

Extremal Combinatorics Notes

Wilson Pan

February 18, 2026

Contents

1 Turán Numbers and Mantel's Theorem (Jan 12)	3
1.1 Basic Definitions	3
1.2 Crude Turán's Theorem	3
1.3 Mantel's and Turán's Theorems	3
2 Turán Density and Supersaturation (Jan 14)	5
2.1 Inductive Proof of Turán's Theorem	5
2.2 Monotonicity of Extremal Density	5
2.3 Counting Copies Above Threshold	5
3 Erdős–Stone and Kővári–Sós–Turán (Jan 16)	7
3.1 Erdős–Stone–Simonovits Theorem	7
3.2 Kővári–Sós–Turán Bound	7
3.3 Unit Distance Problem	8
3.4 KST for 3-Uniform Hypergraphs	9
4 Supersaturation and Erdős–Stone (Jan 21)	10
4.1 Supersaturation Lemma	10
4.2 Hypergraph Kővári–Sós–Turán	10
4.3 The Erdős–Stone Theorem	11
4.4 Degenerate Turán Numbers (Bipartite Graphs)	11
4.5 Extremal Numbers for Trees	12
5 Cycles, Girth, and Paths (Jan 23)	14
5.1 Turán Numbers for Cycles and Girth	14
5.2 Turán Numbers for Paths	14
6 Projective Planes and C_{2k}-Free Graphs (Jan 26)	16
6.1 Projective Plane over a Field	16
6.2 C_4 -Free Graph from $\mathbb{P}^2(\mathbb{F}_q)$	16
6.3 General Projective Planes	17
6.4 C_{2k} -Free Graphs for $k = 2, 3, 5$	17
7 Algebraic Construction for $ex(n, K_{s,t})$ (Jan 28)	20
7.1 Random Polynomial Construction	20
8 Regularity for Binary Words (Feb 4)	22
8.1 Definitions and Chernoff Bound	22
9 Regularity Lemma for Binary Words (Feb 6)	23
9.1 The Regularity Lemma	23
9.2 Twin Subsequences	24
10 Szemerédi's Graph Regularity Lemma (Feb 13)	25
10.1 ϵ -Regular Pairs and Partitions	25
10.2 Proof of the Regularity Lemma	25

11 Triangle Counting and Removal Lemmas (Feb 16)	27
11.1 Triangle Counting Lemma	27
11.2 Triangle Removal Lemma	27
12 Roth's Theorem(Feb 18)	28

1 Turán Numbers and Mantel's Theorem (Jan 12)

1.1 Basic Definitions

Definition 1.1. *Turan Numbers (Forbidden Subgraph Problems):* F is a graph and G graph is F -free if G contains no copy of F as a subgraph.

We want to maximize the size of G subject to G being F -free. Where $\text{size}=e(G) = \# \text{ edges in } G$

Definition 1.2. $ex(n, F) = \max\{e(G) | G \text{ is } F\text{-free and } G \text{ is } n\text{-vertex graph}\}$

1.2 Crude Turán's Theorem

Theorem 1.3. *Crude Turán's Theorem:* If G is n -vertex graph,

$$e(G) = \epsilon \binom{n}{2}.$$

then G contains an independent set of size greater than or equal $\frac{1}{2} \frac{1}{\epsilon} + \frac{1}{2}$
In particular, if $e(G) \geq (1 - \epsilon) \binom{n}{2}$ edges, then G contains a clique on greater than or equal $\frac{1}{2\epsilon} + \frac{1}{2}$ vertices.

$$ex(n, K_t) \leq \left(1 - \frac{1}{2t-1}\right) \binom{n}{2}.$$

Proof. Pick k vertices in G at random without replacement. Say we pick v_1, \dots, v_k
For each edge between v_i and v_j , delete v_i .

Let $X = \# \text{ deleted vertices}$

Note: $X \leq e(G[v_1, \dots, v_k]) =: Y$

$$\mathbb{E}[Y] = \sum_{1 \leq i \leq j \leq k} \mathbb{P}[v_i v_j \in e(G)] = \binom{k}{2} \frac{\epsilon \binom{n}{2}}{\binom{n}{2}} s = \epsilon \binom{k}{2}.$$

Thus, there is a choice of v_1, \dots, v_k such that $Y \leq \epsilon \binom{k}{2}$. Fix it. Let $\alpha(G) = \text{size of largest independent set in } G$ satisfies $\alpha(G) \geq k - Y \geq k - \epsilon \binom{k}{2}$

$$\left(k - \epsilon \binom{k}{2}\right)' = 1 + \frac{\epsilon}{2} - \epsilon k.$$

Set $k = \frac{1+\epsilon}{2} + \delta$ for some $|\delta| \leq \frac{1}{2}$.

$$\begin{aligned} \alpha(G) &\geq \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon \left(\left(\frac{1}{\epsilon} + \frac{1}{2} + \delta \right) \left(\frac{1}{\epsilon} - \frac{1}{2} + \delta \right) \right) / 2 \\ &= \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon \frac{(\frac{1}{\epsilon} + \delta)^2 - \frac{1}{4}}{2} \\ &= \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon \frac{(1/\epsilon)^2 + 2\frac{1}{\epsilon}\delta + \delta^2 - \frac{1}{4}}{2} \\ &= \frac{1}{2\epsilon} + \frac{1}{2} + \frac{\frac{1}{4} - \delta^2}{2} \geq \frac{1}{2\epsilon} + \frac{1}{2} \end{aligned}$$

□

1.3 Mantel's and Turán's Theorems

Remark 1.4. Recall: k -uniform hypergraph on vertex set V is a subset of

$$\binom{V}{k} := \{e \subseteq V : |e| = k\}.$$

Example 1.5. In k -uniform hypergraph on vertex with $\epsilon \binom{n}{k}$ edges there is an independent set of size
 $\geq c_k \frac{1}{\epsilon}^{\frac{1}{k-1}}$

Theorem 1.6. Mantel's Theorem: $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$

More generally, define Turan's graph $T_{n,r}$ to be the n -vertex complete r -partite graph with part sizes as equal as possible.

Note: $K_{r+1} \not\subseteq T_{n,r}$ because among $r+1$ vertices some pair is in same part.

Theorem 1.7. Turan's Theorem: $ex(n, K_{r+1}) = e(T_{n,r})$ and furthermore $T_{n,r}$ is the only extremizer.

$$e(T_{n,r}) = \left(1 - \frac{1}{r}\right) \binom{n}{2} + O(n).$$

Proof. Let G be a graph on n vertices that is K_{r+1} -free and has the maximum possible number of edges (an extremizer). We proceed in two steps.

Suppose G is not a complete multipartite graph. Then non-adjacency is not an equivalence relation, implying there exist two non-adjacent vertices x, y that do not have identical neighborhoods.

Assume $\deg(x) \leq \deg(y)$. Delete x and then clone y to obtain y' . The new graph G' has

$$e(G') = e(G) - \deg(x) + \deg(y) \geq e(G).$$

Also, G' is K_{r+1} -free because any copy of K_{r+1} has at most one of y or y' (since they are non-adjacent). If a clique contains y' , it can be swapped for y (since $N(y') \subseteq N(y)$), which would imply $K_{r+1} \subseteq G$, a contradiction.

Repeating this process transforms G into a complete multipartite graph without decreasing edges. Thus, the extremizer must be a complete multipartite graph.

Since G is K_{r+1} -free, it must be k -partite with $k \leq r$. To maximize edges, we set $k = r$. Let the parts have sizes n_1, \dots, n_r .

$$e(G) = \binom{n}{2} - \sum_{i=1}^r \binom{n_i}{2}$$

To maximize $e(G)$, we must minimize the subtracted sum of pairs inside the parts. This occurs when n_i are as equal as possible ($|n_i - n_j| \leq 1$). If $n_i \geq n_j + 2$, moving a vertex from part i to part j increases the edge count by $(n_i - 1) - n_j > 0$.

Thus, the unique extremizer is the balanced complete r -partite graph, $T_{n,r}$. □

2 Turán Density and Supersaturation (Jan 14)

2.1 Inductive Proof of Turán's Theorem

Proof. Second proof of Turan's Theorem with induction on r .

(Induction Step) Let x be a vertex of largest degree, let $A = N(x)$ be its neighborhood.

$$B := V(G) \setminus A.$$

We have $e(G) = e(A) + e(A, B) + e(B)$ and A is K_{r-1} -free so $e(A) \leq e(T_{|A|, r})$

Each vertex in B has $\deg \leq \deg(x) = |A|$

So,

$$e(A, B) + e(B) \leq \sum_{v \in B} \deg(v) \leq |B||A|.$$

So,

$$e(G) \leq e(T_{|A|, r}) + |A||B| \leq e(T_{n, r}).$$

The right hand side is equal to number of edges in complete r -partite graph with one part of size $|B|$ and other being as equal as possible.

