AMATH 573
Solitons and nonlinear waves
Bernard Deconinck
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Homework 3

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1. **The KdV equation for ion-acoustic waves in plasmas.** Ion-acoustic waves are low-frequency electrostatic waves in a plasma consisting of electrons and ions. We consider the case with a single ion species.

   Consider the following system of one-dimensional equations

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial}{\partial z}(nv) &= 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial z} &= -\dfrac{e}{m}\dfrac{\partial \phi}{\partial z} \\ \dfrac{\partial^2 \phi}{\partial z^2} &= \dfrac{e}{\varepsilon_0}\left[N_0 \exp\left(\dfrac{e\phi}{\kappa T_e}\right) - n\right] \end{cases}$$

   Here n denotes the ion density, v is the ion velocity, e is the electron charge, m is the mass of an ion, ϕ is the electrostatic potential, ε_0 is the vacuum permittivity, N_0 is the equilibrium density of the ions, κ is Boltzmann's constant, and T_e is the electron temperature.

   (a) Verify that $c_s = \sqrt{\dfrac{\kappa T_e}{m}}$, $\lambda_{De} = \sqrt{\dfrac{\varepsilon_0 \kappa T_e}{N_0 e^2}}$, and $\omega_{pi} = \sqrt{\dfrac{N_0 e^2}{\varepsilon_0 m}}$ have dimensions of velocity, length and frequency, respectively. These quantities are known as the ion acoustic speed, the Debye wavelength for the electrons, and the ion plasma frequency.

   (b) Nondimensionalize the above system, using

$$n = N_0 n^*, \quad v = c_s v^*, \quad z = \lambda_{De} z^*, \quad t = \frac{t^*}{\omega_{pi}}, \quad \phi = \frac{\kappa T_e}{e}\phi^*.$$

(c) You have obtained the system

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial}{\partial z}(nv) & = 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial z} & = -\dfrac{\partial \phi}{\partial z} \\ \dfrac{\partial^2 \phi}{\partial z^2} & = \exp(\phi) - n \end{cases}$$

for the dimensionless variables. Note that we have dropped the $*$'s, to ease the notation. Find the linear dispersion relation for this system, linearized around the trivial solution $n = 1$, $v = 0$, and $\phi = 0$.

(d) Rewrite the system using the "stretched variables"

$$\xi = \epsilon^{1/2}(z - t), \quad \tau = \epsilon^{3/2}t.$$

Given that we are looking for low-frequency waves, explain how these variables are inspired by the dispersion relation.

(e) Expand the dependent variables as

$$n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \ldots, v = \epsilon v_1 + \epsilon^2 v_2 + \ldots, \phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots.$$

Using that all disturbances return to their equilibrium values as $\xi \to \pm\infty$, $\tau \to \infty$, find a governing equation which determines how ϕ_1 depends on ξ and τ.

Solution:

(a) The parameters have the following units:

Parameter	κ	T_c	m	ϵ_0	N_0	e
dimension	$\frac{N\ m}{K}$	K	kg	$\frac{C^2}{N\ m^2}$	$\frac{1}{m^3}$	C

With $N = \frac{kg\ m}{s^2}$ (Newton), we then have for the dimensions:

$$c_s = \sqrt{\frac{N\ m\ K}{K\ kg}} = \sqrt{\frac{kg\ m^2 K}{s^2 K\ kg}} = \frac{m}{s},$$

$$\lambda_{De} = \sqrt{\frac{C^2\ N\ m\ K}{N\ m^2 K \frac{1}{m^3} C^2}} = m,$$

$$\omega_{pi} = \frac{c_s}{\lambda_{De}} = \frac{1}{s}.$$

(b) The equation becomes

$$\begin{cases} \dfrac{\partial N_0 n^*}{\partial t^*}\dfrac{\partial t^*}{\partial t} + \dfrac{\partial}{\partial z^*}(N_0 n^* c_s v^*)\dfrac{\partial z^*}{\partial z} & = 0 \\ \dfrac{\partial c_s v^*}{\partial t^*}\dfrac{\partial t^*}{\partial t} + c_s v^* \dfrac{\partial c_s v^*}{\partial z^*}\dfrac{\partial z^*}{\partial z} & = -\dfrac{e}{m}\dfrac{\kappa T_c}{e}\dfrac{\partial \phi^*}{\partial z^*}\dfrac{\partial z^*}{\partial z} \\ \dfrac{\kappa T_c}{e}\dfrac{\partial^2 \phi^*}{\partial z^{*2}}\dfrac{\partial^2 z^*}{\partial z^2} & = \dfrac{e}{\varepsilon_0}\left[N_0 \exp(\phi^*) - N_0 n^*\right] \end{cases}$$

Here we have that $\frac{\partial t^*}{\partial t} = \omega_{pi}$ and $\frac{\partial z^*}{\partial z} = \dfrac{1}{\lambda_{De}}$. While dropping the star, we get

$$\begin{cases} \omega_{pi} N_0 \dfrac{\partial n}{\partial t} + \dfrac{N_0 c_s}{\lambda_{De}} \dfrac{\partial}{\partial z}(nv) & = 0 \\ \omega_{pi} c_s \dfrac{\partial v}{\partial t} + \dfrac{c_s^2}{\lambda_{De}} v \dfrac{\partial v}{\partial z} & = -\dfrac{\kappa T_c}{m \lambda_{De}} \dfrac{\partial \phi}{\partial z} \\ \dfrac{\kappa T_c}{e \lambda_{De}^2} \dfrac{\partial^2 \phi}{\partial z^2} & = \dfrac{e N_0}{\varepsilon_0} \left[\exp(\phi) - n\right] \end{cases}$$

Using the equations for c_s, λ_{De} and ω_{pi}, and noting that $\omega_{pi} = \frac{c_s}{\lambda_{De}}$, we have,

$$\begin{cases} \omega_{pi} N_0 \dfrac{\partial n}{\partial t} \omega_{pi} N_0 \dfrac{\partial}{\partial z}(nv) & = 0 \\ \omega_{pi} c_s \dfrac{\partial v}{\partial t} + \omega_{pi} c_s v \dfrac{\partial v}{\partial z} & = -\dfrac{\kappa T_c \sqrt{N_0 e^2}}{m \sqrt{\varepsilon_0 \kappa T_e}} \dfrac{\partial \phi}{\partial z} = -\omega_{pi} c_s \dfrac{\partial \phi}{\partial z} \\ \dfrac{e N_0}{\varepsilon_0} \dfrac{\partial^2 \phi}{\partial z^2} = \dfrac{\kappa T_c N_0 e^2}{e \varepsilon_0 \kappa T_e} \dfrac{\partial^2 \phi}{\partial z^2} & = \dfrac{e N_0}{\varepsilon_0} \left[\exp(\phi) - n\right] \end{cases}$$

In conclusion, we get:

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial}{\partial z}(nv) & = 0 \\ \dfrac{\partial v}{\partial t} + v \dfrac{\partial v}{\partial z} & = -\dfrac{\partial \phi}{\partial z} \\ \dfrac{\partial^2 \phi}{\partial z^2} & = \exp(\phi) - n \end{cases}$$

(c) Instead of the used notation in the exercises, we opt to use $\tilde{}$ to denote these solutions. Linearizing $\tilde{n}$ around 1 and the rest around 0 gives $\tilde{n} = 1 + n\epsilon + \mathcal{O}(\epsilon^2)$, $\tilde{v} = v\epsilon + \mathcal{O}(\epsilon^2)$ and $\tilde{\phi} = \phi\epsilon + \mathcal{O}(\epsilon^2)$. Filling these in gives

$$\begin{cases} \dfrac{\partial}{\partial t}(1 + \epsilon n + \mathcal{O}(\epsilon^2)) + \epsilon \dfrac{\partial v}{\partial z} + \mathcal{O}(\epsilon^2) & = 0 \\ \epsilon \dfrac{\partial v}{\partial t} + \mathcal{O}(\epsilon^2) & = -\epsilon \dfrac{\partial \phi}{\partial z} + \mathcal{O}(\epsilon^2) \\ \epsilon \dfrac{\partial^2 \phi}{\partial z^2} + \mathcal{O}(\epsilon^2) & = \exp(\epsilon\phi + \mathcal{O}(\epsilon^2)) - 1 - \epsilon n + \mathcal{O}(\epsilon^2) \end{cases}$$

Further simplifying this (and using Taylor polynomials), gives

$$\begin{cases} \epsilon \dfrac{\partial n}{\partial t} + \epsilon \dfrac{\partial v}{\partial z} + \mathcal{O}(\epsilon^2) & = 0 \\ \epsilon \dfrac{\partial v}{\partial t} + \mathcal{O}(\epsilon^2) & = -\epsilon \dfrac{\partial \phi}{\partial z} + \mathcal{O}(\epsilon^2) \\ \epsilon \dfrac{\partial^2 \phi}{\partial z^2} + \mathcal{O}(\epsilon^2) & = 1 + \epsilon\phi - 1 - \epsilon n + \mathcal{O}(\epsilon^2) \end{cases}$$

