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Cauchy data for Levin's method

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In this paper, we describe the Cauchy data that gives rise to slowly oscillating solutions to the Levin equation. We present a general result on the existence of a unique minimizer of $\|Bx\|$ subject to the constraint $Ax = y$, where A, B are linear, but not necessarily bounded operators on a complex Hilbert space. This result is used to obtain the solution to the Levin equation, both in the univariate and multivariate case, which minimizes the mean-square of the derivative over the domain. The Cauchy data that generates this solution is then obtained, and this can be used to supplement the Levin equation in the computation of highly oscillatory integrals in the presence of stationary points.

Keywords: oscillatory integral; Levin method.

1. Introduction

We are primarily interested in integrals of the form

$$\int_a^b e^{\mathrm{i}\omega g(x)} f(x)\,\mathrm{d}x$$

and their higher dimensional analogue. Here we refer to f as the amplitude, the real valued function g as the phase function and ω as the frequency. Such integrals become difficult to evaluate by standard methods, e.g., quadrature, as ω becomes large.

There are many well-known methods for the numerical computation of integrals of this form, e.g., Filon-type (Filon, 1930; Iserles, 2004a,b,Iserles *et al.*, 2005), Levin-type (Levin, 1982, 1997), numerical steepest descent (Huybrechs & Vandewalle, 2006), complex Gaussian quadrature (Deano *et al.*, 2017) and variants of. Many advancements of these ideas were made in the work of Sheehan Olver (Olver, 2006, 2008). An excellent account of a wide array of approaches can be found in the recent book (Deano & Huybrechs, 2009), which served as the inspiration for this work.

The standard Levin method seeks an approximate solution $u = u(x)$ to the differential equation, which we refer to as Levin's equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\mathrm{i}\omega g(x)} u\right) = e^{\mathrm{i}\omega g(x)} f(x), \quad \text{i.e.,} \quad u' + \mathrm{i}\omega g' u = f$$

and the integral is then computed via the fundamental theorem of calculus. To find u, confluent collocation methods are used. It can be shown that this will yield a slowly oscillating solution provided ω is sufficiently large and there are no stationary points. In Levin's original paper (Levin, 1982), he requires that $f(h(\xi))/g'(h(\xi))$ is slowly oscillating, where h is the inverse of the monotone function g.

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His definition of slowly oscillating requires the Fourier transform of the function to have small support relative to ω, i.e.,

$$\frac{f(h(\xi))}{g'(h(\xi))} = \int_{-\omega_0}^{\omega_0} e^{\mathrm{i}\omega\xi t} H(t)\,\mathrm{d}t, \quad \omega_0 \ll \omega.$$

The traditional Levin method breaks down in the presence of stationary points, i.e., points at which $g' = 0$. An alternative approach can be found in Olver (2010), which introduces the use of confluent collocation points and examines their effect on the asymptotic approximation error in the overall method.

With H implicitly defined above, it is straightforward to show that the Levin equation admits the following solution (assuming $\omega_0 < 1$)

$$u(x;\omega) = \frac{1}{\mathrm{i}\omega}\int_{-\omega_0}^{\omega_0} \frac{H(t)}{1+t}\, e^{\mathrm{i}\omega g(x) t}\,\mathrm{d}t.$$

It can be shown that for ω large this solution satisfies

$$\|u'\|_{L^2}^2 = O\left(\frac{\log\omega}{\omega}\right),$$

and stronger estimates are available if H is sufficiently regular. This encapsulates the slow-oscillatory nature of the solution. In fact, we see that as ω becomes larger the average amplitude of the derivative becomes more controlled. If a collocation approach is used to solve Levin's equation, then this approach 'picks out' a nonoscillatory solution of this form. However, the collocation equations break down if g' vanishes anywhere in the domain.

In the absence of stationary points and when f is smooth, it is straightforward to obtain an appropriate initial condition for Levin's equation. The general solution to Levin's equation is

$$u(x;\omega) = e^{-\mathrm{i}\omega g(x)}\left[u(a)\, e^{\mathrm{i}\omega g(a)} + \int_a^x e^{\mathrm{i}\omega g(t)} f(t)\,\mathrm{d}t\right].$$

If $g' \neq 0$, we can continue to integrate by parts (cf. Iserles *et al.*, 2005)

$$\int_a^x e^{\mathrm{i}\omega g(t)} f(t)\,\mathrm{d}t = \sum_{n=0}^{N} \frac{1}{(\mathrm{i}\omega)^{n+1}}\left[\frac{e^{\mathrm{i}\omega g(t)} f_n(t)}{g'(t)}\right]_{t=a}^{t=x} + \frac{R_{N+1}(x;\omega)}{(\mathrm{i}\omega)^{N+1}},$$

where $f_0 = f, f_{n+1} = -(f_n/g')'$ for $n \geq 0$ and we have defined

$$R_{N+1}(x;\omega) = \int_a^x e^{\mathrm{i}\omega g(t)} f_{N+1}(t)\,\mathrm{d}t.$$

This results in the solution

$$u(x;\omega) = e^{-\mathrm{i}\omega g(x)}\left[u(a) - \sum_{n=0}^{N} \frac{1}{(\mathrm{i}\omega)^{n+1}} \frac{f_n(a)}{g'(a)}\right] e^{\mathrm{i}\omega g(a)} + \epsilon_{N+1}(x;\omega),$$

where

$$\epsilon_{N+1}(x;\omega) = \frac{e^{-\mathrm{i}\omega g(x)} R_{N+1}(x;\omega)}{(\mathrm{i}\omega)^{N+1}} + \sum_{n=0}^{N} \frac{1}{(\mathrm{i}\omega)^{n+1}} \frac{f_n(x)}{g'(x)}.$$

It is straightforward to establish the following estimates for the x-derivatives of ϵ_{N+1} relevant for large ω

$$\left\| \epsilon_{N+1}^{(j)}(\cdot;\omega) \right\|_\infty = o(1) \quad \text{for } j = 0, \ldots, N+1.$$

So this particular solution can be made as nonoscillatory as we please (by choosing N large enough), and it is obtained by using the initial condition

$$u(a) = \sum_{n=0}^{N} \frac{1}{(\mathrm{i}\omega)^{n+1}} \frac{f_n(a)}{g'(a)}.$$

However, this analysis breaks down if there is a stationary point in $[a, b]$. In this case, we *don't* expect to find a particular solution that has well controlled derivatives of arbitrarily high order. But we may ask: how well controlled can a particular solution be and how should one obtain it?

In this paper, we aim to construct slowly-oscillating solutions to the Levin differential equation in the presence of stationary points by solving the minimization problem

$$\text{minimize} \int_a^b |u'|^2 \, \mathrm{d}x \quad \text{subject to } u' + \mathrm{i}\omega g' u = f. \tag{1.1}$$

It is shown that a solution exists for which $\|u'\|_{L^2} = o(1)$ as $\omega \to \infty$. The precise nature of the decay is dictated by the order of the stationary point, if present. By specifying the initial data that characterizes such a solution, it can be easily obtained numerically, e.g., by explicit multi-step methods. These results can also be used in conjunction with other works in this area to speed-up or increase regularity of existing numerical methods, e.g., Molabahrami (2017); Ma & Liu (2018); Wang & Xiang (2022a).

To compare to Levin's original method, let us assume for simplicity that $u(a) = u(b)$. Then by Parseval's theorem

$$\|u'\|_{L^2}^2 = \sum_m |m|^2 |\hat{u}(m)|^2.$$

This gives the heuristic link between the two approaches: by minimizing the left-hand side, we are not imposing that the support of the Fourier transform of u (denoted $\hat{u}$) above) lie in some $[-\omega_0, \omega_0]$ for $\omega_0 \ll \omega$, but rather that the contributions from the higher frequency parts of u are negligible.

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The ideas presented in the one-dimensional problem can be extended to higher dimensional oscillatory integrals. If $\Omega \subset \mathbf{R}^n$ is a simplex and we wish to numerically compute integrals of the form

$$\int_\Omega f(\mathbf{x})\, e^{\mathrm{i}\omega g(\mathbf{x})}\, \mathrm{vol}_\Omega(\mathbf{x}),$$

then it is natural to consider solutions $\mathbf{u} : \Omega \to \mathbf{C}^n$ to the equation

$$\nabla \cdot \left(e^{\mathrm{i}\omega g(\mathbf{x})}\mathbf{u}\right) = e^{\mathrm{i}\omega g(\mathbf{x})} f(\mathbf{x}), \quad \text{i.e.,} \quad \nabla \cdot \mathbf{u} + \mathrm{i}\omega \nabla g \cdot \mathbf{u} = f.$$

We cannot arbitrarily impose boundary conditions for $\mathbf{u}$ on $\partial\Omega$ because of the constraint

$$\int_{\partial\Omega} e^{\mathrm{i}\omega g}(\mathbf{u}\cdot\mathbf{n})\mathrm{vol}_{\partial\Omega} = \int_\Omega e^{\mathrm{i}\omega g} f\, \mathrm{vol}_\Omega.$$

However, if we look for solutions of the form $\mathbf{u} = \mathbf{c}u$, where $\mathbf{c} \in \mathbf{R}^n \setminus \{0\}$ is a constant and $u : \Omega \to \mathbf{C}$, then we get a hyperbolic problem of the form

$$\mathbf{c}\cdot\nabla u + \mathrm{i}\omega(\mathbf{c}\cdot\nabla g)u = f.$$

The vector field $\mathbf{c}$ generates the characteristic curves for this hyperbolic problem and by specifying data on a noncharacteristic portion of $\partial\Omega$, this will generate a solution throughout Ω. If we denote the faces of the n-simplex Ω to be $\Sigma_0, \ldots, \Sigma_n$, then after an application of the divergence theorem

$$\int_\Omega e^{\mathrm{i}\omega g} f\, \mathrm{vol}_\Omega = \sum_{i=0}^{n} \int_{\Sigma_i} e^{\mathrm{i}\omega g}(\mathbf{n}\cdot\mathbf{c}u)\, \mathrm{vol}_{\Sigma_i}.$$

In the absence of points for which $\mathbf{c}\cdot\nabla g = 0$, we can write down the general solution to the hyperbolic problem and integrate by parts in much the same way as the one-dimensional problem. This gives rise to a choice of Cauchy data on one of the faces of the the simplex that will generate a solution having as many controlled derivatives as we wish. Again, the question is what to do in the presence of stationary points.

In the presence of stationary points, we take our lead from the one-dimensional problem. We fix u by solving the problem

$$\text{minimize} \int_\Omega |\mathbf{c}\cdot\nabla u|^2\, \mathrm{vol}_\Omega \quad \text{subject to} \quad \mathbf{c}\cdot\nabla u + \mathrm{i}\omega(\mathbf{c}\cdot\nabla g)u = f. \tag{1.2}$$

This gives rise to a solution that is easily obtained numerically, e.g., by the method of lines (Schiesser, 2012), because the derivative of u is so well-controlled along the characteristic curves. The resulting integrals over the faces are of the form

$$\int_\Sigma e^{\mathrm{i}\omega g} f\, \mathrm{vol}_\Sigma,$$

where Σ is an $(n-1)$-simplex and $f = \mathbf{n}\cdot\mathbf{c}u$ is a function that is not rapidly oscillating. That is, we have reduced the integral over an n-simplex to one over an $(n-1)$ simplex and this provides the inductive step for our computation of the initial integral.

The purpose of this paper is to establish the technical results that uniquely classify the ‘optimal’ solution (in the sense of Theorem 2.1 in the next section) to the Levin equation in both the univariate and

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multivariate cases. By deriving the Cauchy data that determines these solutions, we will be able to obtain, *a priori*, regularity estimates. These technical results can be used in a multitude of different settings, both in the computation of oscillatory integrals and elsewhere. Numerical work will be presented elsewhere.

Notation For a manifold $\Lambda \subset \mathbf{R}^n$, we denote the natural volume form on Λ by vol_Λ. For example, if Λ is an n-simplex in $\mathbf{R}^n$ and $\{x_i\}_{i=1}^n$ are standard Cartesian coordinates, then

$$\mathrm{vol}_\Lambda(\mathbf{x}) = \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

The Hilbert Space of square integrable functions from $f : \Lambda \to \mathbf{C}$ will be denoted $L^2(\Lambda)$ with standard inner product

$$\langle f, g\rangle_{L^2(\Lambda)} = \int_\Lambda f(\mathbf{x})\overline{g(\mathbf{x})}\,\mathrm{vol}_\Lambda(\mathbf{x}), \quad \|f\|_{L^2(\Lambda)} = \sqrt{\langle f, f\rangle}.$$

When there is no confusion, we will simply write L^2 in place of $L^2(\Lambda)$. Occasionally, it is necessary to use notation such as

$$\|f(\cdot, \mathbf{y})\|^2_{L^2(\Lambda, \mathrm{d}\mathbf{y})} \equiv \int_\Lambda |f(\cdot, \mathbf{y})|^2\,\mathrm{vol}_\Lambda(\mathbf{y})$$

to highlight which variable is being integrated over. For a closed subset $K \subset \mathbf{R}^n$, we use $C^r(K)$ to denote the Banach space of r-times continuously differentiable functions equipped with the standard norm. We write $A \lesssim B$ if there is a positive constant C such that $A \le CB$. $A = O(B)$ iff $A \lesssim B$ and $a = o(b)$ if $a/b \to 0$ as some parameter becomes large, which will be obvious from the context.

Organization of paper In §2, we present a general result that characterizes the minimizer of $\|Bx\|_H$ subject to $Ax = y$, where H is a complex Hilbert space and A, B are linear differential operators defined on a subspace of H. The minimizer, x^* say, is characterized by the condition $\langle Bx^*, Bx\rangle_H = 0$ for all $x \in \ker A$.

In §3 and §4, we use the abstract result from §2 to solve the minimization problems in (1.1) and (1.2). The relevant orthogonality condition will then be used to characterize the initial data that generates the solution to Levin's equation that minimizes $\|u'\|_{L^2}$ and $\|\mathbf{c} \cdot \nabla u\|_{L^2}$, respectively. For this solution, it is shown that the L^2 norm of the relevant derivative decays for large ω.

In §5, we describe a collocation approach that can be used in conjunction with the previous results to numerically solve Levin's equation in the one-dimensional case.

2. The minimization problem

In both the one-dimensional and higher dimensional problems, we will need to solve a minimization problem of the form

$$\text{minimize } \|Bx\|_H^2 \quad \text{subject to} \quad Ax = y,$$

where A, B are linear differential operators and H is an appropriate Hilbert space of square integrable functions. If $\ker A$ is finite dimensional, say $\ker A = \mathrm{span}\{x_1, \ldots, x_N\}$, then this question is easy to

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answer. Without loss of generality, we can assume $\left\langle Bx_i, Bx_j\right\rangle_H = 0$ for $i \neq j$ and $\|Bx_i\|_H = 1$ for each i. Fix $x \in A^{-1}y$ and consider the polynomial

$$
\begin{aligned}
(t_1, \ldots, t_N) \mapsto & \left\| \sum_{i=1}^{N} t_i Bx_i + Bx \right\|_H^2 - \|Bx\|_H^2 \\
&= \sum_{i=1}^{N} \left(|t_i|^2 + 2\mathrm{Re}\, t_i \langle Bx_i, Bx\rangle_H \right) \\
&= \sum_{i=1}^{N} \left(\left| t_i + \langle Bx_i, Bx\rangle_H \right|^2 - \left| \langle Bx_i, Bx\rangle_H \right|^2 \right).
\end{aligned}
$$

Since $x + \sum_i t_i x_i$ represents the most general solution to $Ax = y$, we can now choose the t_i appropriately to get the unique minimizer. Denoting the minimizer by x^*, it quickly follows that $\langle Bx^*, Bx_i\rangle_H = 0$ for each i and that this completely characterizes the minimizer x^*. However, if $\ker A$ is not finite dimensional, more technical machinery is needed. To this end, we offer the following general result.

