Assignments 1

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1. Given the following problem:

$$(\mathbf{P}_1)\begin{cases} -\left(\frac{\partial}{\partial x}(k_1(x,y)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(k_2(x,y)\frac{\partial u}{\partial y})\right) + \beta_1\frac{\partial u}{\partial x} + \beta_2\frac{\partial u}{\partial y} = f(x,y) & \text{in } \Omega \\ n_x k_1(x,y)\frac{\partial u}{\partial x} + n_y k_2(x,y)\frac{\partial u}{\partial y} + \alpha u = 0 & \text{on } \partial\Omega \end{cases}$$

Here $\Omega \subset \mathbb{R}^2$ is bounded by $\partial\Omega$. The closure of Ω is denoted by $\overline{\Omega}$. The outward unit normal vector is denoted by $\mathbf{n} = [n_x, n_y]^T$. We consider solutions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Further, k_1, k_2 and $f(x,y)$ are given functions and α, β_1 and β_2 are given real-valued constants.

1. Let $k_1(x,y)$, $k_2(x,y) > 0$ on $\overline{\Omega}$ and let $\alpha = 0, \beta_1 = \beta_2 = 0$. Which condition should be satisfied by $f(x,y)$ for the existence of a solution. Is the solution uniquely defined? Motivate your answers.

Solution:

The problem becomes: $(\mathbf{P}_1)\begin{cases} -\frac{\partial}{\partial x}(k_1(x,y)\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(k_2(x,y)\frac{\partial u}{\partial y}) = f(x,y) & \text{in } \Omega \\ n^T\begin{bmatrix} k_1(x,y)\frac{\partial u}{\partial x} & k_2(x,y)\frac{\partial u}{\partial y}\end{bmatrix}^T = 0 & \text{on } \partial\Omega \end{cases}$

There holds:

$$-\frac{\partial}{\partial x}\left(k_1(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_2(x,y)\frac{\partial u}{\partial y}\right) = -\nabla \cdot \begin{bmatrix} k_1\frac{\partial u}{\partial x} \\ k_2\frac{\partial u}{\partial y}\end{bmatrix}$$

Thus:

$$\int_\Omega -\nabla \cdot \begin{bmatrix} k_1\frac{\partial u}{\partial x} \\ k_2\frac{\partial u}{\partial y}\end{bmatrix} = \int_\Omega f(x,y)d\Omega$$

The Gauss divergence theorem says:

$$\int_\Omega f(x,y)d\Omega = -\int_\Omega \text{div}\begin{bmatrix} k_1\frac{\partial u}{\partial x} \\ k_2\frac{\partial u}{\partial y}\end{bmatrix} = -\int_{\partial\Omega}\begin{bmatrix} k_1\frac{\partial u}{\partial x} \\ k_2\frac{\partial u}{\partial y}\end{bmatrix}\cdot n\ d\Gamma = 0,$$

because

$$n^T\begin{bmatrix} k_1(x,y)\frac{\partial u}{\partial x} & k_2(x,y)\frac{\partial u}{\partial y}\end{bmatrix}^T = 0.$$

Thus, for a solution to exist, $\int_\Omega f(x,y)d\Omega = 0$

The solution is not unique. Take, for example, a solution u_1 and we add a constant $u_2 = u_1 + c$ ($c \in \mathbb{R}$). Then

$$\begin{cases} -\frac{\partial}{\partial x}(k_1(x,y)\frac{\partial u_2}{\partial x}) - \frac{\partial}{\partial y}(k_2(x,y)\frac{\partial u_2}{\partial y}) = f(x,y) & \text{in } \Omega \\ n\begin{bmatrix} k_1(x,y)\frac{\partial u_2}{\partial x} & k_2(x,y)\frac{\partial u_2}{\partial y}\end{bmatrix}^T = 0 & \text{on } \partial\Omega \end{cases}$$

$$\iff \begin{cases} -\frac{\partial}{\partial x}(k_1(x,y)\frac{\partial(u_1+c)}{\partial x}) - \frac{\partial}{\partial y}(k_2(x,y)\frac{\partial(u_1+c)}{\partial y}) = f(x,y) & \text{in } \Omega \\ n\begin{bmatrix} k_1(x,y)\frac{\partial(u_1+c)}{\partial x} & k_2(x,y)\frac{\partial(u_1+c)}{\partial y}\end{bmatrix}^T = 0 & \text{on } \partial\Omega \end{cases}$$

$$\iff \begin{cases} -\frac{\partial}{\partial x}(k_1(x,y)\frac{\partial u_1}{\partial x}) - \frac{\partial}{\partial y}(k_2(x,y)\frac{\partial u_1}{\partial y}) = f(x,y) & \text{in } \Omega \\ n\left[k_1(x,y)\frac{\partial u_1}{\partial x} \quad k_2(x,y)\frac{\partial u_1}{\partial y}\right]^T = 0 & \text{on } \partial\Omega \end{cases}$$

We see that u_2 is another solution of this problem. The solution is thus not uniquely defined.

