AMATH 573
Solitons and nonlinear waves
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Homework 5

due: December 2, 2022

1. Show that

$$X = \begin{pmatrix} -i\zeta & q \\ \pm q^* & i\zeta \end{pmatrix}, T = \begin{pmatrix} -i\zeta^2 \mp \frac{i}{2}|q|^2 & q\zeta + \frac{i}{2}q_x \\ \pm q^*\zeta \mp \frac{i}{2}q_x^* & i\zeta^2 \pm \frac{i}{2}|q|^2 \end{pmatrix}$$

are Lax Pairs for the Nonlinear Schrödinger equations

$$iq_t = -\frac{1}{2}q_{xx} \pm |q|^2 q.$$

Here the top (bottom) signs of one matrix correspond to the top (bottom) signs of the other. In other words, show that the X, T with the top (bottom) sign are a Lax pair for the Nonlinear Schrödinger equation with the top (bottom) sign.

Solution:

We want to calculate the compatibility condition $X_t + XT = T_x + TX$, therefore, assuming ζ is independent of x and t. We will prove why time independence must hold, assume therefore that $\zeta_t \neq 0$. The following matrices will be useful:

$$X_t = \begin{pmatrix} -i\zeta_t & q_t \\ \pm q_t^* & i\zeta_t \end{pmatrix},$$

$$T_x = \begin{pmatrix} \mp i\frac{1}{2}(|q|^2)_x & q_x\zeta + \frac{i}{2}q_{xx} \\ \pm q_x^*\zeta \mp \frac{i}{2}q_{xx}^* & \pm i\frac{1}{2}(|q|^2)_x \end{pmatrix},$$

$$\begin{aligned} XT &= \begin{pmatrix} -\zeta^3 \mp \frac{1}{2}|q|^2\zeta \pm q^*q\zeta \mp \frac{i}{2}q_x^*q & -iq\zeta^2 + \frac{1}{2}q_x\zeta + i\zeta^2 q \pm \frac{i}{2}|q|^2 q \\ \mp i\zeta^2 q^* - \frac{i}{2}|q|^2 q^* \pm iq^*\zeta^2 \pm \frac{1}{2}q_x^*\zeta & \pm q^*q\zeta \pm \frac{i}{2}q^*q_x - \zeta^3 \mp \frac{1}{2}|q|^2\zeta \end{pmatrix} \\ &= \begin{pmatrix} -\zeta^3 \mp \frac{1}{2}|q|^2\zeta \pm q^*q\zeta \mp \frac{i}{2}q_x^*q & \frac{1}{2}q_x\zeta \pm \frac{i}{2}|q|^2 q \\ -\frac{i}{2}|q|^2 q^* \pm \frac{1}{2}q_x^*\zeta & \pm q^*q\zeta \pm \frac{i}{2}q^*q_x - \zeta^3 \mp \frac{1}{2}|q|^2\zeta \end{pmatrix} \end{aligned}$$

$$\begin{aligned} TX &= \begin{pmatrix} -\zeta^3 \mp \frac{1}{2}|q|^2\zeta \pm q^*q\zeta \pm \frac{i}{2}q_x q^* & -i\zeta^2 q \mp \frac{i}{2}|q|^2 q + iq\zeta^2 - \frac{1}{2}q_x\zeta \\ \mp iq^*\zeta^2 \mp \frac{1}{2}q_x^*\zeta \pm i\zeta^2 q^* + \frac{i}{2}|q|^2 q^* & \pm qq^*\zeta \mp \frac{i}{2}qq_x^* - \zeta^3 \mp \frac{1}{2}|q|^2\zeta \end{pmatrix} \\ &= \begin{pmatrix} -\zeta^3 \mp \frac{1}{2}|q|^2\zeta \pm q^*q\zeta \pm \frac{i}{2}q_x q^* & \mp \frac{i}{2}|q|^2 q - \frac{1}{2}q_x\zeta \\ \mp \frac{1}{2}q_x^*\zeta + \frac{i}{2}|q|^2 q^* & \pm qq^*\zeta \mp \frac{i}{2}qq_x^* - \zeta^3 \mp \frac{1}{2}|q|^2\zeta \end{pmatrix} \end{aligned}$$