THEOREM 2.1 Let H be a complex Hilbert space, $X \subset H$ a linear subspace and $A : X \to H$, $B : X \to H$ linear operators (not necessarily bounded). If A is a closed operator and

$$
\|x\|_H \lesssim \|Ax\|_H + \|Bx\|_H \quad \forall x \in X
$$

then for each $y \in \operatorname{im} A \subset H$ there exists a unique $x^* \in X$ such that

$$
x^* = \arg\left(\inf_{x \in A^{-1}y} \|Bx\|_H \right).
$$

This minimizer can be characterized by the orthogonality result

$$
x^* = \arg\inf_{x \in A^{-1}y} \|Bx\|_H \quad \Leftrightarrow \quad \langle Bx^*, Bx\rangle_H = 0 \quad \forall x \in \ker A.
$$

Recall that a linear operator $A : X \subset H \to H$ is *closed* if the graph of A

$$
\mathrm{graph}(A) = \{(x, Ax) : x \in X\}
$$

is closed in $H \times H$. We note that if $A : X \to H$ is closed then the pre-image of any singleton $y \in \operatorname{im} A$ is closed. Indeed, $H \times \{y\}$ is closed in $H \times H$ so

$$
\mathrm{graph}\, A \cap (H \times \{y\}) = \{A^{-1}y\} \times \{y\}
$$

is closed in $H \times H$, being the intersection of two closed sets. We conclude that the pre-image $A^{-1}y$ is closed in H.

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Proof of Theorem 1. On existence, first note that for $y \in \operatorname{im} A$ the pre-image $K = A^{-1}y \subset X$ is nonempty and convex. It is also closed, by the observation above. Suppose $\{x_n\} \in K \subset X$ is a minimizing sequence, i.e.,

$$\|Bx_n\|_H \to \alpha = \inf_{x\in K} \|Bx\|_H.$$

By the parallelogram law,

$$\|Bx_n - Bx_m\|_H^2 = 2\|Bx_n\|_H^2 + 2\|Bx_m\|_H^2 - 4\left\|B\left[\frac{1}{2}\left(x_n + x_m\right)\right]\right\|_H^2.$$

Since $\frac{1}{2}(x_n + x_m) \in K$, the final term cannot exceed $-4\alpha^2$. Hence,

$$\|Bx_n - Bx_m\|_H^2 \le 2\|Bx_n\|_H^2 + 2\|Bx_m\|_H^2 - 4\alpha^2 \to 0.$$

Under our hypothesis on the operators A, B, we find

$$\|x_n - x_m\|_H \lesssim \|Ax_n - Ax_m\|_H + \|B(x_n - x_m)\|_H \to 0$$

since $Ax_n = y$ for each n and the latter term vanishes in the limit by the previous observation. So this Cauchy sequence converges to some $x^* \in K$, since K is closed. This establishes the existence of the minimizer in K, i.e.,

$$x^* = \arg\left(\inf_{x\in K} \|Bx\|_H\right).$$

Now suppose x^* is the minimizer. Then any other element of $A^{-1}y$ differs from x^* by some $x \in \ker A$. For $x \in \ker A$, consider

$$\mathbf{R} \ni t \mapsto \|Bx^* - tBx\|_H^2 - \|Bx^*\|_H^2 = t^2\|Bx\|_H^2 - 2t\,\mathrm{Re}\,\left\langle Bx^*, Bx\right\rangle_H \ge 0,$$

meaning $\mathrm{Re}\,\langle Bx^*, Bx\rangle = 0$. Replacing $x \mapsto ix$ gives the same result with the imaginary part replacing the real part. Since $x \in \ker A$ was arbitrary, we conclude

$$\left\langle Bx^*, Bx\right\rangle_H = 0 \quad \forall x \in \ker A.$$

Conversely, if $\langle Bx^*, Bx\rangle_H = 0$ for each $x \in \ker A$, then

$$\|Bx^* - Bx\|_H^2 = \|Bx\|_H^2 + \|Bx^*\|_H^2 \ge \|Bx^*\|_H^2,$$

which shows that

$$x^* = \arg\left(\inf_{x\in A^{-1}y} \|Bx\|_H\right).$$

On uniqueness, if x_1^* and x_2^* are both minimizers, then by linearity

$$\langle Bx, B\left(x_1^* - x_2^*\right)\rangle_H = 0, \quad \forall x \in \ker A.$$

But $x_1^* - x_2^* \in \ker A$ so taking $x = x_1^* - x_2^*$ in the above we see

$$\left\| B\left(x_1^* - x_2^*\right)\right\|_H^2 = \langle Bx, B\left(x_1^* - x_2^*\right)\rangle_H = 0,$$

meaning $x_1^* = x_2^* \bmod \ker B$ and $x_1^* = x_2^* \bmod \ker A$. So the solution is unique modulo $\ker A \cap \ker B$, but this is just $\{0\}$, which follows from the estimate in the hypothesis $\|x\|_H \lesssim \|Ax\|_H + \|Bx\|_H$. $\square$

In many instances, the closed nature of the operator $A : X \subset H \to H$ causes no obstacle. For instance, in the present applications, the operator A will be a skew-symmetric differential operator that can be defined on a dense subspace of H. Such operators are always *closable*, meaning there is a closed extension of A on a larger domain. One then deals with the closure of A, being the closed extension with smallest domain.

REMARK 2.2 The result can be used in a variety of settings disconnected with the numerical computation of oscillatory integrals. For instance, in the numerical study of solutions to inhomogeneous problems of the form

$$\frac{\partial \mathbf{v}}{\partial t} - M\mathbf{v} = \mathbf{f}, \quad \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}),$$

it is very often the case that solutions to the *homogeneous* problem are well understood, but computation of a particular integral is difficult, perhaps because of generic oscillatory behaviour. If we instead find a particular integral that minimizes $\|\dot{\mathbf{v}}\|_{L^2}$, we can then simply add an appropriate solution to the homogeneous problem to fix the initial data.

3. One dimension

Our aim is to numerically approximate integrals of the form

$$\int_a^b e^{\mathrm{i}\omega g(x)} f(x)\,\mathrm{d}x.$$

There are a plethora of cheap and effective means of doing this in the absence of stationary points, i.e., $g' \neq 0$ on $[a, b]$. We confine attention to when there is a single stationary point in $[a, b]$ and without loss of generality we assume $g'(b) = 0$. This is no barrier, since if there is a stationary point at $c \in (a, b)$, then we can write the integral as

$$\int_a^c f(x)\, e^{\mathrm{i}\omega g(x)}\,\mathrm{d}x + \int_c^b f(b - x + c)\, e^{\mathrm{i}\omega g(b-x+c)}\,\mathrm{d}x$$

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and both these integrals have the associated stationary point at the end point of the interval of integration. To this end, we assume $g \in C^\infty[a,b]$ and

$$g'(x) = v(x)(b-x)^k, \quad \text{where } v \in C^\infty[a,b] \text{ and } v(x) \neq 0.$$

For economy of presentation, we will also assume that the amplitude $f \in C^\infty[a,b]$. This is of no real obstacle since, if f were merely continuous, say, then it could be approximated arbitrarily well by polynomials and our methods would then apply to the approximation. There is no doubt that similar results to those obtained here will still hold with significantly less regularity, but that will be investigated elsewhere. Throughout this section, we will refer to the operators

$$A_\omega = \frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\omega g', \quad B = \frac{\mathrm{d}}{\mathrm{d}x}.$$

When considering solutions to $A_\omega u = f$, we will abuse notation and write $u = u(x)$ rather than $u(x;\omega)$. It should be understood that $u(x)$ implicitly depends on the ω that appears in A_ω.

LEMMA 3.1 Let $H = L^2[a,b]$, $X = C^1[a,b]$. Then $A_\omega : X \subset H \to H$ is closed and we have the estimate

$$\|u\|_{L^2} \lesssim \|A_\omega u\|_{L^2} + \|Bu\|_{L^2} \quad u \in X.$$

As was observed in the previous section, because $\ker A_\omega$ is one dimensional we don't actually need to call upon Theorem 2.1. The existence and uniqueness of the minimizer is easily established in this case. However, it is instructive to establish this result in the simpler one-dimensional case because it serves as a foundation for the ideas needed for the analogous result in the higher dimensional case, where an appeal to Theorem 2.1 will be necessary. The proof is contained in the appendix.

REMARK 3.2 The proof of this result is remarkably simple if it is known that $u(c) = 0$ for some $c \in [a,b]$. In this case, we have

$$e^{\mathrm{i}\omega g(x)} u(x) = \int_c^x e^{\mathrm{i}\omega g(t)} A_\omega u(t)\,\mathrm{d}t, \quad u(x) = \int_c^x Bu(t)\,\mathrm{d}t,$$

which leads to a pair of stronger estimates:

$$\|u\|_{L^2} \lesssim \|A_\omega u\|_{L^2} \quad \text{and} \quad \|u\|_{L^2} \lesssim \|Bu\|_{L^2},$$

respectively. In the case where there is no such $c \in [a,b]$, we can only say

$$\left(e^{\mathrm{i}\omega g(x)} - e^{\mathrm{i}\omega g(c)}\right) u(x) = \int_c^x \left(e^{\mathrm{i}\omega g(t)} A_\omega u(t) - e^{\mathrm{i}\omega g(c)} Bu(t)\right)\,\mathrm{d}t.$$

The result of Lemma 3.1 tells us that the hypothesis of Theorem 2.1 holds for our problem of interest. By noting that $\ker A_\omega = \operatorname{span}\{e^{-\mathrm{i}\omega g}\}$, we immediately arrive at the following corollary:

COROLLARY 3.3 The solution to the problem

$$\text{minimize } \|u'\|_{L^2} \quad \text{subject to} \quad A_\omega u = f$$

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exists and is unique in $C^1[a,b]$. It is completely characterized by

$$A_\omega u = f \quad \text{and} \quad \int_a^b e^{\mathrm{i}\omega g} u' g' \, \mathrm{d}x = 0.$$

REMARK 3.4 It is possible to obtain this result by extremizing the Lagrangian

$$L[x,u,u';\lambda] = \int_a^b \left(\left|u'\right|^2 + \lambda(x) \left(u' - \mathrm{i}\omega g' u - f\right)\right) \mathrm{d}x$$

with free boundary conditions on u and λ being a Lagrange multiplier.

LEMMA 3.5 The unique solution to the problem

$$\text{minimize } \|u'\|_{L^2} \quad \text{subject to} \quad A_\omega u = f$$

satisfies the initial condition

$$u(a) = \frac{f(a)}{\mathrm{i}\omega g'(a)} + O\left(\frac{1}{\omega^2}\right).$$

Note that if we choose to solve $A_\omega u = f$ on $[a,c]$ with $c < b$ then there are no stationary points and we expect the integral, at least for large ω, to be determined by the integration-by-parts asymptotic approach. The above gives the first term from *one* of the endpoints via $u(a)\exp(\mathrm{i}\omega g(a))$.

Proof. The general solution to the equation $A_\omega u = f$ is

$$u(x) = e^{\mathrm{i}\omega g(a) - \mathrm{i}\omega g(x)} u(a) + \int_a^x e^{\mathrm{i}\omega g(t) - \mathrm{i}\omega g(x)} f(t)\, \mathrm{d}t.$$

Substituting this into the integral constraint gives

$$0 = \int_a^b \left(\omega^2 g'(x)^2 \left[u(a) e^{\mathrm{i}\omega g(a)} + \int_a^x e^{\mathrm{i}\omega g(t)} f(t)\, \mathrm{d}t \right] + \mathrm{i}\omega g'(x) f(x)\, e^{\mathrm{i}\omega g(x)} \right) \mathrm{d}x.$$

By writing (for $x < b$)

$$\int_a^x e^{\mathrm{i}\omega g(t)} f(t)\, \mathrm{d}t = \frac{f(x)\, e^{\mathrm{i}\omega g(x)}}{\mathrm{i}\omega g'(x)} - \frac{f(a)\, e^{\mathrm{i}\omega g(a)}}{\mathrm{i}\omega g'(a)} - \int_a^x \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{f(t)}{\mathrm{i}\omega g'(t)}\right] e^{\mathrm{i}\omega g(t)}\, \mathrm{d}t,$$

we see that the identity becomes

$$\left[u(a)\, e^{\mathrm{i}\omega g(a)} - \frac{f(a)\, e^{\mathrm{i}\omega g(a)}}{\mathrm{i}\omega g'(a)} \right] \|g'\|_{L^2}^2| = \int_a^b g'(x)^2 \left[\int_a^x \frac{f'(t) g'(t) - f(t) g''(t)}{\mathrm{i}\omega g'(t)^2}\, e^{\mathrm{i}\omega g(t)}\, \mathrm{d}t \right] \mathrm{d}x. \tag{3.1}$$

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Writing $\psi(t) = f'(t)g'(t) - f(t)g''(t)$, $F(x) = -\int_x^b g'(t)^2\, \mathrm{d}t$ and integrating by parts the right-hand side becomes

$$-\frac{1}{\mathrm{i}\omega}\int_a^b \frac{F(x)}{F'(x)}\psi(x)\, e^{\mathrm{i}\omega g(x)}\, \mathrm{d}x.$$

Note that if $g'(t) = (b-t)^k v(t)$ with $v(b) \neq 0$ then

$$\left|\frac{F(x)}{F'(x)}\right| \lesssim (b-x), \quad |\psi(x)| \lesssim (b-x)^{k-1}.$$

Consequently,

$$x \mapsto q(x) = \frac{F(x)}{F'(x)}\frac{\psi(x)}{g'(x)}$$

is integrable on $[a,b]$ and in fact it is straightforward to show that with our assumptions on g and f that $q \in C^\infty[a,b]$. So we may write

$$-\frac{1}{\mathrm{i}\omega}\int_a^b \frac{F(x)}{F'(x)}\psi(x)\, e^{\mathrm{i}\omega g(x)}\, \mathrm{d}x = \frac{1}{\omega^2}\int_a^b q(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\mathrm{i}\omega g(x)}\right)\mathrm{d}x.$$

Integrating by parts we see that the right-hand side is $O(1/\omega^2)$. We deduce

$$u(a) - \frac{f(a)}{\mathrm{i}\omega g'(a)} = O\left(\frac{1}{\omega^2}\right),$$

which is the desired result. □

REMARK 3.6 It is clear from the preceding analysis that the *exact* initial condition for the solution to $A_\omega u = f$ that minimizes $\|u'\|_{L^2}$ satisfies

$$u(a) = \frac{f(a)}{\mathrm{i}\omega g'(a)} + \frac{e^{-\mathrm{i}\omega g(a)}\int_a^b q(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\mathrm{i}\omega g(x)}\right)\mathrm{d}x}{\omega^2\|g'\|_{L^2}^2},$$

where $q \in C^\infty[a,b]$ is defined above. On continued integration by parts, we can get the $O(1/\omega^N)$ correction for any $N > 1$. These terms become more and more complicated and depend on nonlocal quantities such as $\|g'\|_{L^2}$ and on k, the order of the stationary point. For example, it can be shown

$$u(a) = \frac{f(a)}{\mathrm{i}\omega g'(a)} + \frac{1}{\omega^2}\left[\frac{(f/g')'(a)}{g'(a)} - \frac{kf(b)\, e^{\mathrm{i}\omega(g(b)-g(a))}}{(2k+1)\|g'\|_{L^2}^2}\right] + \cdots$$

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In the absence of stationary points ($k = 0$), on integrating Levin's equation

$$\int_a^b e^{\mathrm{i}\omega g(x)} f(x)\,\mathrm{d}x = u(b)\,e^{\mathrm{i}\omega g(b)} - u(a)\,e^{\mathrm{i}\omega g(a)}$$
$$= u(b)\,e^{\mathrm{i}\omega g(b)} - \left[\frac{f(a)}{\mathrm{i}\omega g'(a)} + \frac{(f/g')'(a)}{\omega^2 g'(a)} + \cdots\right] e^{\mathrm{i}\omega g(a)}.$$

The latter term agrees exactly with the asymptotic approach seen in Iserles *et al.* (2005) and $u(a)$ matches that highlighted in the discussion in the introduction.

Note that if we solve

$$A_\omega v = f, \quad v(a) = \frac{f(a)}{\mathrm{i}\omega g'(a)}$$

then since $u - v \in \ker A_\omega$ we have

$$u(x) - v(x) = [u(a) - v(a)]\,e^{\mathrm{i}\omega g(a) - \mathrm{i}\omega g(x)} = O\left(\frac{1}{\omega^2}\right)$$

uniformly in $x \in [a, b]$ and by differentiating we get the estimate

$$\|v'\|_{L^2} \leq \|u' - v'\|_{L^2} + \|u'\|_{L^2} = O\left(\frac{1}{\omega}\right) + \|u'\|_{L^2}$$

i.e., $\|v'\|_{L^2}^2 \approx \|u'\|_{L^2}^2$, so upto an error that decreases with ω, v is as well controlled as u. So rather than trying to find the solution to the exact solution to the problem for u in Corollary 3.3, we can simply solve

$$A_\omega v = f, \quad v(a) = \frac{f(a)}{\mathrm{i}\omega g'(a)}.$$

What remains: quite *how* oscillatory is the solution to the problem

$$\text{minimize} \int_a^b |u'|^2\,\mathrm{d}x \quad \text{subject to } u' + \mathrm{i}\omega g' u = f. \tag{3.2}$$

THEOREM 3.7 If $v \in C^\infty[a, b]$, $g'(x) = (b - x)^k v(x)$, $v \neq 0$ on $[a, b]$ and

$$A_\omega u = f \quad \text{and} \quad \int_a^b e^{\mathrm{i}\omega g} u' g'\,\mathrm{d}x = 0,$$

then $\|u'\|_{L^2}^2 = O\left(\omega^{-1/(k+1)}\right)$.

