AMATH 573
Solitons and nonlinear waves
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Homework 2

due: 10/21/2022

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1. **The Benjamin-Ono equation**

$$u_t + uu_x + \mathcal{H}u_{xx} = 0$$

is used to describe internal waves in deep water. Here $\mathcal{H}f(x,t)$ is the spatial Hilbert transform of $f(x,t)$:

$$\mathcal{H}f(x,t) = \frac{1}{\pi}⨍_{-\infty}^{\infty} \frac{f(z,t)}{z-x}dz,$$

and $⨍$ denotes the Cauchy principal value integral. Write down the linear dispersion relationship for this equation linearized about the zero solution.

Solution:

Linearizing around the zero solution gives $u(x,t) = \epsilon v(x,t) + \mathcal{O}(\epsilon^2)$
Filling this in gives,

$$\begin{aligned}
&\epsilon v_t(x,t) + (\epsilon v(x,t) + \mathcal{O}(\varepsilon^2))(\epsilon v_x(x,t) + \mathcal{O}(\varepsilon^2)) + \mathcal{H}(\epsilon v_{xx}(x,t) + \mathcal{O}(\varepsilon^2))\\
=&\epsilon v_t(x,t) + \epsilon\mathcal{H}v_{xx}(x,t) + \mathcal{O}(\epsilon^2)\\
=&0
\end{aligned}$$

If we now derive it by ϵ and fill in 0, we get

$$v_t(x,t) + \mathcal{H}v_{xx}(x,t) = 0$$

Suppose now that $v(x,t) = e^{ikx-i\omega t}$, this gives us

$$-i\omega e^{ikx-i\omega t} - k^2\mathcal{H}e^{ikx-i\omega t} = 0$$

We thus want to find,

$$\mathcal{H}e^{ikx-i\omega t} = \frac{1}{\pi}⨍_{-\infty}^{\infty} \frac{e^{ikz-i\omega t}}{z-x}dz = \frac{e^{-i\omega t}}{\pi}⨍_{-\infty}^{\infty} \frac{e^{ikz}}{z-x}dz.$$

Since x is a simple pole of $\frac{e^{ikz}}{z-x}$, we get that the residual in x is equal to $\lim_{z\to x}\frac{z-x}{z-x}e^{ikz} = e^{ikx}$. Using Jordan's lemma and since $x\in\mathbb{R}$, we know that $⨍_{-\infty}^{\infty}\frac{e^{ikz}}{z-x} = \pi i$"Res of $\frac{e^{ikz}}{z-x}$ in x". Thus,

$$\mathcal{H}e^{ikx-i\omega t} = \frac{e^{-i\omega t}}{\pi}⨍_{-\infty}^{\infty}\frac{e^{ikz}}{z-x}dz = e^{ikx-i\omega t}i.$$

In conclusion:

$$-i\omega e^{ikx-i\omega t} - k^2 ie^{ikx-i\omega t} = 0 \Rightarrow \omega = -k^2$$

2. Derive the linear dispersion relationship for the one-dimensional surface water wave problem by linearizing around the trivial solution $\zeta(x,t) = 0$, $\phi(x,z,t) = 0$:

$$\begin{aligned} \nabla^2\phi &= 0, && -h < z < \zeta(x,t) \\ \phi_z &= 0, && z = -h \\ \zeta_t + \phi_x\zeta_x &= \phi_z, && z = \zeta(x,t) \\ \phi_t + g\zeta + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) &= T\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}, && z = \zeta(x,t) \end{aligned}$$

Here $z = \zeta(x,t)$ is the surface of the water, $\phi(x,z,t)$ is the velocity potential so that $v = \nabla\phi$ is the velocity of the water, g is the acceleration of gravity, and $T > 0$ is the coefficient of surface tension.