Equality holds iff B is independent and A must be a copy of $e(T_{|A|, r-1})$. Hence G is $T_{n, r}$ for equality to hold. \square

2.2 Monotonicity of Extremal Density

Theorem 2.1. *Observe that if F is a graph then*

$$\frac{ex(n, F)}{\binom{n}{2}} \geq \frac{ex(n+1, F)}{\binom{n+1}{2}}.$$

Proof. Pick n vertices in an F -free $(n+1)$ vertex graph G at random without replacement; let G' be the induced graph on these n vertices.

$$e(G') \leq ex(n, F).$$

$$\mathbb{E}(e(G')) = \binom{n}{2} \frac{e(G)}{\binom{n+1}{2}} \implies \frac{e(G)}{\binom{n+1}{2}} \leq \frac{ex(n, F)}{\binom{n}{2}}.$$

\square

Definition 2.2.

$$\pi(F) := \lim_{n \rightarrow \infty} \frac{ex(n, F)}{\binom{n}{2}}.$$

The Turan density of F is

$$\pi(K_{r+1}) = 1 - \frac{1}{r}.$$

2.3 Counting Copies Above Threshold

Theorem 2.3. *Let F be any graph and let $\epsilon > 0$. Let G be an n -vertex graph. There is a $\delta > 0$ such that*

$$e(G) \geq (\pi(F) + \epsilon) \binom{n}{2} \text{ and } n \geq n_0(F, \epsilon).$$

then there are at least $\delta n^{|V(F)|}$ copies of F .

Proof. Let n_0 be large enough such that $\frac{ex(n_0, F)}{\binom{n_0}{2}} \leq \pi(F) + \frac{\epsilon}{2}$. Given n -vertex G as in theorem. Pick n_0 vertices at random without replacement. Let G' be the n_0 -vertex graph induced on these chosen vertices.

$$\mathbb{E}[e(G')] = \frac{e(G)}{\binom{n}{2}} \binom{n_0}{2} \geq (\pi(F) + \epsilon) \binom{n_0}{2}.$$

Say G' is good if

$$e(G') \geq \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2}.$$

$$\begin{aligned} \mathbb{E}(e(G')) &\leq \mathbb{P}(G' \text{ is bad}) \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2} + \mathbb{P}(G' \text{ is good}) \binom{n_0}{2} \\ &\leq \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2} + \mathbb{P}(G' \text{ is good}) \binom{n_0}{2} \end{aligned}$$

Then $\frac{\epsilon}{2} \binom{n_0}{2} \leq \mathbb{P}(G' \text{ is good}) \binom{n_0}{2}$ so $P(G' \text{ is good}) \geq \frac{\epsilon}{2}$
Pick $|V(F)|$ many vertices without replacement inside G' , call the subgraph they induce G'' . So

$$G \supset G' \supset G''.$$

$$\mathbb{P}(G'' \text{ contains } F) \geq \mathbb{P}(G' \text{ is good}) \cdot \mathbb{P}(G'' \text{ contains } F | G' \text{ is good}) \geq \frac{\epsilon}{2} \frac{1}{\binom{n_0}{|V(F)|}}.$$

Note that G'' is uniformly randomly chosen $|V(F)|$ -vertex subset of G . This implies the # copies of F in $G \geq \mathbb{P}(G'' \text{ contains } F) \binom{n}{|V(F)|} \geq P(\dots) \frac{n^{|V(F)|} (1-o(1))}{|V(F)|!}$. We can choose $\delta = \frac{1}{\binom{n_0}{|V(F)|}}$. \square

3 Erdős–Stone and Kővári–Sós–Turán (Jan 16)

3.1 Erdős–Stone–Simonovits Theorem

Theorem 3.1 (Erdős–Stone–Simonovits). *Let $T_{n,r}$ denote the Turán graph. We know that:*

$$ex(n, T_{s,(r+1)}) = \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}.$$

This implies that if H is any graph with chromatic number $\chi(H) = r + 1$, then

$$ex(n, H) \leq \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}.$$

Conversely, if $\chi(H) > r$ (i.e., H is not r -partite), then $T_{n,r}$ is H -free. Hence,

$$ex(n, H) \geq e(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}.$$

Combining these, for any non-bipartite graph H :

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) \binom{n}{2}.$$

It is important to note that this theorem holds for bipartite graphs as well, where $\chi(H) = 2$. However, substituting $r = \chi(H) - 1 = 1$ gives a coefficient of $1 - 1/1 = 0$. This implies that $ex(n, H) = o(n^2)$, but it does not provide the precise asymptotic behavior (e.g., $n^{2-\epsilon}$). Therefore, while valid, the Erdős–Stone–Simonovits theorem is not the primary tool for determining extremal numbers of bipartite graphs; we rely on theorems like Kővári–Sós–Turán for those bounds.

3.2 Kővári–Sós–Turán Bound

Theorem 3.2 (Kővári–Sós–Turán 1954).

$$ex(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{\frac{1}{s}} \cdot n^{2-\frac{1}{s}} + O(n).$$

Proof. Let G be a $K_{s,t}$ -free graph with n vertices. We will count X , the number of copies of $K_{1,s}$ (stars with s leaves).



We count this in two ways:

1. Since there is no $K_{s,t}$, any set of s vertices can have at most $t-1$ common neighbors. Thus:

$$X \leq \binom{n}{s} (t-1).$$

2. Let d_v be the degree of vertex v . Then:

$$X = \sum_{v \in V(G)} \binom{d_v}{s}.$$

A simple-minded application of Jensen's inequality here fails because the binomial coefficient $\binom{x}{s}$ is not convex for all x . To fix this, we define the function:

$$f_s(x) = \begin{cases} \binom{x}{s} & \text{if } x \geq s - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f_s(x)$ is convex for all x because its derivative:

$$\frac{d}{dx} f_s(x) = \begin{cases} \left(\frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x-s+1} \right) \binom{x}{s} & \text{if } x > s - 1 \\ 0 & \text{otherwise} \end{cases}$$

is non-decreasing. Since d_v is an integer, $\binom{d_v}{s} = f_s(d_v)$. We can now validly apply Jensen's inequality:

$$X = \sum_{v \in V} f_s(d_v) \geq n f_s(d_{\text{avg}}) = n f_s \left(\frac{2m}{n} \right),$$

where $m = e(G)$.

Combining (1) and (2):

$$n f_s \left(\frac{2m}{n} \right) \leq \binom{n}{s} (t - 1).$$

If $m \leq \frac{1}{2}(s - 1)n$, the bound holds trivially. Otherwise, we approximate $\binom{x}{s} \approx \frac{x^s}{s!}$:

$$n \frac{\left(\frac{2m}{n} - s \right)^s}{s!} \leq (t - 1) \frac{n^s}{s!}.$$

Rearranging yields:

$$\frac{2m}{n} \leq (t - 1)^{\frac{1}{s}} n^{1 - \frac{1}{s}} + s \implies m \leq \frac{1}{2} (t - 1)^{\frac{1}{s}} n^{2 - \frac{1}{s}} + O(n).$$

□

3.3 Unit Distance Problem

Example 3.3 (Unit Distance Problem). *How can we place n points in \mathbb{R}^2 to maximize the number of pairs (p, q) such that $\|p - q\| = 1$?*

A strong construction is a $\sqrt{n} \times \sqrt{n}$ grid. Rather than scaling the grid to unit step size, we scale it such that the distance 1 corresponds to a distance d in the integer grid that occurs most frequently.

The possible distances in an unscaled integer grid are $\sqrt{a^2 + b^2}$ for $0 \leq a, b < \sqrt{n}$. By number theory (Landau-Ramanujan), integers representable as a sum of two squares are those where primes $p \equiv 3 \pmod{4}$ appear with even exponents.

To get a better bound, we look for "highly composite" integers. Specifically, if we choose a number d composed of a product of many distinct primes $p \equiv 1 \pmod{4}$, the number of ways to write d as a sum of two squares is very large. Erdős used this to show there exists a distance occurring:

$$n^{1 + \frac{c}{\log \log n}}.$$

This is significantly larger than the $n\sqrt{\log n}$ bound derived from the Pigeonhole Principle on generic integers.

Theorem 3.4 (Upper Bound for Unit Distances). *The number of unit distances is $O(n^{3/2})$.*

Proof. Given n points in \mathbb{R}^2 , construct a graph where vertices are points and $v \sim u$ if $\|v - u\| = 1$. Observe the graph is $K_{2,3}$ -free. Geometrically, this is because two circles with radius 1 intersect at most 2 times; thus, no 2 vertices can share 3 common neighbors.

By KST with $s = 2, t = 3$:

$$e(G) \leq \frac{1}{2}\sqrt{3-1} \cdot n^{2-\frac{1}{2}} + o(n^2) = \frac{1}{\sqrt{2}}n^{\frac{3}{2}} + o(n^2).$$

□

3.4 KST for 3-Uniform Hypergraphs

Definition 3.5. Define $K_{s_1, s_2, t}^{(3)}$ to be the complete 3-uniform, 3-partite hypergraph with parts of size s_1, s_2, t .

