Partiële differentiaalvergelijkingen (3341), 2020/2021

Instruction problems 5: Parabolic problems

A. Separation of variables, Dirichlet boundary conditions
The separation of variables can be applied to more general, linear and homogeneous problems. In this sense, an example is given below:

1. With given $D > 0$ and $k \in \mathbb{R}$, consider the diffusion/heat problem with reaction

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} - ku & \text{for } 0 < x < 1, t > 0, \\ u(0,t) = 0,\ u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = x(1-x), & \text{for } 0 < x < 1. \end{cases}$$

a) Determine the problems for the separated variables X satisfying the boundary conditions and subsequently find the family of solutions for u; disregard here the initial condition.

Solution. **Step 1: Transforming PDE in two ODE (seperation of variables)**
Let us assume that the solution is of the form

$$u(x,t) = X(x)\ T(t) \tag{1}$$

The given PDE in (P) becomes

$$\begin{aligned} & X(x)\ T'(t) = D\ X''(x)T(t) - k\ X(x)\ T(t), \\ \Longrightarrow\ & \frac{T'(t)}{D\ T(t)} = \frac{X''(x)}{X(x)} - \frac{k}{D} \quad \text{[Dividing both sides by } D\ X(x)\ T(t)], \\ \Longrightarrow\ & \frac{1}{D}\left(\frac{T'}{T} + k\right) = \frac{X''(x)}{X(x)}. \end{aligned}$$

Let

$$\frac{X''(x)}{X(x)} = \frac{1}{D}\left(\frac{T'}{T} + k\right) = -\beta \in \mathbb{R}. \tag{2}$$

Step 2: Solving the eigenvalue problem The given boundary conditions in (P) are

$$
\begin{aligned}
& u(0,t) = 0, \\
\implies & X(0)T(t) = 0, \\
\implies & X(0) = 0, \quad [\text{Since} T(t) \equiv 0 \text{ would result in the trivial solution } u(x,t) \equiv 0] \\
& \text{and} \\
& u(1,t) = 0, \\
\implies & X(1)T(t) = 0, \\
\implies & X(1) = 0, \quad [\text{Since} T(t) \equiv 0 \text{ would result in the trivial solution } u(x,t) \equiv 0].
\end{aligned}
$$

Hence we have to solve the following eigenvalue problem

$$X''(x) + \beta\, X(x) = 0, \quad [\text{from } 2], \tag{3}$$

$$X(0) = 0 = X(1). \tag{4}$$

Remark: The values of β for which the BVP in 3-4 has non-trivial solutions are called the **eigenvalues**. The corresponding solutions $X(x)$ are called the **eigenfunctions**.

Case 1 ($\beta < 0$)**:** In this case the general solution (G.S.) of 3 is

$$X(x) = A\, e^{\sqrt{-\beta}\, x} + B\, e^{-\sqrt{-\beta}\, x},$$

where A and B are integrating constant. Now, the BC in 4 implies that only trivial is possible. Hence, we disregard this case.

Case 2 ($\beta = 0$)**:** In this case 3 becomes

$$
\begin{aligned}
& X''(x) = 0, \\
\implies & X(x) = A\, X + B.
\end{aligned}
$$

Then the BC in 4 implies that only trivial is possible. Hence, we again disregard this case.

Case 3 ($\beta > 0$)**:** In this case the general solution (G.S.) of 3 is

$$X(x) = A\, Cos\left(\sqrt{\beta}\, x\right) + B\, Sin\left(\sqrt{\beta}\, x\right), \tag{5}$$

Using the first BC in 4, we get

$$
\begin{aligned}
& X(0) = 0, \\
\implies & A\, Cos(0) + B\, Sin(0) = 0, \\
\implies & A = 0.
\end{aligned} \tag{6}
$$

Using the second BC in 4 and setting $A = 0$, we can write

$$
\begin{aligned}
&X(1) = 0,\\
\Longrightarrow\ & B\ Sin\left(\sqrt{\beta}\right) = 0,\\
\Longrightarrow\ & Sin\left(\sqrt{\beta}\right) = 0,\ \text{[Here } B \neq 0,\ \text{since } B \equiv 0 \text{ will result trivial solution]}.
\end{aligned}
$$

Since $n = 0$ implies trivial solution. Also $Sin(-x) = -Sin(x)$ which implies that $Sin(x)$ is an odd function. Hence,

$$\beta = (n\pi)^2,\ n = 1, 2, 3, \cdots. \tag{7}$$

We set

$$B = 1 \text{ for convenience.} \tag{8}$$

Hence, the G.S. in 5 becomes

$$X_n(x) = Sin\,(n\ \pi\ x)\,,\ n = 1, 2, 3, \cdots, \quad \text{[using6} - 7 \text{ in 5]}. \tag{9}$$

Step 3: Solving the time dependent ODE From 2, we can write

$$T'(t) + (\beta\ D + k)\ T(t) = 0.$$

Hence the G.S. of this ODE is

$$
\begin{aligned}
T(t) &= C\ e^{-(k+\beta\ D)\ t},\\
T_n(t) &= C\ e^{-(k+n^2\ \pi^2\ D)\ t},\ n = 1, 2, 3, \cdots \text{ [Using 7]}.
\end{aligned} \tag{10}
$$

where C is an integrating constant.

