Partiële differentiaalvergelijkingen (3341), 2020/2021

Instruction problems 2: Parabolic problems

A. Tricks to reduce some equations to the standard heat equation
For the equations below, apply the suggested transformations to bring them to the standard form. $D > 0$ is a given diffusion coefficient.

1. Given $a \in \mathbb{R}$ and with $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ solving

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = D\dfrac{\partial^2 u}{\partial x^2}, & \text{for } x > 0, t > 0, \\ u(x, 0) = u_0(x), & \text{for } x \in \mathbb{R}, \end{cases}$$

find the problem solved by $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ defined by $v(x, t) = u(x + at, t)$.

Solution. By definition,

$$v(x,t) := u(x + a\ t, t). \tag{1}$$

$$\begin{aligned} \frac{\partial}{\partial x} v(x,t) &= \frac{\partial}{\partial x} u(x + a\ t, t)\ \frac{\partial}{\partial x}(x + a\ t), \\ &= \frac{\partial}{\partial x} u(x + a\ t, t). \\ \frac{\partial^2}{\partial x^2} v(x,t) &= \frac{\partial^2}{\partial x^2} u(x + a\ t, t)\ \frac{\partial}{\partial x}(x + a\ t), \\ &= \frac{\partial^2}{\partial x^2} u(x + a\ t, t). \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{\partial}{\partial t} v(x,t) &= \frac{\partial}{\partial x} u(x + a\ t, t)\ \frac{\partial}{\partial t}(x + a\ t) + \frac{\partial}{\partial t} u(x + a\ t, t), \\ &= a\ \frac{\partial}{\partial x} u(x + a\ t, t) + \frac{\partial}{\partial t} u(x + a\ t, t), \\ &= D\ \frac{\partial^2}{\partial x^2} u(x + a\ t, t) \text{ (Given PDE)}, \\ &= D\ \frac{\partial^2}{\partial x^2} v(x,t). \end{aligned} \tag{3}$$

$\square$

2. Extend the situation above by considering a function $a : [0, \infty) \to R$ instead of a real constant a. Do so by defining $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ as $v(x, t) = u\Big(x + \int_0^t a(s)ds, t\Big)$.

Solution. By definition,

$$v(x, t) := u\Big(x + \int_0^t a(s)\ ds, t\Big). \tag{4}$$

Hence, we have

$$\begin{aligned} \frac{\partial}{\partial x} v(x, t) &= \frac{\partial u}{\partial x}\ \frac{\partial}{\partial x}\Big(x + \int_0^t a(s)\ ds\Big) = \frac{\partial u}{\partial x}. \\ \frac{\partial^2}{\partial x^2} v(x, t) &= \frac{\partial^2 u}{\partial x^2}\ \frac{\partial}{\partial x}\Big(x + \int_0^t a(s)\ ds\Big) = \frac{\partial^2 u}{\partial x^2}. \end{aligned} \tag{5}$$

We note that

$$\begin{aligned} & x'(t) = a(t), \\ \Longrightarrow & \int_0^t x'(s)\ ds = \int_0^t a(s)ds, \\ \Longrightarrow & \ x(t) = x(0) + \int_0^t a(s)ds. \end{aligned} \tag{6}$$

Hence, we get

$$\begin{aligned} \frac{\partial}{\partial t} v(x, t) &= \frac{\partial u}{\partial x}\ \frac{\partial}{\partial t}\Big(x + \int_0^t a(s)\ ds\Big) + \frac{\partial u}{\partial t}, \\ &= a\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}, \\ &= D\ \frac{\partial^2 u}{\partial x^2}\ \text{(Given PDE)}, \\ &= D\ \frac{\partial^2}{\partial x^2} v(x, t). \end{aligned} \tag{7}$$

□

3. Given $\lambda \in \mathbb{R}$ and with u solving

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \lambda u,$$

show that v defined by $v(x, t) = u(x, t)e^{-\lambda t}$ satisfies the heat equation (hence without the last term, the ”reaction term”).

