Instruction 1: warming up
Course: Partial differential equation (3341), 2020/21

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A. **Ordinary differential equations**

1. **Linear, first order equations.** Use integrating factors for finding the general solution to the equations below ($a \in \mathbf{R}$ is a constant). If applicable, find the solution that satisfies the initial condition:

   a) $u'(t) + a\ t\ u(t) = 0$;

   Solution. **Standard Form (S.F.):**

$$u'(t) + a\ t\ u(t) = 0. \tag{1}$$

Integrating Factor (I.F.):

$$I.F. = e^{\int a\ t\ dt} = e^{\frac{a\ t^2}{2}}. \tag{2}$$

Remark: Using product rule, chain rule and fundamental theorem of calculus, we can write

$$\begin{aligned} \frac{d}{dt}\left(e^{\frac{a\ t^2}{2}}\ u(t)\right) &= u(t)\ \frac{d}{dt}\left(e^{\frac{a\ t^2}{2}}\right) + e^{\frac{a\ t^2}{2}}\ \frac{d}{dt}u(t), \\ &= u(t)\ e^{\frac{a\ t^2}{2}} \frac{d}{dt}\left(\frac{a\ t^2}{2}\right) + e^{\frac{a\ t^2}{2}}\ u'(t), \\ &= u(t)\ e^{\frac{a\ t^2}{2}}\left(\frac{2\ a\ t}{2}\right) + e^{\frac{a\ t^2}{2}}\ u'(t), \\ &= a\ t\ u(t)\ e^{\frac{a\ t^2}{2}} + e^{\frac{a\ t^2}{2}}\ u'(t), \\ &= e^{\frac{a\ t^2}{2}}\Big(u'(t) + a\ t\ u(t)\Big). \end{aligned} \tag{3}$$

Multiplying S.F. by I.F. :

$$\begin{aligned} & e^{\frac{a\ t^2}{2}}\Big(u'(t) + a\ t\ u(t)\Big) = 0, \\ \Longrightarrow\ & e^{\frac{a\ t^2}{2}} \frac{d}{dt}u(t) + u(t)\ e^{\frac{a\ t^2}{2}} \frac{d}{dt}\left(\frac{a\ t^2}{2}\right) = 0, \\ \Longrightarrow\ & \frac{d}{dt}\left(e^{\frac{a\ t^2}{2}}\ u(t)\right) = 0. \end{aligned} \tag{4}$$

Integration with respect to (w.r.t.) t :

$$\begin{aligned}
&\int \frac{d}{dt}\left(e^{\frac{a\,t^2}{2}}\ u(t)\right)\ dt = \int 0\ dt,\\
\Longrightarrow\ & e^{\frac{a\,t^2}{2}}\ u(t) = C,\\
&\text{where } C \text{ is a integrating constant,}\\
\Longrightarrow\ & u(t) = C\ e^{-\ \frac{a\,t^2}{2}}.
\end{aligned} \tag{5}$$

□

b) $u'(t) + a\ t\ u(t) = t;$

Solution. **Standard Form (S.F.):**

$$u'(t) + a\ t\ u(t) = t. \tag{6}$$

Integrating Factor (I.F.):

$$I.F. = e^{\int a\ t\ dt} = e^{\frac{a\,t^2}{2}}.$$

Multiplying S.F. by I.F. :

$$\begin{aligned}
& e^{\frac{a\,t^2}{2}}\left(u'(t) + a\ t\ u(t)\right) = t\ e^{\frac{a\,t^2}{2}},\\
\Longrightarrow\ & e^{\frac{a\,t^2}{2}}\frac{d}{dt}u(t) + u(t)\ e^{\frac{a\,t^2}{2}}\frac{d}{dt}\left(\frac{a\ t^2}{2}\right) = t\ e^{\frac{a\,t^2}{2}},\\
\Longrightarrow\ & \frac{d}{dt}\left(e^{\frac{a\,t^2}{2}}\ u(t)\right) = t\ e^{\frac{a\,t^2}{2}}.
\end{aligned} \tag{7}$$

Integration with respect to (w.r.t.) t :

$$\begin{aligned}
&\int \frac{d}{dt}\left(e^{\frac{a\,t^2}{2}}\ u(t)\right)\ dt = \int t\ e^{\frac{a\,t^2}{2}}\ dt,\\
\Longrightarrow\ & e^{\frac{a\,t^2}{2}}\ u(t) = \int \frac{1}{a}\frac{d}{dt}\left(e^{\frac{a\,t^2}{2}}\right)\ dt + C,\\
&\text{where } C \text{ is a integrating constant,}\\
\Longrightarrow\ & e^{\frac{a\,t^2}{2}}\ u(t) = \frac{1}{a}\ e^{\frac{a\,t^2}{2}} + C,\\
\Longrightarrow\ & u(t) = \frac{1}{a} + C\ e^{-\ \frac{a\,t^2}{2}}.
\end{aligned} \tag{8}$$

