AMath 574
Homework 1, Wietse Vaes 2224416
Due by 11:00pm on January 17, 2023 (updated)

For submission instructions and some additional information, see:

`http://faculty.washington.edu/rjl/classes/am574w2023/homework1.html`

#1.

The gas dynamics equations (2.38) are written as conservation laws for the mass and momentum, which are conserved quantities. Often the so-called "primitive variables" $p(x,t) =$ pressure and $u(x,t) =$ velocity are more natural to use, e.g. in the acoustic equations we consider perturbations of p and u modeled by the linear system (2.5), rather than the linear system with matrix (2.46) modeling perturbations in mass and momentum.

(a) Starting from (2.38), derive the following nonlinear equations for the pressure and velocity:

$$\begin{aligned} p_t + up_x + \rho P'(\rho)u_x &= 0, \\ u_t + (1/\rho)p_x + uu_x &= 0. \end{aligned} \tag{1}$$

Note that these equations involve $\rho(x,t)$ which must now be determined from $p(x,t)$ using the equation of state, by inverting $p = P(\rho)$ for ρ as a function of p. (Exactly what this gives depends on the particular equation of state, so you don't need to go farther.)

Hint: Use for example $(\rho u)_t = \rho_t u + \rho u_t$ in the conservation law for momentum and then use the conservation law for ρ_t, and also note $P(\rho)_x = P'(\rho)\rho_x$, etc.

(b) The equations (1) can be written in the form

$$q_t(x,t) + A(q(x,t))q_x(x,t) = 0,$$

a non-conservative nonlinear system. Determine the matrix $A(q)$ and the eigenvalues of the matrix.

The system is said to be hyperbolic if the matrix is diagonalizable with real eigenvalues. Confirm that the resulting condition on $P'(\rho)$ (and the eigenvalues) agree with what we found from the conservative form (2.38). Also note that if we linearize (1) about

$$p \approx p_0, \quad u \approx u_0, \quad \rho P'(\rho) \approx \rho_0 P'(\rho_0) \equiv K_0,$$

then the linearized equations agree with the acoustics equations (2.50).

Note: This gives a nice derivation of the linear acoustics equations, but when solving a nonlinear gas dynamics problem it is generally necessary to use the conservative form in order to get proper modeling of shock waves and other nonlinear phenomena. Smooth solutions should agree between the two formulations.

Solution:

(a) System of equations (2.38) are:

$$\rho_t + (\rho u)_x = 0,$$
$$(\rho u)_t + (\rho u^2 + P(\rho))_x = 0.$$

This becomes, given that ρ is sufficiently smooth and non-zero.

$$\begin{cases} \rho_t & = -(\rho u)_x, \\ \rho_t u + \rho u_t + \rho_x u^2 + 2\rho u u_x + P(\rho)_x & = 0. \end{cases}$$
$$\Rightarrow \begin{cases} \rho_t & = -(\rho u)_x, \\ \rho_t u + (\rho u)_x u + \rho u_t + \rho u u_x + P(\rho)_x & = 0. \end{cases}$$
$$\Rightarrow \begin{cases} \rho_t & = -(\rho u)_x, \\ \rho u_t + \rho u u_x + P(\rho)_x & = 0. \end{cases}$$
$$\Rightarrow \begin{cases} P'(\rho)\rho_t + P'(\rho)(\rho u)_x & = 0, \\ u_t + u u_x + (1/\rho) p_x & = 0. \end{cases}$$
$$\Rightarrow \begin{cases} P(\rho)_t + P'(\rho)\rho_x u + P'(\rho)\rho u_x & = 0, \\ u_t + u u_x + (1/\rho) p_x & = 0. \end{cases}$$
$$\Rightarrow \begin{cases} P(\rho)_t + P(\rho)_x u + \rho P'(\rho) u_x & = 0, \\ u_t + u u_x + (1/\rho) p_x & = 0. \end{cases}$$
$$\Rightarrow \begin{cases} p_t + p_x u + \rho P'(\rho) u_x & = 0, \\ u_t + u u_x + (1/\rho) p_x & = 0. \end{cases}$$

Note that in the second to third step, we use the first equation in the second one, and in the third to fourth step we multiply the first equation by $P'(\rho)$. In conclusion: from (2.38) we can get (1).

