Instruction 3: Parabolic problems
Course: Partial differential equation (3341), 2020/20201

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A. **Parabolic problems in unbounded domains**

1. Given the constants $D > 0, V \in \mathbb{R}$ and the continuous function $g : [0; 1) \to \mathbb{R}$, consider the initial value problem

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} + g(t)u, \qquad \text{in } \mathbb{R} \times (0, \infty), \tag{1}$$

$$u(x, 0) = \delta, \qquad \text{for } x \in \mathbb{R}, \tag{2}$$

where δ is the Dirac distribution. Find the fundamental solution of the problem above.

Hint: You may apply appropriate transformations for bringing first the equation to the standard form. then you can use directly the fundamental solution for the heat equation.

Solution. **Transformation 1:** We define

$$v(x, t) := u(x + V\,t,\, t). \tag{3}$$

Applying chain rules, we get

$$\begin{aligned} \frac{\partial v(x,t)}{\partial x} &= \frac{\partial u(x + V\,t,\, t)}{\partial x}\,\frac{\partial}{\partial x}(x + V\,t), \\ \Longrightarrow \frac{\partial v(x,t)}{\partial x} &= \frac{\partial u(x + V\,t,\, t)}{\partial x}, \end{aligned} \tag{4}$$

and again applying chain rules, we obtain

$$\begin{aligned} \frac{\partial^2 v(x,t)}{\partial x^2} &= \frac{\partial^2 u(x + V\,t,\, t)}{\partial x^2}\,\frac{\partial}{\partial x}(x + V\,t), \\ \Longrightarrow \frac{\partial^2 v(x,t)}{\partial x^2} &= \frac{\partial^2 u(x + V\,t,\, t)}{\partial x^2}. \end{aligned} \tag{5}$$

Similarly, for the partial derivatives w.r.t. t, we get

$$\begin{aligned}
\frac{\partial v(x,t)}{\partial t} &= \frac{\partial u(x+V\,t,\,t)}{\partial x}\,\frac{\partial}{\partial t}(x+V\,t) + \frac{\partial u(x+V\,t,\,t)}{\partial t}, \\
&= V\,\frac{\partial u(x+V\,t,\,t)}{\partial x} + \frac{\partial u(x+V\,t,\,t)}{\partial t}, \\
&= D\frac{\partial^2 u(x+V\,t,\,t)}{\partial^2 x} + g(t)\,u(x+V\,t,\,t), && \text{[using (1)]}, \\
\frac{\partial v(x,t)}{\partial t} &= D\,\frac{\partial^2 v(x,t)}{\partial x^2} + g(t)\,v(x,t), && \text{[using (5) and (3)}.
\end{aligned}$$

Second IVP: Hence, $v(x,t)$ satisfies the following IVP,

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + g(t)v, \qquad \text{in } \mathbb{R}\times(0,\infty), \tag{6}$$

$$v(x,0) = \delta, \qquad \text{for } x\in\mathbb{R}, \tag{7}$$

Transformation 2: We define the second transformation as

$$w(x,t) := v(x,t)\,e^{-\int_0^t g(s)\,ds}. \tag{8}$$

Applying chain rules, we can write

$$\begin{aligned}
\frac{\partial w(x,t)}{\partial x} &= \frac{\partial v(x,t)}{\partial x}\,e^{-\int_0^t g(s)\,ds}, \\
\implies \frac{\partial^2 w(x,t)}{\partial x^2} &= \frac{\partial^2 v(x,t)}{\partial x^2}\,e^{-\int_0^t g(s)\,ds},
\end{aligned} \tag{9}$$

