AMATH 568: Homework 1
Winter Quarter 2023
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Due: January 11, 2023

(a) Determine the eigenvalues and eigenvectors (real solutions), (b) sketch the behavior and classify the behavior.

1. $\overrightarrow{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \overrightarrow{x}$
2. $\overrightarrow{x}' = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix} \overrightarrow{x}$
3. $\overrightarrow{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \overrightarrow{x}$
4. $\overrightarrow{x}' = \begin{pmatrix} 2 & -5/2 \\ 9/5 & -1 \end{pmatrix} \overrightarrow{x}$
5. $\overrightarrow{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \overrightarrow{x}$
6. $\overrightarrow{x}' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \overrightarrow{x}$
7. $\overrightarrow{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \overrightarrow{x}$

Solution:

1.a say that the equation is of the form

$$\overrightarrow{x}' = A\overrightarrow{x}.$$

Solving this comes down to finding eigenvalues and eigenvectors. In order to find the eigenvalues, we search for all λ, where the following holds

$$\det(A - \lambda I_2) = 0.$$

This as we want non-zero eigenvectors. Note that

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - bc.$$

Knowing this, we get the following eigenvalues:

$$0 = \det(A - \lambda I_2) = \left|\begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix}\right| = \lambda^2 - 4 + 5 = \lambda^2 + 1 \Rightarrow \lambda_\pm = \pm i.$$

In order to find the eigenvector, we find a vector v, such that

$$Av = \lambda v \Rightarrow (A - \lambda I_2)v = 0,$$

thus,

$$(A - iI_2)v_+ = 0 \Rightarrow \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} v_{+1} \\ v_{+2} \end{pmatrix} = 0 \Rightarrow \begin{cases} (2-i)v_{+1} = 5v_{+2} \\ v_{+1} = (2+i)v_{+2} \end{cases}.$$

Since the determinant of the matrix is zero, we know that there are none or an infinite amount of solutions. Since $(0,0)^T$ is a solution, we have an infinite amount of solution. For an eigenvector, we want a non-zero vector. Choosing $v_{+2} = 1$, we get a possible eigenvector $v_+ = (2+i, 1)^T$. Multiples of this vector are also eigenvectors of course. The other eigenvector can be found as follows,

$$(A + iI_2)v_- = 0 \Rightarrow \begin{pmatrix} 2+i & -5 \\ 1 & 2-i \end{pmatrix}\begin{pmatrix} v_{-1} \\ v_{-2} \end{pmatrix} = 0 \Rightarrow \begin{cases} (2+i)v_{-1} = 5v_{-2} \\ v_{-1} = (2-i)v_{-2} \end{cases}.$$

Choosing $v_{-2} = 1$, we get eigenvector $v_- = (2-i, 1)^T$. Thus, we get the following eigenvalues with respective eigenvectors:

$$\left(i, \begin{pmatrix} 2+i \\ 1 \end{pmatrix}\right) \text{ and } \left(-i, \begin{pmatrix} 2-i \\ 1 \end{pmatrix}\right).$$

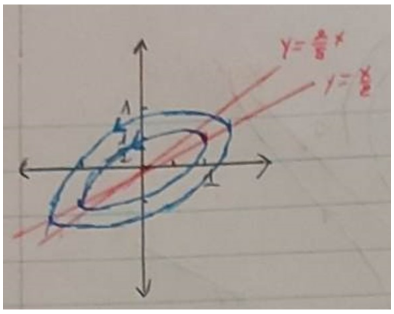
The second eigenvector could have been found via another way, which we will use later.

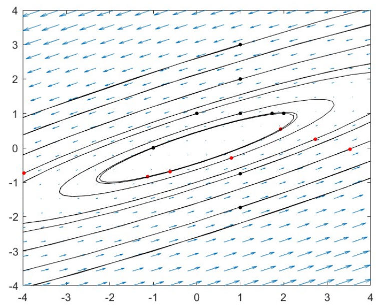
1.b Now that we have two different eigenvalues (λ_1 and λ_2) and eigenvectors (v_1 and v_2), we know that the general form of the real part of the solutions:

$$\overrightarrow{x} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{-\lambda_2 t},$$

with the choice of c_1 and c_2 depending on the initial condition. Note that if the initial condition were to be a multiple of one of the eigenvectors, the other coefficient would be zero and therefore the solution would stay in the direction of this eigenvector.
Note that the eigenvalues decide what the behavior of the solution will be. When looking at the general solution of a linear PDE, it becomes clear why the solutions have certain forms corresponding to certain eigenvalues. Since these eigenvalues are purely imaginary we know that the behavior will be **center shaped** ($e^{it} = \cos(t) + i\sin(t)$). Also, since it's a linear PDE, we know that the origin is the equilibrium, thus it's a center around the origin.
The easiest way to decide how oval it is, is to look at where x' and y' equal zero when $\overrightarrow{x} = (x, y)^T$. $x' = 0$ when $y = \frac{2}{5}x$ and $y' = 0$ when $y = \frac{1}{2}x$. On these lines the solution should be horizontal, respectively, vertical.
Whether the solutions move clockwise or counter-clockwise can be found when filling in $(0, 1)$ into the system. In this case we have that $\overrightarrow{x}' = (-5, -2)$. Because $x' < 0$, we know that it moves **counter-clockwise**. another way of telling is if $A_{2,1}$ is positive or negative. If it's negative we have that it rotates clockwise, if it's positive, it rotates counter clockwise.
The figure will always be color-coded as follows: axis (black), solutions (blue/black), eigenvector influence (green), derivatives being zero (red).