Theorem 3.6 (Generalization of KST to Hypergraphs). *For the 3-uniform case:*

$$ex(n, K_{s_1, s_2, t}^{(3)}) \leq C_{s_1, s_2, t} n^{3-\frac{1}{s_1 s_2}} + O(n^2).$$

Proof. We will count the number of stars $K_{1,1,s_1}$ (sets of edges sharing a common pair of vertices). Let X be this number. For any pair $\{v_1, v_2\}$, let d_{v_1, v_2} be the co-degree (number of edges containing $\{v_1, v_2\}$).

1. **Upper Bound on X:** Fix a set U of size s_1 . Define an auxiliary graph G_U on $V(G)$ where $v_1 \sim v_2$ in G_U if $\{v_1, v_2, u\} \in E(G)$ for all $u \in U$. If G_U contains a copy of $K_{s_2, t}$, then combined with U , we form a $K_{s_1, s_2, t}^{(3)}$. Thus, G_U must be $K_{s_2, t}$ -free.

$$X \leq \binom{n}{s_1} ex(n, K_{s_2, t}) \leq C \binom{n}{s_1} n^{2-\frac{1}{s_2}}.$$

2. **Lower Bound on X:** Summing over pairs:

$$X = \sum_{\{v_1, v_2\} \in \binom{V}{2}} \binom{d_{v_1, v_2}}{s_1} \geq \binom{n}{2} f_{s_1} \left(\frac{3m}{\binom{n}{2}} \right).$$

Combining bounds (assuming m is large enough):

$$\binom{n}{2} \frac{\left(\frac{3m}{n^2/2}\right)^{s_1}}{s_1!} \lesssim \frac{n^{s_1}}{s_1!} n^{2-\frac{1}{s_2}}.$$

Simplifying:

$$n^2 \left(\frac{m}{n^2}\right)^{s_1} \lesssim n^{s_1+2-\frac{1}{s_2}} \implies m^{s_1} \lesssim n^{3s_1-\frac{1}{s_2}}.$$

Taking the s_1 -th root:

$$m \leq C n^{3-\frac{1}{s_1 s_2}}.$$

□

4 Supersaturation and Erdős–Stone (Jan 21)

4.1 Supersaturation Lemma

The classical Turán problem asks for the maximum number of edges in a graph avoiding a subgraph F . Supersaturation asks: what happens if we have just *slightly* more edges than this threshold? It turns out we don't just get one copy of F , but exponentially many.

Lemma 4.1 (Supersaturation Lemma). *If G is a graph on n vertices with $e(G) \geq (\pi(F) + \epsilon) \binom{n}{2}$ edges, then G contains at least $\delta(\epsilon)n^{|V(F)|}$ copies of F , where $\delta(\epsilon) > 0$ is a constant independent of n .*

Corollary 4.2. *For the complete graph K_{r+1} , we know $\pi(K_{r+1}) = 1 - \frac{1}{r}$. Thus:*

$$e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right) \binom{n}{2} \implies G \text{ contains } \delta(\epsilon)n^{r+1} \text{ copies of } K_{r+1}.$$

4.2 Hypergraph Kővári-Sós-Turán

We generalize the KST theorem from bipartite graphs to r -uniform hypergraphs. We are looking for the Turán number of the complete r -partite hypergraph $K_{s_1, \dots, s_r}^{(r)}$. Note that the theorem statement often simplifies the last parameter to t .

Theorem 4.3 (Generalized Kővári-Sós-Turán). *Let $K = K_{s_1, s_2, \dots, s_{r-1}, t}^{(r)}$ be the complete r -partite r -uniform hypergraph with part sizes s_1, \dots, s_{r-1}, t . Then:*

$$ex(n, K) = O_{s,t} \left(n^{r - \frac{1}{s_1 s_2 \cdots s_{r-1}}} \right).$$

Proof. We proceed by induction on the uniformity r . The base case $r = 2$ is the standard graph KST theorem. Assume the bound holds for $(r-1)$ -uniform hypergraphs.

Let G be an r -uniform hypergraph with n vertices and m edges. We define a “hypergraph star” $K_{1,1,\dots,1,s_1}^{(r)}$ as a set of s_1 edges that all share a common core of $r-1$ vertices.

Let X be the number of such stars (copies of $K_{1,1,\dots,1,s_1}^{(r)}$) in G . We double count X .

1. Lower Bound (Convexity): For every subset $S \in \binom{V(G)}{r-1}$ of size $r-1$, let the *co-degree* d_S be the number of edges in G containing S . A set S contributes $\binom{d_S}{s_1}$ stars to the count.

$$X = \sum_{S \in \binom{V(G)}{r-1}} \binom{d_S}{s_1}.$$

Using Jensen's Inequality (and the convexity of $\binom{x}{s_1}$), we bound this sum:

$$X \geq \binom{n}{r-1} \binom{\bar{d}}{s_1}, \quad \text{where } \bar{d} = \frac{\sum d_S}{\binom{n}{r-1}} = \frac{rm}{\binom{n}{r-1}}.$$

Approximating for large n : $\bar{d} \approx \frac{r!m}{n^{r-1}}$ and $\binom{x}{k} \approx \frac{x^k}{k!}$, we get:

$$X \gtrsim n^{r-1} \cdot \left(\frac{m}{n^{r-1}} \right)^{s_1}. \tag{1}$$

2. Upper Bound (Induction): Fix a set of vertices $V_1 = \{v_1, \dots, v_{s_1}\}$. We ask: how many ways can we complete this into a star?

Define an auxiliary $(r-1)$ -uniform hypergraph G_{V_1} on $V(G)$. A set e' of size $r-1$ is an edge in G_{V_1} if $e' \cup \{v_i\} \in E(G)$ for all $i = 1, \dots, s_1$. In other words, e' is in the common neighborhood of all v_i .

If G_{V_1} contains a copy of $K_{s_2, \dots, s_{r-1}, t}^{(r-1)}$, then combined with V_1 , we would form a copy of the forbidden $K_{s_1, \dots, t}^{(r)}$ in G . Thus, G_{V_1} must be free of $K_{s_2, \dots, t}^{(r-1)}$. By the inductive hypothesis:

$$e(G_{V_1}) \leq C \cdot n^{(r-1) - \frac{1}{s_2 \dots s_{r-1}}}.$$

Summing over all choices of V_1 (there are $\binom{n}{s_1}$ such sets):

$$X \leq \binom{n}{s_1} \cdot \max_{V_1} e(G_{V_1}) \lesssim n^{s_1} \cdot n^{r-1 - \frac{1}{s_2 \dots s_{r-1}}}. \quad (2)$$

3. Conclusion: Comparing (1) and (2):

$$n^{r-1} \left(\frac{m}{n^{r-1}} \right)^{s_1} \lesssim n^{s_1 + r - 1 - \frac{1}{s_2 \dots s_{r-1}}}.$$

Rearranging for m :

$$m^{s_1} \lesssim n^{(r-1)s_1} \cdot n^{s_1 - \frac{1}{s_2 \dots s_{r-1}}} = n^{rs_1 - \frac{1}{s_2 \dots s_{r-1}}}.$$

Taking the s_1 -th root:

$$m \lesssim n^{r - \frac{1}{s_1 s_2 \dots s_{r-1}}}.$$

□

4.3 The Erdős-Stone Theorem

Definition 4.4 (Blow-up). A **blow-up** of a graph H , denoted $H(s)$, is obtained by replacing each vertex $v \in V(H)$ with an independent set I_v of size s , and replacing each edge $(u, v) \in E(H)$ with a complete bipartite graph between I_u and I_v .

The Turán graph $T_{n,r}$ is essentially a blow-up of K_r with parts of size n/r .

Theorem 4.5 (Erdős-Stone). If $e(G) \geq (1 - \frac{1}{r} + \epsilon) \binom{n}{2}$ and n is sufficiently large, then G contains $T_{s,(r+1)}$ (a blow-up of K_{r+1} with parts of size s).

Consequently, G contains any graph H with $\chi(H) = r+1$ (since any such H is a subgraph of a large enough blow-up of K_{r+1}).

Proof. **Step 1: Find many cliques.** By the Supersaturation Corollary, since the edge density is greater than the Turán threshold for K_{r+1} , G contains many copies of K_{r+1} :

$$\#K_{r+1} \geq \delta n^{r+1}.$$

Step 2: Construct Auxiliary Hypergraph. Define an $(r+1)$ -uniform hypergraph \mathcal{H} with vertex set $V(G)$. Let a set of $r+1$ vertices form an edge in \mathcal{H} if and only if they form a K_{r+1} in the original graph G .

From Step 1, we know $e(\mathcal{H}) \geq \delta n^{r+1}$. Since δ is constant, this is a dense hypergraph.

Step 3: Find a dense structure in the hypergraph. We apply the **Hypergraph KST Theorem** to \mathcal{H} . Since \mathcal{H} is dense (order n^{r+1}), it exceeds the KST threshold (which is roughly $n^{(r+1)-\epsilon}$). Therefore, \mathcal{H} must contain a copy of the complete $(r+1)$ -partite hypergraph $K_{s,s,\dots,s}^{(r+1)}$.

