilinear form on $S\mathfrak{g}$ restricts to the given bilinear form B on $\mathfrak{g} = S^1\mathfrak{g}$. Recall that $I_S(\mathfrak{g}) = (S\mathfrak{g})^{\mathfrak{g}}$ denotes the $\mathfrak{g}$-invariant subalgebra.

Definition 11.3 A polynomial $p \in S\mathfrak{g}$ is called *harmonic* if $\iota_S(q)p = 0$ for all $q \in I_S^+(\mathfrak{g})$. The space of harmonic polynomials is denoted $\mathrm{Har}_S(\mathfrak{g})$.

Lemma 11.4 *The product map $I_S(\mathfrak{g}) \otimes \mathrm{Har}_S(\mathfrak{g}) \to S\mathfrak{g}$ is surjective.*

Proof Define the graded subspaces $I = I_S(\mathfrak{g})$, $H = \mathrm{Har}_S(\mathfrak{g})$ of $S = S\mathfrak{g}$, and let I^+, H^+, S^+ be the sums of components of positive degree. Under the bilinear form B_S on $S\mathfrak{g}$, the operator of contraction by $q \in I_S^+(\mathfrak{g})$ is dual to the operator of multiplication by q. Hence, the orthogonal complement of H^k in S^k is $S^k \cap I^+S$. This shows that

$$S = I^+S + H. \tag{11.15}$$

Iterating, we obtain

$$S = (I^+)^i S + (I^+)^{i-1} H + \cdots + I^+H + H$$

for all $i \geq 1$. For any given degree k, we have $(I^+)^i S \cap S^k = 0$ if $i > k$. Hence

$$S^k = S^k \cap \big((I^+)^k H + \cdots + I^+H + H\big) = S^k \cap IH,$$

which verifies that $S = IH$. $\square$

Remarks 11.6

1. By a classical result of Kostant [85], the map in this lemma is in fact an isomorphism.
2. For the exterior algebra, one has a similar surjective map

$$I(\mathfrak{g}) \otimes (\wedge\mathfrak{g})_{P\text{-hor}} \to \wedge\mathfrak{g}.$$

   The proof is parallel to that for the symmetric algebra. This map, however, is *not* an isomorphism (already in the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$).

Consider the extension of the map $\lambda\colon\ \mathfrak{g} \to \wedge^2\mathfrak{g}$ to an algebra morphism $\lambda\colon S\mathfrak{g} \to \wedge\mathfrak{g}$.

Lemma 11.5 *The algebra morphism $\lambda\colon\ S\mathfrak{g} \to \wedge\mathfrak{g}$ satisfies*

$$\lambda(S\mathfrak{g}) = \lambda(\mathrm{Har}_S(\mathfrak{g})). \tag{11.16}$$

Proof Since $\lambda(\mathfrak{g}) \subseteq \mathrm{ran}(\mathrm{d})$, we have $\lambda(S^+\mathfrak{g}) \subseteq \mathrm{ran}(\mathrm{d})$. It follows that $\lambda(I_S^+(\mathfrak{g})) \subseteq I^+(\mathfrak{g}) \cap \mathrm{ran}(\mathrm{d}) = 0$. By Lemma 11.4, it follows that the image of $S(\mathfrak{g})$ equals that of $\mathrm{Har}_S(\mathfrak{g})$. □

Let $P_S(\mathfrak{g}) \subseteq I_S^+(\mathfrak{g})$ be the orthogonal complement of $(I_S^+(\mathfrak{g}))^2$, so that

$$I_S^+(\mathfrak{g}) = P_S(\mathfrak{g}) \oplus (I_S^+(\mathfrak{g}))^2,$$

similar to the decomposition $I^+(\mathfrak{g}) = P(\mathfrak{g}) \oplus I^+(\mathfrak{g})^2$ for the exterior algebra. Kostant calls $P_S(\mathfrak{g})$ the *Dynkin subspace*, after [50].

Remark 11.7 The subspace $P_S(\mathfrak{g})$ is different, in general, from the complement defined by the distinguished cochains of transgression, cf. Remark 10.7.

By Chevalley's Theorem 8.1, the inclusion $P_S(\mathfrak{g}) \hookrightarrow I_S(\mathfrak{g})$ extends to an isomorphism $S(P_S(\mathfrak{g})) \to I_S(\mathfrak{g})$, and the transgression map restricts to an isomorphism

$$\mathsf{t}\colon\ P_S(\mathfrak{g}) \to P(\mathfrak{g}).$$

Consider the linear map

$$I_S(\mathfrak{g}) \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, S\mathfrak{g}),\ \ p \mapsto \big(\xi \mapsto \iota_S(\xi)p\big). \tag{11.17}$$

Proposition 11.10 [86] *The isomorphism* $\mathsf{t}\colon\ P_S(\mathfrak{g}) \to P(\mathfrak{g})$ *is a composition of three isomorphisms,*

$$P_S(\mathfrak{g}) \xrightarrow{\text{(i)}} \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Har}_S(\mathfrak{g})) \xrightarrow{\text{(ii)}} \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \lambda(S\mathfrak{g})) \xrightarrow{\text{(iii)}} P(\mathfrak{g}).$$

Here (i) *is the restriction of* (11.17), (ii) *is defined by the map* $\lambda\colon\ S\mathfrak{g} \to \wedge\mathfrak{g}$, *and the map* (iii) *takes* $f \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \lambda(S^m(\mathfrak{g})))$ *to*

$$\phi = \frac{(m!)^2}{(2m+1)!} \sum_{a=1}^n e^a \wedge f(e_a).$$

The inverse map to (iii) *takes* $\phi \in P^{2m+1}(\mathfrak{g})$ *to* $f \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \lambda(S^m(\mathfrak{g})))$ *where*

$$f(\xi) = \frac{1}{2m+1}\iota(\xi)\phi.$$

Proof We first verify that $\iota_S(\xi)p \in \mathrm{Har}_S(\mathfrak{g})$ for all $\xi \in \mathfrak{g}$, $p \in P_S(\mathfrak{g})$. Given $q, r \in I_S^+(\mathfrak{g})$, we have

$$B_S(r, \iota_S(q)p) = (\iota_S(rq)p)_{[0]} = B_S(rq, p) = 0,$$

so that $\iota_S(q)p \in (I_S^+(\mathfrak{g}))^\perp$. Since q and p are invariant, one also has $\iota_S(q)p \in I_S(\mathfrak{g})$. Thus $\iota_S(q)p \in \mathbb{C}$, and for all $\xi \in \mathfrak{g}$,

$$\iota_S(q)\iota_S(\xi)(p) = \iota_S(\xi)\iota_S(q)(p) = 0,$$

proving $\iota_S(\xi)p \in \mathrm{Har}_S(\mathfrak{g})$.

The formula for transgression (Proposition 6.18) shows that $\mathsf{t}: P_S(\mathfrak{g}) \to P(\mathfrak{g})$ is a composition of the three maps (i), (ii) and (iii). Since this composition is an isomorphism, we see in particular that (i) and the composition of (i) and (ii) are injective. By (11.16), the map (ii) is surjective.

In his 1963 paper [85], Kostant established that the adjoint representation appears in $\mathrm{Har}_S(\mathfrak{g}))$ with multiplicity l. (The proof uses techniques from algebraic geometry that are beyond the scope of this book.) As a consequence, the map (i) is an isomorphism. Since the composition of (i) and (ii) is injective, it also follows that (ii) is an isomorphism. Since (i), (ii) and $\mathsf{t}: P_S(\mathfrak{g}) \to P(\mathfrak{g})$ are all isomorphisms, the map (iii) is an isomorphism as well. Recall finally that by Proposition 6.19, the contractions of transgressed elements $\phi = \mathsf{t}(p) \in P^{2m+1}(\mathfrak{g})$ are given by the formula

$$\iota(\xi)\mathsf{t}(p) = \frac{(m!)^2}{(2m)!}\lambda(\iota_S(\xi)p),$$

for $p \in (S^{m+1}\mathfrak{g})^{\mathfrak{g}}$. Thus, up the factor $2m+1$, this map defines an inverse to the isomorphism $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \lambda(S\mathfrak{g})) \to P(\mathfrak{g})$. $\square$

11.4.2 The space $\mathbf{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g}))$

The algebra morphism $\gamma: U\mathfrak{g} \to \mathrm{Cl}\mathfrak{g}$ is, in some respects, the "quantum counterpart" to the algebra morphism $\lambda: S\mathfrak{g} \to \wedge\mathfrak{g}$. Indeed, since $\gamma(\xi) = q(\lambda(\xi))$ on linear elements $\xi \in \mathfrak{g}$, and since the quantization map intertwines products "modulo lower order terms", one has

$$p \in S^k\mathfrak{g} \;\Rightarrow\; \gamma(q_U(p)) = q(\lambda(p)) \mod \mathrm{Cl}^{(2k-2)}\mathfrak{g}.$$

However,

$$\gamma(U\mathfrak{g}) \neq q(\lambda(S\mathfrak{g}))$$

in general. Let us also note that in contrast to the algebra isomorphism $\mathrm{Cl}\mathfrak{g} = (\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} \otimes \gamma(U\mathfrak{g})$, the map $(\wedge\mathfrak{g})^{\mathfrak{g}} \otimes \lambda(S\mathfrak{g}) \to \wedge\mathfrak{g}$ is not an isomorphism (unless $\mathfrak{g}$ is Abelian): Note e.g., that its image is contained in $\ker(\mathrm{d}_\wedge) \subseteq \wedge\mathfrak{g}$.

Theorem 11.7 below shows that nevertheless, q gives an isomorphism from $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \lambda(S\mathfrak{g}))$ to $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g}))$. One ingredient in the proof of this result is a dimension count. We have $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \lambda(S\mathfrak{g})) = l$, and likewise:

Proposition 11.11 [86] *The multiplicity of the adjoint representation inside*

$$\mathrm{End}(V(\rho)) \cong \gamma(U\mathfrak{g})$$

equals

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g})) = l.$$

In [86], this was obtained from the multiplicity of the adjoint representation in $\wedge\mathfrak{g} \cong \mathrm{Cl}(\mathfrak{g})$, due independently to Kostant [86] and Reeder [110]. In Appendix B, we give an elementary direct proof (cf. Proposition B.20).

A second ingredient is the following bilinear form. (Note the similarity of this result with Proposition 11.9.) Let e_a be a basis of $\mathfrak{g}$, and e^a the B-dual basis.

Proposition 11.12 [86] *The space* $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g}))$ *has a non-degenerate symmetric bilinear form,*

$$(F,\ F') = \sum_{a=1}^{n} F(e_a)F'(e^a),$$

where the right-hand side is given by Clifford multiplication.

Note that the right-hand side is independent of the choice of basis: Viewing $\mathrm{id}_{\mathfrak{g}} \in \mathrm{End}(\mathfrak{g})$ as an element of $\mathfrak{g} \otimes \mathfrak{g}^*$, the right-hand side is the image of the element $(F \otimes (F' \circ B^\sharp))(\mathrm{id}_{\mathfrak{g}}) \in \gamma(U\mathfrak{g}) \otimes \gamma(U\mathfrak{g})$ under Clifford multiplication.

Proof A priori, the right-hand side lies in $\gamma(U\mathfrak{g}) \subseteq \mathrm{Cl}\mathfrak{g}$. But since F and F' are equivariant maps, the expression on the right-hand side is $\mathfrak{g}$-invariant. Hence it lies in $\gamma(U\mathfrak{g}) \cap (\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}$. In terms of the identification $\gamma(U\mathfrak{g}) = \mathrm{End}(V(\rho))$, this scalar may be written in terms of the trace

$$\frac{1}{\dim V(\rho)} \sum_{a=1}^{n} \mathrm{tr}_{V(\rho)}(F(e_a)F'(e^a)),$$

from which it is immediate that the bilinear form is symmetric. Taking the e_a to be an orthonormal real basis of $\mathfrak{g}_{\mathbb{R}}$ (recall that B is positive definite on $\mathfrak{g}_{\mathbb{R}}$), so that $e^a = e_a$, we see that $(F, F^*) \geq 0$ for all $F \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g}))$. Hence $(\cdot, \cdot)$ is non-degenerate. $\square$

For $p \in P_S(\mathfrak{g})$, define maps $\lambda_p \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \lambda(S\mathfrak{g}))$ and $\gamma_p \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g}))$ by

$$\lambda_p(\xi) = \lambda(\iota_S(\xi)p), \quad \gamma_p(\xi) = \gamma(q_U(\iota_S(\xi)p)), \quad \xi \in \mathfrak{g}.$$

If p is homogeneous of degree $m+1$, then $\lambda_p(\xi)$ is homogeneous of degree $2m$, while $\gamma_p(\xi)$ has filtration degree $2m$. Furthermore, since $q \circ \lambda$ and $\gamma \circ q_U$ coincide modulo lower order terms,

$$p \in P_S^{m+1}(\mathfrak{g}) \ \Rightarrow \ \gamma_p(\xi) = q(\lambda_p(\xi)) \mod \mathrm{Cl}^{(2m-2)}\mathfrak{g}. \tag{11.18}$$

Proposition 11.13 [86] *The map* $p \mapsto \gamma_p$ *defines an isomorphism*

$$P_S(\mathfrak{g}) \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g})).$$

Proof By Proposition 11.10, the map $p \mapsto \lambda_p$ is an isomorphism. If $p \in P_S^{m+1}(\mathfrak{g})$ with $\gamma_p = 0$, then (11.18) shows that $\lambda_p = 0$, hence $p = 0$. Hence the map $p \mapsto \gamma_p$ is injective, and since $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g})) = l = \dim P_S(\mathfrak{g})$ by Proposition 11.11, it is an isomorphism. $\square$

Theorem 11.7 [86] *For all $p \in P_S(\mathfrak{g})$,*

$$q(\lambda_p(\xi)) = \gamma_p(\xi).$$

Proof By Proposition 6.19,

$$\iota(\xi)\mathrm{t}(p) = \frac{(m!)^2}{(2m)!}\lambda_p(\xi),$$

for $p \in P_S^{m+1}(\mathfrak{g})$. Hence the theorem amounts to the formula

$$\iota(\xi)q(\mathrm{t}(p)) = \frac{(m!)^2}{(2m)!}\gamma_p(\xi).$$

Let $m_1, \ldots, m_l$ be the exponents, with $m_1 \le m_2 \le \cdots \le m_l$. We will give the proof for a basis $p_1 \ldots, p_l$ of $P_S(\mathfrak{g})$, where p_i has degree $m_i + 1$. Let $\gamma_i = \gamma_{p_i} \in \mathrm{Cl}^{(2m_i)}(\mathfrak{g})$ be the resulting basis of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g}))$, and $\phi_i = \mathrm{t}(p_i) \in P^{2m_i+1}(\mathfrak{g})$ the basis of $P(\mathfrak{g})$. Our goal is to show that

$$\iota(\xi)q(\phi_i) = \frac{(m_i!)^2}{(2m_i)!}\gamma_i(\xi).$$

Denote by $\gamma^1, \ldots, \gamma^l$ the dual basis with respect to the bilinear form $(\cdot, \cdot)$ on $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g}))$. Note that γ^i again has filtration degree $2m_i$. Let ϕ^i be the dual basis to ϕ_i relative to the bilinear form $B_\wedge$.

The identity map of $\mathfrak{g}$ can be regarded as an element of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Cl}\mathfrak{g})$; let $c_i \in (\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}}$ be its coefficients relative to the decomposition

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Cl}\mathfrak{g}) = (\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} \otimes \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \gamma(U\mathfrak{g}))$$

and the basis γ^i. Thus

$$\xi = \sum_{i=1}^{l} c_i \gamma^i(\xi), \tag{11.19}$$

which gives $c_i = \sum_{a=1}^n e_a \gamma_i(e^a)$. Since $\gamma_i(e^a) = \gamma(q_U(\iota_S(e^a)p))$ equals $q(\lambda_i(e^a))$ modulo lower order terms, where $\lambda_i(\xi) = \lambda(\iota_S(\xi)p)$, we obtain $c_i \in (\mathrm{Cl}^{(2m_i+1)}\mathfrak{g})^{\mathfrak{g}}$, and in fact

$$c_i = \frac{(2m_i+1)!}{(m_i!)^2} q(\phi_i) \mod (\mathrm{Cl}^{(2m_i-1)}\mathfrak{g})^{\mathfrak{g}}.$$

Since the space $(\mathrm{Cl}^{(2m+1)}\mathfrak{g})^{\mathfrak{g}}$ is spanned by products $q(\phi_{i_1})\cdots q(\phi_{i_r})$ with $m_{i_1} + \cdots + m_{i_r} \le m$, and since $[q(\phi_r), q(\phi^s)] = 2B_\wedge(\phi_r, \phi^s) = 2\delta_r^s$, we note that $[c_j, q(\phi^i)] = 0$ if $m_i < m_j$, and

$$[c_j, q(\phi^i)] = 2\frac{(2m_j+1)!}{(m_j!)^2}\delta_j^i \quad \text{if } m_i \ge m_j. \tag{11.20}$$

Let $f^i,\ f_i \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Cl}^{(2m_i)}\mathfrak{g})$ be defined as

$$f^i(\xi) = \iota(\xi)q(\phi^i), \quad f_i(\xi) = \iota(\xi)q(\phi_i).$$

By Proposition 11.9,

$$(f^i, f_j) := \sum_{a=1}^{n} f^i(e_a) f_j(e^a) = (2m_i + 1)\delta^i_j. \tag{11.21}$$

We have $\iota(\xi) = \frac{1}{2}[\xi, \cdot]$; hence, using (11.19), $f^i(\xi) = \frac{1}{2}\sum_{j=1}^{l} \gamma^j(\xi)[c_j, q(\phi^i)]$. By (11.20), only indices j with $m_i \le m_j$ can make a contribution, and the contribution for $m_i = m_j$ is known. Thus f^i is of the form

$$f^i = \frac{(2m_i+1)!}{(m_i!)^2}\gamma^i + \sum_{m_j > m_i} d^i_j \gamma^j$$

for some coefficients $d^i_j \in (\mathrm{Cl}\,\mathfrak{g})^{\mathfrak{g}}$. We now want to show that these coefficients are all zero, and that for all $i = 1, \dots, l$,

$$f^i(\xi) = \frac{(2m_i+1)!}{(m_i!)^2}\gamma^i(\xi). \tag{11.22}$$

We will simultaneously show that

$$f_i(\xi) = \frac{(m_i!)^2}{(2m_i)!}\gamma_i(\xi). \tag{11.23}$$

Equation (11.22) holds true for $i = l$, since the sum over $m_j > m_i$ is empty in that case. Suppose by induction that Eq. (11.22), with i replaced by k, have been proved for all $k > i$. The orthogonality property (11.21) then shows that also (11.23), with i replaced by k, holds true for all $k > i$. Let us prove (11.22) for the index i. For all k with $m_k > m_i$, we obtain, using (11.21),

$$\begin{aligned}
0 &= (f_k, f^i) \\
&= \frac{(m_k!)^2}{(2m_k)!}\frac{(2m_i+1)!}{(m_i!)^2}\Big(\gamma_k,\ \gamma^i + \sum_{m_j > m_i} d^i_j \gamma^j\Big) \\
&= \frac{(m_k!)^2}{(2m_k)!}\frac{(2m_i+1)!}{(m_i!)^2} d^i_k.
\end{aligned}$$

Hence $d^i_k = 0$ for all $m_k > m_i$, proving (11.22). □

Corollary 11.1 [86, Theorem 89] *Let $\phi_i \in P^{2m_i+1}(\mathfrak{g})$, $i = 1, \dots, l$, be a homogeneous basis of $P(\mathfrak{g})$, and ϕ^i the $B_\wedge$-dual basis. Then $\iota(\xi)q(\phi^i) \in \gamma(U\mathfrak{g}) \cong$* $\mathrm{End}(V(\rho))$, *and one has the formula*

$$\xi = \sum_{i=1}^{l} q(\phi_i)\,\iota(\xi) q(\phi^i)$$

for all $\xi \in \mathfrak{g}$.

