AMATH 568: Homework 8
Winter Quarter 2023
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1. Consider the inverted pendulum dynamics:

$$y'' + (\delta + \epsilon \cos(\omega t)) \sin(y) = 0.$$

   (a) Perform a Floquet analysis (computationally) of the pendulum with continous forceing $\cos(\omega t)$.
   (b) Evaluate for what values of δ, ϵ and ω the pendulum is stabilized.

Solutions:

(a) Assume we have two solutions with different initial conditions:

$$y_1 = 1 \text{ and } \dot{y}_1 = 0,$$

and

$$y_2 = 0 \text{ and } \dot{y}_2 = 1.$$

Note that with these initial conditions $W[y_1, y_2](0) = 1$, and thus it's not outrageous to assume that the solutions will be linearly independent. Note now that $\cos(\omega t) = \cos\left(\omega\left[t + \frac{2\pi}{\omega}\right]\right)$. And therefore, if $p(t) = \delta + \epsilon\cos(\omega t)$,

$$y''(t + \frac{2\pi}{\omega}) + p(t + \frac{2\pi}{\omega})\sin(y(t + \frac{2\pi}{\omega})) = y''(t + \frac{2\pi}{\omega}) + p(t)\sin(y(t + \frac{2\pi}{\omega})) = 0.$$

Therefore, we get that these solutions $y_1(t + \frac{2\pi}{\omega})$ and $y_2(t + \frac{2\pi}{\omega})$ are fundamental solutions of the original equation of the inverted pendulum. Thus, because of some theoretical stuff and that the original solutions are linearly independent, we know that a linear transform exists to express these solutions with solutions $y_1(t)$ and $y_2(t)$:

$$\begin{cases} y_1(t+T) = a_{11}y_1(t) + a_{12}y_2(t) \\ y_2(t+T) = a_{21}y_1(t) + a_{22}y_2(t) \end{cases} \Rightarrow \mathbf{y}(t+T) = A\mathbf{y}(t).$$

It's now pretty clear that at $t = 0$

$$\begin{cases} y_1(T) = a_{11} \\ y_2(T) = a_{21} \end{cases}.$$

Deriving both sides by t and filling in $t = 0$ again, now gives

$$\begin{cases} y_1'(T) = a_{12} \\ y_2'(T) = a_{22} \end{cases}.$$

Thus,

$$A = \begin{bmatrix} y_1(T) & y_1'(T) \\ y_2(T) & y_2'(T) \end{bmatrix}.$$

Say we can diagonilze it, such that a matrix V of eigenvectors and Λ of eigenvalues such that, if we say that $y(t) = Vv(t)$:

$$Vv(t + \frac{2\pi}{\omega}) = AVv(t) \Rightarrow v(t + \frac{2\pi}{\omega}) = V^{-1}V\Lambda V^{-1}Vv(t) = \Lambda v(t).$$

This is a system of equations which are independent of each other. Note now that

$$|v\left(t_0 + n\frac{2\pi}{\omega}\right)| = |\Lambda^n||v(t_0)|,$$

this is clearly unstable whenever the modulus of one of the eigenvalues is higher than one and stable when it's lower than or equal to one. Calculating the eigenvalues gives us the quadratic equation

$$\lambda^2 - (y_1(T) + y_2'(T))\lambda + W[y_1, y_2](T) = 0.$$

Assume the Wronskain was constant for all time, or at least periodic as well, then $W[y_1, y_2](T)$, we then have that the eigenvalues, with $\alpha = y_1(T) + y_2'(T)$, is given by

$$\lambda_\pm = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

If $\alpha > 2$ ($\alpha < -2$), we then have that the absolute value of λ_+ (λ_-) is larger than one. Thus, for $|\alpha|> 2$, the solution is unstable. For $\alpha \leq 2$ ($\alpha \geq -2$), then

$$|\lambda_\pm| \overset{4-\alpha^2 \geq 0}{=} \left| \frac{\alpha \pm i\sqrt{4 - \alpha^2}}{2} \right| = \sqrt{\frac{1}{4}(\alpha^2 + 4 - \alpha^4)} = 1,$$

thus it's stable. In conclusion, for stability we want $|\alpha|\leq 2$. In this case we get the Floquet discriminant

$$|\Gamma|= |y_1(T) + y_2'(T)|\leq 2.$$

Note now that the solution can always be described in $[-\frac{\pi}{2}, \frac{3\pi}{2}]$, both mathematically because we can translate it by 2π and the y'' won't be effected and $\sin(y)$ as well, but also physically as the pendulum is just going in a circular moition and 0 and 2π are the same. As a side note, this tells us that the solution isn't unique, oh well. Moreover, we can approximate y around 0 and π now:

$$y \approx 0: \quad \sin(y) = y - \frac{y^3}{3!} + \mathcal{O}(y^5),$$

$$y_1 \approx \pi: \quad \sin(y_1) = \sin(y + \pi) = -\sin(y) = -y + \frac{y^3}{3!} + \mathcal{O}(y^5).$$

