AMATH 567
Applied Complex Variables
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Homework 3

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Unless specified otherwise, denote the half of a circle with center 0 and radius R that is on the positive imaginary plane by C_R^+, the line on the real axis from $-R$ to R by l_R and the union of both these by C_R.

1. (c) Evaluate
$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}, \qquad a^2, b^2 > 0.$$
   (d) Evaluate
$$\int_0^\infty \frac{dx}{x^6+1}.$$

Solution:

(c) Evaluate
$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \int_0^\infty f(x)dx, \qquad a^2, b^2 > 0.$$
First we find
$$\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^\infty \frac{dx}{(x-ai)(x+ai)(x-bi)(x+bi)}$$
It is clear that $f(-x) = f(x)$, thus $\int_{-\infty}^\infty f(x)dx = 2\int_0^\infty f(x)dx$. Now take $R > a, b \in \mathbb{R}$. Since
$$\begin{aligned}\lim_{|z|\to\infty} |zf(z)| &= \lim_{|z|\to\infty} \left| z\frac{1}{(z-ai)(z+ai)(z-bi)(z+bi)} \right| \\ &= \lim_{|z|\to\infty} \frac{|z|}{|z^4+(a^2+b^2)z^2+a^2b^2|} \\ &\leq \left| \lim_{|z|\to\infty} \frac{|z|}{|z^4|-|(a^2+b^2)z^2+a^2b^2|} \right| = \lim_{|z|\to\infty} \frac{|z|}{|z|^4} \\ &= 0,\end{aligned}$$

we know that

$$\int_{C_R^+} \frac{dz}{(z^2+a^2)(z^2+b^2)} = 0,$$

and since $\oint_{C_R} = \int_{C_R^+} + \int_{l_R}$, we now have that

$$\oint_{C_R} \frac{dz}{(z^2+a^2)(z^2+b^2)} = \int_{l_R} \frac{dz}{(z^2+a^2)(z^2+b^2)}$$

and thus

$$\lim_{R\to\infty} \oint_{C_R} \frac{dz}{(z^2+a^2)(z^2+b^2)} = \int_{-\infty}^{\infty} \frac{dz}{(z^2+a^2)(z^2+b^2)}.$$

Note how $ai, -ai, bi$ and $-bi$ are simple poles, and for $R > a, b$ we have that ai and bi are inside C_R. Taking $R > a, b$ is not a serious constraint since we want to take $R \to \infty$. Using the residue theorem, we get

$$\oint_{C_R} \frac{dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i \left(\mathrm{Res}(ai) + \mathrm{Res}(bi)\right)$$

Since ai and bi are simple poles, we have

$$\mathrm{Res}(ai) = \lim_{z\to ai} \frac{z-ai}{(z-ai)(z+ai)(z-bi)(z+bi)} = \frac{1}{2ai(b^2-a^2)},$$

and

$$\mathrm{Res}(bi) = \lim_{z\to bi} \frac{z-bi}{(z-ai)(z+ai)(z-bi)(z+bi)} = \frac{-1}{2bi(b^2-a^2)}.$$

Thus,

$$\oint_{C_R} \frac{dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i \left(\frac{b-a}{2abi(b^2-a^2)}\right) = \frac{\pi}{ab(b+a)}$$

In conclusion,

$$\int_0^{\infty} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx = \frac{1}{2}\lim_{R\to\infty} \oint_{C_R} f(z)dz = \frac{\pi}{2ab(a+b)}$$

(d) Evaluate

$$\int_0^{\infty} \frac{dx}{x^6+1} = \int_0^{\infty} f(x)dx.$$

Once again, since $f(-x) = f(x)$ we have that $\int_0^{\infty} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$. Since

$$\lim_{|z|\to\infty} |zf(z)| = \lim_{|z|\to\infty} \left|\frac{z}{z^6+1}\right| \leq \left|\lim_{|z|\to\infty} |\frac{|z|}{|z|^6-1}\right| = \left|\lim_{|z|\to\infty} |\frac{|z|}{|z|^6}\right| = 0$$

Taking $R > 1$, we have

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \oint_{C_R} f(z)dz = 2\pi i \sum_{j=1}^{3} B_i$$

with B_i the residues of the poles on the positive imaginary plane. Speaking of, since $-1 = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$, with $k \in \mathbb{Z}$, we have that $a_k = \sqrt[6]{-1} = \cos(\frac{\pi}{6} +$

