AMATH 573
Solitons and nonlinear waves
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Homework 4

Due: Friday, November 18, 2022

Feel free to use maple or mathematica or other software. If you do, you can upload your worksheets on Canvas.

1. Consider the Modified Vector Derivative NLS equation

$$\boldsymbol{B}_t + (\|\boldsymbol{B}\|^2\boldsymbol{B})_x + \gamma(\boldsymbol{e}_1 \times \boldsymbol{B}_0)\,(\boldsymbol{e}_1 \cdot (\boldsymbol{B}_x \times \boldsymbol{B}_0)) + \boldsymbol{e}_1 \times \boldsymbol{B}_{xx} = 0.$$

This equation describes the transverse propagation of nonlinear Alfvén waves in magnetized plasmas. Here $\boldsymbol{B} = (0, u, v)$, $\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{B}_0 = (0, B_0, 0)$, and γ is a constant. The boundary conditions are $\boldsymbol{B} \to \boldsymbol{B}_0$, $\boldsymbol{B}_x \to 0$ as $|x| \to \infty$. By looking for stationary solutions $\boldsymbol{B} = \boldsymbol{B}(x - Wt)$, one obtains a system of ordinary differential equations. Integrating once, one obtains a first-order system of differential equations for u and v.

   a) Show that this system is Hamiltonian with canonical Poisson structure, by constructing its Hamiltonian $H(u, v)$.
   b) Find the value of the Hamiltonian such that the boundary conditions are satisfied. Then $H(u, v)$ equated to this constant value defines a curve in the (u, v)-plane on which the solution lives. In the equation of this curve, let $U = u/B_0$, $V = v/B_0$, and $W_0 = W/B_0^2$. Now there are only two parameters in the equation of the curve: W_0 and γ.
   c) With $\gamma = 1/10$, plot the curve for $W_0 = 3$, $W_0 = 2$, $W_0 = 1.1$, $W_0 = 1$, $W_0 = 0.95$, $W_0 = 0.9$. All of these curves have a singular point at $(1, 0)$. This point is an equilibrium point for the Hamiltonian system, corresponding to the constant solution which satisfies the boundary condition. The curves beginning and ending at this equilibrium point correspond to soliton solutions of the Modified Vector Derivative NLS equation. How many soliton solutions are there for the different velocity values you considered? Draw a qualitatively correct picture of the solitons for all these cases.

Solution:

(a) First we find he system of first order differential equations for u and v, therefore we find that

$$e_1 \times \boldsymbol{B}_0 = \det\left(\begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & B_0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ B_0 \end{bmatrix},$$

$$\boldsymbol{B}_x \times \boldsymbol{B}_0 = \det\left(\begin{bmatrix} i & j & k \\ 0 & u_x & v_x \\ 0 & B_0 & 0 \end{bmatrix}\right) = -\begin{bmatrix} B_0 v_x \\ 0 \\ 0 \end{bmatrix},$$

and

$$e_1 \times \boldsymbol{B}_{xx} = \det\left(\begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & u_{xx} & v_{xx} \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -v_{xx} \\ u_{xx} \end{bmatrix},$$

Therefore we have:

$$\begin{bmatrix} 0 \\ u_t \\ v_t \end{bmatrix} + \left((u^2+v^2)\begin{bmatrix} 0 \\ u \\ v \end{bmatrix}\right)_x + \gamma B_0 v_x \begin{bmatrix} 0 \\ 0 \\ B_0 \end{bmatrix} + \begin{bmatrix} 0 \\ -v_{xx} \\ u_{xx} \end{bmatrix} = 0$$

Thus, we get the system

$$\begin{cases} 0 & = 0 \\ u_t + (u(u^2+v^2))_x - v_{xx} & = 0 \\ v_t + (v(u^2+v^2))_x - \gamma B_0^2 v_x + u_{xx} & = 0 \end{cases}$$

Since we're looking for stationary solutions, we look at $u = u(x - Wt)$ and $v = v(x - Wt) = v(z)$ and then get

$$\begin{cases} -Wu' + (u(u^2+v^2))' - v'' & = 0 \\ -Wv' + (v(u^2+v^2))' - \gamma B_0^2 v' + u'' & = 0 \end{cases}.$$

Integrating this over z gives

$$\begin{cases} -Wu + u(u^2+v^2) & = C_1 + v' \\ -Wv + v(u^2+v^2) - \gamma B_0^2 v & = C_2 - u' \end{cases} \Rightarrow \begin{cases} v' & = u(u^2+v^2-W) - C_1 \\ u' & = -v(u^2+v^2-\gamma B_0^2 - W) + C_2 \end{cases}.$$

The boundary conditions are, for t fixed:

$$\begin{matrix} u(z) = B_0 & v(z) = 0 \\ u'(z) = 0 & v'(z) = 0 \end{matrix} \quad \text{as } |z| \to \infty.$$

Therefore,

$$C_1 = B_0(B_0^2 - W) \text{ and } C_2 = 0$$

Thus the first order system of differential equations for u and v is given by

$$\begin{cases} v' & = u(u^2+v^2-W) - B_0(B_0^2 - W) \\ u' & = -v(u^2+v^2-\gamma B_0^2 - W) \end{cases},$$

with boundary conditions

$$u(z) = B_0 \quad v(z) = 0 \qquad \text{as } |z| \to \infty.$$

If this system is Hamiltonian with canonical Poisson structure, it should be of the form

$$\begin{cases} v' &= \frac{\partial \mathcal{H}}{\partial u} \\ u' &= -\frac{\partial \mathcal{H}}{\partial v} \end{cases}.$$

So we have that

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial u} &= u(u^2+v^2-W) - B_0(B_0^2 - W) \\ \frac{\partial \mathcal{H}}{\partial v} &= v(u^2+v^2-\gamma B_0^2 - W) \end{cases}$$
$$\Rightarrow \begin{cases} \mathcal{H}(u,v) &= \frac{1}{4}u^4 + \frac{1}{2}(uv)^2 - \frac{W}{2}u^2 - B_0(B_0^2-W)u + K_1(v) \\ \mathcal{H}(u,v) &= \frac{1}{2}(uv)^2 + \frac{1}{4}v^4 - \frac{\gamma B_0^2 - W}{2}v^2 + K_2(v) \end{cases}.$$

