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On graceful and harmonious labelings of trees

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Abstract

We prove the Kotzig-Ringel and the Graham-Sloane conjectures, respectively known as the Graceful and Harmonious Labeling Conjectures and derive from the results the spectra of two second order constructs.

1 Introduction

The Kotzig-Ringel conjecture [R64], also known as the *Graceful Labeling Conjecture* (or GLC for short) [Gal05] asserts that every tree admits a graceful labeling. A graph labeling is graceful if it assigns to distinct vertices of a graph distinct integers from a set of consecutive integers so as to induce a bijection between vertex labels and edge labels. The induced edge labeling assigns to each edge the absolute difference of spanning vertex labels. For convenience we reformulate the GLC in terms of functions in $\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$ having a unique fixed point. We express the attractive fixed point condition for a given $f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$, via the size of the image by f^{n-1} as

$$\left|f^{n-1}\left(\{0,\cdots,n-1\}\right)\right|=1, \tag{1}$$

where

$$\forall\, i\in\{0,\cdots,n-1\}\,,\ f^{0}\left(i\right):=i,\ \text{and}\ \forall\, k\geq 0,\ f^{k+1}\left(i\right)=f^{k}\left(f\left(i\right)\right)=f\left(f^{k}\left(i\right)\right).$$

To an arbitrary $f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$ we associate a *functional directed graph*

$$G_f:=\Big(V:=\{0,\cdots,n-1\}\,,\ E:=\{(i,\,f\left(i\right))\}_{0\leq i<n}\Big).$$

The GLC is thus equivalent to the assertion that for all function $f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$ subject to the attractive fixed point condition (1), there exist at least one choice of fixed permutations σ and γ of integers in the domain of f such that

$$f\left(i\right)\in\sigma^{-1}\left(\sigma\left(i\right)\pm\gamma\left(\sigma\left(i\right)\right)\right),\quad \forall\ 0\leq i<n. \tag{2}$$

More generally, a functional directed graph G_g associated with an arbitrary function $g\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$ is graceful iff there exist fixed permutations $\sigma\in {}^{S_n}\!/\!_{\mathrm{Aut}(G_g)}$ and $\gamma\in S_n$ such that

$$g\left(i\right)\in\sigma^{-1}\left(\sigma\left(i\right)\pm\gamma\left(\sigma\left(i\right)\right)\right),\quad \forall\ 0\leq i<n.$$

We denote by $\mathrm{GrL}(G_g)$ the set of distinct graceful labelings of the functional directed graph G_g. The induced edge label sequence of a graph refers to the non-decreasing sequence of edge labels obtained by taking absolute differences of vertex labels spanned by each edge. For instance the function in Figure 1

$$f\,:\,\{0,1,2,3,4,5\}\rightarrow\{0,1,2,3,4,5\}$$

defined by

$$f\left(0\right)=0,\ f\left(1\right)=0,\ f\left(2\right)=0,\ f\left(3\right)=0,\ f\left(4\right)=3,\ f\left(5\right)=3,$$

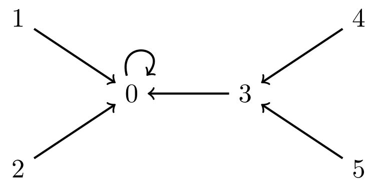

Figure 1: A functional directed graph on 6 vertices.

is a functional spanning subtree of the complete graph (or functional tree for short) on 6 vertices. The attractive fixed point condition (1) is met since $f^5(\{0,1,2,3,4,5\}) = \{0\}$. The edge set of G_f is $\{(0,0), (1,0), (2,0), (3,0), (4,3), (5,3)\}$ and the corresponding induced edge label sequence is $(0,1,1,2,2,3)$.
The GLC is easily verified for the families of star and path functional trees respectively illustrated by

$$g,\, h\,:\,\{0,\cdots,n-1\} \rightarrow \{0,\cdots,n-1\}\,,$$

$$\forall\, 0 \le i < n, \quad g(i) = 0 \;\text{ and }\; h(i) = \begin{cases} i+1 & \text{if } 0 \le i < n-1 \\ n-1 & \text{otherwise} \end{cases}.$$

In particular $|\mathrm{GrL}(G_g)| = \left\lfloor \frac{n}{2} \right\rfloor + \left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right)$. Our main results are proofs of Composition Lemmas. Proofs of the GLC, the strong GLC [GW18], follow as corollaries of the weak and strong graceful Composition Lemmas respectively. A variant of the weak graceful Composition Lemma yields a proof of the Harmonious Labeling Conjecture. We conclude the paper by showing that proofs of the strong GLC and the Harmonious Labeling Conjecture provide concrete illustrations of spectra for two families of non-diagonalizable second order constructs as introduced in [GG18].

This article is accompanied by an extensive SageMath[S18] graceful graph package which implements the symbolic constructions described here. The package is made available at the link:
`https://github.com/gnang/Graceful-Graphs-Package`

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2 Determinantal Certificate of Gracefulness and Harmony

Recall that two multivariate polynomials $F(\mathbf{x})\,,\, G(\mathbf{x}) \in \mathbb{C}[x_0,\cdots,x_{n-1}]$ which split into irreducible factors of the form

$$F(\mathbf{x}) = \prod_{0\le i<m} (P_i(\mathbf{x}))^{\alpha_i}\,, \quad G(\mathbf{x}) = \prod_{0\le i<m} (P_i(\mathbf{x}))^{\beta_i}$$

for non negative integers $\{\alpha_i,\beta_i\}_{0\le i<m}$, where each non identically constant factor $P_i(\mathbf{x})$ is a multivariate polynomial of degree at most 1 in each individual variable and has no common root with any other factor when viewed as a polynomial over $\mathbb{C}[x_0,\cdots,x_{k-1},x_{k+1},\cdots,x_{n-1}][x_k]$ for each variable x_k then

$$\begin{array}{c} \prod_{0\le i<m} (P_i(\mathbf{x}))^{\max(\alpha_i,\beta_i)} = \mathrm{LCM}\{F(\mathbf{x})\,,\, G(\mathbf{x})\} \\ \\ \prod_{0\le i<m} (P_i(\mathbf{x}))^{\min(\alpha_i,\beta_i)} = \mathrm{GCD}\{F(\mathbf{x})\,,\, G(\mathbf{x})\} \end{array}. \tag{3}$$

Recall also that for an arbitrary multivariate polynomial $H(\mathbf{x}) \in \mathbb{C}[x_0,\cdots,x_{n-1}]$ we have that

$$\left(H(\mathbf{x}) \bmod \left\{\prod_{0\le j<n}(x_i - j)\right\}_{0\le i<n}\right) = \sum_{f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}} H(f(0),\cdots,f(n-1)) \prod_{0\le k<n}\,\prod_{0\le j_k \ne f(k)<n} \left(\frac{x_k - j_k}{f(k) - j_k}\right). \tag{4}$$

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Consequently, the polynomial

$$H\left(\mathbf{x}\right) \bmod \left\{ \prod_{0\le j<n} \left(x_i - j\right) \right\}_{0\le i<n}$$

can be obtained via Lagrange interpolation as prescribed or alternatively via Euclidean division irrespective of the order of division by each of the univariate polynomials $\left\{ \prod_{0\le j<n} \left(x_i - j\right) \right\}_{0\le i<n}$.

We describe here a determinental construction for establishing the gracefulness of a given functional directed graph G_f associated with an arbitrary $f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$.

