Applied Asymptotic Analysis Lecture Notes

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Preface

First and foremost: Very much a work in progress and is based on a course taught from Miller's Applied Asymptotic Analysis [3]. These notes are in no way meant as a replacement for this text.

Chapter 1
Preliminaries

1.1 ▪ Basic asymptotic questions

The following gives a sampling of the types of asymptotic questions that are common throughout the mathematical and physical sciences.

1.1.1 ▪ Mathematics

Consider

$$\hat{f}(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x) \mathrm{d}x.$$

How can we characterize $\hat{f}(k)$ as $k \to \infty$? Can we characterize it beyond just its rate of decay?

1.1.2 ▪ Physics

The time-dependent Schrödinger equation is given by

$$\mathrm{i}\hbar\Psi_t(x,t) = \left(-\frac{\hbar^2}{2m}\Delta + V(x)\right)\Psi(x,t),$$

where Δ is the Laplacian. After rescaling, to non-dimensionalize, and separating variables, one is left considering

$$\left(-\frac{\epsilon^2}{2}\Delta + V(x)\right)u(x) = \lambda u(x),$$

where we are purposefully ignoring boundary conditions. So, because ϵ is small, one consider the asymptotic analysis of eigenfunctions of

$$\mathcal{L} = -\frac{\epsilon^2}{2}\Delta + V,$$

as $\epsilon \to 0$. We will touch on this question.

1.1.3 ▪ Water waves/Fluid dynamics

The classical (inviscid) Burgers' equation for $u = u(x,t)$ is given by

$$u_t + uu_x = 0.$$

It is the prototypical model for advective fluids that develop shocks, discontinuities, in finite time. On the other hand, the viscous Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0,$$

does not develop shocks. So, what happens to solutions as $\nu \downarrow 0$? This is discussed in Miller's text although

Similarly, one can consider a dispersive perturbation to regularize the shocks:

$$u_t + uu_x = \epsilon u_{xxx}, \quad \epsilon \neq 0.$$

The story here as $\epsilon \to 0$ is much more complicated.

1.2 ▪ Difficulty "levels" for asymptotics problems

Suppose we want to know the asymptotics of a quantity f_ϵ which itself may, or may not be, a function of other variables (e.g., $f_\epsilon = f_\epsilon(x,t)$).

Level 1: Suppose an explicit formula for f_ϵ is available. It is explicit in the sense that we have:

- An explicit algebraic formula.
- An integral formula
- ...

Level 2: Suppose f_ϵ is not know explicitly but it is defined implicitly as the solution of a linear problem. For example:

- It is the solution of a linear system.
- It is the solution of a differential equation
- ...

Level 3: Suppose f_ϵ is not know explicitly but it is defined implicitly as the solution of a nonlinear problem. For example:

- It is a root of a function.
- It is the solution of a nonlinear differential equation.
- ...

How the material in this course is presented is that we want a fair amount of rigor in our calculations for Level 1. We want to know that our asymptotic approximations are accurate and we want to know precisely what the error term looks like. For Level 2, we still want some rigor when possible. And for the most part, we'll still have it, but we will have to rely on more abstract theorems. Adding rigor to Level 3 is difficult, and we largely ignore it here. We will take what we've learned from Levels 1 & 2 and apply it to Level 3. This is what is called "formal" asymptotics.

1.3 ▪ Big-oh and little-oh

We now include some important definitions.

Definition 1.1 (Big-oh). *Let f and g be two complex-valued functions on $D \subset \mathbb{C}$. Then we write*

$$f(z) = O(g(z)), \quad z \in D,$$

if there exists $K > 0$ such that

$$|f(z)| \leq K|g(z)|, \quad z \in D.$$

The previous definition is a global one, so we include a local version

Definition 1.2 (Big-oh near z_0). *Let f and g be two complex-valued functions on $D \subset \mathbb{C}$ and suppose z_0 is in the closure of D. Then we write*

$$f(z) = O(g(z)), \quad \text{as} \quad z \to z_0 \quad \text{from} \quad D,$$

if there exists $\delta > 0$ such that

$$f(z) = O(g(z)), \quad z \in D \text{ and } 0 < |z - z_0| < \delta.$$

Remark 1.3. *It should e clear that this definition really has nothing to do with $\mathbb{C}$. We could have f, g as functions from a metric space to a normed vector space.*

Remark 1.4. *This definition state that "f grows at worst like g" and, in particular, if* $\lim_{z\to z_0} g(z) = 0$ *then the same must be true of f.*

Definition 1.5 (Big-oh at ∞). *Let f and g be two complex-valued functions defined in an unbounded set $D \subset \mathbb{C}$. Then we write*

$$f(z) = O(g(z)), \quad \text{as} \quad z \to \infty \quad \text{from} \quad D,$$

if there exists $M > 0$ such that

$$f(z) = O(g(z)), \quad z \in D \text{ and } |z| > M.$$

Example 1.6. Consider the integral

$$I_n = \int_{-1}^{1} \mathrm{e}^{-n(x+1)}\,\mathrm{d}x.$$

Then

$$I_n = O(n^{-1}), \quad n \to \infty, \quad n \in \mathbb{N}.$$

This is simple to see because we can compute the integral:

$$I_n = -\frac{1}{n}\,\mathrm{e}^{-n(x+1)}\Big|_{-1}^{1} = \frac{1}{n} - \frac{1}{n}\,\mathrm{e}^{-2n}\,.$$

Then, it is clear that $|I_n| \leq 1/n$ and we can choose $M = 0$, $K = 1$.

Example 1.7 (A function of multiple variables). Consider

$$f_n(z) = \int_{-1}^{1} \frac{\mathrm{e}^{-n(x+1)}}{x-z}\mathrm{d}x, \quad z \notin [-1,1].$$

Our goal is to understand the behavior of $f_n(z)$ as $z, n \to \infty$. We integrate by parts:

$$\begin{aligned} f_n(z) &= -\frac{1}{n}\frac{\mathrm{e}^{-n(x+1)}}{x-z}\Big|_{-1}^{1} - \frac{1}{n}\int_{-1}^{1}\frac{\mathrm{e}^{-n(x+1)}}{(x-z)^2}\mathrm{d}x \\ &= -\frac{1}{n}\frac{1}{1+z} - \frac{1}{n}\frac{\mathrm{e}^{-2n}}{1-z} - \frac{1}{n}\int_{-1}^{1}\frac{\mathrm{e}^{-n(x+1)}}{(x-z)^2}\mathrm{d}x. \end{aligned}$$

Intuitively, for n fixed, the first two terms are of the same order, $O(1/z)$, but the e^{-2n} makes the second much smaller. We need to bound the integral term:

$$\left|\int_{-1}^{1}\frac{\mathrm{e}^{-n(x+1)}}{(x-z)^2}\mathrm{d}x\right| \leq \frac{1}{\operatorname{dist}(z,[-1,1])^2}\int_{-1}^{1}\mathrm{e}^{-n(x+1)}\,\mathrm{d}x \leq \frac{1}{n\operatorname{dist}(z,[-1,1])^2}.$$

So, we can write

$$\left|f_n(z) + \frac{1}{n}\frac{1}{1+z}\right| \leq \frac{\mathrm{e}^{-2n}}{n}\frac{1}{|1-z|} + \frac{1}{n^2\operatorname{dist}(z,[-1,1])^2}.$$

From the exercise that follows have that there exists M, K such that if $|z| > M$, then

$$\frac{1}{|1-z|} \leq \frac{K}{1+|z|}, \quad \frac{1}{\operatorname{dist}(z,[-1,1])} \leq \frac{K}{1+|z|}.$$

So, because $\mathrm{e}^{-2n} \leq n^{-1}$, $n \in \mathbb{N}$,

$$\left|f_n(z) + \frac{1}{n}\frac{1}{1+z}\right| \leq \frac{K}{n^2}\frac{1}{1+|z|}\left(1 + \frac{K}{1+|z|}\right).$$

Then because

$$\left(1 + \frac{K}{1+|z|}\right) \leq 1 + K,$$

we have

$$f_n(z) + \frac{1}{n}\frac{1}{1+z} = O\left(\frac{1}{n^2}\frac{1}{1+|z|}\right), \quad \text{as } n, z \to \infty, \quad n \in \mathbb{N}, z \in \mathbb{C}.$$

If we want to include another term in the expansion, we find

$$f_n(z) + \frac{1}{n}\frac{1}{1+z} + \frac{1}{n}\,\mathrm{e}^{-2n}\,\frac{1}{1-z} = O\left(\frac{1}{n^2}\frac{1}{(1+|z|)^2}\right), \quad \text{as } n, z \to \infty, \quad n \in \mathbb{N}, z \in \mathbb{C}.$$

Exercise 1.1. *Show that as $z \to \infty$, $z \in \mathbb{C}$, and $a \in \mathbb{C}$ fixed,*

$$\frac{1}{z-a} = O\left(\frac{1}{1+|z|}\right),$$
$$\frac{1}{\operatorname{dist}(z,[-1,1])} = O\left(\frac{1}{1+|z|}\right).$$

Definition 1.8 (Little-oh). *Let f and g be two complex-valued functions defined in some set $D \subset C$. Suppose z_0 is in the closure of D. Then we write*

$$f(z) = o(g(z)), \quad \textit{as} \quad z \to z_0 \quad \textit{from} \quad D,$$

if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that

$$|f(z)| \le \epsilon |g(z)|, \quad z \in D \textit{ and } 0 < |z - z_0| < \delta.$$

Definition 1.9 (Little-oh at ∞). *Let f and g be two complex-valued functions defined in an unbounded set $D \subset C$. Suppose z_0 is in the closure of D. Then we write*

$$f(z) = o(g(z)), \quad \textit{as} \quad z \to \infty \quad \textit{from} \quad D,$$

if for any $\epsilon > 0$ there exists $M = M(\epsilon)$ such that

$$|f(z)| \le \epsilon |g(z)|, \quad z \in D \textit{ and } |z| > M.$$

1.3.1 ▪ Properties of big-oh and little-oh

- If $f(z) = o(1)$ as $z \to z_0$ from D then $\lim_{z \to z_0} f(z) = 0$.
- If $f(z) = o(g(z))$ as $z \to z_0$ from D then $f(z) = O(g(z))$ as $z \to z_0$ from D.
- If $f(z) = O(g(z))$ and $g(z) = O(h(z))$ as $z \to z_0$ from D then $f(z) = O(h(z))$ as $z \to z_0$ from D. The same holds for little-oh.
- The order relation $f(z) = o(g(z))$ is often written $f(z) \ll g(z)$. The chain of inequalities is consistent:

$$f(z) \ll g(z) \ll h(z).$$

- Finite linear combinations: Suppose $f_n(z) = O(g_n(z))$ for $n = 1, 2, \ldots, N$. Then

$$\sum_{j=1}^{N} a_n f_n(z) = O\left(\sum_{j=1}^{N} |a_n||g_n(z)|\right).$$

The reader is invited to convince themselves that the absolute values are necessary inside the big-oh.

1.3.2 ▪ Calculus and order relations

Question: If D is open and f, g are both differentiable in D, then as $z \to z_0$, $z \in D$, does

$$f(z) = O(g(z)) \Rightarrow f'(z) = O(g'(z))?$$

The answer is no, largely because of oscillations: $f(z) = \sin(1/z)z$ and $g(z) = z$.

Question: Suppose f, g are continuous on $\mathbb{R}$. Then as $z \to \infty$, $z \in \mathbb{R}$ does

$$f(z) = O(g(z)) \Rightarrow \int_0^z f(z')\mathrm{d}z' = O\left(\int_0^z g(z')\mathrm{d}z'\right)?$$

The answer is again no, $f(z) = 1$, $g(z) = \mathrm{e}^{\mathrm{i}z}$.

The issue here appears due to cancellations. So, maybe we can fix it?

Question: Suppose f, g are continuous on $\mathbb{R}$ and $g(z) \geq 0$. Then as $z \to \infty$, $z \in \mathbb{R}$ does

$$f(z) = O(g(z)) \Rightarrow \int_0^z f(z')\mathrm{d}z' = O\left(\int_0^z g(z')\mathrm{d}z'\right)?$$

There exists $M > 0$ such that

$$|f(x)| \leq K|g(z)| = Kg(z), \quad z > M.$$

Then for $z > M$

$$\left|\int_0^z f(z')\mathrm{d}z'\right| \leq \int_0^z |f(z')|\mathrm{d}z' \leq \int_0^M |f(z')|\mathrm{d}z' + K\int_M^z |g(z')|\mathrm{d}z',$$
$$\int_0^z f(z')\mathrm{d}z' = O\left(1 + \int_M^z g(z')\mathrm{d}z'\right) = O\left(1 + \int_0^z g(z')\mathrm{d}z'\right).$$

In order to reach the conclusion we want, we need that for some constant K'

$$1 \leq K' \int_0^z g(z')\mathrm{d}z',$$

and since g is assumed continuous, it suffices to have that g does not vanish identically.

Function of orders

Suppose $F(z, \theta)$ is defined for $z \in D, \theta \in \mathbb{C}$ with z_0 in the closure of D. Then for functions f, g defined on D, we write

$$f(z) = F(z, O(g(z))), \quad \text{as} \quad z \to z_0, \quad z \in D,$$

if there exists $\delta > 0$ and a function h such that

$$f(z) = F(z, h(z)) \quad \text{and} \quad h(z) = O(g(z)), \quad z \to z_0.$$

And a similar statement will be understood for little-oh.

Example 1.10. This idea helps us understand what we could mean by statements like:

$$\begin{aligned} f(z) &= 1 + o(1), \\ f(z) &= \frac{1}{1 + O(z^{-1})}, \\ f(z) &= e^{1+O(z^{-1})}. \end{aligned}$$

1.4 ▪ Absolute and relative errors

For the following to definitions $f, \tilde{f} : D \to \mathbb{C}$ and z_0 is in the closure of D.

Definition 1.11 (Absolute error). *$\tilde{f}$ approximates f in the sense of absolute error if*

$$f(z) = \tilde{f}(z) + o(1) \quad as \quad z \to z_0, \quad z \in D.$$

If $e(z) = o(1)$ we might also have

$$f(z) = \tilde{f}(z) + O(e(z)).$$

While it might seem trivial, the previous example can be understood in terms of functions of orders using $F(z, \theta) = \tilde{f}(z) + \theta$.

Definition 1.12 (Relative error). *$\tilde{f}$ approximates f in the sense of absolute error if*

$$f(z) = \tilde{f}(z)(1 + o(1)) \quad as \quad z \to z_0, \quad z \in D.$$

If $e(z) = o(1)$ we might also have

$$f(z) = \tilde{f}(z)(1 + O(e(z))).$$

In terms of functions of orders, we use $F(z, \theta) = \tilde{f}(z)(1 + \theta)$.

Important 1.13. *Note that if one has approximation in terms of relative error, as, say $z \to \infty$, it implies that the roots of $\tilde{f}$ and f coincide exactly for z sufficiently large.*

1.5 ▪ Convergent versus asymptotic series

Suppose $f(z)$ is infinitely differentiable on an open set $D \subset \mathbb{C}$, $z_0 \in D$. Recall how one version of Taylor's theorem is proved:

$$
\begin{aligned}
f(z) &= f(z_0) + \int_{z_0}^{z} f'(s_1)\mathrm{d}s_1, \\
&= f(z_0) + f'(z_0)(z-z_0) + \int_{z_0}^{z}\int_{z_0}^{s_1} f''(s_2)\mathrm{d}s_2\mathrm{d}s_1, \\
&= f(z_0) + f'(z_0)(z-z_0) + f''(z_0)\frac{(z-z_0)^2}{2} + \int_{z_0}^{z}\int_{z_0}^{s_1}\int_{z_0}^{s_2} f'''(s_3)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1, \\
&= \sum_{j=0}^{N} f^{(j)}(z_0)\frac{(z-z_0)^j}{j!} + \underbrace{\int_{z_0}^{z}\int_{z_0}^{s_1}\cdots\int_{z_0}^{s_N} f^{(N+1)}(s_{N+1})\mathrm{d}s_{N+1}\cdots\mathrm{d}s_1}_{R_N(z)}.
\end{aligned}
$$

The calculation becomes useful once we estimate R_N:

$$|R_N(z)| \le \frac{|z-z_0|^{N+1}}{(N+1)!}\max_s |f^{(N+1)}(s)|.$$

Note that we have two asymptotic parameters, z and N! First,

$$R_N(z) = O((z-z_0)^{N+1}), \quad z \to z_0.$$

And if for fixed z,

$$R_N(z) = o(1), \quad N \to \infty,$$

the Taylor series converges to $f(z)$.

In the case where it may fail to converge to $f(z)$, for $z \neq z_0$, we write

$$f(z) \sim \sum_{j=0}^{\infty} f^{(j)}(x_0)\frac{(z-z_0)^j}{j!}, \quad \text{as} \quad z \to z_0.$$

This means that for each N (and N held fixed!),

$$f(z) = \sum_{j=0}^{N} f^{(j)}(x_0)\frac{(z-z_0)^j}{j!} + \underbrace{O((z-z_0)^{N+1})}_{\text{order of first neglected term}}, \quad z \to z_0.$$

Example 1.14. Consider the complementary error function

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s.$$

Our goal is to analyze it asymptotically as $|z| \to \infty$. We can tell immediately that there will not be a convergent expansion of this function in powers of $1/z$ as $|z| \to \infty$ because its behavior at $\pm\infty$ is different. How do we expand it? Integrate by parts:

$$\begin{aligned}
I(z) &= \int_z^\infty \frac{-2s}{-2s}\,\mathrm{e}^{-s^2}\,\mathrm{d}s = -\frac{1}{2s}\,\mathrm{e}^{-s^2}\Big|_z^\infty - \frac{1}{2}\int_z^\infty \frac{1}{s^2}\,\mathrm{e}^{-s^2}\,\mathrm{d}s \\
&= \frac{1}{2z}\,\mathrm{e}^{-z^2} + \int_z^\infty \frac{-2s}{4s^3}\,\mathrm{e}^{-s^2}\,\mathrm{d}s \\
&= \frac{1}{2z}\,\mathrm{e}^{-z^2} + \frac{1}{4s^3}\,\mathrm{e}^{-s^2}\Big|_z^\infty + \frac{3}{4}\int_z^\infty \frac{1}{s^4}\,\mathrm{e}^{-s^2}\,\mathrm{d}s \\
&= \frac{1}{2z}\,\mathrm{e}^{-z^2} - \frac{1}{4z^3}\,\mathrm{e}^{-z^2} + \frac{3}{4}\int_z^\infty \frac{-2s}{-2s^5}\,\mathrm{e}^{-s^2}\,\mathrm{d}s \\
&= \frac{1}{2z}\,\mathrm{e}^{-z^2} - \frac{1}{4z^3}\,\mathrm{e}^{-z^2} + \frac{3}{8z^5}\,\mathrm{e}^{-z^2} - \frac{15}{8}\underbrace{\int_z^\infty \frac{1}{s^6}\,\mathrm{e}^{-s^2}\,\mathrm{d}s}_{=\mathrm{e}^{-z^2}O(z^{-6})}.
\end{aligned}$$

Claim: (see [4])

$$\operatorname{erfc}(z) \sim \frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}}\sum_{j=0}^{\infty}(-1)^j\frac{\left(\frac{1}{2}\right)_j}{z^{2j+1}}, \qquad \operatorname{Re} z > 0,$$

where

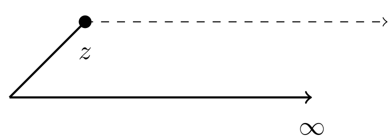
$$(y)_a = \frac{\Gamma(y+a)}{\Gamma(y)}$$

is the Pochhammer symbol. If $a = j$ is an integer, it has the alternate expression

$$\begin{cases} (y)_j = y(y+1)\cdots(y+j-1), \\ (y)_0 = 1. \end{cases}$$

The series on the right-hand side does not converge!

But let us verify that the error term after integration by parts, is truly lower order. We first need to use Cauchy's theorem to deform the integral to the dashed contour



Then consider

$$\int_z^\infty \mathrm{e}^{-s^2}\,s^\ell\mathrm{d}s \underset{s=z+x}{=} \int_0^\infty \frac{\mathrm{e}^{-z^2-x^2-2zx}}{(z+x)^\ell}\mathrm{d}x = \mathrm{e}^{-z^2}\int_0^\infty \frac{\mathrm{e}^{-x^2-2zx}}{(z+x)^\ell}\mathrm{d}x.$$

Then if $\operatorname{Re} z > 0$, since $x \geq 0$, we have that $|\,\mathrm{e}^{-2zx}\,| \leq 1$ and

$$\frac{1}{|z+x|} = \frac{1}{\sqrt{(\operatorname{Im} z)^2 + (\operatorname{Re} z + x)^2}} \leq \frac{1}{\sqrt{(\operatorname{Im} z)^2 + (\operatorname{Re} z)^2}} = \frac{1}{|z|}.$$

Then we can bound

$$\left| \mathrm{e}^{-z^2} \int_0^\infty \frac{\mathrm{e}^{-x^2-2zx}}{(z+x)^\ell} \mathrm{d}x \right| \leq \frac{1}{|z|^\ell} \int_0^\infty \mathrm{e}^{-x^2} = \frac{1}{|z|^\ell} \frac{\sqrt{\pi}}{4}.$$

This shows that the next neglected term in our integration by parts will be lower order. It also follows that $\operatorname{erfc}(z)$ has a different asymptotic expansion in the left-half plane $\operatorname{Re} z < 0$. This is an instance of Stokes' phenomenon.

Note: $\operatorname{erfc}(z)$ is a solution of $y'' + 2zy' = 0$.

1.6 ▪ Asymptotic expansions

We now formalize some of the observations we have from the preceding section.

Definition 1.15. *A sequence of functions $\{\psi_n(z)\}_{n=0}^\infty$ is called an asymptotic sequence as $z \to z_0$ (or ∞) from D if whenever $n > m$, we have*

$$\phi_n(z) = o(\phi_m(z)), \quad z \to z_0, \quad z \in D.$$

This simply states that each function in the sequence is smaller, asymptotically, than those that preceeded it:

$$\phi_n(z) \ll \phi_{n-1}(z) \ll \cdots \ll \phi_m(z).$$

Example 1.16. In the case of Taylor's theorem, we used $\phi_n(z) = (z - z_0)^n$ as $z \to z_0$.

Example 1.17. In the case of $\operatorname{erfc}(z)$, we used $\phi_n(z) = \mathrm{e}^{-z^2} z^{-n}$, as $z \to \infty$, $\operatorname{Re} z > 0$.

Definition 1.18. *Let $\{\phi_n(z)\}_{n=0}^\infty$ be an asymptotic sequence as $z \to z_0$ (or ∞), $z \in D$. Then the sum*

$$\sum_{n=0}^N a_n \phi_n(z),$$

is an asymptotic approximation as $z \to z_0$, $z \in D$, of a function $f(z)$ if

$$f(z) - \sum_{n=0}^N a_n \phi_n(z) = O(\phi_{N+1}(z)), \quad z \to z_0, \quad z \in D.$$

If $\{a_n\}_{n=0}^\infty$ is an infinite sequence of constants and the above is true for each N, then the formal infinite series

$$\sum_{n=0}^\infty a_n \phi_n(z),$$

is called an asymptotic series and is said to be an asymptotic expansion of $f(z)$ as

$z \to z_0$, from D. We then write

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z), \quad z \to z_0, \quad z \in D.$$

We now discuss some examples to highlight a few of the nuances of asymptotic expansions.

Example 1.19. Consider the function

$$f(z) = \frac{1}{z} + \mathrm{e}^{-z^2} \frac{1}{1-z}, \quad z > 1.$$

Then as $z \to \infty$, with respect to $\{z^{-n}\}_{n=0}^{\infty}$

$$f(z) \sim \frac{1}{z}.$$

But with respect to $\{z^{-1}\} \cup \{\mathrm{e}^{-z^2} z^{-n}\}_{n=0}^{\infty}$ we have

$$f(z) \sim \frac{1}{z} + \sum_{n=0}^{\infty} \mathrm{e}^{-z^2} \frac{1}{z^n}, \quad z \to \infty.$$

Because $\mathrm{e}^{-z^2} = o(z^{-j})$ for all $j > 0$, $z > 0$, it is "invisible" to $\{z^{-n}\}_{n=0}^{\infty}$. Here we say that it is *beyond all order*, or BAO.

And we also note that

$$g(z) = \frac{1}{z} + \mathrm{e}^{-z} \frac{1}{1-z}, \quad z > 1,$$

has the same asymptotic expansion as $f(z)$ with respect to $\{z^{-n}\}_{n=0}^{\infty}$.

Important 1.20. *The asymptotic series for a function is unique, once the $\{\phi_n\}_{n=0}^{\infty}$ are specified.*

Definition 1.21 (Asymptotic sum). *Given a set D in the complex plane whose closure contains z_0, the asymptotic sum of an asymptotic series corresponding to a sequence $\{\phi_n(z)\}_{n=0}^{\infty}$, asymptotic as $z \to z_0$ from D, and the coefficients $\{a_n\}_{n=0}^{\infty}$ is the equivalence class of functions $f(z)$ defined in D that satisfy*

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z), \quad z \to z_0, \quad z \in D.$$

That is, f and g are equivalent if

$$f(z) - g(z) \sim 0, \quad z \to z_0, \quad z \in D,$$

with respect to $\{\phi_n(z)\}_{n=0}^{\infty}$.

Then, it is important that every sequence $\{a_n\}_{n=0}^{\infty}$ corresponds to a function.

Proposition 1.22 (Asymptotic summability). *Let $\{\phi_n\}_{n=0}^{\infty}$ be an asymptotic sequence as $z \to z_0$, $z \in D$, and let $\{a_n\}_{n=0}^{\infty}$ be an arbitrary sequence of complex constants. Then the asymptotic series*

$$\sum_{n=0}^{\infty} a_n \phi_n(z),$$

has an asymptotic sum. That is, there is at least one function $f(z)$, defined for $z \in D \setminus \{z_0\}$ for which

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z), \quad z \to z_0, \quad z \in D.$$

1.7 ▪ Asymptotic root finding

The problem of asymptotic root finding can be summarized as the characterization of the roots of a function $f(x, \epsilon)$ when ϵ is a small parameter. More specifically, we wish to "find" $\{x_j\}_{j=1}^{N(\epsilon)}$ such that

$$f(x_j(\epsilon), \epsilon) = 0,$$

for $\epsilon \ll 1$. But what does "find" mean in this context? For our purposes, we want the existence of $x_j(\epsilon)$ and then we want to find an asymptotic approximation as $\epsilon \to 0$, to some desired order.

Theorem 1.23 (Analytic implicit function theorem). *Suppose that x_0 is a complex number such that the equation $f(x_0, 0) = 0$ holds, and suppose that $f(x, \epsilon)$ is analytic as a function of two variables at $(x, \epsilon) = (x_0, 0)$. If, in addition*

$$\frac{\partial f}{\partial x}(x_0, 0) \neq 0,$$

then there are constants $\alpha, \beta > 0$ such that for each ϵ in the disk $|\epsilon| < \alpha$,

$$f(x, \epsilon) = 0$$

has a unique simple root $x = x(\epsilon)$, $x(0) = x_0$ that itself lies in the disk

$$|x - x_0| < \beta.$$

Moreover, $x(\epsilon)$ is an analytic function of ϵ for $|\epsilon| < \alpha$.

The intuition for this comes from the chain rule:

$$f(x(\epsilon), \epsilon) = 0,$$
$$\frac{\partial f}{\partial x} x'(\epsilon) + \frac{\partial f}{\partial \epsilon} = 0.$$

And to be able to solve for $x'(\epsilon)$, we need that $\frac{\partial f}{\partial x} \neq 0$.

