# 2D Lab Assignment Finite Elements

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## 1 Problem statement

We consider a square reservoir (a porous medium) with several wells where water is extracted. This is an important application in countries like Bangladesh where fresh water is extracted from the subarea. Far away from the reservoir, the water pressure is equal to the hydrostatic pressure. Since we are not able to consider an infinite domain, we use a mixed boundary condition which models the transfer of water over the boundary to locations far away. To this extent, we consider a square domain with length 2 in meter, that is $\Omega = (-1,1) \times (-1,1)$ with its boundary $\Gamma$. In this assignment, we consider a steady-state equilibrium determined by Darcy's Law for fluid velocity, given by

$$\mathbf{v} = -\frac{k}{\mu}\nabla p \tag{1}$$

where, $p, k, \mu$ and $\mathbf{v}$, respectively denote the fluid pressure, permeability of the porous medium, viscosity of water and the fluid flow velocity. Since we only consider a plane section of the reservoir in this assignment, the effect of gravity is not important. Next to Darcy's Law, we consider incompressibility, where the extraction wells are treated as point sinks (this assumption can be justified by the fact that the well diameter is much smaller than the dimensions of the porous medium), that extract at the same rate in each direction, hence

$$\nabla \cdot \mathbf{v} = -\sum_{p=1}^{n_{well}} Q_p \delta(\mathbf{x} - \mathbf{x}_p), \qquad (x,y) \in \Omega \tag{2}$$

where $Q_p$ denotes the water extraction rate by well $k$, which is located at $\mathbf{x}_p$. We use the convention $\mathbf{x} = (x,y)$ to represent the spatial coordinates. Further, $\delta(.)$ represents the Dirac Delta Distribution, which is characterised by

$$\begin{cases} \delta(\mathbf{x}) = 0, & \text{if } \mathbf{x} \neq 0 \\ \displaystyle\int_\Omega \delta(\mathbf{x}) d\Omega = 1, & \text{where } \Omega \text{ contains the origin} \end{cases} \tag{3}$$

Next to the above partial differential equation, we consider the boundary condition

$$\mathbf{v} \cdot \mathbf{n} = K(p - p^H), \qquad (x,y) \in \Gamma \tag{4}$$

Here $K$ denotes the transfer rate coefficient of the hormon between the boundary of the domain and its surroundings, and $p^H$ represents the hydrostatic pressure. For the computations, we use the following values:

| Symbol | Value | Unit |
|---|---|---|
| $Q_p$ | 50 | $m^2/s$ |
| $k$ | $10^{-7}$ | $m^2$ |
| $\mu$ | $1.002 \cdot 10^{-3}$ | $Pa \cdot s$ |
| $K$ | 10 | $m/s$ |
| $p^H$ | $10^6$ | $Pa$ |

We consider six wells, located at

$$\begin{cases} x_p = 0.6\cos\left(\frac{2\pi(p-1)}{5}\right) \\ \\ y_p = 0.6\sin\left(\frac{2\pi(p-1)}{5}\right) \end{cases}$$

$p \in \{1, \dots, 5\}$ and for $p = 6$, we have $x_6 = 0$ and $y_6 = 0$

## 2 Questions

In order to solve this problem, one needs to consider the following questions:

1. Give the partial differential equation and boundary condition in terms of the pressure $p$.

   **Solution:**

$$\begin{cases} -\frac{k}{\mu}\Delta p = \nabla[-\frac{k}{\mu}\nabla p] = \nabla\cdot\mathbf{v} = -\sum_{p=1}^{n_{well}} Q_p\delta(\mathbf{x}-\mathbf{x}_p) & (x,y)\in\Omega \\ -\frac{k}{\mu}\frac{\partial p}{\partial \mathbf{n}} = \mathbf{v}\cdot\mathbf{n} = K(p-p^H) & (x,y)\in\Gamma \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} -\frac{k}{\mu}\Delta p = -\sum_{p=1}^{n_{well}} Q_p\delta(\mathbf{x}-\mathbf{x}_p) & (x,y)\in\Omega \\ \frac{k}{\mu}\frac{\partial p}{\partial \mathbf{n}} + Kp = Kp^H & (x,y)\in\Gamma \end{cases}$$

2. Give the weak formulation of the problem (partial differential equation + boundary condition).

   **Solution:**
   Assume that $\varphi \in H^1(\Omega)$ is a test function

$$\begin{aligned} & -\frac{k}{\mu}\Delta p = -\sum_{i=1}^{n_{well}} Q_p\delta(\mathbf{x}-\mathbf{x}_i) \\ \Rightarrow & -\int_\Omega \frac{k}{\mu}\varphi\Delta p d\Omega = \int_\Omega -\varphi\sum_{i=1}^{n_{well}} Q_p\delta(\mathbf{x}-\mathbf{x}_i)d\Omega \\ \Rightarrow & -\int_\Omega \frac{k}{\mu}\nabla\cdot(\varphi\nabla\cdot p) - \frac{k}{\mu}\nabla\varphi\nabla p d\Omega = \int_\Omega -\varphi\sum_{i=1}^{n_{well}} Q_p\delta(\mathbf{x}-\mathbf{x}_i)d\Omega \\ \Rightarrow & -\int_\Gamma \frac{k}{\mu}\varphi\nabla p\cdot\mathbf{n}d\Gamma + \frac{k}{\mu}\int_\Omega \nabla\varphi\nabla p d\Omega = -Q_p\sum_{i=1}^{n_{well}}\varphi(\mathbf{x}_i) \\ \Rightarrow & \int_\Gamma \varphi K(p-p^H)d\Gamma + \frac{k}{\mu}\int_\Omega \nabla\varphi\nabla p d\Omega = -Q_p\sum_{i=1}^{n_{well}}\varphi(\mathbf{x}_i) \\ \Rightarrow & \underline{K\int_\Gamma \varphi p d\Gamma} + \frac{k}{\mu}\int_\Omega \nabla\varphi\nabla p d\Omega = -Q_p\sum_{i=1}^{n_{well}}\varphi(\mathbf{x}_i) + \underline{Kp^H\int_\Gamma \varphi d\Gamma} \end{aligned}$$

   There are only <u>natural</u> boundary conditions.
   Therefore the weak formulation is:

   Find $p \in H^1(\Omega)$ such that: $K\int_\Gamma \varphi p d\Gamma + \frac{k}{\mu}\int_\Omega \nabla\varphi\nabla p d\Omega = -Q_p\sum_{i=1}^{n_{well}}\varphi(\mathbf{x}_i) + Kp^H\int_\Gamma \varphi d\Gamma$, with $\varphi \in H^1(\Omega)$.

