Numerieke methoden 2

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Any questions? Do not hesitate to contact us!

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Exercise 5.1: (Least-squares method)

a) Let $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = n$. The condition number of A is defined by

$$\kappa_2(A) := \frac{\max\limits_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}}{\min\limits_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}}. \tag{1}$$

1) Show that when A is invertible (so that $m = n$), $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$, so the definitions of condition numbers coincide for quadratic matrices.

Answer. By definition we have

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

and

$$\kappa_2(A) = \frac{\max\limits_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}}{\min\limits_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}} = \|A\|_2 \cdot \frac{1}{\min\limits_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}}$$

But

$$\begin{aligned}
\|A^{-1}\|_2 &= \max_{x\neq 0} \frac{\|A^{-1}x\|_2}{\|x\|_2} \\
&= \max_{x\neq 0} \frac{\|A^{-1}x\|_2}{\|AA^{-1}x\|_2} \\
&= \max_{A^{-1}x\neq 0} \frac{\|A^{-1}x\|_2}{\|AA^{-1}x\|_2} \\
&= \max_{y\neq 0} \frac{\|y\|_2}{\|Ay\|_2} \\
&= \frac{1}{\min\limits_{y\neq 0} \frac{\|Ay\|_2}{\|y\|_2}}
\end{aligned}$$

and we can conclude that $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$.

2) Prove from (1) that for any $x, y \in \mathbb{R}^n$

$$\frac{\|y\|_2}{\|x\|_2} \leq \kappa_2(A)\frac{\|Ay\|_2}{\|Ax\|_2}. \tag{2}$$

Answer. We know that

$$\kappa_2(A) = \frac{\max\limits_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}}{\min\limits_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}} = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \cdot \max_{y\neq 0} \frac{\|y\|_2}{\|Ay\|_2}$$
$$= \max_{x,y\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \frac{\|y\|_2}{\|Ay\|_2}$$

Then we can conclude that

$$\kappa_2(A) \geq \frac{\|Ax\|_2}{\|x\|_2} \frac{\|y\|_2}{\|Ay\|_2} \quad \forall x, y \in \mathbb{R}^n - \{0\}$$

with this we have

$$\frac{\|y\|_2}{\|x\|_2} \leq \kappa_2(A) \frac{\|Ay\|_2}{\|Ax\|_2}.$$

b) Let $\bar{x}$ be the least-squares solution to $\|Ax - b\|_2 \to \min$. Show that the vector $A\bar{x} - b$ is perpendicular to the hyperplane spanned by the Ax's, i.e.

$$A\bar{x} - b \perp Ax, \quad \forall x \in \mathbb{R}^n$$

and therefore $\|A\bar{x} - b\|_2^2 + \|A\bar{x}\|_2^2 = \|b\|_2^2$ and $\cos(\theta) := \frac{\|A\bar{x}\|_2}{\|b\|_2} \leq 1$ is well-defined.

Answer.

$$\begin{aligned}
A\bar{x} - b &\perp Ax \\
\langle Ax, A\bar{x} - b\rangle &= 0 \iff \\
(Ax)^t(A\bar{x} - b) &= 0 \iff \\
x^t A^t (A\bar{x} - b) &= 0 \iff \\
x^t A^t A\bar{x} - x^t A^t b &= 0 \iff \\
x^t (A^t A\bar{x} - A^t b) &= 0 \ (\textbf{Stelling 7})
\end{aligned}$$

Take the particular $\bar{x}$, we just proved that $(A\bar{x}, A\bar{x} - b) = 0$. Then

$$\begin{aligned}
||A\bar{x} - b||_2^2 + ||A\bar{x}||_2^2 &= (A\bar{x} - b, A\bar{x} - b) + (A\bar{x}, A\bar{x}) \\
&= (A\bar{x}, A\bar{x}) - 2(A\bar{x}, b) + (b, b) + (A\bar{x}, A\bar{x}) \\
&= 2(A\bar{x}, A\bar{x}) - 2(A\bar{x}, b) + (b, b) \\
&= 2(A\bar{x}, A\bar{x} - b) + (b, b) \\
&= (b, b) = ||b||_2^2
\end{aligned}$$

Using Pitagoras' theorem we have $\cos(\theta) := \frac{\|A\bar{x}\|_2}{\|b\|_2} \leq \frac{\|b\|_2}{\|b\|_2} = 1$.

c) Notice that Δx is the least-square solution to system $Ay = \Delta b$. Then applying a) and b) above, you can now prove

$$\frac{\|\Delta x\|_2}{\|\bar{x}\|_2} \leq \frac{\kappa_2(A)}{\cos(\theta)} \frac{\|\Delta b\|_2}{\|b\|_2}.$$

Answer. Using the part (1) we have

$$\frac{\|\Delta x\|_2}{\|\bar{x}\|_2} \leq \kappa_2(A) \frac{\|A\Delta x\|_2}{\|A\bar{x}\|_2}.$$

Moreover,

$$\begin{aligned}
\frac{\|\Delta x\|_2}{\|\bar{x}\|_2} &\leq \kappa_2(A)\frac{\|A\Delta x\|_2}{\|A\bar{x}\|_2} \\
&= \kappa_2(A)\frac{\|A\Delta x\|_2}{\|A\bar{x}\|_2}\frac{\|A\bar{x}\|_2}{\|b\|_2}\frac{\|b\|_2}{\|A\bar{x}\|_2} \\
&= \kappa_2(A)\frac{\|A\Delta x\|_2}{\|b\|_2}\frac{\|b\|_2}{\|A\bar{x}\|_2} \\
&= \frac{\kappa_2(A)}{\cos(\theta)}\frac{\|A\Delta x\|_2}{\|b\|_2} \\
&\leq \frac{\kappa_2(A)}{\cos(\theta)}\frac{\|\Delta b\|_2}{\|b\|_2}
\end{aligned}$$

Exercise 5.2: (Least-squares method)

a) Find a solution to $\|Ax - b\|_2 \to \min$ by solving the equation $A^T A\bar{x} = A^T b$ for

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}.$$

Compute the least-square error $\|A\bar{x} - b\|_2$ associated with the least-squares solution you found.