In conclusion, we get the hand drawn version (left) and MATLAB checkup (right):





For MATLAB we used the ode45 function and a couple chosen starting points.

2.a Like 1.a, we get the following eigenvalues:

$$0 = \det(A - \lambda I_2) = \left| \begin{pmatrix} -1-\lambda & -1 \\ 0 & -.25-\lambda \end{pmatrix} \right| = (1+\lambda)(.25+\lambda) \Rightarrow (\lambda_1, \lambda_2) = (-1, -.25).$$

We also could have just said that, because it is a triangular matrix, the eigenvalues are on the diagonal. The eigenvectors are given by

$$(A + I_2)v_1 = 0 \Rightarrow \begin{pmatrix} 0 & -1 \\ 0 & .75 \end{pmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = 0 \Rightarrow v_{1,2} = 0.$$

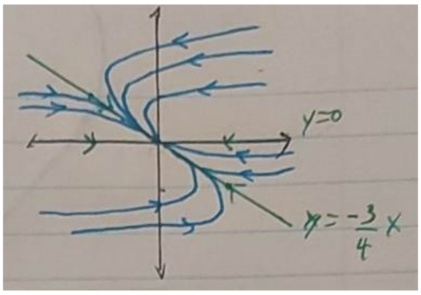
Choosing $v_{1,1} = 1$, we get eigenvector $v_1 = (1, 0)^T$. Furthermore,

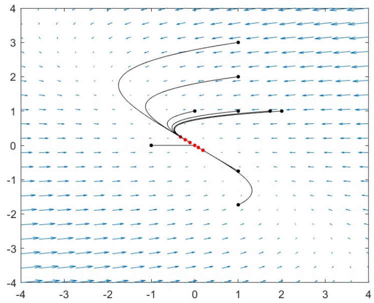
$$(A + I_2/4)v_2 = 0 \Rightarrow \begin{pmatrix} -.75 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = 0 \Rightarrow \frac{3}{4}v_{2,1} = -v_{2,2}.$$

Choosing $v_{2,1} = 1$, we get eigenvector $v_2 = (1, -.75)^T$. Thus, we get the following eigenvalues with respective eigenvectors:

$$\left(-1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \text{ and } \left(-.25, \begin{pmatrix} 1 \\ -.75 \end{pmatrix}\right)$$

2.b Since we have two different real negative eigenvalues, we have that this has a **stable equilibrium**. The equilibrium is once again the origin. As it's real, we could have an initial condition which is a multiple of the eigenvector. This means that solutions stay as multiples of this eigenvalues (if you start on a green line, you stay there). The highest eigenvalue decides which eigenvector has the strongest influence on the solutions. In this case, this is $\lambda = -.25$. Therefore, we can sketch the behavior as follows:





3.a Like 1.a, we get the following eigenvalues:

$$0 = \det(A - \lambda I_2) = \left| \begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} \right| = -(3-\lambda)(1+\lambda) + 4 \Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 \Rightarrow \lambda = 1.$$

Since we only have one eigenvalue, we also have to look if there exist two linearly independent eigenvectors. We have,

$$(A - I_2)v = 0 \Rightarrow \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_1 = 2v_2.$$

Choosing $v_2 = 1$, we get eigenvector $v = (2,1)^T$. There are no other eigenvectors. Thus, we get the following eigenvalue and eigenvectors:

$$\left(1, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

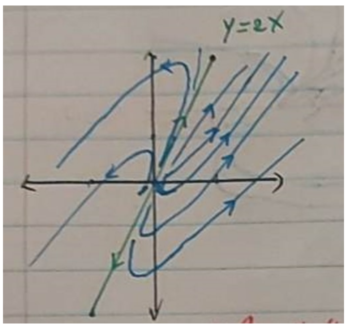
3.b Here we have one real positive eigenvalue with one eigenvector. Therefore the solution is of the form

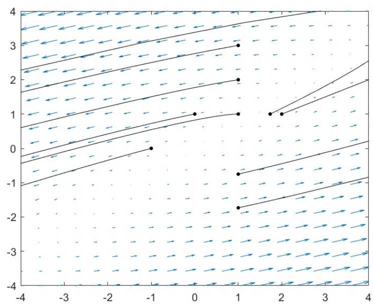
$$\overrightarrow{x} = c_1 v e^{\lambda t} + c_2 \left[v t e^{\lambda t} + \eta e^{\lambda t} \right],$$

with η being

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \eta = v \Rightarrow \eta_1 = 1 + 2\eta_2.$$

Choosing $\eta_2 = 0$, we get that $\eta = (1,0)^T$. It is thus an **unstable improper node** (solutions tend away from the equilibrium). Therefore, we can sketch the behavior as follows:





4.a Like 1.a, we get the following eigenvalues:

$$0 = \det(A - \lambda I_2) = \left| \begin{pmatrix} 2-\lambda & -5/2 \\ 9/5 & -1-\lambda \end{pmatrix} \right| = (\lambda+1)(\lambda-2) + \frac{9}{2} \Rightarrow \lambda^2 - \lambda + \frac{5}{2} = 0 \Rightarrow \left(\lambda - \frac{1-3i}{2}\right)\left(\lambda - \frac{1+3i}{2}\right)$$

$$\Rightarrow (\lambda_1, \lambda_2) = \left(\frac{1-3i}{2}, \frac{1+3i}{2}\right).$$

