Supplementary material Partial Differential Equations (3341)

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Introduction

This document provides supplementary information to be used when studying the subject. This completes the suggested reading in the book [1]. It respects the weekly planning. Please, be aware that this document is subject to changes, so you may want to check for new versions on Blackboard. Feel free to contact the teachers for additional explanations, information, etc.

Week 1: Method of characteristics

We present here the *method of characteristics*, which can be employed to solve certain categories of partial differential equations (PDEs). More precisely, the idea is assume a certain dependency on the independent variables (in the examples below the time t and the space x, or x, y, etc.) to reduce the PDE to an ordinary differential equation (ODE) that can be solved explicitly. This general idea to reduce the PDE to an ODE will be applied more often in this lecture, and in various situations (characteristics, similarity, travelling waves).

The linear wave equation

Here we start with the *method of characteristics* (MOC) aplied to the first order hyperbolic equation (*the wave equation*)

$$\partial_t u + a\partial_x u = 0, \text{ where } x \in \mathbb{R} \text{ and } t > 0. \tag{1}$$

Here $a \in \mathbb{R}$ is a given number. Further, as we will see later, to solve this equation one also needs to know the *initial condition*, namely the value of u at $t = 0$

$$u(x, 0) = u_0(x), \text{ for } x \in \mathbb{R}. \tag{2}$$

Again, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function. Clearly, the initial condition can be stated for another $t = t_0$ but, for the ease of presentation, we will always assume that the intitial time is at $t = 0$.

The idea in the MOC is to assume that x is a function of t, and to choose this function properly so that the PDE reduces to an ODE. More exactly, if $x = x(t)$ (the precise function will be specified later), one can define the new function $v : [0, \infty) \to \mathbb{R}$, $v(t) = u(x(t), t)$. Then, with this choice one has for all $t \geq 0$

$$v'(t) = \frac{du}{dt}(x(t), t) = x'(t)\partial_x u(x(t), t) + \partial_t u(x(t), t), \tag{3}$$

where in the above we have applied the chain rule. After comparing this with the left hand side of (1), it looks that choosing the function $x(\cdot)$ s.t. $x'(t) = a$ allows writing

$$v'(t) = \left(\partial_t u + a\partial_x u\right)\big|_{(x(t),t)}. \tag{4}$$

Observe that the expression on the right was evaluated in the point (x, t) where $x = x(t)$. Therefore, using (1) one obtains that $v'(t) = 0$ for all $t \geq 0$. In other words, the function $v(\cdot)$ is constant in time, which means that $u(x(t), t)$ is constant along the *characteristic* curve in the $x - t$ plane. This characteristic is defined as

$$\{(x, t) \in \mathbb{R} \times [0, \infty), x = x(t)\}, \tag{5}$$

where $x(\cdot)$ is a function satisfying $x'(t) = a$.

Observe that solving the latter ODE gives the characteristic curve as the line in the $x - t$ plane

$$x(t) = x_0 + at, \tag{6}$$

where $t \geq 0$ and x_0 is the point where the characteristic interesects the x-axis.

Since $v'(t) = 0$ for all $t \geq 0$ one has that $v(t) = v(0)$, and recalling the definition of v it follows that

$$u(x(t), t) = v(t) = v(0) = u(x(0), 0) = u_0(x_0), \tag{7}$$

whith $x_0 = x(t) - at$. In other words, for finding $u(x, t)$ one has to find the characteristig through the point (x, t), determine the point x_0 where this characteristic interesects the x-axis and then set $u(x, t) = u_0(x_0)$. In this case,

$$u(x, t) = u_0(x - at). \tag{8}$$

Observe that, in this case, the solution is simply a translation of the initial data u_0. One can say that the initial data travels in time with a fixed velocity. This is, in fact, the simplest example of a *travelling wave* (TW). In general, for a given equation depending on both space and time, a TW is a solution $u(x, t)$ that can be written as

$$u(x, t) = v(x - at),$$

for a properly chosen $a \in \mathbb{R}$. Extensions to the multi-dimensional case are possible.

Remark 1 *Note that the characteristic in* (6) *is a line due to the fact that the factor multiplying $\partial_x u$ is a constant a. Later examples will include nonconstant factors (and it is possible even to take u-dependent factors as well), and the characteristics will not remain straight lines. Also, the fact that v remains constant (and, consequently, that u remains constant along the characteristic) is due to the fact that the right-hand side in* (1) *is 0. Also this aspect will be changed in further examples.*

Remark 2 *Another important aspect here is the fact that the characteristic does intersect the x-axis, so an appropriate initial value can be found. This situation could even be extended to the case where the function u is assumed known along a curve, but one has to be sure that any characteristic does intersect that curve to provide the value of u. If the characteristics would be parallel to the curve along which u is known, than one* *cannot find the appropriate value of u anymore. Finally, a crucial aspect in applying the MOC is that for each point (x, t) where the solution needs to be known, there exists a unique characteristic passing through it. Combined with the observation before that the characteristic does intersect the x-axis (or any other curve along which u is given) the existence is needed to obtain the value $u(x, t)$. Uniqueness is equally important: if there were two different characteristics through (x, t), and they would intersect the x-axis in two different points, which of the two values of u_0 should be used for $u(x, t)$?*

The method of characteristics

We can extend this to a more general situation, to solve the equation

$$\partial_t u + a(x, t)\partial_x u = b(x, t, u), \text{ where } x \in \mathbb{R} \text{ and } t > 0, \tag{9}$$

with given initial condition $u_0 : \mathbb{R} \to \mathbb{R}$. Here $a : \mathbb{R}^2 \to \mathbb{R}$ and $b : \mathbb{R}^3 \to \mathbb{R}$ are given, sufficiently smooth functions (continuously differentiable). Repeating the previous ideas, we seek the charactiristics as solutions to the Cauchy problem

$$\left\{ \begin{array}{rcl} x'(t) & = & a(x(t), t), \quad \text{for } t \geq 0, \\ x(0) & = & x_0. \end{array} \right. \tag{10}$$

Observe that, for every $x_0 \in \mathbb{R}$, (10) has a unique solution at least locally (explain why!). This leads to the following

Definition 1 *A characteristic curve of the equation in* (9) *is any curve in the $x - t$ plane defined as*

$$\{(x, t) \in \mathbb{R} \times [0, \infty), \ \textit{with } x = x(t)\}, \tag{11}$$

where the function $x(\cdot)$ solves (10) *with an arbitrary initial value $x_0 \in \mathbb{R}$.*

Observe that, in general, the characteristic curve needs not to be a line, as previously.

As before, we observe that the function $v(t) = u(x(t), t)$ satisfies for all $t \geq 0$

$$v'(t) = \frac{du}{dt}(x(t), t) = \partial_t u + a(x(t), t)\partial_x u = b(x(t), t, u(x(t), t)) = b(x(t), t, v(t)). \tag{12}$$

Observe that v solves the Cauchy problem

$$\left\{ \begin{array}{rcl} v'(t) & = & b(x(t), t, v(t)), \quad \text{for } t \geq 0, \\ v(0) & = & u_0(x_0). \end{array} \right. \tag{13}$$

Again, due to the assumptions on b, (13) has a unique solution, at least locally, and this also gives the solution u along the characteristic.

From a practical point of view, with $(\tilde{x}, \tilde{t}) \in \mathbb{R} \times \mathbb{R}_+$, to find $u(\tilde{x}, \tilde{t})$ solving (9) with the given initial data u_0, one can apply the following procedure:

1. Find the (general) characteristics by solving (10), for arbitrary $x_0 \in \mathbb{R}$;

2. Solve (13), again, for arbitrary $x_0 \in \mathbb{R}$;

3. Find the characteristic passing through $(\tilde{x}, \tilde{t})$. In other words, find $x_0 \in \mathbb{R}$ s.t. the solution to (10) satisfies $x(\tilde{t}) = \tilde{x}$. With this, set $u(\tilde{x}, \tilde{t}) = u_0(x_0)$.

In the above we have used $(\tilde{x}, \tilde{t})$ to identify the point in the $x - t$ plane only to avoid any confusion between x as a real number, and the function $x(\cdot)$ solving (10). If there is no confusion, one may simply use again x and t.

Remark 3 *The function a in (9) does not depend on the unknown u. In other words, the expression on the left is linear w.r.t. u. This is essential to guarantee the existence and uniqueness of a characteristic curve passing through a given point (x, t). Of course, one can apply the same method also if the function a does depend on u, but only for points (x, t) for which the existence and uniqueness of a characteristic curve through them is guaranteed. As explained, there is no charactersitic, or if there are more than one, no unique value can be assigned to u and solutions in classical sense, namely a function u that has first order partial derivatives and satisfies the equation in each point, cannot be found. Such problems are considered in the general of hyperbolic conservation laws. An example in this sense is the equation*

$$\partial_t u + \partial_x f(u) = 0,$$

with f a given, nonlinear function. Discussing such problems exceeds the framework of this course.

Remark 4 *It is straghtforward to extend the MOC to problems involving multiple spatial dimensions. An example in this sense is*

$$\partial_t u(x, y, t) + a(x, y, t)\partial_x u(x, y, t) + b(x, y, t)\partial_y u(x, y, t) = c(x, y, t, u),$$

with given functions a, b, and given initial data u_0. Now the characteristics are triplets in $\mathbb{R}^3$

$$\{(x, y, t)/t \geq 0, x = x(t), y = y(t)\},$$

where the functions $x(\cdot)$ and $y(\cdot)$ solve the system of ODE's

$$\begin{cases} x'(t) & = & a(x, y, t), \\ y'(t) & = & b(x, y, t), \end{cases}$$

for $t \geq 0$ and with the initial data (x_0, y_0). Convince yourself that $v(t) = u(x(t), y(t), t)$ solving the equation $v'(t) = c(x(t), y(t), t, v)$ also provides the solution u.

Examples

We apply the steps explained before to find the solution for some equations. We omit the details, which you are asked to work out yourself.

A) With given u_0, find u solving the equation

$$\partial_t u + x\partial_x u = 0, \text{ for } x \in \mathbb{R}, t \geq 0,$$

and satisfying $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}$.
The characteristic curves are solutions to

$$x'(t) = x \text{ for } t \geq 0,$$

and with $x(0) = x_0$. Clearly, this gives $x(t) = x_0 e^t$. Observe that these curves are not lines anymore, since the factor multiplying $\partial_x u$ is not constant. With this, let $v(t) = u(x(t),t) = u\big(x_0 e^t, t\big)$. It solves $v'(t) = 0$, thus $v(t) = v(0) = u_0(x_0)$.
Given now $(\tilde{x}, \tilde{t}) \in \mathbb{R} \times \mathbb{R}_+$, one has

$$u(\tilde{x}, \tilde{t}) = v(\tilde{t}) = u_0\big(x_0 e^{\tilde{t}}\big),$$

where x_0 is s.t. $x(t) = \tilde{x}$. This gives $x_0 = \tilde{x}e^{-\tilde{t}}$, and thus $u(\tilde{x}, \tilde{t}) = u_0\big(\tilde{x}e^{-\tilde{t}}\big)$. Leaving the $\tilde{}$'s out, we obtain the solution

$$u(x,t) = u_0\big(xe^{-t}\big).$$

B) We modify now the previous example by considering a non-zero right-hand side,

$$\partial_t u + x\partial_x u = x, \text{ for } x \in \mathbb{R}, t \geq 0.$$

As before, we seek u and satisfying the equation and the initial condition $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}$.
Since the left-hand side of the equation was not changed, the characteristics remain the same, $x(t) = x_0 e^t$. The difference is for $v(t) = u(x(t),t) = u\big(x_0 e^t, t\big)$, which now solves $v'(t) = x(t)$, thus $v'(t) = x_0 e^t$, and with the initial condition $v(0) = u(x(0),0) = u_0(x_0)$. Observe that the function v will not be constant anymore. One gets $v(t) = u_0(x_0) + x_0\big(e^t - 1\big)$.
For $(\tilde{x}, \tilde{t}) \in \mathbb{R} \times \mathbb{R}_+$ we have

$$u(\tilde{x}, \tilde{t}) = u_0(x_0) + x_0\big(e^{\tilde{t}} - 1\big),$$

and x_0 is s.t. $x(\tilde{t}) = \tilde{x}$. As before, $x_0 = \tilde{x}e^{-\tilde{t}}$, yielding

$$u(\tilde{x}, \tilde{t}) = u_0\big(\tilde{x}e^{-\tilde{t}}\big) + \tilde{x}e^{-\tilde{t}}\big(e^{\tilde{t}} - 1\big).$$

In a simpler writing,

$$u(x,t) = u_0\big(xe^{-t}\big) + xe^{-t}\big(e^t - 1\big) = u_0\big(xe^{-t}\big) + x\big(1 - e^{-t}\big).$$

Please, check that this is, indeed, the solution!