Step 4: Map back to G . Let the parts of this hypergraph copy be U_1, \dots, U_{r+1} , each of size s . The definition of the complete hypergraph implies that for any choice of vertices $u_1 \in U_1, \dots, u_{r+1} \in U_{r+1}$, the set $\{u_1, \dots, u_{r+1}\}$ is an edge in \mathcal{H} .

By definition of \mathcal{H} , this means $\{u_1, \dots, u_{r+1}\}$ forms a clique K_{r+1} in G . If every possible tuple forms a clique, then every pair of vertices in distinct parts U_i, U_j must be connected in G .

Thus, $G[U_1 \cup \dots \cup U_{r+1}]$ contains the complete multipartite graph with parts of size s , which is exactly $T_{s(r+1)}$. □

4.4 Degenerate Turán Numbers (Bipartite Graphs)

When H is bipartite ($r = 1$), the Erdős-Stone bound gives $1 - 1/1 = 0$, which is trivial. We rely on KST bounds $ex(n, K_{s,t}) = O(n^{2-1/s})$. We examine the sharpness of these bounds.

Theorem 4.6 (Tightness of KST).

1. **Case $s = 1$ (Trees/Forests):**

$$ex(n, K_{1,t}) \leq \frac{1}{2}(t-1)n.$$

This is tight for a graph consisting of disjoint copies of K_t (or a matching if $t=1$).

2. **Case $s = 2$ (C_4 -free graphs):** We know $ex(n, K_{2,2}) = O(n^{3/2})$. This is tight.

Construction (Finite Geometry): Let q be a prime power. Construct a bipartite graph G with parts A and B , where $|A| = |B| = q^2$.

- Identify vertices in A and B with points in the affine plane \mathbb{F}_q^2 .
- Define edge $(a,b) \sim (x,y)$ if $ax+by=1$, where $(a,b) \in A$ and $(x,y) \in B$.

For a fixed vertex $v = (a,b) \in A$, its neighborhood is the set of solutions (x,y) to the linear equation $ax+by=1$. This describes a line in \mathbb{F}_q^2 , which contains q points (unless $(a,b) = (0,0)$, which we can discard).

$$e(G) \approx q^2 \cdot q = q^3.$$

Since $n = 2q^2$, we have $q \approx \sqrt{n/2}$, so $e(G) \approx (n/2)^{3/2} = \Theta(n^{3/2})$.

Why is it C_4 -free? A C_4 corresponds to two vertices in A sharing two common neighbors in B . Geometrically, this means two distinct lines intersect at two distinct points. In a plane, two distinct lines intersect at most at one point. Thus, no C_4 exists.

3. **Case $s = 3$ ($K_{3,3}$ -free):** Tightness $ex(n, K_{3,3}) = \Theta(n^{5/3})$ is known.

Construction Sketch: Vertices are points in \mathbb{F}_q^3 . Adjacency is defined by $(x,y,z) \sim (a,b,c)$ if $(x-a)^2 + (y-b)^2 + (z-c)^2 = 1$. Geometrically, the neighborhood of a point is a unit sphere. If G contained $K_{3,3}$, three vertices in one part would share three common neighbors. This would imply that three unit spheres intersect at three points. In Euclidean geometry, three spheres intersect at most at 2 points (a circle intersects a sphere at 2 points). This logic carries over to \mathbb{F}_q provided -1 is not a square.

4. **Case $s = 4$ ($K_{4,4}$ -free):** It is an **open problem** whether the KST bound $O(n^{2-1/4})$ is tight for $K_{4,4}$.

4.5 Extremal Numbers for Trees

Lemma 4.7. If a graph G has average degree d , it contains a subgraph H with minimum degree $\delta(H) \geq d/2$.

Proof. Let $n = |V(G)|$. We have $e(G) = \frac{dn}{2}$. Iteratively delete any vertex with degree strictly less than $d/2$. Let S be the set of removed vertices. The number of edges removed is strictly less than $|S| \cdot \frac{d}{2}$. Even if we remove $n-1$ vertices, the number of edges removed is $< (n-1)\frac{d}{2} < \frac{dn}{2}$. Thus, the edge set cannot be emptied. The process must stop at a non-empty subgraph where every remaining vertex has degree $\geq d/2$. \square

Theorem 4.8. If T is a tree with r edges (so $r+1$ vertices), then:

$$ex(n, T) \leq rn.$$

Proof. Suppose $e(G) > rn$. Then the average degree of G is $d(G) = \frac{2e(G)}{n} > 2r$. By the Lemma, there exists a subgraph $H \subseteq G$ with minimum degree $\delta(H) > r$. Since degrees are integers, $\delta(H) \geq r$.

We embed T into H greedily:

1. Order the vertices of T as v_1, v_2, \dots, v_{r+1} such that each v_i (for $i > 1$) has exactly one neighbor among $\{v_1, \dots, v_{i-1}\}$. (This is always possible by doing a BFS/DFS from a root).

2. Map v_1 to any vertex $u_1 \in V(H)$.
3. Suppose we have embedded v_1, \dots, v_{i-1} as u_1, \dots, u_{i-1} . Let v_j be the parent of v_i in the tree ($j < i$).
4. We need to map v_i to a neighbor of u_j in H .
5. The vertex u_j has at least r neighbors in H . We have used at most $i - 1$ vertices so far. Since $i \leq r + 1$, we have used at most r vertices.
6. We only care that u_j has neighbors not already used in the current tree embedding. The degree of u_j is $\geq r$. The number of currently used vertices is $i - 1 \leq r$.
7. The bound requires strict inequality or careful counting. If $\delta(H) \geq r$, u_j has r neighbors. We have used $i - 1$ vertices total. If $i - 1 < r$, there is space. If $i - 1 = r$, we are placing the last vertex. The neighbors of u_j could be occupied by the other $r - 1$ vertices of the tree.
8. However, the conjecture is actually $\frac{1}{2}(r)n$. The bound rn is loose enough. If $e(G) > rn$, $\text{avg deg} > 2r$, $\min \deg \geq r + 1$. Then u_j has $r + 1$ neighbors, but the tree only has $r + 1$ vertices total. At step i , only $i - 1 \leq r$ vertices are occupied. Thus there is always at least one free neighbor for u_j .

□

5 Cycles, Girth, and Paths (Jan 23)

5.1 Turán Numbers for Cycles and Girth

Definition 5.1. If \mathcal{F} is a family of graphs, $ex(n, \mathcal{F}) = \text{maximum number of edges in an } n \text{ vertex graph } G \text{ that contains no } F \in \mathcal{F}$.

Lemma 5.2. $ex(n, \mathcal{F}) = \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}$ if $r = \min_{F \in \mathcal{F}} \chi(F)$

Proof.

$$ex(n, \mathcal{F}) \leq \min_{F \in \mathcal{F}} ex(n, F).$$

By Erdos Stone, we're done. \square

Theorem 5.3. Let $\mathcal{F} = \{C_3, \dots, C_{2k}\}$ then $ex(n, \mathcal{F}) \leq n^{1+\frac{1}{k}}$

Lemma 5.4. G is \mathcal{F} free iff G has girth $\geq 2k + 1$

Lemma 5.5. Every graph G contains a subgraph G' that is bipartite with $e(G') \geq \frac{1}{2}e(G)$.

Proof. Color vertices red or blue uniformly and independently. Then delete the edges that are monochromatic. By independence, for any $e \in E(G)$

$$\Pr[e \in E(G')] = \frac{1}{2} \implies E[e(G')] = \frac{1}{2}e(G).$$

\square

We can also prove this with the greedy algorithm and will achieve a slightly better bound.

Proof. To prove theorem 5.3:

Suppose G has at least $n^{1+\frac{1}{k}}$, by lemma 5.2 we may replace G by a sub-graph of $\min \deg d \geq n^{1+\frac{1}{k}}$. We can do a depth first search by picking a root vertex v . Let V_i be the set of vertices at distance i from v . So $V_0 = \{v\}$. Observe that $G[V_0, V_1, \dots, V_k]$ forms a tree with root v and V_i being the descendent at level i else we get a cycle of length at most $2k$. So we have

$$|V_0| + |V_1| + \dots + |V_k| \geq 1 + d + (d-1)d + d(d-1)^2 + \dots + d(d-1)^{k-1} = 1 + d^k > n.$$

So, we have a contradiction. \square

Theorem 5.6. $ex(n, C_{2k}) \leq c_k n^{1+\frac{1}{k}}$

Theorem 5.7. $ex(n, \{C_4, C_6, \dots, C_{2k}\}) \leq 2n^{1+\frac{1}{k}}$

Remark 5.8. The exponent $1 + \frac{1}{k}$ is sharp if $k = 2, 3, 5$

5.2 Turán Numbers for Paths

Theorem 5.9. Let P_k be a path with k edges then

$$ex(n, P_k) \leq \frac{k}{2}n.$$

Proof. By induction on n it is enough to work with connected graph as we can just split into connected components. Delete vertices of degree $\leq \frac{k}{2}$. If $e(G) > \frac{k}{2}n$ then this also holds after deletions. So WLOG G has minimum degree at least $\frac{k}{2}$.