(b) With $q = (p, u)^T$, it is easy to see that

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} u & P^{-1}(p)P'(P^{-1}(p)) \\ 1/P^{-1}(p) & u \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0$$

Which, if we were to keep that $\rho = P^{-1}(p)$ for notation, we get the asked form

$$q_t(x,t) + A(q(x,t))q_x(x,t) = 0,$$

with

$$A(q(x,t)) = \begin{bmatrix} u & \rho P'(\rho) \\ 1/\rho & u \end{bmatrix}.$$

It eigenvalues are calculated as follows:

$$\det(A - \lambda I_2) = \left| \begin{bmatrix} u-\lambda & \rho P'(\rho) \\ 1/\rho & u-\lambda \end{bmatrix} \right| = \lambda^2 - 2u\lambda + u^2 - P'(\rho) = 0.$$

Therefore, the eigenvalues are given by

$$\lambda = u \pm \frac{\sqrt{4u^2 - 4u^2 + 4P'(\rho)}}{2} = u \pm \sqrt{P'(\rho)}.$$

In order to get a hyperbolic system, we thus want $P'(\rho) > 0$. This is exactly what we found (look at example 2.1).

On the other hand, linearizing the equations around

$$p \approx p_0, \quad u \approx u_0, \quad \rho P'(\rho) \approx \rho_0 P'(\rho_0) \equiv K_0, ,$$

we get:

$$\begin{aligned} p_t + (u_0 + (u - u_0))(p_0 + (p - p_0))_x + (K_0 + ...)(u_0 + (u - u_0)_x = 0, \\ u_t + (1/\rho_0 - 1/\rho_0^2(\rho - \rho_0) + ...)(p_0 + (p - p_0))_x + (u_0 + (u - u_0)(u_0 + (u - u_0)_x = 0. \end{aligned}$$

If we were to ignore multiples of two functions or squares of said functions, we get the linearized equations:

$$\begin{aligned} p_t + u_0 p_x + K_0 u_x \approx 0, \\ u_t + \frac{1}{\rho_0} p + u_0 u \approx 0. \end{aligned}$$

These are exactly the acoustics equations (2.50).

Problem #2.7 in the book:
Show that the $p-$system,

$$v_t - u_x = 0,$$
$$u_t + p(v)_x = 0,$$

is hyperbolic provided the function $p(v)$ satisfies $p'(v) < 0$ for all v.

Solution:

Given that v is sufficiently smooth, the $p-$system can be written as

$$\begin{cases} v_t - u_x & = 0, \\ u_t + p'(v)v_x & = 0, \end{cases}$$
$$\Rightarrow \begin{bmatrix} v \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}_x = 0.$$

In order for the system to be hyperbolic, we want the two by two matrix to be diagonalizable with real eigenvalues. The eigenvalues of said matrix are,

$$\left| \begin{bmatrix} -\lambda & -1 \\ p'(v) & -\lambda \end{bmatrix} \right| = \lambda^2 + p'(v) = 0 \Rightarrow \lambda = \pm\sqrt{-p'(v)}.$$

In conclusion: we have two non-zero real eigenvalues if $p'(v) < 0$ for all v. This would mean that the $p-$system is hyperbolic.

Problem #2.8 in the book:
Isothermal flow is modeled by the system,

$$\rho_t + (\rho u)_x = 0,$$
$$(\rho u)_t + (\rho u^2 + \boldsymbol{P}(\rho))_x = 0,$$

with $\boldsymbol{P}(\rho) = a^2\rho$, where a is constant.

(a) Determine the wave speeds of the linearized equations,

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0,$$

in this case.

(b) The Lagrangian form of the isothermal equations have $p(V) = a^2/V$. Linearize the $p-$system,

$$V_t - U_\xi = 0,$$
$$U_t + p(V)_\xi = 0,$$

in this case about V_0, U_0, and compute the wave speeds for Lagrangian acoustics. Verify that these are what you expect in relation to the eulerian acoustic wave speeds.