Proposition 1a : (*Determinental gracefulness certificate*) G_f associated with $f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$ is graceful iff

$$0 \not\equiv \text{LCM}\left\{ \prod_{0\le i<j<n} \left(x_j - x_i\right), \prod_{0\le i<j<n} \left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right) \right\} \bmod \left\{ \prod_{0\le j<n} \left(x_i - j\right) \right\}_{0\le i<n}$$

Proof : The proof of sufficiency follows from the observation that the only possible roots to the multivariate polynomial

$$\text{LCM}\left\{ \prod_{0\le i<j<n} \left(x_j - x_i\right), \prod_{0\le i<j<n} \left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right) \right\} \bmod \left\{ \prod_{0\le j<n} \left(x_i - j\right) \right\}_{0\le i<n}$$

arise from vertex label assignments in which either distinct vertex variables are assigned the same label or distinct edges are assigned the same induced edge label. Consequently, the congruence identity

$$0 \equiv \text{LCM}\left\{ \prod_{0\le i<j<n} \left(x_j - x_i\right), \prod_{0\le i<j<n} \left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right) \right\} \bmod \left\{ \prod_{0\le j<n} \left(x_i - j\right) \right\}_{0\le i<n}$$

implies that G_f admits no graceful labeling. On the other hand the proof of necessity follows from the fact that every graceful labeling of G_f yields an assignment to the vertex variables $\{x_i\}_{0\le i<n}$ such that

$$\prod_{0\le i<j<n} \left(x_j - x_i\right)\left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right) \in \pm \prod_{0\le i<j<n} \left(j-i\right)^2\left(j+i\right),$$

thus completing the proof.□

Note that the polynomial construction above is determinental since

$$\det\left(\mathbf{V}\right) = \prod_{0\le i<j<n} \left(x_j - x_i\right)\left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right),$$

where

$$\mathbf{V}\left[i,j\right] = \sum_{0\le k<n} x_i^k \left(x_{f(j)} - x_j\right)^{2k} = \frac{1-\left(x_i\left(x_{f(j)} - x_j\right)^2\right)^n}{1-x_i\left(x_{f(j)} - x_j\right)^2}, \quad \forall\, 0\le i,j<n.$$

We now describe a similar determinental construction for certifying that a functional tree is harmonious. Recall from [GS80] that the a functional directed graph G_f associated with $f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$ is harmonious if

$$\forall\, 0\le i<j<n, \quad x_{f(j)}x_j \not\equiv x_{f(i)}x_i \mod \left\{x_i^n - 1\right\}_{0\le i<n}.$$

In the context of harmonious labelings, each induced edge label is obtained instead by taking products of n-th roots of unity assigned to the vertices spanning the corresponding edge.

Proposition 1b : (*Determinental harmony certificate*) G_f associated with $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ is harmonious iff

$$0 \not\equiv \text{LCM}\left\{ \prod_{0\le i<j<n} (x_j - x_i)\,, \prod_{0\le i<j<n} \left(x_{f(j)}x_j - x_{f(i)}x_i\right) \right\} \text{mod}\left\{x_i^n - 1\right\}_{0\le i<n}$$

Proof : The proof of sufficiency follows from the observation that the only possible roots to the multivariate polynomial

$$\text{LCM}\left\{ \prod_{0\le i<j<n} (x_j - x_i)\,, \prod_{0\le i<j<n} \left(x_{f(j)}x_j - x_{f(i)}x_i\right) \right\} \text{mod}\left\{x_i^n - 1\right\}_{0\le i<n},$$

arise from vertex label assignments in which either distinct vertex variables are assigned the same label or distinct edges are assigned the same induced edge label. Consequently the congruence identity

$$0 \equiv \text{LCM}\left\{ \prod_{0\le i<j<n} (x_j - x_i)\,, \prod_{0\le i<j<n} \left(x_{f(j)}x_j - x_{f(i)}x_i\right) \right\} \text{mod}\left\{x_i^n - 1\right\}_{0\le i<n},$$

implies that G_f admits no harmonious labeling. On the other hand the proof of necessity follows from the fact that every harmonious labeling of G_f yields an assignment to the vertex variables $\{x_i\}_{0\le i<n}$ such that

$$\prod_{0\le i<j<n} (x_j - x_i)\left(x_{f(j)}x_j - x_{f(i)}x_i\right) \in \pm \prod_{0\le i<j<n} \left(\omega_n^j - \omega_n^i\right)^2,$$

where $\omega_n := \exp\left(\frac{2\pi\sqrt{-1}}{n}\right)$. Thus completing the proof.□

Note that the polynomial construction above is also determinental since

$$\det(\mathbf{W}) = \prod_{0\le i<j<n} (x_j - x_i)\left(x_{f(j)}x_j - x_{f(i)}x_i\right),$$

where

$$\mathbf{W}[i,j] = \frac{1 - \left(x_i x_j x_{f(j)}\right)^n}{1 - x_i x_j x_{f(j)}}, \quad \forall\, 0 \le i, j < n.$$

3 Composition Lemmas.

We state here and prove by contradiction the *weak Graceful Composition Lemma*. For notational convenience, let

$$\text{LCM}_{f^t} := \text{LCM}\left\{ \prod_{0\le i<j<n} (x_j - x_i)\,, \prod_{0\le i<j<n} \left(\left(x_{f^t(j)} - x_j\right)^2 - \left(x_{f^t(i)} - x_i\right)^2\right) \right\} \text{mod}\left\{ \prod_{0\le j<n} (x_i - j) \right\}_{0\le i<n}$$

Lemma 2a : (*weak Graceful Composition Lemma*) For $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ such that G_f is a spanning disjoint union of cycles and functional trees,

$$\text{LCM}_f \equiv 0 \implies \text{LCM}_{f^2} \equiv 0.$$

Proof : The arguments splits into two main cases. The first case corresponds to the setting where f either has two or more fixed points or alternatively f has no fixed point at all. It is easy to see that in the first case

$$\forall\, 0 < t < n, \quad \text{LCM}_{f^t} \equiv 0.$$

Consequently, the lemma is trivially true in the first case. The second case corresponds to the setting where f has a unique fixed point. In the second case, the lemma is trivially true when the undirected graph associated with G_f without the loop edge is isomorphic to the undirected graph associated with of G_{f^2} without the loop edge. The second case therefore reduces to sub-cases for which the undirected graphs associated with G_f and G_{f^2} without their respective loop edges are non-isomorphic. Within these sub-cases, the lemma is trivially true when G_f has at least one two cycle. Consequently, it suffices to address the remaining sub-cases where the length of the longest path ending at the fixed point of G_f is greater then 1. Note that for all $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ having a unique fixed point

$$\pm\text{LCM}_f \equiv \prod_{0\le i<j<n} (x_j - x_i) \prod_{\substack{0 \le i < j < n \\ f(i) \ne i,\, f(j) \ne j \\ d_1(i,j) \ge 3}} \left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right) \prod_{d_1(i,f^3(i))=3} \left(\left(x_{f^3(i)} - x_{f^2(i)}\right)^2 - \left(x_{f(i)} - x_i\right)^2\right) \times$$

$$\prod_{d_1(i,f^2(i))=2} \left(x_{f^2(i)} + x_i - 2x_{f(i)}\right) \prod_{\substack{0 \le i < j < n \\ f(i) = f(j)}} \left(2x_{f(j)} - x_j - x_i\right) \bmod \left\{\prod_{0\le j<n} (x_i - j)\right\}_{0\le i<n}, \tag{5}$$

where $d_t(u,v)$ denotes the undirected non-loop edge distance separating vertex u from vertex v in the graph G_{f^t}. Without loss of generality let the fixed point of G_f be specified by $f(0) = 0$. Assume for the sake of establishing a contradiction that

$$\text{LCM}_f \equiv 0 \quad \text{and} \quad \text{LCM}_{f^2} \not\equiv 0.$$

$$\implies 0 \not\equiv \prod_{0\le i<j<n} (x_j - x_i) \prod_{0<i<j<n} \left(\left(x_{f^2(j)} - x_j\right)^2 - \left(x_{f^2(i)} - x_i\right)^2\right) \bmod \left\{\prod_{0\le j<n} (x_i - j)\right\}_{0\le i<n}.$$