Remark 1.24. *If $f(x,\epsilon)$ is only assumed to have continuous mixed partial derivatives through order r, then $x(\epsilon)$ will also have r continuous derivatives.*

Then we obtain the expansion

$$x(\epsilon) \sim \sum_{n=0}^{\infty} x^{(n)}(0)\frac{\epsilon^n}{n!}, \quad \epsilon \to 0,$$

and the coefficients $x^{(n)}(0)$ can be obtained from repeated differentiation of $f(x,\epsilon)$.

Example 1.25. Consider

$$f(x,\epsilon) = x^2 - 1 + \epsilon x.$$

Then

$$\begin{cases} f(\pm 1, 0) = 0, \\ \left|\frac{\partial f}{\partial x}(\pm 1, 0)\right| = 2 \neq 0. \end{cases}$$

So, the implicit function theorem applies and we find

$$x_1(\epsilon) = 1 + O(\epsilon), \quad x_2(\epsilon) = -1 + O(\epsilon).$$

This is called a regular perturbation because setting $\epsilon = 0$ does not change the fundamental character of the problem.

Example 1.26. Consider

$$f(x,\epsilon) = \epsilon(x^2 - 1) + x.$$

Then

$$\begin{cases} f(0,0) = 0, \\ \frac{\partial f}{\partial x}(0,0) = 1. \end{cases}$$

So, for ϵ sufficiently small there exists a simple root in a neighborhood of the origin:

$$\begin{aligned} x(\epsilon) &= O(\epsilon), \\ \frac{\partial f}{\partial x}x'(\epsilon) + \frac{\partial f}{\partial \epsilon}\bigg|_{(x,\epsilon)=(0,0)} &= 0, \\ x'(0) = 1 \Rightarrow x(\epsilon) &= \epsilon + O(\epsilon^2). \end{aligned}$$

But this is a quadratic. What happens to the other root? Well, the two roots are given by

$$x_{\pm}(\epsilon) = \frac{-1 \pm \sqrt{1 + 4\epsilon^2}}{2\epsilon}.$$

Note that

$$\sqrt{1 + 4\epsilon^2} = 1 + 2\epsilon^2 + O(\epsilon^4), \quad \epsilon \to 0.$$

Thus

$$x_+(\epsilon) = \frac{2\epsilon^2}{2\epsilon} + O(\epsilon^3), \quad x_-(\epsilon) = -\frac{1}{\epsilon} + O(\epsilon).$$

So, we see the other root flies off to infinity. We have a formula for it, but how would we analyze it if we did not?

The key is a change of variable:

$$f(x,\epsilon) = \epsilon(x^2-1) + x = 0, \quad \Rightarrow \quad x = y/\epsilon \quad \Rightarrow \quad F(y,\epsilon) := \epsilon f(y/\epsilon, \epsilon) = 0.$$

Then we find that

$$F(y,\epsilon) = y^2 - \epsilon^2 + y = 0.$$

We check

$$\begin{cases} F(-1,0) = 0, \\ \frac{\partial F}{\partial y}(-1,0) = -1 \neq 0. \end{cases}$$

Then we find there is a root $y_1(\epsilon) = -1 + O(\epsilon^2)$. And we check

$$\begin{cases} F(0,0) = 0, \\ \frac{\partial F}{\partial y}(0,0) = -1 \neq 0. \end{cases}$$

Then we find there is a root $y_2(\epsilon) = O(\epsilon^2)$.

The previous example demonstrates what is known as "dominant balance". We have the equation $y^2 - \epsilon^2 + y = 0$.

- If we suppose that y itself is not small, then the terms y^2, y are <u>dominant</u>. And they must then <u>balance</u>. This gives

  $$y^2 + y \approx 0.$$

  Since only the root $y = -1$ doesn't contradict our assumption, we can guess one root that $y_1 \approx -1$, as we see above.

- We also see $y = 0$ as a possibility here, but if y is small, y^2 is even smaller. So $-\epsilon^2 + y = 0$ gives the dominant balance, or $y \approx \epsilon^2$. We can see from above that this is correct, to leading order.

Chapter 2

Asymptotic analysis of integrals

We begin with two basic techniques:

- Suppose that

$$f(x;z) \sim \sum_{n=0}^{\infty} a_n \phi_n(x;z). \tag{2.1}$$

  Then

$$\int_a^b f(x;z)\mathrm{d}x \sim \int_a^b \sum_{n=0}^{\infty} a_n \phi_n(x;z)\mathrm{d}x,$$

  can be integrated term-by-term to generate an expansion if for $n > m$

$$\int_a^b |\phi_n(x;z)|\mathrm{d}x = o\left(\int_a^b \phi_m(x;z)\mathrm{d}z\right).$$

- Integration by parts! We have already seen this in action for erfc.

Example 2.1. Consider the incomplete Gamma function

$$\gamma(z;x) = \int_0^x \mathrm{e}^{-t}\, t^{z-1}\mathrm{d}t, \quad x > 0, z > 0,$$

as $x \to 0$. We first Taylor expand the exponential function

$$\mathrm{e}^{-t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!},$$

and write

$$\gamma(z;x) = \int_0^x \left(\sum_{n=0}^{N} (-1)^n \frac{t^n}{n!} + r_N(t)\right) t^{z-1}\mathrm{d}t, \quad r_N(t) + \sum_{n=N+1}^{\infty} (-1)^n \frac{t^n}{n!}.$$

Now, we estimate $r_N(t)$ via

$$|r_N(t)| \le \sum_{n=N+1}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{n+N+1}}{(n+N+1)!} \le t^{N+1} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{n!}{(n+N+1)!},$$

and a crude estimate is $\frac{n!}{(n+N+1)!} \leq 1$, giving $|r_N(t)| \leq \mathrm{e}^t\, t^{N+1}$. Or, more simply, one can just use Taylor's theorem with remainder. Then, we estimate

$$\left|\int_0^x r_N(t)\mathrm{d}t\right| \leq \mathrm{e}^x \frac{|x|^{N+z+1}}{N+z+1} = O(x^{N+z+1}).$$

Then integration term-by-term of the first terms gives

$$\gamma(z;x) = \sum_{n=0}^{N}(-1)^n \frac{x^{z+n}}{j!(n+z)} + O(x^{N+z+1}),$$
$$\gamma(z;x) \sim \sum_{n=0}^{\infty}(-1)^n \frac{x^{z+n}}{j!(n+z)}, \quad x \to 0, \quad x > 0.$$

2.1 ▪ Exponential integrals I: Watson's Lemma

Our goal will be to consider

$$F(\lambda) = \int_\Gamma \mathrm{e}^{\lambda h(\lambda,t)}\, g(\lambda,t)\mathrm{d}t,$$

where $\lambda > 0$, $\Gamma \subset \mathbb{C}$ is some contour. Here we think of h, g as some functions that vary slowly with respect to λ as λ becomes large. In most of our applications, h, g will have no λ-dependence. Recall the Gamma function

$$\Gamma(z) = \lim_{x\to\infty} \gamma(z;x).$$

Then Watson's lemma tells us how to evaluate the most elementary exponential integrals.

Proposition 2.2 (Watson's Lemma). *Suppose $T > 0$ and $\phi(t)$ is a complex-valued, absolutely integrable function on $[0,T]$. Suppose further that $\phi(t)$ is of the form $\phi(t) = t^\sigma g(t)$ where $\sigma > -1$ and $g(t)$ has an infinite number of continuous derivatives in some neighborhood of $t = 0$. Then*

$$F(\lambda) = \int_0^T \mathrm{e}^{-\lambda t}\, \phi(t)\mathrm{d}t \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \frac{\Gamma(\sigma+n+1)}{\lambda^{\sigma+n+1}},$$

as $\lambda \to \infty$, $\lambda > 0$.

Proof. The proof has 4 steps: (1) localize, (2) estimate remainder, (3) explicitly compute large λ behavior, and (4) assemble expansion.

(1): The s be sufficiently small so that g is infinitely differentiable on $[0,s]$. Then

$$F(\lambda) = \int_0^s \mathrm{e}^{-\lambda t}\, \phi(t)\mathrm{d}t + \underbrace{\int_s^T \mathrm{e}^{-\lambda t}\, \phi(t)\mathrm{d}t}_{\ell(s)}.$$

And we can bound

$$|\ell(s)| \leq \mathrm{e}^{-\lambda s} \int_0^T |\phi(t)| \mathrm{d}t = 0(\lambda^{-N}), \quad \lambda \to \infty,$$

for any $N > 0$, provided s is fixed.

(2): We then use Taylor's theorem to get a remainder estimate for $g(t)$:

$$g(t) = \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!} t^n + r_N(t), \quad 0 \leq t \leq s$$

and

$$|r_N(t)| \leq C_N t^{N+1}.$$

So, we find that

$$F(\lambda) = \sum_{j=0}^{N} \frac{g^{(n)}(0)}{n!} \int_0^s t^{n+\sigma} \mathrm{e}^{-\lambda t} \, \mathrm{d}t + O\left(\int_0^s t^{N+1+\sigma} \mathrm{e}^{-\lambda t} \, \mathrm{d}t \right).$$

(3): We need to evaluate integrals of the form

$$\int_0^s t^{j+\sigma} \mathrm{e}^{-\lambda t} \, \mathrm{d}t \underset{u=\lambda t}{=} \frac{1}{\lambda^{j+\sigma+1}} \int_0^{\lambda s} u^{j+\sigma} \mathrm{e}^{-u} \, \mathrm{d}u = \frac{1}{\lambda^{j+\sigma+1}} \gamma(j+\sigma+1, \lambda s).$$

We see that Watson's lemma is stated in terms of Gamma functions, not incomplete Gamma functions, we we have a remaining estimate to perform

$$\gamma(j+\sigma+1, \lambda s) = \Gamma(j+\sigma+1) - \int_{\lambda s}^{\infty} u^{j+\sigma} \mathrm{e}^{-u} \, \mathrm{d}u.$$

Then

$$\int_{\lambda s}^{\infty} u^{j+\sigma} \mathrm{e}^{-u} \, \mathrm{d}u = \int_{\lambda s}^{\infty} u^{j+\sigma} \mathrm{e}^{-u/2} \mathrm{e}^{-u/2} \, \mathrm{d}u \leq \mathrm{e}^{-\lambda s/2} \int_0^{\infty} \int_{\lambda s}^{\infty} u^{j+\sigma} \mathrm{e}^{-u/2} \, \mathrm{d}u = O(\lambda^{-N}),$$

for any $N > 0$. The conclusion is that

$$\int_0^s t^{j+\sigma} \mathrm{e}^{-\lambda t} \, \mathrm{d}t = \frac{1}{\lambda^{j+\sigma+1}} \Gamma(j+\sigma+1) + \mathrm{BAO}, \left| \int_0^s r_N(t) \, \mathrm{e}^{-\lambda t} \, \mathrm{d}t \right| = O\left(\frac{1}{\lambda^{N+\sigma+2}} \right).$$

(4): Then we just assemble everything to see the result holds. ■

Important 2.3. *An important generalization of Watson's lemma is: Suppose $T = \infty$ that $\phi(t)$ is integrable on any finite subinterval of $[0, \infty)$, with $\phi(t) = O(\mathrm{e}^{\mu t} G(t))$, $t \to \infty$, for some $\mu > 0$ and an function G that is integrable on $[0, \infty)$. Suppose further that $\phi(t)$ is of the form $\phi(t) = t^\sigma g(t)$ where $\sigma > -1$ and $g(t)$ has an infinite number of continuous derivatives in some neighborhood of $t = 0$. The conclusion of Watson's lemma holds.*

To see that this holds, we only need to repeat the localization step. Let $T > s$ be such that $|\phi(t)| \leq K\,\mathrm{e}^{\mu t}\,|G(t)|$ for $t > T$

$$\int_s^\infty \mathrm{e}^{-\lambda t}\,\phi(t) = \underbrace{\int_s^T \mathrm{e}^{-\lambda t}\,\phi(t)\mathrm{d}t}_{\text{Watson's lemma applies}} + \int_T^\infty \mathrm{e}^{-\lambda t}\,\phi(t)\mathrm{d}t.$$

Then estimating, for $\lambda > \mu$,

$$\left|\int_T^\infty \mathrm{e}^{-\lambda t}\,\phi(t)\mathrm{d}t\right| \leq K\int_T^\infty \mathrm{e}^{(\mu-\lambda)t}\,|G(t)|\mathrm{d}t \leq K\,\mathrm{e}^{(\mu-\lambda)T}\int_T^\infty |G(t)|\mathrm{d}t = O(\lambda^{-N}),$$

for any $N > 0$.

The following is an important generalization of Watson's lemma.

Proposition 2.4 (Asymptotics of Gaussian integrals). *Consider Suppose $\phi(t)$ is a complex-valued function defined on $\mathbb{R}$, integrable on any bounded subinterval of $\mathbb{R}$, with an infinite number of continuous derivatives in a neighborhood of $t = 0$. Then if $\phi(t) = O(\mathrm{e}^{\mu t^2}\,g(t)))$, $\mu > 0$, $t \in \mathbb{R}$, g is integrable,*

$$F(\lambda) = \int_{-\infty}^\infty \mathrm{e}^{-\lambda t^2}\,\phi(t)\mathrm{d}t \sim \sum_{n=0}^\infty \frac{\phi^{(2n)}(0)}{(2n)!\lambda^{n+\frac{1}{2}}}\Gamma(n+1/2), \quad \lambda\to\infty, \quad \lambda > 0.$$

Using properties of the Gamma function this can be further simplified to

$$F(\lambda) \sim \sqrt{\frac{\pi}{\lambda}}\sum_{n=0}^\infty \frac{\phi^{(2n)}(0)}{\lambda^n 2^{2n} n!}, \quad \lambda\to\infty, \quad \lambda > 0.$$

Proof. The proof again has 4 steps: (1) localize, (2) estimate remainder, (3) explicitly compute large λ behavior, and (4) assemble expansion.

(1): First, we localize the integral. And because we get to choose how we localize, we make things easier by choosing a symmetric interval. For any $\epsilon > 0$, fixed,

$$F(\lambda) = \int_{-\epsilon}^\epsilon \mathrm{e}^{-\lambda t^2}\,\phi(t)\mathrm{d}t + \int_{|t|\geq\epsilon} \mathrm{e}^{-\lambda t^2}\,\phi(t)\mathrm{d}t.$$

The bound $|\phi(t)| \leq C\,\mathrm{e}^{\mu t^2}$ is then used to show that the second term in is $O(\lambda^{-N})$ for any $N > 0$, i.e., it is BAO.

(2): We then use Taylor's theorem with remainder on the first term

$$\int_{-\epsilon}^\epsilon \mathrm{e}^{-\lambda t^2}\,\phi(t)\mathrm{d}t = \int_{-\epsilon}^\epsilon \mathrm{e}^{-\lambda t^2}\left(\sum_{n=0}^N \phi^{(n)}(0)\frac{t^n}{n!} + R_N(t)\right)\mathrm{d}t, \tag{2.2}$$

where $R_N(t) = O(t^{N+1})$, $|t| \leq \epsilon$. We assume N is odd.

(3): To find the large λ limits, we consider

$$\int_{-\epsilon}^{\epsilon} \mathrm{e}^{-\lambda t^2}\, t^\ell \mathrm{d}t \underset{s=\sqrt{\lambda}t}{=} \frac{1}{\lambda^{\ell/2+1/2}} \int_{-\sqrt{\lambda}\epsilon}^{\sqrt{\lambda}\epsilon} \mathrm{e}^{-s^2}\, s^\ell \mathrm{d}s.$$

If ℓ is odd, this vanishes. Now for $\ell = 2n$,

$$\int_{-\sqrt{\lambda}\epsilon}^{\sqrt{\lambda}\epsilon} \mathrm{e}^{-s^2}\, s^\ell \mathrm{d}s = \int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\, s^\ell \mathrm{d}s - 2\int_{\sqrt{\lambda}\epsilon}^{\infty} \mathrm{e}^{-s^2}\, \mathrm{d}s.$$

We claim that this last integral is BAO. One way to do this is to use asymptotics of incomplete Gamma functions, or $\operatorname{erfc}(z)$! But we aim to do this from first principles. So,

$$\begin{aligned}
\int_{\sqrt{\lambda}\epsilon}^{\infty} \mathrm{e}^{-s^2}\, \mathrm{d}s &\underset{s=\sqrt{\lambda}\epsilon+x}{=} \int_0^\infty \mathrm{e}^{-\lambda\epsilon^2 - x^2 - 2\sqrt{\lambda}\epsilon x} (\sqrt{\lambda}\epsilon + x)^\ell \mathrm{d}x \\
&\le \mathrm{e}^{-\lambda^2\epsilon} \int_0^\infty \mathrm{e}^{-x^2/2} \left(\underbrace{\sqrt{\lambda}\epsilon\, \mathrm{e}^{-x^2/(2\ell)}}_{\le\sqrt{\lambda}\epsilon} + \underbrace{x\, \mathrm{e}^{-x^2/(2\ell)}}_{\le C} \right)^\ell \mathrm{d}x \\
&\le \mathrm{e}^{-\lambda\epsilon^2} (\sqrt{\lambda}\epsilon + C)^\ell \int_0^\infty \mathrm{e}^{-x^2/2}\, \mathrm{d}x = O(\lambda^{-N}), \quad \text{for all } N > 0.
\end{aligned}$$

Then for $\ell = 2n$ we evaluate

$$\int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\, s^\ell \mathrm{d}s = 2\int_0^\infty \mathrm{e}^{-s^2}\, s^\ell \mathrm{d}s \underset{t=s^2}{=} \int_0^\infty \mathrm{e}^{-t}\, t^{\ell/2-1/2} \mathrm{d}t = \Gamma(\ell/2 + 1/2).$$

(4): Assemble! ■

2.2 ▪ Exponential integrals II: Laplace's method

We have now seen, in effect, how to analyze integrals of the form

$$F(\lambda) = \int_a^b \mathrm{e}^{\lambda R(t)}\, g(t) \mathrm{d}t,$$

where

- $R(t) = -t$, $a = 0$, and
- $R(t) = -t^2$, $a < 0 < b$.

Throughout, we assume that $R : [a, b] \to (-\infty, c]$ is a smooth and $g(t)$ is absolutely integrable. We will also introduce additional assumptions on local smoothness of g as necessary. We do not state any of the forthcoming results in this section as theorems/lemmas/propositions because there are so many cases to consider. Furthermore, even if we did state it as a theorem, in practice, one would effectively have to work through all of the calculations anyhow to compute all of the quantities involved.

We also only treat the case where $R(t)$ has a single global maximum on $[a, b]$. In the case of multiple global maxima, the results would be applied locally to each one, and all terms combined into a single asymptotic expansion.

2.2.1 ▪ Case 1: Maximum at an endpoint

Suppose that $R(a) = c$. Suppose further that $R'(a) \neq 0$ (note that this implies $R'(a) < 0$). The main idea is to transform the integral as

$$\int_a^b \mathrm{e}^{\lambda R(t)}\, g(t)\mathrm{d}t = c(\lambda) \int_0^d \mathrm{e}^{-\lambda s}\, G(s)\mathrm{d}s,$$

for $d > 0$ and an λ-dependent function $c(\lambda)$.

Step 1: Capture exponential growth or decay by writing

$$\tilde{R}(t) = R(t) - R(a),$$

so that $\tilde{R}(t) < 0$ on $(a, b]$.

Step 2: Next, we need to change variables to write it in the form above. Specifically, we would like to write $t = t(s)$ so that

$$\tilde{R}(t(s)) = -s.$$

This is a job for the implicit function theorem! Define

$$f(t, s) = \tilde{R}(t) + s.$$

Then

$$\begin{cases} f(a, 0) = 0, \\ \frac{\partial f}{\partial t}(a, 0) = \tilde{R}'(a) = R'(a) < 0. \end{cases}$$

The implicit function theorem applies and gives us a smooth root $t(s)$ for $|s| \leq \epsilon$, $f(t(s), s) = 0$. It is important to note that this is a local statement. But then we must localize to use this, but we have seen that localization is an important part of the calculations. Let's see what we know about $t(s)$:

$$\begin{aligned} t(0) &= a, \\ \tilde{R}'(a)t'(0) + 1 &= 0, \\ t'(0) &= -\frac{1}{\tilde{R}'(a)} > 0. \end{aligned}$$

By possibly decreasing ϵ, we can assume that $t(\epsilon) > a$. Then because a is assumed to be the global maximum of R, we know that $\tilde{R}(t)$ attains its maximum on $[t(\epsilon), b]$ and this maximum satisfies $m = \max_{t(\epsilon) \leq t \leq b} \tilde{R}(t) < 0$. Then

$$\int_a^b \mathrm{e}^{\lambda R(t)}\, g(t)\mathrm{d}t = \underbrace{\mathrm{e}^{\lambda R(a)}}_{\text{May grow or decay}} \left[\int_a^{t(\epsilon)} \mathrm{e}^{\lambda \tilde{R}(t)}\, g(t)\mathrm{d}t + \underbrace{\int_{t(\epsilon)}^b \mathrm{e}^{\lambda \tilde{R}(t)}\, g(t)\mathrm{d}t}_{\leq \mathrm{e}^{m\lambda} \int_a^b |g(t)|\mathrm{d}t} \right].$$

So, the second integral is BAO and we can focus on the first

$$\int_a^{t(\epsilon)} \mathrm{e}^{\lambda \tilde{R}(t)}\, g(t)\mathrm{d}t = \int_0^\epsilon \mathrm{e}^{-\lambda s}\, \underbrace{g(t(s))t'(s)}_{\phi(s)}\, \mathrm{d}s.$$

Step 3: We then apply Watson's lemma. But note that this becomes more complicated because we now have the implicitly defined function $t(s)$ involved:

$$\begin{aligned}\phi(0) &= g(a)t'(0) = -\frac{g(a)}{R'(a)},\\ \phi'(0) &= g(a)t''(0) + g(a)[t'(0)]^2 = -g(a)\frac{R''(a)}{[R'(a)]^2} + g'(a)\frac{1}{[R'(a)]^2}.\end{aligned}$$

This last formula comes from the fact that

$$\begin{cases} R'(t)t'(s) + 1 = 0, \\ R'(t)t''(s) + R''(t)[t'(s)]^2 = 0. \end{cases}$$

Upon assembling the expansion, we find

$$\begin{aligned}\int_a^b \mathrm{e}^{\lambda R(t))}\, g(t)\mathrm{d}t = \mathrm{e}^{\lambda R(a)}&\left[\frac{g(a)}{R'(a)}\frac{1}{\lambda} + \frac{1}{[R'(a)]^2}\left[g'(a) - g(a)\frac{R''(a)}{R'(a)}\right]\frac{1}{\lambda^2}\right]\\ &+ O(\lambda^{-3}\,\mathrm{e}^{\lambda R(a)}), \quad \lambda \to \infty, \lambda > 0.\end{aligned}$$

Important 2.5. *If $R(t)$ obtains its unique global maximum at $t = b$, first perform $t \mapsto -t$.*

2.2.2 ▪ Case 2: Maximum in the interior

We now consider the case where the maximum of $R(t)$ lies in (a, b). Near t^* we know

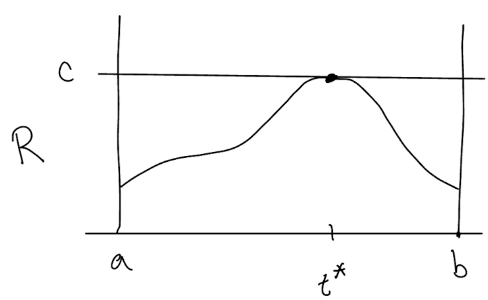


Figure 2.1: An interior maximum.

that for some $c > 0$

$$R(t) = R(t^*) + c(t - t^*)^{2j} + O((t - t^*)^{2j+1}, \quad t \to t^*.$$

From the homework, you certainly see that things are different depending on j. So, we treat the case $j = 1$ first, and leave the reader to fill in the details for $j > 1$.

Step 1: This proceeds as in the previous section without a hitch:

$$\int_a^b \mathrm{e}^{\lambda R(t)}\, g(t)\mathrm{d}t = \mathrm{e}^{\lambda R(t^*)} \int_a^b \mathrm{e}^{\lambda \tilde{R}(t)}\, g(t)\mathrm{d}t, \quad \tilde{R}(t) = R(t) - R(t^*).$$

Step 2: We now look for a (local) change of variables to put things in the form where the proposition on Gaussian integrals applies. We first attempt to use the implicit function theorem

$$f(t,s) = \tilde{R}(t) + s^2 = 0.$$

We find

$$\begin{cases} f(t^*,0) = 0, \\ \frac{\partial f}{\partial t}(t^*,0) = \tilde{R}'(t^*) = 0. \end{cases}$$

The implicit function theorem does not apply! The problem is that we are dealing with a quadratic and we have two branches (non-uniqueness) not that we do not have something that works (non-existence), as the following example indicates.

Example 2.6. Consider the case where

$$\tilde{R}(t) = -(t-t^*)^2 - (t-t^*)^4 = -s^2$$

We can solve directly for $(t-t^*)^2$,

$$(t-t^*)^2 = \frac{-1 \pm \sqrt{1+4s^2}}{2}.$$

Since when $s=0$, we should have $t=t^*$, we see that we need to choose the $+$ sign. And then we find

$$t(s) = t^* \pm \sqrt{\frac{-1 \pm \sqrt{1+4s^2}}{2}}.$$

If we look closely, we can see that for s small, $t(s)$ is a smooth function of s. So, the implicit function theorem fails here because of non-uniqueness. This is good news.

But how do we just take one of the branches? We first calculate

$$\frac{-1 \pm \sqrt{1+4s^2}}{2} = s^2 + O(s^4), \quad s \to 0.$$

Then

$$t(s) = t^* \pm s(1+o(1)),$$

where the $\pm$ sign is creating the problem.

