AMATH 481/581 - Autumn 2022 Homework #3

Presentation skill: discussing problems from a physical perspective.

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Introduction

The transport of certain substances is an important aspect of life. Therefore it would be useful if this could be modelled. One way of doing this is through the one-dimensional advection equation, otherwise known as the transport equation, given by

$$u_t + c(x,t)u_x = 0, \qquad \text{for } -\infty < x < \infty, t > 0. \tag{1}$$

Given an initial condition $u(x,0) = u_0$, the function $u(x,t)$ is the specific quantity of the substance which we would like to model under the influence of a velocity field, c. This field can be constant or dependant on the temporal and/or spatial field. Since this is the only parameter in this equation questions can arise such as: how does this affect the solution for specific c?

Results and discussion

Looking at the advection equation from a physical perspective, u_t denotes the change of u in time and u_x the spatial change of u. Therefore, since

$$u_t = -cu_x,$$

the velocity field c indicates the rate at which u_t changes over u_x. Because of the minus sign indicates that it changes in the opposite direction. Another way of looking at it, is to look at a certain point (x^*, t^*) where $u(x^*, t^*) = u^*$. If time increases, the spatial point where u reaches u^* changes at rate c.

Now we look at two concrete examples of the velocity field. Given that the initial condition is

$$u(x,0) = e^{-(x-5)^2},$$

a Gaussian-like function. We will consider two velocity fields:

$$c(x,t) = -0.5, \text{ and } c(x,t) = -(1 + 2\sin(5t) - H(x-4)),$$

with $H(x)$ the heaviside function.

Using fourth-order Runga-Kuttah and the second-order central difference method over a spatial grid of 200 equally spaced point for $-10 \leq x \leq 10$ and periodic boundary conditions, we can get an approximation of the solution of the transport equation (1) for $t \in [0, 10]$. For the first velocity field the solution can be seen in figure 1.

