Porous medium equation

- Gas flow in a porous medium (Ω)
- Mass balance

$$\partial_t(\phi \rho) + \nabla\cdot(\rho \vec{u}) = 0 \quad , t>0,\ x\in\Omega$$

Unknown: $\boxed{\rho}$ - gas density $\left[\frac{kg}{m^3}\right]$, $\boxed{\vec{u}}$ - gas velocity $\left[\frac{m}{s}\right]$

- Darcy law:

$$\vec{u} = -\frac{1}{\mu} K \nabla p$$

Unknown: $\boxed{p}$ - gas pressure $[Pa] \sim \left[\frac{kg}{m s^2}\right]$

Material/gas properties:

μ - viscosity $\left[\frac{kg}{m s}\right]$, K - permeability $[m^2]$ (assumed scalar)

ϕ - porosity $[-]$

Ideal gas: (adiabatic) $\quad p = p_0 \rho^{\gamma}$, with $\gamma \geq 0$

$(p V^{\gamma} = \text{const.})$

This gives:

$$\partial_t(\phi\rho) = \nabla\cdot\left[\frac{1}{\mu} K p_0 \rho \nabla \rho^{\gamma}\right]$$

$$= \nabla\cdot\left[\frac{1}{\mu} K p_0 \gamma \rho^{\gamma} \nabla\rho\right]$$

$$= \nabla\cdot\left[\frac{1}{\mu} K p_0 \frac{\gamma}{\gamma+1} \nabla\left(\rho^{\gamma+1}\right)\right]$$

A-dimensionalisation: ρ_R, T_R, L_R - reference quantities (density, time, length)

$\rho = \rho_R \tilde{\rho}$, $\tilde{\rho}$ $[-]$

$x = L_R \tilde{x}$, $\tilde{x}$ $[-]$

$t = T_R \tilde{t}$, $\tilde{t}$ $[-]$

$$\frac{\phi}{T_R}\partial_{\tilde{t}}\tilde{\rho} = \frac{\rho_R^{\gamma}}{L_R^2}\frac{p_0}{\mu}\frac{\gamma}{\gamma+1} K \tilde{\Delta}\tilde{\rho}^{\gamma+1}$$

Assume T_R s.t. $\frac{T_R}{\phi}\cdot\frac{\rho_R^{\gamma}}{L_R^2} K \frac{p_0}{\mu}\frac{\gamma}{\gamma+1} = 1$, and

leaving out "~" gives: $\partial_t \rho = \Delta\left(\rho^{\gamma+1}\right)$

Here: $m = \gamma+1 \geq 1$

Simplified form: $\partial_t u = \partial_{xx} u^m$

(here $u = \tilde{\varrho}$, $m = \gamma + 1 \geq 1$)

Note: · equivalent form is $\partial_t u = \partial_x (D(u)\partial_x u)$,

where $D(u) = \begin{cases} m u^{m-1} & \text{if } u \geq 0 \\ 0 & \text{, if } u < 0. \end{cases}$

($u = 0$ means no gas is present, $u < 0$ is physically meaningless. Therefore the law $p = p_0 \varrho^\gamma$ only makes sense for $\varrho \geq 0$.

"Mathematically": $p = \begin{cases} p_0 \varrho^\gamma & \text{, if } \varrho \geq 0 \\ 0 & \text{, if } \varrho < 0 \end{cases}$)

· The equation is parabolic, but degenerates ($D = 0$) if $u = 0$. The boundary of the support of u changes in time (free boundary!)

Idea: find ~~similarity~~ fundamental solutions

$u(x,t) = t^\alpha f(\eta)$, with $\eta = x t^{-\beta}$.

The exponents $\alpha, \beta \in \mathbb{R}$ are determined s.t.

(1) u - solves the equation $\partial_t u = \partial_{xx}(u^m)$

(2) $\lim_{t \searrow 0} u(x,t) = 0 \quad \forall x \in \mathbb{R}^*$ (thus $x \neq 0$)

(3) $\lim_{x \to \pm\infty} u(x,t) = 0 \quad \forall t > 0$

(4) $\int_{\mathbb{R}} u(x,t) = 1 \quad \forall t > 0$

(these properties were satisfied by the fundamental solution to the diffusion equation)

Following the arguments in the book one gets

$\alpha = -\frac{1}{m+1}$, $\beta = \frac{1}{m+1}$, and f solves

$$(f^m)'' + \frac{1}{m+1}(\eta f)' = 0 \quad (11.11)$$

for all $\eta \in \mathbb{R}$.

The derivation in the book (p. 216 - 217) can be followed without any difficulty. One remark can be made. We have

$u(x,t) = t^{\alpha} f(\eta)$, with $\eta = x/t^{\beta}$

and $\alpha = -\beta = -\frac{1}{m+1}$.

We seek solutions u s.t. $\lim_{x \to \pm\infty} u(x,t) = 0$ for all $t > 0$. This is similar to saying: $\lim_{\eta \to \pm\infty} t^{-\frac{1}{m+1}} f(\eta) = 0$,

implying $\lim_{\eta \to \pm\infty} f(\eta) = 0$

Also, for $x \neq 0$, the limit before gives

$0 = x \lim_{\eta \to \pm\infty} t^{-\frac{1}{m+1}} f(\eta) = \lim_{\eta \to \pm\infty} \eta f(\eta)$, since $\eta = x t^{-\frac{1}{m+1}}$. This means that:

$$\lim_{\eta \to \pm\infty} f(\eta) = \lim_{\eta \to \pm\infty} \eta f(\eta) = 0.$$

Also, since $m > 0$, $\lim_{\eta \to \pm\infty} f^{m}(\eta) = 0$

With this, one integrates (11.11) to obtain $\left(f^{m}(\eta)\right)' + \frac{1}{m+1} \eta f(\eta) = A \in \mathbb{R}$

for all $\eta \in \mathbb{R}$. At this point it is not clear why $A = 0$. To see this we observe that for the function $f^{m}(\cdot)$ we have: $\lim_{\eta \to \pm\infty} f^{m}(\eta) = 0$

(see above) $\lim_{\eta \to \pm\infty} \left(f^{m}(\eta)\right)' = \lim_{\eta \to \pm\infty} \left[A - \frac{1}{m+1} \eta f(\eta)\right] = A$

By the Proposition in the ~~last~~ Instruction 8, it follows that $A = 0$, leading to (11.12)