AMATH 567

Applied Complex Variables

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Homework Assignment 2

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1. Evaluate $\oint_C f(z)dz$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:

   (a) e^{iz}

   (b) e^{z^2}

   (c) $\frac{1}{z-1/2}$

   (d) $\frac{1}{z^2-4}$

   (e) $\frac{1}{2z^2+1}$

   (f) $\sqrt{z-4}$, $0 \leq \arg(z-4) < 2\pi$

   Solution A circle with center z_0 and radius δ can be characterized by $z_0 + \delta e^{it}$ with $0 \leq t < 2\pi$, thus:

   (a) Since $e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$ and $\oint_C az^n dz = 0$, $\forall n \in \mathbb{N}$ with $a \in \mathbb{C}$ (see notes), and since $\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$ is convergent for $|z| < \infty$,
   $$\oint_C e^{iz} dz = \oint_C \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \oint_C \frac{(iz)^n}{n!} = 0$$
   Note here that the sum and integral sign can be switched. This since the series is convergent and a complex integral can be defined using line integrals over real numbers.

   (b) $\oint_C e^{z^2} dz = \oint_C \sum_{n=0}^{\infty} \frac{(z)^{2n}}{n!} dz = \sum_{n=0}^{\infty} \oint_C \frac{(z)^{2n}}{n!} dz = 0$

(c) $\frac{1}{z-1/2}$ is analytical everywhere except for $z_0 = 1/2$:

$$\lim_{\Delta z \to 0} \frac{\frac{1}{z+\Delta z-1/2} - \frac{1}{z-1/2}}{\Delta z} = \lim_{\Delta z \to 0} \frac{-\Delta z}{(z-1/2)(z-\Delta z-1/2)\Delta z} = \frac{-1}{(z-1/2)^2}$$

and since $|z_0|= 1/2 < 1$, we know that $\frac{1}{z-1/2}$ is not analytical everywhere inside C. Using the method learned in Lecture 6 we know that $\oint_C \frac{1}{z-1/2} = \oint_{C_1} \frac{1}{z-1/2}$ with C_1 a circle with center $1/2$ and radius $\delta < 1/2$ with characterisation $1/2 + \delta e^{it}$ and $0 \leq t < 2\pi$. Thus

$$\oint_C \frac{1}{z-1/2} = \int_0^{2\pi} \frac{1}{1/2 + \delta eit - 1/2} i\delta e^{it} dt = i\int_0^{2\pi} dt = 2\pi i$$

(d) Note that $\frac{1}{z^2-4}$ is analytical everywhere except for -2 and 2:

$$\lim_{\Delta z \to 0} \frac{\frac{1}{(z+\Delta z)^2-4} - \frac{1}{z^2-4}}{\Delta z} = \lim_{\Delta z \to 0} \frac{-2\Delta z z - \Delta z^2}{((z+\Delta z)^2-4)(z^2-4)\Delta z} = \frac{-2z}{(z^2-4)^2}$$

Since $\frac{1}{z+2}$ and $\frac{1}{z-2}$ are analytical except for points -2 and 2, which are outside of C, we know that $\oint_C \frac{1}{z^2-4} = 0$ (Cauchy's theorem).

(e) Note that $\frac{1}{2z^2+1} = \frac{1}{2}\left(\frac{1}{z^2+1/2}\right) = \frac{1}{2}\frac{1}{(z-i/\sqrt{2})(z+i/\sqrt{2})}$. This function is thus analytical everywhere except for points $z_0 = \frac{i}{\sqrt{2}}$ and $z_1 = \frac{-i}{\sqrt{2}}$ (it can be proven just like in exercise (d)). Since $|z_0|= |z_1|= \frac{1}{\sqrt{2}} < 1$, they are on the inside of C. Using the technique in lecture 6, we define C_0 as a circle around z_0 with radius $\delta_0 < 1 - \frac{\sqrt{2}}{2}$ and C_1 as a circle around z_1 with radius $\delta_1 < 1 - \frac{\sqrt{2}}{2}$. Note that $\frac{1}{2z^2+1} = \frac{i}{\sqrt{2}}\left(\frac{1}{z-\frac{i}{\sqrt{2}}} - \frac{1}{z+\frac{i}{\sqrt{2}}}\right)$, thus:

$$\begin{aligned}\oint_C \frac{dz}{2z^2+1} &= \frac{\sqrt{2}}{4i}\left(\oint_{C_0} \frac{dz}{z-\frac{i}{\sqrt{2}}} - \oint_{C_0} \frac{dz}{z+\frac{i}{\sqrt{2}}} + \oint_{C_1} \frac{dz}{z-\frac{i}{\sqrt{2}}} - \oint_{C_1} \frac{dz}{z+\frac{i}{\sqrt{2}}}\right) \\ &= \frac{\sqrt{2}}{4i}\left(\oint_{C_0} \frac{dz}{z-\frac{i}{\sqrt{2}}} - \oint_{C_1} \frac{dz}{z+\frac{i}{\sqrt{2}}}\right) = \frac{\sqrt{2}}{4i}\left(\int_0^{2\pi} i\frac{\delta_0 e^{it}}{\delta_0 e^{it}} dt - \int_0^{2} i\frac{i\delta_1 e^{it}}{\delta_1 e^{it}} dt\right) \\ &= 0\end{aligned}$$