Proof Write $\phi^i = \mathfrak{t}(p^i)$ with $p^i \in P_S^{2m_i}(\mathfrak{g})$. Theorem 11.7 shows in particular that $\iota(\xi)q(\phi^i) = \iota(\xi)q(t(p^i)) \in \gamma(U\mathfrak{g})$ is a multiple of $\gamma^i(\xi)$ (using the notation of the proof). In particular, these elements are linearly independent. By (11.19), we have

$$\xi = \sum_{i=1}^{l} u_i \, \iota(\xi)q(\phi^i)$$

with odd coefficients $u_i \in (\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} = \mathrm{Cl}(P(\mathfrak{g}))$. Taking the commutator with $q(\phi^j)$, we get

$$2\iota(\xi)q(\phi^j) = [q(\phi^j), \xi] = \sum_{i=1}^{l} [q(\phi^j), u_i]\iota(\xi)q(\phi^i).$$

This shows that

$$[q(\phi_j), u_i] = 2\delta_i^j.$$

Since $[q(\phi_j), \cdot]$ acts as a derivation on $(\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} = \mathrm{Cl}(P(\mathfrak{g}))$, and since the u_i are odd, this implies $u_i = q(\phi_i)$. □

Example 11.2 illustrates the ρ-decomposition of linear elements for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$:

$$h = q(\phi)\, \gamma(h), \quad e = q(\phi)\, \gamma(e), \quad f = q(\phi)\, \gamma(f).$$

11.5 The Harish-Chandra projection of $q(P(\mathfrak{g})) \subseteq \mathrm{Cl}\mathfrak{g}$

Recall that by Theorem 11.2, the Harish-Chandra projection for $\mathrm{Cl}\mathfrak{g}$ restricts to an algebra isomorphism

$$\mathsf{hc}_{\mathrm{Cl}} : \ (\mathrm{Cl}\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\cong} \mathrm{Cl}(\mathfrak{t}). \tag{11.24}$$

The following result (and its proof) was shown to us by Kostant in 2003; a written proof was given by Bazlov [22].

Theorem 11.8 (Bazlov, Kostant) *The isomorphism* (11.24) *takes the quantization of the primitive subspace* $P(\mathfrak{g})$ *isometrically onto* $\mathfrak{t} \subseteq \mathrm{Cl}(\mathfrak{t})$.

Proof Let $\phi \in P(\mathfrak{g})$. By Theorem 11.7, we have $\iota(\xi)q(\phi) \in \gamma(U\mathfrak{g})$ for all $\xi \in \mathfrak{g}$. By Proposition 8.8, $\mathsf{hc}_{\mathrm{Cl}}(\gamma(U\mathfrak{g})) = \mathbb{C}$. Hence, if $\xi \in \mathfrak{t} \subseteq \mathfrak{g}$,

$$\iota(\xi)\mathsf{hc}_{\mathrm{Cl}}(q(\phi)) = \mathsf{hc}_{\mathrm{Cl}}(\iota(\xi)q(\phi)) \in \mathbb{C}.$$

Since $\mathsf{hc}_{\mathrm{Cl}}(q(\phi))$ is odd, this implies that $\mathsf{hc}_{\mathrm{Cl}}(q(\phi))$ is linear. Let $\phi, \psi \in P(\mathfrak{g})$ and $\mu = \mathsf{hc}_{\mathrm{Cl}}(q(\phi))$, $\nu = \mathsf{hc}_{\mathrm{Cl}}(q(\psi))$. Since $\mathsf{hc}_{\mathrm{Cl}}$ is a ring homomorphism on invariants, it follows that

$$2B(q(\phi), q(\psi)) = \mathsf{hc}_{\mathrm{Cl}}([q(\phi), q(\psi)]) = [\mu, \nu] = 2B(\mu, \nu).$$

That is, $\mathsf{hc}_{\mathrm{Cl}} \circ q$ restricts to an isometry from $P(\mathfrak{g})$ onto $\mathfrak{t}$. □

Example 11.4 For the cubic element $\phi \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$ from Section 7.1,

$$\mathsf{hc}_{\mathrm{Cl}}(q(\phi)) = B^\sharp(\rho) \in \mathfrak{t};$$

see Section 8.2.

More generally, we have the following explicit formula. Consider the map

$$S^+\mathfrak{g} \to S\mathfrak{g} \otimes \mathfrak{g}, \quad p \mapsto p' := \sum_a \iota_S(e^a) p \otimes e_a.$$

We think of p' as a $\mathfrak{g}$-valued element of $S\mathfrak{g}$.

Proposition 11.14 *Suppose* $p \in P_S^{m+1}(\mathfrak{g})$, *and let* $\phi = \mathsf{t}(p) \in P^{2m+1}(\mathfrak{g})$ *be its transgression. Then*

$$\mathsf{hc}_{\mathrm{Cl}}(q(\phi)) = \frac{(m!)^2}{(2m)!}(\mathrm{ev}_\rho \circ \mathsf{hc}_U \circ q_U \otimes \mathrm{id}_{\mathfrak{g}})(p'), \tag{11.25}$$

where $\mathsf{hc}_U : U\mathfrak{g} \to U\mathfrak{t}$ *is the Harish-Chandra projection for the enveloping algebra, and* ev_ρ *is evaluation at* ρ.

Proof For $\xi \in \mathfrak{t}$, we obtain, using Theorem 11.7 and $\mathsf{hc}_{\mathrm{Cl}} \circ \gamma = \mathrm{ev}_\rho \circ \mathsf{hc}_U$ (Proposition 8.8), that

$$\begin{aligned}
\iota(\xi)\mathsf{hc}_{\mathrm{Cl}}(q(\phi)) &= \mathsf{hc}_{\mathrm{Cl}}(q(\iota(\xi)\phi)) \\
&= \frac{(m!)^2}{(2m)!}\mathsf{hc}_{\mathrm{Cl}}(\gamma(q_U(\iota_S(\xi)p))) \\
&= \frac{(m!)^2}{(2m)!}\mathrm{ev}_\rho(\mathsf{hc}_U(q_U(\iota_S(\xi)p))).
\end{aligned}$$

Since $\mathsf{hc}_{\mathrm{Cl}}(q(\phi))$ is linear, this proves the desired formula. □

The isomorphism $P(\mathfrak{g}) \to \mathfrak{t}$, $\phi \mapsto \mathsf{hc}_{\mathrm{Cl}}(q(\phi))$ defines a grading on $\mathfrak{t}$, where

$$\mathfrak{t}^m = \mathsf{hc}_{\mathrm{Cl}}(q(P^{2m+1})) \tag{11.26}$$

is spanned by the elements (11.25) with $p \in P_S^{m+1}(\mathfrak{g})$. The subspace $\mathfrak{t}^m$ is non-zero if and only if m is an exponent of $\mathfrak{g}$, and its dimension is the multiplicity of the exponent. Since the decomposition $P(\mathfrak{g}) = \bigoplus_m P^{2m+1}(\mathfrak{g})$ is orthogonal, the decomposition $\mathfrak{t} = \bigoplus_m \mathfrak{t}^m$ is again orthogonal. The space $\mathfrak{t}^0$ is just the center $\mathfrak{z} = \mathfrak{g}$. The space $\mathfrak{t}^1$ corresponds to the cubic primitive elements in $\mathfrak{g}$, hence its dimension is the number of simple summands of $\mathfrak{g}$. Due to the orthogonality, the grading $\mathfrak{t} = \bigoplus_m \mathfrak{t}^m$ may be recovered from the associated filtration

$$\mathfrak{t}^{(m)} = \mathsf{hc}_{\mathrm{Cl}}(q(P^{(2m+1)}(\mathfrak{g})))$$

as $\mathfrak{t}^m = \mathfrak{t}^{(m)} \cap (\mathfrak{t}^{(m-1)})^\perp$. The following result describes this filtration without reference to Clifford algebras.

Proposition 11.15 (Alekseev–Moreau [9]) *For all m,*

$$\mathfrak{t}^{(m)} = (\mathrm{ev}_\rho \circ \mathsf{hc}_U \otimes \mathrm{id}_\mathfrak{g})((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^\mathfrak{g}). \tag{11.27}$$

Proof Recall again that $\mathrm{ev}_\rho \circ \mathsf{hc}_U = \mathsf{hc}_{\mathrm{Cl}} \circ \gamma$. By Kostant's result from Section 11.4.2, the space $(\gamma(U^{(m)}\mathfrak{g}) \otimes \mathfrak{g})^\mathfrak{g} \cong \mathrm{Hom}_\mathfrak{g}(\mathfrak{g}, \gamma(U^{(m)}\mathfrak{g}))$ is spanned by the set of all $\sum_{a=1}^n \iota(e^a)q(\mathsf{t}(p)) \otimes e_a$ with $p \in P_S^{(m+1)}(\mathfrak{g})$. Hence the right-hand side of (11.27) becomes

$$\Big\{\sum_{a=1}^n \mathsf{hc}_{\mathrm{Cl}}\big(\iota(e^a)q(\mathsf{t}(p))\big) \otimes e_a \Big|\ p \in S^{(m+1)}(\mathfrak{g})\Big\}.$$

If e_α is a root vector for a positive (resp. negative) root, then $\iota(e_\alpha)q(\mathsf{t}(p)) \in \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$ (resp. in $\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})$), since $q(\mathsf{t}(p))$ is invariant. Hence these terms vanish under $\mathsf{hc}_{\mathrm{Cl}}$, and we may restrict the sum to a basis e_i, $i = 1, \ldots, l$ of $\mathfrak{t}$. But contraction by $e_i \in \mathfrak{t}$ commutes with $\mathsf{hc}_{\mathrm{Cl}}$. We hence obtain

$$\Big\{\sum_{i=1}^l \iota(e^i)\mathsf{hc}_{\mathrm{Cl}}(q(\mathsf{t}(p))) \otimes e_i \Big|\ p \in S^{(m+1)}(\mathfrak{g})\Big\}.$$

Since $\mathsf{hc}_{\mathrm{Cl}}(q(\mathsf{t}(p))) \in \mathfrak{t}$, this is exactly the space $\mathsf{hc}_{\mathrm{Cl}}(P^{(2m+1)}(\mathfrak{g}))$. □

Since $\mathsf{hc}_U \circ q_U : S\mathfrak{g} \to U\mathfrak{t} = S\mathfrak{t}$ coincides with the orthogonal projection $\mathsf{p}_S : S\mathfrak{g} \to S\mathfrak{t}$ up to lower order terms, Proposition 11.15 has the immediate consequence,

$$\mathfrak{t}^{(m)} = (\mathrm{ev}_\rho \circ \mathsf{p}_S \otimes \mathrm{id}_\mathfrak{g})(S^{(m)}\mathfrak{g} \otimes \mathfrak{g})^\mathfrak{g}.$$

The following is a slightly more precise version of a result of Rohr [112].

Proposition 11.16 *Let $\phi_1, \ldots, \phi_l$ be an orthogonal basis of $P(\mathfrak{g})$, with $\phi_i = \mathsf{t}(p_i)$ for elements $p_i \in (S^{m_i+1}\mathfrak{g})^\mathfrak{g}$, $m_1 \le \cdots \le m_l$. Let $\nu_i = \mathsf{hc}_{\mathrm{Cl}}(\mathsf{t}(p_i)) \in \mathfrak{t}$ be the resulting orthogonal basis of $\mathfrak{t}$. Then $\nu_1, \ldots, \nu_l$ is obtained by applying the Gram–Schmidt orthogonalization procedure to the basis $\tilde{\nu}_1, \ldots, \tilde{\nu}_l$, given as*

$$\tilde{\nu}_i = \frac{(m_i!)^2}{(2m_i)!}\mathrm{ev}_\rho(p_i') \in \mathfrak{t}.$$

Proof We have, $\mathsf{hc}_U(q_U(p_i')) = \mathsf{p}_S(p_i') \mod (S^{(m_i-1)}\mathfrak{g} \otimes \mathfrak{g})^\mathfrak{g}$. Hence, Proposition 11.14 shows that $\nu_i = \tilde{\nu}_i \mod \mathfrak{t}^{(m_i-1)}$. Since $\mathfrak{t}^{(m_i-1)}$ is spanned by ν_j with $m_j < m_i$, it follows that the Gram–Schmidt procedure takes the ordered basis $\tilde{\nu}_1, \ldots, \tilde{\nu}_l$ to $\nu_1, \ldots, \nu_l$. □

11.6 Relation with the principal TDS

In 2003, Kostant made an interesting conjecture relating the decomposition $\mathfrak{t} = \bigoplus_m \mathfrak{t}^m$ to the theory of the *principal TDS* [83]. The conjecture was settled in 2011 through work of Joseph and Alekseev–Moreau. In this section we give a brief survey of these developments, referring to the original articles for proofs and further details.

To state the conjecture, we need to review some background material from [83]. (A nice discussion may be also found in Vogan's article [118].) Let $\mathfrak{g}$ be a semisimple Lie algebra, with positive Weyl chamber $\mathfrak{t}_+$. Up to conjugacy, there is a finite number of Lie algebra morphisms $\iota:\ \mathfrak{sl}(2,\mathbb{C})\to\mathfrak{g}$. If ι is non-zero, then its image is called a "TDS", an acronym for "three-dimensional (Lie) subalgebra". Let $e,\ f,\ h\in\mathfrak{sl}(2,\mathbb{C})$ be the standard basis, so that $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$. We will identify these elements with their images in $\mathfrak{g}$. Since the element h is semisimple in $\mathfrak{g}$, it is conjugate to some element of $\mathfrak{t}_+$. The conjugacy class of any TDS is uniquely determined by the element of $\mathfrak{t}_+$ obtained in this way.

It turns out that there is one conjugacy class of TDS's for which the semisimple element h is conjugate to $\rho^\vee$, the half-sum of positive co-roots. Equivalently [83], the nilpotent element e is *principal nilpotent*. Any element of this conjugacy class of TDS's is called a *principal TDS*. To be explicit, let $e_1,\ldots,e_l$ be root vectors for the simple roots $\alpha_1,\ldots,\alpha_l$, and let $f_1,\ldots,f_l$ be the root vectors for the roots $-\alpha_1,\ldots,-\alpha_l$, normalized by the condition $[e_i,f_i]=h_i:=\alpha_i^\vee$. For $i\neq j$ one has $[e_i,f_j]=0$ (since the difference of simple roots is never a root). Let $r_i\in\mathbb{C}$ be determined by the equation $\sum_{i=1}^l r_i\alpha_i^\vee=\rho^\vee$ (i.e., $r_i=\langle\varpi_i,\rho^\vee\rangle$, where ϖ_i are the fundamental weights). Then

$$e=\sum_{i=1}^l e_i,\quad f=\sum_{i=1}^l r_i f_i,\quad h=\rho^\vee$$

spans a principal TDS. For instance,

$$[h,e]=2\sum_{i=1}^l\langle\alpha_i,\rho^\vee\rangle e_i=2e$$

since $\langle\alpha_i,\rho^\vee\rangle=1$ for $i=1,\ldots,l$. We will refer to this choice as *the* principal TDS, denoted $\mathfrak{s}\subseteq\mathfrak{g}$. In [83], Kostant proved a wealth of beautiful properties of the principal TDS. One of his results is

Theorem 11.9 (Kostant [83]) *Let $\mathfrak{g}$ be a complex simple Lie algebra, with exponents $m_1\le\cdots\le m_l$. Under the adjoint action of the principal TDS $\mathfrak{s}\subseteq\mathfrak{g}$, the Lie algebra $\mathfrak{g}$ breaks up into l irreducible components $\mathfrak{g}=\bigoplus_{i=1}^l\mathfrak{g}_i$, where* $\dim\mathfrak{g}_i=2m_i+1$.

Note that this result "explains" the coincidence $\sum_{i=1}^l(2m_i+1)=n$. Since the $\mathfrak{g}_i$ are all odd-dimensional, their zero weight spaces $\mathfrak{g}_i\cap\mathfrak{t}$ are 1-dimensional. We may define a grading on $\mathfrak{t}$, by taking the m-th graded component to be the sum over all zero weight spaces for the $\mathfrak{g}_i$ with $m_i=m$. Let us refer to this grading, and the associated filtration, as the *TDS grading (resp. filtration) of* $\mathfrak{t}$ *with respect to* $\mathfrak{g}$.

It is reminiscent of the grading $\mathfrak{t} = \bigoplus_m \mathfrak{t}^m$ defined in (11.26), since $\dim \mathfrak{t}^m$ is the multiplicity of the exponent m.

There is one caveat, however. Suppose temporarily that $\mathfrak{g}$ is simple. Then $\mathfrak{g}_1 \cong \mathfrak{s}$ has its zero weight space spanned by $h = \rho^\vee$. On the other hand, we saw that $\mathfrak{t}^1$ is spanned by the element ρ, as opposed to $\rho^\vee$. This suggests passing to the dual Lie algebra $\mathfrak{g}^\vee$. Recall that the Cartan subalgebra of $\mathfrak{g}^\vee$ is the dual $\mathfrak{t}^*$ of the Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$, and that the roots of $\mathfrak{g}^\vee$ are the co-roots of $\mathfrak{g}$ and vice versa. Using the bilinear form B on $\mathfrak{g}$, we identify $\mathfrak{t}$ and $\mathfrak{t}^*$.

Theorem 11.10 (Kostant Conjecture) *The grading $\mathfrak{t} = \bigoplus \mathfrak{t}^m$, where*

$$\mathfrak{t}^m = \mathsf{hc}_{\mathrm{Cl}}(q(P^{2m+1}(\mathfrak{g}))),$$

coincides with the TDS grading of $\mathfrak{t}$ with respect to the dual Lie algebra $\mathfrak{g}^\vee$. That is, $\mathfrak{t}^m$ equals the sum over zero weight spaces $\mathfrak{g}_i^\vee \cap \mathfrak{t}$ with $m_i = m$.

This result was proved in Bazlov's thesis [21] for Lie algebras of type A, by explicit computation of both sides. The general case was established by A. Joseph, in combination with results of Rohr and Alekseev–Moreau.

The proof involves the following "generalized Harish-Chandra projections" of Khoroshkin, Nazarov and Vinberg [78]. Suppose V is a finite-dimensional completely reducible $\mathfrak{g}$-representation. Consider $U(\mathfrak{g}) \otimes V$ as a representation of $\mathfrak{n}_- \times \mathfrak{n}_+$, where $\mathfrak{n}_-$ acts by multiplication on $U(\mathfrak{g})$ from the left, and $\mathfrak{n}_+$ acts via $e_\alpha.(x \otimes v) = x \otimes e_\alpha.v - x e_\alpha \otimes v$. One has

$$U\mathfrak{g} \otimes V = S\mathfrak{t} \otimes V \oplus (\mathfrak{n}_- \times \mathfrak{n}_+).(U\mathfrak{g} \otimes V).$$

Let hc_{KNV} be the map given as projection to the first summand. It takes invariants into the subspace $S\mathfrak{t} \otimes V^{\mathfrak{t}}$:

$$\mathsf{hc}_{KNV}:\ (U\mathfrak{g} \otimes V)^{\mathfrak{g}} \to S\mathfrak{t} \otimes V^{\mathfrak{t}}.$$

The main result of [78] describes the image of this "generalized Harish-Chandra projection" in terms of *Zhelobenko projectors* (generalizations [121] of the extremal projectors from Section 11.2).

The result applies in particular to the case $V = \mathfrak{g}$, giving a projection

$$\mathsf{hc}_{KNV}:\ (U\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}} \to S\mathfrak{t} \otimes \mathfrak{t}$$

(different, in general, from $\mathsf{hc}_U \otimes \mathrm{id}_{\mathfrak{g}}$).

After some additional work on Zhelobenko operators [71], Joseph [72] proved the following result:

Theorem 11.11 (Joseph) *The TDS filtration of $\mathfrak{t}$ with respect to $\mathfrak{g}^\vee$, with m-th component the sum over zero weight spaces $\mathfrak{g}_i^\vee \cap \mathfrak{t}$ with $m_i \leq m$, coincides with the filtration by the subspaces*

$$(\mathrm{ev}_\rho \otimes \mathrm{id}_{\mathfrak{g}}) \circ \mathsf{hc}_{KNV}((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) \subseteq \mathfrak{t}.$$

In light of the description (11.27) of the filtration $\mathfrak{t}^{(m)}$, this would establish the Kostant Conjecture if one could replace hc_{KNV} with $\mathsf{hc}_U \otimes \mathrm{id}_{\mathfrak{g}}$. Alekseev and Moreau [9] proved that indeed

$$(\mathrm{ev}_\rho \otimes \mathrm{id}_{\mathfrak{g}})\mathsf{hc}_{KNV}((U^{(m)}\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}) = \mathfrak{t}^{(m)},$$

which shows that Joseph's Theorem implies the Kostant Conjecture. Shortly later, Joseph [70] gave a direct argument for the equality of the TDS filtration with (11.27), thus providing another proof.