Therefore

$$\begin{aligned} \mathcal{H}(u,v) &= \frac{1}{4}u^4 + \frac{1}{2}(uv)^2 - \frac{W}{2}u^2 - B_0(B_0^2-W)u + \frac{1}{4}v^4 - \frac{\gamma B_0^2 - W}{2}v^2 + K \\ &= \frac{1}{4}(u^2+v^2)^2 - \frac{W}{2}(u^2+v^2) - B_0(B_0^2-W)u - \frac{\gamma B_0^2}{2}v^2 + K, \end{aligned}$$

with K a constant, which we may ignore.
In conclusion the Hamiltonian with canonical Poisson structure of this system is

$$H = \int \frac{1}{4}(u^2+v^2)^2 - \frac{W}{2}(u^2+v^2) - B_0(B_0^2-W)u - \frac{\gamma B_0^2}{2}v^2 dz.$$

(b) Note that the boundary conditions here are $u(x) = B_0$ and $v(x) = 0$ as $|x| \to \infty$. Therefore the value of the Hamiltonian such that the boundary conditions are satisfied is

$$\mathcal{H}(B_0, 0) = \frac{B_0^4}{4} - \frac{W}{2}B_0^2 - B_0^4 + WB_0^2 = \frac{WB_0^2}{2} - \frac{3}{4}B_0^4$$

Substituting $U = u/B_0$, $V = v/B_0$ and $W_0 = W/B_0^2$, we get that $\mathcal{H}$ becomes

$$\mathcal{H}(U,V) = \frac{B_0^4}{4}(U^2+V^2)^2 - \frac{W_0}{2}B_0^4(U^2+V^2) - B_0^4(1-W_0)U - \frac{\gamma}{2}B_0^4V^2,$$

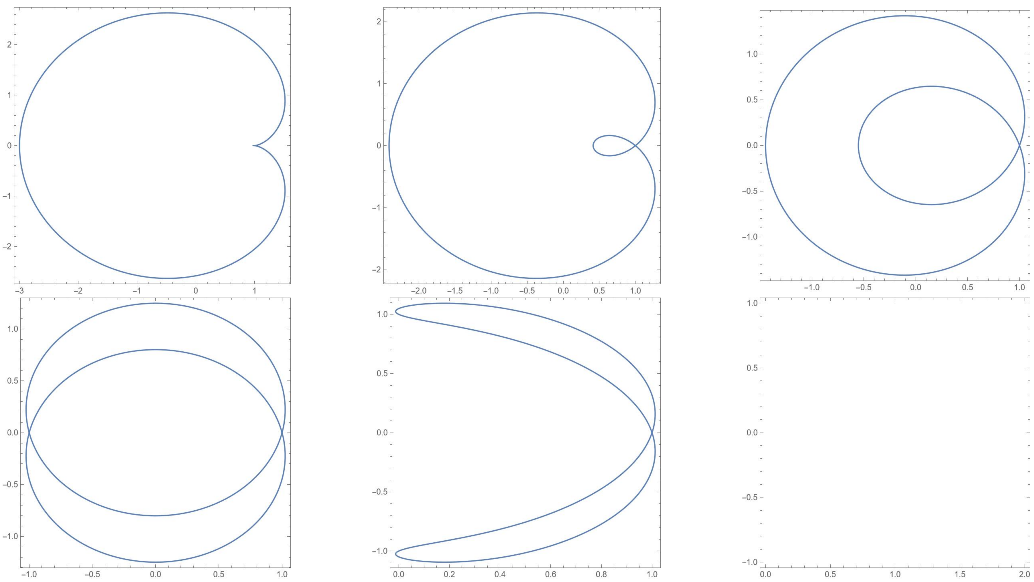
and the constant becomes

$$\mathcal{H}(1,0) = \frac{W_0}{2}B_0^4 - \frac{3}{4}B_0^4.$$

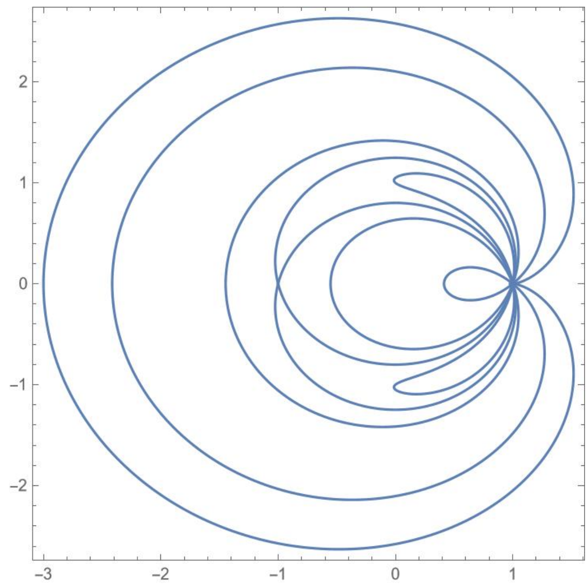
We get the equation of the curve when we set these last two things as equal. Dividing everything by B_0^4 gives

$$\frac{1}{4}(U^2+V^2)^2 - \frac{W_0}{2}(U^2+V^2) - (1-W_0)U - \frac{\gamma}{2}V^2 = \frac{W_0}{2} - \frac{3}{4}$$

(c) Plotting all curves separately, with U on the horizontal axis and V the vertical, for, respectively, $W_0 = 3,\ 2,\ 1.1,\ 1,\ .95\ \&\ .9$, gives



Note that nothing was plotted for $W_0 = .9$. We assume this was because of it becoming a difficult thing to calculate. However, plotting all of these together gives us a better view of what it should be.



