Functional- and Fourieranalysis 19/20 Exercise sheet 2

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The lecture notes for this course can be found on Blackboard. First read them carefully, and then do the exercises below. If you have questions, do not hesitate to contact us.

Note: All exercises marked with [$\star$] are (partially) based on the content of the lecture on Thursday, 12-03-2020. Therefore, we recommend to work on them after that lecture.

Exercise 6: (Hilbert spaces and Projections)

a) Prove Lemma 3 (Parallelogram identity).

b) Show that $l^2(\mathbb{K})$ with $(x, y) := \sum_{k\in\mathbb{N}} x_k \overline{y_k}$ is a Hilbert space.

c) Prove for all $p \in [1, \infty] \setminus \{2\}$ and $n > 1$, that the norm $\|\cdot\|_p$ on $\mathbb{R}^n$ is not induced by a scalar product.
Hint: Parallelogram identity and orthogonal vectors $x, y \in \mathbb{R}^n$.

d) Let $\mathcal{X}$ be a Hilbert space, and let $\mathcal{Y}$ be a non-empty and closed subspace of $\mathcal{X}$. Further let $x \in \mathcal{X} \setminus \mathcal{Y}$, and denote $\tau := \inf_{y\in\mathcal{Y}} \|x - y\|$. Prove that there must hold $\tau > 0$.

Exercise 7: (Orthogonality [$\star$])

a) *Pythagoras' theorem*: Let $\mathcal{X}$ be a Hilbert space and let $x, y \in \mathcal{X}$ be orthogonal.
Show that there holds $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

b) *Bessel's inequality*: Let $\mathcal{X}$ be a Hilbert space and let $(\mathbf{e}_i)_{i\in\mathbb{N}}$ be a orthonormal sequence in $\mathcal{X}$. We denote by $U_n := \operatorname{span}(\mathbf{e}_1, \dots, \mathbf{e}_n)$ for $n \in \mathbb{N}$ the closed subspace spanned by the first n elements. Show that there holds

$$\|x\|^2 = \|\Pi_{U_n}(x)\|^2 + \|x - \Pi_{U_n}(x)\|^2 ,$$

for all $x \in \mathcal{X}$, and that this implies $\sum_{i\in\mathbb{N}} |\,(x, \mathbf{e}_i)\,|^2 \leq \|x\|^2$.

c) *Orthogonal complement*: Let $\mathcal{Y}$ be a closed subspace of a Hilbert space $\mathcal{X}$. We denote by $\mathcal{Y}^\perp := \{x \in \mathcal{X} \mid \forall y \in \mathcal{Y} : (x, y) = 0\}$ the orthogonal complement of $\mathcal{Y}$. Show that for any $x \in \mathcal{X}$, there exist unique elements $y_0 \in \mathcal{Y}$ and $y_1 \in \mathcal{Y}^\perp$, such that $x = y_0 + y_1$.
Remark: This means that $\mathcal{X}$ is the direct sum of $\mathcal{Y}$ and $\mathcal{Y}^\perp$, i.e., $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^\perp$, where the direct sum of two vector spaces $\mathcal{Z}_1$ and $\mathcal{Z}_2$ (which are disjunct up to zero, i.e., $\mathcal{Z}_1 \cap \mathcal{Z}_2 = \{0\}$, and are subspaces of some larger normed space $\mathcal{Z}$) is defined as

$$\mathcal{Z}_1 \oplus \mathcal{Z}_2 := \{z := z_1 + z_2, z_1 \in \mathcal{Z}_1, z_2 \in \mathcal{Z}_2\}.$$

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Exercise 8: (Fourier transform as orthogonal projection [$\star$])

a) Let $\mathbf{e}_k(t) := e^{ikt}$ for $k \in \mathbb{Z}$ be as in Example 6. Show that

$$(\mathbf{e}_k, \mathbf{e}_l)_{L^2_\circ} = \delta_{k,l} := \begin{cases} 1 & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases}.$$

Remark: This means that $(\mathbf{e}_k)_{k\in\mathbb{Z}}$ is a orthonormal sequence in $L^2_\circ$.

b) Define $U_n := \operatorname{span}(\mathbf{e}_k,\ -n \leq k \leq n)$ as in Example 6 and consider the rectangle function $f(x) = \chi_{(-1,1)}(x)$. Compute $\Pi_{U_n}(f)$ for any $n \geq 0$ and plot it for $n \in \{1, 5, 10, 50\}$ (by using e.g. Matlab).

Exercise 9: (Properties of Fourier coefficients [$\star$])

Prove Lemma 6: Let $f \in L^2_\circ$. The following properties hold:

a) If f is a real-valued function, then $\widehat{f}(k) = \overline{\widehat{f}(-k)}$.

b) If f is an even function, then $\widehat{f}(k) \in \mathbb{R}$.
Hint: A function $f : \mathbb{R} \to \mathbb{C}$ is even iff $f(x) = \overline{f(-x)}$ for all $x \in \mathbb{R}$.

c) If f is an odd function, then $\widehat{f}(k) \in i\,\mathbb{R}$.
Hint: A function $f : \mathbb{R} \to \mathbb{C}$ is odd iff $f(x) = -\overline{f(-x)}$ for all $x \in \mathbb{R}$.

Exercise 10: (Impossible projection [optional, bonus points])

a) The space $\mathcal{X} = (C^0([0,1]), \|\cdot\|_{\mathcal{X}})$ with $\|f\|_{\mathcal{X}} := \sup_{x\in[0,1]} |f(x)| + \int_0^1 |f(x)| \mathrm{d}x$ is a Banach space. Show that the set $\mathcal{M} := \{f \in C^0([0,1]) \mid f(0) = 0\}$ is a closed subspace of $\mathcal{X}$.

b) Prove that the distance of the constant function 1 and $\mathcal{M}$ is one, i.e., $\inf\limits_{f\in\mathcal{M}} \|1 - f\|_{\mathcal{X}} = 1$.

c) Show that the distance of the constant function 1 and any function $f \in \mathcal{M}$ is larger than one, i.e., $\|1 - f\|_{\mathcal{X}} > 1$ for all $f \in \mathcal{M}$.

d) Prove that the projection $\Pi_{\mathcal{M}}(1)$ of the constant function 1 according to Definition 7 does not exist in $\mathcal{X}$. Why does this not contradict Lemma 4?