Deriving by ϵ and filling in 0 finally gives,

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial v}{\partial z} &= 0 \\ \dfrac{\partial v}{\partial t} &= -\dfrac{\partial \phi}{\partial z} \\ \dfrac{\partial^2 \phi}{\partial z^2} &= \phi - n \end{cases}$$

Now filling in $n = a_n e^{k_n iz - \omega_n it}$, $v = a_v e^{k_v iz - \omega_v it}$ and $\phi = a_\phi e^{k_\phi iz - \omega_\phi it}$, we then have

$$\begin{cases} -\omega_n a_n i e^{k_n iz - \omega_n it} + k_v a_v i e^{k_v iz - \omega_v it} &= 0 \\ -\omega_v a_v i e^{k_v iz - \omega_v it} &= -k_\phi i a_\phi e^{k_\phi iz - \omega_\phi it} \\ -k_\phi^2 a_\phi e^{k_\phi iz - \omega_\phi it} &= a_\phi e^{k_\phi iz - \omega_\phi it} - a_n e^{k_n iz - \omega_n it} \end{cases}$$

From this we get

$$\begin{cases} a_n \omega_n e^{k_n iz - \omega_n it} &= a_v k_v e^{k_v iz - \omega_v it} \\ a_v \omega_v e^{k_v iz - \omega_v it} &= a_\phi k_\phi e^{k_\phi iz - \omega_\phi it} \\ a_\phi k_\phi^2 e^{k_\phi iz - \omega_\phi it} &= a_n e^{k_n iz - \omega_n it} - a_\phi e^{k_\phi iz - \omega_\phi it} \end{cases}$$

From the first two equations we have that

$$k_n = k_v = k_\phi = k \text{ and } \omega_n = \omega_v = \omega_\phi = \omega.$$

Thus, dividing by $e^{kiz - \omega it}$ gives,

$$\begin{cases} a_n \omega = a_v k \\ a_v \omega = a_\phi k \\ a_\phi k^2 = a_n - a_\phi \end{cases} \Rightarrow \begin{cases} \frac{a_n}{a_\phi}\frac{\omega}{k} = \frac{a_v}{a_v}\frac{k}{\omega} \\ k^2 = \frac{a_n}{a_\phi} - 1 \end{cases} \Rightarrow k^2 = \frac{k^2}{\omega^2} - 1 \Rightarrow \omega^2 = \frac{k^2}{1+k^2}.$$

The linear dispersion relation is thus

$$\omega^2 = \frac{k^2}{1+k^2}.$$

(d) For small frequencies (ω), we have that $\omega^2 \approx k^2$, because of the "1+" in the denominator. therefore the phase velocity is given by $v_p = \omega/k = \pm 1$. We will chose the positive sign, since the negative one is just a wave in the opposite direction. Thus, if we want to look at a low-frequency waves, we would like to look at a variable in the form of $z - t$. Furthermore, the powers of ϵ are chosen in such a way that we magically find the KdV equation in the next exercise. It makes us look at a shorter spatial scale and a slower time.

We now want to rewrite the system with those variables. Note that $\frac{\partial}{\partial t} = -\epsilon^{1/2}\frac{\partial}{\partial \xi} + \epsilon^{3/2}\frac{\partial}{\partial \tau}$, $\frac{\partial}{\partial z} = \epsilon^{1/2}\frac{\partial}{\partial \xi}$ and $\frac{\partial^2}{\partial z^2} = \epsilon\frac{\partial^2}{\partial \xi^2}$. If we rewrite the system, we get

$$\begin{cases} -\epsilon^{1/2}\dfrac{\partial n}{\partial \xi} + \epsilon^{3/2}\dfrac{\partial n}{\partial \tau} + \epsilon^{1/2}\dfrac{\partial (nv)}{\partial \xi} & = 0 \\ -\epsilon^{1/2}\dfrac{\partial v}{\partial \xi} + \epsilon^{3/2}\dfrac{\partial v}{\partial \tau} + \epsilon^{1/2} v\dfrac{\partial v}{\partial \xi} & = -\epsilon^{1/2}\dfrac{\partial \phi}{\partial \xi} \\ \epsilon\dfrac{\partial^2 \phi}{\partial \xi^2} & = \exp(\phi) - n \end{cases}$$

This turns into

$$\begin{cases} \epsilon\dfrac{\partial n}{\partial \tau} + \dfrac{\partial n(v-1)}{\partial \xi} & = 0 \\ \dfrac{1}{2}\dfrac{\partial (v-1)^2}{\partial \xi} + \epsilon\dfrac{\partial v}{\partial \tau} & = -\dfrac{\partial \phi}{\partial \xi} \\ \epsilon\dfrac{\partial^2 \phi}{\partial \xi^2} & = \exp(\phi) - n \end{cases}$$

(e) Expanding the dependent variables gives,

$$\begin{cases} \epsilon\frac{\partial 1+\epsilon n_1+\epsilon^2 n_2+\dots}{\partial \tau} + \frac{\partial (1+\epsilon n_1+\epsilon^2 n_2+\dots)(\epsilon v_1+\epsilon^2 v_2+\dots-1)}{\partial \xi} & = 0 \\ \frac{1}{2}\frac{\partial (\epsilon v_1+\epsilon^2 v_2+\dots-1)^2}{\partial \xi} + \epsilon\frac{\partial \epsilon v_1+\epsilon^2 v_2+\dots}{\partial \tau} & = -\frac{\partial \epsilon\phi_1+\epsilon^2\phi_2+\dots}{\partial \xi} \\ \epsilon\frac{\partial^2 \epsilon\phi_1+\epsilon^2\phi_2+\dots}{\partial \xi^2} & = e^{\epsilon\phi_1+\epsilon^2\phi_2+\dots} - (1+\epsilon n_1+\epsilon^2 n_2+\dots) \end{cases}$$

Simplifying, using Taylor and using a different notation for the partial derivatives gives,

$$\begin{cases} \epsilon^2 n_{1\tau} + \epsilon^3 n_{2\tau} + \epsilon(v_{1\xi} - n_{1\xi}) + \epsilon^2(v_{2\xi} - n_{2\xi} + (n_1 v_1)_\xi) + \dots & = 0 \\ -\epsilon v_{1\xi} + \epsilon^2(-v_{2\xi} + v_1 v_{1\xi} + v_{1\tau}) + \dots & = -\epsilon\phi_{1\xi} - \epsilon^2\phi_{2\xi} + \dots \\ \epsilon^2\phi_{1\xi\xi} + \epsilon^3\phi_{2\xi\xi} + \dots & = 1 + \epsilon\phi_1 + \epsilon^2\left(\frac{\phi_1^2}{2} + \phi_2\right) + \dots \\ & \quad -1 - \epsilon n_1 - \epsilon^2 n_2 + \dots \end{cases}$$

The coefficients of ϵ thus give

$$\begin{cases} v_{1\xi} & = n_{1\xi} \\ -v_{1\xi} & = -\phi_{1\xi} \\ \phi_1 & = n_1 \end{cases}$$

From the first of these equations we get that $v_1(\xi,\tau) + g(\tau) = n_1(\xi,\tau)$ (integrating over ξ). Since, for τ randomly fixed, $\lim_{\xi\to\pm\infty} v_1 = \lim_{\xi\to\pm\infty} n_1 = 0$, we have that $g(\tau) = 0,\ \forall\tau > 0$. Thus,

$$v_1 = n_1 = \phi_1.$$

The coefficients of ϵ^2 give

$$\begin{cases} n_{1\tau} + v_{2\xi} - n_{2\xi} + (n_1 v_1)_\xi & = 0 \\ -v_{2\xi} + \frac{1}{2}(v_1^2)_\xi + v_{1\tau} & = -\phi_{2\xi} \\ \phi_{1\xi\xi} & = \frac{1}{2}\phi_1^2 + \phi_2 - n_2 \end{cases} \Rightarrow \begin{cases} \phi_{1\tau} + v_{2\xi} - n_{2\xi} + (\phi_1^2)_\xi & = 0 \\ -v_{2\xi} + \frac{1}{2}(\phi_1^2)_\xi + \phi_{1\tau} & = -\phi_{2\xi} \\ \phi_{1\xi\xi} & = \frac{1}{2}\phi_1^2 + \phi_2 - n_2 \end{cases}$$

The second equation gives

$$v_{2\xi} = \frac{1}{2}(\phi_1^2)_\xi + \phi_{1\tau} + \phi_{2\xi}.$$

Taking the derivative of the third equation with respect to ξ gives

$$-n_{2\xi} = \phi_{1\xi\xi\xi} - \frac{1}{2}(\phi_1^2)_\xi - \phi_{2\xi}.$$

Putting both these equations into the first equation of the system gives,

$$\phi_{1\tau} + \frac{1}{2}(\phi_1^2)_\xi + \phi_{1\tau} + \phi_{2\xi} + \phi_{1\xi\xi\xi} - \frac{1}{2}(\phi_1^2)_\xi - \phi_{2\xi} + (\phi_1^2)_\xi = 0.$$

In conclusion, we get the equation

$$2\phi_{1\tau} + 2\phi_1\phi_{1\xi} + \phi_{1\xi\xi\xi} = 0,$$

which is the KdV equation (the coefficients do not matter) and we know the solution to this, thus we know how ϕ_1 depends on ξ and τ.