and again applying chain rules, we obtain

$$\begin{aligned}
\frac{\partial w(x,t)}{\partial t} &= \frac{\partial v(x,t)}{\partial t}\,e^{-\int_0^t g(s)\,ds} + e^{-\int_0^t g(s)\,ds}\,\frac{\partial}{\partial t}\left(-\int_0^t g(s)\,ds\right)\,v(x,t), \\
&= \frac{\partial v(x,t)}{\partial t}\,e^{-\int_0^t g(s)\,ds} - g(t)\,e^{-\int_0^t g(s)\,ds}\,v(x,t), \\
&= \left[\frac{\partial v(x,t)}{\partial x} - g(t)\,v(x,t)\right]\,e^{-\int_0^t g(s)\,ds}, \\
&= D\frac{\partial^2 v(x,t)}{\partial x^2}\,e^{-\int_0^t g(s)\,ds}\ \text{[using (6)]}, \\
\implies \frac{\partial w(x,t)}{\partial t} &= D\,\frac{\partial^2 w(x,t)}{\partial x^2}\ \text{[using (9)]}.
\end{aligned}$$

Third IVP: Hence, $w(x,t)$ satisfies the following IVP,

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2}, \qquad \text{in } \mathbb{R}\times(0,\infty), \tag{10}$$

$$w(x,0) = \delta, \qquad \text{for } x\in\mathbb{R}, \tag{11}$$

where the PDE above is in the standard form of the heat equation. Then from the fundamental solution of the standard heat equation, we obtain

$$
\begin{aligned}
w(x,t) &= \frac{1}{2\sqrt{\pi\ D\ t}}\ e^{-\frac{x^2}{4\ D\ t}}, \\
\Longrightarrow\ v(x,t) &= \frac{e^{-\frac{x^2}{4\ D\ t}}\ e^{\int_0^t g(s)\ ds}}{2\sqrt{\pi\ D\ t}}\ \text{[using (8)]}, \\
&= \frac{e^{\left(\int_0^t g(s)\ ds\ -\ \frac{x^2}{4\ D\ t}\right)}}{2\sqrt{\pi\ D\ t}}, \\
\Longrightarrow\ u(x,t) &= \frac{1}{2\sqrt{\pi\ D\ t}} e^{\left(\int_0^t g(s)\ ds\ -\ \frac{(x-V\ t)^2}{4\ D\ t}\right)}\ \text{[using (3)]}.
\end{aligned}
$$

□

2. Consider the heat equation in the semi-infinite interval:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, && x > 0, t > 0, && (12) \\
u(x,0) &= 0, && x > 0, && (13) \\
u(0,t) &= t^{-\frac{1}{2}}, && t > 0. && (14)
\end{aligned}
$$

Seek solutions in the form:

$$
u(x,t) = t^{-\alpha}\ f(\eta) \quad \text{with} \quad \eta = x\ t^{-\beta}. \qquad (15)
$$

a) Determine the constants $\alpha, \beta \in \mathbb{R}$ that allow eliminating the variables t and x, and bringing the original equation determine to an ordinary differential equation in f and the variable η. Which problem solves f ? What are the boundary values?

Solution. From (15), we have

$$
\begin{aligned}
u(x,t) &= t^{-\alpha}\ f(x\ t^{-\beta}), \\
\Longrightarrow\ \frac{\partial u}{\partial x} &= t^{-\alpha}\ f'(x\ t^{-\beta})\ t^{-\beta}, \\
&= t^{-(\alpha+\beta)}\ f'(x\ t^{-\beta}), \\
\Longrightarrow\ \frac{\partial^2 u}{\partial x^2} &= t^{-(\alpha+\beta)}\ f''(x\ t^{-\beta})\ t^{-\beta}, \\
&= t^{-(\alpha+2\beta)}\ f''(x\ t^{-\beta}), \\
&= t^{-(\alpha+2\beta)}\ f''(\eta),
\end{aligned}
$$

and

$$\begin{aligned}\frac{\partial u}{\partial t} &= t^{-\alpha}\ f'(x\ t^{-\beta})\ \left(-x\ \beta\ t^{-(\beta+1)}\right) - \alpha\ t^{-(\alpha+1)}\ f(x\ t^{-\beta}),\\ &= -\ \left(x\ t^{-\beta}\right)\ \beta\ t^{-(\alpha+1)}\ f'(\eta) - \alpha\ t^{-(\alpha+1)}\ f(\eta)\\ &= -\ \eta\ \beta\ t^{-(\alpha+1)}\ f'(\eta) - \alpha\ t^{-(\alpha+1)}\ f(\eta), \quad \text{Using} \quad (15)\\ &= -\ t^{-(\alpha+1)}\Big(\ \eta\ \beta\ f'(\eta) + \alpha\ f(\eta)\Big)\end{aligned}$$

