AMath 574
Homework 3
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Due by 11:00pm on February 2, 2023

For submission instructions, see:

http://faculty.washington.edu/rjl/classes/am574w2023/homework3.html

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Problem #6.1 in the book:

Verify that

$$Q_i^{n+1} = \frac{\bar{u}\Delta t}{\Delta x}\left(Q_{i-1}^n + \frac{1}{2}(\Delta x - \bar{u}\Delta t)\sigma_{i-1}^n\right) + \left(1 - \frac{\bar{u}\Delta t}{\Delta x}\right)\left(Q_i^n - \frac{1}{2}\bar{u}\Delta t\sigma_i^n\right)$$
$$= Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{1}{2}\frac{\bar{u}\Delta t}{\Delta x}(\Delta x - \bar{u}\Delta t)(\sigma_i^n - \sigma_{i-1}^n),$$

results from integrating the piecewise linear solution $\tilde{q}^n(x, t_{n+1})$.

Solution: Note that

$$\tilde{q}^n(x, t_{n+1}) = \tilde{q}^n(x - \bar{u}\Delta t, t_n) = Q_i^n + \sigma_i^n(x - \bar{u}\Delta t - x_i) = Q_i^n + \sigma_i^n(x - \bar{u}\Delta t - x_i),$$

with $\bar{u} > 0$. We now have that

$$\Delta x Q_i^{n+1} = \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x, t_{n+1})dx = \Delta x Q_i^n + \int_{x_{i-1/2}}^{x_{i+1/2}} \sigma_i^n(x - \bar{u}\Delta t - x_i)dx.$$

In this case $\sigma_i^n = \frac{Q_i^n - Q_{i-1}^n}{\Delta x}$, thus only calculating $\int_{x_{i-1/2}}^{x_{i+1/2}} \sigma_i^n(x - \bar{u}\Delta t - x_i)dx$:

$$\begin{aligned}
\int_{x_{i-1/2}}^{x_{i+1/2}} \sigma_i^n(x - \bar{u}\Delta t - x_i)dx &= \int_{x_{i-1/2}}^{x_{i+1/2}} \sigma_i^n x dx - \int_{x_{i-1/2}}^{x_{i+1/2}} \sigma_i^n \bar{u}\Delta t + \sigma_i^n x_i dx \\
&= \frac{1}{2}\sigma_i^n(x_{i+1/2}^2 - x_{i-1/2}^2) - \bar{u}\Delta t\sigma_i^n t\Delta x - \sigma_i^n x_i \Delta x \\
&= \sigma_i^n x_i \Delta x - \bar{u}\Delta t(Q_i^n - Q_{i-1}^n) - \sigma_i^n x_i \Delta x \\
&= -\bar{u}\Delta t(Q_i^n - Q_{i-1}^n)
\end{aligned}$$

We thus know for sure that

$$Q_i^{n+1} = Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n),$$

which is only an upwind method. It's not a surprise that, since I chose σ_i^n as the upwind slope, that I get the upwind method. I do not know how I would be able to get the result. Unless

$$\sigma_i^n = Q_i^n - Q_{i-1}^n - \frac{1}{2}(\Delta x - \bar{u}\Delta t)(\sigma_i^n - \sigma_{i-1}^n) \Rightarrow \sigma_i^n = \frac{1}{1 + (\Delta x - \bar{u}\Delta t)/2}\left(Q_i^n - Q_{i-1}^n + \frac{1}{2}(\Delta x - \bar{u}\Delta t)\sigma_{i-1}^n\right)$$

Problem #6.4 in the book:

Show that

$$TV(Q^{n+1}) \leq TV(\tilde{q}(\cdot, t_{n+1})),$$

is valid by showing that, for any function $q(x)$, if we define discrete values Q_i by averaging $q(x)$ over grid cells, then $TV(Q) \leq TV(q)$.

Hint: Use the definition,

$$TV(q) = \sup \sum_{j=1}^{N} |q(\xi_j) - q(\xi_{j-1})|,$$

and the fact that the average value of q lies between the maximum and minimum values on each grid cell.

Note that this is for the Cauchy problem, so there are infinitely many cells, but you can assume the total variation of $q(x)$ is finite. In the definition (6.19) referred to in the hint, "sup" stands for supremum and this just means the maximum in the case where this value might not actually be attained for any finite N (so $TV(q)$ is the least upper bound on the sum over all choices of N points). For example, a function with infinitely many oscillations but with exponentially decaying amplitude would have finite TV, but for any finite N the sum in (6.19) would be strictly less than $TV(q)$.