Solution: Linearizing around the trivial solution gives $\zeta(x,t) = \epsilon v(x,t) + \mathcal{O}(\epsilon^2)$ and $\phi(x,z,t) = \epsilon u(x,z,t) + \mathcal{O}(\epsilon^2)$. We then get

$$\begin{aligned} \nabla^2(\epsilon u + \mathcal{O}(\epsilon^2)) &= 0, && -h < z < \epsilon v(x,t) + \mathcal{O}(\epsilon^2) \\ \epsilon u_z + \mathcal{O}(\epsilon^2) &= 0, && z = -h \\ \epsilon v_t + (\epsilon u_x)(\epsilon v_x) + \mathcal{O}(\epsilon^2) &= \epsilon u_z + \mathcal{O}(\epsilon^2), && z = \epsilon v(x,t) + \mathcal{O}(\epsilon^2) \\ \epsilon u_t + g\epsilon v + \frac{1}{2}\left((\epsilon u_x)^2 + (\epsilon u_z)^2\right) + \mathcal{O}(\epsilon^2) &= T\frac{\epsilon v_{xx} + \mathcal{O}(\epsilon^2)}{(1 + (\epsilon v_x)^2 + \mathcal{O}(\epsilon^3))^{3/2}}, && z = \epsilon v(x,t) + \mathcal{O}(\epsilon^2) \end{aligned}$$

Note here that

$$u_z = u_z(x,z,t)$$

at $z = \epsilon v(x,t) + \mathcal{O}(\epsilon^2)$, thus

$$u_z = u_z(x, \epsilon v(x,t) + \mathcal{O}(\epsilon^2), t) = u_z(x,0,t) + u_z(x,0,t)(\epsilon v(x,t) + \mathcal{O}(\epsilon^2)) + \mathcal{O}(\epsilon^2),$$

and the same can be done for u_t. This causes us to be able to change the boundary to $z = 0$ eventually.

We now want to derive everything by ϵ and let it go to 0, the difficult part is, deriving $f(\epsilon) = T\frac{\epsilon v_{xx} + \mathcal{O}(\epsilon^2)}{(1+(\epsilon v_x)^2 + \mathcal{O}(\epsilon^3))^{3/2}}$:

$$\begin{aligned} \lim_{\epsilon\to 0} f'(\epsilon) =& \lim_{\epsilon\to 0} T\frac{(v_{xx} + \mathcal{O}(\epsilon))\left(1 + (\epsilon v_x)^2 + \mathcal{O}(\epsilon^3)\right)^{3/2}}{\left(1 + (\epsilon v_x)^2 + \mathcal{O}(\epsilon^3)\right)^3} \\ &- \frac{(\epsilon v_{xx} + \mathcal{O}(\epsilon^2))\,\frac{3}{2}(1 + (\epsilon v_x)^2 + \mathcal{O}(\epsilon^3))^{1/2}\big)\,(2\epsilon v_x + \mathcal{O}(\epsilon^2)}{\left(1 + (\epsilon v_x)^2 + \mathcal{O}(\epsilon^3)\right)^3} \\ =& T v_{xx} \end{aligned}$$

Thus, everything becomes:

$$\begin{aligned} \nabla^2 u &= 0, && -h < z < v(x,t) \\ u_z &= 0, && z = -h \\ v_t &= u_z, && z = 0 \\ u_t + gv &= T v_{xx}, && z = 0 \end{aligned}$$

We will now try and solve for u for this PDE. Suppose that $u(x,z,t) = f(z)e^{i\hat{k}x - i\hat{\omega}t}$, note that since $u_z(-h) = 0$, $f'(-h) = 0$. From the first equation we get that

$$f"(z)e^{i\hat{k}x-i\hat{\omega}t} - \hat{k}^2 f(z)e^{i\hat{k}x-i\hat{\omega}t} = 0 \Rightarrow f"(z) = \hat{k}^2 f(z)$$

Since e^{kz} and e^{-kz} are solutions, the solution is a linear combination of them: $f(z) = Ae^{kz}+Be^{-kz}$. Since $f'(-h) = k\tilde{A}e^{-kh} - k\tilde{B}e^{kh} = 0$, we know that $\tilde{A} = \tilde{B}e^{2kh}$. Thus:

$$f(z) = \tilde{B}e^{kz+2kh} + \tilde{B}e^{-kz} = \frac{2\tilde{B}}{e^{-kh}}\frac{e^{k(z+h)} + e^{-k(z+h)}}{2} = A\cosh(k(z+h))$$

