Laplace Transform

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Outline

- Fourier Series
- Fourier Transform
- Laplace Transform

Why do we like $e^{j\omega t}$?

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- From Taylor Series: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$\begin{aligned} e^{ix} &= \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = \sum_{k=0}^{\infty} i^{2k} \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i^{2k+1} \frac{x^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i(-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ &= \cos x + i \sin x \end{aligned}$$

Why do we like $e^{j\omega t}$?

$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

Assume we have the following sinusoid:

$a_1 \cos(\omega t + \theta_1)$ $\quad \cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$= \boxed{a_1 \cos(\theta_1)\cos(\omega t) - a_1 \sin(\theta_1)\sin(\omega t)}$

$$a_1 \cos(\omega t + \theta_1) = C e^{j\omega t} + \bar{C} e^{-j\omega t}$$

$$C = \frac{1}{2} a_1 (\cos(\theta_1) + j\sin(\theta_1))$$

$$e^{j\theta_1} e^{j\omega t} = e^{j(\omega t + \theta)} = \frac{1}{2} a_1 e^{j\theta_1}$$

Consider the following expression, where C is complex

C_n $\quad C_{-n}$

$C e^{j\omega t} + \bar{C} e^{-j\omega t}$

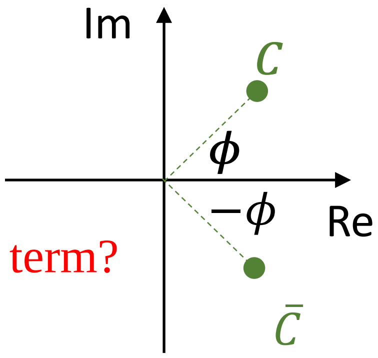
$= (c_1 + jc_2)(\cos(\omega t) + j\sin(\omega t)) + (c_1 - jc_2)(\cos(\omega t) - j\sin(\omega t))$

$= \boxed{c_1 \cos(\omega t) - c_2 \sin(\omega t)} + j(c_2 \cos(\omega t) + c_1 \sin(\omega t))$

$+ c_1 \cos(\omega t) - c_2 \sin(\omega t) - j(c_2 \cos(\omega t) + c_1 \sin(\omega t))$

$= \boxed{2c_1 \cos(\omega t) - 2c_2 \sin(\omega t)}$

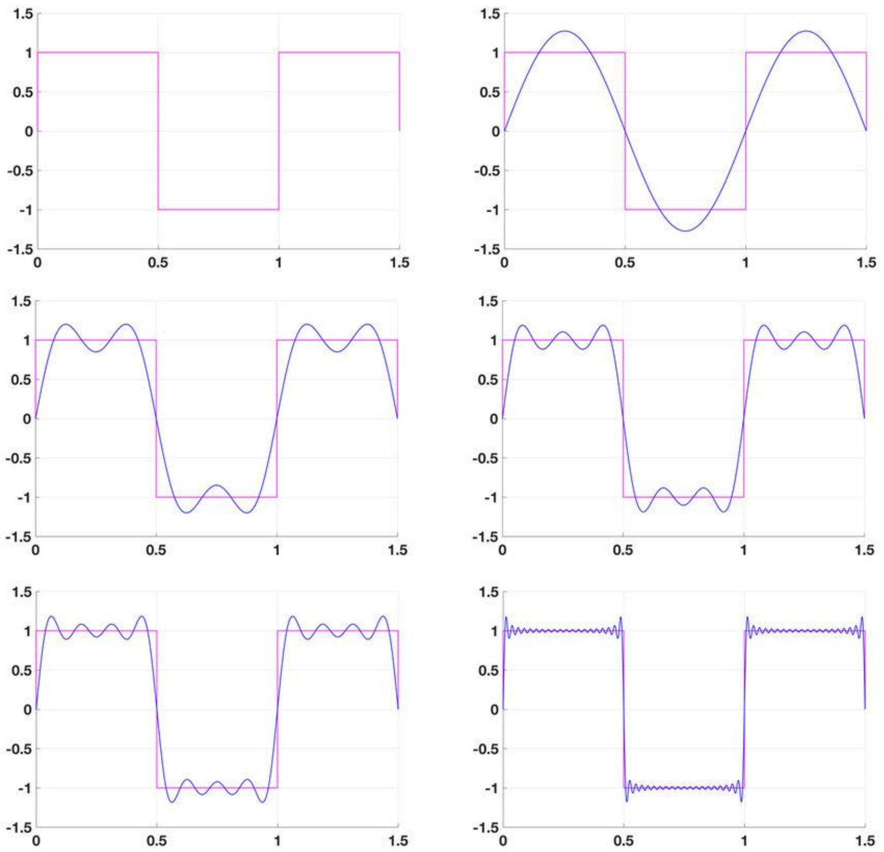
How can we get rid of the complex term?



Fourier Series

Any periodic signal, with period T_0, can be represented by the expression

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega_0 t}\,; \qquad \omega_0 = \frac{2\pi}{T_0}$$



Why do we have negative frequencies?