From (12), we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2},\\ \Longrightarrow\ -\ t^{-(\alpha+1)}\Big(\ \eta\ \beta\ f'(\eta) + \alpha\ f(\eta)\Big) &= t^{-(\alpha+2\beta)}\ f''(\eta).\end{aligned} \tag{16}$$

We can eliminate t from (16), if and only if

$$\begin{aligned}\alpha + 1 &= \alpha + 2\beta,\\ \Longrightarrow \beta &= \frac{1}{2}.\end{aligned}$$

Then from (16) and setting $\beta = 1/2$, we can write

$$\begin{aligned}-\Big(\frac{1}{2}\ \eta\ \ f'(\eta) + \alpha\ f(\eta)\Big) &= f''(\eta),\\ \Longrightarrow f''(\eta) + \frac{1}{2}\ \eta\ f'(\eta) + \alpha\ f(\eta) &= 0.\end{aligned} \tag{17}$$

From (15), we know

$$\begin{aligned}\eta &= x\ t^{-\beta},\\ \Longrightarrow \text{ At } \quad x &= 0, \eta = 0,\\ \text{and} \quad u(0,t) &= t^{-\alpha}\ f(0),\\ \Longrightarrow\ t^{-\frac{1}{2}} &= t^{-\alpha}\ f(0).\end{aligned}$$

If $f(0) = 1$, then

$$\alpha = \frac{1}{2}. \tag{18}$$

Hence

$$u(x,t) = \frac{1}{\sqrt{t}}\ f(\frac{x}{\sqrt{t}}). \tag{19}$$

Using (18) in (17),

$$f''(\eta) + \frac{1}{2}\,\eta\, f'(\eta) + \frac{1}{2}\, f(\eta) = 0,$$
$$\implies f''(\eta) + \frac{1}{2}\left(\eta\, f(\eta)\right)' = 0. \tag{20}$$

□

b) Rewrite the equation for f in a form involving only derivatives of $f(\eta)$ and of $\eta\, f(\eta)$, which may be integrated once.

Solution. Integrating (20), we get

$$f'(\eta) + \frac{1}{2}\,\eta\, f(\eta) = C, \tag{21}$$

where C is the integrating constant. □

c) Use the initial condition for u to determine the behavior of $f(\eta)$ and of $\eta\, f(\eta)$ as $\eta \to \infty$.

Solution. If $x > 0, t \to 0$, then $\eta = x\, t^{-\beta} \implies \eta \to \infty$.

$$\lim_{t\to 0} u(x,t) = u(x,0),$$
$$\lim_{t\to 0} \frac{1}{\sqrt{t}}\, f(\frac{x}{\sqrt{t}}) = 0,$$
$$\implies \lim_{t\to 0} \frac{x}{\sqrt{t}}\, f(\frac{x}{\sqrt{t}}) = 0, \quad \text{for} \quad x > 0$$
$$\implies \lim_{\eta\to\infty} \eta\, f(\eta) = 0.$$

Hence

$$\lim_{\eta\to\infty} \eta\, f(\eta) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} f(\eta) = 0.$$

□

d) Use this behavior to conclude that the derivative $f'(\eta)$ vanishes at ∞, and to conclude that the integration constant in the first order equation above is 0.