Assuming now that $v = e^{ikx-i\omega t}$, we get from the second to last equation that

$$-i\omega e^{ikx-i\omega t} = A\hat{k}\sinh(\hat{k}h)e^{i\hat{k}x-i\hat{\omega}t}.$$

Therefore $\hat{k} = k$, $\hat{\omega} = \omega$ and thus also

$$A = -i\omega(k\sinh(kh))^{-1}.$$

Filling u and v in in the last equation gives,

$$-Ai\omega\cosh(kh)e^{ikx-i\omega t} + ge^{ikx-i\omega t} = -Tk^2 e^{ikx-i\omega t} \Rightarrow A = \frac{Tk^2+g}{\omega\cosh(kh)i}$$

From the last two equations, we get that

$$-i\omega(k\sinh(kh))^{-1} = \frac{Tk^2+g}{\omega\cosh(kh)i} \Rightarrow \omega^2 = (gk+T^3)\frac{\sinh(kh)}{\cosh(kh)} = (gk+Tk^3)\tanh(kh)$$

In conclusion

$$\omega^2 = (gk+Tk^3)\tanh(kh)$$

3. Having found that for the surface water wave problem without surface tension the linear dispersion relationship is $\omega^2 = gk\tanh(kh)$, find the group velocities for the case of long waves in shallow water (kh small), and for the case of deep water (kh big).

   Solution: We know that the group velocity is given by $v_g = \frac{d\omega}{dk}$. First we find the group velocity for long waves in shallow water (kh small). Since kh is small, we have $\tanh(kh) \approx kh$, thus $\omega_\pm = \pm|k|\sqrt{gh}$, thus

$$v_g \frac{d\omega_\pm}{dk} = \pm\text{sgn}(k)\sqrt{gh}$$

   Suppose now that we have the case of deep water. Since $|kh| \gg 1$, we can approximate $\tanh(kh) \approx \text{sgn}(k)1$, thus $\omega_\pm = \pm\sqrt{\text{sgn}(k)gk}$, thus

$$v_g \frac{d\omega_\pm}{dk} = \pm\frac{g}{2\sqrt{gk}} = \pm\frac{1}{2}\sqrt{\text{sgn}(k)\frac{g}{k}}$$

4. Whitham wrote down what is now known as **the Whitham equation** to incorporate the full effect of water-wave dispersion for waves in shallow water by modifying the KdV equation $u_t + vu_x + uu_x + \gamma u_{xxx} = 0$ (where we have included the transport term) to

$$u_t + uu_x + \int_{-\infty}^{\infty} K(x-y)u_y(y,t)dy = 0,$$

where

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk,$$

and $c(k)$ is the positive phase speed for the water-wave problem: $c(k) = \sqrt{g\tanh(kh)/k}$.

(a) What is the linear dispersion relation of the Whitham equation?

(b) Show that the dispersion relation of the KdV equation is an approximation to that of the Whitham equation for long waves, *i.e.*, for $k \to 0$. What are v and γ?

Note that using this process of "Whithamization", one could construct a KdV-like equation (*i.e.*, an equation with the KdV nonlinearity) that has any desired dispersion relation. Similar procedures can be followed for other equations, like the NLS equation *etc.*

Solution:

(a) Linearizing around 0 gives $u = \epsilon v(x,t) + \mathcal{O}(\epsilon^2)$, filling this in gives,

$$\begin{aligned}
&\epsilon v_t(x,t) + \mathcal{O}(\epsilon^2) + (\epsilon v(x,t) + \mathcal{O}(\epsilon^2))(\epsilon v_x(x,t) + \mathcal{O}(\epsilon^2)) + \int_{-\infty}^{\infty} K(x-y)(\epsilon v_y(y,t) + \mathcal{O}(\epsilon^2))dy \\
=&\epsilon v_t(x,t) + \epsilon\int_{-\infty}^{\infty} K(x-y)v_y(y,t)dy + \mathcal{O}(\epsilon^2) \\
=&0.
\end{aligned}$$