□

c) $t^2\ u'(t) + 2\ t\ u(t) = t^2$ with $u(1) = 1;$

Solution. **Standard Form (S.F.):**

$$u'(t) + \frac{2}{t}\, u(t) = 1. \tag{9}$$

Integrating Factor (I.F.):

$$I.F. = e^{\int \frac{2}{t}\, dt} = e^{2\, ln(t)} = e^{ln(t^2)} = t^2. \tag{10}$$

Multiplying S.F. by I.F. :

$$\begin{aligned} & t^2 \left(u'(t) + \frac{2}{t}\, u(t)\right) = t^2, \\ \Longrightarrow\ & t^2\, \frac{d}{dt} u(t) + u(t)\ 2\ t = t^2, \\ \Longrightarrow\ & t^2\, \frac{d}{dt} u(t) + u(t)\ \frac{d}{dt}\left(t^2\right) = t^2, \\ \Longrightarrow\ & \frac{d}{dt}\left(t^2\, u(t)\right) = t^2. \end{aligned} \tag{11}$$

Integration with respect to (w.r.t.) t **:**

$$\begin{aligned} & \int \frac{d}{dt}\left(t^2\, u(t)\right)\ dt = \int t^2\, dt, \\ \Longrightarrow\ & t^2\, u(t) = \frac{t^3}{3} + C, \\ & \text{where } C \text{ is a integrating constant,} \\ \Longrightarrow\ & u(t) = \frac{t}{3} + \frac{C}{t^2}, \end{aligned} \tag{12}$$

which is the general solution.

Particular solution: At t=1, we can write,

$$\begin{aligned} & u(t=1) = \frac{1}{3} + C, \\ & \text{Given, initial condition is } u(1) = 1, \\ \Longrightarrow\ & 1 = \frac{1}{3} + C, \\ \Longrightarrow\ & C = \frac{2}{3}. \end{aligned} \tag{13}$$

Hence the particular solution is $u(t) = \frac{t}{3} + \frac{2}{3\, t^2}$.

□

d) $t^2\, u'(t) + a\ t\ u(t) = t^2$ with $u(0) = 1$.

Solution. **General solution:**

$$u(t) = \frac{t}{3} + \frac{C}{t^2}, \\ \Longrightarrow C = t^2\, u(t) - \frac{t^3}{3}. \tag{14}$$

Particular solution: At t=0,

$$C = 0^2\, u(t)|_{t=0} - \frac{0^3}{3} = 0. \tag{15}$$

Hence the particular solution is $u_p(t) = \frac{t}{3}$.
Since given initial condition is $u(0) = 1$ and at $t = 0$, $u_p(0) = 0$, which is a contradiction. Hence we can not find a particular solution with the given initial condition $u(0) = 1$.

Remark:

$$\frac{d}{dt}u(t) = 1/3 - \frac{C}{t} \tag{16}$$

which is not defined for $t = 0$. This is why we can not find a particular solution at $t = 0$, with $u(0) = 1$.

□

2. **Linear, second order equations with constant coefficients.** Find the solution to the equations below:

a) $u''(t) - 3\, u'(t) + 2\, u(t) = 0$ with $u(0) = 1$ and $u'(1) = 1$;

Solution. **Characteristic equation:** We have

$$u''(t) - 3\, u'(t) + 2\, u(t) = 0. \tag{17}$$

Let us consider that

$$u(t) = e^{r\, t},$$

is a solution of (17). Then using

$$u'(t) = r\, e^{r\, t},\ u''(t) = r^2\, e^{r\, t}, \tag{18}$$

in (17), we can write

$$r^2\, e^{r\, t} - 3\, r\, e^{r\, t} + 2\, e^{r\, t} = 0, \\ \Longrightarrow\ e^{r\, t}\, \left(r^2 - 3\, r + 2\right) = 0. \tag{19}$$

Hence the characteristic equation can be written as

$$r^2 - 3\, r + 2 = 0, \tag{20}$$

with a=-3, b=2.
Here $\Delta = a^2 - 4\, b = 1 > 0$.

General Solution: The roots of (20) are

$$\begin{aligned} r_1 &= \frac{-a+\sqrt{\Delta}}{2} = 2, \\ r_2 &= \frac{-a-\sqrt{\Delta}}{2} = 1. \end{aligned} \tag{21}$$

which are real and distinct.
Then we can write the general solution as

$$\begin{aligned} u(t) &= c_1\, e^{r_1\, t} + c_2\, e^{r_2\, t}, \\ \Longrightarrow u(t) &= c_1\, e^{2\, t} + c_2\, e^{t}, \end{aligned} \tag{22}$$

where c_1, c_2 are constant.
Here

$$u'(t) = 2\, c_1\, e^{2\, t} + c_2\, e^{t}. \tag{23}$$

Particular solution: Given initial conditions are $u(0) = 1$ and $u'(1) = 1$.