Step 4: Principle of superposition Using 9-10 in 1, we have found infinitely many separated solutions of the form

$$u_n(x, t) = C_n\ Sin\,(n\ \pi\ x)\ \ e^{-(k+\beta\ D)\ t},\ n = 1, 2, 3, \cdots,$$

where C_n are constant. Since all $u_n(x, t)$ are solutions of our original BVP (P) and the original PDE in (P) is a 2nd order linear homogeneous PDE. Also, any linear combination of these solutions is also a solution, hence

$$
\begin{aligned}
& u(x, t) = \sum_{n=1}^{\infty} u_n(x, t),\\
\Longrightarrow\ \ & u(x, t) = \sum_{n=1}^{\infty} C_n\ Sin\,(n\ \pi\ x)\ \ e^{-(k+n^2\ \pi^2\ D)\ t},
\end{aligned} \tag{11}
$$

is also a solution.

$\square$

b) Determine the solution u satisfying the given initial condition.

Solution. Now we have to find the constant C_n in 11 such that it satisfies the IC of (P). Given IC of (P) is

$$u(x,0) = x\,(1-x),$$
$$\implies \sum_{n=1}^{\infty} C_n\ Sin\,(n\ \pi\ x) = x\,(1-x) = \phi(x),\ \text{[letting } \phi(x) = x(1-x)\text{]}. \tag{12}$$

Hence the solution of the IBVP of (P) is 11, where

$$\begin{aligned}
C_n &= 2\int_0^1 x(1-x)\ Sin\,(n\ \pi\ x)\,dx,\\
&= \left[2x(1-x)\left(-\frac{Cos(n\pi x)}{n\pi}\right)\right]_0^1 - 2\int_0^1 (1-2x)\left(-\frac{Cos(n\pi x)}{n\pi}\right)dx,\\
&= 0 + \frac{2}{n\pi}\int_0^1 (1-2x)\ Cos(n\pi x)dx,\\
&= \frac{2}{n\pi}\left[(1-2x)\left(\frac{Sin(n\pi x)}{n\pi}\right)\right]_0^1 - \frac{2}{n\pi}\int_0^1 (-2)\ \left(\frac{Sin(n\pi x)}{n\pi}\right)dx,\\
&= 0 + \frac{4}{(n\pi)^2}\int_0^1 Sin(n\pi x)dx,\\
&= \frac{4}{(n\pi)^2}\left[-\frac{Cos(n\pi x)}{n\pi}\right]_0^1,\\
&= \frac{4}{(n\pi)^3}\left[-(-1)^n + 1\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
C_n &= \frac{4}{(n\pi)^3}\left[1-(-1)^n\right],\\
\implies C_{2p} &= 0,\ \forall p \geq 1,\\
\text{and } C_{2p+1} &= \frac{8}{[(2p+1)\pi]^3},\ \forall p \geq 0.
\end{aligned}$$

This gives

$$u(x,t) = \sum_{p\geq 0} \frac{8}{[(2p+1)\pi]^3}\ Sin((2p+1)\pi x)\ e^{-(k+(2p+1)^2\ \pi^2\ D)t}.$$

Remark 2: Can $\phi(x)$ be represented as an infinite series as in 12?
Answer remark 2: Let us assume that $\phi(x)$ is continuous. From the preposition of orthogonality we know that

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{L}{2}, & \text{if } m = n > 0, \end{cases}$$

From 12, we have

$$\begin{aligned}
\phi(x) &= \sum_{n=1}^{\infty} C_n\ Sin\left(\frac{n\ \pi\ x}{L}\right),\ [\text{here } L = 1], \\
\implies \phi(x) &= C_1\ Sin\left(\frac{\pi\ x}{L}\right) + C_2\ Sin\left(\frac{2\ \pi\ x}{L}\right) + C_3\ Sin\left(\frac{3\ \pi\ x}{L}\right) + \cdots, \\
\implies \phi(x) Sin\left(\frac{m\ \pi\ x}{L}\right) &= C_1 Sin\left(\frac{\pi\ x}{L}\right)\ Sin\left(\frac{m\ \pi\ x}{L}\right) + C_2 Sin\left(\frac{2\ \pi\ x}{L}\right) Sin\left(\frac{m\ \pi\ x}{L}\right) \\
&\quad + C_3 Sin\left(\frac{3\ \pi\ x}{L}\right) Sin\left(\frac{m\ \pi\ x}{L}\right) + \cdots, \\
\implies \int_0^L \phi(x) Sin\left(\frac{m\ \pi\ x}{L}\right) dx &= \int_0^L C_1 Sin\left(\frac{\pi x}{L}\right) Sin\left(\frac{m\pi x}{L}\right) dx + \int_0^L C_2 Sin\left(\frac{2\pi x}{L}\right) Sin\left(\frac{m\pi x}{L}\right) dx \\
&\quad + \int_0^L C_3 Sin\left(\frac{3\ \pi\ x}{L}\right) Sin\left(\frac{m\ \pi\ x}{L}\right) dx + \cdots, \\
\implies \int_0^L \phi(x) Sin\left(\frac{m\ \pi\ x}{L}\right) dx &= C_m\ \frac{L}{2}\ [\text{Using orthogonality and setting } m = n], \\
\implies C_m &= \frac{2}{L} \int_0^L \phi(x)\ Sin\left(\frac{m\ \pi\ x}{L}\right) dx.
\end{aligned}$$

