AMATH 568: Homework 4
Winter Quarter 2023
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Due: Febuary 3, 2023

1. Consider the weakly nonlinear oscillator:

$$\frac{d^2y}{dt^2} + y + \epsilon y^5 = 0$$

with $y(0) = 0$ and $y'(0) = A > 0$ and with $0 < \epsilon \ll 1$.

(a) Use a regular perturbation expansion and calculate the first two terms.

(b) Determine at what time the approximation of part (a) fails to hold.

(c) Use a Poincare-Lindstedt expansion and determine the first two terms and frequency corrections.

(d) For $\epsilon = .1$ plot the numerical solution (from MATLAB), the regular expansion solution, and the Poincare-Lindstedt solution for $0 \le t \le 20$.

Solution:

(a) Say that $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$, filling this in gives,

$$\begin{aligned} 0 &= \frac{d^2 y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots}{dt^2} + y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots + \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots)^5 \\ &= \frac{d^2 y_0}{dt^2} + \epsilon\frac{d^2 y_1}{dt^2} + \epsilon^2\frac{d^2 y_2}{dt^2} + y_0 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon y_0^5 + 5\epsilon^2 y_1 y_0^4 + \dots \end{aligned}$$

Therefore we have:

$$\mathcal{O}(\epsilon^0): \qquad \frac{d^2y_0}{dt^2} + y_0 = 0$$

$$\mathcal{O}(\epsilon^1): \qquad \frac{d^2y_1}{dt^2} + y_1 + y_0^5 = 0.$$

Assume that these ϵ can vary and we still have that y must fulfill the initial conditions. Since $y(0) = 0$ holds for all ϵ, we have that $y_i = 0,\ \forall i \ge 0$. Furthermore since $y'(0) = A$ for all ϵ, we have that $y_0'(0) = A$ and $y_i'(0) = 0,\ \forall i > 0$. Therefore, we can solve the first equation:

$$\frac{d^2y_0}{dt^2} + y_0 = 0.$$

The solution is generally of form $y_0 = a\cos(t) + b\sin(t)$. Since $y_0(0) = 0$, we have that $a = 0$, since $y_0'(0) = b\cos(0) = A$, we have that $b = A$ and thus

$$y_0 = A\sin(t).$$

Now that we know y_0, we can find y_1 :

$$\frac{d^2y_1}{dt^2} + y_1 = -A^5\sin(t)^5.$$

First we calculate the homogeneous solution: $\frac{d^2y_1}{dt^2} + y_1 = 0$. Once again, this has the form $y_1 = A\cos(t) + B\sin(t)$. But since $y_1(0) = y_1'(0) = 0$, we have that $y_1 = 0$ is the only homogeneous solution. We now try to calculate the particular solution. Using Wolfram Mathematica, we get that the general solution is:

$$y_1 = \left(\frac{5}{16}A^5 t + c_1\right)\cos(t) - \frac{1}{192}\left[47 + 14\cos(2t) - \cos(4t)\right]A^5\sin(t) + c_2\sin(t).$$

We find c_1 and c_2 by using the conditions: $y_1(0) = 0$ gives us that $c_1 = 0$ and $\frac{dy_1}{dt}(0) = 0$ gives us,

$$\frac{dy_1}{dt}(0) = \frac{5}{16}A^5\cos(0) - \frac{1}{192}[47 + 14\cos(0) - \cos(0) + \frac{192}{A^5}c_2]A^5\cos(0) = \frac{5}{16}A^2 - \frac{5}{16}A^2 + c_2 = c_2,$$

therefore $c_2 = 0$. Finaly we have y_1:

$$y_1 = \frac{A^5}{16}\left(5t\cos(t) - \frac{1}{12}[47 + 14\cos(2t) - \cos(4t)]\sin(t)\right).$$

In conclusion, the solution to two terms using regular perturbation expansion is given by

$$y = A\sin(t) + \epsilon\frac{A^5}{16}\left(5t\cos(t) - \frac{1}{12}[47 + 14\cos(2t) - \cos(4t)]\sin(t)\right)$$

(b) One can show that

$$\sin^5(t) = \frac{10\sin(t) - 5\sin(3t) + \sin(5t)}{16},$$

A solvability condition is that the expression to the right must be orthogonal to the nullspace of the adjoint of the operate to the left of the equation (in this case $L(y) = \frac{d^2y}{dt^2} + y$). Note that we know that this is self adjoint (see HW 1) and it is easy to see that $\sin(t)$ is the basis of the nullspace of this operator in these settings (it's the same as how we calculated y_0 since it's self adjoint). Therefore, we must have that

$$\langle -A^5\sin^5(t), \sin(t)\rangle = 0,$$

but

$$\langle -A^5\sin^5(t), \sin(t)\rangle = \langle -A^5\frac{10\sin(t) - 5\sin(3t) + \sin(5t)}{16}, \sin(t)\rangle = \langle -\frac{5A^5}{8}\sin(t), \sin(t)\rangle \neq 0.$$

The Fredholm alternative theorem cannot be satisfied. Since we have no variables left to choose, for example a frequency for a frequency shift, we know that this is going to be problematic at infinity or even at $t = \frac{1}{\epsilon}$. Indeed, even the solution hints at this since we have the red part

$$y = A\sin(t) + \epsilon\frac{A^5}{16}\left(5t\cos(t) - \frac{1}{12}[47 + 14\cos(2t) - \cos(4t)]\sin(t)\right).$$

t will blow up the solution, and even when $t \geq \frac{1}{\epsilon}$, we no longer have something that works with small perturbations.

(c) If we were to implement $\tau = (\omega_0 + \epsilon\omega_1 + \dots)t = \omega(\epsilon)t$ we get that $\frac{d\tau}{dt} = \omega(\epsilon)$ therefore, we have that

$$\omega^2\frac{d^2y}{d\tau^2} + y + \epsilon y^5 = 0.$$

Let's take $\omega_0 = 1$ for the ease of writing, it is easy to see that we can get back to the general case by just multiplying the new τ by the ω_0 that we want. If we were to now also use $y = y_0 + \epsilon y_1 + \dots$, we get (not calculating past ω_1 and y_1):

$$\omega_0^2\frac{d^2y_0}{d\tau^2} + \epsilon(\omega_0^2\frac{d^2y_1}{d\tau^2} + 2\omega_0\omega_1\frac{d^2y_0}{d\tau^2}) + y_0 + \epsilon y_1 + \epsilon y_0^5 + \dots = 0,$$

thus,

$$\mathcal{O}(\epsilon^0): \ \frac{d^2y_0}{d\tau^2} + y_0 = 0$$
$$\mathcal{O}(\epsilon^1): \ \frac{d^2y_1}{d\tau^2} + y_1 = -(y_0^5 + 2\omega_1\frac{d^2y_0}{d\tau^2}).$$

