# AMATH 573
# Solitons and nonlinear waves
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# **Homework 1**

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1. The Korteweg-deVries (KdV) equation

$$u_t = uu_x + u_{xxx} \tag{1}$$

is often written with different coefficients. By using a scaling transformation on all variables (dependent and independent), show that the choice of the coefficients is irrelevant: by choosing a suitable scaling, we can use any coefficients we please. Can you say the same for the modified KdV (mKdV) equation

$$u_t = u^2 u_x + u_{xxx} \tag{2}$$

### Solution:

Suppose $v$ is a solution to the KdV equation (1). By choosing the right scaling transformations $v(ax, t), v(x, bt)$ and $cv(x, t)$, with $a, b, c \in \mathbb{R}_0$, we want $cv(ax, bt)$ to be a solution of $Au_t = Buu_x + Cu_{xxx}$ with $A, B, C \in \mathbb{R}_0$ randomly fixed. By filling in $cv(ax, bt)$, we get

$$Abcv_t = Bac^2 vv_x + Ca^3 cv_{xxx} \Rightarrow Abv_t = Bacvv_x + Ca^3 v_{xxx}$$

By now choosing

$$\begin{cases} a = \frac{1}{\sqrt[3]{C}} \\ b = \frac{1}{A} \\ c = \frac{\sqrt[3]{C}}{B} \end{cases}$$

We get

$$v_t = vv_x + v_{xxx}$$

Which is true, since $v$ is a solution to the KdV equation. Say now that $u$ is the solution to (1), then $\frac{1}{c}u\left(\frac{x}{a}, \frac{t}{b}\right)$ is the solution to $Au_t = Buu_x + Cu_{xxx}$. **We**

**can conclude that by using a scaling transformation on all variables, the coefficients are irrelevant.**

Suppose on the other hand that $v$ is now a solution of the mKdV equation (2). We get

$$Abcv_t = Bac^3v^2v_x + Ca^3cv_{xxx} \Rightarrow Abv_t = Bac^2v^2v_x + Ca^3v_{xxx}$$

Thus, we should chose

$$\begin{cases} a = \frac{1}{\sqrt[3]{C}} \\ b = \frac{1}{A} \\ c = \sqrt{\frac{\sqrt[3]{C}}{B}} \end{cases}$$

in order to get

$$v_t = v^2v_x + v_{xxx}$$

This choice, however, poses a problem when $B$ and $C$ differ in sign. A solution for this might be to choose $c = \sqrt{\left|\frac{\sqrt[3]{C}}{B}\right|}$ and using $\text{sng}\left(\frac{B}{C}\right)$, but this would also not work.

Since $c$ is the problem, maybe it can be done by only using scaling transformations $v(ax, t)$ and $v(x, bt)$. Denoting $\alpha$ and $\beta$ by, respectively, $\frac{B}{A}$ and $\frac{C}{A}$, we get

$$bv_t = \alpha av^2v_x + \beta a^3v_{xxx} \Rightarrow v_t = \alpha\frac{a}{b}v^2v_x + \beta\frac{a^3}{b}v_{xxx}$$

Therefore we want

$$\begin{cases} a = \sqrt{\frac{\alpha}{\beta}} \\ b = \sqrt{\frac{\alpha^3}{\beta}} \end{cases}$$

Once again, if $\alpha$ and $\beta$ have opposite signs there is a problem. But, by filling these in with absolute values, we get

$$v_t = \frac{\alpha}{|\alpha|}vv_x + \frac{\beta}{|\beta|}v_{xxx}$$

Absolute values and a sgn() would thus also not help. **We can conclude that the coefficients of the mKdV equation do in fact matter especially their signs.**

2. (Use Maple or Mathematica or other symbolic computing software for this problem.) Consider the KdV equation $u_t + uu_x + u_{xxx} = 0$. Show that

$$u = 12\partial_x^2 \ln\left(1 + e^{k_1x - k_1^3t + \alpha}\right)$$

is a one-soliton solution of the equation (i.e., rewrite it in the familiar $\text{sech}^2$ form). Now check that

$$u = 12\partial_x^2 \ln\left(1 + e^{k_1x-k_1^3t+\alpha} + e^{k_2x-k_2^3t+\beta} + \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 e^{k_1x-k_1^3t+\alpha+k_2x-k_2^3t+\beta}\right)$$

is also a solution of the equation. It is a two-soliton solution, as we will verify later. By changing $t$, we can see how the two solitons interact. With $\alpha = 0$ and $\beta = 1$, examine the following three regions of parameter space: (a) $k_1/k_2 > \sqrt{3}$, (b) $\sqrt{3} > k_1/k_2 > \sqrt{(2+\sqrt{5})/2}$, (c) $k_1/k_2 < \sqrt{(2+\sqrt{5})/2}$. Discuss the different types of collisions. Here "examine" and "discuss" are to be interpreted in an experimental sense: play around with this solution and observe what happens. The results you observe are the topic of the second part of Lax's seminal paper.