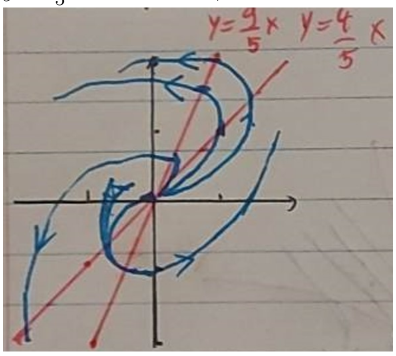
Note that $\lambda_2 = \overline{\lambda_1}$. The eigenvector for λ_1 is given by

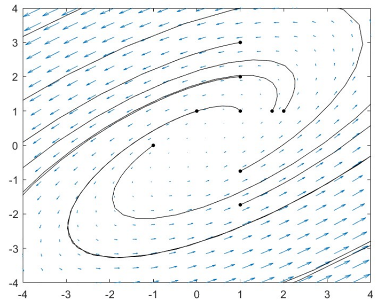
$$(A + \frac{1-3i}{2}I_2)v_1 = 0 \Rightarrow \begin{pmatrix} 3\frac{1+i}{2} & -5/2 \\ 9/5 & -3\frac{1-i}{2} \end{pmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = 0 \Rightarrow \begin{cases} 3\frac{1+i}{2}v_{1,1} = 5/2v_{1,2} \\ 9/5v_{1,1} = 3\frac{1-i}{2}v_{1,2} \end{cases}.$$

Choosing $v_{1,2} = 1$, we get eigenvector $v_1 = (5\frac{1-i}{6}, 1)^T$. Furthermore, we know that since $Av_1 = \lambda_1 v_1$, that $A\overline{v_1} = \overline{Av_1} = \overline{\lambda_1 v_1} = \overline{\lambda_1}\,\overline{v_1} = \lambda_2\overline{v_1}$. Thus, we get the following eigenvalues with respective eigenvectors:

$$\left(\frac{1}{2}(1-3i), \begin{pmatrix} \frac{5}{6}(1-i) \\ 1 \end{pmatrix}\right) \text{ and } \left(\frac{1}{2}(1+3i), \begin{pmatrix} \frac{5}{6}(1+i) \\ 1 \end{pmatrix}\right)$$

4.b Here we have 2 complex eigenvalues with non-zero real parts. As it has an imaginary part, it will rotate and since it has a non-zero real part, it will **spiral**. Because these real parts are positive, the equilibrium is unstable. Since $A_{1,2}$ is positive, it rotates **counter-clockwise**. In order to know how oval-ish the spiral is, we look at where x' and y' are equal to zero. In this case, x' is equal to zero when $y = \frac{4}{5}x$ and y' equals zero when $y = \frac{9}{5}x$. Therefore, we can sketch the behavior as follows:





5.a Like 1.a, we get the following eigenvalues:

$$0 = \det(A - \lambda I_2) = \left|\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix}\right| = \lambda^2 - 1 \Rightarrow (\lambda_1, \lambda_2) = (1, -1).$$

The eigenvectors are given by

$$(A - I_2)v_1 = 0 \Rightarrow \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = 0 \Rightarrow v_{1,1} = v_{1,2}.$$

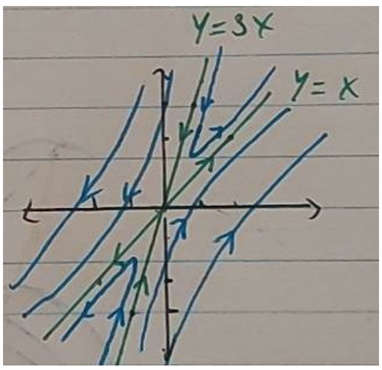
Choosing $v_{1,1} = 1$, we get eigenvector $v_1 = (1, 1)^T$. Furthermore,

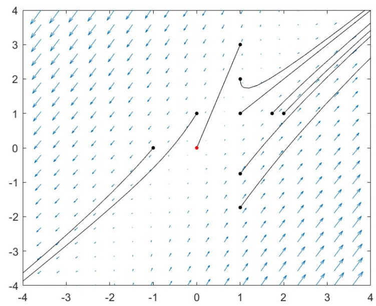
$$(A + I_2)v_2 = 0 \Rightarrow \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = 0 \Rightarrow 3v_{2,1} = v_{2,2}.$$

Choosing $v_{2,1} = 1$, we get eigenvector $v_2 = (1, 3)^T$. Thus, we get the following eigenvalues with respective eigenvectors:

$$\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \text{ and } \left(-1, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$$

5.b Since we have two real eigenvalues with opposing signs, we have that the equilibrium is a **saddle**. It acts stable in the direction of the eigenvector corresponding to the negative eigenvalue and unstable in the direction of the one of the positive eigenvalue. Since the value (define value of a as $|a|$) of the eigenvalues are the same, the eigenvectors have the same effect on the solution. Therefore, we can sketch the behavior as follows:





6.a Like 1.a, we get the following eigenvalues:

$$0 = \det(A - \lambda I_2) = \left| \begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{pmatrix} \right| = \lambda^2 - 4 \Rightarrow (\lambda_1, \lambda_2) = (2, -2).$$

The eigenvectors are given by

$$(A - 2I_2)v_1 = 0 \Rightarrow \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = 0 \Rightarrow v_{1,1} = \sqrt{3} v_{1,2}.$$

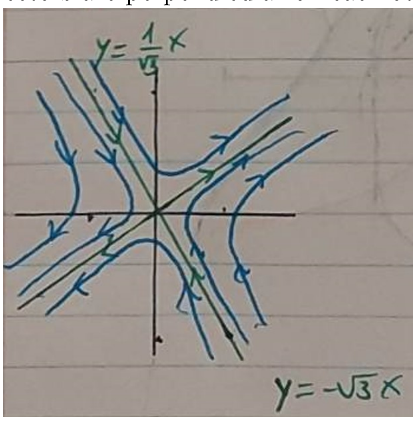
Choosing $v_{1,2} = 1$, we get eigenvector $v_1 = (\sqrt{3}, 1)^T$. Furthermore,

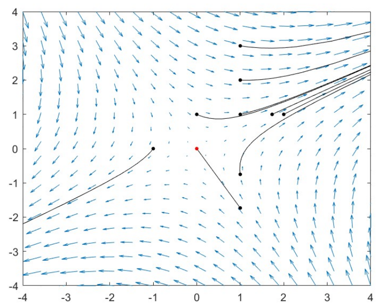
$$(A + 2I_2)v_2 = 0 \Rightarrow \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = 0 \Rightarrow \sqrt{3} v_{2,1} = -v_{2,2}.$$

Choosing $v_{2,1} = 1$, we get eigenvector $v_2 = (1, -\sqrt{3})^T$. Thus, we get the following eigenvalues with respective eigenvectors:

$$\left(2, \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}\right) \text{ and } \left(-2, \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}\right)$$

6.b Once again, we have a **saddle equilibrium** where the eigenvalues have the same value. Note here that the eigenvectors are perpendicular on each other. Therefore, we can sketch the behavior as follows:





7.a Like 1.a, we get the following eigenvalues:

$$0 = \det(A - \lambda I_2) = \left| \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \right| = \lambda^2 - \lambda - 2 \Rightarrow (\lambda_1, \lambda_2) = (2, -1).$$

The eigenvectors are given by

$$(A - 2I_2)v_1 = 0 \Rightarrow \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = 0 \Rightarrow v_{1,1} = 2v_{1,2}.$$

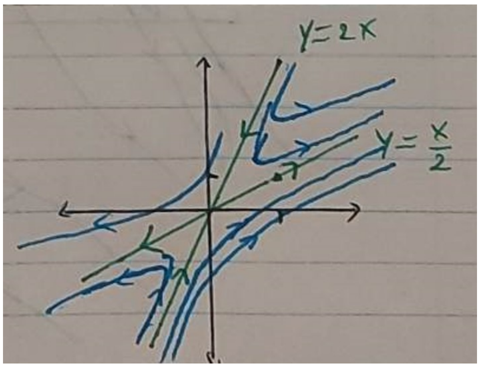
Choosing $v_{1,2} = 1$, we get eigenvector $v_1 = (2, 1)^T$. Furthermore,

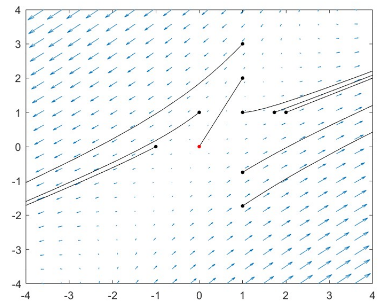
$$(A + 1I_2)v_2 = 0 \Rightarrow \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = 0 \Rightarrow 2v_{2,1} = v_{2,2}.$$

Choosing $v_{2,1} = 1$, we get eigenvector $v_2 = (1, 2)^T$. Thus, we get the following eigenvalues with respective eigenvectors:

$$\left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) \text{ and } \left(-1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

7.b It is once again a **saddle**, but now with different values. Therefore the biggest value has the most effect. In this case, that is eigenvector $(2, 1)^T$. Therefore, we can sketch the behavior as follows:





8. Consider $x' = -(x-y)(1-x-y)$ and $y' = x(2+y)$ and plot the solutions. Verify your qualitative dynamics with MATLAB/Python/fortran.

Solution:

We thus want to have a qualitative solution for different initial values of the system

$$\begin{cases} x' &= -(x-y)(1-x-y) = F(x,y), \\ y' &= x(2+y) = G(x,y). \end{cases}$$

These functions $F(x,y)$ and $G(x,y)$ can be approximated by using Taylor approximation around point (x_0, y_0) as follows:

$$\begin{cases} F(x_0+x, y_0+y) &= F(x_0,y_0) + xF_x(x_0,y_0) + yF_y(x_0,y_0) + \mathcal{O}(x^2) + \mathcal{O}(y^2) \\ G(x_0+x, y_0+y) &= G(x_0,y_0) + xG_x(x_0,y_0) + yG_y(x_0,y_0) + \mathcal{O}(x^2) + \mathcal{O}(y^2) \end{cases}$$

