assignments 2

Wietse Vaes & Lori Trimpeneers

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1. We consider the following Poisson equation on a unit square $\Omega = (0,1) \times (0,1)$

$$(\mathbf{P}_1) \begin{cases} -\Delta u = f(x,y) & (x,y) \in \Omega \\ u = 0 & (x,y) \in \partial\Omega \end{cases}$$

Here $\partial\Omega$ denotes the boundary of Ω. Further, $f(x,y)$ is a given smooth function, which is strictly positive $f(x,y) > 0$ for $(x,y) \in \Omega$ and we are looking for a numerical approximation of the solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

I. Prove that $u > 0$ for $(x,y) \in \Omega$

Solution:
Because $f(x,y) > 0$ for $(x,y) \in \Omega$, we have that $-\Delta u > 0$ and $u|_{\partial\Omega} = 0$.
We have to prove that $u > 0$.
Suppose there is an $\underline{\hat{x}} = (\hat{x}, \hat{y}) \in \Omega$ such that $u(\underline{\hat{x}}) < 0$.
Then, there exists a global minimum, so $\exists \underline{\tilde{x}} = (\tilde{x}, \tilde{y}) : \ u(\underline{\tilde{x}}) \leq u(\underline{x}) \quad , \forall \underline{x} = (x,y) \in \Omega$.
Because $u(\underline{\hat{x}}) < 0$, is $u(\underline{\tilde{x}}) < 0$.
Because $u(\underline{\tilde{x}})$ is a global minimum, there holds

$$\left(\frac{\partial^2 u}{\partial x \partial y}(\underline{\tilde{x}})\right)^2 - \frac{\partial^2 u}{\partial x^2}(\underline{\tilde{x}})\frac{\partial^2 u}{\partial y^2}(\underline{\tilde{x}}) < 0$$

and $\frac{\partial^2 u}{\partial x^2}(\underline{\tilde{x}}) > 0$.
Because $\left(\frac{\partial^2 u}{\partial x \partial y}(\underline{\tilde{x}})\right)^2 > 0$ and $\frac{\partial^2 u}{\partial x^2}(\underline{\tilde{x}}) > 0$, there holds that $\frac{\partial^2 u}{\partial y^2}(\underline{\tilde{x}}) > 0$.
So $\frac{\partial^2 u}{\partial x^2}(\underline{\tilde{x}}) > 0$ and $\frac{\partial^2 u}{\partial y^2}(\underline{\tilde{x}}) > 0$

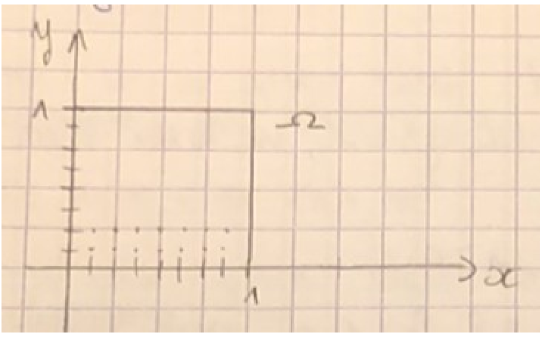
$$\begin{aligned} &\Rightarrow \Delta u > 0 \\ &\Rightarrow -\Delta u < 0 \end{aligned}$$

This is a contradiction, because it is given that $-\Delta u > 0$.
We can conclude that $u > 0, \forall (x,y) \in \Omega$

We discretize problem (P_1) using the finite difference method, in which we use an equidistant (rectangular) grid with m internal gridpoints per direction (so we end up with $n = m^2$ internal gridpoints). We use a horizontal number of the unknowns. Let h be the distance between adjacent gridpoints.

II. give the discretization at an internal point (away from $\partial\Omega$). Show that the truncation error is $\mathcal{O}(h^2)$.

Solution:
In general there holds:



On the boundary, the solution is known.

With the girdpoints:

$$0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < x_{j+1} < \cdots < x_n < x_{n+1} = 1$$
$$0 = y_0 < y_1 < \cdots < y_{k-1} < y_k < y_{k+1} < \cdots < y_m < y_{m+1} = 1$$
$$\begin{cases} x_j - x_{j-1} = \Delta x \\ y_k - y_{k-1} = \Delta y \end{cases}$$
$$x_j = j\Delta x \qquad x_{n+1} = (n+1)\Delta x = 1 \Rightarrow \Delta x = \frac{1}{n+1}$$
$$y_k = k\Delta y \qquad y_{m+1} = (m+1)\Delta y = 1 \Rightarrow \Delta y = \frac{1}{m+1}$$

Define the notatation $u(x_j, y_k) = u_{jk}$.
Like in 1D, there holds

$$\frac{\partial^2 u}{\partial x^2}(x_j, y_k) = \frac{u_{j-1\ k} - 2u_{jk} + u_{j+1\ k}}{\Delta x^2} + \tilde{\mathcal{K}} \cdot \Delta x^2$$
$$\frac{\partial^2 u}{\partial y^2}(x_j, y_k) = \frac{u_{j\ k-1} - 2u_{jk} + u_{j\ k+1}}{\Delta y^2} + \hat{\mathcal{K}} \cdot \Delta y^2$$

Using the PDE gives:

$$\frac{-u_{j-1\ k} + 2u_{jk} - u_{j+1\ k}}{\Delta x^2} + \frac{-u_{j\ k-1} + 2u_{jk} - u_{j\ k+1}}{\Delta y^2} = f(x_j, y_k) + \tilde{\mathcal{K}} \cdot \Delta x^2 + \hat{\mathcal{K}} \cdot \Delta y^2$$

We want to write this as $A\underline{w} = \underline{b}$, with $\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}, \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$

With $N = n \cdot m$.
Using horizontal number of the unknowns, the index (j, k) goes to $l = j + (k-1) \cdot n$.
So in the previous formula u_{jk} goes to w_l:

$$\frac{-w_{l-1} + 2w_l - w_{l+1}}{\Delta x^2} + \frac{-w_{l-n} + 2w_l - w_{l+n}}{\Delta y^2} = f(x_l, y_l) + \tilde{\mathcal{K}} \cdot \Delta x^2 + \hat{\mathcal{K}} \cdot \Delta y^2 \quad , \text{with}(x_j, y_k) \to (x_l, y_l)$$