For a specific n, we have $C_{-n} = \bar{C}_n$

therefore, $\underbrace{C_n e^{jn\omega_0 t} + C_{-n} e^{j(-n\omega_0)t}}$ is a real function

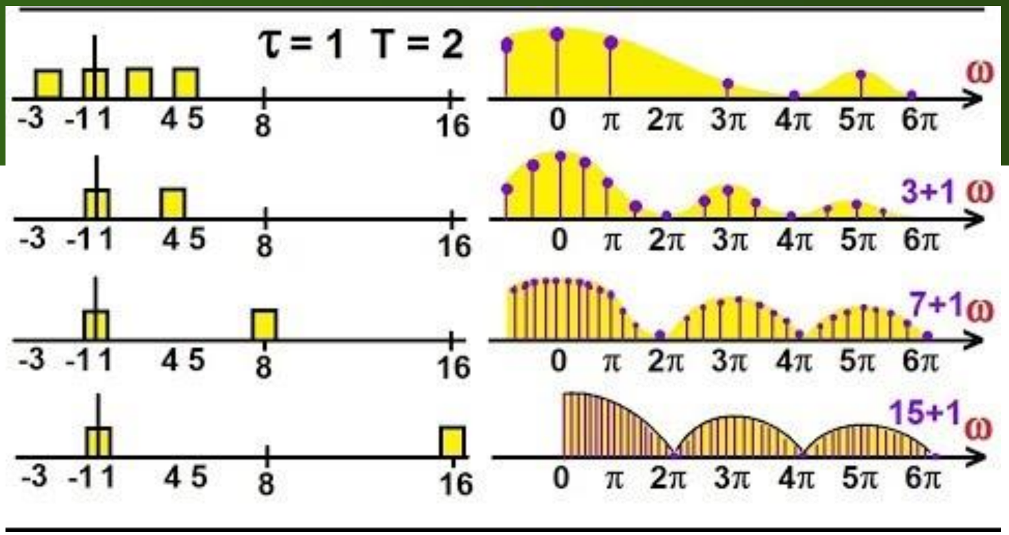
How can we find C_n?

$$C_n = \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jn\omega_0 t}dt$$

Fourier Series

- Why sinusoids?
- Does not change shape when applied to an LTI system, only changes Amplitude and phase.
  - This system increased gain by X and shifted in phase by ϕ at frequency f.

Fourier Transform



What if our function is not periodic?

- Take the period $T_0 \to \infty$

Fourier Series

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega_0 t}; \qquad \omega_0 = \frac{2\pi}{T_0}$$

Fourier Series Coefficients

$$C_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t) e^{-jn\omega_0 t} dt$$

$T_0 \to \infty$

$T_0 \to \infty$

Inverse Fourier Transform $\mathcal{F}^{-1}$

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{C(\omega)}_{F(\omega)} e^{j\omega t} d\omega$$

Fourier Transform $\mathcal{F}$

$$\mathcal{F}\{f(t)\} = \underbrace{C(\omega)}_{F(\omega)} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Laplace Transform $\mathcal{L}\{\, f(t)\, \}$

- Let us multiply $f(t)$ with $e^{-\sigma t}$ and take the Fourier Transform

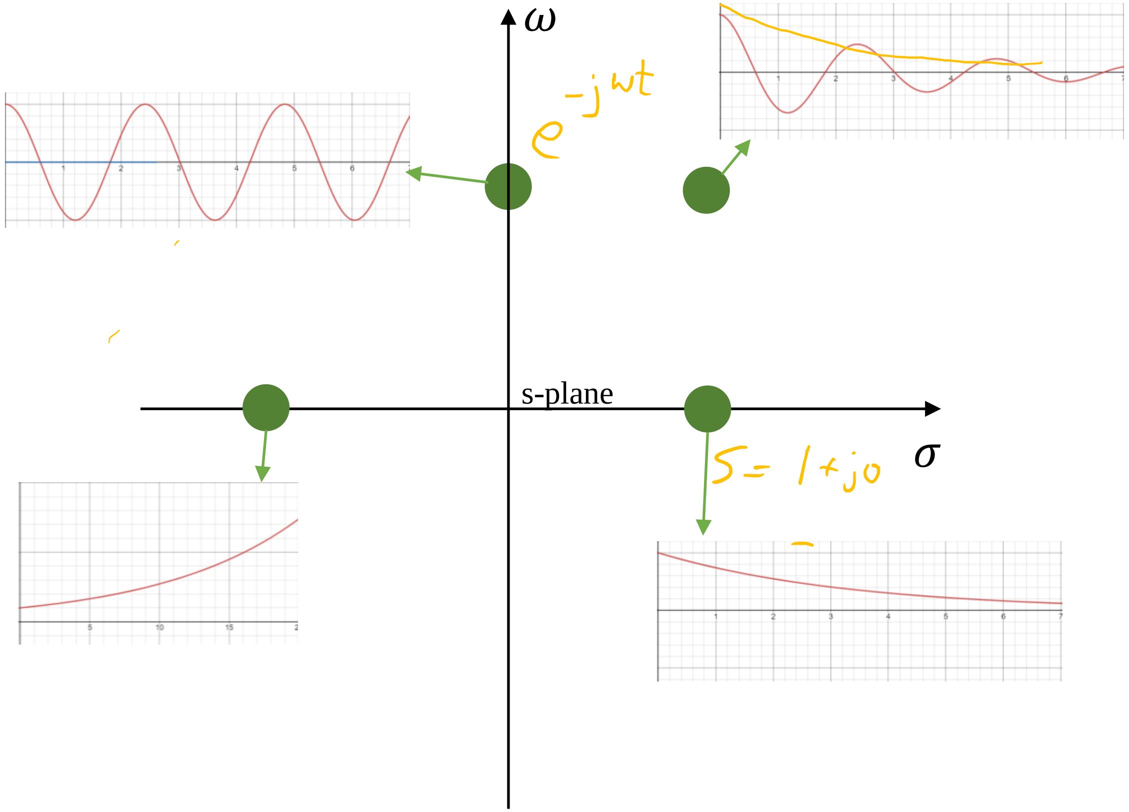
$$\mathcal{L}\{\, f(t)\} = \mathcal{F}\{\, f(t)e^{-\sigma t}\, \}$$
$$= \int_{-\infty}^{\infty} (\, f(t)e^{-\sigma t})e^{-j\omega t}dt$$
$$= \int_{-\infty}^{\infty} f(t)e^{-\sigma t-j\omega t}dt$$
$$= \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}dt$$
$$= \int_{-\infty}^{\infty} f(t)e^{-st}dt \qquad s = \sigma + j\omega$$

