AMATH 567
Applied Complex Variables
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Homework 5

Due: Wednesday, November 9, 2022
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Problem 1.
Evaluate the integrals

$$\frac{1}{2\pi i}\oint_C f(z)dz$$

where C is the unit circle centered at the origin with $f(z)$ gives below. Do these problems by both

(i) enclosing the singular points inside C

(ii) enclosing the singular points outside C (by including the point at infinity)

Show that you obtain the same result in both cases.

(a) $\dfrac{z^2+1}{z^2-a^2}$, $a^2 < 1$.

(b) $\dfrac{z^2+1}{z^3}$.

(c) $z^2 e^{-1/z}$

Solution:

(a.i) We have 2 poles which are visibly inside C, since $a^2 < 1$ ($|a| < 1$): a and $-a$. These are clearly single poles, using Residue theorem, we get

$$\frac{1}{2\pi i}\oint_C f(z)dz = \text{Res}(a) + \text{Res}(-a)$$

Now we find these residues:

$$\text{Res}(a) = \lim_{z \to a} (z-a)\frac{z^2+1}{(z-a)(z+a)} = \frac{a^2+1}{2a}$$

$$\text{Res}(-a) = \lim_{z \to -a} (z+a)\frac{z^2+1}{(z-a)(z+a)} = -\frac{a^2+1}{2a}$$

Thus,

$$\frac{1}{2\pi i}\oint_C f(z)dz = 0$$

(a.ii) There are no immediate singularities visible outside of C, therefore we look at if $z = \infty$ is a singularity. We do this by substituting z by $\frac{1}{t}$ and looking if $t = 0$ is a singularity:

$$\frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-a^2} = \frac{\frac{t^2+1}{t^2}}{\frac{1-(at)^2}{t^2}} = \frac{t^2+1}{1-(at)^2}$$

Therefore

$$\frac{1}{2\pi i}\oint_C f(z)dz = -\frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_C f(\frac{1}{t})\frac{dt}{t^2} = \frac{1}{2\pi i}\oint_C \frac{t^2+1}{t^2(1-(at)^2)}dt$$

Here we have a singularity at $t = 0, \frac{1}{a}, -\frac{1}{a}$, only the first point is in C, thus we find it's residue (it's a double pole):

$$\text{Res}(0) = \lim_{t \to 0} \frac{d}{dt}\frac{t^2+1}{1-(at)^2} = \lim_{t \to 0} \frac{2(a^2+1)t}{(1-(at)^2)^2} = 0$$

In conclusion

$$\frac{1}{2\pi i}\oint_C f(z)dz = -\frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_C f(\frac{1}{t})\frac{dt}{t^2} = \text{Res}(0) = 0$$

(b.i) We have a singularity of order 3 in $z = 0$. Finding the residue is, however, rather easy since

$$\frac{z^2+1}{z^3} = \frac{1}{z} + \frac{1}{z^3}$$

The residue is the coefficient of z^{-1}, thus in this case 1. In conclusion:

$$\frac{1}{2\pi i}\oint_C f(z)dz = \text{Res}(0) = 1$$

(b.ii) Once again, we substitute $z = \frac{1}{t}$ and get

$$\frac{\frac{1}{t^2}+1}{\frac{1}{t^3}} = \frac{\frac{1+t^2}{t^2}}{\frac{1}{t^3}} = \frac{\frac{1}{t^2}+1}{\frac{1}{t^3}} = t + t^3$$

Therefore

$$\frac{1}{2\pi i}\oint_C f(z)dz = -\frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_C f(\frac{1}{t})\frac{dt}{t^2} = \frac{1}{2\pi i}\oint_C \frac{1+t^2}{t}dt$$