2. **Obtaining the KdV equation from the NLS equation.** We have shown that the NLS equation may be used to describe the slow modulation of periodic wave trains of the KdV equation. In this problem we show that the KdV equation describes the dynamics of long-wave solutions of the NLS equation.

   Consider the defocusing NLS equation

$$ia_t = -a_{xx} + |a|^2 a.$$

   (a) Let

$$a(x,t) = e^{i\int V\,dx}\rho^{1/2}.$$

   Derive a system of equations for the phase function $V(x,t)$ and for the amplitude function $\rho(x,t)$, by substituting this form of $a(x,t)$ in the NLS equation, dividing out the exponential, and separating real and imaginary parts. Write your equations in the form $\rho_t = \ldots$, and $V_t = \ldots$. Due to their similarity with the equations of hydrodynamics, this new form of the NLS equation is referred to as its *hydrodynamic form*.

   (b) Find the linear dispersion relation for the hydrodynamic form of the defocusing NLS equation, linearized around the trivial solution $V = 0$, $\rho = 1$. In other words, we are examining perturbations of the so-called Stokes wave solution of the NLS equation, which is given by a signal of constant amplitude.

   (c) Rewrite the system using the "stretched variables"

$$\xi = \epsilon(x - \beta t), \quad \tau = \epsilon^3 t.$$

   Given that we are looking for long waves, explain how these variables are inspired by the dispersion relation. What should the value of β be?

   (d) Expand the dependent variables as

$$V = \epsilon^2 V_1 + \epsilon^4 V_2 + \ldots, \rho = 1 + \epsilon^2\rho_1 + \epsilon^4\rho_2 + \ldots.$$

   Using that all disturbances return to their equilibrium values as $\xi \to \pm\infty$, $\tau \to \infty$, find a governing equation which determines how V_1 depends on ξ and τ. This equation should be equivalent to the KdV equation.

Solution:

(a) Substituting $a(x,t)$ in the NLS equation gives

$$\begin{aligned} & -\frac{\partial\int V dx}{\partial t} e^{i\int V dx}\rho^{1/2} + i\frac{1}{2}e^{i\int V dx}\rho^{-1/2}\rho_t \\ = & -(iVe^{i\int V dx}\rho^{1/2} + e^{i\int V dx}\rho^{-1/2}\rho_x/2)_x + |\rho|e^{i\int V dx}\rho^{1/2}. \end{aligned}$$

Dividing by $e^{i\int V dx}$, this becomes

$$
\begin{aligned}
&-\frac{\partial \int V dx}{\partial t}\rho^{1/2} + i\frac{1}{2}\rho^{-1/2}\rho_t \\
&= -iV_x\rho^{1/2} + V^2\rho^{1/2} - \frac{i}{2}V\rho^{-1/2}\rho_x - iV\rho^{-1/2}\rho_x/2 + \rho^{-3/2}\rho_x^2/4 - \rho^{-1/2}\rho_{xx}/2 + |\rho|\rho^{1/2} \\
&= \rho^{1/2}\left(V^2 + \left(\frac{\rho_x}{2\rho}\right)^2 - \frac{\rho_{xx}}{2\rho} + |\rho|\right) - \left(V_x\rho^{1/2} + V\rho^{-1/2}\rho_x\right)i \\
&= \rho^{1/2}\left(V^2 + \left(\frac{\rho_x}{2\rho}\right)^2 - \frac{\rho_{xx}}{2\rho} + |\rho|\right) - \rho^{-1/2}\left(V\rho\right)_x i.
\end{aligned}
$$

Splitting up the real and imaginary part gives

$$-\int V_t dx = V^2 + \left(\frac{\rho_x}{2\rho}\right)^2 - \frac{\rho_{xx}}{2\rho} + \rho,$$

(here ρ is assumed positive, since $a = e^{\cdots}\rho^{1/2}$) and

$$\rho_t = -2(V\rho)_x.$$

In conclusion

$$
\begin{cases}
V_t &= -\left(V^2 + \left(\frac{\rho_x}{2\rho}\right)^2 - \frac{\rho_{xx}}{2\rho} + \rho\right)_x \\
\rho_t &= -2(V\rho)_x.
\end{cases}
$$

(b) Suppose $V = \epsilon V_1 + \mathcal{O}(\epsilon^2)$ and $\rho = 1 + \epsilon\rho_1 + \mathcal{O}(\epsilon^2)$. Filling this into the hydrodynamic form gives

$$
\begin{cases}
\epsilon V_{1t} + \mathcal{O}(\epsilon^2) = -\left((\epsilon V_1 + \mathcal{O}(\epsilon^2))^2 + \frac{1}{4}\left(\frac{\epsilon\rho_{1x} + \mathcal{O}(\epsilon^2)}{1+\epsilon\rho_1 + \mathcal{O}(\epsilon^2)}\right)^2 - \frac{1}{2}\frac{\epsilon\rho_{1xx} + \mathcal{O}(\epsilon^2)}{1+\epsilon\rho_1+\mathcal{O}(\epsilon^2)} + 1 + \epsilon\rho_1 + \mathcal{O}(\epsilon^2)\right)_x \\
\epsilon\rho_{1t} + \mathcal{O}(\epsilon^2) = -2\left(\epsilon V_1 + \mathcal{O}(\epsilon^2)\right)_x.
\end{cases}
$$

First we Derive the things by ϵ and then let $\epsilon \to 0$. Note in particular the following derivatives:

$$\frac{d}{d\epsilon}\frac{1}{4}\left(\frac{\epsilon\rho_{1x} + \mathcal{O}(\epsilon^2)}{1+\epsilon\rho_1+\mathcal{O}(\epsilon^2)}\right)^2 = \frac{1}{4}\frac{2\epsilon\rho_{1x} + \mathcal{O}(\epsilon)}{(1+\epsilon\rho_1+\mathcal{O}(\epsilon))^3} \overset{\epsilon\to 0}{\to} 0,$$

and

$$\frac{d}{d\epsilon}\frac{1}{2}\frac{\epsilon\rho_{1xx} + \mathcal{O}(\epsilon^2)}{1+\epsilon\rho_1+\mathcal{O}(\epsilon^2)} = \frac{1}{2}\frac{\rho_{1xx} + \mathcal{O}(\epsilon)}{(1+\epsilon\rho_1+\mathcal{O}(\epsilon))^2} \overset{\epsilon\to 0}{\to} \frac{1}{2}\rho_{1xx}.$$

Thus, while losing the 1 in the subscript,

$$
\begin{cases}
V_t = \frac{1}{2}\rho_{xxx} - \rho_x \\
\rho_t = -2V_x.
\end{cases}
$$

Suppose now that $\rho = a_\rho e^{ik_\rho x - i\omega_\rho t}$ and $V = a_V e^{ik_V x - i\omega_V t}$, we get

$$
\begin{cases}
-ia_V\omega_V e^{ik_V x - i\omega_V t} &= -\frac{1}{2}ia_\rho k_\rho^3 e^{ik_\rho x - i\omega_\rho t} - ia_\rho k_\rho e^{ik_\rho x - i\omega_\rho t} \\
-ia_\rho\omega_\rho e^{ik_\rho x - i\omega_\rho t} &= -2a_V k_V i e^{ik_V x - i\omega_V t}.
\end{cases}
$$

From the second equation we know that $k_\rho = k_V (= k)$ and $\omega_\rho = \omega_V (= \omega)$, thus, by dividing by $e^{ikx - i\omega t}$, we get

$$\begin{cases} a_V \omega = \frac{1}{2} a_\rho k^3 + a_\rho k \\ a_\rho \omega = 2 a_V k \end{cases} \Rightarrow \begin{cases} \frac{a_V}{a_\rho}\omega = \frac{1}{2}k^3 + k \\ \frac{a_V}{a_\rho} = \frac{\omega}{2k} \end{cases} \Rightarrow \omega^2 = k^4 + 2k^2.$$

In conclusion, the dispersion relation is

$$\omega^2 = (k^2+2)k^2.$$

(c) Given that we are looking at large waves, $|k|$ will be small, and thus $\omega^2 \approx 2k^2$. Therefore the phase velocity is $\pm\sqrt{2}$. Once again, we will go with the positive one, since the negative one is just the opposite direction. Therefore $\beta = \sqrt{2}$. Note that the order of ϵ are implemented in order to look at a shorter spatial scale and a slower time. We do not yet replace β by $\sqrt{2}$ for the ease of our eyes.