Week 4: Parabolic equations in bounded domains

Comparison principle, uniqueness of a solution

Please study Chapter 8.1 in [1]. One key assumption here is that the solution is sufficiently smooth everywhere, twice continuously differentiable w.r.t. the spatial variables and once in time ($u \in C^{2,1}(\Omega \times [0, \infty))$). In the proof the Divergence Theorem is used. this result has been taught in the Vector Calculus course, but can also be found as Prop. 9.1.3 [1]. I further emphasise the following arguments used in the proof:

a. With reference to equality (9.6) in Prop. 9.1.3, in the present context the role of the functions u and v are taken by $v = u_1 - u_2$ and by $j(v)$. Also observe that $\nabla j(v) = j'(v)\nabla v$ (by the chain rule). Moreover, since $j' \geq 0$ for any argument, one has $\nabla v \cdot \nabla j(v) = j'(v)\nabla v \cdot \nabla v = j'(v)|\nabla v|^2 \geq 0$.

b. The boundary integral on $\partial\Omega_1$ is 0 because on $\partial\Omega_1$ one has $v = u_{r1} - u_{r2} \leq 0$ and, by the definition of j, $j(v) = 0$.

c. The boundary integral on $\partial\Omega_2$ is positive as, by assumption, $q_{r2} - q_{r1} \geq 0$, and by definition $j(v) \geq 0$.

d. Recalling that $J \geq 0$ for any argument, it follows that the first integral in (8.6) is positive. Since all other integrals are poitive as well, from (8.6) one gets $\int_\Omega J(v(x,t))dx = 0$ for all t. Using now the *Vanishing Lemma* we get $J(v(x,t)) = 0$ for all $x \in \Omega$ and $t > 0$.

Boundedness, maximum principle

We use here the comparison principle stated in Lemma 8.1.1. to prove that, under certain assumptions, the solution u to (8.1)–(8.4) is bounded by the extreme values of the initial and boundary data. This can be resumed as:
The solution to (8.1)-(8.4) attains its extreme values on the parabolic boundary of the cylinder $\Omega \times [0, \infty)$, namely $(\Omega \times \{0\}) \cup (\partial\Omega \times (0, \infty))$.
The result is called "maximum principle" although it concerns the minimum as well.

Lemma 1 *Maximum principle Let u be a solution to the problem*

$$\begin{aligned} \partial_t u &= \Delta u && \text{in } \Omega \times (0, \infty), \\ u &= u_D && \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{14}$$

Assume that the functions u_D and u_0 are bounded, namely there exists $m, M \in \mathbb{R}$ s.t. $m \leq u_D(x,t) \leq M$ for all $x \in \partial\Omega$ and $t > 0$, respectively $m \leq u_0(x) \leq M$ for all $x \in \Omega$. Then for the solution u the same bounds hold,

$$m \leq u(x,t) \leq M \textit{ for all } x \in \Omega, t > 0.$$

Proof. We may use here the comparison principle stated in Lemma 8.1.1. In this sense, observe that the constant function $u_m \equiv m$ (i.e. $u_m(x,t) = m$ for all $x \in \Omega$ and $t > 0$) is a solution to the equation in (14). Moreover, by the assumptions on the initial and boundary data one has $m \leq u_D$ (at the boundary) and $m \leq u_0$ (for $t = 0$). I this case one can apply Lemma 8.1.1. with $u_1 = u_m$ and $u_2 = u$ to conclude that $m \leq u$ in $\Omega \times 90, \infty)$.
The upper bound is analogous. □

Remark 5 *The result is obtained for Dirichlet type boundary conditions. However, it can be extended to other types of boundary conditions, including mixed ones (as considered in Lemma 8.1.1). For example, the proof remains almost unchanged if homogeneous Neumann type of conditions are assumed instead of Dirichlet ones,*

$$-\vec{n} \cdot \nabla u = 0 \ \text{on} \ \partial\Omega \times [0, \infty).$$

In this case, the extreme values are taken at $t = 0$. Also, adding a "source term" f is possible, but this may change the extreme values. For example, consider homogeneous Dirichlet and/or Neumann boundary conditions and assume that the initial data u_0 is positive. Given a positive source term ($f \geq 0$ overall), for the solution u of

$$\partial_t u = \Delta u + f \ \text{in} \ \Omega \times (0, \infty)$$

with the conditions mentioned above, one can prove that $u \geq 0$ (a "minimum principle"). However, it is not necessary true that u has a maximal value.

Separation of variables

This is a technique suited for solving certain types of linear equations. In a nutshell, the idea is to seek a solution in *separated form.* More precisely, if the partial differential equation is in n variables ($n \geq 1$), then one seeks a solution $u = u(x_1, \ldots, x_n)$ in the form of a product of n functions in one variable:

$$u(x_1, \ldots, x_n) = X_1(x_1)X_2(x_2)\ldots X_n(x_n).$$

By an abuse of notation, one may interpret x_n as the time t. The idea is to determine the functions $X_1, \ldots, X_n$ separately, and solving simpler, ordinary differential equations.
Before going into details we recall that by now we discussed the existence of a solution in unbounded intervals (similarity solution, as well as by means of a fundamental solution). Also, the boundedness of a solution and the comparison principle were addressed. Similarly, for bounded domains we discussed the comparison principle, the uniqueness and boundedness, and obtained energy estimates. The issue of existence of a solution was left open. Here we address it in a particular way, namely by finding a solution explicitly (or as a series of solutions) and for a particular spatial domain, a finite interval. General existence results are beyond the scope of this curse as it involves an abstract mathematical framework.

Separation of variables for the diffusion/heat equation

With given $D > 0$ and $L > 0$, here we concentrate on parabolic equations like

$$\partial_t u = D\partial_{xx} u, \text{ for } x \in (0, L), t > 0 \tag{15}$$

In the spirit of the above, we seek solutions in the form $u(x, t) = X(x)T(t)$. In other words, we find the functions X and T s.t. $u = XT$ solves the given equation. Moreover, the boundary conditions should be satisfied.
Please study now Chapter 8.5 in [1]. Observe that the first step is to "homogenize" the boundary conditions, namely to subtract a properly function from the solution s.t. the resulting takes the value 0 at the boundary. Additionally, I mention the following:

a. The reason to work with homogeneous boundary conditions will be given later.

b. By choosing $\bar{C}(x) = Ax + B$ one not only reduces the boundary conditions to homogeneous ones, but also leaves the equation unchanged. In other words, $u = C - \bar{C}$ still solves the same equation as C. On the other hand, one could also take $\bar{C}(x) = x^2 + Ax + B$ (with properly chosen A and B). Although this also leads to a function $u = C-\bar{C}$ that satisfies homogeneous boundary conditions, the eqation satisfied by u changes into $\partial_t u = D\partial_{xx} u + 2D$. This particular equation can be reduced to the standard heat equation (as we have seen in Exercise 1 from Instruction 2), but this means an additional step that can be avoided by the choice in the book. Moreover, other choices may lead to equations that can not be simplified anymore.

c. Similar ideas are applied for Neumann boundary conditions, or combinations. For example, if one has $C(0, L) = C_\ell$ and $\partial_x C(L, t) = C_r$, then $\bar{C}(x, t) = C_\ell + C_r x$ is a good choice, and $u = C - \bar{C}$ satisfies the homogeneous boundary conditions $u(0, t) = 0$ and $\partial_x u(L, t) = 0$, as well as the standard heat equation.

d. Pay attention to the initial data, as we now get $u(x, 0) = C_0(x) - \bar{C}(x)$.

e. From $XT' = DX"T$ one divides by the product DXT (assuming that it never becomes 0) to obtain the *separated equation*

$$\frac{1}{D}\frac{T'}{T} = \frac{X"}{X}, \text{ for } x \in (0, L), t > 0. \tag{16}$$

Observe that the expression on the left depends only on t, whereas the one on the right on x. Since x and t are independent variables, this equality can only be valid if both expressions are constant, say β. It is unclear at this point why taking the constant negative, $-\lambda^2$. To see this we recall that we are looking for *non-trivial* solutions, i.e. solutions that are not constant 0. We now proeed by taking $\beta \in \mathbb{R}$ instead of $-\lambda^2$ in (16). This gives

$$X" - \beta X = 0, \text{ for } x \in (0, L).$$

Furthermore, since $u(x,t) = X(x)T(t)$, to assure that $u(0,t) = 0$ for all $t > 0$ while avoiding that $T \equiv 0$ one needs to impose $X(0) = 0$. By a similar argument, $X(L) = 0$. In this way one ends up with the boundary value problem

$$\begin{cases} X\text{"} - \beta X = 0, & \text{for } x \in (0, L), \\ X(0) = 0, & X(L) = 0. \end{cases} \tag{17}$$

Its characteristic equation is $r^2 - \beta = 0$. For $\beta > 0$ we have the general solution

$$X(x) = ae^{\sqrt{\beta}x} + be^{-\sqrt{\beta}x},$$

with $a, b \in \mathbb{R}$ Using now the boundary values $X(0) = 0$ and $X(L) = 0$ one obtains $a = b = 0$, leading to the trivial solution $X \equiv 0$. This also implies $u \equiv 0$, and unless the transformed initial condition $u(x, 0) \equiv 0$ (which would only be the case if $C(x, 0) = \bar{C}(x)$), this is not a solution to the original problem.

In a similar manner one shows that taking $\beta = 0$ also leads to the trivial solution. In conclusion, to obtain non-trivial solutions, one needs to take $\beta = -\lambda^2$ with $\lambda > 0$.

f. The solutions u_n ($n \in \mathbb{N}$) in (8.27) do all satisfy the equation and the homogeneous boundary conditions. However, excepting the particular cases when the (transformed) initial condition is $u(x, 0) = \sin\left(\frac{n\pi x}{L}\right)$, these functions do not satisfy the initial condition and therefore they are not solution to the problem (8.23). Due to the linearity of the equation, any linear combination of these solutions satisfy the equation, and we will see under which conditions the same holds for series too.

Moreover, any linear combination of the functions u_n also satisfies the homogeneous boundary condition. This is why the solution u is sought as the series

$$u(x,t) = \sum_{n=0}^{\infty} c_n \sin\big(\frac{n\pi x}{L}\big) e^{-\frac{n^2\pi^2}{L^2}Dt}.$$

If instead of homogeneous boundary conditions one would take inhomogeneous ones (as in the original problem (8.22)), then it is unclear what boundary conditions to choose in (17). If these are taken homogeneous, then this leads to a solution also satisfying homogeneous boundary conditions, thus not the ones in (8.22). On the other hand, if non-zero values are taken in (17), after determining the functions X_n and the corresponding constants β_n, followed by T_n, it is unclear which linear combination of solutions $X_n T_n$ could be taken s.t. the resulting satisfies both the boundary condition and the initial one. This is why (8.22) is first transformed into (8.22) before applying the technique of separation of variables.

Finally, the coefficients $c_n \in \mathbb{R}$ are obtained s.t. the initial condition is satisfied,

$$u_0(x) = u(x, 0) = \sum_{n=0}^{\infty} c_n \sin\big(\frac{n\pi x}{L}\big)$$

Week 5: Separation of variables for the diffusion equation: convergence

We discussed previously the *separation of variables* method. With the exception of some particular cases, this is a method based on series of functions. The solution is constructed in a formal manner, and the natural question one may ask is whether this can be made mathematically rigorous. This is done below.

Preliminary results

We start with some notions and results, the latter being stated without proof. We restrict to the case of Fourier series for periodic functions with period $2L$. For details you may consult [1] and [2], or the lecture notes for the course *Fourier analysis* given by Jochen Schütz. We mention that a piecewise continuous function also has bounded variation.

$$f:[-L,L]\to\mathbb{R},\quad f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\cos\big(\frac{k\pi x}{L}\big)+b_k\sin\big(\frac{k\pi x}{L}\big)\right], \tag{18}$$

where $a_k, k\in\mathbb{N}$ and $b_k, k\in\mathbb{N}_0$ are the Fourier coefficients of f

$$a_k=\frac{1}{L}\int_{-L}^{L}f(x)\cos\big(\frac{k\pi x}{L}\big)dx,\quad b_k=\frac{1}{L}\int_{-L}^{L}f(x)\sin\big(\frac{k\pi x}{L}\big)dx. \tag{19}$$

The ideas can be extended to more general situations in an analogous way, if other functions are required in the separation of variables (such examples will be seen later). The infinite sum in (18) is approximated by the *partial sum*

$$s_N:[-L,L]\to\mathbb{R},\quad s_N(x)=\frac{a_0}{2}+\sum_{k=1}^{N}\left[a_k\cos\big(\frac{k\pi x}{L}\big)+b_k\sin\big(\frac{k\pi x}{L}\big)\right]. \tag{20}$$

One important question is related to ***convergence***: letting $N\to\infty$, will s_N converge and will f be the limit? In which sense? To answer this, we use the definitions below. We mention that the convergence of the series is equivalent to the convergence of the partial sum, namely we say that the series converges if the limit $\lim_{N\to\infty}s_N$ exists. The series converges to f if the limit before exists and it is equal to f.
Below we use notions from the measure theory, like *almost everywhere (a.e.)*, the *essential supremum*, and the L^∞ and L^2 norms

$$\|f\|_\infty=\operatorname{ess\,sup}|f(x)|,\quad \|f\|_{L^2(-L,L)}=\big(\int_{-L}^{L}f^2(x)dx\big)^{\frac{1}{2}}, \tag{21}$$

where the integral should be understood in Lebesgue sense.