3. Give the Galerkin equations (the system of linear equations).

**Solution:**
Assume that $p \approx p^n = \sum_{j=1}^{n} c_j \varphi_j$, with $c_j \in \mathbb{R}$ and $\varphi_j \in H^1(\Omega)$, we now want:

$$K \int_\Gamma \varphi \sum_{j=1}^{n} c_j \varphi_j d\Gamma + \frac{k}{\mu} \int_\Omega \nabla\varphi \nabla \sum_{j=1}^{n} c_j \varphi_j d\Omega = -Q_p \sum_{i=1}^{n_{well}} \varphi(\mathbf{x}_i) + K p^H \int_\Gamma \varphi d\Gamma$$

$$\sum_{j=1}^{n} c_j \left( K \int_\Gamma \varphi\varphi_j d\Gamma + \frac{k}{\mu} \int_\Omega \nabla\varphi\nabla\varphi_j d\Omega \right) = -Q_p \sum_{i=1}^{n_{well}} \varphi(\mathbf{x}_i) + K p^H \int_\Gamma \varphi d\Gamma$$

Because this holds $\forall \varphi \in H^1(\Omega)$, it will also hold for a $\varphi_i$ which is used as a basis when making $p^n$:

$$\sum_{j=1}^{n} c_j \left( K \int_\Gamma \varphi_i\varphi_j d\Gamma + \frac{k}{\mu} \int_\Omega \nabla\varphi_i\nabla\varphi_j d\Omega \right) = -Q_p \sum_{l=1}^{n_{well}} \varphi_i(\mathbf{x}_l) + K p^H \int_\Gamma \varphi_i d\Gamma$$

We thus define the following matrices: $S_{ij} = K \int_\Gamma \varphi_i\varphi_j d\Gamma + \frac{k}{\mu} \int_\Omega \nabla\varphi_i\nabla\varphi_j d\Omega$, $\underline{p_j} = c_j$ and $\underline{f_i} = -Q_p \sum_{i=1}^{n_{well}} \varphi_i(\mathbf{x}_i) + K p^H \int_\Gamma \varphi_i d\Gamma \quad \forall i,j \in \{1,\dots,n\}$
The Galerkin equations then become:

$$S\underline{p} = \underline{f}$$

4. Give the element matrix and element vector for the internal elements. Distinguish between cases where the point sink lies inside or outside the considered element.

**Solution:**
Suppose we use triangular elements $e_k$ with vertices $(\mathbf{x}_{k1}, \mathbf{x}_{k2}, \mathbf{x}_{k3})$. We can define parts of 3 test functions inside $e_k$ as: $\varphi_i|_{e_k} := \varphi_{ki} = \alpha_{ki} + \beta_{ki}x + \gamma_{ki}y$, with the condition that $\varphi_{ki}(\mathbf{x}_{kj}) = \delta_{ij}$ $(i,j \in \{1,2,3\})$.
(Note that $\nabla\varphi_{ki} = \begin{bmatrix}\beta_{ki} & \gamma_{ki}\end{bmatrix}^T$ inside the triangle)

Suppose now that we only look at internal points, thus $\Gamma \cap e_k = \emptyset$. This implicates that: $K \int_{e_k} \varphi_i\varphi_j d\Gamma = 0$ and $Kp^H \int_{e_k} \varphi_i d\Gamma = 0$.
We now distinguish between the cases when a (or multiple) point sink lies inside or outside the considered element ($\Delta = |e_k|$, the area of the triangle):

inside: $S_{ij}^{e_k} = \frac{k}{\mu} \int_{e_k} \nabla\varphi_{ki} \cdot \nabla\varphi_{kj} d\Omega = \frac{k}{\mu} \int_{e_k} \begin{bmatrix}\beta_{ki} & \gamma_{ki}\end{bmatrix} \begin{bmatrix}\beta_{kj} \\ \gamma_{kj}\end{bmatrix} d\Omega = \frac{k\Delta}{\mu}(\beta_i\beta_j + \gamma_i\gamma_j)$ for $i,j \in \{1,2,3\}$
And $\underline{f}_i^{e_k} = -Q_p \sum_{l=1}^{n_{wellsin\Delta}} \varphi_{ki}(x_l)$, $\quad i \in \{1,2,3\}$ ($n_{wellsin\Delta}$ depends on how many well points are in the triangle $e_k$)

So:

$$S^{e_k} = \frac{k\Delta}{\mu} \begin{bmatrix} \beta_{k1}^2 + \gamma_{k1}^2 & \beta_{k1}\beta_{k2} + \gamma_{k1}\gamma_{k2} & \beta_{k1}\beta_{k3} + \gamma_{k1}\gamma_{k3} \\ \beta_{k1}\beta_{k2} + \gamma_{k1}\gamma_{k2} & \beta_{k2}^2 + \gamma_{k2}^2 & \beta_{k2}\beta_{k3} + \gamma_{k2}\gamma_{k3} \\ \beta_{k1}\beta_{k3} + \gamma_{k1}\gamma_{k3} & \beta_{k2}\beta_{k3} + \gamma_{k2}\gamma_{k3} & \beta_{k3}^2 + \gamma_{k3}^2 \end{bmatrix} \qquad \underline{f}^{e_k} = \begin{bmatrix} -Q_p \sum_{l=1}^{n_{wellsin\Delta}} \varphi_{k1}(x_l) \\ -Q_p \sum_{l=1}^{n_{wellsin\Delta}} \varphi_{k2}(x_l) \\ -Q_p \sum_{l=1}^{n_{wellsin\Delta}} \varphi_{k3}(x_l) \end{bmatrix}$$

outside: $S^{e_k}$ stays the same and $\underline{f}_i^{e_k} = 0$ for $i \in \{1,2,3\}$, thus:

$$\underline{f}^{e_k} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

5. Give the element matrix and element vector for the boundary elements.

**Solution:**
We don't distinguish between the cases as in exercise 4, as there are not on the boundary. We will only calculate $K \int_{e_k} \varphi_i\varphi_j d\Gamma$ and $Kp^H \int_{e_k} \varphi_i d\Gamma$ for the lines on the border and we will be using the theorem of Holand and Bell (we will denote this by using $=^*$)

$x_{k1}$ and $x_{k2}$ are on the boundary:

$K\int_{e_k}\varphi_{k1}^2 d\Gamma =^* K\frac{\|x_{k1}-x_{k2}\|2!0!}{(1+2+0)!} = \frac{\|x_{k1}-x_{k2}\|}{3} = K\int_{e_k}\varphi_{k2}^2 d\Gamma$
$K\int_{e_k}\varphi_{k1}\varphi_{k2} d\Gamma =^* K\frac{\|x_{k1}-x_{k2}\|}{6} = K\int_{e_k}\varphi_{k2}\varphi_{k1} d\Gamma$
$\underline{f}_1^{e_k} = Kp^H\int_{e_k}\varphi_{k1} d\Gamma =^* Kp^H\frac{\|x_{k1}-x_{k2}\|}{2} = \underline{f}_2^{e_k}$ Thus:

$$S^{e_k} = K\|x_{k1}-x_{k2}\|\begin{bmatrix}\frac{1}{3} & \frac{1}{6} & 0\\ \frac{1}{6} & \frac{1}{3} & 0\\ 0 & 0 & 0\end{bmatrix} + \frac{k\Delta}{\mu}\begin{bmatrix}\beta_{k1}^2+\gamma_{k1}^2 & \beta_1\beta_2+\gamma_1\gamma_2 & \beta_1\beta_3+\gamma_1\gamma_3\\ \beta_1\beta_2+\gamma_1\gamma_2 & \beta_{k2}^2+\gamma_{k2}^2 & \beta_2\beta_3+\gamma_2\gamma_3\\ \beta_1\beta_3+\gamma_1\gamma_3 & \beta_2\beta_3+\gamma_2\gamma_3 & \beta_{k3}^2+\gamma_{k3}^2\end{bmatrix} \qquad \underline{f}^{e_k} = Kp^H\|x_{k1}-x_{k2}\|\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix}$$

$x_{k1}$ and $x_{k3}$ are on the boundary:

$$S^{e_k} = K\|x_{k1}-x_{k3}\|\begin{bmatrix}\frac{1}{3} & 0 & \frac{1}{6}\\ 0 & 0 & 0\\ \frac{1}{6} & 0 & \frac{1}{3}\end{bmatrix} + \frac{k\Delta}{\mu}\begin{bmatrix}\beta_{k1}^2+\gamma_{k1}^2 & \beta_1\beta_2+\gamma_1\gamma_2 & \beta_1\beta_3+\gamma_1\gamma_3\\ \beta_1\beta_2+\gamma_1\gamma_2 & \beta_{k2}^2+\gamma_{k2}^2 & \beta_2\beta_3+\gamma_2\gamma_3\\ \beta_1\beta_3+\gamma_1\gamma_3 & \beta_2\beta_3+\gamma_2\gamma_3 & \beta_{k3}^2+\gamma_{k3}^2\end{bmatrix} \qquad \underline{f}^{e_k} = Kp^H\|x_{k1}-x_{k3}\|\begin{bmatrix}\frac{1}{2}\\ 0\\ \frac{1}{2}\end{bmatrix}$$

$x_{k2}$ and $x_{k3}$ are on the boundary:

$$S^{e_k} = K\|x_{k2}-x_{k3}\|\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{1}{3} & \frac{1}{6}\\ 0 & \frac{1}{6} & \frac{1}{3}\end{bmatrix} + \frac{k\Delta}{\mu}\begin{bmatrix}\beta_{k1}^2+\gamma_{k1}^2 & \beta_1\beta_2+\gamma_1\gamma_2 & \beta_1\beta_3+\gamma_1\gamma_3\\ \beta_1\beta_2+\gamma_1\gamma_2 & \beta_{k2}^2+\gamma_{k2}^2 & \beta_2\beta_3+\gamma_2\gamma_3\\ \beta_1\beta_3+\gamma_1\gamma_3 & \beta_2\beta_3+\gamma_2\gamma_3 & \beta_{k3}^2+\gamma_{k3}^2\end{bmatrix} \qquad \underline{f}^{e_k} = Kp^H\|x_{k2}-x_{k3}\|\begin{bmatrix}0\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

In the code, we will first treat the points as internal points and once that's done, we will update $S^{e_k}$ over the boundary (line). We will thus always be defining

$$S_{bd}^{e_k} = \frac{K\|e_k^{bd}\|_2}{3}\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix} \qquad \underline{f}_{bd}^{e_k} = Kp^H\|e_k^{bd}\|_2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

6. In order to solve the problem, you need to determine whether each internal element (triangle) contains a cell. We will determine whether cell with index $p$, with position $\mathbf{x}_p$, is in the element $e_k$ with vertices $\mathbf{x}_{k1}, \mathbf{x}_{k2}$ and $\mathbf{x}_{k3}$. We do so by testing the following criterion:

$$|\Delta(\mathbf{x}_p,\mathbf{x}_{k2},\mathbf{x}_{k3})|+|\Delta(\mathbf{x}_{k1},\mathbf{x}_p,\mathbf{x}_{k3})|+|\Delta(\mathbf{x}_{k1},\mathbf{x}_{k2},\mathbf{x}_p)|: \begin{cases} = |e_k|, & \mathbf{x}_p \in \overline{e}_k\\ > |e_k|, & \mathbf{x}_p \notin \overline{e}_k\end{cases} \tag{5}$$

Here $\Delta(\mathbf{x}_p,\mathbf{x}_q,\mathbf{x}_r)$ denotes the triangle with vertices $\mathbf{x}_p, \mathbf{x}_q$ and $\mathbf{x}_r$, and $|\Delta(\mathbf{x}_p,\mathbf{x}_q,\mathbf{x}_r)|$ denotes its area. Further, $e_k = \Delta(\mathbf{x}_{k1},\mathbf{x}_{k2},\mathbf{x}_{k3})$ represents the triangular element $k$ with vertices $\mathbf{x}_{k1}, \mathbf{x}_{k2}$ and $\mathbf{x}_{k3}$ and $\overline{e}_k$ includes the boundaries of element $e_k$. Express the area of the triangles in terms of the nodal points. To implement the above criterion whether a cell is within an element, use a tolerance of eps in matlab because of possible rounding errors.

