

Extremal Combinatorics Notes

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February 17, 2026

Contents

1 Turán Numbers and Mantel's Theorem (Jan 12)

1.1 Basic Definitions

Definition 1.1. *Turan Numbers (Forbidden Subgraph Problems):* F is a graph and G graph is F -free if G contains no copy of F as a subgraph.

We want to maximize the size of G subject to G being F -free. Where $\text{size} = e(G) = \# \text{ edges in } G$

Definition 1.2. $ex(n, F) = \max\{e(G) \mid G \text{ is } F\text{-free and } G \text{ is } n\text{-vertex graph}\}$

1.2 Crude Turán's Theorem

Theorem 1.3. *Crude Turán's Theorem:* If G is n -vertex graph,

$$e(G) = \epsilon \binom{n}{2}.$$

then G contains an independent set of size greater than or equal $\frac{1}{2\epsilon} + \frac{1}{2}$

In particular, if $e(G) \geq (1-\epsilon)\binom{n}{2}$ edges, then G contains a clique on greater than or equal $\frac{1}{2\epsilon} + \frac{1}{2}$ vertices.

$$ex(n, K_t) \leq \left(1 - \frac{1}{2t-1}\right) \binom{n}{2}.$$

Proof. Pick k vertices in G at random without replacement. Say we pick v_1, \dots, v_k

For each edge between v_i and v_j , delete v_i .

Let $X = \#$ deleted vertices

Note: $X \leq e(G[v_1, \dots, v_k]) =: Y$

$$\mathbb{E}[Y] = \sum_{1 \leq i \leq j \leq k} \mathbb{P}[v_i v_j \in e(G)] = \binom{k}{2} \frac{\epsilon \binom{n}{2}}{\binom{n}{2}} = \epsilon \binom{k}{2}.$$

Thus, there is a choice of v_1, \dots, v_k such that $Y \leq \epsilon \binom{k}{2}$. Fix it. Let $\alpha(G) =$ size of largest independent set in G satisfies $\alpha(G) \geq k - Y \geq k - \epsilon \binom{k}{2}$

$$\left(k - \epsilon \binom{k}{2}\right)' = 1 + \frac{\epsilon}{2} - \epsilon k.$$

Set $k = \frac{1+\epsilon}{2} + \delta$ for some $|\delta| \leq \frac{1}{2}$.

$$\begin{aligned} \alpha(G) &\geq \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon \left(\left(\frac{1}{\epsilon} + \frac{1}{2} + \delta \right) \left(\frac{1}{\epsilon} - \frac{1}{2} + \delta \right) \right) / 2 \\ &= \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon \frac{(\frac{1}{\epsilon} + \delta)^2 - \frac{1}{4}}{2} \\ &= \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon \frac{(1/\epsilon)^2 + 2\frac{1}{\epsilon}\delta + \delta^2 - \frac{1}{4}}{2} \\ &= \frac{1}{2\epsilon} + \frac{1}{2} + \frac{\frac{1}{4} - \delta^2}{2} \geq \frac{1}{2\epsilon} + \frac{1}{2} \end{aligned}$$

□

1.3 Mantel's and Turán's Theorems

Remark 1.4. Recall: k -uniform hypergraph on vertex set V is a subset of

$$\binom{V}{k} := \{e \subseteq V : |e| = k\}.$$

Example 1.5. In k -uniform hypergraph on vertex with $\epsilon \binom{n}{k}$ edges there is an independent set of size $\geq c_k \frac{1}{\epsilon^{\frac{1}{k-1}}}$

Theorem 1.6. *Mantel's Theorem:* $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$

More generally, define Turan's graph $T_{n,r}$ to be the n -vertex complete r -partite graph with part sizes as equal as possible.

Note: $K_{r+1} \not\subseteq T_{n,r}$ because among $r+1$ vertices some pair is in same part.

Theorem 1.7. *Turan's Theorem:* $ex(n, K_{r+1}) = e(T_{n,r})$ and furthermore $T_{n,r}$ is the only extremizer.

$$e(T_{n,r}) = \left(1 - \frac{1}{r}\right) \binom{n}{2} + O(n).$$

Proof. Let G be a graph on n vertices that is K_{r+1} -free and has the maximum possible number of edges (an extremizer). We proceed in two steps.

Suppose G is not a complete multipartite graph. Then non-adjacency is not an equivalence relation, implying there exist two non-adjacent vertices x, y that do not have identical neighborhoods.