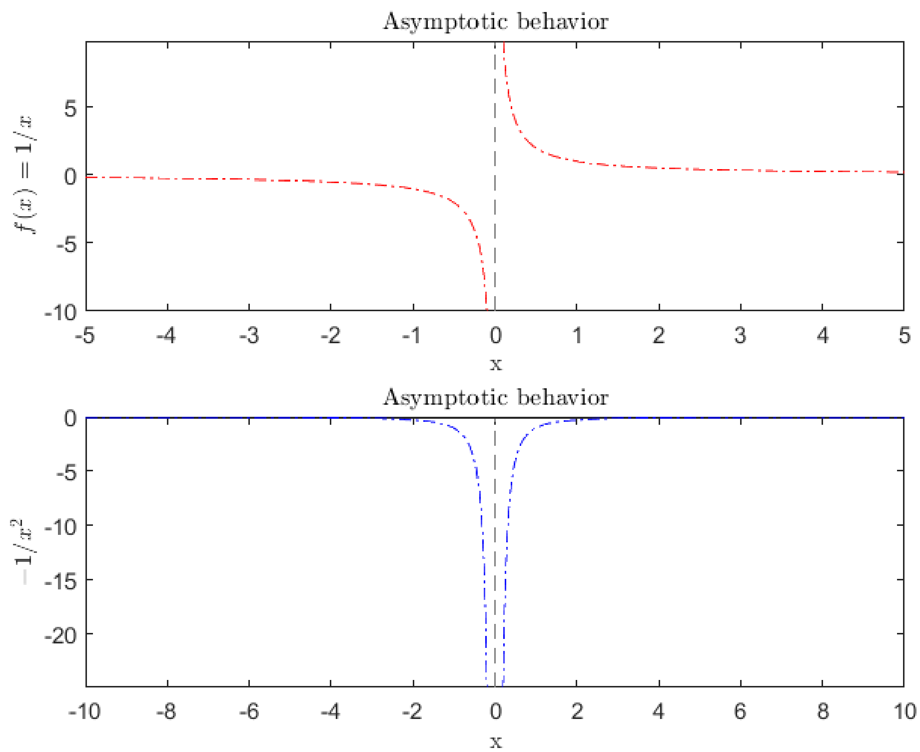
Solution. From (21), we can write

$$f'(\eta) = C - \frac{1}{2}\,\eta\, f(\eta), \tag{22}$$
$$\lim_{\eta\to\infty} f'(\eta) = \lim_{\eta\to\infty}\left(C - \frac{1}{2}\,\eta\, f(\eta)\right), \tag{23}$$
$$\lim_{\eta\to\infty} f'(\eta) = C, \quad \text{Since} \quad \lim_{\eta\to\infty} \eta\, f(\eta) = 0. \tag{24}$$

Since $\lim_{\eta\to\infty} f(\eta) = 0$, thus f has an asymptote. Hence,

$$\lim_{\eta\to\infty} f'(\eta) = 0,$$
$$\Longrightarrow C = 0.$$

Note: More understanding with picture:



$\square$

e) Determine $f = f(\eta)$ and afterwards $u = u(x;t)$.

Solution. Setting $C = 0$ in (21) gives

$$f'(\eta) + \frac{1}{2}\,\eta\, f(\eta) = 0,$$

which is a first order ODE. Using integrating factor method we can solve this ODE with Initial Condition $f(\eta = 0) = 1$.

The integrating factor is

$$e^{\int \frac{\eta}{2}\, d\eta} = e^{\frac{\eta^2}{4}}.$$

Then multiplying the above equation with the I.F. both sides yields

$$e^{\frac{\eta^2}{4}} \left(f'(\eta) + \frac{1}{2}\, \eta\, f(\eta) \right) = 0,$$

$$\implies \frac{d}{d\eta} \left(e^{\frac{\eta^2}{4}}\, f(\eta) \right) = 0.$$

Now integrating both sides gives

$$e^{\frac{\eta^2}{4}}\, f(\eta) = C_2,$$

$$\implies f(\eta) = e^{-\frac{\eta^2}{4}}\, C_2.$$

Since we considered that $f(\eta = 0) = 1$, then we get

$$f(\eta = 0) = e^{-\frac{0^2}{4}}\, C_2,$$

$$\implies 1 = C_2.$$

Hence

$$f(\eta) = e^{-\frac{\eta^2}{4}}. \tag{25}$$

and

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{t}}\, f\!\left(\frac{x}{\sqrt{t}}\right), \\ &= \frac{1}{\sqrt{t}}\, f(\eta), \\ &= \frac{1}{\sqrt{t}}\, e^{-\frac{\eta^2}{4}}, \\ &= \frac{1}{\sqrt{t}}\, e^{-\frac{x^2}{4\,t}}. \end{aligned}$$

