AMATH 568: Homework 5
Winter Quarter 2023
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1. Consider the singular equation:
$$\epsilon y'' + (1+x)^2 y' + y = 0,$$
with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$.
   (a) Obtain a uniform approximation which is valid to $\mathcal{O}(1)$, i.e. determine the leading order behavior.
   (b) Show that assuming the boundary layer to be at $x = 1$ is inconsistent. (Hint: use the stretched inner variable $\xi = (1-x)/\epsilon$).
   (c) Plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution:

(a) The leading order term at first is
$$(1+x)^2 y' + y = 0 \overset{x \neq -1}{\Rightarrow} y' + \frac{1}{(1+x)^2} y = 0 \Rightarrow y = \tilde{A} e^{-\int \frac{1}{(1+x)^2} dx} = A e^{\frac{1}{1+x}}.$$
Note that if $x = -1$ is outside the scope of x. Looking at the second question, I assume the boundary layer will be around $x = 0$. Therefore we calculate A using $y(1) = 1$. Therefore
$$y_{out} = e^{-1/2} e^{\frac{1}{1+x}}.$$
Let's now find what the function will be on the inner layer by using $\xi = \frac{x}{\delta}$, thus $\frac{1}{dx} = \frac{1}{\delta}\frac{1}{d\xi}$:
$$\frac{\epsilon}{\delta^2} y'' + \frac{(1+\delta\xi)^2}{\delta} y' + y = 0 \Rightarrow \frac{\epsilon}{\delta^2} y'' + \frac{1 + 2\delta\xi + (\delta\xi)^2}{\delta} y' + y = 0.$$
If we take $\delta = \epsilon$, this would give us the only non-trivial or existent answer for the leading order term:
$$y''_{in} + y'_{in} = 0$$
We now calculate the solution:
$$e^{\xi} y''_{in} + e^{\xi} y'_{in} = (e^{\xi} y'_{in})' = 0 \Rightarrow y'_{in} = \tilde{A} e^{-\xi} \Rightarrow y_{in} = A e^{-\xi} + B.$$
We now want to make it fulfill the boundary condition on the left side of the domain:
$$y_{in}(0) = A + B = 1.$$
We also want these inner and outer solutions to transfer over smoothly to each other. We do this by calculating $y_{out}(0) = e^{1/2}$ and supposing that this should be equal to $\lim_{\xi\to\infty} y_{in}(\xi) = B$. This, since, it's pretty much like saying: "since ϵ will be small, saying that we want to know what happens at the transference is what happens at infinity for ξ." We now have that
$$B = e^{1/2} \text{ and thus } A = 1 - e^{1/2}.$$
Therefore
$$y_{int} = (1 - e^{1/2}) e^{-\xi} + e^{1/2}.$$
Note that, for the match, $y_{match} = e^{1/2}$ and thus
$$y_0 = (1 - e^{1/2}) e^{-x/\epsilon} + e^{1/2} + e^{-1/2} e^{\frac{1}{1+x}} - y_{match} = (1 - e^{1/2}) e^{-x/\epsilon} + e^{-1/2} e^{\frac{1}{1+x}}$$

(b) Say that the boundary layer is at $x = 1$. Then the outer solution has to fulfill $y_{out}(0) = 1$:

$$y_{out} = e^{\frac{1}{1+x} - 1}.$$

Let's use $\xi = (1-x)/\epsilon$. Then we have that $\frac{d\xi}{dx} = -\frac{1}{\epsilon}$, thus $\frac{1}{dx} = -\frac{1}{\epsilon}\frac{1}{d\xi}$. Therefore the equation becomes

$$\begin{aligned}\frac{1}{\epsilon}y''(\xi) - \frac{(1+x)^2}{\epsilon}y'(\xi) + y(\xi) = 0 &\Rightarrow y''(\xi) - (1+x)^2 y'(\xi) + \epsilon y(\xi) = 0 \\ &\Rightarrow y''(\xi) - (1+1-\epsilon\xi)^2 y'(\xi) + \mathcal{O}(\epsilon) = 0\end{aligned}$$