Using $u(t=0) = 1$ in (22), we can write

$$\begin{aligned} u(0) &= c_1\, e^0 + c_2\, e^0, \\ \Longrightarrow 1 &= c_1 + c_2, \\ \Longrightarrow c_1 &= 1 - c_2. \end{aligned} \tag{24}$$

Using $u'(t=1) = 1$ in (23), we obtain

$$\begin{aligned} u'(1) &= 2\, c_1\, e^2 + c_2\, e, \\ \Longrightarrow 1 &= 2\,(1-c_2)\, e^2 + c_2\, e, \\ \Longrightarrow 1 - 2\, e^2 &= -2\, c_2\, e^2 + c_2\, e, \\ \Longrightarrow 1 - 2\, e^2 &= c_2(-2\, e^2 + e), \\ \Longrightarrow c_2 &= \frac{1-2\, e^2}{e\,(1-2\, e)}. \end{aligned} \tag{25}$$

Using 25 in 24, one gets

$$\begin{aligned} c_1 &= 1 - \frac{1-2\, e^2}{e\,(1-2\, e)}, \\ \Longrightarrow c_1 &= \frac{e - 2\, e^2 - 1 + 2\, e^2}{e\,(1-2\, e)}, \\ \Longrightarrow c_1 &= \frac{(e-1)}{e\,(1-2\, e)}. \end{aligned} \tag{26}$$

Hence, the particular solution is

$$u(t) = \frac{(e-1)}{e\,(1-2\,e)}\; e^{2\,t} + \frac{1-2\,e^2}{e\,(1-2\,e)}\; e^{t}. \tag{27}$$

□

b) $u''(t) - 2\,u'(t) + 2\,u(t) = 0$ with $u(0) = 1$ and $u'(0) = 0$;

Solution. **Characteristic equation:** We have

$$u''(t) - 2\,u'(t) + 2\,u(t) = 0. \tag{28}$$

Let us consider that

$$u(t) = e^{r\,t},$$

is a solution of (28). Then using

$$u'(t) = r\;e^{r\,t},\; u''(t) = r^2\;e^{r\,t}, \tag{29}$$

in (28), we can write

$$\begin{aligned} r^2\;e^{r\,t} - 2\;r\;e^{r\,t} + 2\;e^{r\,t} &= 0, \\ \Longrightarrow e^{r\,t}\;\left(r^2 - 2\;r + 2\right) &= 0, \end{aligned} \tag{30}$$

Hence the characteristic equation can be written as

$$r^2 - 2\;r + 2 = 0, \tag{31}$$

with a= -2, b=2.
Here $\Delta = a^2 - 4\;b = -4 < 0$.

General Solution: The roots of (31) are

$$\begin{aligned} r_1 &= -\frac{a}{2} + \frac{i}{2}\;\sqrt{-\Delta} = 1 + i, \\ r_2 &= -\frac{a}{2} - \frac{i}{2}\;\sqrt{-\Delta} = 1 - i, \end{aligned} \tag{32}$$

which are imaginary and distinct.

Then we can write the general solution as

$$\begin{aligned} u(t) &= e^{-a\,t/2}\left(c_1\;cos(\frac{\sqrt{-\Delta}}{2}\;t) + c_2\;sin(\frac{\sqrt{-\Delta}}{2}\;t)\right), \\ \Longrightarrow u(t) &= e^{t}\left(c_1\;cos(t) + c_2\;sin(t)\right), \end{aligned} \tag{33}$$

where c_1, c_2 are constant.

Here,

$$u'(t) = e^t \left(c_1 \; cos(t) + c_2 \; sin(t)\right) + e^t \left(-c_1 \; sin(t) + c_2 \; cos(t)\right). \tag{34}$$

Particular solution: Initial conditions are $u(0) = 1$ and $u'(0) = 0$.
At $t = 0$, $u(0) = 1$ and from (33),

$$\begin{aligned} & u(0) = e^0 \left(c_1 \; cos(0) + c_2 \; sin(0)\right), \\ \Longrightarrow \; & 1 = c_1 \; 1 + c_2 \; 0, \\ \Longrightarrow \; & c_1 = 1. \end{aligned} \tag{35}$$

At $t = 0$, $u'(0) = 0$ and from (34),

$$\begin{aligned} & u'(0) = e^0 \left(c_1 \; cos(0) + c_2 \; sin(0)\right) + e^0 \left(-c_1 \; sin(0) + c_2 \; cos(0)\right), \\ \Longrightarrow \; & 0 = c_1 \; 1 + c_2 \; 0 - c_1 \; 0 + c_2 \; 1, \\ \Longrightarrow \; & 0 = c_1 + c_2 \\ \Longrightarrow \; & c_2 = -c_1 = -1. \end{aligned} \tag{36}$$

Hence, the particular solution is

$$u(t) = e^t \left(cos(t) - sin(t)\right). \tag{37}$$

□

c) $u''(t) + 4 \; u'(t) + 4 \; u(t) = 0$ with $u'(0) = 0$ and $u(1) = 1$;