Let $F(\mathbf{x}, z)$ denote a polynomial in the n vertex variables occurring as entries of $\mathbf{x}$ and an additional variable z such that

$$F(\mathbf{x}, z) = \prod_{0\le i<j<n} (x_j - x_i) \prod_{0<i<j<n} \Big[\left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right) z^{2^{\text{lex}(i,j)}} +$$

$$\left(x_{f^2(j)} - x_{f(j)}\right)^2 - \left(x_{f^2(i)} - x_{f(i)}\right)^2 + 2\left(x_{f^2(j)} - x_{f(j)}\right)\left(x_{f(j)} - x_j\right) - 2\left(x_{f^2(i)} - x_{f(i)}\right)\left(x_{f(i)} - x_i\right)\Big]$$

$$\bmod \left\{\prod_{0\le j<n} (x_i - j)\right\}_{0\le i<n},$$

where for all $0 < i < j < n$ the integer $\text{lex}(i,j)$ subject to

$$0 \le \text{lex}(i,j) < \binom{n-1}{2},$$

assigns distinct integers to distinct unordered pairs reflecting the lexicographic rank of each pair. Note that

$$F(\mathbf{x}, 1) = \prod_{0\le i<j<n} (x_j - x_i) \prod_{0<i<j<n} \left(\left(x_{f^2(j)} - x_j\right)^2 - \left(x_{f^2(i)} - x_i\right)^2\right) \bmod \left\{\prod_{0\le j<n} (x_i - j)\right\}_{0\le i<n}.$$

Moreover, for every pair $u<v$ subject to $d_1(u,v)=3$ and $d_2(u,v)=2$ we have

$$1\equiv \text{GCD}\left\{(x_v-x_u),\left(\frac{F(\mathbf{x},z)}{\prod\limits_{0\le i<j<n}(x_j-x_i)}\text{mod } z^{2^{\binom{n-1}{2}}-1}\right)\text{mod}\,(z-1)\right\}\text{mod}\left\{\prod_{0\le j<n}(x_i-j)\right\}_{0\le i<n},$$

whenever

$$0\not\equiv\left(\left(\frac{F(\mathbf{x},z)}{\prod\limits_{0\le i<j<n}(x_j-x_i)}\text{mod } z^{2^{\binom{n-1}{2}}-1}\right)\text{mod}\,(z-1)\right)\text{mod}\left\{\prod_{0\le j<n}(x_i-j)\right\}_{0\le i<n}.$$

The assertion right above follows from the fact that we know the factored form of $\frac{F(\mathbf{x},1)}{\prod\limits_{0\le i<j<n}(x_j-x_i)}$ and in particular for every pair $u<v$ subject to $d_1(u,v)=3$ and $d_2(u,v)=2$ we know that

$$1\equiv \text{GCD}\left\{(x_v-x_u),\prod_{\substack{0<i<j<n\\(u,v)\neq(i,j)}}\left(\left(x_{f^2(j)}-x_j\right)^2-\left(x_{f^2(i)}-x_i\right)^2\right)\right\}\text{mod}\left\{\prod_{0\le j<n}(x_i-j)\right\}_{0\le i<n}.$$

Furthermore the polynomial

$$\left(\frac{F(\mathbf{x},z)}{\prod\limits_{0\le i<j<n}(x_j-x_i)}\text{mod } z^{2^{\binom{n-1}{2}}-1}\right)=\sum_{0\le d<2^{\binom{n-1}{2}}-1}P_d(\mathbf{x})\,z^d, \tag{6}$$

expresses a sum over $\left(2^{\binom{n-1}{2}}-1\right)$ multivariate polynomial summands $\left\{P_d(\mathbf{x})\,z^d\right\}_{0\le d<2^{\binom{n-1}{2}}-1}$. For every such pair $u<v$ subject to $d_1(u,v)=3$ and $d_2(u,v)=2$, each non-vanishing summand $P_d(\mathbf{x})\,z^d$ is either a multiple of

$$\left(\left(x_{f(v)}-x_v\right)^2-\left(x_{f(u)}-x_u\right)^2\right)z^{2^{\text{lex}(u,v)}}$$

or multiple of

$$\left(x_{f^2(v)}-x_{f(v)}\right)^2-\left(x_{f^2(u)}-x_{f(u)}\right)^2+2\left(x_{f^2(v)}-x_{f(v)}\right)\left(x_{f(v)}-x_v\right)-2\left(x_{f^2(u)}-x_{f(u)}\right)\left(x_{f(u)}-x_u\right)$$

but never a multiple of both. The number of non-vanishing summands of the form $P_d(\mathbf{x})\,z^d$ in (6) being odd, taken with the fact that for every such pair $u<v$ subject to $d_1(u,v)=3$ and $d_2(u,v)=2$,

$$\left(\left(x_{f(v)}-x_v\right)^2-\left(x_{f(u)}-x_u\right)^2\right)\not\equiv 0\mod\left\{\prod_{0\le j<n}(x_i-j)\right\}_{0\le i<n},$$

we have that

$$\left(\frac{F(\mathbf{x},z)}{\prod\limits_{0\le i<j<n}(x_j-x_i)}\text{mod } z^{2^{\binom{n-1}{2}}-1}\right)\mod\left\{\prod_{0\le j<n}(x_i-j)\right\}_{0\le i<n}$$

can not be a multiple of the polynomial

$$\left(\left(x_{f(v)}-x_v\right)^2-\left(x_{f(u)}-x_u\right)^2\right)z^{2^{\text{lex}(u,v)}}+\left(x_{f^2(v)}-x_{f(v)}\right)^2-\left(x_{f^2(u)}-x_{f(u)}\right)^2+$$

$$2\left(x_{f^2(v)}-x_{f(v)}\right)\left(x_{f(v)}-x_v\right)-2\left(x_{f^2(u)}-x_{f(u)}\right)\left(x_{f(u)}-x_u\right).$$

Furthermore for every such pair $u < v$ subject to $d_1(u,v) = 3$ and $d_2(u,v) = 2$, we have

$$1 \equiv \text{GCD}\left\{(x_v - x_u), \left(\left(x_{f(v)} - x_v\right)^2 - \left(x_{f(u)} - x_u\right)^2\right)\right\} \bmod \left\{\prod_{0\le j<n}(x_i - j)\right\}_{0\le i<n}$$

and

$$1 \equiv \text{GCD}\Big\{(x_v - x_u),\ \left(x_{f^2(v)} - x_{f(v)}\right)^2 - \left(x_{f^2(u)} - x_{f(u)}\right)^2 +$$

$$2\left(x_{f^2(v)} - x_{f(v)}\right)\left(x_{f(v)} - x_v\right) - 2\left(x_{f^2(u)} - x_{f(u)}\right)\left(x_{f(u)} - x_u\right)\Big\} \bmod \left\{\prod_{0\le j<n}(x_i - j)\right\}_{0\le i<n},$$

hence

$$1 \equiv \text{GCD}\left\{(x_v - x_u), \left(\frac{F(\mathbf{x}, z)}{\prod_{0\le i<j<n}(x_j - x_i)} \bmod z^{2^{\binom{n-1}{2}}-1}\right) \bmod (z-1)\right\} \bmod \left\{\prod_{0\le j<n}(x_i - j)\right\}_{0\le i<n},$$

whenever

$$0 \not\equiv \left(\left(\frac{F(\mathbf{x}, z)}{\prod_{0\le i<j<n}(x_j - x_i)} \bmod z^{2^{\binom{n-1}{2}}-1}\right) \bmod (z-1)\right) \bmod \left\{\prod_{0\le j<n}(x_i - j)\right\}_{0\le i<n}.$$

