Homework #6:
Due: tentatively Nov. 26, 2019

NOTE: Condon-Shortley convention should be used unless specified otherwise. Bold symbols denote three component vectors, for example $\boldsymbol{S}$ has three components S_x, S_y, S_z.

1. (5 points) The generators of $SO(3)$ group are $\overleftrightarrow{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathbb{i} \\ 0 & \mathbb{i} & 0 \end{pmatrix}$, $\overleftrightarrow{J}_y = \begin{pmatrix} 0 & 0 & \mathbb{i} \\ 0 & 0 & 0 \\ -\mathbb{i} & 0 & 0 \end{pmatrix}$, $\overleftrightarrow{J}_z = \begin{pmatrix} 0 & -\mathbb{i} & 0 \\ \mathbb{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Consider $\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}} \equiv n_x \overleftrightarrow{J}_x + n_y \overleftrightarrow{J}_y + n_z \overleftrightarrow{J}_z$, where n_x, n_y, n_z are real numbers and $\boldsymbol{n}^2 \equiv n_x^2 + n_y^2 + n_z^2 = 1$.

(a) (3pts) *Compute* $(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^2$ *and* $(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^3$ *explicitly, show that* $(\boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})^3 = \boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}}$.

(b) (2pts) *Use the result of (a) to compute* $\exp(-\mathbb{i}\theta \boldsymbol{n} \bullet \overleftrightarrow{\boldsymbol{J}})$ *explicitly.* [Note: this of course should be $\overleftrightarrow{R}_{\boldsymbol{n}}(\theta)$]

2. (8 points) Schwinger boson. $\hat{b}_1^\dagger$ and $\hat{b}_2^\dagger$ are creation operators for orthonormal boson modes, $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{i,j}$. The occupation basis $|n_1, n_2\rangle \equiv \frac{1}{\sqrt{n_1! n_2!}} (\hat{b}_1^\dagger)^{n_1} (\hat{b}_2^\dagger)^{n_2} |\text{vac}\rangle$ are complete orthonormal basis of the Fock space. Here $|\text{vac}\rangle$ is the boson vacuum, $\hat{b}_i|\text{vac}\rangle = 0$. Denote $|n_1, n_2\rangle$ by $|j, m\rangle$ where $j = \frac{n_1+n_2}{2}$, $m = \frac{n_1-n_2}{2}$. Define three hermitian operators, $\hat{J}_z = \frac{1}{2}(\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2)$, $\hat{J}_x = \frac{1}{2}(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1)$, $\hat{J}_y = \frac{1}{2}(-\mathbb{i}\hat{b}_1^\dagger \hat{b}_2 + \mathbb{i}\hat{b}_2^\dagger \hat{b}_1)$.

(a) (3pts) *Compute the commutators,* $[\hat{J}_x, \hat{J}_y]$, $[\hat{J}_y, \hat{J}_z]$, $[\hat{J}_z, \hat{J}_x]$. The results should be linear combinations of $\hat{J}_{x,y,z}$.

(b) (5pts) *In the fixed total boson number subspace (fixed j quantum number), compute the matrix elements* $(J_x)_{mm'} \equiv \langle j, m|\hat{J}_x|j, m'\rangle$, $(J_y)_{mm'} \equiv \langle j, m|\hat{J}_y|j, m'\rangle$, $(J_z)_{mm'} \equiv \langle j, m|\hat{J}_z|j, m'\rangle$. *Check that these* $(2j+1) \times (2j+1)$ *matrices satisfy the commutation relations in (a).*

3. (17 points) Consider a spin-1 moment, denote the angular momentum operator by $\hat{\boldsymbol{S}}$. Then $[\hat{S}_a, \hat{S}_b] = \sum_c \mathbb{i}\epsilon_{abc}\hat{S}_c$, and $\hat{\boldsymbol{S}}^2 = 1 \cdot (1+1) = 2$ in this 3-dimensional Hilbert space. An obvious complete orthonormal basis is the S_z basis, $|S_z = +1, 0, -1\rangle$.

(a). (3pts) Given unit vector $\boldsymbol{n} = (\sin\eta\cos\phi, \sin\eta\sin\phi, \cos\eta)$, where η, ϕ are real parameters, *compute the eigenvalues of* $\boldsymbol{n} \bullet \hat{\boldsymbol{S}}$. [Hint: eigenvalues can be obtained without calculation, consider $\exp(-\mathbb{i}\theta \boldsymbol{n'} \bullet \hat{\boldsymbol{S}}) \cdot (\hat{\boldsymbol{S}} \bullet \boldsymbol{n}) \cdot \exp(\mathbb{i}\theta \boldsymbol{n'} \bullet \hat{\boldsymbol{S}}) = \hat{\boldsymbol{S}} \bullet \overleftrightarrow{R}_{\boldsymbol{n'}}(\theta) \bullet \boldsymbol{n}$, where $\overleftrightarrow{R}_{\boldsymbol{n'}}(\theta)$ is the $SO(3)$ matrix for rotation around $\boldsymbol{n'}$ by angle θ.]

(b). (5pts) *Use the result of (a) to show that* $(\boldsymbol{n} \bullet \hat{\boldsymbol{S}})^3 = \boldsymbol{n} \bullet \hat{\boldsymbol{S}}$. *Use this fact to compute the* 3×3 *matrix* $\exp(-\mathbb{i}\theta\, \boldsymbol{n} \bullet \hat{\boldsymbol{S}})$ *in terms of real parameters* η, ϕ, θ, *under the* S_z *basis.* [Side remark: this is just $D^{(j=1)}(e^{-\mathbb{i}\theta \boldsymbol{n} \bullet \boldsymbol{\sigma}/2})$]

(c). (3pts) For the $\boldsymbol{n}$ in (a), *Compute the normalized eigenstates of* $\boldsymbol{n} \bullet \hat{\boldsymbol{S}}$. [Hint: can be done by brute-force, or using the result of (b) and the Hint of (a).]

