

A generalized distribution interpolated between the exponential and power law distributions and applied to the walking data of the pill bug (*Armadillidium vulgare*)

Shuji Shinohara^{1*}, Hiroshi Okamoto¹, Toru Moriyama², Yoshihiro Nakajima³, Takaharu Shokaku⁴, Akika Utsumi², Ung-il Chung¹

¹ Department of Bioengineering, Graduate School of Engineering, The University of Tokyo, Tokyo, Japan

² Faculty of Textile Science, Shinshu University, Ueda, Japan

³ Graduate School of Economics, Osaka City University, Osaka, Japan

⁴ Department of Network Design, Meiji University, Tokyo, Japan

* Corresponding author

E-mail: shinohara@bioeng.t.u-tokyo.ac.jp (SS)

AUTHOR CONTRIBUTIONS STATEMENT

Shuji Shinohara: conceptualization, formal analysis, methodology, software, writing — original draft preparation, funding acquisition. Hiroshi Okamoto: Conceptualization, Methodology, writing—review, and Editing Toru Moriyama: Resources, data curation, writing—review, and editing. Yoshihiro Nakajima: Writing—review, and editing. Takaharu Shokaku: Resources, data curation, writing—review, and editing. Akika Utsumi: Resources, data curation, writing—review, and editing. Ung-il Chung: Writing—review and editing, supervision, and project administration.

25 **Abstract**

26 To determine whether the walking pattern of an organism is a Lévy walk or a Brownian walk, it has been
 27 compared whether the frequency distribution of linear step lengths follows a power law distribution or an
 28 exponential distribution. However, there are many cases where actual data cannot be classified into either of
 29 these categories. In this paper, we propose a general distribution that includes the power law and exponential
 30 distributions as special cases. This distribution has two parameters: One represents the exponent, similar to
 31 the power law and exponential distributions, and the other is a shape parameter representing the shape of the
 32 distribution. By introducing this distribution, an intermediate distribution model can be interpolated
 33 between the power law and exponential distributions. In this study, the proposed distribution was fitted to the
 34 frequency distribution of the step length calculated from the walking data of pill bugs. The autocorrelation
 35 coefficients were also calculated from the time-series data of the step length, and the relationship between
 36 the shape parameter and time dependency was investigated. The results showed that individuals whose step-
 37 length frequency distributions were closer to the power law distribution had stronger time dependence.

39 **Keywords**

40 Power law distribution, Exponential distribution, generalized distribution, Weibull distribution, Time
 41 dependence of step length, pill bug, Random walk

42 Introduction

43 Lévy walks are found in the migratory behavior of organisms at various levels, from bacteria and T cells
44 to humans [1–6]. Lévy walks are a type of random walk in which the frequency of occurrence of a linear step
45 length l follows a power law distribution $p(l) \sim l^{-\mu}$, $1 < \mu \leq 3$. Compared to the Brownian walk, which is
46 also a type of random walk (characterized by an exponential distribution $p(l) \sim e^{-\lambda l}$ of the frequency of
47 occurrence of step length l), the Lévy walk is characterized by the occasional appearance of linear movements
48 over very long distances, and why such patterns occur in biological migration has attracted attention [7].

49 To determine whether the gait pattern is a Lévy walk or a Brownian walk, a comparison is made concerning
50 whether the frequency distribution of the linear step length follows a power law distribution or an exponential
51 distribution [5,8–12]. In the comparison, the range of step lengths to be analyzed and the parameters of the
52 model that best fit the data in that range, that is, the exponents μ and λ , are first calculated. The maximum
53 likelihood estimation method is generally used to estimate the parameters. Next, a comparison is made to
54 determine which model fits better, the power law distribution model, or the exponential distribution model.
55 For comparison, Akaike information criteria weights (AICw), which considers the likelihood and number of
56 parameters, are often used [11,12]. To verify whether the model fits the observed data, Clauset et al. [10]
57 proposed goodness-of-fit tests based on the Kolmogorov-Smirnov (KS) statistic. In this way, judgments have
58 been made as to whether the observed data follow a power law distribution or an exponential distribution
59 model, but many actual data cannot be classified as either [3].

60 In this paper, we propose a general distribution such that the power law and exponential distributions are
61 included as special cases. This distribution has two parameters: One represents the exponent, similar to the
62 power law and exponential distributions, and the other is a shape parameter representing the shape of the
63 distribution. In this distribution, if the shape parameter is set to a specific value, it represents the power law
64 distribution, and if it is set to another specific value, it represents an exponential distribution. By introducing

this distribution, distributions that are intermediate between the power law and exponential distributions can be modeled. In this study, the proposed distribution was fitted to the walking data of pill bugs collected by Shokaku et al. [3] as specific observation data. Differently expressed, we estimated the exponent and shape parameters that best fit the observed data.

It has been asserted that there is a time dependency in the time series of the step length in human mobility behavior. For example, Wang et al. [13], Rhee et al. [14], and Zhao et al. [15] demonstrated that the temporal variation of step length is autocorrelated, which means that there is a trend in the time variation of step length, such that short (long) steps are followed by short (long) steps. In addition, it has been emphasized that the time dependence of step length is related to the frequency distribution of the step length following a power law distribution [13].

In this study, to investigate the time dependence of the walking data of the pill bugs described above, we calculated the autocorrelation coefficient between the time series l_t of the step length and the time series $l_{t+\tau}$ with time lag τ . When $\tau=0$, the autocorrelation coefficient is 1 because the two time series are identical. Conversely, when $\tau>0$, any random walk such as the Lévy walk or Brownian walk is theoretically uncorrelated. We investigated the relationship between the autocorrelation coefficient calculated from the time-series data of the linear step length and the shape parameter of the frequency distribution of the linear step length. The results showed that individuals whose frequency distributions were closer to the power law distribution than to the exponential distribution tended to show stronger time dependence.

Methods

A distribution interpolated between the exponential and power law distributions

In this section, we propose a general distribution that includes exponential and power-law distributions as

special cases. First, let us consider the process of elongating the persistence length. This length can be a spatial distance or a time interval. For example, consider a process in which an organism continues to move in a straight line and reaches a distance of l , and then moves in the same straight line by a dl , for the total straight-line distance to increase to $l + dl$. As a discrete example, consider the process of tossing a coin and getting "heads" l times, then getting "heads" again, and extending the period of consecutive "heads" to $l+1$ times.

Suppose that the probability of length l occurring is represented by $p(l)$. Because l is a length, let $l \geq 0$. Because $p(l)$ is a probability distribution, it satisfies $\int_0^\infty p(l)dl = 1$. Let us also denote by $P(l)$ the complementary cumulative frequency distribution (CCDF) in which lengths greater than or equal to l occur. That is,

$$P(l) = \int_l^\infty p(x)dx \quad (1)$$

Based on this definition, $P(0) = \int_0^\infty p(x)dx = 1$ is valid. Similarly, from the definition, the following relationship holds between $p(l)$ and $P(l)$.

$$p(l) = -\frac{dP(l)}{dl} \quad (2)$$

We denote $f(l)$ the probability that the length l will extend to $l + dl$. $f(l)$ can be expressed by:

$$f(l) = \frac{P(l+dl)}{P(l)} \quad (3)$$

In contrast, the probability that length l is reached and ends at l is denoted by $g(l)$. $g(l)$ can be expressed using $p(l)$ and $P(l)$ as follows:

$$g(l) = \frac{p(l)}{P(l)}dl \quad (4)$$

Here, an interval of length l can either extend further from l or end at l , therefore, $f(l)+g(l)=1$ holds. To consider the elongation process in the discrete case, that is, when $dl=1$, let us consider a situation in which a

coin is tossed repeatedly. We assume a situation in which the probability of “heads” in a coin toss is not constant, but varies depending on the number of times the same side appears consecutively. For example, after three consecutive "heads", the probability that the next is also "heads" is represented by $f(3)$. Conversely, the probability that the next is "tails" is expressed as $g(3)=1-f(3)$.