Solution. By definition,

$$v(x,t) := u(x,t)e^{-\lambda t}. \tag{8}$$

Hence, we get

$$\begin{aligned} \frac{\partial}{\partial x}v(x,t) &= \frac{\partial u}{\partial x}\ e^{-\lambda t}. \\ \frac{\partial^2}{\partial x^2}v(x,t) &= \frac{\partial^2 u}{\partial x^2}\ e^{-\lambda t}. \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\partial}{\partial t}v(x,t) &= u\ \frac{\partial}{\partial t}e^{-\lambda t} + e^{-\lambda t}\frac{\partial u}{\partial t}, \\ &= -\lambda\ e^{-\lambda t}\ u + e^{-\lambda t}\frac{\partial u}{\partial t}, \\ &= -\lambda\ v + e^{-\lambda t}\frac{\partial u}{\partial t}, \\ &= -\lambda\ v + e^{-\lambda t}\Big(D\frac{\partial^2 u}{\partial x^2} + \lambda u\Big),\ \text{(Given PDE)}, \\ &= -\lambda\ v + D\ \frac{\partial^2 u}{\partial x^2}e^{-\lambda t} + \lambda\ v, \\ &= D\ \frac{\partial^2 u}{\partial x^2}e^{-\lambda t}, \\ &= D\ \frac{\partial^2}{\partial x^2}v(x,t). \end{aligned} \tag{10}$$

□

4. Generalise this approach for the case where $\lambda : [0,\infty) \to R$ is a given function and u the solution to

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \lambda(t)u.$$

Find an appropriate function v that depends on u but solves the standard heat equation.

Solution. By definition,

$$v(x,t) := u(x,t)e^{-\int_0^t \lambda(s)\, ds}. \tag{11}$$

Hence, we get

$$\begin{aligned} \frac{\partial}{\partial x}v(x,t) &= \frac{\partial u}{\partial x}\ e^{-\int_0^t \lambda(s)\, ds}. \\ \frac{\partial^2}{\partial x^2}v(x,t) &= \frac{\partial^2 u}{\partial x^2}\ e^{-\int_0^t \lambda(s)\, ds}. \end{aligned} \tag{12}$$

$$
\begin{aligned}
\frac{\partial}{\partial t} v(x,t) &= u\ \frac{\partial}{\partial t} e^{-\int_0^t \lambda(s)\,ds} + e^{-\int_0^t \lambda(s)\,ds}\ \frac{\partial u}{\partial t}, \\
&= -u\ e^{-\int_0^t \lambda(s)\,ds}\ \frac{\partial}{\partial t}\left(\int_0^t \lambda(s)\ ds\right) + e^{-\int_0^t \lambda(s)\,ds}\ \frac{\partial u}{\partial t}, \\
&= -\lambda(t)\ e^{-\int_0^t \lambda(s)\,ds}\ u + e^{-\int_0^t \lambda(s)\,ds} \frac{\partial u}{\partial t}, \\
&= e^{-\int_0^t \lambda(s)\,ds}\ \left(\frac{\partial u}{\partial t} - \lambda u\right),\ \text{(Given PDE)}, \\
&= D\ \frac{\partial^2 u}{\partial x^2}\ e^{-\int_0^t \lambda(s)\,ds}, \\
&= D\ \frac{\partial^2}{\partial x^2} v(x,t).
\end{aligned} \tag{13}
$$

□

5. Given $f : [0, \infty)\mathbb{R}$ and with u solving

$$
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(t),
$$

find an appropriate function v, depending on u and f, and solving the heat equation.

Solution. Let us define,

$$
v(x,t) := u(x,t) - \int_0^t f(s)\ ds. \tag{14}
$$

Thus we obtain,

$$
\begin{aligned}
\frac{\partial}{\partial x} v(x,t) &= \frac{\partial u}{\partial x} + 0. \\
\frac{\partial^2}{\partial x^2} v(x,t) &= \frac{\partial^2 u}{\partial x^2}.
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t} v(x,t) &= \frac{\partial u}{\partial t} - \frac{\partial}{\partial t}\left(\int_0^t f(s)\ ds\right), \\
&= \frac{\partial u}{\partial t} - f(t), \\
&= D\ \frac{\partial^2 u}{\partial x^2}\ \text{(Given PDE)}, \\
&= D\ \frac{\partial^2}{\partial x^2} v(x,t).
\end{aligned} \tag{16}
$$

□

6. One can rescale the time or the space to reduce the diffusion coefficient. In this sense, let u solve

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}.$$

Define $\tau = Dt$ and the function v by $v(x,\tau) = u\Big(x, \frac{\tau}{D}\Big) = u(x,t)$. Show that v solves

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

Generalize this approach for the case when $D : [0,\infty) \to \mathbb{R}$ is a given function satisfying $D(t) \geq D_0 > 0$ for all $t \geq 0$, by considering $\tau = \int_0^t D(s)ds$ and $v(x,\tau) = u(x,t)$.