2. Imagine that the signs of k_1 and k_2 are not given. Under which sign combinations is the PDE in $(\mathbf{P}_1)$ elliptic or hyperbolic?

Solution:
We know that:
$$\begin{cases} \frac{\partial}{\partial x}\left(k_1(x,y)\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}k_1(x,y)\frac{\partial u}{\partial x} + k_1(x,y)\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial}{\partial x}\left(k_2(x,y)\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial y}k_2(x,y)\frac{\partial u}{\partial x} + k_2(x,y)\frac{\partial^2 u}{\partial y^2} \end{cases}$$
So:
$$\begin{cases} a_{11} &= -k_1(x,y) \\ a_{22} &= -k_2(x,y) \\ a_{12} &= 0 \end{cases}$$
$\det(A) = a_{11}a_{22} - a_{12}^2 = k_1(x,y)k_2(x,y)$

We can conclude:

1. if $k_1(x,y) > 0$ and $k_2(x,y) > 0 \Rightarrow$ *elliptic*
2. if $k_1(x,y) > 0$ and $k_2(x,y) < 0 \Rightarrow$ *hyperbolic*
3. if $k_1(x,y) < 0$ and $k_2(x,y) > 0 \Rightarrow$ *hyperbolic*
4. if $k_1(x,y) < 0$ and $k_2(x,y) < 0 \Rightarrow$ *elliptic*

3. Let $k_1(x,y),\ k_2(x,y) > 0$ on $\overline{\Omega}$. Let $\alpha > 0, \beta_1 = \beta_2 = 0$, prove that $(\mathbf{P}_1)$ has at most one solution. Existence of a solution does not have to be proved.

Solution:
Assume that there are 2 distrinct solutions u_1 and u_2. Define $v = u_1 - u_2$.
We have,
$$-\left(\frac{\partial}{\partial x}(k_1(x,y)\frac{\partial u_1}{\partial x}) + \frac{\partial}{\partial y}(k_2(x,y)\frac{\partial u_1}{\partial y})\right) = f(x,y)$$
$$-\left(\frac{\partial}{\partial x}(k_1(x,y)\frac{\partial u_2}{\partial x}) + \frac{\partial}{\partial y}(k_2(x,y)\frac{\partial u_2}{\partial y})\right) = f(x,y)$$

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$$-\left(\frac{\partial}{\partial x}\left[k_1(x,y)\left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[k_2(x,y)\left(\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial y}\right)\right]\right) = 0$$

and we have,
$$n_x k_1(x,y)\frac{\partial u_1}{\partial x} + n_y k_2(x,y)\frac{\partial u_1}{\partial y} + \alpha u_1 = 0$$
$$n_x k_1(x,y)\frac{\partial u_2}{\partial x} + n_y k_2(x,y)\frac{\partial u_2}{\partial y} + \alpha u_2 = 0$$

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$$n_x k_1(x,y)\left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right) + n_y k_2(x,y)\left(\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial y}\right) + \alpha(u_1 - u_2) = 0$$

so for v we have,

$$\begin{cases} -\nabla \cdot \begin{bmatrix} k_1(x,y)\frac{\partial v}{\partial x} \\ k_2(x,y)\frac{\partial v}{\partial y} \end{bmatrix} = 0 & \text{, in } \Omega \\ n^T \begin{bmatrix} k_1(x,y)\frac{\partial v}{\partial x} \\ k_2(x,y)\frac{\partial v}{\partial y} \end{bmatrix} = -\alpha v & \text{, on } \partial\Omega \end{cases}$$

If we now define $w = \begin{bmatrix} k_1(x,y)\frac{\partial v}{\partial x} \\ k_2(x,y)\frac{\partial v}{\partial y} \end{bmatrix}$, then
$\nabla \cdot w = 0 \Rightarrow v\nabla \cdot w = 0 \Rightarrow \nabla \cdot (vw) - \nabla(v) \cdot w = 0$ $(\nabla \cdot (vw) = \nabla(v) \cdot w + v\nabla \cdot w)$
Thus,

$$\begin{aligned} 0 &= \int_\Omega \nabla \cdot (vw) - \nabla(v) \cdot w d\Omega \\ &= \int_{\partial\Omega} vw \cdot n d\Gamma - \int_\Omega \nabla(v) \cdot w d\Omega \text{ (Gauss' divergence theorem)} \\ &= -\left(\int_{\partial\Omega} \alpha v^2 d\Gamma + \int_\Omega \begin{bmatrix} k_1(x,y) & k_2(x,y) \end{bmatrix} \begin{bmatrix} \left(\frac{\partial v}{\partial x}\right)^2 \\ \left(\frac{\partial v}{\partial y}\right)^2 \end{bmatrix} d\Omega \right) \text{ (because } w \cdot n = -\alpha v) \end{aligned}$$