We speculate that it will be a curve like for $W_0 = .95$, but tighter. (these plots were made using Wolfram Mathematica file "EX1 WV")

Note in particular the curve where $W_0 = 1$. There seems to be another equilibrium at $(-1, 0)$. This suspicion can be validated if we look back at the original system. When $W_0 = 1$, $W = B_0^2$. If $(U, V) = (-1, 0)$, then $(u, v) = (-B_0, 0)$

and the original system becomes:

$$\begin{cases} v' &= -B_0(B_0^2 - B_0^2) - B_0(B_0^2 - B_0^2) \\ u' &= -0(B_0^2 - \gamma B_0^2 - B_0^2) \end{cases} \Rightarrow \begin{cases} v' &= 0 \\ u' &= 0 \end{cases}.$$

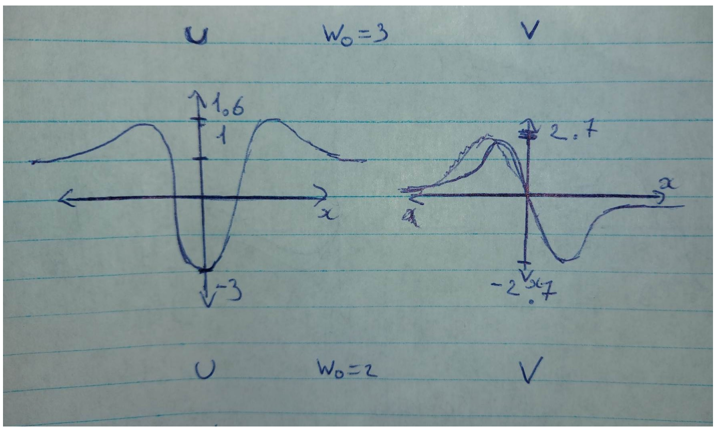
It is thus indeed an equilibrium. Thus, here $u \neq B_0$ for $|z| \to \infty$.

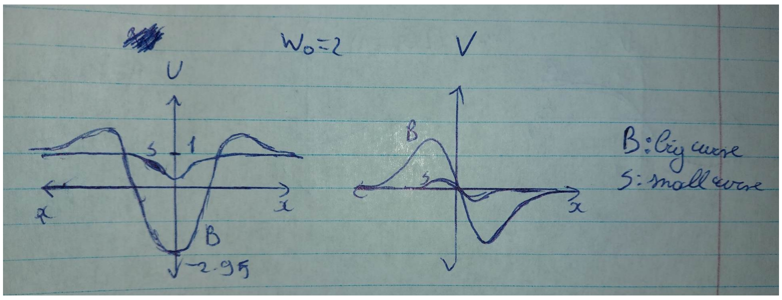
We now try and draw the functions U and V in function of x. We drew these based on how they moved with respect to the point $(1, 0)$. For U we based ourselves on it's horizontal position and V it's vertical position. The soliton solution depends on it's starting point. Most of the W_0 values have 2 soliton solutions each. The exceptions are: $W_0 = 3$ (one soliton each) and $W_0 = 1$ (4 solitons each). Note that most of the time, the direction of the curve can be found because of the fact that $U' \; -V$ (for U and V away from $(1, 0)$), it thus goes to the left when V is positive and right when $Visnegative$.

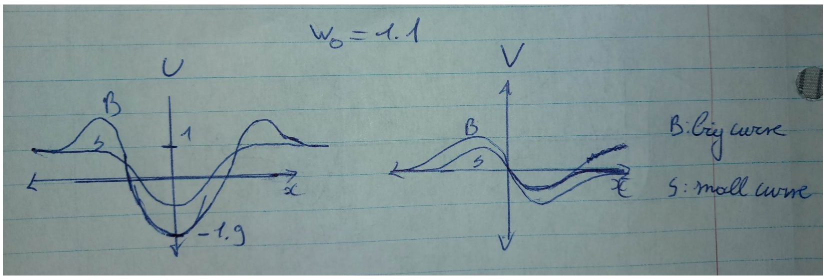
For all the bigger curves (not the ones inside another curve) we see that at the start and the end U goes to 1 from the right ($U > 1$), and in the smaller curves, it goes to 1 from the left ($U < 1$). As W_0 goes to 1, the two curves converge and so do the functions U (the bigger one decreases and smaller one increases). When $W_0 = 1$, it takes an infinite amount of time for U to go to -1, therefore it looks more like a shockwave. If $W_0 < 1$, we have that, when starting with V positive, U starts by overshooting 1 and then goes under 1 to then go back to 1 while it is less than 1. The opposite happens when starting with V negative (it ends up overshooting). $W_0 = .9$ will not be drawn. it looks like $W_0 = .95$, but dampened.

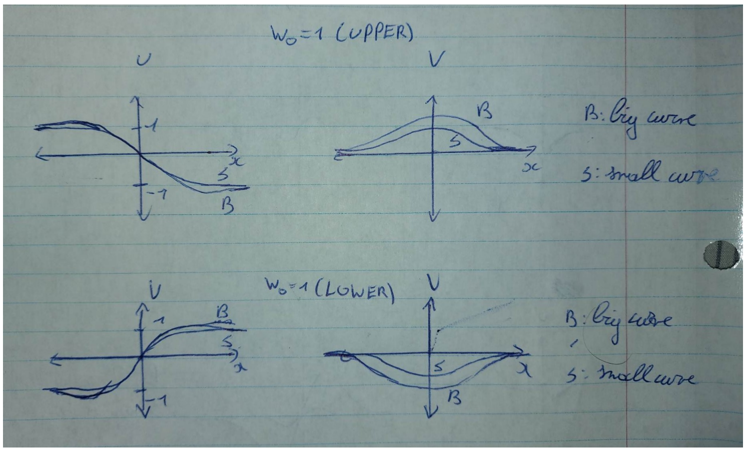
Now we look at what happens to V. While W_0 is bigger than 1, V can always take on both negative and positive signs. Note that when $V = 0$, it's derivative is infinite. Once $W_0 = 1$, it does not change sign and this stay for $W_0 < 1$. Note that there also exists a point where the derivative of V is both 0 and infinite (separate points).