## Solution:

First, we calculate $\partial_x^2 \ln\left(1 + e^{k_1 x - k_1^3 t + \alpha}\right)$:

$$\partial_x \ln\left(1 + e^{k_1 x - k_1^3 t + \alpha}\right) = \frac{k_1 e^{k_1 x - k_1^3 t + \alpha}}{1 + e^{k_1 x - k_1^3 t + \alpha}}$$

Thus,

$$\partial_x^2 \ln\left(1 + e^{k_1 x - k_1^3 t + \alpha}\right) = \frac{k_1^2 e^{k_1 x - k_1^3 t + \alpha}}{\left(1 + e^{k_1 x - k_1^3 t + \alpha}\right)^2}$$

Using $\xi = k_1(x - k_1^2 t) + \alpha$, this becomes

$$u = 12\frac{k_1^2 e^{\xi}}{(1+e^{\xi})^2} = 3k_1^2\left(\frac{2}{e^{-\frac{\xi}{2}} + e^{\frac{\xi}{2}}}\right)^2 = 3k_1^2 \mathrm{sech}^2\left(\frac{\xi}{2}\right) = 3k_1^2 \mathrm{sech}^2\left(\frac{k_1 x - k_1^3 t + \alpha}{2}\right)$$

Using Wolfram Mathematica (no simplifying), we get

$$\begin{aligned}
u_t + uu_x + u_{xxx} = {} & 3k^5 Sech\left(\frac{\xi}{2}\right)^2 Tanh\left(\frac{\xi}{2}\right) \\
& - 9k^5 Sech\left(\frac{\xi}{2}\right)^4 Tanh\left(\frac{\xi}{2}\right) \\
& + 3k^2\left(2k^3 Sech\left(\frac{\xi}{2}\right)^4 Tanh\left(\frac{\xi}{2}\right) - k^3 Sech\left(\frac{\xi}{2}\right)^2 Tanh\left(\frac{\xi}{2}\right)^3\right)
\end{aligned}$$

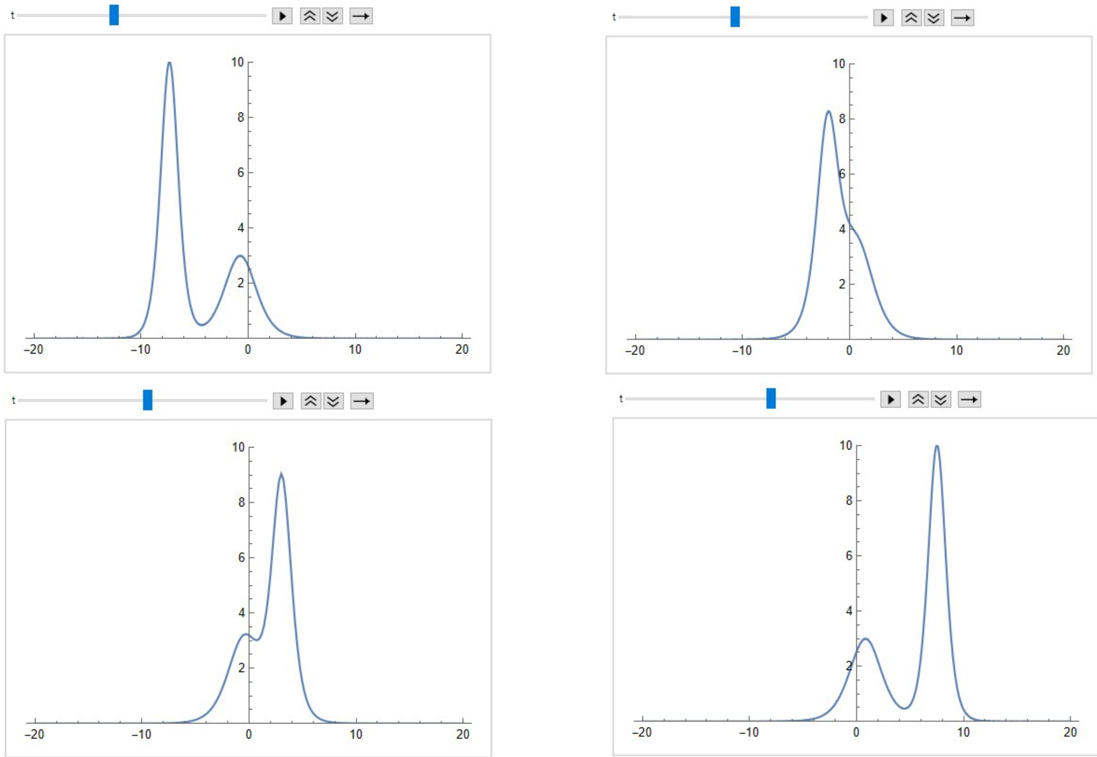
Using $Tanh(x)^2 + Sech^2(x) = 1$ $(Sech(x)^2 = 1 - Tanh(x)^2)$ , we get