Now is $\alpha > 0, v^2 \geq 0, k_1(x,y) > 0$ and $k_2(x,y) > 0$
So, because each part is positive, and the sum has to be 0:

$$\int_{\partial\Omega} \alpha v^2 d\Gamma = 0 \Rightarrow v = 0 \text{ on } \partial\Omega$$

and

$$\int_\Omega \begin{bmatrix} k_1(x,y) & k_2(x,y) \end{bmatrix} \begin{bmatrix} \left(\frac{\partial v}{\partial x}\right)^2 \\ \left(\frac{\partial v}{\partial y}\right)^2 \end{bmatrix} d\Omega = \int_\Omega k_1(x,y)\left(\frac{\partial v}{\partial x}\right)^2 + k_2(x,y)\left(\frac{\partial v}{\partial y}\right)^2 d\Omega = 0 \Rightarrow \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0 \text{ in } \Omega$$

Thus, v is constant in Ω, and because $v = 0$ on $\partial\Omega$ and it's continuous (u_1 and u_2 are continuous), $v = 0$ in Ω, and finally $u_1 = u_2$. Thus, if the solution exists, there is at most one.

2. In this assignment, we prove the following theorem:
Theorem 1: *Let $f \in C^2(\Omega)$, and*

$$\begin{aligned} -u"(x) &= f(x), \quad x \in (0,1) \\ u(0) &= \alpha, \quad u'(1) = \beta \end{aligned}$$

where α, $\beta \in \mathbb{R}$. if all finite-difference approximations for the derivatives give a local truncation error of $\mathcal{O}(h^2)$, where h denotes the gridsize, then the infinity-norm of the difference between the finite-difference solution (that is, the global truncation error), v and exact solution, u, is also of order $\mathcal{O}(h^2)$, that is $\|u - v\|_\infty = \mathcal{O}(h^2)$.
The proof of this theorem is done in sequential small steps.

1. Use an equidistant grid of n unknowns with stepsize h over the interval $(0,1)$, with $x_j = jh, x_n = nh = 1$, with a virtual node next to $x = 1$ to discretize the differential equation including boundary conditions. Use central discretizations and show that the truncation error is $\mathcal{O}(h^2)$. Derive the discretization matrix A such that the numerical solution satisfies

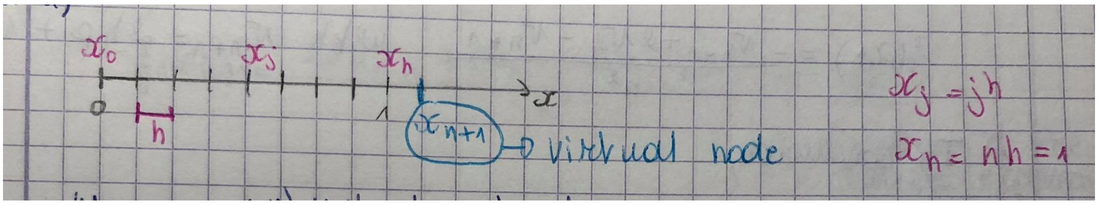
$$A\mathbf{v} = \mathbf{f}$$

One can construct A such that it is symmetric, but for the time being, we will not do so. Motivate that the exact solution represented on the meshpoints satisfies

$$A\mathbf{u} = \mathbf{f} + \mathbf{p}h^2 + \mathbf{q}h$$

Describe the origin of the vectors $\mathbf{p}$ and $\mathbf{q}$.

Solution:



We use central discretization, so:

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h}$$

For the second derivative, we find, because of the taylor expansion,

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u^{(3)}(x) + \frac{h^4}{24}u^{(4)}(\xi_1)$$

$$\text{with } \xi_1 \in (x, x+h)$$

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u^{(3)}(x) + \frac{h^4}{24}u^{(4)}(\xi_2)$$

$$\text{with } \xi_2 \in (x-h, x)$$

$$\Rightarrow \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} = u''(x) + \frac{h^2}{24}(u^{(4)}(\xi_1) + u^{(4)}(\xi_2))$$

So the truncated error is $\mathcal{O}(h^2)$.