Since it is given that the equidistant grid has m internal gridpoints per direction, there follows:

$$N = n \cdot m = m \cdot m = m^2$$

It is also given that h is the distance between adjacent gridpoints, so

$$\Delta x = \Delta y = h$$

So the precious formula becomes:

$$\begin{aligned} \frac{-w_{l-1} + 2w_l - w_{l+1}}{h^2} + \frac{-w_{l-m} + 2w_l - w_{l+m}}{h^2} &= f(x_l, y_l) + \tilde{\mathcal{K}} \cdot h^2 + \hat{\mathcal{K}} \cdot h^2 \\ &= f(x_l, y_l) + h^2(\tilde{\mathcal{K}} + \hat{\mathcal{K}}) \end{aligned}$$

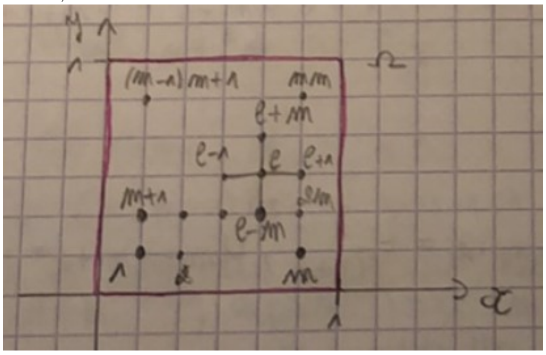
Therefore, the truncated error is $\mathcal{O}(h^2)$. The discretization at an internal point is $(v_l \approx w_l = u_{jk})$

$$\begin{aligned}\frac{-v_{l-1}+2v_l-v_{l+1}}{h^2}+\frac{-v_{l-m}+2v_l-v_{l+m}}{h^2} &= f(x_l,y_l)\\ &= f(\underline{x}_l)\end{aligned}$$

With $\underline{x}_l=(x_l,y_l)$. So if $A\underline{v}\approx\underline{b}$, then $a_{ll-1}=-\frac{1}{h^2}, a_{ll}=\frac{4}{h^2}, a_{ll+1}=-\frac{1}{h^2}, a_{ll-m}=-\frac{1}{h^2}, a_{ll+m}=-\frac{1}{h^2}$ and $\underline{b}_j=f(\underline{x}_j)$

III. Give the discretization at an arbitrary gridpoint adjacent to the boundary.

Solution:
First, we sketch the situation



On the boundary of Ω :
$u=0$

From the precious excersise, we have:

$$\frac{-v_{l-1}+2v_l-v_{l+1}}{h^2}+\frac{-v_{l-m}+2v_l-v_{l+m}}{h^2}=f(\underline{x}_l)$$

For $l=1$:

$$\frac{-0+2v_1-v_2}{h^2}+\frac{-0+2v_1-v_{1+m}}{h^2}=f(\underline{x}_1)$$

We want:$A\underline{v}=\underline{b}\Rightarrow\sum_{j=1}^{m^2}a_{lj}v_j=b_l=f(\underline{x}_l)$
For $l=1:\sum_{j=1}^{m^2}a_{1j}v_j=b_1, a_{1,1}=\frac{2}{h^2}+\frac{2}{h^2}=\frac{4}{h^2}, a_{1,2}=-\frac{1}{h^2}, a_{1,1+m}=-\frac{1}{h^2}$, all other $a_{1,j}=0$.

For $l=2$:

$$\frac{-v_1+2v_2-v_3}{h^2}+\frac{-0+2v_2-v_{2+m}}{h^2}=f(\underline{x}_2)$$

$$\sum_{j=1}^{m^2}a_{2j}v_j=b_2$$

$a_{2,1}=-\frac{1}{h^2}, a_{2,2}=\frac{4}{h^2}, a_{2,3}=-\frac{1}{h^2}, a_{2,2+m}=-\frac{1}{h^2}$, all other $a_{2,j}=0$.

For $l=3,4,\ldots,m-1$:

Almost identical as the case for $l=2$:
$a_{l,l-1}=-\frac{1}{h^2}, a_{l,l}=\frac{4}{h^2}, a_{l,l+1}=-\frac{1}{h^2}, a_{l,l+m}=-\frac{1}{h^2}$, all other $a_{l,j}=0$.

For $l=m$:

$$\frac{-v_{m-1}+2v_m-0}{h^2}+\frac{-0+2v_m-v_{m+m}}{h^2}=f(\underline{x}_m)$$

$$\sum_{j=1}^{m^2}a_{mj}v_j=b_m$$

$a_{m,m-1}=-\frac{1}{h^2}, a_{m,m}=\frac{4}{h^2}, a_{m,m+m}=-\frac{1}{h^2}$, all other $a_{m,j}=0$.

For $l=m+1$:

$$\frac{-0+2v_{m+1}-v_{m+2}}{h^2}+\frac{-v_1+2v_{m+1}-v_{m+1+m}}{h^2}=f(\underline{x}_{m+1})$$

$$\sum_{j=1}^{m^2} a_{m+1,j}v_j = b_{m+1}$$

$a_{m+1,1} = -\frac{1}{h^2}, a_{m+1,m+1} = \frac{4}{h^2}, a_{m+1,m+2} = -\frac{1}{h^2}, a_{m+1,2m+1} = -\frac{1}{h^2}$, all other $a_{m+1j} = 0$.

For $l = 2m+1, 3m+1, \dots, (m-2)m+1$:

Almost identical as the case for $l = m+1$
$a_{l,l-m} = -\frac{1}{h^2}, a_{l,l} = \frac{4}{h^2}, a_{l,l+1} = -\frac{1}{h^2}, a_{l,l+m} = -\frac{1}{h^2}$, all other $a_{l,j} = 0$.

For $l = 2m$:

$$\frac{-v_{2m-1} + 2v_{2m} - 0}{h^2} + \frac{-v_m + 2v_{2m} - v_{3m}}{h^2} = f(\underline{x}_{2m})$$

$$\sum_{j=1}^{m^2} a_{2mj}v_j = b_{2m}$$

$a_{2m,m} = -\frac{1}{h^2}, a_{2m,2m-1} = -\frac{1}{h^2}, a_{2m,2m} = \frac{4}{h^2}, a_{2m,3m} = -\frac{1}{h^2}$ all other $a_{2m,j} = 0$.