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From this and the result in Lemma 3.5, it follows that the solution to

$$A_\omega v = f, \quad v(a) = \frac{f(a)}{\mathrm{i}\omega g'(a)}$$

also has the property that $\|v'\|_{L^2}^2 = O\left(\omega^{-1/(k+1)}\right)$. To prove the theorem, we need the following harmonic-analysis-esque lemma.

LEMMA 3.8 If $u, v \in C^\infty[a,b]$ and $g'(x) = (b-x)^k v(x)$ and $v \neq 0$, then

$$\left\| (b-x)^k \int_a^x u(t)(b-t)^{-k-1}\, e^{\mathrm{i}\omega g(t)}\, \mathrm{d}t \right\|_{L^2(\mathrm{d}x)}^2 = O\left(\frac{1}{\omega^{1/(1+k)}}\right),$$

and

$$\left\| (b-x)^{2k} \int_a^x u(t)(b-t)^{-k-1}\, e^{\mathrm{i}\omega g(t)}\, \mathrm{d}t \right\|_{L^1(\mathrm{d}x)} = O\left(\frac{\log \omega}{\omega}\right).$$

Proof. Fix $x \in (a,b)$ and let $\delta > 0$. We split the integrand into

$$(b-x)^k \int_{x-\delta}^x u(t)(b-t)^{-k-1}\, e^{\mathrm{i}\omega g(t)}\, \mathrm{d}t + (b-x)^k \int_a^{x-\delta} u(t)(b-t)^{-k-1}\, e^{\mathrm{i}\omega g(t)}\, \mathrm{d}t,$$

which we label I_1 and I_2, respectively. For I_1, we have the simple estimate

$$\begin{aligned}
|I_1| &\leq \|u\|_\infty (b-x)^k \int_{x-\delta}^x (b-t)^{-k-1}\, \mathrm{d}t \\
&= \frac{\|u\|_\infty}{k}\left[1 - \left(\frac{b-x}{b-x+\delta}\right)^k\right] \\
&\leq \|u\|_\infty \left(\frac{\delta}{b-x+\delta}\right),
\end{aligned}$$

which follows from the mean value theorem. From this estimate, we quickly see $\|I_1\|_{L^2}^2 = O(\delta)$. For I_2, by the second mean value theorem for integrals

$$|I_2| = \frac{1}{\omega}\left|(b-x)^k \int_a^{x-\delta} \frac{u(t)(b-t)^{-k-1}}{v(t)(b-t)^k}\mathrm{d}\left(e^{\mathrm{i}\omega g}\right)\right| \lesssim \frac{(b-x)^k}{\omega(b-x+\delta)^{2k+1}}.$$

Indeed, Taylor expanding $t \mapsto u(t)/v(t)$ about $t = b$, the integral is

$$\sum_{m=0}^N c_m \int_a^{x-\delta} (b-t)^{m-2k-1}\mathrm{d}\left(e^{\mathrm{i}\omega g}\right) + \int_a^{x-\delta} R_N(t)(b-t)^{-2k-1}\mathrm{d}\left(e^{\mathrm{i}\omega g}\right),$$

with $R_N = O\left((b-t)^{N+1}\right)$ and appropriate constants $\{c_m\}$. The second mean value theorem for integrals can be used in each term of the sum, the first of which dominates the others, giving the desired estimate. The remainder is negligible if we choose N sufficiently large and integrate by parts. Hence,

$$\|I_2\|_{L^2}^2 \lesssim \frac{\delta^{2k-4k-2+1}}{\omega^2} \int_0^{(b-a)/\delta} \tau^{2k}(\tau+1)^{-4k-2}\,\mathrm{d}\tau = O\left(\frac{\delta^{-2k-1}}{\omega^2}\right).$$

Choosing $\delta = \omega^{-1/(1+k)}$, we see that the error terms match and the first estimate follows. The second estimate is obtained in an entirely analogous manner. If I_1 and I_2 play similar roles to the first case, it can be shown that

$$\|I_1\|_{L^1} = O(\delta), \qquad \|I_2\|_{L^1} = O\left(\frac{\log(1/\delta)}{\omega}\right)$$

and the result follows by choosing $\delta = 1/\omega$. □

Proof of Theorem 3.7. The general solution to $A_\omega u = f$ can be written

$$\begin{aligned} u(x) &= e^{\mathrm{i}\omega g(a)-\mathrm{i}\omega g(x)}u(a) + \int_a^x e^{\mathrm{i}\omega g(t)-\mathrm{i}\omega g(x)} f(t)\,\mathrm{d}t \\ &\equiv cw(x) + \int_a^x e^{\mathrm{i}\omega g(t)-\mathrm{i}\omega g(x)} f(t)\,\mathrm{d}t, \end{aligned}$$

where $c = e^{\mathrm{i}g(a)}u(a)$ and $w \in \ker A_\omega$. By Theorem 2.1, since u is the solution to the minimization problem $\langle u', w'\rangle_{L^2} = 0$ for any $w \in \ker A_\omega$. So

$$\begin{aligned} \|u'\|_{L^2}^2 &= \|u' - cw'\|_{L^2}^2 - \|cw'\|_{L^2}^2 \\ &= \left\| f(x) - \mathrm{i}\omega g'(x) e^{-\mathrm{i}\omega g(x)} \int_a^x e^{\mathrm{i}\omega g(t)} f(t)\,\mathrm{d}t \right\|_{L^2(\mathrm{d}x)}^2 - \omega^2 |u(a)|^2 \|g'\|_{L^2}^2. \end{aligned}$$

For $x < b$, we have

$$\begin{aligned} &- \mathrm{i}\omega g'(x) e^{-\mathrm{i}\omega g(x)} \int_a^x e^{\mathrm{i}\omega g(t)} f(t)\,\mathrm{d}t \\ &\quad = -\mathrm{i}\omega g'(x) \left[\frac{f(x)}{\mathrm{i}\omega g'(x)} - \frac{e^{\mathrm{i}\omega g(a)-\mathrm{i}\omega g(x)} f(a)}{\mathrm{i}\omega g'(a)} - \int_a^x \left[\frac{f(t)}{\mathrm{i}\omega g'(t)}\right]' e^{\mathrm{i}\omega g(t)}\,\mathrm{d}t \right] \\ &\quad = -f(x) + \frac{e^{\mathrm{i}\omega g(a)-\mathrm{i}\omega g(x)} g'(x) f(a)}{g'(a)} + g'(x) \int_a^x \left[\frac{f(t)}{g'(t)}\right]' e^{\mathrm{i}\omega g(t)}\,\mathrm{d}t. \end{aligned}$$

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And so

$$\begin{aligned}
&\left\| f(x) - \mathrm{i}\omega g'(x)\, e^{-\mathrm{i}\omega g(x)} \int_a^x e^{\mathrm{i}\omega g(t)} f(t)\,\mathrm{d}t \right\|_{L^2(\mathrm{d}x)}^2 \\
&\quad = \left\| \frac{e^{\mathrm{i}\omega g(a) - \mathrm{i}\omega g(x)} g'(x) f(a)}{g'(a)} + g'(x) \int_a^x \left[\frac{f(t)}{g'(t)}\right]' e^{\mathrm{i}\omega g(t)}\,\mathrm{d}t \right\|_{L^2(\mathrm{d}x)}^2 \\
&\quad = \frac{|f(a)|^2}{|g'(a)|^2} \|g'\|_{L^2}^2 + \left\| g'(x) \int_a^x \left[\frac{f(t)}{g'(t)}\right]' e^{\mathrm{i}\omega g(t)}\,\mathrm{d}t \right\|_{L^2(\mathrm{d}x)}^2 \\
&\qquad + 2\mathrm{Re}\, \frac{f(a)\, e^{\mathrm{i}\omega g(a)}}{g'(a)} \left\langle e^{-\mathrm{i}\omega g(x)} g'(x), g'(x) \int_a^x \left[\frac{f(t)}{g'(t)}\right]' e^{\mathrm{i}\omega g(t)}\,\mathrm{d}t \right\rangle_{L^2(\mathrm{d}x)}.
\end{aligned}$$

If $g'(x) = (b-x)^k v(x)$ with $v(b) \neq 0$ and $k \geq 1$ then for $x < b$

$$g'(x) \int_a^x \left[\frac{f(t)}{g'(t)}\right]' e^{\mathrm{i}g(t)}\,\mathrm{d}t = v(x)(b-x)^k \int_a^x u(t)(b-t)^{-k-1} e^{\mathrm{i}\omega g(t)}\,\mathrm{d}t$$

with $u \in C^\infty[a,b]$ defined accordingly. By Lemma 3.8, we know that

$$\left\| v(x)(b-x)^k \int_a^x u(t)(b-t)^{-k-1}\, e^{\mathrm{i}\omega g(t)}\,\mathrm{d}t \right\|_{L^2(\mathrm{d}x)}^2 \lesssim \omega^{-1/(k+1)}.$$

And from the second part of the same Lemma we have

$$\left\langle e^{-\mathrm{i}\omega g(x)} g'(x), g'(x) \int_a^x \left[\frac{f(t)}{g'(t)}\right]' e^{\mathrm{i}\omega g(t)}\,\mathrm{d}t \right\rangle_{L^2(\mathrm{d}x)} = O\left(\frac{\log\omega}{\omega}\right).$$

Hence,

$$\|u'\|_{L^2}^2 \leq \frac{|f(a)|^2}{|g'(a)|^2} \|g'\|_{L^2}^2 - \omega^2 |u(a)|^2 \|g'\|_{L^2}^2 + O\left(\omega^{-1/(k+1)}\right),$$

and upon using Lemma 3.5

$$\begin{aligned}
\|u'\|_{L^2}^2 &\leq \frac{|f(a)|^2}{|g(a)|^2} \|g'\|_{L^2}^2 - \omega^2 \left| \frac{f(a)}{\mathrm{i}\omega g'(a)} + O\left(\frac{1}{\omega^2}\right) \right|^2 \|g'\|_{L^2}^2 + O\left(\omega^{-1/(k+1)}\right) \\
&= O\left(\omega^{-1/(k+1)}\right).
\end{aligned}$$

□

EXAMPLE 3.9 We take the following example from Deano & Huybrechs (2009, pg. 43). Let $[a,b] = [-1,1]$, $f(x) = e^x$ and $g(x) = x$. It is straightforward to show that the general solution to the

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corresponding Levin equation is

$$u(x) = \left[u(-1) - \frac{e^{-1}}{1+\mathrm{i}\omega}\right] e^{-\mathrm{i}\omega(1+x)} + \frac{e^x}{1+\mathrm{i}\omega}.$$

The authors mention that the slowly oscillating solution can be obtained by choosing $u(-1)$ so that the term in parenthesis vanishes and that this is the solution that the collocation approach would approximate in Levin's method. There are, however, many other slowly oscillating solutions. By Lemma 3.5, we know that by choosing

$$u(-1) = \frac{f(-1)}{\mathrm{i}\omega g'(-1)} = \frac{e^{-1}}{\mathrm{i}\omega}$$

that the resulting solution will satisfy $\|u'\|_{L^2} = o(1)$. Indeed, in this case

$$u(x) = \left[\frac{e^{-1}}{\mathrm{i}\omega(1+\mathrm{i}\omega)}\right] e^{-(1+\mathrm{i}\omega)x} + \frac{e^x}{1+\mathrm{i}\omega}$$

so clearly $u' = O(1/\omega)$ uniformly in $[-1, 1]$. However, higher order derivatives are not as well controlled. To control these, high order terms are needed in the expression for $u(a)$, which naturally coincide with the expansion of $e^{-1}/(1+\mathrm{i}\omega)$ for large ω.

As was highlighted in Remark 3.6, it is possible to obtain higher order terms for the initial data for the solution to the underlying minimization problem (3.2) in the form

$$u(a) = \frac{1}{\mathrm{i}\omega}\sum_{n=0}^{N} \beta_n \omega^{-\alpha_n} + o\left(\frac{1}{\omega^{\alpha_N}}\right),$$

where $\{\alpha_n\}$ is an increasing sequence. The coefficients $\{\beta_n\}$ depend not only on the local behaviour of f and g (and derivatives) at the endpoints, but also global terms such as $\|g'\|_{L^2}$. So there is added numerical work to be done if we use these higher order terms, e.g., standard quadrature would do the job, and in principle terms of arbitrarily high order can be obtained.

It must be stressed that, no matter how many terms are computed in the asymptotic expansion for $u(a)$, it *does not* mean we will get a solution with well controlled derivatives of arbitrarily high order. It will simply give us a solution that is closer to the optimum solution, in the sense of Theorem 2.1. For example, the following simple integral is examined in Wang & Xiang (2022b)

$$\int_0^1 (1+x)\, e^{\mathrm{i}\omega x^2}\, \mathrm{d}x.$$

This can immediately be put in the relevant form for our results and by choosing the appropriate initial condition, it is the case that the corresponding solution to Levin's equation obeys $\|u'\|_{L^2}^2 = O(\omega^{-1/2})$. However, the authors show that *all* the solutions to Levin's equation have the property that $\|u^{(m)}\|_\infty \to \infty$ for $m \geq 2$ as ω becomes large. This doesn't contradict any of our results, but highlights that we can only guarantee control of $\|u'\|_{L^2}$.

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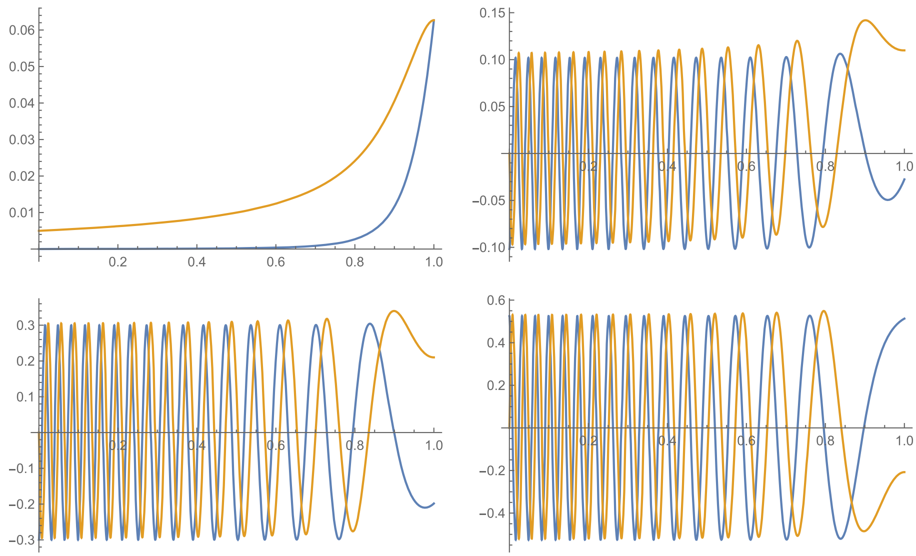

FIG. 1. Real and imaginary parts of solutions to the Levin equation ($\omega = 100$). First uses near-optimal initial data, others have pseudo-random initial data from $[0, 1]$.

EXAMPLE 3.10 Consider the case $[a, b] = [0, 1]$, $g(x) = (1 - x)^2$ and $f(x) = 1$. The associated Levin equation with near-optimal initial condition is

$$u' + 2\mathrm{i}\omega(x - 1)u = 1, \quad u(0) = \frac{f(0)}{\mathrm{i}\omega g'(0)} = \frac{\mathrm{i}}{2\omega}.$$

We see from Fig. 1 that choosing near-optimal initial data has a tremendous effect. In this case, the nonoscillatory nature of the solution makes it much easier to compute numerically.

Standard numerical integrators can obtain the solution to Levin's equation with the near-optimal initial data in very little time to whatever desired accuracy. On the other hand, with random initial data, the oscillatory nature of the solution means the numerical schemes have to work considerably harder. In the case of near-optimal initial data, collocation schemes can even be used to solve Levin's equation (see §5).

