AMATH 568: Homework 2
Winter Quarter 2023
Professor J. Nathan Kutz

wietse vaes 2224416

Due: January 18, 2023

1. Consider the nonhomogeneous problems of Problem 1 and 2: $\vec{x}' = \boldsymbol{A}\vec{x} + \vec{g}(t)$.
   (a) Let $\vec{x} = \boldsymbol{M}\vec{y}$ where the colimns of $\boldsymbol{M}$ are the eigenvectors of the above problems.
   (b) Write the equations in terms of $\vec{y}$ and multiply through by $\boldsymbol{M}^{-1}$.
   (c) Show the resulting equation is
   $$\vec{y}' = \boldsymbol{D}\vec{y} + \vec{h}(t)$$
   where $\boldsymbol{D} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$ is a diagonal matrix whose diagonal elements are the eigenvalues of the problem considered and $\vec{h}(t) = \boldsymbol{M}^{-1}\vec{g}(t)$.
   (d) Show that this system is now *decoupled* so that each component of $\vec{y}$ can be solved independently of the other components.

Solution:

(b) Writing the equation in terms of $\vec{y}$ by multiplying through by $\boldsymbol{M}^{-1}$ just gives:
$$\vec{y}' = (\boldsymbol{M}^{-1}\vec{x})' = \boldsymbol{M}^{-1}\vec{x}' = \boldsymbol{M}^{-1}\boldsymbol{A}\vec{x} + \boldsymbol{M}^{-1}\vec{g}(t) = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}\vec{y} + \boldsymbol{M}^{-1}\vec{g}(t).$$
Note that $\boldsymbol{M}^{-1}$ is also just going to be a constant matrix, that is why we put it into the derivative.

(c) Looking at (b) we see that
$$\vec{y}' = \boldsymbol{D}\vec{y} + \vec{h}(t),$$
with $\boldsymbol{D} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$ and $\vec{h}(t) = \boldsymbol{M}^{-1}\vec{g}(t)$. Note that if we were to use simple diagonalisation of $\boldsymbol{A}$ using eigenvectors and eigenvalues, we get that $\boldsymbol{A} = \boldsymbol{M}\boldsymbol{D}\boldsymbol{M}^{-1}$, with $\boldsymbol{D}$ the diagonal matrix with all the eigenvalues on it.

(d) Since $\boldsymbol{D}$ is a diagonal matrix with elements d_i, $i = 1, \dots, n$, we get that this is equal to
$$\begin{cases} \vec{y}_1' = d_1\vec{y}_1 + \vec{h}_1(t) \\ \vec{y}_2' = d_2\vec{y}_2 + \vec{h}_2(t) \\ \vdots \\ \vec{y}_n' = d_n\vec{y}_n + \vec{h}_n(t) \end{cases}$$
Since $\vec{h}$ does not depend on any components of $\vec{y}$ we get that the derivative of the components rely only on that component itself. It is thus decoupled.

2. Given $L = -d^2/dx^2$ find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -10\exp(x) \qquad y(0) = 0, y'(1) = 0$$

Solution:

First we find the eigenfunctions $u_\lambda(x)$ of operator L:

$$Lu_\lambda(x) = \lambda u_\lambda(x) \Rightarrow -\frac{d^2}{dx^2}u_\lambda(x) = \lambda u_\lambda(x)$$

Therefore, we have solutions

$$u_\lambda(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x).$$

Since $y(0) = 0$, we want $u_\lambda(0) = 0$, thus we get that $c_1 = 0$. Furthermore, since we want $y'(1) = 0$, we have that, for all λ, $u'_\lambda(1) = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}) = 0$. Since we do not want the simple zero solution, we want $c_2 \neq 0$. The only possibility left is for $\lambda = (\frac{(2n+1)\pi}{2})^2$, for $n \in \mathbb{N}$. Note that n does not necisarely need to take on any negative values because of the square. Therefore we redefine the eigenfunctions $u_n = c_n\sin(\frac{(2n+1)\pi}{2}x)$, for $n \in \mathbb{N}$. Multiples of these functions are also solutions. Since we want orthonormality to hold, we will multiply by $\sqrt{2}$. In order to form a solution we sum them while multiplying them by a coefficing c_m. This gives us the solution

$$y = \sum_{n\in\mathbb{N}} c_n\sqrt{2}\sin(\frac{(2n+1)\pi}{2}x).$$

We now want to find these c_n for which the function is a solution. We do this by filling in y and taking the inproduct $\langle u, v\rangle = \int_0^1 uv^*dx$ of both sides with eigenvector u_m. This gives us

$$\begin{aligned}
-10e^x &= -L\sum_{n\in\mathbb{N}} c_n u_n + 2\sum_{n\in\mathbb{N}} c_n u_n \\
&= \sum_{n\in\mathbb{N}}(2-\lambda_n)c_n u_n \\
\langle -10e^x, u_m\rangle &= \langle \sum_{n\in\mathbb{N}}(2-\lambda_n)c_n u_n, u_m\rangle \\
&= (2-\lambda_m)c_m \\
&\Rightarrow \\
c_m &= \frac{\langle -10e^x, u_m\rangle}{(2-\lambda_m)}
\end{aligned}$$