If we discretize it, and knowing that $x_j = x_{j-1} + h$, the problem becomes ($v \approx u$):

$$\begin{cases} \frac{-v_{j-1}+2v_j-v_{j+1}}{h^2} = f(x_j) \\ u(0) = \alpha \text{ and } u'(1) = \beta \end{cases}$$

Now because $u'(1) = \beta$, We have with central discretization:

$$\begin{aligned} \frac{u_{n+1} - u_{n-1}}{2h} &= u'(1) + \frac{h^2}{12}\left(u^{(3)}(\xi_3) + u^{(3)}(\xi_4)\right) \\ &= \beta + \mathcal{O}(h^2) \end{aligned}$$

So, $\frac{v_{n+1}-v_{n-1}}{2h} = \beta$ and also

$$f(x_n) = \frac{-v_{n-1} + 2v_n - v_{n+1}}{h^2} \qquad \text{with } v_{n+1} = 2h\beta + v_{n-1}$$

In conclusion,

$$f(x_n) = \frac{-2v_{n-1} + 2v_n - 2h\beta}{h^2} = \frac{-2v_{n-1} + 2v_n}{h^2} - \frac{2}{h}\beta$$

$$f(x_1) = \frac{-v_0 + 2v_1 - v_2}{h^2} = \frac{-\alpha + 2v_1 - v_2}{h^2}$$

So for the matrix A, we find:

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -2 & 2 \end{bmatrix}$$

So,

$$A \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} f(x_1) + \frac{\alpha}{h^2} \\ \vdots \\ f(x_{n-1}) \\ f(x_n) + \frac{2}{h}\beta \end{bmatrix}$$

(Note for completeness: $v_0 = f(x_0) = \alpha$)
In earlier calculations:

$$\frac{u(x-h) - 2u(x) + u(x+h)}{h^2} = u''(x) + h^2 \frac{u^{(4)}(\xi_1) + u^{(4)}(\xi_2)}{24}$$

We use $\frac{u_{n+1} - u_{n-1}}{2h} = \beta + ch^2$ with $c = \frac{u^{(3)}(\xi_3) + u^{(3)}(\xi_4)}{12}$
$\Rightarrow u_{n+1} = u_{n-1} + 2h\beta + 2ch^3$
So,

$$\begin{aligned} f(x_n) &= \frac{-2u_{n-1} + 2u_n - 2h\beta + 2ch^3}{h^2} + \mathcal{O}(h^2) \\ &= \frac{-2u_{n-1} + 2u_n}{h^2} - \frac{2}{h}\beta + 2ch + \mathcal{O}(h^2) \\ f(x_n) + \frac{2}{h}\beta &= \frac{-2u_{n-1} + 2u_n}{h^2} + 2ch + \mathcal{O}(h^2) \end{aligned}$$

So, if we take $\mathbf{q} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 2c \end{bmatrix}$ and take $\mathbf{p} = \frac{u^{(4)}(\xi_1) + u^{(4)}(\xi_2)}{24}\mathbf{e}$. We become:

$$A\mathbf{u} = \mathbf{f} + \mathbf{p}h^2 + \mathbf{q}h$$

2. Prove the following assertion: *If* $A\mathbf{w} \geq 0$, *then* $\mathbf{w} \geq 0$.

Solution:
Suppose $A\mathbf{w} \geq 0$, thus $\frac{-w_{j-1} + 2w_j - w_{j+1}}{h^2} \geq 0 (\forall j \in \{2, ...n-1\})$, $\frac{2w_1 - w_2}{h^2} \geq 0$ and $\frac{-2w_{n-1} + 2w_n}{h^2} \geq 0$.
So if:

$$\begin{cases} 2w_1 - w_2 \geq 0 \\ -w_{j-1} + 2w_j - w_{j+1} \geq 0 \quad \forall j \in \{2, ...n-1\} \\ -2w_{n-1} + 2w_n \geq 0 \end{cases}$$

Then:

$$\begin{aligned} & w_1 \geq \frac{1}{2}w_2 \\ \Rightarrow\ & \frac{-1}{2}w_2 + 2w_2 - w_3 \geq -w_1 + 2w_2 - w_3 \geq 0 \Rightarrow w_2 \geq \frac{2}{3}w_3 \qquad (-w_1 \leq \frac{-1}{2}w_2) \\ \Rightarrow\ & \frac{-2}{3}w_3 + 2w_3 - w_4 \geq -w_2 + 2w_3 - w_4 \geq 0 \Rightarrow w_3 \geq \frac{3}{4}w_4 \\ \Rightarrow\ & w_{n-1} \geq \frac{n-1}{n}w_n \\ \Rightarrow\ & 2(-\frac{n-1}{n} + 1)w_n = \frac{2}{n}w_n \geq 0 \Rightarrow w_n \geq 0 \end{aligned}$$