On the other hand, the assertion that for every such pair $u < v$ subject to $d_1(u,v) = 3$ and $d_2(u,v) = 2$

$$1 \equiv \text{GCD}\left\{(x_v - x_u), \left(\frac{F(\mathbf{x}, z)}{\prod_{0\le i<j<n}(x_j - x_i)} \bmod z^{2^{\binom{n-1}{2}}-1}\right) \bmod (z-1)\right\} \bmod \left\{\prod_{0\le j<n}(x_i - j)\right\}_{0\le i<n},$$

contradicts the assertion that for every such pair $u < v$ subject to $d_1(u,v) = 3$ and $d_2(u,v) = 2$

$$(x_v - x_u) \equiv \text{GCD}\left\{(x_v - x_u), \frac{F(\mathbf{x}, 1)}{\prod_{0\le i<j<n}(x_j - x_i)}\right\} \bmod \left\{\prod_{0\le j<n}(x_i - j)\right\}_{0\le i<n}$$

as established in (5), in light of the fact that

$$\left(\frac{F(\mathbf{x}, z)}{\prod_{0\le i<j<n}(x_j - x_i)} \bmod z^{2^{\binom{n-1}{2}}-1}\right) \bmod (z-1) \equiv \frac{F(\mathbf{x}, 1)}{\prod_{0\le i<j<n}(x_j - x_i)} \equiv \prod_{0<i<j<n}\left(\left(x_{f^2(j)} - x_j\right)^2 - \left(x_{f^2(i)} - x_i\right)^2\right)$$

$$\bmod \left\{\prod_{0\le j<n}(x_i - j)\right\}_{0\le i<n}$$

following from the fact that the coefficient of the leading term $z^{2^{\binom{n-1}{2}}-1}$ in $F(\mathbf{x}, z)$ is

$$\prod_{0\le i<j<n}(x_j - x_i) \prod_{0<i<j<n}\left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right),$$

known to be congruent to zero according to our premise. We therefore conclude that

$$\text{LCM}_f \equiv 0 \implies \text{LCM}_{f^2} \equiv 0.$$

We prove a similar *Harmonious Composition Lemma*, for functional directed graphs having a unique fixed point. For notational convenience, let

$$\text{LCM}_{f^t} := \text{LCM}\left\{ \prod_{0\le i<j<n} (x_j - x_i), \prod_{0\le i<j<n} \left(x_{f^t(j)} x_j - x_{f^t(i)} x_i\right) \right\} \mod \left\{x_i^n - 1\right\}_{0\le i<n}.$$

Lemma 2b : (*Harmonious Composition Lemma*) For $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ having a unique fixed point and let t be some positive integer less than the length of the longest path ending at the fixed point in G_f, we have

$$\text{LCM}_{f^t} \equiv 0 \implies \text{LCM}_{f^{t+1}} \equiv 0.$$

Proof : Note that

$$\pm\text{LCM}_{f^t} \equiv \prod_{0\le i<j<n} (x_j - x_i) \prod_{\substack{0 \le u < v < n \\ d_t(i,j) \ge 3}} \left(x_{f^t(j)} x_j - x_{f^t(i)} x_i\right) \prod_{d_t(i,f^{3t}(i))=3} \left(x_{f^{3t}(i)} x_{f^{2t}(i)} - x_{f^t(i)} x_i\right) \times$$

$$\left(\prod_{d_t(i,f^2(i))=2} x_{f^t(i)}\right) \left(\prod_{\substack{0 \le i < j < n \\ f^t(i) = f^t(j)}} x_{f^t(j)}\right) \mod \left\{x_i^n - 1\right\}_{0\le i<n}. \tag{7}$$

where $d_t(u,v)$ denotes the undirected non-loop edge distance separating vertex u from vertex v in the graph G_{f^t}. Assume for the sake of establishing a contradiction that

$$\text{LCM}_{f^t} \equiv 0 \quad \text{and} \quad \text{LCM}_{f^{t+1}} \not\equiv 0.$$

$$\implies 0 \not\equiv \prod_{0\le i<j<n} (x_j - x_i)\left(x_{f^{t+1}(j)} x_j - x_{f^{t+1}(i)} x_i\right) \mod \left\{x_i^n - 1\right\}_{0\le i<n}.$$

Let $F(\mathbf{x}, z)$ denote a polynomial in the n vertex variables occurring as entries of $\mathbf{x}$ and an additional variable z such that

$$F(\mathbf{x}, z) = \prod_{0\le i<j<n} (x_j - x_i)\left[\left(x_{f^t(j)} x_j - x_{f^t(i)} x_i\right) z^{2^{\text{lex}(i,j)}} + \left(x_{f^{t+1}(j)} - x_{f^t(j)}\right) x_j - \left(x_{f^{t+1}(i)} - x_{f^t(i)}\right) x_i\right] \mod \left\{x_i^n - 1\right\}_{0\le i<n},$$

where for all $0 \le i < j < n$ the integer $\text{lex}(i,j)$ subject to

$$0 \le \text{lex}(i,j) < \binom{n}{2},$$

assigns distinct integers to distinct unordered pairs reflecting the lexicographic rank of each pair. Note that

$$F(\mathbf{x}, 1) = \prod_{0\le i<j<n} (x_j - x_i)\left(x_{f^{t+1}(j)} x_j - x_{f^{t+1}(i)} x_i\right) \mod \left\{x_i^n - 1\right\}_{0\le i<n}.$$

Moreover, for every pair $u < v$ subject to $d_t(u,v) = 3$ and $d_{t+1}(u,v) = 2$ we have

$$1 \equiv \text{GCD}\left\{ (x_v - x_u), \left(\frac{F(\mathbf{x}, z)}{\prod_{0\le i<j<n} (x_j - x_i)} \mod z^{2^{\binom{n}{2}}-1} \right) \mod (z-1) \right\} \mod \left\{x_i^n - 1\right\}_{0\le i<n},$$

whenever

$$0 \not\equiv \left(\left(\frac{F(\mathbf{x}, z)}{\prod\limits_{0 \le i < j < n} (x_j - x_i)} \bmod z^{2^{\binom{n}{2}}-1} \right) \bmod (z-1) \right) \bmod \{x_i^n - 1\}_{0 \le i < n}.$$

The assertion right above follows from the fact that we know the factored form of $\frac{F(\mathbf{x},1)}{\prod\limits_{0 \le i<j<n}(x_j - x_i)}$ and furthermore

$$1 \equiv \text{GCD} \left\{ (x_v - x_u), \prod_{\substack{0 < i < j < n \\ (u,v) \ne (i,j)}} \left(x_{f^{t+1}(j)} x_j - x_{f^{t+1}(i)} x_i \right) \right\} \bmod \{x_i^n - 1\}_{0 \le i < n}.$$

Moreover

$$\left(\frac{F(\mathbf{x}, z)}{\prod\limits_{0 \le i < j < n} (x_j - x_i)} \bmod z^{2^{\binom{n}{2}}-1} \right) = \sum_{0 \le d < 2^{\binom{n}{2}}-1} P_d(\mathbf{x}) \, z^d, \tag{8}$$

expresses a sum over $\left(2^{\binom{n}{2}} - 1\right)$ multivariate polynomial summands $\left\{P_d(\mathbf{x}) \, z^d\right\}_{0 \le d < 2^{\binom{n}{2}}-1}$. For every such pair $u < v$ subject to $d_t(u,v) = 3$ and $d_{t+1}(u,v) = 2$, each non-vanishing summand $P_d(\mathbf{x}) \, z^d$ is either a multiple of

$$\left(x_{f^t(v)} x_v - x_{f^t(u)} x_u \right) z^{2^{\text{lex}(u,v)}}$$

or a multiple of

$$\left(x_{f^{t+1}(v)} - x_{f^t(v)} \right) x_v - \left(x_{f^{t+1}(u)} - x_{f^t(u)} \right) x_u$$

but never a multiple of both. The number of non-vanishing summands of the form $P_d(\mathbf{x}) \, z^d$ in (8) being odd, taken with the fact that for every such pair $u < v$ subject to $d_t(u,v) = 3$ and $d_{t+1}(u,v) = 2$,

$$\left(x_{f^t(v)} x_v - x_{f^t(u)} x_u \right) \not\equiv 0 \mod \{x_i^n - 1\}_{0 \le i < n},$$

we have that

$$\left(\frac{F(\mathbf{x}, z)}{\prod\limits_{0 \le i < j < n} (x_j - x_i)} \bmod z^{2^{\binom{n}{2}}-1} \right) \mod \{x_i^n - 1\}_{0 \le i < n}$$

can not be a multiple of the polynomial

$$\left(x_{f^t(v)} x_v - x_{f^t(u)} x_u \right) z^{2^{\text{lex}(u,v)}} + \left(x_{f^{t+1}(v)} - x_{f^t(v)} \right) x_v - \left(x_{f^{t+1}(u)} - x_{f^t(u)} \right) x_u.$$