Motivated by this example, suppose that

$$t(s) = t^* \pm s\phi(s), \quad \phi(0) > 0.$$

And ϕ is our new unknown. Now, we get to input which $\pm$ sign we want! Taylor expanding $\tilde{R}(t)$ gives

$$\tilde{R}(t) = \frac{R''(t^*)}{2}(t-t^*)^2 + O((t-t^*)^3), \quad t \to t^*.$$

Setting $t = t^* + s\phi$, we find

$$\tilde{R}(t^* + s\phi) = \frac{R''(t^*)}{2} s^2\phi^2 + O(s^3\phi^3).$$

The leading factor of s^2 is going to cause problems in applying the implicit function theorem. So, define

$$f(\phi, s) = \frac{\tilde{R}(t^* + s\phi)}{s^2} + 1.$$

We define $f(\phi, 0)$ by continuity, and find

$$f(\phi, 0) = \frac{R''(t^*)}{2}\phi^2 + 1 \Rightarrow \phi = \phi^* = \sqrt{\frac{2}{|R''(t^*)|}}.$$

We then apply the implicit function theorem

$$\begin{cases} f(\phi^*, 0) = 0, \\ \frac{\partial f}{\partial \phi}(\phi^*, 0) = R''(t^*)\phi^* \neq 0. \end{cases}$$

This gives the existence of $\phi(s)$ such that $f(\phi(s), s) = 0$ for s sufficiently small. And therefore, we have our change of variables

$$t(s) = t^* + s\phi(s), \quad \phi(s) = \phi^+ + O(s).$$

Now, back to our original problem

$$\int_a^b \mathrm{e}^{\lambda R(t)} g(t)\mathrm{d}t = \mathrm{e}^{\lambda R(t^*)} \left[\int_{t(-\epsilon)}^{t(\epsilon)} \mathrm{e}^{\lambda \tilde{R}(t)} g(t)\mathrm{d}t + \mathrm{BAO} \right].$$

Changing variables gives

$$\int_a^b \mathrm{e}^{\lambda R(t)} g(t)\mathrm{d}t = \mathrm{e}^{\lambda R(t^*)} \left[\int_{-\epsilon}^{\epsilon} \mathrm{e}^{-\lambda s^2} \underbrace{g(t^* + s\phi(s))[\phi(s) + s\phi'(s)]}_{\Phi(s)} \mathrm{d}s + \mathrm{BAO} \right].$$

Step 3: We now apply our generalization of Watson's lemma to Gaussian integrals:

$$\begin{aligned} \int_a^b \mathrm{e}^{\lambda R(t)} g(t)\mathrm{d}t &= \mathrm{e}^{\lambda R(t^*)} g(t^*)\phi(0)\sqrt{\frac{\pi}{\lambda}}(1 + O(\lambda^{-1}), \quad \lambda \to \infty \\ &= \sqrt{\frac{2\pi}{\lambda}} \mathrm{e}^{\lambda R(t^*)} \frac{g(t^*)}{\sqrt{-R''(t^*)}}(1 + O(\lambda^{-1})), \quad \lambda \to \infty. \end{aligned}$$

Example 2.7 (Stirling's formula).

$$\begin{aligned} \Gamma(z) &= \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t, \quad z \to \infty, z > 0 \\ &= \sqrt{2\pi} z^{z-1/2} \mathrm{e}^{-z} \left[1 + \frac{1}{12z} + O(z^{-2}) \right], \quad z \to \infty, z > 0. \end{aligned}$$

Example 2.8.

$$\int_0^\infty f(x) \mathrm{e}^{-\lambda(x^3/3 - x)} \mathrm{d}x.$$

Example 2.9. Suppose $\rho(t) = C_\sigma \, \mathrm{e}^{\sigma^{-2} R(t)} \, g(t)$ is a probability density on $\mathbb{R}$. Suppose, in addition, that $R(t)$ has a unique maximum at $t = t^*$, where $R''(t^*) < 0$ and $g(t^*) > 0$. By moving a constant into C_σ, we can assume that $R(t^*) = 0$. Consider first, the asymptotic behavior of C_σ as $\sigma > 0$ tends to zero:

$$C_\sigma^{-1} = \int_{-\infty}^{\infty} \mathrm{e}^{\sigma^{-2} R(t)} \, g(t) = \sqrt{\pi} \sigma g(t^*) \phi(0) (1 + O(\sigma^2)), \quad \sigma \to 0,$$

where $\phi(0) = \sqrt{-2/R''(t^*)}$. Now, consider, as $\sigma \to 0$,

$$\begin{aligned}
m_\ell &= C_\sigma \int_{-\infty}^{\infty} \left(\frac{t - t^*}{\sigma} \right)^\ell \mathrm{e}^{\sigma^{-2} R(t)} \, g(t) \mathrm{d}t \\
&= C_\sigma \left(\int_{t(-\epsilon)}^{t(\epsilon)} \left(\frac{t - t^*}{\sigma} \right)^\ell \mathrm{e}^{\sigma^{-2} R(t)} \, g(t) \mathrm{d}t + \mathrm{BAO} \right) \\
&= C_\sigma \left(\frac{1}{\sigma^\ell} \int_{-\epsilon}^{\epsilon} s^\ell \phi(s)^\ell \, \mathrm{e}^{-s^2/\sigma^2} \, g(t + s\phi(s)) [\phi(s) + s\phi'(s)] \mathrm{d}s + \mathrm{BAO} \right) \\
&= C_\sigma \left(\frac{g(t^*) \phi(0)^{\ell+1}}{\sigma^\ell} \int_{-\epsilon}^{\epsilon} s^\ell \phi(s)^\ell \, \mathrm{e}^{-s^2/\sigma^2} \, \mathrm{d}s + o(\sigma^{-\ell - 1/2}) \right) \\
&= \frac{\sigma g(t^*) \phi(0)^{\ell+1} \int_{-\infty}^{\infty} y^\ell \, \mathrm{e}^{-y^2} \, \mathrm{d}y + o(\sigma)}{\sqrt{\pi} \sigma g(t^*) \phi(0) + o(\sigma)} \to \frac{\phi(0)^\ell}{\sqrt{\pi}} \int_{-\infty}^{\infty} y^\ell \, \mathrm{e}^{-y^2} \, \mathrm{d}y.
\end{aligned}$$

The using a quick calculation using $\phi(0) y = x$

$$\frac{\phi(0)^\ell}{\sqrt{\pi}} \int_{-\infty}^{\infty} y^\ell \, \mathrm{e}^{-y^2} \, \mathrm{d}y = \frac{\sqrt{|R''(t^*)|}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^\ell \, \mathrm{e}^{-\frac{1}{2} x^2 (\sqrt{|R''(t^*)|})^2},$$

Which we can recognize as the moments of a normal distribution with mean zero and variance $1/|R''(t^*)|$. Thus if T is a random variable with density ρ then, in distribution,

$$\frac{T - t^*}{\sigma} \to \mathcal{N} \left(0, \frac{1}{|R''(t^*)|} \right).$$

Example 2.10. While we will not discuss it in any detail, Laplace's method generalizes to multi-dimensional settings. For example, consider

$$\int_{\mathbb{R}^n} \prod_{i<j} |x_i - x_j|^\beta \, \mathrm{e}^{-\beta \sum_j x_j^2} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

2.3 ▪ Exponential integrals III: Steepest descent

As motivation, consider

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx - k^2 t} \, \hat{f}(k) \mathrm{d}k.$$

For fixed x, we know how to evaluate the large-t asymptotics using Laplace's method. While we should think of e^{ikx} as an oscillatory factor, for this integral it is not really important because of the exponential decay as t increases, and the frequency of the oscillations is fixed.

But instead, suppose we allow $|x|$ to increase along with t. Suppose $s = x/t$ is held fixed, so that

$$u(st,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{t(iks-k^2)}\,\hat{f}(k)\mathrm{d}k.$$

The methods we have derived are unable to handle such an integral.

Example 2.11. The solution of $u_t + u_{xxx} = 0$ on $\mathbb{R}$ can be expressed in the form

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+ik^2 t}\,\hat{f}(k)\mathrm{d}k.$$

Example 2.12. The Hermite polynomials have the integral expression

$$h_k(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (-i\xi)^k\, e^{-\frac{1}{2}(\xi-ix)^2}\,\mathrm{d}\xi.$$

Now, consider $e^{t(iks-k^2)}$. For Laplace's method, we look for places where ht derivative of the exponent vanished. Let us try that again:

$$\begin{aligned} R(k) &= iks - k^2, \\ R'(k) &= is - 2k = 0, \quad k^* = \frac{is}{2}, \\ R''(k^*) &= -2. \end{aligned}$$

Thus we have the exact expression,

$$h(k) = h(k^*) - (k-k^*)^2.$$

Note that the second term here is complex valued.

An idea: If we can deform the integral off the real axis into at contour in the complex plane that passes through k^*, we may be able to apply Laplace's method:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\,\hat{f}(k)\mathrm{d}k = \frac{1}{2\pi}\int_{\mathbb{R}+k^*} e^{ikx-k^2 t}\,\hat{f}(k)\mathrm{d}k.$$

This can be justified if $t > 0$ and $\hat{f}(k)$ is bounded and analytic in a strip about the real axis that contains k^* in its interior. Then we can parameterize the last integral using $k = k^* + y$ giving

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{th(k^*)-ty^2}\,\hat{f}(k^*+y)\mathrm{d}k.$$

Now, we can just apply our generalization of Watson's lemma.

This is one of the simplest implementations of a much more general method called the method of steepest descent for integral. It allows one to handle much more complicated functions in the exponent, under the assumption of analyticity.

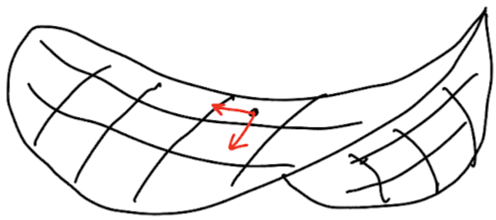


Figure 2.2: A saddle point for $R(x, y)$

2.3.1 ▪ Saddle points and analytic functions

We consider the integral

$$\int_C e^{\lambda h(z)} g(z) dz, \quad \lambda \to \infty, \lambda > 0.$$

We suppose that h, g are analytic functions in a sufficiently large region. More detailed will be incorporated on a case-by-case basis.

The properties of analytic functions will then guide our calculations. Of first importance are the points $z_0, \ldots, z_n$ where,

$$h'(z_j) = 0, \quad h''(z_j) \neq 0.$$

These are called saddle points. But why? The answer is the maximum principle.

First, if $h'(z) = 0$ then

$$\lim_{\epsilon \to 0} \frac{h(z+\epsilon) - h(z)}{\epsilon} = \lim_{\epsilon \to 0} \frac{h(z + i\epsilon) - h(z)}{i\epsilon} = 0.$$

And equality here means that both the real and imaginary parts vanish. Set

$$h(z) = h(x + iy) = R(x, y) + iI(x, y), \quad R(z) = R(x, y), \quad I(z) = I(x, y),$$

where both R, I are real valued. Then if $h'(z) = 0$ we find that

$$\nabla R(x, y) = 0,$$

so that (x, y) is a critical point of the function R. Then, we know that the real and imaginary parts, R, I are harmonic functions: $\Delta R(x, y) = \Delta I(x, y) = 0$. The maximum principle states that harmonic functions cannot have local maxima/minima. So we must have a saddle point. So, starting from the saddle point, we may follow a path so that $R(x, y)$ decreases. This is the direction of minus the gradient, i.e. $-\nabla R(x, y)$.

But, importantly, the properties of analytic functions let us characterize this path nicely and derive a change of variables for it.

Recall the Cauchy–Riemann equations

$$\partial_x R(x, y) = \partial_y I(x, y), \quad \partial_y R = -\partial_x I.$$

Now, consider the level set of values x, y such that

$$I(x, y) = I(x_0, y_0),$$

for a fixed (x_0, y_0). Suppose we parameterize the path as $x(t), y(t)$ satisfying

$$I(x(t), y(t)) = I(x_0, y_0).$$

We find

$$0 = \nabla I \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \partial_x I x'(t) + \partial_y I y'(t).$$

So, this means that ∇I is tangent to this level set. But then the Cauchy–Riemann equations imply that

$$\partial_x I \partial_x R + \partial_y I \partial_y R = \nabla I \cdot \nabla R.$$

This then implies that ∇I and ∇R are orthogonal. So, ∇R must point in the direction of the tangent to the level curves of I!

Important 2.13. *Of the two level curves passing through a saddle point, one curve gives the path of steepest ascent and the other gives the path of steepest descent. And along either curve, the imaginary part of $h(z)$ is constant.*

2.3.2 ▪ Deforming integrals according to saddles

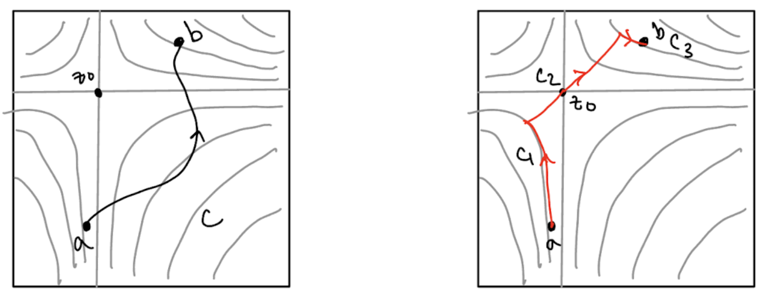


Figure 2.3: Level curves of $R(x, y)$ and the deformation of a contour.

As a cartoon of how this process is going to work, let us consider the integral where C is a path that connects two points a, b as in Figure 2.3. The assumption of analyticity allows us to use Cauchy's theorem to deform contours to essentially any path connecting these two points. In the right panel we have a deformation in which we:

- Follow the path of steepest descent through the saddle as long as possible.
- Do not allow the real part of $h(z)$ to increase when moving away from the saddle.
- Follow paths where the real part is constant to connect to the start/end of the contour.

So, according to the right panel in Figure 2.3 we write

$$\int_C e^{\lambda h(z)} g(z) dz = \sum_{j=1}^{3} e^{\lambda h(z_0)} \int_{C_j} e^{\lambda \tilde{h}(z)} g(z) dz, \quad \tilde{h}(z) = h(z) - h(z_0).$$

Let us first understand the contribution from C_1, C_3. We write for $\tilde{h}(z) = \tilde{R}(z) + \mathrm{i}\tilde{I}(z)$

$$\mathrm{e}^{\lambda h(z_0)} \int_{C_1} \mathrm{e}^{\lambda \tilde{h}(z)}\, g(z)\mathrm{d}z = \mathrm{e}^{\lambda h(z_0)} \int_{C_1} \mathrm{e}^{\lambda \tilde{R}(a)}\, \mathrm{e}^{\mathrm{i}\lambda \tilde{I}(z)}\, g(z)\mathrm{d}z =$$
$$= \mathrm{e}^{\lambda h(z_0)}\, \mathrm{e}^{\lambda \tilde{R}(a)} \int_{C_1} \mathrm{e}^{\mathrm{i}\lambda \tilde{I}(z)}\, g(z)\mathrm{d}z.$$

Since $\tilde{R}(a) < 0$, we this will contribute at an exponentially small scale. A similar calculation holds for C_3.

Now, for the integral on C_2 we write

$$\int_{C_2} \mathrm{e}^{\lambda h(z)}\, g(z)\mathrm{d}z = \mathrm{e}^{\lambda h(z_0)} \int_{C_2} \mathrm{e}^{\lambda \tilde{R}(z)}\, g(z)\mathrm{d}z,$$

because the imaginary part of $h(z)$ is constant on the path of steepest descent. If we can find a change of variables $z(s) = z_0 + s\phi(s)$, we should be able to write

$$\int_{C_2} \mathrm{e}^{\lambda h(z)}\, g(z)\mathrm{d}z = \mathrm{e}^{\lambda h(z_0)} \left[\int_{-\epsilon}^{\epsilon} \mathrm{e}^{-\lambda s^2}\, g(z_0 + s\phi(s))[s\phi(s)]'\mathrm{d}s + \mathrm{BAO} \right].$$

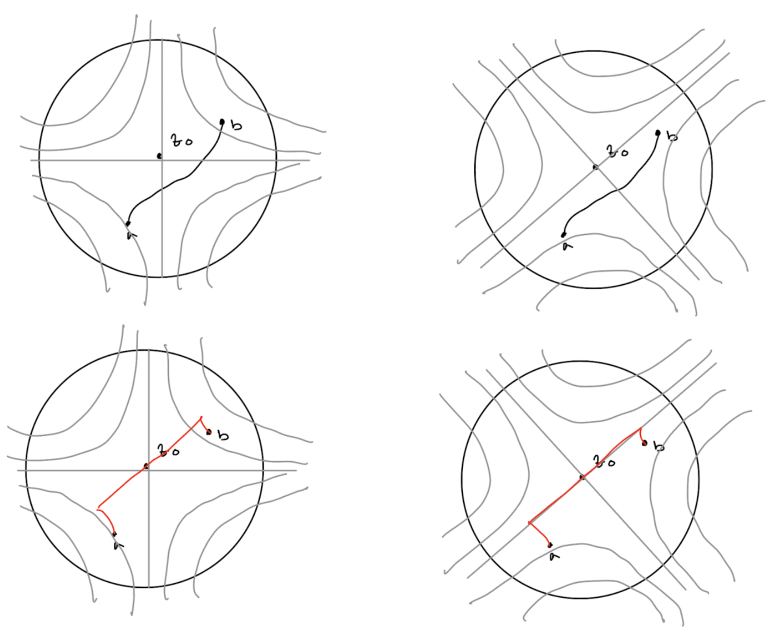


Figure 2.4: Both the real and imaginary parts of $h(z)$ near a saddle and their level curves. Left: Level curves for R. Right: Level curves for I.

Next, we need to understand the function $\phi(s)$ above that is defining the change of variables. But, we have actually already done this, thanks to the analytic implicit function theorem. We want to find $\phi(s)$ such that

$$\frac{\tilde{R}(z_0 + s\phi)}{s^2} + 1 = 0,$$

and to do this, because the imaginary part is constant, it suffices to consider

$$\frac{\tilde{h}(z_0 + s\phi)}{s^2} + 1 = 0.$$

Remark 2.14. *We will no longer have a condition like $\phi(0) > 0$ because the path of steepest descent does not have to leave the saddle point in a horizontal direction.*

Example 2.15. Consider the integral

$$F(\lambda) = \int_{-2\mathrm{i}-2}^{2\mathrm{i}} \mathrm{e}^{\lambda h(z)}, \quad h(z) = (1+\mathrm{i})z + \mathrm{i}z^2/2.$$

Note that this fits the geometry of Figure 2.3. We compute

$$h'(z) = 1 + \mathrm{i} + \mathrm{i}z \Rightarrow z_0 = \mathrm{i}(1+\mathrm{i}) = \mathrm{i} - 1.$$

Then we have that

$$\begin{aligned} F(\lambda) &= \mathrm{e}^{\lambda h(z_0)} \int_{-2\mathrm{i}-2}^{2\mathrm{i}} \mathrm{e}^{\lambda \tilde{h}(z)} \, \mathrm{d}z, \quad \tilde{h}(z) = \frac{\mathrm{i}}{2}(z - z_0)^2 \\ &= \mathrm{e}^{\lambda h(z_0)} \left[\int_{C_2} \mathrm{e}^{\lambda \tilde{h}(z)} \, \mathrm{d}z + \mathrm{BAO} \right]. \end{aligned}$$

In this example, we can roughly use the deformation that is described in Figure 2.3. Next, we change variables using the implicit function theorem applied to the function

$$f(\phi, s) = \frac{\tilde{h}(z_0 + s\phi)}{s^2} + 1.$$

We compute

$$\begin{cases} f(\phi^*, s) = 0 \quad \text{if} \quad \mathrm{i}(\phi^*)^2 + 1 = 0, \\ \frac{\partial f}{\partial \phi}(\phi^*, 0) \neq 0. \end{cases}$$

Then we see that $\phi^* = \pm \, \mathrm{e}^{\mathrm{i}\pi/4}$. And we can choose either. But let's choose $+$. The implicit function theorem then gives us the existence of $\phi(s)$ satisfying $f(\phi(s), s) = 0$ for s sufficiently small. Furthermore $\phi(0) = \phi^*$. Actually, in this case, things are so trivial that $\phi(s) = \phi^*$!

Then we compute

$$\begin{aligned} \mathrm{e}^{\lambda h(z_0)} \int_{C_2} \mathrm{e}^{\lambda \tilde{h}(z)} \, \mathrm{d}z &= \mathrm{e}^{\lambda h(z_0)} \left[\int_{-\epsilon}^{\epsilon} \mathrm{e}^{-\lambda s^2} [s\phi(s)]' + \mathrm{BAO} \right] \\ &= \mathrm{e}^{\lambda h(z_0)} \sqrt{\frac{\pi}{\lambda}} \left[\phi(0) + O(\lambda^{-1}) \right]. \end{aligned}$$

So, we find

$$F(\lambda) = \mathrm{e}^{-\lambda} \sqrt{\frac{\pi}{\lambda}} \left[\mathrm{e}^{\mathrm{i}\pi/4} + O(\lambda^{-1}) \right],$$

where we used that $h(z_0) = -1$.

2.3.3 ▪ The geometry of paths of steepest descent

Suppose $h(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial. Suppose $h'(z_0) = 0, h''(z_0) \neq 0$ and set $\tilde{h}(z) = h(z) - h(z_0)$. Now consider the zero level curve

$$\mathcal{S} = \{z \in \mathbb{C} \mid \operatorname{Im} \tilde{h}(z) = 0\}.$$

We claim:

- $\mathcal{S}$ cannot have any finite closed loops.
- The large z behavior of $\mathcal{S}$ is determined by $a_n z^n$.

To further understand this last statement, consider

$$\operatorname{Im} \tilde{h}(r\,\mathrm{e}^{\mathrm{i}\theta}) = 0 \quad \Leftrightarrow \quad \operatorname{Im} a_n r^n\, \mathrm{e}^{\mathrm{i}n\theta}(1 + O(r^{-1})) = 0.$$

Now, let set $\epsilon = 1/r$,

$$f(\theta, \epsilon) = \operatorname{Im} \epsilon^n \tilde{h}(\epsilon^{-1}\, \mathrm{e}^{\mathrm{i}\theta}) = \operatorname{Im} a_n\, \mathrm{e}^{\mathrm{i}n\theta}(1 + O(\epsilon)) = 0.$$

At $\epsilon = 0$, we find solutions, using $a_n = |a_n|\,\mathrm{e}^{\mathrm{i}\phi}$

$$0 = f(\theta_k, 0) = \operatorname{Im} |a_n|\, \mathrm{e}^{\mathrm{i}n\theta_k + \mathrm{i}\phi} = |a_n| \sin(n\theta_k + \phi),$$

for $n\theta_k + \phi = 0, \pm\pi, \pm 2\pi, \ldots$, i.e.,

$$\theta_k = \frac{k\pi}{n} - \frac{\phi}{n}, \quad k = 0, 1, 2, \ldots, 2n-1.$$

Half of these will be steepest ascent directions at infinity, half will be steepest descent. Now, we apply the implicit function theorem because

$$\frac{\partial f}{\partial \theta}(\theta_k, 0) = n|a_n| \cos(n\theta_k + \phi) \neq 0.$$

Thus there exists $\theta(\epsilon)$ satisfying

$$f(\theta(\epsilon), \epsilon) = 0, \quad \theta(\epsilon) = \theta_k + O(\epsilon) = \theta_k + O(1/r), \quad r \to \infty.$$

Now, we consider an example with a cubic phase function.

Example 2.16. Consider

$$F(\lambda) = \int_{1-2\mathrm{i}}^{1+\mathrm{i}} \mathrm{e}^{\lambda h(z)}\, \mathrm{d}z, \quad h(z) = \mathrm{i}z^2 + z^3.$$

We first compute the saddle points:

$$h'(z) = 2\mathrm{i}z + 3z^2, \quad z = 0, z = -\frac{2}{3}\mathrm{i}.$$

And we consider these points individually.

$z = 0$: We find that $h(0) = 0$, $h''(0) = 2\mathrm{i}$.

$z = -\frac{2}{3}\mathrm{i}$: We find that $h(-\frac{2}{3}\mathrm{i}) = -\mathrm{i}\frac{4}{9} - \mathrm{i}\frac{8}{27}$. This is purely imaginary. Then we compute

$$h''\left(-\frac{2}{3}\mathrm{i}\right) = -2\mathrm{i}.$$

Now, to see if our dominant behavior is going to indeed come from saddle points, we check the endpoints:

$$h(1-2\mathrm{i}) = -7 - \mathrm{i},$$
$$h(1+\mathrm{i}) = -4 + 2\mathrm{i}.$$

Since the real part of h is negative here, the contributions from the endpoints will be exponentially smaller than from the saddles. A schematic of the situation is given in Figure 2.5. We have

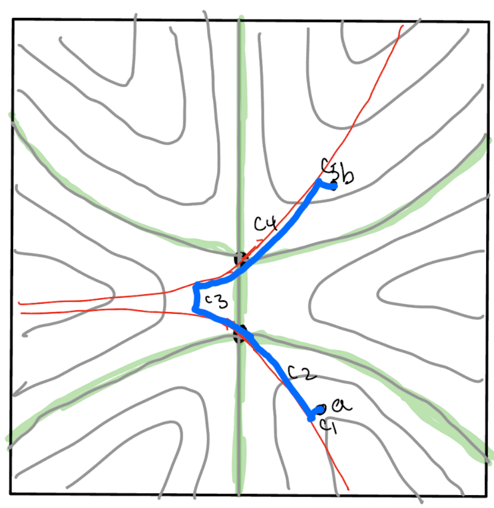


Figure 2.5: The level curves for Example 2.16.

$$F(\lambda) = \mathrm{e}^{\lambda h(0)}\left[\int_{C_4} \mathrm{e}^{\tilde{h}_1(z)}\,\mathrm{d}z + \mathrm{BAO}\right] \qquad (\tilde{h}_1(z) = h(z) - h(0)),$$
$$+\,\mathrm{e}^{\lambda h(-\frac{2}{3}\mathrm{i})}\left[\int_{C_2} \mathrm{e}^{\tilde{h}_2(z)}\,\mathrm{d}z + \mathrm{BAO}\right] \qquad (\tilde{h}_2(z) = h(z) - h(-\frac{2}{3}\mathrm{i})).$$

Now we need to do two changes of variable.

First change of variable:

$$\frac{\tilde{h}_1(0 + s\phi_1)}{s^2} + 1 = 0 \quad \Rightarrow \quad \frac{\tilde{h}_1''(0)}{2}(\phi_1^*)^2 = 0 \quad \Rightarrow \phi_1^* = \mathrm{e}^{\mathrm{i}\pi/4} \quad \Rightarrow \quad \phi_1(s) = \phi_1^* + O(s).$$

Second change of variable:

$$\frac{\tilde{h}_2(-\mathrm{i}\frac{2}{3} + s\phi_2)}{s^2} + 1 = 0 \quad \Rightarrow \quad \frac{\tilde{h}_2''(-\mathrm{i}\frac{2}{3})}{2}(\phi_2^*)^2 = 0 \quad \Rightarrow \phi_2^* = \mathrm{e}^{3\mathrm{i}\pi/4} \quad \Rightarrow \quad \phi_2(s) = \phi_2^* + O(s).$$

We can then assemble the leading-order behavior

$$\begin{aligned} F(\lambda) &= \sqrt{\frac{\pi}{\lambda}}\left[\mathrm{e}^{\lambda h(0)}\left[\phi_1(0)+O(\lambda^{-1})\right]+\mathrm{e}^{\lambda h(-\mathrm{i}\frac{2}{3})}\left[\phi_2(0)+O(\lambda^{-1})\right]\right] \\ &= \sqrt{\frac{\pi}{\lambda}}\left[\mathrm{e}^{\mathrm{i}\pi/4}+\mathrm{e}^{-\mathrm{i}4\lambda/27+\mathrm{i}\frac{3\pi}{4}}+O(\lambda^{-1})\right], \quad \lambda\to\infty, \quad \lambda>0. \end{aligned}$$

Important 2.17. *One needs to check that the parameterization respects the orientation of the deformed contour. The value $\phi_j(0)$ gives the direction the contour leaves the saddle as s increases and this should follow the orientation of the curve. If not, a minus sign must be included. This amounts to taking the other choice for $\phi_j(0)$.*

2.3.4 ▪ A case study: Hermite functions

As noted above, the Hermite polynomials can be expressed as

$$h_k(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-\mathrm{i}\xi)^k\,\mathrm{e}^{-\frac{1}{2}(\xi-\mathrm{i}x)^2}\,\mathrm{d}\xi, \quad k=0,1,2,\ldots.$$

Then define the Hermite wave functions

$$\varphi_k(x) = \frac{1}{\sqrt{k!}}\frac{\mathrm{e}^{-x^2/4}}{(2\pi)^{1/4}}h_k(x), \quad k=0,1,2,\ldots,$$

$$\int_{-\infty}^{\infty}\varphi_k(x)\varphi_\ell(x)\mathrm{d}x = \delta_{k\ell}.$$

The following formulae can be derived using the method of steepest descent for integrals.

