AMATH 567
Applied Complex Variables
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Homework 6

Due: Wednesday, November 16, 2022
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Problem 1.

(a) Let $\hat{f}(s)$ and $\hat{g}(s)$ be the Laplace transforms of one-sided functions $f(t)$ and $g(t)$, respectively. Show that the inverse Laplace transform of $\hat{f}(s)\hat{g}(s)$ is;

$$\int_0^t f(t-\tau)g(\tau)d\tau.$$

(b) Use Laplace transform and the result in (a) to solve the following ordinary differential equation:

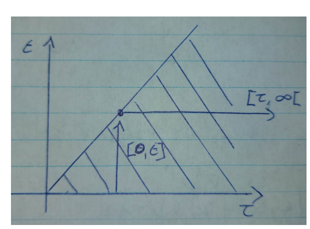
$$\frac{d^2}{dt^2}y + 4y = f(t), \text{ subject to the initial conditions: } y(0) = 0, \frac{dy}{dt}(0) = 0.$$

Solution:

(a) Here we have that

$$\hat{f}(s)\hat{g}(s) = \int_0^\infty e^{-s\xi}f(\xi)d\xi \int_0^\infty e^{-s\tau}g(\tau)d\tau = \int_0^\infty \int_0^\infty e^{-s(\xi+\tau)}f(\xi)g(\tau)d\xi d\tau$$
$$\overset{\xi=t-\tau}{=} \int_0^\infty \int_\tau^\infty e^{-st}f(t-\tau)g(\tau)dtd\tau = \int_0^\infty \int_0^t e^{-st}f(t-\tau)g(\tau)d\tau dt.$$

The second equality is true since neither integrals has the variable over which the other integral integrates. The last equality can be seen when drawing the domain over which we integrate. Both of these domains are the same.



Since e^{-st} does not depend on τ, we can put this in front of the second integral,

$$\hat{f}(s)\hat{g}(s) = \int_0^\infty e^{-st}\int_0^t f(t-\tau)g(\tau)d\tau dt = \mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right].$$

Thus,

$$\mathcal{L}^{-1}[\hat{f}(s)\hat{g}(s)](t) = \mathcal{L}^{-1}\left(\mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]\right)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

In conclusion

$$\mathcal{L}^{-1}[\hat{f}(s)\hat{g}(s)](t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

(b) If $\hat{y}(s) = \mathcal{L}[y](s) = \int_0^\infty e^{-st}y(t)dt$, we have that

$$\mathcal{L}[\frac{d^2}{dt^2}y](s) = \int_0^\infty e^{-st}\frac{d^2}{dt^2}y(t)dt = e^{-st}\frac{d}{dt}y|_0^\infty + s\int_0^\infty e^{-st}\frac{dy}{dt}dt.$$

Assuming that $\frac{dy}{dt} \to 0$ as $t \to \infty$, we have that $e^{-st}\frac{d}{dt}y|_0^\infty = 0$ (second initial condition). Thus,

$$\mathcal{L}[\frac{d^2}{dt^2}y](s) = s\int_0^\infty e^{-st}\frac{dy}{dt}dt = se^{-st}y|_0^\infty + s^2\int_0^\infty e^{-st}ydt = s^2\hat{y}(s).$$

In the last equality we assumed that $y \to 0$ as $t \to \infty$. Thus, denoting the Laplace transformation of function f by $\hat{f}$, the Laplace transform of the given ODE is

$$s^2\hat{y} + 4\hat{y} = \hat{f} \Rightarrow \hat{y} = \frac{\hat{f}}{s^2+4}.$$

Note that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right](t) = \frac{1}{2\pi i}\int_{-\infty i+\alpha}^{\infty i+\alpha}\frac{e^{st}}{s^2+4}ds = \frac{1}{2\pi i}\int_L \frac{e^{st}}{s^2+4}ds,$$

with $\alpha > 0$ fixed. The singularities here are $s = 2i$ and $-2i$, thus all singularities are to the left of the line L. Using Bromwich contour, which we learned about in Lecture 19, we get that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right](t) = \text{Res of } \frac{e^{zt}}{z^2+4} \text{ in } 2i + \text{Res of } \frac{e^{zt}}{z^2+4} \text{ in } -2i, \text{ for } t > 0.$$