Assume $\deg(x) \leq \deg(y)$. Delete x and then clone y to obtain y' . The new graph G' has

$$e(G') = e(G) - \deg(x) + \deg(y) \geq e(G).$$

Also, G' is K_{r+1} -free because any copy of K_{r+1} has at most one of y or y' (since they are non-adjacent). If a clique contains y' , it can be swapped for y (since $N(y') \subseteq N(y)$), which would imply $K_{r+1} \subseteq G$, a contradiction.

Repeating this process transforms G into a complete multipartite graph without decreasing edges. Thus, the extremizer must be a complete multipartite graph.

Since G is K_{r+1} -free, it must be k -partite with $k \leq r$. To maximize edges, we set $k = r$. Let the parts have sizes n_1, \dots, n_r .

$$e(G) = \binom{n}{2} - \sum_{i=1}^r \binom{n_i}{2}$$

To maximize $e(G)$, we must minimize the subtracted sum of pairs inside the parts. This occurs when n_i are as equal as possible ($|n_i - n_j| \leq 1$). If $n_i \geq n_j + 2$, moving a vertex from part i to part j increases the edge count by $(n_i - 1) - n_j > 0$.

Thus, the unique extremizer is the balanced complete r -partite graph, $T_{n,r}$. □

Turán Density and Supersaturation (Jan 14)

2.1 Inductive Proof of Turán's Theorem

Proof. Second proof of Turán's Theorem with induction on r .

(Induction Step) Let x be a vertex of largest degree, let $A = N(x)$ be its neighborhood.

$$B := V(G) \setminus A.$$

We have $e(G) = e(A) + e(A, B) + e(B)$ and A is K_{r-1} -free so $e(A) \leq e(T_{|A|,r})$

Each vertex in B has $\deg \leq \deg(x) = |A|$

So,

$$e(A, B) + e(B) \leq \sum_{v \in B} \deg(v) \leq |B||A|.$$

So,

$$e(G) \leq e(T_{|A|,r}) + |A||B| \leq e(T_{n,r}).$$

The right hand side is equal to number of edges in complete r -partite graph with one part of size $|B|$ and other being as equal as possible.

Equality holds iff B is independent and A must be a copy of $e(T_{|A|,r-1})$. Hence G is $T_{n,r}$ for equality to hold. \square

2.2 Monotonicity of Extremal Density

Theorem 2.1. *Observe that if F is a graph then*

$$\frac{ex(n, F)}{\binom{n}{2}} \geq \frac{ex(n+1, F)}{\binom{n+1}{2}}.$$

Proof. Pick n vertices in an F -free $(n+1)$ vertex graph G at random without replacement; let G' be the induced graph on these n vertices.

$$e(G') \leq ex(n, F).$$

$$\mathbb{E}(e(G')) = \binom{n}{2} \frac{e(G)}{\binom{n+1}{2}} \implies \frac{e(G)}{\binom{n+1}{2}} \leq \frac{ex(n, F)}{\binom{n}{2}}.$$

\square

Definition 2.2.

$$\pi(F) := \lim_{n \rightarrow \infty} \frac{ex(n, F)}{\binom{n}{2}}.$$

The Turán density of F is

$$\pi(K_{r+1}) = 1 - \frac{1}{r}.$$

2.3 Counting Copies Above Threshold

Theorem 2.3. *Let F be any graph and let $\epsilon > 0$. Let G be an n -vertex graph. There is a $\delta > 0$ such that*

$$e(G) \geq (\pi(F) + \epsilon) \binom{n}{2} \text{ and } n \geq n_0(F, \epsilon).$$

then there are at least $\delta n^{|V(F)|}$ copies of F .

Proof. Let n_0 be large enough such that $\frac{ex(n_0, F)}{\binom{n_0}{2}} \leq \pi(F) + \frac{\epsilon}{2}$. Given n -vertex G as in theorem. Pick n_0 vertices at random without replacement. Let G' be the n_0 -vertex graph induced on these chosen vertices.

$$\mathbb{E}[e(G')] = \frac{e(G)}{\binom{n}{2}} \binom{n_0}{2} \geq (\pi(F) + \epsilon) \binom{n_0}{2}.$$