Solution. **Characteristic equation:** We have

$$u''(t) + 4 \; u'(t) + 4 \; u(t) = 0. \tag{38}$$

Let us consider that

$$u(t) = e^{r \, t},$$

is a solution of (38). Then using

$$u'(t) = r \; e^{r \, t}, \; u''(t) = r^2 \; e^{r \, t}, \tag{39}$$

in (38), we can write

$$\begin{aligned} & r^2 \; e^{r \, t} + 4 \; r \; e^{r \, t} + 4 \; e^{r \, t} = 0, \\ \Longrightarrow \; & e^{r \, t} \; \left(r^2 + 4 \; r + 4\right) = 0. \end{aligned} \tag{40}$$

Hence the characteristic equation can be written as

$$r^2 + 4\,r + 4 = 0, \tag{41}$$

with a= 4, b=4.

Here $\Delta = a^2 - 4\,b = 0$.

General Solution: The roots of (41) are

$$r_1 = r_2 = -\frac{a}{2} = -2, \tag{42}$$

which are real and equal.

Then we can write the general solution as

$$\begin{aligned} u(t) &= c_1\, e^{-\frac{a}{2}} + c_2\, t\, e^{-\frac{a}{2}}, \\ \Longrightarrow u(t) &= e^{-2\,t}\left(c_1 + t\, c_2\right), \end{aligned} \tag{43}$$

where c_1, c_2 are constant.

Here,

$$u'(t) = -2\, e^{-2\,t}\left(c_1 + t\, c_2\right) + e^{-2\,t}\left(0 + c_2\right), \tag{44}$$

$$\Longrightarrow u'(t) = -2\, c_1\, e^{-2\,t} + c_2\, \left(1 - 2\,t\right)\, e^{-2\,t}. \tag{45}$$

Particular solution: Given initial conditions are $u'(0) = 0$ and $u(1) = 1$.

At $t = 1$, $u(1) = 1$ and from (43),

$$\begin{aligned} u(1) &= e^{-2}\left(c_1 + c_2\right), \\ \Longrightarrow 1 &= e^{-2}\left(c_1 + c_2\right), \\ \Longrightarrow c_1 + c_2 &= 1/e^{-2} = e^2, \\ \Longrightarrow c_1 &= e^2 - c_2. \end{aligned} \tag{46}$$

At $t = 0$, $u'(0) = 0$ and from (44),

$$\begin{aligned} u'(0) &= -2\, c_1\, e^0 + c_2\, \left(1 - 0\right)\, e^0, \\ \Longrightarrow 0 &= -2\left(e^2 - c_2\right) + c_2, \\ \Longrightarrow c_2 &= \frac{2\, e^2}{3}. \end{aligned} \tag{47}$$

Therefore,

$$c_1 = e^2 - \frac{2\, e^2}{3} = \frac{e^2}{3}. \tag{48}$$

Hence, the particular solution is

$$\begin{aligned} u(t) &= e^{-2\,t} \left(\frac{e^2}{3} + t\, \frac{2\, e^2}{3} \right), \\ \Longrightarrow u(t) &= e^{-2\,t} \left(1 + 2\, t\right)\, \frac{e^2}{3}. \end{aligned} \tag{49}$$

□

B. Partial integration, Gauss/Divergece Theorem Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ of $d = 3$) be a bounded domain with smooth boundary $\partial\Omega$ and $\vec{\nu} = (\nu_1, \ldots, \nu_d)^T \in \mathbb{R}^d$ the unit normal to $\partial\Omega$ pointing outwards Ω.

1. Prove that for any $u \in C^1(\Omega)$ and $v \in C^2(\Omega)$ one has

$$\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \Delta v.$$

Solution. Let us consider, $d = 3$, then we can write

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \tag{50}$$

Hence,

$$\begin{aligned} \vec{\nabla} u &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right), \\ \vec{\nabla} v &= \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \right), \\ \nabla u \cdot \nabla v &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \right) = \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right), \\ \Delta v = \nabla \cdot \nabla v &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \right) = \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\ u\, \vec{\nabla} v &= \left(u\, \frac{\partial v}{\partial x}, u\, \frac{\partial v}{\partial y}, u\, \frac{\partial v}{\partial z} \right). \end{aligned} \tag{51}$$

Therefore,

$$
\begin{aligned}
\vec{\nabla} \cdot \left(u\, \vec{\nabla} v\right) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(u\, \frac{\partial v}{\partial x}, u\, \frac{\partial v}{\partial y}, u\, \frac{\partial v}{\partial z}\right), \\
&= \frac{\partial}{\partial x}\left(u\, \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\, \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(u\, \frac{\partial v}{\partial z}\right), \\
&= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + u\frac{\partial^2 v}{\partial z^2}\right) \text{ (product rule)}, \\
&= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) + u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \\
&= \nabla u \cdot \nabla v + u\, \Delta v.
\end{aligned}
\tag{52}
$$