$$\begin{aligned}
u_t + uu_x + u_{xxx} &= 3k^5 Sech\left(\frac{\xi}{2}\right)^2 Tanh\left(\frac{\xi}{2}\right) - 3k^5 Sech\left(\frac{\xi}{2}\right)^2 Tanh\left(\frac{\xi}{2}\right)^3 \\
&\quad - 9k^5 Sech\left(\frac{\xi}{2}\right)^4 Tanh\left(\frac{\xi}{2}\right) \\
&\quad + 6k^5 Sech\left(\frac{\xi}{2}\right)^4 Tanh\left(\frac{\xi}{2}\right) \\
&= 3k^5 Sech\left(\frac{\xi}{2}\right)^4 Tanh\left(\frac{\xi}{2}\right) - 3k^5 Sech\left(\frac{\xi}{2}\right)^4 Tanh\left(\frac{\xi}{2}\right) \\
&= 0
\end{aligned}$$

With code, which was provided with this pdf ("Ex2a WV"), we also get the exact same results. It is thus clear that $u$ is in fact a one-soliton solution.

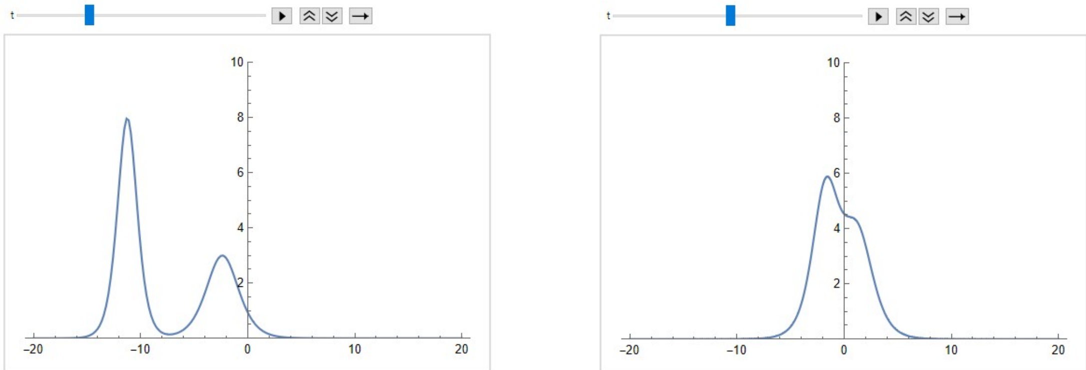
Now, we calculate $\partial_x^2 \ln\left(1 + e^{k_1x - k_1^3t + \alpha} + e^{k_2x - k_2^3t + \beta} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1x - k_1^3t + \alpha + k_2x - k_2^3t + \beta}\right)$, using Wolfram Mathematica (code provided with pdf ("Ex2b WV")). Then we check if the prescribed $u$ is a solution to the KdV equation, which it turns out to be. Thereafter we plot it using 3 different couples of $k_1$ and $k_2$ and the prescribed $\alpha$ and $\beta$.