Definition: The expansion

$$\phi(x) = \sum_{n=1}^{\infty} C_n\ Sin\left(\frac{n\ \pi\ x}{L}\right),$$

where

$$C_n = \frac{2}{L} \int_0^L \phi(x)\ Sin\left(\frac{n\ \pi\ x}{L}\right) dx.$$

is called the Fourier sine expansion of $\phi(x)$. □

c) Alternatively, you can reduce the equation to the standard heat equation by applying an appropriate transformation for u (see also Part A, Problem 3 from the Instruction in Week 2). Compare the results provided by the two methods.

Solution. We define the transformation here as,

$$\begin{aligned} v(x,t) &= u(x,t)e^{kt}, \\ \implies \frac{\partial}{\partial x} v(x,t) &= \frac{\partial u}{\partial x}\ e^{kt}, \\ \implies \frac{\partial^2}{\partial x^2} v(x,t) &= \frac{\partial^2 u}{\partial x^2}\ e^{kt}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} v(x,t) &= u\ \frac{\partial}{\partial t} e^{kt} + e^{kt} \frac{\partial u}{\partial t}, \\ &= k\ e^{kt}\ u + e^{kt} \frac{\partial u}{\partial t}, \\ &= k\ v + e^{kt} \frac{\partial u}{\partial t}, \\ &= k\ v + e^{kt} \Big(D \frac{\partial^2 u}{\partial x^2} - ku \Big), \text{ (Given PDE)}, \\ &= k\ v + D\ \frac{\partial^2 u}{\partial x^2} e^{kt} - k\ v, \\ &= D\ \frac{\partial^2 u}{\partial x^2} e^{kt}, \\ &= D\ \frac{\partial^2}{\partial x^2} v(x,t). \end{aligned}$$

Hence,

$$\frac{\partial v}{\partial t} = D\ \frac{\partial^2 v}{\partial x^2}, \tag{13}$$

$$v(0,t) = 0 = v(1,t), \tag{14}$$

$$v(x,0) = x(1-x). \tag{15}$$

Step 1: Transforming PDE in two ODE Let us assume that the solution is of the form

$$v(x,t) = X(x)\ T(t) \tag{16}$$

Then 13 becomes

$$
\begin{aligned}
& X(x)\ T'(t) = D\ X''(x) T(t), \\
\implies & \frac{T'(t)}{D\ T(t)} = \frac{X''(x)}{X(x)} \quad \text{[Dividing both sides by } D\ X(x)\ T(t)], \\
\implies & \frac{T'}{D\ T} = \frac{X''(x)}{X(x)}.
\end{aligned}
$$

Let

$$
\frac{X''(x)}{X(x)} = \frac{T'(t)}{D\ T(t)} = -\beta \in \mathbb{R}. \tag{17}
$$

Step 2: Solving the eigenvalue problem The given boundary conditions in (TP) are

$$
\begin{aligned}
& v(0,t) = 0, \\
\implies & X(0)T(t) = 0, \\
\implies & X(0) = 0, \quad \text{[Since} T(t) \equiv 0 \text{ would result in the trivial solution } v(x,t) \equiv 0] \\
& \text{and} \\
& v(1,t) = 0, \\
\implies & X(1)T(t) = 0, \\
\implies & X(1) = 0, \quad \text{[Since} T(t) \equiv 0 \text{ would result in the trivial solution } v(x,t) \equiv 0].
\end{aligned}
$$

Hence we have to solve the following eigenvalue problem

$$
X''(x) + \beta\ X(x) = 0, \quad \text{[from 2]}, \tag{18}
$$
$$
X(0) = 0 = X(1). \tag{19}
$$

Case 3 ($\beta > 0$)**:** Similarly like before, the only non-trivial solution exit when $\beta > 0$. In this case the general solution (G.S.) of 18 is

$$
X(x) = A\ Cos\left(\sqrt{\beta}\ x\right) + B\ Sin\left(\sqrt{\beta}\ x\right), \tag{20}
$$

Using the first BC in 19, we get

$$
\begin{aligned}
& X(0) = 0, \\
\implies & A\ Cos(0) + B\ Sin(0) = 0, \\
\implies & A = 0.
\end{aligned} \tag{21}
$$

Using the second BC in 19 and setting $A = 0$, we can write

$$\begin{aligned}
&X(1) = 0, \\
\implies &B\ Sin\left(\sqrt{\beta}\right) = 0, \\
\implies &Sin\left(\sqrt{\beta}\right) = 0, \text{ [Here } B \neq 0, \text{ since } B \equiv 0 \text{ will result trivial solution].}
\end{aligned}$$