□

2. Prove that for any $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ one has

$$\int_\Omega \Delta u(\vec{x}) d\vec{x} = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{\nu}} dS.$$

Solution. We know that for a scalar function $u : \Omega \to \mathbb{R}$,

$$\Delta u := \nabla \cdot (\nabla u) \text{ and } \frac{\partial u}{\partial \vec{\nu}} := \vec{\nu} \cdot \nabla u. \tag{53}$$

Therefore,

$$
\begin{aligned}
\int_\Omega \Delta u(\vec{x}) d\vec{x} &= \int_\Omega \nabla \cdot (\nabla u)(\vec{x})\, d\vec{x}, \\
&= \int_{\partial\Omega} \vec{\nu} \cdot \nabla u\, dS \text{ (By the Gauss divergence theorem)}, \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial \vec{\nu}}\, dS.
\end{aligned}
\tag{54}
$$

□

3. Prove that for any $u, v \in C^1(\Omega) \cap C(\bar{\Omega})$ one has

$$\int_\Omega \frac{\partial u}{\partial x_k} v d\vec{x} = \int_{\partial\Omega} uv\nu_k dS - \int_\Omega u\frac{\partial v}{\partial x_k} d\vec{x}, \quad k = 1, \ldots, d.$$

Hint: Consider the vector-valued function $\vec{F} : \bar{\Omega} \to \mathbb{R}^d, \vec{F} = (f_1, \ldots, f_d)^T$ s.t. $f_i \equiv 0$ (i.e. $f_i(x) = 0$ for all $x \in \bar{\Omega}$) for all $i \neq k$ and take $f_k = uv$.

Solution. Let us consider

$$\vec{F} = (f_1, \dots, f_k, \dots, f_d)^T = (0, \dots, u\ v, \dots, 0)^T, \tag{55}$$

where $f_k := u\ v$, is the $k-$th component of $\vec{F}$. Then,

$$\begin{aligned}\vec{\nabla}\cdot\vec{F} &= \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_k}{\partial x_k} + \cdots + \frac{\partial f_d}{\partial x_d},\\ &= 0 + \cdots + \frac{\partial (u\ v)}{\partial x_k} + \cdots + 0,\\ &= \frac{\partial (u\ v)}{\partial x_k},\end{aligned} \tag{56}$$

and

$$\begin{aligned}\vec{\nu}\cdot\vec{F} &= \nu_1\ f_1 + \cdots + \nu_k\ f_k + \cdots + \nu_d\ f_d,\\ &= 0 + \cdots + \nu_k\ (u\ v) + \cdots + 0,\\ &= \nu_k\ (u\ v).\end{aligned} \tag{57}$$

By the Gauss divergence therorem, we can write,

$$\begin{aligned} &\int_\Omega \nabla\cdot\vec{F}\ d\vec{x} = \int_{\partial\Omega} \vec{\nu}\cdot\vec{F}\ dS,\\ \Longrightarrow &\int_\Omega \frac{\partial (u\ v)}{\partial x_k} = \int_{\partial\Omega} \nu_k\ (u\ v)\ dS,\\ \Longrightarrow &\int_\Omega \left(u\frac{\partial v}{\partial x_k} + \frac{\partial u}{\partial x_k} v\right)\ d\vec{x} = \int_{\partial\Omega} (u\ v)\ \nu_k\ dS,\\ \Longrightarrow &\int_\Omega \frac{\partial u}{\partial x_k} v\ d\vec{x} = \int_{\partial\Omega} (u\ v)\ \nu_k\ dS - \int_\Omega u\ \frac{\partial v}{\partial x_k}\ d\vec{x}.\end{aligned} \tag{58}$$

□

4. Let $\Omega = (a, b)$ be a finite, open interval. Prove that for all $u, v \in C^1([a, b])$ one has

$$\int_a^b u'v dx = u(b)v(b) - u(a)v(a) - \int_a^b uv' dx.$$

Solution. Clearly,

$$\begin{aligned} &\int_a^b \frac{d}{dx}(u\ v)\ dx = \left[(u\ v)\ (x)\right]_{x=a}^{x=b},\\ \Longrightarrow &\int_a^b \left(u\ \frac{dv}{dx} + v\ \frac{du}{dx}\right)\ dx = u(b)v(b) - u(a)v(a),\\ \Longrightarrow &\int_a^b u'v dx = u(b)v(b) - u(a)v(a) - \int_a^b uv' dx.\end{aligned} \tag{59}$$

$\square$

5. Explain why this situation is similar to the one in $\mathbb{R}^d$.

Solution. Let $k = 1$, $\Omega = (a, b)$, $\partial\Omega = \{a\} \cup \{b\}$.
At $x = a$, $\nu_a = -1$ and $x = b$, $\nu_b = 1$.
We showed that for $\mathbb{R}^d$,

$$\int_\Omega \frac{\partial u}{\partial x_k} v \, d\vec{x} = \int_{\partial\Omega} (u \; v) \; \nu_k \; dS - \int_\Omega u \; \frac{\partial v}{\partial x_k} \; d\vec{x} \tag{60}$$