Note that x' equals zero when $x = y$ and $x = 1-y$. Note that solutions crossing line should be vertical. On the other hand, y' equals zero when $x = 0$ and $y = -2$. Note that solutions crossing this line should be horizontal at the intersection. We thus have 4 equilibrium points (when both x' (F) and y' (G) equal zero):

$$(0,0), \quad (0,1), \quad (-2,-2) \text{ and } (3,-2)$$

Thus, when looking at solutions x and y near these equilibrium points (x_0, y_0) and using Taylor approximation, we get the following approximation of the original problem around the equilibrium:

$$\begin{cases} x' &= xF_x(x_0,y_0) + yF_y(x_0,y_0), \\ y' &= xG_x(x_0,y_0) + yG_y(x_0,y_0). \end{cases} \Rightarrow \vec{x}' = \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \vec{x} = \begin{bmatrix} 2x_0 - 1 & 1 - 2y_0 \\ 2 + y_0 & x_0 \end{bmatrix} \vec{x}$$

Plotting the solutions around equilibrium points thus comes down to looking for the eigenvalues/vectors of the Jacobian in the equilibrium points.

(0,0) When filling in the equilibrium point, we get the following eigenvalues of the Jacobian:

$$0 = \left| \begin{bmatrix} -1-\lambda & 1 \\ 2 & -\lambda \end{bmatrix} \right| = \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1) \Rightarrow (\lambda_1, \lambda_2) = (1, -2).$$

The eigenvectors can be found as follows:

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} v_1 = 0 \Rightarrow 2v_{1,1} = v_{1,2}.$$

choosing $v_{1,1} = 1$, we get eigenvector $(1,2)^T$. For the other one:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} v_2 = 0 \Rightarrow v_{2,1} = -v_{2,2}.$$

Choosing $v_{2,1} = 1$, we get eigenvector $(1,-1)^T$. In conclusion: $(0,0)$ is a saddle point, close to it, the line $y = -x$ acts as a sink (stable) and the line $y = 2x$ acts as a source (unstable).

(0,1) When filling in the equilibrium point, we get the following eigenvalues of the Jacobian:

$$0 = \left| \begin{bmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{bmatrix} \right| = \lambda^2+\lambda+3 = (\lambda - \frac{-1+i\sqrt{11}}{2})(\lambda - \frac{-1-i\sqrt{11}}{2}) \Rightarrow (\lambda_1, \lambda_2) = (\frac{-1+i\sqrt{11}}{2}, \frac{-1-i\sqrt{11}}{2}).$$

As they are complex numbers with a negative real part, we have that the solution behaves as a stable spiral near $(0,1)$. Since the real part of these eigenvalues are negative, we have that it is an stable equilibrium, or sink. Since $A_{2,1} > 0$, we have that it is counter-clockwise.

(-2,-2) When filling in the equilibrium point, we get the following eigenvalues of the Jacobian:

$$0 = \left| \begin{bmatrix} -5-\lambda & 5 \\ 0 & -2-\lambda \end{bmatrix} \right| = (\lambda+5)(\lambda+2) \Rightarrow (\lambda_1, \lambda_2) = (-2, -5).$$

As both eigenvalues are negative, we have that this equilibrium is stable. However, we want to find their eigenvectors in order to make a more accurate prediction of the situation around this equilibrium. Since the eigenvector belonging to -2 will have more influence than that of -5.

$$\begin{bmatrix} -3 & 5 \\ 0 & 0 \end{bmatrix} v_1 = 0 \Rightarrow \frac{3}{5} v_{1,1} = v_{1,2},$$

choosing $v_{1,1} = 1$, we get the eigenvector $v_1 = (1, \frac{3}{5})^T$. Thus, the direction $(1, \frac{3}{5})^T$ from the equilibrium point $(-2,-2)$ has the most influence on the solution. As for the other one:

$$\begin{bmatrix} 0 & 5 \\ 0 & 3 \end{bmatrix} v_2 = 0 \Rightarrow v_{2,2} = 0,$$

choosing $v_{2,1} = 1$, we get the eigenvector $v_2 = (1,0)^T$. If the initial condition is in either one of these directions of the equilibrium point close to the equilibrium, they will go to this point (for certain).

(3,-2) When filling in the equilibrium point, we get the following eigenvalues of the Jacobian:

$$0 = \left| \begin{bmatrix} 5-\lambda & 5 \\ 0 & 3-\lambda \end{bmatrix} \right| = (\lambda-5)(\lambda-3) \Rightarrow (\lambda_1, \lambda_2) = (3, 5).$$

As both eigenvalues are positive, we have that this equilibrium is unstable. However, we want to find their eigenvectors in order to make a more accurate prediction of the situation around this equilibrium. Since the eigenvector belonging to 5 will have more influence than that of 3.