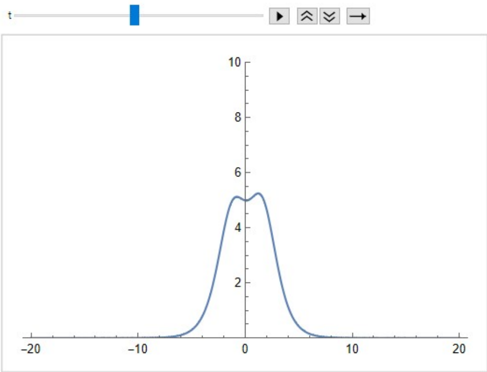
Note that for the results a negative time will also be shown in order to see the collision properly. The results for the first set of $(k_1, k_2) = (\sqrt{3} + 0.1, 1)$ are:

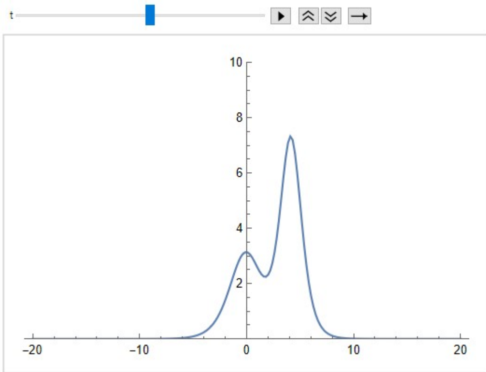



Here we see that a larger wave temporarily dominates over a smaller wave. During it's dominating period, it seems like the larger wave is pushing the smaller one back while also pushing itself forward. It might have something to do with momentum, however, I'm not able to give a correct justification for this hypothesis at this time. After the collision, the two waves act as if nothing happened and move forward like the same waves before the collision.

Now the second set $(k_1, k_2) = (\sqrt{3} - 0.2, 1)$:

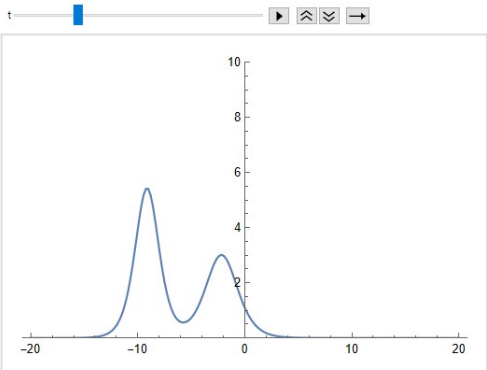
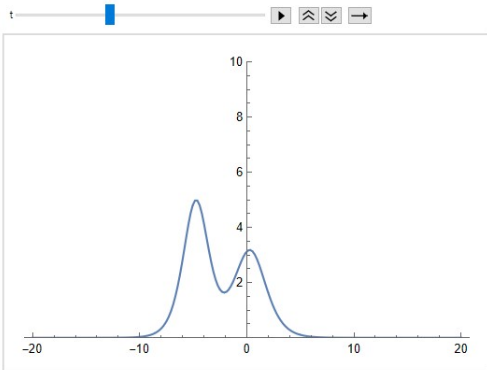
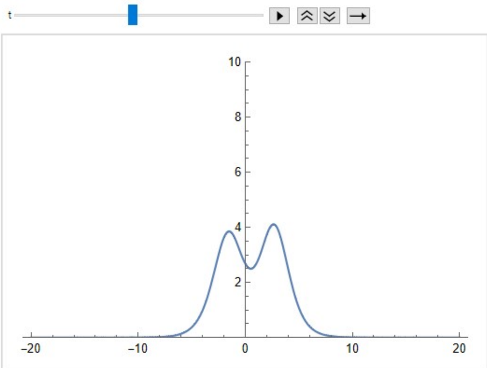
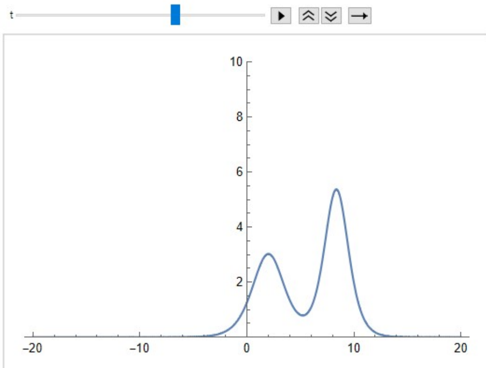







Here we see that a larger incoming wave doesn't dominate the smaller and slower wave, but it pushes it forward and slows itself down. It Exchanges it's characteristics from one wave to the other.

Now the third set $(k_1, k_2) = (\sqrt{(2+\sqrt{5})/2} - 0.1, 1)$:

Here we see the exact same that is happened in the second set, but more clearer. The gap between the two tops is larger and thus the exchange happens quick. In fact, the lower the ratio $k_1/k_2$ goes, the lesser the two waves differ.

