AMATH 573
Solitons and nonlinear waves
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Homework 6

due: December 9, 2022

1. Show that $\bar{N}(x,k)$ is analytic in the open lower-half plane, $\text{Im}k < 0$, by showing $\bar{N}(x,k)$ and $\partial\bar{N}/\partial k$ are bounded there. What are the conditions you need for this to be true?

Solution:

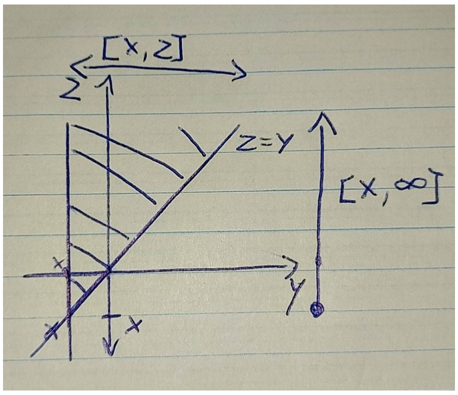
Remind yourself that $\bar{N}(x,k) \to 1$ as $x \to \infty$, furthermore, we assume that it uniformly goes to 1, such that the derivatives with respect to x of $\bar{N}(x,k)$ approaches zero fast enough as $x \to \infty$. Also, this Jost function satisfies the following differential equation:

$$\bar{N}_{xx} - 2ik\bar{N}_x + u\bar{N} = 0.$$

From this, we can now single out n $\bar{N}$:

$$\begin{aligned}
& \bar{N}_{xx} - 2ik\bar{N}_x = -u\bar{N} \\
\Rightarrow\quad & e^{-2ikx}\bar{N}_{xx} - 2ike^{-2ikx}\bar{N}_x = -e^{-2ikx}u\bar{N} \\
\Rightarrow\quad & \left(e^{-2ikx}\bar{N}_x\right)_x = -e^{-2ikx}u\bar{N} \\
\Rightarrow\quad & e^{-2iky}\bar{N}_x = -\int_y^\infty e^{-2ikz}u(z)\bar{N}(z,k)dz \\
\Rightarrow\quad & \bar{N}(x,k) = 1 - \int_x^\infty \int_y^\infty e^{2ik(y-z)}u(z)\bar{N}(z,k)dzdy.
\end{aligned}$$

The bounds are dawn like the figure next to this text. That is why we can change these bounds to the following (and work with it):



$$\begin{aligned}
\bar{N}(x,k) &= 1 - \int_x^\infty \int_y^\infty e^{2ik(y-z)} u(z) \bar{N}(z,k) dz dy \\
&= 1 - \int_x^\infty \int_x^z e^{2ik(y-z)} u(z) \bar{N}(z,k) dy dz \\
&= 1 - \int_x^\infty u(z) \bar{N}(z,k) \int_x^z e^{2ik(y-z)} dy dz \\
&= 1 - \int_x^\infty u(z) \bar{N}(z,k) \frac{1 - e^{2ik(x-z)}}{2ik} dz \\
&= 1 + \int_x^\infty \frac{e^{2ik(x-z)} - 1}{2ik} u(z) \bar{N}(z,k) dz.
\end{aligned}$$

First we want to find an upperbound for $|\bar{N}|$. In order to do that, we solve this integral equation by iteration, using

$$\bar{N}(x,k) = 1 + \sum_{j=1}^{\infty} \bar{N}_j(x,k) = \sum_{j=0}^{\infty} \bar{N}_j(x,k),$$

where $\bar{N}_j(x,k)$ satisfies

$$\bar{N}_j(x,k) = \int_x^\infty \frac{e^{2ik(x-z)} - 1}{2ik} u(z) \bar{N}_{j-1}(z,k) dz,$$

for $j = 1, 2, \ldots$. This recursion formula starts with $\bar{N}_0(x,k) \equiv 1$. We now prove that, if $\mathrm{Im}(k) < 0$ ($k = k_{re} + ik_{im}$, with $k_{im} < 0$) and $x \leq 0$, then

$$\left| \frac{e^{2ikx} - 1}{2ik} \right| \leq \frac{1}{|k|}.$$

This can easily be shown:

$$\begin{aligned}
\left| \frac{e^{2ikx} - 1}{2ik} \right| &= \left| \frac{e^{2ik_{re}x - 2k_{im}x} - 1}{2ik} \right| \\
&\leq \frac{|e2ik_{re}x||e^{-2k_{im}x}| + 1}{2|k|} \\
&= \frac{|e^{-2k_{im}x}| + 1}{2|k|} \\
&\overset{k_{im}x > 0}{\leq} \frac{1}{|k|}.
\end{aligned}$$