We know all of these values except for $-10\langle e^x, u_n\rangle$. We will try and calculate this here:

$$\begin{aligned}
I &= \langle e^x, u_n\rangle \\
&= \int_0^1 e^x\sqrt{2}\sin(\frac{(2n+1)\pi}{2}x)dx \\
&\overset{IBP}{=} e^x\sqrt{2}\sin(\frac{(2n+1)\pi}{2}x)|_0^1 - \frac{(2n+1)\pi}{2}\int_0^1 e^x\sqrt{2}\cos(\frac{(2n+1)\pi}{2}x)dx \\
&\overset{IBP}{=} (-1)^n\sqrt{2}e - \frac{(2n+1)\pi}{2}e^x\sqrt{2}\cos(\frac{(2n+1)\pi}{2}x)|_0^1 - (\frac{(2n+1)\pi}{2})^2\int_0^1 e^x\sqrt{2}\sin(\frac{(2n+1)\pi}{2}x)dx \\
&= (-1)^n\sqrt{2}e + \frac{(2n+1)\pi}{2}\sqrt{2} - \lambda_n I \\
&\Rightarrow \\
I &= \frac{(-1)^n\sqrt{2}e + \frac{(2n+1)\pi}{2}\sqrt{2}}{1+\lambda_n}
\end{aligned}$$

In conclusion, we have that

$$c_m = -10\frac{(-1)^n\sqrt{2}e + \frac{(2n+1)\pi}{2}\sqrt{2}}{(1+\lambda_n)(2-\lambda_n)}.$$

Thus, we have the solution to our ODE:

$$y = \sum_{n\in\mathbb{N}} -10\frac{(-1)^n\sqrt{2}e + \frac{(2n+1)\pi}{2}\sqrt{2}}{(1+\lambda_n)(2-\lambda_n)}\sqrt{2}\sin\left(\frac{(2n+1)\pi}{2}x\right) = \sum_{n\in\mathbb{N}} -10\frac{(-1)^n 2e + (2n+1)\pi}{(1+\lambda_n)(2-\lambda_n)}\sin\left(\frac{(2n+1)\pi}{2}x\right).$$

Note that λ_n will never be 2 or -1.

3. Given $L = -d^2/dx^2$ find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -x \qquad y(0) = 0, y(1) + y'(1) = 0$$

Solution:

The operator is the same, so we have the same kind of eigenfunction u_λ. However, the c_1 and c_2 in the form have changed. We know that, since $y(0) = 0$, $u_\lambda(0) = 0$, thus $c_1 = 0$. Since $y(1) + y'(1) = 0$, we have that $c_2(\sin(\sqrt{\lambda}) + \sqrt{\lambda}\cos(\sqrt{\lambda})) = 0$. This means that $\tan(\sqrt{\lambda}) = -\sqrt{\lambda}$ must hold (since we don't want the trivial solution). Assume we found a countable infinite amount of eigenvalues λ_n for which this holds true (they exist). Do note that $\lambda = 0$ is a solution, but for this we have that the eigenfunction is zero, thus we will not look at this. Here we assume that $n \in \mathbb{N}$ because of the square root. Assume also that we normalized the eigenfunctions with constant $a = \frac{1}{\|u_\lambda\|^2}$, then we get that

$$y = \sum_{n\in\mathbb{N}} c_n a \sin(\sqrt{\lambda_n} x),$$

is a solution to the ODE given correct c_n. We now once again want to find the coefficients c_n. In the same way as last exercise, we get

$$c_n = -\frac{\langle x, u_n\rangle}{(2 - \lambda_n)}.$$

We can calculate $\langle x, u_m\rangle$:

$$\begin{aligned}
\langle x, u_m\rangle &= \int_0^1 xa\sin(\sqrt{\lambda_m} x)dx \\
&= -\frac{a}{\sqrt{\lambda_m}} x\cos(\sqrt{\lambda_m} x)|_0^1 + \frac{a}{\sqrt{\lambda_m}}\int_0^1 \cos(\sqrt{\lambda_m} x)dx \\
&= -\frac{a}{\sqrt{\lambda_m}}\cos(\sqrt{\lambda_m}) + \frac{a}{\lambda_m}\sin(\sqrt{\lambda_m} x)|_0^1 \\
&= \frac{a}{\lambda_m}\sin(\sqrt{\lambda_m}) + \frac{a}{\lambda_m}\sin(\sqrt{\lambda_m}) \\
&= \frac{2a}{\lambda_m}\sin(\sqrt{\lambda_m}).
\end{aligned}$$