The CCDF that the same side will continue for $l+1$ or more consecutively is expressed as follows:

$$P(l+1) = \prod_{i=1}^l f(i) = f(l) \prod_{i=1}^{l-1} f(i) = f(l)P(l) \quad (5)$$

The probability that the same side continues consecutively for l times, that is, the probability of length l occurring, is expressed as follows:

$$p(l) = g(l) \prod_{i=1}^{l-1} f(i) = g(l)P(l) \quad (6)$$

Now, let us consider Brownian and Lévy walks. For the Brownian walk, the probability of a step length l occurring and its CCDF is represented by the following exponential distribution with an exponent β .

$$\begin{cases} p(l) \propto \beta e^{-\beta l} \\ P(l) \propto e^{-\beta l} \end{cases} \quad (7)$$

Conversely, for the Lévy walk, the probability of the occurrence of a step length l and the CCDF is expressed by the following power law distribution with the exponent β .

$$\begin{cases} p(l) \propto \beta l^{-(\beta+1)} \\ P(l) \propto l^{-\beta} \end{cases} \quad (8)$$

If the change rate at length l is defined as $\gamma(l) = \frac{g(l)}{dl} = \frac{p(l)}{P(l)}$, the change rates of the Brownian walk and

the Lévy walk can be expressed as follows:

$$\gamma(l) = \frac{p(l)}{P(l)} = \frac{\beta e^{-\beta l}}{e^{-\beta l}} = \beta = \beta l^{-0} \quad (9)$$

$$\gamma(l) = \frac{p(l)}{P(l)} = \frac{\beta l^{-(\beta+1)}}{l^{-\beta}} = \beta l^{-1} \quad (10)$$

128 We now define the generalized rate of change to include the Brownian and Lévy walk as a special case, as
129 follows:

$$130 \quad \gamma(l) = \frac{p(l)}{P(l)} = \beta l^{-\alpha} \quad (11)$$

131 The case of $\alpha=0$ corresponds to the change rate of the Brownian walk, and the case of $\alpha=1$ corresponds to
132 the change rate of the Lévy walk.

133 Equation (11) can be written as $\gamma(l) = -\frac{dP(l)}{dl} / P(l) = \beta l^{-\alpha}$, and solving it for $P(l)$ yields the following
134 solution:

$$135 \quad P(l) = \exp\left(-\frac{\beta}{-\alpha+1} l^{-\alpha+1}\right) \quad (12)$$

136 If we put $m = -\alpha + 1$, then $P(l)$ is expressed as follows:

$$137 \quad P(l) = \exp\left(-\frac{\beta}{m} l^m\right) \quad (13)$$

138 The probability $p(l)$ of step length l occurring can be written as follows, using equation (2).

$$139 \quad \begin{aligned} p(l) &= -\frac{dP(l)}{dl} \\ &= \beta l^{m-1} \exp\left(-\frac{\beta}{m} l^m\right) \end{aligned} \quad (14)$$

140 In addition, because $-\alpha=m-1$, the change rate in equation (11) can be written as follows:

$$141 \quad \gamma(l) = \beta l^{m-1} \quad (15)$$

142 Let us consider the meanings of m and β . As can be seen from equation (15), the change rate becomes
143 smaller as the value of l increases, since $m-1 < 0$ holds in the range of $m < 1$. Conversely, the longer this persists,
144 the greater the probability that it will continue the next time. When $m > 1$, the change rate increases as l
145 increases. Differently put, the longer the value of l , the higher is the change rate. When $m=1$, the change rate
146 is a constant value β , regardless of the value of l . Thus, m is a parameter that controls how the length of l , or

147 the past history of how long the same condition has lasted, is taken into account.

148 Conversely, for β , if l and m are fixed, the larger β becomes, the higher the change rate becomes. Otherwise
149 expressed, β is a parameter that controls the magnitude of the change rate.

150 For $m = 1$, equation (14) is expressed as follows:

$$151 \quad p(l) = \beta \exp(-\beta l) \quad (16)$$

152 Put differently, it represents an exponential distribution of the exponent $-\beta$. Conversely, when $m = 0$,
153 equation (14) cannot be defined because it involves division by zero. However, when m is sufficiently close
154 to zero, this distribution can be approximated using the Maclaurin expansion, $l^m \approx 1 + m \log l$ as follows:

$$\begin{aligned} p(l) &= \beta l^{m-1} \exp\left(-\frac{\beta}{m} l^m\right) \\ &\approx \beta l^{m-1} \exp\left(-\frac{\beta}{m} (1 + m \log l)\right) \\ &= \beta l^{m-1} \exp\left(-\frac{\beta}{m} - \beta \log l\right) \\ &= \beta l^{m-1} \exp\left(-\frac{\beta}{m}\right) \exp(-\beta \log l) \\ &= \beta l^{m-1} \exp\left(-\frac{\beta}{m}\right) \exp(\log l^{-\beta}) \\ &= \beta l^{m-1} \exp\left(-\frac{\beta}{m}\right) l^{-\beta} \\ &= \beta \exp\left(-\frac{\beta}{m}\right) l^{-(\beta-m+1)} \\ 155 \quad &\approx \beta \exp\left(-\frac{\beta}{m}\right) l^{-(\beta+1)} \end{aligned} \quad (17)$$

156 If we set $Z = \frac{1}{\beta} \exp\left(\frac{\beta}{m}\right)$ as the normalization constant, equation (17) can be rewritten as follows:

$$157 \quad p(l) \approx \frac{1}{Z} l^{-(\beta+1)} \quad (18)$$

158 Otherwise expressed, equation (14) represents the power law distribution of the exponent $-(\beta+1)$ as an
159 approximation. Thus, equation (14) can be said to be a distribution that includes not only the exponential

160 distribution but also the power law distribution as a special case, albeit as an approximation. Therefore, in
161 this study, for convenience, the distribution of equation (14) is named the generalized distribution (GE).
162 Furthermore, parameter m is called the shape parameter in the sense that it represents the shape of the
163 distribution.

164

165 **Relationship between the generalized distribution (GE) and Weibull** 166 **distribution**

167 We discuss the relationship between the GE and the Weibull distribution [16]. The Weibull distribution is
168 used to describe the degradation phenomenon and lifetime of a component statistically, and is expressed as
169 follows:

$$170 \quad p(l) = \frac{m}{\eta} \left(\frac{l}{\eta} \right)^{m-1} \exp \left(- \left(\frac{l}{\eta} \right)^m \right) \quad (19)$$

171 Here, η is the scale parameter. Conversely, m is a parameter that determines the shape of the distribution
172 and is called the shape parameter. The CCDF of the Weibull distribution is expressed as

$$173 \quad P(l) = \exp \left(- \left(\frac{l}{\eta} \right)^m \right) \quad (20)$$

174 In the Weibull distribution, if we set $m=1$ and $\eta=1/\lambda$, an exponential distribution with an exponent λ is
175 obtained as follows:

$$176 \quad p(l) = \frac{m}{\eta} \left(\frac{l}{\eta} \right)^{m-1} \exp \left(- \left(\frac{l}{\eta} \right)^m \right) = \frac{1}{\eta} \exp \left(- \frac{l}{\eta} \right) = \lambda \exp(-\lambda l) \quad (21)$$

177 In addition, if $m=2$ and $\eta = \sqrt{2}\sigma$, the Weibull distribution represents the Rayleigh distribution as shown
178 below:

$$p(l) = \frac{m}{\eta} \left(\frac{l}{\eta} \right)^{m-1} \exp \left(- \left(\frac{l}{\eta} \right)^m \right) = \frac{2}{\eta} \left(\frac{l}{\eta} \right) \exp \left(- \left(\frac{l}{\eta} \right)^2 \right) = \frac{l}{\sigma^2} \exp \left(- \frac{l^2}{2\sigma^2} \right) \quad (22)$$

Thus, the Weibull distribution includes exponential and Rayleigh distributions as special cases. Let us consider the relationship between the Weibull distribution and GE. If we set $\eta = (m/\beta)^{\frac{1}{m}}$, equation (19) can be transformed as follows:

$$\begin{aligned} p(l) &= \frac{m}{\eta} \left(\frac{l}{\eta} \right)^{m-1} \exp \left(- \left(\frac{l}{\eta} \right)^m \right) \\ &= \frac{m}{\eta^m} l^{m-1} \exp \left(- \frac{l^m}{\eta^m} \right) \\ &= \frac{m}{\left((m/\beta)^{\frac{1}{m}} \right)^m} l^{m-1} \exp \left(- \frac{l^m}{\left((m/\beta)^{\frac{1}{m}} \right)^m} \right) \quad (23) \\ &= \frac{m}{(m/\beta)} l^{m-1} \exp \left(- \frac{l^m}{(m/\beta)} \right) \\ &= \beta l^{m-1} \exp \left(- \frac{\beta}{m} l^m \right) \end{aligned}$$