For $l = 3m, 4m, \dots, (m-1)m$:

Almost identical as the case for $l = 2m$
$a_{l,l-m} = -\frac{1}{h^2}, a_{l,l-1} = -\frac{1}{h^2}, a_{l,l} = \frac{4}{h^2}, a_{l,l+m} = -\frac{1}{h^2}$ all other $a_{lj} = 0$.

For $l = (m-1)m+1$:

$$\frac{-0 + 2v_{(m-1)m+1} - v_{(m-1)m+2}}{h^2} + \frac{-v_{(m-2)m+1} + 2v_{(m-1)m+1} - 0}{h^2} = f(\underline{x}_{(m-1)m+1})$$

$$\sum_{j=1}^{m^2} a_{(m-1)m+1,j}v_j = b_{(m-1)m+1}$$

$a_{(m-1)m+1,(m-2)m+1} = -\frac{1}{h^2}, a_{(m-1)m+1,(m-1)m+1} = \frac{4}{h^2}, a_{(m-1)m+1,(m-1)m+2} = -\frac{1}{h^2}$
all other $a_{(m-1)m+1,j} = 0$.

For $l = (m-1)m+2$:

$$\frac{-v_{(m-1)m+1} + 2v_{(m-1)m+2} - v_{(m-1)m+3}}{h^2} + \frac{-v_{(m-2)m+2} + 2v_{(m-1)m+2} - 0}{h^2} = f(\underline{x}_{((m-1)m+2})$$

$$\sum_{j=1}^{m^2} a_{(m-1)m+2,j}v_j = b_{(m-1)m+2}$$

$a_{(m-1)m+2,(m-2)m+2} = -\frac{1}{h^2}, a_{(m-1)m+2,(m-1)m+1} = -\frac{1}{h^2}, a_{(m-1)m+2,(m-1)m+2} = \frac{4}{h^2}$,
$a_{(m-1)m+2,(m-1)m+3} = -\frac{1}{h^2}$ all other $a_{(m-1)m+2,j} = 0$.

For $l = (m-1)m+3, (m-1)m+4, \dots, m^2-1$:

Almost identical as the case for $l = (m-1)m+2$
$a_{l,l-m} = -\frac{1}{h^2}, a_{l,l-1} = -\frac{1}{h^2}, a_{l,l} = \frac{4}{h^2}, a_{l,l+1} = -\frac{1}{h^2}$ all other $a_{lj} = 0$.

For $l = m^2$:

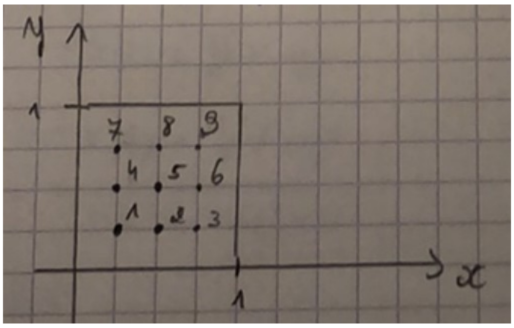
$$\frac{-v_{m^2-1} + 2v_{m^2} - 0}{h^2} + \frac{-v_{m^2-m} + 2v_{m^2} - 0}{h^2} = f(\underline{x}_{m^2})$$

$$\sum_{j=1}^{m^2} a_{m^2,j}v_j = b_{m^2}$$

$a_{m^2,m^2-m} = -\frac{1}{h^2}, a_{m^2,m^2-1} = -\frac{1}{h^2}, a_{m^2,m^2} = \frac{4}{h^2}$ all other $a_{m^2,j} = 0$.

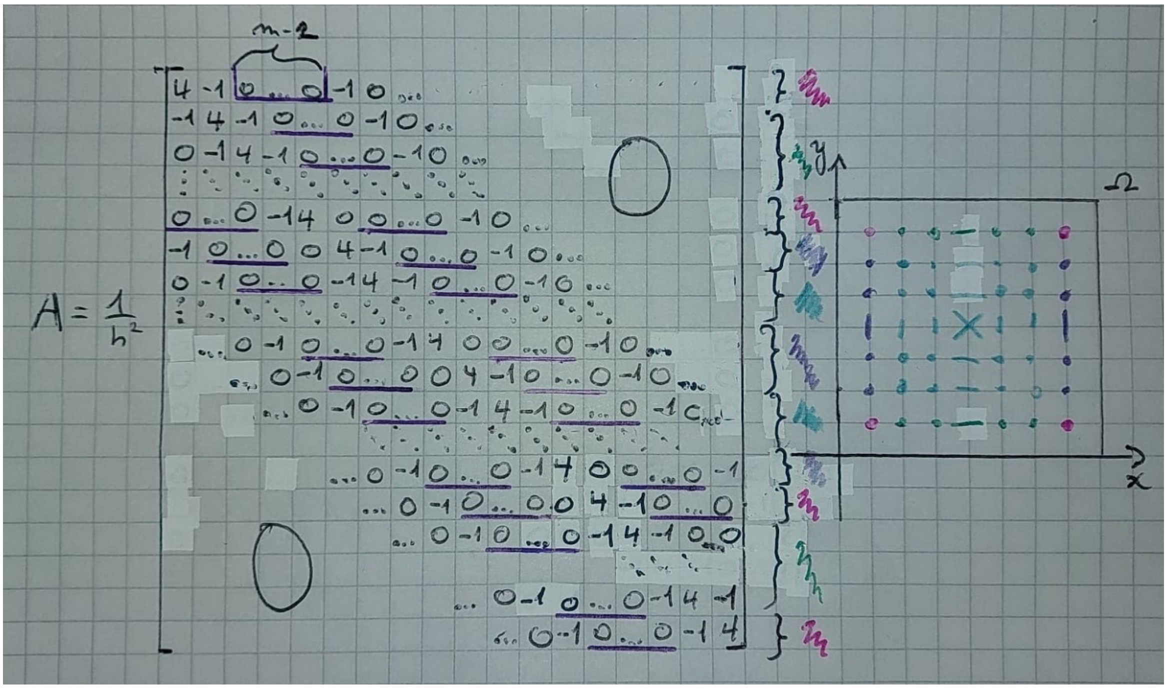
IV. Give the overall discretization matrix $A \in \mathbb{R}^{N\times N}$.