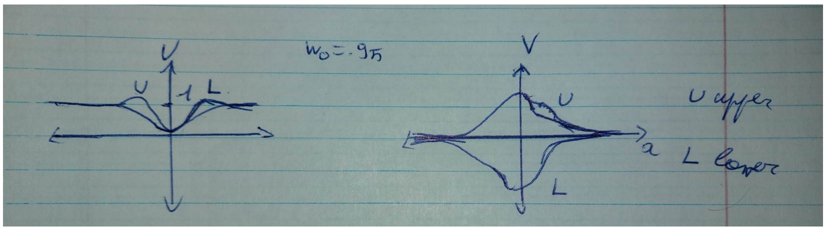
We now try and draw these curves:




W0=2
U
V
1
S
B
-2.95
x
B: big curve
S: small curve


W0 = 1.1
U
V
B
S
1
-1.9
x
B: big curve
S: small curve


W0=1 (UPPER)
U
V
1
-1
x
S
B
B: big curve
S: small curve
W0=1 (LOWER)
B: big curve
S: small curve


W0=.95
U
V
1
L
U upper
L lower

2. Show that the canonical Poisson bracket

$$\{f, g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right)$$

satisfies the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0.$$

Solution:

One of these terms has the form:

$$\begin{aligned}
\{\{f, g\}h\} &= \{\sum_{j=1}^{N} f_{q_j} g_{p_j} - f_{p_j} g_{q_j}, h\} \\
&= \sum_{k=1}^{N} \left[\left(\sum_{j=1}^{N} f_{q_j} g_{p_j} - f_{p_j} g_{q_j} \right)_{q_k} h_{p_k} - \left(\sum_{j=1}^{N} f_{q_j} g_{p_j} - f_{p_j} g_{q_j} \right)_{p_k} h_{q_k} \right] \\
&= \sum_{k=1}^{N} \sum_{j=1}^{N} f_{q_j q_k} g_{p_j} h_{p_k} + f_{q_j} g_{p_j q_k} h_{p_k} - f_{p_j q_k} g_{q_j} h_{p_k} - f_{p_j} g_{q_j q_k} h_{p_k} \\
&\qquad - \left(f_{q_j p_k} g_{p_j} h_{q_k} + f_{q_j} g_{p_j p_k} h_{q_k} - f_{p_j p_k} g_{q_j} h_{q_k} - f_{p_j} g_{q_j p_k} h_{q_k} \right)
\end{aligned}$$

It is possible to pull the sum to the outside since it is a finite sum. Since both j and k go from 1 to N it does not matter if the derivative is to, for example, p_j or p_k, we still have that, for example,

$$\begin{aligned}
\sum_{j=1}^{N} \sum_{k=1}^{N} g_{p_j} h_{p_k} - g_{p_k} h_{p_j} &= \sum_{j=1}^{N} \sum_{k=1}^{N} g_{p_j} h_{p_k} - \sum_{j=1}^{N} \sum_{k=1}^{N} g_{p_k} h_{p_j} \\
&= \left(\sum_{j=1}^{N} g_{p_j} \right) \left(\sum_{k=1}^{N} h_{p_k} \right) - \left(\sum_{k=1}^{N} g_{p_k} \right) \left(\sum_{j=1}^{N} h_{p_j} \right) \\
&= 0
\end{aligned}$$

Assume also that the f, g and h are sufficiently smooth such that the partial derivatives can be switched places, it is then easy to see that

$$\sum_{j=1}^{N} \sum_{k=1}^{N} f_{q_j p_k} - f_{q_k p_j} = 0,$$

since both sum just make every possible combination of j and k and thus everything cancels out. If we now write out the full equation, we get that the terms with the same color cancel each other out, and therefore

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}$$

$$=$$

$$\begin{aligned}
\sum_{k=1}^{N}\sum_{j=1}^{N} \quad & f_{q_jq_k}g_{p_j}h_{p_k}+f_{q_j}g_{p_jq_k}h_{p_k}-f_{p_jq_k}g_{q_j}h_{p_k}-f_{p_j}g_{q_jq_k}h_{p_k} \\
& -\left(f_{q_jp_k}g_{p_j}h_{q_k}+f_{q_j}g_{p_jp_k}h_{q_k}-f_{p_jp_k}g_{q_j}h_{q_k}-f_{p_j}g_{q_jp_k}h_{q_k}\right) \\
& g_{q_jq_k}h_{p_j}f_{p_k}+g_{q_j}h_{p_jq_k}f_{p_k}-g_{p_jq_k}h_{q_j}f_{p_k}-g_{p_j}h_{q_jq_k}f_{p_k} \\
& -\left(g_{q_jp_k}h_{p_j}f_{q_k}+g_{q_j}h_{p_jp_k}f_{q_k}-g_{p_jp_k}h_{q_j}f_{q_k}-g_{p_j}h_{q_jp_k}f_{q_k}\right) \\
& h_{q_jq_k}f_{p_j}g_{p_k}+h_{q_j}f_{p_jq_k}g_{p_k}-h_{p_jq_k}f_{q_j}g_{p_k}-h_{p_j}f_{q_jq_k}g_{p_k} \\
& -\left(h_{q_jp_k}f_{p_j}g_{q_k}+h_{q_j}f_{p_jp_k}g_{q_k}-h_{p_jp_k}f_{q_j}g_{q_k}-h_{p_j}f_{q_jp_k}g_{q_k}\right)
\end{aligned}$$

$$=$$

$$0.$$

In conclusion, the canonical Poisson bracket satisfies the Jacobi identity.

3. Show that the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0$$

is Hamiltonian with canonical Poisson structure and Hamiltonian

$$H = \int \left(\frac{1}{2}p^2 + \frac{1}{2}q_x^2 + 1 - \cos(q) \right) dx,$$

where $q = u$, and $p = u_t$.

Solution:

We have that

$$H = \int \mathcal{H}(q, p, q_x, p_x) dx,$$

with

$$\mathcal{H}(q, p, q_x, p_x) = \frac{1}{2}p^2 + \frac{1}{2}q_x^2 + 1 - \cos(q)$$

For the canonical Poisson structure we have that

$$\begin{cases} q_t = \frac{\delta H}{\delta p} \\ p_t = -\frac{\delta H}{\delta q} \end{cases} \Rightarrow \begin{cases} u_t = \frac{\partial \mathcal{H}}{\partial p} \\ u_{tt} = -\frac{\partial \mathcal{H}}{\partial q} + \partial_x \frac{\partial \mathcal{H}}{\partial q_x} \end{cases} \Rightarrow \begin{cases} u_t = p = u_t \\ u_{tt} = -\sin(q) + \partial_x q_x = -\sin(q) + q_{xx} \end{cases}$$

In conclusion, from this we get that

$$u_{tt} - u_{xx} + \sin(u) = 0,$$

which is the Sine-Gordon equation. We thus verified that the statement in the exercise is true.