Here,

$$\text{Res}(2i) = \lim_{z\to 2i}(z-2i)\frac{e^{zt}}{(z-2i)(z+2i)} = \frac{e^{2it}}{4i} \text{ and } \text{Res}(-2i) = \lim_{z\to -2i}(z+2i)\frac{e^{zt}}{(z-2i)(z+2i)} = \frac{e^{-2it}}{-4i}.$$

Thus,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right](t) = \frac{1}{2}\frac{e^{2it}-e^{-2it}}{2i} = \frac{1}{2}\sin(2t).$$

Assume now that $g(t) = \frac{1}{2}\sin(2t)$, we then know that $\hat{g}(s) = \mathcal{L}\left[\frac{1}{2}\sin(2t)\right](s) = \frac{1}{s^2+4}$. Thus,

$$\hat{y} = \hat{g}\hat{f}.$$

Thus,

$$y(t) = \mathcal{L}^{-1}[\hat{y}](t) = \mathcal{L}^{-1}[\hat{g}\hat{f}](t) \overset{(a)}{=} \int_0^t g(t-\tau)f(\tau)d\tau.$$

In conclusion

$$y(t) = \frac{1}{2}\int_0^t \sin(2(t-\tau))f(\tau)d\tau.$$

Problem 2.
Solve the following Laplace equation

$$\frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi = 0$$

in the upper half plane: $-\infty < x < \infty,\ 0 < y < \infty$, subject to the boundary conditions:

$$\phi \to 0 \text{ as } y \to \infty;\ \phi \to 0 \text{ as } x \to \pm\infty;$$

$$\phi(x,0) = \frac{x}{x^2+a^2}.$$

Solution:

Using a Fourier transform in x gives

$$\Phi(\lambda, y) = \int_{-\infty}^{\infty} e^{i\lambda x}\phi(x,y)dx.$$

Therefore, assuming that the integral is convergent (thus the Fourier transform exists),

$$\mathcal{F}\left(\frac{\partial^2}{\partial y^2}\phi\right) = \frac{\partial^2}{\partial y^2}\Phi,$$

and

$$\begin{aligned}\mathcal{F}\left(\frac{\partial^2}{\partial x^2}\phi\right) &= \int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial^2}{\partial x^2}\phi(x,y)dx = e^{i\lambda x}\frac{\partial}{\partial x}\phi(x,y)\Big|_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial}{\partial x}\phi(x,y)dx \\ &= e^{i\lambda x}\frac{\partial}{\partial x}\phi(x,y)\Big|_{-\infty}^{\infty} - i\lambda e^{i\lambda x}\phi(x,y)\Big|_{-\infty}^{\infty} - \lambda^2\Phi.\end{aligned}$$

Now, since $u \to 0$ when $x \to \pm\infty$ it is not outrageous to assume that $\frac{\partial}{\partial x}u \to 0$ when $x \to \pm\infty$. Assume this holds, then

$$\mathcal{F}\left(\frac{\partial^2}{\partial x^2}\phi\right) = -\lambda^2\Phi.$$

Thus, the Fourier transformation of the Laplace equation becomes

$$\frac{\partial^2}{\partial y^2}\Phi - \lambda^2\Phi = 0 \quad\Rightarrow\quad \frac{\partial^2}{\partial y^2}\Phi = \lambda^2\Phi.$$

Thus, the solution to this is of the form

$$\Phi(\lambda, y) = A(\lambda)e^{\lambda y} + B(\lambda)e^{-\lambda y}.$$

Since $\phi \to 0$ as $y \to \infty$, we have that

$$\lim_{y\to\infty}\Phi(\lambda,y) = \mathcal{L}(\lim_{y\to\infty}\phi(\lambda,y)) = \mathcal{L}(0) = 0.$$

Therefore, since

$$\lim_{y\to\infty}\Phi = A(\lambda)\lim_{y\to\infty}e^{\lambda y},\ \text{when } \lambda > 0,$$

and

$$\lim_{y\to\infty}\Phi = B(\lambda)\lim_{y\to\infty} e^{-\lambda y},\ \text{when } \lambda < 0,$$

we know that

$$A(\lambda) = \begin{cases} 0 & \lambda > 0 \\ A_-(\lambda) & \lambda < 0 \end{cases} \text{ and } B(\lambda) = \begin{cases} B_+(\lambda) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}.$$