Solution:
To give an idea for the structure of the overall discretization matrix $A \in \mathbb{R}^{N\times N}$, suppose that $m = 3(N = m^2 = 9)$.



$$A = \frac{1}{h^2}\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}$$

So, in a more general case:



V. Show that A is a strictly diagonally dominant Z-matrix.

Solution:
A is clearly a $Z-$matrix, since $a_{jk} \leq 0$, if $j \neq k$.
A is also a strictly diagonally dominant matrix, because

$$\frac{4}{h^2} = a_{11} > \sum_{j=2}^{m^2} |a_{1j}| = \frac{|-1|}{h^2} + \frac{|-1|}{h^2} = \frac{2}{h^2}$$

And there exist a k, where there holds

$$\frac{4}{h^2} = a_{kk} \geq \sum_{j=1, j\neq k}^{m^2} |a_{kj}|.$$

Notice that for an internal point: $\frac{4}{h^2} = a_{kk} \geq \frac{|-1|+|-1|+|-1|+|-1|}{h^2} = \frac{4}{h^2}$ (so this is an equality) and for a gridpoint adjacent to the boundary holds a strict inequality ($\frac{4}{h^2} = a_{kk} > \frac{3}{h^2}$ or $\frac{4}{h^2} = a_{kk} > \frac{2}{h^2}$).

Since $f(x,y) > 0$, we have $A\underline{v} > 0$ for the finite difference solution $\underline{v}$.

VI. Argue why the finite difference solution, $\underline{v}$, is strictly positive, that is $\underline{v} > 0$ (Your answer may be short using theorems from the book).

Solution:
This follows from the Discreet Maximum Principe, since A is a strictly diagonally dominant Z-matrix (see Theorem 3.6.1).

VII. Show that

$$u(x,y) = \frac{1}{2}x(1-x)y(1-y),$$

is the solution to problem (P_1) with

$$f(x,y) = x(1-x) + y(1-y).$$

Further, determine the error of the finite difference solution. Give a motivation.

Solution:
We prove that $u(x,y) = \frac{1}{2}x(1-x)y(1-y)$ is a solution of (P_1) with $f(x,y) = x(1-x)+y(1-y)$, by showing that

1. $-\Delta u = f(x,y) \quad , (x,y) \in \Omega$
2. $u = 0 \quad , (x,y) \in \partial\Omega$

For 1.:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2}(1-x)y(1-y) - \frac{1}{2}xy(1-y) \\
\frac{\partial^2 u}{\partial x^2} &= -\frac{1}{2}y(1-y) - \frac{1}{2}y(1-y) = -y(1-y) \\
\frac{\partial u}{\partial y} &= \frac{1}{2}(1-y)x(1-x) - \frac{1}{2}yx(1-x) \\
\frac{\partial^2 u}{\partial y^2} &= -\frac{1}{2}x(1-x) - \frac{1}{2}x(1-x) = -x(1-x)
\end{aligned}$$

So, $-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = x(1-x) + y(1-y) = f(x,y)$

For 2.:
$(x,y) \in \partial\Omega \Rightarrow y = 0, y = 1, x = 0, x = 1$
$u(x,0) = 0, u(x,1) = 0, u(0,y) = 0, u(1,y) = 0, \forall x, y \in (0,1)$

So, $u = 0, \forall (x,y) \in \partial\Omega$

Uniqueness of solution:
Suposse u_1, u_2 are 2 distinct solutions. Define $v = u_1 - u_2$.
We have (in Ω)

$$-\Delta u_1 = f(x,y)$$
$$-\Delta u_2 = f(x,y)$$

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$$-\Delta(u_1 - u_2) = 0$$

and (on $\partial\Omega$)

$$u_1 = 0$$
$$u_2 = 0$$

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$$u_1 - u_2 = 0$$

So,

$$\begin{cases} -\Delta v = 0, & \text{, in } \Omega \\ v = 0 & \text{, on } \partial\Omega \end{cases}$$

Therefore

$$\begin{aligned}
&-\Delta v = 0 \Rightarrow -v\Delta v = 0 \Rightarrow -\int_\Omega v\Delta v d\Omega = 0 \\
&\Rightarrow 0 = -\int_\Omega v\nabla(\nabla v) d\Omega = -\int_\Omega \nabla(v\nabla v) - \nabla v \nabla v d\Omega \\
&\Rightarrow 0 = -\int_\Omega v\nabla v \cdot \underline{n} d\Gamma + \int_\Omega |\nabla v|^2 d\Omega \text{ (Gauss' divergence theorem)} \\
&\Rightarrow \int_\Omega |\nabla v|^2 d\Omega = 0 \text{ (Since } v = 0 \text{ on } \partial\Omega \text{, so } -\int_\Omega v\nabla v \cdot \underline{n} d\Gamma = 0) \\
&\Rightarrow \nabla v = 0 \text{ in } \Omega \\
&\Rightarrow v = c \text{ in } \Omega \text{ and } v \text{ is continuous and } v = 0 \text{ on } \partial\Omega \\
&\Rightarrow v = 0 \Rightarrow u_1 = u_2
\end{aligned}$$

So, u is a solution and the solution is unique, therefore is u the solution.
For the error of the finite difference solution, we have the following. From the first assignment, we have in the 1 dimensional case that

$$\frac{u(x-h) - 2u(x) + u(x+h)}{h^2} = u''(x) + \frac{h^2}{24}(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)),$$

with $\xi_1 \in (x, x+h)$ and $\xi_2 \in (x-h, x)$. Using $x_j = x_{j-1} + h$ gives

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} = u''(x) + \frac{h^2}{24}(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)).$$