4. Higher dimensions

The ideas presented in this section appeal to oscillatory integrals over any n-dimensional simplex. Obviously, the major cases of interest are $n = 2, 3$, but the results hold in more exotic cases. We seek to approximate

$$\int_{\Omega} e^{\mathrm{i}\omega g(\mathbf{x})} f(\mathbf{x}) \, \mathrm{vol}_{\Omega}(\mathbf{x}).$$

Again we will assume that f is smooth. We will also assume that g has a single, isolated stationary point within Ω. Recall that a simplex with vertices at the affinely independent points $\{\mathbf{x}_i\}_{i=0}^{n}$ is the closed

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subset of $\mathbf{R}^n$ consisting of points of the form

$$\sum_{i=0}^{n} \epsilon_i \mathbf{x}_i \quad \text{such that} \quad \sum_{i=0}^{n} \epsilon_i = 1 \text{ and } \epsilon_i \geq 0.$$

We assume that the stationary point, $\mathbf{x}_0$ say, is *nondegenerate* meaning that $\nabla g(\mathbf{x}_0) = 0$ and

$$\det D^2 g(\mathbf{x}_0) \neq 0.$$

Here $D^2 g$ denotes the Hessian matrix with entries $(D^2 g)_{ij} = \partial^2 g / \partial x_i \partial x_j$. This means that in a neighbourhood of the stationary point we have

$$\nabla g(\mathbf{x}) = D^2 g(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + O\left(|\mathbf{x} - \mathbf{x}_0|^2\right).$$

In particular, nondegenerate stationary points are isolated.

The condition that $\mathbf{x}_0$ be a nondegenerate stationary point allows us to invoke the Morse lemma: there exists an open set U containing $\mathbf{x}_0$ and a diffeomorphism $\kappa : U \to \mathbf{R}^n$ so that

$$g(\mathbf{x}) = \mathbf{q}^t J \mathbf{q}, \quad J = \operatorname{diag}(\pm 1, \ldots, \pm 1)$$

in the new coordinates $\mathbf{q} = \kappa(\mathbf{x})$. So, generically speaking, the objects of interest we are computing are of the form

$$\int_{\Sigma} F(\mathbf{q})\, e^{\mathrm{i}\omega \mathbf{q}^t J \mathbf{q}}\, \mathrm{vol}_{\Sigma}(\mathbf{q}),$$

where $\Sigma = \kappa(\Omega)$. So the stationary points are classical centres or saddles. In practice, the construction of the diffeomorphism κ will be nontrivial, so it is not enough to simply construct methods for integrals of the above form.

We can, without loss of generality, assume the stationary point lies at a vertex of Ω. Indeed, if it lies in the interior, we can partition Ω into smaller simplexes by joining each vertex of Ω to the stationary point. See Fig. 2 for details. Similarly, if the stationary point lies on a face. So from now on, we assume the stationary point $\mathbf{x}_0$ lies on a vertex of the simplex. This is analogous to the one-dimensional problem in which we assumed the stationary point was located at the end of the interval, i.e., the vertex of a one-dimensional simplex.

Our aim will be to find a nonoscillatory solution $u = u(\mathbf{x})$ to the equation

$$\mathbf{c} \cdot \nabla u + \mathrm{i}\omega(\mathbf{c} \cdot \nabla g) u = f$$

so that, by the divergence theorem

$$\int_{\Omega} e^{\mathrm{i}\omega g(\mathbf{x})} f(\mathbf{x})\, \mathrm{vol}_{\Omega}(\mathbf{x}) = \int_{\partial\Omega} e^{\mathrm{i}\omega g(\mathbf{x})} (\mathbf{c} \cdot \mathbf{n}) u(\mathbf{x})\, \mathrm{vol}_{\partial\Omega}(\mathbf{x}).$$

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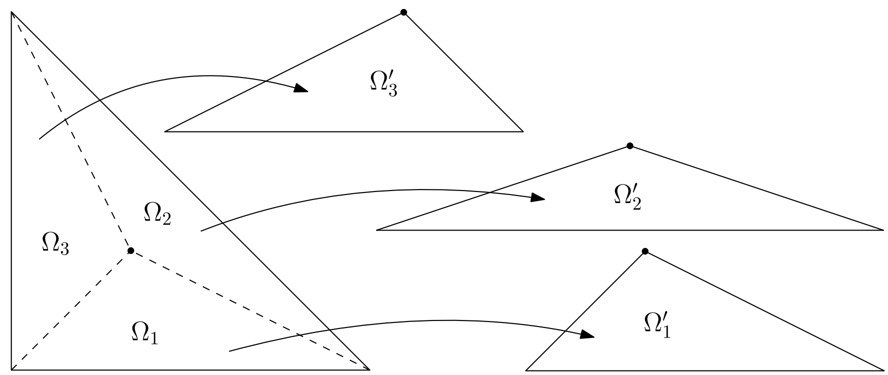

FIG. 2. Simplex decomposition in presence of stationary point.

The vector field $\mathbf{c}$ defines the characteristic curves of this hyperbolic problem. By specifying initial data on one of the faces, Σ say, we can reconstruct the solution throughout Ω so long as every interior point can be connected to Σ through an integral curve of $\mathbf{c}$. We lose no generality by assuming $|\mathbf{c}| = 1$ throughout this section. We will need the following definition.

DEFINITION 4.1 Let $\Omega \subset \mathbf{R}^n$ be an n-simplex with faces $\{\Sigma_i\}_{i=0}^n$ and corresponding outward normals $\{\mathbf{n}_i\}_{i=0}^n$. We say that a constant vector field $\mathbf{c} : \mathbf{R}^n \to \mathbf{R}^n$ *enters* Ω *through* Σ_i if $\mathbf{c} \cdot \mathbf{n}_i < 0$ and $\mathbf{c} \cdot \mathbf{n}_j \geq 0$ for $j \neq i$.

If you imagine a flow of fluid associated with a constant vector field that enters Ω through Σ_i, then this fluid has negative flux over the face Σ_i and non-negative flux over all other faces. It is straightforward to show that these are precisely the constant vector fields permissible if we want data on Σ_i to generate the solution $u = u(\mathbf{x})$ throughout Ω. It's also clear that there are many such vector fields. The following elementary lemma is an immediate consequence of the preceding definition.

LEMMA 4.2 Let $\mathbf{c} : \mathbf{R}^n \to \mathbf{R}^n$ enter the n-simplex through the face Σ_0 opposite the vertex $\mathbf{x}_0$. Then

(a) Each $\mathbf{x} \in \Omega$ can be written uniquely as $\mathbf{x} = \mathbf{y} + \mathbf{c}t$, with $\mathbf{y} \in \Sigma_0$, $t \geq 0$.

(b) For each $\mathbf{y} \in \operatorname{int} \Sigma_0$, there exists a $\tau(\mathbf{y}) > 0$ such that $\mathbf{y} + \mathbf{c}\tau(\mathbf{y}) \in \partial\Omega$ and the function $\tau : \Sigma_0 \to [0, \infty)$ is Lipschitz continuous.

(c) If $\mathbf{x}_0 = \mathbf{y}_0 + \mathbf{c}\tau(\mathbf{y}_0)$, then $\tau(\mathbf{y}) \leq \tau(\mathbf{y}_0)$ for each $\mathbf{y} \in \Sigma_0$.

(d) If $\{\mathbf{o}_i\}_{i=1}^{n-1}$ denotes a basis for the plane $\mathbf{c} \cdot \mathbf{o} = 0$, then any function $m : \Sigma_0 \to \mathbf{C}$ can be written in the form

$$m(\mathbf{y}) = M(\mathbf{o}_1 \cdot \mathbf{x}, \ldots, \mathbf{o}_{n-1} \cdot \mathbf{x}),$$

where $\mathbf{x} = \mathbf{y} + \mathbf{c}t$ and $M : \mathbf{R}^{n-1} \to \mathbf{C}$ is defined appropriately.

(e) There exists an $\rho < 1$ such that

$$\mathbf{c} \cdot (\mathbf{x} - \mathbf{x}_0) \leq \rho|\mathbf{x} - \mathbf{x}_0|, \quad \mathbf{c} \cdot (\mathbf{y} - \mathbf{y}_0) \leq \rho|\mathbf{y} - \mathbf{y}_0|$$

for each $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \Sigma_0$, respectively.

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We omit the proof, other than to point out the correspondence between $\mathbf{x}$ and $\mathbf{y}+\mathbf{c}t$ can be found in a straightforward manner in terms of $\mathbf{c}$, $\mathbf{y}_0$ and $\mathbf{n}_0$, the outward unit normal to Σ_0. The following formulas are relevant

$$t(\mathbf{x}) = \frac{(\mathbf{x}-\mathbf{y}_0)\cdot\mathbf{n}_0}{\mathbf{c}\cdot\mathbf{n}_0}$$

and

$$\mathbf{y}(\mathbf{x}) = \mathbf{y}_0 + \left[\sum_{i=1}^{n-1}[(\mathbf{x}-\mathbf{y}_0)\cdot\mathbf{o}_i]\mathbf{o}_i - \sum_{i=1}^{n-1}[\mathbf{n}_0\cdot\mathbf{o}_i]\left(\frac{(\mathbf{x}-\mathbf{y}_0)\cdot\mathbf{o}_i}{\mathbf{c}\cdot\mathbf{n}_0}\right)\mathbf{c}\right].$$

The latter immediately gives part (d) of the lemma and the former gives

$$\tau(\mathbf{y}_0) = t(\mathbf{x}_0) = \frac{(\mathbf{x}_0-\mathbf{y}_0)\cdot\mathbf{n}_0}{\mathbf{c}\cdot\mathbf{n}_0}.$$

The function $\tau = \tau(\mathbf{y})$ can be thought of as the 'stopping time' for a point on $\mathbf{y}\in\Sigma_0$ to be transported to $\partial\Omega$ along a characteristic curve. Generally,

$$\tau(\mathbf{y}) = \min_{\substack{i\neq 0\\ \mathbf{c}\cdot\mathbf{n}_i\neq 0}} \frac{(\mathbf{x}_0-\mathbf{y})\cdot\mathbf{n}_i}{\mathbf{c}\cdot\mathbf{n}_i}, \quad \mathbf{y}\in\Sigma_0, \tag{4.1}$$

which gives the Lipschitz continuity of $\mathbf{y}\mapsto\tau(\mathbf{y})$ and it also gives rise to the useful estimate $|\tau(\mathbf{y}_0)-\tau(\mathbf{y})| \gtrsim |\mathbf{y}_0-\mathbf{y}|$ for every $\mathbf{y}\in\Sigma_0$.

DEFINITION 4.3 Let $\Omega\subset\mathbf{R}^n$ be a simplex with vertices $\{\mathbf{x}_i\}_{i=0}^n$ and suppose $\mathbf{c}:\mathbf{R}^n\to\mathbf{R}^n$ enters Ω through the face Σ_0 opposite the vertex $\mathbf{x}_0$. We will say the system $(\Omega, g, \mathbf{c}, \mathbf{x}_0)$ is *admissible* if:

(i) $g\in C^\infty(\Omega)$ has a stationary point at $\mathbf{x}_0$;

(ii) $\nabla g\neq 0$ and $\mathbf{c}\cdot\nabla g\neq 0$ on $\Omega\setminus\{\mathbf{x}_0\}$;

(iii) $|\mathbf{c}^t D^2 g(\mathbf{x}_0)(\mathbf{x}_0-\mathbf{x}_i)| > 0$ for $i=1,\ldots,n$.

Not only does (iii) imply the stationary point is nondegenerate, but it also says that the zero set $Z=\{\mathbf{x}\in\mathbf{R}^n : \mathbf{c}\cdot\nabla g(\mathbf{x})=0\}$ is *nontangential* to Ω at $\mathbf{x}=\mathbf{x}_0$. Indeed, the normal to Z at $\mathbf{x}_0$ is parallel to

$$\nabla(\mathbf{c}\cdot\nabla g)(\mathbf{x}_0) = \mathbf{c}^t D^2 g(\mathbf{x}_0).$$

So the condition states that this normal cannot be orthogonal to any of the faces at $\mathbf{x}_0$. See Fig. 3. The central reason for this technical constraint is that it gives rise to the following estimate.

LEMMA 4.4 If $(\Omega, g, \mathbf{c}, \mathbf{x}_0)$ is an admissible system then $\mathbf{c}^t D^2 g(\mathbf{x}_0)\mathbf{c}\neq 0$ and

$$|\mathbf{c}\cdot\nabla g(\mathbf{x})| \gtrsim |\mathbf{x}-\mathbf{x}_0|, \quad \mathbf{x}\in\Omega.$$

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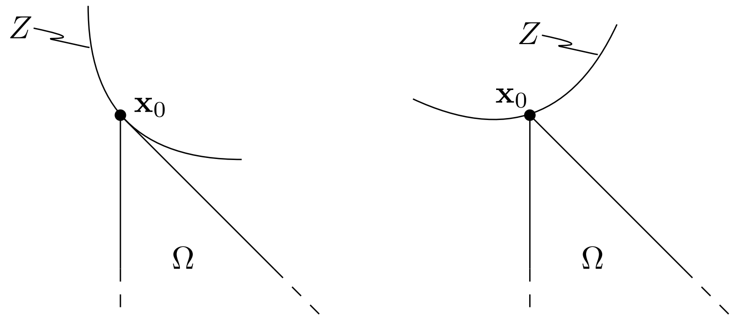

FIG. 3. The zero set $Z = \{\mathbf{x} : \mathbf{c} \cdot \nabla g(\mathbf{x}) = 0\}$ is tangential to Ω at $\mathbf{x}_0$ on the left and nontangential on the right.

Proof. First note that each of $\{\mathbf{c}^t D^2 g(\mathbf{x}_0)(\mathbf{x}_0 - \mathbf{x}_i)\}_{i=1}^n$ have the same sign. Indeed, from (ii), we know that $\Omega \setminus \{\mathbf{x}_0\}$ lies in either $\mathbf{c} \cdot \nabla g(\mathbf{x}) > 0$ or $\mathbf{c} \cdot \nabla g(\mathbf{x}) < 0$. Without loss of generality, suppose it is the former. Then by Taylor's theorem we have for each $\mathbf{x} \in \Omega$

$$0 < \mathbf{c} \cdot \nabla g(\mathbf{x}) = \mathbf{c}^t D^2 g(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + O\left(|\mathbf{x} - \mathbf{x}_0|^2\right),$$

from which it follows that $\mathbf{c}^t D^2 g(\mathbf{x}_0)(\mathbf{x}_i - \mathbf{x}_0) > 0$ for each $i = 1, \dots n$. Let $\delta > 0$ denote the smallest of these numbers. Then for each $\mathbf{x} \in \Omega \setminus \{\mathbf{x}_0\}$ there are $\{\epsilon_i\}_{i=0}^n$ with $\sum_i \epsilon_i = 1$ and $\epsilon_i \geq 0$ such that $\mathbf{x} = \sum_i \epsilon_i \mathbf{x}_i$. Consequently,

$$\mathbf{c}^t D^2 g(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = \sum_{i=0}^n \epsilon_i \mathbf{c}^t D^2 g(\mathbf{x}_0)(\mathbf{x}_i - \mathbf{x}_0) \geq \left(1 - \epsilon_0\right) \delta > 0.$$

This gives $\mathbf{c}^t D^2 g(\mathbf{x}_0)\mathbf{c} < 0$. Also, on sets of the form $\Omega \cap \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| = r\}$, we know by continuity and compactness that the function

$$\mathbf{x} \mapsto \mathbf{c}^t D^2 g(\mathbf{x}_0)\left[\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}\right]$$

is bounded below by a positive constant independent of $r > 0$. So for $\mathbf{x} \in \Omega$ in a sufficiently small neighbourhood of $\mathbf{x}_0$ we have

$$\begin{aligned}\mathbf{c} \cdot \nabla g(\mathbf{x}) &= |\mathbf{x} - \mathbf{x}_0|\left[\mathbf{c}^t D^2 g(\mathbf{x}_0)\left[\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}\right] + O(|\mathbf{x} - \mathbf{x}_0|)\right] \\ &\gtrsim |\mathbf{x} - \mathbf{x}_0|.\end{aligned}$$

The estimate is trivial on the rest of Ω because $\mathbf{c} \cdot \nabla g(\mathbf{x}) > 0$ there. $\square$

Using $\mathbf{x}_0 = \mathbf{y}_0 + \mathbf{c}\tau(\mathbf{y}_0)$, this Lemma gives

$$|\mathbf{c} \cdot \nabla g(\mathbf{y} + \mathbf{c}t)| \gtrsim |(\mathbf{y} - \mathbf{y}_0) - \mathbf{c}(\tau(\mathbf{y}_0) - t)|$$

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and by (e) in Lemma 4.2 and some elementary manipulation we have

$$\begin{aligned}|(\mathbf{y}-\mathbf{y}_0)-\mathbf{c}(\tau(\mathbf{y}_0)-t)|^2 &= |\mathbf{y}-\mathbf{y}_0|^2+|\tau(\mathbf{y}_0)-t|^2-2\mathbf{c}\cdot(\mathbf{y}-\mathbf{y}_0)|\tau(\mathbf{y}_0)-t| \\ &\geq |\mathbf{y}-\mathbf{y}_0|^2+|\tau(\mathbf{y}_0)-t|^2-2\rho|\mathbf{y}-\mathbf{y}_0||\tau(\mathbf{y}_0)-t| \\ &\geq \tfrac{1}{2}\left(1-\rho^2\right)\left(|\mathbf{y}-\mathbf{y}_0|^2+(\tau(\mathbf{y}_0)-t)^2\right) \\ &\gtrsim \left(|\mathbf{y}-\mathbf{y}_0|+(\tau(\mathbf{y}_0)-t)\right)^2.\end{aligned}$$

In summary, we get a lower bound of the form

$$|\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}t)| \gtrsim |\mathbf{y}-\mathbf{y}_0|+(\tau(\mathbf{y}_0)-t). \tag{4.2}$$

This will be used extensively in what follows.