Therefore the leading order term for this is

$$y'' - 4y' = 0.$$

We now want to find the solution to this

$$\begin{aligned} & e^{-4\xi}y'' - 4e^{-4\xi}y' = (e^{-4\xi}y')' = 0 \\ \Rightarrow & y' = \tilde{A}e^{4\xi} \\ \Rightarrow & y_{in} = Be^{4\xi} + C.\end{aligned}$$

We want to know what happens to this function where $x \to 0$, thus $\xi \to \frac{1}{\epsilon}$. Since we want epsilon to go to zero (from the positive side), we want ξ to go to infinity here. Therefore we would like to calculate $\lim_{\xi\to\infty} Be^{4\xi} + C$, which will always be infinity. Except for when B is zero. If this was the case, then y_{in} would be constant, which we do not want. Therefore for at least this choice in ξ, assuming the boundary layer to be at $x = 1$ is inconsistent.

(c) We plot the uniform solution for the different epsilons:

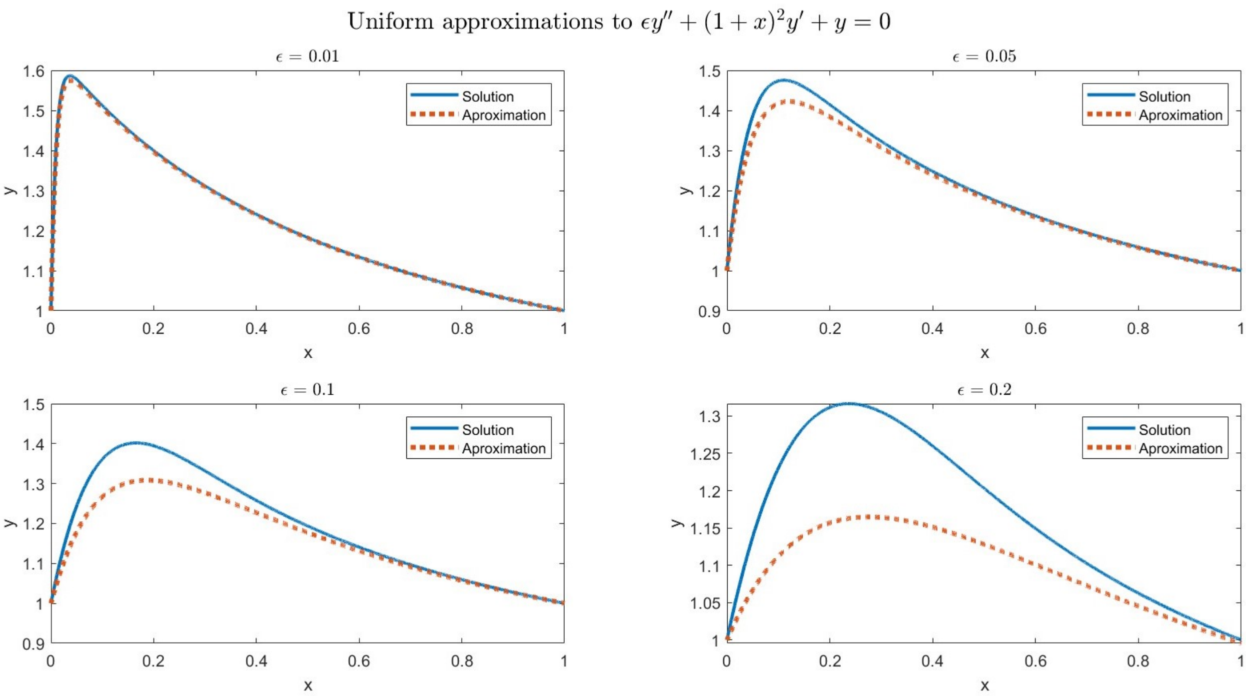

Figure 1: Approximating the solution to $\epsilon y'' + (1+x)^2 y' + y = 0$ using the uniform approximation for different values of ϵ.