Note that $\frac{\partial}{\partial t} = -\beta\epsilon\frac{\partial}{\partial \xi} + \epsilon^3\frac{\partial}{\partial \tau}$ and $\frac{\partial^n}{\partial x^n} = \epsilon^n \frac{\partial^n}{\partial \xi^n}$, using this, we get that the system in the new variables is

$$\begin{cases} -\beta\epsilon V_\xi + \epsilon^3 V_\tau = -\epsilon\left(V^2 + \epsilon^2\left(\frac{\rho_\xi}{2\rho}\right)^2 - \epsilon^2\frac{\rho_{\xi\xi}}{\rho} + \rho\right)_\xi \\ -\beta\epsilon\rho_\xi + \epsilon^3\rho_\tau = -2\epsilon\left(V\rho\right)_\xi \end{cases}$$

$$\Rightarrow \begin{cases} -\beta V_\xi + \epsilon^2 V_\tau = -\left(V^2 + \epsilon^2\left(\frac{\rho_\xi}{2\rho}\right)^2 - \epsilon^2\frac{\rho_{\xi\xi}}{\rho} + \rho\right)_\xi \\ -\beta\rho_\xi + \epsilon^2\rho_\tau = -2\left(V\rho\right)_\xi . \end{cases}$$

(d) Expanding these dependent variables, gives

$$\begin{cases} -\beta(\epsilon^2 V_1 + \epsilon^4 V_2 + \dots)_\xi + \epsilon^2(\epsilon^2 V_1 + \epsilon^4 V_2 + \dots)_\tau \\ \qquad = -[(\epsilon^2 V_1 + \epsilon^4 V_2 + \dots)^2 + \epsilon^2\left(\frac{(1+\epsilon^2\rho_1+\epsilon^4\rho_2+\dots)_\xi}{2(1+\epsilon^2\rho_1+\epsilon^4\rho_2+\dots)}\right)^2 \\ \qquad\quad -\epsilon^2\frac{(1+\epsilon^2\rho_1+\epsilon^4\rho_2+\dots)_{\xi\xi}}{1+\epsilon^2\rho_1+\epsilon^4\rho_2+\dots} + 1 + \epsilon^2\rho_1 + \epsilon^4\rho_2 + \dots]_\xi \\ -\beta(1+\epsilon^2\rho_1+\epsilon^4\rho_2+\dots)_\xi + \epsilon^2(1+\epsilon^2\rho_1+\epsilon^4\rho_2+\dots)_\tau \\ \qquad = -2\left((\epsilon^2 V_1 + \epsilon^4 V_2 + \dots)(1+\epsilon^2\rho_1+\epsilon^4\rho_2+\dots)\right)_\xi \end{cases}$$

$$\Rightarrow \begin{cases} -\beta\epsilon^2 V_{1\xi} + \epsilon^4\left(-\beta V_{2\xi} + V_{1\tau}\right) + \dots & = -[\epsilon^2\rho_{1\xi} + \epsilon^4\left((V_1^2)_\xi - \rho_{1\xi\xi\xi} + \rho_{2\xi}\right)] + \dots \\ -\beta\epsilon^2\rho_{1\xi} + \epsilon^4(-\beta\rho_{2\xi} + \rho_{1\tau}) + \dots & = -2\epsilon^2 V_{1\xi} - 2\epsilon^4\left((V_1\rho_1)_\xi + V_{2\xi}\right) + \dots \end{cases}$$

The coefficients of ϵ^2 give us

$$\begin{cases} \beta V_{1\xi} = \rho_{1\xi} \\ \beta\rho_{1\xi} = 2V_{1\xi} \end{cases} \overset{\beta=\sqrt{2}}{\Rightarrow} \sqrt{2}V_{1\xi} = \rho_{1\xi}.$$

(This step confirms my thought that $\beta = \sqrt{2}$) Integrating by ξ gives us that $\sqrt{2}V_1(\xi,\tau) + g(\tau) = \rho_1(\xi,\tau)$. Since, for τ randomly fixed, $\lim_{\xi\to\pm\infty} V_1 = \lim_{\xi\to\pm\infty}\rho_1 = 0$, we have that $g(\tau) = 0$, $\forall\tau > 0$. Thus

$$\sqrt{2}V_1 = \rho_1.$$

Replacing ρ_1 by $\sqrt{2}V_1$, the coefficients of ϵ^4 give us

$$\begin{cases} \sqrt{2}V_{2\xi} - V_{1\tau} & = (V_1^2)_\xi - \sqrt{2}V_{1\xi\xi\xi} + \rho_{2\xi} \\ \sqrt{2}\rho_{2\xi} - \sqrt{2}V_{1\tau} & = 2(V_1^2)_\xi + 2V_{2\xi}. \end{cases}$$

Multiplying the first equation by $\sqrt{2}$ and summing the two equations gives us

$$2V_{2\xi} + \sqrt{2}\rho_{2\xi} - 2\sqrt{2}V_{1\tau} = \sqrt{2}(\sqrt{2}+1)(V_1^2)_\xi - 2V_{1\xi\xi\xi} + 2V_{2\xi} + \sqrt{2}\rho_{2\xi}.$$

In conclusion:

$$\sqrt{2}V_{1\tau} + \sqrt{2}(\sqrt{2}+1)V_1V_{1\xi} - V_{1\xi\xi\xi} = 0.$$

This is the KdV equation, since the coefficients of it do not matter.

3. Consider the previous problem, but with the focusing NLS equation

$$ia_t = -a_{xx} - |a|^2 a.$$

The method presented in the previous problem does not allow one to describe the dynamics of long-wave solutions of the focusing NLS equation using the KdV equation. How does this show up in the calculations?

Solution:

Given the focusing NLS equation, if we were to let

$$a(x,t) = e^{i\int V dx}\rho^{1/2},$$

substitute this and work everything out (very similar to 2.a), we get

$$\begin{cases} V_t = -\left(V^2 + \left(\frac{\rho_x}{2\rho}\right)^2 - \frac{\rho_{xx}}{2\rho} - \rho\right)_x \\ \rho_t = -2(V\rho)_x. \end{cases}$$

Further linearising V and ρ around, respectively, 0 and 1, we get

$$\begin{cases} V_t = \frac{1}{2}\rho_{xxx} + \rho_x \\ \rho_t = -2V_x. \end{cases}$$

Now we can find the linear relation like in 2.b, this gives

$$\omega^2 = (k^2 - 2)k^2.$$

This is where the problem arises. Since we want to look at long-wave solutions, we have that we want $|k| \ll 1$. This will make $k^2 - 2$ negative and thus will ω be imaginary. Therefore, both V and ρ will explode if we say that $\rho = a_\rho e^{ik_\rho x - i\omega_\rho t}$ and $V = a_V e^{ik_V x - i\omega_V t}$.

4. The mKdV equation considered in the text is known as the *focusing* mKdV equation, because of the behavior of its soliton solutions. This behavior is similar to that of the focusing NLS equation. In this problem, we study the *defocusing* mKdV equation

$$4u_t = -6u^2 u_x + u_{xxx}.$$

You have already seen that you can scale the coefficients of this equation to your favorite values, except for the ratio of the signs of the two terms on the right-hand side.

(a) Examine, using the potential energy method and phase plane analysis, the traveling-wave solutions.

(b) If you have found any heteroclinic connections, find the explicit form of the profiles corresponding to these connections.

Solution:

(a) Take $U(x - vt) = u(t, x)$, with v constant, we then have

$$-4vU' = -6U^2U' + U''' = -2(U^3)' + U'''.$$

Integrating this, with integration constant α, gives

$$-4vU = -2U^3 + U'' + \alpha.$$

Multiplying by U' and integrating again gives

$$-2vU^2 = -\frac{1}{2}U^4 + \frac{1}{2}(U')^2 + \alpha U - \beta.$$

This gives something of the form

$$\frac{1}{2}U'^2 + V(U; v, \alpha) = \beta, \tag{1}$$

with $V(U; v, \alpha) = -\frac{1}{2}U^4 + 2vU^2 + \alpha U = U(-\frac{1}{2}U^3 + 2vU + \alpha)$. There are four forms it can take on, one of these is

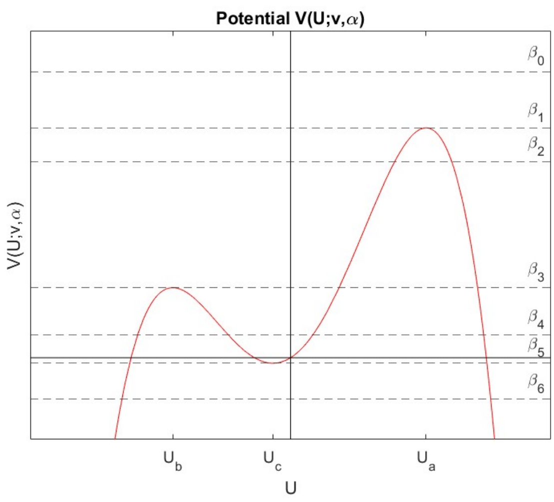

Figure 1: Potential V with $v = 3$ and $\alpha = 4$.

If we were to find the phase plane for the system

$$\begin{cases} u_1' = u_2 \\ u_2' = -\frac{\partial V}{\partial u_1}(U_1; v, \alpha), \end{cases}$$

where $u_1 = U$ and $u_2 = U'$, we have that this solves

$$\frac{1}{2}u_2^2 + V(u_1; v, \alpha) = \beta. \tag{2}$$

(because of (1)).
Here we have 7 interesting cases for β, in particular β_1, β_3 and β_5:

The easiest one to start with is β_5, Particularly at U_c where $V'|_{U_c} = 0$, thus, at $(U_c, 0)$, we have that the system is at an equilibrium. If we start at $(U_c, 0)$, there is no place where U can go. It will thus stay in a singular point. If it starts to the left of U_c, U will be unbounded since U will always start going to the point where $V = \beta_5$ on that side and then bounce back to an infinity. The same happens to the right of U_c where $V = \beta_5$.