$$\boxed{\mathcal{L}\{\, f(t)\} = \int_{0}^{\infty} f(t)e^{-st}dt}$$

We are mainly interested in causal signals
$f(t) = 0$ for $t < 0$

e^{-st}

$e^{-t-j\omega t} = e^{-t} e^{-j\omega t}$



Example 1: unit step

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$$

Find the Laplace transform of:

$$f(t) = 1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$e^{-(\sigma + j\omega)t}$

$t \to \infty$

$\sigma = 0 \quad e^{j\omega t} \to \times$

$\sigma < 0 \quad e^{-\sigma t} e^{-j\omega t}$

$\sigma = -1 \quad e^{t} \to \infty$

$$\mathcal{L}\{1(t)\} = \int_0^{\infty} e^{-st} \mathrm{d}t = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

$\sigma > 0 \quad e^{-t} \to 0$

— this is valid provided $\mathrm{Re}(s) > 0$, so that $e^{-st} \xrightarrow{t \to +\infty} 0$.

Example 2: sinusoid

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

Find the Laplace transform of: $f(t) = \cos t$

$$\mathscr{L}\{\cos t\} = \mathscr{L}\left\{\frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}\right\} \qquad \text{(Euler's formula)}$$

$$= \frac{1}{2}\mathscr{L}\{e^{jt}\} + \frac{1}{2}\mathscr{L}\{e^{-jt}\} \qquad \text{(linearity)}$$

$$\mathscr{L}\{e^{jt}\} = \int_0^{\infty} e^{jt}e^{-st}\mathrm{d}t = \int_0^{\infty} e^{(j-s)t}\mathrm{d}t = \frac{1}{j-s}e^{(j-s)t}\Big|_0^{\infty}$$

$$= -\frac{1}{j-s} \qquad \text{(pole at } s = j\text{)}$$

$$\mathscr{L}\{e^{-jt}\} = \int_0^{\infty} e^{-jt}e^{-st}\mathrm{d}t = \int_0^{\infty} e^{-(j+s)t}\mathrm{d}t = -\frac{1}{j+s}e^{-(j+s)t}\Big|_0^{\infty}$$

$$= \frac{1}{j+s} \qquad \text{(pole at } s = -j\text{)}$$

— in both cases, require $\mathrm{Re}(s) > 0$, i.e., s must lie in the right half-plane (RHP)

Example 2: sinusoid

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

Find the Laplace transform of: $f(t) = \cos t$

$$\begin{aligned}
\mathscr{L}\{\cos t\} &= \frac{1}{2}\mathscr{L}\{e^{jt}\} + \frac{1}{2}\mathscr{L}\{e^{-jt}\} \\
&= \frac{1}{2}\left(-\frac{1}{j-s} + \frac{1}{j+s}\right) \\
&= \frac{1}{2}\left(\frac{-\cancel{j}-s+\cancel{j}-s}{(j-s)(j+s)}\right) \\
&= \frac{1}{2}\left(\frac{-2s}{-1+\cancel{js}-\cancel{js}-s^2}\right) \\
&= \frac{s}{s^2+1} \qquad \text{(poles at } s=\pm j)
\end{aligned}$$

for $\mathrm{Re}(s) > 0$

How does all this relate to dynamic systems?