Analogous for the 1 dimensional case in y

$$\frac{u_{k-1} - 2u_k + u_{k+1}}{h^2} = u''(y) + \frac{h^2}{24}(u^{(4)}(\gamma_1) + u^{(4)}(\gamma_2)),$$

with $\gamma_1 \in (y, y+h)$ and $\gamma_2 \in (y-h, y)$. Therefore in the 2 dimensional case this becomes (and using the given PDE, i.e. $-\Delta u = f(x,y)$)

$$\frac{-u_{j-1k} + 2ujk - u_{j+1k}}{h^2} + \frac{-u_{jk-1} + 2ujk - u_{jk+1}}{h^2} = f(x_j, y_k) + h^2(\tilde{K} + \hat{K}),$$

with $\tilde{K} = \frac{h^2}{24}(\frac{\partial^4 u}{\partial x^4}(\xi_1, y_k) + \frac{\partial^4 u}{\partial x^4}(\xi_2, y_k))$ and $\hat{K} = \frac{h^2}{24}(\frac{\partial^4 u}{\partial y^4}(x_j, \gamma_1) + \frac{\partial^4 u}{\partial x^4}(x_j, \gamma_2))$. Since $u(x,y) = \frac{1}{2}x(1-x)y(1-y)$, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= -y(1-y) \Rightarrow \frac{\partial^3 u}{\partial x^3} = 0 \Rightarrow \frac{\partial^4 u}{\partial x^4} = 0 \\ \frac{\partial^2 u}{\partial y^2} &= -x(1-x) \Rightarrow \frac{\partial^3 u}{\partial y^3} = 0 \Rightarrow \frac{\partial^4 u}{\partial y^4} = 0. \end{aligned}$$

Therefore $\tilde{K} = 0$ and $\hat{K} = 0$, so the error is zero. We can conclude that u can be approximated exactly with a Taylor series up to order 4 and so the finite difference scheme is therefore also exactly.

2. We consider the two-dimensional steady-state heat equation on a unit square $\Omega = (0,1) \times (0,1)$ with closure $\overline{\Omega} = [0,1] \times [0,1]$. The boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ is assembled as follows:

1. $\partial\Omega_1 = \{(x,y) \in \overline{\Omega} : x = 0\} \cup \{(x,y) \in \overline{\Omega} : y = 0\}$
2. $\partial\Omega_2 = \{(x,y) \in \overline{\Omega} : x = 1\} \cup \{(x,y) \in \overline{\Omega} : y = 1\}$

We solve the following 'toy problen':

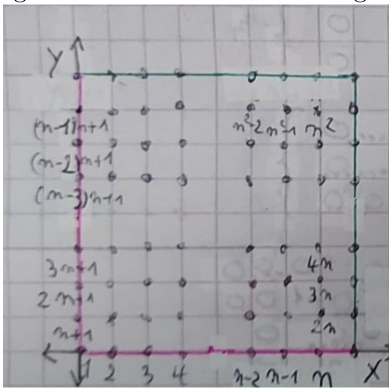
$$(P_2)\begin{cases} -\Delta u = 1, & \mathbf{x} \in \Omega \\ \frac{\partial u}{\partial n} = 0, & \mathbf{x} \in \partial\Omega_1 \\ u = 0, & \mathbf{x} \in \partial\Omega_2 \end{cases}$$

Use the finite-volume method to approximate the solution. Use n unknowns per coordinate direction. Make a countour plot of the solution. Play with several numbers of meshpoints. check whether your numerical approximation falls within

$$\frac{1}{4}(1 - (x^2 + y^2)) \leq u \leq \frac{1}{2}(1 - \frac{x^2 + y^2}{2}) \text{ in } \Omega$$

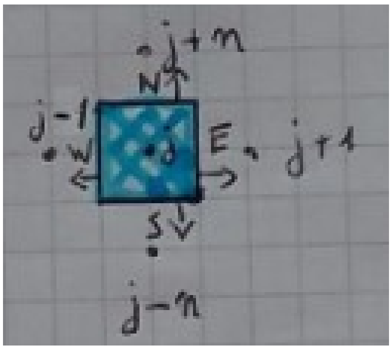
Write a small report of at most two pages about your discretisation and your findings.

Report:
We are going to discretizize the Ω into a grid with $n+1$ grid points horizontally and vertically:



On the boundary $\partial\Omega_1$:
$\frac{\partial u}{\partial n} = 0$
On the boundary $\partial\Omega_2$:
$u = 0$

Due to knowing what u is on $\partial\Omega_2$, we will not be incorporating this into our matrix. The grid will thus only contain n points with spacing $h = \frac{1}{n}(=\Delta x, \Delta y)$. We will thus try to make a matrix $A \in \mathbb{R}^{n^2\times n^2}$ to approximate a solution with $Av = \underline{b}$ (with the finite volume method). Take for instance grid point j with u_j an internal point. We will thus be using a $\Omega_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [x_{y-\frac{n}{2}}, y_{j+\frac{n}{2}}]$.
If we were to integrate the problem over Ω_j, we would get (Note that $\int_{\Omega_j} 1 d\Omega = \Delta x \Delta y = h^2$):