Furthermore for every such pair $u < v$ subject to $d_t(u,v) = 3$ and $d_{t+1}(u,v) = 2$

$$1 \equiv \text{GCD} \left\{ (x_v - x_u), \left(x_{f^t(v)} x_v - x_{f^t(u)} x_u \right) \right\} \bmod \{x_i^n - 1\}_{0 \le i < n}$$

and

$$1 \equiv \text{GCD} \left\{ (x_v - x_u), \left(x_{f^{t+1}(v)} - x_{f^t(v)} \right) x_v - \left(x_{f^{t+1}(u)} - x_{f^t(u)} \right) x_u \right\} \bmod \{x_i^n - 1\}_{0 \le i < n},$$

hence

$$1 \equiv \text{GCD} \left\{ (x_v - x_u), \left(\frac{F(\mathbf{x}, z)}{\prod\limits_{0 \le i < j < n} (x_j - x_i)} \bmod z^{2^{\binom{n}{2}}-1} \right) \bmod (z-1) \right\} \bmod \{x_i^n - 1\}_{0 \le i < n},$$

whenever

$$0 \not\equiv \left(\left(\frac{F(\mathbf{x},z)}{\prod_{0\le i<j<n}(x_j-x_i)} \mod z^{2^{\binom{n}{2}}-1}\right) \mod (z-1)\right) \mod \{x_i^n-1\}_{0\le i<n}.$$

On the other hand, the assertion that for every such pair $u<v$ subject to $d_t(u,v)=3$ and $d_{t+1}(u,v)=2$ we have

$$1 \equiv \text{GCD}\left\{(x_v-x_u),\left(\frac{F(\mathbf{x},z)}{\prod_{0\le i<j<n}(x_j-x_i)} \mod z^{2^{\binom{n}{2}}-1}\right) \mod (z-1)\right\} \mod \{x_i^n-1\}_{0\le i<n},$$

contradicts the assertion that for every such pair $u<v$ subject to $d_t(u,v)=3$ and $d_{t+1}(u,v)=2$

$$(x_v-x_u) \equiv \text{GCD}\left\{(x_v-x_u),\frac{F(\mathbf{x},1)}{\prod_{0\le i<j<n}(x_j-x_i)}\right\} \mod \{x_i^n-1\}_{0\le i<n}$$

as established in (7) in light of the fact that

$$\left(\frac{F(\mathbf{x},z)}{\prod_{0\le i<j<n}(x_j-x_i)} \mod z^{2^{\binom{n}{2}}-1}\right) \mod (z-1) \equiv \frac{F(\mathbf{x},1)}{\prod_{0\le i<j<n}(x_j-x_i)} \equiv \prod_{0\le i<j<n}\left(x_{f^{t+1}(j)}x_j - x_{f^{t+1}(i)}x_i\right) \mod \{x_i^n-1\}_{0\le i<n},$$

following from the fact that the coefficient of the leading term $z^{2^{\binom{n}{2}}-1}$ in $F(\mathbf{x},z)$ is

$$\prod_{0\le i<j<n}(x_j-x_i)\prod_{0\le i<j<n}\left(x_{f^t(j)}x_j - x_{f^t(i)}x_i\right),$$

known to be congruent to zero according to the premise. We therefore conclude that

$$\text{LCM}_{f^t}\equiv 0 \implies \text{LCM}_{f^{t+1}}\equiv 0.$$

4 The Graceful and Harmonious Labeling Theorems.

The relevance of the weak Graceful Composition Lemma to the GLC results from the fact that all functional trees are subject to the fixed point condition (1). The fixed point condition asserts that at most $n-1$ self compositions suffice to transform an arbitrary functional tree on n vertices into one of the n possible constant functions in $\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$. On the other hand recall that constant functions are graceful. The Graceful Labeling Theorem follows from the weak graceful Composition Lemma as follows.

Theorem 3a : (*Graceful Labeling Theorem*) All trees are graceful.

Proof : For any functional tree G_f, the function f^{n-1} corresponds to one of the n possible identically constant function in $\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$. Moreover,

$$0 \not\equiv \text{LCM}\left\{\prod_{0\le i<j<n}(x_j-x_i),\ \prod_{\substack{0\le i<j<n\\ f(i)\ne i,\, f(j)\ne j}}\left(\left(x_{f^{n-1}(j)}-x_j\right)^2-\left(x_{f^{n-1}(i)}-x_i\right)^2\right)\right\} \mod \left\{\prod_{0\le j<n}(x_i-j)\right\}_{0\le i<n}.$$

The claim right above follows from the fact that we can easily identify the only 2 graceful relabelings of any functional star. We also see this from the explicit expression of the reduced LCM of the constant zero function given by

$$\left(\prod_{0<i<n} i\right)\prod_{0<i<j<n}(j-i)^2(j+i)\times$$

$$\left(\sum_{\substack{\sigma \in \mathrm{S}_n \\ \sigma(0)=0}} (-1)^{\binom{n-1}{2}} \operatorname{sgn}\sigma \prod_{0\le i<n} \prod_{0\le j\ne i<n} \left(\frac{x_i - \sigma(j)}{\sigma(i)-\sigma(j)} \right) + \sum_{\substack{\sigma \in \mathrm{S}_n \\ \sigma(0)=n-1}} \operatorname{sgn}\sigma \prod_{0\le i<n} \prod_{0\le j\ne i<n} \left(\frac{x_i - \sigma(j)}{\sigma(i)-\sigma(j)} \right) \right)$$

The desired result therefore follows by repeatedly applying the contrapositive of the weak graceful Composition Lemma.□

Similarly we derive as a corollary of the harmonious Composition Lemma a proof of the Harmonious Labeling Conjecture as follows

Theorem 3b : (*Harmonious Labeling Theorem*) All trees are harmonious.

Proof : For any functional tree G_f, the function f^{n-1} corresponds to one of the n possible identically constant function in $\{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$. Moreover,

$$0 \not\equiv \mathrm{LCM} \left\{ \prod_{0\le i<j<n} (x_j - x_i), \prod_{0\le u<v<n} \left(x_{f^{n-1}(v)} x_v - x_{f^{n-1}(u)} - x_u \right) \right\} \mod \left\{ x_i^n - 1 \right\}_{0\le i<n},$$

The claim above follows from the fact that we can easily identify all harmonious relabelings of any constant function in $\{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$. We also see this from the explicit expression of the reduced LCM of the constant zero function given by

$$x_0^{\binom{n}{2} \mathrm{mod} n} \sum_{\sigma\in \mathrm{S}_n} \operatorname{sgn}\sigma \prod_{0\le i<n} x_{\sigma(i)}^i \equiv \mathrm{LCM} \left\{ \prod_{0\le i<j<n} (x_j - x_i), \prod_{0\le u<v<n} \left(x_{f^{n-1}(v)} x_v - x_{f^{n-1}(u)} - x_u \right) \right\} \mod \left\{ x_i^n - 1 \right\}_{0\le i<n},$$

where

$$\binom{n}{2} \mod n = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \mod 2 \\ 0 & \text{otherwise} \end{cases}$$

The desired result therefore follows by repeatedly applying the contrapositive of the harmonious Composition Lemma. □

5 Strengthening the Composition Lemma.

We state here and prove by contradiction a *strong Composition Lemma* which establishes as a corollary the strong GLC first proposed in [GW18]. For notational convenience, let

$$\mathrm{LCM}_{\ell, f^t} :=$$

$$\mathrm{LCM} \left\{ \prod_{0\le i<j<n} (x_j - x_i), \prod_{\substack{0\le i\ne f(i)<n \\ n-\ell\le j<n}} \left(\left(x_{f^t(i)} - x_i \right)^2 - j^2 \right) \times \right.$$

$$\left. \prod_{0\le u<v<w<n} \left(\left(x_{f^t(w)} - x_w \right)^2 + \left(x_{f^t(v)} - x_v \right)^2 \left(\frac{1+\sqrt{-3}}{2} \right) + \left(x_{f^t(u)} - x_u \right)^2 \left(\frac{1-\sqrt{-3}}{2} \right) \right) \right\}$$

$$\mod \left\{ \prod_{0\le j<n} (x_i - j) \right\}_{0\le i<n}.$$

The polynomial construction above is a variant of the determinantal certificate described in Proposition1a. The main difference being that the multivariate polynomial LCM_{ℓ,f^t} is not congruent to the identically zero polynomial whenever the functional directed graph G_{f^t} admits a labeling in which no two distinct vertices are assigned the same label, no three distinct edges are assigned the same induced edge label and none of the induced edge labels are greater than $(n-\ell-1)$.

