AMATH 568: Homework 7
Winter Quarter 2023
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1. Consider the Optical Parametrix Oscillator,

$$U_t = \frac{i}{2}U_{xx} + VU^* - (1 + i\Delta_1)U$$
$$V_t = \frac{i}{2}\rho V_{xx} - U^2 - (\alpha + i\Delta_2)V + S$$

(a) Assuming slow time $\tau = \epsilon^2 t$ and slow space $\xi = \epsilon x$, derive the Fisher-Kolmogorov equation for the slow evolution of the instability:

$$\varphi_\tau - \varphi_{\xi\xi} \pm \varphi^3 \mp \gamma\varphi = 0.$$

Solutions:

(a) First we see that the stable uniform (x-derivatives zero) steady state (t-derivatives zero) solution for U is simply given by $U = 0$, filling this into the equation for V, we then get that its stable uniform steady state solution must be

$$V = \frac{S}{\alpha + i\Delta_2}.$$

If we now linearize around these solutions, we get

$$U = \epsilon u,$$
$$V = \frac{S}{\alpha + i\Delta_2} + \epsilon v,$$

Filling this in gives us

$$\epsilon u_t = \frac{i}{2}\epsilon u_{xx} + \epsilon\frac{S}{\alpha + i\Delta_2}u^* - \epsilon(1 + i\Delta_1)u + \epsilon^2 vu^*,$$
$$\epsilon v_t = \frac{i}{2}\rho\epsilon v_{xx} - \epsilon^2 u^2 - \epsilon(\alpha + i\Delta_2)v.$$

Later on we will see that the leading order correction term u will be real, therefore $u = u^*$. Let's not think about the contradictory statements that might hold because of circular reasoning. We need it here, along with other assumptions later on. Taking the leading order terms, or linearizing, gives us

$$u_t = \frac{i}{2}u_{xx} + \frac{S}{\alpha + i\Delta_2}u - (1 + i\Delta_1)u,$$
$$v_t = \frac{i}{2}\rho v_{xx} - (\alpha + i\Delta_2)v.$$

Suspect that the solutions are of the form $u = Ae^{\lambda_u t + ik_u x}$ (same for v, different subscripts), *assuming* λ_u and k_u are real for a moment, filling this in gives us

$$\lambda_u = -\frac{i}{2}k_u^2 + \frac{S}{\alpha + i\Delta_2} - (1 + i\Delta_1),$$
$$\lambda_v = -\frac{i}{2}\rho k_v^2 - (\alpha + i\Delta_2).$$

It's clear that here k_u and k_v are zero (only looking at the imaginary part) and thus

$$\lambda_u = \frac{S}{\alpha + i\Delta_2} - (1 + i\Delta_1) \text{ and } \lambda_v = -(\alpha + i\Delta_2).$$

The real part of λ_v is always negative (indeed I know I said they're real, and now it's not, funny how that works), thus this does not blow up, but λ_u on the other hand... It's negative when $S < (\alpha + i\Delta_2)(1 + i\Delta_1)$

and positive if not, which would mean that it's an unstable solution. Let's ignore the fact that this might give rise to complex solutions and that our assumption that these parameters were real, is straight up wrong if Δ_2 and Δ_1 is not purely imaginary. Therefore take

$$S_c = (\alpha + i\Delta_2)(1 + i\Delta_1)$$

as your critical value for stability. Thus, for $|S| > |S_c|$ we can have an unstable solution. We now define the $\tau = \epsilon^2 t$ and $\xi = \epsilon x$ around this S_c (different ϵ as when we were linearizing). where $\delta = |S - S_c|$ small. and $S - S_c = \delta C + \mathcal{O}(\delta^2)$, (let's ignore the higher order terms) with C constant, thus $S = \delta C + (\alpha + i\Delta_2)(1 + i\Delta_1)$. We now expand again about our uniform steady-state solution, but a bit differently:

$$\begin{aligned} U &= \epsilon u(\tau, \xi), \\ V &= \frac{S}{\alpha + i\Delta_2} + \epsilon^2 v(\tau, \xi). \end{aligned}$$