As we are interested in $\phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\Phi(\lambda,y)d\lambda$, an integral, we will not be looking specifically at $\lambda = 0$ since a singular point of the function in the integral is negligible. On the other hand, we have that $\phi(x,0) = \frac{x}{x^2+a^2}$, so

$$\Phi(\lambda,0) = \mathcal{F}[\phi(x,0)](\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x}\frac{x}{x^2+a^2}dx.$$

Note that, on the complex plane, this function has singularities in ai and $-ai$. Now, using Jordan's lemma (Lecture 12, page 1 and 2), we want to calculate this integral. The usage of Jordan's lemma depends on the sign of λ. At least we have that in both cases, on the complex line parameterised by Re^{it} for $t \in [0,\pi]$ or $t \in [\pi, 2\pi]$,

$$\lim_{R\to\infty}\left|\frac{z}{z^2+a^2}\right| = \lim_{R\to\infty}\left|\frac{Re^{it}}{Re^{2it2}+a^2}\right| = \left|\lim_{R\to\infty}\frac{Re^{it}}{Re^{2it2}+a^2}\right| = \left|\lim_{R\to\infty}\frac{Re^{it}}{Re^{2it2}}\right| = \lim_{R\to\infty}\frac{R}{R^2} = 0.$$

Define C_{R+} as the upper semicircle with radius R on the complex plane and C_{R-} as the lower semicircle with radius R on the complex plane. Take $R > a$. We then have, because of Jordan's lemma, that for $\lambda > 0$

$$\int_{-\infty}^{\infty} e^{i\lambda x}\frac{x}{x^2+a^2}dx = \lim_{R\to\infty}\oint_{C_{R+}} e^{i\lambda z}\frac{z}{z^2+a^2}dz = 2\pi i\text{Res}(ai).$$

For $\lambda < 0$, we have that

$$\int_{-\infty}^{\infty} e^{i\lambda x}\frac{x}{x^2+a^2}dx = -\lim_{R\to\infty}\oint_{C_{R-}} e^{i\lambda z}\frac{z}{z^2+a^2}dz = -2\pi i\text{Res}(-ai).$$

These residues can easily be calculated, since they are simple poles:

$$\text{Res}(ai) = \lim_{z\to ai}(z-ai)e^{i\lambda z}\frac{z}{z^2+a^2} = \frac{aie^{-\lambda a}}{2ai} = \frac{e^{-\lambda a}}{2},$$

$$\text{Res}(-ai) = \lim_{z\to -ai}(z+ai)e^{i\lambda z}\frac{z}{z^2+a^2} = \frac{-aie^{\lambda a}}{-2ai} = \frac{e^{\lambda a}}{2}.$$

Therefore we have that

$$\Phi(\lambda,0) = \begin{cases} \pi i e^{-\lambda a} & \lambda > 0 \\ -\pi i e^{\lambda a} & \lambda < 0 \end{cases} = \text{sgn}(\lambda)\pi i e^{-a|\lambda|}.$$

Now, since on one hand

$$\Phi(\lambda,y) = A(\lambda)e^{\lambda y} + B(\lambda)e^{-\lambda y},$$

with

$$A(\lambda) = \begin{cases} 0 & \lambda > 0 \\ A_-(\lambda) & \lambda < 0 \end{cases} \text{ and } B(\lambda) = \begin{cases} B_+(\lambda) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases},$$

thus,

$$\Phi(\lambda,0)=\begin{cases}B_+(\lambda) & \lambda>0\\ A_-(\lambda) & \lambda<0\end{cases},$$

while on the other hand

$$\Phi(\lambda,0)=\begin{cases}\pi i e^{-a\lambda} & \lambda>0\\ -\pi i e^{a\lambda} & \lambda<0\end{cases},$$

we have that

$$\Phi(\lambda,y)=\begin{cases}\pi i e^{-a\lambda}e^{-\lambda y} & \lambda>0\\ -\pi i e^{a\lambda}e^{\lambda y} & \lambda<0\end{cases}=\begin{cases}\pi i e^{-\lambda(y+a)} & \lambda>0\\ -\pi i e^{\lambda(y+a)} & \lambda<0\end{cases}=\operatorname{sgn}(\lambda)\pi i e^{-(y+a)|\lambda|}.$$