Since $n = 0$ implies trivial solution. Also $Sin(-x) = -Sin(x)$ which implies that $Sin(x)$ is an odd function. Hence,

$$\beta = (n\pi)^2,\ n = 1, 2, 3, \cdots. \tag{22}$$

We set

$$B = 1 \text{ for convenience.} \tag{23}$$

Hence, the G.S. in 20 becomes

$$X_n(x) = Sin\,(n\ \pi\ x)\,,\ n = 1, 2, 3, \cdots, \quad \text{[using21} - \text{22 in 20].} \tag{24}$$

Step 3: Solving the time dependent ODE From 17, we can write

$$T'(t) + \beta\ D\ T(t) = 0.$$

Hence the G.S. of this ODE is

$$\begin{aligned}
T(t) &= C\ e^{-\beta\ D\ t}, \\
T_n(t) &= C\ e^{-n^2\ \pi^2\ D\ t},\ n = 1, 2, 3, \cdots \text{ [Using 22].}
\end{aligned} \tag{25}$$

where C is an integrating constant.

Step 4: Compute the solution Hence, using the principle of superposition, we can write

$$v_n(x, t) = \sum_{n=1}^{\infty} C_n\ Sin\,(n\ \pi\ x)\ \ e^{-n^2\ \pi^2\ D\ t}$$

Similarly like before using the IC in 15, we know

$$\begin{aligned}
v(x, 0) &= x(1 - x), \\
\implies \sum_{n=1}^{\infty} C_n\ Sin\,(n\ \pi\ x) &= x\ (1 - x),
\end{aligned}$$

where

$$C_n = 2\int_0^1 x(1-x)\ Sin\,(n\ \pi\ x)\,dx,$$
$$= \begin{cases} 0, & \text{if } n = 2p, \\ \frac{8}{(2p+1)^3\pi^3}, & \text{if } n = 2p+1. \end{cases}$$

Hence

$$v(x,t) = \sum_{p\geq 0} \frac{8}{[(2p+1)\pi]^3}\ Sin((2p+1)\pi x)\ e^{-(2p+1)^2\ \pi^2\ D\ t},$$
$$u(x,t) = \sum_{p\geq 0} \frac{8}{[(2p+1)\pi]^3}\ Sin((2p+1)\pi x)\ e^{-(k+(2p+1)^2\ \pi^2\ D)t}.$$

Comparison of the solutions The solutions of part (b) and (c) are identical.

□

d) Show that problem (P) cannot have two different solutions.

Solution. **(Uniqueness of the solution:)** Let $\bar{u_1}$ and $\bar{u_2}$ are two solutions. Then $\bar{u} = \bar{u_1} - \bar{u_2}$ solves

$$(\bar{P})\begin{cases} \dfrac{\partial \bar{u}}{\partial t} = D\dfrac{\partial^2 \bar{u}}{\partial x^2} - k\bar{u} & \text{for } 0 < x < 1, t > 0, \\ \bar{u}(0,t) = 0,\ \ \bar{u}(1,t) = 0 & \text{for } t > 0, \\ \bar{u}(x,0) = 0, & \text{for } 0 < x < 1. \end{cases} \tag{26}$$

Since $\bar{u_1}$ and $\bar{u_2}$ satisfies the same BC and IC.

Now, we can write

$$\int_0^1 \bar{u}\frac{\partial \bar{u}}{\partial t}dx = D\int_0^1 \bar{u}\frac{\partial^2 \bar{u}}{\partial x^2}dx - k\int_0^1 (\bar{u})^2 dx,$$
$$\Longrightarrow \frac{1}{2}\frac{d}{dt}\int_0^1 (\bar{u})^2 dx = D\left[\bar{u}\frac{\partial \bar{u}}{\partial x}\right]_0^1 - D\int_0^1 \left(\frac{\partial \bar{u}}{\partial x}\right)^2 dx - k\int_0^1 (\bar{u})^2 dx,$$
$$\Longrightarrow \frac{1}{2}\frac{d}{dt}\int_0^1 (\bar{u})^2 dx = 0 - D\int_0^1 \left(\frac{\partial \bar{u}}{\partial x}\right)^2 dx - k\int_0^1 (\bar{u})^2 dx,$$
$$\Longrightarrow \frac{1}{2}\frac{d}{dt}\int_0^1 (\bar{u})^2 dx = -D\int_0^1 \left(\frac{\partial \bar{u}}{\partial x}\right)^2 dx - k\int_0^1 (\bar{u})^2 dx,$$

with

$$E(t) = \int_0^1 (\bar{u})^2 dx.$$

Hence

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E(t) = -D\int_0^1 \left(\frac{\partial \bar{u}}{\partial x}\right)^2 dx - k\, E(t),\\
\Longrightarrow\ & E'(t) + 2k\, E(t) = -2\, D\int_0^1 \left(\frac{\partial \bar{u}}{\partial x}\right)^2 dx,\\
\Longrightarrow\ & E'(t) + 2k\, E(t) \leq 0.
\end{aligned}$$