The initial conditions for y_0 and y_1 have not changed since (a), thus we can just look over there to see that

$$y_0 = A\sin(\tau)$$

Filling into the second equation, we get

$$\frac{d^2y_1}{d\tau^2} + y_1 = -A(A^4\sin(\tau)^5 - 2\omega_1\sin(\tau)).$$

Knowing that

$$\sin^5(\tau) = \frac{10\sin(\tau) - 5\sin(3\tau) + \sin(5\tau)}{16},$$

We want the solution to also be in this form, thus:

$$y_1 = B\sin(\tau) + C\sin(3\tau) + D\sin(5\tau).$$

Substituting all of this gives us the following equations

$$\begin{cases} A(A^4\frac{10}{16} - 2\omega_1) & = -B + B = 0 \\ -9C + C & = \frac{5}{16}A^5 \\ -25D + D & = -\frac{A^5}{16} \end{cases} \Rightarrow \begin{cases} \omega_1 & = A^4\frac{5}{16} \\ C & = -\frac{5}{128}A^5 \\ D & = \frac{1}{384}A^5 \end{cases}$$

Note that the first equation in the system is pretty much the solvability condition. Also note that we did not find a value for B, thus one would not find a unique solution one must think. But we also have our conditions. Because of $y_1'(0) = 0$, we have that

$$B + 3C + 5D = 0 \Rightarrow B = \frac{5}{48}A^5.$$

Therefore, our solution is

$$y_1 = \frac{5}{48}A^5\sin(\tau) - \frac{5}{128}A^5\sin(3\tau) + \frac{1}{384}A^5\sin(5\tau),$$

where $\tau = (1 + A^4\frac{5}{16}\epsilon)t$. Thus the Poincare-Lindsteds expansion for the first two terms and frequency correction is given by.

$$\begin{aligned} y = &A\sin\left((1 + A^4\frac{5}{16}\epsilon)t\right) \\ &+ \epsilon\left[\frac{5}{48}A^5\sin\left((1 + A^4\frac{5}{16}\epsilon)t\right) - \frac{5}{128}A^5\sin\left(3(1 + A^4\frac{5}{16}\epsilon)t\right) + \frac{1}{384}A^5\sin\left(5(1 + A^4\frac{5}{16}\epsilon)t\right)\right] \end{aligned}$$

(d) For $A = 1$, plotting all of them together gives,

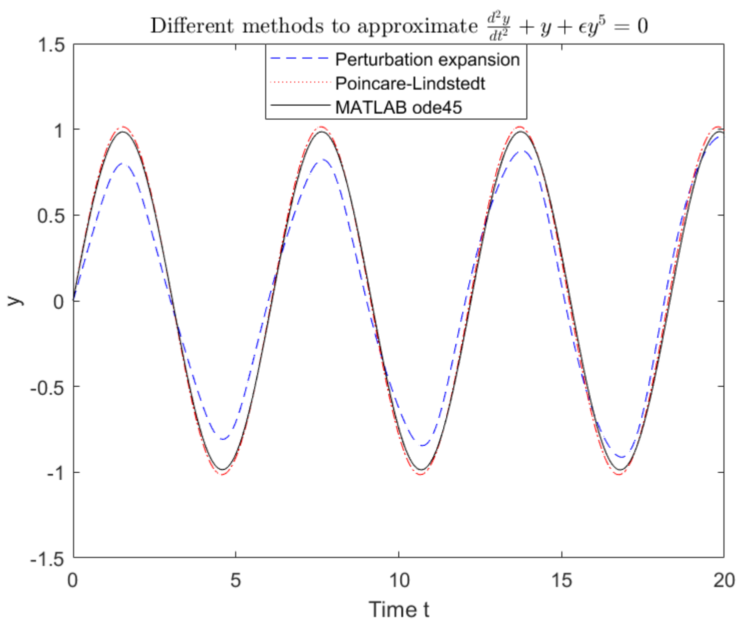


Figure 1: Using the regular perturbation expansion, Poincare-Lindstedt expansion and the built-in MATLAB function "ode45" we approximated the solution to the ode given by $\frac{d^2y}{dt^2} + y + \epsilon y^5 = 0$, with $y(0) = 0$ and $\frac{dy}{dt}(0) = 1$.

As we would think, the Poincare-Lindstedt expansion worked better than the Perturbation expansion.

2. Consider Rayleigh's equation:

$$\frac{d^2y}{dt^2} + y + \epsilon\left[-\frac{dy}{dt} + \frac{1}{3}\left(\frac{dy}{dt}\right)^3\right] = 0$$

which has only one periodic solution called a "limit cycle" ($0 \leq \epsilon \ll 1$). Given

$$y(0) = 0$$

and

$$\frac{dy(0)}{dt} = A$$

(a) Use a multiple scale expansion to calculate the leading order behavior.

(b) Use a Poincare-Lindsted expansion and an expansion of $A = A_0 + A\epsilon A_1 + \ldots$ to calculate the leading-order solution and the first non-trivial frequency shift for the limit cycle.

(c) For $\epsilon = 0.01, 0.1, 0.2$ and 0.3, plot the numerical solution and the multiple scale expansion for $0 \leq t \leq 40$ and for various values of A for your multiple scale solution. Also plot the limit cycle solution calculated from part (b)

(d) Calculate the error

$$E(t) = |y_{numerical}(t) - y_{approximation}(t)|$$

as a function of time ($0 \leq t \leq 40$) using $\epsilon = 0.01, 0.1, 0.2$ and 0.3.

Solution:

(a) Say that $\tau = \epsilon t$, then we can find $y = y_0(t) + \epsilon y_1(t,\tau) + \epsilon^2 y_2(t,\tau) + \ldots$. Before we fill this in, we note that

$$\frac{dy}{dt} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial y}{\partial t} + \epsilon\frac{\partial y}{\partial \tau},$$

and

$$\frac{d^2y}{dt^2} = \frac{\partial^2 y}{\partial t^2} + 2\epsilon\frac{\partial^2 y}{\partial t\partial\tau} + \epsilon^2\frac{\partial^2 y}{\partial \tau^2}.$$

If we were to now fill in the approximation for y, we get

$$\frac{\partial^2 y_0}{\partial t^2} + 2\epsilon\frac{\partial^2 y_0}{\partial t\partial\tau} + \epsilon\frac{\partial^2 y_1}{\partial t^2} + y_0 + \epsilon y_1 + \epsilon\left[-\frac{\partial y_0}{\partial t} + \frac{1}{3}\left(\frac{\partial y_0}{\partial t}\right)^3\right] + \cdots = 0.$$