Therefore, if

$$X_t + XT = T_x + TX$$
$$\Rightarrow$$
$$\begin{pmatrix} -i\zeta_t - \zeta^3 \mp \frac{1}{2}|q|^2\zeta \pm q^*q\zeta \mp \frac{i}{2}q_x^*q & \frac{1}{2}q_x\zeta \pm \frac{i}{2}|q|^2q + q_t \\ -\frac{i}{2}|q|^2q^* \pm \frac{1}{2}q_x^*\zeta \pm q_t^* & i\zeta_t \pm q^*q\zeta \pm \frac{i}{2}q^*q_x - \zeta^3 \mp \frac{1}{2}|q|^2\zeta \end{pmatrix}$$
$$=$$
$$\begin{pmatrix} -\zeta^3 \mp \frac{1}{2}|q|^2\zeta \pm q^*q\zeta \pm \frac{i}{2}q_xq^* \mp i\frac{1}{2}(|q|^2)_x & \mp\frac{i}{2}|q|^2q - \frac{1}{2}q_x\zeta + q_x\zeta + \frac{i}{2}q_{xx} \\ \mp\frac{1}{2}q_x^*\zeta + \frac{i}{2}|q|^2q^* \pm q_x^*\zeta \mp \frac{i}{2}q_{xx}^* & \pm qq^*\zeta \mp \frac{i}{2}qq_x^* - \zeta^3 \mp \frac{1}{2}|q|^2\zeta \pm i\frac{1}{2}(|q|^2)_x \end{pmatrix}$$
$$\Rightarrow$$
$$\begin{pmatrix} -i\zeta_t + \frac{1}{2}(q^*q)_x & q_t \\ \pm q_t^* & i\zeta_t + \frac{1}{2}(q^*q)_x \end{pmatrix}$$
$$=$$
$$\begin{pmatrix} \frac{1}{2}(|q|^2)_x & \mp i|q|^2q + \frac{i}{2}q_{xx} \\ i|q|^2q^* \mp \frac{i}{2}q_{xx}^* & \frac{1}{2}(|q|^2)_x \end{pmatrix}$$
$$\Rightarrow$$
$$\begin{pmatrix} \frac{1}{2}(q^*q)_x & iq_t \\ iq_t^* & \frac{1}{2}(q^*q)_x \end{pmatrix}$$
$$=$$
$$\begin{pmatrix} \frac{1}{2}(|q|^2)_x & -\frac{1}{2}q_{xx} \pm |q|^2q \\ \frac{1}{2}q_{xx}^* \mp |q|^2q^* & \frac{1}{2}(|q|^2)_x \end{pmatrix}.$$

Note that q^* denotes the complex conjugate. Therefore $qq^* = |q|^2$, $iq_t^* = (-iq_t)^*$ and $(|q|^2)^* = |q|^2$, thus we get

$$\begin{pmatrix} \zeta_t & iq_t \\ (iq_t)^* & \zeta_t \end{pmatrix}$$
$$=$$
$$\begin{pmatrix} 0 & -\frac{1}{2}q_{xx} \pm |q|^2q \\ (-\frac{1}{2}q_{xx} \pm |q|^2q)^* & 0 \end{pmatrix}.$$

This equality holds when ζ_t is time independent and

$$iq_t = -\frac{1}{2}q_{xx} \pm |q|^2q.$$

In conclusion, X and T are Lax Pairs for the Nonlinear Schrödinger equations.

2. Let $\psi_n = \psi_n(t)$, $n \in \mathbb{Z}$. Consider the difference equation

$$\psi_{n+1} = X_n \psi_n,$$

and the differential equation

$$\frac{\partial \psi_n}{\partial t} = T_n \psi_n.$$

What is the compatibility condition of these two equations? Using this result, show that

$$X_n = \begin{pmatrix} z & q_n \\ q_n^* & 1/z \end{pmatrix}, T_n = \begin{pmatrix} iq_n q_{n-1}^* - \frac{i}{2}(1/z - z)^2 & \frac{i}{z} q_{n-1} - izq_n \\ -izq_{n-1}^* + \frac{i}{z} q_n^* & -iq_n^* q_{n-1} + \frac{i}{2}(1/z - z)^2 \end{pmatrix}$$

is a Lax Pair for the semi-discrete equation

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - |q_n|^2(q_{n+1} + q_{n-1})$$

Note that this is a discretization of the NLS equation. It is known as the Ablowitz-Ladik lattice. It is an integrable discretization of NLS. For numerical purposes, it is far superior in many ways to the "standard" discretization of NLS:

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - 2|q_n|^2 q_n.$$

Solution:

First we take the partial derivative to t of the first equation:

$$\begin{aligned} \frac{\partial \psi_{n+1}}{\partial t} &= \frac{\partial}{\partial t}(X_n \psi_n) \\ T_{n+1}\psi_{n+1} &= (X_n)_t \psi_n + X_n \frac{\partial \psi_n}{\partial t} \\ T_{n+1} X_n \psi_n &= ((X_n)_t + X_n T_n)\psi_n. \end{aligned}$$