Solution. By definition

$$v(x,\tau) := u(x, \frac{\tau}{D}). \tag{17}$$

Hence we have

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial x}. \\ \frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2}. \end{aligned} \tag{18}$$

and

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \frac{\partial u}{\partial t}\ \frac{\partial}{\partial \tau}\Big(\frac{\tau}{D}\Big)\ \text{(Since } t = \frac{\tau}{D}) \\ &= \frac{\partial u}{\partial t}\ \frac{1}{D} \\ &= D\ \frac{\partial^2 u}{\partial x^2}\ \frac{1}{D}\ \text{(Given PDE)} \\ &= \frac{\partial^2 u}{\partial x^2} \\ &= \frac{\partial^2 v}{\partial x^2}. \end{aligned} \tag{19}$$

Hence

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}. \tag{20}$$

For the generalization, we define,

$$\begin{aligned} \tau &:= \int_0^t D(s)\ ds,\ \text{with } t := t(\tau), \\ v(x,\tau) &:= u(x, t(\tau)). \end{aligned} \tag{21}$$

Then, we can write,

$$\begin{aligned} \implies \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial x} \\ \implies \frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2} \end{aligned} \tag{22}$$

and

$$\begin{aligned} \implies \frac{\partial v}{\partial \tau} &= \frac{\partial u}{\partial t}\frac{\partial t}{\partial \tau} \quad \text{(Since } t = t(\tau)\text{)} \\ &= \frac{\partial u}{\partial t}\frac{1}{\frac{\partial \tau}{\partial t}} \\ &= \frac{\partial u}{\partial t}\frac{1}{\tau'(t)} \\ &= \frac{\partial u}{\partial t}\frac{1}{D(t)} \\ &= D\frac{\partial^2 u}{\partial x^2}\frac{1}{D(t)} \quad \text{(Given PDE)} \\ &= \frac{\partial^2 u}{\partial x^2} \\ &= \frac{\partial^2 v}{\partial x^2} \end{aligned} \tag{23}$$

Hence, we find

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}. \tag{24}$$

$\square$

B. Solving parabolic equations in unbounded domains

Find a solution to the problems below.

1. The reaction-diffusion problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \sin t, & -\infty < x < \infty, t > 0, \\ u(x,0) = \begin{cases} 0, & x < 1, \\ 1, & x \geq 1. \end{cases} \end{cases}$$

Note that the initial condition is discontinuous at $x = 1$!

a) Using an appropriate function f and the transformation $v(x,t) = u(x,t) + f(t)$, transform the equation into

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

Solution. We define

$$v(x,t) = u(x,t) + cos(t). \tag{25}$$

Then we get

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial x}. \\ \frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2}. \end{aligned} \tag{26}$$

and

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial u}{\partial t} - sin(t), \\ &= \frac{\partial^2 u}{\partial x^2} \text{ (Given PDE)}, \\ \implies \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2}. \end{aligned} \tag{27}$$

Given $u(x,0) = \begin{cases} 0, & x < 1, \\ 1, & x \geq 1. \end{cases}$.

Since $v(x,0) = u(x,0) + cos(0) = u(x,0) + 1$.

Hence

$$v(x,0) = \begin{cases} 1, & x < 1, \\ 2, & x \geq 1. \end{cases} \tag{28}$$

□

b) Determine the similarity solution (gelijkvormigheidsoplossing) v for the transformed problem (in v) and next determine u.

Solution. We define

$$w(x,t) := v(x+1,t). \tag{29}$$

Thus we obtain,

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial v}{\partial x}. \\ \frac{\partial^2 w}{\partial x^2} &= \frac{\partial^2 v}{\partial x^2}. \end{aligned} \tag{30}$$

and

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial v}{\partial t}, \\ &= \frac{\partial^2 v}{\partial x^2}, \\ &= \frac{\partial^2 w}{\partial x^2}, \\ \implies \frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2}.\end{aligned} \tag{31}$$

Since $w(x,0) = v(x+1,0) = 1 + u(x+1,0) = \begin{cases} 1, & x+1 < 1, \\ 2, & x+1 \geq 1. \end{cases}$.