Let's go forward with the inverted pendulum dynamics (focusing around π), focusing around 0 is similar. We now approximated the dynamics by equation

$$y'' - (\delta + \epsilon \cos(\omega t))y = 0.$$

Let's now shift time by $\frac{\pi}{2\omega}$: $\tau = t - \frac{\pi}{2\omega}$. Note that $d\tau = dt$, thus

$$y'' - (\delta + \epsilon \cos(\omega\tau + \frac{\pi}{2}))y = y'' - (\delta + \epsilon \sin(\omega\tau))y = 0.$$

We could do all of the derivations above for this τ, but it would just come down to the same thing. Now we can use a different approximation to $\sin(\omega\tau)$, mainly, denoting T as $\frac{2\pi}{\omega}$:

$$\sin(\omega\tau) = \begin{cases} 1 & 0 < \tau < T/2 \\ -1 & T/2 \leq \tau \leq T \end{cases}.$$

Using this we get a final approximation of the ODE, give by

$$\begin{cases} y'' - (\delta + \epsilon)x = 0 & 0 < \tau < T/2 \\ y'' - (\delta - \epsilon)x = 0 & T/2 < \tau < T. \end{cases}$$

Since we have the minus sign from the inverted pendulum we get "exponential" solutions. However, since we have $\delta - \epsilon$ it might turn into sines and cosines. Computationally we can get the Floquet discriminant to the linearlized ODE:

$$\Gamma = 2\cosh\left(\sqrt{\delta + \epsilon}\frac{T}{2}\right)\cosh\left(\sqrt{\delta - \epsilon}\frac{T}{2}\right) + \left(\frac{\sqrt{\delta + \epsilon}}{\sqrt{\delta - \epsilon}} + \frac{\sqrt{\delta - \epsilon}}{\sqrt{\delta + \epsilon}}\right)\sinh\left(\sqrt{\delta + \epsilon}\frac{T}{2}\right)\sinh\left(\sqrt{\delta - \epsilon}\frac{T}{2}\right)$$

(b) For ω large enough, T small, we have that

$$\Gamma = 2\cos\left(\sqrt{\epsilon - \delta}\frac{\pi}{\omega}\right)$$

(from the book). It's clear that $|\Gamma| \leq 2|\cos\left(\sqrt{\epsilon - \delta}\frac{\pi}{\omega}\right)| \leq 2$, which means that, for ω large enough, the inverted pendulum is stabilized for all ϵ and δ. As the dynamics mostly depend on if δ is larger than or lower than ϵ, we'll only plot Γ for one case for those situations with variable ω. We plot this for both the Γ we derived using the approximated ODE and by solving the actual nonlinear ODE using RK45 in figure 1:

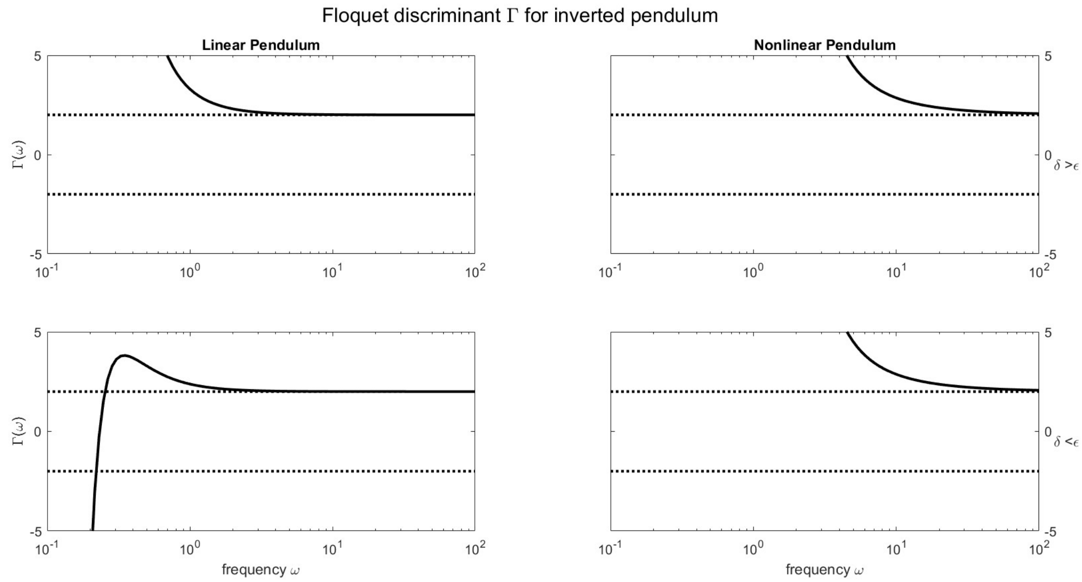

Figure 1: The Floquet discriminant in function of frequency oscillation support ω for both the linearized pendulum (first column) and nonlinear pendulum (second column). In the top row, the oscillation magnitude ϵ is smaller than the natural oscillation frequency δ. The bottom row shows the opposite. The dashed line represents the values 2 and -2, stability holds in between these two lines.

As seen before, for ω large, we get to about 2. For $\delta > \epsilon$, we also have a small interval of the frequency where the inverted pendulum is stable if we were to look at the linear Pendulum, or the approximation of the nonlinear ODE.

Thank you for the fun quarter.