AMATH 567
Applied Complex Variables
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Homework 7

Due: **Wednesday, December 7, 2022**

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Problem 1.

(a) Construct the bilinear transformation

$$w(z) = \frac{az+b}{cz+d}$$

that maps the region between the two circles $|z-\frac{1}{4}|=\frac{1}{4}$ and $|z-\frac{1}{2}|=\frac{1}{2}$ into an infinite strip bounded by the vertical lines $u = \text{Re}\{w\} = 0$ and $u = \text{Re}\{w\} = 1$. To avoid ambiguity suppose that the outer circle is mapped to $u = 1$.

(b) Upon finding the appropriate transformation w; carefully show that the image of the inner circle under w is the vertical line $u = 0$, and similarly for the outer circle.

Solution:

(a) Note first that

$$w(z) = \frac{a}{c}\frac{z+\frac{b}{a}}{z+\frac{d}{c}} = A\frac{z+z_0}{z+z_1}.$$

Here we will find A, z_0 and z_1. We want to go from circles to lines, thus we want a point on the circle to be a singularity. Since we can only use one bilinear transformation and want to go from two circles to two lines, we want the singularity to be a point that is on both circles. There is only one point which is on both circles: $z = 0$. Therefore we immediately have that

$z_1 = 0$. Since $z \in \mathbb{C}$, we can say that $z = x + yi$, with $x, y \in \mathbb{R}$. Thus,

$$\begin{aligned} w(x+yi) &= \frac{Ax + Ayi + Az_0}{x+yi} = \frac{(A(x+yi) + Az_0)(x-yi)}{x^2+y^2} \\ &= \frac{A(x^2+y^2) + Az_0(x+yi)}{x^2+y^2} \\ &= A + \frac{Az_0}{x^2+y^2}(x-yi). \end{aligned}$$

Since $A, z_0 \in \mathbb{C}$ we can say that $A = A_{re} + A_{im}i$ and $z_0 = z_{0re} + z_{0im}i$, with $A_{re}, A_{im}, z_{0re}, z_{0im} \in \mathbb{R}$. We then have that

$$\mathrm{Re}(w(z)) = A_{re} + \frac{(A_{re}z_{0re} - A_{im}z_{0im})x + (A_{re}z_{0im} + A_{im}z_{0re})y}{x^2+y^2}.$$

Note now that $z = \frac{1}{2}$ is on $|z - \frac{1}{4}| = \frac{1}{4}$ and $z = 1$ and $z = \frac{1}{2} + \frac{1}{2}i$ are on $|z - \frac{1}{2}| = \frac{1}{2}$. Thus, $\mathrm{Re}(w(\frac{1}{2})) = 0$ and $\mathrm{Re}(w(1)) = \mathrm{Re}(w(\frac{1}{2} + \frac{1}{2}i)) = 1$. In other words

$$\begin{cases} A_{re} + 2(A_{re}z_{0re} - A_{im}z_{0im}) = 0 \\ A_{re} + A_{re}z_{0re} - A_{im}z_{0im} = 1 \\ A_{re} + (A_{re}z_{0re} - A_{im}z_{0im}) + (A_{re}z_{0im} + A_{im}z_{0re}) = 1 \end{cases}$$

Multiplying the second equation by 2 and then calculating the difference between the first and second equation gives us A_{re}, filling this in gives,

$$\begin{cases} A_{re} = 2 \\ 2z_{0re} - A_{im}z_{0im} = -1 \\ (2z_{0re} - A_{im}z_{0im}) + (2z_{0im} + A_{im}z_{0re}) = -1. \end{cases}$$

Then replacing the third equations by the difference between the third and second equation gives,

$$\begin{cases} A_{re} = 2 \\ 2z_{0re} - A_{im}z_{0im} = -1 \\ 2z_{0im} + A_{im}z_{0re} = 0. \end{cases}$$

We thus have

$$\begin{cases} 2z_{0re} - A_{im}z_{0im} = -1 \\ 2z_{0im} + A_{im}z_{0re} = 0 \end{cases} \Rightarrow \begin{cases} 2z_{0re} - A_{im}z_{0im} = -1 \\ 2z_{0im} + A_{im}z_{0re} = 0 \end{cases}$$
$$\Rightarrow \begin{cases} 2z_{0re} + \frac{A_{im}^2 z_{0re}}{2} = -1 \\ z_{0im} = -\frac{A_{im}z_{0re}}{2} \end{cases} \Rightarrow \begin{cases} z_{0re} = -\frac{2}{4+A_{im}^2} \\ z_{0im} = \frac{A_{im}}{4+A_{im}^2} \end{cases}$$

It's logical that we can choose at least one parameter since mobiüs transformations are not unique. It isn't difficult to see that a circle can be transformed into a line in multiple ways using the same kind of function, it can be transformed at different rates. In conclusion, the general form for all bilinear transformations who do the thing asked is, with $A_{im} \in \mathbb{R}$,

$$w(z) = (2 + A_{im}i)\frac{z - \left(\frac{2 - A_{im}}{4 + A_{im}^2}\right)}{z} = \frac{(2 + A_{im}i)z - 1}{z}.$$

(b) Writing $a \in \mathbb{R}$ instead of A_{im}, first we find z in function of w:

$$w = \frac{(2+ai)z - 1}{z} \Rightarrow wz = (2+ai)z - 1 \Rightarrow z = \frac{1}{(2+ai) - w}.$$