Clearly, at $t = 0$ one has

$$E(0) = \int_0^1 (\bar{u}(x,0))^2 dx = 0,$$

and for any $t > 0$, $E(t) \geq 0$. □

Since $e^{2kt} \geq 0 \forall t$, then we can write

$$\begin{aligned}
& e^{2kt}E'(t) + 2k\, e^{2kt}\, E(t) \leq 0,\\
\Longrightarrow\ & \left(e^{2kt}\, E(t)\right)' \leq 0, \forall t \geq 0,\ \text{Hence it is a decreasing}\\
\Longrightarrow\ & e^{2kt}\, E(t) \leq e^{2k\,0}\, E(0) = 0,\\
\Longrightarrow\ & E(t) = 0 \forall t \geq 0, [\text{Since } E(t) \geq 0],\\
\Longrightarrow\ & \int_0^1 (\bar{u}(x,t))^2 dx = 0,\\
\Longrightarrow\ & \bar{u}(x,t) = 0 \forall x \in (0,1), t > 0,\ [\text{By the vanishing lemma}].
\end{aligned}$$

Hence the solution is unique.

B. Separation of variables, other boundary conditions

As discussed in the lecture, the boundary conditions are determining the eigenpairs (λ, X). Some examples are considered below:

1. With given $D > 0$ and $k \in \mathbb{R}$, consider the diffusion/heat problem with reaction

$$(P) \begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} - 1 & \text{for } 0 < x < 1, t > 0, \\ \dfrac{\partial u}{\partial x}(0,t) = 0,\ \dfrac{\partial u}{\partial x}u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = x(1-x), & \text{for } 0 < x < 1. \end{cases}$$

Note that the boundary conditions are of Neumann type and that an inhomogeneous term appears on the right of the equation.

a) Reduce Problem (P) to the standard heat equation by using the transform $v(x,t) = u(x,t) + f(x,t)$ with an appropriate function f (see also Part A, Problem 5 from the Instruction in Week 2). What are the initial and the boundary conditions?

Solution. We define the transformation here as,

$$\begin{aligned} v(x,t) &= u(x,t) + t, \\ \implies \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial x}, \\ \implies \frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial u}{\partial t} + 1, \\ \implies \frac{\partial v}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} - 1 + 1, \\ \implies \frac{\partial v}{\partial t} &= D\frac{\partial^2 u}{\partial x^2}, \\ \implies \frac{\partial v}{\partial t} &= D\frac{\partial^2 v}{\partial x^2} \end{aligned}$$

Hence the transform problem is

$$(TP1)\begin{cases} \dfrac{\partial v}{\partial t} = D\dfrac{\partial^2 v}{\partial x^2} & \text{for } 0 < x < 1, t > 0, \\ \dfrac{\partial}{\partial x}v(0,t) = 0,\ \dfrac{\partial}{\partial x}v(1,t) = 0 & \text{for } t > 0, \\ v(x,0) = x(1-x), & \text{for } 0 < x < 1. \end{cases} \tag{27}$$

□

b) Determine the solution v and subsequently u satisfying the given initial condition.

Solution. **Step 1: Transforming PDE in two ODE** Let us assume that the solution is of the form

$$v(x,t) = X(x)\ T(t) \tag{28}$$

Then the PDE in 27 becomes

$$\begin{aligned} & X(x)\ T'(t) = D\ X''(x)T(t), \\ \implies & \frac{T'(t)}{D\ T(t)} = \frac{X''(x)}{X(x)} \quad \text{[Dividing both sides by } D\ X(x)\ T(t)], \\ \implies & \frac{T'}{D\ T} = \frac{X''(x)}{X(x)}. \end{aligned}$$

Let

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{D\ T(t)} = -\beta \in \mathbb{R}. \tag{29}$$

Step 2: Solving the eigenvalue problem The given boundary conditions in $(TP1)$ are

$$\begin{aligned} & v'(0,t) = 0, \\ \implies & X'(0)T(t) = 0, \\ \implies & X'(0) = 0, \quad \text{[Since} T(t) \equiv 0 \text{ would result in the trivial solution } v(x,t) \equiv 0] \\ & \text{and} \\ & v'(1,t) = 0, \\ \implies & X'(1)T(t) = 0, \\ \implies & X'(1) = 0, \quad \text{[Since} T(t) \equiv 0 \text{ would result in the trivial solution } v(x,t) \equiv 0]. \end{aligned}$$

Hence we have to solve the following eigenvalue problem

$$X''(x) + \beta\, X(x) = 0, \quad \text{[from 29]}, \tag{30}$$
$$X'(0) = 0 = X'(1). \tag{31}$$