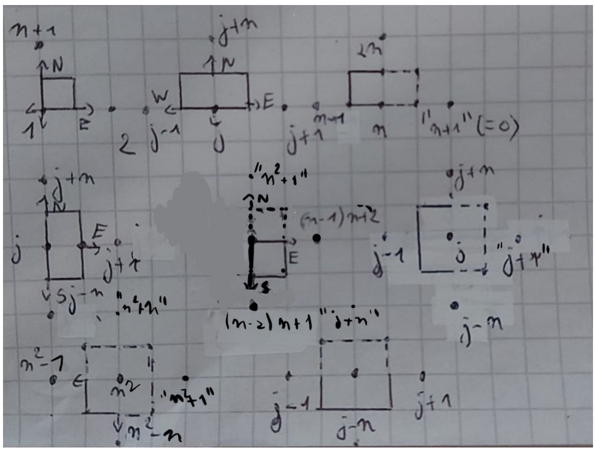


$$
\begin{aligned}
h^2 &= \int_{\Omega_j} -\Delta u d\Omega \\
&= -\int_{\partial\Omega_j} \underline{n}\nabla u d\Gamma \qquad \text{(Divergence theorem)} \\
&= -\sum_{p\in\{S,W,N,E\}} \int_{\partial\Omega_j^p} \underline{n}\nabla u d\Gamma \\
&= -\int_{\partial\Omega_j^S} \underline{n}\nabla u d\Gamma - \int_{\partial\Omega_j^W} \underline{n}\nabla u d\Gamma - \int_{\partial\Omega_j^N} \underline{n}\nabla u d\Gamma - \int_{\partial\Omega_j^E} \underline{n}\nabla u d\Gamma \qquad \text{(Midpoint rule} \rightarrow\text{)} \\
&\approx \frac{\partial u}{\partial y}(x_j, y_{j-\frac{h}{2}})\Delta x + \frac{\partial u}{\partial x}(x_{j-\frac{h}{2}}, y_j)\Delta y - \frac{\partial u}{\partial y}(x_j, y_{j+\frac{h}{2}})\Delta x - \frac{\partial u}{\partial x}(x_{j+\frac{h}{2}}, y_j)\Delta y \\
&\approx \frac{\Delta x}{\Delta y}(v_j - v_{j-n}) + \frac{h}{h}(v_j - v_{j-1}) - (v_{j+n} - v_j) - (v_{j+1} - v_j) \\
&\text{(since } \Delta x = h = \Delta y \text{ and } \frac{h}{h} = 1) \\
&= -v_{j-n} - v_{j-1} + 4v_j - v_{j+1} - v_{j+n}
\end{aligned}
$$

With $v_j \approx u_j$ $(= u(x,y)$ at grid point $j)$. We can thus conclude for all internal points that $a_{j,j-n} = -1$, $a_{j,j-1} = -1$, $a_{j,j} = 4$, $a_{j,j+1} = -1$, $a_{j,j+n} = -1$ and $b_j = h^2$.

We will now look at the boundary points:

The Ω_j become, respectively:
We will be using imaginary grid points where the dotted lines are, this because we know these points are 0.
Note that the volume of Ω_1 is $\frac{h^2}{4}$ and the volume of the second, third and fourth Ω_j is $\frac{h^2}{2}$. These volumes are equal to $\int_{\Omega_j} d\Omega$ Thus the b_j's will change aswell.



For v_1:
This is similar to the internal points, except: $\int_{\partial\Omega_1^W} \underline{n}\nabla u d\Gamma = \int_{\partial\Omega_1^W} \frac{\partial u}{\partial n} d\Gamma = 0$, $\int_{\partial\Omega_1^S} \underline{n}\nabla u d\Gamma = \int_{\partial\Omega_1^S} \frac{\partial u}{\partial n} d\Gamma = 0$, $\int_{\partial\Omega_j^E} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial x}(x_{1+\frac{h}{2}}, y_1)\frac{\Delta y}{2} \approx \frac{(v_{1+1}-v_1)}{2}$ and $\int_{\partial\Omega_1^N} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial y}(x_j, y_{j+\frac{h}{2}})\frac{\Delta x}{2} \approx \frac{(v_{1+n}-v_1)}{2}$.
So,

$$\frac{h^2}{4} \approx 0 + 0 - \frac{v_{1+n}-v_1}{2} - \frac{v_{1+1}-v_1}{2} = v_1 - \frac{v_2}{2} - \frac{v_{1+n}}{2}$$

Thus, $a_{1,1} = 1, a_{1,2} = \frac{-1}{2}$, $a_{1,1+n} = \frac{-1}{2}$ and $b_1 = \frac{h^2}{4}$.

For $v_j, \quad \forall j \in \{2, 3 \dots n-1\}$:
This is similar to the internal points, except: $\int_{\partial\Omega_j^S} \underline{n}\nabla u d\Gamma = \int_{\partial\Omega_j^S} \frac{\partial u}{\partial n} d\Gamma = 0$, $-\int_{\partial\Omega_j^W} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial x}(x_{j-\frac{h}{2}}, y_j)\frac{\Delta y}{2} \approx \frac{(v_j - v_{j-1})}{2}$ and $\int_{\partial\Omega_j^E} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial x}(x_{j+\frac{h}{2}}, y_j)\frac{\Delta y}{2} \approx \frac{(v_{j+1}-v_j)}{2}$.
So,

$$\frac{h^2}{2} \approx \frac{(v_j - v_{j-1})}{2} - (v_{j+n} - v_j) - \frac{(v_{j+1}-v_j)}{2} = -\frac{v_{j-1}}{2} + 2v_j - \frac{v_{j+1}}{2} - v_{j+n}$$

Thus, $a_{j,j-1} = \frac{-1}{2}$, $a_{j,j} = 2$, $a_{j,j+1} = \frac{-1}{2}$, $a_{j,j+n} = -1$ and $b_j = \frac{h^2}{2}$.

For v_n:
Here we change (using imaginary grid points): $\int_{\partial\Omega_n^S} \underline{n}\nabla u d\Gamma = \int_{\partial\Omega_n^S} \frac{\partial u}{\partial n} d\Gamma = 0$, $-\int_{\partial\Omega_n^W} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial x}(x_{n-\frac{h}{2}}, y_n)\frac{\Delta y}{2} \approx \frac{(v_n - v_{n-1})}{2}$ and $\int_{\partial\Omega_n^E} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial x}(x_{n+\frac{h}{2}}, y_n)\frac{\Delta y}{2} \approx \frac{(0-v_n)}{2}$.
So,

$$\frac{h^2}{2} \approx \frac{(v_n - v_{n-1})}{2} - (v_{2n} - v_j) - \frac{(0 - v_n)}{2} = -\frac{v_{n-1}}{2} + 2v_n - v_{2n}$$

Thus, $a_{n,n-1} = \frac{-1}{2}$, $a_{n,n} = 2$, $a_{n,2n} = -1$ and $b_n = \frac{h^2}{2}$.