Expressed differently, it is consistent with GE. However, these two are not equivalent. As shown in equation (16), when $m=1$, GE represents an exponential distribution. In addition, if $m=2$ and $\beta = 1/\sigma^2$, then GE represents the Rayleigh distribution.

$$p(l) = \beta l \exp \left(- \frac{\beta}{2} l^2 \right) = \frac{l}{\sigma^2} \exp \left(- \frac{l^2}{2\sigma^2} \right) \quad (24)$$

Thus, for $m=1$ and $m=2$, the GE is similar to the Weibull distribution. However, when m is close to zero, a difference is observed. When m is close to zero, the Weibull distribution can be approximated using the Maclaurin expansion $l^m \approx 1 + m \log l$ as follows:

$$\begin{aligned}
 p(l) &= \frac{m}{\eta} \left(\frac{l}{\eta} \right)^{m-1} \exp \left(- \left(\frac{l}{\eta} \right)^m \right) \\
 &\approx \frac{m}{\eta^m} l^{m-1} \exp \left(- \frac{1 + m \log l}{\eta^m} \right) \\
 &= \frac{m}{\eta^m} l^{m-1} \exp \left(- \frac{1}{\eta^m} - \frac{m}{\eta^m} \log l \right) \\
 &= \frac{m}{\eta^m} l^{m-1} \exp \left(- \frac{1}{\eta^m} \right) \exp \left(- \frac{m}{\eta^m} \log l \right) \\
 &= \frac{m}{\eta^m} l^{m-1} \exp \left(- \frac{1}{\eta^m} \right) \exp \left(\log l^{-\frac{m}{\eta^m}} \right) \\
 &= \frac{m}{\eta^m} l^{m-1} \exp \left(- \frac{1}{\eta^m} \right) l^{-\frac{m}{\eta^m}} \\
 &= \frac{m}{\eta^m} \exp \left(- \frac{1}{\eta^m} \right) l^{m-1-\frac{m}{\eta^m}} \\
 &\approx \frac{m}{\eta^m} \exp \left(- \frac{1}{\eta^m} \right) l^{-1}
 \end{aligned} \tag{25}$$

If we set $Z = \frac{\eta^m}{m} \exp \left(\frac{1}{\eta^m} \right)$ as the normalization constant, equation (25) can be rewritten as follows:

$$p(l) \approx \frac{1}{Z} l^{-1} \tag{26}$$

When m is sufficiently close to zero, the Weibull distribution can be approximated as a power law distribution with an exponent of -1. Put differently, the Weibull distribution only represents the power law distribution with the exponent -1, regardless of the value of η . Conversely, GE can approximate the power law distribution of any exponent with $-(\beta+1)$ as a parameter, as shown in equation (18).

Parameter Estimation

The GE has two parameters: m and β . In this section, we describe a method for estimating these parameters from the observed data. The first objective of this section is to find the minimum \hat{l}_{\min} and maximum \hat{l}_{\max} of the observed data that should be fitted to the GE model. The second objective is to find the parameters

203 \hat{m} and $\hat{\beta}$ of the GE model that best fit the data in the range $\hat{l}_{\min} \leq l \leq \hat{l}_{\max}$. Suppose we are given N
 204 observed data $D = \{l_1, l_2, \dots, l_N\}$ in the range of $l_{\min} \leq l \leq l_{\max}$. The model for the data must be satisfied
 205 $\int_{l_{\min}}^{l_{\max}} p(l)dl = 1$. For this reason, we multiply equation (14) by a constant term and redefine the GE model
 206 as follows:

$$\begin{aligned} p(l; \beta, m, l_{\min}, l_{\max}) &= \frac{1}{\int_{l_{\min}}^{l_{\max}} p(x)dx} p(l) \\ &= \frac{1}{\int_{l_{\min}}^{l_{\max}} \beta x^{m-1} \exp\left(-\frac{\beta}{m} x^m\right) dx} \beta l^{m-1} \exp\left(-\frac{\beta}{m} l^m\right) \\ &= \frac{1}{\exp\left(-\frac{\beta}{m} l_{\min}^m\right) - \exp\left(-\frac{\beta}{m} l_{\max}^m\right)} \beta l^{m-1} \exp\left(-\frac{\beta}{m} l^m\right) \end{aligned} \quad (27)$$

208 When the observed data are discrete, such as natural numbers, they are defined as follows:

$$\begin{aligned} p(l; \beta, m, l_{\min}, l_{\max}) &= \frac{1}{\sum_{i=l_{\min}}^{l_{\max}} p(i)} p(l) \\ &= \frac{1}{\sum_{i=l_{\min}}^{l_{\max}} \beta i^{m-1} \exp\left(-\frac{\beta}{m} i^m\right)} \beta l^{m-1} \exp\left(-\frac{\beta}{m} l^m\right) \end{aligned} \quad (28)$$

210 In addition, because the CCDF of $p(l; \beta, m, l_{\min}, l_{\max})$ must satisfy $P(l_{\min}) = 1$, we redefine it as follows:

$$P(l; \beta, m, l_{\min}, l_{\max}) = \frac{\exp\left(-\frac{\beta}{m} l^m\right) - \exp\left(-\frac{\beta}{m} l_{\max}^m\right)}{\exp\left(-\frac{\beta}{m} l_{\min}^m\right) - \exp\left(-\frac{\beta}{m} l_{\max}^m\right)} \quad (29)$$

212 When the observed data is discrete, the CCDF is defined as follows:

$$P(l; \beta, m, l_{\min}, l_{\max}) = \frac{\sum_{i=l}^{l_{\max}} \beta i^{m-1} \exp\left(-\frac{\beta}{m} i^m\right)}{\sum_{i=l_{\min}}^{l_{\max}} \beta i^{m-1} \exp\left(-\frac{\beta}{m} i^m\right)} \quad (30)$$

214 The log-likelihood of the observed data D was calculated using the following equation (27) as follows:

$$\begin{aligned}
 \ln L(\beta, m; l_{\min}, l_{\max}) &= \sum_{i=1}^N \ln p(l_i; \beta, m, l_{\min}, l_{\max}) \\
 &= \sum_{i=1}^N \left(-\ln \left(\exp \left(-\frac{\beta}{m} l_{\min}^m \right) - \exp \left(-\frac{\beta}{m} l_{\max}^m \right) \right) + \ln \beta + \ln l_i^{m-1} + \ln \exp \left(-\frac{\beta}{m} l_i^m \right) \right) \\
 &= \sum_{i=1}^N \left(-\ln \left(\exp \left(-\frac{\beta}{m} l_{\min}^m \right) - \exp \left(-\frac{\beta}{m} l_{\max}^m \right) \right) + \ln \beta + (m-1) \ln l_i - \frac{\beta}{m} l_i^m \right) \quad (31) \\
 &= \sum_{i=1}^N \left(C + (m-1) \ln l_i - \frac{\beta}{m} l_i^m \right) \\
 &= NC + \sum_{i=1}^N \left((m-1) \ln l_i - \frac{\beta}{m} l_i^m \right)
 \end{aligned}$$

Here $C = -\ln \left(\exp \left(-\frac{\beta}{m} l_{\min}^m \right) - \exp \left(-\frac{\beta}{m} l_{\max}^m \right) \right) + \ln \beta$. The log-likelihood when the observed data are discrete can be replaced by $C = -\ln \left(\sum_{i=l_{\min}}^{l_{\max}} \beta i^{m-1} \exp \left(-\frac{\beta}{m} i^m \right) \right) + \ln \beta$.

The model parameters that best fit the observed data are \hat{m} and $\hat{\beta}$, which maximize the log-likelihood. In this paper, m varies from 0.01 to 1.0 and β from 0.01 to 4.0 in increments of 0.01, and the parameters \hat{m} and $\hat{\beta}$ for which equation (31) is maximized are obtained numerically. When $m = 0$, equation (31) cannot be defined; therefore, we set the number sufficiently close to 0, $m = 0.01$.

