# Assignments 3

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1.We consider the following boundary value problem for the solution $u = u(\mathbf{x})$ to be determined in the bounded domain $\Omega$ in $\mathbb{R}^2$ (bounded by $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$):

$$\begin{cases} -\nabla \cdot [\mathbf{K}\nabla u] = f(\mathbf{x}), & \text{in } \Omega \\ u = g_1(\mathbf{x}), & \text{on } \Gamma_1 \\ \mathbf{n} \cdot \mathbf{K}\nabla u + Au = g_2(\mathbf{x}), & \text{on } \Gamma_2 \end{cases}$$

Here $\mathbf{K}(\mathbf{x}) = \begin{pmatrix} k_{11}(\mathbf{x}) & k_{12}(\mathbf{x}) \\ k_{21}(\mathbf{x}) & k_{22}(\mathbf{x}) \end{pmatrix}$ denotes the diffusivity tensor (matrix) with entries that depend on the location $\mathbf{x}$. Further, $f(\mathbf{x}), g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ represent given functions and $A \geq 0$ is a constant.

1. Derive the weak formulation in which the order of spatial derivatives is minimized.

   **Solution:**
   There holds

$$\begin{aligned}
& -\nabla \cdot [\mathbf{K}\nabla u] = f(\mathbf{x}) \\
& \Rightarrow -\varphi\nabla \cdot [\mathbf{K}\nabla u] = \varphi f(\mathbf{x}) \text{ (with } \varphi \text{ test function and } \varphi|_{\Gamma 1} = 0) \\
& \Rightarrow -\int_\Omega \varphi\nabla \cdot [\mathbf{K}\nabla u] d\Omega = \int_\Omega \varphi f(\mathbf{x}) d\Omega \\
& \Rightarrow -\int_{\partial\Omega} \varphi\mathbf{n}[\mathbf{K}\nabla u] d\Gamma + \int_\Omega [\nabla \cdot \varphi][\mathbf{K}\nabla u] d\Omega = \int_\Omega \varphi f(\mathbf{x}) d\Omega \\
& \text{(using } \varphi\nabla \cdot [\mathbf{K}\nabla u] = \nabla \cdot [\varphi\mathbf{K}\nabla u] - [\nabla\varphi][\mathbf{K}\nabla u] \text{ and Gauss' divergence theorem)} \\
& \Rightarrow -\int_{\Gamma_1} \varphi\mathbf{n}[\mathbf{K}\nabla g_1(\mathbf{x})] d\Gamma - \int_{\Gamma_2} \varphi(g_2(\mathbf{x}) - Au) d\Gamma + \int_\Omega [\nabla\varphi][\mathbf{K}\nabla u] d\Omega = \int_\Omega \varphi f(\mathbf{x}) d\Omega.
\end{aligned}$$

   Since $\varphi|_{\Gamma 1} = 0$, this can be written as:

$$-\int_{\Gamma_2} \varphi(g_2(\mathbf{x}) - Au) d\Gamma + \int_\Omega [\nabla\varphi][\mathbf{K}\nabla u] d\Omega = \int_\Omega \varphi f(\mathbf{x}) d\Omega$$

   and $u|_{\Gamma 1} = g_1(\mathbf{x})$. So the weak formulation is:

$$\begin{aligned}
& \text{Find } u \in H^1(\Omega), \text{ such that:} \\
& \begin{cases} \int_{\Gamma_2} \varphi Au d\Gamma + \int_\Omega [\nabla\varphi][\mathbf{K}\nabla u] d\Omega = \int_\Omega \varphi f(\mathbf{x}) d\Omega + \int_{\Gamma_2} \varphi g_2(\mathbf{x}) d\Gamma \\ u|_{\Gamma 1} = g_1(\mathbf{x}) \end{cases} \\
& \text{holds } \forall \varphi \in H^1(\Omega), \text{ for which } \varphi|_{\Gamma_1} = 0.
\end{aligned}$$

2. State for each boundary condition whether it is essential or natural. Motivate your answer.

**Solution:**
The natural boundary condition is: $\mathbf{n} \cdot \mathbf{K}\nabla u + Au = g_2(\mathbf{x})$ on $\Gamma_2$. Because it has become a part of the weak formulation (it will follow from the blue parts of the weak formulation).
The essential boundary condition is: $u|_{\Gamma 1} = g_1(\mathbf{x})$. Because this boundary condition must be satisfied both by the minimization problem and the boundary value problem (differential equation).

3. Derive the Galerkin Equations to the weak form in part a.

   **Solution:**
   We have that:

$$u(\mathbf{x}) \approx u^n(\mathbf{x}) = \sum_{j=1}^{n} c_j \varphi_j(\mathbf{x}) + u_B(\mathbf{x}) \text{ (by the method of Ritz),}$$

   with $\varphi_j|_{\Gamma_1} = 0$ and $u_B(\mathbf{x})$ is added since $u|_{\Gamma_1} = g_1(\mathbf{x}) \neq 0$. Now choose $\{\varphi_i\}_{i=1}^n$, so $\varphi = \varphi_i(\mathbf{x})$ and $\varphi_i|_{\Gamma_1} = 0$. Filling all this in in the weak formulation gives:

$$\begin{aligned}&\sum_{j=1}^{n} c_j \left( \int_{\Gamma_2} \varphi_i A \varphi_j d\Gamma + \int_\Omega [\nabla\varphi_i][\mathbf{K}\nabla\varphi_j] d\Omega \right) \\ &= \int_\Omega \varphi_i f(\mathbf{x}) d\Omega + \int_{\Gamma_2} \varphi_i g_2(\mathbf{x}) d\Gamma - \left[ \int_{\Gamma_2} \varphi_i A u_B(\mathbf{x}) d\Gamma + \int_\Omega [\nabla\varphi_i][\mathbf{K}\nabla u_B(\mathbf{x})] d\Omega \right], \text{ for } i = 1, ..., n.\end{aligned}$$

   Now, we define:

$$\begin{aligned}S_{ij} &:= \int_{\Gamma_2} \varphi_i A \varphi_j d\Gamma + \int_\Omega [\nabla\varphi_i][\mathbf{K}\nabla\varphi_j] d\Omega \text{ and} \\ b_i &:= \int_\Omega \varphi_i f(\mathbf{x}) d\Omega + \int_{\Gamma_2} \varphi_i g_2(\mathbf{x}) d\Gamma - \left[ \int_{\Gamma_2} \varphi_i A u_B(\mathbf{x}) d\Gamma + \int_\Omega [\nabla\varphi_i][\mathbf{K}\nabla u_B(\mathbf{x})] d\Omega \right].\end{aligned}$$

   Therefore the Galerkin Equations to the weak form are:

$$\begin{cases} S_{ij} = \int_{\Gamma_2} \varphi_i A \varphi_j d\Gamma + \int_\Omega [\nabla\varphi_i][\mathbf{K}\nabla\varphi_j] d\Omega \text{ and} \\ b_i = \int_\Omega \varphi_i f(\mathbf{x}) d\Omega + \int_{\Gamma_2} \varphi_i g_2(\mathbf{x}) d\Gamma - \left[ \int_{\Gamma_2} \varphi_i A u_B(\mathbf{x}) d\Gamma + \int_\Omega [\nabla\varphi_i][\mathbf{K}\nabla u_B(\mathbf{x})] d\Omega \right] \end{cases}.$$

   From this follows the next system:

$$S\mathbf{c} = \mathbf{b}.$$

4. We use linear triangular elements to solve the problem. The gradients in the answers may contain $\alpha_i$, $\beta_i$ and $\gamma_i$ from the form $\varphi_i = \alpha_i + \beta_i x + \gamma_i y$.