Case 1 ($\beta < 0$)**:** In this case the general solution (G.S.) of 3 is

$$\begin{aligned} X(x) &= A\, e^{\sqrt{-\beta}\, x} + B\, e^{-\sqrt{-\beta}\, x}, \\ \implies X'(x) &= A\ \sqrt{-\beta}\, e^{\sqrt{-\beta}\, x} - B\ \sqrt{-\beta}\, e^{-\sqrt{-\beta}\, x}, \end{aligned}$$

where A and B are integrating constant. Since,

$$\begin{aligned} &X'(0) = A\ \sqrt{-\beta}\, e^{\sqrt{-\beta}\, 0} - B\ \sqrt{-\beta}\, e^{-\sqrt{-\beta}\, 0}, \\ &\implies 0 = \sqrt{-\beta}(A - B), \\ &\implies A = B. \end{aligned}$$

and

$$\begin{aligned} &X'(1) = A\ \sqrt{-\beta}\, e^{\sqrt{-\beta}} - B\ \sqrt{-\beta}\, e^{-\sqrt{-\beta}}, \\ &\implies 0 = A\ \sqrt{-\beta}\left(e^{\sqrt{-\beta}} - \frac{1}{e^{\sqrt{-\beta}}}\right), \\ &\implies A = 0. \end{aligned}$$

Now, the BC in 31 implies that $A = B = 0$, hence the solution is trivial. Hence, we disregard this case.

Case 2 ($\beta = 0$)**:** In this case 30 becomes

$$\begin{aligned} X''(x) &= 0, \\ \implies X'(x) &= A, \\ \implies X(x) &= A\ X + B. \end{aligned}$$

Since,

$$\begin{aligned} &X'(0) = A, \\ &\implies 0 = A, \\ &\implies A = 0. \end{aligned}$$

Hence

$$X(x) = B = B_0.$$

Case 3 ($\beta > 0$)**:** In this case the general solution (G.S.) of 30 is

$$X(x) = A\, Cos\left(\sqrt{\beta}\, x\right) + B\, Sin\left(\sqrt{\beta}\, x\right), \tag{32}$$

$$\Longrightarrow \; X'(x) = -A\, \sqrt{\beta}\, Sin\left(\sqrt{\beta}\, x\right) + B\, \sqrt{\beta}\, Cos\left(\sqrt{\beta}\, x\right), \tag{33}$$

Using the first BC in 31, we get

$$\begin{aligned} &X'(0) = 0,\\ \Longrightarrow\; &-A\, \sqrt{\beta}\, 0 + B\, \sqrt{\beta}\, 1 = 0,\\ \Longrightarrow\; &B = 0. \end{aligned} \tag{34}$$

Using the second BC in 31 and setting $B = 0$, we can write

$$\begin{aligned} &X'(1) = 0,\\ \Longrightarrow\; &A\, Sin\left(\sqrt{\beta}\right) = 0,\\ \Longrightarrow\; &Sin\left(\sqrt{\beta}\right) = 0\ [\text{Here } A \neq 0,\ \text{since } A = 0 \text{ will result trivial solution}],\\ \Longrightarrow\; &\sqrt{\beta} = n\, \pi, \quad n = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots. \end{aligned}$$

Since $n = 0$ implies trivial solution. Also $Sin(-x) = -Sin(x)$ which implies that $Sin(x)$ is an odd function. Hence,

$$\beta = (n\pi)^2,\ n = 1, 2, 3, \cdots. \tag{35}$$

Hence, the G.S. in 32 becomes

$$X_n(x) = Cos\,(n\, \pi\, x)\,,\ n = 1, 2, 3, \cdots. \tag{36}$$

Step 3: Solving the time dependent ODE From 29, we can write

$$T'(t) + \beta\, D\, T(t) = 0.$$

Hence the G.S. of this ODE is

$$\begin{aligned} T(t) &= C\, e^{-\beta\, D\, t},\\ T_n(t) &= C_n\, e^{-n^2\, \pi^2\, D\, t},\ n = 1, 2, 3, \cdots\ [\text{Using } 35]. \end{aligned} \tag{37}$$

where C_n are an integrating constant.

Step 4: Compute the solution (Principle of superposition)

$$v(x,t) = B_0 + \sum_{n=1}^{\infty} C_n e^{-n^2\,\pi^2\,D\,t}\;Cos\,(n\;\pi\;x)$$