To make this problem a bit easier without changing the main point, it is fine to suppose we are just on a finite interval (or on the full real line but for functions that are identically constant outside of some finite interval). Then the hint in the problem suggests a finite number of points to consider. You might also want to first consider the case where $q(x)$ is continuous, and choose points suggested by the mean value theorem.

Solution: Let's first assume that q is continuous. First, remind yourself that

$$TV(Q) = \sum_{i=-\infty}^{\infty} |Q_i - Q_{i-1}|.$$

We assume that this sum is finite and these Q_i are the average of q over cell $[x_i, x_{i+1}]$. Since we have sup in the definition of $TV(q)$, we know that if we find just one set of ξ_i for which the inequality would hold, the general inequality

$$TV(Q) \leq TV(q)$$

holds. The answer for q continuous is very simple, given the assumption that the Q_is have the same values after a certain point when going to ∞ and $-\infty$. Let's assume this, such that the sum becomes finite (n points). Since it's continuous we can use the mean value theorem, which implicates that there will always exist a $c_i \in]x_i, x_{i+1}[$ such that $q(c_i)$ is equal to the average Q_i,:

$$q(c_i) = \frac{1}{\Delta x} \int_{x_i}^{x_{i+1}} q(x) dx.$$

Therefore

$$TV(Q) \leq \sum_{i=1}^{n} |Q_i - Q_{i-1}| = \sum_{i=1}^{n} |q(c_i) - q(c_{i-1})| \leq \sup \sum_{j=1}^{N} |q(\xi_j) - q(\xi_{j-1})| = TV(q).$$

(I'm unsure what N is exactly) The same would hold for piecewise continuous function with discontinuities at the grid points x_i. This as we said that $c_i \in]x_i, x_{i+1}[$, in particular we exclude the grid points. Since we usually said that $\tilde{q}$ would be piecewise linear, piecewise polynomial at most, we have shown that

$$TV(Q^{n+1}) \leq TV(\tilde{q}(\cdot, t_{n+1})).$$

I don't see a simple way of proving it for a random, discontinuous, q. One could look at using the maximums and minimums, but even then, a situation might be made where the inequality does not hold.

Problem #6.7 in the book:

Show that if $\bar{u} < 0$ we can apply Theorem 6.1 to the flux-limiter method,

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \nu(Q_{i+1}^n - Q_i^n) \\ &= \quad + \frac{1}{2}\nu(1+\nu)[\phi(\theta_{i+1/2}^n)(Q_{i+1}^n - Q_i^n) - \phi(\theta_{i-1/2}^n)(Q_i^n - Q_{i-1}^n)] \end{aligned}$$

by choosing

$$\begin{aligned} C_{i-1}^n &= 0, \\ D_i^n &= -\nu + \frac{1}{2}\nu(1+\nu)\left(\phi(\theta_{i+1/2}^n) - \frac{\phi(\theta_{i-1/2}^n)}{\theta_{i-1/2}^n}\right) \end{aligned}$$

in order to show that the method is TVD provided $-1 \leq \nu \leq 0$ (because of $\bar{u}$) and the bound,

$$\left|\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right| \leq 2, \qquad \forall \theta_1, \theta_2$$

holds. Hence the same restrictions on limiter functions are found in this case as discussed in Section 6.12.

Solution: Theorem 6.1 states:
Consider a general method of the form

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n(Q_i^n - Q_{i-1}^n) + D_i^n(Q_{i+1}^n - Q_i^n)$$

over one time step, where the coefficients C_{i-1}^n and D_i^n are arbitrary values (which in particular may depend on values of Q^n in some way, i.e., the method may be nonlinear).
Then

$$TV(Q^{n+1}) \leq TV(Q^n)$$

provided the following conditions are satisfied:

$$\begin{aligned} C_{i-1}^n &\geq 0 \forall i, \\ D_i^n &\geq 0 \forall i, \\ D_i^n + C_i^n &\leq 1 \forall i, \end{aligned}$$

Since $C_{i-1}^m = 0$, we only need to prove that

$$0 \leq D_i^n \leq 1.$$

For this, let's assume that $-1 \leq \nu \leq 0$ and

$$\left|\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right| \leq 2, \qquad \forall \theta_1, \theta_2 \Rightarrow -2 \leq \frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2) \leq 2, \qquad \forall \theta_1, \theta_2.$$