Therefore we want

$$((X_n)_t + X_n T_n - T_{n+1} X_n)\psi_n = 0.$$

Assuming that solutions $\psi_n(t)$ form a complete set in a suitable space, since this operator, acting on all ψ_n, gives a zero result, the operator must be zero. Thus the compatibility condition is

$$(X_n)_t = T_{n+1} X_n - X_n T_n.$$

Assume z is independant of x and t (proof why z must be time independent is similar to exercise 1). In order to prove that the given matrices are a Lax Pair for the first discretization of the NLS equation, we calculate the matrices:

$$(X_n)_t = \begin{pmatrix} 0 & (q_n)_t \\ (q_n^*)_t & 0 \end{pmatrix}$$

$$T_{n+1}X_n = \begin{pmatrix} izq_{n+1}q_n^* - \frac{i}{2}z(1/z-z)^2 + \frac{i}{z}q_nq_n^* - izq_n^*q_{n+1} & iq_{n+1}q_n^*q_n - \frac{i}{2}q_n(1/z-z)^2 + \frac{i}{z^2}q_n - iq_{n+1} \\ -iz^2q_n^* + iq_{n+1}^* - iq_{n+1}^*q_nq_n^* + \frac{i}{2}q_n^*(1/z-z)^2 & -izq_nq_n^* + \frac{i}{z}q_{n+1}^*q_n - \frac{i}{z}q_{n+1}^*q_n + \frac{i}{2z}(1/z-z)^2 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{i}{2}z(1/z-z)^2 + \frac{i}{z}|q_n|^2 & iq_{n+1}|q_n|^2 - \frac{i}{2}q_n(1/z-z)^2 + \frac{i}{z^2}q_n - iq_{n+1} \\ -iz^2q_n^* + iq_{n+1}^* - iq_{n+1}^*|q_n|^2 + \frac{i}{2}q_n^*(1/z-z)^2 & -iz|q_n|^2 + \frac{i}{2z}(1/z-z)^2 \end{pmatrix}$$

and

$$X_nT_n = \begin{pmatrix} izq_nq_{n-1}^* - \frac{i}{2}z(1/z-z)^2 - izq_{n-1}^*q_n + \frac{i}{z}|q_n|^2 & iq_{n-1} - iz^2q_n - i|q_n|^2q_{n-1} + \frac{i}{2}q_n(1/z-z)^2 \\ i|q_n|^2 - \frac{i}{2}q_n^*(1/z-z)^2 - iq_{n-1}^* + \frac{i}{z^2}q_n^* & \frac{i}{z}q_{n-1}q_n^* - iz|q_n|^2 - \frac{i}{z}q_n^*q_{n-1} + \frac{i}{2z}(1/z-z)^2 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{i}{2}z(1/z-z)^2 + \frac{i}{z}|q_n|^2 & iq_{n-1} - iz^2q_n - i|q_n|^2q_{n-1} + \frac{i}{2}q_n(1/z-z)^2 \\ i|q_n|^2 - \frac{i}{2}q_n^*(1/z-z)^2 - iq_{n-1}^* + \frac{i}{z^2}q_n^* & -iz|q_n|^2 + \frac{i}{2z}(1/z-z)^2 \end{pmatrix}$$

Suppose that

$$\begin{aligned} F &= -i(q_{n+1} + q_{n-1}) + iq_n\left((1/z^2 + z^2) - (1/z-z)^2\right) + i|q_n|^2(q_{n+1} + q_{n-1}) \\ &= -i(q_{n+1} + q_{n-1}) + 2iq_n + i|q_n|^2(q_{n+1} + q_{n-1}) \end{aligned}$$

If we calculate the difference of the last two matrices, we get

$$T_{n+1}X_n - X_nT_n$$

$$=$$

$$\begin{pmatrix} 0 & F \\ (F)^* & 0 \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} 0 & -i(q_{n+1} + q_{n-1}) + 2iq_n + i|q_n|^2(q_{n+1} + q_{n-1}) \\ \left(-i(q_{n+1} + q_{n-1}) + 2iq_n + i|q_n|^2(q_{n+1} + q_{n-1})\right)^* & 0 \end{pmatrix}$$

Since $(q_n^*)_t = ((q_n)_t)^*$, we have that the compatibility condition gives that

$$(q_n)_t = -i(q_{n+1} + q_{n-1}) + 2iq_n + i|q_n|^2(q_{n+1} + q_{n-1}),$$

and thus

$$i(q_n)_t = q_{n+1} - 2q_n + q_{n-1} - |q_n|^2(q_{n+1} + q_{n-1})$$

In conclusion, the given pair X_n and T_n is a Lax Pair for the first given discretization of the NLS equation.

3. For the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ with initial condition $u(x, 0) = 0$ for $x \in (-\infty, -L) \cup (L, \infty)$, and $u(x, 0) = d$ for $x \in (-L, L)$, with L and d both positive, consider the forward scattering problem.