The leading order term is

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0,$$

with $y_0(0) = 0$ and $\frac{dy_0}{dt}(0) = \frac{\partial y_0}{\partial t}(0) + \epsilon\frac{\partial y_0}{\partial \tau}(0) = A, \quad \forall\epsilon$, thus $\frac{\partial y_0}{\partial t}(0) = A$. Note that now we would like to solve this, the general form is

$$y_0 = B\cos(t) + C\sin(t),$$

but note here that B and C are independent of t, but not of τ. We ignore that τ depends on t for a moment. Thus,

$$y_0 = B(\tau)\cos(t) + C(\tau)\sin(t).$$

Now that we know this, we can go to the next order term, given by

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = \frac{\partial y_0}{\partial t} - \frac{1}{3}\left(\frac{\partial y_0}{\partial t}\right)^3 - 2\frac{\partial^2 y_0}{\partial t\partial\tau},$$

filling in the previous solution, we get

$$
\begin{aligned}
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= (-B + 2B_\tau)\sin(t) + (C - 2C_\tau)\cos(t) + \frac{1}{3}\left(B\sin(t) - C\cos(t)\right)^3 \\
&= (-B + 2B_\tau)\sin(t) + (C - 2C_\tau)\cos(t) \\
&\quad + \frac{1}{3}\left(B^3\sin(t)^3 - 3B^2C\sin(t)^2\cos(t) + 3BC^2\sin(t)\cos(t)^2 - C^3\cos(t)^3\right) \\
&= (-B + 2B_\tau)\sin(t) + (C - 2C_\tau)\cos(t) \\
&\quad + \frac{1}{3}\left(B^3\sin(t)^3 - 3B^2C\cos(t) + 3B^2C\cos(t)^3 + 3BC^2\sin(t) - 3BC^2\sin(t)^3 - C^3\cos(t)^3\right) \\
&= (2B_\tau - B + BC^2)\sin(t) + (-2C_\tau + C - B^2C)\cos(t) \\
&\quad + \frac{1}{3}\left((B^3 - 3BC^2)\frac{3\sin(t) - \sin(3t)}{4} + (3B^2C - C^3)\frac{3\cos(t) + \cos(3t)}{4}\right) \\
&= (2B_\tau - B + \frac{1}{4}B^3 + BC^2 - \frac{3}{4}BC^2)\sin(t) - (2C_\tau - C + B^2C - \frac{3}{4}B^2C + \frac{1}{4}C^3)\cos(t) \\
&\quad - \frac{B^3/3 - BC^2}{4}\sin(3t) + \frac{B^2C - C^3/3}{4}\cos(3t) \\
&= (2B_\tau - B + \frac{1}{4}B^3 + \frac{1}{4}BC^2)\sin(t) - (2C_\tau - C + \frac{1}{4}C^3 + \frac{1}{4}B^2C)\cos(t) \\
&\quad - \frac{B^3/3 - BC^2}{4}\sin(3t) + \frac{B^2C - C^3/3}{4}\cos(3t)
\end{aligned}
$$

Since we want the solvability condition to hold (see previous exercise), we want

$$
\begin{aligned}
&\begin{cases} 2B_\tau &= B - B^3/4 - BC^2/4 \\ 2C_\tau &= C - C^3/4 - B^2C/4 \end{cases} \\
\Rightarrow &\begin{cases} 2BB_\tau &= B^2 - B^4/4 - B^2C^2/4 \\ 2CC_\tau &= C^2 - C^4/4 - B^2C^2/4 \end{cases} \\
\Rightarrow &(B^2 + C^2)_\tau = (B^2 + C^2) - \frac{B^4}{4} - \frac{C^2B^2}{2} - \frac{C^4}{4} \\
\Rightarrow &(B^2 + C^2)_\tau = (B^2 + C^2) - \frac{1}{4}(B^2 + C^2)^2 \\
\Rightarrow &B^2 + C^2 = \frac{4\alpha^2}{\alpha^2 + (4 - \alpha^2)e^{-\tau}}.
\end{aligned}
$$

Let's find α. Since $y_0(0) = 0$, $B(0) = 0$ and since $\frac{\partial y_0}{\partial t}(0) = A$, $C(0) = A$. Thus, $B(0)^2 + C(0)^2 = A^2$ and $B(0)^2 + C(0)^2 = \alpha^2$, thus $\alpha = \pm A$ (C will have to fulfill its condition, thus $\alpha = A$). Let's chose $B = 0$, such that the conditions are satisfied. We then we have that

$$
C = \frac{2A}{(A^2 + (4 - A^2)e^{-\tau})^{1/2}}.
$$

In conclusion:

$$
y_0 = \frac{2A}{(A^2 + (4 - A^2)e^{-\epsilon t})^{1/2}}\sin(t)
$$

(b) Say that $A = A_0 + \epsilon A_1 + \dots$, we now have that, because of the conditions,

$$
\frac{dy_0}{dt}(0) + \epsilon\frac{dy_1}{dt}(0) + \dots = A_0 + \epsilon A_1 + \dots.
$$

Thus,

$$
\frac{dy_0}{dt}(0) = A_0,\ \frac{dy_1}{dt}(0) = A_1,\ \frac{dy_j}{dt}(0) = A_j,\ \forall j.
$$

Say now that we use $\tau = (1 + \omega_1\epsilon + \dots)t = \omega(\epsilon)t$, note that we took $\omega_0 = 1$ already (explanation in the previous exercise). Substituting this and the $y = y_0 + y_1\epsilon + \ldots$, gives us

$$\begin{aligned}
0 &= \frac{d^2y_0}{d\tau^2} + \epsilon(\frac{d^2y_1}{d\tau^2} + 2\omega_1\frac{d^2y_0}{d\tau^2}) + \epsilon^2\left((\omega_1^2 + 2\omega_2)\frac{d^2y_0}{d\tau^2} + 2\omega_1\frac{d^2y_1}{d\tau^2} + \frac{d^2y_2}{d\tau^2}\right) \\
&+ y_0 + \epsilon y_1 + \epsilon^2 y_2 \\
&+ \epsilon\left[-\frac{dy_0}{d\tau} - \epsilon\omega_1\frac{dy_0}{d\tau} - \epsilon\frac{dy_1}{d\tau} + \frac{1}{3}\left(\frac{dy_0}{d\tau}\right)^3 + \epsilon\omega_0^2\omega_1\left(\frac{dy_0}{d\tau}\right)^3\right] + \ldots.
\end{aligned}$$

It is self explanatory that the leading order solution is now

$$y_0 = A_0\sin(\tau).$$

The next term gives rise to the following equation

$$\frac{d^2y_1}{d\tau^2} + y_1 = \frac{dy_0}{d\tau} - \frac{1}{3}\left(\frac{dy_0}{d\tau}\right)^3 - 2\omega_1\frac{d^2y_0}{d\tau^2}.$$

We know that $\cos^3(\tau) = \frac{3\cos(\tau)+\cos(3\tau)}{4}$, filling everything in, gives us

$$\frac{d^2y_1}{d\tau^2} + y_1 = A_0\cos(\tau) - A_0^3\frac{3\cos(\tau) + \cos(3\tau)}{12} + 2\omega_1 A_0\sin(\tau).$$

In order for the solvability condition (spoken of in 1b) to hold, we need to have that

$$\langle A_0\cos(\tau) - A_0^3\frac{3\cos(\tau) + \cos(3\tau)}{12} + 2\omega_1 A_0\sin(\tau), \sin(\tau)\rangle = 2\omega_1 A_0\langle\sin(\tau), \sin(\tau)\rangle = 0.$$