Case 1. Oscillatory behavior.

$$x = 2\cos\varphi, \qquad 0<\varphi<\pi. \tag{2.3}$$

$$n^{\frac{1}{4}}\psi_{n+p}\left(x\sqrt{n}\right) = \frac{1}{\sqrt{\pi\sin\varphi}}\left(\cos\left[n\left(\varphi-\frac{1}{2}\sin 2\varphi\right)+\left(p+\frac{1}{2}\right)\varphi-\frac{\pi}{4}\right]+O(n^{-1})\right). \tag{2.4}$$

The convergence is uniform for φ in a closed subset of $(0,\pi)$.

Case 2. Exponential decay.

$$|x| = 2\cosh\varphi, \qquad 0<\varphi. \tag{2.5}$$

$$\begin{aligned} n^{\frac{1}{4}}\psi_{n+p}\left(x\sqrt{n}\right) &= \frac{(\mathrm{sgn}(x))^{n+p}}{2n^{\frac{1}{4}}}\frac{\mathrm{e}^{(p+1/2)\varphi}\,\mathrm{e}^{-\frac{n}{2}\left(\frac{x^2}{2}-1-\mathrm{e}^{-2\varphi}-2\varphi\right)}}{\sqrt{\pi\sinh\varphi}}\left(1+O(n^{-1})\right), \\ &= \frac{(\mathrm{sgn}(x))^{n+p}}{2n^{\frac{1}{4}}}\frac{\mathrm{e}^{(p+1/2)\varphi}\,\mathrm{e}^{-\frac{n}{2}(\sinh(2\varphi)-2\varphi)}}{\sqrt{\pi\sinh\varphi}}\left(1+O(n^{-1})\right). \end{aligned}$$

The convergence is uniform for φ in a closed subset of $(0,\infty)$. Observe that $\sinh(2\varphi)-2\varphi>0$ when $\varphi>0$, ensuring exponential decay.

Case 3. The transition region.

$$x = 2\sqrt{n} + \frac{s}{n^{\frac{1}{6}}} \quad s \in \mathbb{C}, \tag{2.6}$$

$$n^{\frac{1}{12}} \psi_{n+p}(x\sqrt{n}) = \mathrm{Ai}(s) + n^{-1/3}\left(\frac{1}{2} - p\right) \mathrm{Ai}'(s) + O(n^{-2/3}). \tag{2.7}$$

The convergence is uniform for s in a compact subset of $\mathbb{C}$.

We do not present the derivation of these formulae here, but considering $h_n(x\sqrt{n})$ can be reduced to the study of

$$I_n(x) = \int_0^\infty \mathrm{e} -nh(t)\mathrm{d}t, \quad h(t) = \frac{1}{2}(t - \mathrm{i}x)^2 - \log t.$$

The three cases above are determined by the configuration of the saddle points:

Case 1. For $|x| < 2$

$$t_\pm = \frac{\mathrm{i}x \pm \sqrt{4 - x^2}}{2}, \quad |t_\pm| = 1.$$

Case 2. For $|x| > 2$

$$t_\pm = \frac{\mathrm{i}}{2}\left(x \pm \sqrt{x^2 - 4}\right), \quad |t_+| > 1, \quad |t_-| < 1.$$

Case 3. For $|x| \approx 2$, the critical points degenerate to $\mathrm{sgn}(x)\mathrm{i}$.

We describe loosely how the calculation proceeds.

Case 1. Both saddle points contribute, leading to oscillatory behavior.

Case 2. Only one point contributes, leading to exponential behavior.

Case 3. The coalescing of the saddle points implies that $g''(t_\pm) \to 0$ in the limit.

We expand upon Case 3 a bit more:

$$h_n(x\sqrt{n}) = (-\mathrm{i}\sqrt{n})^n \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} (-\mathrm{i}t)^n \,\mathrm{e}^{-\frac{n}{2}(t-\mathrm{i}x)^2}\, dt.$$

For $x \approx 2$ we then rescale

$$t = \mathrm{i} + \frac{r}{n^{1/3}}, \quad x\sqrt{n} = 2\sqrt{n} + \frac{s}{n^{1/6}}.$$

giving

$$\begin{aligned} h_n(x\sqrt{n}) &= (-\mathrm{i}\sqrt{n})^n \frac{n^{1/6}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{nH(r)}\, \mathrm{d}r, \\ H(r) &= \log\left(\mathrm{i} + \frac{r}{n^{1/3}}\right) - \frac{1}{2}\left(\left(\mathrm{i} + \frac{r}{n^{1/3}}\right) - \mathrm{i}\left(2 + \frac{s}{n^{2/3}}\right)\right)^2 \\ &= \frac{1}{2} + \log \mathrm{i} + \frac{s}{n^{2/3}} + \frac{1}{n}\left(\mathrm{i}sr + \mathrm{i}\frac{r^3}{3}\right) + \frac{s^2}{2n^{1/4}} + O(n^{-3/4}r^4). \end{aligned}$$

This leads to the guess

$$h_n(x\sqrt{n}) \approx \frac{n^{\frac{n}{2}+\frac{1}{6}}}{\sqrt{2\pi}} \mathrm{e}^{n/2} \mathrm{e}^{sn^{1/3}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr+\mathrm{i}\frac{r^3}{3}} \,\mathrm{d}r.$$

And indeed

$$\begin{aligned} h_n(x\sqrt{n}) &= \sqrt{2\pi} n^{\frac{n}{2}+\frac{1}{6}} \mathrm{e}^{n/2} \mathrm{e}^{sn^{1/3}} (\mathrm{Ai}(s) + O(n^{-1/3})), \\ \mathrm{Ai}(s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr+\mathrm{i}\frac{r^3}{3}} \,\mathrm{d}r. \end{aligned}$$

Here $\mathrm{Ai}(s)$ is the Airy function and it satisfies $\mathrm{Ai}''(s) = s\,\mathrm{Ai}(s)$ with specific exponential asymptotics as $s \to +\infty$ [4]. In this way, we see that $\mathrm{Ai}(s)$ is somehow a universal function that describes the coalescing of saddle points.

2.4 ▪ Exponential integrals IV: Method of stationary phase

Consider

$$F(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda I(t)} \, g(t) \mathrm{d}t.$$

The main idea is that in regions where $I(t)$ varies, cancellation will occur and lead to small contributions. Contributions are smaller if I varies more rapidly. And the dominant contributions will come from stationary phase points $I'(t) = 0$. These can also be thought of as saddle points, but the method of stationary phase uses contour deformations and the geometry of paths of steepest descent in the background.

Suppose $I'(t^*) = 0$ and t^* is the only such point where this occurs. Let us also suppose that $g(t)$ is zero outside a bounded interval $[a, b]$ that contains t^* in its interior. We Taylor expand

$$\tilde{I}(t) = I(t) - I(t^*) = \frac{I''(t^*)}{2}(t - t^*)^2 + O((t - t^*)^3).$$

We look for a change of variables using

$$f(s, \phi) = \frac{\tilde{I}(t^* + s\phi)}{s^2} + \sigma = 0, \quad \sigma = \pm 1.$$

At $s = 0$, by continuity

$$f(0, \phi) = \frac{I''(t^*)}{2}\phi^2 + \sigma = 0 \quad \Rightarrow \quad \phi^2 = -\frac{2\sigma}{I''(t^*)}.$$

And σ is chosen so that this quantity is positive.

Now, for convenience suppose that if $t(s) = t^* + s\phi(s)$ is our change of variables, then $a = t(s_-), b = t(s_+)$, $s_- < 0 < s_+$, so that

$$F(\lambda) = \int_{s_-}^{s_+} \mathrm{e}^{-\mathrm{i}\lambda\sigma s^2} \underbrace{g(t^* + s\phi(s))[s\phi(s)]'}_{K(s)} \mathrm{d}s.$$

The method of stationary phase works by writing

$$K(s) = p(s) + [K(s) - p(s)],$$

for a polynomial $p(s)$ that approximates $K(s)$ — not just a Taylor polynomial.

Set $R_1(s) = (s_+ - s)^N(s - s_+)^N$, and Taylor expand $1/R_1(s)$ at $s = 0$:

$$\frac{1}{R_1(s)} = R_2(s) + O(s^{2N}), \quad s \to 0.$$

So, now $R_1(s)R_2(s)$ is a polynomial of degree $4N - 1$ that vanishes at $s = s_\pm$. Now write

$$K(s) = Q(s) + O(s^{2N}),$$

via a Taylor expansion at $s = 0$. Let $P(s) = Q(s)R_1(s)R_2(s)$ and write

$$\begin{aligned} F(\lambda) &= \mathrm{e}^{\mathrm{i}\lambda I(t^*)} \int_{s_-}^{s_+} P(s)\, \mathrm{e}^{-\mathrm{i}\lambda\sigma s^2}\, \mathrm{d}s \\ &\quad + \mathrm{e}^{\mathrm{i}\lambda I(t^*)} \int_{s_-}^{s_+} [K(s) - P(s)]\, \mathrm{e}^{\mathrm{i}\lambda\sigma s^2}\, \mathrm{d}s. \end{aligned}$$

The method of steepest descent applies to the first term, and the second term can be shown to be lower order using integration by parts. If there are multiple stationary points, one has to add the contributions from each. Miller gives a thorough treatment of this, including the use of "cut-off" functions.

2.5 ▪ Comments on numerical evaluation

Many of the methods we have discussed so far can be implemented numerically, where one captures the leading-order behavior analytically, but solves for the remainder numerically (as opposed to computing terms in its expansion). An interesting implementation of the method of steepest descent for integrals is given in the package [1].

But also, differential equations can directly assist in the numerical evaluation of integrals. Consider the differential equation

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x;\lambda) + \lambda h'(x)\mu(x;\lambda) = g(x), \quad \mu(x_0) = 0.$$

Then by the method of integrating factor

$$\left(\mu(x;\lambda)\, \mathrm{e}^{\lambda h(x)}\right)_x = \mathrm{e}^{\lambda h(x)} \left[\frac{\mathrm{d}\mu}{\mathrm{d}x}(x;\lambda) + \lambda h'(x)\mu(x;\lambda)\right].$$

Then using t to denote the integration variable

$$\int_{x_0}^{x} \left(\mu(t;\lambda)\, \mathrm{e}^{\lambda h(t)}\right)_t \mathrm{d}t = \int_{x_0}^{x} \mathrm{e}^{\lambda h(t)}\, g(t)\mathrm{d}t.$$

Imposing the boundary condition, we find

$$\mu(x;\lambda)\, \mathrm{e}^{\lambda h(x)} = \int_{x_0}^{x} \mathrm{e}^{\lambda h(t)}\, g(t)\mathrm{d}t.$$

Choose $x_0 = -\infty$

$$\mu_-(x;\lambda)\, \mathrm{e}^{\lambda h(x)} = \int_{-\infty}^{x} \mathrm{e}^{\lambda h(t)}\, g(t)\mathrm{d}t.$$

Then choose $x_0 = +\infty$

$$\mu_+(x;\lambda)\,\mathrm{e}^{\lambda h(x)} = -\int_x^\infty \mathrm{e}^{\lambda h(t)}\, g(t)\mathrm{d}t.$$

And we find

$$\mathrm{e}^{\lambda h(x)}[\mu_-(x;\lambda) - \mu_+(x;\lambda)] = \int_{-\infty}^\infty \mathrm{e}^{\lambda h(t)}\, g(t)\mathrm{d}t.$$

And the idea of Levin [2] was to solve for $\mu_\pm$ numerically!

Chapter 3

Second-order linear ordinary differential equations: Beyond existence & uniqueness

Consider the differential equation

$$p_2(z)w''(z) + p_1(z)w'(z) + p_0(z)w(z) = 0.$$

We assume that p_j is a polynomial of some degree. If you follow typical existence and uniqueness theorems, this would be written as

$$w''(z) + \frac{p_1(z)}{p_2(z)}w'(z) + \frac{p_0(z)}{p_2(z)}w(z) = 0,$$

and then maybe as a first-order system

$$x_1(z) = w(z), \quad x_2(z) = w'(z),$$
$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} x_1(z) \\ x_2(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{p_0(z)}{p_2(z)} & -\frac{p_1(z)}{p_2(z)} \end{bmatrix}\begin{bmatrix} x_1(z) \\ x_2(x) \end{bmatrix}.$$

Then the "smoothness" of the coefficient function $\frac{p_0(z)}{p_2(z)}$ and $\frac{p_1(z)}{p_2(z)}$ will determine how long you have a solution.

Example 3.1. Consider

$$w'(z) + \frac{1}{z-1}w(z) = 0.$$

We can write this as

$$\frac{w'(z)}{w(z)} = \frac{1}{1-z} \Rightarrow \frac{\mathrm{d}w}{w} = \frac{1}{1-z}\mathrm{d}z.$$

Upon integrating, we have

$$\log w(z) = -\log(1-z) + C,$$

giving

$$w(z) = \frac{C_0}{1-z}.$$

One interpretation: If the initial condition is given at $z = 0$, $w(0) = C_0$, then the solution ceases to be valid at $z = 1$.

Our interpretation: The solution is an analytic function of z away from $z = 1$. We won't let singular coefficients stop us!

3.1 ▪ Reduction to canonical form

A crucial technique in dealing with second-order linear ODEs is the reduction of the equation one in canonical form. Write

$$w''(z) + \underbrace{\frac{p_1(z)}{p_2(z)}}_{p(z)} w'(z) + \underbrace{\frac{p_0(z)}{p_2(z)}}_{q(z)} w(z) = 0.$$

Now, suppose that $w(z)$ is a solution and consider the function

$$y(z) = w(z) \exp\left(\frac{1}{2}\int_a^z p(\xi)\mathrm{d}\xi\right), \quad z \in D.$$

Since $p(z)$ may have poles, this integral is not path-independent and to make sense of this integral, for z in the complex plane we have to restrict ourselves to a simply connected region on which p is analytic.

Then compute:

$$\begin{aligned} E(z)w''(z) + p(z)E(z)w'(z) + q(z)E(z)w(z) = 0, \\ y'(z) = E'(z)w(z) + E(z)w'(z). \end{aligned}$$

Using the form of $E(z)$

$$\begin{aligned} y'(z) &= E(z)\left(\frac{1}{2}p(z)w(z) + w'(z)\right), \\ y''(z) &= E(z)\left(\frac{1}{4}p(z)^2 w(z) + \frac{1}{2}p(z)w'(z)\right) + E(z)\left(\frac{1}{2}p'(z)w(z) + \frac{1}{2}p(z)w'(z) + w''(z)\right) \\ &= E(z)\left(\frac{1}{4}p(z)^2 w(z) + \frac{1}{2}p'(z)w(z) + p(z)w'(z) + w''(z)\right). \end{aligned}$$

We then use the differential equation to eliminate higher derivatives on the right-hand side, giving

$$y''(z) = E(z)\left(\frac{1}{4}p(z)^2 w(z) + \frac{1}{2}p'(z)w(z) + p(z)w'(z) - p(z)w'(z) - q(z)w(z)\right).$$

Here we see that the $w'(z)$ term drops out, and we have the equation

$$\begin{aligned} &y''(z) - \frac{1}{4}p(z)^2 y(z) - \frac{1}{2}p'(z)y(z) + q(z)y(z) = 0, \\ &y''(z) + f(z)y(z) = 0, \quad f(z) = q(z) - \frac{1}{4}p(z)^2 - \frac{1}{2}p'(z). \end{aligned}$$

Definition 3.2. *A point $z_0 \in \mathbb{C}$ such that $f(z)$ is analytic in a neighborhood of z_0 is called an ordinary point. All other points are singular points.*

3.2 ▪ Second-order linear ODEs with rational coefficients

As we have seen, it suffices to consider

$$y''(z) + f(z)y(z) = 0,$$

where $f(z)$ is a rational function of z. At ordinary points there exists two linearly independent analytic solutions,

$$\begin{cases} y_1(z_0) = y_2'(z_0) = 1, \\ y_1'(z_0) = y_2(z_0) = 0. \end{cases} \tag{3.1}$$

The following fact about analytic functions is useful.

Proposition 3.3. *Suppose $F(w_1, \dots, w_n)$ is an entire function of n complex variables. For $D \subset \mathbb{C}$, open, suppose $g_1, \dots, g_n$ are analytic in D and*

$$F(g_1(z), \dots, g_n(z)) = 0, \quad z \in D.$$

Lastly, suppose $\tilde{g}_1, \dots, \tilde{g}_n$ are analytic continuations of $g_1, \dots, g_n$, respectively to $\tilde{D}$, open, with $\tilde{D} \cap D \neq \varnothing$. Then

$$F(\tilde{g}_1(z), \dots, \tilde{g}_n(z)) = 0, \quad z \in \tilde{D}.$$

The main use of this proposition is as follows. Suppose $y(z)$ and $f(z)$ are analytic in $\tilde{D} \cup D$, $\tilde{D} \cap D \neq \varnothing$, and y solves,

$$y''(z) + f(z)y(z) = 0, \quad z \in D,$$

then y is a solution in all of $\tilde{D} \cup D$. This also implies that the linear independence of solutions is also preserved under analytic continuation. But to fully complete this, we need the following theorem.

Theorem 3.4. *Suppose $f(z)$ is analytic at a point z_0 and its power series at this point has a radius of convergence*[a] *of R. Then there exists two solutions $y_1(z), y_2(z)$ of*

$$y''(z) + f(z)y(z) = 0,$$

satisfying (3.1) that are both analytic for $|z - z_0| < R$.

[a] For this, recall that it suffices that the distance from z_0 to the nearest singularity of $f(z)$ is at least R.

This theorem states that if we stay a finite distance away from singular points, we can always find locally analytic solutions that are analytic in an open disk with a radius of convergence that is bounded below, say by $\epsilon > 0$. So, we can march along any curve that is bounded away from the singular points, covering it with open disks of radius ϵ, centered at points z_j, constructing analytic solutions satisfying

$$\begin{cases} y_1(z_j) = y_2'(z_j) = 1, \\ y_1'(z_j) = y_2(z_j) = 0. \end{cases}$$

But the uniqueness of solutions of the differential equation, we can match on the overlaps, and construct an analytic continuation of the solution on the entire curve. Thus solutions can be continued along any curve containing only regular points.

3.2.1 ▪ Monodromy

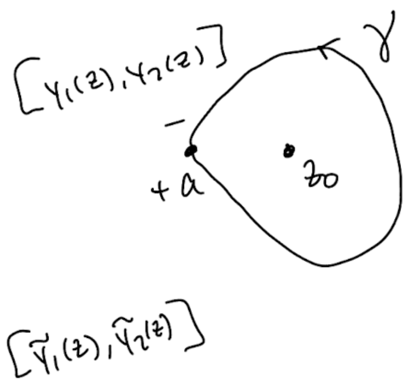


Figure 3.1: A path γ around a singular point.

Suppose z_0 is a singular point. Suppose we start from a regular point $z = a$ and continue the solution around this singular point, and only this singular point, with a counter-clockwise orientation, following a closed curve γ. Then in the notation of Figure 3.1 we have

$$\begin{bmatrix} y_1(a) \\ y_2(a) \end{bmatrix}_- = C_\gamma \begin{bmatrix} y_1(a) \\ y_2(a) \end{bmatrix}_+ .$$

Why? Because we are guaranteed that we still have solutions, when we return to $z = a$, but we are not guaranteed that it is the same solution. But if it is a different pair of solutions, they must be linearly independent and therefore can be expressed it this form. The 2×2 matrix C_γ is called the monodromy matrix associated to γ.

We can then argue by analytic continuation that this matrix C_γ actually only depends on z_0 (any path gives the same matrix) and the choice for the initial conditions at $z = a$. Furthermore, changing initial conditions amounts to just a similarity change for C_γ and we then see that the eigenvalues of C_γ depend only on z_0! They are invariants of the singular point.

Now, decompose

$$C_\gamma = V \Lambda V^{-1}, \quad \Lambda = \begin{bmatrix} \omega_1(z_0) & 0 \\ ? & \omega_2(z_0) \end{bmatrix}.$$

Then consider

$$\begin{bmatrix} \check{y}_1(z) \\ \check{y}_2(z) \end{bmatrix} = V \begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix}.$$

On γ:

$$\begin{bmatrix} \check{y}_1(z) \\ \check{y}_2(z) \end{bmatrix}_- = V \begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix}_- = VC_\gamma \begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix}_+ = \underbrace{VC_\gamma V^{-1}}_{\Lambda} \begin{bmatrix} \check{y}_1(z) \\ \check{y}_2(z) \end{bmatrix}_+.$$

Now suppose that Λ is diagonal with distinct diagonal entries,

$$\omega_1 = \mathrm{e}^{2\mathrm{i}\pi\rho_1}, \quad \omega_2 = \mathrm{e}^{2\mathrm{i}\pi\rho_2},$$

where ρ_j need not be real.

Again, around γ, taking it to be a circle, for simplicity, consider

$$\left[(z-z_0)^{-\rho_1}\check{y}_1(z)\right]_- = \mathrm{e}^{2\pi\mathrm{i}\rho_1}(z-z_0)_-^{-\rho_1}\check{y}_1(z)_+.$$

But then for $z = z_0 + \epsilon\,\mathrm{e}^{\mathrm{i}\theta}$,

$$(z-z_0)_+^{-\rho_1} = \epsilon^{-\rho_1}\,\mathrm{e}^{-\mathrm{i}\rho_1\theta_0}, \quad (z-z_0)_- = \epsilon^{-\rho_1}\,\mathrm{e}^{-\mathrm{i}\rho_1\theta_0 - 2\pi\mathrm{i}\rho_1}.$$

Thus

$$\left[(z-z_0)^{-\rho_1}\check{y}_1(z)\right]_+ = \left[(z-z_0)^{-\rho_1}\check{y}_1(z)\right]_-.$$

This then actually implies that $(z-z_0)^{-\rho_1}\check{y}_1(z)$ has a convergent Laurent series near z_0:

$$\check{y}_1(z) = (z-z_0)^{\rho_1}\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j.$$

Similar considerations follow for $\check{y}_2$.

Next, we consider the case where

$$\Lambda = \begin{bmatrix} \omega & 0 \\ 1 & \omega \end{bmatrix}, \quad \omega = \mathrm{e}^{2\pi\mathrm{i}\rho}$$

so that the geometric multiplicity is only one. Because we have

$$[\check{y}_1(z)]_- = [\check{y}_1(z)]_+\,\omega,$$

we know what happens to $\check{y}_1$. But $\check{y}_2$ is now different. We have

$$[\check{y}_2(z)]_- = [\check{y}_1(z)]_+ + [\check{y}_2(z)]_+\,\omega.$$

We write this as

$$\left[\frac{\check{y}_2(z)}{\check{y}_1(z)}\right]_- = \left[\frac{\check{y}_2(z)}{\check{y}_1(z)}\right]_+ + \frac{1}{\omega}.$$

Now, set $\omega^{-1} = 2\pi\mathrm{i}k$ and define

$$V(z) = \frac{\check{y}_2(z)}{\check{y}_1(z)} - k\log(z-z_0).$$

Using similar arguments, winding around z_0, it can be shown that $V(z)$ now has a convergent Laurent series

$$\begin{aligned} \check{y}_2(z) &= \check{y}_1(z)V(z) + k\log(z-z_0)\check{y}_1(z) \\ &= (z-z_0)^\rho\sum_{j=-\infty}^{\infty} b_j(z-z_0)^j + k\log(z-z_0)(z-z_0)^\rho\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j. \end{aligned}$$

Definition 3.5 (Regular and irregular singular points). *A singular point $z = z_0$ of $y''(z) + f(z)y(z) = 0$ is called regular if the series for $\check{y}_1$, $\check{y}_2$ contain only a finite number of negative powers. A singular point that is not regular is called irregular.*

We note that if z_0 is regular then

$$\begin{aligned}\check{y}_1(z) &= (z - z_0)^\rho \sum_{j=-k}^{\infty} a_j (z - z_0)^j \\ &= a_{-k}(z - z_0)^{\rho - k}(1 + O((z - z_0))), \quad z \to z_0.\end{aligned}$$

If $k = \infty$ so that z_0 is irregular, then no such approximation exists.

3.3 ▪ Computing with ordinary and regular singular points

To make the theory just developed implementable, we need to (1) compute ρ and (2) find the coefficients a_j. To simplify the process we make a crucial observation.

Proposition 3.6 (Reduction of order). *Consider $y''(z) + f(z)y(z) = 0$. Suppose $y_1(z)$ is a solution. Then*

$$y_2(z) = y_1(z) \int_a^z \frac{\mathrm{d}\xi}{y_1(\xi)^2},$$

is another solution.

Proof. We can just check:

$$\begin{aligned}y_2'(z) &= y_1'(z) \int_a^z \frac{\mathrm{d}\xi}{y_1(\xi)^2} + \frac{1}{y_1(z)}, \\ y_2''(z) &= y_1''(z) \int_a^z \frac{\mathrm{d}\xi}{y_1(\xi)^2} + \frac{y_1'(z)}{y_1(z)^2} - \frac{y_1'(z)}{y_1(z)^2} = y_1''(z) \int_a^z \frac{\mathrm{d}\xi}{y_1(\xi)^2}.\end{aligned}$$

So,

$$y_2''(z) + f(z)y_2(z) = [y_1''(z) + f(z)y_1(z)] \int_a^z \frac{\mathrm{d}\xi}{y_1(\xi)^2} = 0.$$

■

We include an alternate proof because it explains the name of the proposition.

Alternate proof of Proposition 3.6. We seek a solution of the form

$$y_2(z) = y_1(z)v(z).$$

Then, of course,

$$\begin{aligned}y_2'(z) &= y_1(z)v'(z) + y_1'(z)v(z), \\ y_2''(z) &= y_1(z)v''(z) + 2y_1'(z)v'(z) + y_1''(z)v(z).\end{aligned}$$

We then use the differential equation to eliminate y_1'', giving

$$\begin{aligned} y_2''(z) &= y_1(z)v''(z) + 2y_1'(z)v'(z) - f(z)y_1(z)v(z) \\ &= y_1(z)v''(z) + 2y_1'(z)v'(z) - f(z)y_2(z). \end{aligned}$$

So, if we can choose $v(z)$ so that

$$y_1(z)v''(z) + 2y_1'(z)v'(z) = 0,$$

we will have that y_2 solve the differential equation. This initially looks like a second-order differential equations for $v(z)$, but upon setting $w(z) = v'(z)$, we are looking to solve

$$w'(z) = -\frac{2y_1'(z)}{y_1(z)}w(z),$$

where y_1 is treated as a known function. We have reduced the order. So, a particular solution is

$$w(z) = \exp\left(-\int_a^z \frac{2y_1'(\xi)}{y_1(\xi)}\mathrm{d}\xi\right) = A\exp(-2\log y_1(z)) = A\frac{1}{y_1(z)^2}.$$

And upon integrating one more time to find $v(z)$, the proposition follows. ■

3.3.1 ▪ Series solutions at ordinary points

Here we suppose that, for simplicity, $z_0 = 0$. And let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

We seek a solution of the form

$$y(z) = \sum_{n=0}^{\infty} y_n z^n.$$

We have

$$y''(z) = \sum_{n=0}^{\infty} n(n-1)y_n z^{n-2} = \sum_{n=2}^{\infty} n(n-1)y_n z^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)y_{n+2}z^n,$$

where $n \mapsto n+2$ is used in the last sum. Then we need to simplify the product

$$\begin{aligned} f(z)y(z) &= \left(\sum_{n=0}^{\infty} f_n z^n\right)\left(\sum_{n=0}^{\infty} y_m z^m\right) \\ &= \sum_{n,m} f_n y_m z^{n+m}. \end{aligned}$$

We set $j = n+m$, $k = n$ and reorder the sum (because it is absolutely convergent) to find

$$f(z)y(z) = \sum_{j=0}^{\infty} z^j \sum_{k=0}^{j} f_k y_{j-k}.$$

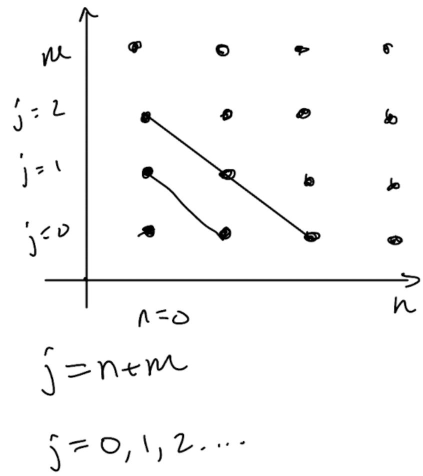


Figure 3.2: The ordering of the product of two power series.

