Numerieke technieken en optimalisatie

Prof. Dr. Jochen Schütz, Manuela Bastidas Olivares

Any questions? Do not hesitate to contact us!

Jochen:	C265	Tel 8315	jochen.schuetz@uhasselt.be
Manuela:	C101A	Tel 8296	manuela.bastidas@uhasselt.be

All computation may be done with Matlab.

Exercise 2.1: QU Decomposition in MATLAB

Consider the following 4×4 matrix A

$$A = \begin{pmatrix} 4 & 1 & -2 & 2 \\ 1 & 2 & 0 & 1 \\ -2 & 0 & 3 & 2 \\ 2 & 1 & -2 & -1 \end{pmatrix}.$$

Use MATLAB to perform *Givens rotations*, and bring the matrix A to QU form. Check your results against MATLAB.

Answer.

```
% A = [4 1 -2 2; 1 2 0 1; -2 0 3 2; 2 1 -2 -1];
% A = [1 2 0; 1 1 1; 2 1 0];

A = [ 1 2 ; 0 3; 0 4];
n  = size(A,1);

U = A;
Q = eye(n);

for i=1:n-1 %Columns
for j= i+1:n % rows (to be zero)

if U(j,i) ~= 0 % Only for non-zero
r = sqrt(U(i,i)^2+U(j,i)^2);
c = U(i,i)/r;
s = U(j,i)/r;

% Rotation matrix
G = eye(n);
G([i,j],[i,j]) = [c s; -s c];

% Matrices QU (see definition)
U = G*U;
Q = Q*G';
end
```

```
end
end

%% Checking it

A1 = Q*U;
difer = norm(A1-A);
disp(difer)

[Q1,U1] = qr(A);
```

Exercise 2.2: (Householder reflection matrix)

For any vector $v \in \mathbb{R}^n$, one defines a matrix

$$Q_v := I_n - \frac{2}{\|v\|_2^2} vv^T.$$

a) Show that

1) Q_v is a symmetric, orthogonal matrix and $Q_v^2 = I_n$.

Answer.

- Symmetry:

$$\begin{aligned} Q_v^T &= \left(I_n - \frac{2}{\|v\|_2} vv^T\right)^T \\ &= (I_n)^T - \left(\frac{2}{\|v\|_2} vv^T\right)^T \\ &= I_n - \frac{2}{\|v\|_2} (v^T)^T v^T \\ &= I_n - \frac{2}{\|v\|_2} vv^T = Q_v \end{aligned}$$

- Orthogonality:

$$\begin{aligned} Q_v Q_v^T &= \left(I_n - \frac{2}{\|v\|_2} vv^T\right)\left(I_n - \frac{2}{\|v\|_2} vv^T\right) \\ &= I_n - \frac{2}{\|v\|_2} vv^T - \frac{2}{\|v\|_2} vv^T + \left(\frac{2}{\|v\|_2} vv^T\right)\left(\frac{2}{\|v\|_2} vv^T\right) \\ &= I_n - \frac{4}{\|v\|_2^2} vv^T + \frac{4}{\|v\|_2^2}\left(vv^T\right)\left(vv^T\right) = I_n \end{aligned}$$

- $Q_v^2 = I_n$:

$$\begin{aligned} Q_v^2 &= Q_v Q_v \\ &= (Q_v)^T Q_v \\ &= (Q_v)^{-1} Q_v = I_n \end{aligned}$$

2) For any $u \in \mathbb{R}^n$ that is orthogonal to v (i.e. in the hyperplane orthogonal to v),

$$Q_v u = u, \qquad Q_v v = -v.$$

Answer.

$$\begin{aligned} Q_v u &= \left(I_n - \frac{2}{\|v\|_2} v v^T\right) u \\ &= I_n u - \frac{2}{\|v\|_2} v(v^T u) = u \end{aligned}$$

$$\begin{aligned} Q_v v &= \left(I_n - \frac{2}{\|v\|_2} v v^T\right) v \\ &= I_n v - \frac{2}{\|v\|_2} v(v^T v) = -v \end{aligned}$$

3) Q_v has only eigenvalues ± 1 and $\det(Q_v) = -1$.