It is important to note that these are not two individual waves interacting with each other. This becomes clear when we disable one of the exponentials by multiplying it by 0. The last exponential terms seems to set some sort of interaction between two waves.

3. **Cole-Hopf transformation**. Show that every non-zero solution of the heat equation $\theta_t = \nu\theta_{xx}$ gives rise to a solution of the dissipative Burgers' equation $u_t + uu_x = \nu u_{xx}$, through the mapping $u = -2\nu\theta_x/\theta$

## Solution:

The mapping is possible as $\theta$ is a non-zero solution. Since $u = -2\nu\theta_x/\theta$, we get that

$$u_x = 2\nu\frac{\theta_x^2 - \theta_{xx}\theta}{\theta^2}, \qquad u_{xx} = 2\nu\frac{3\theta\theta_x\theta_{xx} - \theta^2\theta_{xxx} - 2\theta_x^3}{\theta^3} \quad \text{and} \quad u_t = 2\nu\frac{\theta_x\theta_t - \theta_{xt}\theta}{\theta^2}.$$

Assuming $\theta$ is continuous: $\theta_{tx} = \theta_{xt} = \nu\theta_{xxx}$. Thus,

$$\begin{aligned} u_t + uu_x &= 2\nu\frac{\theta_x\theta_t - \theta_{xt}\theta}{\theta^2} - (2\nu)^2\frac{\theta_x}{\theta}\frac{\theta_x^2 - \theta_{xx}\theta}{\theta^2} \\ &= 2\nu^2\frac{\theta_x\theta_{xx} - \theta\theta_{xxx}}{\theta^2} - 4\nu^2\frac{\theta_x^3 - \theta\theta_x\theta_{xx}}{\theta^3} \\ &= 2\nu^2\frac{\theta\theta_x\theta_{xx} + 2\theta\theta_x\theta_{xx} - \theta^2\theta_{xxx} - 2\theta^3}{\theta^3} \\ &= \nu u_{xx} \end{aligned}$$

**Every non-zero solution of the heat equation does in fact give rise to a solution of the dissipative Burgers' equation through the used mapping.**

4. From the previous problem, you know that every solution of the heat equation $\theta_t = \nu\theta_{xx}$ gives rise to a solution of the dissipative Burgers' equation $u_t + uu_x = \nu u_{xx}$, through the mapping $u = -2\nu\theta_x/\theta$

   (a) Check that $\theta = 1 + \alpha e^{-kx+\nu k^2 t}$ is a solution of the heat equation. What solution of Burgers" equation does it correspond to? Describe this solution qualitatively (velocity, amplitude, steepness, *etc*) in terms of its parameters.

   (b) Check that $\theta = 1 + \alpha e^{-k_1 x+\nu k_1^2 t} + \beta e^{-k_2 x+\nu k_2^2 t}$ is a solution of the heat equation. What solution of Burgers' equation does it correspond to? Describe the dynamics of this solution, i.e., how does it change in time?

## Solution:

(a) For both sections of this exercise, the code "Ex4 WV" was used. Since

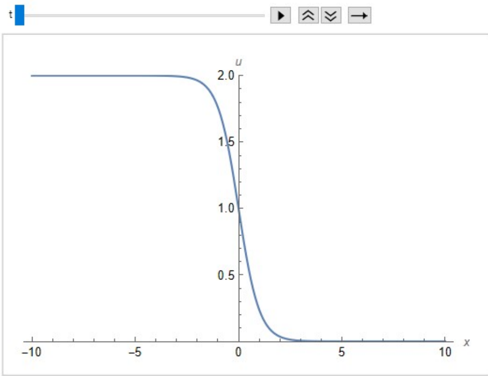
$$\theta_t = \alpha\nu k^2 e^{-kx+\nu k^2 t} \qquad \text{and} \qquad \theta_{xx} = \alpha(-k)^2 e^{-kx+\nu k^2 t} = \alpha k^2 e^{-kx+\nu k^2 t},$$

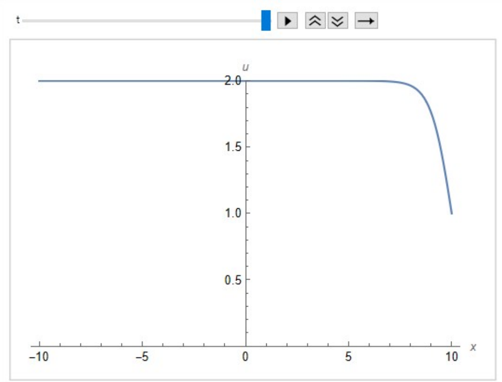
we know that $\theta$ is obvisouly a solution of the heat equation.