Let v_0, v_1, \dots, v_r be the longest path in G . The neighbors of v_0, v_r are in the path otherwise you can extend the path. If $v_0 \sim v_r$ then $v_0v_1\dots v_r$ is a cycle and so $v_iv_{i+1}\dots v_rv_0\dots v_{i-1}$ is also a path of same length. Thus, the neighbors of v_i are inside the path. So $\{v_0, v_1, \dots, v_r\}$ is a connected component of G and thus $v \geq k + 1$ else $e(G) \leq \frac{(r-1)n}{2}$. Otherwise, if $v_0 \not\sim v_r$ then there exist i such that $v_0 \sim v_{i+1}$ and $v_r \sim v_i$ then we can produce a cycle $v_0v_{i+1}v_{i+2}\dots v_rv_iv_{i-1}\dots v_0$ and this is of length r , so it becomes the previous case. By pigeonhole, such a i exist as $\{i : v_0v_i + 1 \in E(G)\}$ and $\{i : v_rv_i \in E(G)\}$ must intersect as $r \leq k$ by $v_0v_r \notin E(G)$ \square

6 Projective Planes and C_{2k} -Free Graphs (Jan 26)

6.1 Projective Plane over a Field

Definition 6.1. *Projective Plane over a Field.* Let \mathbb{F} be a field.

1. *Points:* The set of lines through 0 in \mathbb{F}^3 ; equivalently, the 1-dimensional subspaces.
2. *Lines:* The set of planes through 0 in \mathbb{F}^3 ; equivalently, the 2-dimensional subspaces.
3. *Incidence:* A point P lies on a line ℓ if $P \subset \ell$.

Observe:

1. Any two points determine a unique line.
2. Any two lines intersect in a unique point.
3. There exist 4 points no three of which are collinear.

Inside a projective plane over \mathbb{F} , there is a copy of the affine plane \mathbb{F}^2 embedded as $\{z = 1\}$. We denote this by $\mathbb{P}^2(\mathbb{F})$.

We say $p \sim \ell$ if $p \in \ell$ for p a point and ℓ a line of $\mathbb{P}^2(\mathbb{F}_q)$.

6.2 C_4 -Free Graph from $\mathbb{P}^2(\mathbb{F}_q)$

Remark 6.2. There are $q^3 - 1$ nonzero vectors in $\mathbb{F}_q^3 \setminus \{0\}$. We identify points of $\mathbb{P}^2(\mathbb{F}_q)$ with equivalence classes of nonzero vectors under

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \quad \text{for } \lambda \in \mathbb{F}_q^\times.$$

There are $q - 1$ nonzero choices for λ , so the number of points in $\mathbb{P}^2(\mathbb{F}_q)$ is

$$\frac{q^3 - 1}{q - 1} = q^2 + q + 1.$$

By a similar argument, the number of lines is also $q^2 + q + 1$.

Each line contains exactly $q + 1$ points and dually, each point lies on exactly $q + 1$ lines. There is a natural bijection (called a polarity) between points and lines, preserving incidence: given a point p , its corresponding line is the orthogonal complement p^\perp .

Theorem 6.3. Construction of a C_4 -free graph:

We construct an improved C_4 -free graph as follows:

- The vertices are the points of $\mathbb{P}^2(\mathbb{F}_q)$.
- Points p_1 and p_2 are adjacent if and only if $p_1 \in p_2^\perp$ (i.e., p_1 lies on the line associated to p_2 's orthogonal complement).

For any pair p_1, p_2 , their common neighborhood is

$$p_1^\perp \cap p_2^\perp,$$

which is a single point. Thus, the graph is C_4 -free (contains no 4-cycles). This gives $q^2 + q + 1$ vertices with average degree $\approx q$.

Note: There exist p such that $p \subseteq p^\perp$.

6.3 General Projective Planes

Definition 6.4. General Projective Planes. If P (points) and L (lines) are sets with an incidence relation such that:

1. For any two points, there is a unique line through them.
2. Any two lines meet at a unique point.
3. There exist four points, no three collinear.



Example: The Fano plane (order 2) with 7 points and 7 lines. Each line contains 3 points, and each point lies on 3 lines.

Definition 6.5. Order of a Projective Plane. The order of a projective plane is the number of points per line minus one (so q if each line has $q + 1$ points).

Remark 6.6. Are there planes of order which are not prime powers?

6.4 C_{2k} -Free Graphs for $k = 2, 3, 5$

Theorem 6.7. Construction of C_{2k} -free graphs for $k = 2, 3, 5$:

Let q be a prime power and let \mathbb{F}_q be the finite field with q elements. Define a graph G on the vertex set \mathbb{F}_q^k (the set of k -tuples over \mathbb{F}_q) as follows:

For $\mathbf{a} = (a_0, a_1, \dots, a_{k-1})$ and $\mathbf{b} = (b_0, b_1, \dots, b_{k-1})$ in \mathbb{F}_q^k , declare $\mathbf{a} \sim \mathbf{b}$ if and only if for all $j = 0, 1, \dots, k-2$,

$$b_j = a_j + a_{j+1} \cdot b_{k-1}$$

where indices are taken modulo k (so $a_k = a_0$).

Then G is C_{2k} -free, i.e., it contains no cycle of length $2k$.

Proof. Suppose for contradiction that G contains a cycle of length $2k$:

$$\mathbf{a}^1, \mathbf{b}^1, \mathbf{a}^2, \mathbf{b}^2, \dots, \mathbf{a}^k, \mathbf{b}^k,$$

with $\mathbf{a}^i \sim \mathbf{b}^i \sim \mathbf{a}^{i+1}$ for all i , indices modulo k .

Step 1: System of equations. For each $i = 1, \dots, k$ and $j = 0, \dots, k-2$, adjacency gives:

$$\begin{aligned} b_j^i &= a_j^i + a_{j+1}^i \cdot b_{k-1}^i && \text{(from } \mathbf{a}^i \sim \mathbf{b}^i\text{)} \\ b_j^i &= a_{j+1}^{i+1} + a_{j+1}^{i+1} \cdot b_{k-1}^i && \text{(from } \mathbf{b}^i \sim \mathbf{a}^{i+1}\text{)} \end{aligned}$$

Subtract:

$$(a_j^i - a_j^{i+1}) + (a_{j+1}^i - a_{j+1}^{i+1}) \cdot b_{k-1}^i = 0$$

or

$$(a_j^{i+1} - a_j^i) + (a_{j+1}^{i+1} - a_{j+1}^i) \cdot b_{k-1}^i = 0$$

Summing over i :

$$\sum_{i=1}^k (a_j^{i+1} - a_j^i) + \sum_{i=1}^k (a_{j+1}^{i+1} - a_{j+1}^i) b_{k-1}^i = 0$$

The first sum telescopes to zero (since indices are cyclic), so for each $j = 0, \dots, k-2$,

$$\sum_{i=1}^k (a_{j+1}^{i+1} - a_{j+1}^i) b_{k-1}^i = 0 \quad (3)$$

Step 2: Specialize $j = k-2$. Let $\Delta_i = a_{k-1}^{i+1} - a_{k-1}^i$ and $x_i = b_{k-1}^i$. Then:

$$\sum_{i=1}^k \Delta_i x_i = 0$$

and

$$\sum_{i=1}^k \Delta_i = 0$$

by telescoping.

Step 3: Linear algebra. The system (3) for $j = 0, \dots, k-2$ can be written as a $(k-1) \times k$ Vandermonde-type matrix V applied to the vector $(\Delta_1, \dots, \Delta_k)$:

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ \vdots & \vdots & & \vdots \\ x_1^{k-2} & x_2^{k-2} & \cdots & x_k^{k-2} \end{pmatrix}$$

The equations become $V \cdot (\Delta_1, \dots, \Delta_k)^\top = 0$. Since $\sum_{i=1}^k \Delta_i = 0$, the vector is in the kernel of the row $(1, 1, \dots, 1)$.

Claim 6.8. For each i , $x_i \neq x_{i+1}$ (indices modulo k).

Step 4: Key property of the x_i .

Proof. If $x_i = x_{i+1}$, then $b_{k-1}^i = b_{k-1}^{i+1}$. By the adjacency relations, this forces $\mathbf{b}^i = \mathbf{b}^{i+1}$, so $\mathbf{a}^{i+1} = \mathbf{a}^{i+2}$, contradicting that the cycle is proper. \square

Step 5: Contradiction for $k = 2, 3, 5$. Since not all Δ_i are zero (otherwise the cycle is trivial), and $\sum_{i=1}^k \Delta_i = 0$, equation $\sum_{i=1}^k \Delta_i x_i = 0$ forces a linear dependence among the x_i (not all distinct). Thus, for these small k we can classify possibilities:

Claim 6.9. For $k = 2, 3, 5$, if $x_i = x_j$ for some $i \neq j$, then $x_i = x_{i+1}$ for some i , contradicting the above claim.