where

$$B_0 = \frac{1}{L}\int_0^L \phi(x)dx,$$

$$C_n = \frac{2}{L}\int_0^L \phi(x)\;Cos\left(\frac{n\;\pi\;x}{L}\right)dx.$$

Here $L = 1$ and

$$v(x,0) = x(1-x) = \phi(x),$$

$$\Longrightarrow\; B_0 + \sum_{n=1}^{\infty} C_n\;Cos\,(n\;\pi\;x) = \phi(x).$$

Determine B_0

$$B_0 = \int_0^1 x(1-x)dx,$$

$$\Longrightarrow B_0 = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1,$$

$$\Longrightarrow B_0 = \frac{1}{2} - \frac{1}{3},$$

$$\Longrightarrow B_0 = \frac{1}{6}.$$

Determine C_n

$$
\begin{aligned}
C_n &= 2\int_0^1 x(1-x)\, Cos\,(n\ \pi\ x)\, dx,\\
&= \left[2x(1-x)\left(\frac{Sin(n\pi x)}{n\pi}\right)\right]_0^1 - 2\int_0^1 (1-2x)\left(\frac{Sin(n\pi x)}{n\pi}\right) dx,\\
&= 0 - \frac{2}{n\pi}\int_0^1 (1-2x)\ Cos(n\pi x) dx,\\
&= -\frac{2}{n\pi}\left[(1-2x)\left(-\frac{Cos(n\pi x)}{n\pi}\right)\right]_0^1 - \left(-\frac{2}{n\pi}\right)\int_0^1 (-2)\ \left(-\frac{Cos(n\pi x)}{n\pi}\right) dx,\\
&= \frac{2}{(n\pi)^2}\left[(1-2)\ Cos(n\pi) - (1-0)Cos(0)\right] + \frac{4}{(n\pi)^2}\int_0^1 Cos(n\pi x) dx,\\
&= \frac{2}{(n\pi)^2}\left[-Cos(n\pi) - 1\right] + \frac{4}{(n\pi)^2}\left[\frac{Sin(n\pi x)}{n\pi}\right]_0^1,\\
&= \frac{2}{(n\pi)^2}\left[-(-1)^n - 1\right] + \frac{4}{(n\pi)^3}\left[Sin(n\pi) - Sin(0)\right],\\
&= -\frac{2}{(n\pi)^2}\left[1 + (-1)^n\right] + 0,\\
&= -\frac{2}{(n\pi)^2}\left[1 + (-1)^n\right].
\end{aligned}
$$

$$
\begin{aligned}
\implies\ C_{2p+1} &= 0, \forall p \geq 0,\\
\text{and } C_{2p} &= -\frac{4}{[(2p)\pi]^2}, \forall p \geq 1.
\end{aligned}
$$

Determine the solution

$$
\begin{aligned}
v(x,t) &= B_0 + \sum_{n=1}^{\infty} C_n e^{-n^2\ \pi^2\ D\ t}\ Cos\,(n\ \pi\ x)\,,\\
\implies v(x,t) &= \frac{1}{6} - \frac{4}{[(2p)\pi]^2}\sum_{p\geq 1} e^{-(2p)^2\ \pi^2\ D\ t}\ Cos\,(2\ p\ \pi\ x)\,,\\
\implies u(x,t) &= v(x,t) - t.
\end{aligned}
$$

$\square$

2. Consider the parabolic (heat) problem ($D > 0$ being a given constant)

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < 1, t > 0, \\ \dfrac{\partial u}{\partial x}(0,t) = 1,\ u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = x - 1 + \cos\left(\frac{\pi}{2}x\right) & \text{for } 0 < x < 1. \end{cases}$$

Note that the boundary conditions are not homogeneous.

a) Reduce Problem (P) to a problem involving homogeneous boundary conditions in $x = 0$ and $x = 1$. Use the transform $v(x,t) = u(x,t) + f(x,t)$ with an appropriate function f.

Solution. We define the transformation here as,

$$\begin{aligned} v(x,t) &= u(x,t) - x + 1, \\ \implies \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial x} - 1, \\ \implies \frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial u}{\partial t}, \\ \implies \frac{\partial v}{\partial t} &= D\frac{\partial^2 u}{\partial x^2}, \\ \implies \frac{\partial v}{\partial t} &= D\frac{\partial^2 v}{\partial x^2} \end{aligned}$$

Now,

$$\begin{aligned} \frac{\partial}{\partial x}v(0,t) = \frac{\partial}{\partial x}u(0,t) - 1 = 1 - 1 = 0, \\ v(1,t) = u(1,t) - 1 + 1 = 0, \\ \text{and} \\ v(x,0) = u(x,0) - x + 1 = x - 1 + Cos(\pi x/2) - x + 1 = Cos\left(\frac{\pi x}{2}\right). \end{aligned}$$

Hence the transform problem is

$$(TPB2)\begin{cases} \dfrac{\partial v}{\partial t} = D\dfrac{\partial^2 v}{\partial x^2} & \text{for } 0 < x < 1, t > 0, \\ \dfrac{\partial}{\partial x}v(0,t) = 0,\ v(1,t) = 0 & \text{for } t > 0, \\ v(x,0) = Cos\left(\frac{\pi x}{2}\right) = \phi(x), & \text{for } 0 < x < 1. \end{cases} \tag{38}$$

□

b) Determine the problems for the separated variables X corresponding to the transformed problem (including the homogeneous boundary conditions) and subsequently find the family of solutions for v. Disregard here the initial condition.

c) Determine the solution v of the transformed problem and further determine u.

Solution. **Step 1: Transforming PDE in two ODE** Let us assume that the solution is of the form

$$v(x,t) = X(x)\ T(t) \tag{39}$$

Then the PDE in 38 becomes

$$\begin{aligned} & X(x)\ T'(t) = D\ X''(x)T(t), \\ \implies & \frac{T'(t)}{D\ T(t)} = \frac{X''(x)}{X(x)} \quad \text{[Dividing both sides by } D\ X(x)\ T(t)\text{]}, \\ \implies & \frac{T'}{D\ T} = \frac{X''(x)}{X(x)}. \end{aligned}$$

Let

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{D\ T(t)} = -\beta \in \mathbb{R}. \tag{40}$$