I'm sure we could have figured out where these powers of Epsilon all come from, but let's not do that here. Taking $\delta = \epsilon^2$ and filling this in, gives us

$$\begin{aligned} \epsilon^3 u_\tau &= \frac{i}{2}\epsilon^3 u_{\xi\xi} + \epsilon \frac{\epsilon^2 C + (\alpha + i\Delta_2)(1 + i\Delta_1)}{\alpha + i\Delta_2} u^* - \epsilon(1 + i\Delta_1)u + \epsilon^3 v u^*, \\ \epsilon^4 v_\tau &= \frac{i}{2}\rho\epsilon^4 v_{\xi\xi} - \epsilon^2 u^2 - \epsilon^2(\alpha + i\Delta_2)v. \end{aligned}$$

This turns into

$$\begin{aligned} (1 + i\Delta_1)(u - u^*) &= \epsilon^2 \left(\frac{i}{2} u_{\xi\xi} - u_\tau + \frac{C}{\alpha + i\Delta_2} u^* + v u^* \right), \qquad (1) \\ (\alpha + i\Delta_2)v &= -u^2 + \epsilon^2 \left(\frac{i}{2}\rho v_{\xi\xi} - v_\tau \right). \end{aligned}$$

Which turns into

$$\begin{aligned} u^* &= u - \frac{\epsilon^2}{1 + i\Delta_1} \left(\frac{i}{2} u_{\xi\xi} - u_\tau + \frac{C}{\alpha + i\Delta_2} u^* + v u^* \right), \\ v &= -\frac{u^2}{\alpha + i\Delta_2} + \frac{\epsilon^2}{\alpha + i\Delta_2} \left(\frac{i}{2}\rho v_{\xi\xi} - v_\tau \right). \qquad (2) \end{aligned}$$

Note here that the leading order solution gives us $u^* = u$, therefore u is real! Let's not use that just yet. Multiplying these gives us

$$vu^* = -\frac{u^2}{\alpha + i\Delta_2} u + \frac{\epsilon^2}{\alpha + i\Delta_2} \left(\frac{i}{2}\rho v_{\xi\xi} - v_\tau \right) u + \frac{\epsilon^2}{1 + i\Delta_1} \left(\frac{i}{2} u_{\xi\xi} - u_\tau + \frac{C}{\alpha + i\Delta_2} u^* + v u^* \right) \frac{u^2}{\alpha + i\Delta_2} + \mathcal{O}(\epsilon^4)$$

note now that, according to (2):

$$\begin{aligned} v_\tau &= -\frac{(u^2)_\tau}{\alpha + i\Delta_2} + \mathcal{O}(\epsilon^2), \\ v_{\xi\xi} &= -\frac{(u^2)_{\xi\xi}}{\alpha + i\Delta_2} + \mathcal{O}(\epsilon^2). \end{aligned}$$

Therefore, also because of (1),

$$vu^* = -\frac{u^2}{\alpha + i\Delta_2} u + \frac{\epsilon^2}{(\alpha + i\Delta_2)^2} \left((u^2)_\tau - \frac{i}{2}\rho (u^2)_{\xi\xi} \right) u + \frac{1}{1 + i\Delta_1} \left((1 + i\Delta_1)(u - u^*) \right) \frac{u^2}{\alpha + i\Delta_2} + \mathcal{O}(\epsilon^4),$$

which eventually becomes

$$vu^* = -\frac{1}{\alpha + i\Delta_2} |u|^2 u + \frac{\epsilon^2}{(\alpha + i\Delta_2)^2} \left((u^2)_\tau - \frac{i}{2}\rho (u^2)_{\xi\xi} \right) u + \mathcal{O}(\epsilon^4).$$

Filling this into (1), gives

$$R = (1+i\Delta_1)(u-u^*) = \epsilon^2 \left(\frac{i}{2} u_{\xi\xi} - u_\tau + \frac{C}{\alpha + i\Delta_2} u^* - \frac{1}{\alpha + i\Delta_2} |u|^2 u \right) + \frac{\epsilon^4}{(\alpha + i\Delta_2)^2} \left((u^2)_\tau - \frac{i}{2} \rho (u^2)_{\xi\xi} \right) u + \mathcal{O}(\epsilon^6).$$