Now deriving by ϵ and filling in $\epsilon = 0$, gives

$$v_t(x,t) + \int_{-\infty}^{\infty} K(x-y)v_y(y,t)dy = 0.$$

Assuming that $v = e^{ikx-i\omega t}$, we get

$$-i\omega e^{ikx-i\omega t} + ike^{-i\omega t}\int_{-\infty}^{\infty} K(x-y)e^{iky}dy = 0.$$

Thus,

$$\omega = k\int_{-\infty}^{\infty} K(x-y)e^{-ik(x-y)}dy = -k\int_{\infty}^{-\infty} K(z)e^{-ikz}dz = k\mathcal{F}[K(x)](k),$$

but notice how $K(x) = \mathcal{F}^{-1}[c(k)](x)$ with $\mathcal{F}$ the Fourier transform. Thus,

$$\omega = kc(k) = \sqrt{kg\tanh(kh)}$$

(b) Suppose that we have long waves, thus kh is small. We can approximate $\tanh(kh)$ by its Taylor approximation: $\tanh(kh) = kh - \frac{(kh)^3}{3} + \mathcal{O}(k^5)$. We then get that the dispersion relation for the Whitham equation is

$$\omega \approx \sqrt{kg(kh - \frac{kh^3}{3})} = k\sqrt{gh}\sqrt{1 - \frac{(kh)^2}{3}}.$$

This is assuming that k is positive, which is justified by h and g being positive. Now we approximate $\sqrt{1 - \frac{(kh)^2}{3}} = 1 - \frac{(kh)^2}{6} + \mathcal{O}(k^4)$. Thus,

$$\omega \approx \sqrt{gh}k - \frac{\sqrt{gh}h^2}{6}k^3.$$

In finding the dispersion relation of the linearized prescribed KdV equation $u_t + vu_x + uu_x + \gamma u_{xxx} = 0$, we get:

$$-i\omega + vik - \gamma ik^3 = 0 \Rightarrow \omega = vk - \gamma k^3.$$

If now $v = \sqrt{gh}$ and $\gamma = \frac{\sqrt{gh}h^2}{6}$ we get that these dispersion relations approximate each other.

5. Consider the linear free Schrödinger ("free", because there's no potential) equation

$$i\psi_t + \psi_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0, \quad \psi \to 0 \text{ as } |x| \to \infty,$$

with $\psi(x,0) = \psi_0(x)$ such that $\int_{-\infty}^{\infty} |\psi_0|^2 dx < \infty$.

(a) Using the Fourier transform, write down the solution of this problem.

(b) Using the Method of Stationary Phase, find the dominant behavior as $t \to \infty$ of the solution, along lines of constant x/t.

(c) With $\psi_0(x) = e^{-x^2}$, the integral can be worked out exactly. Compare (graphically or other) this exact answer with the answer you get from the Method of Stationary Phase. Use the lines $x/t = 1$ and $x/t = 2$ to compare.

(d) Use your favorite numerical integrator (write your own, or use maple, mathematica or matlab) to compare (graphically or other) with the exact answer and the answer you get from the Method of Stationary Phase.

Solution:

(a) Guess that $\psi = e^{ikx - i\omega t}$, we then have

$$\omega e^{ikx-i\omega t} - k^2 e^{ikx-i\omega t} = 0 \Rightarrow \omega = k^2,$$

now we define $\psi_k = e^{ikx - ik^2 t}$, which are all solutions to the PDE. Then

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)\psi_k dk, \qquad \text{, with } a(k) = \int_{-\infty}^{\infty} e^{-ikx}\psi_0(x)dx,$$

is the solution using the Fourier transform. It should be possible to calculate this since $\psi_0 \in L_2(\mathbb{R})$

(b) Now let $\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{i\phi(k)t}$ with $\phi(k) = k\frac{x}{t} - k^2$. We want to find the dominant behavior so we look at where $\phi'(k) = 0$ which is $k_0 = \frac{x}{2t}$. We now have that

$$\psi \approx \frac{1}{2\pi}\int_{x/(2t)-\delta}^{x/(2t)+\delta} a(k_0)e^{i\phi(k_0)t + i(k-k_0)^2/2\phi''(k_0)t}dk = \frac{a(k_0)e^{ix^2/(4t)}}{\pi}\int_{x/(2t)}^{x/(2t)+\delta} e^{-it(k-k_0)^2}dk$$

$$\underset{\xi^2=t(k-k_0)^2}{\overset{t \text{ big}}{=}} \frac{a(k_0)e^{ix^2/(4t)}}{\pi}\int_0^{\infty} e^{-i\xi^2}\frac{d\xi}{\sqrt{t}} = \frac{a(k_0)e^{ix^2/(4t)-i\pi/4}}{2\sqrt{\pi t}}$$