Now that we know how this behaves, it is easy to discover how the system behaves for β_4. This will become a circle around the point $(U_c, 0)$ if the initial value is near U_c. If it's further away from (on the other segments) we have that it once again becomes unbounded "bouncing away" from the place where $V = \beta_4$ (not near U_c).

Now consider β_3. Let's Start in between U_b and the closest point to the right of U_b where V intersects β_3, say point $\tilde{U}$. Since $V'(\tilde{U}) \neq 0$ we have that it will bounce back, but since $V'(U_b) = 0$, it won't bounce back, in fact, it will take an endless amount of time for U to go to U_b from that side. Looking to the left (not equal) of U_b we get that it will take an endless amount of time for U to go to U_b, but it is unbounded. Starting on $U = U_b$ it will be a stationary point. The last segment is to the far right of U_b, this part is unbounded and will go to at least where U is equal to where $V = \beta_3$

Now we look at β_2 and β_6. In both of these cases we have two segments where U will go to where V and $\beta_{2/6}$ intersect and then be unbounded.

Now we look at β_1. Here it will take an endless amount of time to go to U_a from both sides, but be unbounded for the rest.

The directions of the curves are pretty easy to find. Since $u_1' = u_2$, if it is under the U_1 axis ($U_2 < 0$), the curve goes to the left (U_1 decreases), if it is above, it goes to the right. This can be used for the phase plane analysis of all following forms of V.

Lastly, β_0: Solutions exist and are fully unbounded, not bouncing from any point in particular. A plot of the phase plane and points where (2) is fulfilled for these β's, confirms all of my suspicions:

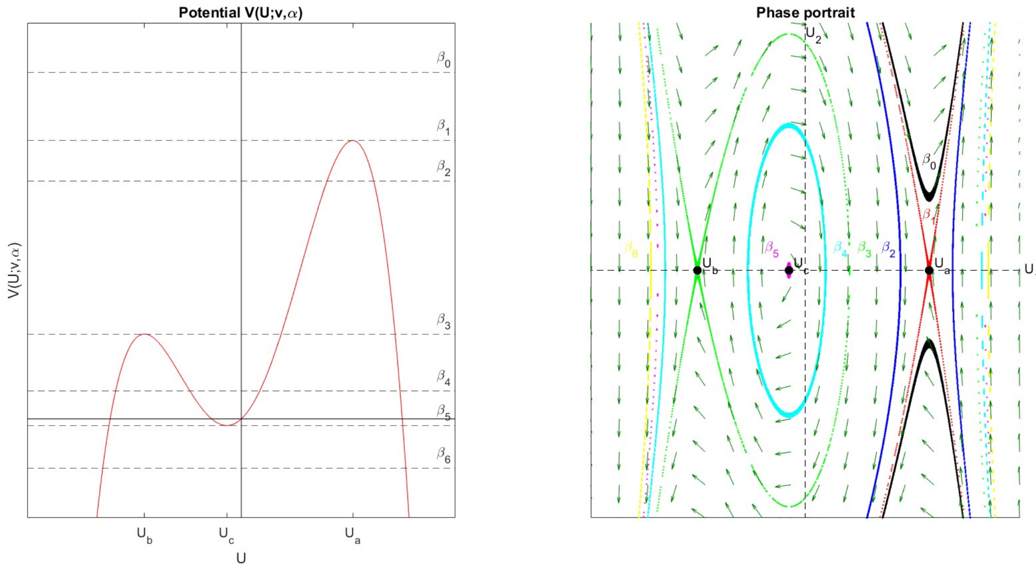

Figure 2: Potential V with $v = 0$ and $\alpha = 12$.

Note that all of the plots are made by looking at couples (U_1, U_2) which fulfill (2) with a tiny plausible error. That is why some curves are thick. These should just be lines. Another form is:

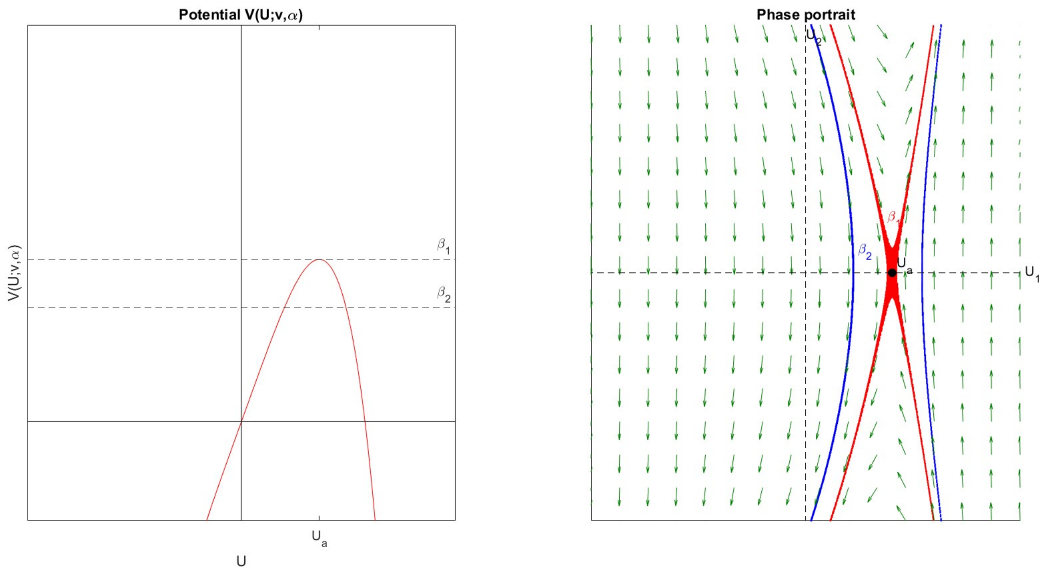

Figure 3: Potential V with $v = 0$ and $\alpha = 12$.

The β's here are pretty much the same as β_1 or β_6 of the previous form. β_1 from this form looks the same as β_1 from the last one and all other imaginable β's look like β_6 of the last form. There might be a bit of a flattening at some point because a flatting in V, but the it isn't extreme.

The third form is:

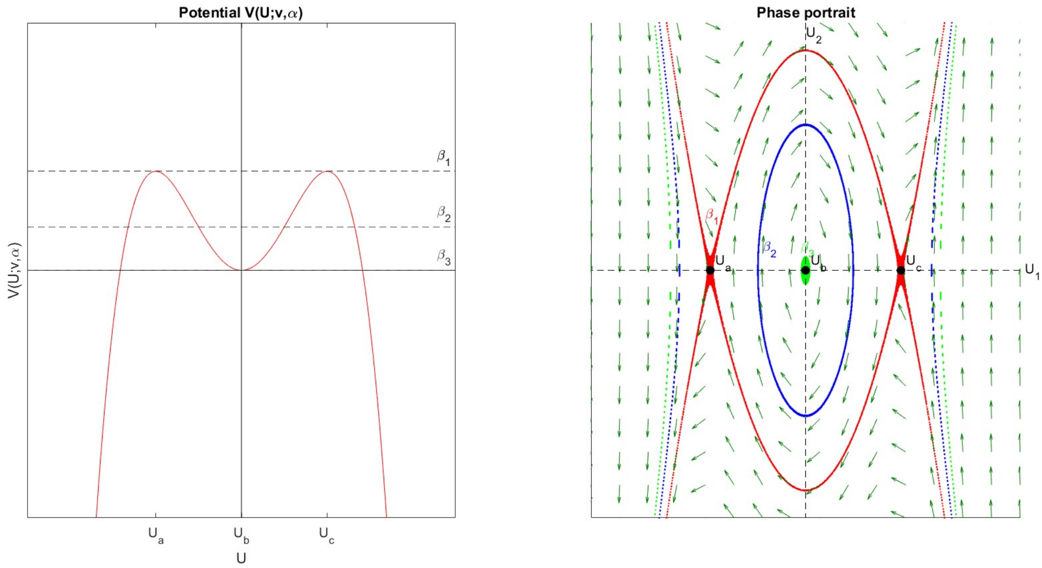

Figure 4: Potential V with $v = 2$ and $\alpha = 0$.

Here we have that the same thing happens for β_2 and β_3 as it did for, respectively, β_4 and β_5 in figure 2. The thing that happened to β_3 on one side in figure 2, now happens to β_1 on the two sides. These three curves can be explained in the same matter as we did for the first form of V. Note that it takes an endless amount of time for U to go to either U_a or U_b.