Appendix A
Graded and filtered super spaces

A.1 Super vector spaces

A *super vector space* is a vector space E equipped with a $\mathbb{Z}_2$-grading,

$$E = E^{\bar{0}} \oplus E^{\bar{1}}.$$

Elements in $E^{\bar{0}}$ will be called even; elements of $E^{\bar{1}}$ are called odd. We will denote by $|v| \in \{\bar{0}, \bar{1}\}$ the parity of homogeneous elements $v \in E$. (Whenever this notation is used, it is implicitly assumed that v is homogeneous.) A *morphism of super vector spaces* $\phi: E \to F$ is a linear map preserving $\mathbb{Z}_2$-gradings. We will denote by $\mathrm{Hom}(E, F)$ the space of *all* linear maps $E \to F$, not necessarily preserving $\mathbb{Z}_2$-gradings. It is itself a super vector space with

$$\mathrm{Hom}(E, F)^{\bar{0}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{0}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{1}}),$$
$$\mathrm{Hom}(E, F)^{\bar{1}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{1}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{0}}).$$

The space of morphisms of super vector spaces $E \to F$ is thus the even subspace $\mathrm{Hom}(E, F)^{\bar{0}}$. Direct sums and tensor products of super vector spaces are just the usual tensor products of vector spaces with $\mathbb{Z}_2$-gradings

$$(E \oplus F)^{\bar{0}} = E^{\bar{0}} \oplus F^{\bar{0}}, \quad (E \oplus F)^{\bar{1}} = E^{\bar{1}} \oplus F^{\bar{1}},$$

respectively

$$(E \otimes F)^{\bar{0}} = (E^{\bar{0}} \otimes F^{\bar{0}}) \oplus (E^{\bar{1}} \otimes F^{\bar{1}}),$$
$$(E \otimes F)^{\bar{1}} = (E^{\bar{0}} \otimes F^{\bar{1}}) \oplus (E^{\bar{1}} \otimes F^{\bar{0}}).$$

If E is a super vector space, and $n \in \mathbb{Z}$, we denote by $E[n]$ the same vector space with $\mathbb{Z}_2$-grading shifted by n mod 2. The "*super sign convention*" which we adopt asserts that the interchange of any two odd objects introduces a minus sign. We will take the categorical viewpoint, advocated in [44], that the super-sign convention is built into the definition of a commutativity isomorphisms. In particular, for the tensor product commutativity isomorphism, one has

$$\mathscr{T}: \ E \otimes F \mapsto F \otimes E, \ \ v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

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The category of super vector spaces is a *tensor category*, with direct sums, tensor products, and a commutativity isomorphism as defined above. One may then define super algebras, super Lie algebras, super coalgebras, etc. as the algebra objects, Lie algebra objects, coalgebra objects etc. in this tensor category. Similarly, various standard constructions with these objects are naturally defined in terms of "categorical constructions".

For example, a *super Lie algebra* is a super vector space $\mathfrak{g}$ together with a bracket $[\cdot,\cdot]\colon\ \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ satisfying the following axioms:

$$[\cdot,\cdot]\circ\mathscr{T}=-[\cdot,\cdot]$$

(skew symmetry) and

$$[\cdot,\cdot]\circ(\mathrm{id}\otimes[\cdot,\cdot])=[\cdot,\cdot]\circ([\cdot,\cdot]\otimes\mathrm{id})+[\cdot,\cdot]\circ(\mathrm{id}\otimes[\cdot,\cdot])\circ(\mathscr{T}\otimes\mathrm{id})$$

(Jacobi identity). On homogeneous elements, the two conditions read

$$[u,v]=-(-1)^{|u||v|}[v,u],$$
$$[u,[v,w]]=[[u,v],w]+(-1)^{|u||v|}[v,[u,w]].$$

Super algebras are $\mathbb{Z}_2$-graded algebras $\mathscr{A}$, such that the multiplication map $m_{\mathscr{A}}\colon \mathscr{A}\otimes\mathscr{A}\to\mathscr{A}$ preserves $\mathbb{Z}_2$-gradings. The super-sign convention makes its appearance when we define the tensor product $\mathscr{A}\otimes\mathscr{B}$ of two such algebras. By definition the multiplication of the tensor algebra is

$$m_{\mathscr{A}\otimes\mathscr{B}}=(m_{\mathscr{A}}\otimes m_{\mathscr{B}})\circ(\mathrm{id}\otimes\mathscr{T}\otimes\mathrm{id}),$$

a composition of maps $\mathscr{A}\otimes\mathscr{B}\otimes\mathscr{A}\otimes\mathscr{B}\to\mathscr{A}\otimes\mathscr{A}\otimes\mathscr{B}\otimes\mathscr{B}\to\mathscr{A}\otimes\mathscr{B}$. Writing $(m_{\mathscr{A}}\otimes m_{\mathscr{B}})((x\otimes y)\otimes(x'\otimes y'))=(x\otimes y)(x'\otimes y')$ the multiplication map is

$$(x\otimes y)(x'\otimes y')=(-1)^{|x'||y|}\,xx'\otimes yy'.$$

The sign convention also shows up when one writes the definition of the commutator $[\cdot,\cdot]=m_{\mathscr{A}}-m_{\mathscr{A}}\otimes\mathscr{T}$ for a super algebra $\mathscr{A}$:

$$[x,y]=xy-(-1)^{|x||y|}yx.$$

This bracket makes $\mathscr{A}$ into a super Lie algebra. The center $\mathrm{Cent}(\mathscr{A})$ of the super algebra $\mathscr{A}$ is the collection of elements x such that $[x,y]=0$ for all $y\in\mathscr{A}$. The super algebra $\mathscr{A}$ is called commutative if $\mathrm{Cent}(\mathscr{A})=\mathscr{A}$, i.e. , $[\mathscr{A},\mathscr{A}]=0$. A *trace* on a super algebra is a morphism $\mathrm{tr}\colon\ \mathscr{A}\to\mathbb{K}$ that vanishes on $[\mathscr{A},\mathscr{A}]\subseteq\mathscr{A}$. An endomorphism $D\in\mathrm{End}(\mathscr{A})$ is called a *derivation* of the super algebra if

$$D(xy)=(Dx)y+(-1)^{|D||x|}x(Dy) \tag{A.1}$$

for all homogeneous elements x,y. The space $\mathrm{Der}(\mathscr{A})$ of such derivations is a super Lie subalgebra of $\mathrm{End}(\mathscr{A})$. Some basic properties of derivations of a super algebra are:

1. Any $D\in\mathrm{Der}(\mathscr{A})$ vanishes on scalars $\mathbb{K}\subseteq\mathscr{A}$. This is immediate from the definition (A.1), applied to $x=y=1$.
2. Derivations are determined by their values on algebra generators.

3. $\mathrm{Der}(\mathscr{A})$ is a left module over the center $\mathrm{Cent}(\mathscr{A})$.
4. The map $x \mapsto [x, \cdot]$ defines a morphism of super Lie algebras $\mathscr{A} \to \mathrm{Der}(\mathscr{A})$. Derivations of this type are called *inner*.

One similarly defines tensor products, the center, and derivations of super Lie algebras.

Remark A.1 Super algebras can be viewed as ordinary algebras by forgetting the $\mathbb{Z}_2$-grading. To avoid misunderstandings, one sometimes refers to commutators, the center, traces, and derivations of a super algebra as super commutators, the super center, super traces, and super derivations. (The terms "anti-commutator" and "anti-derivation" have been used in the older literature, but their use has become less common.)

For any super vector space E, the tensor algebra $T(E)$ is a super algebra, characterized by the universal property:

Proposition A.1 *For any super algebra $\mathscr{A}$ and morphism of super vector spaces $E \to \mathscr{A}$, there is a unique extension to a morphism of super algebras $T(E) \to \mathscr{A}$.*

The symmetric algebra $S(E)$ is the quotient of the tensor algebra by the two-sided ideal generated by all $v \otimes w - (-1)^{|v||w|} w \otimes v$ for homogeneous elements $v, w \in E$. $S(E)$ has the following universal property.

Proposition A.2 *For any commutative super algebra $\mathscr{A}$ and morphism of super vector spaces $E \to \mathscr{A}$, there is a unique extension to a morphism of super algebras $S(E) \to \mathscr{A}$.*

A.2 Graded super vector spaces

A *graded vector space* is a vector space equipped with a $\mathbb{Z}$-grading $E = \bigoplus_{k \in \mathbb{Z}} E^k$. (One can consider gradings by more general Abelian groups, but we will not need it.) The degree of a homogeneous element v is denoted by $|v| \in \mathbb{Z}$. A morphism of graded vector spaces is a degree-preserving linear map. The direct sum of two graded vector spaces E, F is graded as $(E \oplus F)^k = E^k \oplus F^k$, while the tensor product is graded as

$$(E \otimes F)^k = \bigoplus_{i \in \mathbb{Z}} E^i \otimes F^{k-i}.$$

One can make graded vector spaces into a tensor category in two ways. Taking the commutativity isomorphism $E \otimes F \to F \otimes E$ to be $v \otimes w \mapsto w \otimes v$, one obtains what we will call the *category of graded vector spaces*. Taking the isomorphism to be $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$, one obtains the *category of graded super vector spaces*. In the second case, the $\mathbb{Z}_2$-grading is just the mod 2 reduction of the $\mathbb{Z}$-grading.

The algebra objects, Lie algebra objects, and so on in the category of graded vector spaces will be called *graded algebras*, *graded Lie algebras*, and so on, while those in the category of graded super vector spaces will be called *graded super algebras*, *graded super Lie algebras*, and so on. For instance, a graded (super) Lie algebra is a graded (super) vector space, which is also a (super) Lie algebra, with the property $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$.

Example A.1 The exterior algebra $\wedge(V) = \bigoplus_k \wedge^k(V)$ over a vector space V is an example of a commutative graded super algebra. Under direct sum, $\wedge(V \oplus W) = \wedge(V) \otimes \wedge(W)$ as graded super algebras (but not as ordinary algebras). On the other hand, $S(V)$ is a commutative graded algebra. One has $S(V \oplus W) = S(V) \otimes S(W)$ as graded algebras (but not as super algebras).

It is often convenient to regard graded spaces as graded super spaces by *degree doubling*. For example, the symmetric algebra $S(V) = \bigoplus_k S^k(V)$ over a vector space V is a commutative graded algebra. It becomes a commutative graded superalgebra for the double grading

$$S(V)^{2k} = S^k(V), \quad S(V)^{2k+1} = 0.$$

Given a graded vector space E, and any $n \in \mathbb{Z}$, one defines a new graded vector space $E[n]$ by degree shift:

$$(E[n])^k = E^{n+k}.$$

Thus, if $v \in E$ has degree k, then its degree in $E[n] \cong E$ is $k-n$. For a graded super vector space, this operation changes the $\mathbb{Z}_2$-grading by $(-1)^n$. Note also that

$$E[n] = E \otimes \mathbb{K}[n],$$

where $\mathbb{K}$ is the 1-dimensional graded vector space given by a copy of $\mathbb{K}$ in degree $-n$.

If E and F are graded (super) vector spaces, define $\mathrm{Hom}^k(E, F)$ as the set of linear maps $\phi : E \to F$ of degree k, i.e., such that $\phi(E^i) \subseteq F^{k+i}$. Equivalently, $\mathrm{Hom}^k(E, F)$ consists of morphisms of graded vector spaces $E \to F[k]$. One has

$$\bigoplus_k \mathrm{Hom}^k(E, F) \subseteq \mathrm{Hom}(E, F) \subseteq \prod_k \mathrm{Hom}^k(E, F).$$

In general, each of the two inclusions can be strict.

Example A.2 Suppose E, F are graded (super) vector spaces. Then

$$E^* = \mathrm{Hom}(E, \mathbb{K}) = \prod_k \mathrm{Hom}^k(E, \mathbb{K}) = \prod_k (E^{-k})^*$$

while

$$F \cong \mathrm{Hom}(\mathbb{K}, F) = \bigoplus_k \mathrm{Hom}^k(\mathbb{K}, F) = \bigoplus_k F^k.$$

If $F = E$ we write $\mathrm{End}^k(E) = \mathrm{Hom}^k(E, E)$. Then $\bigoplus_k \mathrm{End}^k(E)$ is a graded Lie algebra under commutation. Similarly, if E is a graded super vector space, then $\bigoplus_k \mathrm{End}^k(E)$ is a graded super Lie algebra.

Suppose $\mathscr{A}$ is a graded algebra, and let $\mathrm{Der}(\mathscr{A})$ be the Lie algebra of derivations. The elements of

$$\mathrm{Der}^k(\mathscr{A}) = \mathrm{Der}(\mathscr{A}) \cap \mathrm{End}^k(\mathscr{A})$$

are called *derivations of degree k of the graded algebra* $\mathscr{A}$. The direct sum of these spaces is a graded Lie subalgebra

$$\bigoplus_k \mathrm{Der}^k(\mathscr{A}) \subseteq \bigoplus_k \mathrm{End}^k(\mathscr{A}).$$

In a similar way, one defines *derivations of degree k of a graded super algebra* by taking $\mathrm{Der}(\mathscr{A})$ to be the derivations as a super algebra.

If E is a graded super vector space, then the tensor algebra $T(E)$ and the symmetric algebra $S(E)$ acquire the structure of graded super algebras, in such a way that the inclusion map $E \to T(E)$ resp. $E \to S(E)$ is a morphism of graded super vector spaces. These *internal gradings* are not to be confused with the *external gradings* $T(E) = \bigoplus_{k \geq 0} T^k(E)$ resp. $S(E) = \bigoplus_{k \geq 0} S^k(E)$. Sometimes, we also consider the *total gradings*, given by the internal grading plus twice the external grading. Then $S(E)$ and $T(E)$ are also graded superalgebras relative to the total grading.

A.3 Filtered super vector spaces

A *filtered vector space* is a vector space E together with a sequence of subspaces $E^{(k)}$, $k \in \mathbb{Z}$, such that $E^{(k)} \subseteq E^{(k+1)}$ and

$$\bigcap_k E^{(k)} = 0, \quad \bigcup_k E^{(k)} = E.$$

A *morphism of filtered vector spaces* is a linear map $\phi : E \to F$ taking $E^{(k)}$ to $F^{(k)}$, for all k. Direct sums of filtered vector spaces are filtered in the obvious way, while tensor products are filtered as

$$(E \otimes F)^{(k)} = \bigoplus_i E^{(i)} \otimes F^{(k-i)}.$$

Given a filtered vector space E, and $n \in \mathbb{Z}$, one defines a new filtered vector space $E[n]$ to be the space E with the shifted filtration $E[n]^{(k)} = E^{(n+k)}$.

Filtered vector spaces form a tensor category, hence we can speak of filtered algebras, filtered Lie algebras, and so on by requiring that the relevant structure maps be morphisms. A typical example of a filtered algebra is the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra. If E and F are filtered vector spaces, we define $\mathrm{Hom}^{(k)}(E, F)$ to be the space of linear maps $E \to F$ raising the filtration degree by k. Note that the union over these spaces is usually smaller than $\mathrm{Hom}(E, F)$, since a general linear map $E \to F$ need not have any finite filtration degree.

We put $\mathrm{End}^{(k)}(E) = \mathrm{Hom}^{(k)}(E, E)$. If $\mathscr{A}$ is a filtered algebra, we define

$$\mathrm{Der}^{(k)}(\mathscr{A}) = \mathrm{Der}(\mathscr{A}) \cap \mathrm{End}^{(k)}(\mathscr{A}).$$

Suppose E is a filtered vector space, and let $E^* = \mathrm{Hom}(E, \mathbb{K})$ be its dual. A linear map $E \to \mathbb{K}$ has filtration degree k if and only if it takes $E^{(l)}$ to 0 for all $l < -k$. That is,

$$(E^*)^{(k)} = \mathrm{ann}(E^{(-k-1)}).$$

If the filtration of E is bounded below in the sense that $E^{(l)} = 0$ for some l, then any element of E^* has finite filtration degree. Hence, in this case the $(E^*)^{(k)}$ define a filtration of E^*.

Suppose E is a filtered vector space. The *associated graded vector space* $\mathrm{gr}(E)$ is defined as the direct sum over $\mathrm{gr}^k(E) = E^{(k)}/E^{(k-1)}$. A morphism ϕ of filtered vector spaces induces a morphism $\mathrm{gr}(\phi)$ of associated graded spaces. In this way, gr becomes a functor from the tensor category of filtered vector spaces to the tensor category of graded vector spaces. The associated graded object to a filtered algebra is a graded algebra, the associated graded object to a filtered Lie algebra is a graded Lie algebra, and so on. If E has a filtration, which is bounded in the sense that $E^{(l)} = 0$ and $E^{(m)} = E$ for some l, m, then the filtration on E^* is bounded, and $\mathrm{gr}(E^*) = \mathrm{gr}(E)^*$.

Any graded vector space $E = \bigoplus_k E^k$ can be regarded as a filtered vector space, by putting $E^{(k)} = \bigoplus_{i \le k} E^i$. In this case, $\mathrm{gr}(E) \cong E$.

A *filtered super vector space* [90] is a super vector space E, provided with a filtration $E^{(k)}$ by super subspaces, such that

$$(E^{(k)})^{\bar{0}} = (E^{(k+1)})^{\bar{0}}, \quad \text{for } k \text{ even},$$

$$(E^{(k)})^{\bar{1}} = (E^{(k+1)})^{\bar{1}}, \quad \text{for } k \text{ odd}.$$

Equivalently, the $\mathbb{Z}_2$-grading on the associated graded space $\mathrm{gr}(E)$ is the mod 2 reduction of the $\mathbb{Z}$-grading, making $\mathrm{gr}(E)$ into a graded super vector space. If E is a filtered super vector space with a bounded filtration, then E^* is again a filtered super vector space. An example of a filtered super space is the Clifford algebra $\mathrm{Cl}(V; B)$ of a vector space V with bilinear form B. The filtered super vector spaces form a tensor algebra; hence there are notions of filtered super algebras, filtered super Lie algebras, a space $\mathrm{Der}^{(k)}(\mathscr{A})$ of degree k derivations of a filtered super algebra, and so on.

If E is a filtered super vector space, then the tensor algebra $T(E)$ and the symmetric algebra $S(E)$ acquire the structure of filtered super algebras, in such a way that the inclusion map $E \to T(E)$ resp. $E \to S(E)$ is a morphism of filtered super vector spaces. We will refer to this as the *internal filtration*. Sometimes we also consider the *total filtration*, obtained by adding twice the external filtration degree. The total filtration is such that the map $E[-2] \to T(E)$ resp. $E[-2] \to S(E)$ is filtration-preserving.

Appendix B
Reductive Lie algebras

Throughout this section $\mathbb{K}$ denotes a field of characteristic zero, and all Lie algebras are taken to be finite-dimensional. We soon specialize to the case $\mathbb{K} = \mathbb{C}$. Standard references for the material below are Bourbaki [26–28] and Humphreys [67].

B.1 Definitions and basic properties

A Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is called *simple* if it is non-Abelian and has no ideals other than $\{0\}$ and $\mathfrak{g}$. A Lie algebra is called *semisimple* if it is a direct sum of simple Lie algebras. A Lie algebra is *reductive* if it is the direct sum of a semisimple and an Abelian Lie algebra. These conditions can be expressed in a number of equivalent ways. Most importantly, a Lie algebra is semisimple if and only if the *Killing form*

$$B_{\mathrm{Kil}}(\xi, \zeta) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_\xi \mathrm{ad}_\zeta)$$

is non-degenerate (cf. [26, I. §6.1]). Reductive Lie algebras are characterized by the existence of a finite-dimensional representation $\pi : \mathfrak{g} \to \mathrm{End}(V)$ such that the bilinear form $B(\xi, \zeta) = \mathrm{tr}_V(\pi(\xi)\pi(\zeta))$ is non-degenerate (cf. [26, I. §6.4]). For instance, one may take V to be the direct sum of the adjoint representation of the semisimple part and a diagonalizable faithful representation of the center.