Definition 2 *The Fourier series* ***converges pointwise*** *to f if for all $x\in[-L,L]$ one has $\lim\limits_{N\to\infty}s_N(x)=f(x)$. If the equality of the limit only holds for a.e. $x\in[-L,L]$ (and not for all x) then the convergence is a.e..*

Definition 3 *The Fourier series* ***converges uniformly*** *to f on $[-L, L]$ if one has* $\lim_{N\to\infty} \|s_N - f\|_\infty = 0$.

Definition 4 *The Fourier series* ***converges in L^2 sense*** *to f on $[-L, L]$ if one has* $\lim_{N\to\infty} \|s_N - f\|_{L^2(-L,L)} = 0$.

Clearly, the uniform convergence implies the pointwise convergence, which implies the convergence a.e.. Also, the uniform convergence implies the L^2 convergence. The converse results do not hold.
Recalling that *piecewise continuous* on $[-L, L]$ means that the interval can be partitioned in sub-intervals, in which the function is continuous, we introduce the following spaces of functions

$$\begin{aligned} &PC[-L, L] = \{f : [-L, L] \to \mathbb{R}, f \text{ is piecewise continuous and 2L - periodic}\}, \\ &PC^1[-L, L] = \{f \in PC[-L, L], f' \in PC[-L, L]\}. \end{aligned} \tag{22}$$

In the second space f is piecewise continuously differentiable, and the derivative f' exists has left and right limits everywhere. We have the following convergence results.

Theorem 1 *Let $f \in L^2(-L, L)$, then the Fourier series converges in L^2-sense to f and one has*

$$\|f\|^2_{L^2(-L,L)} = \frac{L}{2} a_0^2 + L \sum_{k=1}^{\infty} \big(a_k^2 + b_k^2\big).$$

The equality above is called Parseval's equality.

Theorem 2 *Let $f \in PC^1[-L, L]$, then the Fourier series converges pointwise and for all $x \in [-L, L]$ one has*

$$\lim_{N\to\infty} s_N(x) = \frac{1}{2}\big(f(x-0) + f(x+0)\big).$$

Remark 6 *In the above, $f(x \pm 0)$ stand for the left and the right limits of f in x. At $x = L$, since f is assumed $2L$ - periodic, instead of $f(L + 0)$ we take $f(-L + 0)$, and $f(L - 0)$ for $f(-L - 0)$. If f is continuous then the series converges pointwise to f.*

Theorem 3 *Let $f \in PC^1[-L, L] \cap C[-L, L]$ be s.t. $f(-L) = f(L)$. Then the Fourier series converges uniformly to f.*

Remark 7 *In the situation stated in Theorem 3, the Fourier series can be differentiated term by term, and $\{s'_N, N \in \mathbb{N}\}$ are the partial sums of the Fourier series of f'.*

Theorem 4 *Let $f \in PC[-L, L]$. Then $S_N(x) = \int_{-L}^{x} s_N(z)dz$ is the partial sum of the Fourier series of the primitive of f defined by $F(x) = \int_{-L}^{x} f(z)dz$.*

Convergence proof for the separation of variables

We consider the parabolic problem discussed first as an example for the separation of variables,

$$(P) \begin{cases} \partial_t u = D\partial_{xx} u, & \text{for } x \in (0,L), t > 0 \\ u(0,t) = 0, \quad u(L,t) = 0, & \text{for } t > 0, \\ u(x,0) = u_0(x), & \text{for } x \in (0,L). \end{cases} \tag{23}$$

The initial condition u_0 and the diffusion $D > 0$ are given (as, of course, $L > 0$). We further assume that $u_0 \in L^2(0,L)$.
By applying the separation of variables we obtain the functions

$$u_k(x,t) = e^{-D\left(\frac{k\pi}{L}\right)^2 t} \sin\left(\frac{k\pi x}{L}\right), \quad k \in \mathbb{N}_0 \tag{24}$$

satisfying the equation and the boundary conditions as in Problem P, stated in (23)). With the Fourier coefficients

$$b_k = \frac{2}{L}\int_0^L u_0(x) \sin\left(\frac{k\pi x}{L}\right) dx, \quad k \in \mathbb{N}_0, \tag{25}$$

we claim that $u(x,t) = \sum_{k=1}^{\infty} b_k u_k(x,t)$ is the solution of Problem P. To prove this claim, we use the coefficients b_k from above and the partial sums

$$s_N(x,t) = \sum_{k=1}^{N} b_k u_k(x,t), \quad N \in \mathbb{N}_0. \tag{26}$$

Observe that only the sin functions are involved in this construction, while the results in the provious section are stated for the general Fourier series. However, since the boundary conditions here are of Dirichlet type (and homogeneous), one can extend the problem straightforwardly to $[-L,L]$ in an odd way. More precisely, if u solves Problem P on $[0,L]$, then the extension $\tilde{u}$ of u satisfying $\tilde{u}(x,t) = -u(-x,t)$ for $x \in [-L,0]$ will also solve the same problem, but now on $[-L,L]$, and for the odd extension of the initial condition to $[-L,L]$. Since $\tilde{u}$ is odd, its Fourier coefficients corresponding to the cos functions will be 0, therefore only the sin functions will appear. For $\tilde{u}$ one can apply the results for Fourier series to obtain all convergence results in terms of sin series. On the other hand, these results are equivalent to the ones for u, but on the interval $[0,L]$. The main result is stated in

Theorem 5 *Let $u_0 \in L^2(0,L)$. Then the function $u(x,t) = \sum_{k=1}^{\infty} b_k u_k(x,t)$ with the coefficients b_k in (25) is a solution to Problem P. More precisely, u solves the equation $(23)_1$ for all $t > 0$ and $x \in (0,L)$, and satisfies the homogeneous Dirichlet boundary conditions in $(23)_2$. Also, it satisfies the initial condition in the L^2 sense,*

$$\lim_{t \searrow 0} \|u(\cdot,t) - u_0\|_{L^2(0,L)} = 0.$$

Finally, u is smooth for all $t > 0$, namely $u \in C^\infty([0, L] \times (0, \infty))$.

Note: Here $t \searrow 0$ means that t approaches 0 from above (the right limit) and $u(\cdot, t)$ is viewed as a function in x.

Proof. We observe that, since $u_0 \in L^2(0, L)$, by Theorem 1 one has

$$\frac{L}{2} \sum_{k=1}^{\infty} b_k^2 = \|u_0\|^2_{L^2(0,L)} < \infty.$$

Therefore the series $\sum_{k=1}^{\infty} b_k^2$ is convergent, implying that

$$\lim_{k\to\infty} |b_k| = 0, \quad \text{and} \quad \lim_{k\to\infty} \sum_{p=k}^{\infty} b_p^2 = 0. \tag{27}$$

We use these limits to prove the result. This is done in two steps: in the first we prove the uniform convergence of s_N, and in the second the properties of the limit u

Step 1. Given any $M > 0$, form (27) it follows that a $k_M \in \mathbb{N}_0$ exists s.t. $|b_k| < M$ for all $k \geq k_M$. Taking M large enough, one gets $|b_k| \leq M$ for all $k \in \mathbb{N}_0$. This implies that, for all $x \in [0, L]$ and $t > 0$,

$$|b_k u_k(x, t)| \leq M e^{-D\left(\frac{k\pi}{L}\right)^2 t}.$$

Let now $\delta > 0$ arbitrary small, then

$$|b_k u_k(x, t)| \leq M e^{-D\left(\frac{k\pi}{L}\right)^2 \delta}$$

for all $x \in [0, L]$ and $t \geq \delta$. Applying now the Weierstraß M-test, since the series $\sum_{k=1}^{\infty} M e^{-D\left(\frac{k\pi}{L}\right)^2 \delta}$ is convergent, it follows that, with u given in the hypothesis of the theorem,

$$s_N \to u \quad \text{uniformly in } [0, L] \times [\delta, \infty). \tag{28}$$

Step 2. This step has three parts: proving the smoothness of u, that it satisfies the equation and the boundary conditions, and the initial condition in the L^2 sense.

Step 2A. Since $s_N \in C^\infty([0, L] \times [\delta, \infty))$ for all $N \in \mathbb{N}_0$ and $\delta > 0$, in view of the uniform convergence to u proved at *Step 1* one obtains that $u \in C^\infty([0, L] \times [\delta, \infty))$. Since $\delta > 0$ is arbitrary, it follows that $u \in C^\infty([0, L] \times (0, \infty))$.

Step 2B. Clearly, $s_N(0, t) = s_N(L, t) = 0$, and the same holds for u due to the convergence of the function series. Further, for every $x \in (0, L)$ and $t > 0$ one can differentiate the series term by term to obtain

$$\partial_t u - D\partial_{xx} u = (\partial_t - D\partial_{xx})(\lim_{N\to\infty} s_N) = \lim_{N\to\infty} \left[(\partial_t - D\partial_{xx}) s_N\right] = 0.$$

Step 2C. It only remains to give the proof for the initial condition, namely

$$\lim_{t \searrow 0} \|u(\cdot,t) - u_0\|^2_{L^2(0,L)} = 0$$

(the square in the norm makes no difference). Observe that, by Theorem 1 one has

$$\begin{aligned}
\|u(\cdot,t) - u_0\|^2_{L^2(0,L)} &= \int_0^L |u(x,t) - u_0(x)|^2 dx \\
&= \int_0^L \sum_{k=1}^{\infty} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1 \right]^2 \sin^2\big(\frac{k\pi x}{L}\big) dx \\
&= \frac{L}{2} \sum_{k=1}^{\infty} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1 \right]^2 . \qquad (29)
\end{aligned}$$

We prove that the series on the right in (29) converges to 0 as $t \searrow 0$. To this aim we use the elementary inequalities (please, check!)

$$0 \le 1 - e^{-z} \le z, \qquad \text{for all } z \ge 0. \qquad (30)$$

Since $u_0 \in L^2$, from (27) it follows that, for any $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}_0$ s.t. $\sum\limits_{k=N_\varepsilon+1}^{\infty} b_k^2 < \frac{\varepsilon}{L}$, implying that, for all $t > 0$,

$$\frac{L}{2} \sum_{k=N_\varepsilon+1}^{\infty} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1 \right]^2 < \frac{\varepsilon}{2}. \qquad (31)$$

To deal with the remaining terms in (29) we observe that, by (30) one has

$$\left[1 - e^{-D\left(\frac{k\pi}{L}\right)^2 t} \right]^2 \le \left[D\big(\frac{k\pi}{L}\big)^2 t \right]^2 .$$

For $k \le N_\varepsilon$ this gives

$$\frac{L}{2} \sum_{k=1}^{N_\varepsilon} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1 \right]^2 \le D^2 \big(\frac{N_\varepsilon \pi}{L}\big)^4 t^2 \ \frac{L}{2} \sum_{k=1}^{N_\varepsilon} b_k^2 \le D^2 \big(\frac{N_\varepsilon \pi}{L}\big)^4 t^2 \|u_0\|^2_{L^2(0,L)},$$

where we have used, again, that u_0 is an L^2 function. Finally, with $t_\varepsilon = \frac{1}{D\|u_0\|_{L^2(0,L)}} \big(\frac{L}{N_\varepsilon \pi}\big)^2 \sqrt{\frac{\varepsilon}{2}}$, for any $t \in (0, t_\varepsilon)$ one gets

$$\frac{L}{2} \sum_{k=1}^{N_\varepsilon} b_k^2 \left[e^{-D\left(\frac{k\pi}{L}\right)^2 t} - 1 \right]^2 < \frac{\varepsilon}{2}. \qquad (32)$$

From (31) and (32) it follows that, for any $\varepsilon > 0$, a $t_\varepsilon > 0$ exists s.t.

$$\|u(\cdot,t) - u_0\|^2_{L^2(0,L)} < \varepsilon$$

for all $t \in (0, t_\varepsilon)$. This gives the desired convergence and the theorem is proved. $\square$

Remark 8 *Since $u_0 \in L^2(0,L)$, the initial condition is satisfied in L^2 sense. If u_0 is better, e.g. $C^k[0,L]$ and the restriction of an odd, $2L$ periodic function, then one can prove that u has the same type of smoothness up to $t=0$, and that $u(\cdot,t)$ converges to u_0 pointwise as $t \searrow 0$.*

We conclude this section with the result presenting the behaviour of the solution as $t \to \infty$.

