Partiële differentiaalvergelijkingen (3341), 2020/2021

Instruction problems 4: Parabolic problems

A. Comparison principle, uniqueness, energy estimates
In this section we let Ω be a bounded domain in $\mathbb{R}^d$ ($d \in \mathbb{N}_0$) and $D > 0$ a given diffusion coefficient. Further, we let $u_0 : \Omega \to [0, \infty)$ be a positive initial condition.

1. Let $u_D : \partial\Omega \times [0, \infty) \to [0, \infty)$ be a positive boundary conditions, and $f : \Omega \times (0, \infty) \to [0, \infty)$ be a *source term* that is assumed (for simplicity) continuous on the closure of the parabolic cylinder $\Omega \times [0, \infty)$. Consider the diffusion problem with source

$$(P) \begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + f & \text{for } x \in \Omega, t > 0, \\ u(x,t) = u_D(x,t), & \text{for } x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega, \end{cases}$$

and assume that it admits a solution that is sufficiently smooth.

a) (Boundedness): Prove that $u(x,t) \geq 0$ for all $x \in \Omega$ and $t \geq 0$.

Solution. Let

$$k(u) := \begin{cases} u, & u < 0, \\ 0, & u \geq 0. \end{cases}$$

and

$$K(u) := \begin{cases} \frac{u^2}{2}, & u < 0, \\ 0, & u \geq 0. \end{cases}$$

Given diffusion problem with source is

$$\frac{\partial u}{\partial t} = D\Delta u + f.$$

Multiplying the above equation with $u(x,t)$ and integrating over Ω, we obtain for all $x \in \Omega$ and $t \geq 0$,

$$\int_\Omega u \, \frac{\partial u}{\partial t} \, dx - \int_\Omega D \, u \, \Delta u \, dx = \int_\Omega u \, f \, dx$$

Rewriting the first integral and applying the Gauss divergence theorem in the second integral, we can write for all $x \in \Omega$ and $t \geq 0$,

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \frac{u^2(x,t)}{2} \, dx - D \int_{\partial\Omega} u \, \partial_n u \, ds + D \int_\Omega |\nabla u(x,t)|^2 \, dx = \int_\Omega u \, f \, dx, \\
\implies &\frac{d}{dt} \int_\Omega K(u) \, dx - D \int_{\partial\Omega} u \, \partial_n u \, ds + D \int_\Omega |\nabla u(x,t)|^2 \, dx = \int_\Omega u \, f \, dx.
\end{aligned} \tag{1}$$

We denote the second, the third and the right hand side integral as I_2, I_3, I_r respectively and we get

$$\begin{aligned}
I_2 : &= D \int_{\partial\Omega} u \, \partial_n u \, ds, \\
&= D \int_{\partial\Omega} k(u) \, \partial_n u \, ds, \\
&= 0 \quad \text{[Since } u = u_D \text{ on } \partial\Omega \text{ and } u_D \geq 0, \text{hence } k(u) = 0 \text{ on } \partial\Omega], \\
I_3 : &= D \int_\Omega |\nabla u(x,t)|^2 \, dx \\
&\geq 0, \\
I_r : &= \int_\Omega u \, f \, dx, \\
&= \int_{-\infty}^0 k(u) \, f \, dx + \int_0^\infty k(u) \, f \, dx, \\
&\leq 0 \quad \text{[Since } f > 0 \text{ and } k(u) \leq 0 \text{ in the first integral]}.
\end{aligned}$$

Using this in 1, we can write for all $x \in \Omega$ and $t \geq 0$,

$$\frac{d}{dt} \int_\Omega K(u) \, dx \leq 0,$$

If the time derivative of a function is negative that means it is decreasing over all the time and it is also less than the initial condition. Hence,

$$\int_\Omega K(u(x,t))\,dx \le 0,$$
$$\implies \int_\Omega K(u(x,t))\,dx \le \int_\Omega K(u_0(x))\,dx,$$
$$\implies \int_\Omega K(u(x,t))\,dx \le 0 \text{ [Since } u_0 \ge 0,\ K(u_0) = 0 \text{ and } \int_\Omega K(u_0(x))\,dx = 0],$$

The positivity of $K(u(x,t))$ implies that the above inequality can only be true iff

$$K(u) \equiv 0,$$
$$\implies u(x,t) \ge 0.$$

□

b) (Uniqueness): Use energy estimates to prove that (P) has at most one solution. More precisely, prove that if $u = u_1 - u_2$ is the difference of two solutions to (P), the associated energy is 0.

