AMATH 567
Applied Complex Variables
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Homework 4

Due: Wednesday, November 2, 2022

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1. Using residue calculus, calculate

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{\sinh x} dx.$$

Solution

We know that $\sin x = \text{Im}(e^{ix})$ and $\sinh x = \frac{e^x - e^{-x}}{2}$. Thus we want to evaluate:

$$I = 2\text{Im} \int_{-\infty}^{\infty} \frac{e^{ix}}{e^x - e^{-x}} dx.$$

Poles to this are where $e^x = e^{-x}$. These are equal when $x = k\pi i$ with $k \in \mathbb{Z}$. However, since $\lim_{R\to\infty} \dfrac{1}{e^{Re^{it}} - e^{-Re^{it}}} = 0$ (if $2\pi k \leq t \leq 2\pi(k+1)$ with k even, the limit is $\dfrac{1}{\infty - 0} = 0$, if k is odd, the limit is $\dfrac{1}{0-\infty} = 0$), we can use Jordan's lemma to get that

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{e^z - e^{-z}} dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iz}}{e^z - e^{-z}} dz = \lim_{R\to\infty} \oint_{C_R} \frac{e^{iz}}{e^z - e^{-z}} dz$$

with C_R the closed semicircle on the positive half of the complex plane. Using the residue theorem, we're only interested in the residues with a positive complex coefficient and where $x = 0$, note that this last one is on C_{R+}. For $R \to \infty$ we get,

$$\lim_{R\to\infty} \oint_{C_R} \frac{e^{iz}}{e^z - e^{-z}} dz = \pi i \text{Res}(0) + 2\pi i \sum_{k=1}^{\infty} \text{Res}(k\pi i)$$

All of these poles are simple poles, thus we use the method of the lecture: if $f(z) = P(z)/Q(z)$ with $P(z)$ (in this case e^{iz}) analytic in z_0 and $Q(z)$ (in this case $e^z - e^{-z}$) has a simple root in z_0, then $\text{Res}(z_0) = \lim_{z\to z_0} \frac{P(z)}{Q'(z)}$. Thus,

$$\text{Res}(k\pi i) = \lim_{z\to k\pi i} \frac{e^{iz}}{e^z + e^{-z}} = \frac{e^{-k\pi}}{e^{ik\pi} + e^{ik\pi}} = \frac{e^{-k\pi}}{2\cos(k\pi)} = \frac{(-1)^k e^{-k\pi}}{2}.$$

Thus,

$$\begin{aligned}\lim_{R\to\infty}\oint_{C_R}\frac{e^{iz}}{e^z-e^{-z}}dz &= \frac{\pi}{2}i+\pi i\sum_{k=1}^{\infty}\left(-e^{-\pi}\right)^k = -\frac{\pi}{2}i+\pi i\sum_{k=0}^{\infty}\left(-e^{-\pi}\right)^k = \pi i\left(\frac{1}{1+e^{-\pi}}-\frac{1}{2}\right)\\ &= \frac{\pi}{2}\frac{1-e^{-\pi}}{1+e^{-\pi}}i = \frac{\pi}{2}\frac{e^{\pi/2}-e^{-\pi/2}}{e^{\pi/2}+e^{-\pi/2}}i\\ &= \frac{\pi}{2}\tanh\left(\frac{\pi}{2}\right)i.\end{aligned}$$

In conclusion:

$$\int_{-\infty}^{\infty}\frac{\sin x}{\sinh x}dx = 2\mathrm{Im}\int_{-\infty}^{\infty}\frac{e^{ix}}{e^x-e^{-x}}dx = 2\mathrm{Im}\lim_{R\to\infty}\oint_{C_R}\frac{e^{iz}}{e^z-e^{-z}}dz = \pi\tanh\left(\frac{\pi}{2}\right)$$

2. Using residue calculus, calculate

$$I = \int_{-\infty}^{\infty} \frac{1+\cos(x)}{(x-\pi)^2} dx.$$

Solution

Say that C_{R+} is parameterised by Re^{it}, with $t \in [0, \pi]$ and l_R is the line between $-R$ and R on the real axis, Assume for both that $R > \pi$. We then have the upper semicircle C_R as the union between both. Note also that this is a proper integral. Since π is the **only** pole and it is on C_R we know that, from lecture 13, that

$$I = P\int_{-\infty}^{\infty} \frac{1+\cos(x)}{(x-\pi)^2} dx = \mathrm{Re} P\int_{-\infty}^{\infty} \frac{1+e^{ix}}{(x-\pi)^2} dx = \mathrm{Re} \lim_{R\to\infty} P \oint_{C_R} \frac{1+e^{iz}}{(z-\pi)^2} dz = \mathrm{Re}(\pi i \mathrm{Res}(\pi))$$