   (a) Compute the element matrix and element vector for an internal triangle.
   Use Newton-Cotes integration if you cannot evaluate the integrals exactly.

   **Solution:**
   Since linear triangular elements are used to solve the problem, an element $e_k$ is defined as the triangle of $\mathbf{x}_{k_1}$, $\mathbf{x}_{k_2}$ and $\mathbf{x}_{k_3}$. Suppose now that we only look at internal points, thus $\Gamma \cap e_k = \emptyset$. This implicates that: $\int_{\Gamma_2} \varphi_i A \varphi_j d\Gamma = 0$, $\int_{\Gamma_2} \varphi_i g_2(\mathbf{x}) d\Gamma = 0$ and $\int_{\Gamma_2} \varphi_i A u_B(\mathbf{x}) d\Gamma = 0$. Also is $\int_\Omega [\nabla\varphi_i][\mathbf{K}\nabla u_B(\mathbf{x})] d\Omega = 0$, since $u_B(\mathbf{x})$ was only added so $u = g_1(\mathbf{x})$ on the boundary $\Gamma_1$. For the element matrix $S_{ij}^{e_k}$ holds that:

$$S_{ij} = \int_\Omega [\nabla\varphi_i][\mathbf{K}\nabla\varphi_j] d\Omega \approx \sum_{k=1}^{n_T} \int_{e_k} [\nabla\varphi_i][\mathbf{K}\nabla\varphi_j] d\Omega, \text{ with } n_T = \text{the number of triangles,}$$

   therefore

$$S_{ij}^{e_k} = \int_{e_k} [\nabla\varphi_i][\mathbf{K}\nabla\varphi_j] d\Omega.$$

Since we use linear triangular elements, there holds that $\varphi_i(\mathbf{x}) = \alpha_i + \beta_i x + \gamma_i y$. On $e_k$ holds that $\varphi_i(\mathbf{x}_j) = \delta_{ij}, i, j \in \{k_1, k_2, k_3\}$, therefore the coefficients $\alpha_i$, $\beta_i$ and $\gamma_i$ can be calculated. This is done as follows:

$$
\begin{aligned}
\varphi_{k_1}(\mathbf{x}_{k_1}) = 1 &\Rightarrow \alpha_{k_1} + \beta_{k_1} x_{k_1} + \gamma_{k_1} y_{k_1} = 1\\
\varphi_{k_1}(\mathbf{x}_{k_2}) = 0 &\Rightarrow \alpha_{k_1} + \beta_{k_1} x_{k_2} + \gamma_{k_1} y_{k_2} = 0\\
\varphi_{k_1}(\mathbf{x}_{k_3}) = 0 &\Rightarrow \alpha_{k_1} + \beta_{k_1} x_{k_3} + \gamma_{k_1} y_{k_3} = 0\\
\\
\varphi_{k_2}(\mathbf{x}_{k_1}) = 0 &\Rightarrow \alpha_{k_2} + \beta_{k_2} x_{k_1} + \gamma_{k_2} y_{k_1} = 0\\
\varphi_{k_2}(\mathbf{x}_{k_2}) = 1 &\Rightarrow \alpha_{k_2} + \beta_{k_2} x_{k_2} + \gamma_{k_2} y_{k_2} = 1\\
\varphi_{k_2}(\mathbf{x}_{k_3}) = 0 &\Rightarrow \alpha_{k_2} + \beta_{k_2} x_{k_3} + \gamma_{k_2} y_{k_3} = 0\\
\\
\varphi_{k_3}(\mathbf{x}_{k_1}) = 0 &\Rightarrow \alpha_{k_3} + \beta_{k_3} x_{k_1} + \gamma_{k_3} y_{k_1} = 0\\
\varphi_{k_3}(\mathbf{x}_{k_2}) = 0 &\Rightarrow \alpha_{k_3} + \beta_{k_3} x_{k_2} + \gamma_{k_3} y_{k_2} = 0\\
\varphi_{k_3}(\mathbf{x}_{k_3}) = 1 &\Rightarrow \alpha_{k_3} + \beta_{k_3} x_{k_3} + \gamma_{k_3} y_{k_3} = 1.
\end{aligned}
$$

This gives:

$$
\begin{aligned}
\begin{pmatrix} 1 & x_{k_1} & y_{k_1} \\ 1 & x_{k_2} & y_{k_2} \\ 1 & x_{k_3} & y_{k_3} \end{pmatrix} \begin{pmatrix} \alpha_{k_1} \\ \beta_{k_1} \\ \gamma_{k_1} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\\
\begin{pmatrix} 1 & x_{k_1} & y_{k_1} \\ 1 & x_{k_2} & y_{k_2} \\ 1 & x_{k_3} & y_{k_3} \end{pmatrix} \begin{pmatrix} \alpha_{k_2} \\ \beta_{k_2} \\ \gamma_{k_2} \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\\
\begin{pmatrix} 1 & x_{k_1} & y_{k_1} \\ 1 & x_{k_2} & y_{k_2} \\ 1 & x_{k_3} & y_{k_3} \end{pmatrix} \begin{pmatrix} \alpha_{k_3} \\ \beta_{k_3} \\ \gamma_{k_3} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}
$$

In conclusion:

$$
\begin{pmatrix} 1 & x_{k_1} & y_{k_1} \\ 1 & x_{k_2} & y_{k_2} \\ 1 & x_{k_3} & y_{k_3} \end{pmatrix} \begin{pmatrix} \alpha_{k_1} & \alpha_{k_2} & \alpha_{k_3} \\ \beta_{k_1} & \beta_{k_2} & \beta_{k_3} \\ \gamma_{k_1} & \gamma_{k_2} & \gamma_{k_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

When these systems are solved, the coefficients $\alpha_i$, $\beta_i$ and $\gamma_i$ for $i \in \{k_1, k_2, k_3\}$ are known. There also holds that:

$$
\nabla \varphi_i = \begin{bmatrix} \frac{\partial \varphi_i}{\partial x} \\ \frac{\varphi_i}{\partial y} \end{bmatrix} = \begin{bmatrix} \beta_i \\ \gamma_i \end{bmatrix}.
$$

For the element matrix holds:

$$
\begin{aligned}
S_{ij}^{e_k} &= \int_{e_k} [\nabla \varphi_i][\mathbf{K} \nabla \varphi_j] d\Omega = \int_{e_k} \begin{bmatrix} \beta_i \\ \gamma_i \end{bmatrix} \left[ \begin{pmatrix} k_{11}(\mathbf{x}) & k_{12}(\mathbf{x}) \\ k_{21}(\mathbf{x}) & k_{22}(\mathbf{x}) \end{pmatrix} \begin{bmatrix} \beta_j \\ \gamma_j \end{bmatrix} \right] d\Omega\\
&= \int_{e_k} \begin{bmatrix} \beta_i \\ \gamma_i \end{bmatrix} \cdot \begin{bmatrix} k_{11}(\mathbf{x}) \beta_j + k_{12}(\mathbf{x}) \gamma_j \\ k_{21}(\mathbf{x}) \beta_j + k_{22}(\mathbf{x}) \gamma_j \end{bmatrix} d\Omega\\
&= \int_{e_k} \left( \beta_i \beta_j k_{11}(\mathbf{x}) + \beta_i \gamma_j k_{12}(\mathbf{x}) + \gamma_i \beta_j k_{21}(\mathbf{x}) + \gamma_i \gamma_j k_{22}(\mathbf{x}) \right) d\Omega\\
&= \beta_i \beta_j \int_{e_k} k_{11}(\mathbf{x}) d\Omega + \beta_i \gamma_j \int_{e_k} k_{12}(\mathbf{x}) d\Omega + \gamma_i \beta_j \int_{e_k} k_{21}(\mathbf{x}) d\Omega + \gamma_i \gamma_j \int_{e_k} k_{22}(\mathbf{x}) d\Omega.
\end{aligned}
$$