Examples B.1

1. The 3-dimensional Lie algebra $\mathfrak{g}$ with basis e, f, h and bracket relations
$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f \tag{B.1}$$
is semisimple. It is isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbb{K})$ of tracefree 2×2-matrices under the identification
$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
2. The Lie algebra $\mathfrak{gl}(n, \mathbb{K}) = \mathrm{Mat}_n(\mathbb{K})$ of $n \times n$-matrices is reductive, and its subalgebra $\mathfrak{sl}(n, \mathbb{K})$ of trace-free matrices is semisimple.

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3. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the Lie algebra $\mathfrak{so}(n, \mathbb{K})$ is semisimple for $n \geq 3$.
4. The Lie algebra $\mathfrak{g}$ of any compact real Lie group G is reductive. It is semisimple if and only if the center of G is finite.

Definition B.1 A $\mathfrak{g}$-representation $\pi : \mathfrak{g} \to \mathrm{End}(V)$ is called *irreducible* if V does not contain $\mathfrak{g}$-invariant subspaces other than V or $\{0\}$. It is called *completely reducible* if it is a direct sum of irreducible subspaces.

We will use the terms $\mathfrak{g}$-*representation* and $\mathfrak{g}$-*module* interchangeably.

According to *Weyl's Theorem* (cf. [26, I. §6.1]), any finite-dimensional representation V of a semisimple Lie algebra $\mathfrak{g}$ is completely reducible. If $\mathfrak{g}$ is only reductive, the complete reducibility of $\mathfrak{g}$-representations is not automatic: A counterexample is provided by the representation of the Lie algebra $\mathfrak{g} = \mathbb{K}$ on $V = \mathbb{K}^2$ as strictly upper triangular 2×2-matrices. It does hold provided the center acts by diagonalizable endomorphisms.

A Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ is called *compact* if it admits an invariant symmetric bilinear form that is positive definite. This is equivalent to the existence of a compact Lie group G having $\mathfrak{g}$ as its Lie algebra (cf. [28, IX. §1.3]).

Suppose $\mathfrak{g}$ is a Lie algebra over $\mathbb{K} = \mathbb{C}$. A Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is called a *real form* of $\mathfrak{g}$ if $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$. A complex Lie algebra $\mathfrak{g}$ is reductive if and only if it admits a compact real form $\mathfrak{g}_{\mathbb{R}}$ (cf. [28, IX. §3.3]).

B.2 Cartan subalgebras

A Lie algebra $\mathfrak{g}$ is *nilpotent* if the series of ideals

$$\mathfrak{g},\ [\mathfrak{g},\mathfrak{g}],\ [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]],\ [\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]],\ \ldots$$

is eventually zero. A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra that is equal to its own normalizer, i.e., $\mathrm{ad}_\xi(\mathfrak{h}) \subseteq \mathfrak{h} \Rightarrow \xi \in \mathfrak{h}$.

Examples B.2

1. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$. Then the subalgebra $\mathfrak{h}$ of diagonal matrices is a Cartan subalgebra.
2. For any $\xi \in \mathfrak{g}$, let $\mathfrak{g}^0(\xi)$ be the generalized eigenspace of ad_ξ for the eigenvalue 0. (I.e., $\zeta \in \mathfrak{g}^0(\xi)$ if and only if $\mathrm{ad}_\xi^n \zeta = 0$ for n sufficiently large.) The *rank* of $\mathfrak{g}$ is defined as
$$\mathrm{rank}(\mathfrak{g}) = \min\{\dim \mathfrak{g}^0(\xi) |\ \xi \in \mathfrak{g}\}.$$
If $\dim \mathfrak{g}^0(\xi) = \mathrm{rank}(\mathfrak{g})$, then $\mathfrak{g}^0(\xi)$ is a Cartan subalgebra. Cf. [28, VII. §2.3]. Thus, the rank of $\mathfrak{g}$ is the minimal dimension of a Cartan subalgebra.
3. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$ has rank 1. The element h satisfies $\dim \mathfrak{g}^0(h) = 1$; it hence spans a Cartan subalgebra.

4. If $\mathfrak{g}$ is complex reductive, and $\mathfrak{t}_{\mathbb{R}} \subseteq \mathfrak{g}_{\mathbb{R}}$ is a maximal Abelian subalgebra of a compact real form of $\mathfrak{g}$, then $\mathfrak{t} = \mathfrak{t}_{\mathbb{R}} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$.

Cartan subalgebras of semisimple Lie algebras are always commutative, and consist of semisimple elements. Cf. [28, VII. §2.4]. If $\mathfrak{g}$ is semisimple and $\mathbb{K} = \mathbb{C}$, any two Cartan subalgebras are conjugate in $\mathfrak{g}$. The same holds for compact Lie algebras over $\mathbb{K} = \mathbb{R}$. In this case, the Cartan subalgebras are exactly the maximal Abelian subalgebras. (This is not true in general: E.g., for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$, the span of f is maximal Abelian but is not a Cartan subalgebra.)

B.3 Representation theory of $\mathfrak{sl}(2, \mathbb{C})$

We need some facts from the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Let e, f, h be the standard basis as above.

Theorem B.1 *Up to isomorphism, there is a unique $k+1$-dimensional irreducible representation $V(k)$ of $\mathfrak{sl}(2, \mathbb{C})$, for any $k \geq 0$. It admits a basis $v_0, \ldots, v_k$ such that for all $j = 0, \ldots, k$,*

$$\pi(f)v_j = (j+1)v_{j+1}, \quad \pi(h)v_j = (k-2j)v_j, \quad \pi(e)v_j = (k-j+1)v_{j-1}$$

with the convention $v_{k+1} = 0$, $v_{-1} = 0$.

Proof It is straightforward to verify that these formulas define a representation of $\mathfrak{sl}(2, \mathbb{C})$. Since $\pi(e)^{k+1} = 0$, the operator $\pi(e)$ has a non-zero kernel on every invariant subspace of $V(k)$. But $\ker(\pi(e))$ is spanned by v_0. It follows that every invariant subspace contains v_0, and hence also contains the vectors $v_j = \frac{1}{j!}\pi(f)^j v_0$. This shows that $V(k)$ is irreducible.

Suppose conversely that $\pi : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}(V)$ is any finite-dimensional irreducible representation. Let $v_0 \in V$ be an eigenvector of $\pi(h)$, with eigenvalue $s_0 \in \mathbb{C}$ chosen in such a way that $s_0 + 2$ is not an eigenvalue of $\pi(h)$. (This is possible since $\dim V < \infty$.) The calculation

$$\pi(h)\pi(e)v_0 = \pi([h, e])v_0 - \pi(e)\pi(h)v_0 = (s_0 + 2)\pi(e)v_0$$

shows that $\pi(e)v_0 = 0$, since otherwise it would be an eigenvector with eigenvalue $s_0 + 2$. Define

$$v_j = \frac{1}{j!}\pi(f)^j v_0, \quad j = 0, 1, \ldots.$$

Arguing as above, v_j is an eigenvector of $\pi(h)$ with eigenvalue $s_0 - 2j$, provided that it is non-zero. Hence, the sequence of v_j's is eventually 0, and the non-zero v_j are linearly independent. Let $k \geq 0$ be defined by $v_k \neq 0$, $v_{k+1} = 0$. By construction, the span of $v_0, \ldots, v_k$ is invariant under $\pi(f)$ and under $\pi(h)$. The calculation

$$\pi(e)v_{j+1} = \frac{1}{j+1}\pi(e)\pi(f)v_j = \frac{1}{j+1}(\pi(h) + \pi(f)\pi(e))v_j$$

and induction on j shows that $\pi(e)v_{j+1} = (s_0 - j)v_j$. Taking $j = k$ this identity shows that $s_0 = k$. Hence, $v_0, \ldots, v_k$ span a copy of $V(k)$. Since V is irreducible, this span coincides with all of V. $\square$

Remark B.1 The representation $V(k)$ admits a concrete realization as the k-th symmetric power of the defining representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathbb{C}^2$.

We will often use the following simple consequence of these formulas:

Corollary B.1 *Let* $\pi:\ \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}(V)$ *be a finite-dimensional* $\mathfrak{sl}(2, \mathbb{C})$*-representation. Then the operator* $\pi(h)$ *on* V *is diagonalizable and its eigenvalues are integers. Moreover, letting* $V_r = \ker(\pi(h) - r)$, *we have*:

$$r > 0 \Rightarrow \pi(f):\ V_r \to V_{r-2} \quad \textit{is injective},$$
$$r < 0 \Rightarrow \pi(e):\ V_r \to V_{r+2} \quad \textit{is injective}.$$

Proof The statements hold true for all irreducible components, hence also for their direct sum. $\square$

B.4 Roots

Assume for the remainder of this appendix that $\mathbb{K} = \mathbb{C}$. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$, with a given compact real form $\mathfrak{g}_{\mathbb{R}}$, and let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Cartan subalgebra obtained by complexification of a maximal Abelian subalgebra $\mathfrak{t}_{\mathbb{R}} \subseteq \mathfrak{g}_{\mathbb{R}}$. The choice of $\mathfrak{g}_{\mathbb{R}}$ determines a complex conjugation map $\mathfrak{g} \to \mathfrak{g}$, $\xi \mapsto \xi^c$, with $\mathfrak{g}_{\mathbb{R}}$ as its fixed point set. It is convenient to introduce the conjugate linear anti-automorphism

$$*:\ \mathfrak{g} \to \mathfrak{g}, \quad \xi \mapsto \xi^* = -\xi^c,$$

so that $\mathfrak{g}_{\mathbb{R}}$ corresponds to skew-adjoint elements. We will also fix an invariant positive definite symmetric bilinear form B on $\mathfrak{g}_{\mathbb{R}}$, and use the same notation for its complexification to a bilinear form on $\mathfrak{g}$. The resulting bilinear form on $\mathfrak{g}^*$ will be denoted by B^*. For any $\alpha \in \mathfrak{t}^*$, define the subspace

$$\mathfrak{g}_\alpha = \{\zeta \in \mathfrak{g} |\ \xi \in \mathfrak{t} \Rightarrow \mathrm{ad}_\xi \zeta = \langle \alpha, \xi \rangle \zeta\ \}.$$

Then $\mathfrak{g}$ is a direct sum over the non-zero subspaces $\mathfrak{g}_\alpha$. Elementary properties of these subspaces are

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta},$$
$$\mathfrak{g}_\alpha^c = \mathfrak{g}_{-\alpha},$$
$$\mathfrak{g}_0 = \mathfrak{t}.$$

Furthermore, $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha + \beta \neq 0$, while $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are non-singularly paired for all $\alpha \neq 0$.

A non-zero element $\alpha \in \mathfrak{t}^*$ is called a *root* of $\mathfrak{g}$ if $\mathfrak{g}_\alpha \neq 0$; the corresponding subspace $\mathfrak{g}_\alpha$ is called a *root space*. The set of roots is denoted by $\mathfrak{R}$. Thus

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathfrak{R}} \mathfrak{g}_\alpha.$$

Example B.3 Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with its standard basis e, f, h, and with $\mathfrak{t} = \mathrm{span}(h)$. Then $\alpha(h) = 2$ (resp. $\alpha(h) = -2$) defines a root α, with corresponding root space $\mathfrak{g}_\alpha = \mathrm{span}(e)$ (resp. $\mathrm{span}(f)$).

Let $B^\flat\colon\ \mathfrak{g} \to \mathfrak{g}^*,\ \xi \mapsto B(\xi, \cdot)$ be the isomorphism defined by B, and let $B^\sharp :$ $\mathfrak{g}^* \to \mathfrak{g}$ be its inverse.

Proposition B.1 *For all $\alpha \in \mathfrak{R}$, the space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is 1-dimensional, and is spanned by $B^\sharp(\alpha)$. More precisely, if $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$, then*

$$[e, f] = B(e, f)\ B^\sharp(\alpha).$$

Proof For $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{t}$ we have

$$B(\mathrm{ad}(e) f, h) = -B(f, \mathrm{ad}(e)h) = B(e, f)\langle \alpha, h\rangle.$$

Since $\mathrm{ad}(e) f \in \mathfrak{t}$, this implies $[e, f] = B(e, f)\ B^\sharp(\alpha)$. □

Definition B.2 For any root $\alpha \in \mathfrak{R}$, the *co-root* $\alpha^\vee \in \mathfrak{t}$ is the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\langle \alpha, \alpha^\vee\rangle = 2$. The set of co-roots is denoted by $\mathfrak{R}^\vee$.

In terms of the bilinear form B one has

$$\alpha^\vee = 2\frac{B^\sharp(\alpha)}{B^*(\alpha, \alpha)}.$$

Proposition B.2 *Let $e_\alpha \in \mathfrak{g}_\alpha$ with normalization $B(e_\alpha, e_\alpha^c) = \frac{2}{B(\alpha^\vee, \alpha^\vee)}$, and put $f_\alpha = e_\alpha^c$, $h_\alpha = \alpha^\vee$. Then $e_\alpha, f_\alpha, h_\alpha$ are the standard basis for an $\mathfrak{sl}(2, \mathbb{C})$-subalgebra $\mathfrak{sl}_\alpha \subseteq \mathfrak{g}$.*

Proof We have $[h_\alpha, e_\alpha] = \langle \alpha, h_\alpha\rangle e_\alpha = 2e_\alpha$ and similarly $[h_\alpha, f_\alpha] = -2 f_\alpha$. Furthermore,

$$[e_\alpha, f_\alpha] = B(e_\alpha, f_\alpha) B^\sharp(\alpha) = \alpha^\vee = h_\alpha.$$

This shows that the elements $e_\alpha, f_\alpha, h_\alpha$ satisfy the bracket relations (B.1). □

The representation theory of $\mathfrak{sl}(2, \mathbb{C})$, applied to the Lie subalgebras $\mathfrak{sl}_\alpha$, implies the basic properties of root systems.

Proposition B.3 *Let $\alpha \in \mathfrak{R}$ be a root. Then*:

1. $\dim(\mathfrak{g}_\alpha) = 1$.
2. *If $\beta \in \mathfrak{R}$ is a multiple of α, then $\beta = \pm\alpha$.*
3. *Let $\beta \in \mathfrak{R}$, with $\beta \neq \pm\alpha$. Then*
$$\langle \beta, \alpha^\vee \rangle < 0 \Rightarrow \alpha + \beta \in \mathfrak{R}.$$
4. (*Root strings.*) *Given $\beta \in \mathfrak{R}$, with $\beta \neq \pm\alpha$, there exist integers $q, p \geq 0$ such that for any integer r, $\beta + r\alpha \in \mathfrak{R}$ if and only if $-q \leq r \leq p$. These integers satisfy*
$$q - p = \langle \beta, \alpha^\vee \rangle.$$
The direct sum $\bigoplus_{r=-q}^{p} \mathfrak{g}_{\beta+r\alpha}$ is an irreducible $\mathfrak{sl}_\alpha$-representation of dimension $p + q + 1$.
5. *If $\alpha, \beta, \alpha + \beta$ are all roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Proof We will regard $\mathfrak{g}$ as an $\mathfrak{sl}_\alpha$-representation by restricting the adjoint representation of $\mathfrak{g}$. That is, the standard basis elements h_α, e_α and f_α act as $\mathrm{ad}(h_\alpha)$, $\mathrm{ad}(e_\alpha)$ and $\mathrm{ad}(f_\alpha)$, respectively.

1. Since $\mathrm{ad}(h_\alpha)$ acts on $\mathfrak{g}_{-\alpha}$ as a non-zero scalar -2, Corollary B.1 shows that the map $\mathrm{ad}(e_\alpha) : \mathfrak{g}_{-\alpha} \to \mathfrak{g}_0$ is injective. On the other hand, Proposition B.1 shows that its range is 1-dimensional. Hence $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 1$.
2. We may assume that $t\alpha$ is not a root for $|t| < 1$. We will show that it is not a root for $|t| > 1$. Suppose on the contrary that $t\alpha$ is a root for some $t > 1$, and take the smallest such t. The operator $\mathrm{ad}(h_\alpha)$ acts on $\mathfrak{g}_{t\alpha}$ as a positive scalar $2t > 0$. By Corollary B.1, it follows that $\mathrm{ad}(f_\alpha) : \mathfrak{g}_{t\alpha} \to \mathfrak{g}_{(t-1)\alpha}$ is injective. Since $t > 1$, and since there are no smaller multiples of α that are roots, other than α itself, this implies that $t = 2$, and $\mathrm{ad}(f_\alpha) : \mathfrak{g}_{2\alpha} \to \mathfrak{g}_\alpha$ is injective. But this is impossible, since $\mathfrak{g} = \mathfrak{sl}_\alpha \oplus \mathfrak{sl}_\alpha^\perp$ is an $\mathfrak{sl}_\alpha$-invariant decomposition with $\mathfrak{g}_\alpha \subseteq \mathfrak{sl}_\alpha$, $\mathfrak{g}_{2\alpha} \subseteq \mathfrak{sl}_\alpha^\perp$.
3. Suppose α, β are distinct roots with $\langle \beta, \alpha^\vee \rangle < 0$. Since $\mathrm{ad}(h_\alpha)$ acts on $\mathfrak{g}_\beta$ as a negative scalar $\langle \beta, \alpha^\vee \rangle < 0$, Corollary B.1 shows that $\mathrm{ad}(e_\alpha) : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$ is injective. Since $\alpha + \beta \neq 0$ it is thus an isomorphism. In particular, $\mathfrak{g}_{\alpha+\beta}$ is non-zero.
4. If $\mathfrak{g}_{\beta+r\alpha} \neq 0$, then $\mathrm{ad}(h_\alpha)$ acts on it as a scalar $\langle \beta, \alpha^\vee \rangle + 2r$. Let q and p be the largest integers such that $\mathfrak{g}_{\beta+p\alpha} \neq 0$, respectively $\mathfrak{g}_{\beta-q\alpha} \neq 0$. Then $\mathfrak{g}_{\beta+p\alpha} \subseteq \ker(\mathrm{ad}(e_\alpha))$. Consequently,
$$V = \bigoplus_{j \geq 0} \mathrm{ad}^j(f_\alpha)\mathfrak{g}_{\beta+p\alpha}$$
is an irreducible $\mathfrak{sl}_\alpha$-representation $V \subseteq \mathfrak{g}$. Its dimension is $k + 1$ where $k = \langle \beta, \alpha^\vee \rangle + 2p$ is the eigenvalue of $\mathrm{ad}(h_\alpha)$ on $\mathfrak{g}_{\beta+p\alpha}$. In particular, all subspaces $\mathfrak{g}_{\beta+r\alpha}$ for $-k \leq \langle \beta, \alpha^\vee \rangle + 2r \leq k$ are non-zero. By a similar argument, we see that $\mathfrak{g}_{\beta-q\alpha}$ and its images under $\mathrm{ad}^j(e_\alpha)$, $j = 0, 1, 2, \ldots$ span an irreducible representation V' of dimension $k' + 1$, where $k' = 2q - \langle \beta, \alpha^\vee \rangle$. Since all root spaces $\mathfrak{g}_{\beta+r\alpha}$ with $\langle \beta, \alpha^\vee \rangle + 2r > 0$ are contained in V, we must have $V = V'$, $k = k'$. In particular, $\langle \beta, \alpha^\vee \rangle + 2p = 2q - \langle \beta, \alpha^\vee \rangle$, hence $q - p = \langle \beta, \alpha^\vee \rangle$ and $k = q + p$.

5. This follows from item B.3.4, since $\mathrm{ad}(e_\alpha): \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+\alpha}$ is an isomorphism if $\mathfrak{g}_\beta$ and $\mathfrak{g}_{\beta+\alpha}$ are non-zero. $\square$

Definition B.3 (Lattices) Let $\mathfrak{g}$ be a complex reductive Lie algebra, as above.