Hint: Read Section 2.13 first and note that when linearizing about a constant specific volume V_0, the relation (2.102) between ξ and x is roughly $\xi = (x - x_0)/V_0$.

Solution:

(a) First we calculate the eigenvalues of the two by two matrix:

$$\left| \begin{bmatrix} u_0 - \lambda & K_0 \\ 1/\rho_0 & u_0 - \lambda \end{bmatrix} \right| = \lambda^2 - 2u_0\lambda + u_0^2 - \frac{K_0}{\rho_0} = 0 \Rightarrow \lambda = u \pm \sqrt{\frac{K_0}{\rho_0}}.$$

Since the eigenvalues should be of the form $\lambda = u \pm c$, with c the wave speeds, we know that the wave speeds are $\pm\sqrt{\frac{K_0}{\rho_0}}$

(b) Linearizing around U_0 and V_0, we get

$$\begin{aligned} V_t - U_\xi &= 0, \\ U_t + (\frac{a^2}{V})_\xi &= U_t + (\frac{a^2}{V_0} - \frac{a^2}{V_0^2}(V - V_0) + \dots)_\xi \approx U_t - \frac{a^2}{V_0^2}V_\xi \approx 0, \end{aligned}$$

Therefore, we get the following linearized equations:

$$\begin{bmatrix} V \\ U \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ -\frac{a^2}{V_0^2} & 0 \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix}_x = 0.$$

Calculating the eigenvalues of the two by two matrix gives us:

$$\left| \begin{bmatrix} -\lambda & -1 \\ -\frac{a^2}{V_0^2} & -\lambda \end{bmatrix} \right| = \lambda^2 - \left(\frac{a}{V_0}\right)^2 \Rightarrow \lambda = \pm\frac{a}{V_0}.$$

Here the speeds are thus $\pm\frac{a}{V_0}$. However, we can also look at the Eulerian equations. Since $x = \xi V_0 + x_0$, we can put the system back into x coordinates:

$$\begin{aligned} V_t - V_0 U_x &= 0, \\ U_t + V_0 p(V)_x &= 0, \end{aligned}$$

Linearizing this around V_0 and U_0 gives us

$$\begin{aligned} V_t - V_0 U_x &= 0, \\ U_t + V_0(\frac{a^2}{V})_x &= U_t + V_0(\frac{a^2}{V_0} - \frac{a^2}{V_0^2}(V - V_0) + \dots)_x \approx U_t - \frac{a^2}{V_0}V_x \approx 0. \end{aligned}$$

Therefore, we get the following linearized equations:

$$\begin{bmatrix} V \\ U \end{bmatrix}_t + \begin{bmatrix} 0 & -V_0 \\ -\frac{a^2}{V_0} & 0 \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix}_x = 0.$$

Calculating the eigenvalues of the two by two matrix gives us:

$$\left| \begin{bmatrix} -\lambda & -V_0 \\ -\frac{a^2}{V_0} & -\lambda \end{bmatrix} \right| = \lambda^2 - a^2 \Rightarrow \lambda = \pm a.$$

We thus have the wave speeds $\pm a$, for the Eulerian equations. In conclusion, the Lagrangian acoustic wave speeds are $\pm \frac{a}{V_0}$, which is just the Eulerian wave speeds divided by V_0.

Problem #3.1(d,e,f) in the book You might want to do Problem 3.2 first.
For each of the Riemann problems below sketch the solution in the phase plane, and sketch $q^1(x,t)$ and $q^2(x,t)$ as functions of x at some fixed time t:

$$(d)\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad q_l = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad q_r = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$(e)\quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad q_l = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(f)\quad \begin{bmatrix} 2 & 1 \\ 10^{-4} & 2 \end{bmatrix}, \quad q_l = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution:

(d) First we find the eigenvalues and eigenvectors of the matrix:

$$\left| \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \right| = \lambda^2 - 2\lambda = 0 \Rightarrow (\lambda^1, \lambda^2) = (0, 2).$$

