Prof. Dr. Jochen Schütz C265 jochen.schuetz@uhasselt.be Tel 8315
Stephan Lunowa, M.Sc. C101a stephan.lunowa@uhasselt.be

The lecture notes for this course can be found on Blackboard. First read them carefully, and then do the exercises below. If you have questions, do not hesitate to contact us.

Exercise 1: ($\mathbf{C^0([a,b])}$ is infinite-dimensional)

In Example 2 of the lecture notes, $\dim(C^0([a,b])) = \infty$ is claimed. Prove the missing step:

Let $n \in \mathbb{N}$ and n pairwise distinct points $x_i \in [a,b]$, $1 \leq i \leq n$, be given. Further, denote $\varepsilon := \frac{1}{2}\min_{i\neq j}|x_i - x_j| > 0$. Show, that there exist continuous functions $f_i \in C^0([a,b])$ with the following properties:

- $\operatorname{supp}(f_i) \subset (x_i - \varepsilon, x_i + \varepsilon)$
- $f_i(x_i) = 1$.

Also prove the following consequences:

- $f_i(x_j) = 0$ for any $j \neq i$,
- the f_i are linearly independent,
- $\dim(C^0([a,b])) = \infty$.

Exercise 2: (Norms)

Show the following statements:

a) *Reversed triangle inequality:* Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. Show, that there holds

$$\|x - y\| \geq |\,\|x\| - \|y\|\,| \qquad \text{for all } x, y \in \mathcal{X}\,.$$

b) $(C^0([0,1]), \|\cdot\|_*)$ with $\|f\|_* := \sup\limits_{x\in[0,1]} x|f(x)|$ is a normed space.

c) $(C^0((0,1]), \|\cdot\|_\infty)$ with $\|f\|_\infty := \sup\limits_{x\in(0,1]} |f(x)|$ is **not** a normed space.

d) $(C^1([-1,1]), \|\cdot\|_{**})$ with $\|f\|_{**} := |f(0)| + \sup\limits_{x\in[-1,1]} |f'(x)|$ is a normed space.
Hint: Fundamental theorem of calculus.

Exercise 3: (Completeness)

Show the following statements:

a) Every finite-dimensional normed space is complete.
Hint: First show that for $e_i, 1 \leq i \leq n$, a basis and $x = \sum\limits_{i=1}^{n} \xi_i e_i$, the function $\|\cdot\| : x \mapsto \sum\limits_{i=1}^{n} |\xi_i|$ is a norm. You may use that on finite-dimensional spaces, norms are equivalent.

Prof. Dr. Jochen Schütz C265 jochen.schuetz@uhasselt.be Tel 8315
Stephan Lunowa, M.Sc. C101a stephan.lunowa@uhasselt.be

b) $C^0([0,1])$ with the norm $\|f\|_* := \sup\limits_{x\in[0,1]} x|f(x)|$ from Exercise 2 is not complete.
Hint: Find a Cauchy-sequence that has the limit $f(x) = \ln(x)$ or $f(x) = x^{-1/2}$.

c) $(C^1([-1,1]), \|\cdot\|_{**})$ with $\|f\|_{**} := |f(0)| + \sup\limits_{x\in[-1,1]} |f'(x)|$ from Exercise 2 is a Banach space.

Exercise 4: (Absolute convergence and completeness)

Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. Show that the following statements are equivalent:

i) $(\mathcal{X}, \|\cdot\|)$ is a Banach space.

ii) Each absolutely convergent series is convergent, i.e. for all $(x_i)_{i\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$ with $\sum\limits_{i\in\mathbb{N}} \|x_i\| < \infty$, there exists an $x \in \mathcal{X}$ with $\sum\limits_{i\in\mathbb{N}} x_i = x$.

Hint: For the proof of (i) $\Rightarrow$ (ii) look at the sequence $y_n := \sum\limits_{i=1}^{n} x_i$. For the proof of (ii) $\Rightarrow$ (i) use the following Lemma: Let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy-Sequence. Then, there exists a sub-sequence $(x_{i_k})_{k\in\mathbb{N}}$, such that $\|x_{i_{k+1}} - x_{i_k}\| \le 2^{-k}$.

Exercise 5: (Hölder's inequality for Lebesgue functions and sequences)

Let $p, q \in \mathbb{R}^{\ge 1} \cup \{\infty\}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Note the convention for p or q equal to infinity.

a) Let $\Omega \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) be an open set. The $L^p(\Omega)$-norm is defined by

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p \mathrm{d}x\right)^{1/p} & \text{for } p < \infty, \\ \operatorname*{ess\,sup}\limits_{x\in\Omega} |f(x)| & \text{for } p = \infty. \end{cases}$$

Further, let $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$. Prove that $uv \in L^1(\Omega)$ with

$$\|uv\|_{L^1(\Omega)} \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Hint: The *essential supremum* $\operatorname{ess\,sup}(\cdot)$ is the supremum $\sup(\cdot)$ up to values on a set of measure zero, e.g. $\operatorname{ess\,sup}_{x\in\mathbb{R}} f(x) = 1$ for

$$f(x) := \begin{cases} x & \text{for } x \in \mathbb{Q}, \\ 10^{10} & \text{for } x = \pi, \\ \frac{1}{1+|x|} & \text{for } x \in \mathbb{R} \setminus (\mathbb{Q} \cup \{\pi\}). \end{cases}$$

b) Let $u \in l^p(\mathbb{K})$ and $v \in l^q(\mathbb{K})$ (see Example 2 in the lecture notes for the definition). Prove that $uv \in l^1(\mathbb{K})$ with

$$\|uv\|_{l^1(\mathbb{K})} \le \|u\|_{l^p(\mathbb{K})} \|v\|_{l^q(\mathbb{K})}.$$