Lemma 4 : (*strong Graceful Composition Lemma*) For $f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$ having a unique fixed point, let t be some positive integer less than the length of the longest path ending at the fixed point in G_f and an integer $0 \le \ell < \lceil \frac{n-1}{2} \rceil$, we have

$$\mathrm{LCM}_{\ell,f^t} \equiv 0 \implies \mathrm{LCM}_{\ell,f^{t+1}} \equiv 0$$

Proof :Without loss of generality let the fixed point of G_f be specified by $f(0)=0$. Assume for the sake of establishing a contradiction that

$$\mathrm{LCM}_{\ell,f^t} \equiv 0 \quad \text{and} \quad \mathrm{LCM}_{\ell,f^{t+1}} \not\equiv 0$$

$$\implies 0 \not\equiv \prod_{0\le i<j<n} (x_j - x_i) \prod_{\substack{0<i<n \\ n-\ell<j<n}} \left(\left(x_{f^{t+1}(i)} - x_i\right)^2 - j^2\right) \times$$

$$\prod_{0<u<v<w<n} \left(\left(x_{f^{t+1}(w)} - x_w\right)^2 + \left(x_{f^{t+1}(v)} - x_v\right)^2 \left(\frac{1+\sqrt{-3}}{2}\right) + \left(x_{f^{t+1}(u)} - x_u\right)^2 \left(\frac{1-\sqrt{-3}}{2}\right)\right)$$

$$\mathrm{mod} \left\{ \prod_{0\le j<n} (x_i - j) \right\}_{0\le i<n}.$$

Let $F_\ell(\mathbf{x}, z)$ denote a polynomial in the n vertex variables in the entries of $\mathbf{x}$ and a new additional variable z such that

$$F_\ell(\mathbf{x}, z) =$$

$$\prod_{0\le i<j<n} (x_j - x_i) \prod_{\substack{0<i<n \\ n-\ell\le j<n}} \left[\left(\left(x_{f^t(i)} - x_i\right)^2 - j^2\right) z^{2^{\mathrm{lex}(i,j)}} + \left(x_{f^{t+1}(i)} - x_{f^t(i)}\right)\left(\left(x_{f^{t+1}(i)} - x_{f^t(i)}\right) + 2\left(x_{f^t(i)} - x_i\right)\right)\right] \times$$

$$\prod_{0<u<v<w<n} \left[\left(\sum_{\tau\in\{u,v,w\}} \left(x_{f^t(\tau)} - x_\tau\right)^2 \omega^{\underset{\{u,v,w\}}{\mathrm{lex}}(\tau)}\right)^2 z^{2^{(n-1)\ell+\mathrm{lex}(u,v,w)}} + \right.$$

$$\left. \sum_{\tau\in\{u,v,w\}} \left(\left(x_{f^{t+1}(\tau)} - x_{f^t(\tau)}\right)^2 + 2\left(x_{f^{t+1}(\tau)} - x_{f^t(\tau)}\right)\left(x_{f^t(\tau)} - x_\tau\right)\right) \omega^{\underset{\{u,v,w\}}{\mathrm{lex}}(\tau)} \right] \mathrm{mod} \left\{ \prod_{0\le j<n} (x_i - j) \right\}_{0\le i<n}$$

where

$$\omega = \frac{1+\sqrt{-3}}{2} \text{ and } \forall\, 0<u<v<w<n \implies \underset{\{u,v,w\}}{\mathrm{lex}}(u) = 0,\ \underset{\{u,v,w\}}{\mathrm{lex}}(v) = 1,\ \underset{\{u,v,w\}}{\mathrm{lex}}(w) = 2.$$

For any triplet α, β, γ such that

$$\begin{cases} 3 & = & \max\{d_t(\alpha,\beta), d_t(\beta,\gamma), d_t(\gamma,\alpha)\} \\ & \text{and} & \\ 2 & = & \max\{d_{t+1}(\alpha,\beta), d_{t+1}(\beta,\gamma), d_{t+1}(\gamma,\alpha)\} \end{cases},$$

we have

$$1 \equiv \mathrm{GCD}\left\{ \sum_{\tau\in\{\alpha,\beta,\gamma\}} \left(x_{f^{t+1}(\tau)} - x_\tau\right)^2 \omega^{\underset{\{u,v,w\}}{\mathrm{lex}}(\tau)}, \left(\frac{F_\ell(\mathbf{x}, z)}{\prod_{0\le i<j<n} (x_j - x_i)} \mathrm{mod}\, z^{2^{\binom{n-1}{3}+(n-1)\ell}-1}\right) \mathrm{mod}\,(z-1) \right\} \mathrm{mod} \left\{ \prod_{0\le j<n} (x_i - j) \right\}_{0\le i<n},$$

whenever

$$0 \not\equiv \left(\left(\frac{F_\ell(\mathbf{x}, z)}{\prod_{0 \le i < j < n} (x_j - x_i)} \text{mod } z^{2^{\binom{n-1}{3}+(n-1)\ell}-1} \right) \text{mod}\,(z-1) \right) \text{mod} \left\{ \prod_{0 \le j < n} (x_i - j) \right\}_{0 \le i < n}$$

This follows from the same argument used in the proof of the Lemma 2a and 2b. On the other hand, the assertion that

$$1 \equiv \text{GCD} \left\{ \sum_{\tau \in \{\alpha,\beta,\gamma\}} \left(x_{f^{t+1}(\tau)} - x_\tau\right)^2 \omega^{\text{lex}_{\{\alpha,\beta,\gamma\}}(\tau)}, \left(\frac{F_\ell(\mathbf{x}, z)}{\prod_{0 \le i < j < n} (x_j - x_i)} \text{mod } z^{2^{\binom{n-1}{3}+(n-1)\ell}-1} \right) \text{mod}\,(z-1) \right\} \text{mod} \left\{ \prod_{0 \le j < n} (x_i - j) \right\}_{0 \le i < n},$$

contradicts the known fact that for any such α, β, γ such that

$$\begin{cases} 3 & = & \max\left\{d_t(\alpha,\beta), d_t(\beta,\gamma), d_t(\gamma,\alpha)\right\} \\ & \text{and} & \\ 2 & = & \max\left\{d_{t+1}(\alpha,\beta), d_{t+1}(\beta,\gamma), d_{t+1}(\gamma,\alpha)\right\} \end{cases},$$

there must be a non-zero constant κ such that

$$\kappa \sum_{\tau \in \{\alpha,\beta,\gamma\}} \left(x_{f^{t+1}(\tau)} - x_\tau\right)^2 \omega^{\text{lex}_{\{\alpha,\beta,\gamma\}}(\tau)} \equiv \text{GCD} \left\{ F_\ell(\mathbf{x}, 1), \sum_{\tau \in \{\alpha,\beta,\gamma\}} \left(x_{f^{t+1}(\tau)} - x_\tau\right)^2 \omega^{\text{lex}_{\{\alpha,\beta,\gamma\}}(\tau)} \right\} \text{mod} \left\{ \prod_{0 \le j < n} (x_i - j) \right\}_{0 \le i < n}$$

in light of the fact that

$$\left(\frac{F_\ell(\mathbf{x}, z)}{\prod_{0 \le i < j < n} (x_j - x_i)} \text{mod } z^{2^{\binom{n-1}{3}+(n-1)\ell}-1} \right) \text{mod}\,(z-1) \equiv$$

$$\prod_{\substack{0 < i < n \\ n-\ell \le j < n}} \left(\left(x_{f^{t+1}(i)} - x_i\right)^2 - j^2 \right) \times$$