Then for $k = 1$,

$$\begin{aligned} &\int_a^b \frac{\partial u}{\partial x} v \; d\vec{x} = u(b) \; v(b) \; \nu_b + u(a) \; v(a) \nu_a - \int_a^b u \; \frac{\partial v}{\partial x} \; d\vec{x}, \\ \implies &\int_a^b \frac{\partial u}{\partial x} v \; d\vec{x} = u(b) \; v(b) - u(a) \; v(a) - \int_a^b u \; \frac{\partial v}{\partial x} \; d\vec{x}. \end{aligned} \tag{61}$$

$\square$

6. Prove that for all $u \in C^1(\Omega) \cap C(\bar{\Omega})$ and $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ one has

$$\int_\Omega \nabla u \nabla v d\vec{x} = \int_{\partial\Omega} u \partial_{\vec{\nu}} v dS - \int_\Omega u \Delta v d\vec{x}.$$

Hint: Use the Divergence Theorem for de vector-valued function $\vec{F} = u\nabla v$.

Solution. From problem 1, we can write,

$$\begin{aligned} \int_\Omega \nabla u \; \nabla v \; d\vec{x} + \int_\Omega u \; \Delta v \; d\vec{x} &= \int_\Omega \nabla \cdot (u \; \nabla v) \; d\vec{x}, \\ &= \int_{\partial\Omega} \vec{\nu} \cdot (u \; \nabla v) \; dS \text{ (Gauss divergence)}, \\ &= \int_{\partial\Omega} u(\vec{\nu} \cdot \nabla v) \; dS. \end{aligned} \tag{62}$$

We know that for a scalar function $u : \Omega \to \mathbb{R}$,

$$\frac{\partial u}{\partial \vec{\nu}} := \vec{\nu} \cdot \nabla u. \tag{63}$$

Hence,

$$\begin{aligned} &\int_\Omega \nabla u \; \nabla v \; d\vec{x} + \int_\Omega u \; \Delta v \; d\vec{x} = \int_{\partial\Omega} u \; \partial_{\vec{\nu}} v \; dS, \\ \implies &\int_\Omega \nabla u \; \nabla v \; d\vec{x} = \int_{\partial\Omega} u \; \partial_{\vec{\nu}} v \; dS - \int_\Omega u \; \Delta v \; d\vec{x}. \end{aligned} \tag{64}$$

$\square$

C. First order PDEs Below $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is a given function. Solve the first order equations below with the given initial conditions (do not forget to check your answers!):

$$\begin{cases} \dfrac{\partial u}{\partial t} + 3x^{\frac{4}{3}} \dfrac{\partial u}{\partial x} &= f, \qquad \text{for } x > 0, t > 0, \\ u(x, 0) &= x^{\frac{1}{3}}, \qquad \text{for } x > 0. \end{cases}$$

Consider three situations:

1. $f(x, t, u) = 0$,

Solution. **PDE:** Given PDE,

$$\begin{aligned} \frac{\partial u}{\partial t} + 3x^{\frac{4}{3}} \frac{\partial u}{\partial x} &= 0, && \text{for } x > 0, t > 0, \\ u(x, 0) = x^{\frac{1}{3}} &= u_0(x), && \text{for } x > 0 \text{ (Initial condition)}. \end{aligned} \tag{65}$$

Step1: ODE (non-linear): Let $x(t)$ be the solution of the following ODE s.t.

$$\frac{dx(t)}{dt} = 3x^{\frac{4}{3}}, \quad x(0) = x_0 \text{ (characteristic curve)}. \tag{66}$$

Step2: Solving ODE: Integrating,

$$\begin{aligned} &\frac{1}{3} \int x^{-\frac{4}{3}}\, dx = \int 1\, dt \\ \implies &\frac{1}{3} \frac{x^{-\frac{1}{3}}}{(-1/3)} = t + C, \\ \implies &- x^{-\frac{1}{3}} - t = C. \end{aligned} \tag{67}$$

At $t = 0$,

$$\begin{aligned} &- x_0^{-\frac{1}{3}} - 0 = C, \\ \implies &C = -x_0^{-\frac{1}{3}}. \end{aligned} \tag{68}$$