Solution. To prove the uniqueness we assume that u_1 and u_2 are two solutions of P.
Then $\bar{u} = u_1 - u_2$ solves

$$(\bar{P})\begin{cases} \dfrac{\partial \bar{u}}{\partial t} = D\Delta\bar{u} & \text{for } x \in \Omega, t > 0, \\ \bar{u}(x,t) = 0, & \text{for } x \in \partial\Omega, t > 0, \\ \bar{u}(x,0) = 0, & \text{for } x \in \Omega, \end{cases}$$

since u_1 and u_2 satisfies the same BC and IC. First we define the energy as $E(t) = \int_\Omega (\bar{u}(x,t))^2\,dx$. Then we can use the same argument as in $1(a)$, to show that

$$E(t) = 0, \forall t \ge 0$$
$$\implies \int_\Omega (\bar{u}(x,t))^2\,dx = 0, \forall x \in \Omega, t > 0.$$

By the Vanishing lemma,

$$\bar{u}(x,t) = 0.$$

□

2. We derive different energy estimates for the diffusion problem with zero source and homogeneous Dirichlet boundary conditions,

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u & \text{for } x \in \Omega, t > 0, \\ u(x,t) = 0, & \text{for } x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega. \end{cases}$$

Assume that it admits a solution that is sufficiently smooth. With $p \geq 2$ and for $t \geq 0$ we define the "energy"

$$E^p(t) = \frac{1}{p}\int_\Omega u^p(x,t)dx + \frac{4(p-1)D}{p^2}\int_0^t \int_\Omega \left|\nabla\left(u^{\frac{p}{2}}(x,s)\right)\right|^2 dxds.$$

a) Explain why the first integral in the definition of E makes sense even for non-integer values of p (i.e. why is u^p well defined).

Solution. Since $u(x,t) \geq 0$ for all $x \in \Omega$ and $t > 0$ and it is a solution to the problem (P).
Hence u^p is well defined for non-integer values of p. □

b) Show that $(E^p)'(t) = 0$ for all $t > 0$.
Hint: Observe that $\partial_t(u^p) = pu^{p-1}\partial_t u$ and $\nabla\left(u^{\frac{p}{2}}(x,t)\right) = \frac{p}{2}\left(u^{\frac{p}{2}-1}(x,t)\right)\nabla u$.

Solution. Given diffusion problem is

$$\frac{\partial u}{\partial t} = D\Delta u.$$

Multiplying the above equation with $u^p(x,t)$ and integrating over Ω, we obtain for all $x \in \Omega$ and $t \geq 0$,

$$\int_\Omega u^p\ \frac{\partial u}{\partial t}\ dx - \int_\Omega D\ u^p\ \Delta u\ dx = 0,$$

$$\frac{d}{dt}\int_\Omega \frac{u^{p+1}(x,t)}{p+1}\ dx + D\int_\Omega |\nabla u(x,t)|^{p+1}\ dx = 0, \tag{2}$$

Here

$$
\begin{aligned}
D\int_\Omega |\nabla u|^{p+1}\,dx &= D\int_\Omega |\nabla u|\,|\nabla u^P|\,dx,\\
&= p\,D\int_\Omega u^{p-1}|\nabla u|^2\,dx,\ [\text{Since } \nabla u^p = pu^{p-1}\nabla u],\\
&= \frac{4\,p\,D}{(p+1)^2}\int_\Omega \left|\ \frac{(p+1)}{2}u^{\frac{p-1}{2}}\ \nabla u\ \right|^2\ dx,\\
&= \frac{4\,p\,D}{(p+1)^2}\int_\Omega \left|\ \nabla u^{\frac{p+1}{2}}\ \right|^2\ dx,\ [\text{Since } \nabla u^{\frac{p+1}{2}} = \frac{p+1}{2}u^{\frac{p-1}{2}}\nabla u]
\end{aligned}
$$