$$\prod_{0 < u < v < w < n} \left(\left(x_{f^{t+1}(w)} - x_w\right)^2 + \left(x_{f^{t+1}(v)} - x_v\right)^2 \left(\frac{1+\sqrt{-3}}{2}\right) + \left(x_{f^{t+1}(u)} - x_u\right)^2 \left(\frac{1-\sqrt{-3}}{2}\right) \right) \text{ mod} \left\{ \prod_{0 \le j < n} (x_i - j) \right\}_{0 \le i < n}.$$

which follows from the fact that the coefficient of the leading term $z^{2^{\binom{n-1}{3}+(n-1)\ell}-1}$ in $F_\ell(\mathbf{x}, 1)$ is

$$\prod_{0 \le i < j < n} (x_j - x_i) \prod_{\substack{0 < i < n \\ n-\ell \le j < n}} \left(\left(x_{f^t(i)} - x_i\right)^2 - j^2 \right) \times$$

$$\prod_{0 < u < v < w < n} \left(\left(x_{f^t(w)} - x_w\right)^2 + \left(x_{f^t(v)} - x_v\right)^2 \left(\frac{1+\sqrt{-3}}{2}\right) + \left(x_{f^t(u)} - x_u\right)^2 \left(\frac{1-\sqrt{-3}}{2}\right) \right)$$

known to be congruent to zero according to the premise. We therefore conclude that

$$\text{LCM}_{\ell, f^t} \equiv 0 \implies \text{LCM}_{\ell, f^{t+1}} \equiv 0.$$

6 Strengthening the Graceful Labeling Theorem.

The strong Composition Lemma yields as a corollary the strong GLC first proposed in [GW18], stated as follows

Theorem 5 : (*strong Graceful Labeling Theorem*) Induced edge label sequences of identically constant functions in $\{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ are common to all functional trees on n vertices.

Proof : Assume for notational convenience and without loss of generality that f is non increasing. Consequently f^{n-1} denotes the identically constant zero function. Having proved that all trees are graceful we now focus on the remaining $\lfloor \frac{n}{2} \rfloor + \left(n - 2 \lfloor \frac{n}{2} \rfloor\right) - 1$ induced edge label sequences associated with identically constant functions in $\{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ each of which is determined by congruence identities of the form

$$0 \not\equiv \text{LCM}_{\ell, f^{n-1}},$$

$0 \le \ell < \lceil \frac{n-1}{2} \rceil$. Seen from the fact that we can easily identify the only 2 relabelings associated with any one of the $\lfloor \frac{n}{2} \rfloor + \left(n - 2 \lfloor \frac{n}{2} \rfloor\right)$ possible induced edge label sequences of a constant functions in $\{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$. The desired result follows by repeatedly applying the contrapositive of the strong Graceful Composition Lemma. □

7 Graceful and Harmonious labelings as spectra of second order constructs.

The induced edge label sequence of a given functional tree G_f associated with $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ is common to all functional trees in $\{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ iff

$$\left\{ f^{k+1}(z) - f^k(z) = \lambda\, f^k(z) \right\}_{0 \le k < n}, \tag{9}$$

for some function $\lambda \in \{1-n, \cdots, -1, 0, 1, \cdots, n-1\}^{\{0,\cdots,n-1\}}$. More specifically,

$$\lambda \in \bigcup_{0 \le j < \lceil \frac{n}{2} \rceil} \text{SP}_{j,n}$$

where

$$\lambda \in \text{SP}_{j,n} :=$$

$$\left\{ g \,|\, \forall\, i \in \{0, \cdots, n-1\},\ 0 \le (i + g(i)) = i + \mathfrak{s}(i)\,(j - \gamma(i)) < n, \text{for some } \begin{array}{c} \mathfrak{s} \in \{-1, 1\}^{\{0,\cdots,n-1\}} \\ \gamma \in \text{S}_n \end{array} \right\}. \tag{10}$$

We describe here how the constraint (9) expresses a construct eigenvalue-eigenvector problem as recently introduced in [GG18].

Recall that second order constructs are matrices whose entries are morphisms. The algebra of *constructs* is prescribed by a *combinator* noted Op, and a *composer* noted $\mathcal{F}$. The *composer* specifies the rule for composing entry morphisms while the *combinator* specifies the rule for combining the compositions of entry morphisms. Natural choices for a *combinator* include for instance

$$\sum_{0 \le j < k},\ \prod_{0 \le j < k},\ \max_{0 \le j < k},\ \min_{0 \le j < k}.$$

For example, the product of second-order constructs $\mathbf{A}$ and $\mathbf{B}$ of size respectively $n_0 \times k$ and $k \times n_1$ results in a construct noted $\text{GProd}_{\text{Op},\mathcal{F}}(\mathbf{A}, \mathbf{B})$ of size $n_0 \times n_1$ specified entry-wise by

$$\text{GProd}_{\text{Op},\mathcal{F}}(\mathbf{A}, \mathbf{B})[i_0, i_1] = \underset{0 \le j < k}{\text{Op}}\ \mathcal{F}(\mathbf{A}[i_0, j], \mathbf{B}[j, i_1]),\ \forall \begin{cases} 0 \le i_0 < n_0 \\ 0 \le i_1 < n_1 \end{cases}.$$

For instance, the product of 2×2 constructs is given by

$$\text{GProd}_{\text{Op},\mathcal{F}}\left(\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \right) = \begin{pmatrix} \text{Op}(\mathcal{F}(a_{00}, b_{00}), \mathcal{F}(a_{01}, b_{10})) & \text{Op}(\mathcal{F}(a_{00}, b_{01}), \mathcal{F}(a_{01}, b_{11})) \\ \text{Op}(\mathcal{F}(a_{10}, b_{00}), \mathcal{F}(a_{11}, b_{10})) & \text{Op}(\mathcal{F}(a_{10}, b_{01}), \mathcal{F}(a_{11}, b_{11})) \end{pmatrix}.$$

Let the composer $\mathcal{F}$ and its dual $\mathcal{G}$ be set to

$$\mathcal{F}\left(f\left(z\right),g\left(z\right)\right) := f\left(g\left(z\right)\right) \quad \text{and} \quad \mathcal{G}\left(f\left(z\right),g\left(z\right)\right) := g\left(f\left(z\right)\right),$$

where for functions $f\left(z\right),g\left(z\right)$. Let the combinator be set to

$$\underset{0\le j<k}{\mathrm{Op}} := \sum_{0\le j<k}.$$

Recall that one of the two possible ways of defining the construct eigenvalue-eigenvector equation is

$$\mathrm{GProd}_{\sum,\mathcal{F}}\left(\mathbf{A}\left(z\right),\ \mathbf{v}\left(z\right)\right) = \mathrm{GProd}_{\sum,\mathcal{F}}\left(\lambda\left(z\right)\mathbf{I}_n,\ \mathbf{v}\left(z\right)\right), \tag{11}$$

where

$$\mathbf{A}\left(z\right) \in \left(\left\{1-n,\cdots,-1,0,1,\cdots,n-1\right\}^{\left\{1-n,\cdots,-1,0,1,\cdots,n-1\right\}}\right)^{n\times n}$$

and

$$\mathbf{v}\left(z\right) \in \left(\left\{1-n,\cdots,-1,0,1,\cdots,n-1\right\}^{\left\{1-n,\cdots,-1,0,1,\cdots,n-1\right\}}\right)^{n\times 1}.$$

There is a construct $\mathbf{A}\left(z\right)$ whose spectra includes all labeled functional trees having their induced edge label sequence in common with an identically constant function in $\left\{0,\cdots,n-1\right\}^{\left\{0,\cdots,n-1\right\}}$. In fact

$$\mathbf{A}\left(z\right) = z\,\text{Incidence Matrix of}\left(G_h\right),$$

where

$$h \in \left\{0,\cdots,n-1\right\}^{\left\{0,\cdots,n-1\right\}} \text{ such that } h\left(i\right) = \begin{cases} i+1 & \text{if } 0\le i<n-1 \\ n-1 & \text{otherwise} \end{cases} \quad \forall\, 0\le i<n\,.$$