Note that the second to last equality holds because of the chosen eigenvalues. Therefore we have that

$$c_n = \frac{2a\sin(\sqrt{\lambda_n})}{\lambda_n(2 - \lambda_n)}.$$

In conclusion, the solution to this is

$$y = \sum_{n\in\mathbb{N}} \frac{2a^2\sin(\sqrt{\lambda_n})}{\lambda_n(2 - \lambda_n)}\sin(\sqrt{\lambda_n} x).$$

At first the fact that λ_n could be equal to zero may look like a problem, but we said that we are not going to look at that eigenvalue since it would give us eigenfunction zero.

4. Consider the Sturm-Liouville eigenvalue problem:

$$Lu = -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = \lambda\rho(x)u \qquad 0 < x < L$$

with the boundary conditions

$$\alpha_1 u(0) - \beta_1 u'(0) = 0$$
$$\alpha_2 u(L) - \beta_2 u'(L) = 0$$

and with $p(x) > 0$, $\rho(x) > 0$, and $q(x) \geq 0$ and with $p(x), \rho(x), q(x)$ and $p'(x)$ continuous over $0 < x < L$. With the inner product $(\phi, \psi) = \int_0^L \rho(x)\phi(x)\psi^*(x)dx$, show the following:

(a) L is a self-adjoint operator.

(b) Eigenfunctions corresponding to different eigenvalues are orthogonal, i.e. $(u_n, u_m) = 0$.

(c) Eigenvalues are real, non-negative and eigenfunctions may be chosen to be real valued.

(d) Each eigenvalue is simple, i.e. it only has one eigenfunction. (Hint: recall that for each eigenvalue, there can be at most two linearly independent solutions – calculate the Wronskian of these two solutions and see what it implies.)

Solution:

(a) In order to prove that L is a self-adjoint operator, we have to prove that $\langle Lu, v\rangle = \langle u, Lv\rangle$, $\langle\cdot,\cdot\rangle$ being the normal inproduct. We prove this (ignoring the notation of x dependence):

$$\begin{aligned}
\langle Lu, v\rangle &= \int_0^L -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]v^* + q(x)uv^* dx \\
&= -\int_0^L \frac{d}{dx}\left[p(x)\frac{du}{dx}\right]v^* dx + \int_0^L uq(x)v^* dx \\
&= -p(x)\frac{du}{dx}v^*\Big|_0^L + \int_0^L p(x)\frac{du}{dx}\frac{d}{dx}v^* dx + \int_0^L uq(x)v^* dx \\
&= -p(x)\frac{du}{dx}v^*\Big|_0^L + p(x)u\frac{d}{dx}v^*\Big|_0^L - \int_0^L u\frac{d}{dx}\left[p(x)\frac{d}{dx}v^*\right]dx + \int_0^L uq(x)v^* dx \\
&= \left(p(x)u\frac{d}{dx}v^* - p(x)\frac{du}{dx}v^*\right)\Big|_0^L + \langle u, Lv\rangle \\
&= J + \langle u, Lv\rangle
\end{aligned}$$

Assuming now that u and v are solutions to the problem, we can use the boundary conditions of the problem to look further at J:

$$\begin{aligned}
J &= \left[p(x)\left(u\frac{dv^*}{dx} - \frac{du}{dx}v^*\right)\right]\Big|_0^L \\
&= p(L)\left(u(L)\frac{dv^*}{dx}(L) - \frac{du}{dx}(L)v^*(L)\right) - p(0)\left(u(0)\frac{dv^*}{dx}(0) - \frac{du}{dx}(0)v^*(0)\right) \\
&= \frac{\alpha_2}{\beta_2}p(L)(u(L)v^*(L) - v^*(L) - u(L)) - \frac{\alpha_1}{\beta_1}p(0)(u(0)v^*(0) - v^*(0) - u(0)) \\
&= 0
\end{aligned}$$

In conclusion:

$$\langle Lu, v\rangle = \langle u, Lv\rangle.$$

(b) Since $\rho(x)$ is continuous over $]0, L[$, it is not outrageous to assume that ρ is bounded by a certain $0 < m$ and M. Thus,

$$0 \leq m\langle u, v\rangle \leq (u, v) \leq M\langle u, v\rangle.$$

Therefore they are just equivalent, or the norms from this inproduct are equivalent. I will thus mostly be working with the normal inproduct. Thus, if we prove that $\langle u, v\rangle = 0$, we also have that