The eigenvectors are

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = (a, -a)^T,$$

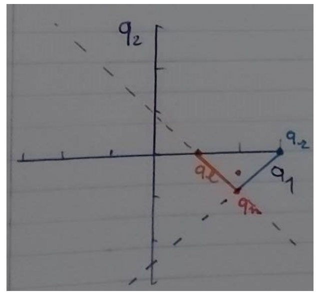
choose $a = 1$. The other eigenvector is given by:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_2 = 0 \Rightarrow v_2 = (a, a)^T,$$

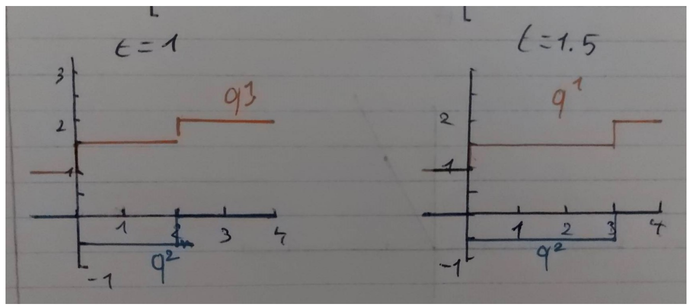
while again choosing $a = 1$. Note now that $q_r - q_l = (1,0)^T = \frac{1}{2}v_1 + \frac{1}{2}v_2 = a_1 v_1 + a_2 v_2$. Thus we have that

$$q_m = q_l + a_1 v_1 = \begin{bmatrix} 1 + \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}.$$

We thus know what value the functions are going to take on in the middle of the two jumps. The phase plane for this problem is thus:



These jumps happen at $x = 0$ initially, but after time t, they happen at 0 and $2t$. Drawing this at time $t = 1$ and $t = 1.5$, we get:



(e) First we find the eigenvalues and eigenvectors of the matrix:

$$\left\| \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} \right\| = (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2.$$

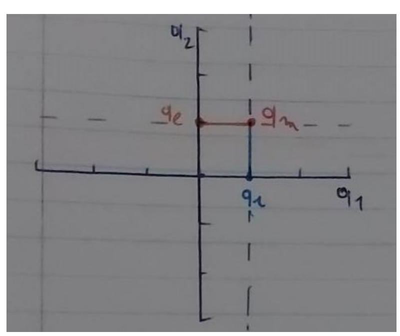
The eigenvectors are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = (a, b)^T,$$

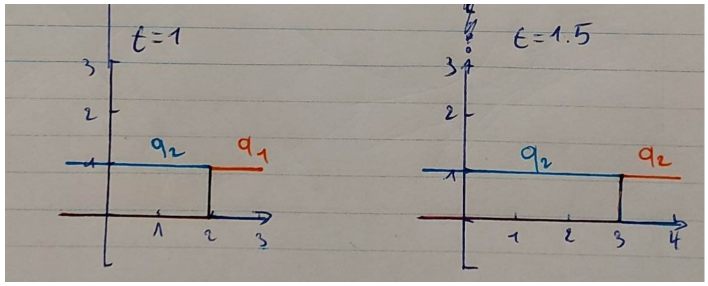
choose $a = 1$ and $b = 0$ for one eigenvector v_1 and $a = 0$ and $b = 1$ for the other v_2, linear independent, eigenvector. Note now that $q_r - q_l = (1, -1)^T = v_1 - v_2 = a_1 v_1 + a_2 v_2$. Thus we have that

$$q_m = q_l + a_1 v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We thus know what value the functions are going to take on in the middle of the two jumps. The phase plane for this problem is thus:



These jumps happen at 0 initially, but after time t, they happen at $2t$. Drawing this at time $t = 1$ and $t = 1.5$, we get:



(f) First we find the eigenvalues and eigenvectors of the matrix:

$$\left|\begin{bmatrix} 2-\lambda & 1 \\ 10^{-4} & 2-\lambda \end{bmatrix}\right| = \lambda^2 - 4\lambda + 4 - 10^{-4} = 0 \Rightarrow (\lambda^1, \lambda^2) = (2 - 10^{-2}, 2 + 10^{-2}).$$