1. The lattice
$$Q = \mathrm{span}_{\mathbb{Z}} \mathfrak{R} \subseteq \sqrt{-1}\mathfrak{t}^*_{\mathbb{R}}$$
spanned by the roots is called the *root lattice*.
2. The lattice
$$Q^\vee = \mathrm{span}_{\mathbb{Z}} \mathfrak{R}^\vee \subseteq \sqrt{-1}\mathfrak{t}_{\mathbb{R}}$$
spanned by the co-roots is called the *co-root lattice*.
3. The lattice
$$P = \{\mu \in \sqrt{-1}\mathfrak{t}^*_{\mathbb{R}} |\ \xi \in Q^\vee \Rightarrow \langle \mu, \xi \rangle \in \mathbb{Z}\}$$
dual to the co-root lattice is called the *weight lattice*.
4. The lattice
$$P^\vee = \{\xi \in \sqrt{-1}\mathfrak{t}_{\mathbb{R}} |\ \mu \in Q \Rightarrow \langle \mu, \xi \rangle \in \mathbb{Z}\}$$
dual to the root lattice is called the *co-weight lattice*.

Proposition B.3.4 shows in particular that $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \mathfrak{R}_+$. Hence
$$Q \subseteq P,$$
and dually $Q^\vee \subseteq P^\vee$.

B.5 Simple roots

We will find it convenient to work with the bilinear form $(\cdot|\cdot) = -B$ on $\mathfrak{g}$, given by $(\xi_1|\xi_2) = -B(\xi_1, \xi_2)$, so that $(\cdot, \cdot)$ is positive definite on $\sqrt{-1}\mathfrak{t}_{\mathbb{R}}$. The same notation $(\cdot|\cdot)$ will be used for the dual bilinear form on $\mathfrak{g}^*$. Then $(\alpha|\alpha) > 0$ for all roots α.

Fix $u \in \sqrt{-1}\mathfrak{t}$ with $(\alpha|u) \neq 0$ for all $\alpha \in \mathfrak{R}$. The choice of u determines a decomposition
$$\mathfrak{R} = \mathfrak{R}_+ \cup \mathfrak{R}_-$$
into *positive roots* and *negative roots*, where $\mathfrak{R}_+$ (resp. $\mathfrak{R}_-$) consists of all roots such that $(\alpha|u) > 0$ (resp. < 0). Note that $\alpha \in \mathfrak{R}_- \Leftrightarrow -\alpha \in \mathfrak{R}_+$, and that the sum of two positive roots (resp. of two negative roots) is again positive (resp. negative).

Definition B.4 A positive root is called *simple* if it cannot be written as a sum of two positive roots.

Proposition B.4 (Simple roots) *The set* $\{\alpha_1, \ldots, \alpha_l\}$ *of simple roots has the following properties*:

1. *The simple roots $\alpha_1, \dots \alpha_l$ form a basis of the root lattice Q.*
2. *A root $\alpha \in \mathfrak{R}$ is positive if and only if all coefficients in the expansion $\alpha = \sum_i k_i \alpha_i$ are ≥ 0. It is negative if and only if all coefficients are ≤ 0.*
3. *One has $(\alpha_i|\alpha_j) \leq 0$ for $i \neq j$.*

Proof We begin by proving B.4.3. If α_i and α_j are distinct simple roots, then their difference $\alpha_i - \alpha_j$ is *not* a root. (Otherwise, either $\alpha_i = \alpha_j + (\alpha_i - \alpha_j)$ or $\alpha_j = \alpha_i + (\alpha_j - \alpha_i)$ would be a sum of two positive roots.) By Proposition B.3, it follows that $\langle \alpha_i, \alpha_j^\vee \rangle \leq 0$, proving B.4.3.

We next show that the α_i are linearly independent. Indeed suppose $\sum_i k_i \alpha_i = 0$. Let

$$\mu = \sum_{k_i > 0} k_i \alpha_i,$$

hence $\mu = -\sum_{k_j < 0} k_j \alpha_j$. Taking the scalar product of μ with itself, and using B.4.3 we obtain

$$0 \leq (\mu|\mu) = -\sum_{k_i > 0,\ k_j < 0} k_i k_j (\alpha_i|\alpha_j) \leq 0.$$

Hence $\mu = 0$, which shows that all $k_i = 0$, proving B.4.1.

We claim that any $\alpha \in \mathfrak{R}_+$ can be written in the form $\alpha = \sum k_i \alpha_i$ for some $k_i \in \mathbb{Z}_{\geq 0}$. Otherwise, let α be a counterexample with $(\alpha|u)$ as small as possible. Since α is not a simple root, it can be written as a sum $\alpha = \alpha' + \alpha''$ of two positive roots α' and α''. Then $(\alpha'|u)$ and $(\alpha''|u)$ are both strictly smaller than $(\alpha|u)$. Hence, neither is a counterexample, and each can be written as a linear combination of α_i's with non-negative coefficients. Hence the same is true of α. This proves B.4.2. □

B.6 The Weyl group

Associated to any root α is a reflection $w_\alpha \in \mathrm{GL}(\mathfrak{t})$ given by

$$w_\alpha(\xi) = \xi - \langle \alpha, \xi \rangle \alpha^\vee.$$

Dually, one has a reflection $w_\alpha \in \mathrm{GL}(\mathfrak{t}^*)$ given by

$$w_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha.$$

Thus $\langle w_\alpha(\mu),\ w_\alpha(\xi) \rangle = \langle \mu, \xi \rangle$ for all $\mu \in \mathfrak{t}^*$, $\xi \in \mathfrak{t}$. The reflection $w_\alpha \in \mathrm{GL}(\mathfrak{t})$ admits an extension to a Lie algebra automorphism $\mathfrak{g}$, as follows. Let $e_\alpha, h_\alpha, f_\alpha$ be the basis for $\mathfrak{sl}_\alpha$, as defined in Proposition B.2.

Proposition B.5 *The transformation*

$$\theta_\alpha = \exp(\mathrm{ad}(e_\alpha)) \exp(-\mathrm{ad}(f_\alpha)) \exp(\mathrm{ad}(e_\alpha)) \in \mathrm{GL}(\mathfrak{g})$$

is a well-defined Lie algebra automorphism of $\mathfrak{g}$. It has the property $\theta_\alpha|_\mathfrak{t} = w_\alpha$, and restricts to isomorphisms $\mathfrak{g}_\beta \to \mathfrak{g}_{w_\alpha(\beta)}$ for all roots β.

Proof Since $\mathrm{ad}(e_\alpha)$ and $\mathrm{ad}(f_\alpha)$ are nilpotent Lie algebra derivations of $\mathfrak{g}$, the exponentials are well-defined Lie algebra automorphisms of $\mathfrak{g}$. If $h \in \ker(\alpha) \subseteq \mathfrak{t}$, then $\mathrm{ad}(e_\alpha)h = 0 = \mathrm{ad}(f_\alpha)h$, hence $\theta_\alpha(h) = h$. On the other hand, if $h = h_\alpha$ we may replace $\mathfrak{g}$ with $\mathfrak{sl}_\alpha$, and hence assume $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, with $e_\alpha = e$, $f_\alpha = f$, $h_\alpha = h$ given by matrices as in Example B.1. Then $\theta_\alpha = \theta$ is the transformation

$$\theta = \mathrm{Ad}(\exp(e)\exp(-f)\exp(e))$$

using exponentials of matrices. We compute

$$\exp(e)\exp(-f)\exp(e) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

conjugation of $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ by this matrix gives $-h$. For the second part, let $v \in \mathfrak{g}_\beta$. The calculation

$$\mathrm{ad}(h)\theta_\alpha(v) = \theta_\alpha(\mathrm{ad}(w_\alpha^{-1}h)v) = \langle \beta, w_\alpha^{-1}h\rangle\theta_\alpha(v) = \langle w_\alpha(\beta), h\rangle\theta_\alpha(v)$$

for all $h \in \mathfrak{t}$ shows that $\theta_\alpha(v) \in \mathfrak{g}_{w_\alpha(\beta)}$. □

Definition B.5 The subgroup $W \subseteq \mathrm{GL}(\mathfrak{t})$ generated by the reflections $w_\alpha, \alpha \in \mathfrak{R}$, is called the *Weyl group* of the pair $(\mathfrak{g}, \mathfrak{t})$ (or of the root system $\mathfrak{R} \subseteq \mathfrak{t}^*$).

Corollary B.2 *The action of the Weyl group W on $\mathfrak{t}$ preserves the sets $\mathfrak{R}^\vee \subseteq Q^\vee \subseteq P^\vee$. The dual action on $\mathfrak{t}^*$ preserves the sets $\mathfrak{R} \subseteq Q \subseteq P$.*

Proof Consider first the dual action of w_α on $\mathfrak{t}^*$. By the proposition, if $\mathfrak{g}_\beta$ is nonzero, then so is $\mathfrak{g}_{w_\alpha(\beta)}$. Hence w_α preserves the set $\mathfrak{R}$ of roots, hence also their $\mathbb{Z}$-span Q. It is immediate from the definition that w_α preserves P.

Using the formula $\beta^\vee = 2(\beta|\cdot)/(\beta|\beta)$ for the co-roots, we see that the action of w_α on $\mathfrak{t}$ is just the orthogonal reflection for α, with respect to $(\cdot|\cdot)$, and that $w_\alpha(\beta^\vee) = (w_\alpha\beta)^\vee$. In particular, the set of coroots $\mathfrak{R}^\vee$ is w_α-invariant as well, and so is $Q^\vee \subseteq P^\vee$. □

We have $w_\alpha w_\beta w_\alpha = w_{\beta'}$ where $\beta' = w_\alpha(\beta)$. Hence, any $w \in W$ can be written as a product of at most $|\mathfrak{R}_+|$ reflections w_α. In particular, $|W| < \infty$.

Proposition B.6 *The reflection $w_i = w_{\alpha_i}$ defined by a simple root α_i permutes the set $\mathfrak{R}_+\backslash\{\alpha_i\}$.*

Proof Suppose $\alpha \in \mathfrak{R}_+\backslash\{\alpha_i\}$. Write $\alpha = \sum_j k_j\alpha_j \in \mathfrak{R}_+$, so that all $k_j \geq 0$. The root

$$w_i\alpha = \alpha - \langle \alpha, \alpha_i^\vee\rangle\alpha_i = \sum_j k'_j\alpha_j \tag{B.2}$$

has coefficients $k'_j = k_j$ for $j \neq i$, while $k'_i = k_i - \langle \alpha, \alpha_i^\vee\rangle$. Since α is not a multiple of α_i it follows that $k'_j = k_j > 0$ for some $j \neq i$. This shows that $w_i\alpha$ is positive. □

For any element $\alpha = \sum_i k_i \alpha_i \in Q$, one defines its *height* as

$$\mathrm{ht}(\alpha) = \sum_i k_i .$$

The formula (B.2) for $w_i\alpha$ shows that if $\langle \alpha, \alpha_i^\vee \rangle > 0 \Rightarrow \mathrm{ht}(w_i\alpha) < \mathrm{ht}(\alpha)$.

Proposition B.7 *Any Weyl group element w can be written as a product of simple reflections*

$$w = w_{i_1} \dots w_{i_r} . \tag{B.3}$$

Proof It suffices to show that every w_α, $\alpha \in \mathfrak{R}_+$ can be written in this form. The proof uses induction on $k = \mathrm{ht}(\alpha)$. Since $0 < (\alpha|\alpha) = \sum_i k_i (\alpha|\alpha_i)$, there is at least one i with $(\alpha|\alpha_i) > 0$. As remarked above, this implies that $\mathrm{ht}(w_i\alpha) < \mathrm{ht}(\alpha)$. We have

$$w_\alpha = w_i \, w_{\alpha'} \, w_i$$

with $\alpha' = w_i\alpha$; by induction $w_{\alpha'}$ is a product of simple reflections. □

Definition B.6 The *length* $l(w)$ of a Weyl group element $w \in W$ is the smallest number r such that w can be written in the form (B.3). If $r = l(w)$, the expression (B.3) is called *reduced*.

It is immediate that

$$l(w^{-1}) = l(w), \quad l(ww') \le l(w) + l(w').$$

Proposition B.8 *For any Weyl group element w, and any simple root α_i, we have $l(ww_i) = l(w) + 1$ if $w\alpha_i$ is positive, and $l(ww_i) = l(w) - 1$ if $w\alpha_i$ is negative.*

Proof Write $w = w_{i_1} \cdots w_{i_r}$ with $r = l(w)$. Suppose $w\alpha_i$ is negative. Then there is $m \le r$ such that

$$\beta := w_{i_m} \cdots w_{i_r}\alpha \in \mathfrak{R}_+, \quad w_{i_{m-1}} \cdots w_{i_r}\alpha \in \mathfrak{R}_- .$$

That is, w_{i_m} changes the positive root β to a negative root. The only positive root with this property is $\beta = \alpha_{i_m}$. With $u = w_{i_m} \cdots w_{i_r} \in W$ we have $\alpha_{i_m} = u\alpha_i$, thus $w_{i_m} = uw_iu^{-1}$. Multiplying from the right by u, and from the left by $w_{i_1} \cdots w_{i_{m-1}}$, we obtain

$$w_{i_1} \cdots w_{i_{m-1}} w_{i_{m+1}} \cdots w_{i_r} = ww_i .$$

This shows that $l(ww_i) = l(w) - 1$. The case that $w\alpha_i$ is positive is reduced to the previous case, since $w'\alpha_i$ with $w' = ww_i$ is negative. □

For any $w \in W$, let $\mathfrak{R}_{+,w}$ be the set of positive roots that are made negative under w^{-1}. That is,

$$\mathfrak{R}_{+,w} = \mathfrak{R}_+ \cap w\mathfrak{R}_- . \tag{B.4}$$

By Proposition B.6, $\mathfrak{R}_{+,w_i} = \{\alpha_i\}$.

Proposition B.9 *If* $w = w_{i_1} \cdots w_{i_r}$, $r = l(w)$ *is a reduced expression,*

$$\mathfrak{R}_{+,w} = \{\alpha_{i_1},\ w_{i_1}\alpha_{i_2},\ \ldots,\ w_{i_1}\cdots w_{i_{r-1}}\alpha_{i_r}\}. \tag{B.5}$$

Hence

$$l(w) = |\mathfrak{R}_{+,w}|. \tag{B.6}$$

By this proposition, the choice of a reduced expression of w determines an ordering of the set $\mathfrak{R}_{+,w}$.

Proof Consider $w' = ww_i$. Suppose a positive root α is made negative under $(w')^{-1} = w_i w^{-1}$. Since w_i changes the sign of $\pm\alpha_i$, and preserves both $\mathfrak{R}_+\backslash\{\alpha_i\}$ and $\mathfrak{R}_-\backslash\{-\alpha_i\}$, we see that $w^{-1}\alpha$ is negative if and only if $w^{-1}\alpha \neq -\alpha_i$. That is,

$$\mathfrak{R}_{+,ww_i} = \begin{cases} \mathfrak{R}_{+,w} \cup \{w\alpha_i\} & \text{if } \ w\alpha_i \in \mathfrak{R}_+, \\ \mathfrak{R}_{+,w}\backslash\{-w\alpha_i\} & \text{if } \ w\alpha_i \in \mathfrak{R}_-. \end{cases}$$

(B.5) now follows by induction on $l(w)$, using that for a reduced expression $w = w_{i_1}\cdots w_{i_r}$, all $w_{i_1}\cdots w_{i_{k-1}}\alpha_{i_k}$ are positive. □

Let us now introduce the half-sum of positive roots

$$\rho := \frac{1}{2}\sum_{\alpha\in\mathfrak{R}_+}\alpha \in \frac{1}{2}\mathrm{span}_{\mathbb{Z}}(\mathfrak{R}_+).$$

Lemma B.1 *For all* $w \in W$,

$$\rho - w\rho = \sum_{\alpha\in\mathfrak{R}_{+,w}}\alpha.$$

Proof By definition $\mathfrak{R}_{+,w} = \mathfrak{R}_+ \cap w\mathfrak{R}_-$, with complement in $\mathfrak{R}_+$ given as $\mathfrak{R}'_{+,w} = \mathfrak{R}_+ \cap w\mathfrak{R}_+$. Hence $w\mathfrak{R}_+ = \mathfrak{R}'_{+,w} \cup (-\mathfrak{R}_{+,w})$, which gives

$$w\rho = \frac{1}{2}\sum_{\alpha\in\mathfrak{R}'_{+,w}}\alpha - \frac{1}{2}\sum_{\alpha\in\mathfrak{R}_{+,w}}\alpha = \rho - \sum_{\alpha\in\mathfrak{R}_{+,w}}\alpha$$

as claimed. □

B.7 Weyl chambers

Let $\mathfrak{t}_{\mathrm{reg}} \subseteq \mathfrak{t}_{\mathbb{R}}$ be the set of regular elements in $\mathfrak{t}_{\mathbb{R}}$, i.e., those elements for which $\ker(\mathrm{ad}_\xi) = \mathfrak{t}$. It is the complement of the union of root hyperplanes

$$H_\alpha = \left\{\xi \in \mathfrak{t}_{\mathbb{R}} \,\middle|\, \frac{1}{2\pi\sqrt{-1}}\langle\alpha,\xi\rangle = 0\right\},\ \ \alpha \in \mathfrak{R}.$$

The components of $\mathfrak{t}_{\mathrm{reg}}$ are called the *open Weyl chambers*. We will refer to the closures of the open chambers as the *closed Weyl chambers*, or simply *Weyl chambers*. We let

$$\mathfrak{t}_+ = \left\{ \xi \in \mathfrak{t}_{\mathbb{R}} \,\middle|\, \frac{1}{2\pi\sqrt{-1}} \langle \alpha, \xi \rangle \geq 0 \;\; \forall \alpha \in \mathfrak{R}_+ \right\}$$

be the *positive Weyl chamber*. The action of the Weyl group permutes the chambers.

We say that a root hyperplane H_α *separates the chambers* C, C' if for points x and x' in the interior of the chambers, $\frac{1}{2\pi\sqrt{-1}}\langle \alpha, x\rangle$ and $\frac{1}{2\pi\sqrt{-1}}\langle \alpha, x'\rangle$ have opposite signs, but $\frac{1}{2\pi\sqrt{-1}}\langle \beta, x\rangle$ and $\frac{1}{2\pi\sqrt{-1}}\langle \beta, x'\rangle$ have equal signs for all roots $\beta \neq \pm\alpha$.

Proposition B.10 *The Weyl group W acts simply transitively on the set of Weyl chambers. That is, every Weyl chamber is of the form $w\mathfrak{t}_+$ for a unique $w \in W$.*

Proof Since the Weyl group action preserves the set of roots, it also preserves the union of hyperplanes H_α. Hence W acts by permutations on the set of Weyl chambers. Any two adjacent Weyl chambers are separated by some root hyperplane H_α, and the reflection w_α interchanges the two Weyl chambers. By induction, it follows that any Weyl chamber is taken to $\mathfrak{t}_+$ by a finite number of Weyl reflections. An element $w \in W$ fixes $\mathfrak{t}_+$ only if it preserves the set $\mathfrak{R}_+$ of positive roots, if and only if $\mathfrak{R}_{+,w} = \emptyset$. But this means $l(w) = 0$, i.e., $w = \mathrm{id}$. □

The length of a Weyl group element has the following interpretation in terms of the Weyl chambers.

Proposition B.11 *The length $l(w)$ is the number of root hyperplanes crossed by a line segment from the interior of $\mathfrak{t}_+$ to a point in the interior of $w\mathfrak{t}_+$.*

Proof Let $x \in \mathrm{int}(\mathfrak{t}_+)$, $x' \in \mathrm{int}(w\mathfrak{t}_+)$. The line segment

$$x_t = (1-t)x + tx', \;\; t \in [0, 1]$$

meets the hyperplane H_α, $\alpha \in \mathfrak{R}_+$ if and only if

$$\frac{1}{2\pi\sqrt{-1}} \langle \alpha, x' \rangle < 0.$$

Write $\langle \alpha, x' \rangle = \langle w^{-1}\alpha, w^{-1}x' \rangle$. Since $w^{-1}x' \in \mathrm{int}(\mathfrak{t}_+)$, the condition means $w^{-1}\alpha \in \mathfrak{R}_-$, i.e., $\alpha \in \mathfrak{R}_{+,w}$. Hence the number of hyperplanes crossed equals $|\mathfrak{R}_{+,w}| = l(w)$. □

Remark B.2 For generic choices of points in the interiors of $\mathfrak{t}_+$ and $w\mathfrak{t}_+$, the line segment connecting the two points does not meet $H_\alpha \cap H_\beta$ for $\alpha \neq \beta$. The sequence of chambers crossed by this line segment corresponds to a reduced expression of w.