Say G' is good if

$$e(G') \geq \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2}.$$

$$\begin{aligned}\mathbb{E}(e(G')) &\leq \mathbb{P}(G' \text{ is bad}) \left(\pi(F) + \frac{\epsilon}{2} \right) \binom{n_0}{2} + \mathbb{P}(G' \text{ is good}) \binom{n_0}{2} \\ &\leq \left(\pi(F) + \frac{\epsilon}{2} \right) \binom{n_0}{2} + \mathbb{P}(G' \text{ is good}) \binom{n_0}{2}\end{aligned}$$

Then $\frac{\epsilon}{2} \binom{n_0}{2} \leq \mathbb{P}(G' \text{ is good}) \binom{n_0}{2}$ so $\mathbb{P}(G' \text{ is good}) \geq \frac{\epsilon}{2}$
Pick $|V(F)|$ many vertices without replacement inside G' , call the subgraph they induce G'' . So

$$G \supset G' \supset G''.$$

$$\mathbb{P}(G'' \text{ contains } F) \geq \mathbb{P}(G' \text{ is good}) \cdot \mathbb{P}(G'' \text{ contains } F | G' \text{ is good}) \geq \frac{\epsilon}{2} \frac{1}{\binom{n_0}{|V(F)|}}.$$

Note that G'' is uniformly randomly chosen $|V(F)|$ -vertex subset of G . This implies the # copies of F in $G \geq \mathbb{P}(G'' \text{ contains } F) \binom{n}{|V(F)|} \geq P(\dots) \frac{n^{|V(F)|(1-o(1))}}{|V(F)|!}$. We can choose $\delta = \frac{1}{\binom{n_0}{|V(F)|}}$ \square

3 Erdős–Stone and Kővári–Sós–Turán (Jan 16)

3.1 Erdős–Stone–Simonovits Theorem

Theorem 3.1 (Erdős–Stone–Simonovits). *Let $T_{n,r}$ denote the Turán graph. We know that:*

$$ex(n, T_{s,(r+1)}) = \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}.$$

This implies that if H is any graph with chromatic number $\chi(H) = r + 1$, then

$$ex(n, H) \leq \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}.$$

Conversely, if $\chi(H) > r$ (i.e., H is not r -partite), then $T_{n,r}$ is H -free. Hence,

$$ex(n, H) \geq e(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}.$$

Combining these, for any non-bipartite graph H :

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) \binom{n}{2}.$$

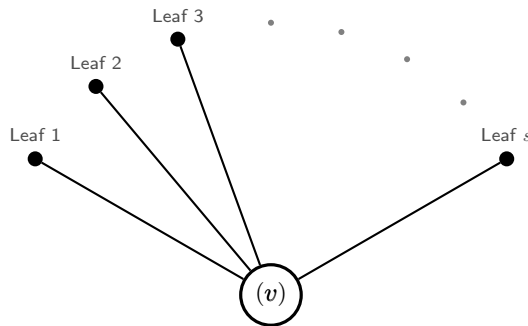
It is important to note that this theorem holds for bipartite graphs as well, where $\chi(H) = 2$. However, substituting $r = \chi(H) - 1 = 1$ gives a coefficient of $1 - 1/1 = 0$. This implies that $ex(n, H) = o(n^2)$, but it does not provide the precise asymptotic behavior (e.g., $n^{2-\epsilon}$). Therefore, while valid, the Erdős–Stone–Simonovits theorem is not the primary tool for determining extremal numbers of bipartite graphs; we rely on theorems like Kővári–Sós–Turán for those bounds.

3.2 Kővári–Sós–Turán Bound

Theorem 3.2 (Kővári–Sós–Turán 1954).

$$ex(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{\frac{1}{s}} \cdot n^{2-\frac{1}{s}} + O(n).$$

Proof. Let G be a $K_{s,t}$ -free graph with n vertices. We will count X , the number of copies of $K_{1,s}$ (stars with s leaves).



We count this in two ways:

1. Since there is no $K_{s,t}$, any set of s vertices can have at most $t - 1$ common neighbors. Thus:

$$X \leq \binom{n}{s} (t - 1).$$

2. Let d_v be the degree of vertex v . Then:

$$X = \sum_{v \in V(G)} \binom{d_v}{s}.$$

A simple-minded application of Jensen's inequality here fails because the binomial coefficient $\binom{x}{s}$ is not convex for all x . To fix this, we define the function:

$$f_s(x) = \begin{cases} \binom{x}{s} & \text{if } x \geq s-1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f_s(x)$ is convex for all x because its derivative:

$$\frac{d}{dx}f_s(x) = \begin{cases} \left(\frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x-s+1}\right) \binom{x}{s} & \text{if } x > s-1 \\ 0 & \text{otherwise} \end{cases}$$

is non-decreasing. Since d_v is an integer, $\binom{d_v}{s} = f_s(d_v)$. We can now validly apply Jensen's inequality:

$$X = \sum_{v \in V} f_s(d_v) \geq n f_s(d_{\text{avg}}) = n f_s\left(\frac{2m}{n}\right),$$

where $m = e(G)$.