Proposition 1 *Let u solve Problem P stated in* (23)*. For any $x \in [0,L]$ one has*

$$\lim_{t\to\infty} u(x,t) = 0.$$

Proof. For any $N \in \mathbb{N}_0$ one has $|s_N(x,t)| \leq \sum_{k=1}^{N} |b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 t}$ for all $x \in [0,L]$ and $t > 0$. Similarly, for all $x \in [0,L]$ and $t > 0$

$$|u(x,t)| \leq \sum_{k=1}^{\infty} |b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 t}.$$

Take now a $\tau > 0$ fixed (arbitrary) and rewrite the above as

$$|u(x,t)| \leq \sum_{k=1}^{N} |b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 \tau} e^{-D\left(\frac{k\pi}{L}\right)^2 (t-\tau)}.$$

We recall that the coefficients b_k are bounded, see Step 1 in the proof of Theorem 5. Using the fact that the exponential function grows more rapid than the quadratic one, from (27) one obtains the existence of a $k^* \in \mathbb{N}_0$ s.t.

$$|b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 \tau} \leq \frac{M}{k^2},$$

for all $k \geq k^*$. Using now the convergence of the harmonic series and that k^* is fixed, finite, it follows that the series below is convergent, namely that

$$R := \sum_{k=1}^{N} |b_k| e^{-D\left(\frac{k\pi}{L}\right)^2 \tau} < \infty.$$

Finally, for all $t > \tau$ one has obviously $e^{-D\left(\frac{k\pi}{L}\right)^2 (t-\tau)} < e^{-D\left(\frac{\pi}{L}\right)^2 (t-\tau)}$ for all $k \in \mathbb{N}_0$, and therefore

$$|u(x,t)| \leq R e^{-D\left(\frac{\pi}{L}\right)^2 (t-\tau)}.$$

The result follows immediately, by letting $t \to \infty$. $\square$

Week 6: Elliptic equations, separation of variables

In this part we use examples to discuss some additional aspects related to the separation of variables. As for parabolic problems, the basic idea is to seek solutions in a *separated form.* We restrict ourselves to the case of two spatial dimensions, but the ideas can be extended in an absolutely analogous manner to multiple dimensions. As for the parabolic case, we assume that u can be written as

$$u(x, y) = X(x)Y(y). \tag{33}$$

Observe that this only works for Cartesian domains, namely $(x, y) \in \Omega = \Omega_x \times \Omega_y$, where Ω_x and Ω_y are intervals (why?). We use the form stated in (33) to determine X and Y, and actually write u as a series of functions $u_k = X_k Y_k$, $k \in \mathbb{N}$.
Later we consider domains with radial symmetry. We use a similar approach, but rewrite the problem in polar coordinates.
In any case, we will use "as much as possible" homogeneous boundary conditions. Later we make the statement "as much as possible" clear.

Elliptic equation in a strip

First study Section 9.2 in the book. We now consider the elliptic equation:

$$\left\{\begin{array}{rcll} \Delta u &=& \partial_x u & \text{for } (x, y) \in (0, \infty) \times (0, 1), \\ u(x, 0) &=& 0, \;\; \partial_y u(x, 1) = 1 & \text{for } x > 0, \\ \partial_x u(0, y) &=& \sin\left(\frac{\pi}{2} y\right) - \sin\left(\frac{3\pi}{2} y\right) & \text{for } y \in (0, 1). \end{array}\right. \tag{34}$$

Observe that, compared to the example in Section 9.2, the strip is now horizontal. Also, the boundary conditions are not homogeneous. We will see below that it is easier to to focus now the attention on the y-direction, namely to find Y in the separated form (33). Therefore we homogenise the conditions for $y = 1$ by considering e.g.

$$v(x, y) = u(x, y) - y,$$

which satisfies the problem in (34) but with the homogeneous condition $\partial_y v(x, 1) = 0$ for all $x > 0$:

$$\left\{\begin{array}{rcll} \Delta v &=& \partial_x v & \text{for } (x, y) \in (0, \infty) \times (0, 1), \\ v(x, 0) &=& 0, \;\; \partial_y v(x, 1) = 0 & \text{for } x > 0, \\ \partial_x v(0, y) &=& \sin\left(\frac{\pi}{2} y\right) - \sin\left(\frac{3\pi}{2} y\right) & \text{for } y \in (0, 1). \end{array}\right. \tag{35}$$

Now we assume v in a separated form, $v(x, y) = X()x)Y(y)$ (see (33)). Using this in the equation of (34) gives $X''Y + XY'' = X'Y$, leading to (see also (16))

$$\frac{X'' - X'}{X} = -\frac{Y''}{Y}, \text{ for } x > 0, y \in (0, 1). \tag{36}$$

As before, we conclude that the terms in the left and right of the equality equal a constant, $\beta \in \mathbb{R}$, and seek β in such a way that non-trivial solutions are obtained.
From (36) and using the boundary conditions at $y = 0$ and $y = 1$ one ends up with

$$\begin{cases} Y'' + \beta Y = 0, & \text{for } y \in (0,1), \\ Y(0) = 0, & Y'(1) = 0. \end{cases} \tag{37}$$

As argued for parabolic equations, to obtain nontrivial solutions one needs to take $\beta > 0$. In this case one obtains the solution

$$Y(y) = a\cos\left(\sqrt{\beta}y\right) + b\sin\left(\sqrt{\beta}y\right), \text{ for } a, b \in \mathbb{R}.$$

From $Y(0) = 0$ one obtains $a = 0$. Next, $Y'(1) = 0$ gives the pairs $\{(\beta_k, Y_k), k \in \mathbb{N}\}$

$$\beta_k = \left[\frac{(2k+1)\pi}{2}\right]^2, \quad Y_k(y) = \sin\left(\frac{(2k+1)\pi}{2}y\right), \qquad . \tag{38}$$

We use the β_k above to find the corresponding X_k. From (36) one obtains ($k \in \mathbb{N}$)

$$X_k'' - X_k' - \beta_k X_k = 0, \text{ for } x > 0.$$

The solutions to the characteristic equation $r^2 - r - \beta_k = 0$ are

$$\delta_k = \frac{1}{2}\left(1 - \sqrt{1 + 4\beta_k}\right), \gamma_k = \frac{1}{2}\left(1 + \sqrt{1 + 4\beta_k}\right), \tag{39}$$

providing the solutions ($k \in \mathbb{N}$)

$$X_k(x) = c_k e^{\delta_k x} + d_k e^{\gamma_k x}, \qquad \text{for } x > 0. \tag{40}$$

Here $c_k, d_k \in \mathbb{R}$ are arbitrary and will be determined in such a way that they lead to a solution that satisfies the condition at $x = 0$.
At this point we observe that (38) and (40) provide two families of functions ($k \in \mathbb{N}$)

$$w_k(x,y) = e^{\delta_k x}\sin\left(\beta_k y\right), \text{ and } z_k(x,y) = e^{\gamma_k x}\sin\left(\beta_k y\right) \tag{41}$$

satisfying all the equation in (35) and the boundary conditions for $y = 0$ and $y = 1$.
We seek now the function v as the series

$$v(x,y) = \sum_{k\in\mathbb{N}} \left\{c_k w_k(x,y) + d_k z_k(x,y)\right\}. \tag{42}$$

In other words we identify the coefficients c_k, d_k so that the v also satisfies the remaining boundary condition, for $x = 0$.
From (41) and (42) one obtains

$$\partial_x v(0,y) = \sum_{k\in\mathbb{N}} \left\{(c_k\delta_k + d_k\gamma_k)\sin\left(\beta_k y\right)\right\}$$

The boundary condition for $x = 0$ immediately implies

$$c_0\delta_0 + d_0\gamma_0 = 1, c_1\delta_1 + d_1\gamma_1 = -1, \text{ and } c_k\delta_k + d_k\gamma_k = 0 \text{ for } k > 1. \tag{43}$$

Observe that one can have multiple solutions. For example, one can take $c_k = d_k = 0$ for $k > 1$, $d_0 = d_1 = 0$ and $c_0 = \frac{1}{\delta}_0 = -\frac{2}{\sqrt{\pi^2+1}-1}$, $c_1 = -\frac{1}{\delta}_1 = \frac{2}{\sqrt{9\pi^2+1}-1}$, yielding

$$\begin{aligned} v(x,y) = \quad & -\frac{2}{\sqrt{\pi^2+1}-1} e^{-\dfrac{\sqrt{\pi^2+1}-1}{2}x} \sin\left(\frac{\pi}{2}y\right) \\ & + \frac{2}{\sqrt{9\pi^2+1}-1} e^{-\dfrac{\sqrt{9\pi^2+1}-1}{2}x} \sin\left(\frac{3\pi}{2}y\right), \end{aligned} \tag{44}$$

and finally $u(x,y) = v(x,y) + y$.
However, one can find another solution, e.g. by taking $d_0 = 1$, $c_0 = \frac{1-\gamma_0}{\delta_0}$, and the other coefficients as above. Also, the coefficients c_k, d_k can be chosen differently for $k > 1$ as long as they satisfy $c_k\delta_k = -d_k\gamma_k$. One may be puzzled, is this in contradiction with the uniqueness result, as following from Lemma 9.1.1 (see Corollary 9.1.2)? The answer is no, as these results only apply for bounded domains, whereas the strip considered here is unbounded in the x direction.
Clearly, to identify a solution uniquely, one needs an additional criterion. In doing so we observe that δ_k, γ_k introduced in (39) have different signs, $\delta_k < 0 < \gamma_k$ for all $k \in \mathbb{N}$. Consequently, the function w_k defined in (41) is bounded, whereas z_k not. This suggests the following selection principle:

Boundedness: Find the solution u to (34) that is bounded for all $(x,y) \in (0,\infty) \times (0,1)$.
Clearly, u is bounded if and only if v is bounded. Since $\gamma_k > 0$ for all k, the functions z_k are not bounded. To find bounded solutions, one needs to take $d_k = 0$ in (42). Together with the equalities for c_k, d_k emerging from the condition at $x = 0$ one identifies then c_k uniquely. In the present case, this is the choice made for the solution found in (44).

Elliptic equation in a rectangle

We consider the elliptic equation in a rectangle

$$\begin{cases} \Delta u &= 0 & \text{for } (x,y) \in (0,L) \times (0,H), \\ u(x,0) &= f_1(x), \ \partial_y u(x,1) = f_2(x) & \text{for } x \in (0,L), \\ u(0,y) &= f_3(y), \ u(L,y) = 0 & \text{for } y \in (0,H), \end{cases} \tag{45}$$

where f_1, f_2 and f_3 are given functions. The solution in this case is slightly different than in the case of a strip. Instead of homogenising the boundary conditions, we use the linearity of the problem and decompose the solution into

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y).$$

The functions u_1, u_2 and u_3 are all solutions to the elliptic equation (i.e. $\Delta u_j = 0$ for $j = 1, 2, 3$) and satisfy homogeneous conditions excepting on one part of the boundary. More precisely, for u_1 we assume

$$u_1(x, 0) = f_1(x), \partial_y u_1(x, H) = 0 \text{ for } x \in (0, L), u_1(0, y) = u_1(L, y) = 0 \text{ for } y \in (0, H).$$

Similarly, for u_2 and u_3 we assume

$$u_2(x, 0) = 0, \partial_y u_2(x, H) = f_2(x) \text{ for } x \in (0, L), u_2(0, y) = u_2(L, y) = 0 \text{ for } y \in (0, H),$$

respectively

$$u_3(x, 0) = \partial_y u_3(x, H) = 0 \text{ for } x \in (0, L), u_3(0, y) = f_3(y), u_3(L, y) = 0 \text{ for } y \in (0, H),$$

(why did we not consider a fourth function u_4?)
The remaining reduces to finding the functions u_j, $j = 1, 2, 3$. This can be done by separation of variables, and accounting for the specific boundary conditions. For each k, the separated equations become

$$\frac{X''}{X} = -\frac{Y''}{Y},$$

implying that both sides of the equality are equal to a constant β
For $j = 1$ and $j = 2$ the boundary conditions at $x = 0$ and $x = L$ are homogeneous. Therefore in these cases we find β s.t. the problems

$$\begin{cases} X'' - \beta X = 0, & \text{for } x \in (0, L), \\ X(0) = 0, & X(L) = 0. \end{cases}$$

have non-trivial solutions. As already seen, we end up with $\beta_k = -\left(\frac{k\pi}{2}\right)^2$ and $X_k(x) = \sin\left(\frac{k\pi x}{2}\right)$, $k \in \mathbb{N}_0$. Next we find Y_k as solutions to

$$Y_k'' + \beta_k Y_k = 0, \text{ for } y \in (0, H),$$

namely

$$Y_k(y) = a_k e^{-\frac{k\pi y}{2}} + b_k e^{\frac{k\pi y}{2}}.$$

Finally, we determine $a_k, b_k \in \mathbb{R}$ s.t.

$$u_1(x, y) = \sum_{k \in \mathbb{N}_0} \left\{ a_k e^{-\frac{k\pi y}{2}} + b_k e^{\frac{k\pi y}{2}} \right\} \sin\left(\frac{k\pi x}{2}\right)$$

satisfies the boundary conditions for $y = 0$ and $y = H$, and in a similar fashion u_2.
We note that the only difference between u_1 and u_2 is in the boundary conditions for $y = 0$ and $y = H$. The series used for determining u_1 and u_2 being the same, one can find directly a function $\tilde{u}$ that satisfies both boundary conditions at $y = 0$ and $y = H$ simultaneously. This means finding a_k, b_k s.t.

$$\tilde{u}(x, 0) = f_1(x), \partial_y \tilde{u}(x, H) = f_2(x) \text{ for } x \in (0, L).$$

Finally, we mention that for determining u_3 the procedure is absolutely analogous, but now one finds first the Y- functions and then X.