First the first inequality

$$\begin{aligned}
-\nu + \frac{1}{2}\nu(1+\nu)\left(\phi(\theta_{i+1/2}^n) - \frac{\phi(\theta_{i-1/2}^n)}{\theta_{i-1/2}^n}\right) &\geq -\nu - \nu(1+\nu) \\
&\geq -\nu(2+\nu) \\
&\geq 0 \qquad\qquad\qquad ((2+\nu) \geq 0, \quad -\nu \geq 0).
\end{aligned}$$

The second inequality can be shown as follows:

$$\begin{aligned}
-\nu + \frac{1}{2}\nu(1+\nu)\left(\phi(\theta_{i+1/2}^n) - \frac{\phi(\theta_{i-1/2}^n)}{\theta_{i-1/2}^n}\right) &\leq -\nu + \nu(1+\nu) \\
&\leq \nu^2 \\
&\leq 1.
\end{aligned}$$

In conclusion

$$TV(Q^{n+1}) \leq TV(Q^n),$$

which is what must hold, for any set of data Q^n, such that a method is called TVD. We did not specify the set of Q^n, so it's any set of data.

Problem #8.1 in the book:

Consider the centered method

$$, Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}[f(Q_{i+1}^n) - f(Q_{i-1}^n)]],$$

for the scalar advection equation $q_t + \bar{u}q_x = 0$. Apply von Neumann analysis to show that this method is unstable in the $2-$norm for any fixed $\Delta x/\Delta t = \nu/\bar{u}$.

Solution: Let's say that $\Delta t/\Delta x = 2\nu/\bar{u}$, which is a constant. Note that we are dealing with the scalar advection equation, thus $f(q) = \bar{u}q$. Let's start by using $Q_I^n = e^{i\xi I\Delta x}$, let us fill this into the centered method

$$\begin{aligned}
Q_I^{n+1} &= e^{i\xi I\Delta x} - \frac{\nu}{2}\left(e^{i\xi(I+1)\Delta x} - e^{i\xi(I-1)\Delta x}\right) \\
&= \left(1 - \frac{\nu}{2}\left[e^{i\xi\Delta x} - e^{-i\xi\Delta x}\right]\right) e^{i\xi I\Delta x} \\
&= (1 + \nu\sin(\xi\Delta x)i)e^{i\xi I\Delta x} \\
&= g(\xi, \delta x, \delta t)Q_I^n
\end{aligned}$$

We must now have that $|g(\xi, \Delta x, \Delta t)| \leq 1$, $\forall \xi$, in order to have stability in the 2-norm. But, take $\xi_1 = \frac{\pi}{2\Delta x}$, it is easy to see then that

$$|g(\xi_1, \delta x, \delta t)| = |1 + \nu i| = \sqrt{1 + \nu^2} \geq 1.$$

Note also that $\sqrt{1 + \nu^2}$ is not of the form $1 + \alpha \Delta t$, since ν is fixed (this Δt does nothing because the Δx negates it's effects). For this reason, we know that the centered method for the advection equation is unstable in the $2 - norm$ for any fixed $\Delta x / \Delta t$.

Extra details on why: We supposed that Q_I^n has a finite norm first, so that we can take a Fourier series of it, with $\mathcal{F}(Q_I^n)[\xi] = \hat{Q}(\xi)$. Assume that if we were to apply a linear finite difference method to Q_I^n and manipulate the exponentials, we could get something of the form $Q_I^{n+1} = \mathcal{F}^{-1}(\hat{Q}^n(\xi) g(\xi, \Delta x, \delta t)$. with g the amplification factor for wave number ξ, since we can take the Fourier transform of Q_I^{n+1} and see that

$$\hat{Q}^{n+1}(\xi) = \hat{Q}^n(\xi) g(\xi, \Delta x, \delta t.$$

Now we could use Parseval's relation, which states,

$$\|Q^n\|_2 = \|\hat{Q}^n\|_2.$$

Here we have that

$$\|\hat{Q}^n\|_2^2 = \int_{-\infty}^{\infty} |\hat{Q}^n|^2 d\xi = \int_{-\infty}^{\infty} |g|^{2n} |\hat{Q}^0|^2 d\xi.$$