Combining (1) and (2):

$$n f_s\left(\frac{2m}{n}\right) \leq \binom{n}{s} (t-1).$$

If $m \leq \frac{1}{2}(s-1)n$, the bound holds trivially. Otherwise, we approximate $\binom{x}{s} \approx \frac{x^s}{s!}$:

$$n \frac{\left(\frac{2m}{n} - s\right)^s}{s!} \leq (t-1) \frac{n^s}{s!}.$$

Rearranging yields:

$$\frac{2m}{n} \leq (t-1)^{\frac{1}{s}} n^{1-\frac{1}{s}} + s \implies m \leq \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + O(n).$$

□

3.3 Unit Distance Problem

Example 3.3 (Unit Distance Problem). *How can we place n points in \mathbb{R}^2 to maximize the number of pairs (p, q) such that $\|p - q\| = 1$?*

A strong construction is a $\sqrt{n} \times \sqrt{n}$ grid. Rather than scaling the grid to unit step size, we scale it such that the distance 1 corresponds to a distance d in the integer grid that occurs most frequently.

The possible distances in an unscaled integer grid are $\sqrt{a^2 + b^2}$ for $0 \leq a, b < \sqrt{n}$. By number theory (Landau-Ramanujan), integers representable as a sum of two squares are those where primes $p \equiv 3 \pmod{4}$ appear with even exponents.

To get a better bound, we look for "highly composite" integers. Specifically, if we choose a number d composed of a product of many distinct primes $p \equiv 1 \pmod{4}$, the number of ways to write d as a sum of two squares is very large. Erdős used this to show there exists a distance occurring:

$$n^{1 + \frac{c}{\log \log n}}.$$

This is significantly larger than the $n\sqrt{\log n}$ bound derived from the Pigeonhole Principle on generic integers.

Theorem 3.4 (Upper Bound for Unit Distances). *The number of unit distances is $O(n^{3/2})$.*

Proof. Given n points in \mathbb{R}^2 , construct a graph where vertices are points and $v \sim u$ if $\|v - u\| = 1$. Observe the graph is $K_{2,3}$ -free. Geometrically, this is because two circles with radius 1 intersect at most 2 times; thus, no 2 vertices can share 3 common neighbors.

By KST with $s = 2, t = 3$:

$$e(G) \leq \frac{1}{2} \sqrt{3-1} \cdot n^{2-\frac{1}{2}} + o(n^2) = \frac{1}{\sqrt{2}} n^{\frac{3}{2}} + o(n^2).$$

□

3.4 KST for 3-Uniform Hypergraphs

Definition 3.5. Define $K_{s_1, s_2, t}^{(3)}$ to be the complete 3-uniform, 3-partite hypergraph with parts of size s_1, s_2, t .

Theorem 3.6 (Generalization of KST to Hypergraphs). *For the 3-uniform case:*

$$ex(n, K_{s_1, s_2, t}^{(3)}) \leq C_{s_1, s_2, t} n^{3 - \frac{1}{s_1 s_2}} + O(n^2).$$

Proof. We will count the number of stars $K_{1,1,s_1}$ (sets of edges sharing a common pair of vertices). Let X be this number. For any pair $\{v_1, v_2\}$, let d_{v_1, v_2} be the co-degree (number of edges containing $\{v_1, v_2\}$).