We can now establish bounds on $M_j(x,k)$. First say that $\mathrm{Im}(k)<0$ and $x\leq 0$, we then have

$$\begin{aligned}|\bar{N}_1(x,k)| &= \left|\int_x^\infty \frac{e^{2ikx}-1}{2ik}u(z)\bar{N}_0(z,k)dz\right| \\ &\leq \int_x^\infty \left|\frac{e^{2ikx}-1}{2ik}\right||u(z)|dz \\ &\leq \frac{1}{|k|}\int_x^\infty |u(z)|dz.\end{aligned}$$

Now define

$$U(x)=\int_x^\infty |u(z)|dz.$$

Assuming that this integral is defined. Next,

$$\begin{aligned}|\bar{N}_2(x,k)| &= \left|\int_x^\infty \frac{e^{2ikx}-1}{2ik}u(z)\bar{N}_1(z,k)dz\right| \\ &\leq \int_x^\infty \left|\frac{e^{2ikx}-1}{2ik}\right||u(z)|\frac{U(z)}{|k|}dz \\ &\leq \frac{1}{|k|^2}\int_x^\infty U'Udz=\frac{U^2(x)}{2|k|^2}.\end{aligned}$$

Like in the notes, it is easy to see that we get the following inequality for $\bar{N}_j$:

$$|\bar{N}_j(x,k)|\leq \frac{U^j(x)}{j!\,|k|^j},$$

for $j=1,2,\ldots$. Putting this together, we obtain

$$|\bar{N}(x,k)|\leq \sum_{j=0}^{\infty}|\bar{N}_j(x,k)|\leq \sum_{j=0}^{\infty}\frac{U^j(x)}{j!\,|k|^j}=e^{U(x)/|k|}.$$

Thus, if we have that

$$\int_{-\infty}^{\infty}|u(x)|dx=U(-\infty)<\infty,$$

then

$$|\bar{N}(x,k)|\leq e^{U(-\infty)/|k|}<\infty,$$

since U is a monotone decreasing function. This means that $\bar{N}$ is uniformly bounded as a function of x. It is also bounded in function of k in the lower-half plane, but not uniformly. It in fact blows up when $k\to 0$.

Now we find a bound for $\frac{\partial \bar{N}}{\partial k}$ using its integral equation. Since

$$\bar{N}(x,k)=1+\int_x^\infty \frac{e^{2ik(x-z)}-1}{2ik}u(z)\bar{N}(z,k)dz,$$

we find that

$$\frac{\partial \bar{N}}{\partial k}=F(k,x)+\int_x^\infty \frac{e^{2ik(x-z)}-1}{2ik}u(z)\frac{\partial \bar{N}}{\partial k}(z,k)dz,$$

with

$$F(x,k) = \int_X^\infty \frac{2ik(x-z)e^{2ik(x-z)} + 1 - e^{2ik(x-z)}}{2ik^2} u(z)\bar{N}(z,k)dz.$$

Note that for $F(x,k)$, $k=0$ is a removable singularity of the kernel of the integral defining $F(x,k)$. Suppose $k = k_{re} + ik_{im}$ with $k_{im} < 0$, we have

$$\begin{aligned}
F(x,k) &= \int_x^\infty \frac{(x-z)e^{2ik(x-z)}}{k} u(z)\bar{N}(z,k)dz - \int_x^\infty \frac{e^{2ik(x-z)} - 1}{2ik^2} u(z)\bar{N}(z,k)dz \\
\Rightarrow \quad |F(z,k)| &\leq \int_x^\infty \frac{e^{-2k_{im}(x-z)} + 1}{2|k|^2} |u(z)|\bar{N}(z,k)|dz + \int_x^\infty \frac{e^{-2k_{im}(x-z)}}{|k|} (z-x)|u(z)||\bar{N}(z,k)|dz \\
&\overset{x-z<0}{\leq} \frac{1}{|k|^2} \int_x^\infty |u(z)||\bar{N}(z,k)dz + \frac{1}{|k|} \int_x^\infty (z-x)|u(z)||\bar{N}(z,k)|dz \\
&\leq \frac{e^{U(-\infty)/|k|}}{|k|} \left(\frac{U(x)}{|k|} + V(x) \right) \\
&\leq \frac{e^{U(-\infty)/|k|}}{|k|} \left(\frac{U(-\infty)}{|k|} + \lim_{x\to-\infty} V(x) \right),
\end{aligned}$$