Now we use the inverse Fourier transform on Φ to find ϕ:

$$\begin{aligned}\phi(x,y)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\Phi(\lambda,y)d\lambda=\frac{i}{2}\left(\int_0^{\infty}e^{-\lambda(ix+y+a)}d\lambda-\int_{-\infty}^{0}e^{\lambda(y+a-ix)}d\lambda\right)\\
&=\frac{i}{2}\left(\frac{-e^{-\lambda(ix+y+a)}}{ix+y+a}\Bigg|_0^{\infty}-\frac{e^{\lambda(y+a-ix)}}{y+a-ix}\Bigg|_{-\infty}^{0}\right)\\
&=\frac{i}{2}\left(\frac{1}{ix+y+a}-\frac{1}{y+a-ix}\right)=\frac{i}{2}\left(\frac{-2ix}{x^2+(y+a)^2}\right)\\
&=\frac{x}{x^2+(y+a)^2}.\end{aligned}$$

We now check that everything, which was given, is correct. The boundary conditions do indeed hold:

$$\lim_{x\to\pm\infty}\frac{x}{x^2+(y+a)^2}=\lim_{x\to\pm\infty}\frac{1}{x}=0,$$

$$\lim_{y\to\infty}\frac{x}{x^2+(y+a)^2}=\lim_{y\to\infty}\frac{1}{y^2}=0,$$

and

$$\phi(x,0)=\frac{x}{x^2+(0+a)^2}=\frac{x}{x^2+a^2}.$$

Now we check if it satisfies the Laplace equation, for this we calculate the partial derivatives:

$$\frac{\partial}{\partial x}\phi=\frac{1}{x^2+(y+a)^2}-\frac{2x^2}{\left(x^2+(y+a)^2\right)^2},\ \text{thus}\ \frac{\partial^2}{\partial x^2}\phi=\frac{8x^3}{\left(x^2+(y+a)^2\right)^3}-\frac{6x}{\left(x^2+(y+a)^2\right)^2},$$

and

$$\frac{\partial}{\partial y}\phi=\frac{-2x(y+a)}{x^2+(y+a)^2},\ \text{thus}\ \frac{\partial^2}{\partial y^2}\phi=\frac{8x\,(y+a)^2}{\left((y+a)^2+x^2\right)^3}-\frac{2x}{\left((y+a)^2+x^2\right)^2}.$$

Therefore

$$\frac{\partial^2}{\partial x^2}\phi+\frac{\partial^2}{\partial y^2}\phi=\frac{8x^3+8x(y+a)^2}{\left(x^2+(y+a)^2\right)^3}-\frac{8x}{\left(x^2+(y+a)^2\right)^2}=\frac{8x^3+8x(y+a)^2-8x((x^2+(y+a)^2)}{\left(x^2+(y+a)^2\right)^3}=0$$

In conclusion, the solution to this system is

$$\phi(x,y)=\frac{x}{x^2+(y+a)^2}.$$

Problem 3.
Use Fourier transform to solve the following wave equation:

$$\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u,\ -\infty < x < \infty, 0 < t < \infty$$

subject to the intial condition: $u(x,0) = 0,\ \frac{\partial}{\partial t}u = \delta(x)$ at $t = 0$.
and the boundary condition: $u(x,t) \to 0$ as $x \to \pm\infty$.