As $\theta_x = -\alpha k e^{-kx+\nu k^2 t}$ and using $\xi = x - \nu kt$, we get

$$u = 2\nu k\frac{\alpha e^{-k\xi}}{1 + \alpha e^{-k\xi}} = \frac{2\nu k}{\alpha^{-1}e^{k\xi} + 1} = \frac{2\nu k}{\alpha^{-1}e^{k(x-\nu kt)} + 1}$$

Assuming that $\alpha$ is always positive and not zero in order to avoid dividing by 0 (or having a simple solution). Plotting this function with $k = 2$, $\nu = .5$ we get the following plot at time $t = 0$ and 10.






It thus looks like some kind of shock moving forward at a speed of $\frac{x_1-x_2}{t_1-t_2}$, where $u(x_1,t_1) = u(x_2,t_2)$, which is $\frac{10}{10}$ (empirically determined).

A more correct approach would be to look at the expression of $u$, where we can clearly see a form of $x-ct$. $c$ would be the velocity in this case. Here the $c = \nu k$, which would indeed be 1. Defining the term "shock wave" as the part of the function which is not almost constant, the direction of this shock wave should also be taken into account. If $c$ is positive, as it is here, the shock wave moves forward when time increases. On the other hand, if $c$ is negative, the shock wave moves back when time increases.

Another thing to notice is the "amplitude" of the wave. First we define the "amplitude" as the value of the function when it is near constant and non-zero. Note how the function seems to be constant at $x \to \pm\infty$. After experimenting with a couple of parameters, we believe that the amplitude is $2k\nu$. A more correct approach would be to look at the limits of $x$ going to $\pm\infty$ for a $t \in \mathbb{R}$ constant. Say that $k$ is positive, then

$$\lim_{x\to\infty} \frac{2\nu k}{\alpha^{-1}e^{k(x-\nu kt)}+1} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{2\nu k}{\alpha^{-1}e^{k(x-\nu kt)}+1} = 2\nu k$$

Say now that $k$ is negative, then

$$\lim_{x\to\infty} \frac{2\nu k}{\alpha^{-1}e^{k(x-\nu kt)}+1} = 2\nu k \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{2\nu k}{\alpha^{-1}e^{k(x-\nu kt)}+1} = 0$$

The amplitude is thus clearly $2\nu k$.

The steepness of the curve can be expressed by $|u_x|$. Which is

$$\begin{aligned} u_x &= \left|\frac{-2\nu k^2 a^{-1} e^{k(x-\nu kt)}}{(1+\alpha^{-1}e^{k(x-\nu kt)})^2}\right| = \frac{k^2}{2}|\nu|\left(\frac{2}{e^{-\frac{1}{2}(k(x-\nu kt)-\ln(\alpha))}+e^{\frac{1}{2}(k(x-\nu kt)-\ln(\alpha))}}\right)^2 \\ &= \frac{|\nu|k^2}{2}\mathrm{Sech}^2\left(\frac{1}{2}(k(x-\nu kt)-\ln(\alpha))\right) \end{aligned}$$

Note that this is a wave as seen in exercise 2. Here we also see two things. Firstly that $\mathrm{sgn}(u_x) = -\nu$, which indicates if the shock wave is increasing ($\nu < 0$) or decreasing ($\nu > 0$). Secondly that $\alpha$ causes a shift in the shock wave of magnitude $\frac{\ln(\alpha)}{2}$ at time $t = 0$.

Lastly we try to find out what happens when $t \to \infty$ with $x$ constant: if $\nu > 0$, then

$$\lim_{t\to\infty} u(x,t) = 2\nu k$$

and if $\nu < 0$, then

$$\lim_{t\to\infty} u(x,t) = 0$$

This result could also have been found logically, as $\nu$ decides if the function is increasing or decreasing. The sign of $k$ might seem like a problem, but since only time changes and the coefficient of $t$ has $k^2$, it is not a problem.
Note that it is crucial that $x$ is constant as the limit for $x$ and $t$ going to $\infty$ may be undefined, for example, if $k > 0$ and $\nu > 0$, then for $x \to \infty$ and $t \to \infty$ $u$ should either go to 0 or $2\nu k$, it all depends on how $x$ and $t$ go to $\infty$.

