AMATH 568: Homework 6
Winter Quarter 2023
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1. Consider the singular equation:
$$\epsilon y'' + (1+x)^2 y' + y = 0$$
with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$.
   (a) Obtain the leading order uniform solution using the WKB method.
   (b) Plot the uniform solution for $\epsilon = 0.1, 0.05, 0.1, 0.2$.

Solution:

(a) Let's say that
$$y = A(x)e^{\frac{S(x)}{\delta}},$$
but let's first say that
$$y(x) = e^{S(x)/\delta} = e^{\frac{S_0(x)+\delta S_1(x)+\delta^2 S_2(x)+\dots}{\delta}} = e^{S_1(x)}e^{\frac{S_0(x)}{\delta}+\delta S_2(x)+\dots}.$$
Let's calculate the derivatives:
$$y' = \left(\frac{S_{0x}(x)+\delta S_{1x}(x)+\delta^2 S_{2x}(x)+\dots}{\delta}\right)e^{\frac{S_0(x)+\delta S_1(x)+\delta^2 S_2(x)+\dots}{\delta}},$$
and
$$y'' = \left(\left(\frac{S_{0x}(x)+\delta S_{1x}(x)+\delta^2 S_{2x}(x)+\dots}{\delta}\right)^2 + \frac{S_{0xx}(x)+\delta S_{1xx}(x)+\delta^2 S_{2xx}(x)+\dots}{\delta}\right)e^{\frac{S_0(x)+\delta S_1(x)+\delta^2 S_2(x)+\dots}{\delta}}.$$
Plugging this in gives us
$$\begin{aligned}
0 &= \epsilon\left(\left(\frac{S_{0x}(x)+\delta S_{1x}(x)+\delta^2 S_{2x}(x)+\dots}{\delta}\right)^2 + \frac{S_{0xx}(x)+\delta S_{1xx}(x)+\delta^2 S_{2xx}(x)+\dots}{\delta}\right)e^{\frac{S_0(x)+\delta S_1(x)+\delta^2 S_2(x)+\dots}{\delta}}\\
&+ (1+x)^2\left(\frac{S_{0x}(x)+\delta S_{1x}(x)+\delta^2 S_{2x}(x)+\dots}{\delta}\right)e^{\frac{S_0(x)+\delta S_1(x)+\delta^2 S_2(x)+\dots}{\delta}}\\
&+ e^{\frac{S_0(x)+\delta S_1(x)+\delta^2 S_2(x)+\dots}{\delta}}\\
&= \epsilon\frac{S_{0x}^2}{\delta^2} + 2\epsilon\frac{S_{1x}S_{0x}}{\delta} + \epsilon\frac{S_{0xx}}{\delta} + \epsilon S_{1xx} + (1+x)^2\frac{S_{0x}}{\delta} + S_{1x} + 1 + \dots\\
&\overset{\delta=\epsilon}{\Rightarrow} S_{0x}^2 + (1+x)^2 S_{0x} + 2\epsilon S_{1x}S_{0x} + \epsilon S_{0xx} + \epsilon(1+x)^2 S_{1x} + \epsilon + \mathcal{O}(\epsilon^2) = 0.
\end{aligned}$$
The leading order term is
$$S_{0x}^2 = -(1+x)^2 S_{0x} \Rightarrow S_{0x} = -(1+x)^2 \text{ or } S_{0x} = 0 \Rightarrow S_0 = \frac{-1}{3}(1+x)^3 \text{ or } S_0 = B$$
Also, the next term is
$$2S_{1x}S_{0x} + S_{0xx} + (1+x)^2 S_{1x} + 1 = 0,$$
for $S_0 = B$, we have
$$(1+x)^2 S_{1x} = -1 \Rightarrow S_1 = -\int \frac{1}{(1+x)^2} = S_1 = \frac{1}{1+x} + C,$$
Therefore, one solution is
$$y = C_1 e^{\frac{1}{1+x}}.$$

For $S_0 = \frac{-1}{3}(1+x)^3$, we have

$$(1+x)^2 S_{1x} = 1 - 2(1+x) \Rightarrow S_{1x} = \frac{1}{(1+x)^2} - \frac{d}{dx}\ln\left((1+x)^2\right) \Rightarrow S_1 = -\frac{1}{1+x} - \ln\left((1+x)^2\right)$$

Therefore we have that

$$y = \frac{C_2}{(1+x)^2} e^{-\frac{1}{1+x} - \frac{1}{3\epsilon}(1+x)^3}.$$

Thus,

$$y = C_1 e^{\frac{1}{1+x}} + \frac{C_2}{(1+x)^2} e^{-\frac{1}{1+x} - \frac{1}{3\epsilon}(1+x)^3}.$$