Elliptic equation in a disc, sector, etc.

Study Section 9.3. There are only a few things to remark here.
First of all, when using Cartesian coordinates the domain appearing in Section 9.3 is not Cartesian, whereas this was a claim when using the separated form in (33). However, when using polar coordinates the domain becomes a rectangle, and the separation of variables may still be considered but in a different form,

$$u(r, \varphi) = R(r)\Phi(\varphi).$$

The equation in polar coordinates has a singularity at the origin, where $r = 0$. There the equation cannot be stated. However, this only appears due to the switch to polar coordinates, as the origin $(0, 0)$ in Cartesian coordinates expands to a segment $\{0\} \times (0, \phi_0)$ in polar coordinates. This makes it impossible to interpret the equation at the origin, whereas in Cartesian coordinates the origin has no particular meaning for the equation: the equation simply holds there as well.
In fact the properties that can be obtained from the problem posed in Cartesian coordinates transfer to the problem in polar coordinates and its solution. We give here two specific examples:

- *Uniqeness*: If uniqueness is obtained for the problem in Cartesian coordinates, it will hold for the translated problem into polar coordinates. Alternatively, if a problem is given in polar coordinates, to prove the uniqueness of a solution one can reformulate it to Cartesian coordinates and prove the uniqueness for this variant.

- *Boundedness/comparison principle* Similarly, to prove the boundedness of a solution for the problem formulated in polar coordinates, or a comparison principle, one may find it convenient to formulate first the problem in Cartesian coordinates. The results obtained in this form are transferred to the problem in polar coordinates. For example, on page 202 of [1] a "physical argument" is used to justify the choice $B = 0$. This is nothing but claiming that the solution must be bounded, which is not so obvious for the problem formulated in polar coordinates. However, if one goes back to the Cartesian coordinates, then the boundedness is guaranteed by Lemma 9.1.1 and its consequences (the maximum principle). Therefore the choice $B = 0$ is justified whenever the origin is included in the domain Ω where the equation should hold, as otherwise the solution would have a singularity in the origin. This singularity would be then artificial, as in Cartesian coordinates the solution remains bounded.

- *Separated solution*: The solution in polar coordinates is sought as $u(r, \varphi) = R(r)\Phi(\varphi)$. Assuming now that the domain Ω is the entire disc (meaning that the solution is periodic w.r.t. φ) and the boundary data f does not depend on φ

(in fact, it is constant), then one may seek first solutions that are radially symmetric, namely $u(r, \varphi) = R(r)$ (thus not depending on φ). This can simplify the solution strategy dramatically.

A similar situation can appear if a sector is considered as in Section 9.3, but the boundary conditions along the sector boundaries $\varphi = 0$ and $\varphi = \varphi_0$ are of homogeneous Neumann type, $\partial_\varphi u(r, 0) = \partial_\varphi u(r, \varphi_0) = 0$.

Week 7: Hyperbolic equations

We consider here second order problems of hyperbolic type. first order problems have already been seen in the first lecture, where the method of characteristics was discussed.

Energy estimates

We consider first the problem in a bounded domain $\Omega \subset \mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \Delta u & \text{for } x \in \Omega \text{ and } t > 0, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega \text{ and } t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & \text{for } x \in \Omega, \end{cases} \tag{46}$$

where $c > 0$ and u_0, u_1 are given. Before deriving an energy estimate we observe that there are two differences when compared to the parabolic and elliptic problems:

1. All derivatives are of second order, like in the case of the Laplace/Poisson equation. However, if all terms of the equation are brought to one side, on sees that the derivatives in t have a different sign than the ones in the spatial variables. Therefore time and space are treated differently.

2. Compared to the diffusion/heat equation, where only one time derivative appears, here we have two time derivatives. Therefore here we also take two initial conditions (the last two equalities in (46)), compared to one in the parabolic case. Recalling the physical motivation, which is the vibration of a membrane, or the oscillation of a spring-mass system, the need to impose two initial conditions is explained as follows. The oscillations can be initiated in two ways: either by an initial displacement from the equilibrium position (the first condition), or by giving an initial impuls (the second condition). Of course, a one can have the two initiators simultaneously.

With reference to equation (10.5) in the book we remark that it is the outcome of

$$\frac{d}{dt} \int_\Omega \left[\left(\frac{\partial u}{\partial t} \right)^2 + c^2 |\nabla u|^2 \right] dx = 2c^2 \int_{\partial\Omega} \frac{\partial u}{\partial t} (\vec{n} \cdot \nabla u) ds. \tag{47}$$

Observe that here we used $|\nabla u|$ and not $\|\nabla u\|$ as in the book. In fact they both mean the same, namely the length of the vector ∇u in the point (x,t). However, the notation $\|\nabla u\|$ is also used for the $L^2(\Omega)$ norm for ∇u, and we want to avoid any possible confusion here.

Defining the *energy* as

$$E(t) = \int_\Omega \left[\left(\frac{\partial u}{\partial t} \right)^2 + c^2 |\nabla u|^2 \right] dx, \tag{48}$$

(47) is telling us that it holds

$$E'(t) = 2c^2 \int_{\partial\Omega} \frac{\partial u}{\partial t} (\vec{n} \cdot \nabla u) ds.$$

As explained in the book, since for all $x \in \partial\Omega$ one has $u(x,t) = 0$ for all $t > 0$, we get $\partial_t u(x,t) = 0$ there, and the term on the right is vanishing. The same would be achieved if the Dirichlet boundary condition with constant value (here 0) would be replaced by a homogeneous Neumann boundary condition ($\vec{n} \cdot \nabla u = 0$) on the boundary $\partial\Omega$, or on a part of it that does not change in time.
With this, we actually have proved

Proposition 2 *Let u be a sufficiently smooth solution to* (46). *Then the energy E defined in* (48) *remains constant in time.*

Clearly, if other terms appear in the first equation of (46), one needs to use further strategies to estimate the energy, and this may not remain constant. For example, if the equation is replaced by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u + f,$$

where $f \in L^\infty(0,T; L^2(\Omega))$ (this means that a $C_f > 0$ exists s.t. $\|f(t)\|_{L^2(\Omega)} \leq C_f$ for a.e. t), repeating the steps in the book one gets

$$\frac{d}{dt} \int_\Omega \left[\left(\frac{\partial u}{\partial t} \right)^2 + c^2 |\nabla u|^2 \right] dx = 2c^2 \int_{\partial\Omega} \frac{\partial u}{\partial t} (\vec{n} \cdot \nabla u) ds + 2 \int_\Omega f \frac{\partial u}{\partial t} dx. \tag{49}$$

using the Cauchy-Schwarz inequality and the inequality of means $2ab \leq a^2 + b^2$ that is valid for any $a, b \in \mathbb{R}$, the last term on the right can be estimated by

$$2 \int_\Omega f \frac{\partial u}{\partial t} dx \leq 2\|f(t)\| \|\partial_t u(t)\| \leq \|f(t)\|^2 + \|\partial_t u(t)\|^2.$$

In the above $\| \cdot \|$ stands for the $L^2(\omega)$ norm, and by $f(t)$ we mean the function in x obtained for a fixed t.
With this, using the definition of the energy in (48) and the boundary conditions stated in (46), from (49) one gets

$$E'(t) \leq \|f(t)\|^2 + \|\partial_t u(t)\|^2 \leq \|f(t)\|^2 + E(t).$$

Now one can use Gronwall's inequality to prove that the energy has a bounded growth in time. This growth is, however, exponential. We leave the details as a homework!
We end this part by mentioning that the uniqueness result stated in Proposition 10.1.1. also holds if an additional term f is added in the equation, as considered above. The proof is absolutely identical.

Separation of variables

Here we only remark that the strategy is as for elliptic or parabolic problems. We solve first the equation in X since here the boundary conditions are given, $X(0) = X(L) = 0$. Observe again that these must be homogeneous! With the values λ_n obtained here we determine the corresponding functions T_n, which are not exponentials anymore, but, like X_n, trigonometric. The initial conditions are now used to determine the coefficients.

The method of D'Alembert (unbounded domains)

We now consider the problem in $\mathbb{R}$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{50}$$

where $c > 0$ and u_0, u_1 are given (observe that, for consistency with the notations used previously, we use here u_0 and u_1 for the initial condition instead of f and g, as in [1]). Clearly, no boundary conditions can be prescribed.

This problem is solved in the book by means of a change of variables (the method of D'Alembert). Here we give an alternative approach, following the ideas presented for first order equations. We start by observing that the equation can be rewritten as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u\right] = 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{51}$$

This suggests using the auxiliary function $v(x,t) = \partial_t u - c\partial_x u$, which satisfies the equation

$$\partial_t v + c\partial_x v = 0,$$

for $x \in \mathbb{R}$ and $t > 0$. Applying the method of characteristics, we obtain that v satisfies

$$v(x,t) = v(x - ct, 0).$$

Using the initial conditions and the definition of v we get

$$v(x,t) = \partial_t u(x - ct, 0) - c\partial_x u(x - ct, 0) = u_1(x - ct) - cu_0'(x - ct) \tag{52}$$

(note that u_0 is a function in one variable, therefore the we speak about u_0' and not about a partial derivative of u_0).

Having found v we draw our attention t u, which satisfies

$$\partial_t u(x,t) - c\partial_x u(x,t) = v(x,t)$$

for all $x \in \mathbb{R}$ and $t > 0$. For the above, the characteristics are defined by $x'(t) = -c$, yielding $x(t) = x_0 - ct$.

Let now x_0 be fixed. Along the characteristic $x(t) = x_0 - ct$, for the function $w : [0, \infty) \to \mathbb{R}$, $w(t) = u(x_0 - ct, t)$ one has

$$w'(t) = -c\partial_x u(x_0 - ct, t) + \partial_t u(x_0 - ct, t) = v(x_0 - ct, t).$$

From this and using (52), after integration one obtains straightforwardly

$$\begin{aligned} u(x_0 - ct, t) = w(t) &= w(0) + \int_0^t v(x_0 - c\tau, \tau) d\tau \\ &= w(0) + \int_0^t \big(u_1(x_0 - c\tau - c\tau) - cu_0'(x_0 - c\tau - c\tau)\big) d\tau \\ &= w(0) + \int_0^t \big(u_1(x_0 - 2c\tau) - cu_0'(x_0 - 2c\tau)\big) d\tau. \end{aligned}$$

Since $w(0) = u(x_0, 0)$, after applying the substitution $y = x_0 - 2c\tau$, the above leads to

$$\begin{aligned} u(x_0 - ct, t) &= u_0(x_0) - \frac{1}{2c} \int_{x_0}^{x_0 - 2ct} \big(u_1(y) - cu_0'(y)\big) dy \\ &= u_0(x_0) + \frac{1}{2}\big[u_0(x_0 - 2ct) - u_0(x_0)\big] + \frac{1}{2c} \int_{x_0 - 2ct}^{x_0} u_1(y) dy \\ &= \frac{1}{2}\big[u_0(x_0) + u_0(x_0 - 2ct)\big] + \frac{1}{2c} \int_{x_0 - 2ct}^{x_0} u_1(y) dy. \end{aligned}$$

Finally, with $x = x_0 - ct$ gives

$$u(x, t) = \frac{1}{2}\big[u_0(x + ct) + u_0(x - ct)\big] + \frac{1}{2c} \int_{x - ct}^{x + ct} u_1(y) dy,$$

which is exactly the solution found in (10.18), but with $f = u_0$ and $g = u_1$.

Remark 9 *If the initial impulse is 0 ($g \equiv 0$) one has the solution*

$$u(x, t) = \frac{1}{2}\big[u_0(x + ct) + u_0(x - ct)\big].$$

In other words, the initial profile is divided by two, and these two parts travel with the velocity c to the left and to the right. A straightforward calculations gives

$$\int_{\mathbb{R}} u(x, t) dx = \frac{1}{2} \left\{ \int_{\mathbb{R}} u_0(x - ct) dx + \int_{\mathbb{R}} u_0(x + ct) dx \right\} = \int_{\mathbb{R}} u_0(y) dy,$$

which means that the initial displacement is conserved.

