oef 116

Wietse Vaes

Zij

$$f :]0,1]\times]0,1] \to \mathbb{R} : (x,y) \mapsto \frac{x^2-y^2}{(x^2+y^2)^2}$$

I) Bereken

$$\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy$$

en

$$\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx$$

$$\begin{aligned}
\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy &= \int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dx\right) dy = \int_0^1 \left(\int_0^1 \frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2}dx\right) dy \\
&= \int_0^1 \left(\int_0^1 \frac{1}{x^2+y^2} - 2y^2\frac{1}{(x^2+y^2)^2}dx\right) dy \\
&= \int_0^1 \left(\frac{\arctan(\frac{x}{y})}{y} - 2y^2\left(\frac{\arctan(\frac{x}{y})}{2y^3} + \frac{x}{2y^2(x^2+y^2)}\right)\right)\Bigg|_0^1 dy = \int_0^1 \left(-\frac{x}{(x^2+y^2)}\Bigg|_0^1\right) dy \\
&= \int_0^1 -\frac{1}{(1^2+y^2)}dy = -\frac{\arctan(\frac{y}{1})}{1}\Bigg|_0^1 = -\arctan(1) = -\frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx &= \int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy\right) dx = \int_0^1 -\left(\int_0^1 \frac{y^2-x^2}{(y^2+x^2)^2}dy\right) dx = \int_0^1 \frac{1}{(x^2+1^2)}dx \\
&= \frac{\arctan(\frac{x}{1})}{1}\Bigg|_0^1 = \arctan(1) = \frac{\pi}{4}
\end{aligned}$$

II) We zien nu in dat $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy \neq \int_0^1 \left(\int_0^1 f(x,y)dy\right) dx$.
Een gevolg van de stelling van Fubini is dat: zij $f \in \mathcal{L}^1(\mathbb{R}^{p+q}, \mathbb{R})$, dan is

$$\int_{\mathbb{R}^{p+q}} f(x,y)d(x,y) = \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}^p} f(x,y)dx\right) dy = \int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^q} f(x,y)dy\right) dx$$

Neem nu p en q gelijk aan 1. Omdat nu $\mathcal{L}^1(]0,1]^2, \mathbb{R}) \subset \mathcal{L}^1(\mathbb{R}^2, \mathbb{R})$, zouden we f kunnen redefinieren als

$$g(x,y) = \begin{cases} f(x,y) & (x,y) \in]0,1]^2 \\ 0 & (x,y) \notin]0,1]^2 \end{cases}$$

Nu geldt (zeker): als $f \in \mathcal{L}^1(]0,1]^2, \mathbb{R})$ dan is $g \in \mathcal{L}^1(\mathbb{R}^2, \mathbb{R})$. Echter nu is

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x,y)dx\right) dy = \int_0^1 \left(\int_0^1 g(x,y)dx\right) dy = -\frac{\pi}{4} \neq \frac{\pi}{4} = \int_0^1 \left(\int_0^1 g(x,y)dy\right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x,y)dy\right) dx$$

Dus $g \notin \mathcal{L}^1(\mathbb{R}^2, \mathbb{R})$ en tenslotte $f \notin \mathcal{L}^1(]0,1]^2, \mathbb{R})$