If $A_0 \neq 0$, we know that $\omega_1 = 0$ must hold. Say that $A = 0$, then $y_0 = 0$ and after that the ode does not change since y_0 is zero. Thus we would only have trivial solutions. The equation then becomes

$$\frac{d^2y_1}{d\tau^2} + y_1 = A_0\cos(\tau) - A_0^3\frac{3\cos(\tau) + \cos(3\tau)}{12}.$$

Since we are being put through Algebra hell, we will be using Wolfram Mathematica. This ODE gives us the general form

$$y_1 = \frac{1}{96}\left((48A_0 - 10A_0^3)\cos(\tau) + A_0^3\cos(3\tau) + 12(4A_0 - A_0^3)t\sin(\tau)\right) + c_1\cos(\tau) + c_2\sin(\tau).$$

Note that there is a τ term, which is probably because $\cos(\tau)$ can be a part of the nullspace of $L = y_{\tau\tau} + y$. From the condition that $y_1(0) = 0$, we get

$$c_1 = -\frac{48A_0 - 9A_0^3}{96}.$$

Now from $\frac{dy_1}{dt}(0) = A_1$, we get

$$c_2 = A_1.$$

Therefore the next term solution is

$$y_1 = \frac{1}{96}\left(-A_0^3\cos(\tau) + A_0^3\cos(3\tau) + 12(4A_0 - A_0^3)\tau\sin(\tau)\right) + A_1\sin(\tau).$$

The derivative is

$$\frac{dy_1}{d\tau} = \frac{1}{96}\left(A_0^3\sin(\tau) - 3A_0^3\sin(3\tau) + 12(4A_0 - A_0^3)\sin(\tau) + 12(4A_0 - A_0^3)\tau\cos(\tau)\right) + A_1\cos(\tau)$$

Now, we just want the first frequency shift that is non-zero, we look at the next order terms ($\omega_1 = 0$):

$$\frac{d^2y_2}{d\tau^2} + y_2 = -2\omega_2\frac{d^2y_0}{d\tau^2} + \frac{dy_1}{d\tau}.$$

Filling everything in gives us

$$\frac{d^2y_2}{d\tau^2} + y_2 = 2\omega_2 A_0 \sin(t) + \frac{1}{96}\left(A_0^3\sin(\tau) - 3A_0^3\sin(3t) + 12(4A_0 - A_0^3)\sin(\tau) + 12(4A_0 - A_0^3)\tau\cos(\tau)\right) + A_1\cos(\tau)$$

$$= \frac{1}{96}\left[192\omega_2 A_0 + A_0^3 + 12(4A_0 - A_0^3)\right]\sin(\tau) - 3A_0^3\sin(3\tau) + 12(4A_0 - A_0^3)t\cos(\tau) + A_1\cos(\tau).$$

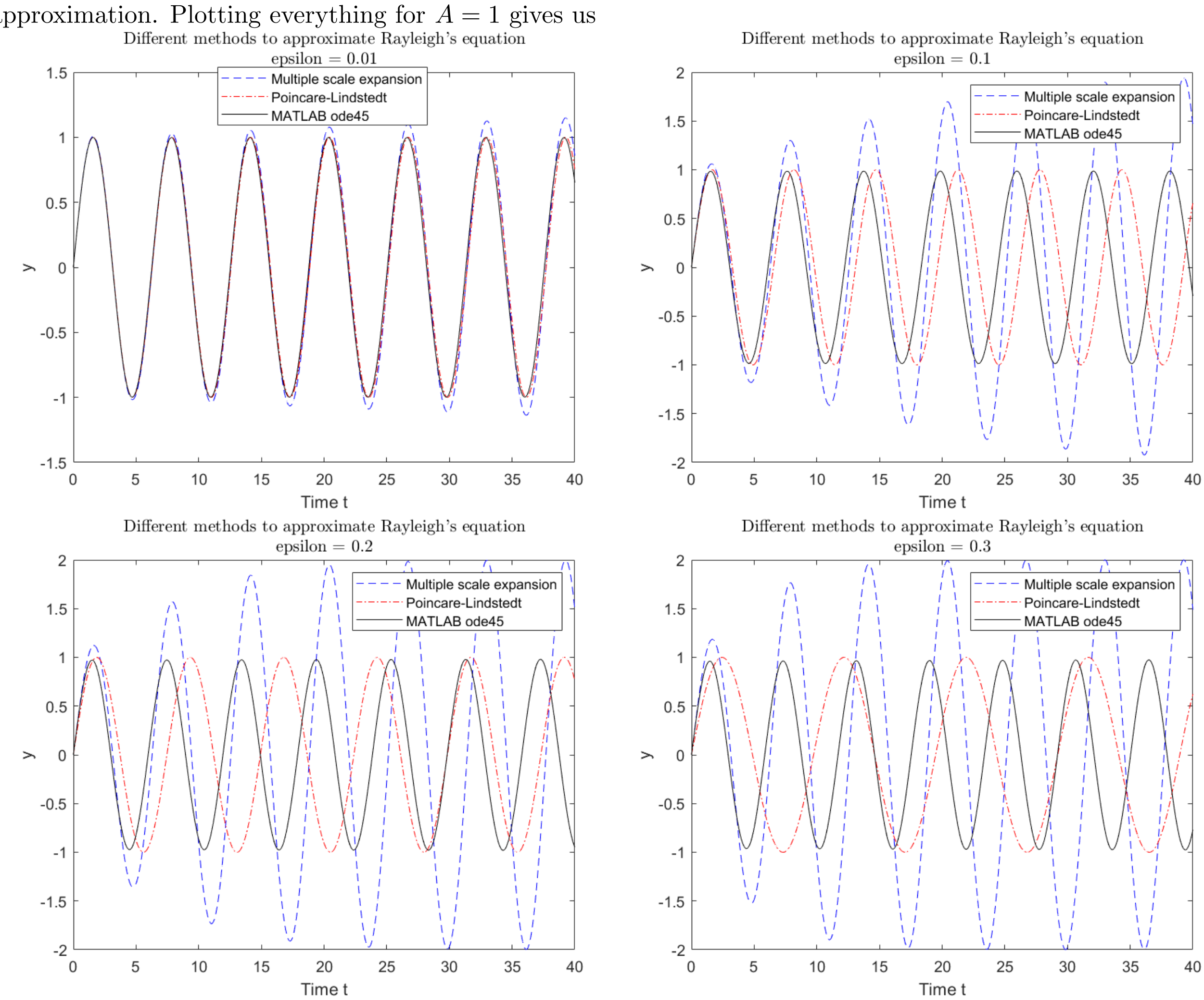
Since we want our solvability condition to hold, we want to be sure that

$$\frac{1}{96}\left[192\omega_2 A_0 + A_0^3 + 12(4A_0 - A_0^3)\right] = 0 \quad \Rightarrow \quad \omega_2 = \frac{1}{192}(11A_0^2 - 48) = \frac{11}{192}A_0^2 - 4.$$