$\frac{k\pi}{3}) + i\sin(\frac{\pi}{6} + \frac{k\pi}{3})$. All of these a_k are simple poles of $f(z)$, the ones on the positive complex plane are:

$$a_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}i \qquad a_1 = i \qquad a_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Using the trick from lecture 11, which states that if $f(z) = P(z)/Q(z)$ (in this case $P(z) = 1$ and $Q(z) = z^6 + 1$) where $P(z)$ is analytic and $Q(z)$ has a simple zero at z_0, then $\text{Res}(z_0) = P(z_0)/Q'(z_0)$, we get,

$$\text{Res}(a_0) = \frac{1}{6(\sqrt{3}/2 + i/2)^5} = \frac{1}{3(-\sqrt{3} + i)} = \frac{-\sqrt{3} - i}{12},$$

$$\text{Res}(a_1) = \frac{1}{6i}$$

and

$$\text{Res}(a_2) = \frac{1}{6(-\sqrt{3}/2 + i/2)^5} = \frac{1}{3(\sqrt{3} + i)} = \frac{\sqrt{3} - i}{12}.$$

Thus, $\oint_{C_R} f(z)dz = 2\pi(\frac{1}{6} + \frac{2}{12}) = \frac{2\pi}{3}$ for $R > 1$. In conclusion

$$\int_0^\infty \frac{dx}{x^6 + 1} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{x^6 + 1} = \lim_{R\to\infty} \oint_{C_R} \frac{dz}{z^6 + 1} = \frac{\pi}{3}$$

2. Evaluate the following integrals:

   (a) $\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+a^2}dx,\ a^2>0$

   (b) $\int_{-\infty}^{\infty} \frac{\cos(kx)dx}{(x^2+a^2)(x^2+b^2)},\ a^2, b^2, k > 0$

   (h) $\int_0^{2\pi} \frac{d\theta}{(5-3\sin(\theta))^2}$

Solution:

(a) Since

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+a^2}dx = \mathrm{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}dx\right)$$

We will be trying to find the latter. First we want to prove that $|\frac{x}{x^2+a^2}| \to 0$ over C_R^+ as $R\to\infty$. Therefore we characterize C_R^+ by Re^{it} for $0\le t\le\pi$, and

$$\lim_{R\to\infty}\left|\frac{z}{z^2+a^2}\right| = \lim_{R\to\infty}\left|\frac{Re^{it}}{R^2e^{2it}+a^2}\right| \le \left|\lim_{R\to\infty}\frac{|R||e^{it}|}{|R^2||e^{2it}|-|a^2|}\right|$$
$$= \left|\lim_{R\to\infty}\frac{R}{R^2-a^2}\right| = 0$$

Jordan's lemma now states that $\int_{C_R^+}\frac{ze^{iz}dz}{z^2+a^2}$ goes to zero as $R\to\infty$. Thus

$$\int_{-\infty}^{\infty}\frac{xe^{ix}dx}{x^2+a^2} = \lim_{R\to\infty}\int_{l_R}\frac{ze^{iz}dz}{z^2+a^2} = \lim_{R\to\infty}\int_{l_R}\frac{ze^{iz}dz}{z^2+a^2} + \int_{C_R^+}\frac{ze^{iz}dz}{z^2+a^2} = \lim_{R\to\infty}\oint_{C_R}\frac{ze^{iz}dz}{z^2+a^2}$$

Take $R>a$, Since ai is clearly the only singularity in the upper complex plane and note that it is a simple pole. Therefore we can use the residue theorem for C_R:

$$\oint_{C_R}\frac{ze^{iz}dz}{z^2+a^2} = 2\pi i\mathrm{Res}(ai)$$

This residue is

$$\mathrm{Res}(ai) = \lim_{z\to ai}\frac{(z-ai)ze^{iz}dz}{z^2+a^2} = \lim_{z\to ai}\frac{ze^{iz}dz}{z+ai} = \frac{e^{-a}}{2}$$

Therefore

$$\oint_{C_R}\frac{ze^{iz}dz}{z^2+a^2} = \pi e^{-a}i,$$

which is only imaginary. In conclusion

$$\int_{-\infty}^{\infty}\frac{x\sin(x)}{x^2+a^2}dx = \pi e^{-a}$$

(b) Since

$$\int_{-\infty}^{\infty} \frac{\cos(kx)dx}{(x^2+a^2)(x^2+b^2)} = \text{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ikx}dx}{(x^2+a^2)(x^2+b^2)}\right),$$

we will be trying to find the latter. First we want to prove that $|\frac{1}{(z^2+a^2)(z^2+b^2)}| \to 0$ over C_R^+ as $R \to \infty$. Therefore we characterize C_R^+ by Re^{it} for $0 \le t \le \pi$, and

$$\begin{aligned} \lim_{R\to\infty} \frac{1}{|(z^2+a^2)(z^2+b^2)|} &\le \lim_{R\to\infty} \frac{1}{|(|R^2||e^{2it}|-|a^2|)(|R^2||e^{2it}|-|b^2|)|} \\ &= \lim_{R\to\infty} \frac{1}{|(R^2-a^2)(R^2-b^2)|} = \lim_{R\to\infty} \frac{1}{R^4} \\ &= 0 \end{aligned}$$