Hence

$$w(x,0) = \begin{cases} 1, & x < 0, \\ 2, & x \geq 0. \end{cases} \tag{32}$$

We know that with the initial condition

$$w(x,0) = \begin{cases} w_-, & x < 0, \\ w_+, & x \geq 0. \end{cases} \tag{33}$$

the similarity solution of

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \tag{34}$$

is

$$\begin{aligned} w(x,t) &= w_+ + \frac{(w_- - w_+)}{2}\left(1 - erf(\frac{x}{2\sqrt{t}})\right), \\ &= 2 - \frac{1}{2}\left(1 - \text{erf}(\frac{x}{2\sqrt{t}})\right), \end{aligned}$$

$$\begin{aligned} \text{Hence, } w(x,t) &= \frac{3}{2} + \frac{1}{2}\,\text{erf}\left(\frac{x}{2\sqrt{t}}\right), \\ \implies v(x,t) &= \frac{3}{2} + \frac{1}{2}\,\text{erf}\left(\frac{x-1}{2\sqrt{t}}\right), \quad (\text{Since } v(x,t) = w(x-1,t)\ ), \\ \implies u(x,t) &= \frac{3}{2} + \frac{1}{2}\,\text{erf}\left(\frac{x-1}{2\sqrt{t}}\right) - cos(t), \quad (\text{Since } u(x,t) = v(x,t) - cos(t)\ ). \end{aligned} \tag{35}$$

Here

$$\text{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-z^2}dz. \tag{36}$$

□

c) Determine the limits $\lim_{x\to-\infty} u(x,t)$ and $\lim_{x\to\infty} u(x,t)$ for $t > 0$.

Solution.

$$
\begin{aligned}
u(x,t) &= \frac{3}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x-1}{2\sqrt{t}}\right) - cos(t),\\
\implies u(x,t) &= \frac{3}{2} + \frac{1}{2}\,\frac{2}{\sqrt{\pi}}\int_0^{\frac{x-1}{2\sqrt{t}}} e^{-z^2}dz - cos(t).
\end{aligned} \tag{37}
$$

Since

$$
\begin{aligned}
\lim_{x\to\infty}\frac{x-1}{2\sqrt{t}} &= +\infty,\\
&\text{and,}\\
\lim_{x\to-\infty}\frac{x-1}{2\sqrt{t}} &= -\infty,
\end{aligned} \tag{38}
$$

Hence

$$
\begin{aligned}
\lim_{x\to\infty} u(x,t) &= \frac{3}{2} + \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-z^2}dz - cos(t),\\
&= \frac{3}{2} + \frac{1}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{2} - cos(t),\\
&= 2 - cos(t).
\end{aligned} \tag{39}
$$

and

$$
\begin{aligned}
\lim_{x\to-\infty} u(x,t) &= \frac{3}{2} + \frac{1}{\sqrt{\pi}}\int_0^{-\infty} e^{-z^2}dz - cos(t),\\
&= \frac{3}{2} + \frac{1}{\sqrt{\pi}}\,\frac{-\sqrt{\pi}}{2} - cos(t),\\
&= 1 - cos(t).
\end{aligned} \tag{40}
$$

□

2. Consider the reaction-diffusion problem in the 1^e quadrant

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + tu, & 0 < x < \infty, t > 0,\\
u(x,0) = 1, & x > 0,\\
u(0,t) = 0, & t > 0.
\end{cases}
$$

a) Find an appropriate function v, depending on u, satisfying the standard heat/diffusion equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

Solution. We define,

$$v(x,t) := u(x,t)e^{-\frac{t^2}{2}}. \tag{41}$$

Thus we get

$$\begin{aligned} &\implies \frac{\partial}{\partial x}v(x,t) = \frac{\partial u}{\partial x}\, e^{-\frac{t^2}{2}}, \\ &\implies \frac{\partial^2}{\partial x^2}v(x,t) = \frac{\partial^2 u}{\partial x^2}\, e^{-\frac{t^2}{2}}, \end{aligned} \tag{42}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}v(x,t) &= u\, \frac{\partial}{\partial t}e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}}\frac{\partial u}{\partial t}, \\ &= -t\, e^{-\frac{t^2}{2}}\, u + e^{-\frac{t^2}{2}}\frac{\partial u}{\partial t}, \\ &= e^{-\frac{t^2}{2}}\left(\frac{\partial u}{\partial t} - t\, u\right), \\ &= e^{-\frac{t^2}{2}}\, \frac{\partial^2 u}{\partial x^2},\ \text{(Given PDE)}, \\ &= \frac{\partial^2}{\partial x^2}v(x,t). \end{aligned} \tag{43}$$

Hence

$$\begin{aligned} \frac{\partial}{\partial t}v(x,t) &= \frac{\partial^2}{\partial x^2}v(x,t),\ x > 0,\ t > 0, \\ v(x,0) &= u(x,0)\, e^0 = 1,\ x > 0, \\ v(0,t) &= 0,\ t > 0. \end{aligned} \tag{44}$$

□

b) Determine the similarity solution (gelijkvormigheidsoplossing) v of the standard problem and then u.

