AMATH 567: Applied Complex Variables. Autumn 2022

Homework Assignment 1

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1. Express each of the following in polar exponential form:

   (a) $-i$

   (b) $1+i$

   (c) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

   ### Solution:

   The polar exponential form of $z = a+bi$ is $z = re^{\theta i} = r(\cos(\theta)+i\sin(\theta))$, with $r = \sqrt{a^2+b^2}$ the magnitude of z and $\theta = \tan^{-1}(\frac{b}{x})$ the angle between the positive x-axis and the line from 0 to z. Here θ can be $\tilde{\theta} = \theta + 2\pi$ as well, however, We will only be calculating the θ in $[-\pi, \pi]$. Thus,

   (a) If $z = -i$, then $r = \sqrt{(-1)^2} = 1$ and, using the fact that θ is the angle between the positive x-axis and the line from 0 to z,$\theta = -\frac{\pi}{2}$. Thus,

   $$z = e^{-\frac{\pi}{2}i}$$

   (b) If $z = 1+i$, then $r = \sqrt{1^2+1^2} = \sqrt{2}$ and $\theta = \tan^{-1}(1) = \frac{\pi}{4}$. Thus,

   $$z = \sqrt{2}e^{\frac{\pi}{4}i}$$

   (c) If $z = \frac{1}{2}+\frac{\sqrt{3}}{2}i$, then $r = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$ and $\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$. Thus,

   $$z = e^{\frac{\pi}{3}i}$$

2. Express each of the following in the form of $a+bi$, where a and b are real.

(a) $e^{2+i\pi/2}$

(b) $\frac{1}{1+i}$

(c) $(1+i)^3$

(d) $|3+4i|$

(e) $\cos(i\pi/4+c)$, with $c \in \mathbb{R}$

Solution:

(a) $e^{2+i\pi/2} = e^2(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = e^2 i$

(b) $\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1}{2} - \frac{1}{2}i$

(c) $(1+i)^3 = 1^3 + 3 * 1^2 i + 3 * 1 i^2 + i^3 = -2 + 2i$

(d) $|3+4i| = \sqrt{3^2+4^2} = 5$

(e) $\cos(i\pi/4+c)$, with $c \in \mathbb{R}$. We know that $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$, then

$$\begin{aligned}
\cos(i\pi/4+c) &= \frac{e^{-\pi/4+ci} + e^{\pi/4-ci}}{2} = \frac{e^{-\pi/4}(\cos(c)+i\sin(c)) + e^{\pi/4}(\cos(c)-i\sin(c))}{2} \\
&= \frac{e^{-\pi/4}+e^{\pi/4}}{2}\cos(c) + \frac{e^{-\pi/4}-e^{\pi/4}}{2}\sin(c)i \\
&= \cosh(\pi/4)\cos(c) - \sinh(\pi/4)\sin(c)i
\end{aligned}$$

3. Solve for the roots of the following equation:

(a) $z^3 = 4$

(b) $z^4 = -1$

Solution:

(a) This problem comes down to $z^3 - 4 = 0$. Since $z = \sqrt[3]{4}$ is clearly is root, we get (using Horner): $z^3 - 4 = (z - \sqrt[3]{4})(x^2 + \sqrt[3]{4}x + 2\sqrt[3]{2}) = 0$. Finding roots of the second part is easy: $D = -6\sqrt[3]{2} \Rightarrow z = \frac{-\sqrt[3]{4} \pm \sqrt{6}\sqrt[6]{2}i}{2} = \frac{-\sqrt[3]{4} \pm \sqrt[6]{2^4 3^3}i}{2} = -\frac{\sqrt[3]{4}}{2} \pm \frac{\sqrt[3]{4}\sqrt{3}}{2}i$. The roots are thus,

$$z = \sqrt[3]{4}, \qquad -\frac{\sqrt[3]{4}}{2} + \frac{\sqrt[3]{4}\sqrt{3}}{2}i \quad \text{and} \quad -\frac{\sqrt[3]{4}}{2} - \frac{\sqrt[3]{4}\sqrt{3}}{2}i$$

(b) This problem comes down to $z^4+1=0$, but $z^4+1=(z^2)^2-i^2=(z^2-i)(z^2+i)$. We thus want to solve $z^2=-i$ and $z^2=i$. In order to find the square root of i and $-i$, we convert them to their polar form:

$$i=\cos(\frac{\pi}{2})+\sin(\frac{\pi}{2})i \qquad \text{and} \qquad -i=\cos(-\frac{\pi}{2})+\sin(-\frac{\pi}{2})i$$

We know that if $z=r(\cos(\theta)+i\sin(\theta)$, that $z^p=r^p(\cos(p(\theta+2k\pi))+i\sin(p(\theta+2k\pi))$ with $p\in\mathbb{R}$ and $k\in\mathbb{N}$. Thus,

$$\sqrt{i}=\cos(\frac{\pi}{4}+k\pi)+\sin(\frac{\pi}{4}+k\pi)i \qquad \text{and} \qquad \sqrt{-i}=\cos(-\frac{\pi}{4}+k\pi)+\sin(-\frac{\pi}{4}+k\pi)i, k\in\mathbb{N}$$

The roots are thus

$$z=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, -(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i), \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i \text{ and } -\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$$

4. Establish the following result:

(a) $(z+w)^*=z^*+w^*$

(b) $\mathrm{Re}(z)\leq|z|$

(c) $|wz^*+w^*z|\leq 2|wz|$

(d) $|z_1z_2|=|z_1||z_2|$

Solution:

(a) Suppose that $z=a+bi$ and $w=c+di$. We then have,

$$(z+w)^*=(a+c+(b+d)i)^*=a+c-(b+d)i=a-bi+c-di=z^*+w^*$$

(b) Suppose that $z=a+bi$, then $\mathrm{Re}(z)=a$ and $|z|=\sqrt{a^2+b^2}$. We now have,

$$|z|=\sqrt{a^2+b^2}\geq\sqrt{a^2}=|a|\geq a=\mathrm{Re}(z)$$

(c) Suppose that $z=a+bi$ and $w=c+di$. Then,

$$|wz^*+w^*z|=|(c+di)(a-bi)+(c-di)(a+bi)|=|2ac+2db|=2|ac+db|$$

On the other hand we have

$$\begin{aligned}2|wz|=2|ac+db+(ad-bc)i|&=2\sqrt{(ac+db)^2+((ad-bc))^2}\\&\geq 2\sqrt{(ac+db)^2}=2|ac+db|=|wz^*+w^*z|\end{aligned}$$

(d) Suppose that $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. We then have,

$$\begin{aligned}
|z_1 z_2| &= |(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2} \\
&= \sqrt{(a_1 a_2)^2 + (b_1 b_2)^2 + (a_1 b_2)^2 + (a_2 b_1)^2} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \\
&= \sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2} = |z_1||z_2|
\end{aligned}$$