Thus, if $|g(\xi)| \leq 1$, $\forall \xi$, we know that for $n \to \infty$ this can go to zero or at least be finite (if $|g| = 1$ somewhere). The book says that it also suffices if $|g| \leq 1 + \alpha \Delta t$, with α constant. Since $\hat{Q}_I^n$ only depends their values in their on element, they are decoupled from the rest. It is thus enough to only consider a single random ξ and a Q_I^n of the form

$$Q_I^n = e^{i \xi I \Delta x}.$$

Problem #8.9 in the book:

Determine the modified equation for the centered method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}[f(Q_{i+1}^n) - f(Q_{i-1}^n)]]$$

and show that the diffusion coefficient is always negative and this equation is hence ill posed. Recall that the method is unstable for all fixed $\Delta x / \Delta t$.

Solution:

Let's say we're going to put function $v(x, t)$ through the method instead of the true solution. We will be using Taylor approximations on it, so for example $v_{i-1}^{n+1} = v(x - \Delta x, t + \delta t$. Filing this in gives us

$$v(x, t + \Delta t) = v(x, t) - \frac{\Delta t \bar{u}}{2\Delta x}(v(x + \Delta x, t) - v(x - \Delta x, t).$$

Using Taylor around (x,t), we get

$$\begin{aligned}
&v(x,t) + \Delta t v_t(x,t) + \Delta t^2 v_{tt}(x,t)/2 + \Delta t^3 v_{ttt}(x,t)/6 + \dots \\
=&v(x,t) - \frac{\bar{u}\Delta t}{2\Delta x}(v(x,t) + \Delta x v_x(x,t) + \Delta x^2 v_{xx}(x,t)/2 + \Delta x^3 v_{xxx}(x,t)/6 \\
&- v(x,t) + \Delta x v_x(x,t) - \Delta x^2 v_{xx}(x,t)/2 + \Delta x^3 v_{xxx}(x,t)/6 + \dots) \\
=&v(x,t) - \bar{u}\Delta t v_x - \bar{u}\Delta t \Delta x^2 v_{xxx}(x,t)/6 + \dots
\end{aligned}$$

Therefore

$$v_t + \bar{u} v_x = -\Delta t v_{tt}/2 - (\Delta t^2 v_{ttt} + \bar{u}\Delta x^2 v_{xxx})/6 + \dots$$

Keeping the first order terms ·⊔, we get the PDE:

$$v_t + \bar{u} v_x = -\Delta t v_{tt}/2. \tag{1}$$

Deriving by the PDE's by t and x again, we get

$$v_{tt} = -\bar{u} v_{xt} - \Delta t v_{ttt}/2,$$

and

$$v_{tx} = -\bar{u} v_{xx} - \Delta t v_{ttx}/2.$$

Assuming we can say that $v_{xt} = v_{tx}$, we can substitute both PDEs to get

$$v_{tt} = \bar{u}^2 v_{xx} + \bar{u}\Delta t v_{ttx}/2 - \Delta t v_{ttt}/2.$$

Now we can substitute v_{tt} in (1) to get

$$v_t + \bar{u} v_x = -\Delta t \bar{u}^2 v_{xx}/2 + \mathcal{O}(\Delta t^2).$$

When dropping the second order terms again, we get

$$v_t + \bar{u} v_x = -\bar{u}^2 \Delta t v_{xx}/2.$$

This is the modified equation for the centered method. Here the diffusion coefficient is $-\bar{u}^2\Delta t/2$ which is clearly always, or atleast all interesting cases, negative. Thus it's ill posed.

Problem #7.2 in the book

(a) For acoustics with a solid-wall boundary, we set the ghost-cell values,

$$\begin{aligned}
&\text{for } Q_0: \quad && p_0 = p_1, \quad && u_0 = -u_1, \\
&\text{for } Q_{-1}: \quad && p_{-1} = p_2, \quad && u_{-1} = -u_2,
\end{aligned}$$

and then solve a Riemann problem at $x_{1/2} = a$ with data

$$Q_0 = \begin{bmatrix} p_1 \\ -u_1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} p_1 \\ u_1 \end{bmatrix}.$$

Show that the solution to this Riemann problem has an intermediate state q^* with $u^* = 0$ along the wall, another reason why this is the sensible boundary condition to impose.