Here we have a simple pole at $t=0$. Since

$$\frac{1+t^2}{t} = \frac{1}{t} + t,$$

the residue is

$$\text{Res}(0) = 1.$$

In conclusion

$$\frac{1}{2\pi i}\oint_C f(z)dz = \text{Res}(0) = 1.$$

(c.i) We have one clear pole inside C: $z=0$. Taylor expanding $e^{-1/z} = \sum_{n=0}^{\infty}\frac{(-z)^{-n}}{n!} = 1 - \frac{1}{z} + \frac{1}{2z^2} - \frac{1}{6z^3} + \frac{1}{24z^4} + ...$, thus:

$$z^2 e^{-1/z} = \sum_{n=0}^{\infty}\frac{(-z)^{-n+2}}{n!}$$

Thus, the residue can be found at $n=3$, and this is

$$\text{Res}(0) = \frac{-1}{6}$$

In conclusion

$$\frac{1}{2\pi i}\oint_C f(z)dz = \text{Res}(0) = \frac{-1}{6}$$

(c.ii) Once again, we substitute $z = \frac{1}{t}$ and get

$$\frac{e^{-t}}{t^2}$$

$t=0$ is clearly a singularity. We now want to find

$$\frac{1}{2\pi i}\oint_C \frac{e^{-t}}{t^2}\frac{1}{t^2}dt.$$

Calculating the Taylor polynomial of e^{-t} around $t=0$, gives

$$\frac{1}{2\pi i}\oint_C \frac{e^{-t}}{t^4}dt = \frac{1}{2\pi i}\oint_C \frac{\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}}{t^4}dt = \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty}\frac{(-t)^{n-4}}{n!}dt = \frac{-1}{6}$$

The last equation holds since the residue of $\sum_{n=0}^{\infty}\frac{(-t)^{n-4}}{n!}$ in $t=0$ is clearly the coefficient of $\frac{1}{z}$, which is when $n=3$, thus the coefficient is $\frac{-1}{6}$.
In conclusion:

$$\frac{1}{2\pi i}\oint_C \frac{e^{-t}}{t^4}dt = \text{Res}(0) = \frac{-1}{6}$$

Problem 2. Find the Fourier transform of

$$f(t) = \begin{cases} 1 & \text{for } -a < t < a \\ 0 & \text{otherwise} \end{cases}$$

Then, do the inverse transform using techniques of contour integration, e.g. Jordan's lemma, principal values, etc.

Solution:

The Fourier transform of $f(t)$ is,

$$\hat{f}(\lambda) = \mathcal{F}(f(t))(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} f(t)dt = \int_{-a}^{a} e^{i\lambda t}dt = \frac{2}{\lambda}\sin(\lambda a).$$

The Inverse Fourier transform is calculated as

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}\hat{f}(\lambda)d\lambda &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda t}\sin(\lambda a)}{\lambda}d\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i(a-t)\lambda} - e^{-i(a+t)\lambda}}{\lambda}d\lambda \\ &= \frac{1}{2\pi i}\left(P\int_{-\infty}^{\infty}\frac{e^{i(a-t)\lambda}}{\lambda}d\lambda - P\int_{-\infty}^{\infty}\frac{e^{-i(a+t)\lambda}}{\lambda}d\lambda\right).\end{aligned}$$

Note that for $g(\lambda) = \frac{e^{im\lambda}}{\lambda}$, for $m \in \mathbb{R}$, $\lambda = 0$ is a simple pole and

$$\text{Res}(g, \lambda = 0) = \lim_{\lambda \to 0} e^{im\lambda} = 1.$$

Thus, if C_{R+} is the union between the the real line from $-R$ to R and the upper half of the circle with center 0 and radius R and C_{R-} is the union between the the real line from $-R$ to R and the lower half of the circle with center 0 and radius R, we have that, since 0 is on the real axis and thus on both C_{R+} and C_{R-},

$$\oint_{C_{R+}}\frac{e^{im\lambda}}{\lambda}d\lambda = \pi i\text{Res}(0) = \pi i = \pi i\text{Res}(0) = \oint_{C_{R-}}\frac{e^{im\lambda}}{\lambda}d\lambda.$$