Solution:

Using a Fourier transform in x gives

$$U(\lambda, t) = \int_{-\infty}^{\infty} e^{i\lambda x}u(x,t)dx.$$

Therefore, assuming that the integral is convergent (thus the Fourier transform exists),

$$\mathcal{F}\left(\frac{\partial^2}{\partial t^2}y\right) = \frac{\partial^2}{\partial t^2}U,$$

and

$$\begin{aligned}\mathcal{F}\left(\frac{\partial^2}{\partial x^2}u\right) &= \int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial^2}{\partial x^2}t(x,y)dx = e^{i\lambda x}\frac{\partial}{\partial x}u(x,t)|_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial}{\partial x}u(x,t)dx \\ &= e^{i\lambda x}\frac{\partial}{\partial x}u(x,t)|_{-\infty}^{\infty} - i\lambda e^{i\lambda x}u(x,t)|_{-\infty}^{\infty} + (i\lambda)^2U.\end{aligned}$$

Now since $u \to 0$ when $x \to \pm\infty$ it is not outrageous to assume that $\frac{\partial}{\partial x}u \to 0$ when $x \to \pm\infty$. Assume this holds, then, partly because of the boundary condition,

$$\mathcal{F}\left(\frac{\partial^2}{\partial x^2}u\right) = (i\lambda)^2U.$$

The Fourier transformed wave equation thus is

$$\frac{\partial^2}{\partial t^2}U = (c\lambda i)^2U.$$

It is clear that the initial conditions for this are, on one hand on the other

$$U(\lambda, 0) = \mathcal{F}(u(x,0)) = 0,$$

and

$$\frac{\partial}{\partial t}U(\lambda,t) = \frac{\partial}{\partial t}\mathcal{F}(u(x,t)) = \mathcal{F}\left(\frac{\partial}{\partial t}u(x,t)\right) = \mathcal{F}\left(\delta(x)\right) = e^0 = 1, \qquad \text{at } t = 0.$$

Knowing this and knowing that the solution to $\frac{\partial^2}{\partial t^2}U = (c\lambda i)^2U$ is of the form

$$U = A(\lambda)e^{c\lambda it} + B(\lambda)e^{-c\lambda it}.$$

We get that

$$\begin{cases} A(\lambda)e^{c\lambda i0} + B(\lambda)e^{-c\lambda i0} = A(\lambda) + B(\lambda) = 0 \\ c\lambda iA(\lambda) - c\lambda iB(\lambda) = 1 \end{cases} \Rightarrow \begin{cases} A(\lambda) = \frac{1}{2c\lambda i} \\ B(\lambda) = -\frac{1}{2c\lambda i} \end{cases}.$$

Thus,

$$U(\lambda,t)=\frac{1}{c\lambda}\frac{e^{c\lambda it}-e^{-c\lambda it}}{2i}=\frac{\sin(c\lambda t)}{c\lambda}.$$

We now use the inverse Fourier transform to get the solution to the original problem:

$$\begin{aligned}\mathcal{F}^{-1}\left(\frac{\sin(c\lambda t)}{c\lambda}\right)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\frac{\sin(c\lambda t)}{c\lambda}d\lambda=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\frac{e^{ic\lambda t}-e^{-ic\lambda t}}{2ic\lambda}d\lambda\\&=\frac{1}{4ci\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda(x-ct)}-e^{-i\lambda(x+ct)}}{\lambda}d\lambda\\&=\frac{1}{4ci\pi}\left(P\int_{-\infty}^{\infty}\frac{e^{-i\lambda(x-ct)}}{\lambda}d\lambda-P\int_{-\infty}^{\infty}\frac{e^{-i\lambda(x+ct)}}{\lambda}d\lambda\right).\end{aligned}$$

Note that for $g(\lambda)=\frac{e^{im\lambda}}{\lambda}$, for $m\in\mathbb{R}_0$ random, $\lambda=0$ is a simple pole and

$$\operatorname{Res}(g,\lambda=0)=\lim_{\lambda\to 0}e^{im\lambda}=1.$$

Thus, if C_{R+} is the upper semicircle of the complex plane with radius R and C_{R-} is the lower semicircle of the complex plane with radius R , we have that, since 0 is on the real axis and thus on both C_{R+} and C_{R-},

$$\oint_{C_{R+}}\frac{e^{im\lambda}}{\lambda}d\lambda=\pi i\operatorname{Res}(0)=\pi i=\pi i\operatorname{Res}(0)=\oint_{C_{R-}}\frac{e^{im\lambda}}{\lambda}d\lambda.$$