It's clear that the lower the epsilon, the better the approximation (better form), which is what we expected. The "solution" has been calculated using the `bvp5c` function on MATLAB

2. Consider the singular equation:

$$\epsilon y'' - x^2 y' - y = 0,$$

with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$.

(a) With the method of dominant balance, show that there are three distinguished limits: $\delta = \epsilon^{1/2}$, $\delta = \epsilon$, and $\delta = 1$ (the outer problem). Write down each of the problems in the various distinguished limits.

(b) Obtain the leading order uniform approximation (hint: there are boundary layers at $x = 0$ and $x = 1$).

(c) Plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution:

(a) First, let's get the leading order term ($\delta = 1$). This gives us

$$-x^2 y' - y = 0 \overset{x \neq 0}{\Rightarrow} e^{-\frac{1}{x}} y' + \frac{1}{x^2} e^{-\frac{1}{x}} y = 0 \Rightarrow y_{out} = A e^{\frac{1}{x}}$$

Note that We will want to go with x to zero. This is however a problematic point, since it won't exist there, thus we need to assume that our outer solution is zero. Now we substitute $\xi_1 = \frac{x}{\delta}$ to look what happens:

$$\frac{\epsilon}{\delta^2} y'' - \frac{x^2}{\delta} y' - y = 0.$$

Substituting $x = \delta \xi_1$:

$$\epsilon y'' - \delta^3 \xi_1^2 y' - \delta^2 y = 0.$$

Taking $\delta = \sqrt{\epsilon}$, we get interesting dominating terms

$$y'' - y = 0.$$

This layer would be at $x = 0$, thus we have that here $y(0) = 1$. The solution here is

$$y_{1in} = B e^{\xi_1} + C e^{-\xi_1} \Rightarrow B + C = 1,$$

but since we will be wanting to take ξ_1 of y_{1in} to infinity, we cannot have the e^{ξ_1} term, therefore $B = 0$, thus $C = 1$. Now we want to look at the layer at $x = 1$ by substituting $\xi_2 = \frac{1-x}{\delta}$:

$$\frac{\epsilon}{\delta^2} y'' + \frac{x^2}{\delta} y' - y = 0.$$

Substituting $x = 1 - \delta \xi_2$:

$$\epsilon y'' + \delta (1 - \delta \xi_2)^2 y' - \delta^2 y = 0 \Rightarrow \epsilon y'' + \delta y' + \mathcal{O}(\delta^2) = 0.$$

If we now take $\delta = \epsilon$, we have a new, interesting, leading order term:

$$y'' + y' = 0.$$

This gives us the following solution:

$$(e^{\xi_2} y')' = 0 \Rightarrow y' = \tilde{D} e^{-\xi_2} \Rightarrow y_{2in} = D e^{-\xi_2} + E.$$

Here we have the boundary value $y_{2in}(x = 1) = y_{2in}(\xi_2 = 0) = 1$, thus $D + E = 1$.

In conclusion we have

$$\begin{cases} x^2 y' + y = 0 & \text{outer solution,} \\ y'' - y = 0 & \text{inner solution } x = 0, \\ y'' + y' = 0 & \text{inner solution } x = 1, \end{cases}$$

with solutions up until now:

$$\begin{cases} y_{out} = 0 & \text{outer solution,} \\ y_{1in} = C e^{-x/\sqrt{\epsilon}} & \text{inner solution } x = 0, \\ y_{2in} = D e^{-(1-x)/\epsilon} + E & \text{inner solution } x = 1, \end{cases}$$

with $D + E = 1$.