**Solution:**
It is common knowledge that the determinant of 2 non-parallel vectors in a matrix form equals the area of a parallelogram made of these two vectors as sides. The area of a triangle with those two vectors as sides is half of the area of the parallelogram.
Thus: Suppose we have a triangle with vertices $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, then the area of the parallelogram equals:

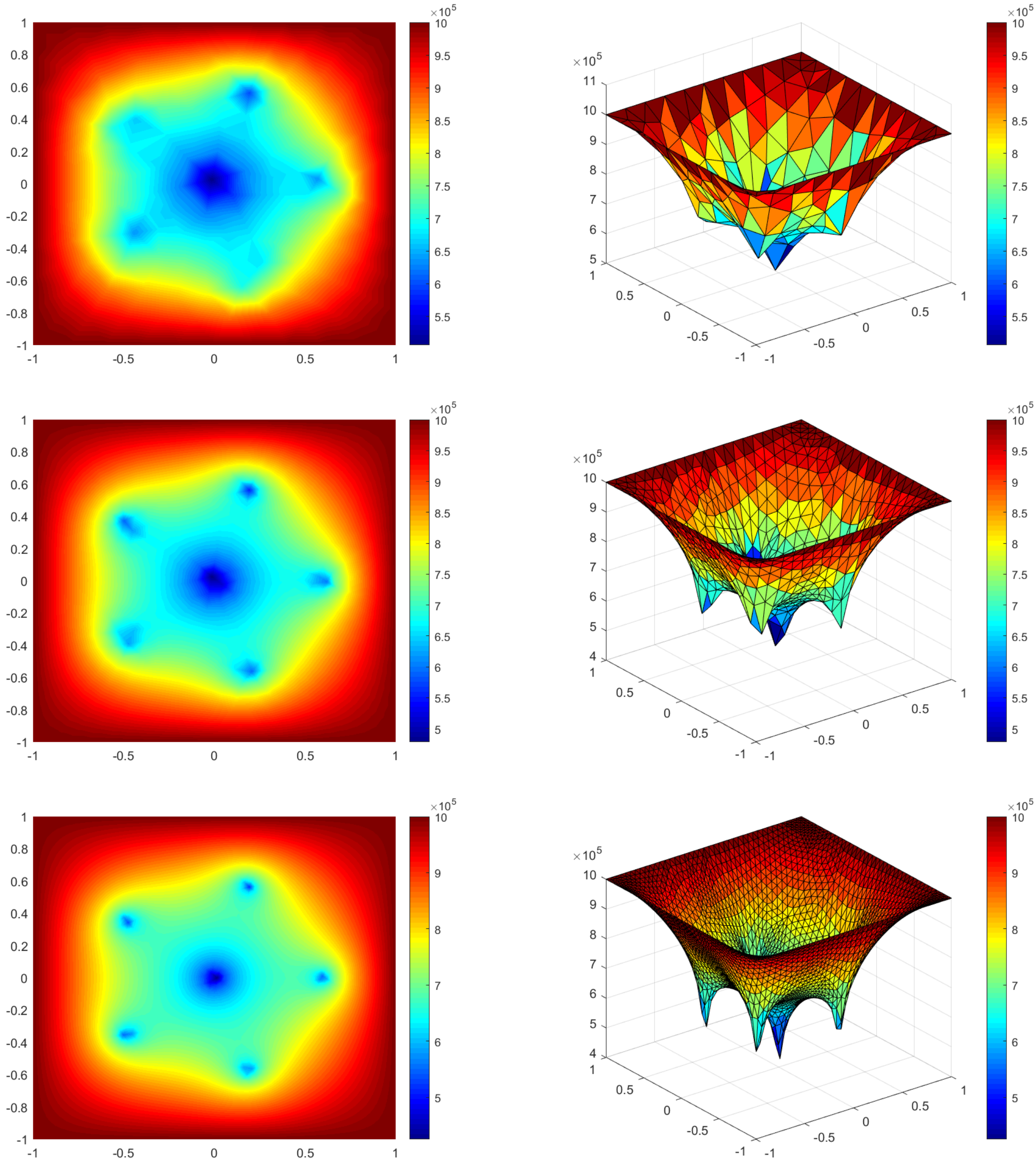
$$A = \left|\det\begin{pmatrix}x_2-x_1 & y_2-y_1\\ x_3-x_1 & y_3-y_1\end{pmatrix}\right| = |(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)|$$

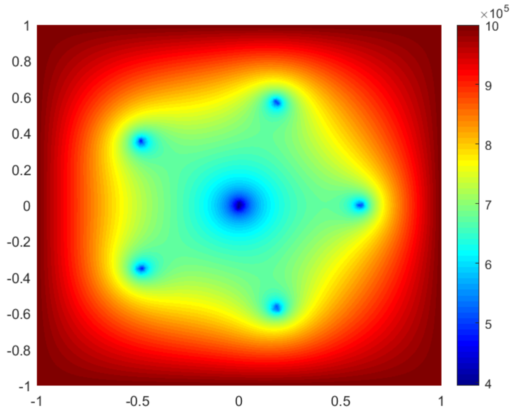
So, $|\Delta(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)| = \frac{A}{2} = \frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{2}$
We will, however, be using the following formula to calculate the area in the code:

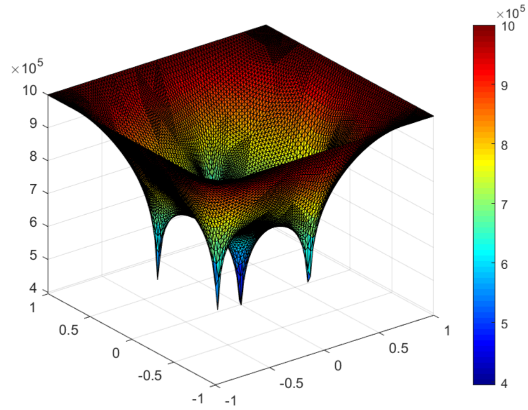
$$A_{par} = \left|\det\begin{pmatrix}1 & x_1 & y_1\\ 1 & x_3 & y_3\\ 1 & x_3 & y_3\end{pmatrix}\right| \qquad A_{triangle} = \frac{A_{par}}{2}$$

7. Program the finite-element code (GenerateElementMatrix, GenerateElementVector, GenerateBoundaryElementMatrix, GenerateBoundaryElementVector), and evaluate the solution. Use mesh refinement to evaluate the quality of the solution. Plot your solution in terms of contour plot and a three-dimensional surface plot.

**Solution:**
The code can be found in the attachments of the mail. We first plot our results for a non refined grid, a refined grid, double refined grid and triple refined grid:






We see that there are indeed "wells" on the 6 spots where the wells should be. This because the pressure should be lower as the water is extracted there. There is one deep well in the middle which was to be expected. The outer "ring" stays mostly the same over all grids, inside the pentagram the most change happens. The maximum value never exceeds $10^6$. Which is normal as it shouldn't get higher than the hydrostatic pressure. The minimum value however decreases as the grid gets more refined, this could be explained by there being more points where the PDE gets updated and thus the descent is more exact.

8. Use Darcy's Law, equation (1), to compute the velocities in both directions, by writing both components of equation (1) in a weak form, and by subsequent derivation of the Galerkin equations. Implement this where you solve the resulting systems of linear equations:

$$M\underline{v}_x = C_x\underline{p}, \qquad M\underline{v}_y = C_y\underline{p} \tag{6}$$

to get $\underline{v}_x$ and $\underline{v}_y$.