The eigenvectors are

$$\begin{bmatrix} 10^{-2} & 1 \\ 10^{-4} & 10^{-2} \end{bmatrix} v_1 = 0 \Rightarrow v_1 = (a, -a10^{-2})^T,$$

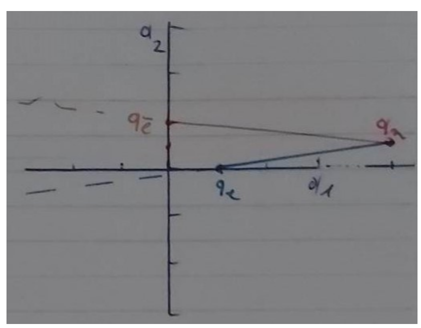
choose $a = 1$. The other eigenvector is given by:

$$\begin{bmatrix} -10^{-2} & 1 \\ 10^{-4} & -10^{-2} \end{bmatrix} v_2 = 0 \Rightarrow v_2 = (a, a10^{-2})^T,$$

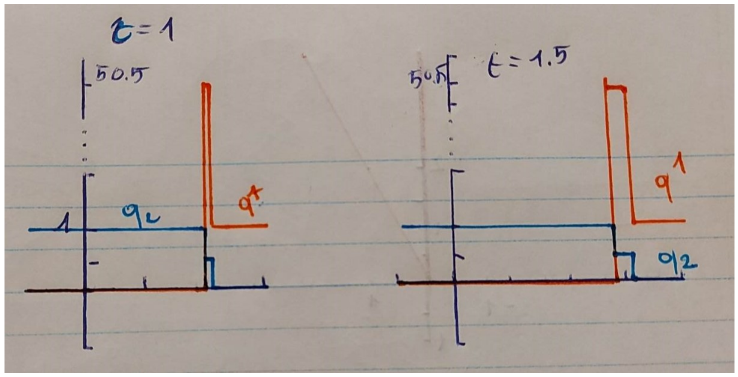
while again choosing $a = 1$. Note now that $q_r - qlr = (1, -1)^T = 50.5v_1 - 49.5v_2 = a_1v_1 + a_2v_2$. Thus we have that

$$q_m = q_l + a_1v_1 = \begin{bmatrix} 50.5 \\ .495 \end{bmatrix}$$

We thus know what value the functions are going to take on in the middle of the two jumps. The phase plane for this problem is thus:



These jumps happen at 0 initially, but after time t, they happen at $(2 - 10^{-2})t$ and $(2 + 10^{-2})t$. Drawing this at time $t = 1$ and $t = 1.5$, we get:



The effect is a bit more exaggerated.

Problem #3.2 in the book:

Write a script in MATLAB or other convenient language that, given any 2×2 matrix A and states q_l and q_r, solves the Riemann problem and produces the plots required for Exercise 3.1. Test it out on the problems of Exercise 3.1 and others.

You can use Matlab for this one, but I suggest you try writing the program in Python. A Jupyter notebook with a partial solution can be found in the class repository to help get you started.

Note that the module `numpy.linalg` contains an `eig` function similar to Matlab.

Solution: The code can be found attached with this homework. It can also be found in the Jupyter hub. This is also the code for problem 3.3.

Problem #3.3 in the book: Solve each of the Riemann problems below. In each case sketch a figure in the $x - t$ plane similar to Figure 3.3, indicating the solution in each wedge.

$$(a)\ A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad q_l = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad q_r = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

$$(b)\ A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad q_l = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

You do not need to draw the dashed lines of Figure 3.3, just the wedges with the correct wave speeds. Sketch it by hand or e.g. in a Jupyter notebook, as you please.