The last form is:

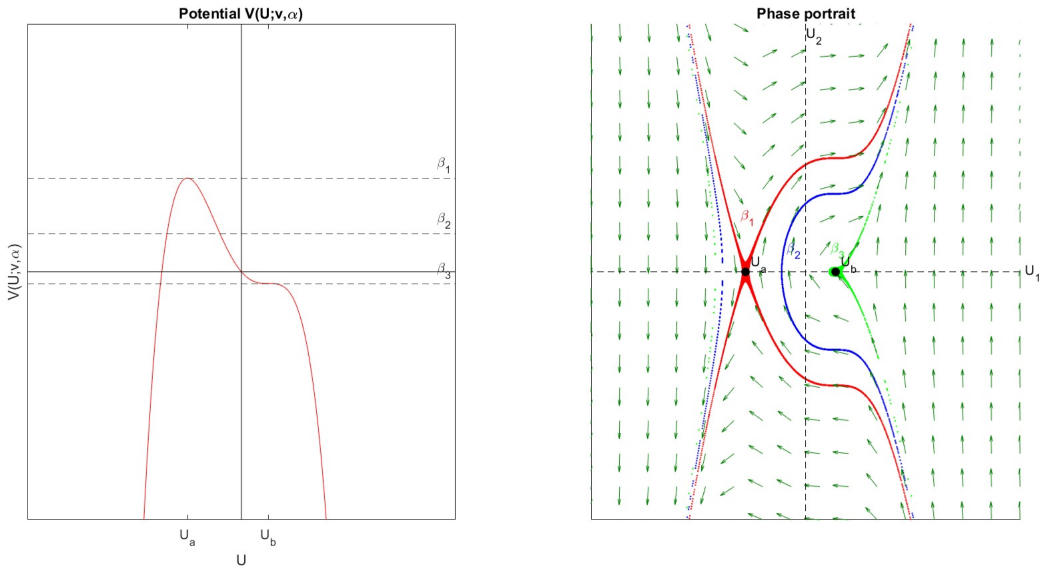

Figure 5: Potential V with $v = \frac{3}{4\sqrt[3]{2}}$ and $\alpha = -1$.

Due to point U_b the whole figure changes. Looking at β_1 and β_2, not a lot should change, but, due to point U_b we have some extreme flattening (the curve goes fully flat for a moment at $U = U_b$). Like β_3 of the first form, it will take an infinite amount of time for U to go to U_3 if $\beta = \beta_3$, but this time U can only reach U_b from the right hand side. That is why it gets this shape and not a cross-like shape. It is also important to not forget that everything is unbounded here and there exists another side where $V < \beta_3$, but nothing too special happens there.

(b) Only the third form has a heteroclinic connection. What makes V have this form is that $\alpha = 0$ and $v > 0$. Thus suppose $\alpha = 0$ and $v > 0$. We also want to find the β for when the heteroclinic connection happens. V reaches β when $\frac{dV}{dU} = 0$ and $U \neq 0$. It thus happens when

$$(-2U^2 + 4v)U = 0 \Rightarrow U^2 = 2v \Rightarrow U = \pm\sqrt{2v}, \qquad (v > 0)$$

Filling this into V gives

$$\beta = -2v^2 + 4v^2 = 2v^2.$$

Thus, we want to find

$$U' = \pm\sqrt{4v^2 + U^4 - 4vU^2} = \pm\sqrt{(2v - U^2)^2}.$$

Since the heteroclinic connection is between $U = -\sqrt{2v}$ and $U = \sqrt{2v}$ (the points which reach a maximum) we have that $2v - U^2 > 0$ in the interested interval, thus

$$U' = \pm(2v - U^2) \Rightarrow \int_{U_0}^{U} \frac{dU}{2v - U^2} = \pm \int_0^z dz = \pm z.$$

It is known that

$$\int \frac{dU}{2v - U^2} = \frac{1}{\sqrt{2v}} \tanh^{-1}\left(\frac{U}{\sqrt{2v}}\right) + C.$$

Thus,

$$\frac{1}{\sqrt{2v}} \tanh^{-1}\left(\frac{U}{\sqrt{2v}}\right) = \pm z + \frac{1}{\sqrt{2v}} \tanh^{-1}\left(\frac{U_0}{\sqrt{2v}}\right).$$

In conclusion

$$U(z) = \sqrt{2v}\tanh\left(\pm\sqrt{2v}z + \tanh^{-1}\left(\frac{U_0}{\sqrt{2v}}\right)\right).$$

This seems correct, since it will take an infinite amount of "time" for U to go to $\pm\sqrt{2v}$

5. Consider the so-called Derivative NLS equation (DNLS)

$$b_t + \alpha \left(b|b|^2\right)_x - i b_{xx} = 0.$$

This equation arises in the description of quasi-parallel waves in space plasmas. Here $b(x,t)$ is a complex-valued function.

(a) Using a polar decomposition

$$b(x,t) = B(x,t)e^{i\theta(x,t)},$$

with B and θ real-valued functions, and separating real and imaginary parts (after dividing by the exponential), show that you obtain the system

$$\begin{aligned} B_t + 3\alpha B^2 B_x + \frac{1}{B}(B^2\theta_x)_x &= 0, \\ \theta_t + \alpha B^2 \theta_x + \theta_x^2 - \frac{1}{B}B_{xx} &= 0. \end{aligned}$$

(b) Assuming a traveling-wave envelope, $B(x,t) = R(z)$, with $z = x - vt$ and constant v, show that $\theta(x,t) = \Phi(z) - \Omega t$, with constant Ω, is consistent with these equations. You can show (but you don't have to) that assuming a traveling-wave amplitude results in only this possibility for $\theta(x,t)$. At this point, we have reduced the problem of finding solutions with traveling envelope to that of finding two one-variable functions $R(z)$ and $\Phi(x)$. The problem also depends on two parameters v (envelope speed) and Ω (frequency like).

(c) Substituting these ansatz in the first equation of the above system, show that

$$\Phi' = \frac{C + vs - 3s^2}{2s},$$

where C is a constant and $s = \alpha R^2/2$.

(d) Lastly, by substituting your results in the second equation of the system, show that $s(z)$ satisfies

$$\frac{1}{2}s'^2 + V(s) = E,$$

the equation for the motion of a particle with potential $V(s)$. Find the expression for $V(s)$ and for E.

You can (don't do this; it would take a lot of work; there's a lot of cases[1]) use this observation to classify the traveling-envelope solutions of the DNLS, in the same vein that we did at the beginning of this chapter for KdV.

Solution:

[1]Two semicolons so close together? Really? And a footnote?

(a) Substituting the expression for b gives,

$$\begin{aligned}
&B_t e^{i\theta} + iB\theta_t e^{i\theta} + \alpha(B^3 e^{i\theta})_x - i(B_x e^{i\theta} + iB\theta_x e^{i\theta})_x \\
&= B_t + iB\theta_t + \alpha(3B_x B^2 + iB^3\theta_x) - iB_{xx} + B_x\theta_x + B_x\theta_x + B\theta_{xx} + iB\theta_x^2 \\
&= 0.
\end{aligned}$$

Splitting up the real and imaginary part (and dividing by B, which is not equal to 0 (polar decomposition)) gives,

$$\begin{cases}
B_t + \alpha 3B^2 B_x + 2B_x\theta_x + B\theta_{xx} = B_t + \alpha 3B^2 B_x + \frac{1}{B}(B^2\theta_x)_x = 0, \\
\theta_t + \alpha B^2\theta_x - \dfrac{B_{xx}}{B} + \theta_x^2 = 0.
\end{cases}$$

In conclusion

$$\begin{cases}
B_t + \alpha 3B^2 B_x + \frac{1}{B}(B^2\theta_x)_x = 0, \\
\theta_t + \alpha B^2\theta_x + \theta_x^2 - \dfrac{B_{xx}}{B} = 0.
\end{cases}$$

(b) Substituting $B(x,t)$ by $R(x-vt)$ gives,

$$\begin{cases}
-vR' + \alpha 3R^2 R' + \frac{1}{R}(R^2\theta_x)_x = 0, \\
\theta_t + \alpha R^2\theta_x + \theta_x^2 - \dfrac{R"}{R} = 0.
\end{cases}$$

We now want to show that, by filling in $\theta(x,t) = \Phi(z) - \Omega t$, the consistency of these equations holds (note that $\frac{\partial}{\partial x} = \frac{\partial}{\partial z}$). This means that no x or t will be present in the equations:

$$\begin{cases}
-vR' + \alpha 3R^2 R' + \frac{1}{R}(R^2\Phi')' = 0, \\
-v\Phi' - \Omega + \alpha R^2\Phi' - \dfrac{R"}{R} + \Phi'^2 = 0.
\end{cases}$$

(c) By multiplying the first equation by R and integrating it, we get

$$\begin{aligned}
\tilde{C} &= \int -vRR' + 3\alpha R^3 R' + (R^2\Phi')' dz = \int -\frac{v}{2}(R^2)' + \frac{3}{4}\alpha(R^4)' + (R^2\Phi')' dz \\
&= -\frac{v}{2}R^2 + \frac{3}{4}\alpha R^4 + R^2\Phi'.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Phi' &= \frac{\tilde{C} + v\frac{R^2}{2} - 3\alpha(\frac{R^2}{2})^2}{R^2} = \frac{\alpha\tilde{C} + v\frac{\alpha R^2}{2} - 3(\frac{\alpha R^2}{2})^2}{2\frac{\alpha R^2}{2}} \\
&\underset{C=\alpha\tilde{C}}{\overset{s=\alpha R^2/2}{=}} \frac{C + vs - 3s^2}{2s}
\end{aligned}$$