**Solution:**
We have the following system:

$$\begin{cases} v_x = -\frac{k}{\mu}\frac{\partial p}{\partial x}, & (x,y) \in \Omega \\ v_y = -\frac{k}{\mu}\frac{\partial p}{\partial y}, & (x,y) \in \Omega \end{cases}$$

The weak forms are now:

$$\text{Find } v_x \in H^1(\Omega) \text{ such that: } \int_\Omega \varphi v_x d\Omega = -\frac{k}{\mu}\int_\Omega \varphi \frac{\partial p}{\partial x} d\Omega, \text{ with } \varphi \in H^1(\Omega)$$

$$\text{Find } v_y \in H^1(\Omega) \text{ such that: } \int_\Omega \varphi v_y d\Omega = -\frac{k}{\mu}\int_\Omega \varphi \frac{\partial p}{\partial y} d\Omega, \text{ with } \varphi \in H^1(\Omega)$$

(we will only describe the process of $v_x$ because $v_y$ is almost the same).

Assume that $p \approx p^n = \sum_{j=1}^n p_j\varphi_j$ and $v_x \approx v_x^n = \sum_{j=1}^n v_j^x\varphi_j$, with the same $\varphi_j$ as before. We get (note that once again $\varphi_j \in H^1(\Omega)$ and thus we may choose $\varphi = \varphi_i \ \forall i \in 1,\ldots,n$):

$$\sum_{j=1} v_j^x \int_\Omega \varphi_i\varphi_j d\Omega = -\frac{k}{\mu}\int \varphi_i \frac{\partial p}{\partial x} d\Omega$$

We define the following matrices: $M_{ij} = \int_\Omega \varphi_i\varphi_j d\Omega$, $(c_x)_j = v_j^x$ and $\underline{f}_i = -\frac{k}{\mu}\int_\Omega \varphi_i\frac{\partial p}{\partial x} d\Omega$. From previous assignments we know that $(c_x)_j = v_x(\mathbf{x}_j)$, so we can say that $\underline{c}_x = \underline{v}_x$.
The Galerkin equtions then become:

$$M\underline{v}_x = \underline{f}$$

.
We will now calculate the element matrix over $e_l$ (using the theorem Holand & Bell with $\Delta$ being the area of the element $e_l$ calculated like in exc. 6):

$$M_{ij}^{e_l} = \frac{\Delta}{24}(1 + \delta_{ij})$$

For the element vector we note that:

$$p(\underline{x})|_{e_l} \approx p(\underline{x}_{l1})\varphi_{l1}(\underline{x}) + p(\underline{x}_{l2})\varphi_{l2}(\underline{x}) + p(\underline{x}_{l3})\varphi_{l3}(\underline{x}) \Rightarrow \frac{\partial p}{\partial x}|_{e_l} \approx p(\underline{x}_{l1})\beta_{l1} + p(\underline{x}_{l2})\beta_{l2} + p(\underline{x}_{l3})\beta_{l3}$$

Thus:

$$\begin{aligned}
\underline{f}_i^{e_l} &= -\frac{k}{\mu}\int_{e_l} \varphi_{li}\frac{\partial p}{\partial x} d\Omega \\
&\approx -\frac{k}{\mu}\int_{e_l} (p(\underline{x}_{l1})\beta_{l1} + p(\underline{x}_{l2})\beta_{l2} + p(\underline{x}_{l3})\beta_{l3})\varphi_{li} d\Omega \\
&= -\frac{k}{\mu}\sum_{j=1}^{3} p(\underline{x}_{lj})\beta_{lj}\int_{e_l} \varphi_{li} d\Omega \\
&= \sum_{j=1}^{3} p(\underline{x}_{lj})\left(-\frac{k\beta_{lj}}{\mu}\frac{\Delta}{6}\right)
\end{aligned}$$

So:

$$\underline{f}^{e_l} = -\frac{k}{\mu}\int_{e_l}\varphi_i\frac{\partial p}{\partial x} d\Omega \approx \frac{-k\Delta}{6\mu}\begin{bmatrix}\beta_{l1} & \beta_{l2} & \beta_{l3}\\ \beta_{l1} & \beta_{l2} & \beta_{l3}\\ \beta_{l1} & \beta_{l2} & \beta_{l3}\end{bmatrix}\begin{bmatrix}p_{l1}\\ p_{l2}\\ p_{l3}\end{bmatrix} = C_x^{e_l}\begin{bmatrix}p_{l1}\\ p_{l2}\\ p_{l3}\end{bmatrix}.$$

(Note that for $v_x$: $C_x^{e_l} := \frac{-k\Delta}{6\mu}\begin{bmatrix}\beta_{l1} & \beta_{l2} & \beta_{l3}\\ \beta_{l1} & \beta_{l2} & \beta_{l3}\\ \beta_{l1} & \beta_{l2} & \beta_{l3}\end{bmatrix}$ and $v_y$ : $C_y^{e_l} := \frac{-k\Delta}{6\mu}\begin{bmatrix}\gamma_{l1} & \gamma_{l2} & \gamma_{l3}\\ \gamma_{l1} & \gamma_{l2} & \gamma_{l3}\\ \gamma_{l1} & \gamma_{l2} & \gamma_{l3}\end{bmatrix}$).

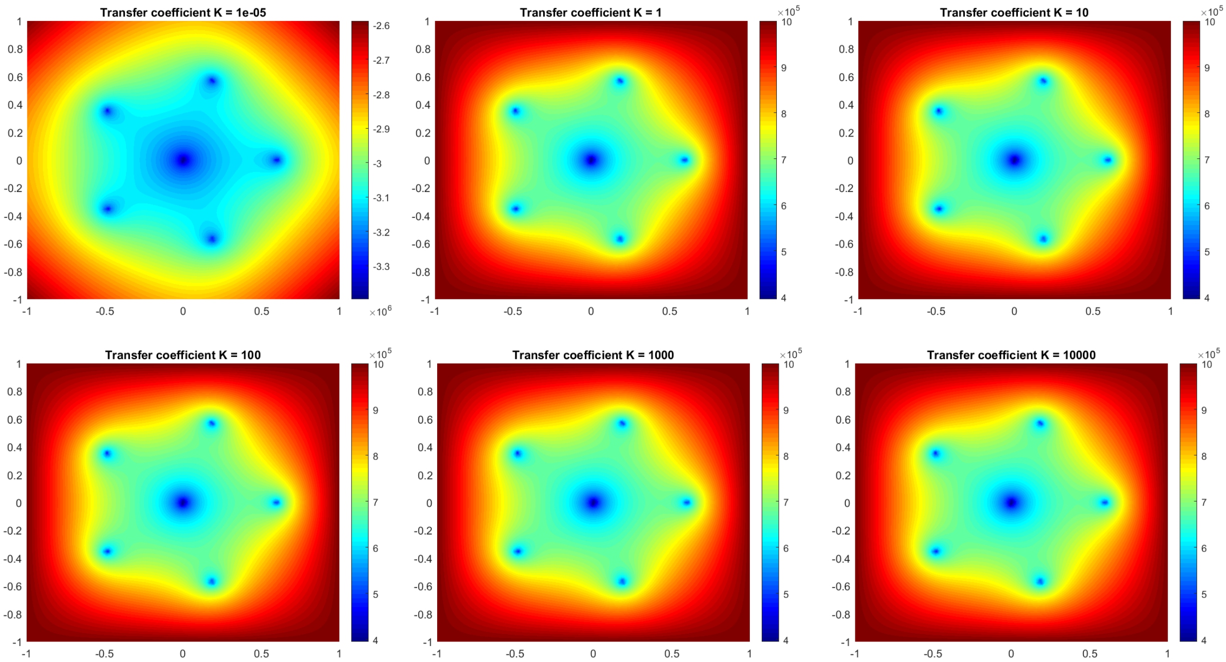
Eventually we get:

$$M\underline{v}_x = C_x\underline{p} \qquad M\underline{v}_y = C_y\underline{p}.$$

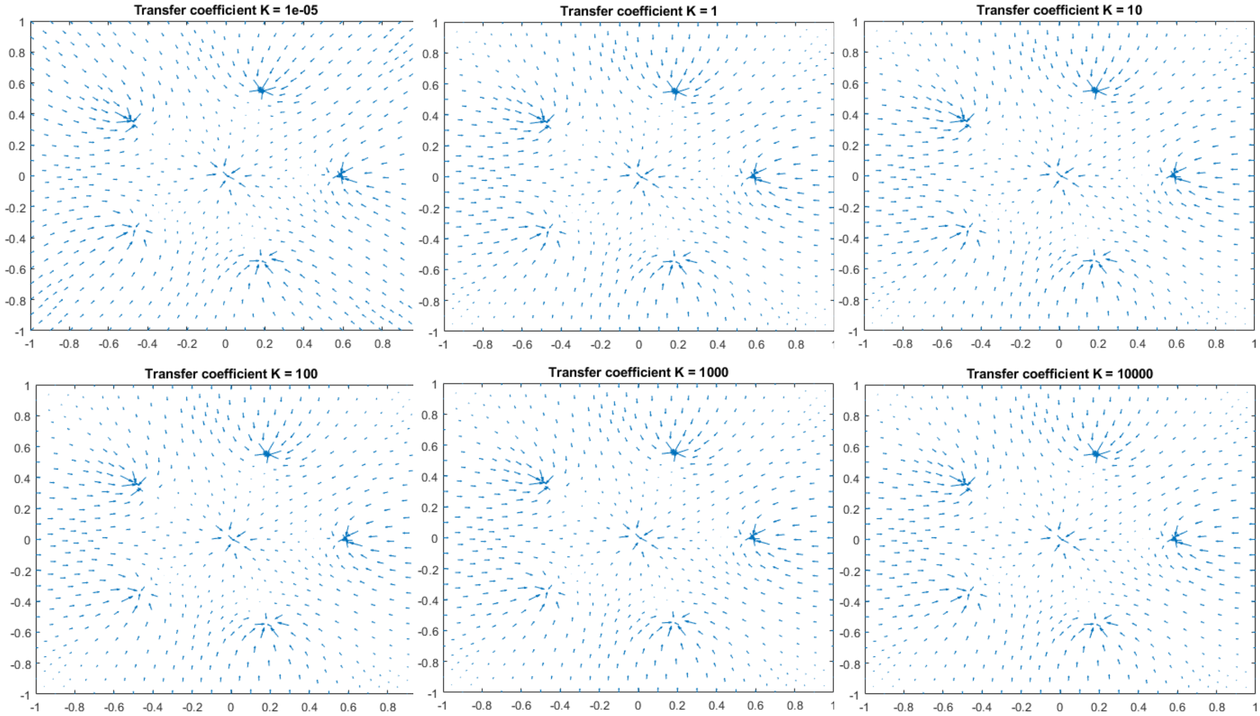
9. Perform various simulations where you let the transfer coefficient $K$ range between 0.00001 and 10000. Show the contour plots, and give the values of the minimal pressure (which is important from an engineering point of view). Explain your results.

**Solution**

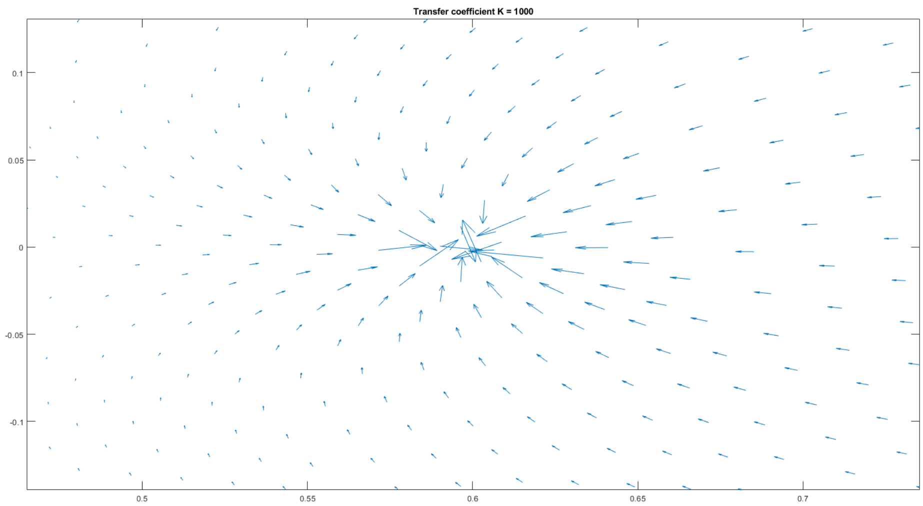
We have chosen to look at the following transfer coefficients: 0.00001, 1, 10, 100, 1000 and 10000. These are the results:



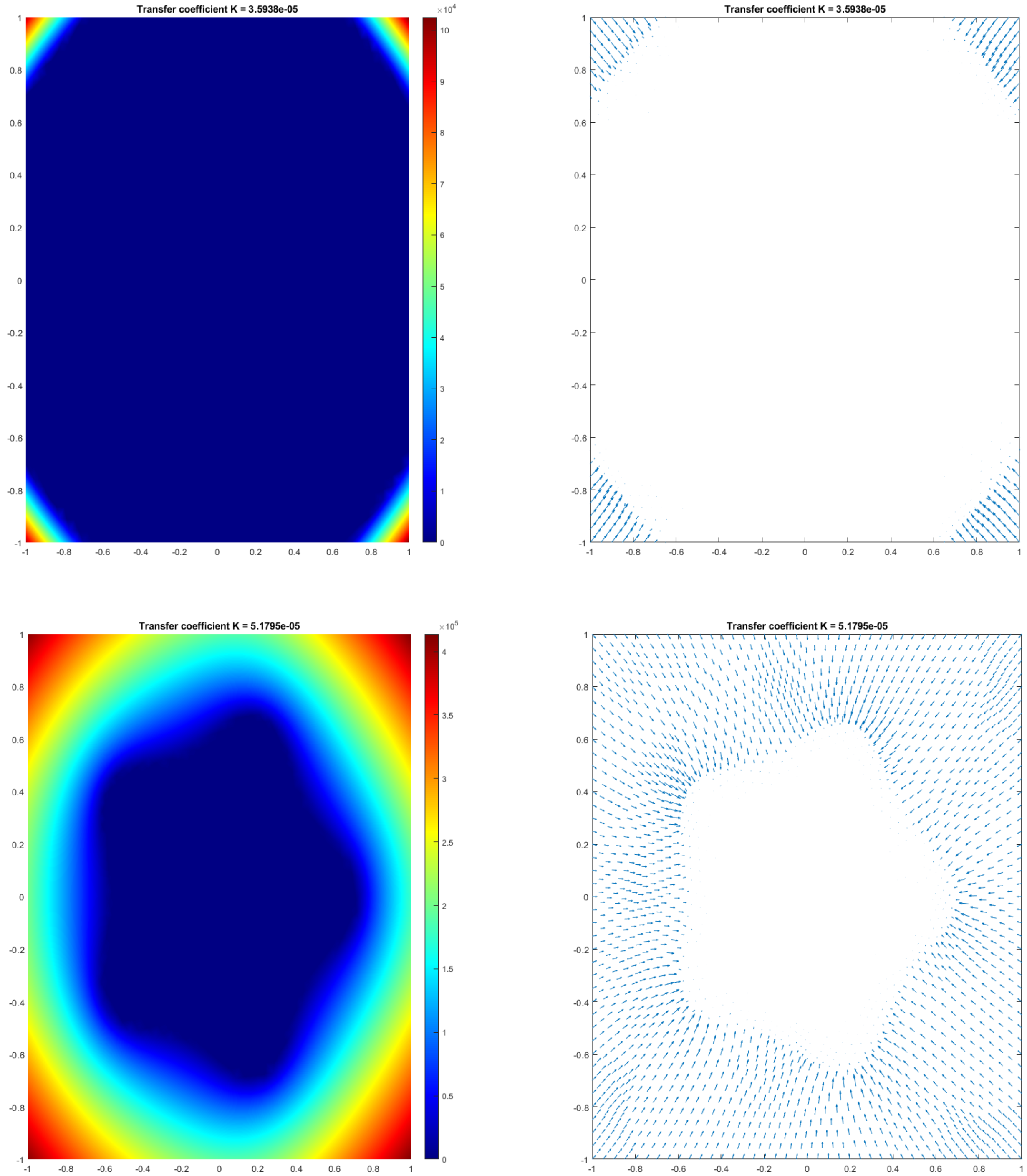

We also plot the velocities for a less refined grid (so the directions are more visible):

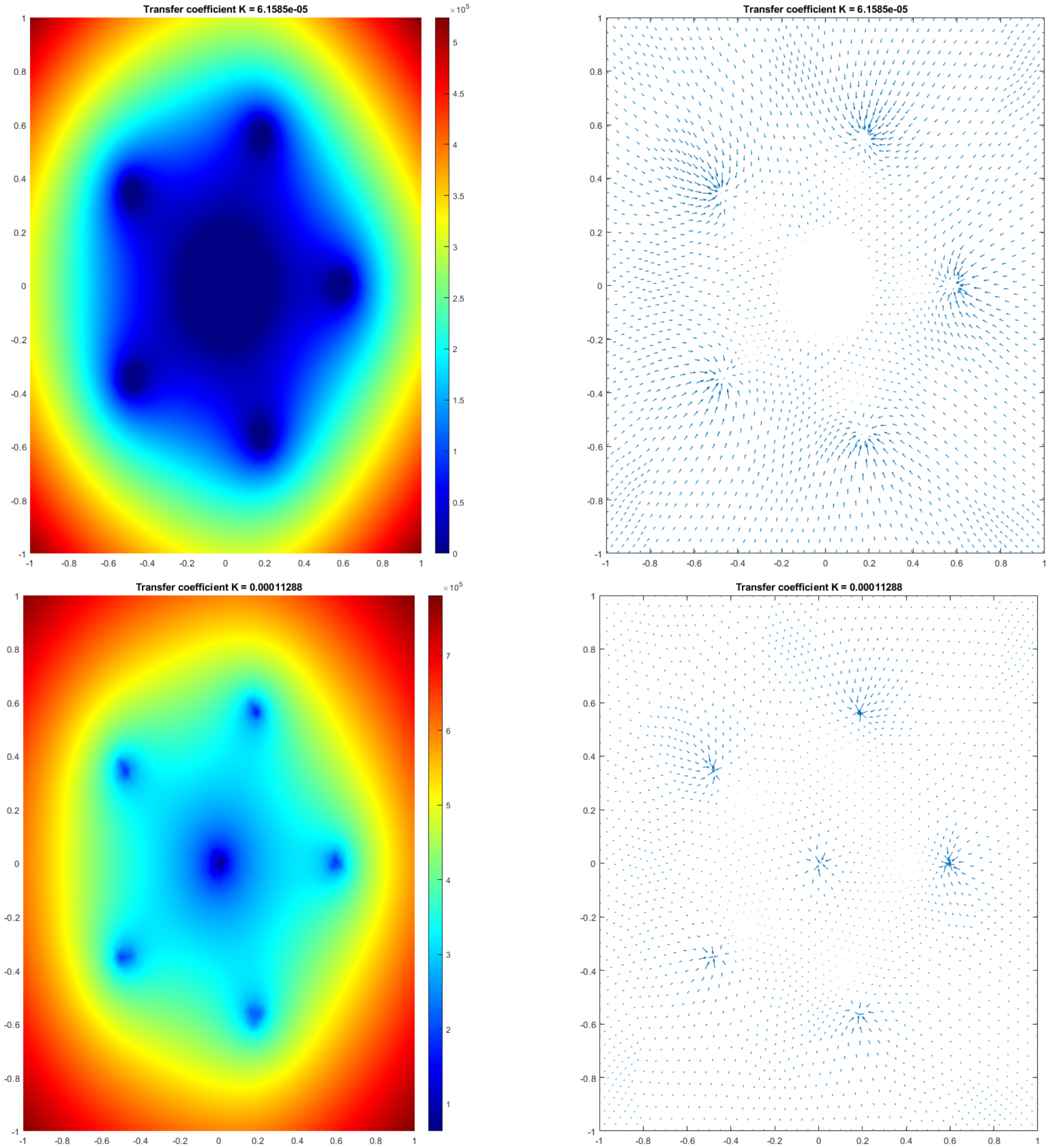



If we zoomed in a on the most right well (while using a more refined grid), we get:




The minimum values are respectively: $-3397480.6\ Pa$, $394787.6\ Pa$, $394830.6\ Pa$, $394834.9\ Pa$, $394835.3\ Pa$ and $394835.4\ Pa$. If the transfer coefficient is extremely low almost no water would be supplied to the domain. Thus the reservoir will dry up faster. However, if the transfer coefficient increases more water will be supplied and the pressure will thus rise. The negative pressure begs the question: Can water pressure be negative? Our naive mathematical brains would tell us no. This because: Suppose the reservoir would empty out almost immediately, then the pressure would be a really big negative number, but could we talk about pressure in that case? If we were to look at points in between $10^{-5}$ and 1 we would see that around 0.00005 the pressure would be 0 at some places, but the shape would be the same. Suppose now that we follow our naive mathematical brains and say that the pressure would be 0 if it was negative. We would get the following (for various transfer coefficients):

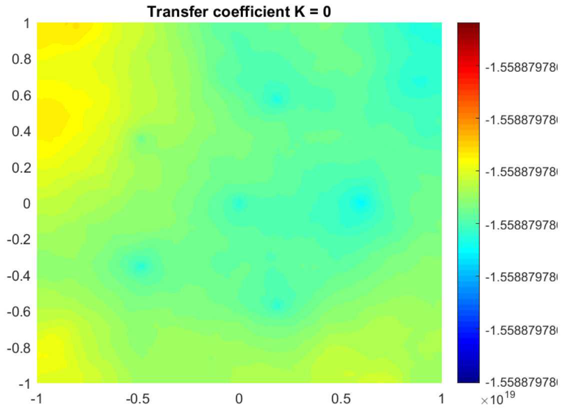

Transfer coefficient K = 3.5938e-05
×10^4
Transfer coefficient K = 3.5938e-05
Transfer coefficient K = 5.1795e-05
×10^5
Transfer coefficient K = 5.1795e-05

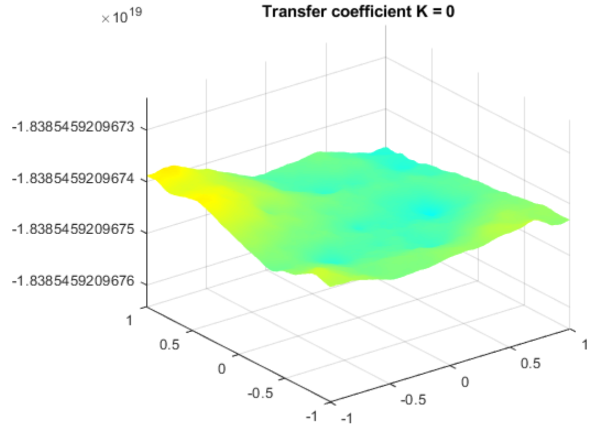



We thus see that from $10^{-4}$ onward the pressure would be positive and there would be a pileup of water. In the other cases, the model predicts a little pileup on some places. It also does not sense any velocities in places with no pressure because there would be next to no water. However we know that water will be flowing through these places. Thus we might want to use the velocities of the model with negative pressure, another model or perhaps other boundary conditions.

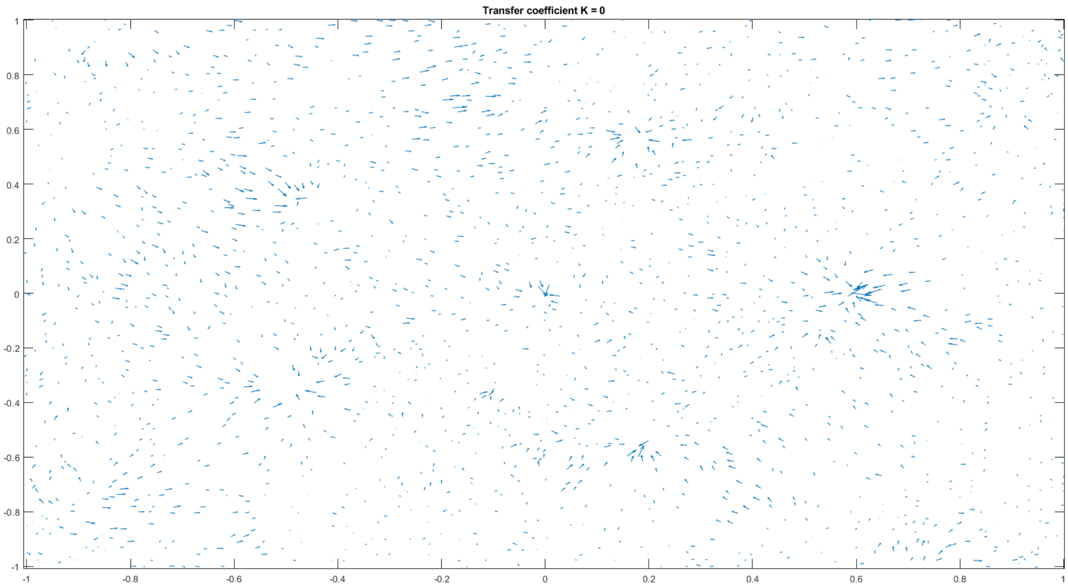
10. What happens if K = 0? Explain the results.

**Solution:**
If the extract coefficient were 0 the magnitude of the pressure would be immense ($10^{20}$) but it would be negative. The meaning of K = 0 is that no water arrives, so there can be no water pressure. Like in assignment 9, if the reservoir were to be empty, the pressure would be negative infinite, but then you really can't talk about pressure anymore. We can thus conclude that the reservoir would be empty. The results are (not using if $p < 0 \rightarrow p = 0$):




If we were to plot the velocities, we would get something quite chaotic:




However, if we would want to use the model as we want (so when $p < 0$, it will be changed to $p = 0$) we get the following plot (which is what we'd expect if the pressure is negative everywhere):