Remark B.3 Proposition B.10 means in particular that there is a unique Weyl group element w_0 with the property $w_0(\mathfrak{t}_+) = -\mathfrak{t}_+$. Equivalently, $\mathfrak{R}_{+,w_0} = \mathfrak{R}_+$, so that w_0 is the unique *longest Weyl group element*:

$$l(w_0) = |\mathfrak{R}_+|.$$

By Proposition B.9, any reduced expression for w_0 defines an ordering of the set $\mathfrak{R}_+$. In [121], it is shown that the orderings are exactly the *normal orderings* of $\mathfrak{R}_+$: That is, whenever $\alpha, \beta, \gamma = \alpha + \beta$ are in $\mathfrak{R}_+$, then γ is listed between α and β.

B.8 Weights of representations

Let $\mathfrak{g}$ be a complex reductive Lie algebra, with Cartan subalgebra $\mathfrak{t}$. Let $\pi : \mathfrak{g} \to \mathrm{End}(V)$ be a representation on a (possibly infinite-dimensional) complex vector space V. An element $\mu \in \mathfrak{t}^*$ will be called a *weight* of V if the space

$$V_\mu = \{v \in V |\ \pi(\xi)v = \langle \mu, \xi \rangle v,\ \ \xi \in \mathfrak{t}\}$$

is non-zero. In this case V_μ is called the weight space. We observe that for all roots $\alpha \in \mathfrak{R}$,

$$\pi(e_\alpha):\ V_\mu \to V_{\mu+\alpha},\ \ \pi(f_\alpha):\ V_\mu \to V_{\mu-\alpha}.$$

Indeed, if $v \in V_\mu$ and $h \in \mathfrak{t}$ we have

$$\pi(h)\pi(e_\alpha)v = \pi(e_\alpha)\pi(h)v + \pi([h, e_\alpha])v = \langle \mu, h \rangle \pi(e_\alpha)v + \langle \alpha, h \rangle \pi(e_\alpha)v$$

proving that $\pi(e_\alpha)v \in V_{\mu+\alpha}$. The set of weights of V is denoted by

$$P(V) = \{\mu \in \mathfrak{t}^* |\ V_\mu \neq \{0\}\},$$

and the dimension of V_μ is the *multiplicity* of the weight μ. If V is a direct sum of its weight spaces, and all weight spaces are finite-dimensional, then the set of weights along with their multiplicities is conveniently encoded in their *formal character*

$$\mathrm{ch}(V) = \sum_{\mu \in P} \dim(V_\mu) \mathrm{e}^\mu$$

an element of the group ring of P. (It may also be viewed as a function on $\mathfrak{t}$, replacing the formal exponent e^μ with the function $\xi \mapsto \mathrm{e}^{\langle \mu, \xi \rangle}$.) The character has the properties $\mathrm{ch}(V \oplus V') = \mathrm{ch}(V) + \mathrm{ch}(V')$, $\mathrm{ch}(V \otimes V') = \mathrm{ch}(V) \otimes \mathrm{ch}(V')$ and $\mathrm{ch}(V^*) = -\mathrm{ch}(V)$.

Examples B.4

1. The weights of the adjoint representation of $\mathfrak{g}$ are

   $$P(\mathfrak{g}) = \mathfrak{R} \cup \{0\}.$$

   The formal character is $\mathrm{ch}(\mathfrak{g}) = \dim \mathfrak{t} + \sum_{\alpha \in \mathfrak{R}_+} \mathrm{e}^\alpha$.

2. The set of weights of the representation of $\mathfrak{g}$ by left-multiplication on $U(\mathfrak{g})$ is empty. Indeed, suppose $x \in U(\mathfrak{g})$ is an element of filtration degree r, such that $hx = \langle \mu, h\rangle x$ for all $h \in \mathfrak{t}$. Then the image $y \in S^r(\mathfrak{g})$ of x satisfies $hy = 0$ for all $h \in \mathfrak{t}$, hence $y = 0$. It follows that x has filtration degree $r - 1$. Proceeding by induction we find $x = 0$.

The second example illustrates that if $\dim V = \infty$, the direct sum of weight spaces may be strictly smaller than V. Furthermore, the example of Verma modules discussed below shows that $P(V)$ need not be a subset of P in that case.

Proposition B.12 *Let V be a finite-dimensional completely reducible $\mathfrak{g}$-representation. Then the set of weights $P(V)$ is contained in P, and is invariant under the action of the Weyl group W. Furthermore,*

$$V = \bigoplus_{\mu \in P(V)} V_\mu.$$

Proof Since V is finite-dimensional, Weyl's Theorem shows that it is completely reducible as an $\mathfrak{sl}_\alpha$-representation for all roots α. In particular, the transformations $\pi(h_\alpha)$, $\alpha \in \mathfrak{R}$, are all diagonalizable. Since these transformations commute, they are in fact *simultaneously* diagonalizable. Since the h_α, together with the center $\mathfrak{z}$, span $\mathfrak{t}$, it follows that V is a direct sum of the weight spaces. Suppose $\mu \in \mathfrak{t}^*$ is a weight, and let $v \in V_\mu$ be non-zero. Then v is in particular an eigenvector of $\pi(h_\alpha)$, with eigenvalue $\langle \mu, \alpha^\vee\rangle$. By the representation theory of $\mathfrak{sl}_\alpha$, the eigenvalues of $\pi(h_\alpha)$ are integers. This shows that $\mu \in P$. For the W-invariance, consider the automorphism

$$\Theta_\alpha = \exp(\pi(e_\alpha)) \exp(-\pi(f_\alpha)) \exp(\pi(e_\alpha)) \in \mathrm{GL}(V). \tag{B.7}$$

(Here the finite-dimensionality of V is used to define the exponential of an endomorphism of V.) Using the same argument as in the proof of Proposition B.5, we see that Θ_α implements w_α:

$$\Theta_\alpha \pi(h) \Theta_\alpha^{-1} = \pi(w_\alpha h), \quad h \in \mathfrak{t}.$$

Hence Θ_α takes V_μ to $V_{w_\alpha(\mu)}$. Since this is true for all $\alpha \in \mathfrak{R}$ and $\mu \in P(V)$, this shows the W-invariance of $P(V)$. □

Since $P(V)$ is W-invariant, it is uniquely determined by its intersection with the set

$$P_+ = \{\mu \in P \mid \forall \alpha \in \mathfrak{R}_+ : \ \langle \mu, \alpha^\vee\rangle \in \mathbb{Z}_{\geq 0}\}$$

of *dominant weights*.

Remark B.4 In the next section we will see that for infinite-dimensional irreducible $\mathfrak{g}$-representations, the set of weights need not be W-invariant.

B.9 Highest weight representations

We keep the assumptions from the last sections; in particular, $\mathfrak{g}$ is a complex reductive Lie algebra and $\mathfrak{t}$ is its Cartan subalgebra. Let $\mathfrak{n}_+$, $\mathfrak{n}_-$ be the nilpotent Lie subalgebras

$$\mathfrak{n}_+ = \bigoplus_{\alpha\in\mathfrak{R}_+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \bigoplus_{\alpha\in\mathfrak{R}_-} \mathfrak{g}_\alpha.$$

Let $\mathfrak{b}_+ = \mathfrak{t} \oplus \mathfrak{n}_+$ and $\mathfrak{b}_- = \mathfrak{n}_- \oplus \mathfrak{t}$ be the *Borel subalgebras*.

Definition B.7 Let V be any $\mathfrak{g}$-representation. A non-zero vector $v \in V$ is called a *highest weight vector* of highest weight $\mu \in \mathfrak{t}^*$, if

$$v \in V_\mu, \quad \pi(\mathfrak{n}_+)v = 0.$$

The representation V is called a *highest weight representation* if there is a highest weight vector v with $V = \pi(U\mathfrak{g})v$.

The highest weight vectors span the subspace

$$V^{\mathfrak{n}_+} = \{v \in V \,|\, \pi(\mathfrak{n}_+)v = 0\}.$$

Note that $V^{\mathfrak{n}_+}$ is invariant under $U(\mathfrak{t})$, and that it is annihilated by all of $U(\mathfrak{g})\mathfrak{n}_+$. Using the direct sum decomposition $U(\mathfrak{g}) = U(\mathfrak{n}_-) \oplus U(\mathfrak{g})\mathfrak{b}_+$, it follows that

$$V = U(\mathfrak{n}_-)V^{\mathfrak{n}_+} = V^{\mathfrak{n}_+} \oplus U^+(\mathfrak{n}_-)V^{\mathfrak{n}_+}.$$

An important example of a highest weight representation is given as follows. Given $\mu \in \mathfrak{t}^*$, define a representation of $\mathfrak{b}_+$ on $\mathbb{C}$ by letting $\xi \in \mathfrak{t}$ act as a scalar $\langle \mu, \xi \rangle$ and letting $\mathfrak{n}_+$ act as zero. Let

$$L(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}$$

be the induced $\mathfrak{g}$-representation (where the $\mathfrak{g}$-action comes from the left-regular representation on $U(\mathfrak{g})$). Equivalently, $L(\mu)$ is the quotient of $U(\mathfrak{g})$ by the left ideal generated by $\mathfrak{n}_+$ together with the set of all $\xi - \langle \mu, \xi \rangle$, $\xi \in \mathfrak{t}$. The image $v \in L(\mu)$ of $1 \in U(\mathfrak{g})$ is a highest weight vector of weight μ. One calls $L(\mu)$ the *Verma module*. It is the universal highest weight module in the following sense.

Proposition B.13 *Let V be a highest weight representation of highest weight $\mu \in \mathfrak{t}^*$. Then there exists a surjective $\mathfrak{g}$-module morphism $L(\mu) \to V$.*

Proof Let $v \in V$ be a highest weight vector. The map $\mathbb{C} \to V$, $t \mapsto tv$ is $U(\mathfrak{b}_+)$-equivariant, since $\pi(\mathfrak{n}_+)v = 0$, $\pi(\xi)v = \langle \mu, \xi \rangle v$, $\xi \in \mathfrak{t}$. Hence, the $\mathfrak{g}$-equivariant surjection $U(\mathfrak{g}) \to V$ $x \mapsto \pi(x)v$ descends to a $\mathfrak{g}$-equivariant surjection $L(\mu) \to V$. $\square$

Denote by $\mathrm{cone}_{\mathbb{Z}}\mathfrak{R}_+$ all sums $\sum_i k_i\alpha_i$ with integers $k_i \geq 0$.

Proposition B.14 *The Verma module $L(\mu)$ is a direct sum of its weight spaces, and has the formal character*

$$\operatorname{ch}(L(\mu)) = \frac{\mathrm{e}^{\mu}}{\prod_{\alpha\in\mathfrak{R}_+}(1-\mathrm{e}^{-\alpha})} = \sum_{\nu\in Q} f(\nu)\mathrm{e}^{\mu-\nu}.$$

Here $\nu \mapsto f(\nu)$ is the Kostant partition function, *i.e., the cardinality of the set of maps $\mathfrak{R}_+ \to \mathbb{Z}_{\geq 0}$, $\alpha \mapsto k_\alpha$ such that $\nu = \sum_\alpha k_\alpha \alpha$. In particular,*

$$P(L(\mu)) = \mu - \operatorname{cone}_{\mathbb{Z}}\mathfrak{R}_+,$$

and the multiplicity of the weight μ equals 1.

Proof Choose an ordering of the set $\mathfrak{R}_+$. By the Poincaré–Birkhoff–Witt Theorem, $U(\mathfrak{n}_-)$ has a basis consisting of ordered products $\prod_{\alpha\in\mathfrak{R}_+} f_\alpha^{k_\alpha}$ labeled by sets $\{k_\alpha, \alpha\in\mathfrak{R}_+\}$ with $k_\alpha \in \mathbb{Z}_{\geq}$. Hence $L(\mu)$ has a basis

$$v_{\{k_\alpha\}} = \Big(\prod_{\alpha\in\mathfrak{R}_+} \pi(f_\alpha)^{k_\alpha}\Big)v, \tag{B.8}$$

where v is the highest weight vector. Since $\pi(f_\alpha)$ shifts weights by $-\alpha$, the basis vectors (B.8) are weight vectors of weight

$$\mu - \sum_{\alpha\in\mathfrak{R}_+} k_\alpha\alpha.$$

The multiplicity of the weight $\mu-\nu$ is the number of ways of writing $\nu = \sum_{\alpha\in\mathfrak{R}_+} k_\alpha\alpha$, which is exactly $f(\nu)$. □

Corollary B.3 *Let V be a highest weight module for $\mu \in \mathfrak{t}^*$. Then*

$$P(V) \subseteq \mu - \operatorname{cone}_{\mathbb{Z}}\mathfrak{R}_+,$$

and all weights have finite multiplicity. The weight μ has multiplicity 1. *If V is irreducible, then $V^{\mathfrak{n}_+} = V_\mu$.*

Proof By Proposition B.13, V is a quotient of the Verma module $L(\mu)$. Hence $P(V) \subseteq P(L(\mu))$, and the multiplicity of the weight $\mu-\nu$ is at most $f(\nu) < \infty$. Suppose V is irreducible, and suppose μ' is another highest weight. Then $P(V) \subseteq \mu' - \operatorname{cone}_{\mathbb{Z}}\mathfrak{R}_+$. Since μ' itself lies in $\mu - \operatorname{cone}_{\mathbb{Z}}\mathfrak{R}_+$, this is impossible unless $\mu' = \mu$. Equivalently, V_μ contains all highest weight vectors. □

The sum of two proper submodules of $L(\mu)$ is again a proper submodule. (Any submodule is a sum of weight spaces; the submodule is proper if and only if μ does not appear as a weight.) Taking the sum of all proper submodules, we obtain a maximal proper submodule $L'(\mu)$. The quotient module

$$V(\mu) = L(\mu)/L'(\mu)$$

is then irreducible. (The preimage of a proper submodule $W \subseteq V(\mu)$ is a proper submodule in $L(\mu)$, hence contained in $L'(\mu)$. Thus $W = 0$.)

Proposition B.15 *Let V be an irreducible highest weight module, of highest weight $\mu \in \mathfrak{t}^*$. Then V is isomorphic to $V(\mu)$; the isomorphism is unique up to a non-zero scalar.*

Proof We will show that if V, V' are two irreducible modules of highest weight $\mu \in \mathfrak{t}^*$, then $V \cong V'$. (Uniqueness of the isomorphism follows since V_μ and V'_μ are 1-dimensional.) Let $v \in V$, $v' \in V'$ be highest weight vectors. Let $S \subseteq V \oplus V'$ be the submodule generated by $s = v \oplus v'$, that is, $S = U(\mathfrak{n}_-)s$. Since $\mathfrak{n}_+$ annihilates s, while $h \in \mathfrak{t}$ acts as a scalar $\langle \mu, h \rangle$, the representation S is again a highest weight module for μ. The projection $p: S \to V$ is $\mathfrak{g}$-equivariant, and hence is surjective:

$$p(S) = p(U(\mathfrak{g})s) = U(\mathfrak{g})p(s) = U(\mathfrak{g})v = V.$$

We claim that p is also injective. Suppose not, so that $\ker(p) = S \cap (0 \oplus V') \subseteq S$ is a non-trivial submodule. The restriction of $p': S \to V'$ to $\ker(p)$ is clearly injective. Since V' is irreducible, this restriction is also surjective, hence $\ker(p) \cong V'$ is a highest weight module of highest weight μ. But S_μ is spanned by $s \notin \ker(p)$, hence $\ker(p)_\mu = 0$. This contradiction shows that $\ker(p) = 0$, hence p is an isomorphism. Likewise $p': S \to V'$ is an isomorphism, proving $V \cong V'$. □

Proposition B.16 *Let V be an irreducible highest weight module, of highest weight $\mu \in \mathfrak{t}^*$. Then*

$$\dim V < \infty \Leftrightarrow \mu \in P_+.$$

Proof Suppose $\mu \in P_+$, and let $v \in V_\mu$ be non-zero. Given $\alpha \in \mathfrak{R}_+$, we have $\pi(e_\alpha)v = 0$ and $\pi(h_\alpha)v = \langle \mu, \alpha^\vee \rangle v$. Then $v_j = \frac{1}{j!}\pi(f_\alpha)^j v$, $j = 0, 1, 2, \ldots$, span an irreducible $\mathfrak{sl}_\alpha$-module $W \subseteq V$. It is finite-dimensional since $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$. The subspace $\pi(\mathfrak{g})W$ is again a finite-dimensional $\mathfrak{sl}_\alpha$-invariant subspace. Indeed, for all $\xi \in \mathfrak{sl}_\alpha$

$$\pi(\xi)\pi(\mathfrak{g})W \subseteq \pi(\mathfrak{g})W + \pi(\mathfrak{g})\pi(\xi)W \subseteq \pi(\mathfrak{g})W.$$

By induction, we hence see that $\pi(U^{(r)}\mathfrak{g})v = \pi(U^{(r)}\mathfrak{g})W$ is a finite-dimensional $\mathfrak{sl}_\alpha$-invariant subspace. This shows that *all* vectors $w \in V$ are contained in some finite-dimensional $\mathfrak{sl}_\alpha$-submodule, and hence that the operators $\pi(e_\alpha)$ and $\pi(f_\alpha)$ are locally nilpotent. (That is, for all $w \in V$ there exists $N > 0$ such that $\pi(e_\alpha)^N w = 0$ and $\pi(f_\alpha)^N w = 0$.) As a consequence, the transformation Θ_α defined in (B.7) is a well-defined automorphism of V, with

$$\Theta_\alpha \circ \pi(h) \circ \Theta_\alpha^{-1} = \pi(w_\alpha h)$$

for all $h \in \mathfrak{t}$. It follows that $P(V)$ is w_α-invariant. Since α was arbitrary, this proves that $P(V)$ is W-invariant. But $P(V) \subseteq \mu - \mathrm{cone}\mathfrak{R}_+$ has compact intersection with P_+. We conclude that $P(V)$ is finite. Since the weights have finite multiplicity, it follows that $\dim V < \infty$. □

In summary, we have proved the following result:

Theorem B.2 (Finite-dimensional irreducible $\mathfrak{g}$-representations) *Let $\mathfrak{g}$ be a complex reductive Lie algebra. Then the finite-dimensional irreducible $\mathfrak{g}$-representations are classified by the set P_+ of dominant weights. More precisely, any such representation is isomorphic to $V(\mu)$, for a unique $\mu \in P_+$.*

Definition B.8 Let V be a $\mathfrak{g}$-representation, and $\mu \in P_+$. The *μ-isotypical subspace* $V_{[\mu]}$ is the direct sum of the irreducible components of highest weight μ.

Equivalently, $V_{[\mu]}$ is characterized as the image of the map

$$\mathrm{Hom}_{\mathfrak{g}}(V(\mu), V) \otimes V(\mu) \to V,$$

or also as the submodule generated by $V_\mu^{\mathfrak{n}_+}$. If $\dim V < \infty$, we obtain

$$V = \bigoplus_{\mu \in P_+} V_{[\mu]}.$$

The *multiplicity* of the representation $V(\mu)$ in V equals

$$\dim \mathrm{Hom}_{\mathfrak{g}}(V(\mu), V) = \dim \mathrm{Hom}_{\mathfrak{g}}(V(\mu), V_{[\mu]}) = \dim V_\mu^{\mathfrak{n}_+}.$$

B.10 Extremal weights

Let V be a completely reducible $\mathfrak{g}$-representation. A weight $\mu \in P(V)$ will be called an *extremal weight* if it satisfies

$$\|\mu + \rho\| \geq \|\nu + \rho\|$$

for all $\nu \in P(V)$.