This reordering is visualized in Figure 3.2. Plugging this all in, we have

$$0 = y''(z) + f(z)y(z) = \sum_{n=0}^{\infty} z^n \left[(n+2)(n+1)y_{n+2}z^n + \sum_{k=0}^{n} f_k y_{n-k} \right].$$

Upon equating coefficients, we have

$$\begin{aligned} n = 0: &\quad 2y_2 + f_0 y_0 = 0, \\ n = 1: &\quad 6y_3 + f_1 y_0 + y_0 f_1 = 0, \\ n = 2: &\quad 12y_4 + f_2 y_0 + f_1 y_1 + f_0 y_2 = 0, \\ &\quad \vdots \end{aligned}$$

The pattern here, is the contributions from products $f_j y_k$ at step n is over all the combinations of j, k such that $j + k = n$. Note that these equations do not determine y_0, y_1 because they are determined by initial conditions.

Not only are we computing the solution formula, we are also computing the leading terms in the expansion of $y(z)$ as $z \to z_0$.

3.3.2 ▪ The method of Frobenius: Series solutions at regular singular points

Again, we suppose that $z_0 = 0$. And suppose it is a regular singular point. This means the Laurent series contribution to the solution expression has a finite number of negative terms. In a sense, this gives us a foothold to start at the most negative term and apply ideas from the previous section. But, we need to know where to start. This will be determined in the process. So look for a solution of the form

$$y(z) = z^{\rho} \sum_{n=0}^{\infty} y_n z^n = \sum_{n=0}^{\infty} y_n z^{n+\rho}, \quad y_0 \neq 0.$$

We compute

$$\begin{aligned} y''(z) &= \sum_{n=0}^{\infty}(\rho+n)(\rho+n-1)y_n z^{n+\rho-2} \\ &= \rho(\rho-1)y_0 + (\rho+1)\rho y_1 + \sum_{n=2}^{\infty}(\rho+n)(\rho+n-1)y_n z^{n+\rho-2} \\ &= \rho(\rho-1)y_0 z^{\rho-2} + (\rho+1)\rho y_1 z^{\rho-1} + \sum_{n=0}^{\infty}(\rho+n+2)(\rho+n+1)y_{n+2}z^{n+\rho}. \end{aligned}$$

The computation of the product of y and f is essentially the same as before, but we have to account for the fact that f has a pole of some order

$$\begin{aligned} f(z)y(z) &= \left(\sum_{n=-N}^{\infty} f_n z^n\right)\left(\sum_{m=0}^{\infty} y_m z^{m+\rho}\right) = z^{\rho} z^{-N}\left(\sum_{n=0}^{\infty} f_{n-N} z^n\right)\left(\sum_{m=0}^{\infty} y_m z^m\right) \\ &= z^{\rho-N}\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} f_{k-N}y_{n-k} \\ &= z^{\rho}\sum_{n=-N}^{\infty} z^n \sum_{k=0}^{n+N} f_{k-N}y_{N-k+n}. \end{aligned}$$

Then enforcing that the differential equation holds amounts to considering

$$\begin{aligned} 0 = \rho(\rho-1)y_0 z^{\rho-2} + (\rho+1)\rho y_1 z^{\rho-1} + \sum_{n=-N}^{-1} z^{n+\rho}\sum_{k=0}^{n+N} f_{k-N}y_{N-k+n} \\ + \sum_{n=0}^{\infty} z^{n+\rho}\left[(\rho+n+2)(\rho+n+1)y_{n+2} + \sum_{k=0}^{n+N} f_{k-N}y_{N-k+n}\right]. \end{aligned}$$

Case 1: $N = 1$ We first note the reduction

$$\sum_{n=-N}^{-1} z^{n+\rho}\sum_{k=0}^{n+N} f_{k-N}y_{N-k+n} = z^{\rho-1}f_{-1}y_0.$$

Upon equating coefficients, we have

$$\begin{aligned} n = \rho-2: &\quad \rho(\rho-1)y_0 = 0, \\ n = \rho-1: &\quad \rho(\rho+1)y_1 + f_{-1}y_0 = 0, \\ n = \rho: &\quad (\rho+2)(\rho+1)y_2 + f_{-1}y_1 + f_0 y_0 = 0, \\ &\quad \vdots \end{aligned}$$

Since, we have assumed that $y_0 \neq 0$, we must have $\rho = 0, 1$. The second equation eliminates $\rho = 0$ as a possibility. Continuing this process then generates a convergent expansion and reduction of order can be used to obtain another linearly independent solution. Then in the use of reduction of order we need to integrate

$$\begin{aligned} \frac{1}{y_1(z)^2} &= \left(zy_0\left[1 + zy_1/y_0 + \cdots\right]\right)^{-2} = \frac{1}{z^2 y_0^2}\left[1 + zy_1/y_0 + \cdots\right]^{-2} \\ &= \frac{1}{z^2y_0^2}\left[1 - 2zy_1/y_2 + O(z^2)\right]. \end{aligned}$$

And it follows that $y_1 \neq 0$. Upon integration of this, a logarithm will be introduced.

Conclusion: We have repeated monodromy eigenvalues with a Jordan block:

$$V^{-1}C_\gamma V = \begin{bmatrix} \omega & 0 \\ 1 & \omega \end{bmatrix}.$$

Case 2: $N = 2$ Upon equating coefficients, we have

$$\begin{aligned} n = \rho - 2 : &\quad \rho(\rho-1)y_0 + f_{-2}y_0 = 0, \\ n = \rho - 1 : &\quad \rho(\rho+1)y_1 + f_{-2}y_1 + f_{-1}y_0 = 0, \\ n = \rho : &\quad (\rho+2)(\rho+1)y_2 + f_{-2}y_2 + f_{-1}y_1 + f_0y_0 = 0, \\ &\quad \vdots \end{aligned}$$

The first equation tells us that

$$I(\rho) := \rho(\rho-1) + f_{-2} = 0 \quad \Longrightarrow \quad \rho = \frac{1 \pm \sqrt{1-4f_{-2}}}{2}.$$

Let ρ_1 be the root with the largest real part. Then the next equation gives

$$I(\rho_1+1)y_1 + f_{-1}y_0 = 0 \quad \Longrightarrow \quad y_1 = \frac{f_{-1}y_0}{I(\rho+1)},$$

because $I(\rho_1+1) \neq 0$. We can continue and compute y_n. The same process applies to ρ_2 provided that $I(\rho_2+n)$ does not vanish for any integer $n > 0$.

Conclusion 1: $N = 2$, $\rho_1 - \rho_2$ is not an integer: The method of Frobenius produces two linearly independent series solutions. To see this, we try to construct the second solution with ρ_2 we find the equations

$$\begin{aligned} n = \rho_2 - 2 : &\quad I(\rho_2)y_0 = 0, \\ n = \rho_2 - 1 : &\quad I(\rho_2+1)y_1 + f_{-1}y_0 = 0, \\ n = \rho_2 : &\quad I(\rho_2+2)y_2 + f_{-1}y_1 + f_0y_0 = 0, \\ &\quad \vdots \end{aligned}$$

And then $\rho_2 + j$, for a positive integer j is never a root of $I(\rho)$ and the method succeeds. We have distinct monodromy eigenvalues

$$V^{-1}C_\gamma V = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}.$$

Conclusion 2: $N = 2$, $\rho_1 - \rho_2$ is an integer: Recall that we must have $\rho_1 - \rho_2 = m > 0$, for a positive integer m. So, we start solving:

$$\begin{aligned}
n = \rho_2 - 2 : &\quad I(\rho_2)y_0 = 0, \\
n = \rho_2 - 1 : &\quad I(\rho_2 + 1)y_1 + f_{-1}y_0 = 0, \\
n = \rho_2 : &\quad I(\rho_2 + 2)y_2 + f_{-1}y_1 + f_0y_0 = 0, \\
&\quad \vdots \\
n = \rho_2 + \ell - 2 : &\quad \underbrace{I(\rho_2 + m)}_{I(\rho_1)=0} y_m + \sum_{j=-1}^{m-2} f_j y_{m-2-j} = 0.
\end{aligned}$$

Two things can now happen. (1) The sum could vanish. In this case there is no equation determining y_m. But the process continues and all remaining coefficients are determined by y_0, y_m. But one can check that this does not contradict that there are only two linearly independent solutions. Indeed, setting $y_0 = 0, y_m = 1$, gives the solution obtained from ρ_1 with $y_0 = 1$. So, we have repeated monodromy eigenvalues, but no Jordan block:

$$V^{-1}C_\gamma V = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}.$$

And (2), the sum does not vanish, and this equation is not able to be satisfied. The method of Frobenius fails to produce a second solution. Reduction of order must be used, and a logarithm is to be expected in the new solution. We have repeated monodromy eigenvalues:

$$V^{-1}C_\gamma V = \begin{bmatrix} \omega & 0 \\ 1 & \omega \end{bmatrix}.$$

Case 3: $N > 2$ Consider, for example $N = 3$.

$$\begin{aligned}
0 = \rho(\rho-1)y_0 z^{\rho-2} + (\rho+1)\rho y_1 z^{\rho-1} + \sum_{n=-3}^{-1} z^{n+\rho} \sum_{k=0}^{n+N} f_{k-N} y_{3-k+n} \\
+ \sum_{n=0}^{\infty} z^{n+\rho} \left[(\rho+n+2)(\rho+n+1)y_{n+2} + \sum_{k=0}^{n+3} f_{k-N} y_{3-k+n} \right].
\end{aligned}$$

The sum in the first like has a term $z^{\rho-3}$ in it. There is no term to balance this term. The method fails. And it is clear that for $N > 3$, the same issue occurs. This actually shows that the series expansion for $y(z)$ cannot have a finite number of negative terms — if it did we would have succeeded.

Theorem-Definition 3.7 (Regular and irregular singular points). *A singular point z_0 of the differential equation $y''(z) + f(z)y(z) = 0$ corresponding to a pole of $f(z)$ is regular if the pole is of order one or two. If the order of the pole is greater than two, then z_0 is an irregular singular point.*

The point at infinity is a regular singular point if for $a \neq 0$, $f(z) = az^{-2} + O(z^{-3})$ or $f(z) = az^{-3} + O(z^{-4})$, $z \to \infty$. It is an irregular singular point if $f(z) =$

$az^p + O(z^{p-1})$, with $a \neq 0$ and $p \geq -1$.

3.4 ▪ Computing with irregular singular points: At infinity

Our approach above fails to construct two linearly independent solutions when we have an irregular singular point. We consider the case when we have an irregular singular point at infinity. To get some intuition in to what can happen here, we note that $f(z) = 1$ produces an irregular singular point at infinity. And we know the linearly independent solutions are

$$y_1(z) = \mathrm{e}^{\mathrm{i}z}, \quad y_2(z) = \mathrm{e}^{-\mathrm{i}z}.$$

While these functions do not have nice power series at infinity, the exponents do! So, we make the change of variable

$$y(z) = \mathrm{e}^{\phi(z)},$$

and seek a power series expansion for $\phi(z)$. We begin differentiating

$$y'(z) = \phi'(z)\,\mathrm{e}^{\phi(z)}, \quad y''(z) = \phi''(z)\,\mathrm{e}^{\phi(z)} + [\phi'(z)]^2\,\mathrm{e}^{\phi(z)}.$$

And

$$y''(z) + f(z)y(z) = 0 \quad \Longrightarrow \quad \phi''(z)\,\mathrm{e}^{\phi(z)} + [\phi'(z)]^2\,\mathrm{e}^{\phi(z)} + f(z)\,\mathrm{e}^{\phi(z)} = 0.$$

We arrive at the Ricatti equation

$$\phi''(z) + [\phi'(z)]^2 + f(z) = 0.$$

And, yes, we have turned our linear problem in to a nonlinear problem. But now, set $u(z) = \phi'(z)$ and we have the first-order nonlinear equation to consider

$$u'(z) + u(z)^2 + f(z) = 0.$$

Recall that because $f(z)$ is a rational function, we have a convergent series at infinity

$$f(z) = z^p \sum_{n=0}^{\infty} f_n z^{-n}, \quad p \geq -1.$$

We look for a series solution for $u(z)$ at infinity:

$$u(z) = z^q \sum_{n=0}^{\infty} u_n z^{-n/2}, \quad u_0 \neq 0.$$

The appearance of $z^{n/2}$ might seem a bit odd here. But, because our equation is nonlinear, we need to account for some other terms possibly being present.

We substitute into the differential equation

$$\underbrace{\sum_{n=0}^{\infty}(q - n/2)u_n z^{q-n/2-1}}_{(*)} + \underbrace{z^{2q}\sum_{n=0}^{\infty} z^{-n/2}\left(\sum_{k=0}^{n} u_k u_{n-k}\right)}_{(**)} + \underbrace{z^p \sum_{n=0}^{\infty} f_n z^{-n}}_{(***)} = 0.$$

We use dominant balance to now determine how the terms must relate to one another.

- Balancing $(*)$ and $(**)$: To leading order we would need $q-1 = 2q$, or $q = -1$. Then $(*)$ and $(**)$ and $O(z^{-2})$. There is then no way to cancel $(***)$ at this order because these terms are not dominant.
- Balancing $(*)$ and $(***)$: We must choose $q-1 = p$. Then $(**)$ is $O(z^{2p+2})$. And we check that $2p+2 > p$ for $p \geq -1$ and this balance is also not dominant.
- Balancing $(**)$ and $(***)$: We need $2q = p$, $q = p/2$. The $(*)$ is $O(z^{p/2-1})$ and we hope that
$$p/2 - 1 < p \quad \Leftrightarrow \quad p > -2.$$
As this holds, we've obtained our dominant balance to begin the computation.

With our dominant balance in hand, we can begin computing terms. Suppose that $p/2 - 1 = p - 1/2$, or $p = -1$. Then the first two equations are

$$\begin{aligned} z^p = z^{-1} :&\quad u_0^2 = -f_0, \\ z^{p-1/2} = z^{-3/2} :&\quad p/2u_0 + 2u_0u_1 = 0. \end{aligned}$$

Giving

$$u_0 = \pm \mathrm{i} f_0^{1/2}, \quad u_1 = 1/4.$$

Specifically, this gives

$$\begin{aligned} u(z) &= \pm \mathrm{i} f_0^{1/2} z^{-1/2} + \frac{1}{4z} + O(z^{-3/2}), \\ \phi(z) &= \pm 2\mathrm{i} f_0^{1/2} z^{1/2} + \frac{1}{4}\log z + O(z^{-1/2}). \end{aligned}$$

And

$$y(z) = \exp\left(\pm 2\mathrm{i} f_0^{1/2} z^{-1/2} + \frac{1}{4}\log z + O(z^{-1/2})\right), \quad z \to \infty.$$

3.5 ▪ An inverse problem associated with ODEs in the complex plane

The theory we have developed can be extended to systems

$$Y'(z) = A(z)Y(z),$$

where A is matrix-valued rational function. To an extent, all the singular points of A are characterized by the monodromy matrice. So, there is a natural inverse problem. From the monodromy matrices around each of the singular points of $A(z)$, is it possible to reconstruct $A(z)$? This is known as an "inverse monodromy" problem, and it is related to Hilbert's 21st problem.

To understand the extent to which this is possible, we ask when is the monodromy matrix M_0 about a singular point z_0 equal to the identity matrix. We know that

$$Y(z) - Y(a) = \int_z^a A(\xi)Y(\xi)\xi.$$

Cauchy's theorem tells us that if $Y(z)$ is analytic in a simply connected region then we have path independence. Conversely, if we have path independence and Y is continuous, then Y is analytic (Morera's theorem). This might seem to imply that $M_0 = I$ iff z_0 is an ordinary point, but this is not true. One could have two linearly independent solutions, one of which is singular at z_0, such that

$$\int_\gamma A(z)Y(z)\mathrm{d}z = 0.$$

This imposes a condition that must hold between the series coefficients of $A(z)$ and that of $Y(z)$. If this holds for all solutions, the monodromy matrix will be trivial — and $A(z)$ will not be recoverable. So, this question has some serious nuance to it.

Chapter 4

Asymptotic analysis of differential equations depending on a parameter

4.1 ▪ Regular perturbations I: Boundary-value problems

Consider the boundary-value problem

$$\begin{cases} y''(x) + y(x) + \epsilon g(y(x)) = 0, \\ y(-1) = 0, \\ y(1) = 1, \end{cases}$$

when $\epsilon \ll 1$, $g : \mathbb{R} \to \mathbb{R}$. Since this is nonlinear, we should not expect to be able to find an explicit expression for the solution. Instead, we suppose there exists a formal expansion

$$y(x;\epsilon) = \sum_{n=0}^{\infty} \epsilon^n y_n(x).$$

We write this knowing that we may not find a convergent expansion and it may end up being an asymptotic expansion. We proceed as we have done before and plug this into the equation, and equating powers of ϵ:

$$\sum_{n=0}^{\infty} \epsilon^n [y_n''(x) + y_n(x)] + \epsilon g\left(\sum_{n=0}^{\infty} \epsilon^n y_n(x)\right) = 0.$$

We obtain the equation

$$\epsilon^0 : \quad y_0''(x) + y_0(x) = 0, \quad y_0(-1) = 1, \quad y_0(1) = 0.$$

The general solution of this problem is

$$y_0(x) = a\cos x + b\sin x.$$

In imposing the boundary conditions, we find the linear system

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 1 & \sin 1 \\ \cos 1 & -\sin 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

We use Cramer's rule to find

$$a = \frac{\det \begin{bmatrix} 1 & \sin 1 \\ 0 & -\sin 1 \end{bmatrix}}{\det \begin{bmatrix} \cos 1 & \sin 1 \\ \cos 1 & -\sin 1 \end{bmatrix}} = \frac{\sin 1}{2\sin 1 \cos 1} = \frac{1}{2\cos 1},$$

$$b = \frac{\det \begin{bmatrix} \cos 1 & 1 \\ \cos 1 & 0 \end{bmatrix}}{\det \begin{bmatrix} \cos 1 & \sin 1 \\ \cos 1 & -\sin 1 \end{bmatrix}} = \frac{1}{2\sin 1}.$$

Then,

$$y_0(x) = \frac{1}{2}\left[\frac{\cos x}{\cos 1} + \frac{\sin x}{\sin 1}\right].$$

At the next order,

$$\epsilon^1 : \quad y_1''(x) + y_1(x) + g(y_0(x)) = 0, \quad y_1(-1) = 0, \quad y_1(1) = 0.$$

To see this, we just have to expand

$$g\left(\sum_{n=0}^{\infty} \epsilon^n y_n(x)\right) = g(y_0(x)) + \epsilon g'(y_0(x))y_1(x) + O(\epsilon^2).$$

So, we need to discuss how to handle inhomogeneous boundary-value problems.

4.1.1 ▪ Variation of parameters

> "Sweet Clyde, use variation of parameters and expand the Wronskian." — Bubblegum on *Futurama*

Consider solving

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x). \tag{4.1}$$

We know that we can construct the general solution of this differential equation using a linear combination of linear indpendent solutions of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \tag{4.2}$$

and any particular solution of (4.1). Variation of parameters gives us a method to find it. Specifically, we look for a solution of the form

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x),$$

where y_1 and y_2 are linearly independent solutions of (4.2). We will impose more and more constraints on v_1, v_2 as we go.

We start differentiating

$$y'(x) = v_1'(x)y_1(x) + v_1(x)y_1'(x) + v_2'(x)y_2(x) + v_2(x)y_2'(x).$$

We want something that is solvable for v_1, v_2, so we impose that $v_1'(x)y_1(x)+v_2'(x)y_2(x) = 0$. This will eliminate second derivatives for v_1, v_2. Thus,

$$\begin{aligned} y'(x) &= v_1(x)y_1'(x) + v_2(x)y_2'(x), \\ y''(x) &= v_1'(x)y_1'(x) + v_1(x)y_1''(x) + v_2'(x)y_2'(x) + v_2(x)y_2''(x). \end{aligned}$$

Next, we use that y_1, y_2 are solutions of (4.2):

$$\begin{aligned} y''(x) = v_1'(x)y_1'(x) &- v_1(x)[p(x)y_1'(x) + q(x)y_1(x)] \\ &+ v_2'(x)y_2'(x) - v_2(x)[p(x)y_2'(x) + q(x)y_2(x)]. \end{aligned}$$

And then we impose that we want y to solve (4.1)

$$\begin{aligned} &v_1'(x)y_1'(x) - v_1(x)[\cancel{p(x)y_1'(x)} + q(x)y_1(x)] + v_2'(x)y_2'(x) - v_2(x)[\cancel{p(x)y_2'(x)} + q(x)y_2(x)] \\ &\qquad + \cancel{p(x)v_1(x)y_1'(x)} + \cancel{p(x)v_2(x)y_2'(x)} + q(x)v_1(x)y_1(x) + q(x)v_2(x)y_2(x) = f(x), \\ &v_1'(x)y_1'(x) - \cancel{v_1(x)q(x)y_1(x)} + v_2'(x)y_2'(x) - \cancel{v_2(x)q(x)y_2(x)} \\ &\qquad + \cancel{q(x)v_1(x)y_1(x)} + \cancel{q(x)v_2(x)y_2(x)} = f(x). \end{aligned}$$

So, this simplifies to the system

$$\begin{aligned} v_1'(x)y_1(x) + v_2'(x)y_2(x) &= 0, \\ v_1'(x)y_1'(x) + v_2'(x)y_2'(x) &= f(x). \end{aligned}$$

Cramer's rule gives

$$v_1'(x) = \frac{\det\begin{bmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{bmatrix}}{\det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}}, \quad v_2'(x) = \frac{\det\begin{bmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{bmatrix}}{\det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}}.$$

This leads us to define the Wronskian

$$W(f,g)(x) = \det\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}.$$

Example 4.1. For $y_1(x) = \cos x, y_2(x) = \sin(x)$ we have

$$W(y_1,y_2)(x) = \det\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = 1.$$

Important 4.2. *Then, the variation of parameters formula for the general solution of* (4.1) *is*

$$y(x) = Ay_1(x) + By_2(x) + y_2(x)\int_{x_1}^{x} f(\xi)\frac{y_1(\xi)\mathrm{d}\xi}{W(y_1,y_2)(\xi)} - y_1(x)\int_{x_2}^{x} f(\xi)\frac{y_2(\xi)\mathrm{d}\xi}{W(y_1,y_2)(\xi)},$$

where y_1, y_2 are linearly independent solutions of (4.2).

In the example above, we find, for $f(x) = g(y_0(x))$

$$y(x) = A\cos x + B\sin x - \cos x\int_{x_1}^{x} f(\xi)\sin\xi\mathrm{d}\xi + \sin x\int_{x_1}^{x} f(\xi)\cos\xi\mathrm{d}\xi,$$

and then x_1, x_2, A, B are chosen to impose the boundary conditions.

4.1.2 ▪ Abel and Liouville's identities

An important but related calculation is the following. Suppose that y_1 and y_2 are two solutions of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

Define

$$W(x) = W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Then

$$\begin{aligned}
W'(x) &= y_1(x)y_2''(x) - y_1''(x)y_2(x) + y_1'(x)y_2'(x) - y_1'(x)y_2'(x) = y_1(x)y_2''(x) - y_1''(x)y_2(x) \\
&= -y_1(x)[p(x)y_2'(x) + q(x)y_2(x)] + y_2(x)[p(x)y_1'(x) + q(x)y_1(x)] \\
&= -y_1(x)p(x)y_2'(x) + y_2(x)p(x)y_1'(x) = -p(x)W(x).
\end{aligned}$$

Therefore,

$$\boxed{W(x) = W(x_0)\,\mathrm{e}^{-\int_{x_0}^{x} p(\xi)\mathrm{d}\xi}\,.}$$

This is Abel's formula or identity.

This generalizes to systems of equations:

$$Y'(x) = A(x)Y(x), \quad Y(x) \in \mathbb{C}^n.$$

Suppose that

$$\Phi(x) = \begin{bmatrix} y_1(x) & y_2(x) & \cdots y_n(x) \end{bmatrix},$$

is a matrix built out of solutions. Then we have the following.

Theorem 4.3 (Liouville's identity). *Suppose Φ is as above and $A(x)$ is sufficiently smooth. Then*

$$\det \Phi(x) = \det \Phi(x_0) \exp\left(\int_{x_0}^{x} \operatorname{tr} A(\xi)\mathrm{d}\xi\right).$$

Proof. Consider the difference $\det \Phi(x+h) - \det \Phi(x)$. Define

$$\Phi_i(x) = \Phi_i(x;h) = \begin{bmatrix} y_1(x) & \cdots y_i(x) & y_i(x+h) & \cdots & y_n(x+h) \end{bmatrix}.$$

Then

$$\det \Phi(x+h) - \Phi(x) = \det \Phi_0 - \det \Phi_n = \sum_{j=0}^{n-1} [\det \Phi_j(x) - \det \Phi_{j+1}(x)].$$

We then examine each individual difference. We use that the if all but one column is fixed, the determinant is linear in the one column that is allowed to vary

$$\begin{aligned}
&\det \Phi_j(x) - \det \Phi_{j+1} \\
&= \det \begin{bmatrix} y_1(x) & \cdots & y_j(x) & y_{j+1}(x+h) - y_{j+1}(x) & y_{j+2}(x+h) & \cdots y_n(x+h) \end{bmatrix}.
\end{aligned}$$

So,

$$\begin{aligned}
\Delta_j(x) &:= \lim_{h\to 0} \frac{\det \Phi_j(x) - \det \Phi_{j+1}}{h} \\
&= \det \begin{bmatrix} y_1(x) & \cdots & y_j(x) & y'_{j+1}(x) & y_{j+2}(x) & \cdots y_n(x) \end{bmatrix} \\
&= \det \begin{bmatrix} y_1(x) & \cdots & y_j(x) & A(x)y_{j+1}(x) & y_{j+2}(x) & \cdots y_n(x) \end{bmatrix} \\
&= \sum_{k=1}^{n} y_{k,j+1}(x) \det \begin{bmatrix} y_1(x) & \cdots & y_j(x) & a_k(x) & y_{j+2}(x) & \cdots y_n(x) \end{bmatrix}
\end{aligned}$$

where $a_k(x)$ is the kth column of $A(x)$. Therefore by Cramer's rule

$$\frac{\mathrm{d}}{\mathrm{d}x} \det \Phi(x) = \sum_{k=1}^{n} \sum_{j=1}^{n} y_{k,j}(x) \det \begin{bmatrix} y_1(x) & \cdots & y_{j-1}(x) & a_k(x) & y_{j+1}(x) & \cdots y_n(x) \end{bmatrix},$$

where $y_{k,j}$ is the kth entry of y_j. Then we note that

$$\det \begin{bmatrix} y_1(x) & \cdots & y_{j-1}(x) & a_k(x) & y_{j+1}(x) & \cdots y_n(x) \end{bmatrix} = e_j^T \operatorname{adj} \Phi(x) a_k(x)$$

Then we compute

$$\sum_{j=1}^{n} y_{k,j}(x) e_j^T \operatorname{adj} \Phi(x) a_k(x) = e_k^T \det \Phi(x) a_k(x).$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}x} \det \Phi(x) = \det \Phi(x) \sum_{k=1}^{n} A_{kk}(x) = \operatorname{tr} A(x) \det \Phi(x).$$

■

4.2 ▪ Connecting regular perturbations to Newton's method

Again, consider the boundary-value problem

$$\begin{cases} y''(x) + y(x) + \epsilon g(y(x)) = 0, \\ y(-1) = 0, \\ y(1) = 1, \end{cases}$$

When $\epsilon = 0$, we find our zeroth-order approximation $y_0(x)$ that solves

$$\begin{cases} y_0''(x) + y_0(x) = 0, \\ y_0(-1) = 1, \\ y_0(1) = 0. \end{cases}$$

Now, if we look to update the solution, supposing the true solution is given by

$$y(x) = y_0(x) + \delta y(x).$$

We would hope that $\delta y(x)$ should satisfy

$$\begin{cases} \delta y''(x) + \delta y(x) + \epsilon g(y_0(x) + \delta y(x)) = -y_0''(x) - y_0(x), \\ \delta y(-1) = 0, \\ \delta y(1) = 0, \end{cases}$$

But solving this is just as hard as solving our original problem. So we relax what we want δy to solve.