   - Find $a(k)$, for all time t.
   - Knowing that the number of solitons emanating from the initial condition is the number of zeros of $a(k)$ on the positive imaginary axis (*i.e.*, $k = i\kappa$, with $\kappa > 0$), discuss how many solitons correspond to the given initial condition, depending on the value of $2L^2d$. You might want to use Maple, Mathematica or Matlab for this.
   - What happens for $d < 0$?
   - In the limit $L \to 0$, but $2dL = \alpha$, $u(x, 0) \to \alpha\delta(x)$. What happens to $a(k)$ when you take this limit? Discuss.

Solution:

- Suppose we start with variable $\tilde{x}$. Substitute $x = -\tilde{x}$ (this has no effect on this exercise), we then have that the KdV equation becomes

$$u_t = 6uu_x + u_{xxx},$$

the KdV equation we used in class to get the equations for the scattering data $a(k)$. The initial conditions don't really change.

We now calculate $a(k, 0)$, but since we know that a does not change over time, it will be $a(k)$ for all time t. Since

$$\alpha(x) = u(x, 0) = \begin{cases} d & x \in (-L, L) \\ 0 & \text{otherwise} \end{cases}$$

we want to construct the scattering data for the ordinary differential equation at the initial timestep

$$\psi_{xx} + (\alpha(x) + k^2)\psi = 0.$$

We start with finding ϕ and φ. First, we determine ϕ. By definition, as $x \to -\infty$, we have that

$$\phi \to e^{-ikx} \text{ and } \bar{\phi} \to e^{ikx}.$$

We thus know that $\phi = e^{-ikx}$ and $\bar{\phi} = e^{ikx}$ for $x < -L$, since this is the solution to the given ODE for $x < -L$,

$$\bar{\phi}_{xx} + k^2\bar{\phi} = 0,$$

and the fact that $\bar{\phi}(x, k) = \phi(x, -k)$. On the other hand we need to look at domain $(-L, L)$ and (L, ∞). We calculate $\phi(x, k)$ for these, respective, PDE's:

$$\phi_{xx} + (d + k^2)\phi = 0 \text{ and } \phi_{xx} + k^2\phi = 0$$

We get that the solution looks like

$$\phi = \begin{cases} e^{-ikx} & x < -L \\ c_1 e^{i\sqrt{d+k^2}x} + c_2 e^{-i\sqrt{d+k^2}x} & -L < x < L \\ c_3 e^{ikx} + c_4 e^{-ikx} & x > L \end{cases}$$

We want the eigenfunctions to be continuous, therefore we take the limits to get that

$$\begin{cases} e^{ikL} & = c_1 e^{-i\sqrt{d+k^2}L} + c_2 e^{i\sqrt{d+k^2}L} \\ c_1 e^{i\sqrt{d+k^2}L} + c_2 e^{-i\sqrt{d+k^2}L} & = c_3 e^{ikL} + c_4 e^{-ikL} \end{cases}.$$

We also want a condition on the derivative of the eigenfunctions. We can obtain the derivative of the eigenfunctions in $-L$ and L by integrating the ODE over a small space around $-L$ and L:

$$\begin{aligned} &\psi_{xx} + (\alpha(x) + k^2)\psi && = 0 \\ \Rightarrow &\int_{-L-\epsilon}^{-L+\epsilon} \psi_{xx} + (\alpha(x) + k^2)\psi dx && = 0 \\ \Rightarrow &\psi_x|_{-L-\epsilon}^{-L+\epsilon} + \int_{-L-\epsilon}^{-L+\epsilon} \alpha(x)\psi dx + k^2 \int_{-L-\epsilon}^{-L+\epsilon} \psi dx && = 0 \\ \Rightarrow &\psi_x(-L+\epsilon) - \psi_x(-L-\epsilon) && = 0 \end{aligned} \tag{1}$$

We took $\epsilon \to 0$ for the last two integrals going into the last line. They equal zero because of continuity in ψ, the finite jump and the fact that the integrals indicate the area under the curve. This conditions just indicates that the derivative also needs to be continuous. It thus gives us (as $\epsilon \to 0$), for eigenfunction ϕ,

$$c_1 i\sqrt{d+k^2} e^{-i\sqrt{d+k^2}L} - c_2 i\sqrt{d+k^2} e^{i\sqrt{d+k^2}L} = -ike^{ikL}.$$

We now use the same method to get the derivative of ψ in the other point of interest (L):

$$\begin{aligned} &\psi_{xx} + (\alpha(x) + k^2)\psi && = 0 \\ \Rightarrow &\int_{L-\epsilon}^{L+\epsilon} \psi_{xx} + (\alpha(x) + k^2)\psi dx && = 0 \\ \Rightarrow &\psi_x|_{L-\epsilon}^{L+\epsilon} + \int_{L-\epsilon}^{L+\epsilon} \alpha(x)\psi dx + k^2 \int_{L-\epsilon}^{L+\epsilon} \psi dx && = 0 \\ \Rightarrow &\psi_x(L+\epsilon) - \psi_x(L-\epsilon) && = 0 \end{aligned} \tag{2}$$