- **Case $k = 2$:** Only possible if $x_1 = x_2$, violating $x_1 \neq x_2$.
- **Case $k = 3$:** If $x_1 = x_2$, $x_2 = x_3$, or $x_3 = x_1$, we violate the claim.
- **Case $k = 5$:** If $x_i = x_j$ for $i \neq j$, by relabeling, could assume $x_1 = x_j$ for $j = 2, 3, 4, 5$:
 - $j = 2$ or 5 gives $x_1 = x_2$ or $x_5 = x_1$, directly a violation.

- $j = 3$: $x_1 = x_3$. The system $\sum_{i=1}^5 \Delta_i = 0$, $\sum_{i=1}^5 \Delta_i x_i = 0$ with $x_1 = x_3$ and $x_i \neq x_{i+1}$ for any i only allows further coincidences that eventually force $x_i = x_{i+1}$.
 - $j = 4$: similarly, $x_1 = x_4$ eventually forces a pair $x_i = x_{i+1}$.
- **Case $k = 4$:** The pattern $x_1 = x_3 \neq x_2 = x_4$ is possible with no consecutive equal x_i , which is why this construction fails for $k = 4$.

Therefore, for $k = 2, 3, 5$, no proper $2k$ -cycle exists in G . \square

7 Algebraic Construction for $ex(n, K_{s,t})$ (Jan 28)

7.1 Random Polynomial Construction

Recall: $ex(n, K_{s,t}) \leq c_{s,t}n^{2-\frac{1}{s}} + o(n)$. We will see that $ex(n, K_{s,t}) = \theta\left(n^{2-\frac{1}{s}}\right)$ for $t \geq t_0(s)$.

Theorem 7.1. *Construction of the extremal graph for $ex(n, K_{s,t})$.*

Proof. We will first consider a random graph by picking edges with probability $p = n^{-\frac{1}{s}}$. Let $x^{(1)}, \dots, x^{(s)}$ be s vertices chosen uniformly at random without replacement on the left. Let $N(x^{(1)}, \dots, x^{(s)})$ that are the common neighbors with $d(x^{(1)}, \dots, x^{(s)}) = |N(\dots)|$. Let our random variable be $d(x^{(1)}, \dots, x^{(s)})$ and is distributed according to

$$Binomial(n, p^s) = Binomial\left(n, \frac{1}{n}\right) \sim Poisson(1).$$

So $\mathbb{E}[d(x^{(1)}, \dots, x^{(s)})] = 1$ and $\mathbb{E}[d(\dots)^k] = o_k(1)$. With probability $\frac{1}{n^\epsilon}$ there is at least $\epsilon \log n$ neighbors. Yet there are $\binom{n}{s}$ such s tuples $(x^{(1)}, \dots, x^{(s)})$. and then $\frac{n}{s}$ disjoint. This does not work.

Consider another construction of $K_{s,t}$ -free graph with $\Omega(n^{2-\frac{1}{s}})$ edges.

For a bipartite graph with \mathbb{F}_p^s where $n = 2p^s$ on each side we say $x \sim y$ if $f(x, y) = 0$. Pick equation f uniformly at random among all polynomials in s, t of degree at most d = in each of x and y . It would be

$$f = \sum_{|\alpha|, |\beta| \leq d} c_{\alpha\beta} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} y_1^{\beta_1} y_2^{\beta_2} \dots y_s^{\beta_s}. \quad (c_{\alpha\beta} \in \mathbb{F}_p, |\alpha| = \alpha_1 + \dots + \alpha_d)$$

Part I: This random graph is locally (neighborhood look "similar") similar to uniform random graph.

Part II: Globally very different/more discrete. \square

Lemma 7.2. *For any fixed $x, y \in \mathbb{F}_p^s$*

$$\Pr[x \sim y] = \frac{1}{p}.$$

Proof. We have

$$f = c_{0,0} + \sum_{\alpha, \beta \neq 0} c_{\alpha\beta} x^\alpha b^\beta.$$

For any choice of $c_{\alpha,\beta}$ fir $(\alpha, \beta) \neq (0, 0)$ there is a unique choice of $c_{0,0}$ such that $f(x, y) = 0$.

$$\mathbb{E}[e(G)] = \frac{1}{p}n^2 = c_s n^{2-\frac{1}{s}}.$$

\square

Let $x^{(1)}, \dots, x^{(s)}$ be s vertices on say left, $N(x^{(1)}, \dots, x^{(s)})$ be the common neighbors with of $x^{(1)}, \dots, x^{(s)}$ and this is same as $\{y \in \mathbb{F}_p^s : f(x^{(1)}, y) = 0, \dots, f(x^{(s)}, y) = 0\}$.

Lemma 7.3. *If g is a random polynomial of degree of D in m variables and $z^{(1)}, \dots, z^{(t)} \in \mathbb{F}_p^m$ then the random variables $g(z^{(1)}), \dots, g(z^{(t)})$ are independent if $D > t$ and $p > t^2$.*

Warm up: When $m = 1$, write $g(x) = \underbrace{a_0 + a_1 x + \dots + a_{t-1} x^{t-1}}_{g_{small}(x)} + \underbrace{\dots + a_D x^D}_{g_{large}(x)}$.

For any choice of $g_{large}(x)$ and any values b_1, \dots, b_t there is a unique polynomial g_{small} such that $g(z^{(i)}) = g_{small}(z^{(i)}) + g_{large}(z^{(i)}) = b_i$ for $i = 1, \dots, t$ if and only if $g_{small}(z^{(i)}) = b_i - g_{large}(z^{(i)})$ by lagrange interpolation.

For the general case, if $z_1^{(1)}, \dots, z_1^{(t)}, \dots, z_m^{(1)}, \dots, z_m^{(t)}$ are distinct, then $g(x_1, \dots, x_m) = \bar{g}(x_1) +$ other terms and use $m = 1$ = case. Pick a invertible linear transform $f : \mathbb{F}_p^s \rightarrow \mathbb{F}_p^s$ such that $Tz^{(1)}, \dots, Tz^{(t)}$ are distinct. To pick such a T , let $T(x_1, \dots, x_s) = (x_1 + S(x_2, \dots, x_s), x_2, \dots, x_s)$. For $z^{(i)}, z^{(j)}$ distinct, the

space of S for which $T(z^{(i)}) = T(z^{(j)})$ is of co-dimension 1, so a random S has probability $\frac{1}{p}$ of satisfying this. So $\Pr[\exists i, j \text{ such that } T(z^{(i)}) = T(z^{(j)})] \leq \frac{1}{p} \binom{t}{2} < 1$.

We can now conclude from

$$\Pr[f(x^{(1)}, y) = 0, \dots, f(x^{(s)}, y) = 0] = \frac{1}{p^s}.$$

If $d > s$ and $p > s^2$ then

$$\mathbb{E}[N(x^{(1)}, \dots, x^{(s)})] = 1.$$

Furthermore, if α is a positive integer and $p \cdot s < d$ then

$$\mathbb{E} \left[|N(x^{(1)}, \dots, x^{(s)})|^p \right] = \mathbb{E} \left[\left(\sum_{y \in \mathbb{F}_p^s} R(y) \right)^\alpha \right] = \mathbb{E} \left[\sum_{y^{(1)}, \dots, y^{(\alpha)}} R(y^{(1)}) \dots R(y^{(\alpha)}) \right].$$

$$\begin{aligned} \text{Where } R(y) &= \begin{cases} 1 & \text{if } y \in N(x^{(1)}, \dots, x^{(s)}) \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{y^{(1)}, \dots, y^{(\alpha)}} p^{-s \# \text{distinct } y's} \leq \sum_k \sum_{y^{(1)}, \dots, y^{(k)} \in \mathbb{F}_p^s} p^{-sk} = \sum_k p^{sk} p^{-sk} \cdot \# \text{ways to partition } s \text{ into } k \text{ bins}. \end{aligned}$$

Last equality from having to choose $y^{(i)}$ such that k are distinct. This is in $O(1)$. We want to assign values to $y^{(i)}$ such that exactly k are distinct. Equivalently, choose k distinct values in \mathbb{F}_p^s . Then choose which of the $y^{(i)}$ go to which bin.

$$\Pr[|N(x^{(1)}, \dots, x^{(s)})| \geq T] \leq \frac{\mathbb{E} [|N(x^{(1)}, \dots, x^{(s)})|^\alpha]}{T^\alpha} = \frac{C_{\alpha, s}}{T^\alpha}.$$

The set $N(x^{(1)}, \dots, x^{(s)})$ is contained in $\{y \in \mathbb{F}_p^s : f(x^{(1)}, y) = 0, \dots, f(x^{(s)}, y) = 0\}$. A vector subspace V of \mathbb{F}_p^s of dimension D has p^D elements.

Theorem 7.4. Lang-Weil Theorem: If g_1, \dots, g_r are polynomials of degree $\leq D$ in s variables over \mathbb{F}_p , then the set $\{y \in \mathbb{F}_p^s : g_1(y) = \dots = g_r(y) = 0\}$ has either $O_{D, r}(1)$ points, or more than $\frac{p}{2}$ points.

So $\Pr[\text{some } s \text{ vertices have neighborhood of size } \geq O_{s, \alpha}(1)] \leq C_{s, \alpha} \left(\frac{p}{2}\right)^{-\alpha} \binom{p^s}{s}$. Choose $\alpha = s^2 + 1$. To ensure this probability goes to 0, we need to choose $d = s^2 + 1$.