(c) Using $\psi_0(x) = e^{-x^2}$ we get that $a(k) = \int_{-\infty}^{\infty} e^{-xk-x^2}dx = \sqrt{\pi}e^{-k^2/4}$ (using Wolfram Mathematica). Now we want to calculate $\psi(x,t)$ where $x/t = 1$ and $x/t = 2$. Thus,

$$\psi(x,t) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-k^2/4}e^{ik(x/t-k)t}dk$$

Filling in x/t and using Wolfram Mathematica, we get for $x/t = 1$ and $x/t = 2$, respectively,

$$\frac{e^{\frac{t^2}{-1-4it}}}{\sqrt{1+4it}} \qquad \text{and} \qquad \frac{e^{\frac{4t^2}{-1-4it}}}{\sqrt{1+4it}}$$

Plotting the real parts of these functions, respectively, gives:

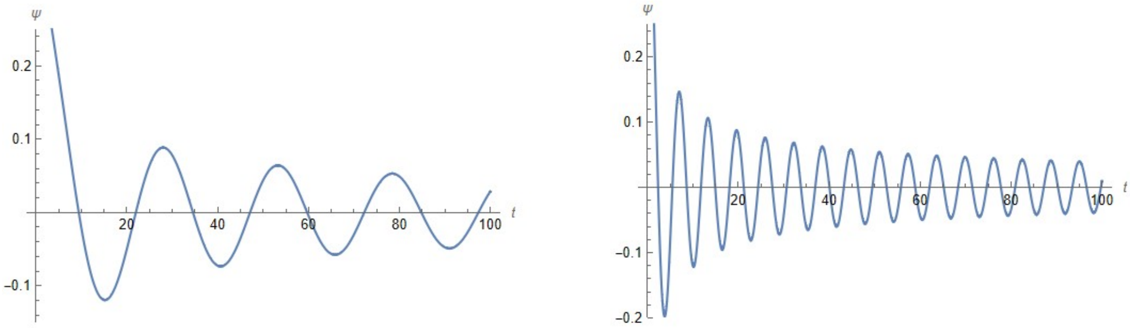

Figure 1: Exact solution to the free linear Schrödinger equation with $x/t = 1$ and 2.

Using the $a(k)$ in (b) we get that the dominant behavior would be

$$\frac{e^{-1/16}e^{i/4(t-\pi)}}{2\sqrt{t}} \qquad \text{and} \qquad \frac{e^{-1/4}e^{it-i\pi/4}}{2\sqrt{t}}$$

Since we're only interested in the real part, we get

$$\frac{e^{-1/16}\cos((t-\pi)/4)}{2\sqrt{t}} \qquad \text{and} \qquad \frac{e^{-1/4}\cos(t-\pi/4)}{2\sqrt{t}}$$

Plotting these functions, respectively, gives:

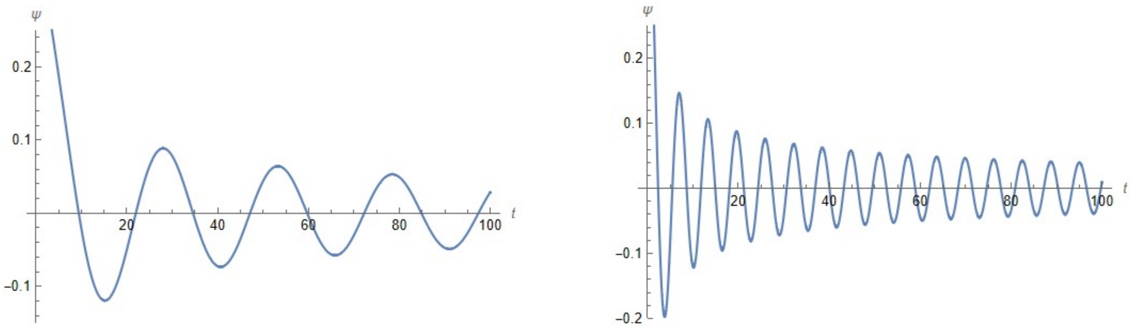

Figure 2: Dominant behavior of the free linear Schrödinger equation with $x/t = 1$ and 2.