Supposing that $|\delta y| \ll 1$ we expand

$$g(y_0(x) + \delta y(x)) = g(y_0(x)) + g'(y_0(x))\delta y(x) + \cdots .$$

and we instead impose that

$$\begin{cases} \delta y''(x) + \delta y(x) + \epsilon g'(y_0(x))\delta y(x) = -y_0''(x) - y_0(x) - \epsilon g(y_0(x)), \\ \delta y(-1) = 0, \\ \delta y(1) = 0, \end{cases}$$

Note that if drop the term $\epsilon g'(y_0(x))\delta y(x)$, this amounts to the equation satisfied by $\epsilon y_1(x)$ above.

But without dropping this term, we can abstract this equation as

$$\mathcal{L}\delta y = F(y_0).$$

And Newton's method is simply the iteration

$$\begin{aligned} &\text{Solve: } \mathcal{L}\delta y = F(y_0), \\ &\text{Set: } y_0 \leftarrow y_0 + \delta y, \\ &\text{Repeat until convergence.} \end{aligned}$$

Remark 4.4. *Regular perturbation theory for this boundary-value problem, when fully justified, amounts to establishing an implicit function theorem for function spaces.*

4.3 ▪ Singular perturbations I: The WKB method

In this section, we consider the ODE:

$$\epsilon^2 y''(x) + g(x;\epsilon)y(x) = 0, \quad 0 < \epsilon \ll 1,$$

where, of course, y depends implicitly on ϵ, and we suppose that $g(x;\epsilon) = O(1)$. We say this problem singularly perturbed because if $\epsilon = 0$, one cannot impose any boundary conditions.

The Wentzel—Kramers—Brillouin (WKB) method mirrors what occurs for an irregular singular point in $y''(z) + f(z)y(z) = 0$ to generate an expansion. And recall that we went through some of this calculation for an irregular singular point at infinity. So, we will exchange our small parameter for a large one, setting

$$\lambda = \frac{1}{\epsilon}.$$

We write

$$\begin{cases} y''(x) + f(x;\lambda)y(x) = 0, \\ f(x;\lambda) \sim \lambda^2 \displaystyle\sum_{n=0}^{\infty} f_n(x)\lambda^{-n}. \end{cases}$$

One way to motivate the WKB method is to consider the simple case $f(x;\lambda) = \lambda^2 f_0$, where f_0 is a constant. Then

$$y(x) = \begin{cases} c_1\,\mathrm{e}^{\lambda \mathrm{i} f_0^{1/2} x} + c_2 \mathrm{i}\,\mathrm{e}^{-\lambda \mathrm{i} f_0^{1/2} x} & f_0 > 0, \\ c_1\,\mathrm{e}^{\lambda(-f_0)^{1/2} x} + c_2\,\mathrm{e}^{-\lambda(-f_0)^{1/2} x} & f_0 < 0. \end{cases}$$

And a degeneracy occurs for $f_0 = 0$. So, like in the case of an irregular singular point we are led to change our unknown:

$$y(x) = \mathrm{e}^{\phi(x)},$$

giving a Ricatti equation for $u(x) = \phi'(x)$:

$$u'(x) + u(x)^2 + f(x;\lambda) = 0.$$

We then suppose

$$u(x;\lambda) \sim \lambda \sum_{n=0}^{\infty} u_n(x) b_n(\lambda), \quad b_0(\lambda) = O(1),$$

for a yet-to-be determined asymptotic sequence $\{b_n(\lambda)\}$. Next, suppose we look for a valid approximation on some interval (α, β).

Substitute in our asymptotic expression,

$$\underbrace{\lambda \sum_{n=0}^{\infty} u_n'(x) b_n(\lambda)}_{(*)} + \underbrace{\lambda^2 u_0(x)^2 b_0(\lambda)^2 + 2\lambda^2 u_0(x) u_1(x) b_0(\lambda) b_1(\lambda) + \cdots}_{(**)}$$

$$+ \underbrace{\lambda^2 \sum_{n=0}^{\infty} f_n(x)\lambda^{-n}}_{(***)} = 0.$$

We now consider dominant balance.

- $(*)$ and $(**)$ balance: Need $\lambda b_0(\lambda) \approx \lambda^2 b_0(\lambda)^2$, and this implies $b_0(\lambda) = O(\lambda^{-1})$ and $(***)$ is dominant.
- $(*)$ and $(***)$ balance: $\lambda b_0(\lambda) \approx \lambda^2$, and this implies that $b_0(\lambda) \approx \lambda$ and $(**)$ is dominant.
- $(**)$ and $(***)$ balance: We find $\lambda^2 b_0(\lambda)^2 \approx \lambda^2$, $b_0(\lambda) = 1 + o(1)$ and $(*)$ is lower order.

So, we just choose $b_0(\lambda) = 1$ and find, at the leading order,

$$u_0(x)^2 + f_0(x) = 0,$$

$$u_0(x) = \begin{cases} \pm \mathrm{i} f_0(x)^{1/2} & f_0(x) > 0,\ x \in (\alpha,\beta), \\ \pm(-f_0(x))^{1/2} & f_0(x) < 0,\ x \in (\alpha,\beta), \\ ?? & \text{otherwise}. \end{cases}$$

We then go to the next order, with the equation

$$\lambda u_0'(x)b_0(\lambda) + 2\lambda^2 u_0(x)u_1(x)b_0(\lambda)b_1(\lambda) + \lambda f_1(x) = 0.$$

Using $b_0(\lambda) = 1$, this reduces to

$$\lambda u_0'(x) + 2\lambda^2 u_0(x)u_1(x)b_1(\lambda) + \lambda f_1(x) = 0.$$

For this to balance, we need to choose $b_1(\lambda) = O(\lambda^{-1})$ and just set $b_1(\lambda) = \lambda^{-1}$. Thus

$$u_1(x) = -\frac{1}{2u_0(x)}\left[u_0'(x) + f_1(x)\right].$$

And we then continue,

$$u_1'(x) + \lambda^2\left[u_0(x)u_2(x)b_0(\lambda)b_2(\lambda) + u_1(x)^2 b_1(\lambda)^2 + u_2(x)u_0(x)b_2(\lambda)b_0(\lambda)\right] + f_2(x) = 0.$$

We choose $b_2(\lambda) = \lambda^{-2}$ and find

$$\begin{aligned} u_1'(x) + \left[2u_0(x)u_2(x) + u_1(x)^2\right] + f_2(x) &= 0, \\ u_2(x) &= -\frac{1}{2u_0(x)}\left[u_1'(x) + u_1(x)^2 + f_2(x)\right]. \end{aligned}$$

The nth equation, using $b_n(\lambda) = \lambda^{-n}$, gives

$$u_n(z) = -\frac{1}{2u_0(x)}\left[u_{n-1}'(x) + \sum_{k=1}^{n-1} u_k(x)u_{n-k}(x) + f_n(x)\right].$$

WKB recap: Upon considering

$$y''(x) + f(x;\lambda)y(x) = 0, \quad y(x) = \mathrm{e}^{\phi(x)}, \quad u(x) = \phi'(x),$$

with

$$u(x) \sim \lambda\sum_{n=0}^{\infty} u_n(x)\lambda^{-n}, \quad \phi(x) \sim \lambda\sum_{n=0}^{\infty}\lambda^{-n}\int_{x_0}^{x} u_n(\xi)\mathrm{d}\xi.$$

We compute

$$\begin{aligned} \int_{x_0}^{x} u_0(\xi)\mathrm{d}\xi &= \begin{cases} \pm\mathrm{i}\int_{x_0}^{x} f_0(\xi)^{1/2}\mathrm{d}x & f_0(\xi) > 0, \\ \pm\int_{x_0}^{x}(-f_0(\xi))^{1/2}\mathrm{d}x & f_0(\xi) < 0, \end{cases} \\ \int_{x_0}^{x} u_1(x) &= -\frac{1}{2}\int_{x_0}^{x}\left[\frac{u_0'(\xi)}{u_0(\xi)} + \frac{f_1(\xi)}{u_0(\xi)}\right]\mathrm{d}\xi \\ &= -\frac{1}{2}\left[\log u_0(x) - \log u_0(x_0)\right] - \frac{1}{2}\int_{x_0}^{x}\frac{f_1(\xi)}{u_0(\xi)}\mathrm{d}\xi \\ \int_{x_0}^{x}\frac{f_1(\xi)}{u_0(\xi)}\mathrm{d}\xi &= \begin{cases} \mp\mathrm{i}\int_{x_0}^{x}\frac{f_1(\xi)}{(f_0(\xi))^{1/2}}\mathrm{d}x & f_0(\xi) > 0, \\ \pm\int_{x_0}^{x}\frac{f_1(\xi)}{(-f_0(\xi))^{1/2}}\mathrm{d}x & f_0(\xi) < 0, \end{cases} \end{aligned}$$

And we form two sets of asymptotic solutions based on the sign of $f_0(x)$:

$$y(x) = \begin{cases} y_{\mathrm{osc}}^{\pm}(x) := \frac{1}{|f_0(x)|^{1/4}}\exp\left(\pm\mathrm{i}\lambda\int_{x_0}^{x}(f_0(\xi))^{1/2}\mathrm{d}\xi \mp \frac{\mathrm{i}}{2}\int_{x_0}^{x}\frac{f_1(\xi)}{f_0(\xi)^{1/2}}\mathrm{d}\xi + O(\lambda^{-1})\right) \\ \qquad f_0(\xi) > 0, \\ y_{\exp}^{\pm}(x) = \frac{1}{|f_0(x)|^{1/4}}\exp\left(\pm\lambda\int_{x_0}^{x}(-f_0(\xi))^{1/2}\mathrm{d}\xi \pm \frac{1}{2}\int_{x_0}^{x}\frac{f_1(\xi)}{(-f_0(\xi))^{1/2}}\mathrm{d}\xi + O(\lambda^{-1})\right) \\ \qquad f_0(\xi) < 0. \end{cases}$$

Important 4.5. *These expansions can be rigorously justified, see [3]. But it is important to understand what this means: In any region (α, β) where $f_0(x)$ is bounded away from zero, there exists two solutions of $y''(x) + f(x;\lambda)y(x) = 0$ that have the described asymptotics.*

To explain the connection to specific initial conditions, we write the expansions in the general form

$$y_{\mathrm{osc/exp}}(x) \sim \exp\left(\lambda \sum_{n=0}^{\infty} \int_{x_0}^{x} u_n^{\pm}(\xi)\mathrm{d}\xi\right).$$

The results in [3] show that the oscillatory solutions have the behavior one might expect. If $y(x)$ is a solution of $y''(x) + f(x;\lambda)y(x) = 0$ with initial conditions

$$y(x_0) = 1, \quad y'(x_0) = \lambda \sum_{n=0}^{N} u_n^{\pm}(x_0) + O(\lambda^{-N}),$$

when $f_0(x_0) > 0$, then

$$y(x) = \exp\left(\lambda \sum_{n=0}^{N} \int_{x_0}^{x} u_n^{\pm}(\xi)\mathrm{d}\xi\right)\left(1 + O(\lambda^{-N})\right), \quad x \in (x_0 - \epsilon, x_0 + \epsilon).$$

But the exponential solutions have a directional character to them, in the sense that only exponentially growing solutions can be guaranteed to have accuracy. If $f_0(x_0) < 0$ and $y(x)$ has initial condition

$$y(x_0) = 1, \quad y'(x_0) = \lambda \sum_{n=0}^{N} u_n^{+}(x_0) + O(\lambda^{-N}),$$

then

$$y(x) = \exp\left(\lambda \sum_{n=0}^{N} \int_{x_0}^{x} u_n^{+}(\xi)\mathrm{d}\xi\right)\left(1 + O(\lambda^{-N})\right), \quad x \in [x_0, x_0 + \epsilon).$$

Similarly, if $f_0(x_0) < 0$ and $y(x)$ has initial condition

$$y(x_0) = 1, \quad y'(x_0) = \lambda \sum_{n=0}^{N} u_n^{-}(x_0) + O(\lambda^{-N}),$$

then

$$y(x) = \exp\left(\lambda \sum_{n=0}^{N} \int_{x_0}^{x} u_n^{-}(\xi)\mathrm{d}\xi\right)\left(1 + O(\lambda^{-N})\right), \quad x \in (x_0 - \epsilon, x_0].$$

The intuition for the directional character is as follows. Suppose we have a differential equation with two solutions. Suppose one solution grows exponentially with growth rate increasing as λ increases. And the other is $O(1)$, or decays. We wish to generate a solution that does not grow. If we make any amount of error, even of $O(\lambda^{-N})$ in selecting initial conditions, we will obtain a solution that grows exponentially — "most" solutions grow exponentially.

4.3.1 ▪ Turning points

Turning points for $y''(x)+f(x;\lambda)y(x)=0$ are points where $f_0(x)$ vanishes and the WKB asymptotics ceases to be valid. If the root is simple, as one crosses a turning point, the oscillatory solutions turn to exponential solutions, or vice versa. Understanding how a given solution passes through the turning point is called a connection problem.

Suppose x^* is a turning point and $f_0(x)<0$ for $x<x^*$. Then the rough idea is that for $x<x^*$

$$y(x) = a_+ y^+_{\text{exp}}(x) + a_- y^-_{\text{exp}}(x).$$

And for $x>x^*$

$$y(x) = b_+ y^+_{\text{osc}}(x) + b_- y^-_{\text{osc}}(x).$$

Note that when we write this, we are saying the exists bonefide solutions $y^{\pm}_{\text{exp/osc}}$ of $y''(x)+f(x;\lambda)y(x)=0$ that satisfy the same asymptotics as the expressions above.

We want

1. $T=T(\lambda)$ such that $\begin{bmatrix} a_+ \\ a_- \end{bmatrix} = T(\lambda)\begin{bmatrix} b_+ \\ b_- \end{bmatrix}$.
2. Asymptotics for $y(x)$ that are valid near x^*.

We note that the first problem is considered ill-posed as stated in the sense that solving it would require specific information about the specific solutions in the relation, not just the asymptotics of the solutions under consideration. To expand upon this further, we note that for each $N>0$,

$$y^{\pm}_{\text{osc/exp}}(x) = |f_0(x)|^{-1/4}\exp\left(\lambda \sum_{\substack{n=0\\ n\neq 1}}^{N} \lambda^{-n}\int_{x_*}^{x} u_n^{\pm}(\xi)\mathrm{d}\xi\right)(1+O(\lambda^{-N})).$$

This implies that $y^-_{\text{exp}}(x)$ grows exponentially for $x<x^*$, while $y^+_{\text{exp}}(x)$ decays. So, if we have the two relations

$$\begin{aligned} y^-_{\text{exp}}(x) &= c_+ y^+_{\text{osc}}(x) + c_- y^-_{\text{osc}}(x),\\ y^+_{\text{exp}}(x) &= d_+ y^+_{\text{osc}}(x) + d_- y^-_{\text{osc}}(x). \end{aligned}$$

Then for any constant σ

$$y^-_{\text{exp}}(x) + \sigma y^+_{\text{exp}}(x) = (c_+ + \sigma d_+) y^+_{\text{osc}}(x) + (c_- + \sigma d_-) y^-_{\text{osc}}(x).$$

But

$$y^-_{\text{exp}}(x) + \sigma y^+_{\text{exp}}(x) = y^-_{\text{exp}}(x)(1+O(\lambda^{-N})),$$

for any $N>0$. This implies that the coefficients $c_\pm$ cannot be determined by asymptotics alone. The same is not true of the coefficients $d_\pm$.

From [3], the validity of the asymptotic expressions can be extend to domains that shrink towards the turning point. More specifically,

$$y_{\text{exp}}^{\pm}(x) \text{ holds for } x < x^* - \lambda^{-p}, \quad 0 < p < 2/3,$$
$$y_{\text{osc}}^{\pm}(x) \text{ holds for } x > x^* - \lambda^{-p}, \quad 0 < p < 2/3.$$

Then, next, we condider our equation near the turning point

$$y''(x) + f(x;\lambda)y(x) = 0, \quad |x - x^*| = O(\lambda^{-2/3}).$$

Here we will just do a leading-order calculation. We have

$$y''(x) + \lambda^2 f_0(x)y(x) = O(\lambda)y(x).$$

Set $x = x_0 + \alpha t$, $Y(t) = y(x_0 + \alpha t)$ and compute

$$Y''(t) = \alpha^2 y''(x_0 + \alpha t),$$

giving

$$\alpha^{-2}Y(t) + \lambda^2 f_0(x_0 + \alpha t)Y(t) = O(\lambda)Y(t)$$
$$f(x_0 + \alpha t) = f'(x_0)\alpha t + O(\alpha^2 t^2).$$

Set $\nu = f'(x_0) > 0$ and we have

$$Y''(t) + \nu\lambda^2\alpha^3 tY(t) = O(\lambda^2\alpha^4 t^2 + \alpha^2\lambda)Y(t).$$

Now, we choose $\alpha = \nu^{-1/3}\lambda^{-2/3}$ so that, to leading order, we have

$$Y''(t) + tY(t) \approx 0.$$

This is nothing other than the Airy equation. The error term is

$$\lambda^2\alpha^4 t^2 = O(\alpha t^2) = O(\lambda^{-3/2}t^2),$$

which is small provided $t \ll \lambda^{3/4}$, or

$$|x - x^*| = |\alpha t| \ll \lambda^{-2/3}\lambda^{3/4} = \lambda^{5/12}.$$

The important thing here is we can take $t \ll \lambda^{\delta}$, $\delta > 0$ so that we have Airy asymptotics for

$$|x - x^*| \ll \lambda^{2/3+\delta}.$$

We have determined an overlap region where both asymptotics are valid and we can construct a uniformly valid expression. We will discuss this more thoroughly in the next section when we consider boundary layers.

4.3.2 ▪ A toy example

To demonstrate the solution of the connection problem, we consider the singularly perturbed Airy equation (after $x \mapsto -x$),

$$y''(x) + \lambda^2 xy(x) = 0,$$

which clearly has a turning point at $x = 0$, with exponential behavior for $x < 0$. The WKB method gives solutions

$$\begin{aligned} y^{\pm}_{\mathrm{osc}}(x) &= \frac{1}{|x|^{1/4}} \exp\left(\pm \mathrm{i}\lambda \int_0^x \xi^{1/2} \mathrm{d}\xi + O(\lambda^{-1})\right), \quad (x > 0), \\ y^{\pm}_{\mathrm{exp}}(x) &= \frac{1}{|x|^{1/4}} \exp\left(\pm \lambda \int_0^x (-\xi)^{1/2} \mathrm{d}\xi + O(\lambda^{-1})\right), \quad (x < 0). \end{aligned}$$

Our goal is to compute constants $c^{\pm}(\lambda)$ such that if

$$y(x) = y^{+}_{\mathrm{exp}}(x), \quad x < 0,$$

then

$$y(x) = c^{+}(\lambda) y^{+}_{\mathrm{osc}}(x) + c^{-}(\lambda) y^{-}_{\mathrm{osc}}(x), \quad x > 0.$$

Connection to solutions near the turning point

The solution near the turning point are solutions of

$$\begin{cases} Y''(t) + tY(t) = 0, \\ t = \lambda^{2/3} x. \end{cases}$$

The two linearly independent solutions can be taken to be $\mathrm{Ai}(-t), \mathrm{Bi}(-t)$. In a general problem, you would not have purely Ai, Bi, but some error terms.

The connection problem is then stated as

$$\begin{aligned} y^{+}_{\mathrm{exp}}(x) &= \frac{1}{|x|^{1/4}} \exp\left(\lambda \int_0^x (-\xi)^{1/2} \mathrm{d}\xi\right) (1 + O(\lambda^{-1})) \\ &= c_A(\lambda) \mathrm{Ai}(-\lambda^{2/3} x) + c_B(\lambda) \mathrm{Bi}(-\lambda^{2/3} x). \end{aligned}$$

Recall that if

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

then

$$c_1 = \frac{W(y, y_2)}{W(y_1, y_2)}, \quad c_2 = \frac{W(y, y_1)}{W(y_2, y_1)} = -\frac{W(y, y_1)}{W(y_1, y_2)}.$$

We need two expressions

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} y^{+}_{\mathrm{exp}}(x) &= \lambda |x|^{1/4} \exp\left(\lambda \int_0^x (-\xi)^{1/2} \mathrm{d}\xi\right) (1 + O(\lambda^{-1})), \\ W(\mathrm{Ai}(-\lambda^{2/3} \cdot), \mathrm{Bi}(-\lambda^{2/3} \cdot)) &= -\frac{\lambda^{2/3}}{\pi}. \end{aligned}$$

How does one derive this? Since the Wronskian is known to be independent of x by Abel's identity, one evaluates it as $x \to \pm\infty$. See [4]. Then we can compute

$$\begin{aligned} c_A(\lambda) = -\frac{\pi}{\lambda^{2/3}} \Bigg[&\frac{1}{|x|^{1/4}} \exp\left(\lambda \int_0^x (-\xi)^{1/2} \mathrm{d}\xi\right) (-\lambda^{2/3}) \mathrm{Bi}'(-\lambda^{2/3} x)(1 + O(\lambda^{-1})) \\ &- \lambda |x|^{1/4} \exp\left(\lambda \int_0^x (-\xi)^{1/2} \mathrm{d}\xi\right) \mathrm{Bi}(-\lambda^{2/3} x)(1 + O(\lambda^{-1})) \Bigg] \end{aligned}$$

Using the known asymptotics for Bi [4] we find, for $x < 0$

$$\frac{\pi}{\lambda^{2/3}}\lambda|x|^{1/4}\exp\left(\lambda\int_0^x(-\xi)^{1/2}\mathrm{d}\xi\right)\mathrm{Bi}(-\lambda^{2/3}x) = \sqrt{\pi}\lambda^{1/6}(1+o(1)),$$
$$-\frac{\pi}{\lambda^{2/3}}\frac{1}{|x|^{1/4}}\exp\left(\lambda\int_0^x(-\xi)^{1/2}\mathrm{d}\xi\right)(-\lambda^{2/3})\mathrm{Bi}'(-\lambda^{2/3}x) = \sqrt{\pi}\lambda^{1/6}(1+o(1)).$$

And one finds

$$\boxed{c_A(\lambda) = 2\sqrt{\pi}\lambda^{1/6}(1+o(1)).}$$

Then

$$\begin{aligned} c_B(\lambda) = \frac{\pi}{\lambda^{2/3}}\Bigg[&\frac{1}{|x|^{1/4}}\exp\left(\lambda\int_0^x(-\xi)^{1/2}\mathrm{d}\xi\right)(-\lambda^{2/3})\mathrm{Ai}'(-\lambda^{2/3}x)(1+O(\lambda^{-1})) \\ &-\lambda|x|^{1/4}\exp\left(\lambda\int_0^x(-\xi)^{1/2}\mathrm{d}\xi\right)\mathrm{Ai}(-\lambda^{2/3}x)(1+O(\lambda^{-1}))\ \Bigg]\end{aligned}$$

For $x < 0$ we get decay from essentially all terms, and we find

$$c_B(\lambda) = O(\lambda^{-N}),$$

for any $N > 0$.

Connection to the oscillatory region

We need to compute the coefficients $c_A^{\pm}$ in

$$\mathrm{Ai}(-\lambda^{2/3}x) = c_A^{+}(\lambda)y_{\mathrm{osc}}^{+}(x) + c_A^{-}(\lambda)y_{\mathrm{osc}}^{-}(x).$$

Typically, one has to compute with Wronskians here again, but we can read things off from asymptotics, for $x > 0$

$$\mathrm{Ai}(-\lambda^{2/3}x) = \frac{1}{\sqrt{\pi}}\frac{1}{x^{1/4}}\frac{1}{\lambda^{1/6}}\left[\cos\left(\lambda\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right)+O(\lambda^{-1})\right].$$

Because

$$\begin{aligned} y_{\mathrm{osc}}^{\pm} &= \frac{1}{|x|^{1/4}}\exp\left(\pm\mathrm{i}\lambda\int_0^x\xi^{1/2}\mathrm{d}\xi + O(\lambda^{-1})\right) \\ &= \frac{1}{|x|^{1/4}}\exp\left(\pm\mathrm{i}\lambda\frac{2}{3}x^{3/2}\right)(1+O(\lambda^{-1})).\end{aligned}$$

Then we can check that

$$\begin{aligned} &\frac{1}{2\sqrt{\pi}}\frac{1}{\lambda^{1/6}}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\,y_{\mathrm{osc}}^{+}(x) + \frac{1}{2\sqrt{\pi}}\frac{1}{\lambda^{1/6}}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\,y_{\mathrm{osc}}^{-}(x) \\ &\qquad = \frac{1}{\sqrt{\pi}}\frac{1}{x^{1/4}}\frac{1}{\lambda^{1/6}}\left[\cos\left(\lambda\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right)+O(\lambda^{-1})\right].\end{aligned}$$

Thus it suffices to choose

$$c_A^{\pm}(\lambda) = \frac{1}{2\sqrt{\pi}}\frac{1}{\lambda^{1/6}}\,\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{4}}\,.$$

So, the solution of the connection problem is given by

$$\boxed{c^{\pm}(\lambda) = c_A(\lambda)c_A^{\pm}(\lambda) = \mathrm{e}^{\mp\mathrm{i}\frac{\pi}{4}}(1+o(1)).}$$

While we performed this calculation in a special case, the result is actually universal and applies to connect through general turning points.

4.3.3 ▪ An application to the time-dependent Schrödinger equation

Consider

$$\mathrm{i}\hbar\Psi_t + \frac{\hbar^2}{2}\Psi_{xx} - V(x)\Psi = 0, \quad x, t \in \mathbb{R}.$$

In quantum mechanics, $|\Psi|^2$ represents a probability density, so one should have

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \mathrm{d}x = 1 < \infty.$$

We perform separation of variables and set $\Psi(x,t) = \psi(x)T(t)$

$$\begin{aligned} \mathrm{i}\hbar T'(t) + \frac{\hbar^2}{2}\psi''(x)T(t) - V(x)\psi(x)T(t) &= 0, \\ \mathrm{i}\hbar\frac{T'(t)}{T(t)} + \underbrace{\frac{\hbar^2}{2}\psi''(x) - V(x)\psi(x)}_{=\text{const.}} &= 0, \end{aligned}$$

So, letting this constant be E we have

$$-\frac{\hbar^2}{2}\psi''(x) + V(x)\psi(x) = E\psi(x). \tag{4.3}$$

And we are interested in solutions of this such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x < \infty,$$

and wish to know for which values of E we might have such a solution (i.e. eigenvalues).