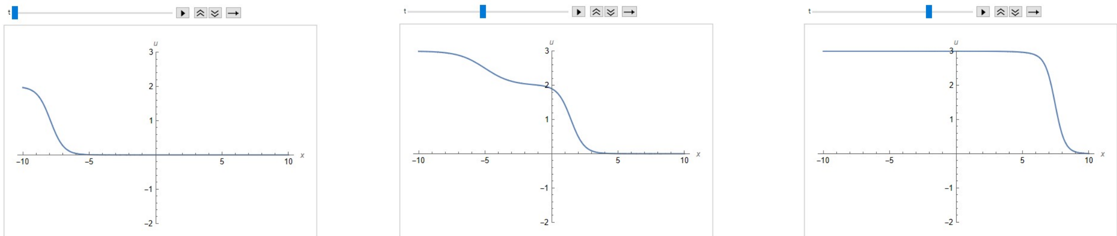
(b) Since

$$\theta_t = \nu(\alpha k_1^2 e^{-k_1 x + \nu k_1^2 t} + \beta k_2^2 e^{-k_2 x + \nu k_2^2 t}) \qquad \text{and} \qquad \theta_{xx} = \alpha k_1^2 e^{-k_1 x + \nu k_1^2 t} + \beta k_2^2 e^{-k_2 x + \nu k_2^2 t},$$

we know that $\theta$ is obviously a solution of the heat equation.
As $\theta_x = -(\alpha k_1 e^{-k_1 x + \nu k_1^2 t} + \beta k_2 e^{-k_2 x + \nu k_2^2 t})$ and using $\xi = x - \nu k_1 t$ and $\zeta = x - \nu k_2 t$, we get

$$u = 2\nu \frac{\alpha k_1 e^{-k_1 \xi} + \beta k_2 e^{-k_2 \zeta}}{1 + \alpha e^{-k_1 \xi} + \beta e^{-k_2 \zeta}}$$

In order to avoid getting a simple solution, diving by zero or the last exercise, we assume that $\alpha$ and $\beta$ are both non-zero. If we were to plot this with $k_1 = 3$, $k_2 = 2$, $\nu = .5$, $\alpha = 2$ and $\beta = 2^6$, we can pinpoint 3 interesting timestamps:



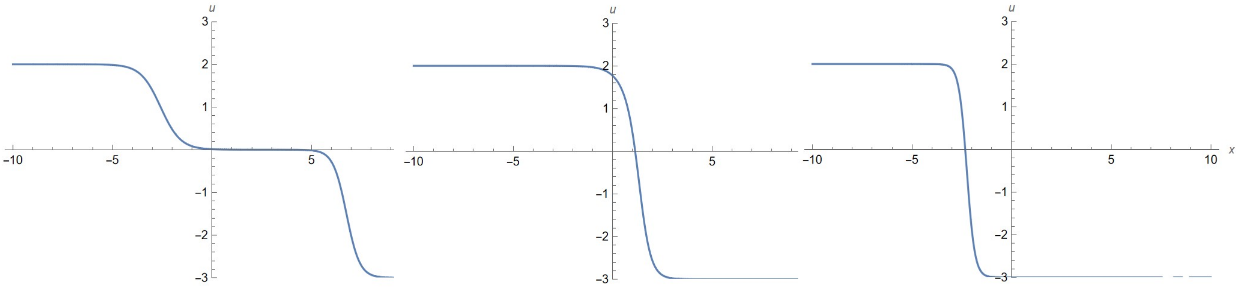
It seems like we have 2 different interacting shock waves with height 3 and 2, all of the previously described characteristics of these shock waves seem to hold separately. Since we chose $\beta$ to be exponentially bigger than $\alpha$ the two waves were clearly visible at first. Since we know how these waves will act separately, the interesting part is how they interact.
Suppose $k_1$ and $k_2$ are the same sign and $\nu > 0$, so both shock waves move in the same direction. If $\nu < 0$ time should be seen as reversed and the amplitudes have the opposite sign. Separately both waves would have amplitude $2\nu k_1$ and $2\nu k_2$, if the bigger waves hasn't "caught up" with the smaller wave (as seen above) both amplitudes can be seen. Once the bigger amplitude has caught up, it dominates in the function's graph. This can be algebraically shown by (suppose both $k_1$ and $k_2$ are positive and $x$ fixed):

$$\lim_{t\to\infty} u(x,t) = 2\nu \max(k_1, k_2) \qquad \text{and} \qquad \lim_{t\to-\infty} u(x,t) = 0$$