More explicitly we have that

$$\forall\, 0\le i,j<n,\quad \mathbf{A}\left(z\right)\left[i,j\right] = \begin{cases} -z & \text{if } 0\le i=j<n-1 \\ z & \text{or } j=i+1 \\ 0 & \text{otherwise} \end{cases}$$

For instance, when $n=5$

$$\mathbf{A}\left(z\right) = \begin{pmatrix} -z & z & 0 & 0 & 0 \\ 0 & -z & z & 0 & 0 \\ 0 & 0 & -z & z & 0 \\ 0 & 0 & 0 & -z & z \\ 0 & 0 & 0 & 0 & -z+z \end{pmatrix}.$$

Consequently, $\mathbf{A}\left(1\right)$ is a non diagonalizable matrix in Jordan normal form whose characteristic polynomial is

$$x\left(x+1\right)^{n-1} = \det\left(x\mathbf{I}_n - \mathbf{A}\left(1\right)\right).$$

Eigenvalue-eigenvector pairs of $\mathbf{A}(1)$ determine two of the construct eigenvalue-eigenvector pairs for $\mathbf{A}(z)$

$$\begin{cases} \text{GProd}_{\sum,\mathcal{F}}\left(\mathbf{A}(z), z\,\mathbf{1}_{n\times 1}\right) & = & \text{GProd}_{\sum,\mathcal{F}}\left(0\,\mathbf{I}_n, z\,\mathbf{1}_{n\times 1}\right) \\ \\ \text{GProd}_{\sum,\mathcal{F}}\left(\mathbf{A}(z), z\,\mathbf{I}_n[:,0]\right) & = & \text{GProd}_{\sum,\mathcal{F}}\left(-z\,\mathbf{I}_n, z\,\mathbf{I}_n[:,0]\right) \end{cases}.$$

The corresponding two construct eigenvalue-eigenvector pairs are

$$\begin{pmatrix} f^0 \\ f^1 \\ \vdots \\ f^{n-2} \\ f^{n-1} \end{pmatrix} = z\mathbf{1}_{n\times 1} \implies \begin{cases} f(z) = z \\ \\ \lambda(z) = 0 \end{cases} \quad \text{and} \quad \begin{pmatrix} f^0 \\ f^1 \\ \vdots \\ f^{n-2} \\ f^{n-1} \end{pmatrix} = z\mathbf{I}_n[:,0], \implies \begin{cases} f(z) = 0 \\ \\ \lambda(z) = -z \end{cases}.$$

The strong Graceful Labeling Theorem establishes other eigenvalue-eigenvector pairs whose entries are compositions of some function

$$f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}.$$

In the setting of harmonious labelings of functional trees, the combinator is set to

$$\underset{0\le j<k}{\text{Op}} := \prod_{0\le j<k}.$$

Harmoniously labeled functional trees are solutions to constraints of the form

$$\left\{f^{k+1}(z)\cdot f^k(z) = f^k\lambda(z)\right\}_{0\le k<n} \text{ where } f \in \left\{\omega_n^0, \cdots, \omega_n^{n-1}\right\}^{\left\{\omega_n^0,\cdots,\omega_n^{n-1}\right\}} \text{ and } \omega_n = \exp\left(\frac{2\pi\sqrt{-1}}{n}\right) \tag{12}$$

Such constraints illustrate the second possible formulation of construct eigenvalue-eigenvector equation for some function $\lambda \in \text{S}_n$. The corresponding construct eigenvalue-eigenvector equation is of the form

$$\text{GProd}_{\prod,\mathcal{F}}\left(\mathbf{A}(z), \mathbf{v}(z)\right) = \text{GProd}_{\prod,\mathcal{G}}\left(\left(\lambda(z)\right)^{\circ\mathbf{I}_n}, \mathbf{v}(z)\right), \tag{13}$$

where

$$\mathbf{A}(z) \in \left(\left\{\omega_n^0, \cdots, \omega_n^{n-1}\right\}^{\left\{\omega_n^0,\cdots,\omega_n^{n-1}\right\}}\right)^{n\times n}$$

and

$$\mathbf{v}(z) \in \left(\left\{\omega_n^0, \cdots, \omega_n^{n-1}\right\}^{\left\{\omega_n^0,\cdots,\omega_n^{n-1}\right\}}\right)^{n\times 1}.$$

There is construct $\mathbf{A}(z)$ whose spectra includes all harmoniously labeled functional trees in $\left\{\omega_n^0, \cdots, \omega_n^{n-1}\right\}^{\left\{\omega_n^0,\cdots,\omega_n^{n-1}\right\}}$. In fact

$$\mathbf{A}(z) = z^{\circ\text{Unsigned Incidence Matrix of}(G_h)},$$

where

$$h \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}} \text{ such that } h(i) = \begin{cases} i+1 & \text{if } 0 \le i < n-1 \\ n-1 & \text{otherwise} \end{cases} \quad \forall\, 0 \le i < n,$$

and the matrix

$$z^{\circ\text{Unsigned Incidence Matrix of}(G_h)}$$

describes the result of the entry-wise exponentiation by z. More explicitly,

$$\forall\, 0 \le i, j < n, \quad \mathbf{A}\left(z\right)\left[i,j\right] = \begin{cases} z^1 & \text{if } 0 \le i = j < n-1 \text{ or } j = i+1 \\ z^2 & \text{if } i = j = n-1 \\ z^0 & \text{otherwise} \end{cases} .$$

For instance when $n = 5$ we have

$$\mathbf{A}\left(z\right) = \left(\begin{array}{ccccc} z & z & 1 & 1 & 1 \\ 1 & z & z & 1 & 1 \\ 1 & 1 & z & z & 1 \\ 1 & 1 & 1 & z & z \\ 1 & 1 & 1 & 1 & z^2 \end{array}\right)$$

Just as in the previous case the unsigned incidence matrix of G_h is a non diagonalizable matrix already in Jordan normal form whose characteristic polynomial is

$$\left(x-2\right)\left(x-1\right)^{n-1} = \det\left(x\mathbf{I}_n - \text{Unsigned Incidence Matrix of}\left(G_h\right)\right).$$

Eigenvalue-eigenvector pairs of the unsigned incidence matrix of G_h determine two of the construct eigenvalue-eigenvector pairs for $\mathbf{A}\left(z\right)$

$$\begin{cases} \text{GProd}_{\prod,\mathcal{F}}\left(\mathbf{A}\left(z\right), z^{\circ\mathbf{1}_{n\times 1}}\right) & = & \text{GProd}_{\prod,\mathcal{G}}\left(\left(z^2\right)^{\circ\mathbf{I}_n}, z^{\circ\mathbf{1}_{n\times 1}}\right) \\ \text{GProd}_{\prod,\mathcal{F}}\left(\mathbf{A}\left(z\right), z^{\circ\mathbf{I}_n[:,0]}\right) & = & \text{GProd}_{\prod,\mathcal{G}}\left(z^{\circ\mathbf{I}_n}, z^{\circ\mathbf{I}_n[:,0]}\right) \end{cases}.$$

Which determines the construct eigenvectors

$$\left(\begin{array}{c} f^0 \\ f^1 \\ \vdots \\ f^{n-2} \\ f^{n-1} \end{array}\right) = z^{\circ\mathbf{1}_{n\times 1}} \implies \begin{cases} f\left(z\right) & = & z \\ \lambda\left(z\right) & = & z^2 \end{cases} \quad \text{and} \quad \left(\begin{array}{c} f^0 \\ f^1 \\ \vdots \\ f^{n-2} \\ f^{n-1} \end{array}\right) = z^{\circ\mathbf{I}_n[:,0]}, \implies \begin{cases} f\left(z\right) = z^0 \\ \lambda\left(z\right) = z \end{cases}.$$

The Harmonious Labeling Theorem establishes other eigenvalue-eigenvector pairs whose entries are compositions of some function

$$f \in \left\{\omega_n^0, \cdots, \omega_n^{n-1}\right\}^{\left\{\omega_n^0, \cdots, \omega_n^{n-1}\right\}}.$$

Consequently, unlike the matrix case, the number of distinct eigenvalues for an $n \times n$ construct can be exponential in n.

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