Which looks near identical to the solution.

(d) Since my life motto is "if it ain't broke (yet), don't fix it (yet)" I chose for the standard integrate function of MATLAB. Plotting the approximation (using the analytic form of $a(k)$), solution and dominant behavior over each other gives us, for $x/t = 1$,

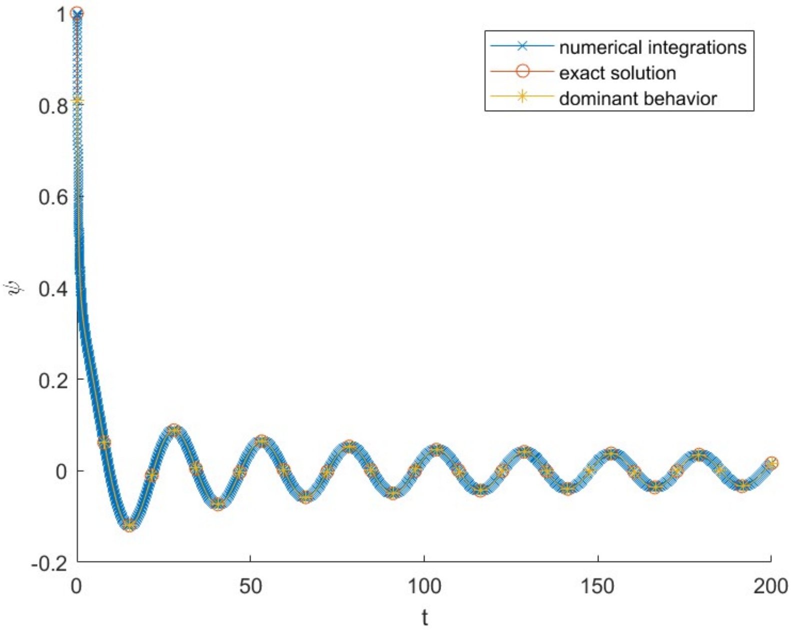

Figure 3: The approximation, solution and dominant behavior of the linear free Schrödinger equation with $x/t = 1$.

and for $x/t = 2$,

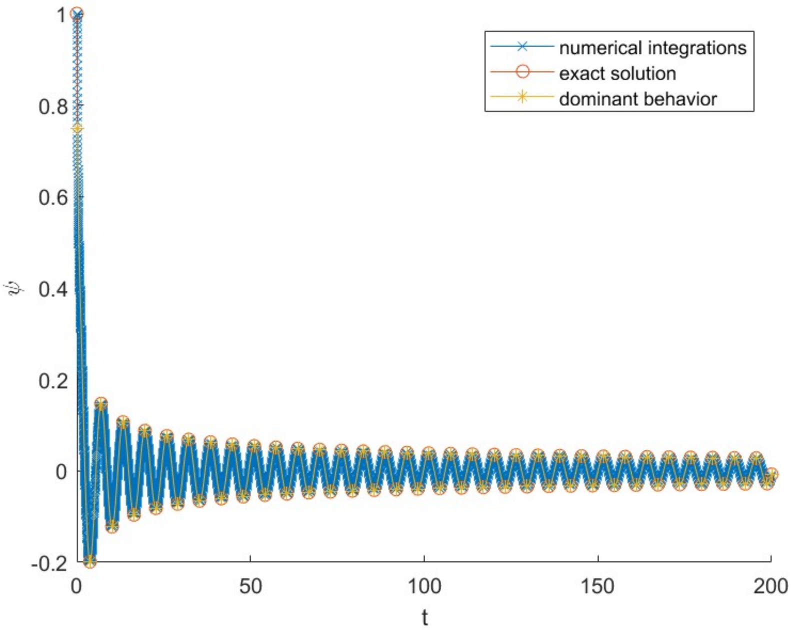

Figure 4: The approximation, solution and dominant behavior of the linear free Schrödinger equation with $x/t = 2$.