This gives us

$$ikc_3 e^{ikL} - ikc_4 e^{-ikL} = c_1 i\sqrt{d+k^2} e^{i\sqrt{d+k^2}L} - c_2 i\sqrt{d+k^2} e^{-i\sqrt{d+k^2}L}.$$

Using Wolfram Mathematica ("EX3 WV.nb"), we solve the system

$$\begin{cases} c_1 i\sqrt{d+k^2}e^{-i\sqrt{d+k^2}L} - c_2 i\sqrt{d+k^2}e^{i\sqrt{d+k^2}L} & = -ike^{ikL} \\ c_1 e^{-i\sqrt{d+k^2}L} + c_2 e^{i\sqrt{d+k^2}L} & = e^{ikL} \\ c_1 i\sqrt{d+k^2}e^{i\sqrt{d+k^2}L} - c_2 i\sqrt{d+k^2}e^{-i\sqrt{d+k^2}L} & = ikc_3 e^{ikL} - ikc_4 e^{-ikL} \\ c_1 e^{i\sqrt{d+k^2}L} + c_2 e^{-i\sqrt{d+k^2}L} & = c_3 e^{ikL} + c_4 e^{-ikL} \end{cases}$$

Using the asymptotic behavior and the same method used in finding ϕ, we find the form for φ:

$$\varphi = \begin{cases} b_3 e^{ikx} + b_4 e^{-ikx} & x < -L \\ b_1 e^{i\sqrt{d+k^2}x} + b_2 e^{-i\sqrt{d+k^2}x} & -L < x < L \\ e^{ikx} & x > L \end{cases}$$

Here we clearly have that $\phi(-x,k) = \varphi(x,k)$, it has the same conditions (function and its derivative are continuous). Thus we have that $b_3 = c_4$, $b_4 = c_3$, $b_1 = c_2$ and $b_2 = c_1$. In Wolfram Mathematica, we calculated the coefficients using the conditions instead of using this. These equalities do hold.

We can now calculate $a(k)$ by using the Wronskian relations over all the parts of the x-axis. If $x > L$:

$$\begin{aligned} a(k) &= \frac{W(\varphi,\phi)}{2ik} \\ &= \frac{1}{2ik}W(e^{ikx}, c_3e^{ikx} + c_4e^{-ikx}) \\ &= \frac{c_4}{2ik}W(e^{ikx}, e^{-ikx}) \\ &= c_4 \end{aligned}$$

If $x < -L$

$$\begin{aligned} a(k) &= \frac{W(\varphi,\phi)}{2ik} \\ &= \frac{1}{2ik}W(b_3e^{ikx} + b_4e^{-ikx}, e^{-ikx}) \\ &= b_3 = c_4 \end{aligned}$$

If $-L < x < L$, note that the Wronskian is bilinear.

$$\begin{aligned}
a(k) &= \frac{W(\varphi,\phi)}{2ik} \\
&= \frac{1}{2ik} W(b_1 e^{i\sqrt{d+k^2}x} + b_2 e^{-i\sqrt{d+k^2}x}, c_1 e^{i\sqrt{d+k^2}x} + c_2 e^{-i\sqrt{d+k^2}x}) \\
&= \frac{1}{2ik}\left(W(b_2 e^{-i\sqrt{d+k^2}x}, c_1 e^{i\sqrt{d+k^2}x}) + W(b_1 e^{i\sqrt{d+k^2}x}, c_2 e^{-i\sqrt{d+k^2}x})\right) \\
&= \frac{1}{2ik}\left(-2b_2c_1 i\sqrt{d+k^2} + 2b_1c_2 i\sqrt{d+k^2}\right) \\
&= \frac{\sqrt{d+k^2}}{k}(b_1c_2 - b_2c_1) \\
&= \frac{\sqrt{d+k^2}}{k}(c_2^2 - c_1^2)
\end{aligned}$$

Using Wolfram Mathematica, we verified that these are all equal, thus we know that this is our $x-$independent $a(k)$.