Solution. We know that with the initial condition

$$w(x,0) = w_+,\ x > 0. \tag{45}$$

the similarity solution of

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \tag{46}$$

is

$$\begin{aligned}
w(x,t) &= w_+ - \left(1 - \operatorname{erf}(\frac{x}{2\sqrt{t}})\right), \\
\text{Hence, } v(x,t) &= 1 - \left(1 - \operatorname{erf}(\frac{x}{2\sqrt{t}})\right), \\
\implies v(x,t) &= \operatorname{erf}(\frac{x}{2\sqrt{t}}), \\
\implies u(x,t) &= \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)e^{\frac{t^2}{2}}, \ \text{(Since } v(x,t) = u(x,t)e^{-\frac{t^2}{2}} \text{).}
\end{aligned} \tag{47}$$

□

3. Let $n \in \mathbb{N}$ be a natural number, $D > 0$ and let u_n be the solution of the diffusion problem

$$\left\{\begin{array}{rcl}
\dfrac{\partial u_n}{\partial t} &=& D\dfrac{\partial^2 u_n}{\partial x^2}, \quad -\infty < x < \infty, t > 0, \\
u_n(x,0) &=& \left\{\begin{array}{ll} 0, & x < -\frac{1}{n}, \text{ or } x > \frac{1}{n}, \\ \dfrac{n}{2}, & -\frac{1}{n} < x < \frac{1}{n}. \end{array}\right.
\end{array}\right.$$

a) Determine the similarity solution u_n.
Hint: Use the Mean Value Theorem (middelwaardestelling) for integrals: let $f : [a,b] \to \mathbb{R}$ be continuous, then there exists a $\xi \in (a,b)$ s.t. $\int_a^b f(z)dz = (b-a)f(\xi)$.

Solution. We know that with the initial condition

$$w(x,0) = \left\{\begin{array}{ll} w_-, & x < 0, \\ w_+, & x \geq 0. \end{array}\right. \tag{48}$$

the similarity solution of

$$\frac{\partial w}{\partial t} = D\,\frac{\partial^2 w}{\partial x^2} \tag{49}$$

is

$$
\begin{aligned}
w(x,t) &= w_+ + \frac{(w_- - w_+)}{2}\left(1-\operatorname{erf}\left(\frac{x}{2\,D\,\sqrt{t}}\right)\right),\\
w(x,t) &= \frac{(w_+ + w_-)}{2} + \frac{(w_+ - w_-)}{2}\,\operatorname{erf}\left(\frac{x}{2\,D\,\sqrt{t}}\right).
\end{aligned}
\tag{50}
$$

We define

$$
\begin{aligned}
&u = w_1 + w_2,\\
&\text{where,}\\
&w_1(x,0) = \begin{cases} 0, & x < -\frac{1}{n}\\ \frac{n}{2}, & x > -\frac{1}{n}. \end{cases}\\
&w_2(x,0) = \begin{cases} 0, & x < \frac{1}{n}\\ -\frac{n}{2}, & x > \frac{1}{n}. \end{cases}
\end{aligned}
\tag{51}
$$

Note:

$$
w_1(x,0) + w_2(x,0) = u_n(x,0)! \tag{52}
$$

Then,

$$
\begin{aligned}
w_1(x,t) &= \frac{n}{4} + \frac{n}{2\,\sqrt{\pi}}\int_0^{\frac{\left(x+\frac{1}{n}\right)}{2\sqrt{D\,t}}} e^{-z^2}\,dz,\\
w_2(x,t) &= -\frac{n}{4} - \frac{n}{2\,\sqrt{\pi}}\int_0^{\frac{\left(x-\frac{1}{n}\right)}{2\sqrt{D\,t}}} e^{-z^2}\,dz,
\end{aligned}
\tag{53}
$$