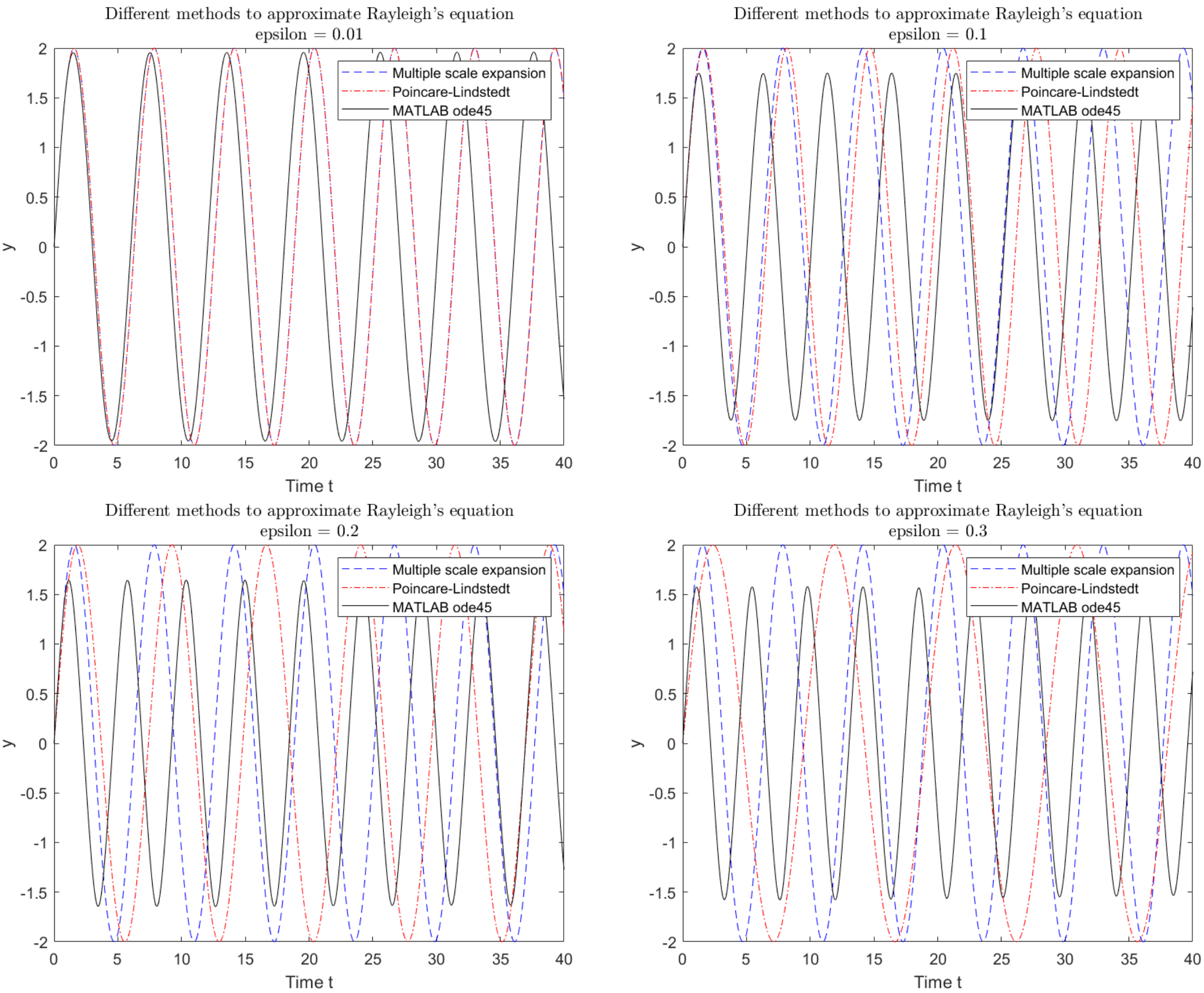
In conclusion, the leading order term is

$$y_0 = A_0\sin(\tau) = A_0\sin([1 + \epsilon^2(\frac{11}{192}A_0^2 - 4)]t), \text{ where } \omega_2 = \frac{11}{192}A_0^2 - 4$$

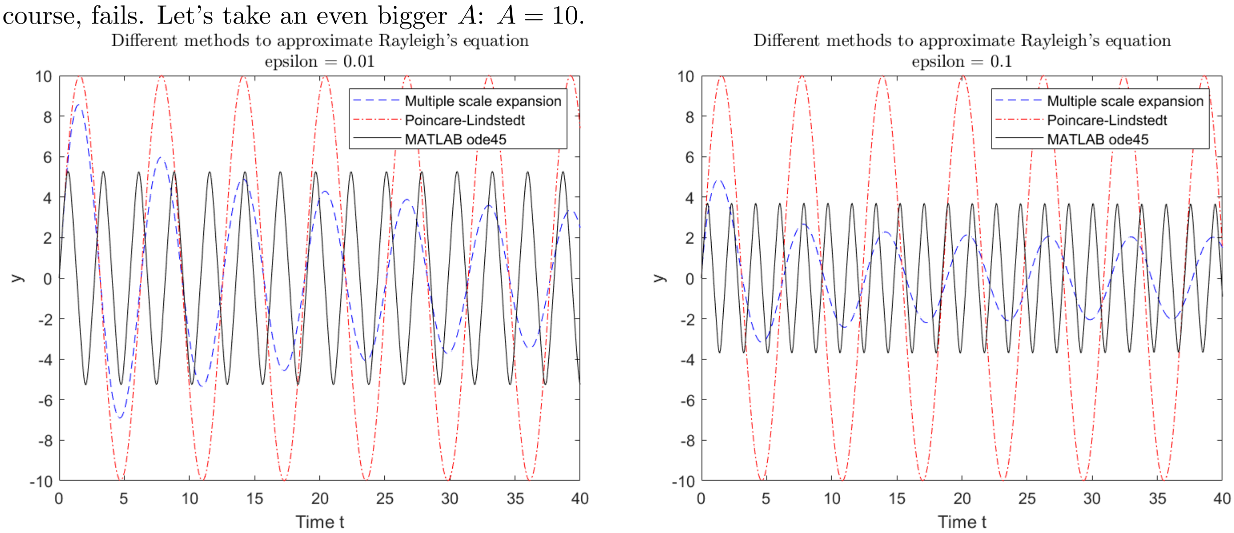
(c) Say we take $A = 0$, then we would of course just get the zero solution for all of them. An interesting A would be $A = \pm 2$. When looking at (b) and y_1, we see that there is an explicit τ in there, the only way it would disappear is if $A = 0$ or $A = \pm 2$. However, we do not include this order solution in the approximation. Plotting everything for $A = 1$ gives us