Since $a(k)$ is independent of time, we have that $a(k)$ for all time t is

$$a(k) = \frac{e^{2ikL}}{2k\sqrt{d+k^2}}\left(2k\sqrt{d+k^2}\cos(2\sqrt{d+k^2}L) - i(d+2k^2)\sin(2\sqrt{d+k^2}L)\right)$$

- We now want to look at the roots of $a(k)$ on the positive imaginary axis, therefore we look at the numerator of $a(i\kappa)$. This narrows our search down to

  $$2\kappa\sqrt{d-\kappa^2}\cos(2\sqrt{d-\kappa^2}L) - (d-2\kappa^2)\sin(2\sqrt{d-\kappa^2}L) = 0$$

  Which gives us

  $$\tan\left(2\sqrt{d-\kappa^2}L\right) = \frac{2\kappa\sqrt{d-\kappa^2}}{d-2\kappa^2}, \tag{3}$$

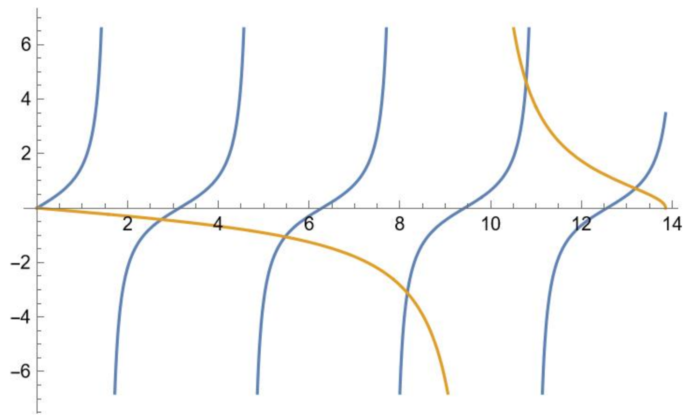
  We now substitute

  $$x = 2\sqrt{d-\kappa^2}L \Leftrightarrow \sqrt{d-\left(\frac{x}{2L}\right)^2} = \kappa, \qquad (\kappa > 0 \Rightarrow x > 0, \quad \text{if } x \in \mathbb{R})$$

  to get

  $$\tan(x) = -\frac{Lx\sqrt{4d-\left(\frac{x}{L}\right)^2}}{2dL^2 - x^2} = g(x) \tag{4}$$

  Assuming $d > 0$, $g(x)$ exists if $0 < x < 2\sqrt{d}L$, with $x \neq \sqrt{2dL^2}$, where its vertical asymptote is. Note that the positive restriction holds because we want $\kappa > 0$. The periodicity of $\tan(x)$ is π. We always have one point where the equality is fulfilled: $x = 0$ $(\kappa = \sqrt{d})$. Plotting both functions gives



Note here that almost every tangent (blue) crosses $g(x)$ (yellow) once, except for one tangent which *might* cross it twice. $g(x)$ does not cross a blue curve twice if its asymptote is at $x = (2l+1)\frac{\pi}{2}$, with $l \in \mathbb{N}$. Also note that it might or might not hit the last tangent in $[0, 2\sqrt{d}L]$. If we look at periods starting at $x = 0$, even if 0.99...8 of the last period in $[0, 2\sqrt{d}L]$ has passed, $g(x)$ won't cross the last tangent. That is why we will use the floor function $\lfloor \cdot \rfloor$.

We now calculate how many points fulfill equality (4). We simply do this by calculating how many tangents are fully in $[0, 2\sqrt{d}L]$. An example of a full tangent here is defined as $\tan x$ for $x \in [0, \pi]$. Note that we always have at least one point where the equality holds: $x = 0$. We now calculate how many full tangents are in $[0, 2\sqrt{d}L]$ by

$$\left\lfloor \frac{2\sqrt{d}L}{\pi} \right\rfloor .$$

If we now have that the asymptote of function $g(x)$ is an odd multiple of $\frac{\pi}{2}$, then asymptotes collide and $a(k)$ has a root less. Lastly, note that $2\sqrt{d}L = \sqrt{2}\sqrt{2dL^2}$

In conclusion:

* If $2L^2 d = \left((2l+1)\frac{\pi}{2}\right)^2$, for $l \in \mathbb{N}$,

$$\text{Number of roots: } \left\lfloor \frac{\sqrt{2}}{\pi}\sqrt{2dL^2} \right\rfloor + 1, \text{ or } \left\lfloor \frac{2l+1}{\sqrt{2}} \right\rfloor + 1.$$

* If $2L^2 d \neq \left((2l+1)\frac{\pi}{2}\right)^2$, for $l \in \mathbb{N}$,

$$\text{Number of roots: } \left\lfloor \frac{\sqrt{2}}{\pi}\sqrt{2dL^2} \right\rfloor + 2$$

(A check: the figure was made with $d = 3$ and $L = 4$. This gives us 6 roots of $a(k)$, as we can see.)