Note that we also calculated $t \to -\infty$ in case $\nu < 0$. The first limit holds as the exponential with the highest coefficient of t in it will dominate. If both $k_1$ and $k_2$ are negative we have:

$$\lim_{t\to\infty} u(x,t) = -2\nu \max(|k_1|, |k_2|) \qquad \text{and} \qquad \lim_{t\to-\infty} u(x,t) = 0$$

Note that the signs of $k_1$ and $k_2$ are not of importance when time changes (the coefficients are $k_1^2$ and $k_2^2$). Suppose now that $k_1$ and $k_2$ have the opposite sign of each other, for example $k_1 = -3$ and $k_2 = 2$. What we see then is two shock waves colliding:




Here we think that the wave with the largest magnitude of speed dominates. Indeed, if $s$ equals the sign of the $k$ with the largest magnitude, we get:

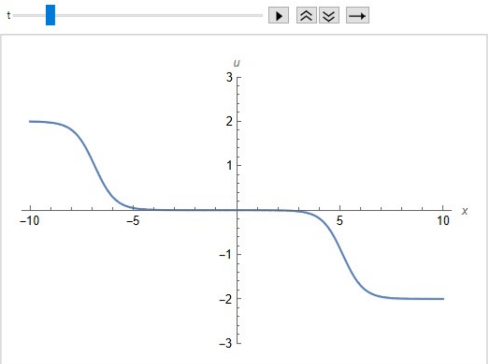
$$\lim_{t\to\infty} u(x,t) = 2\nu s \max(|k_1|, |k_2|) \qquad \text{and} \qquad \lim_{t\to-\infty} u(x,t) = -s2\nu \min(|k_1|, |k_2|)$$

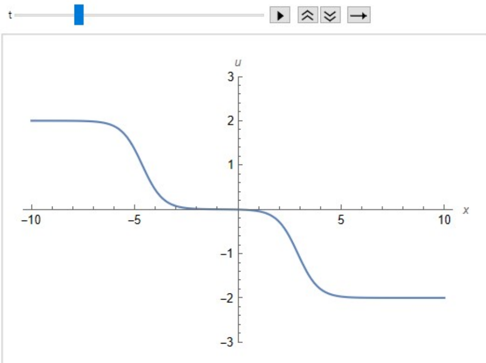
Lastly, what if $|k_1| = |k_2|$. If $k_1$ and $k_2$ have the same sign, the solution will be of the same form as exercise 4a:

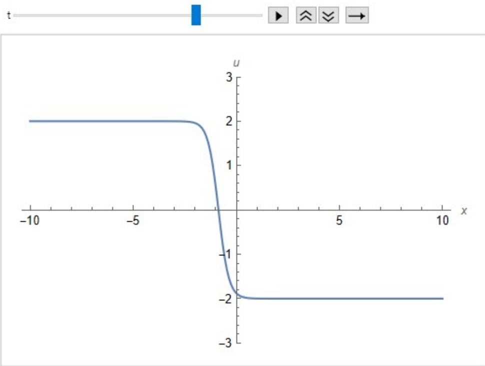
$$u(x,t) = 2\nu \frac{(\alpha+\beta)e^{-k_1(x-\nu k_1 t)}}{1 + (\alpha+\beta)e^{-k_1(x-\nu k_1 t)}}$$

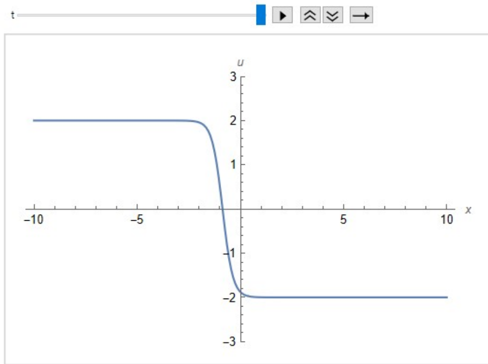
The "two" waves will thus become one and have the same amplitude and speed as the two waves separately. The only thing that happens is a shift.

Lastly, suppose that $k_1$ and $k_2$ have the opposite sign. If this is the case, the function becomes almost stationary after a while. This can be seen here:






The position is determined by the $\alpha$ and $\beta$, since one might get a "head start" on the other for the race to $x = 0$.