Using Newton-Cotes, there holds that:

$$
\begin{aligned}
\int_{e_k} k_{11}(\mathbf{x}) d\Omega &\approx \frac{|\Delta|}{6} \sum_{p \in k_1, k_2, k_3} k_{11}(\mathbf{x}_p), \int_{e_k} k_{12}(\mathbf{x}) d\Omega \approx \frac{|\Delta|}{6} \sum_{p \in k_1, k_2, k_3} k_{12}(\mathbf{x}_p),\\
\int_{e_k} k_{21}(\mathbf{x}) d\Omega &\approx \frac{|\Delta|}{6} \sum_{p \in k_1, k_2, k_3} k_{21}(\mathbf{x}_p), \int_{e_k} k_{22}(\mathbf{x}) d\Omega \approx \frac{|\Delta|}{6} \sum_{p \in k_1, k_2, k_3} k_{22}(\mathbf{x}_p),
\end{aligned}
$$

with

$$\Delta = det \begin{bmatrix} 1 & x_{k_1} & y_{k_1} \\ 1 & x_{k_2} & y_{k_2} \\ 1 & x_{k_3} & y_{k_3} \end{bmatrix} = \|(\mathbf{x}_{k_2} - \mathbf{x}_{k_1}) \times (\mathbf{x_{k_3}} - \mathbf{x_{k_1}})\|.$$

So,

$$S^{e_k} = \begin{bmatrix} S^{e_k}_{k_1k_1} & S^{e_k}_{k_1k_2} & S^{e_k}_{k_1k_3} \\ S^{e_k}_{k_2k_1} & S^{e_k}_{k_2k_2} & S^{e_k}_{k_2k_3} \\ S^{e_k}_{k_3k_1} & S^{e_k}_{k_3k_2} & S^{e_k}_{k_3k_3} \end{bmatrix},$$

with

$$S^{e_k}_{ij} = \beta_i\beta_j\frac{|\Delta|}{6}\sum_{p\in k_1,k_2,k_3} k_{11}(\mathbf{x}_p) + \beta_i\gamma_j\frac{|\Delta|}{6}\sum_{p\in k_1,k_2,k_3} k_{12}(\mathbf{x}_p) + \gamma_i\beta_j\frac{|\Delta|}{6}\sum_{p\in k_1,k_2,k_3} k_{21}(\mathbf{x}_p)$$
$$+ \gamma_i\gamma_j\frac{|\Delta|}{6}\sum_{p\in k_1,k_2,k_3} k_{22}(\mathbf{x}_p).$$

For the element vector $b^{e_k}_i$ holds that:

$$b_i = \int_\Omega \varphi_i f(\mathbf{x})d\Omega \approx \sum_{k=1}^{n_T}\int_{e_k} \varphi_i f(\mathbf{x})d\Omega, \text{ with } n_T = \text{the number of triangles on } \Omega,$$

therefore

$$b^{e_k}_i = \int_{e_k} \varphi_i f(\mathbf{x})d\Omega.$$

Using Newton-Cotes gives:

$$\begin{aligned} b^{e_k}_i &= \int_{e_k} \varphi_i f(\mathbf{x})d\Omega \\ &\approx \frac{|\Delta|}{6}\sum_{p\in\{k_1,k_2,k_3\}} \varphi_i(\mathbf{x}_p)f(\mathbf{x}_p) \text{ (Newton-Cotes)} \\ &= \frac{|\Delta|}{6} f(\mathbf{x}_i), \text{ for } i \in \{k_1,k_2,k_3\} \text{ (since } \varphi_i(\mathbf{x}_p) = \delta_{ip}), \end{aligned}$$

with

$$\Delta = det \begin{bmatrix} 1 & x_{k_1} & y_{k_1} \\ 1 & x_{k_2} & y_{k_2} \\ 1 & x_{k_3} & y_{k_3} \end{bmatrix} = \|(\mathbf{x}_{k_2} - \mathbf{x}_{k_1}) \times (\mathbf{x_{k_3}} - \mathbf{x_{k_1}})\|.$$

So,

$$b^{e_k} = \begin{bmatrix} \frac{|\Delta|}{6} f(\mathbf{x}_{k_1}) \\ \frac{|\Delta|}{6} f(\mathbf{x}_{k_2}) \\ \frac{|\Delta|}{6} f(\mathbf{x}_{k_3}). \end{bmatrix}$$

(b) Compute the element matrix and element vector for a boundary element.
Use Newton-Cotes integration if you cannot evaluate the integrals exactly.

**Solution:**
Since linear triangular elements are used to solve the problem, an element $e_k$ is defined as the triangle of $\mathbf{x}_{k_1}$, $\mathbf{x}_{k_2}$ and $\mathbf{x}_{k_3}$. For a boundary element holds that:

$$S^{bd}_{ij} = \int_{\Gamma_2} \varphi_i A\varphi_j d\Gamma + \int_\Omega [\nabla\varphi_i][\mathbf{K}\nabla\varphi_j]d\Omega.$$

Because the second integral for an element is already approximated in exercise (a), only the first integral needs to be approximated for an element. Denote $bd_{e_k}$ as the line element on $\Gamma_2$, so $bd_{e_k}$ is the line between $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$, with $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$ boundary elements on $\Gamma_2$. There follows that

$$\int_{\Gamma_2} \varphi_i A \varphi_j d\Gamma \approx \sum_{k=1}^{n_B} \int_{bd_{e_k}} \varphi_i A \varphi_j d\Gamma, \text{ with } n_B = \text{the number of lines on } \Gamma_2.$$

There holds:

$$\begin{aligned}
\int_{bd_{e_k}} \varphi_i A \varphi_j d\Gamma = A \int_{bd_{e_k}} \varphi_i \varphi_j d\Gamma \\
&= A \frac{|bd_{e_k}|}{(1+1+1)!}(1+\delta_{ij}), \text{ with } |bd_{e_k}| = \|\mathbf{x}_{k_2} - \mathbf{x}_{k_1}\| \\
&\text{(because of the theorem of Holand and Bell)} \\
&= A \frac{|bd_{e_k}|}{6}(1+\delta_{ij}), \text{ with } i,j = k_1, k_2.
\end{aligned}$$