Finally: $w_1 \geq \frac{1}{2}w_2 \geq \frac{1 \cdot 2}{2 \cdot 3}w_3 \geq \frac{1}{4}w_4 \geq \cdots \geq \frac{1}{n-1}w_{n-1} \geq \frac{1}{n}w_n \geq 0$
We conclude: $\mathbf{w} \geq 0$

3. Let $A\mathbf{z} = \|\mathbf{b}\|_\infty \mathbf{e}$, with $\|\mathbf{b}\|_\infty := \max_j |b_j|$ and $\mathbf{e} := [1\ 1\ \dots\ 1]^T$, and let $A\mathbf{w} = \mathbf{b}$. Show that $\|\mathbf{w}\|_\infty \leq \|\mathbf{z}\|_\infty$.

Solution:
We have that : $\|\mathbf{b}\|_\infty \mathbf{e} \geq \mathbf{b}$

$\Rightarrow A\mathbf{z} \geq A\mathbf{w}$
$\Rightarrow A\mathbf{z} - A\mathbf{w} \geq 0$
$\Rightarrow A(\mathbf{z} - \mathbf{w}) \geq 0$
$\Rightarrow \mathbf{z} - \mathbf{w} \geq 0$ (because of exercise 2 b)
$\Rightarrow \mathbf{z} \geq \mathbf{w}$
$\Rightarrow \|\mathbf{z}\|_\infty \geq \|\mathbf{w}\|_\infty$ $\square$

The error is defined by $\varepsilon := \mathbf{u} - \mathbf{v}$, we decompose the error into

$$\varepsilon = \varepsilon_1 + \varepsilon_2, \text{ with } A\varepsilon_1 = h^2\mathbf{p} \text{ and } A\varepsilon_2 = h\mathbf{q}$$

4. Let $A\mathbf{v}^{(1)} = \mathbf{e}$, where $\mathbf{e} = [1\ 1\ \ldots\ 1]^T$, prove that $v_j^{(1)} = \frac{1}{2}jh(2-jh), j \in \{1,\ldots,n\}$, and show that $\|\mathbf{v}^{(1)}\|_\infty \leq \frac{1}{2}$.

   Solution:
   If $A\mathbf{v}^{(1)} = \mathbf{e}$, then:
   $$\begin{cases} 2v_1 - v_2 = h^2 \\ -v_{j-1} + 2v_j - v_{j+1} = h^2 \quad \forall j \in \{2, ...n-1\} \\ 2v_n - 2v_{n-1} = h^2 \end{cases}$$
   Now we check if we fill in $v_j^{(1)} = v_j = \frac{1}{2}jh(2-jh), j \in \{1,\ldots,n\}$, the eqations from above still hold. We see that:
   $$\begin{aligned} 2v_1 - v_2 &= (2-h)h - (2-2h)h \text{ (because: } 2v_1 = 2\frac{1}{2}1h(2-h)) \\ &= 2h - 2h - h^2 + 2h^2 \\ &= h^2 \end{aligned}$$
   $$\begin{aligned} -v_{j-1} + 2v_j - v_{j+1} &= -\frac{1}{2}(j-1)h(2-(j-1)h) + jh(2-jh) - \frac{1}{2}(j+1)h(2-(j+1)h) \\ &= \frac{1}{2}\left(-2jh + j^2h^2 - jh^2 + 2h - jh^2 + h^2 + 4jh - 2j^2h^2 - 2jh + j^2h^2 + jh^2 - 2h + jh^2 + h^2\right) \\ &= \frac{1}{2}(2h^2) = h^2 \end{aligned}$$
   $$\begin{aligned} -2v_{n-1} + 2v_n &= -(n-1)h(2-(n-1)h) + nh(2-nh) \qquad (nh = 1) \\ &= -1 + h - 2h + h + h^2 + 1 \\ &= h^2 \end{aligned}$$
   Now we show that $\|\mathbf{v}^{(1)}\|_\infty \leq \frac{1}{2}$:
   We know that $v_n^{(1)} = \frac{1}{2}nh(2-nh) = \frac{1}{2}$ (because $nh = 1$) and if we define $f : [0, n[\to \mathbb{R} : j \mapsto \frac{1}{2}jh(2-jh)$ (≥ 0),then $f'(j) = h - h^2j = h(1-hj) = h(1-\frac{j}{n}) > 0$. Also $f'(n) = h * 0 = 0$, so $v_n^{(1)}$ is the maximum. We can conclude that $\|v^{(1)}\|_\infty = \max_j |v_j^{(1)}| \leq v_n^{(1)} = \frac{1}{2}$.