4. Check explicitly that the conserved quantities $F_{-1} = \int u dx$, $F_0 = \int \frac{1}{2}u^2 dx$, $F_1 = \int \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right) dx$, $F_2 = \int \left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2\right) dx$ are mutually in involution with respect to the Poisson bracket defined by the Poisson structure given by ∂_x.

Solution:

We want to check that

$$\{F_i, F_j\} = 0, \qquad \forall i, j = -1, 0, 1, 2$$

We already know that $\{F_i, F_i\} = 0$, $\forall i$. The Poisson bracket of two functionals is defined by

$$\{F, G\} = \int \frac{\delta F}{\delta u} B \frac{\delta G}{\delta u} dx$$

Note that we have an integral going from $-\infty$ to ∞ or something periodic. Note also that u is equal and behaves the same at the bounds, thus if we find that what's inside the integral, is of the form $f_x(u, u_x, ...)$, we then have that the Poisson bracket is zero. First we calculate the variational derivatives

$$\frac{\delta F_{-1}}{\delta u} = 1$$

$$\frac{\delta F_0}{\delta u} = u$$

$$\frac{\delta F_1}{\delta u} = \frac{u^2}{2} + u_{xx}$$

$$\frac{\delta F_2}{\delta u} = \frac{1}{6}u^3 - \frac{1}{2}u_x^2 + \partial_x(uu_x) + \partial_{xx}(\frac{3}{5}u_{xx}) = \frac{1}{6}u^3 + \frac{1}{2}u_x^2 + uu_{xx} + \frac{3}{5}u_{4x}$$

Since $\partial_x \frac{\delta F_{-1}}{\delta u} = 0$, everything with F_{-1} is zero:

$$\{F_j, F_{-1}\} = 0, \qquad \forall j = 0, 1, 2$$

Note that, since $\{F, G\} = -\{G, F\}$, it is sufficient to only check half of the cases. Now we check F_1 and F_2 with F_0:

$$\{F_1, F_0\} = \int \frac{\delta F_1}{\delta u} \partial_x \frac{\delta F_0}{\delta u} dx = \int \frac{u^2}{2}u_x + u_x u_{xx} dx = \int \left(\frac{u^3}{6} + \frac{1}{2}u_x^2\right)_x dx = 0,$$

and

$$\{F_2, F_0\} = \int \frac{\delta F_2}{\delta u} \partial_x \frac{\delta F_0}{\delta u} dx = \int \frac{1}{6}u^3 u_x + \frac{1}{2}u_x^3 + uu_x u_{xx} + \frac{3}{5}u_x u_{4x} dx$$

Note here that $\frac{1}{2}\left(uu_x^2\right)_x = \frac{1}{2}u_x^3 + uu_x u_{xx}$ and $(uu_{4x})_x = u_{5x} + u_x u_{4x}$ (thus $u_x u_{4x} = (uu_{4x})_x - u_{5x}$). Thus,

$$\begin{aligned}
\{F_2, F_0\} &= \int \frac{\delta F_2}{\delta u} \partial_x \frac{\delta F_0}{\delta u} dx \\
&= \int \frac{1}{24}\left(u^4\right)_x + \frac{1}{2}\left(uu_x^2\right)_x + \frac{3}{5}\left(uu_{4x}\right)_x - \frac{3}{5}u_{5x} dx \\
&= \int \left(\frac{1}{24}u^4 + \frac{1}{2}uu_x^2 + \frac{3}{5}uu_{4x} - \frac{3}{5}u_{4x}\right)_x dx \\
&= 0
\end{aligned}$$

Lastly we check F_2 with F_1:

$$\begin{aligned}
\int \frac{\delta F_2}{\delta u}\partial_x \frac{\delta F_1}{\delta u}dx &= \int \left(\frac{1}{6}u^3 + \frac{1}{2}u_x^2 + uu_{xx} + \frac{3}{5}u_{4x}\right)(uu_x + u_{xxx})\,dx \\
&= \int \frac{1}{6}u^4u_x + \frac{1}{2}uu_x^3 + u^2u_xu_{xx} + \frac{3}{5}uu_xu_{4x} \\
&\quad + \frac{1}{6}u^3u_{xxx} + \frac{1}{2}u_x^2u_{xxx} + uu_{xx}u_{xxx} + \frac{3}{5}u_{xxx}u_{4x}dx \\
&= \int \frac{1}{30}(u^5)_x + \frac{1}{2}uu_x^3 + u^2u_xu_{xx} + \frac{3}{5}uu_xu_{4x} \\
&\quad + \frac{1}{6}u^3u_{xxx} + \frac{1}{2}u_x^2u_{xxx} + uu_{xx}u_{xxx} + \frac{3}{10}(u_{xxx}^2)_xdx \\
&= \int \frac{1}{2}uu_x^3 + u^2u_xu_{xx} + \frac{3}{5}uu_xu_{4x} + \frac{1}{6}u^3u_{xxx} + \frac{1}{2}u_x^2u_{xxx} + uu_{xx}u_{xxx}dx
\end{aligned}$$

Note that $\frac{1}{4}(u^2u_x^2)_x = \frac{1}{2}uu_x^3 + \frac{1}{2}u^2u_xu_{xx}$ and $\frac{1}{6}(u^3u_{xx})_x = \frac{1}{2}u^2u_xu_{xx} + \frac{1}{6}u^3u_{xxx}$, thus

$$\left(\frac{1}{4}u^2u_x^2 + \frac{1}{6}u^3u_{xx}\right)_x = \frac{1}{2}uu_x^3 + u^2u_xu_{xx} + \frac{1}{6}u^3u_{xxx}$$

Thus,

$$\{F_2, F_1\} = \int \frac{3}{5}uu_xu_{4x} + \frac{1}{2}u_x^2u_{xxx} + uu_{xx}u_{xxx}dx$$

Since,

$$\begin{aligned}
(uu_xu_{xxx})_x &= u_x^2u_{xxx} + uu_{xx}u_{xxx} + uu_xu_{4x}, \\
(u_x^2u_{xx})_x &= 2u_xu_{xx}^2 + u_x^2u_{xxx},
\end{aligned}$$

and

$$(uu_{xx}^2)_x = u_xu_{xx}^2 + 2uu_{xx}u_{xxx},$$

we have that

$$\left(-\frac{1}{10}u_x^2u_{xx} + \frac{1}{5}uu_{xx}^2 + \frac{3}{5}uu_xu_{xxx}\right)_x = \frac{2}{5}uu_xu_{4x} + \frac{1}{2}u_x^2u_{xxx}$$

In conclusion,

$$\{F_2, F_1\} = \int g(u, u_x, u_{xx}, u_{xxx})_xdx = 0.$$

Therefore, the conserved quantities in this exercise are all mutually in involution with respect to the Poisson bracket defined by the Poisson structure given by ∂_x.