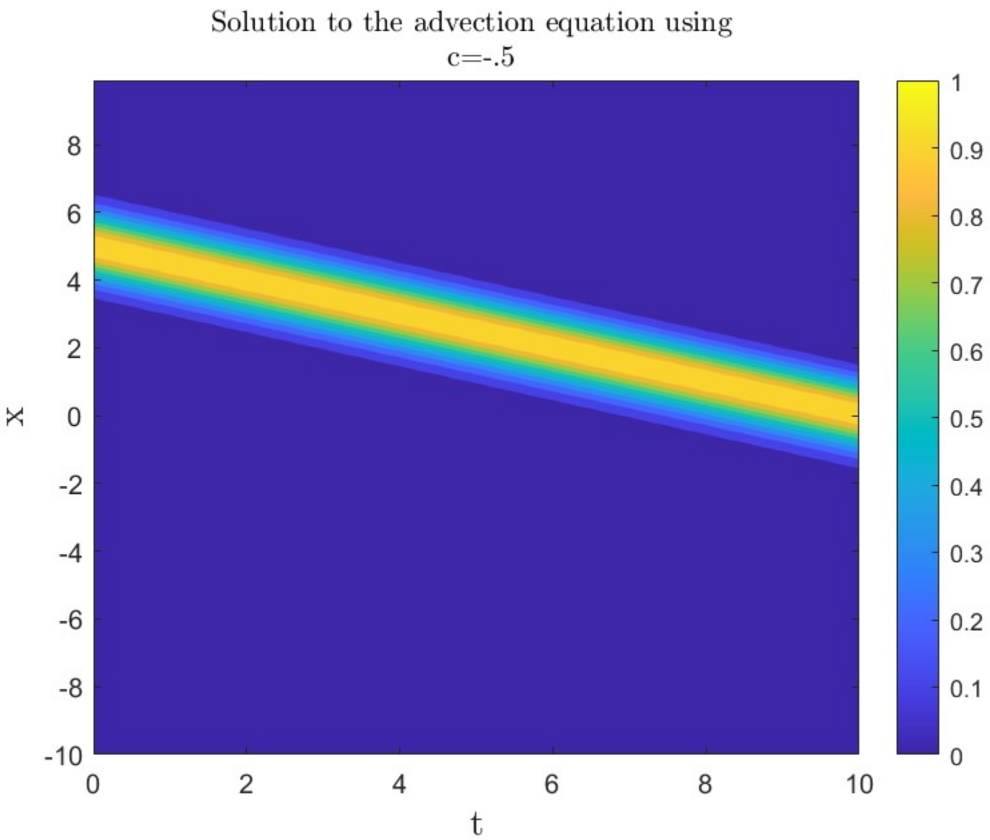

Figure 1: An approximation of the solution to the advection equation with velocity field $c = -0.5$.

It shows exactly what we expected. The function, a sort of wave, moves down, x decreases, as time increases. Since c is negative, x decreases. The speed and direction of change does not change since c is constant.

A more interesting case of c is when $c(x,t) = -(1 + 2\sin(5t) - H(x-4))$, since it is time and space dependant. The solution to this can be seen in Figure 2.

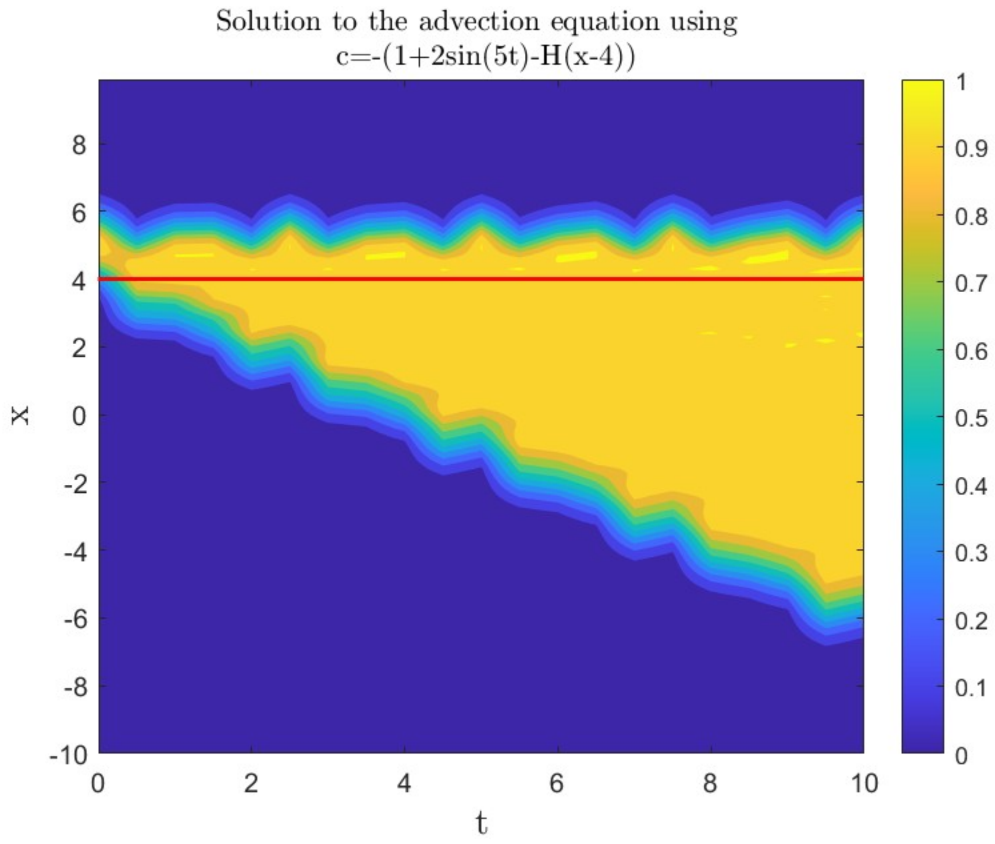

Figure 2: An approximation of the solution to the advection equation with velocity field $c = -(1 + 2\sin(5t) - H(x-4))$. A red line is made at $x = 4$ to signify where the Heaviside function changes from 0 to 1.