8 Regularity for Binary Words (Feb 4)

8.1 Definitions and Chernoff Bound

Definition 8.1. *Binary word is sequence $w_1, w_2, \dots, w_n \in \{0, 1\}$ and write $w = w_1 w_2 \dots, w_n$. The density of w is*

$$d(w) = \frac{\#\text{1's in } w}{n}.$$

Definition 8.2. *Word w is ϵ -regular if for any subword w' of w of length $\geq \epsilon \cdot \text{len}(w)$ satisfies*

$$|d(w) - d(w')| \leq \epsilon.$$

Lemma 8.3. *By Chernoff bound,*

$$\Pr[|d(w') - d(w)| \geq \epsilon] \leq e^{-c_\epsilon n}.$$

9 Regularity Lemma for Binary Words (Feb 6)

9.1 The Regularity Lemma

Definition 9.1. Partition \mathcal{P}' is a refinement of partition \mathcal{P} if each $w' \in \mathcal{P}'$ is a sub-word of some word in \mathcal{P}

Theorem 9.2. (Regularity Lemma for Binary Words)

For all $\epsilon > 0$, every word $w \in \{0, 1\}^n$ can be partitioned into $\leq M(\epsilon)$ many sub-words $w^{(1)}, \dots, w^{(m)}$ (i.e. $w = w^{(1)} \dots w^{(m)}$). So that the total length of $w^{(i)}$'s that are ϵ -irregular is at most ϵn .

Proof. For a partition \mathcal{P} of w into sub-words $\mathcal{P} = \{w^{(1)}, \dots, w^{(m)}\}$ we define $f(\mathcal{P}) = \sum_{w' \in \mathcal{P}} d(w)^2 |w'| / |w|$. Alternatively, we pick a symbol of w at random, say $i \in [n]$ uniformly, then pick another symbol at random from the same part as 1'st, say j is picked.

$$\Pr[w_i = w_j] = \sum_{w' \in \mathcal{P}} \frac{|w'|}{|w|} (d(w')^2 + (1 - d(w'))^2) = 2f(\mathcal{P}) + \sum_{w'} \frac{|w'|}{|w|} - 2 \sum_{w'} d(w') \frac{|w'|}{|w|} = 2f(\mathcal{P}) + 1 - 2d(w).$$

Lemma 9.3. If \mathcal{P}' is a refinement of \mathcal{P} then

$$f(\mathcal{P}') = f(\mathcal{P}) + \sum_{w' \in \mathcal{P}'} (d(w') - d(\text{parent}(w')))^2 \frac{|w'|}{|w|}.$$

Proof. Expanding the second term we have

$$\sum_{w' \in \mathcal{P}'} d(w') \frac{|w'|}{|w|} - 2d(w')d(\text{parent}(w')) + d(\text{parent}(w'))^2 \frac{|w'|}{|w|} = \sum_1 + \sum_2 + \sum_3.$$

We have $\sum_1 = f(\mathcal{P}')$ and

$$\begin{aligned} \sum_2 &= \sum_{\tilde{w} \in \mathcal{P}} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} \left(-2d(w')d(\tilde{w}) \frac{|w'|}{|w|} \right) \\ &= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w}) \frac{1}{|w|} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} d(w') |w'| \\ &= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w}) \frac{1}{|w|} \cdot d(\tilde{w}) |\tilde{w}| \\ &= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w})^2 \frac{|\tilde{w}|}{|w|} \\ &= -2f(\mathcal{P}) \end{aligned}$$

Then for the third sum

$$\begin{aligned} \sum_3 &= \sum_{\tilde{w} \in \mathcal{P}} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} d(\tilde{w})^2 \frac{|w'|}{|w|} \\ &= \sum_{\tilde{w} \in \mathcal{P}} d(\tilde{w})^2 \frac{|\tilde{w}|}{|w|} = f(\mathcal{P}) \end{aligned}$$

So we have $\sum_1 + \sum_2 + \sum_3 = f(\mathcal{P}') - f(\mathcal{P})$

□

Start with the trivial partition (i.e. $\mathcal{P} = \{w\}$) and repeat while $\sum_{w' \in \mathcal{P}, w' \text{ is } \epsilon\text{-irregular}} |w'| \leq \epsilon |w|$. For $w' \in \mathcal{P}$ that is ϵ -irregular, find a way to write as $w' = w^{(1)}w^{(2)}w^{(3)}$ where $|d(w^{(2)}) - d(w^{(1)})| \geq \epsilon$ and $|w^{(2)}| \geq \epsilon |w'|$. Replace w' with the three words $w^{(1)}, w^{(2)}, w^{(3)}$ to obtain a new partition of w , call it \mathcal{P}' . Repeat with \mathcal{P} replaced by \mathcal{P}' .

To analyze this algorithm, if we drop some words we have

$$f(\mathcal{P}') \geq f(\mathcal{P}) + \sum_{\substack{w' \in \mathcal{P}' \\ w' \text{ is } \epsilon\text{-irregular}}} \sum_{w^{(2)}} (d(w^{(2)}) - d(w')) \frac{|w^{(2)}|}{|w|} \geq f(\mathcal{P}) + \sum_{\substack{w' \in \mathcal{P}' \\ w' \text{ is } \epsilon\text{-irregular}}} \epsilon^2 \frac{\epsilon |w'|}{|w|} = f(\mathcal{P}) + \epsilon^4.$$

Note: $f(\mathcal{P}) \leq 1$ for any partition \mathcal{P} .

So the number of steps in the algorithm is $\leq \frac{1}{\epsilon^4}$. The theorem holds with $M(\epsilon) = 3^{1/\epsilon^4}$

□

9.2 Twin Subsequences

Example 9.4. Define a twin in a word $w \in \{0,1\}^n$ is a pair of subsequences $x, y \in w$ that are disjoint (no symbol in both x and y) and are equal as word.

We define

$$t(w) = \text{the length of longest pair twins in } w \text{ and } t(n) = \min_{w \in \{0,1\}^n} t(w).$$

We have the trivial bounds $\frac{n}{4} \leq t(n) \leq \frac{n}{2}$.

Theorem 9.5.

$$t(n) \geq \left(\frac{1}{2} - o(1)\right)n.$$

Proof. Pick small $\epsilon > 0$ and cut w into ϵ -regular words $t \leq \epsilon n$ junk symbols. Enough $t(w) \geq (\frac{1}{2} - c\epsilon) |w|$ for ϵ -regular w .

Cut w into $\frac{1}{\epsilon}$ equally long sub-words $w^{(1)}, \dots, w^{(m)}$ where $m = \frac{1}{\epsilon}$.

The first twin is 0's from $w^{(1)}$ + 1's from $w^{(2)}$ + 0's from $w^{(3)}$...

The second twin is 1's from $w^{(1)}$ + 0's from $w^{(2)}$ + 1's from $w^{(3)}$...

Choose twins then

$$|\# 0's in w^{(i)}| - |\# 1's in w^{(2)}| \leq \epsilon |w^{(1)}|.$$

So, # symbols not in either of the twins is less than or equal to

$$\epsilon |w| (1's in w^{(1)}) + \epsilon |w| (1's in w^{(2)}) + \sum_i 4\epsilon |w^{(i)}|.$$

□

10 Szemerédi's Graph Regularity Lemma (Feb 13)

10.1 ϵ -Regular Pairs and Partitions

Definition 10.1. For disjoint sets $U, V \subset V(G)$, the density between U and V is

$$d(U, V) = \frac{e(U, V)}{|U||V|}.$$

Definition 10.2. A pair (U, V) of disjoint subsets of $V(G)$ is ϵ -regular if $\forall U' \subseteq U, V' \subseteq V$ such that $|U'| \geq \epsilon|U|$ and $|V'| \geq \epsilon|V|$ we have

$$|d(U, V) - d(U', V')| \leq \epsilon.$$

Definition 10.3. We say a partition $V(G) = V_1 \cup \dots \cup V_k \cup J$ is ϵ -regular partition when

1. $|J| \leq \epsilon|V(G)|$
2. $|V_1| = |V_2| = \dots = |V_k|$
3. All except $\leq \epsilon k^2$ pairs (V_i, V_j) are ϵ -regular

10.2 Proof of the Regularity Lemma

Theorem 10.4. (Szemerédi's Regularity Lemma) $\forall \epsilon > 0, m$ there exist a constant $M = M(\epsilon, m)$ such that every graph admits an ϵ -regular partition $V(G) = V_1 \cup \dots \cup V_k \cup J$ where the number of parts $m \leq k \leq M$.