Step 2: Solving the eigenvalue problem The given boundary conditions in $(TPB2)$ are

$$\begin{aligned}
& v'(0,t) = 0, \\
\implies & X'(0)T(t) = 0, \\
\implies & X'(0) = 0, \quad [\text{Since} T(t) \equiv 0 \text{ would result in the trivial solution } v(x,t) \equiv 0] \\
& \text{and} \\
& v(1,t) = 0, \\
\implies & X(1)T(t) = 0, \\
\implies & X(1) = 0, \quad [\text{Since} T(t) \equiv 0 \text{ would result in the trivial solution } v(x,t) \equiv 0].
\end{aligned}$$

Hence we have to solve the following eigenvalue problem

$$X''(x) + \beta\, X(x) = 0, \quad [\text{from } 40], \tag{41}$$

$$X'(0) = 0 = X(1). \tag{42}$$

Similarly like other exercise (check it yourself), the solutions are trivial for $\beta < 0$ and $\beta = 0$.

Case 3 ($\beta > 0$)**:** In this case the general solution (G.S.) of 41 is

$$X(x) = A\ Cos\left(\sqrt{\beta}\, x\right) + B\ Sin\left(\sqrt{\beta}\, x\right), \tag{43}$$

$$\implies X'(x) = -A\ \sqrt{\beta}\ Sin\left(\sqrt{\beta}\, x\right) + B\ \sqrt{\beta}\ Cos\left(\sqrt{\beta}\, x\right), \tag{44}$$

Using the first BC in 42, we get

$$\begin{aligned}
& X'(0) = 0, \\
\implies & -A\ \sqrt{\beta}\ 0 + B\ \sqrt{\beta}\ 1 = 0, \\
\implies & B = 0.
\end{aligned} \tag{45}$$

Using the second BC in 42 and setting $B = 0$, we can write

$$\begin{aligned}
& X(1) = 0, \\
\implies & A\ Cos\left(\sqrt{\beta}\right) = 0, \\
\implies & Cos\left(\sqrt{\beta}\right) = 0 \ [\text{Here } A \neq 0, \text{ since } A = 0 \text{ will result trivial solution}], \\
\implies & \sqrt{\beta} = \frac{(2\,n+1)\,\pi}{2}, \quad n = 0, 1, 2, 3, \cdots.
\end{aligned}$$

Hence, the eigenvalues are

$$\beta = \frac{(2\ n+1)^2\ \pi^2}{4},\ n = 0, 1, 2, 3, \cdots, \tag{46}$$

and the corresponding eigenfunctions are

$$X_n(x) = Cos\left(\frac{(2\ n+1)\ \pi\ x}{2}\right),\ n = 1, 2, 3, \cdots. \tag{47}$$

Step 3: Solving the time dependent ODE From 40, we can write

$$T'(t) + \beta\ D\ T(t) = 0.$$

Hence the G.S. of this ODE is

$$\begin{aligned} T(t) &= C\ e^{-\beta\ D\ t}, \\ T_n(t) &= C_n\ e^{-\frac{(2\ n+1)^2 \pi^2}{4}\ D\ t},\ n = 0, 1, 2, 3, \cdots \text{ [Using 46]}. \end{aligned} \tag{48}$$

where C_n are an integrating constant.

Step 4: Compute the solution (Principle of superposition)

$$v(x, t) = \sum_{n=0}^{\infty} C_n\ e^{-\frac{(2\ n+1)^2 \pi^2\ D\ t}{4}}\ Cos\left(\frac{(2\ n+1)\ \pi\ x}{2}\right)$$

where

$$C_n = \frac{2}{L} \int_0^L \phi(x)\ \ Cos\left(\frac{(2\ n+1)\ \pi\ x}{L}\right) dx.$$

Here $L = 2$ and

$$\begin{aligned} v(x, 0) = Cos\left(\frac{\pi x}{2}\right) &= \phi(x), \\ \implies \sum_{n=0}^{\infty} C_n\ Cos\left(\frac{(2\ n+1)\ \pi\ x}{2}\right) &= \phi(x). \end{aligned}$$

Determine C_n

$$C_n = \frac{2}{2} \int_0^2 Cos\left(\frac{\pi x}{2}\right)\ Cos\left(\frac{(2\ n+1)\ \pi\ x}{2}\right) dx = 0, \quad \text{for } n \geq 1.$$

and

$$
\begin{aligned}
C_0 &= \frac{2}{2}\int_0^2 Cos\left(\frac{\pi x}{2}\right)\ Cos\left(\frac{2\pi\ x}{2}\right)dx,\\
&= \int_0^2 Cos\left(\frac{\pi x}{2}\right)\ Cos\left(\frac{\pi x}{2}\right)dx,\\
&= 1.
\end{aligned}
$$

Determine the solution

$$
\begin{aligned}
& v(x,t) = C_0\ e^{-\frac{\pi^2 D\,t}{4}}\ Cos\left(\frac{\pi x}{2}\right),\\
\Longrightarrow\ & v(x,t) = e^{-\frac{\pi^2 D\,t}{4}}\ Cos\left(\frac{\pi x}{2}\right),\\
\Longrightarrow\ & u(x,t) = v(x,t) + x - 1.
\end{aligned}
$$

□