We now fill in z into $|z - b|= b$, with $b \in \mathbb{R}_0$,

$$b = |z-b|= \left|\frac{1 - b((2+ai) - w)}{(2+ai) - w}\right| \Rightarrow 1 = \left|\frac{\frac{1}{b} - (2+ai) + w}{(2+ai) - w}\right| \Rightarrow |(2+ai)-w|= |\frac{1}{b}-(2+ai)+w|.$$

Since $|z|^2= z\overline{z}$ with $\overline{z}$ the complex conjugate of z, we have that

$$\begin{aligned}
&((2-ai) - \overline{w})((2+ai) - w) = (\frac{1}{b} - (2-ai) + \overline{w})(\frac{1}{b} - (2+ai) + w)\\
\Rightarrow &((2-ai) - \overline{w})((2+ai) - w) = \left(\frac{1}{b}\right)^2 + ((2-ai) - \overline{w})((2+ai) - w) - \frac{1}{b}((2-ai) - \overline{w}) - \frac{1}{b}((2+ai) - w)\\
\Rightarrow &\frac{1}{b} - (2-ai) + \overline{w} - (2+ai) + w = 0\\
\Rightarrow &\frac{1}{b} + (w + \overline{w}) - 4 = 0\\
\Rightarrow &(w + \overline{w}) = 4 - \frac{1}{b}
\end{aligned}$$

It is clear to see that, if $w = u + vi$, with $u, v \in \mathbb{R}$; $w + \overline{w} = 2u = 2\mathrm{Re}(w)$. Thus,

$$\mathrm{Re}(w) = 2 - \frac{1}{2b}.$$

if $b = \frac{1}{4}$, z is on the inner circle, we have

$$u = \mathrm{Re}(w) = 0,$$

The image of the inner circle under w is the vertical line $u = 0$. If $b = \frac{1}{2}$, z is on the outer circle, we have

$$u = \mathrm{Re}(w) = 1.$$

The image of the outer circle under w is the vertical line $u = 1$.

Problem 2.

Use the result of **Problem 1** to find the steady state temperature $T(x, y)$ in the region bounded by the two circles, where the inner circle is maintained at $T = 0°C$ and the outer circle at $T = 100°C$. Assume T satisfies the two-dimensional Laplace equation.

Solution:

With C_1 being the inner circle and C_2 the outer circle, we have the following partial differential equation

$$\begin{cases} \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \\ T(x,y) = 0, & (x,y) \in C_1 \\ T(x,y) = 100, & (x,y) \in C_2 \end{cases}$$

This is clearly discontinuous in $(0, 0)$, we will thus ignore this point. Note that if

$$w(z) = \frac{(2+ai)z - 1}{z},$$

with $z = x + yi$, the bilinear transformation stated in the previous exercise, we have that

$$w'(z) = \frac{(2+ai)z}{z^2} - \frac{(2+ai)z - 1}{z^2} = \frac{1}{z^2}.$$

Since $w'(z) \neq 0, \forall z \in \mathbb{C}_0$, we have that $w(z)$ is a conformal map. If now $w(z) = u + vi$, we have, due to the invariance of the Laplace equation under conformal mapping (lecture 22), that the PDE becomes

$$\begin{cases} \frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0 \\ T(u,v) = 0, & u = 0 \\ T(u,v) = 100, & u = 1. \end{cases}$$

Since the boundary conditions are only stated for values for u, we have that T is constant over v, there we have that

$$\begin{cases} \frac{\partial^2 T}{\partial u^2} = 0 \\ T(u) = 0, & u = 0 \\ T(u) = 100, & u = 1 \end{cases}$$

The solution to this is

$$T(u) = au + b.$$

Using the boundary conditions, we get

$$\begin{cases} 0 = T(0) = b \\ 100 = T(1) = a \end{cases} \Rightarrow \begin{cases} b = 0 \\ a = 100 \end{cases}$$

Thus,

$$T(u,v) = 100u$$

. We now want to find $\text{Re}(w)$:

$$u = \text{Re}(w(z)) = \text{Re}\left(\frac{(2+ai)z - 1}{z}\right) = \text{Re}\left(\frac{(2+ai)|z|^2 - \overline{z}}{|z|^2}\right) = 2 - \frac{x}{x^2 + y^2}$$

Thus,

$$T(x,y) = 100\left(2 - \frac{x}{x^2+y^2}\right).$$

We now check this solution:

$$\frac{\partial T}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \text{ and } \frac{\partial^2 T}{\partial x^2} = -\frac{2x\cdot(x^2-3y^2)}{(x^2+y^2)^3},$$

and

$$\frac{\partial T}{\partial y} = \frac{2xy}{(y^2+x^2)^2} \text{ and } \frac{\partial^2 T}{\partial y^2} = \frac{2x\cdot(x^2-3y^2)}{(x^2+y^2)^3}.$$

We thus clearly see that

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

Note that, with $b \in \mathbb{R}_0$. we have that if $|z-b|= b$:

$$|z-b|= |x-b+yi|= \sqrt{(x-b)^2+y^2} = b \Rightarrow x^2-2bx+b^2+y^2 = b^2 \Rightarrow x^2+y^2 = 2bx.$$

Thus on C_1 ($b = \frac{1}{4}$), without $(x,y) = (0,0)$, we have that

$$T(x,y) = 100\left(2 - \frac{4}{2}\right) = 0,$$

and on C_1 ($b = \frac{1}{2}$), we have that

$$T(x,y) = 100\left(2 - \frac{2}{2}\right) = 100.$$

The boundary conditions are thus fulfilled. In conclusion the solution to this problem is

$$T(x,y) = 100\left(2 - \frac{x}{x^2+y^2}\right).$$