Recitation 2

Problem 1.

Prove the theorem

for all real values with first the well ordering principle, then induction.

*Proof.* By contradiction and use of the Well Ordering Principle. Assume that the theorem is false. Then, some nonnegative integers serve as counterexamples. They can be collected in the set

By our assumption, *C* is a nonempty set, and by the Well Ordering Principle it has a minimum element, call it *c*. But holds. So *c* must be greater than zero. Thus is a nonnegative integer and must be true. So

But then when we add the *c*th term to both sides we get

which means that does hold, which creates a contradiction, completing the proof.

*Proof.* By induction. Let be the theorem above for all nonnegative integers . We must show that holds by reaching

*Base Case*.

The theorem holds for zero and we are done.

*Ind. Step.* We assume and attempt to show . Given

we can add the next term to both sides getting

and we are done.∎

Problem 2.

*Proof.* By induction. Let be that on day the following are true:

1. If then no one will leave the island that day.
2. If , then contestants will leave the island that day.
3. If , then all contestants will have already left the island, and no more will leave that day.

*Base case.* On the first day, given that there is at least one person with a purple eye, there are two possible cases. In the first case, there is one purple eye, and . Since that person can see no purple eyes, and that person knows there is at least one purple eye, then that person must conclude that they have a purple eye, and therefore contestants leave the island that day, holding part two of the theorem. In the second case, there are more than one contestants with purple eyes. Each of these contestants can see at least one other purple eye. If one contestant sees only one other purple eye, then it is still possible for that person to be leaving the island later that day, and thus it is still possible that the second person does not see a purple eye on the first person’s forehead. Otherwise, the contests with purple eyes all see several contestants with purple eyes, and it is still logical for any of them to presume they might still have a red eye. The second part of the theorem then holds. Thus all cases hold and the base case is true.

*Inductive Step*. Let be the inductive hypothesis. We will show that holds true for all three of its cases.

Case 1. It is the day and . If , then contestants would have left the island that day. Each contestant with a purple eye sees exactly contestants with purple eyes. But, if there were contestants with purple eyes, those contestants would have left the island the day before, according to . Therefore there must be more than purple eyes, and the only additional person who could have a purple eye is the person the observing contestant cannot observe: itself. Being master logicians, every purple-eyed contestant must come to this conclusion, and all contestants leave the island, and part 2 of holds.

Case 2. It is the day and Each purple-eyed contestant can see at least other purple eyes. Being master logicians, each observing purple-eyed contestant has considered Case 1, and each knows that even if there were purple eyes and the observer was the nd purple eye that it could not know until tomorrow that Case 1 was not true. So any purple-eyed contestants must wait one more day before possibly leaving, and no one leaves the island in this case, holding part 1 of .

Case 3. It is the day and +1 According to part 3 of , for all values of less than all of the purple eyes would have already left the island. By part 2 of , if then all of the purple eyes would have left yesterday. Since , there is no amount of for which any purple-eyed contestants would be remaining on this day. Therefore part 3 of holds.

As all cases hold, shows and the proof is complete.