For $v_j, \quad \forall j \in \{n+1, 2n+1 \dots (n-2)n+1\}$:
This is similar to the internal points, except: $\int_{\partial\Omega_j^W} \underline{n}\nabla u d\Gamma = \int_{\partial\Omega_j^W} \frac{\partial u}{\partial n} d\Gamma = 0$, $-\int_{\partial\Omega_j^S} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial y}(x_j, y_{j-\frac{h}{2}})\frac{\Delta x}{2} \approx \frac{(v_j - v_{j-n})}{2}$ and $\int_{\partial\Omega_j^N} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial y}(x_j, y_{j+\frac{h}{2}})\frac{\Delta x}{2} \approx \frac{(v_{j+n}-v_j)}{2}$.
So,

$$\frac{h^2}{2} \approx \frac{(v_j - v_{j-n})}{2} + 0 - (v_{j+1} - v_j) - \frac{(v_{j+n} - v_j)}{2} = \frac{-v_{j-n}}{2} + 2v_n - v_{j+1} - \frac{v_{j+n}}{2}$$

Thus, $a_{j,j-n} = \frac{-1}{2}$, $a_{j,j} = 2,$, $a_{j,j+1} = -1$ $a_{j,j+n} = \frac{-1}{2}$ and $b_j = \frac{h^2}{2}$.

For $v_{(n-1)n+1}$:
Here we change (using imaginary grid points): $\int_{\partial\Omega^W_{(n-1)n+1}} \underline{n}\nabla u d\Gamma = \int_{\partial\Omega^W_{(n-1)n+1}} \frac{\partial u}{\partial n} d\Gamma = 0$, $\int_{\partial\Omega^N_{(n-1)n+1}} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial y}(x_{(n-1)n+1}, y_{(n-1)n+1+\frac{h}{2}})\frac{\Delta x}{2} \approx \frac{(0 - v_{(n-1)n+1})}{2}$ and $\int_{\partial\Omega^S_{(n-1)n+1}} \underline{n}\nabla u d\Gamma \approx \frac{\partial u}{\partial x}(x_{(n-1)n+1}, y_{(n-1)n+1-\frac{h}{2}})\frac{\Delta y}{2} \approx \frac{(v_{(n-1)n+1} - v_{(n-2)n+1})}{2}$.
So,

$$\frac{h^2}{2} \approx \frac{(v_{(n-1)n+1} - v_{(n-2)n+1})}{2} + 0 - \frac{(0 - v_{(n-1)n+1})}{2} - (v_{(n-1)n+2} - v_{(n-1)n+1}) = -v_{(n-2)n+1} + 2v_{(n-1)n+1} - \frac{v_{(n-1)n+2}}{2}$$

Thus, $a_{(n-1)n+1,(n-2)n+1} = \frac{-1}{2}$, $a_{(n-1)n+1,(n-1)n+1} = 2$, $a_{(n-1)n+1,(n-1)n+2} = -1$ and $b_j = \frac{h^2}{2}$.

For $v_j, \quad \forall j \in \{2n, 3n \dots (n-1)n\}$:

$$h^2 \approx (v_j - v_{j-n}) + (v_j - v_{j-1}) - (v_{j+n} - v_j) - (0 - v_j) = -v_{j-n} - v_{j-1} + 4v_j - v_{j+n}$$

Thus, $a_{j,j-n} = -1$, $a_{j,j-1} = -1$, $a_{j,j} = 4$, $a_{j,j+n} = -1$ and $b_j = h^2$.

For v_{n^2}:

$$h^2 \approx (v_{n^2} - v_{n^2-n}) + (v_{n^2} - v_{n^2-1}) - (0 - v_{n^2}) - (0 - v_{n^2}) = -v_{n^2-n} - v_{n^2-1} + 4v_{n^2}$$

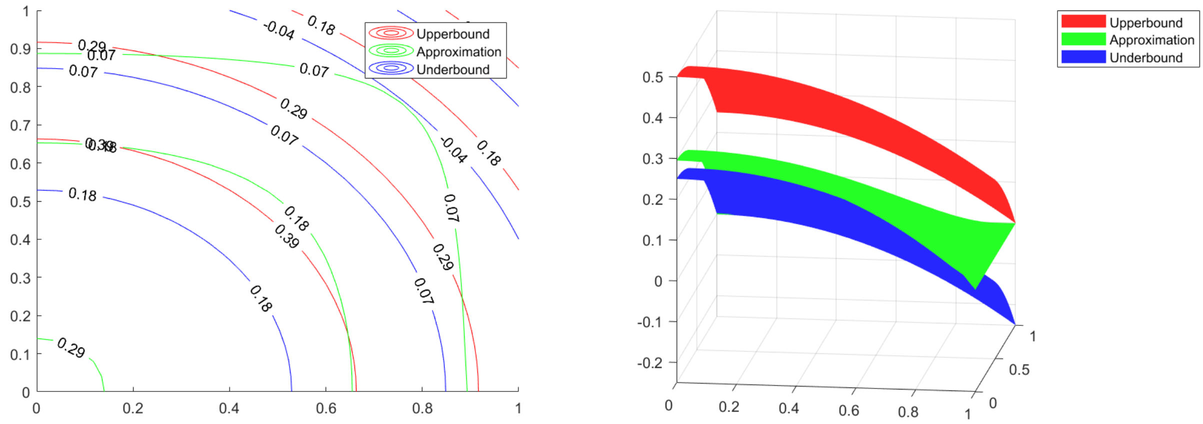
Thus, $a_{n^2,n^2-n} = -1,\ a_{n^2,n^2-1} = -1,\ a_{n^2,n^2} = 4$ and $b_{n^2} = h^2$.

For $v_j, \quad \forall j \in \{(n-1)n+2, (n-1)n+3 \ldots n^2-1\}$:

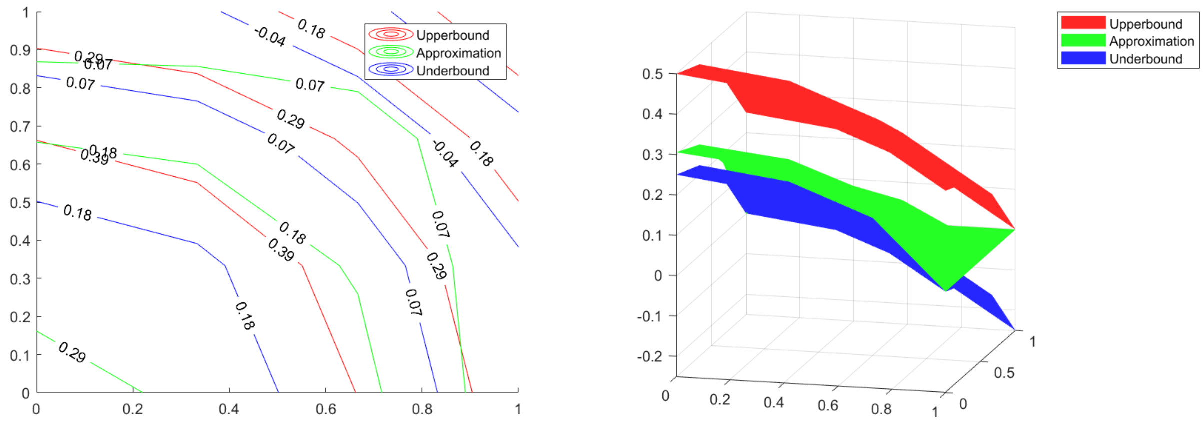
$$h^2 \approx (v_j - v_{j-n}) + (v_j - v_{j-1}) - (0 - v_j) - (v_{j+1} - v_j) = -v_{j-n} - v_{j-1} + 4v_j - v_{j+1}$$