Proposition B.17 (Extremal weights) *Suppose μ is an extremal weight of V. Then μ is a dominant weight, $\mu \in P_+$, and*

$$V_\mu \subseteq V^{\mathfrak{n}_+}.$$

Thus $V(\mu)$ has multiplicity $\dim V_\mu$ *in V. In particular, if V is irreducible, then the highest weight of V is the unique extremal weight.*

Proof Suppose first that $V = V(\mu)$ is irreducible of highest weight μ. We will show that μ is the unique extremal weight of $V(\mu)$, that is, $\|\nu + \rho\| \leq \|\mu + \rho\|$ for all $\nu \in P(V(\mu))$, with equality if and only if $\nu = \mu$. Given $\nu \in P(V(\mu))$, choose $w \in W$ such that $w^{-1}(\nu + \rho) \in P_+$. Since $w^{-1}\nu \in P(V(\mu))$ we have $\mu - w^{-1}\nu \in \mathrm{cone}_{\mathbb{Z}}\mathfrak{R}_+$. Similarly $\rho - w^{-1}\rho \in \mathrm{cone}_{\mathbb{Z}}\mathfrak{R}_+$. Hence, both have non-negative inner product with $w^{-1}(\nu + \rho)$. Writing

$$\mu + \rho = (\mu - w^{-1}\nu) + (\rho - w^{-1}\rho) + w^{-1}(\nu + \rho),$$

we obtain

$$\begin{aligned}\|\mu+\rho\|^2 = \|(\mu-w^{-1}\nu)+(\rho-w^{-1}\rho)\|^2+\|\nu+\rho\|^2 \\ +2\big((\mu-w^{-1}\nu)+(\rho-w^{-1}\rho)|\, w^{-1}(\nu+\rho)\big) \\ \ge \|\nu+\rho\|^2,\end{aligned}$$

with equality if and only if $(\mu-w^{-1}\nu)+(\rho-w^{-1}\rho)=0$. Since both summands are in the positive root cone, each of them has to vanish, giving $\nu = w\mu$ and $\rho = w\rho$. Since the W-stabilizer of ρ is trivial, this implies that $w=1$ and $\mu=\nu$.

This proves the proposition for $V=V(\mu)$. The general case follows by decomposing V into irreducible components. □

B.11 Multiplicity computations

The following result is often used to calculate multiplicities in given representations.

Proposition B.18 *Let V and V' be finite-dimensional completely reducible $\mathfrak{g}$-representations, with formal character* $\mathrm{ch}(V)$, $\mathrm{ch}(V')$. *Then the dimension of the space* $\mathrm{Hom}_{\mathfrak{g}}(V, V')$ *of intertwining operators equals* $\frac{1}{|W|}$ *times the multiplicity of the* 0 *weight in*

$$\mathrm{ch}(V')\mathrm{ch}(V)^* \prod_{\alpha\in\mathfrak{R}}(1-\mathrm{e}^{\alpha}). \tag{B.9}$$

Proof It suffices to check for the cases that $V=V(\mu)$ and $V'=V(\mu')$ are irreducible representations of highest weights $\mu,\mu'\in P_+$. By the Weyl character formula (cf. Eq. (8.10)),

$$\mathrm{ch}(V(\mu)) = \frac{\sum_{w\in W}(-1)^{l(w)}\mathrm{e}^{w(\mu+\rho)-\rho}}{\prod_{\alpha\in\mathfrak{R}_+}(1-\mathrm{e}^{-\alpha})},$$

and similarly for $V(\mu')$. The conjugation of $\mathrm{ch}(V(\mu))$ amounts to sign changes of all weights:

$$\mathrm{ch}(V(\mu))^* = \frac{\sum_{w\in W}(-1)^{l(w)}\mathrm{e}^{-w(\mu+\rho)+\rho}}{\prod_{\alpha\in\mathfrak{R}_-}(1-\mathrm{e}^{-\alpha})}.$$

Hence the right-hand side of (B.9) becomes

$$\sum_{w\in W}\sum_{w'\in W}(-1)^{l(w)+l(w')}\mathrm{e}^{w'(\mu'+\rho)-w(\mu+\rho)}.$$

Consider the coefficient of the zero weight. Since μ,μ' are dominant, we have $w'(\mu'+\rho)-w(\mu+\rho)=0$ only if $\mu=\mu'$ and $w=w'$. Hence the coefficient of e^0 is $|W|\delta_{\mu,\mu'}$. □

Proposition B.19 *The character of the ρ-representation is given by*

$$\mathrm{ch}(V(\rho)) = \mathrm{e}^{\rho}\prod_{\alpha\in\mathfrak{R}_+}(1+\mathrm{e}^{-\alpha}).$$

It satisfies $\mathrm{ch}(V(\rho))^* = \mathrm{ch}(V(\rho))$.

Proof This result may be obtained, for instance, from the Weyl character formula. Alternatively, it follows from the interpretation of the ρ-representation in terms of the spin representation; see the computation in Remark 8.9. □

The following result will be needed in Section 11.4.2

Proposition B.20 *View* $\mathrm{End}(V(\rho))$ *as a* $\mathfrak{g}$*-representation. Then*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{End}(V(\rho))) = l,$$

the rank of $\mathfrak{g}$.

Proof By Proposition B.18, the left-hand side equals $|W|^{-1}$ times the multiplicity of the zero weight in

$$\mathrm{ch}(\mathfrak{g})\mathrm{ch}(\mathrm{End}(V(\rho))) \prod_{\alpha\in\mathfrak{R}} (1-\mathrm{e}^{\alpha}). \tag{B.10}$$

We have $\mathrm{End}(V(\rho)) = V(\rho) \otimes V(\rho)$; hence

$$\mathrm{ch}(\mathrm{End}(V(\rho))) = \mathrm{ch}(V(\rho))^2 = \prod_{\alpha\in\mathfrak{R}} (1+\mathrm{e}^{\alpha}).$$

Consequently,

$$\begin{aligned}\mathrm{ch}(\mathrm{End}(V(\rho))) \prod_{\alpha\in\mathfrak{R}} (1-\mathrm{e}^{\alpha}) &= \prod_{\alpha\in\mathfrak{R}_+} (1+\mathrm{e}^{\alpha})(1+\mathrm{e}^{-\alpha})(1-\mathrm{e}^{\alpha})(1-\mathrm{e}^{-\alpha}) \\ &= \prod_{\alpha\in\mathfrak{R}_+} (2-(\mathrm{e}^{2\alpha}+\mathrm{e}^{-2\alpha})).\end{aligned}$$

The right-hand side of this expression is supported on $2Q$, twice the root lattice. On the other hand, the character for the adjoint representation is

$$\mathrm{ch}(\mathfrak{g}) = l + \sum_{\beta\in\mathfrak{R}} \mathrm{e}^{\beta}.$$

But $\beta \notin 2Q$ for all roots β. (This is obvious for simple roots, since they are basis elements of Q, and hence it holds true in general because any root is a simple root for a suitable choice of basis.) Hence, the multiplicity of the zero weight in (B.10) equals the constant term of $\mathrm{ch}(\mathfrak{g})$ (i.e., the rank l of $\mathfrak{g}$) times the multiplicity of the zero weight in

$$\mathrm{ch}(\mathrm{End}(V(\rho))) \prod_{\alpha\in\mathfrak{R}} (1-\mathrm{e}^{\alpha}).$$

By Proposition B.18 this multiplicity equals $|W|$ times the multiplicity of the trivial representation on $\mathrm{End}(V(\rho))$; but the latter is 1 by Schur's Lemma. □

Appendix C
Background on Lie groups

In this appendix we review some basic material on Lie groups. Standard references include [30] and [48].

C.1 Preliminaries

A (real) *Lie group* is a group G, equipped with a (real) manifold structure such that the group operations of multiplication and inversion are smooth. For example, $\mathrm{GL}(N,\mathbb{R})$, with manifold structure as an open subset of $\mathrm{Mat}_N(\mathbb{R})$, is a Lie group. According to a theorem of E. Cartan, any topologically closed subgroup H of a Lie group G is a Lie subgroup: the smoothness is automatic. Hence, it is immediate that, $\mathrm{SO}(n)$, $\mathrm{GL}(N,\mathbb{C})$, $\mathrm{U}(n)$, etc. are again Lie groups. A related result is that if G_1 and G_2 are Lie groups, then any continuous group homomorphism $G_1 \to G_2$ is smooth. Consequently, a given topological group cannot have more than one smooth structure making it into a Lie group.

For $a \in G$, let $\mathscr{A}^L(a)$ be the diffeomorphism of G given by left-multiplication, $g \mapsto ag$. A vector field $X \in \mathfrak{X}(G)$ is called *left-invariant* if it is invariant under all $\mathscr{A}^L(a)$, $a \in G$. Equivalently, a vector field is left-invariant if and only if its action on functions commutes with pull-back under $\mathscr{A}^L(a)$, for all a. It is immediate that the Lie bracket of two left-invariant vector fields is again left-invariant. Let $\mathfrak{X}^L(G) \subseteq \mathfrak{X}(G)$ denote the Lie algebra of left-invariant vector fields. Any element of $\mathfrak{X}^L(G)$ is determined by its value at the group unit $e \in G$. This gives a vector space isomorphism $T_eG \to \mathfrak{X}^L(G)$, $\xi \mapsto \xi^L$. One calls

$$\mathfrak{g} = T_eG \cong \mathfrak{X}^L(G),$$

with Lie bracket induced from that on $\mathfrak{X}^L(G)$, the *Lie algebra of G*. *Lie's Third Theorem* asserts that for any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, there is a unique connected, simply connected Lie group G having $\mathfrak{g}$ as its Lie algebra.

If $G = \mathrm{GL}(N,\mathbb{R})$, the tangent space $\mathfrak{g} = T_eG$ is canonically identified with the space $\mathrm{Mat}_N(\mathbb{R})$ of $N \times N$-matrices, and one may verify that the Lie bracket is simply the commutator of matrices. (This is the main reason for working with $\mathfrak{X}^L(G)$

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rather than $\mathfrak{X}^R(G)$, since the latter choice would have resulted in *minus* the commutator.)

C.2 Group actions on manifolds

An *action of a Lie group G on a manifold M* is a group homomorphism $\mathscr{A}:\ G \to \mathrm{Diff}(M)$ into the group of diffeomorphisms of M, with the property that the action map $G \times M \to M$, $(g,x) \mapsto \mathscr{A}(g)(x)$ is smooth. It induces actions on the tangent bundle and on the cotangent bundle, and hence there are notions of invariant vector fields $\mathfrak{X}(M)^G$, invariant differential forms $\Omega(M)^G$, and so on.

Example C.1 There are three important actions of a Lie group on itself: The actions by left- and right-multiplication, and the adjoint (or conjugation) action:

$$\mathscr{A}^L(g)(a) = ga,\ \ \mathscr{A}^R(g)(a) = ag^{-1},\ \ \mathrm{Ad}(g)(a) = gag^{-1}.$$

An *action of a Lie algebra $\mathfrak{g}$ on a manifold M* is a Lie algebra homomorphism $\mathscr{A}:\ \mathfrak{g} \to \mathfrak{X}(M)$ such that the map $\mathfrak{g} \times M \to TM$, $(\xi, x) \mapsto \mathscr{A}(\xi)(x)$ is smooth.

Given a Lie group action

$$\mathscr{A}:\ G \to \mathrm{Diff}(M),$$

its differential at the group unit defines an action of the Lie algebra $\mathfrak{g}$ (which we denote by the same letter). In terms of the action of vector fields on functions,

$$\mathscr{A}(\xi) f = \frac{\partial}{\partial t}\Big|_{t=0} \exp(-t\xi)^* f, \quad \xi \in \mathfrak{g}.$$

One calls $\xi_M = \mathscr{A}(\xi)$ the generating vector fields[1] for the G-action.

Example C.2 The generating vector fields $\mathrm{Ad}(\xi) \in \mathfrak{X}(\mathfrak{g})$ for the adjoint action of G on $\mathfrak{g}$ are

$$\mathrm{Ad}(\xi)|_\mu = \mathrm{ad}_\mu(\xi)$$

(using the identifications $T_\mu\mathfrak{g} = \mathfrak{g}$). The generating vector fields for the three natural actions of G on itself are

$$\mathscr{A}^L(\xi) = -\xi^R,\ \ \mathscr{A}^R(\xi) = \xi^L,\ \ \mathrm{Ad}(\xi) = \xi^L - \xi^R.$$

(Note, e.g., that the vector field $\mathscr{A}^R(\xi)$ must be left-invariant, since the action $\mathscr{A}^R(g)$ commutes with the left-action.) We have $[\xi^L, \zeta^R] = 0$, since the left- and right-actions commute.

[1] Some authors use opposite sign conventions, so that ξ_M is an anti-homomorphism.

The action of G on M lifts to an action on the tangent bundle TM. Given a fixed point $x \in M$ of a G-action, so that $\mathscr{A}(g)x = x$ for all $g \in G$, the action preserves the fiber T_xM, defining a linear representation $G \to \mathrm{GL}(T_xM)$. In particular, the adjoint action

$$\mathrm{Ad}: \ G \to \mathrm{Diff}(G)$$

fixes e, and hence induces a linear action on $T_eG = \mathfrak{g}$, denoted by the same letter:

$$\mathrm{Ad}: \ G \to \mathrm{GL}(\mathfrak{g}).$$

Since the adjoint action on G is by group automorphisms, its linearization acts by Lie algebra automorphisms of $\mathfrak{g}$. One also defines an infinitesimal adjoint action of elements $\mu \in \mathfrak{g}$ by

$$\mathrm{ad}_\mu: \ \mathfrak{g} \to \mathfrak{g}, \ \ \mathrm{ad}_\mu(\xi) = [\mu, \xi]_{\mathfrak{g}}.$$

Then $\mathrm{ad}: \ \mu \mapsto \mathrm{ad}_\mu$ is a Lie algebra homomorphism

$$\mathrm{ad}: \ \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$$

into the Lie algebra of derivations of $\mathfrak{g}$. (A linear map $A \in \mathrm{End}(\mathfrak{g})$ is a derivation of the Lie bracket if and only if $A[\xi_1, \xi_2] = [A\xi_1, \xi_2] + [\xi_1, A\xi_2]$.)

This adjoint representation $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ of the Lie algebra is the differential of the adjoint representation $G \to \mathrm{Aut}(\mathfrak{g})$ of the Lie group (note that $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g})$).

C.3 The exponential map

Any $\xi \in \mathfrak{g} = T_eG$ determines a unique 1-parameter subgroup $\phi_\xi: \mathbb{R} \to G$ such that

$$\phi_\xi(t_1 + t_2) = \phi_\xi(t_1)\phi_\xi(t_2), \quad \phi_\xi(0) = e, \quad \left.\frac{\partial \phi_\xi}{\partial t}\right|_{t=0} = \xi.$$

In fact, ϕ_ξ is a solution curve of the left-invariant vector field ξ^L. One defines the *exponential map*

$$\exp: \ \mathfrak{g} \to G, \ \xi \mapsto \phi_\xi(1).$$

For matrix Lie groups, the abstract exponential map coincides with the usual exponential of a matrix as a Taylor series. The 1-parameter subgroup may be written in terms of the exponential map as $\phi_\xi(t) = \exp(t\xi)$.

The exponential map is natural with respect to Lie morphisms. Hence, if $\phi:$ $G \to H$ is a morphism of Lie groups, and denoting by the same letter its differential $\phi: \ \mathfrak{g} \to \mathfrak{h}$, we have $\exp(\phi(\xi)) = \phi(\exp(\xi))$. In particular, this applies to the adjoint representation $\mathrm{Ad}: \ G \to \mathrm{GL}(\mathfrak{g})$ and its differential $\mathrm{ad}: \ \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. That is,

$$\mathrm{Ad}(\exp(\mu)) = \exp(\mathrm{ad}_\mu) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{ad}_\mu^n.$$

Since $(\mathrm{d}_0 \exp)(\xi) = \frac{d}{dt}|_{t=0} \exp(t\xi) = \xi$, the differential of the exponential map at the origin is the identity $\mathrm{d}_0 \exp = \mathrm{id}$. Hence, by the implicit function theorem the exponential map restricts to a diffeomorphism from an open neighborhood of 0 in $\mathfrak{g}$ to an open neighborhood of e in G. We are interested in the differential of $\exp: \mathfrak{g} \to G$ at *any* given point $\mu \in \mathfrak{g}$. It is a linear operator $\mathrm{d}_\mu \exp: \mathfrak{g} = T_\mu \mathfrak{g} \to T_g G$. Since $\mathfrak{g}$ is a vector space, $T_\mu \mathfrak{g} \cong \mathfrak{g}$ canonically. On the other hand, we may use the left-action to obtain an isomorphism, $\mathrm{d}_e \mathscr{A}^L(g): \mathfrak{g} \to T_g G$, and hence an isomorphism $TG = G \times \mathfrak{g}$ by *left-trivialization*.

Theorem C.1 *The differential of the exponential map* $\exp: \mathfrak{g} \to G$ *at* $\mu \in \mathfrak{g}$ *is the linear operator* $\mathrm{d}_\mu \exp: \mathfrak{g} \to T_{\exp(\mu)}\mathfrak{g}$ *given by the formula*

$$\mathrm{d}_\mu \exp = j^L(\mathrm{ad}_\mu),$$

where we use left-trivialization to identify $T_{\exp(\mu)}\mathfrak{g} \cong \mathfrak{g}$.

Here $j^L(z) = \frac{1-\mathrm{e}^{-z}}{z}$ is the holomorphic function introduced in Section 4.3.5.

Proof For $\zeta \in \mathfrak{g}$, we have the following identities of operators on $C^\infty(G)$,

$$\frac{\partial}{\partial s}(\mathscr{A}^R_{\exp(s\zeta)})^* = \zeta^L \circ (\mathscr{A}^R_{\exp(s\zeta)})^* = (\mathscr{A}^R_{\exp(s\zeta)})^* \circ \zeta^L. \tag{C.1}$$

These are proved by evaluating the two sides of

$$\frac{\partial}{\partial u}\Big|_{u=0}(\mathscr{A}^R_{\exp((s+u)\zeta)})^* = \frac{\partial}{\partial u}\Big|_{u=0}(\mathscr{A}^R_{\exp(u\zeta)})^*(\mathscr{A}^R_{\exp(s\zeta)})^*.$$

The operator $j^L(\mathrm{ad}_\mu)$ may be written as an integral $\int_0^1 \mathrm{d}s\; \exp(-s\mathrm{ad}_\mu)$. We have to show that for all $\xi \in \mathfrak{g}$,

$$(\mathrm{d}_\mu \exp)(\xi) \circ (\mathscr{A}^L_{\exp(-\mu)})^* = \int_0^1 \mathrm{d}s\; (\exp(-s\mathrm{ad}_\mu)\xi)$$

as operators on functions $f \in C^\infty(G)$. To compute the left-hand side, write

$$\begin{aligned}(\mathrm{d}_\mu \exp)(\xi) \circ (\mathscr{A}^L_{\exp(-\mu)})^* f &= \frac{\partial}{\partial t}\Big|_{t=0}((\mathscr{A}^L_{\exp(-\mu)})^* f)(\exp(\mu + t\xi)) \\ &= \frac{\partial}{\partial t}\Big|_{t=0} f(\exp(-\mu)\exp(\mu + t\xi)) \\ &= \Big(\frac{\partial}{\partial t}\Big|_{t=0}(\mathscr{A}^R_{\exp(-\mu)})^*(\mathscr{A}^R_{\exp(\mu+t\xi)})^* f\Big)\Big|_e .\end{aligned}$$

Using (C.1), we compute

$$\begin{aligned}\frac{\partial}{\partial t}\Big|_{t=0}(\mathscr{A}^R_{\exp(-\mu)})^*(\mathscr{A}^R_{\exp(\mu+t\xi)})^* &= \int_0^1 \mathrm{d}s\; \frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}(\mathscr{A}^R_{\exp(-s\mu)})^*(\mathscr{A}^R_{\exp(s(\mu+t\xi))})^* \\ &= \int_0^1 \mathrm{d}s\; \frac{\partial}{\partial t}\Big|_{t=0}(\mathscr{A}^R_{\exp(-s\mu)})^*(t\xi)^L(\mathscr{A}^R_{\exp(s(\mu+t\xi))})^*\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \mathrm{d}s\ (\mathscr{A}^R_{\exp(-s\mu)})^*\ \xi^L\ (\mathscr{A}^R_{\exp(s(\mu))})^* \\
&= \int_0^1 \mathrm{d}s\ (\mathrm{Ad}_{\exp(-s\mu)}\xi)^L \\
&= \int_0^1 \mathrm{d}s\ (\exp(-s\,\mathrm{ad}_\mu)\xi)^L.
\end{aligned}$$

Applying this result to f at e, we obtain $\int_0^1 \mathrm{d}s\ (\exp(-s\,\mathrm{ad}_\mu)\xi)(f)$ as desired. □

Corollary C.1 *The exponential map has maximal rank at $\mu \in \mathfrak{g}$ if and only if* ad_μ *has no eigenvalue in the set* $2\pi i\mathbb{Z}\backslash\{0\}$.