The red line on the figure is where $x = 4$. This signals where the Heaviside function starts contributing to the velocity field. Looking at the time before the non-zero part of the function hits the red line, a clear wave can be seen. Once it hit the red line it fragments into a fan, the only explanation for this behavior is the discontinuity in c. The time dependence of the velocity field is also clearly visible. Defining the "edges" of a function as the place where a lot of change happens, a sort of periodic behavior can be seen on the edges of the function. Since $-1 \leq \sin(x) \leq 1$, the velocity field can be both positive and negative. This means that the function can move up and down the $x-$axis, which can clearly be seen in the wavy pattern at the edges.

Conclusion

The advection equation (1) tries to model the transport of substances over time. Here the velocity field c influences the solution of the equation a great deal. A point, where $u(x,t) = u^*$, changes spatially at rate c when time changes. Next to it being able to be constant, it can also be dependent on it's spatial and temporal location. Clear influences can be seen when approximating and plotting the solution. This analysis was, however, hand wavy. A more correct approach to knowing how c will influence the solution is to look at it in a more mathematical standpoint. Other velocity fields could be considered, but this would result in another hand wavy analysis.

Appendix

A Code first presentation skill

```
%% Problem 1
clear all
%Parameters
f = @(x) exp(-(x-5).^2);
L = 10;
T = 10;
Deltax = 0.1;

%Discretisation
x = linspace(-L,L-Deltax,200);
N = length(x);

%Calculating A, could be done better with diag
A = sparse(N,N);
for index1 = 1:N
    if index1 ~= 1 && index1 ~= N
        A(index1,index1-1) = -1;A(index1,index1+1) = 1;
    elseif index1 == N
        A(index1,index1-1) = -1;A(index1,1) = 1;
    elseif index1 == 1
        A(index1,index1+1) = 1; A(index1,N) = -1;
    end
end

A = A/(Deltax*2);

%Deliverable
A1=full(A);
```

```
%Initial condition
y0 = f(x);

%Calculating y without for c constant
[t, y1] = ode45(@(t, x) advec1(t, x, A), linspace(0,T,21), y0);
A2 = y1';

%Calculating y without for c nonconstant
xx=x;
[t, y2] = ode45(@(t, x) advec2(t, x, xx), linspace(0,T,21), y0);

A3 = y2';
%Plotting
figure(1);
[XX,TT] = meshgrid(t,x);
contourf(XX,TT,y1','edgecolor','none'); xlabel('t','Interpreter','Latex','FontSize',14);ylabel('x','Interpreter','Latex','FontSize',14); title(sprintf('Solution to the advection equation using \n c=-.5'),...
        'Interpreter','Latex'); colorbar();
figure(2);
contourf(XX,TT,y2','edgecolor','none'); xlabel('t','Interpreter','Latex','FontSize',14);ylabel('x','Interpreter','Latex','FontSize',14); title(sprintf('Solution to the advection equation using \n c=-(1+2sin(5t)-H(x-4))'),...
        'Interpreter','Latex'); colorbar();
hold on; yline(4,'r-','Linewidth',1.5)

%% Functions
function u_t = advec1(t, x, A)
c = -0.5;
```

```
u_t = -c*A*x;
end

function u_t = advec2(t, x, xx)
    N = length(xx);
    A = sparse(N,N);
     Deltax = 0.1;
     for index1 = 1:N
         c = (1-heaviside(xx(index1)-4)+2*sin(5*t));
         if index1 ~= 1 && index1 ~= N
             A(index1,index1-1) = -c;A(index1,index1+1) = c;
         elseif index1 == N
             A(index1,index1-1) = -c;A(index1,1) = c;
         elseif index1 == 1
             A(index1,index1+1) = c; A(index1,N) = -c;
         end
     end
    A = A/(2*Deltax);
     u_t = A*x;
end

function H = heaviside(x)
    H = (x > 0) + 0.5*(x == 0);
end
```