(d) Note that $2s = \alpha R^2$. When multiplying the second equation by $s' = \alpha R R'$, we get

$$-v\Phi' s' - \Omega s' + 2s\Phi' s' + \Phi'^2 s' - \frac{R"}{R} s' = 0.$$

Note that

$$\frac{R"}{R} s' = \frac{R"}{R} \alpha R R' = \frac{\alpha}{2} (R'^2)'.$$

Thus, we get

$$\begin{aligned}
&- \frac{v}{2}(C\frac{s'}{s} + vs' - 3ss') - \Omega s' + Cs' + vss' - 3s^2 s' \\
&+ \frac{C^2 + 2Cvs - 6Cs^2 + (vs)^2 - 6vs^3 + 9s^4}{4s^2} s' + \frac{\alpha}{2}(R'^2)' \\
= &- \frac{v}{2}(C\ln(s)' + vs' - \frac{3}{2}(s^2)') - \Omega s' + Cs' + \frac{v}{2}(s^2)' - (s^3)' \\
&- (\frac{C^2}{4s})' + (\frac{Cv}{2}\ln(s))' - \frac{3}{2}Cs' + \frac{v^2}{4}s' - \frac{3}{4}(s^2)' + \frac{3}{4}(s^3)' + \frac{\alpha}{2}(R'^2)' \\
= &\left(\frac{1}{4s}\left(-C^2 - (v^2 + 4\Omega - C)s^2 + (5v-3)s^3 - s^4\right)\right)' + \frac{\alpha}{2}\left(R'^2\right)' \\
&= 0.
\end{aligned}$$

Say now that

$$4sF(s) = -C^2 - (v^2 + 4\Omega - C)s^2 + (5v-3)s^3 - s^4.$$

Integrating everything gives

$$F(s) + \frac{\alpha}{2}(R')^2 = \tilde{C}.$$

Note now that, since $s' = \alpha R R'$, $R'^2 = \left(\frac{s'}{\alpha R}\right)^2 = \frac{1}{\alpha s} s'^2$. Thus,

$$\frac{1}{2}s'^2 + sF(s) = \tilde{C}s.$$

In conclusion, if $V(s) = \frac{1}{4}\left(-4\tilde{C}s - (v^2 + 4\Omega - C)s^2 + (5v-3)s^3 - s^4\right)$, we have:

$$\frac{1}{2}s'^2 + V(s) = C^2 = E.$$

With the energy the square of the C in Φ'.

6. Consider example 5.2 in the notes. Check that $y = x^2/t$ and $t^{1/2}q$ are both scaling invariant. Find the ordinary differential equation satisfied by $G(y)$, for similarity solutions of the form $q(x,t) = t^{-1/2}G(y)$. Show that this results in the same similarity solutions as in the example.

Solution:

Let's scale the x, t and q by a, a^2, and a (from example 5.1, since it needs to be scaled so that the NLS equation is unchanged):

$$x = \frac{\hat{x}}{a}, \qquad t = \frac{\hat{t}}{a^2} \text{ and } q = a\hat{q}.$$

It is clear that

$$\frac{x^2}{t} = \frac{\frac{\hat{x}^2}{a^2}}{\frac{\hat{t}}{a^2}} = \frac{\hat{x}^2}{\hat{t}} \text{ and } t^{\frac{1}{2}}q = \frac{\hat{t}^{\frac{1}{2}}}{a}a\hat{q} = \hat{t}^{\frac{1}{2}}\hat{q}.$$

The NLS equation is

$$iq_t = -q_{xx} + \sigma|q|^2 q.$$

First note that $\frac{\partial y}{\partial t} = -\left(\frac{x}{t}\right)^2$ and $\frac{\partial y}{\partial x} = 2\frac{x}{t}$ we find the derivatives:

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial t}(t^{-1/2}G(y)) = -\left(\frac{1}{2}t^{-3/2}G(y) + \left(\frac{x}{t}\right)^2 t^{-1/2}G'(y)\right),$$

and

$$\frac{\partial q}{\partial x} = 2\frac{x}{t}t^{-1/2}G'(y) \Rightarrow \frac{\partial^2 q}{\partial x^2} = 2\left(t^{-3/2}G'(y) + 2\left(\frac{x}{t}\right)^2 t^{-1/2}G''(y)\right).$$

Note that

$$t^{3/2}\frac{\partial q}{\partial t} = -\left(\frac{1}{2}G(y) + yG'(y)\right),$$

and

$$t^{3/2}\frac{\partial^2 q}{\partial x^2} = 2\left(G'(y) + 2yG''(y)\right).$$

Thus, the NLS equation times $t^{3/2}$ is,

$$\begin{aligned} -i\left(\frac{1}{2}G + yG'\right) &= -2\left(G' + 2yG''\right) + \sigma|G|^2 G \\ \Rightarrow \left(\frac{-i}{2}G + 2G'\right) + 2y\left(-\frac{i}{2}G' + 2G''\right) &= \sigma|G|^2 G. \end{aligned}$$

In the example we had $z = \frac{x}{t^{1/2}}$ $(y^{1/2} = z)$, thus $\frac{dz}{dy} = \frac{y^{-1/2}}{2} = \frac{1}{2z}$ and $\frac{d^2 z}{dy^2} = -\frac{y^{-3/2}}{4} = -\frac{1}{4z^3}$. Therefore

$$\frac{d}{dy} = \frac{1}{2z}\frac{d}{dz},$$

and

$$\frac{d^2}{dy^2} = \frac{d^2 z}{dy^2}\frac{d}{dz} + \frac{dz}{dy}^2\frac{d^2}{dz^2} = -\frac{1}{4z^3}\frac{d}{dz} + \frac{1}{4z^2}\frac{d^2}{dz^2}.$$

. In conclusion, when working with variable z, we get

$$
\begin{aligned}
&\frac{-i}{2}G + \frac{1}{z}G' + 2z^2\left(\frac{-i}{4z}G' - \frac{2}{4z^3}G' + \frac{2}{4z^2}G"\right) = \sigma|G|^2 G \\
\Rightarrow\ &\frac{-i}{2}\left(G + zG'\right) + G" = \sigma|G|^2 G \\
\Rightarrow\ &\left(G' - \frac{i}{2}zG\right)' = \sigma|G|^2 G.
\end{aligned}
$$

7. One way to write the **Toda Lattice** is

$$\frac{da_n}{dt} = a_n(b_{n+1} - b_n),$$
$$\frac{db_n}{dt} = 2(a_n^2 - a_{n-1}^2),$$

where a_n, b_n, $n \in \mathbb{Z}$, are functions of t.

(a) Find a scaling symmetry of this form of the Toda lattice, *i.e.*, let[2] $a_n = \alpha A_n$, $b_n = \beta B_n$, $t = \gamma\tau$, and determine relations between α, β and γ so that the equations for the Toda lattice in the (A_n, B_n, t) variables are identical to those using the (a_n, b_n, τ) variables.

(b) Using this scaling symmetry, find a two-parameter family of similarity solutions of the Toda lattice. If necessary, find relations among the parameters that guarantee the solutions you found are real for all n and for $t > 0$.

The Toda Lattice was introduced originally by Toda in 1967 in the form

$$\frac{dq_n}{dt} = p_n,$$
$$\frac{dp_n}{dt} = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)},$$

where q_n, p_n, $n \in \mathbb{Z}$, are functions of t. It is clear that this form does not lend itself to a scaling symmetry: the quantities q_n show up as arguments of the exponential function, and they cannot be scaled. This can be remedied by returning to the physical setting of the derivation, where a constant would multiply these exponents. This constant, being a dimensional quantity, scales in its own way under a scaling transformation.

Solution:

(a) Note that $\frac{d\tau}{dt} = \frac{1}{\gamma}$, thus, $\frac{d}{dt} = \frac{1}{\gamma}\frac{d}{d\tau}$. Substituting everything now gives

$$\frac{\alpha}{\gamma}\frac{dA_n}{d\tau} = \alpha\beta A_n(B_{n+1} - B_n),$$
$$\frac{\beta}{\gamma}\frac{db_n}{dt} = 2\alpha^2(A_n^2 - A_{n-1}^2).$$

Thus,

$$\begin{cases} \dfrac{1}{\gamma} = \beta \\ \dfrac{\beta}{\gamma} = \alpha^2 \end{cases} \Rightarrow \begin{cases} \dfrac{1}{\gamma} = \beta \\ \beta^2 = \alpha^2 \end{cases} \Rightarrow \begin{cases} \gamma = \pm\dfrac{1}{\alpha} \\ \beta = \pm\alpha \end{cases}.$$

Here you should chose + or - for both, not one sign being + and the other -. For (b) we only use the case where the sign is +, since $-$ is nearly the same.

[2] In principle, we could let α, β and γ depend on n. If you did this, you quickly discover that a scaling symmetry only exists if they do not.