5. Deduce $\|\varepsilon_1\|_\infty \leq \frac{1}{2}h^2\|\mathbf{p}\|_\infty$.

   Solution:
   We have,
   $$A\mathbf{u} = \mathbf{f} + \mathbf{p}h^2 + \mathbf{q}h.$$

So,

$$A\varepsilon = A(\mathbf{u}-\mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{f} + \mathbf{p}h^2 + \mathbf{q}h - \mathbf{f} = \mathbf{p}h^2 + \mathbf{q}h.$$

Also,

$$A\varepsilon_1 = h^2\mathbf{p} \text{ and } A\varepsilon_2 = h\mathbf{q}.$$

Now,

$$\begin{aligned} A\varepsilon_1 &= h^2\mathbf{p} \\ \Rightarrow A\frac{\varepsilon_1}{h^2\mathbf{p}} &= \mathbf{e} \text{ (where } \frac{1}{\mathbf{p}} \text{ means we divide element wise)} \end{aligned}$$

From the previous exercise we find,

$$\begin{aligned} \|\frac{\varepsilon_1}{h^2\mathbf{p}}\|_\infty &\leq \frac{1}{2} \\ \Rightarrow \frac{\|\varepsilon_1\|_\infty}{h^2\|\mathbf{p}\|_\infty} &\leq \frac{1}{2} \\ \Rightarrow \|\varepsilon_1\|_\infty &\leq \frac{1}{2}h^2\|\mathbf{p}\|_\infty \end{aligned}$$

6. Let $A\mathbf{v}^{(2)} = \frac{1}{h}[0\ 0\ \ldots\ 0\ 1]^T$, show that $v_j^{(2)} = \frac{1}{2}jh$, and show that $\|\varepsilon_{(2)}\|_\infty \leq \frac{1}{2}h^2|q|$.

   Solution:
   If $A\mathbf{v}^{(2)} = \frac{1}{h}[0\ 0\ \ldots\ 0\ 1]^T$, then:

$$\begin{cases} 2v_1 - v_2 = 0 \\ -v_{j-1} + 2v_j - v_{j+1} = 0 \quad \forall j \in \{2, ...n-1\} \\ 2v_n - 2v_{n-1} = h \end{cases}$$

   Now we check if we fill in $v_j^{(2)} = v_j = \frac{1}{2}jh, j \in \{1, \ldots, n\}$, the eqautions from above still hold. We see that:

$$\begin{aligned} 2v_1 - v_2 &= h - \frac{1}{2}2h \\ &= 0 \end{aligned}$$

$$\begin{aligned} -v_{j-1} + 2v_j - v_{j+1} &= -\frac{1}{2}(j-1)h + jh - \frac{1}{2}(j+1)h \\ &= -\frac{1}{2}jh + \frac{1}{2}h + jh - \frac{1}{2}jh - \frac{1}{2}h \\ &= 0 \end{aligned}$$

$$\begin{aligned} -2v_{n-1} + 2v_n &= -(n-1)h + nh \\ &= h \end{aligned}$$

   It is easy to see that $\|v^{(2)}\|_\infty \leq \frac{1}{2}$. $(\frac{1}{2} = \frac{1}{2}nh = \frac{1}{2}\frac{n}{n} > \frac{1}{2}\frac{j}{n} \geq 0, \quad \forall j \in \{0, \cdots, n-1\})$.
   We now know that,

$$A\varepsilon_2 = h\mathbf{q} \Rightarrow A\frac{\varepsilon_2}{h^2\mathbf{q}} = \frac{1}{h}[0\ 0\ \ldots\ 0\ 1]^T,$$

   again where $\frac{1}{\mathbf{q}}$ means we divide element wise, if $q_j = 0$, then we define $\frac{1}{q_j} = 0$.
   So from above we find,

$$\|\frac{\varepsilon_2}{h^2\mathbf{q}}\|_\infty \leq \frac{1}{2} \Rightarrow \frac{\|\varepsilon_2\|_\infty}{h^2|q|} \leq \frac{1}{2} \Rightarrow \|\varepsilon_2\|_\infty \leq \frac{1}{2}h^2|q|$$

