A counterexample to a conjecture of Ghosh

Hung Hua, Elliot Krop, and Christopher Raridan

ABSTRACT. We answer two questions of Shamik Ghosh in the negative. We show that there exists a lobster tree of diameter less than 6 which accepts no α -labeling with two central vertices labeled by the critical number and the maximum vertex label. We also show a simple example of a tree of diameter 4, with an even degree central vertex which does not accept a maximum label in any graceful labeling.

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1. Introduction

For basic graph theoretic notation and definitions we refer to West [4]. The well-known graceful tree conjecture states that for any tree T on n vertices, there exists an injective vertex-labeling from $\{0, \ldots, n-1\}$, so that the set of edge-weights, defined as absolute difference on the labels on incident vertices for each edge, is $\{1, \ldots, n-1\}$.

We call a labeling f bipartite, if there exists and integer c such that for any edge uv, either $f(u) \le c < f(v)$ or $f(v) \le c < f(u)$. A bipartite labeling that is graceful is called an α -labeling. We call the number c, the critical number.

For any tree T, let P be a longest path in T and call T a k-distant tree if all of its vertices are a distance at most k from P. At this time, the conjecture is unknown for any trees with tree distance at least 2. For 2-distant trees, or *lobsters*, the problem is known as Bermond's conjecture [1].

Recently, in the pursuit of obtaining graceful lobsters from smaller graceful lobster, Ghosh [2] asked two questions, which if answered in the affirmative, could have led to the solution of Bermond's conjecture. We address both of these questions:

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QUESTION 1. [2] Does there exist an α -labeling of a lobster of diameter at most 5 such that the central vertices are labeled by the critical number and the maximum label?

QUESTION 2. [2] For any tree T of diameter 4, with central vertex v of even degree, does there exist a graceful labeling of T such that the label on v is the maximum label?

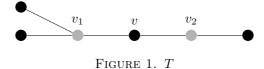
Note: The trees considered in Question 1 were those with two central vertices. In the argument below we answer this as well as another related question.

Definition 1.1. A vertex v in a tree T is an almost central vertex, if v is adjacent to a central vertex and lies on a longest path of T.

QUESTION 3. Does there exist an α -labeling of a lobster of diameter at most 5 such that the central vertex and an almost central vertex is labeled by the critical number and the maximum label, respectively?

2. An example

Let T be the following simple 1-distant tree.



Van Bussel [3] showed that T is not 0-centered, that is, does not accept a graceful labeling with central vertex v labeled 0. By considering the complementary labelings (vertices relabeled by n-1 minus their old labels), we conclude that the central vertex cannot be labeled by the maximum label, n-1. This observation gives a negative answer to Question 2.

We consider Question 3 and again refer to T. If this question is to be answered in the affirmative, v must be labeled by the critical number c. Since we are looking for an α -labeling, and if v is in the lesser labeled partite set, we must have c=3. There are only a few cases left to consider; either v_1 or v_2 is labeled by the maximum label 5. Since the vertex with the maximum label must be adjacent to the vertex labeled 0, it is easy to verify that there is no graceful labeling of T whether we label v_1 or v_2 by 5.

To answer Question 1, we need another simple example.

Let S be the following simple 1-distant tree.

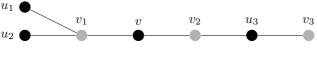


Figure 2. S

We notice that for an α -labeling of S, the bipartitions are $\{u_1, u_2, v, u_3\}$ and $\{v_1, v_2, v_3\}$.

Assume first that v is labeled by the maximum label, 6. In this case the critical number is 2, so to satisfy the conditions of Question 1, label v_2 by 2. Since the vertex with the maximum label must be adjacent to the vertex labeled 0 in any graceful labeling, v_1 must be labeled 0. Since v_3 is in the bipartition with the lower labels, the label 1 is forced on v_3 . Notice that in this case, u_3 cannot be labeled 5, otherwise the weight 4 would be on two edges. Furthermore, it is easy to check that either a label of 3 or 5 on u_3 would produce two edges of the same weight in S.

Next, consider the case when v_2 is labeled by 6. In this case, the critical number is 3, so we label v by 3. Arguing as above, u_3 must be labeled 0, which leads to labeling u_1 and u_2 by 1 and 2. This leaves only two possibilities. Either v_1 is labeled 4 or 5, and in either case the labeling is not graceful.

We note, as suggested by the anonymous referee, that this example is vertex minimum in the sense of Question 1. That is, it is impossible to delete u_1 (for example) since this would produce P_6 , which does accept an α -labeling such that the central vertices are labeled by the critical number and the maximum label. Furthermore, there are α -labelings of S in which v_2 receives either the critical number or the maximum label.

3. Acknowledgements

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Hung Hua hhua@student.clayton.edu

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ELLIOT KROP ELLIOTKROP@CLAYTON.EDU

Christopher Raridan christopherraridan@clayton.edu