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

$$s = \sigma + j\omega$$

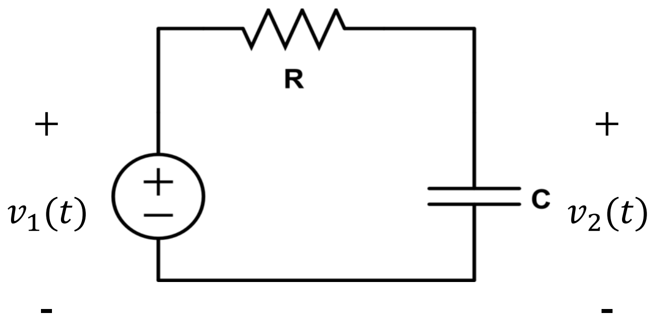
- Dynamic systems are modelled as differential equations.
- Linear differential equations with constant coefficients have solutions that are either exponentials or sinusoids.

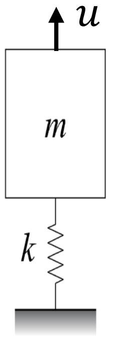
$\frac{d}{dt}(x) = -ax$

$$\frac{d}{dt}(\underbrace{Ce^{-at}}_{x}) = -a(\underbrace{Ce^{-at}}_{x})$$

$$\dot{x} = -ax \quad \Rightarrow \quad x(t) = Ce^{-at}$$

$$\ddot{x} = -bx \quad \Rightarrow \quad x(t) = C\sin(\sqrt{b}t)$$





Modelling an Electric System

Consider the RC circuit shown here

Using KVL, we can obtain

$$v_1(t) = R\,i(t) + v_2(t)$$

We also have the relationship

$$v_2(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$$

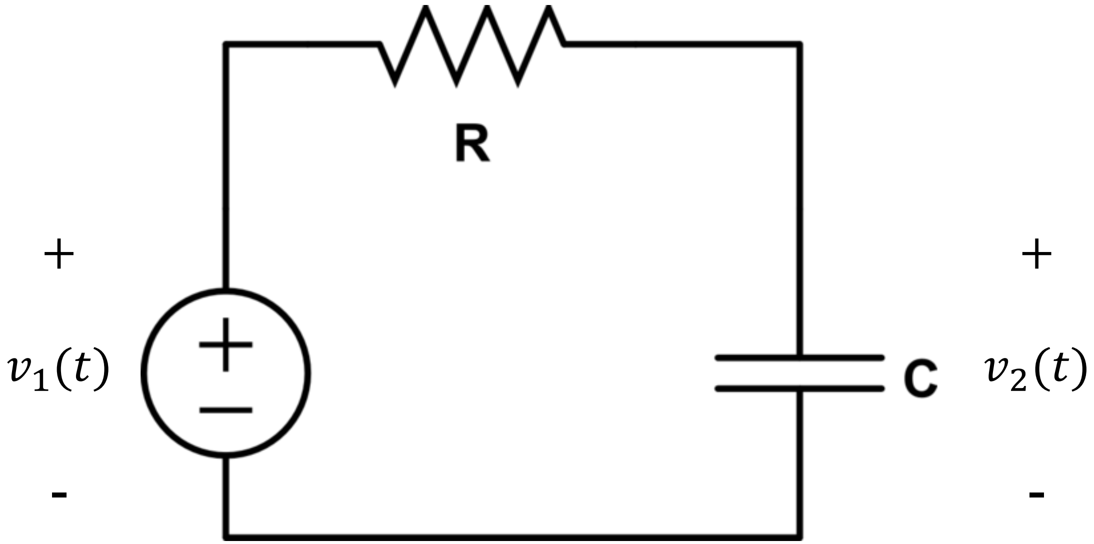
(assuming no initial charge)

Differentiate v_2

$$\dot{v}_2 = \frac{1}{C} i(t)$$

Substitute in $i(t)$

$$v_1(t) = RC\dot{v}_2 + v_2(t)$$



Modelling a Mechanical System

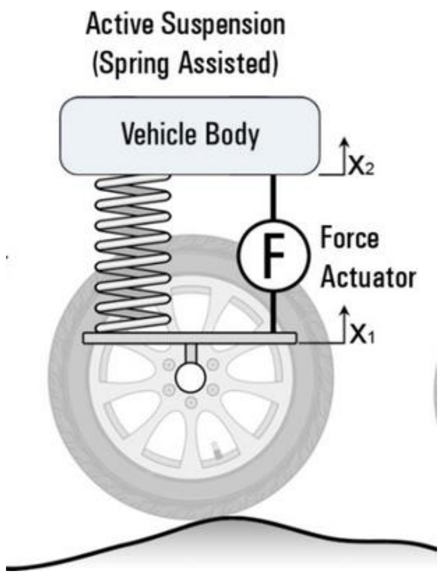
Consider the simple spring mass system

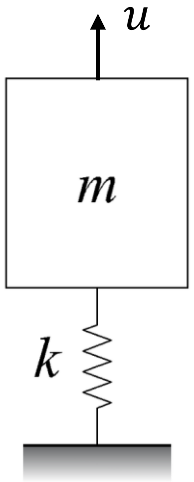
Newton's second law (translational motion):

$$F = ma = spring\ force\ +\ external\ force$$

$$spring\ force = -kx$$

$$m\ddot{x} = -kx + u$$





How does all this relate to dynamic systems?