Hence form 2, we can write

$$
\frac{d}{dt}\int_\Omega \frac{u^{p+1}(x,t)}{p+1}\,dx + \frac{4\,p\,D}{(p+1)^2}\int_\Omega |\nabla u^{\frac{p+1}{2}}|^2\,dx = 0, \tag{3}
$$

For

$$
\begin{aligned}
&E^{p+1}(t) = \frac{1}{p+1}\int_\Omega u^{p+1}(x,t)dx + \frac{4\,p\,D}{(p+1)^2}\int_0^t\int_\Omega |\nabla u^{\frac{p+1}{2}}|^2\,dx\,dt,\\
&\implies (E^{p+1}(t))' = 0, [\text{immediately follow from 3}].
\end{aligned}
$$

□

c) Which condition has to be fulfilled by u_0 so that the energy $E^p(t)$ is finite? Think at L^p spaces.

Solution. If $u_0 \in C^1$, then for any $t > 0$,

$$
\int_\Omega |\nabla u(x,t)|^2\,dx \le \int_\Omega |\nabla u_0|^2\,dx.
$$

If $u_0 \in C(\overline{\Omega})$ then $\int_\Omega u_0^{p+1}\,dx$ must be finite. □

B. Fourier series (recap/training)

1. With $L > 0$ given, check that the trigonometric functions

$$
\left\{\cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right)/k = 0, 1, \dots\right\}
$$

are orthogonal w.r.t. the inner product on $(-L, L)$:

$$\int_{-L}^{L} \cos\Big(\frac{k\pi x}{L}\Big)\cos\Big(\frac{p\pi x}{L}\Big)\,dx = \begin{cases} 0, & \text{if } k \neq p, \\ 2L, & \text{if } k = p = 0, \\ L, & \text{if } k = p > 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{k\pi x}{L}\Big)\sin\Big(\frac{p\pi x}{L}\Big)\,dx = \begin{cases} 0, & \text{if } k \neq p, \\ L, & \text{if } k = p > 0, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{k\pi x}{L}\Big)\sin\Big(\frac{p\pi x}{L}\Big) = 0, \text{ for all } k, p.$$

Hint: Use the identities $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$ and $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$.

Solution. We know that

$$\cos\Big(\frac{k\pi x}{L}\Big)\cos\Big(\frac{p\pi x}{L}\Big) = \frac{1}{2}\left\{\cos\Big(\frac{(k+p)\pi x}{L}\Big) + \cos\Big(\frac{(k-p)\pi x}{L}\Big)\right\}.$$

If $k \neq p$, then $k + p \neq 0$ and $k - p \neq 0$ for any $k, p \geq 0$ then

$$\begin{aligned}
&\int_{-L}^{L} \cos\Big(\frac{k\pi x}{L}\Big)\cos\Big(\frac{p\pi x}{L}\Big)\,dx \\
&= \frac{1}{2}\int_{-L}^{L} \left\{\cos\Big(\frac{(k+p)\pi x}{L}\Big) + \cos\Big(\frac{(k-p)\pi x}{L}\Big)\right\}\,dx, \\
&= \frac{1}{2}\frac{L}{(k+p)\,\pi}\left[sin\left(\frac{(k+p)\,\pi\,x}{L}\right)\right]_{-L}^{L} + \frac{1}{2}\frac{L}{(k-p)\,\pi}\left[sin\left(\frac{(k-p)\,\pi\,x}{L}\right)\right]_{-L}^{L}, \\
&= 0.
\end{aligned}$$

If $k = p = 0$ then $\cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{p\pi x}{L}\right) = 1$. Hence

$$\int_{-L}^{L} \cos\Big(\frac{k\pi x}{L}\Big)\cos\Big(\frac{p\pi x}{L}\Big)\,dx = \int_{-L}^{L} 1\,dx = 2\,L.$$