Note that this is true since,

$$0 \leq \lim_{|z|\to\infty} |z\frac{1+e^{iz}}{(z-\pi)^2}| \leq \lim_{|z|\to\infty} \frac{|z|(|1|+|e^{iz}|)}{|(z-\pi)^2|} \leq \lim_{|z|\to\infty} 2\frac{|z|}{|z|^2} = 0,$$

thus we have that,

$$\lim_{R\to\infty} \int_{C_{R+}} \frac{1+e^{iz}}{(z-\pi)^2} dz = 0.$$

Since π is a pole, we have that

$$\mathrm{Res}(\pi) = \lim_{z\to\pi} \left((z-\pi)\frac{1+e^{iz}}{(z-\pi)^2} \right) = \lim_{z\to\pi} \frac{1+e^{iz}}{z-\pi} = ie^{i\pi} = -i$$

(l'Hopital's rule).
In conclusion

$$I = \mathrm{Re}(-\pi i^2) = \pi.$$

3. Evaluate the following integral using residue calculus:

$$I = \int_0^\infty \frac{x^a}{1 + 2x\cos(b) + x^2}dx,$$

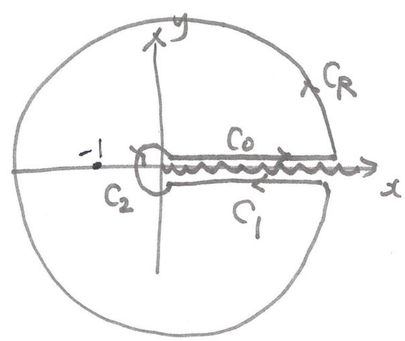
where $-1 < a < 1,\ a \neq 0$, and $-\pi < b < \pi,\ b \neq 0$. Justify all key steps. Do not use the general formula for this integral.

Solution

First we find the poles:

$$D = 4(\cos^2(b) - 1) = -4\sin^2(b) \Rightarrow z_\pm = \frac{-2(\cos(b) \pm i\sin(b))}{2} = -e^{\pm ib}$$

They are both simple poles since $b \in]-\pi, \pi[$ and they always have an imaginary component since $b \neq 0, \pi$ and $-\pi$ too. If we were to go to a complex variables function, we would get a multi valued function since $a \in]-1, 1[$. For example, if $z = -1 = e^{\pi i + 2k\pi i}$, with $k \in \mathbb{Z}$, we have that $z^a = e^{a\pi i + 2ka\pi i}$ where there exists multiple at least 2 k's where $2ka \in [0, 2[$ $(k = 0, \pm 1)$. Thus, for the following we state that $\arg(z) \in [0, 2\pi[$ (setting a branch cut). Using the notations used in lecture 14 page 3, we have $C = C_0 + C_1 + C_2 + C_R$ (take $R > 1$):



First we calculate

$$\oint_C \frac{z^a}{1 + 2z\cos(b) + z^2}dz$$

Since neither poles are on the C_R if $R > 1$ and the radius δ of C_2 is small enough, we have that

$$\oint_C \frac{z^a}{1 + 2z\cos(b) + z^2}dz = 2\pi i \left(\text{Res}(-e^{ib}) + \text{Res}(-e^{-ib})\right)$$

Since these are simple poles, they can easily be found by

$$\text{Res}(-e^{ib}) = \lim_{z\to -e^{ib}} \frac{z^a}{z + e^{-ib}} = -\frac{-1^a e^{iab}}{e^{ib} - e^{-ib}} \text{ and } \text{Res}(-e^{-ib}) = \lim_{z\to -e^{-ib}} \frac{z^a}{z + e^{ib}} = \frac{-1^a e^{-iab}}{e^{ib} - e^{-ib}}.$$