Remark 10.5. # edges inside parts + # edges adjacent to J + # edges in ϵ -irregular pairs is at most

$$k \binom{n/k}{2} + (\epsilon n) \cdot n + \epsilon k^2 (n/k)^2 \leq \frac{1}{k} n^2 + \epsilon n^2 + \epsilon n^2.$$

Proof. For a partition \mathcal{P} of $V := V(G)$, we define

$$f(\mathcal{P}) = \sum_{U, W \in \mathcal{P}, U \neq W} d(U, W)^2 \underbrace{\frac{|U||W|}{|V|^2}}_{f(U, W)}.$$

Lemma 10.6. Suppose $A, B \subset V$ are disjoint and we have partitions $A = A_1 \cup \dots \cup A_k$ and $B = B_1 \cup \dots \cup B_l$ then

$$\sum_{i,j} f(A_i, B_j) = f(A, B) + \sum_{i,j} (d(A, B) - d(A_i, B_j))^2 \frac{|A_i||B_j|}{|V|^2}.$$

Proof is exactly the same as the proof of the regularity lemma for binary words.

Algorithm: To find an ϵ -regular partition

1. Start with any equipartition of V into m parts.
2. We will start with partitions of the form $V = V_1 \cup V_2 \cup \dots \cup V_k \cup J_1 \cup \dots \cup J_\ell$
3. For each ϵ -irregular pair (V_i, V_j) there exist partitions $V_i = V_{i,1}^{(i,j)} \cup V_{i,2}^{(i,j)}$ and $V_j = V_{j,1}^{(i,j)} \cup V_{j,2}^{(i,j)}$ such that $|d(V_i, V_j) - d(V_{i,1}^{(i,j)}, V_{j,1}^{(i,j)})| \geq \epsilon$ and $|V_{i,1}^{(i,j)}| \geq \epsilon|V_i|$ and $|V_{j,1}^{(i,j)}| \geq \epsilon|V_j|$
Note: We write is as such since there may be many ϵ -irregular pairs (V_i, V_j) and we want to keep track of them.

4. For each i consider all partitions of V_i as $V_i = V_{i,1} \cup V_{i,2}$ and take the common refinement. Two V, V' are in the same part of the refinement iff V, V' are in the same part in each of the partitions. Observe: Each V_i is cut into $\leq 2^k$ parts.

Let \mathcal{P} be the old partition and \mathcal{P}_{new} be the new partition.

$$\begin{aligned}
f(\mathcal{P}_{\text{new}}) &= \sum_{\substack{A,B \in \mathcal{P}_{\text{new}} \\ A \neq B}} f(A, B) \\
&\geq \sum_i f(J_i, ?) + \sum_{i,j} \sum_{\substack{A,B \\ \text{parent}(A)=V_i \\ \text{parent}(B)=V_j}} f(A, B) \\
&\geq f(\text{Junk}) + \sum_{i \neq j} \left(f(V_i, V_j) + \epsilon^2 \frac{\epsilon^2 |V_i||V_j|}{|V|^2} \right) \\
&= f(\mathcal{P}) + \epsilon^4 \left(\frac{|V| - |\text{Junk}|}{|V|} \right)^2 \\
&\geq f(\mathcal{P}) + \frac{\epsilon^4}{4}
\end{aligned}$$

5. Cut each of the parts in \mathcal{P}_{new} into equally big parts so total # parts is $\leq k \cdot 4^k$ and some left over where # of leftovers is $\leq k2^k$ and the size of them $\leq k2^k \cdot \frac{|V|}{k4^k} \leq \frac{|V|}{2^k}$

So the total junk size is $\leq \sum_{k \geq m} \frac{|V|}{2^k}$

The total number of iterations is at most $\frac{4}{\epsilon^4}$, and let $g(k) = k \cdot 4^k$. The total number of parts at the end is at most

$$M \leq g^{(\lceil 1/\epsilon^5 \rceil)}(m)$$

where $g^{(t)}$ denotes g composed with itself t times, starting from m ; that is, we iterate g at most $\frac{1}{\epsilon^5}$ times as $\epsilon k^2 \cdot \frac{\epsilon^4}{k^2} = \epsilon^5$.

So using $g(k) \leq 4^{4^k}$ we have

$$M \leq 4^{4^{\dots 4^m}} \left\{ \frac{2}{\epsilon^5} \right\} \text{ times.}$$

□

11 Triangle Counting and Removal Lemmas (Feb 16)

11.1 Triangle Counting Lemma

Lemma 11.1 (Triangle Counting Lemma). Let $X, Y, Z \subset V(G)$ be disjoint with $d(X, Y), d(X, Z), d(Y, Z) \geq 2\epsilon$, and suppose $(X, Y), (X, Z), (Y, Z)$ are ϵ -regular. Then the number of XYZ -triangles is at least

$$(1 - 2\epsilon)(d(X, Y) - \epsilon)(d(X, Z) - \epsilon)(d(Y, Z) - \epsilon)|X||Y||Z|$$

Proof. Observe the following:

1. $Y_{\text{bad}} = \{x \in X : \deg_Y(x) < (d(X, Y) - \epsilon)|Y|\}$
2. $Z_{\text{bad}} = \{x \in Z : \deg_Z(x) < (d(X, Z) - \epsilon)|Z|\}$

Then $|Y_{\text{bad}}|, |Z_{\text{bad}}| \leq \epsilon|X|$.

Consider Y_{bad} ,

$$e(Y_{\text{bad}}, Y) = \sum_{x \in Y_{\text{bad}}} \deg_Y(x) < |Y_{\text{bad}}|(d(X, Y) - \epsilon)|Y|.$$

By ϵ -regularity we have $d(X, Y) - \epsilon$ is within ϵ of $d(X, Y)$ so a contradiction if $|Y_{\text{bad}}| > \epsilon|X|$.

To prove the Triangle Counting Lemma, let $X' := X \setminus (Y_{\text{bad}} \cup Z_{\text{bad}})$ then $|X'| \geq (1 - 2\epsilon)|X|$ by observation.

Let $x \in X'$ then $N_Y(x) := N(x) \cap Y$ and $N_Z(x) := N(x) \cap Z$.

The number of triangles containing x is

$$e(N_Y(x), N_Z(x)) \geq |N_Y(x)||N_Z(x)|(d(Y, Z) - \epsilon). \quad (*)$$

By ϵ -regularity of (Y, Z) and definition of bad. Additionally

$$(*) \geq (d(X, Y) - \epsilon)(d(X, Z) - \epsilon)(d(Y, Z) - \epsilon)|Y||Z|.$$

So

$$\#\Delta = \sum_{x \in X'} (*) \geq (*) (1 - 2\epsilon)|X|.$$

□

11.2 Triangle Removal Lemma

Lemma 11.2. *Triangle Removal Lemma:* $\forall \epsilon > 0, \exists \delta > 0$ such that every graph with $\leq \delta n^3$ triangles can be made triangle free by deleting only ϵn^2 edges.

Proof. Pick $\epsilon' = \frac{\epsilon}{4}$ and $m = \frac{4}{\epsilon}$ and apply szemerédi's regularity lemma to G to get an ϵ -regular partition $V(G) = V_1 \cup \dots \cup V_k \cup J$ where $m \leq k \leq M(\epsilon', m)$ and $|J| \leq \epsilon'n$. To clean up the graph

1. Delete all edges incident to J as we do not know anything about J
2. Delete all edges inside ϵ -irregular parts
3. Delete all edges inside a single V_i
4. Delete edges between V_i and V_j if $d(V_i, V_j) < 2\epsilon'$

If at least one Δ survives, say xyz with $x \in X, y \in Y, z \in Z$ with $X, Y, Z \in \{V_1, \dots, V_k\}$ then X, Y, Z satisfy the condition of triangle counting lemma. So we have

$$\#\Delta \in G \geq \#XYZ\Delta \geq (1 - 2\epsilon')(d(X, Y) - \epsilon)(d(X, Z) - \epsilon)(d(Y, Z) - \epsilon)|X||Y||Z| \geq (1 - 2\epsilon)\epsilon^3 \left(\frac{|V(G)|(1 - \epsilon')}{M} \right)^3.$$

So the total # edges deleted in steps (1)-(4) is at most

$$n|J| + \#\epsilon'\text{-irregular pairs} \cdot \left(\frac{n}{k} \right)^2 + k \left(\frac{n}{k} \right)^2 + k^2 \cdot \epsilon' \left(\frac{n}{k} \right)^2 \leq \epsilon'n^2 + \epsilon'k^2 \left(\frac{n}{k} \right)^2 + \frac{n^2}{m} + \epsilon'n^2.$$

So the bound we get is $\delta \geq \frac{1}{\text{tower}(\frac{c}{\epsilon^5})}$

□

12 Roth's Theorem(Feb 18)

Theorem 12.1. *Roth's Theorem (1953): If $A \subset [n]$ of density at least $\alpha > 0$ (i.e. $|A| \geq \alpha n$) and $n \geq n_0(\alpha)$ then A contains a 3-term arithmetic progression i.e., a set of the form $x, x + d, x + 2d$ with $d \neq 0$*

Remark 12.2. *Same holds for $A \subset P$ for any n -term arithmetic progression P .*

Theorem 12.3. *Szemerédi Theorem (1974): $\forall \alpha > 0, \forall k$ if $n \geq n_0(\alpha, k)$ then $|A| \geq \alpha n \implies A$ contains a k -term arithmetic progression.*