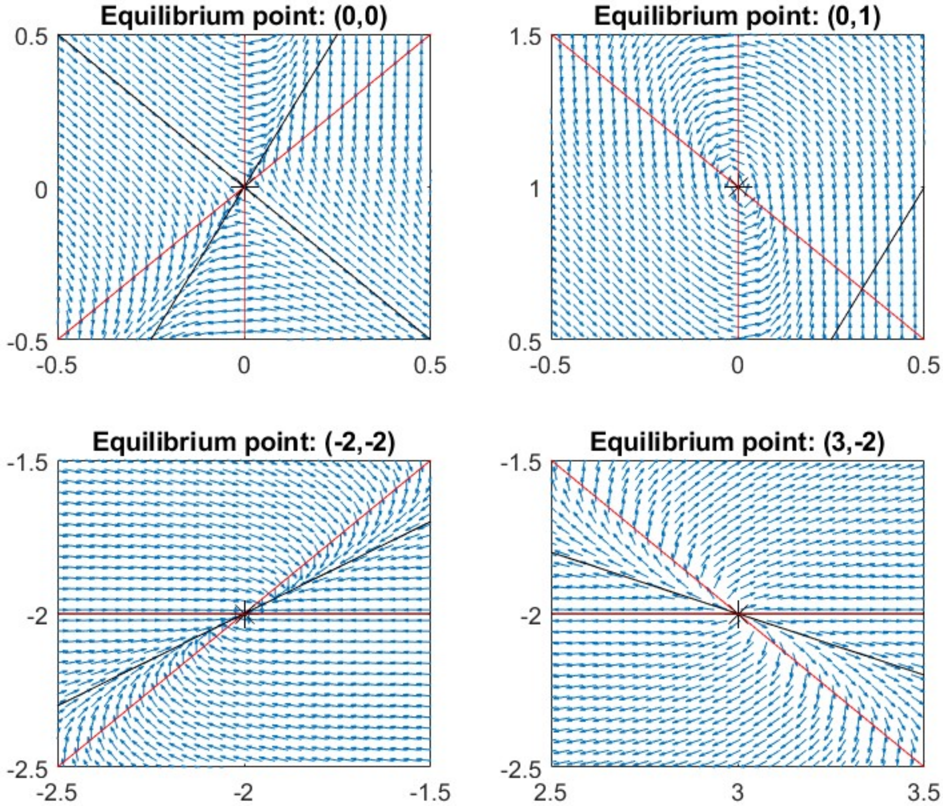
$$\begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix} v_1 = 0 \Rightarrow -\frac{2}{5} v_{1,1} = v_{1,2},$$

choosing $v_{1,1} = 1$, we get the eigenvector $v_1 = (1, -\frac{2}{5})^T$. As for the other one:

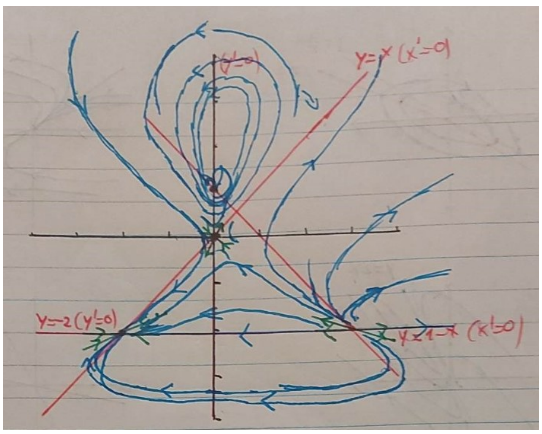
$$\begin{bmatrix} 0 & 5 \\ 0 & -2 \end{bmatrix} v_2 = 0 \Rightarrow v_{2,2} = 0,$$

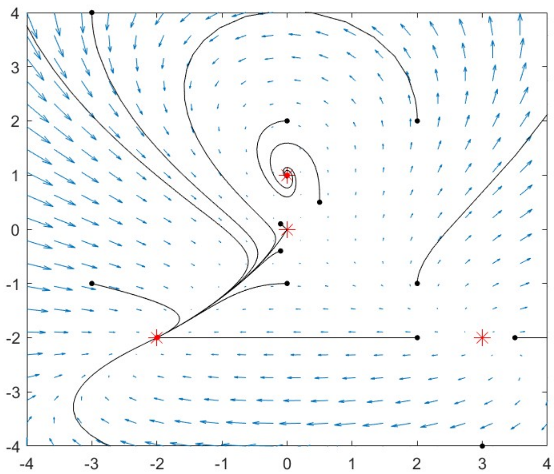
choosing $v_{2,1} = 1$, we get the eigenvector $v_2 = (1,0)^T$. Thus, the direction $(1,0)^T$ from the equilibrium point $(3,-2)$ has the most influence on the solution. If the initial condition is in either one of these directions of the equilibrium point close to the equilibrium, they will go to this point.

First we plot the phase plane around the equilibria;



Now we we try to sketch the solutions and use MATLAB to confirm our suspicions:





9. Consider $x' = x - y^2$ and $y' = y - x^2$ and plot the solutions. Verify your qualitative dynamics with MATLAB/Python/fortran.

Solution:

Using the same method as last exercise, we calculate the Jacobian:

$$\begin{bmatrix} 1 & -2y \\ -2x & 1 \end{bmatrix}$$

The eigevalues of this Jacobian is given by

$$0 = \left| \begin{bmatrix} 1-\lambda & -2y \\ -2x & 1-\lambda \end{bmatrix} \right| = \lambda^2 - 2\lambda + 1 - 4xy \Rightarrow (\lambda_1, \lambda_2) = (1 - 2\sqrt{xy}, 1 + 2\sqrt{xy})$$

Next we find the equilibria: $x' = 0$ when $x = y^2$ and $y' = 0$ when $y = x^2$. Thus we have the following equilibria:

$$(0,0) \text{ and } (1,1)$$

We now look at what kind of equilibria they are:

(0,0) There is only one eigenvalue: $\lambda = 1$. Since this is positive, we know that $(0,0)$ is an unstable equilibrium. We now need to know if it is a proper or improper node. Thus, we have to look at how many eigenvectors it has. We have that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} v = 0,$$

This means that v can be whatever it wants to be as an eigenvector. Thus, we can have 2 linear independent eigenvectors, for example: $(1,0)^T$ and $(0,1)^T$. Therefore, $(0,0)$ is an unstable proper node, otherwise known as a star node.