Answer. Define the new system

$$A^T A = \begin{pmatrix} 3 & 3 \\ 3 & 11 \end{pmatrix}$$

and

$$A^T b = \begin{pmatrix} 6 \\ 14 \end{pmatrix}$$

Then

$$A^T A\bar{x} = A^T b \longrightarrow \bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now we can compute the error

$$A\bar{x} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

then

$$\|A\bar{x} - b\|_2 = \| \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \|_2 = \sqrt{6} = 2.4495$$

b) Use the factorization $A = QU$ to find the least-squares solution $\|Ax - b\|_2 \to \min$ for

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}.$$

Answer. Using Given's rotations the factorization of A is $A = QU$ with

$$Q = \begin{pmatrix} -0.5774 & 0.7071 & -0.4082 \\ -0.5774 & -0.7071 & -0.4082 \\ -0.5774 & 0 & -0.8165 \end{pmatrix}$$

and

$$U = \begin{pmatrix} -1.7321 & -1.7321 \\ 0 & 2.8284 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

then

$$\hat{b} = Q^T b = \begin{pmatrix} -3.4641 \\ 2.828 \\ \mathbf{2.4495} \end{pmatrix}$$

$$\begin{pmatrix} -1.7321 & -1.7321 \\ 0 & 2.8284 \end{pmatrix} \bar{x} = \begin{pmatrix} -3.4641 \\ 2.828 \end{pmatrix} \longrightarrow \bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now we can compute the error

$$\|A\bar{x} - b\|_2 = \| - \sqrt{\mathbf{6}}\|_2 = 2.4495$$

Exercise 5.3: Non-uniqueness of minimalization problems

Consider the problem

$$\left\| \begin{matrix} x-1 \\ x \end{matrix} \right\|_1 \to \min .$$

Does it have a solution? Is it unique?

Answer.

Consider the problem

$$\left\| \begin{matrix} x-1 \\ x \end{matrix} \right\|_1 \to \min .$$

First we can compute the norm-1 of the vector

$$\left\| \begin{matrix} x-1 \\ x \end{matrix} \right\|_1 = |x-1| + |x|$$

- Case 1. If $x \leq 0$ then $|x-1| + |x| = -x + 1 - x = 1 - 2x \geq 1$.
- Case 2. If $0 < x < 1$ then $|x-1| + |x| = x + 1 - x = 1$.
- Case 3. If $x \geq 1$ then $|x-1| + |x| = x - 1 + x = 2x - 1 \geq 1$.

And we can conclude

$$\left\| \begin{matrix} x-1 \\ x \end{matrix} \right\|_1 = |x-1| + |x| \geq 1 := \min, \quad \forall x$$

but the solution is not unique, because for every $x \in [0, 1]$

$$\left\| \begin{matrix} x-1 \\ x \end{matrix} \right\|_1 = 1 = \min .$$

Exercise 5.4: Linear regression

Assume you are given a set of points (x_i, y_i), $1 \le i \le N$, with $x_i \neq x_j, i \neq j$, and you want to find $m \in \mathbb{R}$ and $b \in \mathbb{R}$ such that, for $f(x) := mx + b$, these parameters minimize

$$\sum_{i=1}^{N} |f(x_i) - y_i|^2.$$

Show how to obtain m and b.

Answer.

Using ex.4.2 we know that $||Ax - c||_2 \to \min$ by solving $A^t Ax = A^t c$ and now we want to minimize the following problem

$$\min_{m,b\in\mathbb{R}} \|mx - b - y\|_2^2$$

Define

$$A = \begin{pmatrix} x_1 & \cdots & 1 \\ \vdots & & \vdots \\ x_N & \cdots & 1 \end{pmatrix} \text{ and } c = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$$

then

$$A^t A = \begin{pmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & N \end{pmatrix} \text{ and } A^t c = \begin{pmatrix} \sum x_i y_i \\ \sum y_i \end{pmatrix}$$

We can solve $A^t Ax = A^t c$ using Gaussian reduction:

$$\begin{pmatrix} \sum x_i^2 & \sum x_i & \sum x_i y_i \\ \sum x_i & N & \sum y_i \end{pmatrix} \underbrace{\rightarrow}_{R_1:N\cdot R_1 \text{ and } R_2:-\sum x_i \cdot R_2} \begin{pmatrix} N\sum x_i^2 & N\sum x_i & N\sum x_i y_i \\ -(\sum x_i)^2 & -N\sum x_i & -\sum x_i \sum y_i \end{pmatrix}$$

$$\underbrace{\rightarrow}_{R_2:R_1+R_2 \text{ and } R_1:1/N\cdot R_1} \begin{pmatrix} \sum x_i^2 & \sum x_i & \sum x_i y_i \\ N\sum x_i^2 - (\sum x_i)^2 & 0 & N\sum x_i y_i - \sum x_i \sum y_i \end{pmatrix}$$

Then $m = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{N\sum x_i^2 - \left(\sum x_i\right)^2}$ and $b = \frac{\sum y_i - \left(m \sum x_i\right)}{N}$.