7. Give an upper bound of the infinite-norm of the error and finish the conclusions in the proof of Theorem 1.
   Solution:

$$\|\varepsilon\|_\infty = \|\varepsilon_1 + \varepsilon_2\|_\infty \leq \|\varepsilon_1\|_\infty + \|\varepsilon_2\|_\infty \leq \frac{1}{2}h^2(\|p\|_\infty + |q|)$$

   Thus:

$$\|\varepsilon\|_\infty = \mathcal{O}(h^2)$$

3. **Programming assignment:** consider the following one-dimensional boundary value problem

$$(\mathbf{P}_2): \begin{cases} -u" + u = 1 & \text{in } (0,1) \\ u(0) = 0, & u'(1) = 0 \end{cases}$$

We use an equidistant grid of n unknowns with stepsize h over the interval (0,1), with $x_j = jh, x_n = nh = 1$, with a virtual node next to $x = 1$ to discretize the differential equation including boundary conditions (see assignment 2). Program the discretization and plot the numerical solution for several stepsizes in one figure with the exact solution, which is given by

$$u(x) = 1 - \frac{e^x + e^{2-x}}{1 + e^2}$$

Use Richardson's extrapolation (use subsequent halving of the grid size) to determine the order of the maxnorm of the error.

Write a small report of at most two pages about your discretisation and your findings.

Report:
We shall first try and find a matrix A, with which we shale find discrete points of the approximation:
We use central discretization, so:

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h}$$

For the second derivative, we find, because of the taylor expansion,

$$\begin{aligned}
u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u^{(3)}(x) + \frac{h^4}{24}u^{(4)}(\xi_1) \\
&\text{with } \xi_1 \in (x, x+h) \\
u(x-h) &= u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u^{(3)}(x) + \frac{h^4}{24}u^{(4)}(\xi_2) \\
&\text{with } \xi_2 \in (x-h, x) \\
&\Rightarrow \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} = u''(x) + \mathcal{O}(h^2)
\end{aligned}$$

So $-u''(x) + u(x) \approx \frac{-u(x-h)+(h^2+2)u(x)-u(x+h)}{h^2}$.

If we discretize it, and knowing that $x_j = x_{j-1} + h$, the problem becomes ($v \approx u$):

$$\begin{cases} \frac{-v_{j-1}(h^2+2)v_j - v_{j+1}}{h^2} = 1 \\ u(0) = 0 \text{ and } u'(1) = 0 \end{cases}$$

Now because $u'(1) = 0$, We have with central discretization:

$$\begin{aligned}
\frac{u_{n+1} - u_{n-1}}{2h} &= u'(1) + \frac{h^2}{12}\left(u^{(3)}(\xi_3) + u^{(3)}(\xi_4)\right) \\
&= \mathcal{O}(h^2)
\end{aligned}$$

So, $\frac{v_{n+1}-v_{n-1}}{2h} = 0$ and also

$$1 = \frac{-v_{n-1} + (h^2+2)v_n - v_{n+1}}{h^2} \qquad \text{with } v_{n+1} = (2h)0 + v_{n-1}$$

In conclusion,

$$1 = \frac{-2v_{n-1} + (h^2+2)v_n}{h^2}$$

$$1 = \frac{-v_0 + (h^2+2)v_1 - v_2}{h^2} = \frac{(h^2+2)v_1 - v_2}{h^2}$$

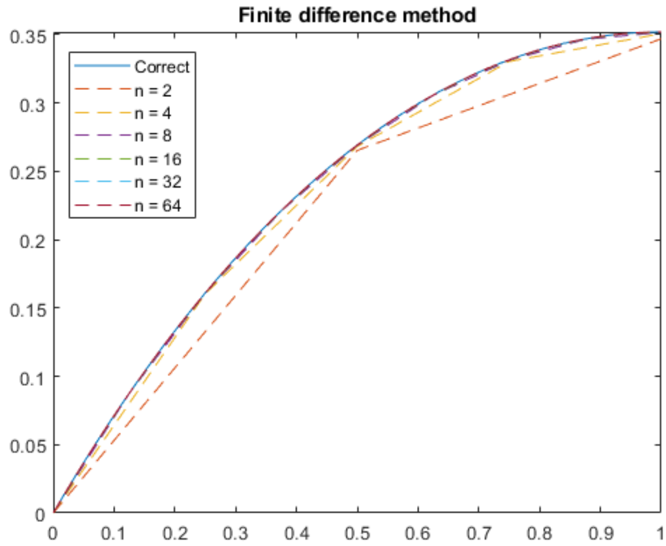
So for the matrix A, we find:

$$A = \frac{1}{h^2}\begin{bmatrix} (h^2+2) & -1 & 0 & \cdots & \cdots & 0 \\ -1 & (h^2+2) & -1 & \cdots & \cdots & 0 \\ 0 & -1 & (h^2+2) & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & (h^2+2) & -1 \\ 0 & 0 & \cdots & 0 & -2 & (h^2+2) \end{bmatrix}$$