where we have defined

$$V(x) = \int_x^\infty (z-x)|u(z)|dz,$$

which is a monotone decreasing function of x. *Assume that this integral and limit exists*, then this is a uniform bound on $F(x,k)$. Now, since $F(x,k)$ is uniformly bounded, if $U(-\infty)$ and $\lim_{x\to-\infty} V(x)$ are finite, we have that $\frac{\partial k\bar{N}}{\partial k}$ is also bounded:

$$\left| \frac{\partial \bar{N}}{\partial k} \right| \leq \frac{e^{2U(-\infty)/|k|}}{|k|} \left(\frac{U(-\infty)}{|k|} + \lim_{x\to-\infty} V(x) \right).$$

(course notes)
Clearly, one has to, once again, be cautious around $k=0$. Since $\bar{N}$ and its derivative are bounded in the lower-half complex plane, we have that $\bar{N}$ is analytic in that region.

In conclusion, if,

$$\int_{-\infty}^\infty |u(x)|dx < \infty, \text{ and } \lim_{x\to-\infty} \int_x^\infty (z-x)|u(z)|dz < \infty,$$

we have that $\bar{N}$ is analytic in the open lower-half complex plane for k.

2. Recall the second member of the stationary KdV hierarchy from HW4:

$$30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x} + c_1(6uu_x + u_{xxx}) + c_0u_x = 0.$$

Integrating once, it can be rewritten as

$$(10u^3 + 10u_x^2 + 10uu_{xx} - 5u_x^2 + u_{xxxx}) + c_1(3u^2 + u_{xx}) + c_0u + c_{-1} = 0. \quad (1)$$

This is an ordinary differential equation for u as a function of x. You already know that the two soliton is a solution of this. You know this equation can be written as

$$\frac{\delta T_2}{\delta u} = 0 \quad \Longleftrightarrow \quad \frac{\delta}{\delta u}\left(F_2 + c_1F_1 + c_0F_0 + c_{-1}F_{-1}\right) = 0, \quad (2)$$

where F_k, $k = -1, 0, \ldots$ are the conserved quantities of the KdV equation. These conserved quantities are in involution,

$$\{F_j, F_k\} = 0 \quad \Rightarrow \{T_j, F_k\} = 0,$$

for $j, k = -1, 0, \ldots$. From HW4, you know that this implies (using $j = 2$)

$$\frac{\delta T_2}{\delta u}\frac{d}{dx}\frac{\delta F_k}{\delta u} = \frac{dH_k}{dx}, \quad k = 0, 1. \quad (3)$$

For some functions H_0 and H_1. Thus, H_0 and H_1 are conserved quantities of (1)[1].

(a) Find H_0 and H_1 explicitly.

(b) Check explicitly, by taking an x derivative, that H_0 and H_1 are conserved along solutions of (1).

Reinterpreting what you just did: $\delta F_0/\delta u$ and $\delta F_1/\delta u$ are integrating factors for (1): factors with which to multiply the equation so it can be integrated. In other words, (1) has two different ways in which it can be made *exact*!

Since (2) shows that (1) is a Lagrangian equation, we expect it to be Hamiltonian. Novikov & Bogoyavlenski (1974) showed this is indeed the case. In fact, the Hamiltonian system has canonical Poisson structure. Usually, one uses the Legendre transformation to go from a Lagrangian to a Hamiltonian system. Here the more general Ostrogradski transformation has to be used. In addition, using the ideas outlined above, they showed that the resulting Hamiltonian system is completely integrable, as it has enough conserved quantities. Indeed, (1) is 4-th order, thus it will be a Hamiltonian system with q_1, q_2 and p_1, p_2. We found two conserved quantities for this system, which is the required number, following the Liouville-Arnol'd theorem.

Solution:

[1] Why did we not include $k = -1$?