5. Find the fourth conserved quantity for the KdV equation $u_t = uu_x + u_{xxx}$, *i.e.*, the conserved quantity which contains $\frac{1}{24}\int u^4 dx$.

Solution:

We know that $[u^4] = 8$, other possible terms with weight 8 are u^2u_{xx}, uu_x^2, u_{xx}^2, u_xu_{3x}, uu_{4x} and u_{6x}. We know that

$$\left(u^2u_x\right)_x = 2uu_x^2 + u^2u_{xx}, \quad \left(u_xu_{xx}\right)_x = u_{xx}^2 + u_xu_{xxx}, \quad \text{and} \quad \left(uu_{xxx}\right)_x = u_xu_{xxx} + uu_{4x}$$

Therefore uu_x^2 and u^2u_{xx} are equivalent, up to an $x-$derivative $((u^2u_x)_x)$. On the other hand, u_{xx}^2, u_xu_{xxx} and uu_{4x} are equivalent, up to an $x-$derivative $((u_xu_{xx} + uu_{xxx})_x)$. Therefore,

$$F_2 = \int \left(a_1u^4 + a_2uu_x^2 + a_3u_{xx}^2\right) dx = \int f(x,t)dx,$$

should be able to define the whole fourth conserved quantity. We now want to find the coefficients a_1, a_2 and a_3. First we find f_t:

$$f_t = 4a_1u^3u_t + a_2u_tu_x^2 + 2a_2uu_xu_{xt} + 2a_3u_{xx}u_{xxt}.$$

Deriving the KdV equation by x twice, gives us what u_t, u_{xt} and u_{xxt} are. These are

$$u_{xt} = u_x^2 + uu_{xx} + u_{4x},$$

and

$$u_{xxt} = 2u_xu_{xx} + u_xu_{xx} + uu_{xxx} + u_{5x} = 3u_xu_{xx} + uu_{xxx} + u_{5x}.$$

Filling this in gives,

$$\begin{aligned}
f_t &= 4a_1u^4u_x + 4a_1u^3u_{xxx} \\
&\quad + a_2uu_x^3 + a_2u_x^2u_{xxx} + 2a_2uu_x^3 + 2a_2u^2u_xu_{xx} + 2a_2uu_xu_{4x} \\
&\quad + 6a_3u_xu_{xx}^2 + 2a_3uu_{xx}u_{xxx} + 2a_3u_{xx}u_{5x} \\
&= \frac{4a_1}{5}(u^5)_x + 4a_1u^3u_{xxx} \\
&\quad + 3a_2uu_x^3 + a_2u_x^2u_{xxx} + 2a_2u^2u_xu_{xx} + 2a_2uu_xu_{4x} \\
&\quad + 6a_3u_xu_{xx}^2 + 2a_3uu_{xx}u_{xxx} + 2a_3u_{xx}u_{5x}
\end{aligned}$$

Since we're dealing with conserved quantities, we want $f_t = g_x$ so that $(F_2)_t = \int f_t(u, u_x, ...)dx = \int g_x(u, u_x, ...)dx = 0$. We can thus ignore $(u^5)_x$. Since,

$$(u_{xx}u_{4x})_x = u_{xxx}u_{4x} + u_{xx}u_{5x} = \frac{1}{2}(u_{xxx})_x + u_{xx}u_{5x},$$

we can also ignore $u_{xx}u_{5x}$. Note now that,

$$(u^3u_{xx})_x = 3u^2u_xu_{xx} + u^3u_{xxx} \Rightarrow u^3u_{xxx} = (u^3u_{xx})_x - 3u^2u_xu_{xx},$$

$$(u^2u_x^2)_x = 2uu_x^3 + 2u^2u_xu_{xx} \Rightarrow uu_x^3 = \frac{1}{2}(u^2u_x^2)_x - u^2u_xu_{xx},$$

The exercise states that $\frac{1}{24}u^4$ is going to be a component, thus we want $a_1 = \frac{1}{24}$. Because of the last two equations, we know that a linear combination of $u^3 u_{xxx}$, uu_x^3 and $u^2 u_x u_{xx}$ can be equivalent to a function derived by x, therefore we want nothing to remain from this linear combination. Thus

$$(-12a_1 - 3a_2 + 2a_2)u^2 u_x u_{xx} = 0 \Rightarrow a_2 = -12a_1 \Rightarrow a_2 = -\frac{1}{2}$$

The following remains to not be a derivative of something:

$$a_2 u_x^2 u_{xxx} + 2a_2 u u_x u_{4x} + 6a_3 u_x u_{xx}^2 + 2a_3 u u_{xx} u_{xxx}.$$

Since

$$(u_x^2 u_{xx})_x = 2u_x u_{xx}^2 + u_x^2 u_{xxx} \Rightarrow u_x^2 u_{xxx} = (u_x^2 u_{xx})_x - 2u_x u_{xx}^2,$$

$$(uu_{xx}^2)_x = u_x u_{xx}^2 + 2u u_{xx} u_{xxx} \Rightarrow u u_{xx} u_{xxx} = \frac{1}{2}(uu_{xx}^2)_x - \frac{1}{2}u_x u_{xx}^2,$$

and

$$\begin{aligned}
(uu_x u_{xxx})_x &= u_x^2 u_{xxx} + u u_{xx} u_{xxx} + u u_x u_{4x} \\
&\Rightarrow \\
u u_x u_{4x} &= (u u_x u_{xxx} - u_x^2 u_{xx} - \frac{1}{2}u u_{xx}^2)_x + \frac{5}{2}u_x u_{xx}^2,
\end{aligned}$$

we have that the only thing that is not a derivative of something and remains, is

$$(-2a_2 + 5a_2 + 6a_3 - a_3)u_x u_{xx}^2$$

Since we cannot form this into the derivative of something, we want the coefficient to be zero. This gives us that

$$(-2a_2 + 5a_2 + 6a_3 - a_3) = 0 \Rightarrow a_3 = -\frac{3}{5}a_2 = \frac{3}{10}$$

In conclusion, the coefficients of F_2 are

$$a_1 = \frac{1}{24}, \quad a_2 = -\frac{1}{2} \text{ and } a_3 = \frac{3}{10},$$

such that

$$F_2 = \int \frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2 dx$$

is the fourth conserved quantity for the KdV equation.