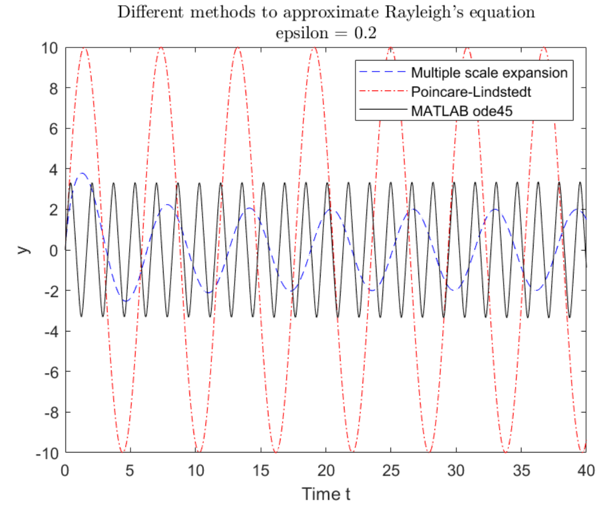


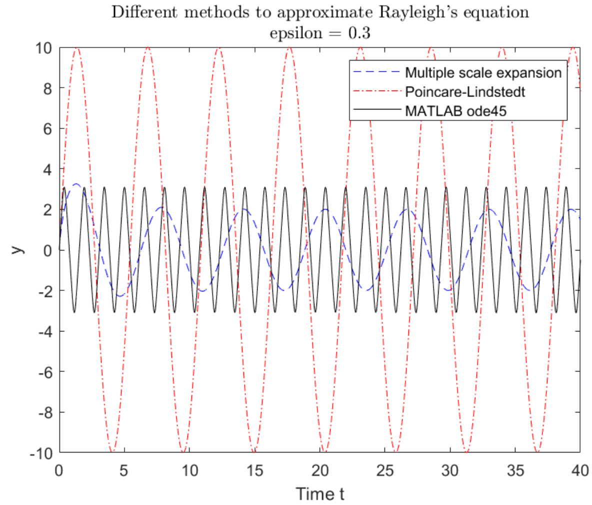
We see here that the multiple scale expansion is getting pretty bad, this is mainly because for τ big, the solution will go to $2\sin(t)$, and the numerical solutions eems to stay at $\sin(t)$ amplitude. The frequency shifting of the Poincare-Lindsted expansion seems to also be failing. Maybe we can ramp up the amplitude of the numerical solution by choosing A bigger, for $A = 2$ we get:



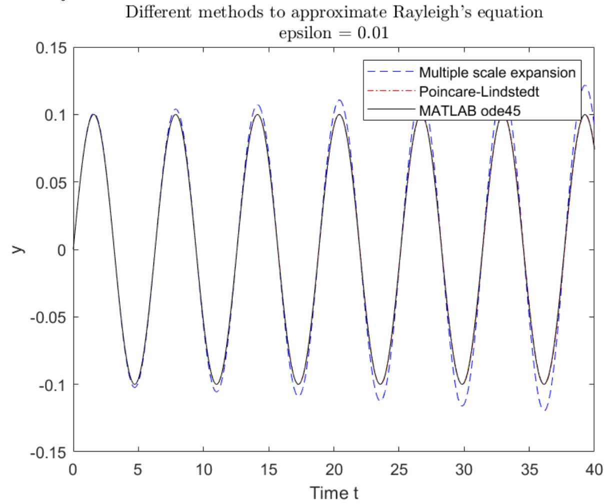
This is better, for epsilon small. The frequencies are problematic here. The higher it gets, the more it, of course, fails. Let's take an even bigger A: $A = 10$.

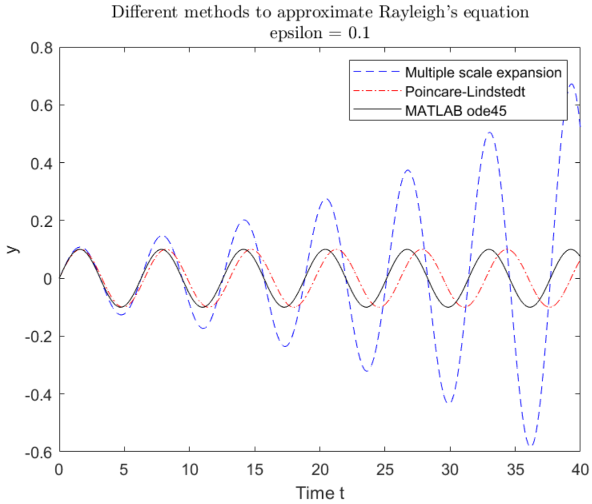


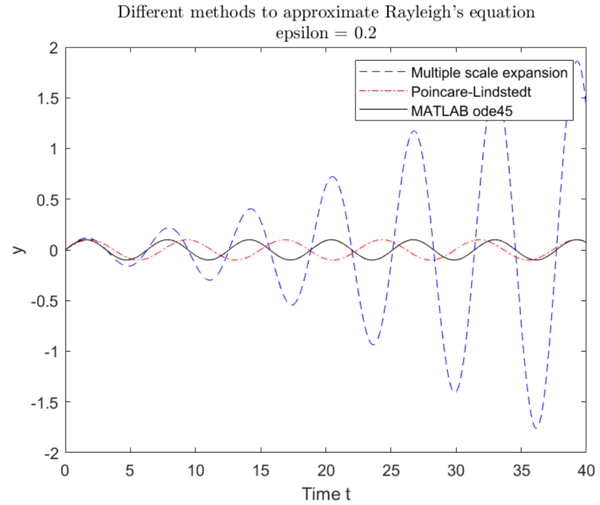


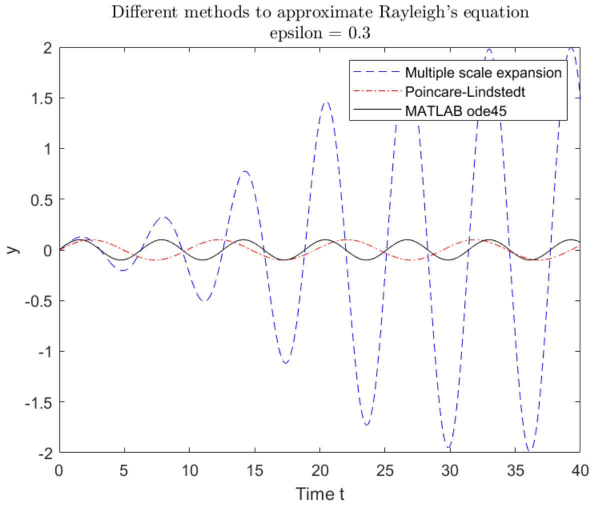


Here everything just fails, the frequency shifting for Pointcare-Linsted fails (and also for multiple scale) and amplitudes are horrible. It almost makes me think that my answers to a and b are wrong. Let's also try a small $A = .1$:



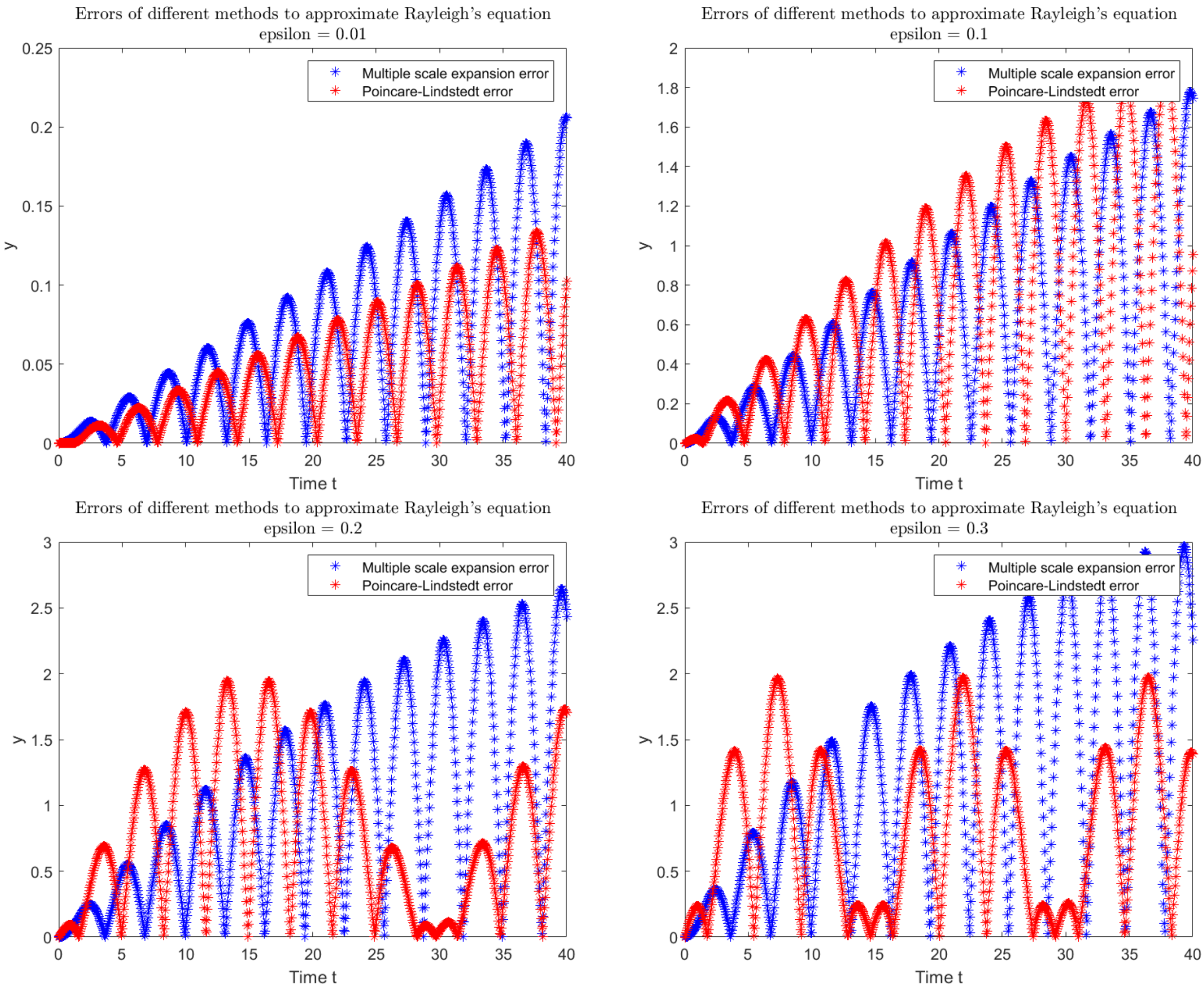




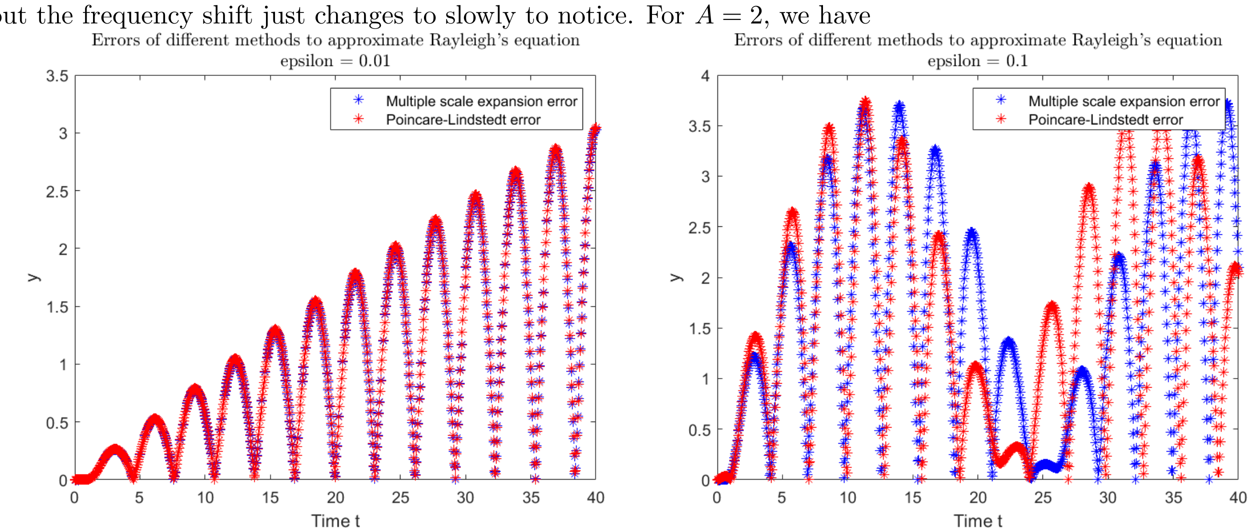


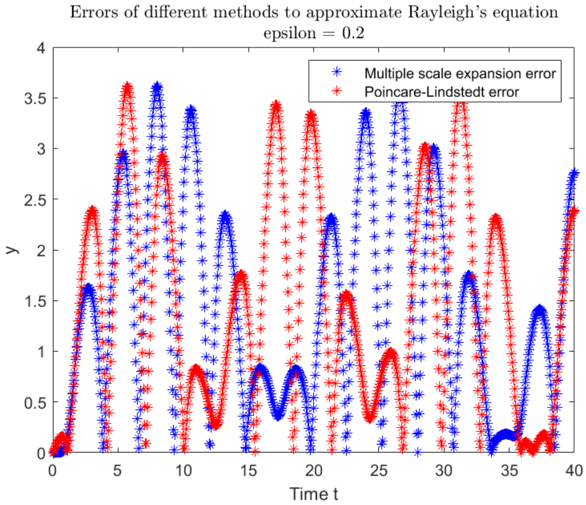
The frequency shifting is definitely not perfect, but it's oke for small epsilon. The amplitude of the multiple scale solution is bad eventually, which is to be expected as it goes to $2\sin(t)$, but it goes slower since A compensates $e^{-\epsilon\tau}$.