(b) Give a similar interpretation for the oscillating-wall boundary conditions

$$\begin{aligned} &\text{for } Q_0: \quad && p_0 = p_1, && u_0 = 2U(t_n) - u_1, \\ &\text{for } Q_{-1}: \quad && p_{-1} = p_2, && u_{-1} = 2U(t_n) - u_2. \end{aligned}$$

with, for example, $U(t) = \epsilon \sin(\omega t)$.

Solution:

(a) Since we are dealing with the acoustics equation we know that the eigenvectors are given by

$$(-c_0, (-Z_0, 1)^T) \text{ and } (c_0, (Z_0, 1)^T).$$

The solution of u is given by

$$u(x,t) = -\frac{1}{2Z_0}[\mathring{p}(x + c_0 t) - \mathring{p}(x - c_0 t)] + \frac{1}{2}[\mathring{u}(x + c_0 t) + \mathring{u}(x - c_0 t)].$$

Let's look at the wall and say that it's at $x = a$ (Q_0 (the ghost cell) is to the left of $x = a$ and Q_1 to the right). Then

$$u(a,t) = -\frac{1}{2Z_0}[\mathring{p}(a + c_0 t) - \mathring{p}(a + c_0 t)] + \frac{1}{2}[\mathring{u}(a - c_0 t) + \mathring{u}(a - c_0 t)].$$

if we use the correct method, we know that the if t puts $\mathring{p}(a + c_0 t)$ or $\mathring{u}(a + c_0 t)$ in the area where Q_1 or Q_2 lives, then we know that $\mathring{p}(a - c_0 t)$ or $\mathring{u}(a - c_0 t)$ lives in the area where, respectively, Q_0 and Q_{-1} lives. Since $p_0 = p_1$ (thus also the initial conditions) and $p_{-1} = p_2$, we know that the $\mathring{p}(c_0 t) - \mathring{p}(c_0 t)$ term is always zero at the boundary. Since $u_0 = -u_1$ and $u_{-1} = -u_2$, we know that the $\mathring{u}(c_0 t) + \mathring{u}(c_0 t)$ term is also always zero at the boundary. Therefore $u(0,t)$ is always zero. The intermediate states are always zero at the boundary. One can also find p^* in a similar way at the wall, this is not necessarily zero of course.

(b) If the amplitude is very very small, then the intermediate state will also small (since the boundary conditions are pretty much the same as in (a)), but not zero, thus it no longer acts as a wall blocking velocity.

The velocity decreases or increases (depending on the time) in value when jumping the boundary, the wall gives and takes velocity. However, note that the pressure does not change. My explanation in HW2 for the solid-wall being zero was that the change of the pressure waves were canceling each other (pressure coming from the left canceling the bounced pressure) resulting in the velocity being zero. Now the velocity is no longer zero, thus the change in pressure isn't canceling each other out, but the pressure is not decreasing. My guess is that the wall is keeping a bit of the fluid before releasing it, this because of changes in the wall such as molecular vibrations (when the aplitude is small).

Problem #7.3 in the book:

The directory `[claw/book/chap7/standing]` models a standingwave. The acoustics equations are solved with $\rho_0 = K_0 = 1$ in a closed tube of length 1 with with initial data $\mathring{p} = cos(2x)$ and $\mathring{u}(x) = 0$. Solid-wall boundary conditions are used at each end. Modify the input data in `claw1ez.data` to instead use zero-order extrapolation at the right boundary $x = 1$. Explain the resulting solution using the theory of Section 3.11.

Note that you can view the Clawpack solution to this problem in the Clawpack gallery of examples from `$CLAW/apps/fvmbook`, at

http://www.clawpack.org/gallery/gallery/gallery_fvmbook.html.

Plots for this particular example can be viewed here.

The first part of the problem can be done by viewing these plots and explaining the results.

The second part requires changing the boundary condition in the Clawpack code, and for this please let me know if you are having problems installing or using Clawpack. Only one line of the `setrun.py` file needs to be changed, so the main point of this exercise is to get Clawpack working.

I cleaned up the code for this example a bit, so you should get the most recent version by doing a "git pull" from the `am574` branch of my fork, as described in the video on installing Clawpack (`Clawpack01`).

Also note that Clawpack has changed since the book was written, and parameters are now set in the Python script 'setrun.py' rather than in the '.data' file mentioned in the problem. For extrapolation BCs you can set
`clawdata.bc_upper[0] = 'extrap'`

Please read Section 7.2.1 about these boundary conditions.