(a) After using scaling coefficients $u(\frac{x}{\sqrt{6}}, t)$, we get the following conserved quantities from HW4:

$$F_{-1} = \int u dx,$$

$$F_0 = \int \frac{1}{2} u^2 dx,$$

$$F_1 = \frac{1}{6} \int \left(u^3 - \frac{1}{2} u_x^2 \right) dx,$$

$$F_2 = \frac{1}{60} \int \left(\frac{5}{2} u^4 - 5 u u_x^2 + \frac{1}{2} u_{xx}^2 \right) dx.$$

We ignore the factors $\frac{1}{6}$ and $\frac{1}{24}$ (it can be compensated by c_1, c_0 and c_{-1}), we now calculate the variational derivatives:

$$\frac{\delta F_{-1}}{\delta u} = 1,$$

$$\frac{\delta F_0}{\delta u} = u,$$

$$\frac{\delta F_1}{\delta u} = 3u^2 + u_{xx},$$

$$\frac{\delta F_2}{\delta u} = 10u^3 - 5u_x^2 + 10(uu_x)_x + u_{xxxx} = 10u^3 + 5u_x^2 + 10uu_{xx} + u_{xxxx}.$$

Thus,

$$\frac{\delta T_2}{\delta u} = 10u^3 + 5u_x^2 + 10uu_{xx} + u_{xxxx} + c_1(3u^2 + u_{xx}) + c_0 u + c_{-1}.$$

Note that this is the term on the left of (1). We now calculate

$$\begin{aligned}
\frac{\delta T_2}{\delta u} \frac{d}{dx} \frac{\delta F_0}{\delta u} &= 10u^3 u_x + 5u_x^3 + 10uu_x u_{xx} + u_x u_{xxxx} + c_1(3u^2 u_x + u_x u_{xx}) + c_0 u u_x + c_{-1} u_x \\
&= \frac{10}{4}(u^4)_x + 5(uu_x^2)_x + (u_x u_{xxx})_x - u_{xx} u_{xxx} + c_1((u^3)_x + \frac{1}{2}(u_x^2)_x) + \frac{c_0}{2}(u^2)_x + c_{-1} u_x \\
&= \left(\frac{10}{4} u^4 + 5uu_x^2 + u_x u_{xxx} - \frac{1}{2} u_{xx}^2 + c_1(u^3 + \frac{1}{2} u_x^2) + \frac{c_0}{2} u^2 + c_{-1} u \right)_x \\
&= \frac{dH_0}{dx}.
\end{aligned}$$

Therefore

$$H_0 = \frac{10}{4} u^4 + 5uu_x^2 + u_x u_{xxx} - \frac{1}{2} u_{xx}^2 + c_1(u^3 + \frac{1}{2} u_x^2) + \frac{c_0}{2} u^2 + c_{-1} u + C_0,$$

with C_0 a constant.

On the other hand,

$$\begin{aligned}
\frac{\delta T_2}{\delta u}\frac{d}{dx}\frac{\delta F_1}{\delta u} &= \frac{\delta F_2}{\delta u}\frac{d}{dx}\frac{\delta F_1}{\delta u} + c_1\frac{\delta F_1}{\delta u}\frac{d}{dx}\frac{\delta F_1}{\delta u} + c_0\frac{\delta F_0}{\delta u}\frac{d}{dx}\frac{\delta F_1}{\delta u} + c_{-1}\frac{\delta F_{-1}}{\delta u}\frac{d}{dx}\frac{\delta F_1}{\delta u} \\
&= (10u^3 + 5u_x^2 + 10uu_{xx} + u_{xxxx})(6uu_x + u_{xxx}) \\
&\quad + \frac{c_1}{2}\left(\frac{\delta F_1}{\delta u}^2\right)_x + c_0(6u^2u_x + uu_{xxx}) + c_{-1}\left(\frac{\delta F_1}{\delta u}\right)_x \\
&= 12(u^5)_x + 30uu_x^3 + 60uu_xu_{xx} + 6uu_xu_{xxxx} \\
&\quad + 10u^3u_{xxx} + 5u_x^2u_{xxx} + 10uu_{xx}u_{xxx} + \frac{1}{2}(u_{xxx}^2)_x \\
&\quad + \frac{c_1}{2}\left(\frac{\delta F_1}{\delta u}^2\right)_x + c_0(2(u^3)_x + (uu_{xx})_x - \frac{1}{2}(u_x^2)_x) + c_{-1}\left(\frac{\delta F_1}{\delta u}\right)_x.
\end{aligned}$$