(1,1) There are two eigenvalues: $(\lambda_1, \lambda_2) = (-1, 2)$. They have opposite signs. This means that we are dealing with a saddle point. We now want to calculate their eigenvalues:

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} v_1 = 0 \Rightarrow v_{1,1} = v_{1,2},$$

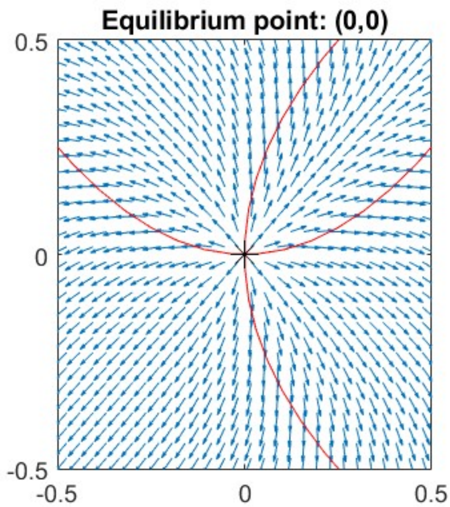
choosing $v_{1,1} = 1$, we get the eigenvector $v_1 = (1,1)^T$. We also have that

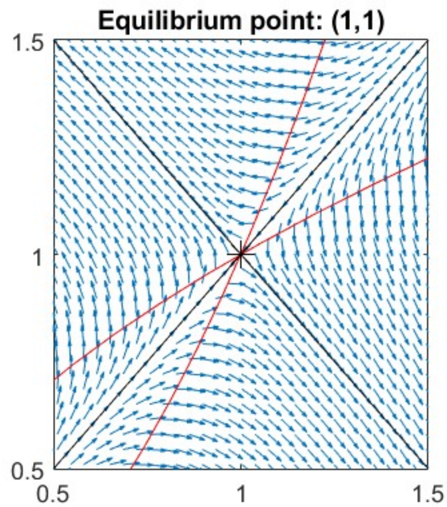
$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} v_2 = 0 \Rightarrow v_{2,1} = -v_{2,2},$$

choosing $v_{2,1} = 1$, we get the eigenvector $v_2 = (1,-1)^T$.
Therefore, the solutions around $(1,1)$ are unstable when in the direction $(1,-1)$ of point $(1,1)$ and stable when in the direction $(1,1)$ of point $(1,1)$.

We thus know that around the equilibrium $(0,0)$, we will get straight-ish lines, but the lines where x' and y' equal zero, will help greatly when drawing qualitatively correct solution. My attempt is as follows:

First we plot the phase plane around the equilibria;





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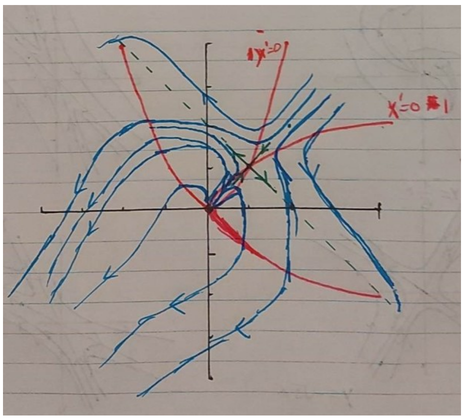
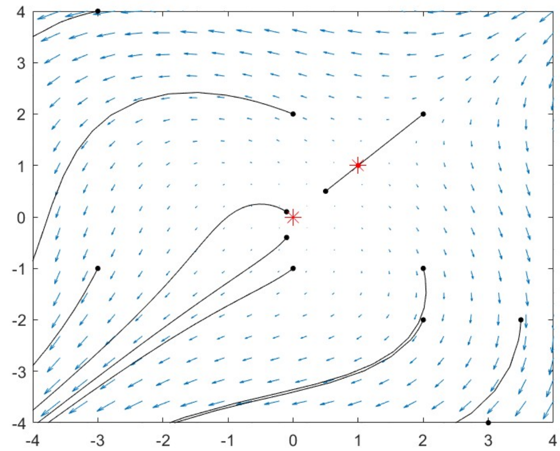




Note that the bottom left part is there because of the fact that these lines are not allowed to cross (uniqueness). Thus they tend to get pushed together near the $x = y$ line.

10. Consider $x' = (2+x)(y-x)$ and $y' = (4-x)(y+x)$ and plot the solutions. Verify your qualitative dynamics with MATLAB/Python/fortran.