If $k = p \geq 1$ then $\cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{p\pi x}{L}\right) = \frac{1}{2}\left\{\cos\left(\frac{2\,k\,\pi x}{L}\right) + \cos\left(0\right)\right\} = \frac{1}{2}\left\{\cos\left(\frac{2\,k\,\pi x}{L}\right) + 1\right\}$. Hence

$$\begin{aligned}
\int_{-L}^{L} \cos\Big(\frac{k\pi x}{L}\Big)\cos\Big(\frac{p\pi x}{L}\Big)\,dx &= \frac{1}{2}\int_{-L}^{L} \left\{\cos\Big(\frac{2\,k\,\pi x}{L}\Big) + 1\right\}\,dx, \\
&= \frac{1}{2}\frac{L}{2\,k\,\pi}\left[sin\left(\frac{2\,k\;\;\pi\,x}{L}\right)\right]_{-L}^{L} + \frac{1}{2}\,2\,L, \\
&= L.
\end{aligned}$$

□

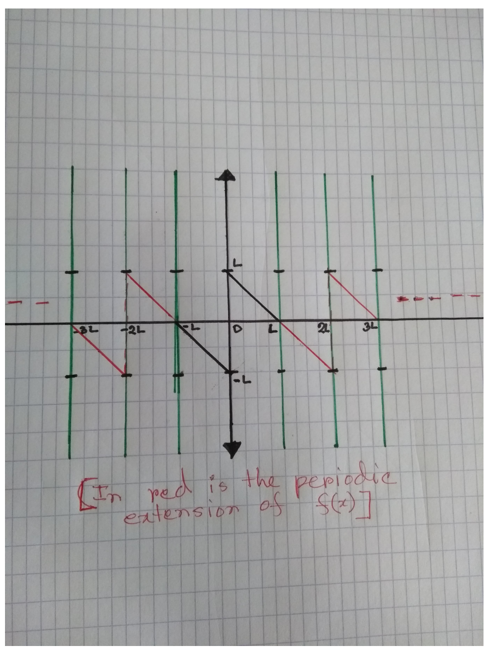
2. Below $f : \mathbb{R} \to \mathbb{R}$ is a given, periodic function with period $2L$. Determine its Fourier coefficients in the following cases:

a) $f(x) = \begin{cases} -L - x, & \text{if } -L < x < 0, \\ L - x, & \text{if } 0 \le x < L; \end{cases}$

Solution. If $x \in (0, L)$ then

$$\begin{aligned} f(-x) &= -L - (-x) \quad [\text{since } -x \in (-L, 0)], \\ &= -(L - x), \\ &= -f(x). \end{aligned}$$

Hence, f is an odd function.



Note: $f(x)$ is piecewise continuous. It has a unique point of (jump) discontinuities at $x = 0$, where

$$f(0-) = -L, \quad f(0+) = L,$$
$$\text{Further, } f(L) = f(-L) = 0.$$

$f'(x)$ is again piecewise continuous. In fact,

$$f'(x) = -1, \quad \text{for} \quad x \neq 0 \text{ (and } x \neq 2L, 4L, \cdots),$$

where $f'(x)$ is not defined since $f(x)$ is discontinuous at $x = 0$.
$f(x)$ is an odd function, hence

$$a_k = 0.$$

Therefore,

$$\int_{-L}^{L} f(x)\, cos\left(\frac{k\,\pi\,x}{L}\right)\, dx = \int_{-L}^{0} f(x)\, cos\left(\frac{k\,\pi\,x}{L}\right)\, dx + \int_{0}^{L} f(x)\, cos\left(\frac{k\,\pi\,x}{L}\right)\, dx \tag{4}$$

Now,

$$\begin{aligned}\int_{-L}^{0} f(x)\, cos\left(\frac{k\,\pi\,x}{L}\right)\, dx &= \int_{L}^{0} f(-z)\, cos\left(-\frac{k\,\pi\,z}{L}\right)\, (-dz), \text{ [letting } z = -x], \\ &= \int_{0}^{L} -f(z)\, cos\left(\frac{k\,\pi\,z}{L}\right)\, dz, \\ &= -\int_{0}^{L} f(z)\, cos\left(\frac{k\,\pi\,z}{L}\right)\, dz\end{aligned}$$