We can find C_1 and C_2 using the boundary conditions:

$$\begin{cases} C_1 e + C_2 e^{-1-\frac{1}{3\epsilon}} & = 1 \\ C_1\sqrt{e} + \frac{C_2}{4} e^{-1/2-8/(3\epsilon)} & = 1 \end{cases} \Rightarrow \begin{cases} C_1 & = \frac{1}{\sqrt{e}} + \frac{1-\sqrt{e}}{e-4e^{\frac{7}{3\epsilon}}} \\ C_2 & = -\frac{4(-1+\sqrt{e})e^{1+\frac{8}{3\epsilon}}}{-e+4e^{\frac{7}{3\epsilon}}} \end{cases}$$

In conclusion the solution using the WKB method is

$$y = \left(\frac{1}{\sqrt{e}} + \frac{1-\sqrt{e}}{e-4e^{\frac{7}{3\epsilon}}}\right) e^{\frac{1}{1+x}} - \frac{1}{(1+x)^2} \frac{4(-1+\sqrt{e})e^{1+\frac{8}{3\epsilon}}}{-e+4e^{\frac{7}{3\epsilon}}} e^{-\frac{1}{1+x} - \frac{1}{3\epsilon}(1+x)^3}$$

(b) Plotting this monstrosity for the epsilon values using MATLAB and comparing it to a solution gotten from the `bvp5c()` function, we get

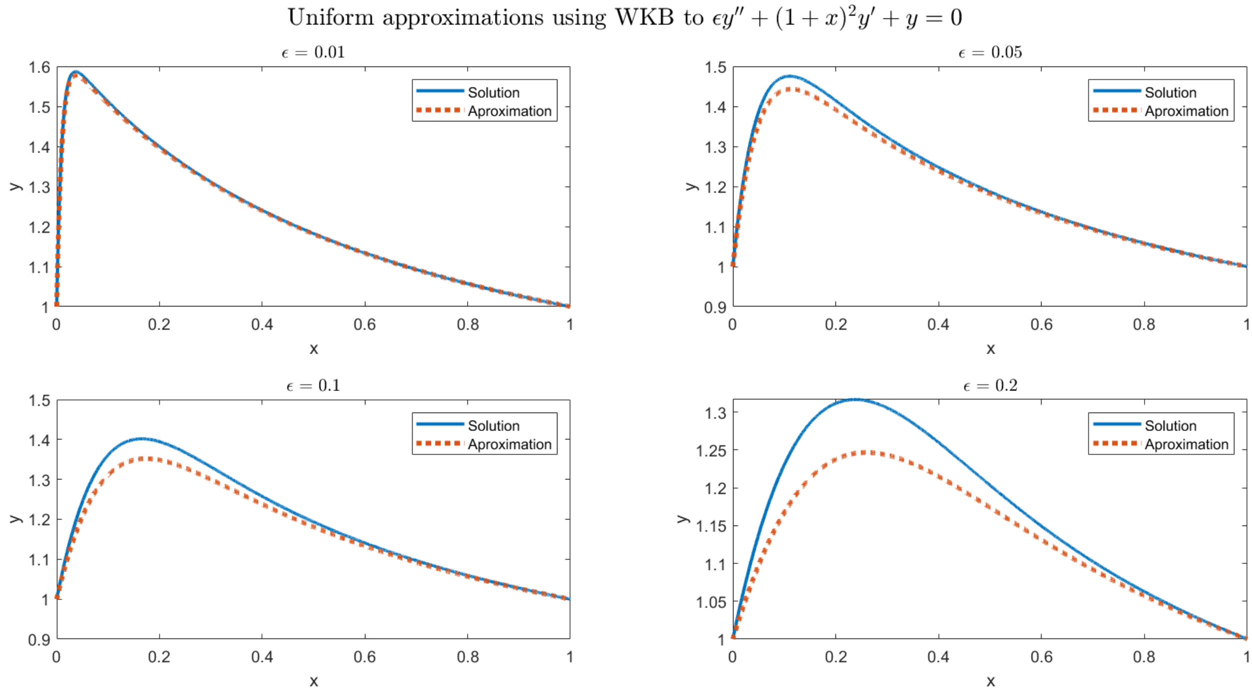

Figure 1: Approximating the solution to $\epsilon y'' + (1+x)^2 y' + y = 0$ with the uniform approximation using the WKB method for different values of ϵ.

Here we see once again that the smaller the epsilon, the better the approximation. This method does do a better job than in homework 5 and certainly for bigger epsilon.