6. **Recursion operator** For a Bi-Hamiltonian system with two Poisson structures given by B_0, B_1, one defines a recursion operator $R = B_1 B_0^{-1}$, which takes one element of the hierarchy of equations to the next element. For the KdV equation with $B_0 = \partial_x$ and $B_1 = \partial_{xxx} + \frac{1}{3}(u\partial_x + \partial_x u)$, we get $B_0^{-1} = \partial_x^{-1}$, integration with respect to x. Write down the recursion operator. Apply it to u_x (the zero-th KdV flow) to obtain the first KdV flow. Now apply it to $uu_x + u_{xxx}$ to get (up to rescaling of t_2) the second KdV equation. What is the third KdV equation?

Solution:

The recursion operator is

$$R = B_1 B_0^{-1} = \partial_{xxx}\partial_x^{-1} + \frac{1}{3}u\partial_x\partial_x^{-1} + \frac{1}{3}\partial_x u\partial_x^{-1} = \partial_{xx} + \frac{2}{3}u + \frac{1}{3}u_x\partial_x^{-1}$$

Applying it to u_x gives

$$B_1 B_0^{-1} u_x = u_{xxx} + \frac{2}{3}uu_x + \frac{1}{3}u_x u = u_{xxx} + uu_x,$$

which is the KdV flow.

Applying it to $uu_x + u_{xxx}$, gives

$$\begin{aligned}
B_1 B_0^{-1}(uu_x + u_{xxx}) &= \partial_x(u_x^2 + uu_{xx}) + u_{5x} + \frac{2}{3}u^2 u_x + \frac{2}{3}uu_{xxx} + \frac{1}{3}u_x\partial_x^{-1}(\frac{1}{2}\partial_x(u^2)) + \frac{1}{3}u_x u_{xx} \\
&= 3u_x u_{xx} + uu_{xxx} + u_{5x} + \frac{5}{6}u^2 u_x + \frac{2}{3}uu_{xxx} + \frac{1}{3}u_x u_{xx} \\
&= \frac{5}{6}u^2 u_x + \frac{10}{3}u_x u_{xx} + \frac{5}{3}uu_{xxx} + u_{5x} \\
&= \frac{5}{3}\left(\frac{1}{2}u^2 u_x + 2u_x u_{xx} + uu_{xxx} + \frac{3}{5}u_{5x}\right)
\end{aligned}$$

This is clearly the second KdV equation with a rescaling of $t_2^* = \dfrac{5}{3}t_2$.

In order to find the third KdV equation we apply the recursion operator on $f = \frac{1}{2}u^2 u_x + 2u_x u_{xx} + uu_{xxx} + \frac{3}{5}u_{5x}$. This gives

$$\begin{aligned}
B_1 B_0^{-1}(f) = \partial_x(uu_x^2 &+ \frac{1}{2}u^2 u_{xx} + 2u_{xx}^2 + 2u_x u_{xxx} + u_x u_{xxx} + uu_{4x} + \frac{3}{5}u_{6x}) \\
&+ \frac{2}{3}\left(\frac{1}{2}u^3 u_x + 2uu_x u_{xx} + u^2 u_{xxx} + \frac{3}{5}uu_{5x}\right) \\
&+ \frac{1}{3}u_x\partial_x^{-1}\left(\frac{1}{2}u^2 u_x + 2u_x u_{xx} + uu_{xxx} + \frac{3}{5}u_{5x}\right)
\end{aligned}$$

Note that $(uu_{xx})_x = u_x u_{xx} + uu_{xxx}$, thus

$$
\begin{aligned}
B_1 B_0^{-1}(f) = {} & u_x^3 + 2uu_x u_{xx} + uu_x u_{xx} + \frac{1}{2}u^2 u_{xxx} + 4u_{xx}u_{xxx} + 3u_{xx}u_{xxx} \\
& \quad + 3u_x u_{4x} + u_x u_{4x} + uu_{5x} + \frac{3}{5}u_{7x} \\
& + \frac{1}{3}u^3 u_x + \frac{4}{3}uu_x u_{xx} + \frac{2}{3}u^2 u_{xxx} + \frac{2}{5}uu_{5x} \\
& + \frac{1}{3}u_x \left(\frac{1}{6}u^3 + \frac{1}{2}u_x^2 + uu_{xx} + \frac{3}{5}u_{4x}\right) \\
= {} & u_x^3 + 3uu_x u_{xx} + \frac{1}{2}u^2 u_{xxx} + 7u_{xx}u_{xxx} \\
& \quad + 4u_x u_{4x} + uu_{5x} + \frac{3}{5}u_{7x} \\
& + \frac{1}{3}u^3 u_x + \frac{4}{3}uu_x u_{xx} + \frac{2}{3}u^2 u_{xxx} + \frac{2}{5}uu_{5x} \\
& + \frac{1}{18}u^3 u_x + \frac{1}{6}u_x^3 + \frac{1}{3}uu_x u_{xx} + \frac{1}{5}u_x u_{4x} \\
= {} & \frac{7}{18}u^3 u_x + \frac{7}{6}u_x^3 + \frac{14}{3}uu_x u_{xx} + \frac{7}{6}u^2 u_{xxx} + 7u_{xx}u_{xxx} + \frac{21}{5}u_x u_{4x} + \frac{7}{5}uu_{5x} + \frac{3}{5}u_{7x} \\
= {} & \frac{7}{30}\left(\frac{5}{3}u^3 u_x + 5u_x^3 + 20uu_x u_{xx} + 5u^2 u_{xxx} + 30u_{xx}u_{xxx} + 18u_x u_{4x} + 6uu_{5x} + \frac{18}{7}u_{7x}\right)
\end{aligned}
$$