To evaluate the model's goodness of fit to the observed data, we used the Kolmogorov-Smirnov statistic $D(l_{\min}, l_{\max})$, which represents the distance between the CCDF, $S(l; l_{\min}, l_{\max})$, calculated from the observed data D and the theoretical CCDF expressed in equation (29) or equation (30).

$$D(l_{\min}, l_{\max}) = \max_{l_{\min} \leq l \leq l_{\max}} \left| S(l; l_{\min}, l_{\max}) - P(l; \hat{\beta}, \hat{m}, l_{\min}, l_{\max}) \right| \quad (32)$$

In this paper, let \hat{l}_{\max} be the maximum l_{\max} of the observed data.

229 If $l_{\max} = \hat{l}_{\max}$, then $D(l_{\min}, \hat{l}_{\max})$ can be considered as a function of l_{\min} . \hat{l}_{\min} , which minimizes

230 $D(l_{\min}, \hat{l}_{\max})$, is numerically calculated from within the observed data. That is,

$$231 \quad \hat{l}_{\min} = \arg \min_{l \in D} D(l, \hat{l}_{\max}) \quad (33)$$

232 As shown above, \hat{l}_{\min} , \hat{m} , and $\hat{\beta}$ can be obtained numerically using equations (31) and (33).

233

234 Autocorrelation coefficient

235 Suppose we are given T time-series data (l_1, l_2, \dots, l_T) . In this case, the autocorrelation coefficient $r(\tau)$ with

236 time lag τ is expressed as follows:

$$237 \quad r(\tau) = \frac{\sum_{t=1}^{T-\tau} (l_t - \bar{l}_{1,T-\tau})(l_{t+\tau} - \bar{l}_{1+\tau,T})}{\sqrt{\sum_{t=1}^{T-\tau} (l_t - \bar{l}_{1,T-\tau})^2} \sqrt{\sum_{t=1}^{T-\tau} (l_{t+\tau} - \bar{l}_{1+\tau,T})^2}} \quad (34)$$

$$\bar{l}_{1,T-\tau} = \frac{\sum_{t=1}^{T-\tau} l_t}{T-\tau}$$

$$\bar{l}_{1+\tau,T} = \frac{\sum_{t=1}^{T-\tau} l_{t+\tau}}{T-\tau}$$

238 When $\tau=0$, we take the correlation coefficient between the same time-series data $r(0)=1$. We calculated the
 239 autocorrelation coefficients of the time-series data of linear step length to investigate the time dependence of
 240 the pill bugs' walking data. In this study, the autocorrelation coefficient was obtained in the range of $\tau = 1-$
 241 100. The autocorrelation coefficients for every individual are averaged from $r(1)$ to $r(100)$ and are expressed
 242 as \bar{r} .

$$243 \quad \bar{r} = \frac{\sum_{\tau=1}^{100} r(\tau)}{100} \quad (35)$$

244 The programs for the parameter estimation and autocorrelation coefficient calculation described above
 245 were developed using C++. The compiler was MinGW 8.1.0 64-bit for C++ [17]. The Qt library (Qt version

Qt 5.15.2 MinGW 64-bit) was also used for development [18].

Statistical analyses

Wilcoxon's rank sum test was used to test the difference in means. For all analyses statistical significance was set at $p < 0.01$. The following analysis was performed using the R 3.6.1 statistical software (2019-07-05) [19] unless otherwise specified. We used the R packages of exactRankTests version 0.8.31 for the Wilcoxon rank sum test. The operating system used was Windows 10.

Application

We applied the method described above to the pill bug's gait data. It is known that pill bugs have a habit termed turn alternation, following which they turn to the right (left), left (right), and so on [20]. The mechanism underlying turn alternation is assumed to be based primarily on proprioceptive information from the previous turn and arises from bilaterally asymmetrical leg movements that occur when turning [21]. During one turn, the outer-side legs travel further than the inner ones. After completing the turn, the relatively rested inner-side legs exert more influence on subsequent movements than the outer-side ones and bias the animal to turn in the opposite direction at the next step.

By alternating turns, pill bugs can maintain a straight course to avoid an obstacle. Moving in a straight course is considered the most adaptive strategy when precise information about environmental resources or hazards is absent [22]. However, when pill bugs were examined in successive T-mazes, they sometimes turned in the same direction as they had at the previous junction (turn repetition). For example, in an experiment on 12 pill bugs using 200 successive T-mazes (for approximately 30 min), three individuals maintained a high rate of turn alternation, four a low rate, and the remaining five spontaneously increased

and decreased the rate [23]. Why some pill bugs did not maintain a high rate of turn alternation, that is, generate turn repetition at a rate other than low, is still unclear.

Shokaku et al. developed an automatic turntable-type multiple T-maze device to observe the appearance of turn alternation and turn repetition in pill bugs over a long period and to investigate the effects of these turns on gait patterns [3]. This is a virtually infinite T-maze that uses a turntable. The pill bug turns to the left or right at a T-junction, goes straight ahead, and then crosses another T-junction, and so on. Using this device, Shokaku et al. observed 34 pill bugs for more than 6 h each. An example of a walking pattern in the T-maze is shown in Fig. 1.

Fig. 1 Trajectory of an individual's gait.

When the turns are repeated regularly, alternating left and right, the pill bug is considered to move straight. Conversely, if the same turn is repeated, such as right and right, it is considered to have changed direction.

In this classification of gait patterns, the pill bug is considered to decide whether to continue or abort straight-ahead movement each time it encounters a T-intersection. The straight-line distance l was calculated using the method shown in Fig. 2. Using this method, time-series data of the straight-line distance l for each individual were obtained.

Fig. 2 Sample calculation of step length l . The black polygonal line with the arrow represents a turn; L represents a left turn, R a right turn. The red line represents an approximate linear movement. The L-R-L-R pattern shown in this figure represents linear movement with a step length of 4 ($l=4$).

Results

Figure 3 shows an example of a discrete case, that is, the GE represented by equation (28). Figure 3(a) and 3(b) show the cases of $m=1.0$, and $m=0.01$, respectively. Figure 3(a) shows a single logarithmic graph with the vertical axis on a logarithmic scale and Figure 3(b) shows a double logarithmic graph with both axes on a logarithmic scale. In these figures, the GE for the case $\beta = 0.5, 1.0, 2.0$ is shown. In Fig. 3(a), the exponential approximation curve is shown, and in Fig. 3(b), the power approximation curve is shown. From the figure, we can see that the GE for $m=1.0$, can be approximated by an exponential distribution with an exponent $-\beta$. Conversely, the GE for $m = 0.01$ can be approximated by the power law distribution with exponent $-(\beta+1)$.

Fig. 3 Examples of GE. (a) For $m=1.0$, the GEs are shown for $\beta=0.5, 1.0$ and 2.0 . The vertical axis is shown logarithmically. The exponential approximation curves are also shown. (b) For $m=0.01$, the GE for $\beta=0.5, 1.0$ and 2.0 are shown. Both axes are shown logarithmically. The power approximation curves are also shown.

Figure 4 shows examples of walking data for three individual pill bugs. Figure 4(a) shows the time series of the step length of subject 4 in the experiment of Shokaku et al [3]. Figure 4(b) shows the CCDF of the step length. Figures 4(c) and 4(d) show the data for subject 15, and Figures 4(e) and 4(f) show the data for subject 14. Figures 4(b), (d), and (f) also show the results of fitting the CCDF of the GE model expressed in equation (30) to the observed data. Figures 4(b), 4(d), and 4(f) are double logarithmic graphs with both axes displayed in logarithmic form. Note that in these figures, $P(\hat{l}_{\min}) = 1$ is based on the definition of equation (30).