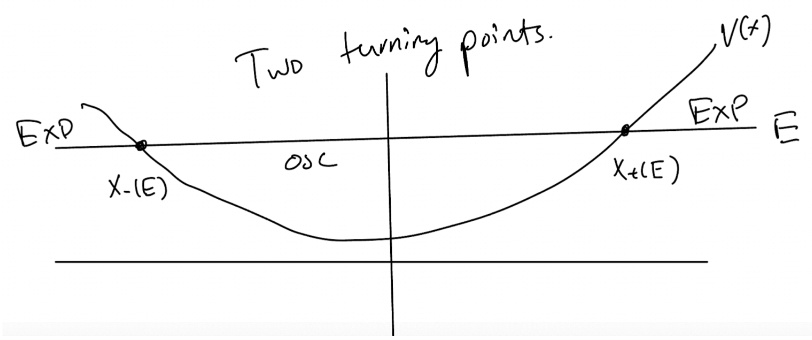


Figure 4.1: A schematic for two turning points.

We suppose $V(x)$ is a convex function such that for some values of E, the function

$$-V(x) + E$$

will have two distinct roots, see Figure 4.1. Label these roots $x_{\pm}(E)$. The WKB method, with $\hbar/\sqrt{2} = \lambda^{-1}$ gives the expansion of decaying solutions to the left and right of the interval $[x_-(E), x_+(E)]$:

$$\psi_{\mathrm{L}}(x) = \frac{1}{(V(x)-E)^{1/4}} \exp\left(\frac{\sqrt{2}}{\hbar}\int_{x_-(E)}^{x} (V(\xi)-E)^{1/2}\mathrm{d}\xi\right)(1+o(1)),$$
$$\psi_{\mathrm{R}}(x) = \frac{1}{(V(x)-E)^{1/4}} \exp\left(-\frac{\sqrt{2}}{\hbar}\int_{x_+(E)}^{x} (V(\xi)-E)^{1/2}\mathrm{d}\xi\right)(1+o(1)).$$

Between the turning points, we use the solution of the connection problem to find

$$\psi_{\mathrm{L}}(x) = \frac{2}{(E-V(x))^{1/4}}\left[\cos\left(\frac{\sqrt{2}}{\hbar}\int_{x_-(E)}^{x}(E-V(\xi))^{1/2}\mathrm{d}\xi - \frac{\pi}{4}\right) + o(1)\right],$$
$$\psi_{\mathrm{R}}(x) = \frac{2}{(E-V(x))^{1/4}}\left[\cos\left(\frac{\sqrt{2}}{\hbar}\int_{x_+(E)}^{x}(E-V(\xi))^{1/2}\mathrm{d}\xi + \frac{\pi}{4}\right) + o(1)\right].$$

We want these two solutions to be proportional, so we need that the arguments of the cosines differ by a multiple of π. It might seem we should enforce it for multiples of 2π, as $\cos(x)$ is 2π-periodic. But since $\cos(x+\pi) = -\cos(x)$, we can account for this by changing one of $\psi_{\mathrm{L/R}}$ by a factor (-1). So,

$$n\pi = \frac{\sqrt{2}}{\hbar}\int_{x_-(E)}^{x}(E-V(\xi))^{1/2}\mathrm{d}\xi - \frac{\pi}{4} - \frac{\sqrt{2}}{\hbar}\int_{x_+(E)}^{x}(E-V(\xi))^{1/2}\mathrm{d}\xi - \frac{\pi}{4}$$
$$\frac{2n+1}{2}\pi = \frac{\sqrt{2}}{\hbar}\int_{x_-(E)}^{x_+(E)}(E-V(\xi))^{1/2}\mathrm{d}\xi =: \frac{\Phi(E)}{\hbar}.$$

Miller makes this calculation using Wronskians, which is the more foolproof way. So, we obtain predictions for the eigenvalues by solving the phase-integral equation

$$\Phi(E_n) = \frac{2n+1}{2}\pi\hbar, \quad n = 0, 1, 2, \ldots.$$

This prediction turns out to be exact in the case of the harmonic oscillator $V(x) = x^2$.

This ideas also allow one to predict for which values of E one might expect eigenvalues/eigenfunctions.

4.4 ▪ Singular perturbations II: Boundary layers

Next, we turn to singularly perturbed boundary-value problems of the form

$$\begin{cases} \epsilon y''(x) + a(x;\epsilon)y'(x) + b(x;\epsilon)y(x) = 0, \\ y(0) = c, \\ y(1) = d. \end{cases}$$

We will suppose that the solution exists and is unique for all $\epsilon > 0$.

4.4.1 ▪ An explicit example

Consider

$$\begin{cases} \epsilon y''(x) - y'(x) - y(x) = 0, \\ y(0) = 1, \\ y(1) = 0. \end{cases} \tag{4.4}$$

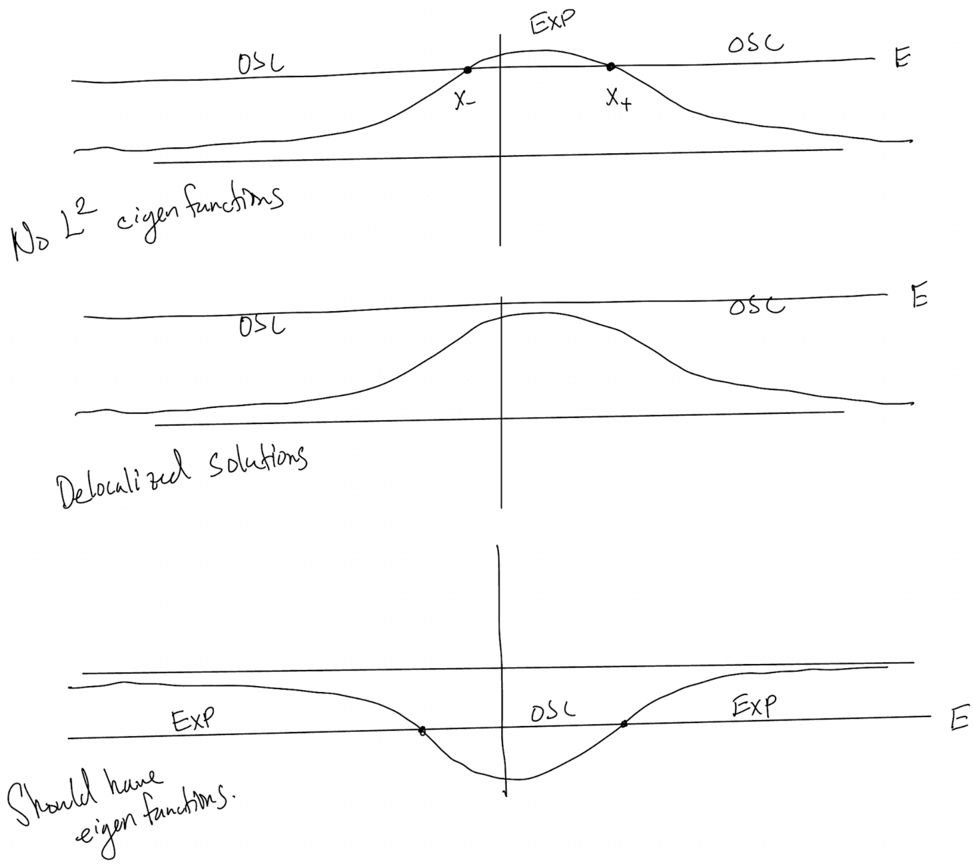


Figure 4.2: For different V and E where one might expect eigenfunctions.

We substitute $y(x) = \mathrm{e}^{mx}$ to find the characteristic equation $\epsilon m^2 - m - 1 = 0$,

$$m_\pm(\epsilon) = \frac{1 \pm \sqrt{1+4\epsilon}}{2\epsilon}.$$

The solution is then a linear combination

$$y(x) = A\,\mathrm{e}^{m_+(\epsilon)x} + B\,\mathrm{e}^{m_-(\epsilon)x}.$$

Upon enforcing the boundary condition, we find the system

$$\begin{aligned} A + B &= 1, \\ A\,\mathrm{e}^{m_+(\epsilon)} + B\,\mathrm{e}^{m_-(\epsilon)} &= 0, \end{aligned}$$

or

$$\begin{bmatrix} 1 & 1 \\ \mathrm{e}^{m_+(\epsilon)} & \mathrm{e}^{m_-(\epsilon)} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

By Cramer's rule, we have

$$A = \frac{\mathrm{e}^{m_-(\epsilon)}}{\mathrm{e}^{m_-(\epsilon)} - \mathrm{e}^{m_+(\epsilon)}}, \quad B = -\frac{\mathrm{e}^{m_+(\epsilon)}}{\mathrm{e}^{m_-(\epsilon)} - \mathrm{e}^{m_+(\epsilon)}}.$$

We plot the solution in Figure 4.3 for decreasing values of ϵ. And, we now do some calculations to see how this observed phenomena manifests itself. As $\epsilon \to 0$

$$\begin{aligned}
\sqrt{1+4\epsilon} &= 1 + 2\epsilon + O(\epsilon^2),\\
m_+(\epsilon) &= \frac{1}{\epsilon} + O(1),\\
m_-(\epsilon) &= -1 + O(\epsilon),\\
\mathrm{e}^{m_+(\epsilon)x} &= \mathrm{e}^{\frac{2x}{\epsilon}+O(1)},\\
\mathrm{e}^{m_-(\epsilon)x} &= \mathrm{e}^{-x+O(\epsilon x)}.
\end{aligned}$$

This allows us to compute the asymptotics of the coefficients A, B also

$$A = O(\mathrm{e}^{-1/\epsilon}), \quad B = 1 + O(\mathrm{e}^{-1/\epsilon}).$$

For $0 \le x \le 1-\delta$ we have

$$y(x) = O(\mathrm{e}^{-1/\epsilon+1/x}) + \mathrm{e}^{m_-(\epsilon)x}(1 + O(\mathrm{e}^{-1/\epsilon})) = \mathrm{e}^{-x} + o(1).$$

This is the "outer expansion", an ϵ independent (to leading-order), slowly varying solution. Now, for x close to the boundary $x=1$, the error terms that were dropped above cannot be assumed to be small. We write

$$\begin{aligned}
y(x) &= \mathrm{e}^{m_+(\epsilon)(x-1)} \frac{\mathrm{e}^{m_-(\epsilon)}}{\mathrm{e}^{m_-(\epsilon)-m_+(\epsilon)} - 1} - \mathrm{e}^{m_-(\epsilon)x} \frac{\mathrm{e}^{m_+(\epsilon)}}{\mathrm{e}^{m_-(\epsilon)} - \mathrm{e}^{m_+(\epsilon)}}\\
&= \mathrm{e}^{\frac{1}{\epsilon}(x-1)+O(x-1)} \frac{\mathrm{e}^{m_-(\epsilon)}}{\mathrm{e}^{m_-(\epsilon)-m_+(\epsilon)} - 1} + \mathrm{e}^{-x}(1 + O(\epsilon)),\\
&= -\,\mathrm{e}^{\frac{1}{\epsilon}(x-1)-1}(1 + O(x-1)) + \mathrm{e}^{-x}(1 + O(\epsilon)).
\end{aligned}$$

We define the inner variable to be

$$Z = \frac{x-1}{\epsilon} \le 0,$$

and find

$$Y(Z) := y(1+\epsilon Z) = -\,\mathrm{e}^{Z-1} + \mathrm{e}^{-1} + O(\epsilon).$$

This is the so-called inner expansion, inside the boundary layer.

This whole process, was fairly cumbersome. But the good news is that we can derive all of this directly from the differential equation, arguing on grounds of dominant balance.

Outer expansion: We suppose the solution is slowly varying, so that everything that is large compared to ϵ. And we need to drop one boundary condition

$$\begin{cases} \cancel{\epsilon y''(x)} - y'(x) - y(x) = 0,\\ y(0) = 1,\\ \cancel{y(1) = 0}. \end{cases} \implies \begin{cases} -y_{\text{out}}'(x) - y_{\text{out}}(x) = 0,\\ y_{\text{out}}(0) = 1. \end{cases}$$

This is easily solved $y_{\text{out}}(x) = \mathrm{e}^{-x}$.

```
In[261]:= mp[e_] := (1 + Sqrt[1 + 4 e]) / (2 e);
          mm[e_] := (1 - Sqrt[1 + 4 e]) / (2 e);
          A[e_] := (Exp[mm[e]]) / (Exp[mm[e]] - Exp[mp[e]]);
          B[e_] := (-Exp[mp[e]]) / (Exp[mm[e]] - Exp[mp[e]]);

In[265]:= y[x_, e_] := A[e] Exp[mp[e] x] + B[e] Exp[mm[e] x];

In[266]:= e = 1.0;
          Table[Plot[y[x, 1 / (4 j)], {x, 0, 1} , PlotRange → All], {j, 1, 10}] // Show
```

Out[267]=

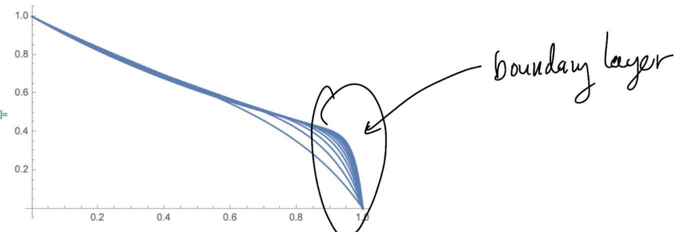


Figure 4.3: The boundary layer that is formed in (4.4).

Outer expansion: We first change variables using

$$\begin{aligned} Y(Z) &= y(1+\epsilon Z), \\ Y'(Z) &= \epsilon y'(1+\epsilon Z), \\ Y''(Z) &= \epsilon^2 y''(1+\epsilon Z). \end{aligned}$$

So that Y should solve

$$\epsilon^{-1} Y''(Z) - \epsilon^{-1} Y'(Z) - Y(Z) = 0,$$

The dominant balance is to have the first two terms cancel, giving

$$\begin{cases} Y_{\text{in}}''(Z) - Y_{\text{in}}'(Z) = 0, \\ Y_{\text{in}}(0) = 0. \end{cases}$$

The general solution is

$$Y_{\text{in}}(Z) = c_1 + c_2 \,\mathrm{e}^{Z}.$$

The boundary condition is enforced via $c_1 + c_2 = 0$. This does not determine both coefficients. How do we determine the other coefficient? Matching!

In the intermediate regime, we want

$$Y_{\text{in}}\left(\frac{x-1}{\epsilon}\right) \approx y_{\text{out}}(x), \quad 0 < c \ll \left|\frac{x-1}{\epsilon}\right| \ll \epsilon^{-1},$$

that is,

$$c_1 + c_2 \,\mathrm{e}^{\frac{x-1}{\epsilon}} \approx \mathrm{e}^{-x} \quad \Longrightarrow \quad c_1 = \mathrm{e}^{-1},$$

and

$$Y_{\text{in}}(Z) = \mathrm{e}^{-1} - \mathrm{e}^{Z-1}.$$

This recovers the calculation from above.

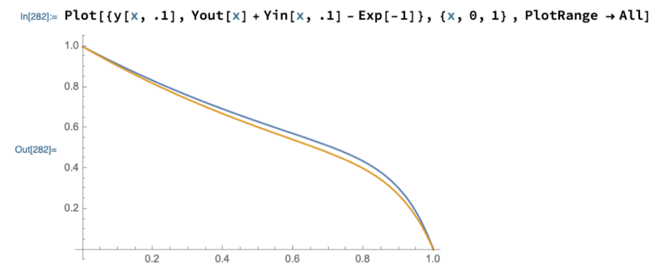


Figure 4.4: The comparison of the uniformly valid approximation with the true solution.

But there is an important question that remains to be answered. By using the explicit solution formula, we knew where to impose the boundary condition for the outer/inner solution. But what if we do not have an explicit solution? We now see what happens if we (mistakenly) look for a boundary layer near $x = 0$.

Set

$$Y(Z) = y(\epsilon Z),$$

giving

$$\begin{cases} Y''(Z) - Y'(Z) - \epsilon Y(Z) = 0, \\ Y(0) = 1. \end{cases}$$

In dropping the $O(\epsilon)$ terms we would get $Y(Z) \approx c_1 + c_2\,\mathrm{e}^Z$. Outside the supposed layer, we have $Z \gg 1$. So, $Y(Z)$ blows up unless $c_2 = 0$. This implies there is no layer after all.

4.4.2 ▪ Uniformly valid approximations

In the previous example, we might hope that we can construct an approximation over the entire interval by adding our two approximations,

$$y_{\text{out}}(x) + Y_{\text{in}}(Z).$$

But if each individual function approximates the solution in the intermediate (matching) region

$$0 < c \ll \left| \frac{x-1}{\epsilon} \right| \ll \epsilon^{-1},$$

we will be doubling up. But, in this region, we have

$$y_{\text{out}}(x) \approx Y_{\text{in}}(Z) \approx \mathrm{e}^{-1} =: y_{\text{match}}(x).$$

So, our uniformly valid approximation becomes

$$y(x) = y(x;\epsilon) \approx y_{\text{out}}(x) + Y_{\text{in}}(Z) - y_{\text{match}}(x).$$

See Figure 4.4 for a demonstration of its accuracy.

4.4.3 ▪ Higher-order corrections

We show how to construct higher-order approximations, still working with our given example, but the procedure is general. Suppose

$$y_{\text{out}}(x) \sim \sum_{n=0}^{\infty} \epsilon^n y_n(x).$$

At zeroth order we find,

$$\begin{cases} y_0'(x) + y_0(x) = 0, \\ y_0(0) = 1. \end{cases}$$

And at all subsequent orders

$$\begin{cases} y_{n-1}''(x) - y_n'(x) - y_n(x) = 0, \\ y_n(0) = 0, \end{cases}$$

where we note that $y_{n-1}''(x)$ should be treated as a known function.

At first, order, we have

$$y_1'(x) + y_1(x) = \mathrm{e}^{-x}.$$

Solving this gives the general solution

$$y_1(x) = c_1 \mathrm{e}^{-x} + x \mathrm{e}^{-x}.$$

The boundary condition implies $c_1 = 0$ and to first order

$$y_{\text{out}}(x) = \mathrm{e}^{-x} + \epsilon x \mathrm{e}^{-x} + O(\epsilon^2).$$

Then we begin work on the inner expansion, supposing that

$$y(1 + \epsilon z) \sim Y_{\text{in}}(Z) \sum_{n=0}^{\infty} \epsilon^n Y_n(Z).$$

For this example, we assume integer powers. But we will also see below that this assumption can be sometimes too restrictive. At zeroth order

$$\begin{cases} Y_0''(Z) - Y_0'(Z) = 0, \\ Y_0(0) = 0, \end{cases} \quad \Rightarrow Y_0(Z) = \mathrm{e}^{-1} - \mathrm{e}^{Z-1},$$

after matching.

At first order,

$$\begin{cases} Y_1''(Z) - Y_1'(Z) = Y_0(Z), \\ Y_1(0) = 0. \end{cases}$$

Using variation of parameters or the method of undetermined coefficients, we find

$$Y_1(Z) = \mathrm{e}^{-1}\left[-1 + \mathrm{e}^Z - Z - \mathrm{e}^Z Z - \tilde{c} + \mathrm{e}^Z \tilde{c}\right].$$

The constant $\tilde{c}$ is fixed by matching. But to complete this matching, we cannot just match at leading order like we did before, we have to introduce a new intermediate variable. Set

$$\omega = \frac{x-1}{\chi(\epsilon)}, \quad \epsilon \ll \chi(\epsilon) \ll 1,$$
$$Z = \frac{x-1}{\epsilon} \Rightarrow \omega = \frac{\epsilon}{\chi(\epsilon)} Z.$$

Note that these assumptions imply that $\omega < 0$, and $\chi(\epsilon)/\epsilon \gg 1$. We then expand

$$\begin{aligned}
y_{\text{out}}(1+\chi(\epsilon)\omega) &= \mathrm{e}^{-1-\chi(\epsilon)\omega} + \epsilon(1+\chi(\epsilon)\omega)\,\mathrm{e}^{-1-\chi(\epsilon)\omega} \\
&= \mathrm{e}^{-1}\left(1+\epsilon-\chi(\epsilon)\omega + O(\epsilon\chi(\epsilon)+\chi(\epsilon)^2)\right), \\
Y_{\text{in}}\left(\frac{\chi(\epsilon)}{\epsilon}\omega\right) &= \mathrm{e}^{-1}\left[1-\mathrm{e}^{\frac{\chi(\epsilon)}{\epsilon}\omega}\right] \\
&\quad + \epsilon\,\mathrm{e}^{-1}\left[-1+\mathrm{e}^{\frac{\chi(\epsilon)}{\epsilon}\omega} - \frac{\chi(\epsilon)}{\epsilon}\omega - \frac{\chi(\epsilon)}{\epsilon}\omega\,\mathrm{e}^{\frac{\chi(\epsilon)}{\epsilon}\omega} - \tilde{c} + \mathrm{e}^{\frac{\chi(\epsilon)}{\epsilon}\omega}\,\tilde{c}\right] \\
&= \mathrm{e}^{-1} - \epsilon\,\mathrm{e}^{-1} - \mathrm{e}^{-1}\,\chi(\epsilon)w - \epsilon\,\mathrm{e}^{-1}\,\tilde{c} + O\left(\mathrm{e}^{\frac{\chi(\epsilon)}{\epsilon}\omega}\right).
\end{aligned}$$

In order for the big-oh term in the first and last equation to truly be an error term, we need that

$$\epsilon \log \epsilon^{-1} \ll \chi(\epsilon)^2 \ll \epsilon.$$

Then

$$y_{\text{match}}(x) = \mathrm{e}^{-1}(1+\epsilon-\chi(\epsilon)\omega) = \mathrm{e}^{-1}(1+\epsilon-(x-1)).$$

In Figure 4.5 we show the improved accuracy.

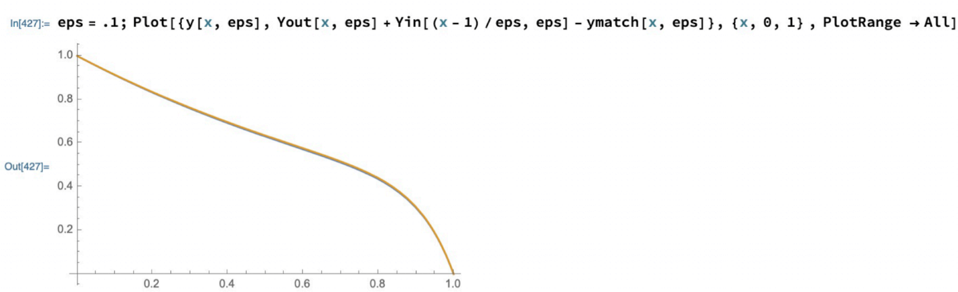


Figure 4.5: A comparison of the true solution with the $O(\epsilon)$ uniform approximation.

4.4.4 ▪ Generalizing: Non-constant coefficients

4.4.5 ▪ Example 1

This theory is really only explained by example, so you can argue that it is not indeed a theory at all. But now we consider

$$\begin{cases} \epsilon y''(x) + (1+x^2)y'(x) + xy(x) = 0, \\ y(0) = 0, \\ y(1) = 2. \end{cases}$$

Our coefficient of $y'(x)$ is $a(x) = 1 + x^2$:

- Since $a(0) > 0$ we expect a boundary layer near $x = 0$.
- Since $a(1) > 0$ we do not expect a boundary later at $x = 1$.

Outer problem: To leading order we have

$$\begin{cases} (1+x^2)y_0'(x) + xy_0(x) = 0, \\ y(1) = 2. \end{cases}$$

Inner problem: Set our inner variable $Z = x/\delta$, $Y(Z) = y(\delta Z)$. The problem in the rescaled variable is

$$\begin{cases} \frac{\epsilon}{\delta^2} Y''(Z) + \frac{1+\delta^2 Z^2}{\delta} Y'(Z) + \delta Z Y(Z) = 0, \\ Y(0) = 0, \end{cases}$$

In this case we can set $\delta = \epsilon$, the leading-order problem is

$$\begin{cases} Y_0''(Z) + Y_0'(Z) = 0, \\ Y_0(0) = 0, \end{cases}$$

We find

y_{out}: Cannot treat coefficients as constants, but the equation is first-order.

Y_{in}: To leading order, coefficients can be taken constent if $a(x;\epsilon) \neq 0$ at the boundary layer.

Example 2

Consider

$$\begin{cases} \epsilon y''(x) + ax^\alpha y'(x) + bx^\beta y(x) = 0, \\ y(0) = 1, \\ y(1) = 0. \end{cases}$$

Because the coefficient of the second term is non-negative, we expect a boundary layer at $x = 0$. We construct the first term for y_{out}:

$$\begin{cases} ax^\alpha y_0'(x) + bx^\beta y_0(x) = 0, \\ y_0(1) = 0. \end{cases}$$

Then we look for the inner solution, Y_{in}, setting $Z = x/\delta$:

$$\begin{cases} \frac{\epsilon}{\delta^2} Y''(Z) + aZ^{\alpha}\delta^{\alpha-1}Y'(Z) + bZ^{\beta}\delta^{\beta}Y(Z) = 0, \\ Y(0) = 1. \end{cases}$$

We need to consider some cases here.

$\alpha = 1$, $\beta \geq 0$: We have

$$\frac{\epsilon}{\delta^2} Y''(Z) + aZY'(Z) + bZ^{\beta}\delta^{\beta}Y(Z) = 0$$

We cannot have the first term drop out, so we need $\delta = \sqrt{\epsilon}$, giving

$$\begin{aligned} Y_0''(Z) + aZY_0'(Z) + bZ^{\beta}Y(Z) = 0, \quad \beta = 0, \\ Y_0''(Z) + aZY_0'(Z) = 0, \quad \beta > 0, \end{aligned}$$

$\alpha = 1$, $\beta = 0$: We find $\delta = \sqrt{\epsilon}$,

$$Y_0''(Z) + bZY_0(Z) = 0.$$

$\alpha = 1$, $\beta = 0$: We find $\epsilon/\delta^2 = \delta$ giving $\delta = \epsilon^{1/3}$, and

$$Y_0''(Z) + aZ^2Y_0'(Z) + bZY_0(Z) = 0.$$

The take away is as follows:

- For general coefficients, the fundamenta character depends mostly on $a(x;\epsilon)$
- One needs to determine the points x^* where boundary/internal layers exist.
- For leading order, use the first term in the Taylor series for $a(x;\epsilon)$ at $x = x^*$
- Beyond first order, the intermediate variables play a crucial role.