So,

$x_{k_1}$ and $x_{k_2}$ are on the boundary: $|bd_{e_k}| = \|\mathbf{x}_{k_2} - \mathbf{x}_{k_1}\|$

$$S^{bd_{e_k}} = \begin{bmatrix} A\frac{\|\mathbf{x}_{k_2}-\mathbf{x}_{k_1}\|}{3} & A\frac{\|\mathbf{x}_{k_2}-\mathbf{x}_{k_1}\|}{6} & 0 \\ A\frac{\|\mathbf{x}_{k_2}-\mathbf{x}_{k_1}\|}{6} & A\frac{\|\mathbf{x}_{k_2}-\mathbf{x}_{k_1}\|}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} S^{e_k}_{k_1k_1} & S^{e_k}_{k_1k_2} & S^{e_k}_{k_1k_3} \\ S^{e_k}_{k_2k_1} & S^{e_k}_{k_2k_2} & S^{e_k}_{k_2k_3} \\ S^{e_k}_{k_3k_1} & S^{e_k}_{k_3k_2} & S^{e_k}_{k_3k_3} \end{bmatrix},$$

with

$$S^{e_k}_{ij} = \beta_i\beta_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{11}(\mathbf{x}_p) + \beta_i\gamma_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{12}(\mathbf{x}_p) + \gamma_i\beta_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{21}(\mathbf{x}_p) + \gamma_i\gamma_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{22}(\mathbf{x}_p).$$

$x_{k_1}$ and $x_{k_3}$ are on the boundary: $|bd_{e_k}| = \|\mathbf{x}_{k_3} - \mathbf{x}_{k_1}\|$

$$S^{bd_{e_k}} = \begin{bmatrix} A\frac{\|\mathbf{x}_{k_3}-\mathbf{x}_{k_1}\|}{3} & 0 & A\frac{\|\mathbf{x}_{k_3}-\mathbf{x}_{k_1}\|}{6} \\ 0 & 0 & 0 \\ A\frac{\|\mathbf{x}_{k_3}-\mathbf{x}_{k_1}\|}{6} & 0 & A\frac{\|\mathbf{x}_{k_3}-\mathbf{x}_{k_1}\|}{3} \end{bmatrix} + \begin{bmatrix} S^{e_k}_{k_1k_1} & S^{e_k}_{k_1k_2} & S^{e_k}_{k_1k_3} \\ S^{e_k}_{k_2k_1} & S^{e_k}_{k_2k_2} & S^{e_k}_{k_2k_3} \\ S^{e_k}_{k_3k_1} & S^{e_k}_{k_3k_2} & S^{e_k}_{k_3k_3} \end{bmatrix},$$

with

$$S^{e_k}_{ij} = \beta_i\beta_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{11}(\mathbf{x}_p) + \beta_i\gamma_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{12}(\mathbf{x}_p) + \gamma_i\beta_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{21}(\mathbf{x}_p) + \gamma_i\gamma_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{22}(\mathbf{x}_p).$$

$x_{k_2}$ and $x_{k_3}$ are on the boundary: $|bd_{e_k}| = \|\mathbf{x}_{k_3} - \mathbf{x}_{k_2}\|$

$$S^{bd_{e_k}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A\frac{\|\mathbf{x}_{k_3}-\mathbf{x}_{k_2}\|}{3} & A\frac{\|\mathbf{x}_{k_3}-\mathbf{x}_{k_2}\|}{6} \\ 0 & A\frac{\|\mathbf{x}_{k_3}-\mathbf{x}_{k_2}\|}{6} & A\frac{\|\mathbf{x}_{k_3}-\mathbf{x}_{k_2}\|}{3} \end{bmatrix} + \begin{bmatrix} S^{e_k}_{k_1k_1} & S^{e_k}_{k_1k_2} & S^{e_k}_{k_1k_3} \\ S^{e_k}_{k_2k_1} & S^{e_k}_{k_2k_2} & S^{e_k}_{k_2k_3} \\ S^{e_k}_{k_3k_1} & S^{e_k}_{k_3k_2} & S^{e_k}_{k_3k_3} \end{bmatrix},$$

with

$$S^{e_k}_{ij} = \beta_i\beta_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{11}(\mathbf{x}_p) + \beta_i\gamma_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{12}(\mathbf{x}_p) + \gamma_i\beta_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{21}(\mathbf{x}_p) + \gamma_i\gamma_j \frac{|\Delta|}{6} \sum_{p\in k_1,k_2,k_3} k_{22}(\mathbf{x}_p).$$

For a boundary element holds that:

$$b_i^{bd} = \int_\Omega \varphi_i f(\mathbf{x}) d\Omega + \int_{\Gamma_2} \varphi_i g_2(\mathbf{x}) d\Gamma - \left[ \int_{\Gamma_2} \varphi_i A u_B(\mathbf{x}) d\Gamma + \int_\Omega [\nabla \varphi_i][\mathbf{K} \nabla u_B(\mathbf{x})] d\Omega \right].$$

The first integral is already approximated in exercise (a), for the second integral holds the following, with $bd_{e_k}$ the line element on $\Gamma_2$, so $bd_{e_k}$ is the line between $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$, with $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$ boundary elements on $\Gamma_2$:

$$\int_{\Gamma_2} \varphi_i g_2(\mathbf{x}) d\Gamma \approx \sum_{k=1}^{n_B} \int_{bd_{e_k}} \varphi_i g_2(\mathbf{x}) d\Gamma, \text{ with } n_B = \text{the number of lines on } \Gamma_2.$$

There holds:

$$\begin{aligned} \int_{bd_{e_k}} \varphi_i g_2(\mathbf{x}) d\Gamma &\approx \frac{|bd_{e_k}|}{2} \sum_{p\{k_1,k_2\}} \varphi_i(\mathbf{x}_p) g_2(\mathbf{x}_p) \text{ (using Newton-Cotes)} \\ &= \frac{|bd_{e_k}|}{2} g_2(\mathbf{x}_i), \text{ with } i \in \{k_1, k_2\} \text{ (since } \varphi_i(\mathbf{x}_p) = \delta_{ip}). \end{aligned}$$

For $u_B$ holds that, since $\varphi_i, i \in \{1, ..., n\}$:

$$u_B(\mathbf{x}) = \sum_{j=n+1}^{n+n_E} g_1(\mathbf{x}_j) \varphi_j(\mathbf{x}),$$

where the $\varphi_j$ are piecewise linear and $\varphi_j|_{\Gamma_1} \neq 0$ also $n_E$ = the number of gridpoints on $\Gamma_1$ (the number of essential boundary points). There holds

$$\begin{aligned} &\int_{\Gamma_2} \varphi_i A u_B(\mathbf{x}) d\Gamma + \int_\Omega [\nabla \varphi_i][\mathbf{K} \nabla u_B(\mathbf{x})] d\Omega \\ &= \left[ \sum_{k=1}^{n_B} \int_{bd_{e_k}} \varphi_i A u_B(\mathbf{x}) d\Gamma \right] + \left[ \sum_{k=1}^{n_T} \int_{e_k} [\nabla \varphi_i][\mathbf{K} \nabla u_B(x)] d\Omega \right], \end{aligned}$$

with $n_B$ = the number of lines on $\Gamma_2$ and $n_T$ = the number of triangles. Next, there holds that:

$$\int_{bd_{e_k}} \varphi_i A u_B(\mathbf{x}) d\Gamma = \sum_{j=1}^{3} g_1(\mathbf{x}_{k_j}) \int_{bd_{e_k}} \varphi_i A \varphi_j d\Gamma = \sum_{j=1}^{3} g_1(\mathbf{x}_{k_j}) A \frac{|bd_{e_k}|}{6} (1 + \delta_{ij}),$$

the last equality follows from the derivation of $S^{bd_{e_k}}$. The $\varphi_i,\ i \in \{1, 2, 3\}$ are the non-zero help function on $e_k$ from the group $\varphi_i,\ i \in \{n+1, \ldots, n+n_E\}$. Also holds:

$$\int_{e_k} [\nabla \varphi_i][\mathbf{K} \nabla u_B(\mathbf{x})] d\Omega = \sum_{j=1}^{3} g_1(\mathbf{x}_{k_j}) \int_{e_k} [\nabla \varphi_i][\mathbf{K} \nabla \varphi_j] d\Omega = \sum_{j=1}^{3} g_1(\mathbf{x}_{k_j}) S_{ij}^{e_k}.$$