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

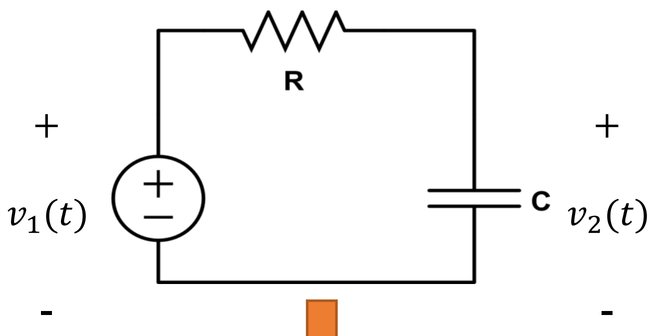
$$s = \sigma + j\omega$$

- Dynamic systems are modelled as differential equations.
- Linear differential equations with constant coefficients have solutions that either exponentials or sinusoids.

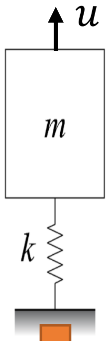
$$\dot{x} = -ax \Rightarrow x(t) = Ce^{-at}$$

$$\ddot{x} = -bx \Rightarrow x(t) = C\sin(\sqrt{b}t)$$

Example: oscillating mass-spring system



$$\dot{v}_2 = -\frac{1}{RC}v_2(t) + \frac{1}{RC}v_1(t)$$



$$m\ddot{x} = -kx + u$$

More on Laplace

What is the relationship between $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{\dot{f}(t)\}$?

$$\mathcal{L}\{\dot{f}(t)\} = \int_0^\infty \dot{f}(t)e^{-st}dt$$

$$= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$$

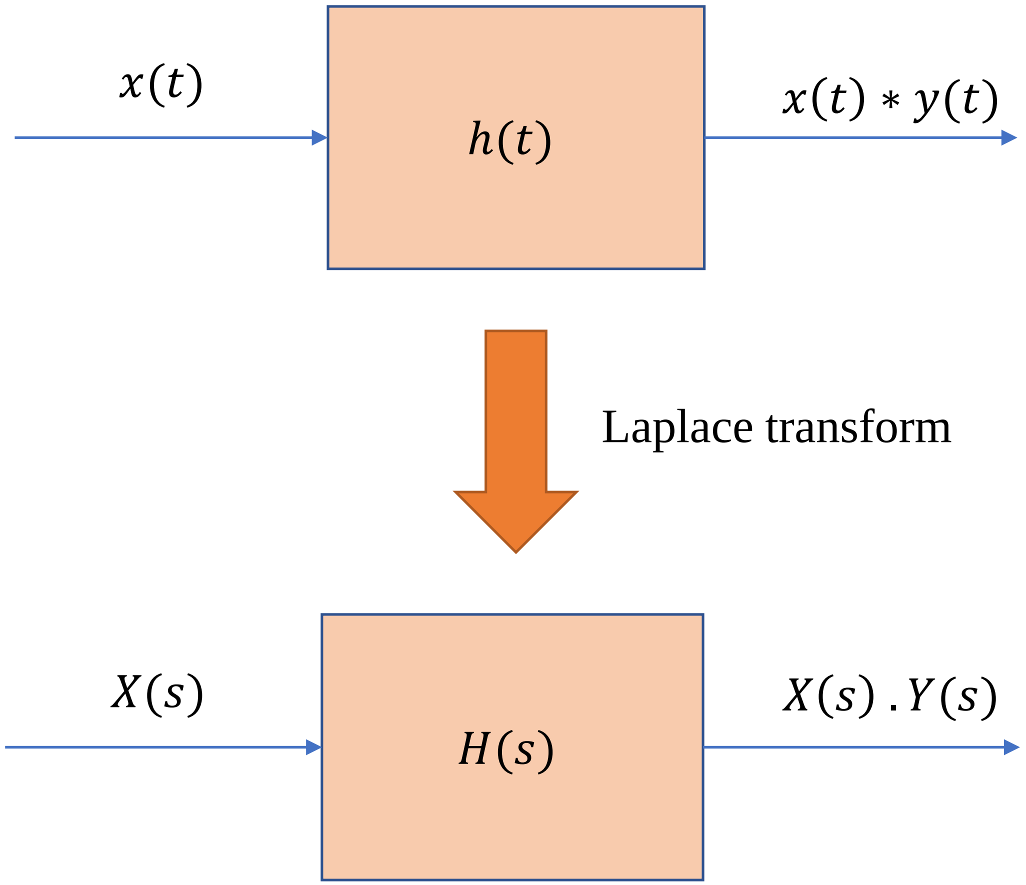
$$= -f(0) + s\int_0^\infty f(t)e^{-st}dt$$