Solution:

(a) First we get the eigenvalues of the matrix A:

$$\left|\begin{bmatrix} -\lambda & 0 & 4 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}\right| = \lambda^2(1-\lambda)-4(1-\lambda) = (\lambda^2-4)(1-\lambda) = 0 \Rightarrow (\lambda^1, \lambda^2, \lambda^3) = (-2, 1, 2).$$

The corresponding eigenvectors can be found as follows:

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} v_1 = 0 \Rightarrow \begin{cases} v_{1,1} = -2v_{1,3} = -2a \\ v_{1,2} = 0 \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} v_2 = 0 \Rightarrow \begin{cases} v_{2,1} = v_{2,3} = 0 \\ v_{2,2} = a \end{cases}$$

$$\begin{bmatrix} -2 & 0 & 4 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix} v_3 = 0 \Rightarrow \begin{cases} v_{3,1} = 2v_{3,3} = 2a \\ v_{3,2} = 0 \end{cases}$$

All of these a's are not necessarily the same, but here we choose $a = 1$. This gives us the following eigenvalues and eigenvector pairs:

$$\left(-2, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right), \qquad \left(1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \text{ and } \left(2, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right).$$

If we now were to calculate $q_r - q_l$, we find the linear combination:

$$q_r - q_l = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{2}v_1 + 3v_2 + \frac{1}{2}v_3 = a_1v_1 + a_2v_2 + a_3v_3.$$

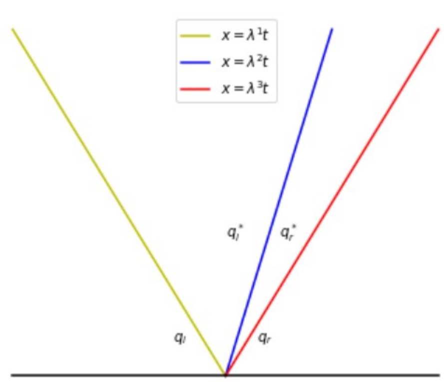
Therefore, we have the two intermediate values to getting from q_l to q_r:

$$q_l^* = q_l + a_1v_1 = \begin{bmatrix} 0 \\ 2 \\ \frac{1}{2} \end{bmatrix},$$

and,

$$q_r^* = q_l^* + a_2v_2 = \begin{bmatrix} 0 \\ 5 \\ \frac{1}{2} \end{bmatrix}$$

Thus, noting the eigenvalues and the intermediate states, we get the following $x-t$ plane:



(b) First we get the eigenvalues of the matrix A:

$$\left|\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right| = (1-\lambda)(2-\lambda)(3-\lambda) = 0 \Rightarrow (\lambda^1, \lambda^2, \lambda^3) = (1, 2, 3).$$

The corresponding eigenvectors can be found as follows:

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} v_1 = 0 \Rightarrow \begin{cases} v_{1,1} = a \\ v_{1,2} = v_{1,3} = 0 \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v_2 = 0 \Rightarrow \begin{cases} v_{2,1} = 2v_{2,3} = 0 \\ v_{2,2} = a \end{cases}$$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_3 = 0 \Rightarrow \begin{cases} v_{3,1} = v_{3,3} = a \\ v_{3,2} = 0 \end{cases}$$

All of these a's are not necessarily the same, but here we choose $a = 1$. This gives us the following eigenvalues and eigenvector pairs:

$$\left(1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \quad \left(2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \text{ and } \left(3, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

If we now were to calculate $q_r - q_l$, we find the linear combination:

$$q_r - q_l = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 0v_1 + 2v_2 + 2v_3 = a_1 v_1 + a_2 v_2 + a_3 v_3.$$

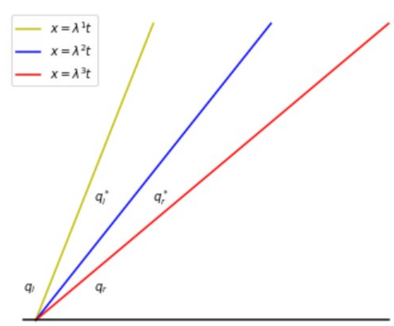
Therefore, we have the two intermediate values to getting from q_l to q_r:

$$q_l^* = q_l + a_1 v_1 = q_l = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and,

$$q_r^* = q_l^* + a_2 v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Thus, noting the eigenvalues and the intermediate states, we get the following $x-t$ plane:



Note that the function does not jump on the $x = \lambda^1 t$ line since $q_l = q_l^*$.