We would now like to use Jordan's lemma in Lecture 12 (particularly page 1 and 2). This, however, depends on the sign of m and thus, in the original problem, on the sign of $a - t$ and $a + t$. Note first that $|\frac{1}{\lambda}|$ goes to 0 on C_{R+} and C_{R-} as $R \to \infty$:

$$\lim_{R\to\infty}\left|\frac{1}{\lambda}\right| = \lim_{R\to\infty}\left|\frac{1}{Re^{it}}\right| = \lim_{R\to\infty}\frac{1}{R} = 0$$

(here $t \in [0, \pi]$ or $t \in [\pi, 2\pi]$). We now look at the three cases:

t > a If $t > a$, then $a + t > 0$ and $t - a > 0$. Using Lecture 12 page 2, we get:

$$P\int_{-\infty}^{\infty}\frac{e^{-i(t-a)\lambda}}{\lambda}d\lambda = -\oint_{C_{R-}}\frac{e^{-i(t-a)z}}{z}dz = -\pi i,$$

and

$$P\int_{-\infty}^{\infty}\frac{e^{-i(a+t)\lambda}}{\lambda}d\lambda = -\oint_{C_{R-}}\frac{e^{-i(a+t)z}}{z}dz = -\pi i,$$

Thus

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}\hat{f}(\lambda)d\lambda = 0, \qquad \text{for } t > a$$

$\boldsymbol{t < -a}$ If $t < -a$, then $-(a+t) > 0$ and $-(t-a) > 0$. Thus,

$$P\int_{-\infty}^{\infty} \frac{e^{-i(t-a)\lambda}}{\lambda} d\lambda = \oint_{C_{R+}} \frac{e^{-i(t-a)z}}{z} dz = \pi i,$$

and

$$P\int_{-\infty}^{\infty} \frac{e^{-i(a+t)\lambda}}{\lambda} d\lambda = \oint_{C_{R+}} \frac{e^{-i(a+t)z}}{z} dz = \pi i,$$

Thus

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}\hat{f}(\lambda) d\lambda = 0, \qquad \text{for } t < -a$$

$\boldsymbol{-a < t < a}$ If $-a < t < a$, then $a+t > 0$ and $-(t-a) > 0$. Thus,

$$P\int_{-\infty}^{\infty} \frac{e^{-i(t-a)\lambda}}{\lambda} d\lambda = \oint_{C_{R+}} \frac{e^{-i(t-a)z}}{z} dz = \pi i,$$

and

$$P\int_{-\infty}^{\infty} \frac{e^{-i(a+t)\lambda}}{\lambda} d\lambda = -\oint_{C_{R-}} \frac{e^{-i(a+t)z}}{z} dz = -\pi i,$$

Thus

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}\hat{f}(\lambda) d\lambda = \frac{2\pi i}{2\pi i} = 1, \qquad \text{for } -a < t < a$$

In conclusion

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}\hat{f}(\lambda) d\lambda = \begin{cases} 1 & \text{for } -a < t < a \\ 0 & \text{otherwise} \end{cases} = f(t)$$

Problem 3. Consider the function

$$f(z) = \ln(z^2 - 1)$$

made single-valued by restricting the angles in the following ways, with $z_1 = z - 1 = r_1 e^{i\theta_1}$, $z_2 = z + 1 = r_2 e^{i\theta_2}$,

(a) $-3\pi/2 < \theta_1 \leq \pi/2, \ -3\pi/2 < \theta_2 \leq \pi/2$

(b) $0 < \theta_1 \leq 2\pi, \ 0 < \theta_2 \leq 2\pi$

(c) $-\pi < \theta_1 \leq \pi, \ 0 < \theta_2 \leq 2\pi$.