$$= s\mathcal{L}\{f(t)\} - f(0)$$

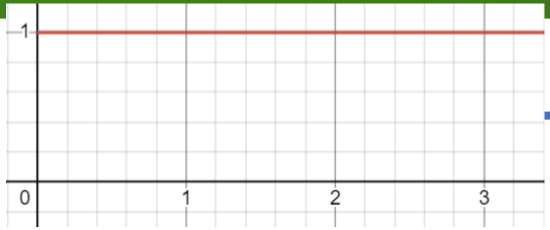
In General

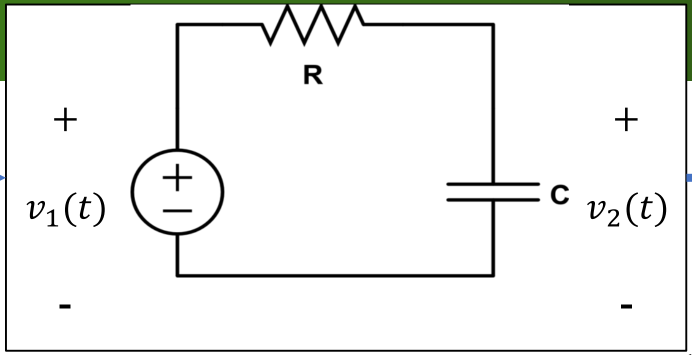
$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} + \sum_{k=0}^{n-1} -s^{n-1-k}f^{(k)}(0)$$

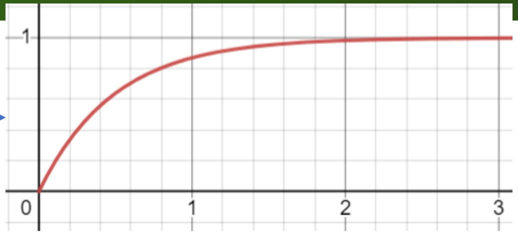
More on Laplace



Example







How would $v_2(t)$ look like if you suddenly give it 1 V?

$v_1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$ and $v_2(0) = 0$?

$$\dot{v}_2 = -\frac{1}{RC} v_2(t) + \frac{1}{RC} v_1(t)$$

We take the Laplace of both sides:

$$sV_2(s) - v_2(0) = -\frac{1}{RC} V_2(s) + \frac{1}{RC} V_1(s)$$

$$\left(s + \frac{1}{RC}\right) V_2(s) = \frac{1}{RC} V_1(s) + v_2(0)$$

We have shown that $V_1(s) = \frac{1}{s}$ for $Re(s) > 0$.

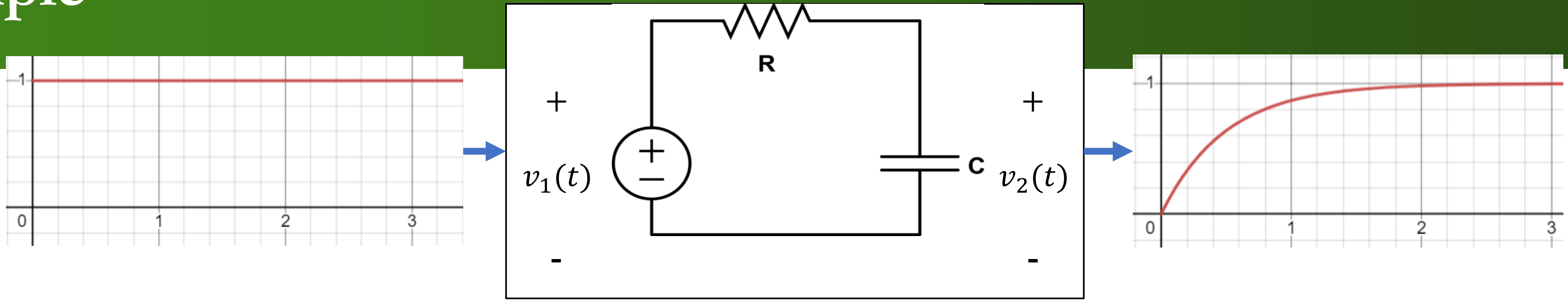
$$\left(s + \frac{1}{RC}\right) V_2(s) = \frac{1}{RCs}$$

$$\Rightarrow V_2(s) = \frac{1}{s(RCs+1)} = \frac{1}{s} - \frac{1}{s + \frac{1}{RC}}$$

$$v_2(t) = \mathcal{L}^{-1}\{V_2(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s + \frac{1}{RC}}\right\}$$
$$= 1 - e^{-\frac{1}{RC}t}$$

Transient response vs steady-state response ($t \to \infty$)

Example



$$\dot{v}_2 = -\frac{1}{RC}v_2(t) + \frac{1}{RC}v_1(t)$$

We take the Laplace of both sides:

$$sV_2(s) - v_2(0) = -\frac{1}{RC}V_2(s) + \frac{1}{RC}V_1(s)$$

$$\left(s + \frac{1}{RC}\right)V_2(s) = \frac{1}{RC}V_1(s) + v_2(0)$$

Is there a way to express the system in general?