Solution:

Using the same method as exercise 8, we calculate the Jacobian:

$$\begin{bmatrix} y-2x-2 & 2+x \\ -y-2x+4 & 4-x \end{bmatrix}$$

Next we find the equilibria: $x' = 0$ when $x = -2$ or $y = x$ and $y' = 0$ when $x = 4$ or $y = -x$. Thus, we have the following equilibria:

$$(0,0), \quad (-2,2) \text{ and } (4,4)$$

We now look at what kind of equilibria they are:

(0,0) The eigenvalues of the Jacobi matrix in this equilibrium becomes,

$$\left| \begin{bmatrix} -2-\lambda & 2 \\ 4 & 4-\lambda \end{bmatrix} \right| = \lambda^2 - 2\lambda - 16 = (\lambda - (1-\sqrt{17}))(1-(1+\sqrt{17})) \Rightarrow (\lambda_1, \lambda_2) = (1-\sqrt{17}, 1+\sqrt{17}).$$

They have opposite signs, thus we're dealing with a saddle equilibrium. Now we find the eigenvectors:

$$\begin{bmatrix} -3+\sqrt{17} & 2 \\ 4 & 3+\sqrt{17} \end{bmatrix} v_1 = 0 \Rightarrow 4v_{1,1} = -(3+\sqrt{17})v_{1,2}.$$

Choosing $v_{1,2} = 1$, we get the eigenvector $v = (-\frac{3+\sqrt{17}}{4}, 1)^T$. On the other hand,

$$\begin{bmatrix} -3-\sqrt{17}) & 2 \\ 4 & 3-\sqrt{17} \end{bmatrix} v_2 = 0 \Rightarrow 4v_{2,1} = -(3-\sqrt{17})v_{2,2}.$$

Choosing $v_{2,2} = 1$, we get the eigenvector $v = (\frac{\sqrt{17}-3}{4}, 1)^T$. In conclusion, around $(0,0)$ we have that the equilibrium is stable in the $(-\frac{3+\sqrt{17}}{4}, 1)^T$ direction and unstable in the $(\frac{\sqrt{17}-3}{4}, 1)^T$ direction.

(-2,2) The eigenvalues of the Jacobi matrix in this equilibrium becomes,

$$\left| \begin{bmatrix} 4-\lambda & 0 \\ 6 & 6-\lambda \end{bmatrix} \right| = 0 \Rightarrow (\lambda_1, \lambda_2) = (4, 6)$$

Both are positive real numbers, so we're dealing with an unstable equilibrium. Now we find the eigenvectors:

$$\begin{bmatrix} 0 & 0 \\ 6 & 2 \end{bmatrix} v_1 = 0 \Rightarrow -3v_{1,1} = v_{1,2},$$

choosing $v_{1,1} = 1$, we get the eigenvector $v_1 = (1, -3)^T$. Furthermore,

$$\begin{bmatrix} -2 & 0 \\ 6 & 0 \end{bmatrix} v_2 = 0 \Rightarrow v_{2,1} = 0,$$

choosing $v_{2,2} = 1$, we get the eigenvector $v_1 = (0, 1)^T$.

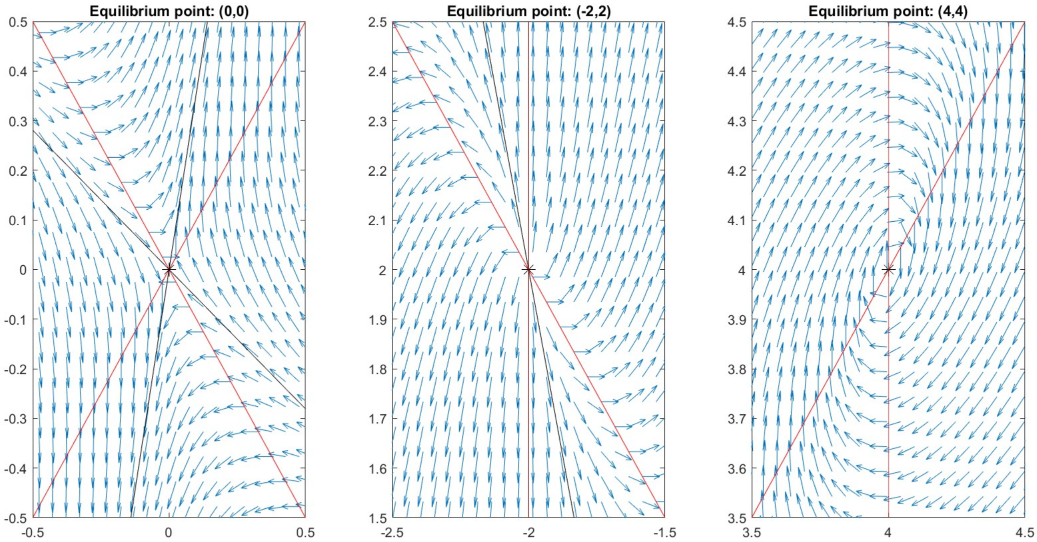
(4,4) The eigenvalues of the Jacobi matrix in this equilibrium becomes,

$$0 = \left| \begin{bmatrix} 6-\lambda & 6 \\ -8 & -\lambda \end{bmatrix} \right| = \lambda^2 - 6\lambda + 48 \Rightarrow (\lambda_1, \lambda_2) = (3+\sqrt{39}i, 3-\sqrt{39}i)$$

Since these are imaginary numbers with a positive real number, we have that this equilibrium is an unstable spiral near this equilibrium. Since $A_{2,1} < 0$, we have that this is clockwise.

An attempt of drawing solutions can be found under here. Once again, I paid particular attention to where the lines are horizontal/vertical and the effect of the eigenvectors found with the first equilibrium.

First we plot the phase plane around the equilibria;



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