We get inspiration from HW4:

$$(15u^2u_x^2 + 10u^3u_{xx})_x = 30uu_x^3 + 60uu_xu_{xx} + 10u^3u_{xxx},$$

and

$$(6uu_xu_{xxx} + 2uu_{xx}^2 - u_x^2u_{xx})_x = 6uu_xu_{xxxx} + 10uu_{xx}u_{xxx} + 5u_x^2u_{xxx}.$$

Therefore,

$$\begin{aligned}
\frac{\delta T_2}{\delta u}\frac{d}{dx}\frac{\delta F_1}{\delta u} &= \left(12u^5 + 15u^2u_x^2 + 10u^3u_{xx} + 6uu_xu_{xxx} + 2uu_{xx}^2 - u_x^2u_{xx}\frac{1}{2}u_{xxx}^2\right)_x \\
&\quad + \frac{c_1}{2}\left(\frac{\delta F_1}{\delta u}^2\right)_x + c_0(2(u^3)_x + (uu_{xx})_x - \frac{1}{2}(u_x^2)_x) + c_{-1}\left(\frac{\delta F_1}{\delta u}\right)_x \\
&= \frac{dH_1}{dx}.
\end{aligned}$$

Thus,

$$\begin{aligned}
H_1 &= 12u^5 + 15u^2u_x^2 + 10u^3u_{xx} + 6uu_xu_{xxx} + 2uu_{xx}^2 - u_x^2u_{xx}\frac{1}{2}u_{xxx}^2 \\
&\quad + \frac{c_1}{2}\left(3u^2 + u_{xx}\right)^2 + c_0\left(2u^3 + uu_{xx} - \frac{1}{2}u_x^2\right) + c_{-1}(3u^2 + u_{xx}) + C_1,
\end{aligned}$$

with C_1 a constant.

In conclusion,

$$H_0 = \frac{10}{4}u^4 + 5uu_x^2 + u_xu_{xxx} - \frac{1}{2}u_{xx}^2 + c_1(u^3 + \frac{1}{2}u_x^2) + \frac{c_0}{2}u^2 + c_{-1}u + C_0,$$

and

$$\begin{aligned}
H_1 &= 12u^5 + 15u^2u_x^2 + 10u^3u_{xx} + 6uu_xu_{xxx} + 2uu_{xx}^2 - u_x^2u_{xx}\frac{1}{2}u_{xxx}^2 \\
&\quad + \frac{c_1}{2}\left(3u^2 + u_{xx}\right)^2 + c_0\left(2u^3 + uu_{xx} - \frac{1}{2}u_x^2\right) + c_{-1}(3u^2 + u_{xx}) + C_1,
\end{aligned}$$

with C_0 and C_1 constant.

(b) We saw that the left side of (1) is equal to $\frac{\delta T_2}{\delta u}$. Thus, along solutions of (1),

$$\frac{\delta T_2}{\delta u} = 0.$$

Then we have that

$$\frac{dH_0}{dx} = \frac{\delta T_2}{\delta u}\frac{d}{dx}\frac{\delta T_0}{\delta u} = 0\frac{d}{dx}\frac{\delta T_0}{\delta u} = 0,$$

and

$$\frac{dH_1}{dx} = \frac{\delta T_2}{\delta u}\frac{d}{dx}\frac{\delta T_1}{\delta u} = 0\frac{d}{dx}\frac{\delta T_1}{\delta u} = 0,$$

are conserved quantities. In conclusion, the solutions to (1) are conserved quantities of both Hamiltonians.

3. **The Ostrovsky equation** is used to model weakly nonlinear long waves in a rotating frame. It is given by

$$(\eta_t + \eta\eta_x + \eta_{xxx})_x = \gamma\eta,$$

with $\gamma \neq 0$. In what follows, we assume that as $|x| \to \infty$, η and its derivatives approach zero as fast as we need them to.