(f) Note that $\sqrt{z-4}$, with $0 \leq \arg(z-4) < 2\pi$ is analytical inside C:

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{\sqrt{z+\Delta z-4} - \sqrt{z-4}}{\Delta z} &= \lim_{\Delta z \to 0} \frac{\sqrt{z+\Delta z-4}^2 - \sqrt{z-4}^2}{\Delta z(\sqrt{z+\Delta z-4} + \sqrt{z-4})} \\ &= \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z(\sqrt{z+\Delta z-4} + \sqrt{z-4})} = \frac{1}{2\sqrt{z-4}}.\end{aligned}$$

Note that it is not analytical on -2 and 2. Thus, Cauchy's theorem states that

$$\oint_C \sqrt{z-4} dz = 0$$

2. We wish to evaluate the integral

$$\int_0^\infty e^{ix^2}$$

Consider the contour

$$I_R = \oint_{C_{(R)}} e^{iz^2} dz$$

Where $C_{(R)}$ is the closed circular sector in the upper half plane with boundary points $0, R$, and $Re^{i\pi/4}$. Show that $I_R = 0$ and that

$$\lim_{R\to\infty} \int_{C_{1(R)}} e^{iz^2} dz = 0,$$

where $C_{1(R)}$ is the line integral along the circular sector from R to $Re^{i\pi/4}$. Then, breaking up the contour $C_{(R)}$ into three component parts, deduce

$$\lim_{R\to\infty} \left(\int_0^R e^{ix^2} dx - e^{i\pi/4} \int_0^R e^{-r^2} dr \right) = 0,$$

and from the well-known result of real integration:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

deduce that $I = e^{i\pi/4}\sqrt{\pi}/2$

Solution

We know that $e^{iz^2} = \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{n!}$ is a convergent series for $|z| < \infty$. Due to e^{iz^2} thus being analytical and Cauchy's theorem, we know that

$$I_R = 0$$

We can characterize $C_{1(R)}$ by Re^{it} with $0 \le t \le \pi/4$. This is possible due to e^{iz^2} being analytical and the integrals are thus path independent. We now have,

$$\begin{aligned}
0 \le |\int_{C_{1(R)}} e^{iz^2} dz| &= |\int_0^{\pi/4} e^{i(Re^{it})^2} iRe^{it} dt| \\
&\le \int_0^{\pi/4} |Re^{iR^2\cos(2t) - R^2\sin(2t)} ie^{it}| dt = \int_0^{\pi/4} Re^{-R^2\sin(2t)} dt \\
&\overset{-\sin(t)\le -2t/\pi}{\le} \int_0^{\pi/4} Re^{-2R^2 t} dt = \frac{1}{2R} - \frac{e^{-R^2\pi/2}}{2R}.
\end{aligned}$$

This last line is true since $0 \le t \le \pi/4 \Rightarrow 0 \le 2t \le \pi/2$. We thus know that

$$0 \le \lim_{R\to\infty} |\int_{C_{1(R)}} e^{iz^2} dz| \le \lim_{R\to\infty} \frac{1}{2R} - \frac{e^{-R^2\pi/2}}{2R} = 0$$

In conclusion $\lim_{R\to\infty}|\int_{C_{1(R)}} e^{iz^2}dz|= 0$, thus

$$\lim_{R\to\infty}\int_{C_{1(R)}} e^{iz^2}dz = 0$$

Now, because of the path Independence, we can break up $C_(R)$ into three component parts, $C_{1(R)}$, $C_{2(R)}$ =$\{(R-t)e^{i\pi/4}|0 \leq t \leq R\}$ and $C_{3(R)} = \{t|0 \leq t \leq R\}$. Note that in the second part the $R-t$ is needed in order for the correct direction. We then have that

$$\oint_{C_{(R)}} = \int_{C_{1(R)}} + \int_{C_{2(R)}} + \int_{C_{3(R)}}$$

Thus:

$$\begin{aligned}
\lim_{R\to\infty} I_R &= \lim_{R\to\infty}\left(\int_{C_{1(R)}} e^{iz^2}dz + \int_{C_{2(R)}} e^{iz^2}dz + \int_{C_{3(R)}} e^{iz^2}dz\right)\\
&= \lim_{R\to\infty}\left(-e^{i\pi/4}\int_0^R e^{i(R-t)^2e^{i\pi/2}}dt + \int_0^R e^{it^2}dt\right)\\
&\overset{r=R-t}{=} \lim_{R\to\infty}\left(\int_0^R e^{it^2}dt + e^{i\pi/4}\int_R^0 e^{i^2r^2}dr\right)\\
&= \lim_{R\to\infty}\left(\int_0^R e^{it^2}dt - e^{i\pi/4}\int_0^R e^{-r^2}dr\right)
\end{aligned}$$

In conclusion, since $I_R = 0$:

$$\lim_{R\to\infty}\left(\int_0^R e^{it}dt - e^{i\pi/4}\int_0^R e^{-r^2}dr\right) = 0$$

and furthermore

$$\int_0 {}^{it^2}dt = e^{i\pi/4}\int_0^\infty e^{-r^2}dr = e^{i\pi/4}\frac{\sqrt{\pi}}{2}$$

3. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}.$$

Show how to evaluate this integral by considering

$$\oint_{C_{(R)}} \frac{dz}{z^2+1},$$

where $C_{(R)}$ is the closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x axis.