Hence,

$$
\begin{aligned}
u_n(x,t) &= w_1(x,t) + w_2(x,t),\\
&= \frac{n}{4} + \frac{n}{2\,\sqrt{\pi}}\int_0^{\frac{\left(x+\frac{1}{n}\right)}{2\sqrt{D\,t}}} e^{-z^2}\,dz - \frac{n}{4} - \frac{n}{2\,\sqrt{\pi}}\int_0^{\frac{\left(x-\frac{1}{n}\right)}{2\sqrt{D\,t}}} e^{-z^2}\,dz,\\
&= \frac{n}{2\,\sqrt{\pi}}\left[\int_0^{\frac{\left(x+\frac{1}{n}\right)}{2\sqrt{D\,t}}} e^{-z^2}\,dz - \int_0^{\frac{\left(x-\frac{1}{n}\right)}{2\sqrt{D\,t}}} e^{-z^2}\,dz\right],\\
\implies u_n(x,t) &= \frac{n}{2\,\sqrt{\pi}}\int_{\frac{\left(x-\frac{1}{n}\right)}{2\sqrt{D\,t}}}^{\frac{\left(x+\frac{1}{n}\right)}{2\sqrt{D\,t}}} e^{-z^2}\,dz.
\end{aligned}
\tag{54}
$$

Note that by the Mean value theorem: Let $f : [a, b] \to R$ be a function, there exist an $\xi \in (a, b)$ such that

$$\int_a^b f(z)\, dz = (b - a)\, f(\xi). \tag{55}$$

Here $f : [\frac{(x+\frac{1}{n})}{2\sqrt{D\,t}}, \frac{(x-\frac{1}{n})}{2\sqrt{D\,t}}] \to R$. Let $\xi_n \in \left(\frac{(x+\frac{1}{n})}{2\sqrt{D\,t}}, \frac{(x-\frac{1}{n})}{2\sqrt{D\,t}}\right)$ such that

$$\begin{aligned}
\int_{\frac{(x-\frac{1}{n})}{2\sqrt{D\,t}}}^{\frac{(x+\frac{1}{n})}{2\sqrt{D\,t}}} e^{-z^2}\, dz &= \left(\frac{\left(x+\frac{1}{n}\right)}{2\sqrt{D\,t}} - \frac{\left(x-\frac{1}{n}\right)}{2\sqrt{D\,t}}\right) e^{-\xi_n^2}, \\
&= \frac{x + \frac{1}{n} - x + \frac{1}{n}}{2\sqrt{D\,t}}\, e^{-\xi_n^2}, \\
\implies \int_{\frac{(x-\frac{1}{n})}{2\sqrt{D\,t}}}^{\frac{(x+\frac{1}{n})}{2\sqrt{D\,t}}} e^{-z^2}\, dz &= \frac{1}{n\,\sqrt{D\,t}}\, e^{-\xi_n^2}.
\end{aligned} \tag{56}$$

Hence

$$\begin{aligned}
u_n(x,t) &= \frac{n}{2\,\sqrt{\pi}} \int_{\frac{(x-\frac{1}{n})}{2\sqrt{D\,t}}}^{\frac{(x+\frac{1}{n})}{2\sqrt{D\,t}}} e^{-z^2}\, dz, \\
&= \frac{n}{2\,\sqrt{\pi}}\, \frac{1}{n\,\sqrt{D\,t}}\, e^{-\xi_n^2}, \\
\implies u_n(x,t) &= \frac{e^{-\xi_n^2}}{2\,\sqrt{D\,\pi\,t}}
\end{aligned} \tag{57}$$

□

b) Fix now $x \in \mathbb{R}$ and $t > 0$. Determine the limit $u(x,t) = \lim_{n\to\infty} u_n(x,t)$. Observe that this limit defines a function u still solving the heat equation. What is the initial condition for u?

Solution. Let $x \in \mathbb{R}$ and $t > 0$.
Since $\frac{(x-\frac{1}{n})}{2\sqrt{D\,t}} < \xi_n < \frac{(x+\frac{1}{n})}{2\sqrt{D\,t}}$.
If $n \to \infty$ then we get $\xi_n \to \frac{x}{2\,\sqrt{D\,t}}$.
Thus we can write

$$\begin{aligned}
u(x,t) &= \lim_{n\to\infty} u_n(x,t), \\
\implies u(x,t) &= \frac{e^{-\frac{x^2}{4\,D\,t}}}{2\,\sqrt{D\,\pi\,t}}
\end{aligned} \tag{58}$$

Initial Condition:

$$
\begin{aligned}
u(x,0) &= \frac{e^{-\frac{x^2}{4\,D\,0}}}{2\,\sqrt{D\,\pi\,0}}, \\
\implies u(x,0) &= \frac{1}{0\,e^{\infty}} = \frac{1}{\infty} = 0.
\end{aligned}
\tag{59}
$$

$\square$