Thus, starting from $uu_x + u_{xxx}$, using the recursion operator, we get that the third KdV equation is (up to rescaling of t_3)

$$
\frac{5}{3}u^3 u_x + 5u_x^3 + 20uu_x u_{xx} + 5u^2 u_{xxx} + 30u_{xx}u_{xxx} + 18u_x u_{4x} + 6uu_{5x} + \frac{18}{7}u_{7x}
$$

7. Consider the function $U(x) = 2\partial_x^2 \ln\left(1 + e^{kx+\alpha}\right)$. Show that for a suitable k, $U(x)$ is a solution of the first member of the stationary KdV hierarchy (as you've already seen, it is the one-soliton solution):

$$6uu_x + u_{xxx} + c_0 u_x = 0.$$

(Note: it may be easier to define c_0 in terms of k, instead of the other way around)

Having accomplished this, let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots)}$. Determine the dependence of α on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations:

$$\begin{aligned} u_{t_1} &= 6uu_x + u_{xxx}, \\ u_{t_2} &= 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x}, \\ u_{t_3} &= 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u_{xx}u_{xxx} + 70u^2u_{xxx} + 42u_xu_{xxxx} + 14uu_{5x} + u_{7x}. \end{aligned}$$

Based on this, write down a guess for the one-soliton solution that solves the entire KdV hierarchy.

Solution:

$$U(x) = 2\partial_x \frac{ke^{kx+\alpha}}{1+e^{kx+\alpha}} = 2\frac{k^2 \mathrm{e}^{kx+\alpha}}{(\mathrm{e}^{kx+\alpha}+1)^2} = \frac{k^2}{2}\left(\frac{2}{e^{\frac{1}{2}(kx+\alpha)} + e^{-\frac{1}{2}(kx+\alpha)}}\right)^2 = \frac{k^2}{2}\mathrm{sech}^2\left(\frac{1}{2}(kx+\alpha)\right)$$

Using Wolfram Mathematica (file "EX 7 WV"), we get that

$$6UU_x + U_{xxx} = -4k^5 \mathrm{csch}(kx+\alpha)^3 \sinh(\frac{1}{2}(kx+\alpha))^4,$$

and

$$c_0 u_x = -4c_0 k^3 \mathrm{csch}(kx+\alpha)^3 \sinh(\frac{1}{2}(kx+\alpha))^4.$$

Therefore, if $c_0 = -k^2$, we get that U would be a solution to

$$6UU_x + U_{xxx} - k^2 u_x = 0$$

Say that $z = kx + \alpha(t_1, t_2, t_3, \ldots)$, using wolfram mathematica, the first KdV equation gives

$$-4\alpha_{t_1} k^2 \mathrm{csch}(z)^3 \sinh(\frac{1}{2}z)^4 = -4k^5 \mathrm{csch}(z)^3 \sinh(\frac{1}{2}z)^4 \Rightarrow \alpha_{t_1} = k^3,$$

the second KdV equation gives

$$-4\alpha_{t_2} k^2 \mathrm{csch}(z)^3 \sinh(\frac{1}{2}z)^4 = -4k^7 \mathrm{csch}(z)^3 \sinh(\frac{1}{2}z)^4 \Rightarrow \alpha_{t_2} = k^5,$$

and the third KdV eqaution gives

$$-4\alpha_{t_3} k^2 \mathrm{csch}(z)^3 \sinh(\frac{1}{2}z)^4 = -4k^9 \mathrm{csch}(z)^3 \sinh(\frac{1}{2}z)^4 \Rightarrow \alpha_{t_3} = k^7.$$

It is pretty clear that $\alpha_{t_j} = k^{1+2j}$ for $j = 1, 2\&3$ ($\alpha_{t_j} = k|c_0|^j$). Therefore we speculate that the same holds for the other t_j's. Also, since $\alpha_{t_j} = k^{1+2j}$, we know that

$$\alpha(t_1, t_2, t_3 \dots) = k^{1+2j} t_j + C(t_1, \dots, t_{j-1}, t_{j+1}, \dots).$$

Thus,

$$\alpha(t_1, t_2, t_3, \dots) = C + \sum_{j=1}^{\infty} k^{1+2j} t_j,$$

with C a constant. Therefore, if

$$\alpha(t_1, t_2, t_3, \dots) = k^3 t_1 + k^5 t_2 + k^7 t_3 + C,$$

$U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\dots)}$ is a solution to the first thee KdV equations.

Filling this into $U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\dots)}$ gives

$$U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\dots)} = 2\partial_x^2 \ln(1 + e^{kx+C+\sum_{j=1}^{\infty} k^{1+2j} t_j})$$

In conclusion, my guess of the one-soliton solution that solves the entire KdV hierarchy is,

$$u(x, t_1, t_2, t_3, \dots) = 2\partial_x^2 \ln(1 + e^{kx+C+\sum_{j=1}^{\infty} k^{1+2j} t_j})$$

8. **Warning: maple/mathematica-intensive.** Consider the function

$$U(x) = 2\partial_x^2 \ln\left(1 + e^{k_1 x + \alpha} + e^{k_2 x + \beta} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1 x + k_2 x + \alpha + \beta}\right).$$

Show that for a suitable k_1, k_2, $U(x)$ is a solution of the second member of the stationary KdV hierarchy:

$$30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x} + c_1(6uu_x + u_{xxx}) + c_0 u_x = 0.$$

(Note: it may be easier to define c_1, c_0 in terms of k_1 and k_2 instead of the other way around)

Having accomplished this, let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots),\beta=\beta(t_1,t_2,t_3,\ldots)}$. Determine the dependence of α and β on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations, given above.

Based on this, write down a guess for the two-soliton solution of the entire KdV hierarchy.

Solution:

I was unable to complete this exercise. My approach to this can be found in file "EX 8 WV". First I tried to see if something like exercise 7 happened: if we made linear combinations of the terms and divided out u_x or $6uu_x$, it would be constant. This was however a hand wavy approach and required some luck. Afterwards I tried to solve the equations using the function "Solve". This, however, gave me ugly expressions, which I didn't think to be the answer.