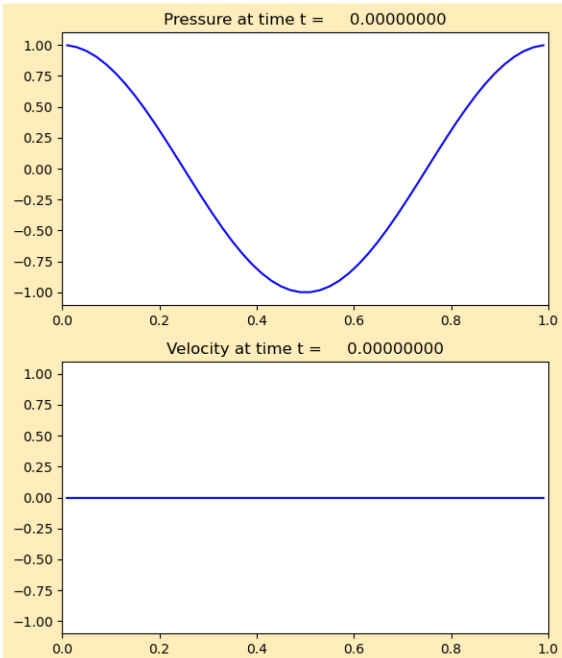
Note also that this code is set up to use
`clawdata.cfl_desired = 1.0`
so that even the upwind method is "exact" for the acoustics equations, so you are seeing a good representation of how the acoustics solution should behave with these BCs.

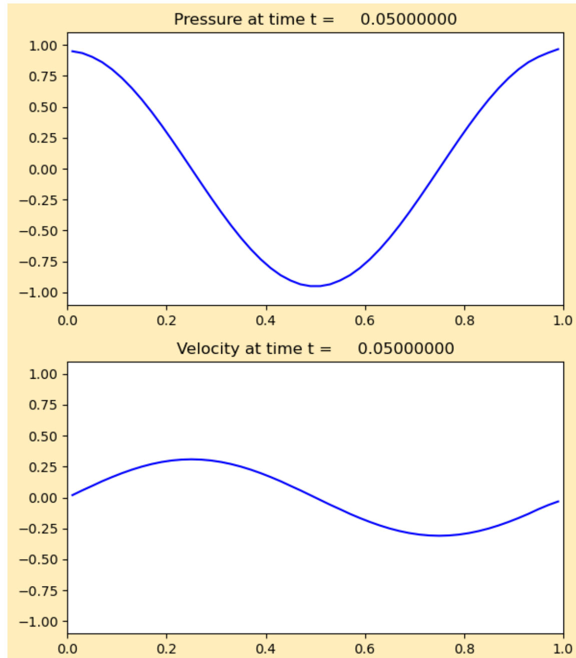
In addition to describing the behavior, please also include the png files from a couple frames you computed at interesting times in relation to your description. Note that if you do
`make .plots`
in this directory then the `_plots` directory contains png files of the plots at each output frame.

Solution:
With an extrapolation method we would have that, for $N+1$ Q_is, Q_N^{i+1} is calculated using Q_{N+1}^{i} or even Q_{N+2}^{i+1} where these are calculated/set in a different manner than the method. For example, extrapolation of the *zero-order* is setting