REMARK 4.5 If we were to drop the nondegeneracy condition and consider a wider class of stationary points, then it will be clear from the analysis that the significantly weaker condition

$$|\mathbf{c}\cdot\nabla g(\mathbf{x})| \gtrsim |\mathbf{x}-\mathbf{x}_0|^k \quad \mathbf{x}\in\Omega$$

for some $k \geq 1$, would be sufficient to derive results more aligned with those in §3. We will leave that extended analysis for another work, but mention in passing that numerical experimentation reinforces this prediction.

For the vector fields $\mathbf{c}$ and stationary points $\mathbf{x}_0$ of the form discussed above, we want to solve the following optimization problem:

$$\text{minimize} \int_\Omega |\mathbf{c}\cdot\nabla u|^2 \operatorname{vol}_\Omega \quad \text{subject to } \mathbf{c}\cdot\nabla u+\mathrm{i}\omega(\mathbf{c}\cdot\nabla g)u=f.$$

Before we appeal to Theorem 2.1, we need the following lemma. The following notation will be used throughout this section

$$A^{\mathbf{c}}_\omega=\mathbf{c}\cdot\nabla+\mathrm{i}\omega(\mathbf{c}\cdot\nabla g), \quad B^{\mathbf{c}}=\mathbf{c}\cdot\nabla.$$

Again we will abuse notation and write $u=u(\mathbf{x})$ rather than $u=u(\mathbf{x};\omega)$ when referring to a solution of $A^{\mathbf{c}}_\omega u=f$.

LEMMA 4.6 Let $(\Omega, g, \mathbf{c}, \mathbf{x}_0)$ be an admissible system and define $H=L^2(\Omega)$ and $X=C^1(\Omega)$. Then $A^{\mathbf{c}}_\omega : X\subset H\to H$ is closed and

$$\|u\|_{L^2} \lesssim \left\|A^{\mathbf{c}}_\omega u\right\|_{L^2}+\|B^{\mathbf{c}}u\|_{L^2} \quad \forall u\in X.$$

The previous result is true under the much weaker assumption that ∇g and $\mathbf{c}\cdot\nabla g$ are nonvanishing on $\Omega\setminus\{\mathbf{x}_0\}$. The proof is left in the appendix.

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COROLLARY 4.7 The solution to the problem

$$\text{minimize } \|\mathbf{c}\cdot\nabla u\|_{L^2} \quad \text{subject to} \quad A^{\mathbf{c}}_{\omega}u = f$$

exists and is unique in $C^1(\Omega)$. It is completely characterized by

$$A^{\mathbf{c}}_{\omega}u = f \quad \text{and} \quad \int_{\Omega}(\mathbf{c}\cdot\nabla u)\left[(\mathbf{c}\cdot\nabla g)\,e^{\mathrm{i}\omega g}w\right]\mathrm{vol}_{\Omega} = 0$$

for all $w = w(\mathbf{x}\cdot\mathbf{o})$ with $\mathbf{c}\cdot\mathbf{o} = 0$.

This is immediate from Theorem 2.1 the observation that $v \in \ker A^{\mathbf{c}}_{\omega}$ iff

$$v(\mathbf{x}) = e^{-\mathrm{i}\omega g(\mathbf{x})}w(\mathbf{x}\cdot\mathbf{o})$$

for arbitrary $w \in C^1(\mathbf{R})$ and $\mathbf{c}\cdot\mathbf{o} = 0$. We know from Lemma 4.2 that the function $\mathbf{x} \mapsto w(\mathbf{x}\cdot\mathbf{o})$ can specify arbitrary behaviour on the face Σ_0.

LEMMA 4.8 Let $(\Omega, g, \mathbf{c}, \mathbf{x}_0)$ be an admissible system and let Σ_0 denote the face of Ω opposite the vertex $\mathbf{x}_0$. Then the unique solution to the problem

$$\text{minimize } \|\mathbf{c}\cdot\nabla u\|_{L^2} \quad \text{subject to} \quad A^{\mathbf{c}}_{\omega}u = f$$

satisfies the initial condition

$$\sup_{\mathbf{y}\in\Sigma_0}\left|u(\mathbf{y}) - \frac{f(\mathbf{y})}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y})}\right| = O\left(\frac{1}{\omega^{3/2}}\right).$$

Proof. This follows in a similar manner to the proof to Lemma 3.5. For $\mathbf{x} \neq \mathbf{x}_0$, the solution to $A^{\mathbf{c}}_{\omega}u = f$ can be written

$$\begin{aligned}
e^{\mathrm{i}\omega g(\mathbf{x})}u(\mathbf{x}) &= e^{\mathrm{i}\omega g(\mathbf{y})}u(\mathbf{y}) + \int_0^t e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}f(\mathbf{y}+\mathbf{c}s)\,\mathrm{d}s \\
&= e^{\mathrm{i}\omega g(\mathbf{y})}u(\mathbf{y}) + \frac{f(\mathbf{x})\,e^{\mathrm{i}\omega g(\mathbf{x})}}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{x})} - \frac{f(\mathbf{y})\,e^{\mathrm{i}\omega g(\mathbf{y})}}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y})} \\
&\quad - \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f(\mathbf{y}+\mathbf{c}s)}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)}\right)e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\,\mathrm{d}s,
\end{aligned}$$

where $t = t(\mathbf{x})$ and $\mathbf{y} = \mathbf{y}(\mathbf{x}) \in \Sigma_0$ are such that $\mathbf{x} = \mathbf{y} + \mathbf{c}t$. Hence,

$$\begin{aligned}
e^{\mathrm{i}\omega g(\mathbf{x})}(\mathbf{c}\cdot\nabla u) &= e^{\mathrm{i}\omega g(\mathbf{x})}f(\mathbf{x}) - \mathrm{i}\omega(\mathbf{c}\cdot\nabla g)\,e^{\mathrm{i}\omega g(\mathbf{x})}u(\mathbf{x}) \\
&= -\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{x})\left[u(\mathbf{y}) - \frac{f(\mathbf{y})}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y})}\right]e^{\mathrm{i}\omega g(\mathbf{y})} \\
&\quad + \mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{x})\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f(\mathbf{y}+\mathbf{c}s)}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)}\right)e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\,\mathrm{d}s.
\end{aligned}$$

And so

$$\int_\Omega (\mathbf{c}\cdot\nabla u)\left[(\mathbf{c}\cdot\nabla g)\, e^{\mathrm{i}\omega g} w\right] \mathrm{vol}_\Omega = 0$$

with $w = w(\mathbf{x}\cdot\mathbf{o})$ and $\mathbf{c}\cdot\mathbf{o} = 0$ iff

$$\begin{aligned}\int_\Omega |\mathbf{c}\cdot\nabla g(\mathbf{x})|^2 \left[u(\mathbf{y}) - \frac{f(\mathbf{y})}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y})}\right] w(\mathbf{x}\cdot\mathbf{o})\,\mathrm{vol}_\Omega(\mathbf{x}) \\ = \int_\Omega |\mathbf{c}\cdot\nabla g(\mathbf{x})|^2 \left[\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f(\mathbf{y}+\mathbf{c}s)}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)}\right) e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\,\mathrm{d}s\right] w(\mathbf{x}\cdot\mathbf{o})\,\mathrm{vol}_\Omega(\mathbf{x}).\end{aligned} \tag{4.3}$$

Observe that for any $\rho : \Omega \to \mathbf{C}$, we can write

$$\int_\Omega \rho(\mathbf{x}) w(\mathbf{x}\cdot\mathbf{o})\,\mathrm{vol}_\Omega(\mathbf{x}) = \int_{\Sigma_0}\left(\int_0^{\tau(\mathbf{y})} \rho(\mathbf{y}+\mathbf{c}t)\,\mathrm{d}t\right) w(\mathbf{y}\cdot\mathbf{o})\,\mathrm{vol}_{\Sigma_0}(\mathbf{y}),$$

where $\tau(\mathbf{y}) > 0$ is as in Lemma 4.2. If we define

$$F(t,\mathbf{y}) = -\int_t^{\tau(\mathbf{y})} |\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)|^2\,\mathrm{d}s$$

and

$$G(\mathbf{x}) = (\mathbf{c}\cdot\nabla g(\mathbf{x}))(\mathbf{c}\cdot\nabla f(\mathbf{x})) - f(\mathbf{x})(\mathbf{c}\cdot\nabla)^2 g(\mathbf{x}),$$

then, on integration by parts in t, the right-hand side of (3) becomes[1]

$$\begin{aligned}&\int_\Omega |\mathbf{c}\cdot\nabla g(\mathbf{x})|^2 \left[\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f(\mathbf{y}+\mathbf{c}s)}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)}\right) e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\,\mathrm{d}s\right] w(\mathbf{x}\cdot\mathbf{o})\,\mathrm{vol}_\Omega(\mathbf{x}) \\ &= \frac{1}{\mathrm{i}\omega}\int_{\Sigma_0}\left(\int_0^{\tau(\mathbf{y})} \frac{F(t,\mathbf{y})}{F_t(t,\mathbf{y})} G(\mathbf{y}+\mathbf{c}t)\, e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)}\,\mathrm{d}t\right) w(\mathbf{y}\cdot\mathbf{o})\,\mathrm{vol}_{\Sigma_0}(\mathbf{y}) \\ &= \frac{1}{(\mathrm{i}\omega)^2}\int_{\Sigma_0}\left(\int_0^{\tau(\mathbf{y})} H(t,\mathbf{y})\frac{\partial}{\partial t}\left(e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)}\right)\mathrm{d}t\right) w(\mathbf{y}\cdot\mathbf{o})\,\mathrm{vol}_{\Sigma_0}(\mathbf{y}) \\ &= \frac{1}{\omega^{3/2}}\int_{\Sigma_0} I(\mathbf{y},\omega)\, w(\mathbf{y}\cdot\mathbf{o})\,\mathrm{vol}_{\Sigma_0}(\mathbf{y}),\end{aligned}$$

[1] The $\int_0^t \cdots$ part is differentiated and $t \mapsto |\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}t)|^2$ is integrated.

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where

$$H(t,\mathbf{y}) = \frac{G(\mathbf{y}+\mathbf{c}t)\int_t^{\tau(\mathbf{y})} |\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)|^2\,\mathrm{d}s}{\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}t)^3}$$

and

$$I(\mathbf{y};\omega) = \frac{1}{\sqrt{\omega}}\int_0^{\tau(\mathbf{y})} H(t,\mathbf{y})\frac{\partial}{\partial t}\left(e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)}\right)\mathrm{d}t.$$

To justify these steps, note that away from the stationary point $\mathbf{x}_0$ we have $|\mathbf{c}\cdot\nabla g(\mathbf{x})| \neq 0$ by our hypothesis on g and $\mathbf{c}$. So H is smooth away from the point $(t,\mathbf{y}) = (\tau(\mathbf{y}_0),\mathbf{y}_0)$. Since $G \in C(\Omega)$, Lemma 4.4 gives

$$|H(t,\mathbf{y})| \lesssim \frac{\int_t^{\tau(\mathbf{y})} |\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)|^2\,\mathrm{d}s}{|\mathbf{y}+\mathbf{c}t-\mathbf{x}_0|^3} \lesssim \frac{\int_t^{\tau(\mathbf{y})} |\mathbf{y}+\mathbf{c}s-\mathbf{x}_0|^2\,\mathrm{d}s}{|\mathbf{y}+\mathbf{c}t-\mathbf{x}_0|^3}.$$

Carrying out the integration, we find

$$|H(t,\mathbf{y})| \lesssim \sum_{j=1}^{3}\left[\frac{\tau(\mathbf{y})-t}{|\mathbf{y}+\mathbf{c}t-\mathbf{x}_0|}\right]^j. \tag{4.4}$$

From Lemma 4.2, we know $\tau(\mathbf{y}) \le \tau(\mathbf{y}_0)$ for each $\mathbf{y} \in \Sigma_0$ and by observing

$$\tau(\mathbf{y}_0) = \frac{(\mathbf{x}_0-\mathbf{y}_0)\cdot\mathbf{n}_0}{\mathbf{c}\cdot\mathbf{n}_0}, \quad t \equiv \frac{(\mathbf{y}+\mathbf{c}t-\mathbf{y}_0)\cdot\mathbf{n}_0}{\mathbf{c}\cdot\mathbf{n}_0}$$

(recall that $\mathbf{n}_0$ denotes the outward normal to Σ_0) we find

$$\frac{|\tau(\mathbf{y})-t|}{|\mathbf{y}+\mathbf{c}t-\mathbf{x}_0|} \le \frac{|\tau(\mathbf{y}_0)-t|}{|\mathbf{y}+\mathbf{c}t-\mathbf{x}_0|} \le \frac{|(\mathbf{y}+\mathbf{c}t-\mathbf{x}_0)\cdot\mathbf{n}_0|}{|\mathbf{c}\cdot\mathbf{n}_0||\mathbf{y}+\mathbf{c}t-\mathbf{x}_0|} \le \frac{1}{|\mathbf{c}\cdot\mathbf{n}_0|},$$

which shows $(t,\mathbf{y}) \mapsto H(t,\mathbf{y})$ is bounded on Ω and continuous away from the point $(t,\mathbf{y}) = (\tau(\mathbf{y}_0),\mathbf{y}_0)$ so the previous manipulations are justified. Form here it can be shown that $I \in C(\Sigma_0)$ and

$$|I(\mathbf{y};\omega)| \lesssim \tau(\mathbf{y}). \tag{4.5}$$

We leave the proof of this estimate to the appendix. In summary

$$\int_\Omega |\mathbf{c}\cdot\nabla g(\mathbf{x})|^2\left[u(\mathbf{y}) - \frac{f(\mathbf{y})}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y})}\right] w(\mathbf{x}\cdot\mathbf{o})\,\mathrm{vol}_\Omega(\mathbf{x}) = \frac{1}{\omega^{3/2}}\int_{\Sigma_0} I(\mathbf{y},\omega)\,w(\mathbf{y}\cdot\mathbf{o})\,\mathrm{vol}_{\Sigma_0}(\mathbf{y})$$

or equivalently

$$\int_{\Sigma_0} |\kappa(\mathbf{y})|^2 \left[u(\mathbf{y}) - \frac{f(\mathbf{y})}{\mathrm{i}\omega \mathbf{c} \cdot \nabla g(\mathbf{y})} \right] w(\mathbf{y} \cdot \mathbf{o}) \,\mathrm{vol}_{\Sigma_0}(\mathbf{y}) = \frac{1}{\omega^{3/2}} \int_{\Sigma_0} I(\mathbf{y}, \omega)\, w(\mathbf{y} \cdot \mathbf{o}) \,\mathrm{vol}_{\Sigma_0}(\mathbf{y})$$

where we have defined

$$\begin{aligned} |\kappa(\mathbf{y})|^2 &= \int_0^{\tau(\mathbf{y})} |\mathbf{c} \cdot \nabla g(\mathbf{y} + \mathbf{c}s)|^2 \,\mathrm{d}s \\ &\gtrsim \int_0^{\tau(\mathbf{y})} |(\mathbf{y} + \mathbf{c}\tau(\mathbf{y}) - \mathbf{x}_0) - \mathbf{c}(\tau(\mathbf{y}) - s)|^2 \,\mathrm{d}s \\ &\gtrsim \int_0^{\tau(\mathbf{y})} \left[|\mathbf{y} + \mathbf{c}\tau(\mathbf{y}) - \mathbf{x}_0|^2 + (\tau(\mathbf{y}) - s)^2 \right] \mathrm{d}s. \end{aligned}$$

In the third line, we used property (e) in Lemma 4.2. Performing the integral

$$|\kappa(\mathbf{y})|^2 \gtrsim \tau(\mathbf{y}) \left[|\mathbf{y} + \mathbf{c}\tau(\mathbf{y}) - \mathbf{x}_0|^2 + \tfrac{1}{3}\tau(\mathbf{y})^2 \right] \gtrsim \tau(\mathbf{y}). \tag{4.6}$$

The final estimate follows since the term in square parenthesis is nonzero for $\mathbf{y} \neq \mathbf{y}_0$ and also for $\mathbf{y} = \mathbf{y}_0$, so by continuity and compactness it is bounded below by some positive constant. If $\{\mathbf{o}_i\}_{i=1}^{n-1}$ are a basis for solutions to $\mathbf{o} \cdot \mathbf{c} = 0$, then we can replace $w(\mathbf{y} \cdot \mathbf{o})$ with any function of the form $w = w(\mathbf{y} \cdot \mathbf{o}_1, \ldots, \mathbf{y} \cdot \mathbf{o}_{n-1})$ and by Lemma 4.2 we know such functions specify arbitrary data on Σ_0. So by the fundamental lemma of the calculus of variations we deduce

$$|\kappa(\mathbf{y})|^2 \left[u(\mathbf{y}) - \frac{f(\mathbf{y})}{\mathrm{i}\omega \mathbf{c} \cdot \nabla g(\mathbf{y})} \right] = \frac{1}{\omega^{3/2}} I(\mathbf{y}; \omega).$$

Now using the estimates (4.5) and (4.6) we deduce our result. □

REMARK 4.9 The proof of the previous result follows in a similar fashion under the significantly weaker assumption

$$|\mathbf{c} \cdot \nabla g(\mathbf{x})| \gtrsim |\mathbf{x} - \mathbf{x}_0|^k, \quad \mathbf{x} \in \Omega$$

for some $k \geq 1$.