1. **Upper Bound on X :** Fix a set U of size s_1 . Define an auxiliary graph G_U on $V(G)$ where $v_1 \sim v_2$ in G_U if $\{v_1, v_2, u\} \in E(G)$ for all $u \in U$. If G_U contains a copy of $K_{s_2, t}$, then combined with U , we form a $K_{s_1, s_2, t}^{(3)}$. Thus, G_U must be $K_{s_2, t}$ -free.

$$X \leq \binom{n}{s_1} ex(n, K_{s_2, t}) \leq C \binom{n}{s_1} n^{2 - \frac{1}{s_2}}.$$

2. **Lower Bound on X :** Summing over pairs:

$$X = \sum_{\{v_1, v_2\} \in \binom{V}{2}} \binom{d_{v_1, v_2}}{s_1} \geq \binom{n}{2} f_{s_1} \left(\frac{3m}{\binom{n}{2}} \right).$$

Combining bounds (assuming m is large enough):

$$\binom{n}{2} \frac{\left(\frac{3m}{\binom{n}{2}} \right)^{s_1}}{s_1!} \lesssim \frac{n^{s_1}}{s_1!} n^{2 - \frac{1}{s_2}}.$$

Simplifying:

$$n^2 \left(\frac{m}{n^2} \right)^{s_1} \lesssim n^{s_1 + 2 - \frac{1}{s_2}} \implies m^{s_1} \lesssim n^{3s_1 - \frac{1}{s_2}}.$$

Taking the s_1 -th root:

$$m \leq C n^{3 - \frac{1}{s_1 s_2}}.$$

□

4 Supersaturation and Erdős–Stone (Jan 21)

4.1 Supersaturation Lemma

The classical Turán problem asks for the maximum number of edges in a graph avoiding a subgraph F . Supersaturation asks: what happens if we have just *slightly* more edges than this threshold? It turns out we don't just get one copy of F , but exponentially many.

Lemma 4.1 (Supersaturation Lemma). *If G is a graph on n vertices with $e(G) \geq (\pi(F) + \epsilon) \binom{n}{2}$ edges, then G contains at least $\delta(\epsilon)n^{|V(F)|}$ copies of F , where $\delta(\epsilon) > 0$ is a constant independent of n .*

Corollary 4.2. *For the complete graph K_{r+1} , we know $\pi(K_{r+1}) = 1 - \frac{1}{r}$. Thus:*

$$e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right) \binom{n}{2} \implies G \text{ contains } \delta(\epsilon)n^{r+1} \text{ copies of } K_{r+1}.$$

4.2 Hypergraph Kővári–Sós–Turán

We generalize the KST theorem from bipartite graphs to r -uniform hypergraphs. We are looking for the Turán number of the complete r -partite hypergraph $K_{s_1, \dots, s_r}^{(r)}$. Note that the theorem statement often simplifies the last parameter to t .

Theorem 4.3 (Generalized Kővári–Sós–Turán). *Let $K = K_{s_1, s_2, \dots, s_{r-1}, t}^{(r)}$ be the complete r -partite r -uniform hypergraph with part sizes s_1, \dots, s_{r-1}, t . Then:*

$$ex(n, K) = O_{s,t} \left(n^{r - \frac{1}{s_1 s_2 \dots s_{r-1}}} \right).$$

Proof. We proceed by induction on the uniformity r . The base case $r = 2$ is the standard graph KST theorem. Assume the bound holds for $(r-1)$ -uniform hypergraphs.

Let G be an r -uniform hypergraph with n vertices and m edges. We define a “hypergraph star” $K_{1,1,\dots,1,s_1}^{(r)}$ as a set of s_1 edges that all share a common core of $r-1$ vertices.

Let X be the number of such stars (copies of $K_{1,1,\dots,1,s_1}^{(r)}$) in G . We double count X .

1. Lower Bound (Convexity): For every subset $S \in \binom{V(G)}{r-1}$ of size $r-1$, let the *co-degree* d_S be the number of edges in G containing S . A set S contributes $\binom{d_S}{s_1}$ stars to the count.

$$X = \sum_{S \in \binom{V(G)}{r-1}} \binom{d_S}{s_1}.$$

Using Jensen's Inequality (and the convexity of $\binom{x}{s_1}$), we bound this sum:

$$X \geq \binom{n}{r-1} \binom{\bar{d}}{s_1}, \quad \text{where } \bar{d} = \frac{\sum d_S}{\binom{n}{r-1}} = \frac{rm}{\binom{n}{r-1}}.$$

Approximating for large n : $\bar{d} \approx \frac{r!m}{n^{r-1}}$ and $\binom{x}{k} \approx \frac{x^k}{k!}$, we get:

$$X \gtrsim n^{r-1} \cdot \left(\frac{m}{n^{r-1}} \right)^{s_1}. \quad (1)$$

2. Upper Bound (Induction): Fix a set of vertices $V_1 = \{v_1, \dots, v_{s_1}\}$. We ask: how many ways can we complete this into a star?