Fig. 4 Step length data for three individuals. (a) Step length time series for individuals of subject 4. (b) CCDF of subject 4. $\hat{l}_{\min} = 3, \hat{l}_{\max} = 30, m = 0.01, \beta = 1.68$ (c) Step length time series for individuals of

314 subject 15. (d) CCDF of subject 15. $\hat{l}_{\min} = 10, \hat{l}_{\max} = 81, m = 0.24, \beta = 0.82$ (e) Step length time series for
 315 individuals of subject 14. (f) CCDF of subject 14. $\hat{l}_{\min} = 7, \hat{l}_{\max} = 127, m = 0.44, \beta = 0.42$

316

317 Figure 5 shows the relationship between the autocorrelation coefficient \bar{r} of the time-series data of step
 318 length and the shape parameter m when the GE fits the frequency distribution of each individual's step length.

319

320 **Fig. 5 Relationship between shape parameters and autocorrelation coefficients.** (a) Scatter plots between
 321 shape parameters and autocorrelation coefficients. The regression line is also shown. (b) Box plot for the case
 322 of grouping individuals with $m=0.01$ and $m>0.01$. (c) Correlogram for each group. The vertical axis
 323 represents the mean value of the autocorrelation coefficient of the individuals in the group. The error bars
 324 represent the standard errors.

325

326 The autocorrelation coefficient shown on the vertical axis represents the average value from $r(1)$ to $r(100)$.
 327 In calculating the shape parameter m , only individuals with $\hat{l}_{\max} - \hat{l}_{\min}$ value of 5 or higher were included in
 328 the analysis. The autocorrelation coefficients were calculated from 1 to 100 for the time lag, and only
 329 individuals with more than 200 time-series data were included in the analysis. A total of 27 individuals were
 330 analyzed. The mean \pm standard deviation (SD) of the number of time-series data of the step lengths of these
 331 individuals was 601.22 ± 285.77 . The minimum and maximum values of the data were 213 and 1244,
 332 respectively. Figure 5(a) shows a scatter plot of the shape parameters and autocorrelation coefficient. There
 333 was no significant correlation between the two parameters ($r=-0.47, p=0.014, n=27$). Figure 5(b) is a box
 334 plot showing the autocorrelation coefficients for each group when the population was divided into two
 335 groups: Group 1 ($n=19$) with $m=0.01$ and Group 2 ($n=8$) with $m>0.01$. The Wilcoxon rank-sum test revealed
 336 a significant difference between the two groups ($p=4.05 \times 10^{-5}, W=145$). Figure 5(c) shows the correlogram
 337 for each group. The horizontal axis represents the time lag τ . The vertical axis represents the average

autocorrelation coefficient for the τ of each individual in each group. The error bars represent the standard error.

Figure 6 shows the relationship between the GE shape parameter m and exponent β for the 27 individuals described above. Figure 6(a) shows a scatter plot for both. There was a significant negative correlation between the two ($r=-0.67$, $p=1.2\times 10^{-4}$, $n=27$). Figure 6(b) contains a box plot showing the exponent of the GE for each group. The Wilcoxon rank-sum test showed that there was a significant difference between them ($p=9.2\times 10^{-4}$, $W=135$).

Fig. 6 Relation between shape parameters and exponent parameters. (a) Scatter plot of the shape parameter and the exponent parameter. The regression line is also shown. (b) Box plot for the case of grouping individuals with $m=0.01$ and $m>0.01$.

Discussion

This paper proposes a generalized distribution that includes exponential and power law distributions as special cases. By using this approach, a model can be created that better fits the observed data than the exponential or power law distributions, that is, a model with a higher likelihood. However, this distribution contains two parameters while the exponential and power law distributions only have one parameter. The model most suitable for the relevant data must be determined comparatively by using AICw that considers both likelihood and the number of parameters of the model.

The proposed model handles the intermediate distribution between the exponential and power law distributions. We defined the change rate $\gamma(l) = \beta l^{m-1}$ as shown in equation (15) to connect the exponential and power law distributions. However, the change rate has countless definitions. Therefore, it is necessary to

361 verify the validity and suitability of this definition in the future.

362 In this study, the proposed GE was applied to the walking data of pill bugs. A significant difference resulted
363 between the autocorrelation coefficients of Group 1, which followed an approximate power law distribution
364 with a shape parameter of $m=0.01$, and Group 2, which followed the other distributions with shape parameters
365 of $m>0.01$, as shown in Fig. 5(b). However, this difference is nontrivial. For example, if we randomly shuffle
366 the order of the time-series data as shown in Fig. 4 (a), the time dependence disappears, but the value of the
367 shape parameter is unchanged because the probability distribution, as shown in Fig. 4 (b), is not affected by
368 the shuffling.

369 As equation (15) shows, the smaller m is, the smaller the change rate becomes as the step length increases.
370 Conversely, the smaller the value of m , the higher is the probability that the step length will be further
371 elongated when the step length increases. This means that when m is small, the occurrence of a long step
372 length becomes more frequent.

373 As shown in Fig. 6(a), there is a negative correlation between the shape parameter and the exponent
374 parameter of the GE. Otherwise expressed, the exponent parameter tends to increase as the shape parameter
375 decreases. As equation (15) shows, when the exponent parameter increases, the overall change rate increases,
376 and the frequency of the short step length increases. Thus, individuals with small shape parameter values can
377 be said to have a relatively higher frequency of short and long step lengths than those with intermediate step
378 lengths.

379 Ross et al. [24] demonstrated that in human hunting behavior, the mode of exploration changes depending
380 on encounters with prey. In particular, they indicated that in response to encounters, hunters more tortuously
381 search areas of higher prey density and spend more of their search time in such areas; however, they adopt
382 more efficient unidirectional, inter-patch movements after failing to encounter prey soon enough. This type
383 of search behavior is called an area-restricted search (ARS) [24,25]. In ARS, searches with short travel
384 distances within patches are combined with searches with long travel distances between patches, so there
385 may be a tendency for relatively short and long straight distances to appear more frequently than intermediate

straight distances, that is, distributions with small shape parameters may be more likely to appear. In addition, the time dependence of the step length may appear because the search within a patch continues for a while after encountering the food, or conversely, the search between patches continues for a while when the food is not found. It may be that the power law distribution and time dependence tend to appear simultaneously when the search between different levels is combined hierarchically, such as intra-patch and inter-patch searches.

The result of the significant difference between the autocorrelation coefficients of Group 1 and Group 2 is consistent with the result of Wang et al. [13], who found time dependence in the time-series data of step length when the frequency distribution of the step length follows a power law distribution. However, it is unclear why the shape parameter is associated with the time dependence. The shape parameter of the distribution takes into account the history of the distance traveled in a straight line, that is, how long the same process has lasted when elongating the straight-line distance, and is related to the process within single straight-line behavior. Conversely, the time dependence of the time-series data of straight-line distance is associated with the relationship between multiple straight-line behaviors. In the future, why this correlation is observed between the two, must be clarified theoretically and experimentally.

It is also unclear why this association was observed in pill bugs and may have a completely different cause than in humans. The question whether this relationship also manifests in the migratory behavior of animals other than pill bugs requires further research.

Acknowledgments

This research was partially supported by the Center of Innovation Program of the Japan Science and Technology Agency (JST). This work was supported by JSPS KAKENHI, Grant Number JP21K12009.