Hence,

$$\begin{aligned}
& - x^{-\frac{1}{3}} - t = -x_0^{-\frac{1}{3}}, \\
& \Longrightarrow \; x^{-\frac{1}{3}} = x_0^{-\frac{1}{3}} - t, \\
& \Longrightarrow \; \frac{1}{x^{\frac{1}{3}}} = \left(x_0^{-\frac{1}{3}} - t\right), \\
& \Longrightarrow \; x = x(t) = \frac{1}{\left(x_0^{-\frac{1}{3}} - t\right)^3},
\end{aligned} \tag{69}$$

which is the solution of the non-linear ODE. **It is the characteristic curves of the original PDE.** Then we get,

$$x_0 = x(0) = \frac{1}{\left(x^{-\frac{1}{3}} + t\right)^3}. \tag{70}$$

Step3: Solution of the original PDE, We define,

$$\begin{aligned}
& v(t) := u(x(t), t), \\
\Longrightarrow \; & v'(t) = \frac{d}{dt} u(x(t), t), \\
& \quad = u_t(x(t), t) + \frac{dx(t)}{dt}\, u_x(x(t), t), \\
& \quad = u_t(x(t), t) + 3\, x^{\frac{4}{3}}\, u_x(x(t), t), \\
& \quad = 0 \quad \text{(given from PDE)}.
\end{aligned} \tag{71}$$

Integrating both sides w.r.t. t,

$$\begin{aligned}
& v(t) = c, \quad \text{where } c \text{ is an integrating constant}, \\
\Longrightarrow \; & u(x(t), t) = c.
\end{aligned} \tag{72}$$

At $t = 0$,

$$\begin{aligned}
& u(x(0), 0) = c = c, \\
\Longrightarrow \; & x_0^{1/3} = c \text{ (initial condition)}.
\end{aligned} \tag{73}$$

Then the solution becomes,

$$u(x(t), t) = x_0^{1/3} = \frac{1}{\left(x^{-\frac{1}{3}} + t\right)}. \tag{74}$$

It is the general solution of the PDE, it is constant on each characteristic curve.

Step4:Validation

Note: Every solution of the PDE is constant on the solution curves of the ODE.

Moral: Solutions of PDEs generally depend on arbitrary functions (instead of arbitrary constants). You need an auxiliary condition if you want to determine a unique solution. Such conditions are usually called initial or boundary conditions.

□

2. $f(x,t,u) = 1$,

Solution. **PDE:** Given PDE,

$$\begin{aligned} \frac{\partial u}{\partial t} + 3x^{\frac{4}{3}}\frac{\partial u}{\partial x} = &\quad 1, \quad \text{for } x > 0, t > 0, \\ u(x,0) = &\quad x^{\frac{1}{3}}, \quad \text{for } x > 0. \end{aligned} \tag{75}$$

Step1: ODE (non-linear)

$$\frac{dx(t)}{dt} = 3x^{\frac{4}{3}},\ x(0) = x_0 \text{ (characteristic curve)}. \tag{76}$$

Step2: Solving ODE In (1.), we show that the solution of the ODE is

$$x = x(t) = \frac{1}{\left(x_0^{-\frac{1}{3}} - t\right)^3} \quad \text{and} \quad x_0 = x(0) = \frac{1}{\left(x^{-\frac{1}{3}} + t\right)^3}. \tag{77}$$

Step3: We define,

$$\begin{aligned} v(t) &:= u(x(t),t), \\ \Longrightarrow\ v'(t) &= \frac{d}{dt}u(x(t),t), \\ &= u_t(x(t),t) + \frac{dx(t)}{dt}\ u_x(x(t),t), \\ &= u_t + 3\ x^{\frac{4}{3}}\ u_x, \\ &= 1 \quad \text{(given PDE).} \end{aligned} \tag{78}$$

Integrating both sides w.r.t. t,

$$\begin{aligned} v(t) &= t + c, \quad \text{where } c \text{ is an integrating constant,} \\ \Longrightarrow u(x(t),t) &= t + c. \end{aligned} \tag{79}$$

At $t = 0$,

$$\begin{aligned} u(x(0), 0) &= c + 0 = c, \\ \Longrightarrow x_0^{1/3} &= c \text{ (initial condition)}. \end{aligned} \tag{80}$$

Then the solution becomes,

$$\begin{aligned} u(x(t), t) &= x_0^{1/3} + t, \\ \Longrightarrow u(x(t), t) &= \frac{1}{\left(x^{-\frac{1}{3}} + t\right)} + t. \end{aligned} \tag{81}$$

Step4:Validation

$\square$

3. $f(x, t, u) = u.$

Solution. **PDE(non-homogeneous):** Given PDE,

$$\begin{aligned} \frac{\partial u}{\partial t} + 3x^{\frac{4}{3}} \frac{\partial u}{\partial x} = & \quad u, \quad \text{for } x > 0, t > 0, \\ u(x, 0) = & \quad x^{\frac{1}{3}} = \phi(x), \quad \text{for } x > 0. \end{aligned} \tag{82}$$

Step1: ODE (non-linear)

$$\frac{dx(t)}{dt} = 3x^{\frac{4}{3}}, \; x(0) = x_0 \text{ (characteristic curve)}. \tag{83}$$

Step2: Solving ODE In (1.), we show that the solution of the ODE is

$$x = x(t) = \frac{1}{\left(x_0^{-\frac{1}{3}} - t\right)^3}, \text{ and } \quad x_0 = x(0) = \frac{1}{\left(x^{-\frac{1}{3}} + t\right)^3}. \tag{84}$$