Solution

Since R will be going to ∞, we will assume that $R > 2$. Let $C_{+(R)}$ be the semicircle which is described above. This can be characterised by Re^{it} with $0 \leq t < \pi$. We then have,

$$\lim_{R\to\infty} \int_{C_{+(R)}} \frac{dz}{z^2+1} = \lim_{R\to\infty} \int_0^{\pi} \frac{iRe^{it}}{R^2e^{2it}+1} dt = \int_0^{\pi} \lim_{R\to\infty} \frac{iRe^{it}}{R^2e^{2it}+1} dt = \int_0^{\pi} 0dt = 0$$

Now we Evaluate $\oint_{C_{(R)}} \frac{dz}{z^2+1}$. Note that $\frac{1}{z^2+1}$ is analytical everywhere except for i and $-i$, with i in $C_{(R)}$ (since $R > 2$). We define C_0 as the circle with center i and radius 1,

$$\oint_{C_{(R)}} \frac{dz}{z^2+1} = \frac{1}{2i} \left(\oint_{C_1} \frac{dz}{z-i} - \oint_{C_1} \frac{dz}{z+i} \right) = \frac{1}{2i} \oint_{C_1} \frac{dz}{z-i} = \frac{1}{2i} \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = \pi$$

Define $l_{(R)}$ as the straight line from $(-R, 0)$ to $(R, 0)$, we then know that

$$\oint_{C_{(R)}} = \int_{C_{+(R)}} + \int_{l_{(R)}}$$

In conclusion:

$$\pi = \lim_{R\to\infty} \oint_{C_{(R)}} \frac{dz}{z^2+1} = \lim_{R\to\infty} \int_{C_{+(R)}} \frac{dz}{z^2+1} + \int_{l_{(R)}} \frac{dz}{z^2+1} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

4. Let

$$f(z) = e^{\frac{t}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n.$$

Show from the definition of Laurent series and using properties of integration that

$$\begin{aligned} J_n(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin(\theta))} d\theta \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos(n\theta - t\sin(\theta)) d\theta. \end{aligned}$$

The functions $J_n(t)$ are called Bessel functions, which are well-known special functions in mathematics and physics.

Solution

From the definition of Laurent series around 0, we know that

$$J_n(t) = \frac{1}{2\pi i}\oint_C \frac{e^{\frac{t}{2}(z-1/z)}}{z^{n+1}} dz$$

With C a curve around 0. Since 0 is the only non analytic point, we can chose C to be $e^{i\theta}$ with $-\pi \leq \theta < \pi$ (method in lecture 6). Note that $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. Substituting this, we get:

$$\begin{aligned} J_n(t) &= \frac{1}{2\pi i}\oint_C \frac{e^{\frac{t}{2}(e^{i\theta}-e^{-i\theta})}}{e^{(n+1)i\theta}} ie^{i\theta} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\frac{t}{2}(e^{i\theta}-e^{-i\theta})-ni\theta} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\frac{(e^{it}-e^{-it})}{2i})} d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin(\theta))} d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(-(n\theta - t\sin(\theta))) + i\sin(-(n\theta - t\sin(\theta))) d\theta \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} \cos(n\theta - t\sin(\theta)) d\theta - i\int_{-\pi}^{\pi} \sin(n\theta - t\sin(\theta)) d\theta\right) \end{aligned}$$

Say that $f(\theta) = \cos(n\theta - t\sin(\theta))$, it is clear that $f(-\theta) = \cos(-n\theta - t\sin(-\theta)) = \cos(-(n\theta - t\sin(\theta))) = f(\theta)$. $f(\theta)$ is an even function, thus $\int_{-a}^{a} f(\theta)d\theta = 2\int_0^a f(\theta)d\theta$.
Say that $g(\theta) = \sin(n\theta - t\sin(\theta))$, it is clear that $g(-\theta) = -g(\theta)$. $g(\theta)$ is an uneven function, thus $\int_{-a}^{a} g(\theta)d\theta = 0$. In conclusion,

$$\begin{aligned} J_n(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin(\theta))} d\theta \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} \cos(n\theta - t\sin(\theta)) d\theta - i\int_{-\pi}^{\pi} \sin(n\theta - t\sin(\theta)) d\theta\right) \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos(n\theta - t\sin(\theta)) d\theta \end{aligned}$$