References

1. Harris TH, Banigan EJ, Christian DA, Konradt C, Wojno EDT, Norose K, et al. Generalized Lévy walks and the role of chemokines in the migration of effector CD8+ T cells. *Nature*. 2012;486: 545–548. DOI: [10.1038/nature11098](https://doi.org/10.1038/nature11098).
2. Ariel G, Rabani A, Benisty S, Partridge JD, Harshey RM, Be’Er A. Swarming bacteria migrate by Lévy walk. *Nat Commun*. 2015;6: 8396. DOI: [10.1038/ncomms9396](https://doi.org/10.1038/ncomms9396).
3. Shokaku T, Moriyama T, Murakami H, Shinohara S, Manome N, Morioka K. Development of an automatic turntable-type multiple T-maze device and observation of pill bug behavior. *Rev Sci Instrum*. 2020;91: 104104. DOI: [10.1063/5.0009531](https://doi.org/10.1063/5.0009531).
4. Humphries NE, Queiroz N, Dyer JRM, Pade NG, Musyl MK, Schaefer KM, et al. The environmental context explains the Lévy and Brownian movement patterns of marine predators. *Nature*. 2010;465: 1066–1069. DOI: [10.1038/nature09116](https://doi.org/10.1038/nature09116).
5. Humphries NE, Weimerskirch H, Queiroz N, Southall EJ, Sims DW. Foraging success of biological Lévy flights recorded in situ. *Proc Natl Acad Sci U S A*. 2012;109: 7169–7174. DOI: [10.1073/pnas.1121201109](https://doi.org/10.1073/pnas.1121201109).
6. Raichlen DA, Wood BM, Gordon AD, Mabulla AZP, Marlowe FW, Pontzer H. Evidence of Lévy walk foraging patterns in human hunter–gatherers. *Proc Natl Acad Sci U S A*. 2014;111: 728–733. DOI: [10.1073/pnas.1318616111](https://doi.org/10.1073/pnas.1318616111).
7. Sims DW, Southall EJ, Humphries NE, Hays GC, Bradshaw CJA, Pitchford JW, et al. Scaling laws of marine predator search behavior *Nature*. 2008;451: 1098–1102. DOI: [10.1038/nature06518](https://doi.org/10.1038/nature06518).
8. Jansen VA, Mashanova A, Petrovskii S. Comment on "Lévy walks evolve through interaction between movement and environmental complexity". *Science*. 2012 Feb 24;335(6071): 918; author reply: 918. doi: [10.1126/science.1215747](https://doi.org/10.1126/science.1215747), PMID: [22362991](https://pubmed.ncbi.nlm.nih.gov/22362991/).

9. White EP, Enquist BJ, Green JL. On estimating the exponent of power-law frequency distributions. Ecology. 2008 Apr;89(4): 905-912. doi: [10.1890/07-1288.1](https://doi.org/10.1890/07-1288.1). Erratum in: Ecology. 2008 Oct;89(4): 905–912. DOI: [10.1890/07-1288.1](https://doi.org/10.1890/07-1288.1), PMID: [18481513](https://pubmed.ncbi.nlm.nih.gov/18481513/).
10. Clauset A, Shalizi CR, Newman MEJ. Power-Law Distributions in Empirical Data. In SIAM Rev. 2009;51(4): 661–703. DOI: [10.1137/070710111](https://doi.org/10.1137/070710111).
11. Edwards AM, Phillips RA, Watkins NW, Freeman MP, Murphy EJ, Afanasyev V, et al. Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer. Nature. 2007 Oct 25;449(7165): 1044-1048. doi: [10.1038/nature06199](https://doi.org/10.1038/nature06199), PMID: [17960243](https://pubmed.ncbi.nlm.nih.gov/17960243/).
12. Shinohara S, Okamoto H, Manome N, Gunji PY, Nakajima Y, Moriyama M, et al. Simulation of foraging behavior using a decision-making agent with Bayesian and inverse Bayesian inference: Lévy and Brownian walk-like search patterns derived from differences in prey distributions. bioRxiv. 2021. doi: [10.1101/2021.06.08.447450](https://doi.org/10.1101/2021.06.08.447450).
13. Wang XW, Han XP, Wang BH. Correlations and scaling laws for human mobility. PLOS ONE. 2014;9: e84954. DOI: [10.1371/journal.pone.0084954](https://doi.org/10.1371/journal.pone.0084954).
14. Rhee I, Shin M, Hong S, Lee K, Kim SJ, Chong S. On the Levy-walk nature of human mobility. IEEE ACM Trans Netw. 2011;19: 630–643. DOI: [10.1109/TNET.2011.2120618](https://doi.org/10.1109/TNET.2011.2120618).
15. Zhao K, Musolesi M, Hui P, Rao W, Tarkoma S. Explaining the power-law distribution of human mobility through transportationmodality decomposition. Sci Rep.: 9136. Sci Rep. 2015;5: 9136. <https://doi.org/10.1038/srep09136>.
16. Weibull W. A Statistical Distribution Function Of Wide Applicability. J Appl Mech. 1951;vol. 18: 293–297. doi:[10.1115/1.4010337](https://doi.org/10.1115/1.4010337).
17. Mingw. Available: <http://www.mingw.org/>. [Accessed 23 January 2021].
18. Qt. Available: <https://www.qt.io/>. [Accessed 23 January 2021].
19. The Comprehensive R Archive Network. Available: [Cran.r-project.org/](http://cran.r-project.org/). (Accessed 15 July 2020).

- 457 20. Moriyama T, Migita M, Mitsuishi M. Self-corrective behavior for turn alternation in pill bugs
458 (*Armadillidium vulgare*). Behav Processes. 2016;122: 98-103. DOI: [10.1016/j.beproc.2015.11.016](https://doi.org/10.1016/j.beproc.2015.11.016).
- 459 21. Hughes RN. Mechanisms for turn alternation in woodlice (Porcellio scaber): The role of bilaterally
460 asymmetrical leg movements. Anim Learn Behav. 1985;13: 253-260. DOI: [10.3758/BF03200018](https://doi.org/10.3758/BF03200018).
- 461 22. Jander R. Ecological aspects of spatial orientation. Annu Rev Ecol Syst. 1975;6: 171-188. DOI:
462 [10.1146/annurev.es.06.110175.001131](https://doi.org/10.1146/annurev.es.06.110175.001131).
- 463 23. Moriyama T. Decision-making and turn alternation in pill bugs (*Armadillidium vulgare*). Int J Comp
464 Psychol. 1999;12: 153-170.
- 465 24. Ross CT, Winterhalder B. Evidence for encounter-conditional, area-restricted search in a preliminary
466 study of Colombian blowgun hunters. PLOS ONE. 2018;13: e0207633. DOI:
467 [10.1371/journal.pone.0207633](https://doi.org/10.1371/journal.pone.0207633).
- 468 25. Hills TT, Kalff C, Wiener JM. Adaptive Lévy Processes and Area-Restricted Search in Human
469 Foraging. PLOS ONE. 2013;8(4): e60488. <https://doi.org/10.1371/journal.pone.0060488>.
- 470

471

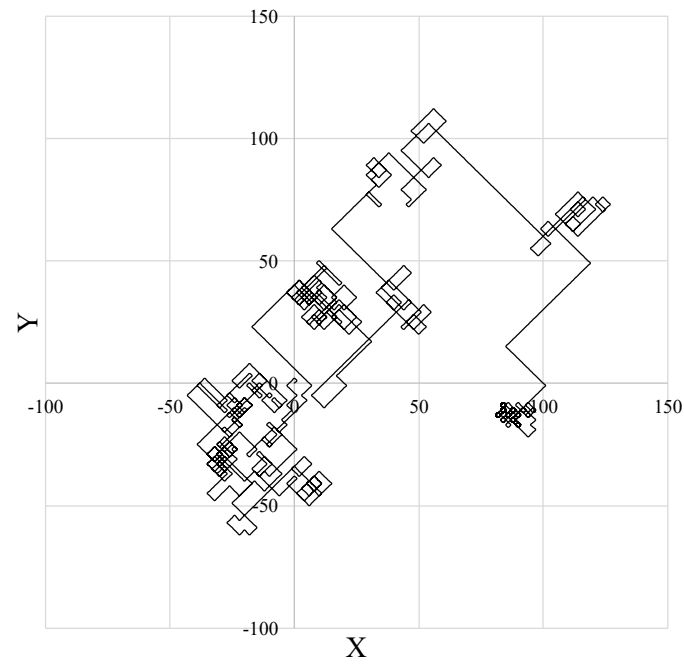


Fig. 1 Trajectory of an individual's gait.

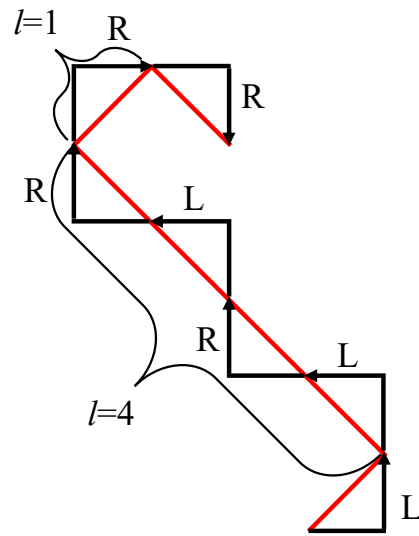


Fig. 2 Sample calculation of step length l . The black polygonal line with the arrow represents a turn; L represents a left turn, R a right turn. The red line represents an approximate linear movement. The L-R-L-R pattern shown in this figure represents linear movement with a step length of 4 ($l=4$).