Define an auxiliary $(r-1)$ -uniform hypergraph G_{V_1} on $V(G)$. A set e' of size $r-1$ is an edge in G_{V_1} if $e' \cup \{v_i\} \in E(G)$ for all $i = 1, \dots, s_1$. In other words, e' is in the common neighborhood of all v_i .

If G_{V_1} contains a copy of $K_{s_2, \dots, s_{r-1}, t}^{(r-1)}$, then combined with V_1 , we would form a copy of the forbidden $K_{s_1, \dots, t}^{(r)}$ in G . Thus, G_{V_1} must be free of $K_{s_2, \dots, t}^{(r-1)}$. By the inductive hypothesis:

$$e(G_{V_1}) \leq C \cdot n^{(r-1) - \frac{1}{s_2 \dots s_{r-1}}}.$$

Summing over all choices of V_1 (there are $\binom{n}{s_1}$ such sets):

$$X \leq \binom{n}{s_1} \cdot \max_{V_1} e(G_{V_1}) \lesssim n^{s_1} \cdot n^{r-1 - \frac{1}{s_2 \dots s_{r-1}}}. \quad (2)$$

3. Conclusion: Comparing (1) and (2):

$$n^{r-1} \left(\frac{m}{n^{r-1}} \right)^{s_1} \lesssim n^{s_1 + r - 1 - \frac{1}{s_2 \cdots s_{r-1}}}.$$

Rearranging for m :

$$m^{s_1} \lesssim n^{(r-1)s_1} \cdot n^{s_1 - \frac{1}{s_2 \cdots s_{r-1}}} = n^{rs_1 - \frac{1}{s_2 \cdots s_{r-1}}}.$$

Taking the s_1 -th root:

$$m \lesssim n^{r - \frac{1}{s_1 s_2 \cdots s_{r-1}}}.$$

□

4.3 The Erdős-Stone Theorem

Definition 4.4 (Blow-up). A **blow-up** of a graph H , denoted $H(s)$, is obtained by replacing each vertex $v \in V(H)$ with an independent set I_v of size s , and replacing each edge $(u, v) \in E(H)$ with a complete bipartite graph between I_u and I_v .

The Turán graph $T_{n,r}$ is essentially a blow-up of K_r with parts of size n/r .

Theorem 4.5 (Erdős-Stone). If $e(G) \geq (1 - \frac{1}{r} + \epsilon) \binom{n}{2}$ and n is sufficiently large, then G contains $T_{s,(r+1)}$ (a blow-up of K_{r+1} with parts of size s).

Consequently, G contains any graph H with $\chi(H) = r + 1$ (since any such H is a subgraph of a large enough blow-up of K_{r+1}).

Proof. Step 1: Find many cliques. By the Supersaturation Corollary, since the edge density is greater than the Turán threshold for K_{r+1} , G contains many copies of K_{r+1} :

$$\#K_{r+1} \geq \delta n^{r+1}.$$

Step 2: Construct Auxiliary Hypergraph. Define an $(r + 1)$ -uniform hypergraph \mathcal{H} with vertex set $V(G)$. Let a set of $r + 1$ vertices form an edge in \mathcal{H} if and only if they form a K_{r+1} in the original graph G .

From Step 1, we know $e(\mathcal{H}) \geq \delta n^{r+1}$. Since δ is constant, this is a dense hypergraph.

Step 3: Find a dense structure in the hypergraph. We apply the **Hypergraph KST Theorem** to \mathcal{H} . Since \mathcal{H} is dense (order n^{r+1}), it exceeds the KST threshold (which is roughly $n^{(r+1)-\epsilon}$). Therefore, \mathcal{H} must contain a copy of the complete $(r + 1)$ -partite hypergraph $K_{s,s,\dots,s}^{(r+1)}$.

Step 4: Map back to G . Let the parts of this hypergraph copy be U_1, \dots, U_{r+1} , each of size s . The definition of the complete hypergraph implies that for *any* choice of vertices $u_1 \in U_1, \dots, u_{r+1} \in U_{r+1}$, the set $\{u_1, \dots, u_{r+1}\}$ is an edge in \mathcal{H} .

By definition of \mathcal{H} , this means $\{u_1, \dots, u_{r+1}\}$ forms a clique K_{r+1} in G . If every possible tuple forms a clique, then every pair of vertices in distinct parts U_i, U_j must be connected in G .