The numerical integrations took a long time to compute, since multiple integrals were needed to for all timestamps. There had to be a fine time grid in order to avoid aliasing. Especially when x/t becomes big, since the oscillations become more rapid. The aliasing becomes clear if we make the time grid more coarse:

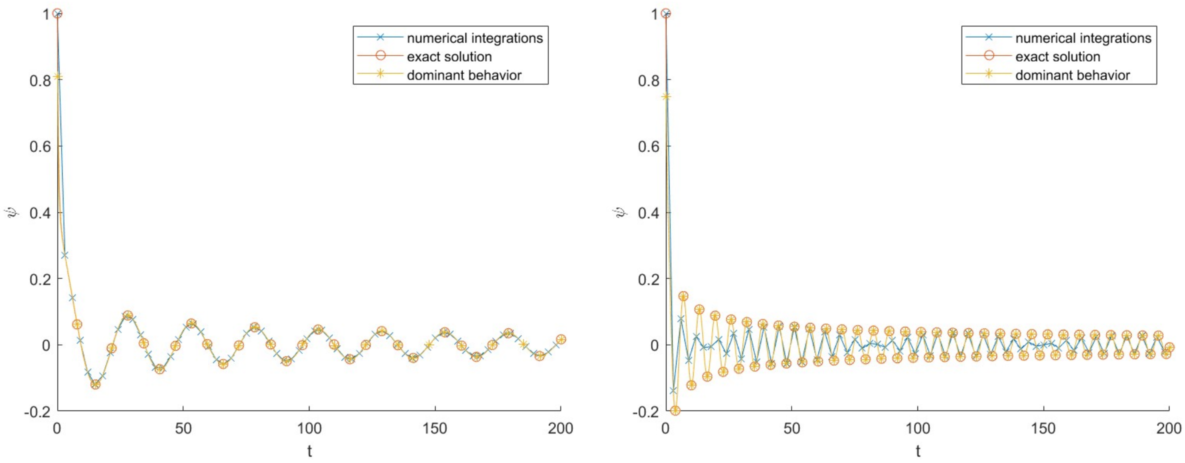

Figure 5: A coarser grid revealing the possibility of aliasing.

In conclusion: the most time efficient and possible method to look at this is the Method of Stationary Phase.

6. Everything that we have done for continuous space equations also works for equations with a discrete space variable. Consider the discrete linear Schrödinger equation:

$$i\frac{d\psi_n}{dt} + \frac{1}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) = 0,$$

where h is a real constant, n is any integer, $t > 0$, $\psi_n \to 0$ as $|n| \to \infty$, and $\psi_n(0) = \psi_{n,0}$ is given.

(a) The discrete analogue of the Fourier transform is given by

$$\psi_n(t) = \frac{1}{2\pi i}\oint_{|z|=1} \hat{\psi}(z,t) z^{n-1} dz,$$

and its inverse

$$\hat{\psi}(z,t) = \sum_{m=-\infty}^{\infty} \psi_m(t) z^{-m}.$$

Show that these two transformations are indeed inverses of each other.

(b) The dispersion relation of a semi-discrete problem is obtained by looking for solutions of the form $\psi_n = z^n e^{-i\omega t}$. Show that for the semi-discrete Schrödinger equation

$$\omega(z) = -\frac{(z-1)^2}{zh^2}.$$

How does this compare to the dispersion relation of the continuous space problem? Specifically, demonstrate that you recover the dispersion relationship for the continuous problem as $h \to 0$.