473
474

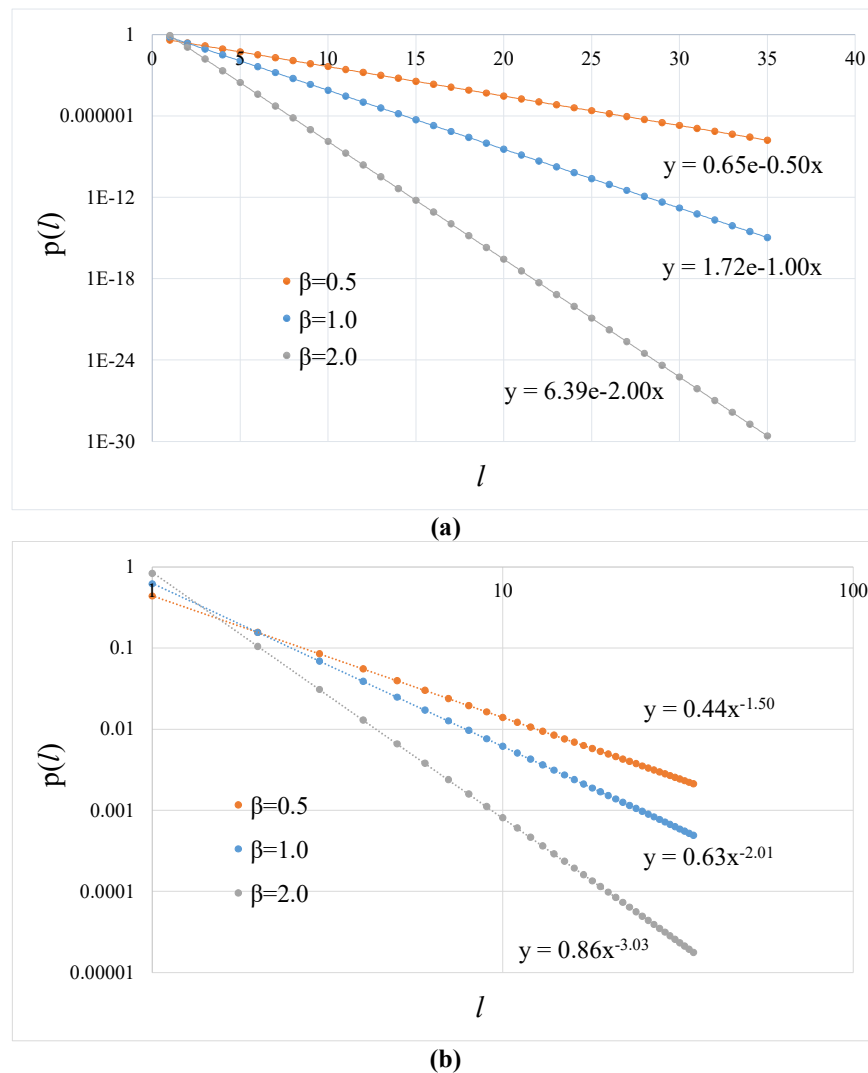


Fig. 3 Examples of GE. (a) For $m=1.0$, the GEs are shown for $\beta=0.5$, 1.0 and 2.0 . The vertical axis is shown logarithmically. The exponential approximation curves are also shown. (b) For $m=0.01$, the GE for $\beta=0.5$, 1.0 and 2.0 are shown. Both axes are shown logarithmically. The power approximation curves are also shown.

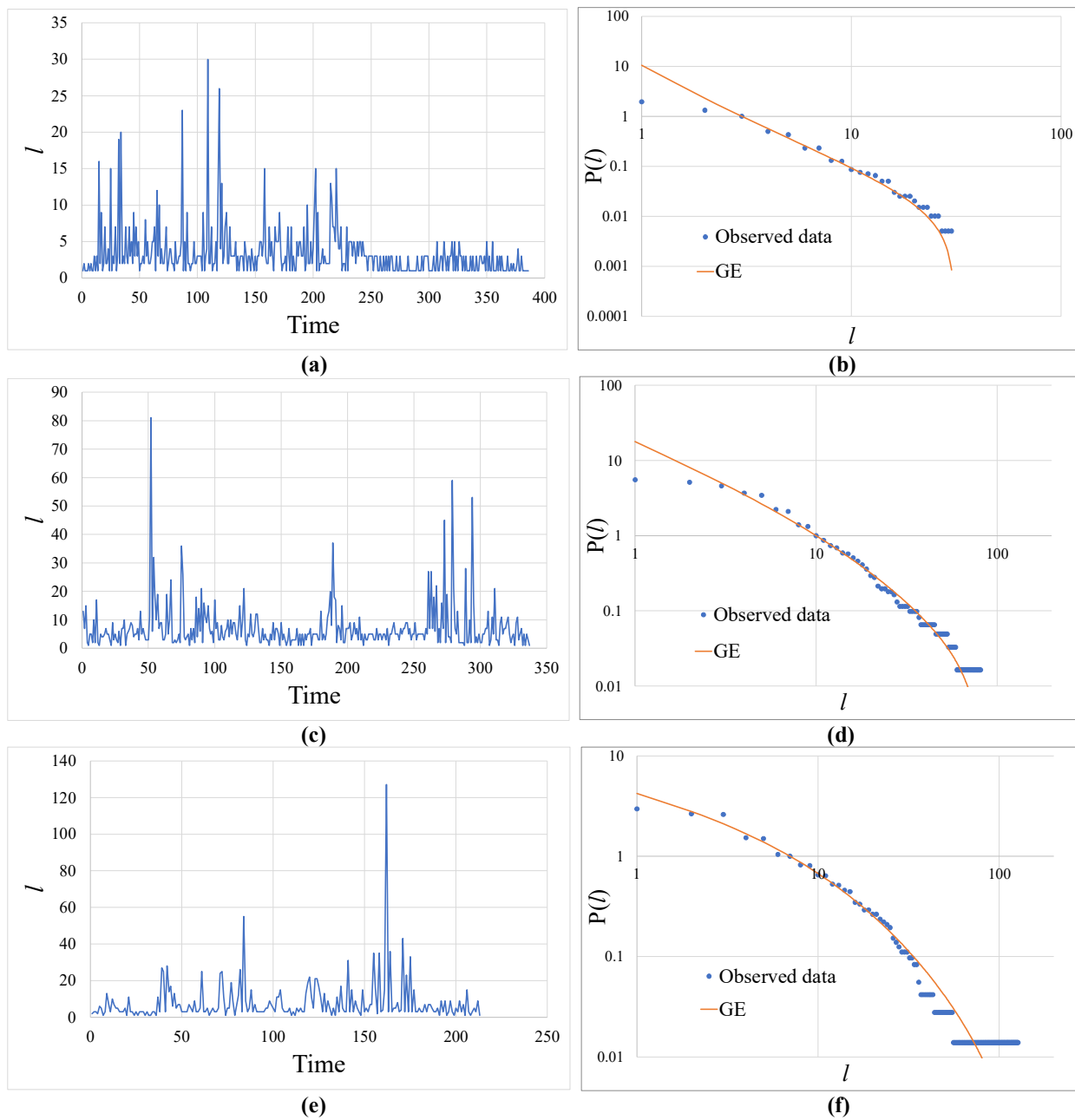
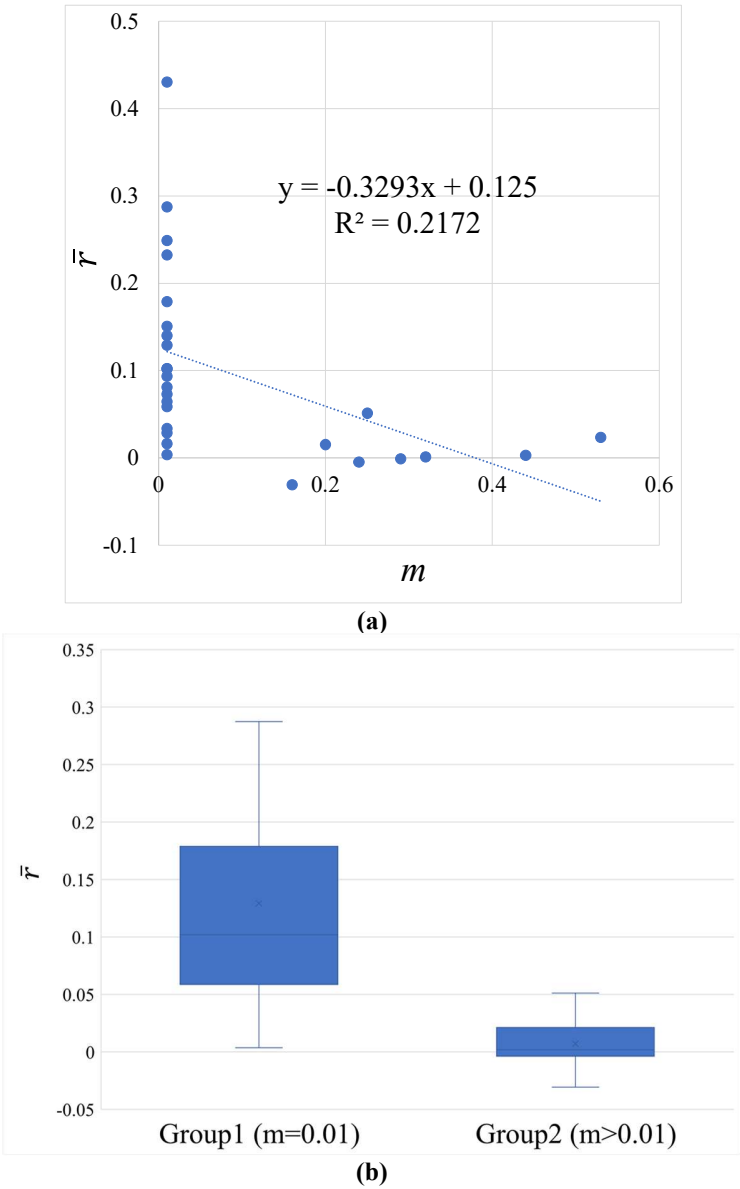