- If $d < 0$, we have that $\sqrt{d - \kappa^2}$ is an imaginary number with no real component, thus from (3), we get

$$\tan\left(2\sqrt{|d-\kappa^2|}Li\right) = \frac{2\kappa\sqrt{|d-\kappa^2|}}{d - 2\kappa^2} i$$

Using the exponential form of the tangent, we get

$$\frac{e^{-2\sqrt{|d-\kappa^2|}L} - e^{2\sqrt{|d-\kappa^2|}L}}{i(e^{-2\sqrt{|d-\kappa^2|}L} + e^{2\sqrt{|d-\kappa^2|}L})} = \frac{2\kappa\sqrt{|d-\kappa^2|}}{d-2\kappa^2}i.$$

Thus,

$$\frac{e^{-2\sqrt{|d-\kappa^2|}L} - e^{2\sqrt{|d-\kappa^2|}L}}{e^{-2\sqrt{|d-\kappa^2|}L} + e^{2\sqrt{|d-\kappa^2|}L}} = \frac{2\kappa\sqrt{|d-\kappa^2|}}{2\kappa^2-d}.$$

Since the right side is positive ($\kappa > 0$), we must have that the left side is positive, this is true when

$$e^{-2\sqrt{|d-\kappa^2|}L} > e^{2\sqrt{|d-\kappa^2|}L} \Rightarrow 4\sqrt{|d-\kappa^2|} < 0$$

which is not possible. In conclusion, if $d < 0$, we have no roots of $a(i\kappa)$ (with $\kappa > 0$) and thus no soliton solutions!

- As a reminder,

$$a(k) = \frac{e^{2ikL}}{2k\sqrt{d+k^2}}\left(2k\sqrt{d+k^2}\cos(2\sqrt{d+k^2}L) - i(d+2k^2)\sin(2\sqrt{d+k^2}L)\right)$$

If $2dL = \alpha$, then $d = \frac{\alpha}{2L}$. Substituting this and using Wolfram Mathematica to calculate the limit, we get

$$a(k) = 1 - \frac{i\alpha}{2k} = \frac{\alpha + 2ik}{2ik},$$

which is the same thing as seen in class. In conclusion $a(k)$ becomes the scattering data for when we immediately use $u(x,0) = \alpha\delta(x)$.

A little about Bäcklund transformations. We have occasionally name-dropped Bäcklund transformations: transformations from one nonlinear equation to another, providing ways to link the solutions of the equations. For instance, the Cole-Hopf transformation is a Bäcklund transformation linking the Burgers equation to the heat equation. Similarly, the Miura transformation links the KdV and mKdV equations. Below you'll work with two more Bäcklund transformations.

4. **The Liouville equation**. Consider the horribly nonlinear[1] PDE

$$u_{xy} = e^u,$$

known as Liouville's equation. Consider the transformation

$$\begin{aligned} v_x &= -u_x + \sqrt{2}e^{(u-v)/2}, \\ v_y &= u_y - \sqrt{2}e^{(u+v)/2}, \end{aligned}$$

where $u(x, y)$ satisfies Liouville's equation above.

(a) Find an equation satisfied by $v(x, y)$: $v_{xy} = \ldots$. Your right-hand side cannot have any u's. Those should all be eliminated.

(b) Write down the general solution for $v(x, y)$ from the equation you obtained.

(c) Use this solution for v in your Bäcklund transformation and solve for u, obtaining the general solution of the Liouville equation!

Solution:

(a) Deriving the first part of the Bäcklund transformation by y and the second one by x gives

$$\begin{aligned} v_{xy} &= -u_{xy} + \frac{u_y - v_y}{2}\sqrt{2}e^{(u-v)/2}, \\ v_{yx} &= u_{yx} - \frac{u_x + v_x}{2}\sqrt{2}e^{(u+v)/2}. \end{aligned}$$

Assuming these functions are smooth enough and thus the order of the partial derivatives can be switched. Summing these two and substituting v_y and v_x gives us

$$\begin{aligned} 2v_{xy} &= \frac{u_y - u_y + \sqrt{2}e^{(u+v)/2}}{2}\sqrt{2}e^{(u-v)/2} - \frac{u_x - u_x + \sqrt{2}e^{(u-v)/2}}{2}\sqrt{2}e^{(u+v)/2} \\ &= e^u - e^u \\ &= 0. \end{aligned}$$

Thus

$$v_{xy} = 0$$

[1]Technical term.

(b) The general solution for $v(x,y)$ is then

$$v(x,y) = f(x) + g(y),$$

with f and g differentiable functions.