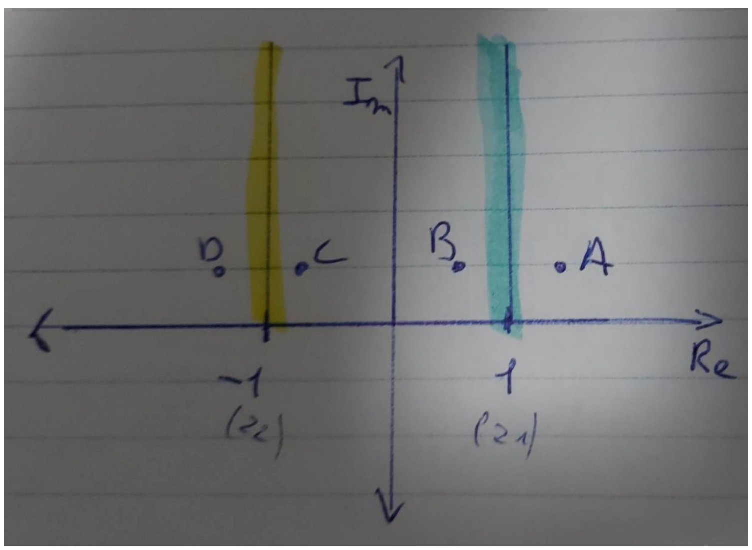
Find where the branch cuts are for each case by locating where the function is discontinuous. Use AB tests and show your results.

Solution:

First we see that

$$f(z) = \ln((z-1)(z+1)) = \ln(z_1 z_2) = \ln(r_1 r_2) + i(\theta_1 + \theta_2)$$

(a) We make 4 points: A, B, C and D like below.



Thus, A and B are extremely close to the real axis, but even closer to the potential branch cut at 1. They are, respectively, to the right and left of 1. Same goes for C and D, but next to the potential branch cut at -1.

At A, $\theta_1 = \frac{1}{2}\pi$ and $\theta_2 = 0$, thus

$$f(z) = \ln(r_1 r_2) + \frac{1}{2}\pi i.$$

At B, $\theta_1 = -\frac{3}{2}\pi$ and $\theta_2 = 0$, thus

$$f(z) = \ln(r_1 r_2) - \frac{3}{2}\pi i.$$

$f(z)$ is discontinuous here, so this branch cut remains.

At C, $\theta_1 = -\pi$ and $\theta_2 = \frac{1}{2}\pi$, thus

$$f(z) = \ln(r_1 r_2) - \frac{1}{2}\pi i.$$

At D, $\theta_1 = -\pi$ and $\theta_2 = -\frac{3}{2}\pi$, thus

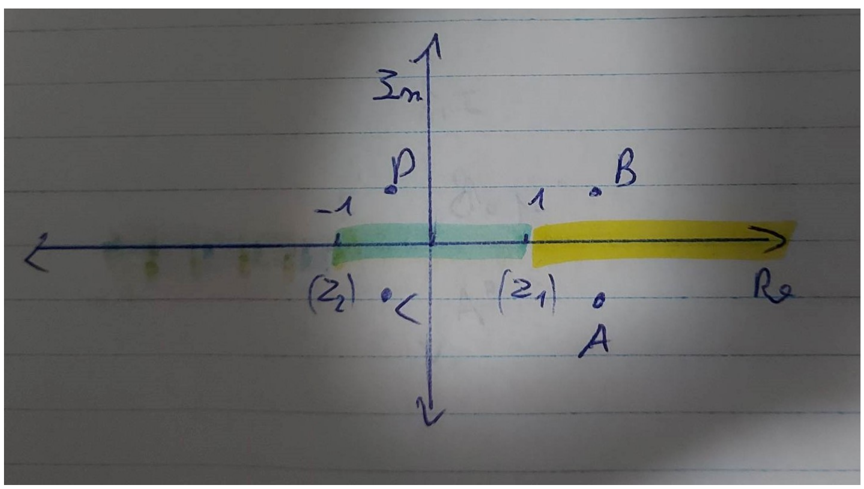
$$f(z) = \ln(r_1 r_2) - \frac{5}{2}\pi i.$$

$f(z)$ is discontinuous here, so this branch cut remains.