Fig. 4 Step length data for three individuals. (a) Step length time series for individuals of subject 4. (b) CCDF of subject 4. $\hat{l}_{\min} = 3, \hat{l}_{\max} = 30, m = 0.01, \beta = 1.68$ (c) Step length time series for individuals of subject 15. (d) CCDF of subject 15. $\hat{l}_{\min} = 10, \hat{l}_{\max} = 81, m = 0.24, \beta = 0.82$ (e) Step length time series for individuals of subject 14. (f) CCDF of subject 14. $\hat{l}_{\min} = 7, \hat{l}_{\max} = 127, m = 0.44, \beta = 0.42$

476



477

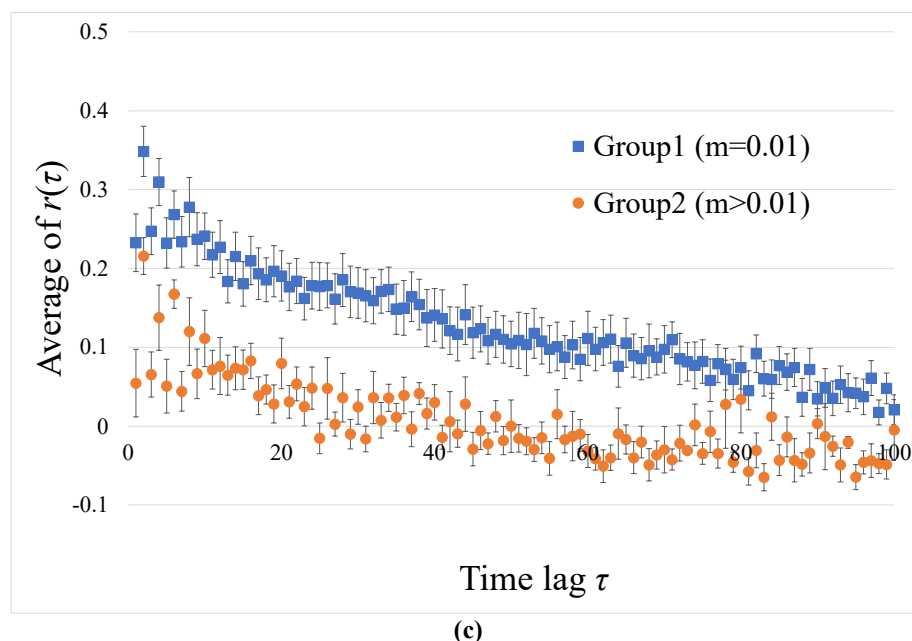


Fig. 5 Relationship between shape parameters and autocorrelation coefficients. (a) Scatter plots between shape parameters and autocorrelation coefficients. The regression line is also shown. (b) Box plot for the case of grouping individuals with $m=0.01$ and $m>0.01$. (c) Correlogram for each group. The vertical axis represents the mean value of the autocorrelation coefficient of the individuals in the group. The error bars represent the standard errors.

478

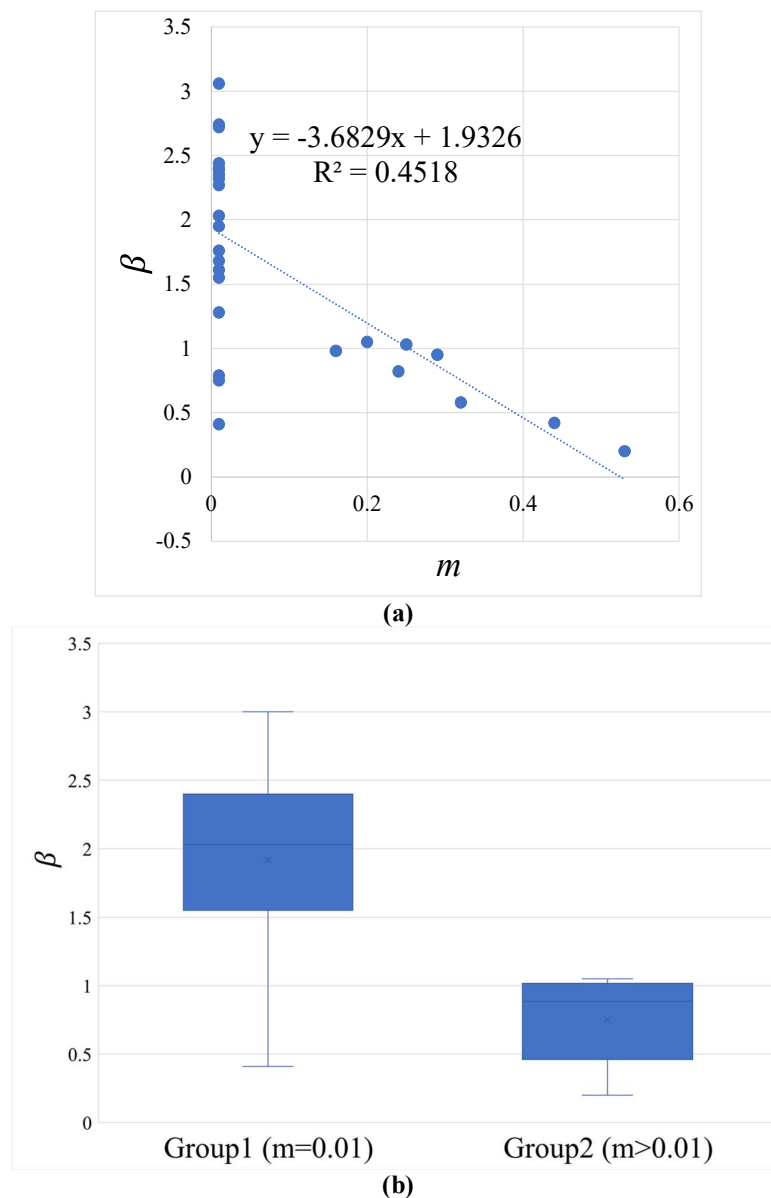


Fig. 6 Relation between shape parameters and exponent parameters. (a) Scatter plot of the shape parameter and the exponent parameter. The regression line is also shown. (b) Box plot for the case of grouping individuals with $m=0.01$ and $m>0.01$.

479
480