$(u, v) = 0$. Take two random different eigenvalues $\lambda_n \neq \lambda_m$, with those corresponding eigenvectors u_n and u_m. We now have that, because of the self-adjoint character,

$$\langle Lu_n, u_m \rangle = \langle u_n, Lu_m \rangle.$$

However, we also have that

$$\langle Lu_n, u_m \rangle = \lambda_n \langle u_n, u_m \rangle \text{ and } \langle u_n, Lu_m \rangle = \lambda_m \langle u_n, u_m \rangle.$$

Because of these two things, we get that

$$(\lambda_n - \lambda_m)\langle u_n, u_m \rangle = 0,$$

but since $\lambda_n \neq \lambda_m$, we most have that $\langle u_n, u_m \rangle = 0$. In conclusion,

$$(u_n, u_m) = 0.$$

(c) Say that we have eigenvalue λ with eigenvector u, then we have that

$$\langle Lu, u \rangle = \lambda \langle u, u \rangle,$$

but also, wile taking the complex conjugate of everything:

$$\langle L^* u^*, u^* \rangle = \langle Lu, u \rangle,$$

because of the self-adjointness, but also

$$\lambda^* \langle u^*, u^* \rangle = \lambda^* \langle u, u \rangle.$$

So:

$$\lambda \langle u, u \rangle = \lambda^* \langle u, u \rangle,$$

and since $\|u\|^2 = \langle u, u \rangle \neq 0$, (definition of norms and eigenfunctions being not zero) we have that

$$\lambda - \lambda^* = 0,$$

thus λ is real. Since λ is real, we can also take real eigenfunctions.

In order to prove that the eigenvalues are non-negative, we can show that the operator is positive definite or that $\langle Lu, u \rangle = 0$. We thus want to prove that $\langle Lu, u \rangle \geq 0$ for the eigenfunctions. Since then $\lambda \|u\|^2 = \lambda \langle u, u \rangle \geq 0$.

$$\begin{aligned}
\langle Lu, u \rangle &= \int_0^L -\frac{d}{dx}\left[p\frac{du}{dx}\right] u^* + quu^* dx \\
&= -p\frac{du}{dx}u^*\Big|_0^L + \int_0^L p\frac{du}{dx}\frac{du}{dx}^* + quu^* dx \\
&= -p\frac{du}{dx}u^*\Big|_0^L + \int_0^L p\left|\frac{du}{dx}\right|^2 + q|u|^2 dx \\
&\geq -p\frac{du}{dx}u^*\Big|_0^L
\end{aligned}$$

Note that this last line works since p and q are positive. Since the eigenfunctions can be real, we can drop the complex conjugate. Filling in the conditions given, we get that

$$\langle Lu, u \rangle \geq p(\frac{\alpha_1}{\beta_1}u^2(0) - \frac{\alpha_2}{\beta_2}u^2(L)) \geq \frac{\alpha_1}{\beta_1}|u(0)|^2 - \frac{\alpha_2}{\beta_2}|u(L)|^2.$$

If this were to be larger than or equal to zero, we would have that the eigenvalues are non-negative.

(d) Suppose we have those distinct eigenfunctions u and v (and solutions) for eigenvalue λ. If we prove that $W[u, v] = 0$, then u and v are linearly dependent and if the eigenfunctions are orthonomal, they are the same. The wronskian is

$$W[u, v] = uv' - u'v.$$

It's easy to see that this is zero at the boundary points because of the boundary conditions. So we know that $W[u, v](0)$ and $W[u, v](L)$ are zero. Let's now look at something that is also zero:

$$vLu - uLv = \lambda vu - \lambda uv = 0.$$

However, we also have that

$$\begin{aligned}
vLu - uLv &= -v\frac{d}{dx}\left[p\frac{du}{dx}\right] + vqu + u\frac{d}{dx}\left[p\frac{dv}{dx}\right] - uqv \\
&= u\frac{d}{dx}\left[p\frac{dv}{dx}\right] - v\frac{d}{dx}\left[p\frac{du}{dx}\right] \\
&= u\frac{d}{dx}\left[p\frac{dv}{dx}\right] + \frac{du}{dx}p\frac{dv}{dx} - \frac{dv}{dx}p\frac{du}{dx} - v\frac{d}{dx}\left[p\frac{du}{dx}\right] \\
&= \frac{d}{dx}\left(p(u\frac{dv}{dx} - v\frac{du}{dx}\right) \\
&= \frac{d}{dx}\left(pW[u, v]\right).
\end{aligned}$$

Thus,

$$\frac{d}{dx}\left(pW[u, v]\right) = 0.$$

Therefore, $pW[u, v]$ is constant, and since $W[u, v]$ is zero at the boundaries, we know that $pW[u, v] = 0$ everywhere. Since $p > 0$, we know that

$$W[u, v] = 0.$$

Therefore, u and v are linearly dependant and since we chose them to be orthonormal, they are the same. A contradiction.

In conclusion, Each eigenvalue is simple.