Step3: We define,

$$
\begin{aligned}
v(t) &:= u(x(t),t),\\
\implies v'(t) &= \frac{d}{dt}u(x(t),t),\\
&= u_t(x(t),t) + \frac{dx(t)}{dt}\,u_x(x(t),t),\\
&= u(x(t),t), \quad \text{given from PDE}\\
&= v(t),\\
\implies v'(t) - v(t) &= 0, \quad \text{Integrating factor is } e^{-t},\\
\implies v(t) &= v(0)e^t,\\
&= u(x(0),0)e^t,\\
&= x_0^{1/3}e^t,\\
\implies u(x(t),t) &= \frac{e^t}{\left(x^{-\frac{1}{3}}+t\right)}.
\end{aligned} \tag{85}
$$

Step4:Validation

$\square$

D. A bit of Analysis:
Prove the 2^{nd} *"Vanishing lemma"*:

Lemma 1. Let $\Omega \subset \mathbb{R}^d$ be a domain and $f : \Omega \to \mathbb{R}$ a continuous, positive function ($f(\vec{x}) \geq 0$ for all $\vec{x} \in \Omega$). If $\int_\Omega f(\vec{x})\, d\vec{x} = 0$ then one has $f \equiv 0$ (f is 0 overall in Ω, or $f(\vec{x}) = 0$ for all $\vec{x} \in \Omega$).

Solution. We assume that $\exists\ \vec{x_0} \in \Omega : f(\vec{x_0}) = f_0 > 0$.
Since f is continuous, by the definition of continuity , $\exists$ a ball $B_{\vec{x_0}} \subseteq \Omega$ s.t. $\forall \vec{x} \in B_{\vec{x_0}}$,

$$
\begin{aligned}
|f(\vec{x}) - f(\vec{x_0})| &< \frac{1}{2}f(\vec{x_0}), \ \text{(here } \frac{1}{2}f(\vec{x_0}) = \epsilon \text{ in the def. of cont.)},\\
-\frac{1}{2}f(\vec{x_0}) < f(\vec{x}) - f(\vec{x_0}) &< \frac{1}{2}f(\vec{x_0}),\\
f(\vec{x_0}) - \frac{1}{2}f(\vec{x_0}) < f(\vec{x}) &< \frac{1}{2}f(\vec{x_0}) + f(\vec{x_0}),\\
\frac{1}{2}f(\vec{x_0}) < f(\vec{x}) &< \frac{3}{2}f(\vec{x_0}),
\end{aligned} \tag{86}
$$

which implying that

$$f(\vec{x}) > \frac{1}{2} f(\vec{x_0}) = \frac{1}{2} f_0 > 0, \ \forall \vec{x} \in B_{\vec{x_0}}. \tag{87}$$

Let us consider that $\chi_{B_{\vec{x_0}}}$ be the characteristic function of $B_{\vec{x_0}}$, i.e. the function

$$\begin{aligned} \chi_{B_{\vec{x_0}}} :& \Omega \longrightarrow \mathbb{R}, \\ & \vec{x} \longmapsto \chi_{B_{\vec{x_0}}}(\vec{x}) = \begin{cases} 1 & \text{if } \ \vec{x} \in B_{\vec{x_0}}, \\ 0, & \text{if } \vec{x} \notin B_{\vec{x_0}}. \end{cases} \end{aligned} \tag{88}$$

Clearly, from (87) and (88), we can write

$$f(\vec{x}) \geq \chi_{B_{\vec{x_0}}}(\vec{x}) \ f(\vec{x}) \geq 0, \ \forall \vec{x} \in \Omega \tag{89}$$

If

$$\begin{aligned} 0 &= \int_{\Omega} f(\vec{x}) \ d\vec{x}, \\ &\geq \int_{\Omega} \chi_{B_{\vec{x_0}}}(\vec{x}) \ f(\vec{x}) \ d\vec{x}, \\ &= \int_{B_{\vec{x_0}}} f(\vec{x}) \ d\vec{x}, \ (\text{Using } \ (88)), \\ &\geq \int_{B_{\vec{x_0}}} \frac{1}{2} f(\vec{x_0}) \ d\vec{x}, \ (\text{Using } \ (87)), \\ &= \frac{1}{2} f(\vec{x_0}) \ \int_{B_{\vec{x_0}}} 1 \ d\vec{x}, \\ &= \frac{1}{2} f(\vec{x_0}) \ |B_{\vec{x_0}}|, \\ &> 0, \ (\text{Since } \frac{1}{2} f(\vec{x_0}) > 0, \ \text{the volume of } B_{\vec{x_0}} = |B_{\vec{x_0}}| > 0), \end{aligned} \tag{90}$$

which is a contradiction.
Hence, if $\int_{\Omega} f(\vec{x}) \ d\vec{x} = 0$ then one has $f \equiv 0$ (f is 0 overall in Ω. □