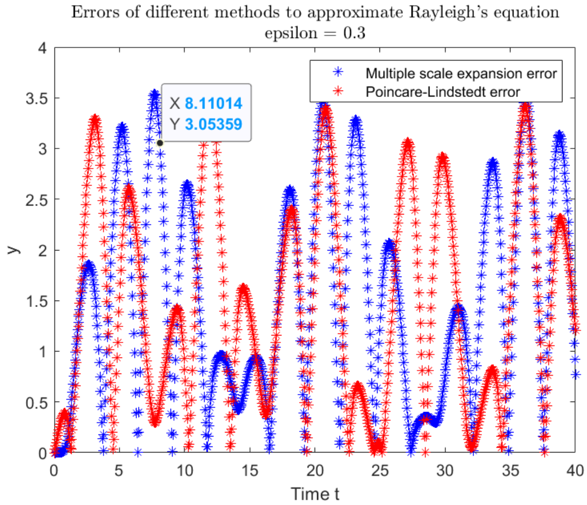
(d) For $A = 1$, we have



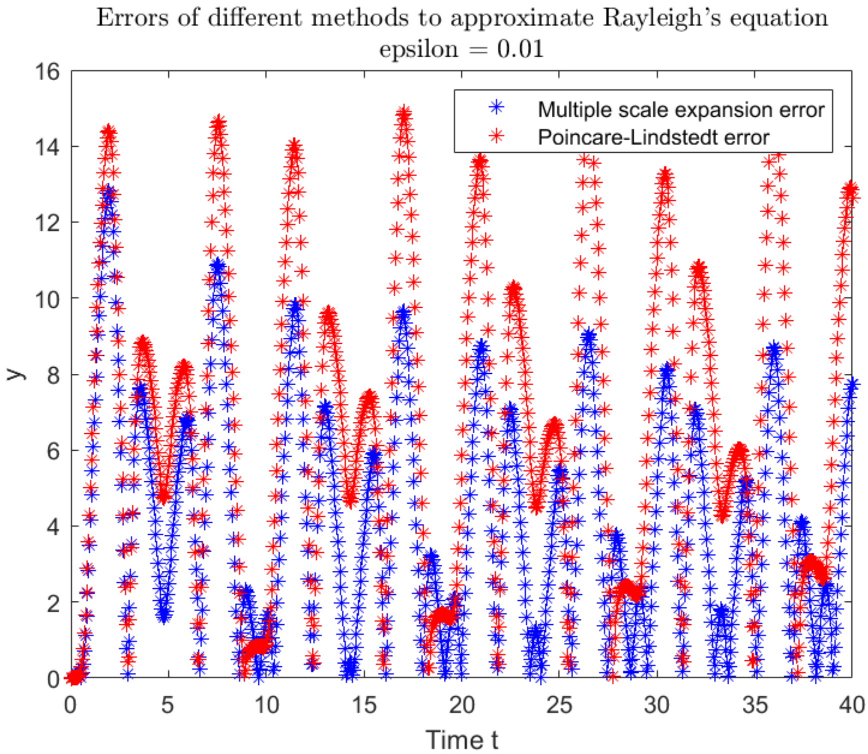
Let's first note that, for epsilon rising, the error will almost always rises (there are expections). It's clear here that the rising amplitudes play a mayor part with the multiple scale expansion error (as well as wrong frequencies). One might think that it also plays a big part with the Poincare-Lindstedt method, but the frequency shift just changes to slowly to notice. For $A = 2$, we have



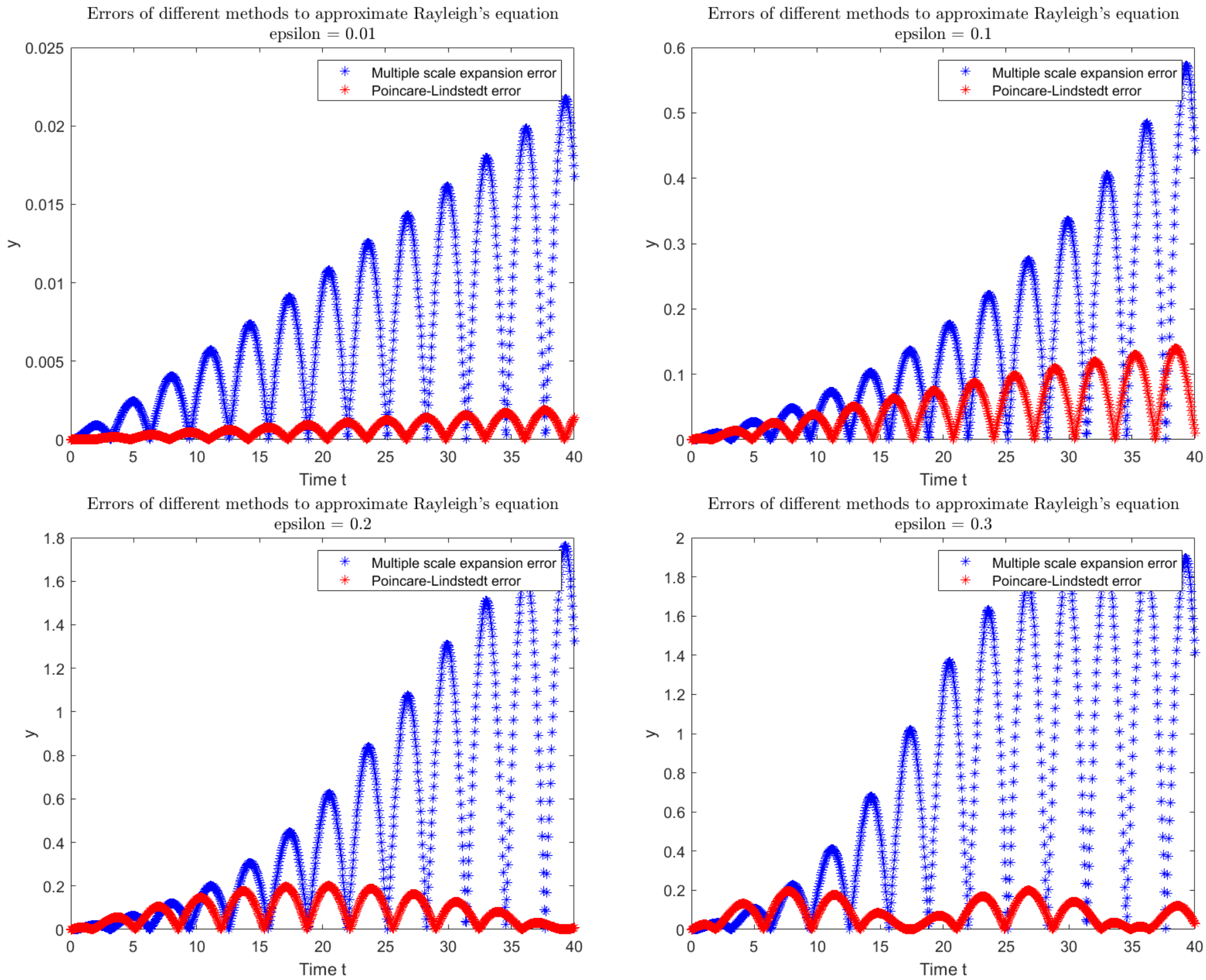




Here we have that the error for both methods are pretty much the same in the beginning. Afterwards the frequencies become the big problem, but they both work about the same. Fpr $A = 10$, we get



This is just a mess. It's always like this. For $A = 1/10$, we get



The error rises quickly as epsilon grows. the Multiple scale expansion error gives an amplification error and the Poincare-Lindstedt error does the frequency shifting not as wel.