Therefore, we can conclude that

$$\begin{aligned} &\int_{bd_{e_k}} \varphi_i A u_B(\mathbf{x}) d\Gamma + \int_{e_k} [\nabla \varphi_i][\mathbf{K} \nabla u_B(\mathbf{x})] d\Omega \\ &= \sum_{j=1}^{3} g_1(\mathbf{x}_{k_j}) \left[ \int_{bd_{e_k}} \varphi_i A \varphi_j d\Gamma + \int_{e_k} [\nabla \varphi_i][\mathbf{K} \nabla \varphi_j] d\Omega \right]. \end{aligned}$$

There holds that:

$$\int_{bd_{e_k}} \varphi_i A \varphi_j d\Gamma + \int_{e_k} [\nabla \varphi_i][\mathbf{K} \nabla \varphi_j] d\Omega = S_{ij}^{bd_{e_k}},$$

with $i,j \in \{1,2,3\}$. We will write: $\sum_{j=1}^{3} g_1(\mathbf{x}_{k_j}) S_{ij}^{bd_{e_k}} = S^{bd_{e_k}} g_1(x_{e_k}) \in \mathbb{R}^3$.

So,
$x_{k_1}$ and $x_{k_2}$ are on the boundary: $|bd_{e_k}| = \|\mathbf{x}_{k_2} - \mathbf{x}_{k_1}\|$

$$b^{bde_k} = \begin{bmatrix} \frac{|\Delta|}{6} f(\mathbf{x}_{k_1}) \\ \frac{|\Delta|}{6} f(\mathbf{x}_{k_2}) \\ \frac{|\Delta|}{6} f(\mathbf{x}_{k_3}) \end{bmatrix} + \begin{bmatrix} \frac{\|\mathbf{x}_{k_2} - \mathbf{x}_{k_1}\|}{2} g_2(\mathbf{x}_{k_1}) \\ \frac{\|\mathbf{x}_{k_2} - \mathbf{x}_{k_1}\|}{2} g_2(\mathbf{x}_{k_2}) \\ 0 \end{bmatrix} - S^{bd_{e_k}} g_1(x_{e_k})$$

$x_{k_1}$ and $x_{k_3}$ are on the boundary: $|bd_{e_k}| = \|\mathbf{x}_{k_3} - \mathbf{x}_{k_1}\|$

$$b^{bd_{e_k}} = \begin{bmatrix} \frac{|\Delta|}{6} f(\mathbf{x}_{k_1}) \\ \frac{|\Delta|}{6} f(\mathbf{x}_{k_2}) \\ \frac{|\Delta|}{6} f(\mathbf{x}_{k_3}) \end{bmatrix} + \begin{bmatrix} \frac{\|\mathbf{x}_{k_3} - \mathbf{x}_{k_1}\|}{2} g_2(\mathbf{x}_{k_1}) \\ 0 \\ \frac{\|\mathbf{x}_{k_3} - \mathbf{x}_{k_1}\|}{2} g_2(\mathbf{x}_{k_3}) \end{bmatrix} - S^{bd_{e_k}} g_1(x_{e_k})$$

$x_{k_2}$ and $x_{k_3}$ are on the boundary: $|bd_{e_k}| = \|\mathbf{x}_{k_3} - \mathbf{x}_{k_2}\|$

$$b^{bde_k} = \begin{bmatrix} \frac{|\Delta|}{6} f(\mathbf{x}_{k_1}) \\ \frac{|\Delta|}{6} f(\mathbf{x}_{k_2}) \\ \frac{|\Delta|}{6} f(\mathbf{x}_{k_3}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\|\mathbf{x}_{k_3} - \mathbf{x}_{k_2}\|}{2} g_2(\mathbf{x}_{k_2}) \\ \frac{\|\mathbf{x}_{k_3} - \mathbf{x}_{k_2}\|}{2} g_2(\mathbf{x}_{k_3}) \end{bmatrix} - S^{bd_{e_k}} g_1(x_{e_k})$$

This all with

$$\Delta = det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \|(\mathbf{x}_2 - \mathbf{x}_1) \times (\mathbf{x}_3 - \mathbf{x}_1)\|.$$

5. We consider the special case that $A = 0$ and $\Gamma = \Gamma_2$. Derive the compatibility condition.

**Solution:**
The problem becomes:

$$\begin{cases} -\nabla \cdot [\mathbf{K}\nabla u] = f(\mathbf{x}), & \text{in } \Omega \\ \mathbf{n} \cdot \mathbf{K}\nabla u = g_2(\mathbf{x}), & \text{on } \Gamma. \end{cases}$$

Now follows that:

$$\begin{aligned} \int_\Omega f(\mathbf{x}) d\Omega &= \int_\Omega -\nabla \cdot [\mathbf{K}\nabla u] \\ &= -\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{K}\nabla u d\Gamma \\ &= -\int_{\partial\Omega} g_2(\mathbf{x}) d\Gamma. \end{aligned}$$

Thus the compatibility condition is: $\int_\Omega f(\mathbf{x}) d\Omega + \int_{\partial\Omega} g_2(\mathbf{x}) d\Gamma = 0$.

2. We use linear, triangular elements to solve a boundary value problem. We use an isoparametric transformation to derive the finite element method. We consider a triangular element $e_k$ in physical space, $e_k \subset \Omega$, and this element has vertices with coordinates $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ in the $x, y$ Cartesian coordinate system. This element is maped onto a reference element $e$, which is the triangle with vertices $(0,0), (1,0)$ and $(0,1)$ in the $s, t$ coordinate system.

1. For the basis functions in the reference element $e$, we use the basis functions $\varphi_1(s,t) = 1 - s - t, \varphi_2(s,t) = s, \varphi_3(s,t) = t$. Let $(s_1, t_1) = (0,0), (s_2, t_2) = (1,0)$ and $(s_3, t_3) = (0,1)$. Further, $\delta_{ij}$ represents the Kronecker Delta. Further, we use $x(s,t) = \sum_{p=1}^{3} x_p \varphi_p(s,t)$. Show that these basis functions satisfy $\varphi_i(s_j, t_j) = \delta_{ij}$.