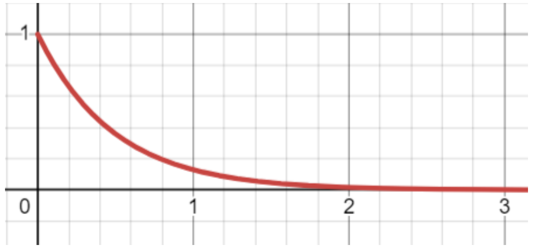
The transfer function:

$$H(s) = \frac{output}{input} = \frac{V_2(s)}{V_1(s)}$$

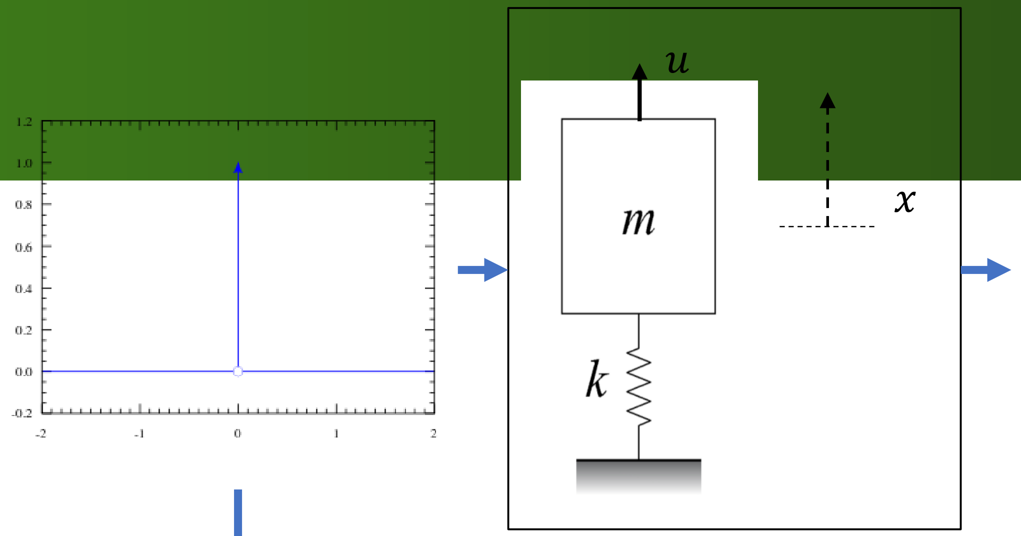
$$H(s) = \frac{1}{RCs + 1}$$

Impulse response:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-\frac{1}{RC}t}$$



Example



$$m\ddot{x} = -kx + u$$

We take the Laplace of both sides:

$$ms^2X(s) - sx(0) - \dot{x}(0) = -kX(s) + U(s)$$

$$(ms^2 + k)X(s) = U(s) + sx(0) + \dot{x}(0)$$

Transfer function:

$$H(s) = \frac{X(s)}{U(s)} = \frac{1}{ms^2 + k}$$

How would $x(t)$ look like if you hit the mass with a hammer?

$u(t) = \delta(t)$ (Dirac Delta function)

$$x(t) = u(t) * h(t) \;\Rightarrow\; X(s) = U(s).H(s)$$

We have $U(s) = 1 \;\Longrightarrow\; X(s) = \frac{1}{ms^2+k} = \frac{\frac{1}{m}}{s^2+\frac{k}{m,}}$

$$x(t) = \frac{1}{\sqrt{mk}}.\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$\cos(\omega t)u(t) \Leftrightarrow \frac{s}{s^2+\omega^2}$$

$$\sin(\omega t)u(t) \Leftrightarrow \frac{\omega}{s^2+\omega^2}$$

System Poles and Zeros

- For a Transfer function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$H(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2)\ldots(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_{n-1})(s - p_n)}$$

- The zeros are the roots of
- The poles are the roots of

$$N(s) = 0 \implies z_i's$$

- The gain constant

$$D(s) = 0 \implies p_i's$$

$$K = b_m / a_n$$

The **poles** and **zeros** are properties of the transfer function, and therefore of the differential equation describing the input-output system dynamics.

Together with the gain constant K **they completely characterize the differential equation, and provide a complete description of the system**.