Homework: Consider the problem (46), but with homogeneous Neumann boundary conditions $\vec{n} \cdot \nabla u = 0$ on $\partial\Omega$ and for all $t \geq 0$. Prove that the total impulse is conserved,

$$\int_{\Omega} \frac{\partial u}{\partial t}(x, t) dx = \int_{\Omega} g(x) dx.$$

Then prove that in the absence of an initial impulse, the total displacement in Ω is conserved.
Hint: In general no explicit solution is available, but you may integrate the equation in (46) to deduce that the total impulse remains 0, and that the total displacement is constant in time.

Week 8/9: Travelling waves

We have already discussed the simplest form of the wave equation

$$\partial_t u + a\partial_x u = 0, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{53}$$

and with a given initial condition $u(x, 0) = u_0(x)$ for $x \in \mathbb{R}$. Here $a \in \mathbb{R}$ is given. The solution to (53) is a *wave* travelling with the *speed* a, $u(x,t) = u_0(x - at)$, see Section . This situation was extended in Section the case of a second order hyperbolic equation, where two waves travelling in opposite direction were encountered. In this section we increase the complexity of the problem en discuss the case of nonlinear equations. We start with a well-known example, *the Burgers equation*

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = \nu\frac{\partial^2 u}{\partial x^2}, \text{ for } x \in \mathbb{R} \text{ and } t > 0. \tag{54}$$

This is an important equation as it can be seen as the one-dimensional counterpart of the Navier-Stokes model, still featuring several of the aspects that makes the latter a challenging mathematical issue. Moreover, when $\nu = 0$, the resulting is a nonlinear hyperbolic equation (the inviscid Burgers equation) which, in case of smooth solutions, can be seen as prototypical for any nonlinear equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}f(u) = 0, \text{ for } x \in \mathbb{R} \text{ and } t > 0, \tag{55}$$

where $f : \mathbb{R} \to \mathbb{R}$ is any twice continuously differentiable function. To see the connection between the two equations, one rewrites $\partial_x f(u) = f'(u)\partial_x u$ and multiplies the resulting (55) by $f''(u)$ to obtain

$$f''(u)\partial_t u + f''(u)f'(u)\partial_x u = 0.$$

Letting now $v = f'(u)$, the latter equation reduces to

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = 0,$$

which is nothing but (54) for $\nu = 0$. Observe that this transformation *is only possible* if the solution u is sufficiently smooth, at least continuously differentiable.
However, this smoothness cannot be guaranteed. One can see it immediately if considering (53) with a non-smooth initial condition. The solution will not become smoother, as it happened for parabolic problems. Moreover, in the nonlinear case the solution may become non-smooth even when starting with a smooth initial data. To see this, recall that the solution to first order hyperbolic problems like (53) can be solved by the method of characteristics. For (55) the characteristics are obtained as solutions to

$$x'(t) = f'(u(x(t), t)), \text{ for } t > 0. \tag{56}$$

For the inviscid Burgers equation one has $x'(t) = u(x(t), t)$. Clearly, in the $x--t$ plane the characteristics for larger values of u are more inclined to the x-axis than those for smaller values of u. Other said, larger values of u will travel faster than smaller ones, and the former can catch up the latter. At points where something like this occurs, the solution u will have different left- and right limits, so it becomes discontinuous. In other words, smooth solutions do not exist any more.
This pleads for a more general concept of solutions, called *weak solutions*, in which derivatives are interpreted in a weak sense by using smooth test functions. More precisely, u will be called a solution if one has

$$\int_0^\infty \int_{\mathbb{R}} u(x,t)\partial_t\varphi(x,t)dxdt + \int_0^\infty \int_{\mathbb{R}} f(u(x,t))\partial_x\varphi(x,t)dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)dx = 0$$

for all *test functions* $\varphi \in C^\infty(\mathbb{R} \times [0,\infty))$ having a compact support in $\mathbb{R} \times [0,\infty)$. Without entering into details, we only mention that this new concept does not require that u is smooth, it allows even solutions that are discontinuous, so it solves the "existence" problem. On the other hand, one is facing the "uniqueness" problem. More precisely, there can be more than one "weak solution" to the same equation, and the question then is how to choose one of them as being physically relevant?
One option is to select the solution as the limit

$$u = \lim_{\nu \searrow 0} u^\nu,$$

where u^ν is the solution to the parabolic equation

$$\frac{\partial u^\nu}{\partial t} + \frac{\partial}{\partial x} f(u^\nu) = \nu \frac{\partial^2 u^\nu}{\partial x^2}, \text{ for } x \in \mathbb{R} \text{ and } t > 0, \tag{57}$$

(the so-called *viscous regularisation* of the hyperbolic equation obtained for $\nu = 0$). This choice is commonly called *entropy solution* and is motivated by physics. More precisely, the hyperbolic equation is a simplified mathematical model for e.g. flow, in which diffusive effects are neglected. However, although these effects are disregarded, to make the correct choice of a solution one has to recall what was neglected when the model was simplified!

Travelling waves for (57)

Remembering the wave-like solutions obtained for the simplified equations like (53), one question appearing naturally is whether such solutions are also possible for (57), at least for suitable initial conditions. In other words, is it possible to have solutions of the form

$$u^\nu(x,t) = v(x - ct), \tag{58}$$

for a function v and a *wave speed* $c \in \mathbb{R}$ that need to be determined? Such solutions are called *travelling waves*.

Clearly, the answer to the question above also depends on the expected behaviour as x, respectively η are approaching $\pm\infty$. From now on we seek solutions satisfying

$$\lim_{x\to-\infty} u^\nu(x,t) = u_\ell, \text{ and } \lim_{x\to\infty} u^\nu(x,t) = u_r, \tag{59}$$

for all $t > 0$, and where $u_\ell, u_r \in \mathbb{R}$ are assumed given. Consequently, one has

$$\lim_{\eta\to-\infty} v(\eta) = u_\ell, \text{ and } \lim_{\eta\to\infty} v(\eta) = u_r. \tag{60}$$

With this we formulate the following

Definition 5 *(Travelling wave)*
Given the left and right states $u_\ell, u_r \in \mathbb{R}$, a travelling wave (TW) solution to (57) is a solution $u^\nu : \mathbb{R}\times[0,\infty) \to \mathbb{R}$ to (57) for which a travelling wave speed $c \in \mathbb{R}$ and a function $v : \mathbb{R} \to \mathbb{R}$ exist s.t.

$$\lim_{\eta\to-\infty} v(\eta) = u_\ell, \lim_{\eta\to\infty} v(\eta) = u_r, \text{ and } u^\nu(x,t) = v(x-ct),$$

for all $x \in \mathbb{R}$ and $t \geq 0$.

Observe that, given $u_\ell, u_r \in \mathbb{R}$, two unknowns have to be determined when speaking of TW solutions: the function v itself, and the wave speed c. For the ease of writing we use $\eta = x - ct$ as new variable. Observe that assuming that u^ν has the form stated in Definition 5 (or in (58)) implicitly relates the originally independent variables x and t and reduces them to only one. We therefore adopt a strategy that has been used before: to assume a certain relation between x and t, and to reduce the partial differential equation to an ordinary one.
We discuss below the existence of a TW solution and how to obtain the function v and the TW velocity c. For obvious reasons we restrict to the case $u_\ell \neq u_r$, the case $u_\ell = u_r$ being trivial. First of all, with η as above one uses the chain rule to obtain

$$\frac{\partial}{\partial t} = -c\frac{d}{d\eta}, \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{d}{d\eta}. \tag{61}$$

With this, since u^ν is a solution to (57), v solves the equation

$$-cv' + \big(f(v)\big)' = \nu v'', \text{ for all } \eta \in \mathbb{R}.$$

In the above, $\frac{d}{d\eta}$ is replaced by the more simple notation $'$. Integrating the above, one obtains that v solves the first order differential equation

$$A - cv + f(v) = \nu v', \text{ for all } \eta \in \mathbb{R}, \tag{62}$$

where $A \in \mathbb{R}$ is an arbitrary constant. As mentioned before, $c \in \mathbb{R}$ is an unknown too.

We now use the stated behaviour for $\eta \to \pm\infty$ to determine the constant A and the TW velocity c. More precisely, we assume the existence of a v solving (62) and satisfying (60), and use the latter to determine A and c. We observe that (62) reduces to

$$v'(\eta) = g(v(\eta)), \text{ for all } \eta \in \mathbb{R} \text{ and with } g(v) = \frac{1}{\nu}\big(f(v) - cv + A\big). \tag{63}$$

The function f is assumed continuous, therefore the same holds for g. Since one has $\lim\limits_{\eta\to\infty} v(\eta) = u_r$ and $\lim\limits_{\eta\to-\infty} v(\eta) = u_\ell$, one obtains that

$$\lim_{\eta\to\infty} g(v(\eta)) = g(u_r), \text{ and } \lim_{\eta\to-\infty} g(v(\eta)) = g(u_\ell).$$

By (63), this immediately shows that v' has limits as $\eta \to \pm\infty$ as well,

$$\lim_{\eta\to\infty} v'(\eta) = g(u_r), \text{ and } \lim_{\eta\to-\infty} v'(\eta) = g(u_\ell).$$

We use now an elementary result, of which proof is left as homework:

Proposition 3 *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function and assume that the following limits exist:*

$$\lim_{\eta\to\infty} h(\eta) = a, \quad \text{and} \quad \lim_{\eta\to\infty} h'(\eta) = b, \quad \text{with } a, b \in \mathbb{R}.$$

Then one has $b = 0$.

Clearly, the same result can be stated for $\eta \to -\infty$. With this, we observe that Proposition 3 can be applied to v to obtain that

$$g(u_\ell) = g(u_r) = 0.$$

From (63) one gets

$$f(u_r) - cu_r + A = 0, \text{ and } f(u_\ell) - cu_\ell + A = 0.$$

This is a linear system of two equations with two unknowns, giving

$$c = \frac{f(u_r) - f(u_\ell)}{u_\ell - u_r}, \text{ and } A = cu_\ell - f(u_\ell) = cu_r - f(u_r).$$

The latter equality follows from the specific form of c. We have therefore proved the following

Proposition 4 *(Necessary condition for the existence of TW solutions)*
Assume that (57) has a TW solution v connecting the given left and right states u_ℓ and u_r ($u_\ell, u_r \in \mathbb{R}$). Then, the corresponding TW velocity is

$$c = \frac{f(u_r) - f(u_\ell)}{u_\ell - u_r}, \tag{64}$$

and v is a solution to

$$v' = \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big), \text{ for all } \eta \in \mathbb{R}. \tag{65}$$

Observe that, by (64), the term on the right in (65) can be replaced by $\frac{1}{\nu}\big(f(v)-f(u_r)-c(v-u_r)\big)$.
Proposition 4 only gives necessary conditions but not sufficient ones for the existence of TW solutions. The existence depends on the properties of f. Moreover, the uniqueness is not true. To see this, observe that, if v is a TW solution and hence solves

$$\begin{cases} v' = \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big), & \text{for all } \eta \in \mathbb{R}, \\ \lim\limits_{\eta\to-\infty} v(\eta) = u_\ell, \quad \text{and } \lim\limits_{\eta\to\infty} v(\eta) = u_r, \end{cases} \tag{66}$$

then, for any $\zeta \in \mathbb{R}$, so does the ζ-translation of v, $v_\zeta(\eta) = v(\eta - \zeta)$. Therefore, to fix the ideas, we will use a *normalisation* of the TW by requiring that

$$v(0) = v_0 \tag{67}$$

for some value v_0 between u_ℓ and u_r. E.g., one may choose $v_0 = (u_\ell + u_r)/2$, but, in some situations, other choices may be more convenient. This shows that, actually, one can reformulate the question of existence to the following
Question. Le $v_0 = (u_\ell + u_r)/2$ (or any value between the two states) be given and v be the solution to the initial value problem

$$\begin{cases} v' &= \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big), \quad \text{for all } \eta \in \mathbb{R}, \\ v(0) &= v_0. \end{cases} \tag{68}$$

Does v satisfy $\lim\limits_{\eta\to-\infty} v(\eta) = u_\ell,\quad$ and $\lim\limits_{\eta\to\infty} v(\eta) = u_r$?
Note that the existence and uniqueness of a solution v to (68) is guaranteed by the standard theory for ordinary differential equations (why?). It only remains to investigate the behaviour of v as $\eta \to \pm\infty$, which can be done by phase line arguments.
We start by observing that (65) is an ordinary differential equation $v' = g(v)$, and that the two states u_ℓ and u_r are equilibrium solutions to it, $g(u_\ell) = g(u_r) = 0$. Since g is a C^1 function and because at $\eta = 0$ the solution lies between u_ℓ and u_r, it follows the same for $v(\eta)$ for all $\eta \in \mathbb{R}$. Also, v should approach u_ℓ, respectively u_r as η approaches $-\infty$, respectively ∞. This rules out the possibility that g has another zero between u_ℓ and u_r. To see this, assume that an u^* between the two states exists s.t. $g(u^*) = 0$. Then u^* is an equilibrium for (65), and then v could never cross it to approach either u_ℓ at $-\infty$, or u_r at ∞. This implies that no TW solutions connecting the given states will exist.
Therefore one concludes that g has constant sign for all arguments between u_ℓ and u_r. Essentially, the existence of a TW solution reduces to the analysis of the (sign of the) function g, in connection with the ordering of u_ℓ and u_r. We have the following cases.