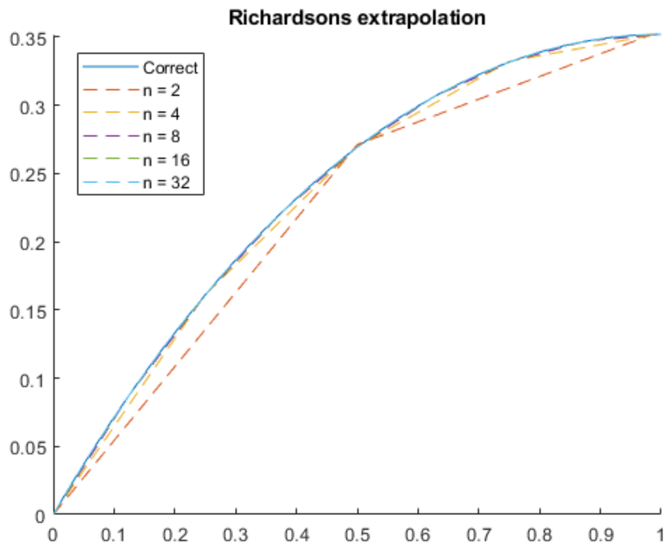
So,

$$A \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

(Note for completeness: $v_0 = 0$)
(**Extra:** We could divide the last row of A and f by $\frac{1}{2}$ to make A symmetric)
We chose to look at the following amounts of unknowns: 2, 4, 8, 16, 32 & 64.
If we were to plot our approximations (with lines connecting the nodes), we would get the following:



Due to u being equal to $A^{-1}f + A^{-1}qh + A^{-1}ph^2 = v + A^{-1}qh + A^{-1}ph^2 = v + \mathcal{O}(h)$. If we were to use Richardson's extrapolation, we would get an order of h^2 for the maxnorm of the error (If we were to listen to Wikipedia). If we were to use subsequent halving of the grid size, the Richardson's extrapolation would be (the used h is in the brackets): $u_j = \frac{2^1 v_j(h/2) - v_j(h)}{2-1}$. We would get the following plot:



If we were to try and calculate the order by $\log_2(\frac{\varepsilon_{2h}}{\varepsilon_h}) = p$. It would be around 1.5 if we used the second norm and 1.9 if we used the max norm.

Code:

```
n = [2,4,8,16,32,64];
f = @(x) 1-(exp(x)+exp(2-x))/(1+exp(2));
figure(7)
fplot(f,'DisplayName','Correct'); xlim([0,1]); ylim([0,max(f(0:0.01:1))])
hold on
e1=[];
all = {};
for i = n
    v = taak1(i,0,0);
    h = 1/i;
    x = 0:h:1;
    plot(x,v,'--','DisplayName',['n = ', num2str(i)]);
    e1 = [e1, max(abs(v-f(x).'))];
    all{log(i)/log(2)} = v;
end
log(e1(1)/e1(2))/log(2)
title('Finite difference method'); legend show
hold off

figure(4)
hold on
fplot(f,'DisplayName','Correct'); xlim([0,1]); ylim([0,max(f(0:0.01:1))])
e2 = [];
for i = 1:(length(all)-1)
    vnew = [];
    h1 = all{i};
    h2 = all{i+1};
    h = (1/(2^i));
    for j = 1:length(h1)
        vnew(j) = 2*h2(j*2-1)-h1(j);
    end
    x = 0:h:1;
    plot(x,vnew,'--','DisplayName',['n = ', num2str(2^i)]);
    e2 = [e2, max(abs(vnew-f(x)))];
end
title('Richardsons extrapolation'); legend show


function v = taak1(n,alpha,beta)
h = 1/n;
A = zeros(n);
A(1,1:2) = [1+2/(h^2), -1/(h^2)];
for i = 2:n-1
    A(i,(i-1):(i+1)) = [-1/(h^2), 1+2/(h^2), -1/(h^2)];
end
A(n,n-1:n) = [-2/(h^2), 1+2/(h^2)];
%A(n,n-1:n) = [-1/(h^2), 1/2+1/(h^2)];
f = ones(n,1);
f(1) = f(1) +alpha/(h^2);
f(n) = f(n) +2*beta/(h^2);
%f(n) = f(n)/2 +beta/(h^2);

v = (A^-1)*f;
v = [0;v];
```