(d). (3pts) The solution of (c) contains the "uniaxial spin nematic state", the eigenstate of $\boldsymbol{n} \bullet \hat{\boldsymbol{S}}$ with eigenvalue 0. Denote this state by $|\boldsymbol{n} \bullet \hat{\boldsymbol{S}} = 0\rangle$. *Compute for* $\boldsymbol{n}$ *along* x, y, z *directions the spin-nematic states, namely* $|S_x = 0\rangle$ *and* $|S_y = 0\rangle$ *and* $|S_z = 0\rangle$, *in terms of the* S_z *basis. Choose their overall complex phase factors carefully so that they are invariant under time-reversal symmetry. Check that they form complete orthonormal basis.* [Hint: time-reversal symmetry action on S_z basis is, $\hat{\mathcal{T}}|S_z\rangle = (-1)^{S_z}|-S_z\rangle$]

(e). (3pts) *Write down the matrix representation of spin operators* $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$, *in the basis of the three spin-nematic states* $|S_x = 0\rangle$ *and* $|S_y = 0\rangle$ *and* $|S_z = 0\rangle$ *solved in (d). Namely compute* $(S_a)_{bc} \equiv \langle S_b = 0|\hat{S}_a|S_c = 0\rangle$. [Note: if you have solved these basis correctly, these three 3×3 hermitian matrices should be purely imaginary, according to time-reversal symmetry properties of spin operators]

4. (20 points) Consider three spin-1/2 moments (labeled by subscripts $i = 1, 2, 3$). Each spin-1/2 has a 2-dimensional Hilbert space with complete orthonormal basis $|s_i = \pm\frac{1}{2}\rangle$, and spin operators $\hat{S}_{i,a} = \frac{1}{2}\sigma_a$, for $a = x, y, z$, under the above basis in the 2-dim'l Hilbert space.

The entire 8-dimensional Hilbert space is the tensor product of the three spin-1/2 Hilbert spaces. The S_z tensor product basis are denoted by $|s_1, s_2, s_3\rangle$ with $s_i = \pm\frac{1}{2}$. Then $\hat{S}_{i,z}|s_1, s_2, s_3\rangle = s_i|s_1, s_2, s_3\rangle$.

The commutation relations between spin operators are $[\hat{S}_{i,a}, \hat{S}_{j,b}] = \delta_{i,j} \sum_c \mathbb{i}\epsilon_{abc}\hat{S}_{i,c}$.

(a). (4pts) Define $\hat{S}_{2+3,a} = \hat{S}_{2,a} + \hat{S}_{3,a}$. *What are the possible values of the total spin* *quantum number for the sum of spin 2 and 3, $\hat{\boldsymbol{S}}_{2+3} = \hat{\boldsymbol{S}}_2 + \hat{\boldsymbol{S}}_3$? Or equivalently what are the possible eigenvalues of $\hat{\boldsymbol{S}}^2_{2+3} \equiv \sum_a \hat{S}^2_{2+3,a}$? Write down the $|S_{2+3}, S_{2+3,z}\rangle$ basis in terms of the S_z tensor product basis $|s_2, s_3\rangle$.*

(b) (8pts) *What are the possible values of total spin quantum number for the sum of the three spins, $\hat{\boldsymbol{S}}_{1+2+3} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2 + \hat{\boldsymbol{S}}_3$? Write down the $|S_{1+2+3}, S_{1+2+3,z}\rangle$ basis in terms of the S_z tensor product basis $|s_1, s_2, s_3\rangle$.* [Hint: the result of (a) may be useful.]

(c). (8pts) Consider the "symmetries" generated by

$C_3: \quad |s_1, s_2, s_3\rangle \mapsto |s_2, s_3, s_1\rangle, \quad \hat{S}_{1,a} \mapsto \hat{S}_{3,a},\ \hat{S}_{2,a} \mapsto \hat{S}_{1,a},\ \hat{S}_{3,a} \mapsto \hat{S}_{2,a}$; and

$\sigma: \quad |s_1, s_2, s_3\rangle \mapsto |s_1, s_3, s_2\rangle, \quad \hat{S}_{1,a} \mapsto \hat{S}_{1,a},\ \hat{S}_{2,a} \mapsto \hat{S}_{3,a},\ \hat{S}_{3,a} \mapsto \hat{S}_{2,a}$.

This is the D_3 group. with 6 group elements $\{\mathbb{1}, C_3, C_3^2, \sigma, \sigma C_3, \sigma C_3^2\}$, classified into 3 conjugacy classes, $\{\mathbb{1}\}$, $\{C_3, C_3^2\}$, $\{\sigma, \sigma C_3, \sigma C_3^2\}$. The character table of its irreducible representations ($\Gamma_{1,2,3}$) is

	$\mathbb{1}$	$2C_3$	3σ
Γ_1	1	1	1
Γ_2	1	1	-1
Γ_3	2	-1	0

Note that $\hat{\boldsymbol{S}}_{1+2+3}$ is invariant under this D_3 group. Therefore we can label states with simultaneous eigenvalues of $\hat{\boldsymbol{S}}^2_{1+2+3}$ and $\hat{S}_{1+2+3,z}$, and D_3 irreducible representations.

Find new complete orthonormal basis of 8-dimensional Hilbert space (in terms of S_z tensor product basis), which form irreducible representations of D_3 group and are simultaneous eigenstates of $\hat{\boldsymbol{S}}^2_{1+2+3}$ and $\hat{S}_{1+2+3,z}$. [Hint: the result of (b) may be helpful.]