Case A. If $u_\ell > u_r$, then v decays from u_ℓ to u_r. Therefore, there exists an $\eta_0 \in \mathbb{R}$ s.t. $v(\eta_0) \in (u_r, u_\ell)$ and $v'(\eta_0) < 0$. This implies that $g(v(\eta_0)) < 0$, and therefore $g(v) < 0$ for all $v \in (u_r, u_\ell)$. If a $v \in (u_r, u_\ell)$ exists s.t. $g(v) \geq 0$ then no TW connecting u_ℓ to u_r exist.

On the other hand, if $u_\ell > u_r$ and g are s.t. $g(v) < 0$ for all $v \in (u_r, u_\ell)$, then by standard ordinary differential equation arguments one obtains that the solution to (68) has the desired behaviour, implying the existence of a TW solution.

Case B. If $u_\ell < u_r$, the analysis is similar. If a $v \in (u_\ell, u_r)$ exists s.t. $g(v) \leq 0$ then no TW connecting u_ℓ to u_r exist. Otherwise, if $u_\ell < u_r$ and g are s.t. $g(v) > 0$ for all $v \in (u_\ell, u_r)$, then a TW solution does exist.
The discussion above can be summarised in the following

Lemma 2 *Let $u_\ell, u_r \in \mathbb{R}$ be given s.t. $u_\ell \neq u_r$ and assume $f \in C^1(\mathbb{R})$. Then* (57) *does admit TW solutions in the sense of Definition 5 if and only if the function $g : \mathbb{R} \to \mathbb{R}$ defined by*

$$g(v) = \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big) \quad \textit{with} \quad c = \frac{f(u_r) - f(u_\ell)}{u_\ell - u_r}$$

has a constant sign for all v between the two states, and this sign is the same as the sign of $u_r - u_\ell$. The TW is unique up to a translation, namely up to a normalisation as in (67).

Remark 10 *For the Burgers equation* (54) *one has $f(u) = \frac{1}{2}u^2$. In this case, a direct calculation shows gives $c = \frac{1}{2}(u_\ell + u_r)$ and $g(v) = \frac{1}{2\nu}(v - u_\ell)(v - u_r)$ (please, check!). Clearly, for any v between u_ℓ and u_r one has $g(v) < 0$, showing that TW solutions are possible if and only if $u_\ell > u_r$.*

Remark 11 *A similar conclusion can be drawn if the function f in* (57) *is convex. To see this, we note that the function $g(v) = \frac{1}{\nu}\big(f(v) - f(u_\ell) - c(v - u_\ell)\big)$ will be convex too. Since $g(u_\ell) = g(u_r) = 0$, the convexity of g implies that $g(v) < 0$ for all arguments v between u_ℓ and u_r (please, give a proof). Therefore, for a convex f TW solutions exist if and only if $u_\ell > u_r$ (again the case $u_\ell = u_r$ is not considered since it is trivial). Similarly if f is concave, then TW solutions exist if and only if $u_\ell < u_r$.*
We finally remark that whether f is convex or concave is not essential, the sign of g. The convexity/concavity of f is only relevant for proving that g is negative/positive. If this can be proved by employing arguments, then the result is still valid.

The arguments above show that the TW is a monotone function and the range is either (u_ℓ, u_r) or (u_r, u_ℓ) (depending on the ordering of the states). To fix the ideas assume that $u_r < u_\ell$ and that $f \in C^1$ is s.t. $g(v) < 0$ for all $v \in (u_r, u_\ell)$, the other case being analogous. Also, assume that the TW is normalised s.t. $v(0) = \frac{u_\ell + u_r}{2}$.
We assume further that the states are approached only asymptotically, i.e.

$$\begin{aligned} &\lim_{\eta \to \infty} v(\eta) = u_r, \quad \lim_{\eta \to -\infty} v(\eta) = u_\ell, \quad \text{and} \\ &u_r < v(\eta) < u_\ell \quad \text{for all } \eta \in \mathbb{R}. \end{aligned} \tag{69}$$

The discussion on when this is fulfilled is postponed, we only mention that, alternatively, one may have e.g. that $v(\eta) > u_r$ for all $\eta < \eta_r$ and $v(\eta) = u_r$ for all $\eta \geq \eta_r$.

Under the assumptions in (69), the TW $v : \mathbb{R} \to (u_r, u_\ell)$ is a bijection, so the inverse function $\eta : (u_r, u_\ell) \to \mathbb{R}$, $\eta = \eta(v)$ is well defined. This is a decreasing function satisfying the equation

$$\eta'(v) = \frac{1}{v'(\eta(v))} = \frac{1}{g(v)},$$

where we have used the rule for differentiating the inverse function, and the equation for v. Moreover, $\eta\left(\frac{u_\ell+u_r}{2}\right) = 0$. From this, by integration one obtains

$$\eta = \int_{\frac{u_\ell+u_r}{2}}^{v} \frac{1}{g(z)} dz,$$

with g as in Lemma 2. In other words, we have determined the inverse of the (normalised) TW. This can be interpreted as the implicit definition of the TW, also called as the ***integral representation*** of the TW,

$$\eta = \int_{\frac{u_\ell+u_r}{2}}^{v(\eta)} \frac{1}{g(z)} dz, \tag{70}$$

for all $\eta \in \mathbb{R}$. If (70) can be solved in terms of $v(\eta)$, one obtains the TW explicitly. However, compared with the integral representation, this last step is not always possible.

As discussed in Remark 10, when considering the Burgers equation (54) one has $f(u) = \frac{1}{2}u^2$ and TW solutions are possible if and only if $u_\ell > u_r$. These solutions travel with the velocity $c = \frac{1}{2}(u_\ell + u_r)$. When considering the normalisation $v(0) = \frac{u_\ell+u_r}{2}$, the TW are solutions to

$$\left\{ \begin{array}{rcll} v' & = & \frac{1}{2\nu}(v - u_\ell)(v - u_r), & \text{for all } \eta \in \mathbb{R}, \\ v(0) & = & \frac{u_\ell+u_r}{2}. & \end{array} \right. \tag{71}$$

From (70), the integral representation of the TW is

$$\eta = \int_{\frac{u_\ell+u_r}{2}}^{v(\eta)} \frac{2\nu}{(z - u_\ell)(z - u_r)} dz. \tag{72}$$

In this case, the TW can be obtained explicitly. Using partial fractions,

$$\frac{1}{(z - u_\ell)(z - u_r)} = \frac{1}{u_\ell - u_r}\left[\frac{1}{z - u_\ell} - \frac{1}{z - u_r}\right],$$

one obtains

$$\eta = \int_{\frac{u_\ell+u_r}{2}}^{v(\eta)} \frac{2\nu}{u_\ell - u_r}\left[\frac{1}{z - u_\ell} - \frac{1}{z - u_r}\right] dz.$$

Since $v(\eta) \in (u_r, u_\ell)$, this gives

$$\eta = \frac{2\nu}{u_\ell - u_r} \ln\left|\frac{z - u_\ell}{z - u_r}\right|\Bigg|_{z=\frac{u_\ell+u_r}{2}}^{v(\eta)} = \frac{2\nu}{u_\ell - u_r} \ln\frac{u_\ell - v(\eta)}{v(\eta) - u_r}.$$

From this, it follows that

$$v(\eta) = u_r + (u_\ell - u_r)\left[1 + e^{\dfrac{u_\ell - u_r}{2\nu}\eta}\right]^{-1},$$

yielding the TW solution

$$u(x,t) = u_r + (u_\ell - u_r)\left[1 + e^{\dfrac{u_\ell - u_r}{2\nu}\left(x - \dfrac{u_\ell + u_r}{2}t\right)}\right]^{-1}. \tag{73}$$

Observe that this solution satisfies all required properties, including that $u(x,t) = \frac{u_\ell + u_r}{2}$ whenever $x = \frac{u_\ell + u_r}{2}t$ (which corresponds to $\eta = 0$).
Recall that in the introduction of this paragraph we discussed the viscous Burgers equation (54), in which $\nu > 0$, as a regularisation of the inviscid Burgers equation obtained for $\nu = 0$. For the latter, as a particular case of (55), we saw that it may admit more than one weak solution, and we mentioned the limit $u = \lim\limits_{\nu \searrow 0} u^\nu$ as the physically relevant solution, with u^ν solving (54). Having now the TW solution computed explicitly, one can pass $\nu \to 0$ in (74) to obtain the limit

$$u(x,t) = \begin{cases} u_\ell, & \text{if } x < \frac{u_\ell + u_r}{2}t, \\ u_r, & \text{if } x > \frac{u_\ell + u_r}{2}t. \end{cases} \tag{74}$$

Observe that this is a discontinuous solution to the inviscid Burgers equation, called a *shock*. Based on the arguments above, such a solution can be obtained (for $f(u) = \frac{1}{2}u^2$) if and only if $u_\ell > u_r$, and will be called *admissible*. This strategy is being adopted for more general situations, in which TW solutions and, more general, solutions to regularised hyperbolic equations, are used to decide whether a shock solution is admissible or not.
We finally come back to the assumption made in (69), namely that the TW only reaches the left and the right states asymptotically. Using the integral representation in (70), it follows that the (improper) integral on the right should diverge if v approaches either u_r or u_ℓ,

$$\lim_{v \searrow u_r} \int_{\frac{u_\ell + u_r}{2}}^{v} \frac{1}{g(z)} dz = \infty, \quad \text{and} \quad \lim_{v \nearrow u_\ell} \int_{\frac{u_\ell + u_r}{2}}^{v} \frac{1}{g(z)} dz = -\infty. \tag{75}$$

Recalling the comparison theorem for integrals and the convergence of p-integrals, it follows that g should have the following asymptotic behaviour

$$\lim_{v \searrow u_r} \frac{g(v)}{(v - u_r)^p} = a, \quad \text{and} \quad \lim_{v \nearrow u_\ell} \frac{g(v)}{(u_\ell - v)^q} = b,$$

for some exponents $p, q \geq 1$ and where $a, b \in \mathbb{R}_0$. E.g. for the Burgers equation one has $p = q = 1$ and $a = b = -\frac{u_\ell - u_r}{2\nu}$.

However, one may have the case when the asymptotic behaviour of g is different, implying that at least one of the integrals in (75) is convergent. To fix the ideas, assume that a (finite!) $\eta_+ \in \mathbb{R}$ exists s.t.

$$\lim_{v \searrow u_r} \int_{\frac{u_\ell + u_r}{2}}^{v} \frac{1}{g(z)} dz = \eta_+,$$

whereas the limit $v \nearrow u_\ell$ is still $-\infty$. Then for the function η in (70) one has $\eta(u_r) = \eta_+$, and its range becomes $(-\infty, \eta_+]$. In other words, we have a bijection $\eta : [u_r, u_\ell) \to (-\infty, \eta_+]$, having an inverse $v : (-\infty, \eta_+] \to [u_r, u_\ell)$. To obtain the TW solution, one needs to extend this function by a constant to the right of η_+, namely $v(\eta) = u_r$ for $\eta \geq \eta_+$. Such an example will be discussed later.

Degenerate Burgers equation

With $\nu > 0$ and $m > 1$ we consider the degenerate Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = \nu \frac{\partial^2 (u^m)}{\partial x^2}, \text{ for } x \in \mathbb{R} \text{ and } t > 0. \tag{76}$$

Given two states $u_\ell, u_r \geq 0$, we seek travelling wave (TW) solutions to (76) connecting them.
Remark: To understand the name "degenerate" we refer to the porous medium equation, where the same nonlinear diffusion term is encountered. In particular, the diffusion is vanishing if $u = 0$.

Problem: Show that travelling waves are possible if $u_\ell \geq u_r$, but not if $u_\ell < u_r$. Then, show that TW solutions exist whenever $u_\ell \geq u_r \geq 0$, and find the TW solution for the case $m = 2$, $u_\ell = 1$ and $u_r = 0$.