**Solution:**
We have to show that:

$$\varphi_i(s_j, t_j) = \delta_{ij} = \begin{cases} 1, & \text{if j=i} \\ 0, & \text{if j} \neq \text{i} \end{cases} .$$

For $\varphi_1$ holds:

$\varphi_1(s_1, t_1) = \varphi_1(0,0) = 1 - 0 - 0 = 1$ , $\varphi_1(s_2, t_2) = \varphi_1(1,0) = 1 - 1 - 0 = 0$ and
$\varphi_1(s_3, t_3) = \varphi_1(0,1) = 1 - 0 - 1 = 0.$

So $\varphi_1(s_j, t_j) = \begin{cases} 1, & \text{if j=1} \\ 0, & \text{if j} \neq 1 \end{cases} .$
For $\varphi_2$ holds:

$$\varphi_2(s_1, t_1) = \varphi_2(0,0) = 0 \ , \ \varphi_2(s_2, t_2) = \varphi_2(1,0) = 1 \ , \ \varphi_2(s_3, t_3) = \varphi_2(0,1) = 0.$$

So $\varphi_2(s_j, t_j) = \begin{cases} 1, & \text{if j=2} \\ 0, & \text{if j} \neq 2 \end{cases} .$
For $\varphi_3$ holds:

$$\varphi_3(s_1, t_1) = \varphi_3(0,0) = 0 \ , \ \varphi_3(s_2, t_2) = \varphi_3(1,0) = 0 \ , \ \varphi_3(s_3, t_3) = \varphi_3(0,1) = 1.$$

So $\varphi_3(s_j, t_j) = \begin{cases} 1, & \text{if j=3} \\ 0, & \text{if j} \neq 3 \end{cases} .$
We can conclude that

$$\varphi_i(s_j, t_j) = \delta_{ij} = \begin{cases} 1, & \text{if j=i} \\ 0, & \text{if j} \neq \text{i} \end{cases} .$$

2. Express the Jacobian matrix $\frac{\partial(x,y)}{\partial(s,t)} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$, and its determinant $\Delta$ in terms of the vertex coordinates of $e_k$.

   **Solution:**
   Suppose that the vertices coordinates of $e_k$ are $(x_{k_1}, y_{k_1})$, $(x_{k_2}, y_{k_2})$ and $(x_{k_3}, y_{k_3})$. There holds that:

$$\mathbf{x} = \mathbf{x}(s,t) = \mathbf{x}_{k_1}(1 - s - t) + \mathbf{x}_{k_2} s + \mathbf{x}_{k_3} t,$$

   because

$$\mathbf{x}(0,0) = \mathbf{x}_{k_1}, \mathbf{x}(1,0) = \mathbf{x}_{k_2}, \mathbf{x}(0,1) = \mathbf{x}_{k_3}.$$

   Therefore:

$$\begin{aligned} x &= x(s,t) = x_{k_1}(1 - s - t) + x_{k_2} s + x_{k_3} t, \\ y &= y(s,t) = y_{k_1}(1 - s - t) + y_{k_2} s + y_{k_3} t. \end{aligned}$$

   So,

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} x_{k_2} - x_{k_1} & x_{k_3} - x_{k_1} \\ y_{k_2} - y_{k_1} & y_{k_3} - y_{k_1} \end{pmatrix},$$

   therefore the determinant $\Delta$ is given by:

$$\Delta = (x_{k_2} - x_{k_1})(y_{k_3} - y_{k_1}) - (y_{k_2} - y_{k_1})(x_{k_3} - x_{k_1}).$$

3. Calculate the Jacobian matrix $\frac{\partial(s,t)}{\partial(x,y)}$. Express your results in terms of the coordinates of the vertices $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ and the determinant $\Delta$ (which is the determinant from assignment 2b).

**Solution:**
Suppose that the vertices coordinates of $e_k$ are $(x_{k_1}, y_{k_1})$, $(x_{k_2}, y_{k_2})$ and $(x_{k_3}, y_{k_3})$. There holds that:

$$\frac{\partial(s,t)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}^{-1} = \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^{-1}.$$

Since for a $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ holds that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, it follows that:

$$\frac{\partial(s,t)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} y_{k_3} - y_{k_1} & x_{k_1} - x_{k_3} \\ y_{k_1} - y_{k_2} & x_{k_2} - x_{k_1} \end{pmatrix}.$$

4. Express $\frac{\partial \varphi_1}{\partial x}$ and $\frac{\partial \varphi_1}{\partial y}$ in terms of $\Delta$ and the coordinate positions of the vertices $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$.

**Solution:**
Suppose that the vertices coordinates of $e_k$ are $(x_{k_1}, y_{k_1})$, $(x_{k_2}, y_{k_2})$ and $(x_{k_3}, y_{k_3})$. From the chain rule follows that:

$$\begin{aligned}
\frac{\partial \varphi_1}{\partial x} &= \frac{\partial \varphi_1}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial \varphi_1}{\partial t}\cdot\frac{\partial t}{\partial x} = -1\cdot\frac{1}{\Delta}(y_{k_3} - y_{k_1}) - 1\cdot\frac{1}{\Delta}(y_{k_1} - y_{k_2}) = \frac{1}{\Delta}(y_{k_1} - y_{k_3} - y_{k_1} + y_{k_2}) \\
&= \frac{1}{\Delta}(y_{k_2} - y_{k_3}), \\
\frac{\partial \varphi_1}{\partial y} &= \frac{\partial \varphi_1}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial \varphi_1}{\partial t}\cdot\frac{\partial t}{\partial y} = -1\cdot\frac{1}{\Delta}(x_{k_1} - x_{k_3}) - 1\cdot\frac{1}{\Delta}(x_{k_2} - x_{k_1}) = \frac{1}{\Delta}(x_{k_3} - x_{k_1} + x_{k_1} - x_{k_2}) \\
&= \frac{1}{\Delta}(x_{k_3} - x_{k_2}).
\end{aligned}$$

5. Let $v_1$ and $v_2$ be given constants, compute $S_{11}^{e_k} = \int_{e^k} |\nabla\varphi_1|^2 + (v_1\frac{\partial \varphi_1}{\partial x} + v_2\frac{\partial \varphi_1}{\partial y})\varphi_1 d\Omega$ in terms of $\Delta$ and the coordinate positions of the vertices $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$.

**Solution:**
Suppose that the vertices coordinates of $e_k$ are $(x_{k_1}, y_{k_1})$, $(x_{k_2}, y_{k_2})$ and $(x_{k_3}, y_{k_3})$. There holds that:

$$\nabla\varphi_1 = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x} \\ \frac{\partial \varphi_1}{\partial y} \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} y_{k_2} - y_{k_3} \\ x_{k_3} - x_{k_2} \end{pmatrix}.$$

So,

$$|\nabla\varphi_1|^2 = ((\nabla\varphi_1\cdot\nabla\varphi_1)^{1/2})^2 = \frac{1}{\Delta^2}\begin{pmatrix} y_{k_2} - y_{k_3} \\ x_{k_3} - x_{k_2} \end{pmatrix}\cdot\begin{pmatrix} y_{k_2} - y_{k_3} \\ x_{k_3} - x_{k_2} \end{pmatrix} = \frac{1}{\Delta^2}\left[(y_{k_2} - y_{k_3})^2 + (x_{k_3} - x_{k_2})^2\right].$$