Thus, $G[U_1 \cup \dots \cup U_{r+1}]$ contains the complete multipartite graph with parts of size s , which is exactly $T_{s(r+1)}$. □

4.4 Degenerate Turán Numbers (Bipartite Graphs)

When H is bipartite ($r = 1$), the Erdős-Stone bound gives $1 - 1/1 = 0$, which is trivial. We rely on KST bounds $ex(n, K_{s,t}) = O(n^{2-1/s})$. We examine the sharpness of these bounds.

Theorem 4.6 (Tightness of KST).

1. **Case $s = 1$ (Trees/Forests):**

$$ex(n, K_{1,t}) \leq \frac{1}{2}(t-1)n.$$

This is tight for a graph consisting of disjoint copies of K_t (or a matching if $t = 1$).

2. **Case $s = 2$ (C_4 -free graphs):** We know $ex(n, K_{2,2}) = O(n^{3/2})$. This is tight.

Construction (Finite Geometry): Let q be a prime power. Construct a bipartite graph G with parts A and B , where $|A| = |B| = q^2$.

- Identify vertices in A and B with points in the affine plane \mathbb{F}_q^2 .
- Define edge $(a, b) \sim (x, y)$ if $ax + by = 1$, where $(a, b) \in A$ and $(x, y) \in B$.

For a fixed vertex $v = (a, b) \in A$, its neighborhood is the set of solutions (x, y) to the linear equation $ax + by = 1$. This describes a line in \mathbb{F}_q^2 , which contains q points (unless $(a, b) = (0, 0)$, which we can discard).

$$e(G) \approx q^2 \cdot q = q^3.$$

Since $n = 2q^2$, we have $q \approx \sqrt{n/2}$, so $e(G) \approx (n/2)^{3/2} = \Theta(n^{3/2})$.

Why is it C_4 -free? A C_4 corresponds to two vertices in A sharing two common neighbors in B . Geometrically, this means two distinct lines intersect at two distinct points. In a plane, two distinct lines intersect at most at one point. Thus, no C_4 exists.

3. **Case $s = 3$ ($K_{3,3}$ -free):** Tightness $ex(n, K_{3,3}) = \Theta(n^{5/3})$ is known.

Construction Sketch: Vertices are points in \mathbb{F}_q^3 . Adjacency is defined by $(x, y, z) \sim (a, b, c)$ if $(x - a)^2 + (y - b)^2 + (z - c)^2 = 1$. Geometrically, the neighborhood of a point is a unit sphere. If G contained $K_{3,3}$, three vertices in one part would share three common neighbors. This would imply that three unit spheres intersect at three points. In Euclidean geometry, three spheres intersect at most at 2 points (a circle intersects a sphere at 2 points). This logic carries over to \mathbb{F}_q provided -1 is not a square.

4. **Case $s = 4$ ($K_{4,4}$ -free):** It is an **open problem** whether the KST bound $O(n^{2-1/4})$ is tight for $K_{4,4}$.

4.5 Extremal Numbers for Trees

Lemma 4.7. If a graph G has average degree d , it contains a subgraph H with minimum degree $\delta(H) \geq d/2$.

Proof. Let $n = |V(G)|$. We have $e(G) = \frac{dn}{2}$. Iteratively delete any vertex with degree strictly less than $d/2$. Let S be the set of removed vertices. The number of edges removed is strictly less than $|S| \cdot \frac{d}{2}$. Even if we remove $n - 1$ vertices, the number of edges removed is $< (n - 1) \frac{d}{2} < \frac{dn}{2}$. Thus, the edge set cannot be emptied. The process must stop at a non-empty subgraph where every remaining vertex has degree $\geq d/2$. \square

Theorem 4.8. If T is a tree with r edges (so $r + 1$ vertices), then:

$$ex(n, T) \leq rn.$$

Proof. Suppose $e(G) > rn$. Then the average degree of G is $d(G) = \frac{2e(G)}{n} > 2r$. By the Lemma, there exists a subgraph $H \subseteq G$ with minimum degree $\delta(H) > r$. Since degrees are integers, $\delta(H) \geq r$.