Solution. TW are solutions in the form

$$u(x,t) = v(\eta), \text{ where } \eta = x - ct.$$

Here $c \in \mathbb{R}$ is a constant TW velocity to be determined. Assuming that such solutions exist, and using the differentiation rules in which $'$ stands for the derivative w.r.t. η

$$\frac{\partial}{\partial t} = -c()' \text{ and } \frac{\partial}{\partial x} = ()',$$

if u is a TW solution to (76) then v must satisfy the equation

$$-cv'(\eta) + \frac{1}{2}(v^2(\eta))' = \nu (v^m(\eta))'', \text{ for all } \eta \in \mathbb{R}.$$

Integrating w.r.t η one gets the ordinary differential equation

$$(v^m(\eta))' = \frac{1}{\nu}\left[A - cv(\eta) + \frac{1}{2}v^2(\eta)\right], \text{ for all } \eta \in \mathbb{R}, \tag{77}$$

where the constant $A \in \mathbb{R}$ is yet unspecified. To determine it explicitly, as well as to find the TW velocity $c \in \mathbb{R}$ we use the behaviour of v at $\pm\infty$.
One has $\lim_{\eta\to\infty} v(\eta) = u_r$, therefore, by (77) the following limits exist:

$$\lim_{\eta\to\infty} v(\eta)^m = u_r^m, \text{ and } \lim_{\eta\to\infty} (v^m(\eta))' = A - cu_r + \frac{1}{2}u_r^2.$$

Using now Lemma 1 in the Instruction 8 for the function $f = v^m$ one obtains that $A = A - cu_r + \frac{1}{2}u_r^2$. Therefore, for all $\eta \in \mathbb{R}$, v solves the TW equation

$$(v^m(\eta))' = g(v(\eta)), \text{ where } g(v) = \frac{1}{\nu}\left[\frac{1}{2}(v^2 - u_r^2) - c(v - u_r)\right]. \tag{78}$$

Furthermore, letting $\eta \to -\infty$ and using the argument above leads to

$$0 = cu_r - \frac{1}{2}u_r^2 - cu_\ell + \frac{1}{2}u_\ell^2,$$

which gives

$$c = \frac{1}{2}\frac{u_\ell^2 - u_r^2}{u_\ell - u_r} = \frac{u_\ell + u_r}{2}. \tag{79}$$

At this point we note that this respects a general property of TW solutions to an equation of the form

$$\partial_t(\alpha(u)) + \partial_x(\beta(u)) = \partial_x(\gamma(u)\partial_x u), \tag{80}$$

where α, β, γ are C^1 functions s.t. α is strictly increasing and γ non-negative. In this case, one has
Proposition. If (80) has TW solutions connecting the (non-equal) states u_ℓ and u_r, then the travel with the velocity

$$c = \frac{\beta(u_\ell) - \beta(u_r)}{\alpha(u_\ell) - \alpha(u_r)}.$$

Homework: Give a proof of this general property.
Remark. The TW velocity in the proposition does not depend on the diffusion γ!

Returning to the TW solution of (76), we observe that for the function g in (78) one has $g(u_\ell) = g(u_r) = 0$, and has constant sign between the two states. In other words, u_ℓ and u_r are two equilibrium solutions for the equation in (78). Since no equilibrium point exists between the two states, a TW *may* exist. It is a solution v to (78) s.t.

$$\lim_{\eta\to-\infty} v(\eta) = u_\ell, \text{ and } \lim_{\eta\to\infty} v(\eta) = u_r. \tag{81}$$

Non-existence. Assume that $u_r > u_\ell \geq 0$. Form (81) it follows that v, hence v^m is increasing at least in an interval of $\mathbb{R}$, and this for some value $v \in (u_\ell, u_r)$. Therefore an $\eta \in \mathbb{R}$ exists s.t. $v(\eta) \in (u_\ell, u_r)$ and $(v^m(\eta))' > 0$, implying that $g(v(\eta)) > 0$. However, this is a contradiction since $g(v) < 0$ for all $v \in (u_\ell, u_r)$. this shows that no TW exist whenever $u_\ell < u_r$.

Existence. The case $u_\ell = u_r$ is trivial, then the equilibrium solution is a TW. We consider separately the cases $u_\ell > u_r > 0$ and $u_\ell > u_r = 0$.
The case $u_\ell > u_r > 0$. Since $u_r \leq v(\eta) \leq u_\ell$ for all $\eta \in \mathbb{R}$, it follows that v is bounded away from 0 everywhere. We rewrite (78) as

$$v'(\eta) = g(v(\eta)), \text{ with } g(v) = \frac{1}{2m\nu}\frac{(v-u_r)(v-u_\ell)}{v^{m-1}}. \tag{82}$$

Clearly, $g \in C^1[u_r, u_\ell]$, and therefore it is Lipschitz continuous in this closed interval. For any initial condition $v_0 \in (u_r, u_\ell)$, the equation in (82) has a unique solution v satisfying $v(0) = v_0$, at least locally. In fact, the existence is global since $g \in C^1$, and the two equilibria u_r and u_ℓ are lower and upper bounds for v. Moreover, $g(v) < 0$ for all $v \in (u_r, u_\ell)$, so v has the behaviour in (81). This means that (76) admits a TW solution connecting u_ℓ to u_r.
Integral representation. The TW can be given in the *integral form.* To fix the ideas, we choose $v_0 = \frac{1}{2}(u_r + u_\ell)$, but any other choice $v_0 \in (u_r, u_\ell)$ is possible. Observe that $v : \mathbb{R} \to (u_r, u_\ell)$ is a bijection, so it make sense to consider the function $\eta : (u_r, u_\ell) \to \mathbb{R}$. Since $\eta'(v) = \frac{1}{v'(\eta(v))}$ from (82) one obtains

$$\eta'(v) = \frac{1}{g(v)} \text{ for all } v \in (u_r, u_\ell).$$

Integrating the above and using the initial condition $v(0) = \frac{1}{2}(u_r + u_\ell)$ one obtains

$$\eta(v) = \int_{\frac{1}{2}(u_r+u_\ell)}^{v} \frac{1}{g(z)} dz. \tag{83}$$

Again, observe that for all $v \in (u_r, u_\ell)$ the integrand in the above is strictly negative and therefore η is well defined. If h is s.t. the integral in (83) can be computed explicitly, one gets the TW in implicit form, as $\eta = H(v)$, with H a primitive of the function $1/g$ s.t. $H\big(\frac{1}{2}(u_r + u_\ell)\big) = 0$. If, further, the inverse of H (say $G = H^{-1}$) can be found explicitly, then v can be obtained as $v = G(\eta)$, and with this the TW solution $u(x,t) = v(x - ct)$ with c given in (79).

Remark 12 *Observe that if $v \nearrow u_\ell$, g behaves asymptotically like $(v - u_\ell)$. Since $g < 0$ in the given interval, the integrand $1/g$ approaches $-\infty$ as $v \nearrow 0$. Moreover, due to the asymptotic behaviour of g in the left neighbourhood of u_ℓ it follows that $1/g \notin L^1((v^*, u_\ell)$ for any $v^* \in (u_r, u_\ell)$, in the sense that any primitive H of $1/g$ blows down to $-\infty$ as $v \nearrow u_\ell$. From (83) it follows that $\eta \to -\infty$ when $v \nearrow u_\ell$.*
Analogously, $\eta \to \infty$ if $v \searrow u_r$. This confirms that the integral representation of the TW has the expected behaviour, approaching u_ℓ, u_r if η goes to $-\infty$, respectively ∞.

The case $u_\ell > u_r = 0$. For simplicity we only consider $u_\ell = 1$. If $u_\ell \neq 1$, the situation can be reduced to this case by rescaling the TW, namely by working with $w(\eta) = \frac{1}{u_\ell} v(\eta)$.

We start by observing that v needs not to be a C^1 function, but v^m. This is important if $m > 1$, as will be seen below. At this point, from (78) it follws that v satisfies

$$2\nu(v^m(\eta))' = v(v(\eta) - u_\ell), \text{ for all } \eta \in \mathbb{R}. \tag{84}$$

Since $m > 1$, this means that for any η one either has $v(\eta) = 0$, or it holds that

$$\frac{2\nu m}{m-1}(v^{m-1}(\eta))' = v(\eta) - 1,$$

and $0 < v(\eta) < 1$.
As before, v is monotone decreasing, and we fix v by taking $v(0) = \frac{1}{2}$. This gives the integral representation of the TW

$$\eta = 2\nu m \int_{\frac{1}{2}}^{v} \frac{z^{m-2}}{z-1} dz. \tag{85}$$

Since $g(z) = 2\nu m(z^{m-2})/(z-1)$ has a constant sign for any argument z between 0 and 1, η is a bijection mapping (0, 1) to its range. It is unclear whether its range is the entire $\mathbb{R}$ or not. Without making the proof rigorous (I invite you to do it!) and in the spirit of Remark 12, we observe the following:

- As $z \nearrow 1$, $1/g(z)$ behaves like $1/(z-1)$. This means that

  $$\lim_{v \nearrow 1} \int_{\frac{1}{2}}^{v} \frac{z^{m-2}}{z-1} dz = -\infty,$$

  showing that $\lim_{v \nearrow 1} \eta = -\infty$. This also implies that the TW v only reaches the left state 1 asymptotically.

- For $z \searrow 0$, $1/g(z)$ behaves like $-z^{m-2}$. If $m \geq 2$, this is continuous up to 0, and therefore the integral in (85) has a finite limit when $v \searrow 0$. More precisely, an $\eta_0 > 0$ exists (why strictly positive?) s.t.

  $$\lim_{v \searrow 0} \int_{\frac{1}{2}}^{v} \frac{z^{m-2}}{z-1} dz = \eta_0.$$

  The situation for $m \in (1, 2)$ is quite similar. Here the reciprocal of h is sigular in 0, but this singularity is integrable since $m - 2 > -1$ and therefore the primitive of z^{m-2} is defined in 0. This means that also in this case an $\eta_0 > 0$ exists as the limit of the integral if $v \searrow 0$.

We have therefore shown that the integral representation in (85) defines the strictly decreasing function $\eta : [0, 1) \to (-\infty, \eta_0]$, which can be inverted to find the TW v defined only on the interval $(-\infty, \eta_0]$. Observe that this TW reaches the right state $u_r = 0$ at a finite η_0.
To extend the TW for arguments $\eta > \eta_0$ we recall that v constructed above was using only one possibility offered by the TW equation (84). The other possibility was simply $v = 0$. This means that the TW solution on the entire $\mathbb{R}$ is the extension of the solution above by 0 for arguments $\eta \geq \eta_0$.

Remark 13 *Clearly, the right limit in η_0 of the TW v' is 0. From the discussion below, for $\eta < \eta_0$ but close, $\eta_0 - \eta$ has the same order as $v(\eta)^{m-1}$. This means that $v(\eta)$ behaves like $(\eta_0 - \eta)^{\frac{1}{m-1}}$. For $m \in (1, 2)$, this power is greater than 1, which means that actually even $v'(\eta)$ approaches 0 when $\eta \nearrow \eta_0$ and implying that the solution constructed above is C^1 everywhere on $\mathbb{R}$. For $m = 2$ the left limit of v' is finite but not 0, so the solution v has a kink at η_0. For $m > 2$ the left limit is $-\infty$. However, for all $m > 1$, v^m is C^1 everywhere, and the equation (84) is satisfied by the TW everwhere in $\mathbb{R}$.*

For the case $m = 2$ the TW can be found explicitly from (85), which becomes

$$\eta = 4\nu \int_{\frac{1}{2}}^{v} \frac{1}{z-1} dz.$$

This gives for all $v \in (0, 1)$

$$\eta = 4\nu\big(\ln|v-1| - \ln(1/2)\big),$$

and therefore

$$v = 1 - \frac{1}{2} e^{\frac{\eta}{4\nu}}.$$

In other words we found the TW v explicitly, but this only holds if $v \in (0, 1)$. Whereas the expression on the right in the above is less than 1 for all values of η, the value 0 is achieved for $\eta_0 = 4\nu \ln 2$. With this we found the TW

$$v(\eta) = \begin{cases} 1 - \frac{1}{2} e^{\frac{\eta}{4\nu}}, & \text{if } \eta \le \eta_0, \\ 0, & \text{if } \eta > \eta_0, \end{cases} \tag{86}$$

where η_0 is given above. Clearly, one has

$$\lim_{\eta \nearrow \eta_0} v'(\eta) = -\frac{1}{4\nu} < 0, \text{ and } \lim_{\eta \nearrow \eta_0} (v^2)'(\eta) = 0.$$

This means that v' has finite left and right limits in η_0, while v^2 is C^1 (see Remark 13). Finally, with the TW velocity $c = \frac{1}{2}$ as given in (79), the TW solution u is

$$u(x,t) = \begin{cases} 1 - \frac{1}{2} e^{\frac{1}{4\nu}(x - \frac{1}{2}t)}, & \text{if } x \le \frac{1}{2}t + 4\nu \ln 2, \\ 0, & \text{if } x > \frac{1}{2}t + 4\nu \ln 2. \end{cases} \tag{87}$$

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