Proof $\mathrm{d}_\mu \exp$ is an isomorphism if and only if $j^L(\mathrm{ad}_\mu)$ is invertible, i.e., has non-zero determinant. The determinant is given in terms of the eigenvalues λ of ad_μ as a product, $\prod_\lambda j^L(\lambda)$. This vanishes if and only if there is an eigenvalue with $j^L(\lambda) = 0$, that is, $\lambda \neq 0$ and $\mathrm{e}^\lambda = 1$. □

Remarks C.1

1. Using instead the right-action to identify $TG \cong G \times \mathfrak{g}$ one obtains
$$\mathrm{d}_\mu \exp = j^R(\mathrm{ad}_\mu).$$
This follows from the formula for the left-trivialization because the adjoint action of $\exp\mu$ on $\mathfrak{g}$ is
$$\mathrm{Ad}(\exp\mu) = \mathrm{d}_e\mathscr{A}^R(\exp\mu)^{-1} \circ \mathrm{d}_a\mathscr{A}^L(\exp\mu),$$
and since
$$\mathrm{Ad}(\exp\mu)\, j^L(\mathrm{ad}_\mu) = \mathrm{e}^{\mathrm{ad}_\mu} j^L(\mathrm{ad}_\mu) = j^R(\mathrm{ad}_\mu).$$
2. In particular, the Jacobian of the exponential map relative to the left-invariant volume form is the function, $\mu \mapsto \det(j^L(\mathrm{ad}_\mu))$, while for the right-invariant volume form one obtains $\det(j^R(\mathrm{ad}_\mu))$. In general, the two Jacobians are not the same: Their quotient is the function
$$\det(\mathrm{e}^{\mathrm{ad}_\mu}) = \mathrm{e}^{\mathrm{tr}(\mathrm{ad}_\mu)}.$$
The function $G \to \mathbb{R}^\times$, $g \mapsto \det(\mathrm{Ad}(g))$ is a group homomorphism called the *modular character*; it relates the left- and right-invariant volume forms Γ^L and Γ^R on G defined by a generator $\Gamma \in \det(\wedge\mathfrak{g}^*)$. A Lie group G is called *unimodular* if the modular character is trivial. For any Lie algebra $\mathfrak{g}$, the Lie algebra morphism $\mathfrak{g} \to \mathbb{R}$, $\mu \mapsto \mathrm{tr}(\mathrm{ad}_\mu)$ is called the *(infinitesimal) modular character*; $\mathfrak{g}$ is called unimodular if the modular character is zero. Clearly, the differential of the modular character of a Lie group G is the modular character of its Lie algebra $\mathfrak{g}$.

   For instance, any compact Lie group, any semisimple Lie group, and more generally any Lie group admitting a bi-invariant pseudo-Riemannian metric, is unimodular. Likewise, any quadratic Lie algebra is unimodular: in that case, ad_μ is skew-adjoint, so its trace vanishes.

An example of a non-unimodular Lie group is the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a \in \mathbb{R} - \{0\},\ b \in \mathbb{R},$$

i.e., the group of affine transformations of $\mathbb{R}$.

3. For a quadratic Lie algebra, the determinants of $j^L(\mathrm{ad}_\mu)$ and $j^R(\mathrm{ad}_\mu)$ coincide, and are equal to

$$J(\mu) := \det j(\mathrm{ad}_\mu) = \det\Big(\frac{\sinh \mathrm{ad}_\mu/2}{\mathrm{ad}_\mu/2}\Big).$$

By our results from Section 4.3.7 this function admits a global analytic square root.

C.4 The vector field $\frac{1}{2}(\xi^L + \xi^R)$

For any $\xi \in \mathfrak{g}$, we will use the same letter to denote the corresponding constant vector field on $\mathfrak{g}$. To avoid confusion with commutators of vector fields, the Lie bracket will be denoted $[\cdot,\cdot]_\mathfrak{g}$. The half-sum $\xi^\sharp = \frac{1}{2}(\xi^L + \xi^R) \in \mathfrak{X}(G)$ has properties similar to that of the constant vector field $\xi \in \mathfrak{X}(\mathfrak{g})$. For example, for $\xi, \zeta \in \mathfrak{g}$ the vector fields $\xi^\sharp, \zeta^\sharp$ "almost" commute in the sense that

$$[\xi^\sharp, \zeta^\sharp] = \frac{1}{4}[\xi,\zeta]^L_\mathfrak{g} - \frac{1}{4}[\xi,\zeta]^R_\mathfrak{g} = \frac{1}{4}\mathrm{Ad}([\xi,\zeta]_\mathfrak{g})$$

vanishes at $e \in G$. Note also that the vector fields $\mathrm{Ad}(\xi) = \xi^L - \xi^R$ satisfy

$$[\mathrm{Ad}(\xi), \zeta^\sharp] = [\xi,\zeta]^\sharp_\mathfrak{g},$$

similar to a property of the constant vector field on $\mathfrak{g}$.

Let $\mathfrak{g}' \subseteq \mathfrak{g}$ denote the subset where the exponential map has maximal rank. As remarked above, this is the subset where $\mathrm{ad}_\mu : \mathfrak{g} \to \mathfrak{g}$ has no eigenvalue of the form $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. Given a vector field $X \in \mathfrak{X}(G)$, one has a well-defined vector field $\exp^*(X) \in \mathfrak{X}(\mathfrak{g}')$ such that $\exp^*(X)_\mu = (\mathrm{d}_\mu \exp)^{-1}(X_{\exp\mu})$ for all $\mu \in \mathfrak{g}^*$. In particular, for $\xi \in \mathfrak{g}$ we can consider

$$\exp^*\xi^L, \quad \exp^*\xi^R, \quad \exp^*\xi^\sharp.$$

Since $T_\mu\mathfrak{g} \cong \mathfrak{g}$, each of these vector fields define elements of $C^\infty(\mathfrak{g}') \otimes \mathfrak{g}$, depending linearly on ξ. The map taking ξ to this vector field is therefore an element of $C^\infty(\mathfrak{g}') \otimes \mathrm{End}(\mathfrak{g})$.

Using left-trivialization of the tangent bundle, we have

$$(\exp^*\xi^L)_\mu = (j^L(\mathrm{ad}_\mu))^{-1}(\xi) = \frac{\mathrm{ad}_\mu}{1 - \mathrm{e}^{-\mathrm{ad}_\mu}}\xi.$$

Similarly,

$$(\exp^* \xi^R)_\mu = (j^R(\mathrm{ad}_\mu))^{-1}(\xi) = \frac{\mathrm{ad}_\mu}{\mathrm{e}^{\mathrm{ad}_\mu} - 1}\xi.$$

The difference with the constant vector field ξ is

$$(\exp^* \xi^L)_\mu - \xi = \mathrm{ad}_\mu f^L(\mathrm{ad}_\mu)(\xi) = \mathrm{Ad}(f^L(\mathrm{ad}_\mu)\xi),$$
$$(\exp^* \xi^R)_\mu - \xi = \mathrm{ad}_\mu f^R(\mathrm{ad}_\mu)(\xi) = \mathrm{Ad}(f^R(\mathrm{ad}_\mu)\xi),$$

where

$$f^L(z) = \frac{1}{1-\mathrm{e}^{-z}} - \frac{1}{z}, \quad f^R(z) = \frac{1}{\mathrm{e}^z - 1} - \frac{1}{z}.$$

Note that $f^L(\mathrm{ad}_\mu)$ and $f^R(\mathrm{ad}_\mu) \in \mathrm{End}(\mathfrak{g})$ are well defined for all $\mu \in \mathfrak{g}'$. The formula shows that the difference between the vector fields $\exp^* \xi^L$, $\exp^* \xi^R$ and the constant vector field ξ is tangent to the adjoint orbits. Put differently, the radial part of these vector fields equals ξ. Finally,

$$(\exp^* \xi^\sharp)_\mu - \xi = f(\mathrm{ad}_\mu)(\mathrm{ad}_\mu \xi) = \mathrm{Ad}\big(f(\mathrm{ad}_\mu)\xi\big)\big|_\mu,$$

where $f = \frac{1}{2}(f^L + f^R)$. That is,

$$f(z) = \frac{1}{2}\left(\frac{1}{\mathrm{e}^z - 1} + \frac{1}{1-\mathrm{e}^{-z}}\right) - \frac{1}{z} = \frac{1}{2}\coth\left(\frac{z}{2}\right) - \frac{1}{z}.$$

Remarks C.2 The function $j^R(z)^{-1} = \frac{z}{\mathrm{e}^z - 1}$ is the well-known generating functions for the *Bernoulli numbers* B_n:

$$j^R(z)^{-1} = \frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

The expansion of the function $f(z)$ reads as

$$f(z) = \frac{1}{2}\coth\left(\frac{z}{2}\right) - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n-1}.$$

C.5 Maurer–Cartan forms

The *left-invariant Maurer–Cartan form* $\theta^L \in \Omega^1(G)^L \otimes \mathfrak{g}$ is defined in terms of its contractions with left-invariant vector fields by

$$\iota(\xi^L)\theta^L = \xi.$$

Similarly, one defines the *right-invariant Maurer–Cartan form* $\theta^R \in \Omega^1(G)^R \otimes \mathfrak{g}$ by

$$\iota(\xi^R)\theta^R = \xi.$$

For matrix Lie groups $G \subseteq \mathrm{Mat}_N(\mathbb{R})$, one has the formulas

$$\theta^L = g^{-1}\mathrm{d}g, \quad \theta^R = \mathrm{d}g\, g^{-1}.$$

(More precisely, $\mathrm{d}g$ is a matrix-valued 1-form on G, to be interpreted as the pull-back of the coordinate differentials on $\mathrm{Mat}_N(\mathbb{R}) \cong \mathbb{R}^{N^2}$ under the inclusion map $G \to \mathrm{Mat}_N(\mathbb{R})$.) Let $\mathrm{Mult}: G \times G \to G$ be the group multiplication, and let $\mathrm{pr}_1, \mathrm{pr}_2: G \times G \to G$ be the projections to the two factors.

Proposition C.1 (Properties of Maurer–Cartan forms)

1. *The Maurer–Cartan forms are related by*

$$\theta^R_g = \mathrm{Ad}_g(\theta^L_g),$$

2. *The differential of* θ^L, θ^R *is given by the* Maurer–Cartan equations

$$\mathrm{d}\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0, \quad \mathrm{d}\theta^R - \frac{1}{2}[\theta^R, \theta^R] = 0.$$

3. *The pull-backs of* θ^L *and* θ^R *under group multiplication are given by the formula*

$$\mathrm{Mult}^*\theta^L = \mathrm{Ad}_{g_2^{-1}}\mathrm{pr}_1^*\theta^L + \mathrm{pr}_2^*\theta^L,$$
$$\mathrm{Mult}^*\theta^R = \mathrm{Ad}_{g_1}\mathrm{pr}_2^*\theta^R + \mathrm{pr}_1^*\theta^L.$$

For matrix Lie groups, all of these results are easily proved from $\theta^L = g^{-1}\mathrm{d}g$ and $\theta^R = \mathrm{d}g g^{-1}$. For instance, $\mathrm{Mult}^*\theta^L$ is computed as follows:

$$(g_1g_2)^{-1}\mathrm{d}(g_1g_2) = g_2^{-1}g_1^{-1}\mathrm{d}g_1g_1^{-1} + g_2^{-1}\mathrm{d}g_2.$$

Applying the de Rham differential to the Maurer–Cartan equations, we obtain

$$[[\theta^L, \theta^L], \theta^L] = 0, \quad [[\theta^R, \theta^R], \theta^R] = 0. \tag{C.2}$$

These equations also follow from the Jacobi identity for $\mathfrak{g}$.

Consider now the pull-back of the Maurer–Cartan forms under the exponential map, $\exp^*\theta^L, \exp^*\theta^R \in \Omega^1(\mathfrak{g}) \otimes \mathfrak{g}$. At any given point $\mu \in \mathfrak{g}$, these are elements of $T^*_\mu\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g}^* \otimes \mathfrak{g}$. Thus, we can view $\exp^*\theta^L, \exp^*\theta^R$ as maps $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

Theorem C.2 *The maps* $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *determined by* $\exp^*\theta^L$ *and* $\exp^*\theta^R$ *are given by*

$$\mu \mapsto j^L(\mathrm{ad}_\mu), \quad \mu \mapsto j^R(\mathrm{ad}_\mu),$$

respectively.

Proof Let $\mu \in \mathfrak{g}$ and $\xi \in T_\mu\mathfrak{g} = \mathfrak{g}$. Then, using Theorem C.1

$$\begin{aligned}\iota(\xi)(\exp^*\theta^L)_\mu &= \iota(\mathrm{d}_\mu\exp(\xi))\theta^L_{\exp\mu}\\ &= j^L(\mathrm{ad}_\mu)(\xi).\end{aligned}$$

This proves the result for θ^L, and also that for θ^R since $\theta^R = \mathrm{Ad}_g\theta^L$. $\square$

Letting $\mathrm{d}\mu \in \Omega^1(\mathfrak{g};\mathfrak{g})$ denote the tautological $\mathfrak{g}$-valued 1-form on $\mathfrak{g}$ (i.e., the Maurer–Cartan form of $\mathfrak{g}$ viewed as an additive Lie group), we may write this as

$$\exp^* \theta^L = j^L(\mathrm{ad}_\mu)(\mathrm{d}\mu), \quad \exp^* \theta^R = j^R(\mathrm{ad}_\mu)(\mathrm{d}\mu).$$

Hence

$$\frac{1}{2}\exp^*(\theta^L + \theta^R) - \mathrm{d}\mu = g(\mathrm{ad}_\mu)\mathrm{ad}_\mu(\mathrm{d}\mu),$$

where $g(z) = z^{-2}(\sinh z - z)$ is the function introduced in Section 4.3.5.

C.6 Quadratic Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A bilinear form B on $\mathfrak{g}$ is called G-invariant if it is invariant under the adjoint action:

$$B(\mathrm{Ad}_g(\xi), \mathrm{Ad}_g(\zeta)) = B(\xi, \zeta)$$

for all ξ, ζ. If B is furthermore non-degenerate, we will refer to G as a *quadratic Lie group*.

Examples C.3

1. Any semisimple Lie group is a quadratic Lie group, taking B to be the Killing form.
2. The group $G = \mathrm{GL}(N, \mathbb{R})$ is quadratic, using the trace form $B(\xi, \zeta) = \mathrm{tr}(\xi\zeta)$ on its Lie algebra.
3. If H is any Lie group, then the semidirect product $H \ltimes \mathfrak{h}^*$ is a quadratic Lie group for the bilinear form on $\mathfrak{h} \ltimes \mathfrak{h}^*$ given by the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

Given an invariant symmetric bilinear form B on $\mathfrak{g}$ (not necessarily non-degenerate), there is an important 3-form on the group called the *Cartan 3-form*:

$$\eta = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) \in \Omega^3(G).$$

Since B is invariant and since $\theta^R = \mathrm{Ad}_g \theta^L$, the 3-form η is also right-invariant. In fact,

$$\eta = \frac{1}{12} B(\theta^R, [\theta^R, \theta^R]).$$

As for any bi-invariant differential form on a Lie-group, it is closed:

$$\mathrm{d}\eta = 0.$$

This may also be seen using the Maurer–Cartan-equation for θ^L,

$$\mathrm{d}\eta = -\frac{1}{24} B([\theta^L, \theta^L], [\theta^L, \theta^L]) = -\frac{1}{24} B(\theta^L, [\theta^L, [\theta^L, \theta^L]]) = 0,$$

using (C.2). The pull-back of $\exp^* \eta$ of the closed 3-form η to $\mathfrak{g}$ is exact. The Poincaré Lemma gives an explicit primitive $\varpi \in \Omega^2(\mathfrak{g})$ with $\mathrm{d}\varpi = \Phi^*\eta$. The identification

$$\Omega^2(\mathfrak{g}) \cong C^\infty(\mathfrak{g}) \otimes \wedge^2 \mathfrak{g}^*$$

takes ϖ_μ to an element of $\wedge^2\mathfrak{g}^*$, which we can view as a skew-adjoint map $\mathfrak{g} \to \mathfrak{g}^*$.

Proposition C.2 *The skew-adjoint map corresponding to*

$$\varpi_\mu \in \wedge^2 T_\mu^* G \cong \wedge^2 \mathfrak{g}^*$$

is $\mu \mapsto B^\flat \circ g(\mathrm{ad}_\mu)$, *where* $g(z) = z^{-2}(\sinh(z) - z)$.

Proof Recall that the homotopy operator for a vector space V is the map

$$h:\ \Omega^\bullet(V) \to \Omega^{\bullet-1}(V),$$

given as pull-back under $H:\ I \times V \to V,\ (t, x) \mapsto tx$ followed by integration over the fibers over the projection $\mathrm{pr}_2:\ I \times X \to X$. It satisfies $\mathrm{d}h + h\mathrm{d} = \mathrm{Id} - \pi^* i^*$, where $i:\ \{0\} \to V$ is the inclusion and $\pi:\ V \to \{0\}$ is the projection. We have

$$\exp^* \theta^L = j^L(\mathrm{ad}_\xi)\mathrm{d}\xi;$$

hence

$$\begin{aligned} H^* \exp^* \theta^L &= j^L(t\mathrm{ad}_\xi)(t\mathrm{d}\xi + \xi\mathrm{d}t) \\ &= t\, j^L(t\mathrm{ad}_\xi)\mathrm{d}\xi + \xi\mathrm{d}t. \end{aligned}$$

Consequently,

$$H^* \exp^* \eta = \frac{1}{4} B\Big(\big[\xi \mathrm{d}t, t\, j^L(t\mathrm{ad}_\xi)\mathrm{d}\xi\big],\ t j^L(t\mathrm{ad}_\xi)\mathrm{d}\xi\Big) + \cdots,$$

where $\ldots$ indicates a term not containing $\mathrm{d}t$. Using the definition of j^L, and using that the transpose of $j^L(\mathrm{ad}_\xi)$ is $j^R(\mathrm{ad}_\xi)$, this may be written as

$$\begin{aligned} H^* \exp^* \eta &= \frac{1}{4}\mathrm{d}t \wedge B\big(t\mathrm{ad}_\xi\, j^L(t\mathrm{ad}_\xi)\mathrm{d}\xi,\ t\, j^L(t\mathrm{ad}_\xi)\mathrm{d}\xi\big) + \cdots \\ &= \frac{1}{4} t\mathrm{d}t \wedge B\Big(\big(1 - \mathrm{e}^{-t\mathrm{ad}_\xi}\big)\mathrm{d}\xi,\ j^L(t\mathrm{ad}_\xi)\mathrm{d}\xi\Big) + \cdots \\ &= \frac{1}{4} t\mathrm{d}t \wedge B\big(j^R(t\mathrm{ad}_\xi)(1 - \mathrm{e}^{-t\mathrm{ad}_\xi})\mathrm{d}\xi,\ \mathrm{d}\xi\big) + \cdots \\ &= \frac{1}{2}\mathrm{d}t \wedge B\Big(\frac{\cosh(t\mathrm{ad}_\xi) - 1}{\mathrm{ad}_\xi}\mathrm{d}\xi,\ \mathrm{d}\xi\Big). \end{aligned}$$

Integrating from $t = 0$ to 1, and using

$$\int_0^1 \frac{\cosh(tz) - 1}{z} = \frac{\sinh(z) - z}{z^2} = g(z),$$

we obtain $(\mathrm{pr}_2)_* H^* \exp^* \eta = \frac{1}{2} B(g(\mathrm{ad}_\xi)\mathrm{d}\xi, \mathrm{d}\xi)$. $\square$

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