Now the first integral can be calculated:

$$\begin{aligned}
\int_{e^k} |\nabla\varphi_1|^2 d\Omega &= \int_e \left[\begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y} \end{pmatrix}\begin{pmatrix} \frac{\partial \varphi_1}{\partial s} \\ \frac{\partial \varphi_1}{\partial t} \end{pmatrix}\right]\cdot\left[\begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y} \end{pmatrix}\begin{pmatrix} \frac{\partial \varphi_1}{\partial s} \\ \frac{\partial \varphi_1}{\partial t} \end{pmatrix}\right] |\Delta| d\Omega_{st} \\
&= \int_e \frac{1}{\Delta^2}\left[(y_{k_2} - y_{k_3})^2 + (x_{k_3} - x_{k_2})^2\right] |\Delta| d\Omega_{st} \\
&= \frac{1}{|\Delta|}\left[(y_{k_2} - y_{k_3})^2 + (x_{k_3} - x_{k_2})^2\right]\int_e d\Omega_{st} \\
&= \frac{1}{|\Delta|}\left[(y_{k_2} - y_{k_3})^2 + (x_{k_3} - x_{k_2})^2\right]\frac{1}{2},
\end{aligned}$$

since $\int_e d\Omega_{st}$ is the surface of the triangle with vertices $(0,0),(1,0),(0,1)$. For the second integral, there holds

$$v_1\frac{\partial\varphi_1}{\partial x}+v_2\frac{\partial\varphi_1}{\partial y}=\frac{v_1}{\Delta}\left(y_{k_2}-y_{k_3}\right)+\frac{v_2}{\Delta}\left(x_{k_3}-x_{k_2}\right).$$

Therefore

$$\begin{aligned}\int_{e^k}(v_1\frac{\partial\varphi_1}{\partial x}+v_2\frac{\partial\varphi_1}{\partial y})\varphi_1 d\Omega &= \int_e(v_1\frac{\partial\varphi_1}{\partial x}+v_2\frac{\partial\varphi_1}{\partial y})\varphi_1|\Delta|d\Omega_{st}\\
&=\frac{|\Delta|}{\Delta}[v_1(y_{k_2}-y_{k_3})+v_2(x_{k_3}-x_{k_2})]\int_e\varphi_1 d\Omega_{st}\\
&=\frac{|\Delta|}{\Delta}[v_1(y_{k_2}-y_{k_3})+v_2(x_{k_3}-x_{k_2})]\int_e(1-s-t)d\Omega_{st}\\
&=\frac{|\Delta|}{\Delta}[v_1(y_{k_2}-y_{k_3})+v_2(x_{k_3}-x_{k_2})]\int_0^1\int_0^{1-s}(1-s-t)dtds\\
&=\frac{|\Delta|}{\Delta}[v_1(y_{k_2}-y_{k_3})+v_2(x_{k_3}-x_{k_2})]\int_0^1[t-st-t^2/2]_0^{1-s}ds\\
&=\frac{|\Delta|}{\Delta}[v_1(y_{k_2}-y_{k_3})+v_2(x_{k_3}-x_{k_2})]\int_0^1\frac{1}{2}-s+\frac{1}{2}s^2ds\\
&=\frac{|\Delta|}{\Delta}[v_1(y_{k_2}-y_{k_3})+v_2(x_{k_3}-x_{k_2})]\left[\frac{1}{2}s-\frac{1}{2}s^2+\frac{1}{6}s^3\right]_0^1\\
&=\frac{|\Delta|}{\Delta}[v_1(y_{k_2}-y_{k_3})+v_2(x_{k_3}-x_{k_2})]\frac{1}{6}.\end{aligned}$$

We can conclude that

$$\begin{aligned}S_{11}^{e_k}&=\int_{e^k}|\nabla\varphi_1|^2+(v_1\frac{\partial\varphi_1}{\partial x}+v_2\frac{\partial\varphi_1}{\partial y})\varphi_1 d\Omega\\
&=\int_{e^k}|\nabla\varphi_1|^2 d\Omega+\int_{e^k}(v_1\frac{\partial\varphi_1}{\partial x}+v_2\frac{\partial\varphi_1}{\partial y})\varphi_1 d\Omega\\
&=\frac{1}{|\Delta|}\left[(y_{k_2}-y_{k_3})^2+(x_{k_3}-x_{k_2})^2\right]\frac{1}{2}+\frac{|\Delta|}{\Delta}[v_1(y_{k_2}-y_{k_3})+v_2(x_{k_3}-x_{k_2})]\frac{1}{6}.\end{aligned}$$

**3.** We consider the following weak formulation for $u\in H_0^1(0,1)$, where $H_0^1(0,1):=\{u\in H^1(0,1):u(0)=u(1)=0\}$, where $H^1(0,1):=\{u\in L^2(0,1):u'\in L^2(0,1)\}$, and $u'=\frac{du}{dx}$:

$$(W):\begin{cases}\text{Find } u\in H_0^1(0,1) \text{ such that } a(u,v)=(f,v),\ \forall\ v\in H_0^1(0,1),\\ \text{where } a(u,v)=\int_0^1 u'v'+vu'\ dx \text{ and } (f,v)=\int_0^1 vf\ dx\end{cases}$$

Note that $a(\cdot,\cdot)$ is a nonsymmetrix bilinear form in $H_0^1(0,1)\times H_0^1(0,1)$.

1. Derive the corresponding boundary value problem for smooth solutions on the above weak formulation.

   **Solution:**

We have that

$$\begin{aligned}
&a(u,v) = (f,v)\\
&\Rightarrow \int_0^1 u'v' + vu'dx = \int_0^1 vfdx\\
&\Rightarrow \int_0^1 u'v' + (uv)' - (uv')dx = \int_0^1 vfdx \text{ (because } (uv)' = u'v + uv'\text{, so } vu' = (uv)' - uv')\\
&\Rightarrow \int_0^1 u'v' - uv'dx + \int_0^1 (uv)'dx = \int_0^1 vfdx\\
&\Rightarrow \int_0^1 u'v' - uv'dx + (uv)|_0^1 = \int_0^1 vfdx\\
&\Rightarrow \int_0^1 u'v' - uv'dx + 0 = \int_0^1 vfdx \text{ (since } u(1) = v(1) = u(0) = v(0) = 0)\\
&\Rightarrow \int_0^1 v'(u' - u)dx = \int_0^1 vfdx.
\end{aligned}$$

Now let $U = u' - u \Rightarrow dU = (u'' - u')dx$ and $dV = v'dx \Rightarrow V = v$, then partial integration on the integral on the left gives:

$$\begin{aligned}
&v(u'-u)|_{x=0}^{x=1} - \int_0^1 v(u''-u')dx = \int_0^1 vfdx\\
&\Rightarrow \int_0^1 v(f + u'' - u')dx = 0 \text{ (since } v(1) = v(0) = 0\text{, so } v(u'-u)|_{x=0}^{x=1} = 0).
\end{aligned}$$

Because $v(1) = v(0) = 0$, it follows from Dubois-Reymond that:

$$f + u'' - u' = 0 \text{ on (0,1)} \Rightarrow -u'' + u' = f$$

So the boundary value problem becomes:

$$\begin{cases} -u'' + u' = f \text{ on (0,1)} \\ u(0) = u(1) = 0 \end{cases}.$$

Let $\Sigma_0^h(0,1)$ be a finite dimensional subset of $H_0^1(0,1)$, then we search the finite element approximation of $u$ in $\Sigma_0^h(0,1)$, given by

$$(W_h): \text{Find } u_h \in \Sigma_0^h(0,1) \text{ such that } a(u_h, v_h) = (f, v_h), \qquad \forall\, v_h \in \Sigma_0^h(0,1).$$

2. Use the form $(W_h)$ and the weak form $(W)$ to show that

$$a(u - u_h, v_h) = 0, \qquad \forall\, v_h \in \Sigma_0^h(0,1)$$

**Solution:**
From $(W_h)$ follows that:

$$a(u_h, v_h) = (f, v_h) \Rightarrow \int_0^1 u_h'v_h' + v_h u_h' dx = (f, v_h).$$