□

3. With given $D > 0$ and $k \in \mathbb{R}$, consider the reaction-diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + k\, u, \qquad x > 0, t > 0, \tag{26}$$

$$u(x, 0) = 1, \qquad x > 0, \tag{27}$$

$$u(0, x) = 0, \qquad t > 0. \tag{28}$$

(a) Apply an appropriate transformation of u into v for bringing the equation for u to the standard type

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

What are the boundary and initial conditions then?

Solution. We define the transformation

$$v(x,t) := u(x,t)\; e^{-k\,t}. \tag{29}$$

Applying chain rules, we can write

$$\begin{aligned} \frac{\partial v(x,t)}{\partial x} &= \frac{\partial u(x,t)}{\partial x}\; e^{-k\,t}, \\ \Longrightarrow \quad \frac{\partial^2 v(x,t)}{\partial x^2} &= \frac{\partial^2 u(x,t)}{\partial x^2}\; e^{-k\,t}, \end{aligned} \tag{30}$$

and again applying chain rules, we obtain

$$\begin{aligned} \frac{\partial v(x,t)}{\partial t} &= \frac{\partial u(x,t)}{\partial t}\; e^{-k\,t} + e^{-k\,t}\; \frac{\partial}{\partial t}\,(-k\;t)\;\; u(x,t), \\ &= \left(\frac{\partial u(x,t)}{\partial t} - k\;t\;u(x,t)\right)\; e^{-k\,t}, \\ &= \frac{\partial^2 u(x,t)}{\partial x^2}\; e^{-k\,t}, \text{ [using (26)]} \\ \Longrightarrow \quad \frac{\partial v(x,t)}{\partial t} &= \frac{\partial^2 v(x,t)}{\partial x^2}, \text{ [using (30)].} \end{aligned}$$

Hence the new IVP in $v(x,t)$ is

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \qquad x > 0, t > 0, \tag{31}$$

$$v(x,0) = 1, \qquad x > 0, \tag{32}$$

$$v(0,t) = 0, \qquad t > 0. \tag{33}$$

□

(b) Find the similarity solution v for the transformed problem and afterwards determine u.

Solution. The similarity solution for the standard heat equation is

$$v(x,t) = \operatorname{erf}(\frac{x}{2\,\sqrt{t}}) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\,\sqrt{t}}} e^{-z^2}\; dz.$$

Hence

$$u(x,t) = \left(\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\,\sqrt{t}}} e^{-z^2}\; dz\right) e^{k\,t} \tag{34}$$

□

(c) Determine the limits $\lim_{t\to\infty} u(x,t)$ for $x > 0$, depending on k.

Solution. Observe that $v(x,t) \in [0,1]$ for any $x > 0, t > 0$.
Case 1: If $k < 0 : e^{k\,t} \to 0$ when $t \to \infty$, then

$$\lim_{t\to\infty} u(x,t) = 0, \quad \text{[By} \quad (34)]$$

Case 2: If $k > 0 : e^{k\,t} \to \infty$ when $t \to \infty$, then

$$\lim_{t\to\infty} u(x,t) = \infty, \quad \text{[By} \quad (34)]$$

Case 3: If $k = 0 : e^{k\,t} \to 1$ when $t \to \infty$, then

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2}\, dz \quad \text{[By} \quad (34)] \\ &= \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \\ \lim_{t\to\infty} u(x,t) &= 1.\end{aligned}$$

□