Thus, $a_{j,j-n} = -1,\ a_{j,j-1} = -1,\ a_{j,j} = 4,\ a_{j,j+1} = -1$ and $b_j = h^2$.

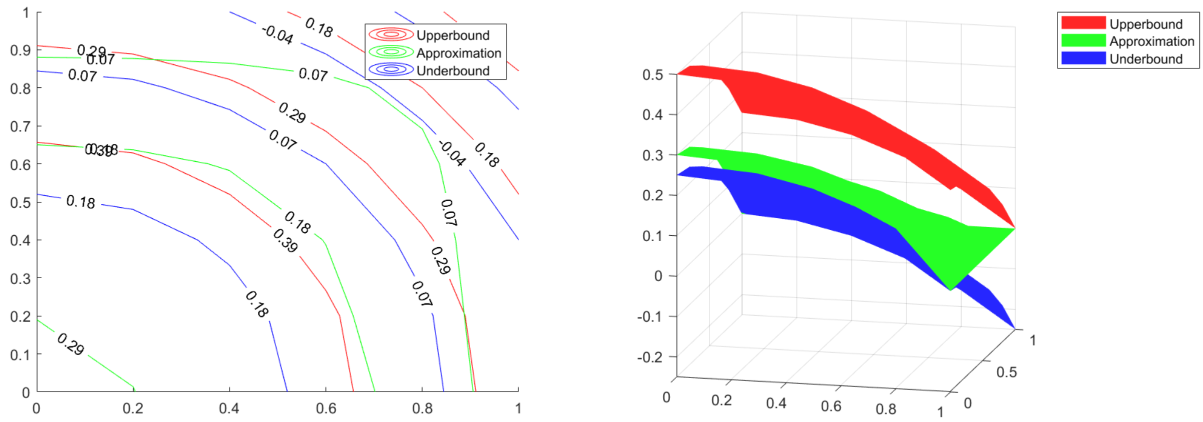
All other elements of A is equal to 0.
If we were to solve the system $A\underline{v} = \underline{b}$. We would get following approximation (n = 100):



If we were to chose n really low, for example 3, we would notice that not a lot has changed:



And if we were to plot n=5, we would see that it mainly smoothes, which was to be expected:



code:

```
function taak2(n)
[A, b] = matrix_maker(n);
points = A^(-1)*b;
pointsreal = reshape(points,n,n);
pointsreal = [pointsreal;zeros(1,n)];
pointsreal = [pointsreal,zeros(n+1,1)];

h = 1/n;
[X,Y] = meshgrid(0:h:1,0:h:1);
f1 = 1/4*(1-(X.^2+Y.^2));
f2 = 1/2*(1-(X.^2+Y.^2)/2);
figure;
surf(X,Y,f2,'EdgeColor', 'none', 'faceColor', [1 0.15 0.15]);
hold on
surf(X,Y,pointsreal, 'EdgeColor', 'none', 'faceColor', [0.15 1 0.15]);
surf(X,Y,f1,'EdgeColor', 'none', 'faceColor', [0.15 0.15 1]);
zlim([min([min(min(pointsreal)),min(min(f1)),min(min(f2))])
     max([max(max(pointsreal)),max(max(f1)),max(max(f2))])])
legend([ "Upperbound", "Approximation", "Underbound"])

randen = round(linspace(min([min(min(pointsreal)),min(min(f1)),min(min(f2))]),
                        max([max(max(pointsreal)),max(max(f1)),max(max(f2))]),8),2);
figure; hold on;
contour(X,Y,f2,randen,'red','ShowText','on');
contour(X,Y,pointsreal,randen, 'green', 'ShowText','on')
contour(X,Y,f1,randen,'blue','ShowText','on');
legend([ "Upperbound", "Approximation", "Underbound"])

function [A,b] = matrix_maker(n)
N = n^2;
A = zeros(N);
b = h^2*ones(n^2,1);
for i = 1:N
    if i == 1
        A(i,i) = 1; A(i,i+1) = -1/2; A(i,i+n) = -1/2; b(i) = b(i)/4;
    elseif i == n
        A(i,i-1) = -1/2; A(i,i) = 2; A(i,i+n) = -1; b(i) = b(i)/2;
    elseif i == (n-1)*n+1
        A(i,i-n) = -1/2; A(i,i) = 2; A(i,i+1) = -1;b(i) = b(i)/2;
    elseif i == n^2
        A(i,i-n) = -1; A(i,i-1) = -1; A(i,i) = 4;
    elseif i <= n-1
        A(i,i-1) = -1/2; A(i,i) = 2; A(i,i+1) = -1/2; A(i,i+n) = -1; b(i) = b(i)/2;
    elseif mod(i,n) == 1
        A(i,i-n) = -1/2; A(i,i) = 2; A(i,i+1) = -1; A(i,i+n) = -1/2; b(i) = b(i)/2;
    elseif mod(i,n) == 0
        A(i,i-n) = -1; A(i,i-1) = -1; A(i,i) = 4; A(i,i+n) = -1;
    elseif i>= (n-1)*n+2
        A(i,i-n) = -1; A(i,i-1) = -1; A(i,i) = 4; A(i,i+1) = -1;
    else
        A(i,i-n) = -1; A(i,i-1) = -1; A(i,i) = 4; A(i,i+1) = -1; A(i,i+n) = -1;
    end
end
```