Solution:

(a)

$$\frac{1}{2\pi i}\oint_{|z|=1}\left(\sum_{m=-\infty}^{\infty} \psi_m(t) z^{-m}\right) z^{n-1} dz = \frac{1}{2\pi i}\oint_{|z|=1} \sum_{m=-\infty}^{\infty} \psi_m(t) z^{n-m-1} dz$$

Assuming that $\hat{\psi}(z,t)$ exists, we know that the series converges and thus the integral sign can be switched. Note that the unit circle can be characterised by $e^i\theta$ for $0 \leq \theta < 2\pi$. Thus,

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{|z|=1} \sum_{m=-\infty}^{\infty} \psi_m(t) z^{n-m-1} dz &= \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty}\int_0^{2\pi} \psi_m(t) e^{(n-m-1)i\theta} i e^{i\theta} d\theta \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \psi_m(t)\int_0^{2\pi} e^{(n-m)i\theta} d\theta
\end{aligned}$$

if $n \neq m$, then $\int_0^{2\pi} e^{(n-m)\theta i} d\theta = 0$
if $n = m$, then $\int_0^{2\pi} d\theta = 2\pi$. Thus,

$$\frac{1}{2\pi i} \oint_{|z|=1} \left(\sum_{m=-\infty}^{\infty} \psi_m(t) z^{-m} \right) z^{n-1} dz = \psi_n(t)$$

The other way around is:

$$\sum_{m=-\infty}^{\infty} \frac{1}{2\pi i} \oint_{|z|=1} \hat{\psi}(z,t) z^{m-1} dz z^{-m}$$

Denoting a_m as $\frac{1}{2\pi i} \oint_{|z|=1} \frac{\hat{\psi}(z,t)}{z^{-m+1}} dz$, we then have

$$\sum_{m=-\infty}^{\infty} \frac{1}{2\pi i} \oint_{|z|=1} \hat{\psi}(z,t) z^{m-1} dz z^{-m} = \sum_{m=-\infty}^{\infty} a_{-m} z^{-m} = \sum_{m=-\infty}^{\infty} a_m z^m$$

This is the definition of the Laurent series of $\hat{\psi}(z,t)$, thus:

$$\sum_{m=-\infty}^{\infty} \frac{1}{2\pi i} \oint_{|z|=1} \hat{\psi}(z,t) z^{m-1} dz z^{-m} = \hat{\psi}(z,t)$$

(b) Filling in $\psi_n = z^n e^{-i\omega t}$ (and dividing by $e^{-i\omega t}$) gives

$$\omega z^n + \frac{1}{h^2} \left(z^{n+1} - 2z^n + z^{n-1} \right) = 0.$$

Thus, assuming $z \neq 0$ (if $z = 0$, we have the trivial solution),

$$\omega(z) = -\frac{1}{h^2} \left(z - 2 + z^{-1} \right) = -\frac{1}{h^2 z} \left(z^2 - 2z + 1 \right) = -\frac{(z-1)^2}{zh^2}$$

In order to see that, as $h \to 0$, $w(z) \to z^2$ we look back at the semi-discrete problem:

$$i\frac{d\psi_n}{dt} + \frac{1}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) = 0$$

It is important to note that $\psi_{n\pm 1}(t) \approx \psi(z_n \pm h, t)$, with z_n a gridpoint, and it will equal as $h \to 0$. We can thus substitute this and use Taylor polynomials:

$$\begin{aligned}
&i\frac{d\psi(z_n,t)}{dt} + \frac{1}{h^2}(\psi(z_n + h, t) - 2\psi(z_n,t) + \psi(z_n - h, t)) \\
=&i\frac{d\psi(z_n,t)}{dt} + \frac{1}{h^2}[\psi(z_n,t) + \frac{d\psi}{dz}(z_n,t)h + \frac{h^2}{2}\frac{d^2\psi}{dz^2}(z_n,t) \\
&- 2\psi(z_n,t) \\
&+ \psi(z_n,t) - \frac{d\psi}{dz}(z_n,t)h + \frac{h^2}{2}\frac{d^2\psi}{dz^2}(z_n,t))] + \mathcal{O}(h) \\
=&i\frac{d\psi}{dt}(z_n,t) + \frac{d^2\psi}{dz^2}(z_n,t) + \mathcal{O}(h)
\end{aligned}$$

Taking $h \to 0$ will make the discrete points form a continues line, thus filling in $\psi = e^{ikz - iwt}$ gives:

$$i\frac{d\psi_n}{dt} + \frac{1}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) \overset{h \to 0}{=} \omega e^{ikz - iwt} - k^2 e^{ikz - iwt} = 0 \Rightarrow \omega = k^2$$