In conclusion:

$$\oint_C \frac{z^a}{1 + 2z\cos(b) + z^2}dz = (-1)^{a+1} 2\pi i \frac{\sin(ab)}{\sin(b)}$$

Since,

$$\lim_{|z|\to\infty} \left| z\frac{z^a}{1 + 2z\cos(b) + z^2} \right| = \lim_{|z|\to\infty} \left| \frac{z^{a+1}}{1 + 2z\cos(b) + z^2} \right| \leq \lim_{|z|\to\infty} \left| \frac{|z|^{a+1}}{-|1 + 2z\cos(b)| + |z|^2} \right|$$
$$= \lim_{|z|\to\infty} \frac{1}{|z|^{1-a}} \overset{1-a>0}{=} 0,$$

we have that $\int_{C_R} \dfrac{z^a}{1+2z\cos(b)+z^2}dz = 0$.

On C_2, we look at what happens when it's radius goes to 0. Thus it is parameterised by δe^{it} for $t \in]0, 2\pi[$ as $\delta \to 0$. We then have

$$\begin{aligned}
\lim_{\delta\to 0}\left|\int_{C_2} \frac{z^a}{1+2z\cos(b)+z^2}dz\right| &= \lim_{\delta\to 0}\left|\int_0^{2\pi} \frac{\delta^a e^{ait}}{1+2\delta e^{it}\cos(b)+\delta^2 e^{2it}}\delta i e^{it}dt\right| \\
&\leq \lim_{\delta\to 0}\int_0^{2\pi} \left|\frac{\delta^{a+1} e^{ait}}{1+2\delta e^{it}\cos(b)+\delta^2 e^{2it}}\right| dt \\
&= \lim_{\delta\to 0}\int_0^{2\pi} \delta^{a+1} \overset{a+1>0}{=} 0
\end{aligned}$$

On C_0 we also want what happens when the radius of C_2 goes to zero. This means that the line C_0 goes to the branch line, or it can be parameterised by $te^{i0} = t$, with $t \in [0, R]$. If we take the limit of $R \to \infty$ we have that

$$\lim_{R\to\infty}\int_{C_0} \frac{z^a}{1+2z\cos(b)+z^2}dz = \lim_{R\to\infty}\int_0^R \frac{t^a}{1+2z\cos(b)+z^2}dt = I$$

On C_1 we also want what happens when the radius of C_2 goes to zero. This means that the line C_1 goes to the branch line from the other side than C_0, or it can be parameterised by $(R-\tilde{t})e^{2\pi i}$, with $\tilde{t} \in [0, R]$. If we take the limit of $R \to \infty$ we have that

$$\begin{aligned}
\lim_{R\to\infty}\int_{C_1} \frac{z^a}{1+2z\cos(b)+z^2}dz &= \lim_{R\to\infty}\int_0^R \frac{-(R-\tilde{t})^a e^{a2\pi i}}{1+2(R-\tilde{t})e^{2\pi i}\cos(b)+(R-\tilde{t})^2 e^{4\pi i}}e^{2\pi i}d\tilde{t} \\
&\overset{R-\tilde{t}=t}{=} -e^{2a\pi i}\lim_{R\to\infty}\int_0^R \frac{t^a}{1+2t\cos(b)+t^2}dt \\
&= -e^{2a\pi i} I
\end{aligned}$$

In conclusion, for $\delta \to 0$

$$(-1)^{a+1}2\pi i\frac{\sin(ab)}{\sin(b)} = \lim_{R\to\infty}\oint_{C_R} \frac{z^a}{1+2z\cos(b)+z^2}dz = (1-e^{2a\pi i})I$$

Thus,

$$I = (-1)^{a+1}2\pi i\frac{\sin(ab)}{\sin(b)(1-e^{2a\pi i})} = (-1)^{a+2}\pi\frac{\sin(ab)}{\sin(b)e^{a\pi i}\dfrac{e^{a\pi i}-e^{-a\pi i}}{2i}} = e^{a\pi i}\pi\frac{\sin(ab)}{\sin(b)\sin(a\pi)}e^{-a\pi i}$$

$$= \pi\frac{\sin(ab)}{\sin(b)\sin(a\pi)}$$

In conclusion:

$$I = \pi\frac{\sin(ab)}{\sin(b)\sin(a\pi)},$$

which is clearly real and exists because of the restrictions on b and a.