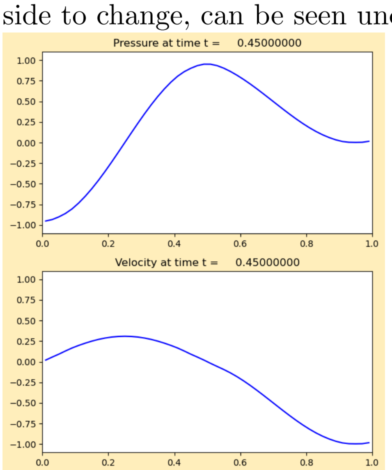
$$Q_{N+1}^n = Q_{N+2}^n = Q_N^n.$$

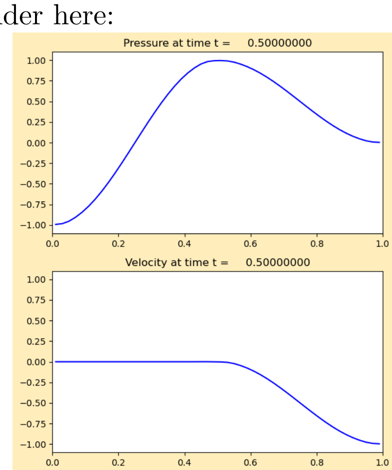
This behaves in roughly the same manner as a standard first-order upwind method. In the code we have that the left side is a wall, which will result in $u(0) = 0$. Since the extrapolation is going to be using 2 constant values always, my guess would be that the right side will kind of stay behind. It will kind of be like a whip, it'll take some effort to change it, but once it's changed it will stay like that and take effort to change again since it uses two constant values at the end point and that end point is then also used to calculate other points. Of course, since one side is a wall and the other is kind of not, more like a penetrable wall, we have that eventually the pressure and velocity will be constant. First we plot the first two frames to see the beginning:

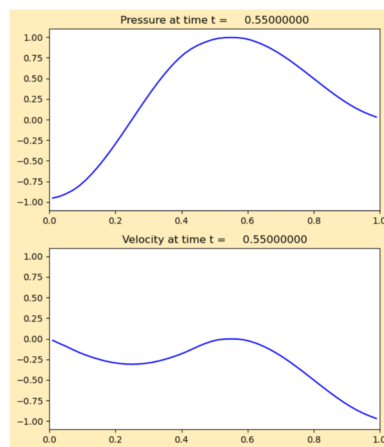




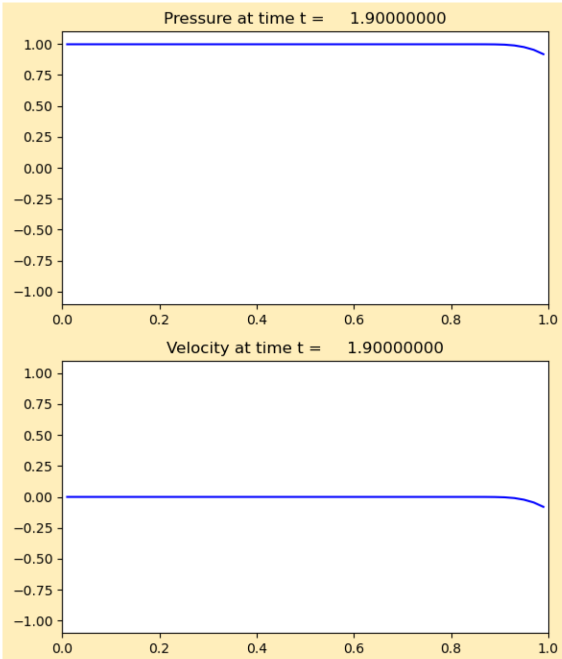
Three interesting consecutive plots, where I think you can see the effort it takes for the right side to change, can be seen under here:

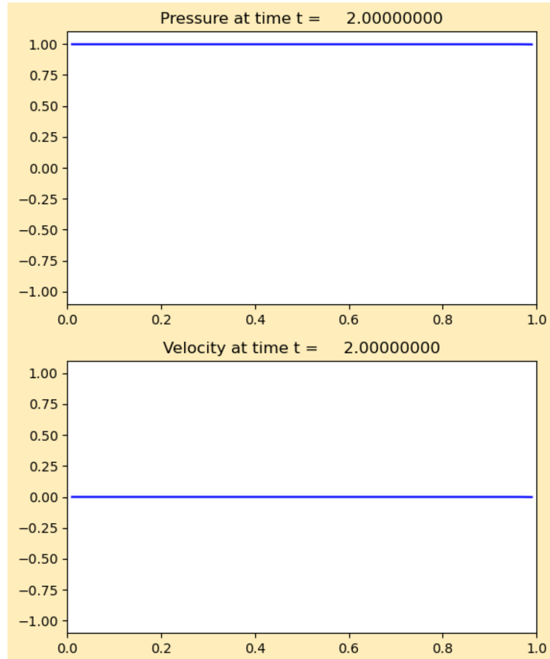






The end situation can be seen under here, we can see that it does turn constant since there's a flow out:


Pressure at time t = 1.90000000
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.0
0.2
0.4
0.6
0.8
1.0
Velocity at time t = 1.90000000


Pressure at time t = 2.00000000
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.0
0.2
0.4
0.6
0.8
1.0
Velocity at time t = 2.00000000