Using the above in 4, we get

$$\int_{-L}^{L} f(x)\, cos\left(\frac{k\,\pi\,x}{L}\right)\, dx = 0. \tag{5}$$

Similarly, we can derive

$$\int_{-L}^{L} f(x)\, sin\left(\frac{k\,\pi\,x}{L}\right)\, dx = 2\int_{0}^{L} f(x)\, sin\left(\frac{k\,\pi\,x}{L}\right)\, dx. \tag{6}$$

If $k \geq 1$ then

$$
\begin{aligned}
b_k &= \frac{2}{L} \int_0^L (L - x)\, sin\left(\frac{k\,\pi\,x}{2}\right)\, dx, \\
&= 2 \int_0^L \; sin\left(\frac{k\,\pi\,x}{L}\right)\, dx - \frac{2}{L} \int_0^L x\, sin\left(\frac{k\,\pi\,x}{L}\right)\, dx, \\
&= \frac{2\,L}{k\,\pi} \left[-cos\left(\frac{k\,\pi\,x}{L}\right)\right]_0^L - \frac{2}{L} \int_0^L x\, \left(-\frac{L}{k\,\pi} cos\left(\frac{k\,\pi\,x}{L}\right)\right)'\, dx, \\
&= \frac{2\,L}{k\,\pi}\, (1 - cos(k\,\pi)) + \frac{2}{k\,\pi} \left[x\, cos\left(\frac{k\,\pi\,x}{L}\right)\right]_0^L - \frac{2}{k\,\pi} \int_0^L cos\left(\frac{k\,\pi\,x}{L}\right)\, dx, \\
&= \frac{2\,L}{k\,\pi} \left[1 - (-1)^k\right] + \frac{2\,L}{k\,\pi}\, (-1)^k - \frac{2}{k\,\pi} \frac{L}{k\,\pi} \left[sin\left(\frac{k\,\pi\,x}{L}\right)\right]_0^L, \\
&= \frac{2\,L}{k\,\pi} \left[1 - (-1)^k + (-1)^k\right], \\
&= \frac{2\,L}{k\,\pi}.
\end{aligned}
$$

Hence

$$S_N(x) = \sum_{k=1}^{N} \frac{2\,L}{k\,\pi}\, sin\left(\frac{k\,\pi\,x}{L}\right).$$

Since $f(x), f'(x)$ are piecewise continuous, therefore

$$f(x) = \sum_{k \geq 1} \frac{2\,L}{k\,\pi}\, sin\left(\frac{k\,\pi\,x}{L}\right), \; \forall x \neq 2\,p\,L, \quad p \in \mathbb{Z}.$$

At $x = 0$, or $x = 2\,p\,L$ (where $p \in \mathbb{Z}$),

$$\sum_{k \geq 1} \frac{2\,L}{k\,\pi}\, sin\left(\frac{k\,\pi}{L}\,x\right) = \frac{1}{2}\,(f(0+) + f(0-)) = \frac{1}{2}\,(L - L) = 0.$$

□

b) $f(x) = \begin{cases} x + L, & \text{if } -L < x < 0, \\ L - x, & \text{if } 0 \leq x < L; \end{cases}$

Solution. Note: Similar calculations and arguments.

But now, $f(x)$ is an even function because if $x \in (0, L)$ then

$$\begin{aligned} f(-x) &= (-x) + L \quad [\text{since } -x \in (-L, 0)], \\ &= (L - x), \\ &= f(x). \end{aligned}$$

Here $f(x)$ is continuous (also it's extension). And

$$f'(x) = \begin{cases} 1, & \text{if } -L < x < 0, \\ -1, & \text{if } 0 < x < L, \end{cases}$$

is piecewise continuous.

Since, $f(x)$ is an even function, then

$$\begin{aligned} b_k &= 0, \\ a_k = \frac{2}{L} \int_0^L f(x)\, cos\left(\frac{k\,\pi\,x}{2}\right)\, dx, \quad k &\geq 0. \end{aligned}$$

Now,

$$\begin{aligned} a_0 &= L, \\ a_k &= \end{aligned}$$

$\square$

c) $f(x) = x$, and f is even;

d) $f(x) = x$, and f is odd;