- Show that $\int_{-\infty}^{\infty} \eta dx = 0$. In other words, not only is $\int_{-\infty}^{\infty} \eta dx$ conserved, but its value is fixed at zero.
- Using this result, show that $\int_{-\infty}^{\infty} \eta^2 dx$ is a conserved quantity. Do this by rewriting the equation in evolution form, with an indefinite integral on the right-hand side.
- Use the definition of the variational derivative to verify that the Ostrovsky equation is Hamiltonian with Poisson operator ∂_x and Hamiltonian

$$H = \frac{1}{2}\int_{-\infty}^{\infty} \left(\eta_x^2 - \frac{1}{3}\eta^3 - \gamma\phi^2\right) dx,$$

where $\phi_x = \eta$.

Solution:

- Since the function η and its derivatives approach zero as fast as we need them to at infinity, we have that

$$\int_{-\infty}^{\infty} \eta dx = \frac{1}{\gamma}\int_{-\infty}^{\infty} (\eta_t + \eta\eta_x + \eta_{xxx})_x dx = \frac{1}{\gamma}\left(\eta_t + \eta\eta_x + \eta_{xxx}\right)|_{-\infty}^{\infty} = 0.$$

Thus,

$$\int_{-\infty}^{\infty} \eta dx = 0.$$

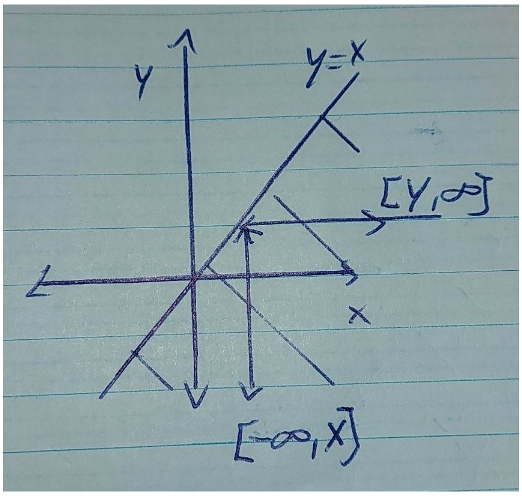
- Taking the integral of the equation over $[-\infty, x]$ and multiplying by η gives,

$$\begin{aligned}
\gamma\eta \int_{-\infty}^{x} \eta(y)dy &= \eta \int_{-\infty}^{x} \left(\eta_t(y) + \eta(y)\eta_x(y) + \eta_{xxx}(y)\right)_x dy \\
&= \eta\eta_t + \eta^2\eta_x + \eta\eta_{xxx} \\
&= \frac{1}{2}(\eta^2)_t + \frac{1}{3}(\eta^3)_x + (\eta\eta_{xx})_x - \eta_x\eta_{xx} \\
&= \frac{1}{2}(\eta^2)_t + \frac{1}{3}(\eta^3)_x + (\eta\eta_{xx})_x - \frac{1}{2}(\eta_x^2)_x.
\end{aligned}$$

Now take the integral over $[-\infty, \infty]$ on both sides. Since η and its derivatives go to zero when x goes to infinity, we get,

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{2}(\eta^2(x))_t dx &= \gamma \int_{-\infty}^{\infty} \eta(x) \int_{-\infty}^{x} \eta(y) dy dx \\
\int_{-\infty}^{\infty} \frac{1}{2}\frac{d}{dt}\eta^2(x) dx &= \gamma \int_{-\infty}^{\infty} \int_{-\infty}^{x} \eta(x)\eta(y) dy dx \\
\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \eta^2(x) dx &= \gamma \int_{-\infty}^{\infty} \int_{y}^{\infty} \eta(x)\eta(y) dx dy \\
&= \gamma \int_{-\infty}^{\infty} \eta(y) \int_{y}^{\infty} \eta(x) dx dy \\
&= \gamma \int_{-\infty}^{\infty} \eta(y) \left(\int_{-\infty}^{\infty} \eta(x) dx - \int_{-\infty}^{y} \eta(x) dx \right) dy \\
&= -\gamma \int_{-\infty}^{\infty} \eta(y) \int_{-\infty}^{y} \eta(x) dx dy
\end{aligned}
$$

The second to third step can easily be seen to be true when picturing the integration domain:



Since we have that

$$\int_{-\infty}^{\infty} \eta(x) \int_{-\infty}^{x} \eta(y) dy dx = -\int_{-\infty}^{\infty} \eta(y) \int_{-\infty}^{y} \eta(x) dx dy,$$

we know that it is equal to zero. Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \eta^2(x) dx = 0$$

In conclusion,

$$\int_{-\infty}^{\infty} \eta^2 dx$$

is a conserved quantity.