Example 3: Bender & Orszag pg. 431

Consider the problem

$$\begin{cases} \epsilon y''(x) + (1+x)y'(x) + y(x) = 0, \\ y(0) = 1, \\ y(1) = 1. \end{cases}$$

Because the coefficient of y' is positive, we expect a boundary layer at $x = 0$. And one can show that

$$y_{\text{out}}(x) \sim \frac{2}{1+x} + \epsilon\left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right] + \epsilon^2\left[\frac{6}{(1+x)^5} - \frac{1}{2(1+x)^3} - \frac{1}{4(1+x)}\right] + \cdots$$

The boundary layer variable is

$$Z = \frac{x}{\epsilon}.$$

And

$$Y_{\text{in}}(x) \sim 1 + A_0(\mathrm{e}^{-Z} - 1) + \epsilon\left[-Z + A_0\left(-\frac{1}{2}Z^2\,\mathrm{e}^{-Z} + Z\right) + A_1(\mathrm{e}^{-Z} - 1)\right]$$
$$+ \epsilon^2\left[Z^2 - 2Z + A_0\left(\frac{1}{8}Z^4\,\mathrm{e}^{-Z} - Z^2 + 2Z\right) + A_1\left(-\frac{1}{2}Z^2\,\mathrm{e}^{-Z} + Z\right) + A_2(\mathrm{e}^{-Z} - 1)\right] + \cdots.$$

In matching these solutions, we have $x \to 0$, so we can Taylor expand each individual term in the expansion for y_{out} at $x = 0$:

$$y_{\text{out}}(x) = 2 - 2x + 2x^2 + \epsilon\left(\frac{3}{2} - \frac{11}{2}x\right) + \frac{21}{4}\epsilon^2 + O(\epsilon^3 + \epsilon^2 x + \epsilon x^2 + x^3),$$

as $\epsilon, x \to 0$. Then for Y_{in} we consider the limit $\epsilon Z = x \to 0$, $Z \to \infty$,

$$\begin{aligned} Y_{\text{in}}(x) = 1 &- A_0 + \epsilon Z + \epsilon A_0 Z - \epsilon A_1 + \epsilon^2 Z^2 - 2\epsilon^2 Z - \epsilon^2 A_0 Z^2 \\ &+ 2\epsilon^2 A_0 Z + \epsilon^2 A_1 Z - \epsilon^2 A_2 + O(\epsilon^3 + \epsilon^3 Z + \epsilon^3 Z^2 + \epsilon^3 Z^3). \end{aligned}$$

To match and produce the requisite constants, we introduce the intermediate variable

$$\omega = \frac{x}{\chi(\epsilon)}, \quad Z = \frac{\chi(\epsilon)}{\epsilon}\omega,$$

and compute

$$\begin{aligned} y_{\text{out}}(x) &= 2 - 2\chi(\epsilon)\omega + 2\chi(\epsilon)^2\omega^2 + \frac{3}{2}\epsilon - \frac{11}{2}\epsilon\chi(\epsilon)\omega + \frac{21}{4}\epsilon^2 + O(\epsilon^3 + \epsilon^2\chi(\epsilon) + \epsilon\chi(\epsilon)^2 + \chi(\epsilon)^3), \\ Y_{\text{in}}(Z) &= 1 - A_0 + \chi(\epsilon)\omega(A_0 - 1) - \epsilon A_1 + \chi(\epsilon)^2\omega^2(1 - A_0) + 2\epsilon\chi(\epsilon)\omega(A_0 - 1 + A_1/2) - \epsilon^2 A_2 \\ &\quad + O(\epsilon^3 + \epsilon^2\chi(\epsilon) + \epsilon\chi(\epsilon)^2 + \chi(\epsilon)^3). \end{aligned}$$

From the error term in the last equation, we see that we should choose $\chi(\epsilon)^3 \ll \epsilon^2$ or $\epsilon \ll \chi(\epsilon) \ll \epsilon^{2/3}$. And we find

$$A_0 = -1, \quad A_1 = -\frac{3}{2}, \quad A_2 = -\frac{21}{4}. \tag{4.5}$$

4.5 ▪ Regular perturbations II: Method of multiple scales

The method of multiple scales[1] is a perturbation expansion method that is typically used on initial value problems as a means to capture/remove so-called secular terms, and extend time horizon over which the perturbation expansion is valid. Arguably, the most famous example where the method of multiple scales succeeds is the Duffing oscillator

$$\begin{cases} y''(x) + y(x) + \epsilon y(x)^3 = 0, \\ y(0) = 1, \\ y'(0) = 0. \end{cases} \tag{4.6}$$

A solution of this problem with $\epsilon = 0.1$ is shown in Figure 4.6.

The first thing one might try is the approach from above, where we seek

$$y(x) = y(x;\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n y_n(x).$$

[1] Also called "two-timing" by the late, great Bob O'Malley

```
ϵ = 0.1;
s = NDSolve[{y''[x] + y[x] + ϵ y[x]^3 == 0, y[0] == 1, y'[0] == 0}, y, {x, 0, 50}];
Plot[Evaluate[y[x] /. s], {x, 0, 50}, PlotRange → All]
```

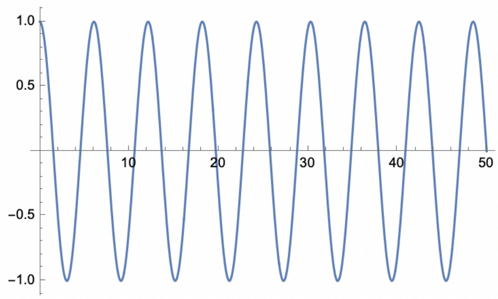


Figure 4.6: The Duffing oscillator with $\epsilon = 0.1$

giving

$$\begin{cases} y_0''(x) + y_0(x) = 0, \\ y_0(0) = 1, \\ y_0'(0) = 0. \end{cases}$$

And it follows that $y_0(x) = \cos(x)$. At the next order we have

$$\begin{cases} y_1''(x) + y_1(x) = -\cos^3(x), \\ y_0(0) = 0, \\ y_0'(0) = 0. \end{cases}$$

We need to simplify the right-hand side a bit here. So

$$\cos^3(x) = \left(\frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2}\right)^3 = \frac{1}{8}\left[\mathrm{e}^{3\mathrm{i}x} + 3\,\mathrm{e}^{\mathrm{i}x} + 3\,\mathrm{e}^{-\mathrm{i}x} + \mathrm{e}^{-3\mathrm{i}x}\right] = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x.$$

We apply variation of parameters with each of these right-hand side functions to construct particular solutions. We choose the linearly independent solutions to be $y_c(x) = \cos x, y_s(x) = \sin x$ so that

$$W(y_c, y_s) = \det\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} = 1.$$

Choosing $x_1 = x_2 = 0$ in the variation of parameters formula, gives that a particular solution of $y''(x) + y(x) = \frac{1}{4}\cos 3x$ is

$$y(x) = \frac{\sin x}{4}\int_0^x \cos(3\xi)\cos(\xi)\mathrm{d}\xi - \frac{\cos x}{4}\int_0^x \cos(3\xi)\sin(\xi)\mathrm{d}\xi.$$

Using the identities

$$2\cos(3\xi)\cos(\xi) = \cos(4\xi) + \cos(2\xi),$$
$$2\cos(3\xi)\sin(\xi) = \sin(4\xi) - \sin(2\xi),$$

we find

$$y_{p,1}(x) = \frac{\sin x}{8}\left[\frac{1}{4}\sin(4x) + \frac{1}{2}\sin(2x)\right] - \frac{\cos x}{8}\left[-\frac{1}{4}\cos(4x) + \frac{1}{2}\cos(2x) - \frac{1}{4}\right]$$

A solution of $y''(x) + y(x) = \frac{3}{4}\cos x$ can be guess (by undetermined coefficients) to be

$$y_{p,2}(x) = Ax\cos x + Bx\sin x \quad \Rightarrow A = 0, B = \frac{3}{8},$$

or

$$y_{p,2}(x) = \frac{3}{8}x\sin x.$$

This gives the perturbation expansion

$$y(x) = \cos x + \epsilon(y_{p,1}(x) + y_{p,2}(x)) + O(\epsilon^2).$$

This is fine, but if our goal is to track the behavior of the system over long time scales, we see that since $y_{p,2}(x)$ can grow as x, our expansion only remains valid if $\epsilon x \ll 1$. The term $y_{p,2}$ is called secular — it excites a resonant frequency.

The method of multiple scales, aims to fix this issue by introducing another scale $X = \epsilon x$ and we treat it as an independent variable, only connected to x via the relation $\mathrm{d}X = \epsilon \mathrm{d}x$. So, we seek a solution of the form

$$y(x) = Y_0(x, X) + \epsilon Y_1(x, X) + O(\epsilon^2).$$

We begin differentiating,

$$\begin{aligned} y'(x) &= \left(\frac{\partial Y_0}{\partial x} + \frac{\partial Y_0}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\right) + \epsilon\left(\frac{\partial Y_1}{\partial x} + \frac{\partial Y_1}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}x}\right) + O(\epsilon^2) \\ &= \frac{\partial Y_0}{\partial x} + \epsilon\left(\frac{\partial Y_0}{\partial X} + \frac{\partial Y_1}{\partial x}\right) + O(\epsilon^2). \end{aligned}$$

And then,

$$y''(x) = \frac{\partial^2 Y_0}{\partial x^2} + \epsilon\left(2\frac{\partial^2 Y_0}{\partial x \partial X} + \frac{\partial^2 Y_1}{\partial x^2}\right) + O(\epsilon^2).$$

At zeroth order we have

$$\frac{\partial^2 Y_0}{\partial x^2} + Y_0 = 0, \quad Y(0,0) = 0.$$

This gives

$$Y_0(x, X) = c(X)\cos x, \quad c(0) = 1.$$

But the calculations are often much easier if one works with complex valued things. We use the notation c.c. to refer to complex conjugate so that

$$a\,\mathrm{e}^{\mathrm{i}x} + c.c. = a\,\mathrm{e}^{\mathrm{i}x} + \bar{a}\,\mathrm{e}^{-\mathrm{i}x}.$$

So, we have, in redefining c

$$Y_0(x, X) = c(X)\,\mathrm{e}^{\mathrm{i}t} + c.c., \quad c(0) = 1/2.$$

At first order,

$$2\frac{\partial^2 Y_0}{\partial x \partial X} + \frac{\partial^2 Y_1}{\partial x^2} + Y_1 + Y_0^3 = 0.$$

Then we rewrite this equation as

$$\begin{aligned}\frac{\partial^2 Y_1}{\partial x^2} + Y_1 = &-2\mathrm{i}c'(X)\,\mathrm{e}^{\mathrm{i}t} + 2\mathrm{i}\bar{c}'(X)\,\mathrm{e}^{-\mathrm{i}t} \\ &- c(X)^3\,\mathrm{e}^{3\mathrm{i}t} - 3c(X)^2\bar{c}(X)\,\mathrm{e}^{\mathrm{i}t} - 3\bar{c}(X)^2 c(X)\,\mathrm{e}^{-\mathrm{i}t} - \bar{c}(X)^3\,\mathrm{e}^{-3\mathrm{i}t}.\end{aligned}$$

To eliminate the secular terms, we need to eliminate the occurance of $\mathrm{e}^{\pm \mathrm{i}t}$ on the right-hand side. So we need to solve

$$-2\mathrm{i}c'(X) - 3c(X)^2\bar{c}(X) = 0$$

It is natural to seek a solution of the form

$$c(X) = A\,\mathrm{e}^{\mathrm{i}\alpha X},$$

giving

$$2\alpha A\,\mathrm{e}^{\mathrm{i}\alpha X} - 3A^2\bar{A}\,\mathrm{e}^{\mathrm{i}\alpha X} = 0 \quad \Longrightarrow \alpha = \frac{3}{2}|A|^2.$$

From the initial condition $c(0) = 1/2$, we find $A = 1/2$ and $\alpha = \frac{3}{8}$. So, the approximate solution expression is

$$y(x) = \frac{1}{2}\,\mathrm{e}^{\mathrm{i}x + \mathrm{i}\frac{3}{8}\epsilon x} + c.c. + O(\epsilon) = \cos\left(x + \frac{3}{8}\epsilon x\right) + O(\epsilon).$$

And this solution will be valid for $t \ll \epsilon^{-2}$, but to see this one has to compute more terms.

Chapter 5

From asymptotics to new expressions

5.1 ▪ Riemann–Hilbert problems

Riemann–Hilbert problems (RHPs) are boundary-value problems in the complex plane for piecewise (or sectionally) analytic functions.

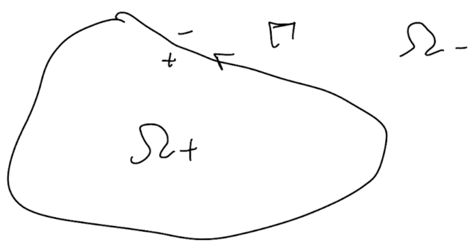


Figure 5.1: The contours and regions for an RHP.

Definition 5.1. *Let Γ be a smooth, simple closed curve that divides the complex plane into two regions $\Omega_\pm$. Suppose $g, f : \Gamma \to \mathbb{C}$ are smooth functions defined on Γ. Then a solution of the RHP (Γ, g, f) is an analytic function $\phi : \mathbb{C} \setminus \Gamma \to \mathbb{C}$ that satisfies*

$$\phi^+(s) = \phi^-(s)g(s) + f(s),$$

where

$$\phi^\pm(s) = \lim_{\substack{z \to s \\ z \in \Omega_\pm}} \phi(z),$$

are continuous on Γ (the limits are taken non-tangentially), and is of finite degree at ∞

$$\phi(z) = O(z^N), \quad z \to \infty, \quad \textit{for some} \quad N > 0.$$

See Figure 5.1 for a description of $\Gamma, \Omega_\pm$.

There are many generalizations of this definition. In particular, one often considers the case where f, g, ϕ are matrix- or vector-valued. For us, we will have $g = 1$ but will allow Γ to be an infinite line. In this case, we just use (Γ, f) to refer to the RHP. Our theoretical developments require Γ to be bounded, but the results extend to the case where Γ is unbounded with additional hypotheses [5].

Definition 5.2. *The solution of an RHP vanishes at infinity if*

$$\phi(z) = o(1), \quad z \to \infty, z \in \mathbb{C} \setminus \Gamma.$$

Lemma 5.3. *Suppose f is a Lipschitz function and $s_0 \in \Gamma$ then the limit*

$$\lim_{\epsilon \downarrow 0} \int_{\Gamma \setminus B(s_0,\epsilon)} \frac{f(z)}{z - s_0} \mathrm{d}z, \quad B(s_0, \epsilon) = \{z \mid |z - s_0| < \epsilon\},$$

exists.

The limit

$$\lim_{\epsilon \downarrow 0} \int_{\Gamma \setminus B(s_0,\epsilon)} \frac{f(z) - f(s_0)}{z - s_0} \mathrm{d}z,$$

exists because the function being integrated is integrable on Γ. And then it follows that the limit

$$\lim_{\epsilon \downarrow 0} \int_{\Gamma \setminus B(s_0,\epsilon)} \frac{f(s_0)}{z - s_0} \mathrm{d}z,$$

also exists, and the proof is complete.

Proof. ■

Proposition 5.4. *There is a unique solution of (Γ, f) that vanishes at infinity if Γ is continuously differentiable and f Lipschitz continuous .*

Proof. Consider the function

$$\phi(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(s)}{s - z} \mathrm{d}s, \quad z \in \mathbb{C} \setminus \Gamma.$$

Let $s_0 \in \Gamma$, and define $\Gamma_\epsilon(z_0) = \Gamma_\epsilon = \Gamma \cap \{z \mid |z - s_0| < \epsilon\}$, for ϵ small so that it matches Figure 5.2. Define

$$\phi_\epsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\epsilon} \frac{f(s)}{s - z} \mathrm{d}s.$$

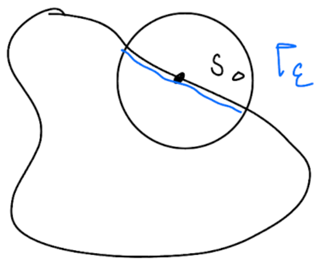


Figure 5.2: The contour Γ and subcontour Γ_ϵ.

It follows that for $\psi_\epsilon(z) := \phi(z) - \phi_\epsilon(z)$

$$\psi_\epsilon^+(s_0) = \psi_\epsilon^-(s_0),$$

because Δ_ϵ is analytic in a neighborhood of s_0. So, now let $\gamma : [\alpha, \beta] \to \Gamma_\epsilon$ be a continuously differentiable parameterization of Γ_ϵ with $\gamma(0) = s_0$. We have

$$\phi_\epsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_\alpha^\beta \frac{F(s)\gamma'(s)}{\gamma(s) - z} \mathrm{d}s, \quad F(s) = f(\gamma(s)).$$

We then have

$$\phi_\epsilon(z) = \underbrace{\frac{F(0)}{2\pi \mathrm{i}} \int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s) - z} \mathrm{d}s}_{\phi_\epsilon^0(z)} + \frac{1}{2\pi \mathrm{i}} \int_\alpha^\beta [F(s) - F(0)] \frac{\gamma'(s)}{\gamma(s) - z} \mathrm{d}s.$$

Then Cauchy's integral formula implies that

$$\phi_\epsilon^0(z_+) - \phi_\epsilon^0(z_-) = F(0) = f(s_0),$$

see Figure 5.3. It remains to show that the limit as we approach s_0 of

$$\phi_\epsilon(z) - \phi_\epsilon^0(z),$$

is the same regardless of the region from which we approach. We then have

$$|F(s) - F(0)| \leq L|\gamma(s) - \gamma(0)|,$$

where L is the Lipschitz constant for f. In Figure 5.4, we demonstrate how the law of sines implies

$$\left| \frac{\gamma(s) - \gamma(0)}{\gamma(s) - z_\pm} \right| = \left| \frac{\sin \delta}{\sin \phi} \right| \leq C.$$

So, we find that

$$\left| \frac{1}{2\pi \mathrm{i}} \int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s) - z} [F(s) - F(0)] \mathrm{d}s \right| \leq \frac{CL}{2\pi} |\Gamma_\epsilon|,$$

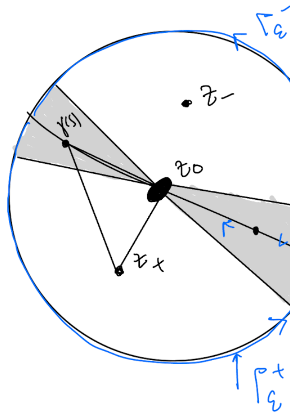


Figure 5.3: The points $z_{\pm}$, and Cauchy's integral formula applies by deforming the integrals to the boundaries $\Gamma_\epsilon^{\pm}$. A non-tangential limit requires us to avoid the shaded, gray regions.

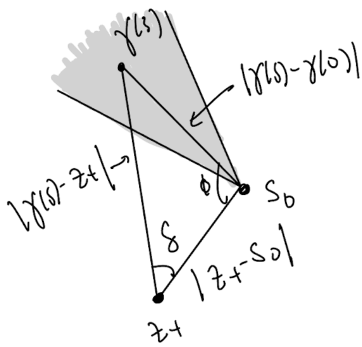


Figure 5.4: The use of the law of sines.

here $|\Gamma_\epsilon|$ is the arclength of the curve. If the following limit exists, we have

$$\begin{aligned}
\limsup_{\substack{z\to s_0\\ z\in\Omega_+}} \phi(z) &= \limsup_{\substack{z\to s_0\\ z\in\Omega_+}} \left[\phi_\epsilon^0(z) + [\phi_\epsilon(z) - \phi_\epsilon^0(z)] + \underbrace{[\phi(z) - \phi_\epsilon(z)]}_{\psi_\epsilon(z)}\right] \\
&\leq \lim_{\substack{z\to s_0\\ z\in\Omega_+}} \phi_\epsilon^0(z) + \psi_\epsilon(s_0) + \limsup_{\substack{z\to s_0\\ z\in\Omega_+}} [\phi_\epsilon(z) - \phi_\epsilon^0(z)]
\end{aligned}$$

And then in taking $\epsilon \to 0$, $|\Gamma_\epsilon| \to 0$ and we find, using Lemma 5.3,

$$\begin{aligned}
\limsup_{\substack{z\to s_0\\ z\in\Omega_+}} \phi(z) &= \limsup_{\epsilon\downarrow 0} \limsup_{\substack{z\to s_0\\ z\in\Omega_+}} \phi(z) \le \limsup_{\epsilon\downarrow 0} \lim_{\substack{z\to s_0\\ z\in\Omega_+}} \phi_\epsilon^0(z) + \limsup_{\epsilon\downarrow 0} \psi_\epsilon(s_0) \\
&= f(s_0) - \lim_{\epsilon\downarrow} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\epsilon^+} \frac{f(z_0)}{z - s_0} \mathrm{d}z + \lim_{\epsilon\downarrow 0} \psi_\epsilon(s_0) \\
&= \frac{f(s_0)}{2} + \lim_{\epsilon\downarrow 0} \psi_\epsilon(s_0).
\end{aligned}$$

Repeating with a $\liminf$ yelds the same result, implying

$$\lim_{\substack{z\to s_0\\ z\in\Omega_+}} \phi(z) = \frac{f(s_0)}{2} + \lim_{\epsilon\downarrow 0} \psi_\epsilon(s_0).$$

And similarly, we obtain

$$\lim_{\substack{z\to s_0\\ z\in\Omega_-}} \phi(z) = -\frac{f(s_0)}{2} + \lim_{\epsilon\downarrow 0} \psi_\epsilon(s_0).$$

It remains to show that boundary values $\psi^\pm$ are continuous. We do not do this here.

Now, suppose that $\psi(z)$ is another solution. Then it follows that $\Delta(z) = \phi(z) - \psi(z)$ is an entire function that decays at infinity, and hence $\Delta(z) = 0$. ∎

5.2 ▪ An alternate representation for erfc

Recall the complementary error function

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s.$$

There is an issue using this formula directly for numerical evaluation, for example, because the function

$$\mathrm{e}^{-s^2},$$

can oscillate and grow wildly. We have already shown that

$$\operatorname{erfc}(z) = \begin{cases} \frac{\mathrm{e}^{-z^2}}{z\sqrt{\pi}} + O\left(\frac{\mathrm{e}^{-z^2}}{z^2}\right) & \operatorname{Re} z \ge 0, \\ 2 + \frac{\mathrm{e}^{-z^2}}{z\sqrt{\pi}} + O\left(\frac{\mathrm{e}^{-z^2}}{z^2}\right) & \operatorname{Re} z \le 0. \end{cases}$$

We are going to write a new equation that is solved by the error term. Define

$$\phi(z) = \begin{cases} -\mathrm{e}^{z^2}\operatorname{erfc}(z) & \operatorname{Re} z > 0, \\ \mathrm{e}^{z^2}(2 - \operatorname{erfc}(z)) & \operatorname{Re} z < 0. \end{cases}$$

Then

$$\Phi(z) = O(z^{-1}), \quad z \in \mathbb{C} \setminus \mathrm{i}R.$$

This implies that ϕ is <u>the</u> solution of the RHP $(\mathrm{i}\mathbb{R}, f)$, $f(s) = 2\,\mathrm{e}^{s^2}$. So, we know that

$$\phi(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \frac{f(s)}{s-z} \mathrm{d}s,$$

giving the new representation

$$\operatorname{erfc}(z) = \begin{cases} -\,\mathrm{e}^{-z^2}\,\phi(z) & \operatorname{Re} z > 0, \\ 2 - \mathrm{e}^{-z^2}\,\phi(z) & \operatorname{Re} z < 0. \end{cases}$$

And computing $\phi(z)$ numerically is efficient [6]. And it turns out that the contour integral defining $\phi(z)$ can be deformed into, say, the left-half plane, and the asymptotic expansion we have derived for erfc can be shown to hold in an enlarged region.

5.3 ▪ A derivation of the Fourier transform

Consider the ODE for $\mu(x;k)$

$$\mu_x(x;k) - \mathrm{i}k\mu(x;k) = f(x), \quad x \in \mathbb{R}, k \in \mathbb{C}. \tag{5.1}$$

When we solve this equation, as we know, we will obtain solutions as integrals against solutions of the problem when $f \equiv 0$. From that, we hope to express $f(x)$, in turn, as an integral. The strategy can be summarized as follows:

1. Find the general solution of (5.1).
2. Analyze the analyticity of the solution.
3. Determine its behavior at infinity.
4. Formulate an RHP.
5. Recover $f(x)$ from the solution of the RHP.

Step 1: We use an integrating factor

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\mathrm{i}kx}\,\mu(x;k)\right) = f(x).$$

With the initial condition $\mu(x_0;k) = 0$, we find

$$\mu(x;k) = \mathrm{e}^{\mathrm{i}kx} \int_{x_0}^{x} \mathrm{e}^{-\mathrm{i}ky}\, f(y)\mathrm{d}y.$$

We think of this as the general solution because x_0 is not fixed. Really, one needs to add on an arbitrary multiple of a solution of the homogeneous equation to get the true general solution.

Step 2: We can write the solution as

$$\mu(x;k) = \int_{x_0}^{x} \mathrm{e}^{\mathrm{i}k(x-y)}\, f(y)\mathrm{d}y.$$

Note that the sign of $(x-y)$ is determined by $x_0 > x$ or $x_0 < x$. So, for $x_0 < x$, $x - y \geq 0$ and this implies the function is analytic, as a function of k in the upper-half plane. The reverse is true for $x < x_0$ where $\mu(x;k)$ is analytic as a function of k in the lower-half plane. So, define

$$\mu_1(x;k) = \int_{-\infty}^{x} \mathrm{e}^{\mathrm{i}k(x-y)}\, f(y)\mathrm{d}y, \quad \mu_2(x;k) = \int_{\infty}^{x} \mathrm{e}^{\mathrm{i}k(x-y)}\, f(y)\mathrm{d}y.$$

which are analytic in $\mathbb{C}^{\pm}$, respectively.

Step 3: To determine the asymptotic behavior, we use integration by parts: For $k \in \mathbb{C}^+$

$$\mu_1(x;k) = -\left.\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{\mathrm{i}k}\right|_{-\infty}^{x} + \frac{1}{\mathrm{i}k}\underbrace{\int_{-\infty}^{x} \mathrm{e}^{\mathrm{i}k(x-y)}\, f'(y)\mathrm{d}y}_{O(1) \quad k\to\infty} = O(k^{-1}), \quad k \to \infty.$$

A similar calculation follows for $\mu_2(x;k)$ for $\operatorname{Im} k < 0$.

Step 4: Consider for x fixed and $k \in \mathbb{R}$

$$\mu_1(x;k) - \mu_2(x;k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k(x-y)}\, f(y)\mathrm{d}y = \mathrm{e}^{\mathrm{i}kx}\, \hat{f}(k), \quad \hat{f}(k) = \int_{\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\, f(x)\mathrm{d}x.$$

So, define

$$\mu(k;x) = \begin{cases} \mu_1(x;k) & \operatorname{Im} k > 0, \\ \mu_2(x;k) & \operatorname{Im} k < 0. \end{cases}$$

Then, for each x fixed, $\mu(k;x)$ is the solution of the RHP $(\mathbb{R}, g)$, $g(k;x) = \mathrm{e}^{\mathrm{i}kx}\, \hat{f}(k)$.

Step 5: So, we know that

$$\mu(k;x) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}sx}\, \hat{f}(s)}{s-k}\mathrm{d}s.$$

We can also show, using integration by parts that $\frac{\mathrm{d}\mu_j}{\mathrm{d}x}(x;k) = o(1)$ as $k \to \infty$ for x fixed. Thus

$$\lim_{k\to\infty} -\mathrm{i}k\mu_1(x;k) = f(x).$$

And then using the new solution representation we have

$$\lim_{k\to\infty} \frac{-\mathrm{i}k}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}sx}\, \hat{f}(s)}{s-k}\mathrm{d}s = f(x).$$

To compute this limit we use the important fact.

Theorem 5.5 (Dominated convergence theorem). *Suppose $f_n(x) \to f(x)$ for each fixed x and there exists $g(x)$, $|f_n(x)| \le g(x)$ satisfying*

$$\int g(x)\mathrm{d}x < \infty.$$

Then

$$\lim_{n\to\infty} f_n(x)\mathrm{d}x = \int f(x)\mathrm{d}x.$$

Then we see that

$$\lim_{k\to\infty} \frac{-k}{s-k} = 1.$$

So, provided we take our limit above along the imaginary axis, for example and $\int_{-\infty}^{\infty} |\hat{f}(k)|\mathrm{d}k < \infty$, we have that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sx}\,\hat{f}(s)\mathrm{d}s.$$

This is the Fourier inversion theorem.

5.4 ▪ A transform for the time-dependent Schrödinger equation

TODO.

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