REMARK 4.10 Writing the solution to $A_\omega^{\mathbf{c}} u = f$ in the form

$$u(\mathbf{x}) = e^{-\mathrm{i}\omega g(\mathbf{x})} \left[e^{\mathrm{i}\omega g(\mathbf{y})} u(\mathbf{y}) + \int_0^t e^{\mathrm{i}\omega g(\mathbf{y} + \mathbf{c}s)} f(\mathbf{y} + \mathbf{c}s) \,\mathrm{d}s \right]$$

and if $\mathbf{c} \cdot \nabla g \neq 0$, it is clear how to proceed in a manner entirely analogous to that discussed in the one-dimensional problem in the absence of stationary points in the introduction to derive an appropriate

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initial condition for Levin's equation. The relevant adjustments would be $f_0 = f$ and

$$f_{n+1} = -(\mathbf{c}\cdot\nabla)\left[\frac{f_n}{(\mathbf{c}\cdot\nabla)g}\right].$$

This leads to a choice of Cauchy data $u(\mathbf{y})$, $\mathbf{y} \in \Sigma_0$, that has well controlled derivatives of arbitrarily high order.

As with the one-dimensional problem, this shows that if v satisfies

$$A^{\mathbf{c}}_{\omega} v = f \quad \text{and} \quad v(\mathbf{y}) = \frac{f(\mathbf{y})}{\mathrm{i}\omega\mathbf{c}\cdot\nabla g(\mathbf{y})} \quad \mathbf{y}\in\Sigma_0$$

and u is as in the previous lemma then $\|\mathbf{c}\cdot\nabla v\|_{L^2} \approx \|\mathbf{c}\cdot\nabla u\|_{L^2}$. Again, the question is how well controlled is $\|\mathbf{c}\cdot\nabla u\|_{L^2}$. Before we proceed, we note that the general solution to $A^{\mathbf{c}}_{\omega}u = f$ can be written

$$u(\mathbf{x}) = e^{-\mathrm{i}\omega g(\mathbf{x})}\left[e^{\mathrm{i}\omega g(\mathbf{y})}u(\mathbf{y}) + \int_0^t e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)} f(\mathbf{y}+\mathbf{c}s)\,\mathrm{d}s\right],$$

where $\mathbf{y} = \mathbf{y}(\mathbf{o}\cdot\mathbf{x})$ and $t = t(\mathbf{x})$ are as in Lemma 4.2. So u takes the form

$$u(\mathbf{x}) = w(\mathbf{x}) + e^{-\mathrm{i}\omega g(\mathbf{x})}\int_0^{t(\mathbf{x})} e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)} f(\mathbf{y}+\mathbf{c}s)\,\mathrm{d}s,$$

where $w \in \ker A^{\mathbf{c}}_{\omega}$.

LEMMA 4.11 If $(\Omega, g, \mathbf{c}, \mathbf{x}_0)$ is an admissible system and $v \in C^\infty(\Omega)$ then

$$\left\|\mathbf{c}\cdot\nabla g(\mathbf{x})\int_0^{t(\mathbf{x})}\frac{v(\mathbf{y}+\mathbf{c}s)}{(\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s))^2}\,e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\,\mathrm{d}s\right\|^2_{L^2(\mathrm{d}\mathbf{x})} = O\left(\frac{\log\omega}{\omega}\right)$$

$$\left\|[\mathbf{c}\cdot\nabla g(\mathbf{x})]^2\int_0^{t(\mathbf{x})}\frac{v(\mathbf{y}+\mathbf{c}s)}{(\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s))^2}\,e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\,\mathrm{d}s\right\|_{L^1(\mathrm{d}\mathbf{x})} = O\left(\frac{1}{\omega}\right).$$

Proof. We start with a little setting up to simplify the problem. Without loss of generality, we assume $\mathbf{c}\cdot\nabla g > 0$ on $\Omega\setminus\{\mathbf{x}_0\}$. Now we fix $\epsilon > 0$ so that $s \mapsto \psi(s) = \mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)$ is decreasing when $\mathbf{y}+\mathbf{c}s \in \Omega_\epsilon = \{\mathbf{x}\in\Omega : (\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}_0 < \epsilon\}$, see Fig. 4. This is possible since

$$\psi'(s) = \mathbf{c}^t D^2 g(\mathbf{x} - \mathbf{c}(t(\mathbf{x}) - s)\mathbf{c}$$

and $\mathbf{c}^t D^2 g(\mathbf{x}_0)\mathbf{c} < 0$ so if $|\mathbf{x}-\mathbf{x}_0|$ is sufficiently small $\mathbf{c}^t D^2 g(\mathbf{x})\mathbf{c} < 0$. If $\mathbf{x}\in\Omega_\epsilon$, then $|\mathbf{x}-\mathbf{x}_0| \lesssim (\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}_0 < \epsilon$. The contribution to the estimates from $\mathbf{x}\in\Omega\setminus\Omega_\epsilon$ are straightforward because they are separated from the stationary point at $\mathbf{x} = \mathbf{x}_0$, integration by parts in the s-integral will do. If $\ell(\mathbf{x})$

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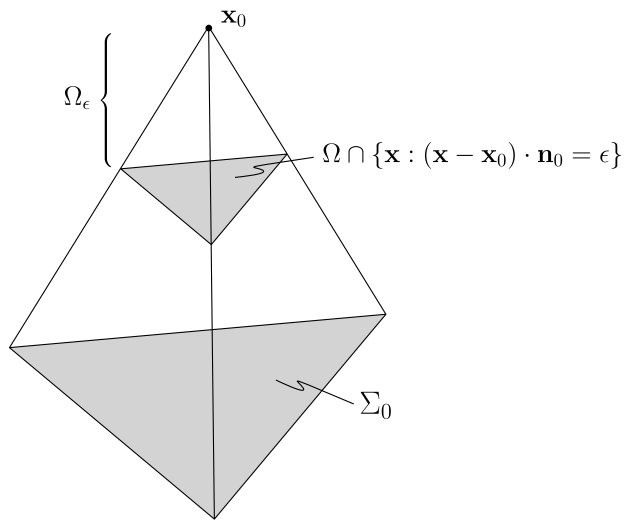

FIG. 4. The decomposition of Ω for the L^1 and L^2 estimates.

denotes the line from $\mathbf{y} \in \Sigma$ to $\mathbf{x} \in \Omega$, then we can write

$$\int_0^{t(\mathbf{x})} \mathrm{d}s \equiv \int_{\ell(\mathbf{x}) \cap (\Omega \setminus \Omega_\epsilon)} \mathrm{d}s + \int_{\ell(\mathbf{x}) \cap \Omega_\epsilon} \mathrm{d}s.$$

In the first integral, $|\mathbf{y} + \mathbf{c}s - \mathbf{x}_0| \gtrsim \epsilon$ so we can integrate by parts as before. So, without loss of generality, we can establish the estimates in the lemma by confining attention to Ω_ϵ, or equivalently we can assume that $s \mapsto \psi(s)$ is monotone decreasing throughout Ω. We will do the latter.

Fix $\delta > 0$, write $\mathbf{x} = \mathbf{y} + \mathbf{c}t$ and make the splitting

$$\mathbf{c} \cdot \nabla g(\mathbf{x}) \left[\int_{t(\mathbf{x})-\delta}^{t(\mathbf{x})} + \int_0^{t(\mathbf{x})-\delta} \right] \frac{v(\mathbf{y} + \mathbf{c}s)}{(\mathbf{c} \cdot \nabla g(\mathbf{y} + \mathbf{c}s))^2}\, e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\, \mathrm{d}s \equiv J_1 + J_2.$$

We have the obvious upper bound

$$|J_1| \leq |\mathbf{c} \cdot \nabla g(\mathbf{x})| \|v\|_\infty \int_{t(\mathbf{x})-\delta}^{t(\mathbf{x})} \frac{1}{(\mathbf{c} \cdot \nabla g(\mathbf{y} + \mathbf{c}s))^2}\, \mathrm{d}s.$$

Making the change of variables $\sigma = \psi(s)$, permissible because of the monotonicity of ψ, we get

$$\left| \mathbf{c} \cdot \nabla g(\mathbf{x}) \int_{\mathbf{c} \cdot \nabla g(\mathbf{x} - \mathbf{c}\delta)}^{\mathbf{c} \cdot \nabla g(\mathbf{x})} \frac{1}{\sigma^2} \frac{\mathrm{d}\sigma}{\psi'(\psi^{-1}(\sigma))} \right| \lesssim \left| \frac{\mathbf{c} \cdot \nabla g(\mathbf{x}) - \mathbf{c} \cdot \nabla g(\mathbf{x} - \mathbf{c}\delta)}{\mathbf{c} \cdot \nabla g(\mathbf{x} - \mathbf{c}\delta)} \right|$$

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so using the estimate in Lemma 4.4 and differentiability of $\mathbf{x} \mapsto \mathbf{c} \cdot \nabla g(\mathbf{x})$

$$|J_1| \lesssim \frac{\delta}{|\mathbf{x} - \mathbf{x}_0 - \mathbf{c}\delta|} \lesssim \frac{\delta}{\sqrt{|\mathbf{x} - \mathbf{x}_0|^2 + \delta^2}}.$$

The latter estimate follows from (e) in Lemma 4.2. So for $n \geq 2$ (recall that n is the dimension of the simplex Ω) we have

$$\|J_1\|_{L^2}^2 \lesssim \delta^2 \int_\Omega \frac{\mathrm{vol}_\Omega(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_0|^2 + \delta^2} \lesssim \delta^2 \int_0^1 \frac{r^{n-1}}{r^2 + \delta^2}\,\mathrm{d}r = O\left(\delta^2 \log\left(\frac{1}{\delta}\right)\right).$$

The logarithmic term is only necessary when $n = 2$. For J_2, we proceed in exactly the same manner as we did when estimating the integral I_2 in Lemma 3.8 using the second mean value theorem for integrals.

$$\begin{aligned} |J_2| &= \left| \frac{\mathbf{c} \cdot \nabla g(\mathbf{x})}{\omega} \int_0^{t(\mathbf{x})-\delta} \frac{v(\mathbf{y} + \mathbf{c}s)}{(\mathbf{c} \cdot \nabla g(\mathbf{y} + \mathbf{c}s))^3} \mathrm{d}\left(e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\right) \right| \\ &\lesssim \frac{|\mathbf{c} \cdot \nabla g(\mathbf{x})|}{\omega |\mathbf{c} \cdot \nabla g(\mathbf{x} - \mathbf{c}\delta)|^3} \\ &\lesssim \frac{|\mathbf{x} - \mathbf{x}_0|}{\omega \left(|\mathbf{x} - \mathbf{x}_0|^2 + \delta^2\right)^{3/2}}. \end{aligned}$$

From this, we get

$$\|J_2\|_{L^2}^2 \lesssim \int_\Omega \frac{|\mathbf{x} - \mathbf{x}_0|^2 \,\mathrm{vol}_\Omega(\mathbf{x})}{\omega^2 \left(|\mathbf{x} - \mathbf{x}_0|^2 + \delta^2\right)^3} \lesssim \frac{1}{\omega^2} \int_0^1 \frac{r^{n+1}}{\left(r^2 + \delta^2\right)^3}\,\mathrm{d}r.$$

It's necessary to split the cases where $n = 2, 3$ and $n \geq 4$. We find

$$\|J_2\|_{L^2}^2 = \begin{cases} O(\delta^{-2}/\omega^2), & n = 2, \\ O(\delta^{-1}/\omega^2), & n = 3, \\ O(1/\omega^2), & n \geq 4. \end{cases}$$

Matching the estimates with $\|J_1\|_{L^2}^2$ gives the desired result ($n = 2$ being the weakest upper bound). The second estimate is obtained in exactly the same manner—employing the same splitting as in the first case one finds $\|J_1\|_{L^1} = O(\delta)$ and $\|J_2\|_{L^1} = O(1/\omega)$, so matching these we get our result. □

THEOREM 4.12 Let $(\Omega, g, \mathbf{c}, \mathbf{x}_0)$ be an admissible system. Then the unique solution to the problem

$$A_\omega^{\mathbf{c}} u = f \quad \text{and} \quad \langle \mathbf{c} \cdot \nabla u, \mathbf{c} \cdot \nabla w \rangle_{L^2} = 0 \quad \forall w \in \ker A_\omega^{\mathbf{c}}$$

satisfies $\|\mathbf{c} \cdot \nabla u\|_{L^2}^2 = O\left(\omega^{-1/2}\right)$.

