Functional- and Fourieranalysis 19/20 Exercise sheet 3

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The lecture notes for this course can be found on Blackboard. First read them carefully, and then do the exercises below. If you have questions, do not hesitate to contact us.

Exercise 11: (Computation of Fourier coefficients)

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function given by $f(x) = (\frac{x}{\pi})^2$ on $]-\pi, \pi[$.

a) Compute the Fourier coefficients $\widehat{f}(k)$.

b) Assume that $s_N \to f$ in $L^2_\circ$. Compute $\|f\|_{L^2_\circ}$ in two different manners and thereby find a formula for $\sum\limits_{k\in\mathbb{N}} \frac{1}{k^4}$.

c) Determine the Fourier series, once with $\mathbf{e}_k(x) := e^{ikx}$, and once with $\sin(kx)$ and $\cos(kx)$.

Exercise 12: (Matlab)

a) Write a Matlab script that can compute $\widehat{f}(k)$ and s_N for an arbitrary function $f \in L^2_\circ$ and an arbitrary value of $N \in \mathbb{N}$. You may use the template (from blackboard) and the Matlab function `integral`.

b) Let $f(x) = \frac{x}{\pi}$ for $x \in]-\pi, \pi[$. Plot s_{10}, s_{20}, s_{100} as function of x.

c) Using f from b), make a logarithmic plot of $\|f - s_N\|_{L^2}$ and $\|f - s_N\|_\infty$ as function of N. Can you see the Gibbs phenomenon?

Exercise 13: (Convergence in different norms)

Read Remark 6 thoroughly. Give an example of a sequence of functions $(f_n) \subset L^2_\circ$, which

a) converges towards a function f in $L^2_\circ$, i.e. $\|f_n - f\|_{L^2_\circ} \to 0$ as $n \to \infty$, but not pointwise.

b) converges towards a function f pointwise, but not in $L^2_\circ$.

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Exercise 14: (Gibbs phenomenon)

This exercise demonstrates that the Fourier series of a non-continuous function cannot converge uniformly. Consider

$$f(x) = \begin{cases} -1, & x \in (-\pi, 0), \\ 1, & x \in (0, \pi). \end{cases}$$

a) Prove that for this f, the s_{2N} ($N \geq 0$) are given by

$$s_{2N}(x) = \frac{4}{\pi} \sum_{k=1}^{N} \frac{\sin((2k-1)x)}{2k-1}.$$

b) Prove that for $N \geq 1$

$$s'_{2N}(x) = \frac{4}{\pi} \frac{\sin(2Nx)}{2\sin(x)},$$

with the zeros $x \in \left\{ \frac{j\pi}{2N} \mid j = \pm 1, \ldots, \pm(2N-1) \right\}$.
Hint: Use induction and the angle-sum identities.

c) Try to compute $s_{2N}\left(\frac{\pi}{2N}\right)$. Remark: $\int\limits_0^\pi \frac{\sin(t)}{t} \mathrm{d}t \approx 1.85194$.
Hint: The occurring sum can be interpreted as a Riemann-sum.

d) What does this have to do with the Gibbs phenomenon?