Jordan's lemma now states that $\int_{C_R^+} \frac{e^{ikz}dz}{(z^2+a^2)(z^2+b^2)}$ goes to zero as $R \to \infty$. Thus

$$\int_{-\infty}^{\infty} \frac{e^{ikx}dx}{(x^2+a^2)(x^2+b^2)} = \lim_{R\to\infty} \int_{l_R} \frac{e^{ikz}dz}{(z^2+a^2)(z^2+b^2)} = \lim_{R\to\infty} \oint_{C_R} \frac{e^{ikz}dz}{(z^2+a^2)(z^2+b^2)}$$

Take $R > a, b$, Since ai and bi are clearly the only singularities in the upper complex plane and note that they are simple poles. Therefore we can use the residue theorem for C_R:

$$\oint_{C_R} \frac{e^{ikz}dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i(\text{Res}(ai) + \text{Res}(bi))$$

These residues are

$$\text{Res}(ai) = \lim_{z\to ai} \frac{e^{ikz}(z-ai)}{(z^2+a^2)(z^2+b^2)} = \lim_{z\to ai} \frac{e^{ikz}}{(z+ai)(z^2+b^2)} = \frac{e^{-ka}}{2ai(b^2-a^2)}$$

and

$$\text{Res}(bi) = \lim_{z\to bi} \frac{e^{ikz}(z-bi)}{(z^2+a^2)(z^2+b^2)} = \lim_{z\to bi} \frac{e^{ikz}}{(z+bi)(z^2+a^2)} = -\frac{e^{-kb}}{2bi(b^2-a^2)}$$

We now have that

$$\oint_{C_R} \frac{e^{ikz}dz}{(z^2+a^2)(z^2+b^2)} = \pi \frac{be^{-ka} - ae^{-kb}}{ab(b^2-a^2)}$$

Since this is only real, we have that

$$\int_{-\infty}^{\infty} \frac{\cos(kx)dx}{(x^2+a^2)(x^2+b^2)} = \pi \frac{be^{-ka} - ae^{-kb}}{ab(b^2-a^2)}$$

(h) Take $z = e^{i\theta}$, we then have that $d\theta = \frac{dz}{iz}$. Since $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$, we have

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin(\theta))^2} = \oint_{|z|=1} \frac{1}{(5-\frac{3(z-z^{-1})}{2i})^2} \frac{dz}{iz} = \oint_{|z|=1} \frac{4zi}{(-3z^2+10iz+3)^2} dz$$

The roots of the denumerator are: $D = -100 + 36 = -64$, thus $z_0 = \frac{-10i+8i}{-6} = \frac{i}{3}$ and $z_1 = 3i$. Thus

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin(\theta))^2} = \oint_{|z|=1} \frac{4zi}{9(z-3i)^2(z-i/3)^2} dz$$

Since both are double poles, we can use the method in lecture 10 to find it's residual. However, we're only interested in z_0 as $|z_0| = 1/3 < 1$. We thus find,

$$\text{Res}(z_0) = \lim_{z \to i/3} \frac{d}{dz}\left((z-i/3)^2 \frac{4zi}{9(z-3i)^2(z-i/3)^2}\right) = \lim_{z \to i/3} \frac{d}{dz}\left(\frac{4zi}{9(z-3i)^2}\right).$$

Here we have that

$$\frac{d}{dz}\left(\frac{4zi}{9(z-3i)^2}\right) = \frac{-36i(z-3i)^2 + 72zi(z-3i)}{81(z-3i)^4} = \frac{-36zi - 108 + 72zi}{81(z-3i)^3} = 4i\frac{z+3i}{9(z-3i)^3}.$$

Thus,

$$\text{Res}(z_0) = 4i\frac{10/3i}{9*(8i/3)^3} = -\frac{5}{64}i$$

Thus,

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin(\theta))^2} = 2\pi i \text{Res}(z_0) = \frac{5\pi}{32}$$

3. Use a sector with radius R centered at the origin with angle $0 \leq \theta \leq \frac{2\pi}{5}$ to find, for $a > 0$,

$$\int_0^\infty \frac{dx}{x^5 + a^5} = \frac{\pi}{5a^4 \sin(\pi/5)}.$$

Solution:

Denote the line Re^{it}, with $0 \leq t \leq \frac{2\pi}{5}$ by $C_{\tilde{R}}$, the line between 0 and $Re^{\frac{2\pi}{5}i}$ by $l_{\tilde{R}}$, the line from 0 to R by l_R and C_R the union between the previous three. First we find

$$\int_{C_R} \frac{dz}{z^5 + a^5},$$

the roots of the denominator are given by $z_k^5 = -a^5 = a^5 e^{i\pi} = a^5(\cos(\pi+2k\pi)+i\sin(\pi+2k\pi))$, thus, $z_k = a(\cos(\frac{\pi}{5} + \frac{2k}{5}\pi) + i\sin(\frac{\pi}{5} + \frac{2k}{5}\pi))$ the root within C_R (if $R > a$) is

$$z_0 = a(\cos(\frac{\pi}{5}) + i\sin(\frac{\pi}{5})) = ae^{\frac{\pi}{5}i}.$$

Since this is a simple pole, we find it by using the trick from lecture 11 (same as in Ex. 1d):

$$\text{Res}(ae^{\frac{\pi}{5}i}) = \frac{1}{5a^4 e^{\frac{4\pi}{5}i}}$$

Thus,

$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{z^5 + a^5} = \frac{2\pi i}{5a^4 e^{\frac{4\pi}{5}i}}$$

Calculating the integral over $C_{\tilde{R}}$ we get

$$|\int_{C_{\tilde{R}}} \frac{dz}{z^5 + a^5}| = |\int_0^{2\pi/5} \frac{Rie^{it}}{R^5 e^{5it} + a^5} dt| \leq \int_0^{2\pi/5} \left| \frac{R}{|R^5| - |a^5|} \right| dt$$

Taking the limit of the function for $R \to \infty$, we get

$$0 \leq \lim_{R\to\infty} |\int_{C_{\tilde{R}}} \frac{dz}{z^5 + a^5}| \leq \int_0^{2\pi/5} \lim_{R\to\infty} \left| \frac{R}{|R^5| - |a^5|} \right| dt = 0$$

Thus,

$$\lim_{R\to\infty} \int_{C_{\tilde{R}}} \frac{dz}{z^5 + a^5} = 0$$

The integral over $l_{\tilde{R}}$, with parametrisation $(1-\tilde{t})Re^{\frac{2\pi}{5}i}$ for $0 \leq \tilde{t} \leq 1$, is

$$\begin{aligned}
\int_{l_{\tilde{R}}} \frac{dz}{z^5 + a^5} &= \int_0^1 \frac{-Re^{\frac{2\pi}{5}i}}{(1-\tilde{t})^5 R^5 e^{2\pi i} + a^5} d\tilde{t} \overset{t=1-\tilde{t}}{=} \int_0^1 \frac{-Re^{\frac{2\pi}{5}i}}{t^5 R^5 + a^5} dt \\
&= \int_0^1 \frac{-R}{R^5 t^5 + a^5} dt = e^{\frac{2\pi}{5}i} \int_0^1 \frac{-R}{R^5 t^5 + a^5} dt \\
&\overset{x=Rt}{=} -e^{\frac{2\pi}{5}i} \int_0^R \frac{dx}{x^5 + a^5}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{2\pi i}{5a^4 e^{\frac{4\pi}{5}i}} &= \lim_{R\to\infty} \oint_{C_R} \frac{dz}{z^5+a^5} = \lim_{R\to\infty}\left(\int_{l_R}\frac{dx}{x^5+a^5}+\int_{l_{\tilde{R}}}\frac{dx}{x^5+a^5}+\int_{C_{\tilde{R}}}\frac{dx}{x^5+a^5}\right)\\
&= \lim_{R\to\infty}\left(\int_0^R \frac{dx}{x^5+a^5} - e^{\frac{2\pi}{5}i}\int_0^R\frac{dx}{x^5+a^5}\right)\\
&= \left(1-e^{\frac{2\pi i}{5}}\right)\int_0^\infty \frac{dx}{x^5+a^5}
\end{aligned}$$

In conclusion

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^5+a^5} &= \frac{2\pi i}{5a^4 e^{\frac{4\pi}{5}i}(1-e^{\frac{2\pi}{5}i})} = \frac{2\pi i}{5a^4(e^{\frac{4\pi}{5}i}-e^{\frac{6\pi}{5}i})} = \frac{2\pi i}{5a^4 e^{\pi i}(e^{\frac{-\pi}{5}i}-e^{\frac{\pi}{5}i})} = \frac{\pi}{5a^4\frac{e^{\frac{\pi}{5}i}-e^{\frac{-\pi}{5}i}}{2i}}\\
&= \frac{\pi}{5a^4\sin(\pi/5)}
\end{aligned}$$