Answer. Let λ and x be the eigenvalues and eigenvectors of Q_v, i.e

$$Q_v x = \lambda x$$

Moreover,

$$\|Q_v x\|^2 = \|\lambda x\|^2 = \lambda^2 \|x\|^2$$

and

$$\|Q_v x\|^2 = (Q_v x)^T (Q_v x) = x^T Q_v^T Q_v x = x^T x = \|x\|^2$$

then

$$\lambda = \pm 1$$

If u is orthogonal to v then $Q_v u = u$, it means that u is an eigenvector with eigenvalue 1. Since there are $n-1$ vectors (L.I) orthogonal to v then 1 is an eigenvalue of multiplicity $n-1$.

$$\det(Q_v) = (1)^{n-1} \cdot (-1) = -1$$

b) Let $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, find the reflect vector $v \in \mathbb{R}^3$ such that $Q_v x = -\|x\|_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Answer. The right reflection is through a hyperplane that bisects the angle between x and $Q_v x$

$$v = \alpha(x - Q_v x) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \|x\|_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \|x\|_2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2.74 \\ 2 \\ 3 \end{pmatrix}$$

Exercise 2.3: (Sparse matrices and the CRS format) Self-study

A matrix $A \in \mathbb{R}^{n\times n}$ is called *sparse* if most of its entries are zero. (More precisely, we say that it has at most $O(n)$ nonzero entries.) We consider the *compressed row storage* (CRS) format. Let A have N nonzero entries, with $N \ll n^2$. Instead of saving the matrix A as a one-dimensional array of size n^2, one uses three arrays: `entries` and `colmns` are arrays of size N, `row_pointer` is an array of size $n+1$. `entries` saves the nonzero entries of the matrix A in the order of their occurrence (left to right, then top to bottom) and `colmns` saves the corresponding column indices. `row_pointer[i]` saves the index of entries where the first entry in row $i-1$ is located, i.e., `entries[row_pointer[i]]` gives the first element of row i (note that array indices start at 0), `row_pointer[n]` gives the size of `entries`.

a) Let

$$A := \begin{pmatrix} 12 & 22 & 0 \\ 0 & 23 & 33 \\ 0 & 0 & 44 \end{pmatrix}.$$

Give the CRS format of A.

Answer.
```
Entries:   [ 12 22 23 33 44 ]
Columns:   [ 0 1 1 2 2 ]
Row_pointer:   [ 0 2 4 5 ]
```

b) Recover the matrix from the CRS format given below
```
entries = [ 5 8 3 6 ]
columns = [ 0 1 2 1 ]
row_pointer = [ 0 0 2 3 4 ]
```

Answer.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 8 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 6 & 0 & 0 \end{pmatrix}$$

c) A matrix resulting from a discretization of a one-dimensional Laplace operator is given as

$$A := \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \cdots & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

for $h := \frac{1}{n+1}$. We assume real numbers are saved as `double`, i.e., required memory is 8 byte; and indices are saved as `int`, i.e., required memory is 4 byte. Take $n = 50,000$ and compute the memory required to save A completely, i.e., including the zero entries or in CRS format.

Answer. - Complete matrix: $(n \times n) \cdot 8 = 2^1 \cdot 10^{10}$ bytes $= 20$ GB.
- CRS: $24n - 16 + 16n - 4 = 40n - 20 = 1.999.980 \approx 2.000.000$ bytes $= 0.002$ GB
+ Non-zeros: $n + 2(n-1) = 3n - 2 \to (3n-2)\cdot 8 = 24n - 16 = 1.199.984$
+ Indices: $n + 2(n-1) + (n+1) = 4n - 1 \to (4n-1)\cdot 4 = 16n - 4 = 799.996$

Exercise 2.4: (Convergence of iterative solvers)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^{n\times n}$ that is induced by a vector norm, i.e.,

$$\|A\| := \sup_{0\neq x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|}.$$

By $\rho(A)$, we denote the *spectral radius* of a matrix A, i.e.,

$$\rho(A) := \max\{|\lambda| \big| \lambda \text{ is an eigenvalue of } A\}.$$

a) Show that for all matrices $A \in \mathbb{R}^{n\times n}$,

$$\rho(A) \leq \|A\|.$$

Answer. Let λ be an eigenvalue of A and $v \neq 0$ be a corresponding eigenvector

$$\lambda v = Av \to ||\lambda v|| = ||Av||$$
$$|\lambda|||v|| = ||Av|| \leq ||A||||v|| \to |\lambda| \leq ||A||$$

In conclusion, for all λ we have $|\lambda| \leq ||A||$ and this implies $\max_\lambda(|\lambda|) \leq ||A||$

b) Compute the spectral radius of the matrix

$$A = \begin{pmatrix} a & 4 \\ 0 & a \end{pmatrix}.$$

When is $\rho(A) < 1$? Check that $\|A^k\|^{\frac{1}{k}}$ can be greater than one in this case.

Answer.

$$\det(A - \lambda I) = (a-\lambda)^2 - 0 = 0 \iff \lambda = a$$

Then $\rho(A) \leq 1 \iff |\lambda| \leq 1 \iff |a| \leq 1$ and with this we can show that:

$$||A||_\infty = |a| + 4 \to 4 < ||A||_\infty < 5$$
$$||A||_1 = |a| + 4 \to 4 < ||A||_1 < 5$$