- Expanding the definition of the variational derivative ($\phi = \partial_x^{-1}\eta$) and using it on the Hamiltonian gives,

$$\frac{\delta H}{\delta \eta} = -\frac{1}{2}\partial_x^{-1}\left(\frac{\partial}{\partial \phi}(-\gamma\phi^2)\right)+\frac{1}{2}\frac{\partial}{\partial \eta}(-\frac{1}{3}\eta^3)-\frac{1}{2}\partial_x\left(\frac{\partial}{\partial \eta_x}(\eta_x^2)\right) = \partial_x^{-1}\gamma\phi-\left(\frac{1}{2}\eta^2+\eta_{xx}\right).$$

We know that taking the partial x derivative of this should land us η_t, thus

$$\eta_t = \partial_x \frac{\delta H}{\delta \eta} = \gamma\phi - (\eta\eta_x + \eta_{xxx}) \Rightarrow \eta_t + \eta\eta_x + \eta_{xxx} = \gamma\phi.$$

Note that the Ostrovsky equation can be written as

$$\eta_t + \eta\eta_x + \eta_{xxx} = \gamma\phi,$$

which is what we get.

In conclusion, the Ostrovsky equation,

$$(\eta_t + \eta\eta_x + \eta_{xxx})_x = \gamma\eta,$$

is Hamiltonian with Poisson operator ∂_x and Hamiltonian

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\left(\eta_x^2 - \frac{1}{3}\eta^3 - \gamma\phi^2\right)dx,$$

4. Using the Painlevé test, discuss the integrability of

$$u_t = u^p u_x + u_{xxx}.$$

Solution:

We are looking for solutions of the form

$$u = \sum_{k=0}^{\infty} \alpha_k(t)(x - x_0(t))^{k-n},$$

where n is an natural number so that we have a pole of order n at $x = x_0$. Here α_k, for $k = 0, 1, \dots$, are complex timedependant functions.

Close to x_0, we have

$$\begin{aligned}
u &\sim \alpha_0(t)(x - x_0(t))^{-n}, \\
u_t &\sim \alpha_0'(x - x_0)^{-n} + n\alpha_0 x_0'(x - x_0)^{-n-1}, \\
u_x &\sim -n\alpha_0(x - x_0)^{-n-1}, \\
u_{xx} &\sim n(n+1)\alpha_0(x - x_0)^{-n-2}, \\
u_{xxx} &\sim -n(n+1)(n+2)\alpha_0(x - x_0)^{-n-3}.
\end{aligned}$$

Thus, around a pole, the PDE becomes

$$\alpha_0'(x-x_0)^{-n}+n\alpha_0 x_0'(x-x_0)^{-n-1} = -n\alpha_0^{p+1}(x-x_0(t))^{-(p+1)n-1}-n(n+1)(n+2)\alpha_0(x-x_0)^{-n-3}.$$

The terms on the right are potentially the most singular (since n and $n+1$ get trumped by $n+3$). If we say that the $-n-3$ is the most singular term, then we want $n = -3$. This is negative (and thus u would not have a pole) and this would also mean that $\alpha_0 = 0$, which we don't want. If $-(p+1)n - 1$ is the most singular, then $n = -\frac{1}{p+1}$, which might be possible, but then α_0, must also equal zero. Thus we want $-(p+1)n - 1$ and $-n - 3$ to be the most singular at the same time, thus

$$-(p+1)n - 1 = -n - 3 \Rightarrow n = \frac{2}{p}.$$

Since we can only have that n can be a natural number (in order to get a pole), we know that, for $p = 1, 2$, the equation is integrable. For the others, we cannot immediately say anything. The α_0 does not pose a problem:

$$-\frac{2}{p}\alpha_0^{p+1}(x - x_0)^{-\frac{2}{p}-3} - \frac{2}{p}(\frac{2}{p} + 1)(\frac{2}{p} + 2)\alpha_0(x - x_0)^{-\frac{2}{p}-3} = 0,$$

has to hold. Thus,

$$\alpha_0 = \sqrt[p]{-(\frac{2}{p} + 1)(\frac{2}{p} + 2)}.$$

In conclusion, we can say that the equation is integrable for $p = 1$ and 2, which are the Kdv and mKdV equation.