We would now like to use Jordan's lemma in Lecture 12 (particularly page 1 and 2). This, however, depends on the sign of m and thus, in the original problem, on the sign of $x-ct$ and $x+ct$. Note first that $|\frac{1}{\lambda}|$ goes to 0 on C_{R+} and C_{R-} as $R\to\infty$:

$$\lim_{R\to\infty}\left|\frac{1}{\lambda}\right|=\lim_{R\to\infty}\left|\frac{1}{Re^{it}}\right|=\lim_{R\to\infty}\frac{1}{R}=0$$

(here $t\in[0,\pi]$ or $t\in[\pi,2\pi]$). Assume $c>0$ ($c<0$ works in roughly the same way), we now look at the three cases:

$\boldsymbol{x-ct>0}$ If $x-ct>0$, then $x+ct>0$ since $t,c>0$. Using Lecture 12 page 2, we get:

$$P\int_{-\infty}^{\infty}\frac{e^{-i(x-ct)\lambda}}{\lambda}d\lambda=-\oint_{C_{R-}}\frac{e^{-i(x-ct)z}}{z}dz=-\pi i,$$

and

$$P\int_{-\infty}^{\infty}\frac{e^{-i(x+ct)\lambda}}{\lambda}d\lambda=-\oint_{C_{R-}}\frac{e^{-i(x+ct)z}}{z}dz=-\pi i,$$

Thus

$$\mathcal{F}^{-1}\left(\frac{\sin(c\lambda t)}{c\lambda}\right)=0,\qquad\text{for } x-ct>0.$$

$\boldsymbol{x+ct<0}$ If $x+ct<0$, then $-(x+ct)>0$ and $-(x-ct)>0$. Thus,

$$P\int_{-\infty}^{\infty}\frac{e^{-i(x-ct)\lambda}}{\lambda}d\lambda=\oint_{C_{R+}}\frac{e^{-i(x-ct)z}}{z}dz=\pi i,$$

and

$$P\int_{-\infty}^{\infty}\frac{e^{-i(x+ct)\lambda}}{\lambda}d\lambda=\oint_{C_{R+}}\frac{e^{-i(x+ct)z}}{z}dz=\pi i,$$

Thus

$$\mathcal{F}^{-1}\left(\frac{\sin(c\lambda t)}{c\lambda}\right)=0,\qquad \text{for } x+ct<0.$$

$\boldsymbol{-ct<x<ct}$ If $-ct<x<ct$, then $x+ct>0$ and $-(x-ct)>0$. Thus,

$$P\int_{-\infty}^{\infty}\frac{e^{-i(x-ct)\lambda}}{\lambda}d\lambda=\oint_{C_{R+}}\frac{e^{-i(x-ct)z}}{z}dz=\pi i,$$

and

$$P\int_{-\infty}^{\infty}\frac{e^{-i(x+ct)\lambda}}{\lambda}d\lambda=-\oint_{C_{R-}}\frac{e^{-i(x+ct)z}}{z}dz=-\pi i,$$

Thus

$$\mathcal{F}^{-1}\left(\frac{\sin(c\lambda t)}{c\lambda}\right)=\frac{1}{2c},\qquad \text{for } |x|<ct.$$

This gives us that

$$\mathcal{F}^{-1}\left(\frac{\sin(c\lambda t)}{c\lambda}\right)=\begin{cases}\frac{1}{2c} & \text{for } |x|<ct\\ 0 & \text{otherwise}\end{cases}.$$

For $c<0$, say that $c=-k$ with $k>0$. The function of which we take the inverse Fourier transform becomes

$$\frac{\sin(c\lambda t)}{c\lambda}=\frac{\sin(k\lambda t)}{k\lambda},\ \text{with } k>0.$$

Thus,

$$\mathcal{F}^{-1}\left(\frac{\sin(c\lambda t)}{c\lambda}\right)=\mathcal{F}^{-1}\left(\frac{\sin(k\lambda t)}{k\lambda}\right)\begin{cases}\frac{1}{2k} & \text{for } |x|<kt\\ 0 & \text{otherwise}\end{cases}=\begin{cases}\frac{-1}{2c} & \text{for } |x|<-ct\\ 0 & \text{otherwise}\end{cases}.$$

In conclusion

$$u(x,t)=\begin{cases}\frac{1}{2|c|} & \text{for } |x|<|c|t\\ 0 & \text{otherwise}\end{cases}.$$

This function satisfies all that is given in the problem.