(b) We use the folowing two-parameter family of similarity solutions:

$$ta_n = x_n \text{ and } tb_n = y_n.$$

As we want them to be invariant, x_n and y_n must be constant, since we would like to make a new variable z, but we could only use t, which makes it impossible to cancel out any α's. This, however, is clearly invariant:

$$ta_n = \alpha\tau\frac{1}{\alpha}A_n = \tau A_n \text{ and } tb_n = \alpha\tau\frac{1}{\alpha}B_n = \tau B_n.$$

This gives $a_n = \frac{x_n}{t}$ and $b_n = \frac{y_n}{t}$. Filling this in gives

$$\begin{aligned} -\frac{x_n}{t^2} &= \frac{x_n}{t}\left(\frac{y_{n+1}}{t} - \frac{y_n}{t}\right) = \frac{x_n}{t^2}\left(y_{n+1} - y_n\right), \\ -\frac{y_n}{t^2} &= \frac{2}{t^2}(x_n^2 - x_{n-1}^2). \end{aligned}$$

Which becomes

$$\begin{aligned} -x_n &= x_n\left(y_{n+1} - y_n\right), \\ -y_n &= 2(x_n^2 - x_{n-1}^2). \end{aligned}$$

If $x_n \neq 0$ (if $x_n = 0$, then y_{n+1} or x_{n+1} must be imaginary) , we have that $y_{n+1} = y_n - 1 = y_0 - (n+1)$, or $y_n = y_{n-1} - 1 = y_0 - n$ (it's the same for negative n). Thus, given y_0, we can find every y_n for $n \in \mathbb{Z}$. After that we can find, for $n > 0$,

$$\begin{aligned} x_n^2 &= x_{n-1}^2 - \frac{y_n}{2} = x_{n-1}^2 - \frac{y_0}{2} + \frac{n}{2} = x_{n-2}^2 - 2\frac{y_0}{2} + \frac{(n)+(n-1)}{2} = x_0^2 - n\frac{y_0}{2} + \frac{\sum_{i=1}^{n} i}{2} \\ &= x_0^2 - n\frac{y_0}{2} + \frac{n(n+1)}{4}, \end{aligned}$$

or (to get it for the negative indices)

$$\begin{aligned} x_0^2 &= x_{-1}^2 - \frac{y_0}{2} = x_{-2}^2 - 2\frac{y_0}{2} - \frac{1}{2} = x_{-n}^2 - n\frac{y_0}{2} - \frac{\sum_{i=1}^{n-1} i}{2} \\ &= x_{-n}^2 - n\frac{y_0}{2} - \frac{n(n-1)}{4}. \end{aligned}$$

Thus,

$$x_{-n}^2 = x_0 + n\frac{y_0}{2} + \frac{n(n-1)}{4} \Rightarrow x_k = x_0 - k\frac{y_0}{2} + \frac{k(k+1)}{4}.$$

Thus we have that

$$\begin{aligned} y_n &= y_0 - n, \\ x_n^2 &= x_0^2 - n\frac{y_0}{2} + \frac{n(1+n)}{4}. \end{aligned}$$

For y_n to be real, we want y_0 to be real. For x_n to be real we want

$$-\frac{n^2}{4} + n(\frac{y_0}{2} - \frac{1}{4}) < x_0^2$$

This cannot be equal since we do not want x_n to be zero. Since both sides of this go to $-\infty$, this is plausible. Moreover, denoting the rounding of of a number as $[y_0]$, the left hand, n dependant, function, reaches it's max in $n = [y_0 - \frac{1}{2}]$. Thus,

$$-\frac{[y_0 - \frac{1}{2}]^2}{4} + \frac{1}{2}[y_0 - \frac{1}{2}](y_0 - \frac{1}{2}) < x_0^2 \neq 0,$$

must hold for all x_n to be real.

8. Consider the equation

$$u_t = 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x},$$

which we will encounter more in later chapters, due to its relation to the KdV equation. Show that it has a scaling symmetry.

When we look for the scaling symmetry of the KdV equation, we have two equations for three unknowns: we have three quantities (x, t, u) to scale, and after normalizing one coefficient to 1, two remaining terms that need to remain invariant. Thus it is no surprise that we find a one-parameter family of scaling symmetries. The above equation has two more terms, and it should be clear that some “luck” is needed in order for there to be a scaling symmetry.

Solution: Consider $x = \frac{\hat{x}}{a}$, $t = \frac{\hat{t}}{b}$ and $u = c\hat{u}$, note that

$$\frac{\partial^n}{\partial x^n} = a^n \frac{\partial}{\partial \hat{x}} \quad \text{and} \quad \frac{\partial}{\partial t} = b\frac{\partial}{\partial \hat{t}}.$$

Substituting everything gives

$$bc\hat{u}_{\hat{t}} = 30ac^3\hat{u}^2\hat{u}_{\hat{x}} + 20a^3c^2\hat{u}_{\hat{x}}\hat{u}_{\hat{x}\hat{x}} + 10a^3c^2\hat{u}\hat{u}_{\hat{x}\hat{x}\hat{x}} + a^5c\hat{u}_{5\hat{x}}$$

$$\Rightarrow$$

$$\hat{u}_{\hat{t}} = 30\frac{ac^2}{b}\hat{u}^2\hat{u}_{\hat{x}} + 20\frac{a^3c}{b}\hat{u}_{\hat{x}}\hat{u}_{\hat{x}\hat{x}} + 10\frac{a^3c}{b}\hat{u}\hat{u}_{\hat{x}\hat{x}\hat{x}} + \frac{a^5}{b}\hat{u}_{5\hat{x}}.$$

There is a scaling symmetry, since

$$\begin{cases} ac^2 = b \\ a^3c = b \\ a^5 = b \end{cases} \Rightarrow \begin{cases} c = a^2 \\ b = a^5 \end{cases}.$$

Therefore there is a scaling symmetry with ansatz

$$x = \frac{\hat{x}}{a}, \qquad t = \frac{\hat{t}}{a^5} \text{ and } u = a^2\hat{u}.$$

9. Consider a Modified KdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0.$$

(a) Find its scaling symmetry.

(b) Using the scaling symmetry, write down an ansatz for any similarity solutions of the equation.

(c) Show that your ansatz is compatible with $u = (3t)^{-1/3} w(z)$, with $z = x/(3t)^{1/3}$.

(d) Use the above form of u to find an ordinary differential equation for $w(z)$. This equation will be of third order. It can be integrated once (do this) to obtain a second-order equation. The second-order equation you obtain this way is known as the second of the Painlevé equations. We will see more about these later.

Solution:

(a) Consider $x = \frac{\hat{x}}{a}$, $t = \frac{\hat{t}}{b}$ and $u = c\hat{u}$, note that

$$\frac{\partial^n}{\partial x^n} = a^n \frac{\partial^n}{\partial \hat{x}^n} \quad \text{and} \quad \frac{\partial}{\partial t} = b\frac{\partial}{\partial \hat{t}}.$$

Substituting everything gives

$$bc\hat{u}_{\hat{t}} - 6ac^3\hat{u}^2\hat{u}_{\hat{x}} + a^3 c\hat{u}_{\hat{x}\hat{x}\hat{x}} = 0.$$

Therefore $b = ac^2$ and $b = a^3$, thus $b = a^3$ and $c = a$. Therefore the scaling symmetry is

$$x = \frac{\hat{x}}{a}, \qquad t = \frac{\hat{t}}{a^3} \text{ and } u = a\hat{u}.$$

(b) From the scaling symmetry it is simple to see that

$$z = \frac{x}{t^{1/3}},$$

and

$$t^{\frac{1}{3}} u(x, t),$$

are both scaling invariant:

$$\frac{x}{t^{1/3}} = \frac{\frac{\hat{x}}{a}}{\frac{\hat{t}^{1/3}}{a}} = \frac{\hat{x}}{\hat{t}^{1/3}} \text{ and } t^{\frac{1}{3}} u = \frac{a}{a}\hat{t}^{\frac{1}{3}}\hat{u}.$$

Thus we also assume that

$$t^{\frac{1}{3}} u(x, t) = w(z) \Rightarrow u = t^{-\frac{1}{3}} w(z).$$

(c) The only thing that changes is that t is now $3t$. Since $3t$ scales the same as t, they are compatible. Thus my previous ansatz is compatible with this one.

(d) We have that $\frac{\partial z}{\partial t} = \frac{-3}{3}\frac{x}{(3t)^{4/3}}$, $\frac{\partial z}{\partial x} = \frac{1}{(3t)^{1/3}}$, thus

$$\frac{\partial u}{\partial t} = -(3t)^{-4/3}w + (3t)^{-1/3}w'\frac{\partial z}{\partial t} = -(3t)^{-4/3}w - x(3t)^{-5/3}w'$$

$$\frac{\partial u}{\partial x} = (3t)^{-1/3}w'\frac{\partial z}{\partial x} = (3t)^{-2/3}w' \Rightarrow \frac{\partial^3 u}{\partial x^3} = (3t)^{-4/3}w'''.$$

By multiplying the Modified Kdv equation above by $(3t)^{4/3}$, we get

$$\begin{aligned} 0 &= -w - \frac{x}{(3t)^{1/3}}w' - 6(3t)^{4/3}(3t)^{-2/3}w^2(3t)^{-2/3}w' + w''' = -w - zw' - 6w^2w' + w''' \\ &= -(zw)' - 2(w^3)' + w''' = (w" - zw - 2w^3)'. \end{aligned}$$

Integrating this becomes the second of the Painlevé equations:

$$w" - zw - 2w^3 = C \Rightarrow w" = 2w^3 + zw + C.$$