(c) Filling in the general solution of v into the Bäcklund transformation, gives

$$f'(x) = -u_x + \sqrt{2}e^{-(f(x)+g(y))/2}e^{u/2},$$
$$g'(y) = u_y - \sqrt{2}e^{(f(x)+g(y))/2}e^{u/2}.$$

Say that $w = e^{-u/2}$, then $u = -2\ln(w)$ and $u_x = -2\frac{w_x}{w}$. We then get the ODEs

$$f'(x) = 2\frac{w_x}{w} + \sqrt{2}e^{-(f(x)+g(y))/2}\frac{1}{w},$$
$$g'(y) = -2\frac{w_y}{w} - \sqrt{2}e^{(f(x)+g(y))/2}\frac{1}{w}.$$

This gives

$$f'(x)w - 2w_x = \sqrt{2}e^{-(f(x)+g(y))/2},$$
$$g'(y)w + 2w_y = -\sqrt{2}e^{(f(x)+g(y))/2}.$$

Multiplying by $-e^{-f(x)/2}$ and $e^{g(y)/2}$ respectively gives,

$$2(e^{-f(x)/2}w)_x = -\sqrt{2}e^{-(f(x)+g(y)/2)},$$
$$2(e^{g(y)/2}w)_y = -\sqrt{2}e^{f(x)/2+g(y)}.$$

Integrating this gives

$$\sqrt{2}e^{-f(x)/2}w = -e^{-g(y)/2}\int_0^x e^{-f(s)}ds + \tilde{C}_1(y),$$
$$\sqrt{2}e^{g(y)/2}w = -e^{f(x)/2}\int_0^y e^{g(s)}ds + \tilde{C}_2(x).$$

Thus,

$$w = -\frac{e^{(f(x)-g(y))/2}}{\sqrt{2}}\int_0^x e^{-f(s)}ds + C_1(y),$$
$$w = -\frac{e^{(f(x)-g(y))/2}}{\sqrt{2}}\int_0^y e^{g(s)}ds + C_1(x).$$

Thus,

$$w = -\frac{e^{(f(x)-g(y))/2}}{\sqrt{2}}\left(\int_0^x e^{-f(s)}ds + \int_0^y e^{g(s)}ds\right) + C,$$

with C constant.

In conclusion,

$$u = -2\ln\left(C - \frac{e^{(f(x)-g(y))/2}}{\sqrt{2}}\left(\int_0^x e^{-f(s)}ds + \int_0^y e^{g(s)}ds\right)\right).$$

5. **The sine-Gordon equation.** Consider the sine-Gordon equation

$$u_{xt} = \sin u,$$

also horribly nonlinear.

(a) Show that the transformation

$$v_x = u_x + 2\sin\frac{u+v}{2},$$
$$v_t = -u_t - 2\sin\frac{u-v}{2},$$

is an *auto-Bäcklund transformation* for the sine-Gordon equation. In other words, v satisfies the same equation as u.

(b) Let $u(x,t)$ be the simplest[2] solution of the sine-Gordon equation. With this $u(x,y)$ solve the auto-Bäcklund transformation for $v(x,t)$, to find a more complicated solution of the sine-Gordon equation. Congratulations! You just found the one-soliton solution of the sine-Gordon equation.

Solution:

(a) Deriving the first part of the transformation by t and the second part by x, gives

$$v_{xt} = u_{xt} + (u_t + v_t)\cos\frac{u+v}{2},$$
$$v_{tx} = -u_{tx} - (u_x - v_x)\cos\frac{u-v}{2},$$

Suppose that u is sufficiently smooth such that the partial derivatives can change place. Summing these two and substituting u_x and u_t gives

$$2v_{xt} = -2\sin(\frac{u-v}{2})\cos(\frac{u+v}{2}) + 2\sin(\frac{u+v}{2})\cos(\frac{u-v}{2}) = 2\sin(\frac{u+v}{2} - \frac{u-v}{2}) = 2\sin(v)$$

Thus we have that

$$v_{xt} = \sin(v),$$

which is the sine-Gordon equation.

(b) The simpelest solution of the sine-Gordon equation is

$$u = 0.$$

Filling this into the auto-Bäcklund transformation gives

$$v_x = 2\sin\frac{v}{2},$$
$$v_t = 2\sin\frac{v}{2}.$$

[2] I mean it.

Since both equations look alike, we only solve one (using Wolfram Mathematica ("EX5 WV.nb")):

$$v_x = 2\sin(\frac{v}{2}) \Rightarrow \int \frac{dv}{\sin(\frac{v}{2})} = \int 2dx \Rightarrow 2\ln\left(\tan(v/4)\right) = 2x + \tilde{C}(t)$$

Thus, $v = 4\tan^{-1}(C_1(t)e^x)$ and $v = 4\tan^{-1}(C_2(x)e^t)$, therefore it is clear to see that

$$v = 4\tan^{-1}(ce^{x+t}),$$

with c a constant. In conclusion, using the auto-Bäcklund transformation, we get that a solution to the sine-Gordon equation is

$$v(x,t) = 4\tan^{-1}(ce^{x+t}).$$