From $(W)$ follows that:

$$a(u, v_h) = (f, v_h) \Rightarrow \int_0^1 u'v_h' + v_h u' dx = (f, v_h).$$

So,

$$\begin{aligned}
a(u-u_h, v_h) &= \int_0^1 (u-u_h)'v_h' + v_h(u-u_h)'dx \\
&= \int_0^1 u'v_h' + v_h u' dx - \int_0^1 u_h' v_h' + v_h u_h' dx \\
&= (f, v_h) - (f, v_h) \text{ (from above)} \\
&= 0.
\end{aligned}$$

3. Let $\|f\|_{L^2(0,1)} := \left[\int_0^1 f^2 dx\right]^{1/2}$, show that

$$0 \leq \|(u-u_h)'\|^2_{L^2(0,1)} = a(u-u_h, u-u_h) = a(u-u_h, u-v_h), \qquad \forall\ v_h \in_0^h (0,1)$$

**Solution:**
There holds that:

$$\begin{aligned}
&\|(u-u_h)'\|^2_{L^2(0,1)} = \int_0^1 [(u-u_h)']^2 dx \\
&\Rightarrow \|(u-u_h)'\|^2_{L^2(0,1)} = \int_0^1 (u'-u_h')^2 dx \\
&\Rightarrow \|(u-u_h)'\|^2_{L^2(0,1)} \geq 0 \text{ (since } (u'-u_h')^2 \geq 0 \Rightarrow \int_0^1 (u'-u_h')^2 dx \geq 0).
\end{aligned}$$

For $a(u-u_h, u-u_h)$ there holds that:

$$\begin{aligned}
a(u-u_h, u-u_h) &= \int_0^1 (u-u_h)'(u-u_h)' + (u-u_h)(u-u_h)'dx \\
&= \int_0^1 (u'-u_h')^2 dx + \int_0^1 (u-u_h)(u-u_h)'dx \\
&= \|(u-u_h)'\|^2_{L^2(0,1)} + \int_0^1 (u-u_h)(u-u_h)'dx.
\end{aligned}$$

Now the second integral is zero, because:

$$\begin{aligned}
\int_0^1 (u-u_h)(u-u_h)'dx &= \int_0^1 uu' - u_h u' - u_h' u + u_h u_h' dx \\
&= \int_0^1 \left(\frac{1}{2}u^2\right)' - (u_h u)' + \left(\frac{1}{2}u_h^2\right)' dx \\
&= \int_0^1 \frac{d}{dx}\left(\frac{1}{2}u^2 - u_h u + \frac{1}{2}u_h^2\right) dx \\
&= \int_0^1 \frac{d}{dx}\left(\frac{1}{2}(u-u_h)^2\right) dx \\
&= \frac{1}{2}(u-u_h)^2|_0^1 \\
&= 0 \text{ (since } u(0) = u(1) = u_h(0) = u_h(1) = 0).
\end{aligned}$$

There also holds that:

$$\begin{aligned}
a(u-u_h, u-u_h) &= \int_0^1 (u-u_h)'(u-u_h)' + (u-u_h)(u-u_h)'dx \\
&= \left(\int_0^1 (u'-u_h')u' + u(u'-u_h')dx\right) - \left(\int_0^1 (u'-u_h')u_h' + u_h(u'-u_h')dx\right) \\
&= a(u-u_h, u) - a(u-u_h, u_h) \\
&= a(u-u_h, u) \text{ (because of b } a(u-u_h, u_h) = 0) \\
&= a(u-u_h, u) - a(u-u_h, v_h) \text{ (since} a(u-u_h, v_h) = 0 \text{ because of b)} \\
&= a(u-u_h, u-v_h).
\end{aligned}$$

We can conclude that:

$$0 \leq \|(u-u_h)'\|^2_{L^2(0,1)} = a(u-u_h, u-u_h) = a(u-u_h, u-v_h).$$

4. Show that there is a $K > 0$ such that

$$|a(u,v)| \leq K\|u'\|_{L^2(0,1)}\|v'\|_{L^2(0,1)}, \qquad \forall\ u,\ v \in H_0^1(0,1)$$

**Solution:**
There holds that:

$$\begin{aligned}
|a(u,v)| &= \left|\int_0^1 u'v' + vu'dx\right| \\
&= \left|\int_0^1 u'(v+v')dx\right| \\
&\leq ||u'\|_{L^2(0,1)}\|v+v'\|_{L^2(0,1)} \text{ (because of Cauchy-Schwartz)} \\
&\leq ||u'\|_{L^2(0,1)} \left(\|v\|_{L^2(0,1)} + \|v'\|_{L^2(0,1)}\right) \text{ (using the triangle inequality).}
\end{aligned}$$

For $\|v\|_{L^2(0,1)}$ there holds that:

$$\begin{aligned}
\|v\|^2_{L^2(0,1)} &= \int_0^1 v^2 dx \\
&\leq \frac{1}{\alpha}\int_0^1 (v')^2 dx \text{ (becuase of Poincaré's inequality, } \alpha > 0).
\end{aligned}$$

Therefore:

$$\begin{aligned}
\|v\|_{L^2(0,1)} &\leq \sqrt{\frac{1}{\alpha}\int_0^1 (v')^2 dx} \\
&= \sqrt{\frac{1}{\alpha}}\|v'\|_{L^2(0,1)}.
\end{aligned}$$

So,

$$|a(u,v)| \leq ||u'\|_{L^2(0,1)} \left(1 + \sqrt{\frac{1}{\alpha}}\right) \|v'\|_{L^2(0,1)}.$$

If we now define: $1 + \sqrt{\frac{1}{\alpha}} = K$, then $K > 0$ and

$$|a(u,v)| \leq K||u'\|_{L^2(0,1)}\ \|v'\|_{L^2(0,1)}.$$

5. Show that the results from assignment 3. and 4. imply that, for $u_h$ satisfying $(W_h)$ and for $u$ satisfying $(W)$ (and $u' \neq u_h'$), there is a $K > 0$ such that

$$0 \leq \|(u-u_h)'\|_{L^2(0,1)} \leq K\|(u-v_h)'\|_{L^2(0,1)}, \qquad \forall\ v_h \in \Sigma_0^h(0,1).$$

**Solution:**
We have :

$$
\begin{aligned}
0 &\leq \|(u-u_h)'\|^2_{L^2(0,1)} \\
&= a(u-u_h, u-v_h) \text{ (this follows from 3.)} \\
&= |a(u-u_h, u-v_h)| \text{ (since } 0 \leq a(u-u_h, u-v_h)) \\
&\leq K\|(u-u_h)'\|_{L^2(0,1)}\|(u-v_h)'\|_{L^2(0,1)} \text{ (this follows from 4.).}
\end{aligned}
$$

So,

$$
\begin{aligned}
&\|(u-u_h)'\|^2_{L^2(0,1)} \leq K\|(u-u_h)'\|_{L^2(0,1)}\|(u-v_h)'\|_{L^2(0,1)} \\
&\Rightarrow \|(u-u_h)'\|_{L^2(0,1)} \leq K\|(u-v_h)'\|_{L^2(0,1)},
\end{aligned}
$$

since $u' \neq u_h'$, so $\|(u-u_h)'\|_{L^2(0,1)}\| \neq 0$. We can conclude that

$$0 \leq \|(u-u_h)'\|_{L^2(0,1)} \leq K\|(u-v_h)'\|_{L^2(0,1)}, \qquad \forall\, v_h \in \Sigma_0^h(0,1).$$