Proof. If u is as in the statement of the theorem, then by the same reasoning as in the proof of Theorem 3.7, we have for $w \in \ker A_\omega^{\mathbf{c}}$

$$\|\mathbf{c}\cdot\nabla u\|_{L^2}^2 = \|\mathbf{c}\cdot\nabla u - \mathbf{c}\cdot\nabla w\|_{L^2}^2 - \|\mathbf{c}\cdot\nabla w\|_{L^2}^2.$$

Taking $w(\mathbf{x}) = u(\mathbf{y})\, e^{\mathrm{i}\omega g(\mathbf{y})-\mathrm{i}\omega g(\mathbf{x})}$, we get

$$\begin{aligned}
\|\mathbf{c}\cdot\nabla u\|_{L^2}^2 &= -\omega^2\|u(\mathbf{y})(\mathbf{c}\cdot\nabla g)\|_{L^2(\mathrm{d}\mathbf{x})}^2 \\
&\qquad + \left\| f(\mathbf{x}) - \mathrm{i}\omega(\mathbf{c}\cdot\nabla g)e^{-\mathrm{i}\omega g(\mathbf{x})}\int_0^{t(\mathbf{x})} e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)} f(\mathbf{y}+\mathbf{c}s)\,\mathrm{d}s \right\|_{L^2(\mathrm{d}\mathbf{x})}^2 \\
&= -\omega^2\|u(\mathbf{y})(\mathbf{c}\cdot\nabla g)\|_{L^2(\mathrm{d}\mathbf{x})}^2 + \left\| \frac{f(\mathbf{y})\mathbf{c}\cdot\nabla g(\mathbf{x})}{\mathbf{c}\cdot\nabla g(\mathbf{y})}\, e^{\mathrm{i}\omega g(\mathbf{y})-\mathrm{i}\omega g(\mathbf{x})} \right. \\
&\qquad \left. -(\mathbf{c}\cdot\nabla g)\, e^{-\mathrm{i}\omega g(\mathbf{x})}\int_0^{t(\mathbf{x})} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f(\mathbf{y}+\mathbf{c}s)}{\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)}\right) e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\,\mathrm{d}s \right\|_{L^2(\mathrm{d}\mathbf{x})}^2 \\
&= \left\| \frac{f(\mathbf{y})}{\mathbf{c}\cdot\nabla g(\mathbf{y})}(\mathbf{c}\cdot\nabla g) \right\|_{L^2(\mathrm{d}\mathbf{x})}^2 - \omega^2\|u(\mathbf{y})(\mathbf{c}\cdot\nabla g)\|_{L^2(\mathrm{d}\mathbf{x})}^2 + \mathcal{E}(\omega)
\end{aligned}$$

with $\mathcal{E}(\omega)$ defined accordingly. From Lemma 4.8, we have

$$\left\| \frac{f(\mathbf{y})}{\mathbf{c}\cdot\nabla g(\mathbf{y})}(\mathbf{c}\cdot\nabla g) \right\|_{L^2(\mathrm{d}\mathbf{x})}^2 - \omega^2\|u(\mathbf{y})(\mathbf{c}\cdot\nabla g)\|_{L^2(\mathrm{d}\mathbf{x})}^2 = O\left(\frac{1}{\omega^{1/2}}\right),$$

so it remains to estimate $\mathcal{E}(\omega)$. We have

$$\begin{aligned}
\mathcal{E}(\omega) = -2\mathrm{Re}&\left\langle \frac{f(\mathbf{y})\, e^{\mathrm{i}\omega g(\mathbf{y})}}{\mathbf{c}\cdot\nabla g(\mathbf{y})}[\mathbf{c}\cdot\nabla g(\mathbf{x})]^2, \int_0^{t(\mathbf{x})} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f(\mathbf{y}+\mathbf{c}s)}{\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)}\right) e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\,\mathrm{d}s \right\rangle_{L^2(\mathrm{d}\mathbf{x})} \\
&+ \left\| \mathbf{c}\cdot\nabla g(\mathbf{x}) \int_0^{t(\mathbf{x})} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f(\mathbf{y}+\mathbf{c}s)}{\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)}\right) e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)}\,\mathrm{d}s \right\|_{L^2(\mathrm{d}\mathbf{x})}^2 .
\end{aligned}$$

These terms are, respectively, the form of those estimated in Lemma 4.11. Applying those results, we find $\mathcal{E}(\omega) = O(\log\omega/\omega)$, completing the proof. □

5. Collocation with initial conditions

It has already been mentioned that once appropriate initial data is specified, standard explicit multi-step methods can be employed to construct a slowly oscillating solution to the Levin equation and hence approximate the underlying oscillatory integral. There are other methods available though.

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To solve the initial value problem

$$u' + \mathrm{i}\omega g' u = f, \quad u(a) = \frac{f(a)}{\mathrm{i}\omega g'(a)},$$

we can invoke a similar idea to that in Levin's original paper, but in our case the crucial difference is that we allow for stationary points. As before we assume $g'(x) = (b-x)^k v(x)$ for some smooth v with $v \neq 0$ on $[a,b]$. Let

$$\Phi = \{\varphi_0, \ldots, \varphi_{N+1}\}$$

be a smooth Chebyshev set (see Powell, 1981, and references therein) on $[a,b]$ and fix $a = x_0 < \ldots < x_{N+1} = b$. Define

$$\Delta_{ij} = \det \begin{pmatrix} \varphi_i'(a) & \varphi_j'(a) \\ \varphi_i'(b) & \varphi_j'(b) \end{pmatrix}.$$

We assume that Φ is chosen so that $\Delta_{0,N+1} \neq 0$. Now write

$$u(x) = \sum_{j=0}^{N+1} q_j \varphi_j(x).$$

Then we arrive at the linear system

$$\sum_{j=0}^{N+1} \left(A_{ij} + \mathrm{i}\omega B_{ij} \right) q_j = f(x_i), \quad i = 1, \ldots, N, \tag{5.1}$$

where $A_{ij} = \varphi_j'(x_i)$ and $B_{ij} = g'(x_i)\varphi_j(x_i)$ for $0 \leq i,j \leq N+1$. The initial condition and stationary point give rise to the endpoint constraints

$$\sum_{j=0}^{N+1} A_{0,j} q_j = 0, \quad \sum_{j=0}^{N+1} A_{N+1,j} q_j = f(x_{N+1}). \tag{5.2}$$

The first of these translates to $u'(a) = 0$, which follows from continuity of the Levin equation at $x = a$ and the initial condition $\mathrm{i}\omega g'(a)u(a) = f(a)$. Using the equations in (5.2) to eliminate q_0 and q_{N+1} from the N equations in (5.1), we find the augmented linear system

$$\sum_{j=1}^{N} \left(\hat{A}_{ij} + \mathrm{i}\omega \hat{B}_{ij} \right) q_j = \hat{f}(x_i), \quad i = 1, \ldots, N,$$

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where $\hat{A}_{ij} = \hat{\varphi}_j'(x_i)$, $\hat{B}_{ij} = g'(x_i)\hat{\varphi}_j(x_i)$ for $1 \leq i, j \leq N$ and we have defined

$$\hat{\varphi}_j(x) = \varphi_j(x) - \frac{\Delta_{j,N+1}}{\Delta_{0,N+1}}\varphi_0(x) - \frac{\Delta_{0,j}}{\Delta_{0,N+1}}\varphi_{N+1}(x),$$
$$\hat{f}(x) = f(x) - \frac{f(x_{N+1})}{\Delta_{0,N+1}} \det \begin{pmatrix} \varphi_0'(a) & \varphi_{N+1}'(a) \\ A_\omega \varphi_0(x) & A_\omega \varphi_{N+1}(x) \end{pmatrix}.$$

We note that if we extend the definition of $\{\hat{\varphi}_j\}$ to include the cases $j = 0, N+1$ then $\hat{\varphi}_0 = \hat{\varphi}_{N+1} = 0$ identically. What is more

$$\hat{\varphi}_j'(a) = \hat{\varphi}_j'(b) = 0, \quad j = 1, \ldots, N.$$

We also see that $\hat{f}(b) = 0$.

If $\hat{\Phi} = \{\hat{\varphi}_1, \ldots, \hat{\varphi}_N\}$ is a Chebyshev system on $[x_1, x_N]$, then B is invertible and so for ω sufficiently large the solution to this linear system exists, and in fact

$$\left(\hat{A} + \mathrm{i}\omega\hat{B}\right)^{-1} = \frac{\hat{B}^{-1}}{\mathrm{i}\omega} + O\left(\frac{1}{\omega^2}\right).$$

However, $\hat{\Phi}$ will *not* be a Chebyshev system in general, as can be seen with the basic example

$$\Phi = \left\{1, x, \ldots, x^{N+1}\right\}.$$

We can circumnavigate this problem in several ways. For instance, if we consider the Chebyshev system of the form

$$\Phi = \left\{p_N, q_0, q_1, \ldots, q_{N-1}, p_{N+1}\right\},$$

where the p_i, q_i are polynomials of degree i, but the p_i unspecified at this stage. Then for $j = 1, \ldots, N$

$$\hat{\varphi}_j(x) = q_{j-1}(x) - \frac{\Delta_{j,N+1}}{\Delta_{0,N+1}} p_N(x) - \frac{\Delta_{0,j}}{\Delta_{0,N+1}} p_{N+1}(x).$$

Now if we are to sample at the points $\{x_i\}_{i=0}^{N+1}$ then we can set

$$p_j(x) = \prod_{i=1}^{j}(x - x_i), \quad \text{for } j = N, N+1.$$

So that for $p_N(x_i) = p_{N+1}(x_i) = 0$ for $i = 1, \ldots, N$ and so for $j = 1, \ldots, N$

$$\varphi_j(x_i) = q_{j-1}(x_i), \quad i = 1, \ldots, N.$$

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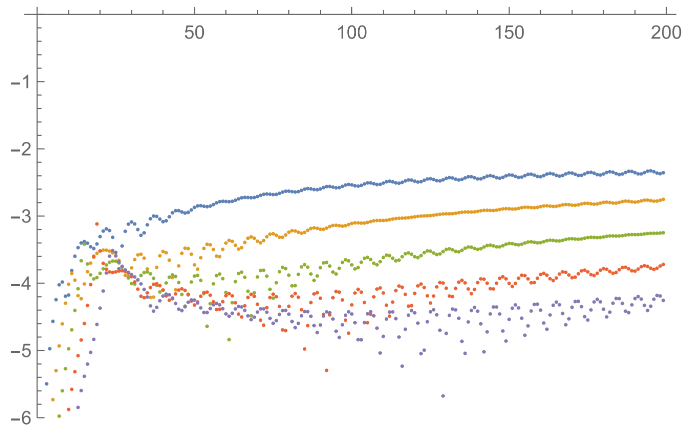

FIG. 5. Plots of $\sqrt{\omega} \times$ error on $\log_{10}$ scale using standard collocation at extreme Chebyshev points and $1 \le \omega \le 200$, ranging from 10 collocation points to 20 collocation points, increasing in increments of 2.

And so long as $\{q_0, \ldots, q_{N-1}\}$ is a Chebyshev on $[x_1, x_N]$ we know that the inverse of $\hat{B}$ exists and the linear problem is well-posed.

This demonstrates that by using the appropriate Cauchy data, the nonoscillatory solution can be obtained by collocation in much the same way as it can in the standard Levin method in the absence of stationary points. The same idea also works for confluent collocation methods where Levin's equation and its derivatives are used.

EXAMPLE 5.1 Using the same setup in Example 3.10, we approximate

$$\int_0^1 e^{\mathrm{i}\omega(1-x)^2}\, \mathrm{d}x = e^{-\mathrm{i}\pi/4}\sqrt{\frac{\pi}{4\omega}}\,\mathrm{Erfi}\left(e^{\mathrm{i}\pi/4}\sqrt{\omega}\right).$$

A rudimentary collocation scheme with Chebyshev polynomials is implemented to solve (5.1)–(5.2) and the results are shown in Figs 5 and 6. The horizontal axes measure $1 \le \omega \le 200$ and the vertical axes measure, on a $\log_{10}$ scale, the error produced after being scaled by $\sqrt{\omega}$.

Using *confluent* collocation yields better accuracy with fewer sample points. This is to be expected. In the asymptotic approach, valid as $\omega \to \infty$, the coefficients of the higher order terms in the asymptotic expansion depend on derivatives of f and g at the stationary point at $x = 1$.

The accuracy of the collocation approach can be easily improved using more sophisticated schemes, e.g., by using confluent collocation with high multiplicity in at $x = 1$ and lower (or zero) multiplicity at other points. A more detailed numerical investigation will be presented elsewhere.

6. Conclusion

The purpose of this paper was to introduce an initial condition for Levin's equation that would pick out a slowly oscillating solution in the presence of stationary points. In Levin's original analysis Levin (1997), an appropriate initial condition is obtained via an iterative procedure, but this procedure fails if a certain matrix fails to be invertible, which is precisely what happens in the current context if g has a stationary point.

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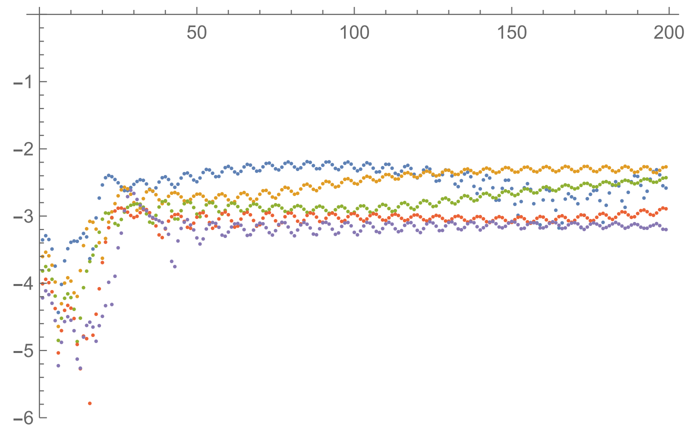

FIG. 6. Plots of $\sqrt{\omega}\times$ error on $\log_{10}$ scale using confluent collocation at extreme Chebyshev points with one derivative taken at each collocation point, ranging from 7 collocations points to 12 collocation points, increasing in increments of 1.

If there are no stationary points, then in both the one-dimensional and higher dimensional problems, it is possible to pick initial data that give rise to a particular solution of the relevant Levin equation that has well controlled derivatives of arbitrarily high order.

When there are stationary points in the domain of integration, we do not expect to find a particular solution to $A^{\mathbf{c}}_{\omega}u = f$ (or $A_{\omega}u = f$) that has well behaved derivatives of arbitrarily high order. This is because the form of asymptotic expansion of terms such as

$$\int_0^{t(\mathbf{x})} e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}s)} f(\mathbf{y}+\mathbf{c}s)\,\mathrm{d}s$$

will *not* be uniform in $\mathbf{x}$. In general, the expansion will have to transition as $|\mathbf{x}-\mathbf{x}_0|$ becomes small and the dominant contribution comes from the stationary point. In finding a solution that minimizes $\|\mathbf{c}\cdot\nabla u\|_{L^2}$, we get a solution whose derivative is well controlled along characteristic curves.

It is natural to ask about the minimization of higher order derivatives. To this end, we would consider the problem

$$\text{minimize } \|u'\|_{H^k} \quad \text{subject to} \quad A_{\omega}u = f,$$

where we refer to the standard Sobolev space

$$H^k[a,b] = \left\{u : [a,b] \to \mathbf{C} : u, u', \ldots, u^{(k)} \in L^2[a,b]\right\},$$

where the derivatives are understood in the distributional sense. The case $k = 0$ has been covered and for $k \geq 1$ it can be shown that if $X = C^{k+1}[a,b]$, $H = H^k[a,b]$ then $A_{\omega} : X \subset H \to H$ is closed and

$$\|u\|_{H^k} \lesssim \|A_{\omega}u\|_{H^k} + \|Bu\|_{H^k} \quad \forall u \in X.$$

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So Theorem 2.1 can be applied and an expression for the initial data that generates the relevant minimizer can be obtained. The higher dimensional case is similar. These ideas will be pursued in a later work.

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Appendix

Here we provide proofs of Lemma 3.1 and its generalization, Lemma 4.6. We also elaborate on the estimate for $I(\mathbf{y};\omega)$ in (4.5).

Proof of Lemma 3.1. That $A : X \subset H \to H$ is closed is a standard exercise in the analysis of unbounded linear operators on Hilbert space, so we focus on the estimate. For ease of notation, we make the linear change of variables

$$x \mapsto \frac{b-x}{b-a},$$

which maps the interval $[a,b]$ to $[0,1]$ and the stationary point gets mapped to the origin. So we can write $g'(x) = x^k v(x)$ with $v \neq 0$ on $[0,1]$. Now fix $\varphi \in C^1[0,1]$ such that $0 \leq \varphi \leq 1$ and

$$\varphi(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{4}, \\ 0, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

For $\epsilon > 0$, write $\varphi_\epsilon(x) = \varphi(x/\epsilon)$ so $\|\varphi_\epsilon'\|_\infty = O(1/\epsilon)$. The purpose for this splitting is as follows: since $(1-\varphi_\epsilon)u$ vanishes identically in a neighbourhood of the origin, we will be able to write

$$(1-\varphi_\epsilon)u = \frac{(1-\varphi_\epsilon)(A_\omega u - Bu)}{\mathrm{i}\omega g'},$$

so we will be able to estimate $\|(1-\varphi_\epsilon)u\|_{L^2}$ in terms of $\|A_\omega u\|_{L^2}$ and $\|Bu\|_{L^2}$. The importance of the term $\varphi_\epsilon u$ is that it vanishes at $x = \epsilon/2$, so we will be able to get an estimate for $\varphi_\epsilon u$ in terms of derivatives of u using a Poincaré-type inequality. This, in turn, will allow us to control the estimates in terms of derivatives and achieve the desired result. By the triangle inequality,

$$\|u\|_{L^2} \leq \|\varphi_\epsilon u\|_{L^2} + \|(1-\varphi_\epsilon)u\|_{L^2}.$$

Since $\varphi_\epsilon = 1$ on a neighbourhood of $x = 0$, we have

$$A_\omega u = u' + \mathrm{i}\omega g' u, \quad \text{i.e.,} \quad (1-\varphi_\epsilon)u = \frac{(1-\varphi_\epsilon)(A_\omega u - Bu)}{\mathrm{i}\omega x^k v}.$$

And so for some $\alpha > 0$ that depends only on v and ω, we have

$$\|(1-\varphi_\epsilon)u\|_{L^2} \leq \alpha\epsilon^{-k}\left(\|A_\omega u\|_{L^2} + \|Bu\|_{L^2}\right).$$