In conclusion both branch cuts remain, the branch cuts are where

$$\theta_1 = \frac{\pi}{2} \text{ and } \theta_2 = \frac{\pi}{2}.$$

(b) We make 4 points: A, B, C and D like below.



Thus, A and B are extremely close to the potential branch cut at -1, one is under -1, the other above. Same goes for C and D, but next to the potential branch cut at 1.

At A, $\theta_1 = 2\pi$ and $\theta_2 = 2\pi$, thus

$$f(z) = \ln(r_1 r_2) + 4\pi i.$$

At B, $\theta_1 = 0$ and $\theta_2 = 0$, thus

$$f(z) = \ln(r_1 r_2).$$

$f(z)$ is discontinuous here, so this branch cut remains.

At C, $\theta_1 = \pi$ and $\theta_2 = 2\pi$, thus

$$f(z) = \ln(r_1 r_2) + 3\pi i.$$

At D, $\theta_1 = \pi$ and $\theta_2 = 0$, thus

$$f(z) = \ln(r_1 r_2) + \pi.$$

$f(z)$ is discontinuous here, so this branch cut remains.

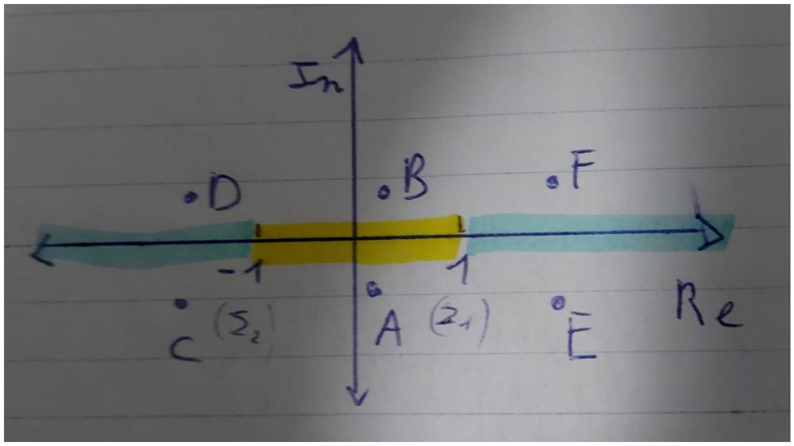
In conclusion both branch cuts remain here, the branch cuts are thus:

$$\theta_1 = 2\pi \text{ and } \theta_2 = 2\pi.$$

or, differently stated,

$$[-1, \infty[\text{ on the real axis.}$$

(c) We make 6 points: A, B, C, D, E and F like below.



Thus, A and B are extremely close to the real axis, but the real part is inbetween -1 and 1, one is above the other under. C and D, close to the real axis, but to the left of -1 (one below, one above). E and F, close to the real axis, but to the right of 1 (one below, one above)

At A, $\theta_1 = -\pi$ and $\theta_2 = 2\pi$, thus

$$f(z) = \ln(r_1 r_2) + \pi i.$$

At B, $\theta_1 = \pi$ and $\theta_2 = 0$, thus

$$f(z) = \ln(r_1 r_2) + \pi i.$$

$f(z)$ is continuous here, so this branch cut cancels out.

At C, $\theta_1 = -\pi$ and $\theta_2 = \pi$, thus

$$f(z) = \ln(r_1 r_2).$$

At D, $\theta_1 = \pi$ and $\theta_2 = \pi$, thus

$$f(z) = \ln(r_1 r_2) + 2\pi i.$$

$f(z)$ is discontinuous here, so this branch cut remains.

At E, $\theta_1 = 0$ and $\theta_2 = 2\pi$, thus

$$f(z) = \ln(r_1 r_2) + 2\pi i.$$

At F, $\theta_1 = 0$ and $\theta_2 = 0$, thus

$$f(z) = \ln(r_1 r_2).$$

$f(z)$ is discontinuous here, so this branch cut remains.

In conclusion the branch cuts are the lines

$$]-\infty, -1] \cup [1, \infty[\text{ on the real axis.}$$