Let's now ignore contributions higher than $\mathcal{O}(\epsilon^6)$ (and honestly $\mathcal{O}(\epsilon^4)$ too). Thus the right hand side R must satisfy Fredholm's alternative theorem, which states that it must be orthogonal to the null space of the adjoint of the leading order behavior:

$$(1 + i\Delta_1)(u - u^*) = 0 \Rightarrow (1 + i\Delta_1)u = ((1 - i\Delta_1)u)^*.$$

Thus, we must have that

$$(1 - i\Delta_1)R + (1 + i\Delta_1)R^* = 0,$$

since, I've taken this from the book and can confirm that there are no contradictions because,

$$(1 - \Delta_1^2)(u - u^*) + (1 - \Delta_1^2)(u^* - u) = 0.$$

Taking the right hand side term and filling it in, we get

$$(1-i\Delta_1) \left(\frac{i}{2} u_{\xi\xi} - u_\tau + \frac{C}{\alpha + i\Delta_2} u - \frac{1}{\alpha + i\Delta_2} u^3 \right) + (1+i\Delta_1) \left(-\frac{i}{2} u_{\xi\xi} - u_\tau + \frac{C}{\alpha - i\Delta_2} u - \frac{1}{\alpha - i\Delta_2} u^3 \right) = 0.$$

This becomes

$$\Delta_1 u_{\xi\xi} - 2u_\tau + \left(\frac{1 - i\Delta_1}{\alpha + i\Delta_2} + \frac{1 + i\Delta_1}{\alpha - i\Delta_2} \right) (Cu - u^3)) = 0,$$

thus

$$\begin{aligned} 0 &= \Delta_1 u_{\xi\xi} - 2u_\tau + \left(\frac{(1 - i\Delta_1)(\alpha - i\Delta_2) + (1 + i\Delta_1)(\alpha + i\Delta_2)}{\alpha^2 + \Delta_2^2} \right) (Cu - u^3)) \\ &= \Delta_1 u_{\xi\xi} - 2u_\tau + 2 \left(\frac{\alpha - \Delta_1\Delta_2}{\alpha^2 + \Delta_2^2} \right) (Cu - u^3)) \\ &= \Delta_1 u_{\xi\xi} - 2u_\tau + 2 \left(\frac{\mathrm{Re}(S_c)}{\alpha^2 + \Delta_2^2} \right) (Cu - u^3)). \end{aligned}$$

Say we divide by -2, substitute $\xi = (\Delta_1/2)^{1/2}\zeta$ and multiplying by $\left(\frac{\Delta_1\Delta_2 - \alpha}{\alpha^2 + \Delta_2^2}\right)^{1/2}$ the equation becomes

$$\left(\frac{\Delta_1\Delta_2 - \alpha}{\alpha^2 + \Delta_2^2} \right)^{1/2} u_\tau - \left(\frac{\Delta_1\Delta_2 - \alpha}{\alpha^2 + \Delta_2^2} \right)^{1/2} u_{\zeta\zeta} - \mathrm{sgn}(\mathrm{Re}(S_c)) \left(\left(\frac{\Delta_1\Delta_2 - \alpha}{\alpha^2 + \Delta_2^2} \right)^{1/2} u \right)^3 + \left(\frac{\Delta_1\Delta_2 - \alpha}{\alpha^2 + \Delta_2^2} \right)^{1/2} uC \left(\frac{\Delta_1\Delta_2 - \alpha}{\alpha^2 + \Delta_2^2} \right)$$

Substituting $\varphi = \left(\frac{\Delta_1\Delta_2 - \alpha}{\alpha^2 + \Delta_2^2}\right)^{1/2} u$, we get

$$\varphi_\tau - \varphi_{\zeta\zeta} - \mathrm{sgn}(\mathrm{Re}(S_c))\varphi^3 + \mathrm{sgn}(C)\gamma\varphi = 0,$$

where

$$\gamma = -|C| \left(\frac{\Delta_1\Delta_2 - \alpha}{\alpha^2 + \Delta_2^2} \right).$$

This is not exactly the same as the book. I cannot get a relation between the signs of the real side of S_c and C, the only thing we know about their relation is that $S - S_c = \epsilon^2 C$, but in my opinion, if the real part is S_c is positive or negative, the sign of C can be both either way. Also, I cannot prove that the gamma from the book is equal to this. I honestly don't think it's correct. But it's been really "engineeringy" and I'm not cut out for that tomfoolery.