Example 1

- Consider a linear system that is described by the differential equation

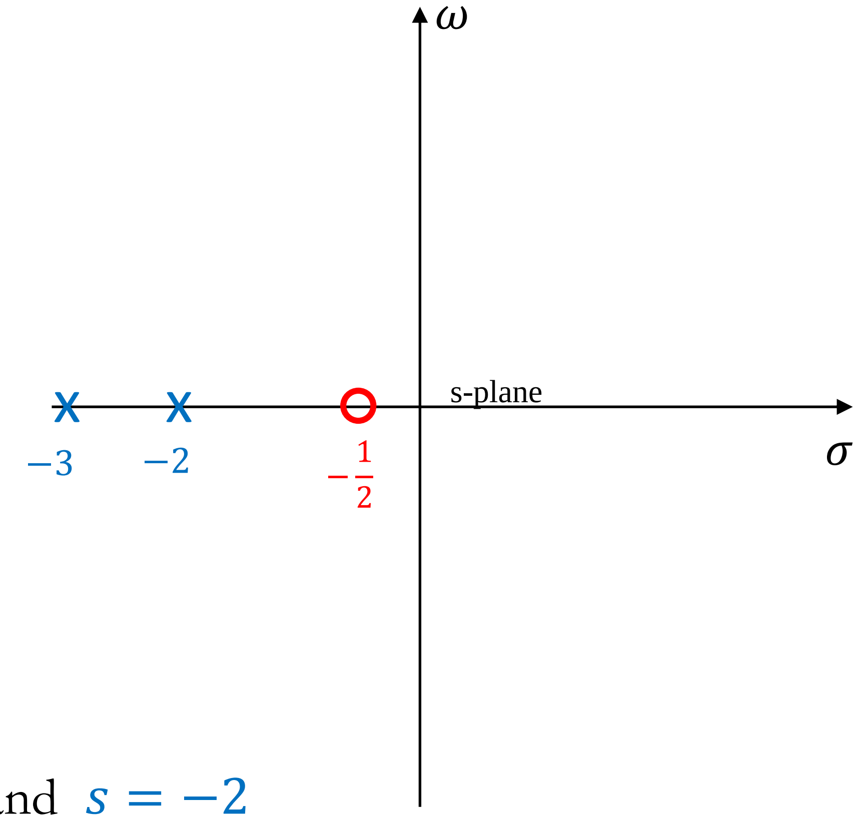
- Find the system poles and zeros

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 2\frac{du}{dt} + u$$

$$H(s) = \frac{2s+1}{s^2+5s+6}$$

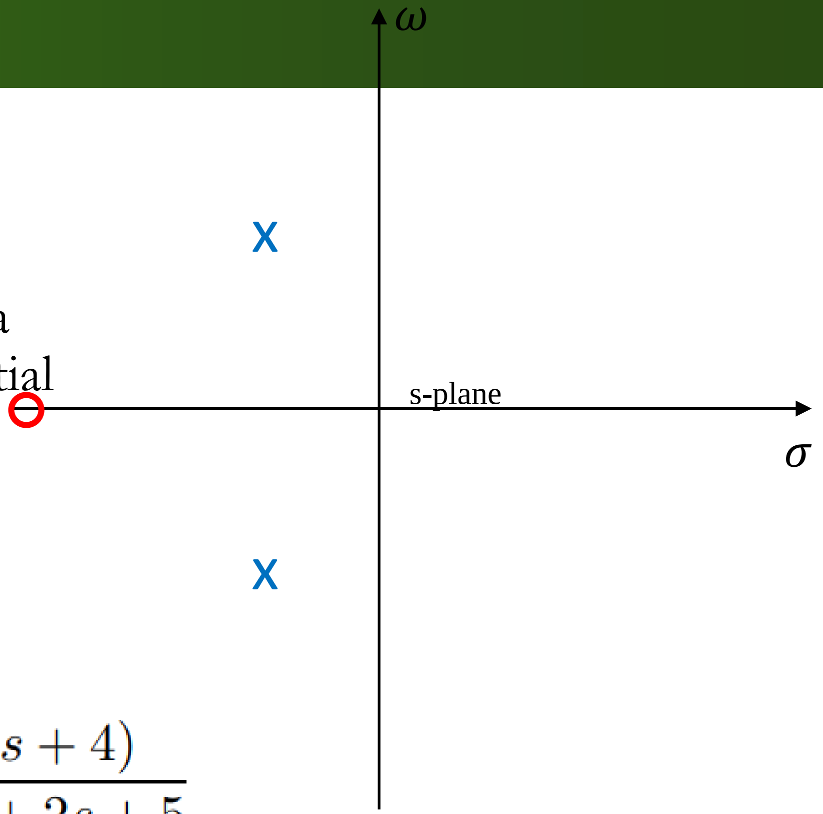
$$= \frac{1}{2}\frac{s+1/2}{(s+3)(s+2)}$$

- The system has a single zero at $s = -\frac{1}{2}$, and a pair of poles at $s = -3$ and $s = -2$



Example 2

- A system has a pair of complex conjugate poles $p_1, p_2 = -1 \pm j2$, a single real zero $z_1 = -4$, and a gain factor $K = 3$. Find the differential equation representing the system.



The transfer function is

$$\begin{aligned} H(s) &= K\frac{s-z}{(s-p_1)(s-p_2)} \\ &= 3\frac{s-(-4)}{(s-(-1+j2))(s-(-1-j2))} = 3\frac{(s+4)}{s^2+2s+5} \end{aligned}$$

and the differential equation is

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=3\frac{du}{dt}+12u$$