(b) from (a) we get the matching terms:

$$\begin{cases} \lim_{x\to 0} y_{out} = \lim_{\xi_1\to\infty} y_{1in} \\ \lim_{x\to 1} y_{out} = \lim_{\xi_2\to\infty} y_{2in} \\ C = 1 \\ D + E = 1 \end{cases} \Rightarrow \begin{cases} 0 & = 0 \\ 0 & = E \\ C & = 1 \\ D + E & = 1 \end{cases} \Rightarrow \begin{cases} B & = 0 \\ E & = 0 \\ C & = 1 \\ D & = 1 \end{cases}.$$

The matching terms are always zero, therefore we have the full solution

$$y = e^{-\xi_1} + e^{-(\xi_2-1)} = e^{-x/\sqrt{\epsilon}} + e^{-(1-x)/\epsilon}$$

(c) We plot the uniform solution for the different epsilons:

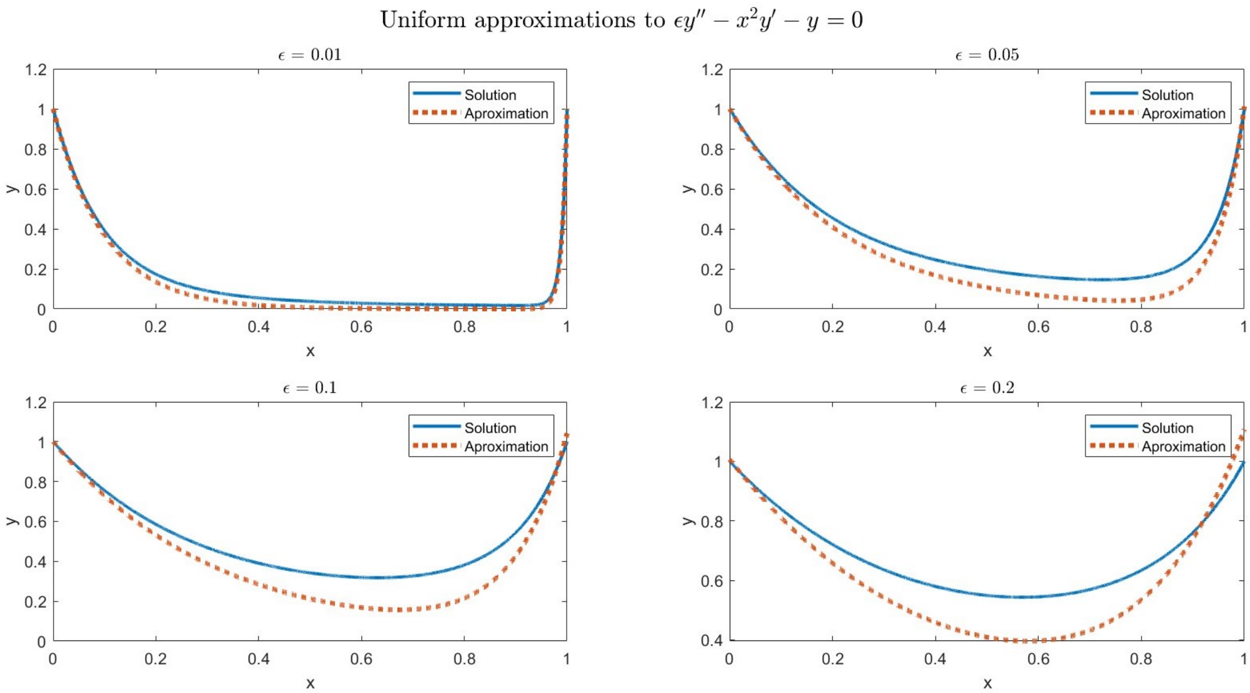

Figure 2: Approximating the solution to $\epsilon y'' - x^2 y' - y = 0$ using the uniform approximation for different values of ϵ.

Here we see that the boundary values are not necessarily fulfilled to the approximations, but the form is kind of oke. The lower the epsilon, the better of course.