Machine Learning homework 6 solution

Constrained Optimization and SVM

Wiktor Jurasz - M.Nr. 03709419

December 1, 2018

1 Problem 1

1.1 Strong Duality

$$f_0(\theta) = \theta_1 - \sqrt{3}\theta_2 \tag{1}$$

$$f_1(\theta) = \theta_1^2 + \theta_2^2 - 4 \tag{2}$$

1. f_0 and f_1 are convex (sum of convex functions)
2. There exists θ (e.g $[0, 0]$) for which $f_1 < 0$

From 1 and 2 $\Rightarrow$ Slater's constraint qualification is fulfilled $\Rightarrow$ Strong duality holds

1.2 f_0 Minimum

$$L(\theta_1, \theta_2, \alpha) = \theta_1 - \sqrt{3}\theta_2 + \alpha(\theta_1^2 + \theta_2^2 - 4) \tag{3}$$

$$\nabla_\theta L(\theta_1, \theta_2, \alpha) = [1 + 2\alpha\theta_1, -\sqrt{3} + 2\alpha\theta_2] \tag{4}$$

$$\nabla_\theta L(\theta_1, \theta_2, \alpha) = 0 \Leftrightarrow \theta_1 = \frac{-1}{2\alpha} \wedge \theta_2 = \frac{\sqrt{3}}{2\alpha} \tag{5}$$

$$\theta^*(\alpha) = \underset{\theta}{\operatorname{argmin}}\, L(\theta_1, \theta_2, \alpha) = [\frac{-1}{2\alpha}, \frac{\sqrt{3}}{2\alpha}] \tag{6}$$

$$g(\alpha) = L(\theta^*(\alpha), \alpha) = -\frac{1}{2\alpha} - \frac{3}{2\alpha} + \frac{1}{4\alpha} + \frac{3}{4\alpha} - 4\alpha = -\frac{1}{\alpha} - 4\alpha \tag{7}$$

$$\begin{gathered} g'(\alpha) = \alpha^{-2} - 4 = 0 \Leftrightarrow \alpha = \frac{1}{2} \\ g''(\alpha) = -2\alpha^{-3} \\ g''(\frac{1}{2}) = -16 < 0 \Rightarrow max(g(\alpha)) = -4 = min_\theta(L(\theta_1, \theta_2, \alpha)) \end{gathered} \tag{8}$$

Now we can go back to the gradient of L to determine θ for which f_0 has minimum.

$$\begin{gathered} \theta_1 = \frac{-1}{2 * \frac{1}{2}} = -1 \\ \theta_2 = \frac{\sqrt{3}}{2 * \frac{1}{2}} = \sqrt{3} \end{gathered} \tag{9}$$

Because Strong duality holds we can say that f_0 subject to $f_1 < 0$ has minimum at $[-1, \sqrt{3}]$ with value -4