On the other hand, since $\varphi_\epsilon(\epsilon/2) = 0$, we have the Poincaré-type inequality valid for $0 \leq x \leq \epsilon/4$

$$\begin{aligned} |\varphi_\epsilon(x)u(x)| &= \left| \left(\int_x^{\epsilon/4} + \int_{\epsilon/4}^{\epsilon/2} \right) \left[\varphi_\epsilon'(t)u(t) + \varphi_\epsilon(t)u'(t)\right] \mathrm{d}t \right| \\ &= \left| \int_x^{\epsilon/4} u'(t)\,\mathrm{d}t + \int_{\epsilon/4}^{\epsilon/2} \left[\varphi_\epsilon'(t)u(t) + \varphi_\epsilon(t)u'(t)\right] \mathrm{d}t \right| \\ &\leq \sqrt{\frac{\epsilon}{4} - x}\|Bu\|_{L^2} + \int_{\epsilon/4}^{\epsilon/2} |\varphi_\epsilon'(t)u(t)|\,\mathrm{d}t + \int_{\epsilon/4}^{\epsilon/2} |\varphi_\epsilon(t)u'(t)|\,\mathrm{d}t. \end{aligned}$$

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For the range $\epsilon/4 < x \leq \epsilon/2$, the same estimate holds with the first term removed. Applying the L^2 norm to this estimate and using the triangle inequality, we find

$$\|\varphi_\epsilon u\|_{L^2} \leq \|Bu\|_{L^2} \left(\int_0^{\epsilon/4} \left(\frac{\epsilon}{4} - x\right) \mathrm{d}x\right)^{1/2} + \frac{\sqrt{\epsilon}}{\sqrt{2}} \left(\int_{\epsilon/4}^{\epsilon/2} |\varphi_\epsilon'(t)u(t)| \,\mathrm{d}t + \int_{\epsilon/4}^{\epsilon/2} |\varphi_\epsilon(t)u'(t)| \,\mathrm{d}t\right)$$

the factor of $\sqrt{\epsilon/2}$ coming from integrating over $[0, \epsilon/2]$. Clearly,

$$\int_{\epsilon/4}^{\epsilon/2} |\varphi_\epsilon(t)u'(t)| \,\mathrm{d}t \leq \left(\int_{\epsilon/4}^{\epsilon/2} |\varphi_\epsilon(t)|^2\right)^{1/2} \|Bu\|_{L^2} \leq \frac{\sqrt{\epsilon}}{2} \|Bu\|_{L^2}$$

and by applying the triangle inequality and Cauchy–Schwarz, we find

$$\begin{aligned}\int_{\epsilon/4}^{\epsilon/2} |\varphi_\epsilon'(t)u(t)| \,\mathrm{d}t &\leq \frac{\|\varphi'\|_\infty}{\epsilon} \int_{\epsilon/4}^{\epsilon/2} \frac{|A_\omega u(t)| + |Bu(t)|}{\omega v(t) t^k} \,\mathrm{d}t \\ &\leq \frac{\beta}{\epsilon^{k+1/2}} \left(\|A_\omega u\|_{L^2} + \|Bu\|_{L^2}\right),\end{aligned}$$

for some $\beta > 0$ that depends only on φ, ω, v and k. Collecting all these together, we find

$$\|\varphi_\epsilon u\|_{L^2} \leq \left(\frac{\epsilon^2}{32} + \frac{\epsilon}{2\sqrt{2}} + \beta\epsilon^{-k-1/2}\right) \|Bu\|_{L^2} + \beta\epsilon^{-k-1/2} \|A_\omega u\|_{L^2}.$$

Combining this with the estimate for $\|(1-\varphi_\epsilon)u\|_{L^2}$, we conclude

$$\begin{aligned}\|u\|_{L^2} &\leq \epsilon^{-k} \left(\frac{\beta}{\sqrt{\epsilon}} + \alpha\right) \|A_\omega u\|_{L^2} + \epsilon^{-k} \left(\frac{\epsilon^{2+k}}{32} + \frac{\epsilon^{1+k}}{2\sqrt{2}} + \frac{\beta}{\sqrt{\epsilon}} + \alpha\right) \|Bu\|_{L^2} \\ &\lesssim \|A_\omega u\|_{L^2} + \|Bu\|_{L^2}.\end{aligned}$$

The absolute constant is unimportant at this stage, so we don't need to choose ϵ to optimize the estimate. For example, we can take $\epsilon = 1$ so that the overall constant just depends on φ and ωg. □

Proof of Lemma 4.6. Again the closed nature of $A_\omega^{\mathbf{c}} : X \to H$ is standard, and to establish the estimate we follow much the same line of reasoning as for the proof to Lemma 3.1. This time we don't keep track of the constants through the estimates. We lose no generality assuming the vertices of the simplex are the standard unit vectors along each coordinate axis and $\mathbf{x}_0 = 0$. Let φ be as in the proof to Lemma 3.1 and define $\theta : \Omega \to [0, 1]$ by

$$\theta(\mathbf{x}) = \varphi(x_1 + x_2 + \ldots + x_n) \equiv \varphi(\mathbf{1} \cdot \mathbf{x}).$$

Fixing $\epsilon > 0$ and writing $\theta_\epsilon(\mathbf{x}) = \theta(\mathbf{x}/\epsilon)$, we make the splitting

$$\|u\|_{L^2} \leq \|(1-\theta_\epsilon)u\|_{L^2} + \|\theta_\epsilon u\|_{L^2}.$$

For the first term, we use

$$(1-\theta_\epsilon)u = \frac{(1-\theta_\epsilon)\left(A_\omega^{\mathbf{c}} u - B^{\mathbf{c}} u\right)}{\mathrm{i}\omega(\mathbf{c} \cdot \nabla g)}.$$

Since $|\mathbf{c}\cdot\nabla g| \gtrsim 1$ on $\mathbf{1}\cdot\mathbf{x} > \epsilon/4$ we immediately get, on an application of Cauchy–Schwarz

$$\|(1-\theta_\epsilon)u\|_{L^2} \lesssim \left\|A^{\mathbf{c}}_\omega u\right\|_{L^2} + \|B^{\mathbf{c}}u\|_{L^2}.$$

Now we examine $\theta_\epsilon u$ within the region $\mathbf{1}\cdot\mathbf{x} \le \epsilon/2$. Since $\mathbf{c}$ enters Ω through the face opposite $\mathbf{x}_0 = 0$, we know that $\mathbf{c}\cdot\mathbf{1} < 0$. So for each $\mathbf{x}\in\Omega$ with $\mathbf{1}\cdot\mathbf{x}\le\epsilon/2$, we can write $\mathbf{x} = \mathbf{x}^\epsilon + \mathbf{c}t$ where $\mathbf{1}\cdot\mathbf{x}^\epsilon = \epsilon/2$ and $t \ge 0$. On noticing that $\theta_\epsilon(\mathbf{x}^\epsilon) = 0$, we get

$$\begin{aligned}|\theta_\epsilon(\mathbf{x})u(\mathbf{x})| &= \left|\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\theta_\epsilon(\mathbf{x}^\epsilon+\mathbf{c}s)u(\mathbf{x}^\epsilon+\mathbf{c}s)\right)\mathrm{d}s\right| \\ &= \left|\int_0^t (\mathbf{c}\cdot\nabla(\theta_\epsilon u))(\mathbf{x}^\epsilon+\mathbf{c}s)\,\mathrm{d}s\right|.\end{aligned}$$

Noting that $\theta_\epsilon(\mathbf{x}^\epsilon+\mathbf{c}s) = 1$ in the region $|\mathbf{1}\cdot\mathbf{c}|s \ge \epsilon/4$ and proceeding in much the same way as we did in the proof to Lemma 3.1 we obtain

$$\|\theta_\epsilon u\|_{L^2} \lesssim \left\|A^{\mathbf{c}}_\omega u\right\|_{L^2} + \|B^{\mathbf{c}}u\|_{L^2}$$

and pairing this with the estimate for $\|(1-\theta_\epsilon)u\|_{L^2}$ gives the result. □

Proof of estimate (4.5). Recall that

$$H(t,\mathbf{y}) = \frac{G(\mathbf{y}+\mathbf{c}t)\int_t^{\tau(\mathbf{y})}|\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}s)|^2\,\mathrm{d}s}{\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}t)^3}$$

and

$$I(\mathbf{y};\omega) = \frac{1}{\sqrt{\omega}}\int_0^{\tau(\mathbf{y})} H(t,\mathbf{y})\frac{\partial}{\partial t}\left(e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)}\right)\mathrm{d}t.$$

Here G is a smooth function and we know that H is bounded on Ω and is smooth away from $(t,\mathbf{y}) = (\tau(\mathbf{y}_0),\mathbf{y}_0)$. We have claimed that $I(\mathbf{y};\omega) \lesssim \tau(\mathbf{y})$ *uniformly* in ω. Here we outline the proof.

Fix $\epsilon > 0$ and define the closed subset of Σ_0

$$\Sigma_0^\epsilon = \{\mathbf{y}\in\Sigma_0 : |\mathbf{y}-\mathbf{y}_0| \le \epsilon\}.$$

Note that $\mathbf{c}\cdot\nabla g(\mathbf{x}) \ne 0$ if $\mathbf{x} = \mathbf{y}+\mathbf{c}t(\mathbf{x})$ and $\mathbf{y}\notin\Sigma_0^\epsilon$. So for $\mathbf{y}$ outside Σ_0^ϵ we can integrate by parts

$$I(\mathbf{y};\omega) = -\frac{1}{\sqrt{\omega}}H(0,\mathbf{y})\,e^{\mathrm{i}\omega g(\mathbf{y})} - \frac{1}{\sqrt{\omega}}\int_0^{\tau(\mathbf{y})}\frac{\partial H}{\partial t}(t,\mathbf{y})\,e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)}\,\mathrm{d}t.$$

The first term is $O(\tau(\mathbf{y})/\sqrt{\omega})$. For the second, it is straightforward to establish

$$\frac{\partial H}{\partial t} = \frac{K(\mathbf{y},t)}{\mathbf{c}\cdot\nabla g(\mathbf{y}+\mathbf{c}t)} \tag{A.1}$$

with $|K| \lesssim 1$. Using (4.2), we get

$$\begin{aligned}\left|\frac{1}{\sqrt{\omega}}\int_0^{\tau(\mathbf{y})}\frac{\partial H}{\partial t}(t,\mathbf{y})\,e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)}\,\mathrm{d}t\right| &\lesssim \frac{1}{\sqrt{\omega}}\int_0^{\tau(\mathbf{y})}\frac{\mathrm{d}t}{|\mathbf{y}-\mathbf{y}_0| + (\tau(\mathbf{y}_0)-t)} \\ &\lesssim \frac{\tau(\mathbf{y})}{\sqrt{\omega}\epsilon}.\end{aligned}$$

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The latter estimate follows since $\mathbf{y} \notin \Sigma_0^\epsilon$ so that $|\mathbf{y} - \mathbf{y}_0| > \epsilon$. In summary, for $\mathbf{y} \in \Sigma_0 \setminus \Sigma_0^\epsilon$, we have $|I(\mathbf{y};\omega)| \lesssim \tau(\mathbf{y})/\sqrt{\omega}\epsilon$.

To estimate $I(\mathbf{y};\omega)$ when $\mathbf{y} \in \Sigma_0^\epsilon$, we will assume that $\epsilon > 0$ is sufficiently small enough so that $\tau(\mathbf{y}) \gtrsim 1$ on Σ_0^ϵ, which is possible since $\mathbf{y} \mapsto \tau(\mathbf{y})$ is continuous and $\tau(\mathbf{y}_0) > 0$. Now we define

$$\tau^* = \min_{\mathbf{y}\in\Sigma_0^\epsilon} \tau(\mathbf{y}) \equiv \tau(\mathbf{y}^*) \quad \text{for some } \mathbf{y}^* \text{ with } |\mathbf{y}^* - \mathbf{y}_0| = \epsilon.$$

That the minimum is achieved on the boundary follows from the observation

$$\tau^* = \min_{\substack{i\neq 0\\ \mathbf{c}\cdot\mathbf{n}_i\neq 0}} \left[\min_{\mathbf{y}\in\Sigma_0^\epsilon} \frac{(\mathbf{x}_0 - \mathbf{y})\cdot\mathbf{n}_i}{\mathbf{c}\cdot\mathbf{n}_i} \right] = \min_{\substack{i\neq 0\\ \mathbf{c}\cdot\mathbf{n}_i\neq 0}} \left[\min_{\mathbf{y}\in\partial\Sigma_0^\epsilon} \frac{(\mathbf{x}_0 - \mathbf{y})\cdot\mathbf{n}_i}{\mathbf{c}\cdot\mathbf{n}_i} \right],$$

which is a consequence of the maximum (minimum) principle for harmonic functions. What is more, we know that the minimum cannot be achieved on $\partial\Sigma_0$ since $\tau(\mathbf{y}) \gtrsim 1$ on Σ_0^ϵ by construction, but if $\mathbf{y}^* \in \partial\Sigma_0$ then it would satisfy $(\mathbf{x}_0 - \mathbf{y}^*)\cdot\mathbf{n}_i = 0$ for some $i \in \{1,\ldots,N\}$ meaning $\tau^* = 0$, a contradiction. So we can conclude that $|\mathbf{y}_0 - \mathbf{y}^*| = \epsilon$. With τ^* defined above and for $\mathbf{y} \in \Sigma_0^\epsilon$, we can write

$$I(\mathbf{y};\omega) = \frac{1}{\sqrt{\omega}} \left(\int_0^{\tau^*} + \int_{\tau^*}^{\tau(\mathbf{y})} \right) H(t,\mathbf{y}) \frac{\partial}{\partial t} \left(e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)} \right) \mathrm{d}t.$$

On the first term, we can integrate by parts

$$\begin{aligned} &\int_0^{\tau^*} H(t,\mathbf{y}) \frac{\partial}{\partial t} \left(e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)} \right) \mathrm{d}t \\ &\qquad = H(\tau^*,\mathbf{y})\, e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}\tau^*)} - H(0,\mathbf{y})\, e^{\mathrm{i}\omega g(\mathbf{y})} - \int_0^{\tau^*} \frac{\partial H}{\partial t}(t,\mathbf{y})\, e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)}\, \mathrm{d}t. \end{aligned}$$

The first two terms are bounded uniformly in ω since H is bounded. Using (A1) and (4.2) and $|\tau(\mathbf{y}_0) - \tau(\mathbf{y}^*)| \gtrsim |\mathbf{y}_0 - \mathbf{y}^*| = \epsilon$, we find

$$\left| \int_0^{\tau^*} \frac{\partial H}{\partial t}(t,\mathbf{y})\, e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)}\, \mathrm{d}t \right| \lesssim \int_0^{\tau^*} \frac{\mathrm{d}t}{|\mathbf{y} - \mathbf{y}_0| + (\tau(\mathbf{y}_0) - t)} \lesssim \frac{1}{\epsilon}.$$

Since $\tau(\mathbf{y}) \gtrsim 1$ on Σ_0^ϵ, we conclude

$$I(\mathbf{y};\omega) = \frac{1}{\sqrt{\omega}} \int_{\tau^*}^{\tau(\mathbf{y})} H(t,\mathbf{y}) \frac{\partial}{\partial t} \left(e^{\mathrm{i}\omega g(\mathbf{y}+\mathbf{c}t)} \right) \mathrm{d}t + O\left(\frac{\tau(\mathbf{y})}{\sqrt{\omega}\epsilon} \right), \quad \mathbf{y} \in \Sigma_0^\epsilon.$$

The first term, which contains the presence of a stationary point, can easily be estimated using the boundedness of H. It is bounded above by

$$\frac{1}{\sqrt{\omega}} \int_{\tau^*}^{\tau(\mathbf{y})} \omega |H(t,\mathbf{y})|\, \mathrm{d}t \lesssim \sqrt{\omega} |\tau(\mathbf{y}) - \tau(\mathbf{y}^*)| \lesssim \sqrt{\omega}\epsilon \lesssim \sqrt{\omega}\epsilon\tau(\mathbf{y}).$$

In summary, we have the following estimates covering all $\mathbf{y} \in \Sigma_0$:

$$|I(\mathbf{y};\omega)| \lesssim \begin{cases} \tau(\mathbf{y})/\sqrt{\omega}\epsilon, & \mathbf{y} \in \Sigma_0 \setminus \Sigma_0^\epsilon \\ \tau(\mathbf{y}) \left(\sqrt{\omega}\epsilon + 1/\sqrt{\omega}\epsilon \right), & \mathbf{y} \in \Sigma_0^\epsilon. \end{cases}$$

Choosing $\epsilon = 1/\sqrt{\omega}$ gives the desired result. □