We embed T into H greedily:

1. Order the vertices of T as v_1, v_2, \dots, v_{r+1} such that each v_i (for $i > 1$) has exactly one neighbor among $\{v_1, \dots, v_{i-1}\}$. (This is always possible by doing a BFS/DFS from a root).
2. Map v_1 to any vertex $u_1 \in V(H)$.
3. Suppose we have embedded v_1, \dots, v_{i-1} as u_1, \dots, u_{i-1} . Let v_j be the parent of v_i in the tree ($j < i$).
4. We need to map v_i to a neighbor of u_j in H .
5. The vertex u_j has at least r neighbors in H . We have used at most $i - 1$ vertices so far. Since $i \leq r + 1$, we have used at most r vertices.
6. We only care that u_j has neighbors not already used in the current tree embedding. The degree of u_j is $\geq r$. The number of currently used vertices is $i - 1 \leq r$.
7. The bound requires strict inequality or careful counting. If $\delta(H) \geq r$, u_j has r neighbors. We have used $i - 1$ vertices total. If $i - 1 < r$, there is space. If $i - 1 = r$, we are placing the last vertex. The neighbors of u_j could be occupied by the other $r - 1$ vertices of the tree.
8. However, the conjecture is actually $\frac{1}{2}(r)n$. The bound rn is loose enough. If $e(G) > rn$, $\text{avg deg} > 2r$, $\text{min deg} \geq r + 1$. Then u_j has $r + 1$ neighbors, but the tree only has $r + 1$ vertices total. At step i , only $i - 1 \leq r$ vertices are occupied. Thus there is always at least one free neighbor for u_j . \square

5 Jan 23

5.1 Turan Number of Cycles and Girth Problem

Definition 5.1. If \mathcal{F} is a family of graphs, $ex(n, \mathcal{F}) =$ maximum number of edges in an n vertex graph G that contains no $F \in \mathcal{F}$.

Lemma 5.2. $ex(n, \mathcal{F}) = \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}$ if $r = \min_{F \in \mathcal{F}} \mathcal{X}(F)$

Proof.

$$ex(n, \mathcal{F}) \leq \min_{F \in \mathcal{F}} ex(n, F).$$

By Erdos Stone, we're done. □

Theorem 5.3. Let $\mathcal{F} = \{C_3, \dots, C_{2k}\}$ then $ex(n, \mathcal{F}) \leq n^{1+\frac{1}{k}}$

Lemma 5.4. G is \mathcal{F} free iff G has girth $\geq 2k + 1$

Lemma 5.5. Every graph G contains a subgraph G' that is bipartite with $e(G') \geq \frac{1}{2}e(G)$.

Proof. Color vertices red or blue uniformly and independent. Then delete the edges that are monochromatic. By independence, for any $e \in E(G)$

$$\Pr[e \in E(G')] = \frac{1}{2} \implies E[e(G')] = \frac{1}{2}e(G).$$

□

We can also prove this with the greedy algorithm and will achieve a slightly better bound.

Proof. To prove theorem 5.3:

Suppose G has at least $n^{1+\frac{1}{k}}$, by lemma 5.2 we may replace G by a sub-graph of min deg $d \geq n^{1+\frac{1}{k}}$.

We can do a depth first search by picking a root vertex v . Let V_i be the set of vertices at distance i from v . So $V_0 = \{v\}$. Observe that $G[V_0, V_1, \dots, V_k]$ forms a tree with root v and V_i being the descendent at level i else we get a cycle of length at most $2k$. So we have

$$|V_0| + |V_1| + \dots + |V_k| \geq 1 + d + (d-1)d + d(d-1)^2 + \dots + d(d-1)^{k-1} = 1 + d^k > n.$$

So, we have a contradiction. □

Theorem 5.6. $ex(n, C_{2k}) \leq c_k n^{1+\frac{1}{k}}$

Theorem 5.7. $ex(n, \{C_4, C_6, \dots, C_{2k}\}) \leq 2n^{1+\frac{1}{k}}$

Remark 5.8. The exponent $1 + \frac{1}{k}$ is sharp if $k = 2, 3, 5$

5.2 Turan Number for Paths

Theorem 5.9. Let P_k be a path with k edges then

$$ex(n, P_k) \leq \frac{k}{2}n.$$

Proof. By induction on n it is enough to work with connected graph as we can just split into connected components. Delete vertices of degree $\leq \frac{k}{2}$. If $e(G) > \frac{k}{2}n$ then this also holds after deletions. So WLOG G has minimum degree at least $\frac{k}{2}$.

Let v_0, v_1, \dots, v_r be the longest path in G . The neighbors of v_0, v_r are in the path otherwise you can extend the path. If $v_0 \sim v_r$ then $v_0 v_1 \dots v_r$ is a cycle and so $v_i v_{i+1} \dots v_r v_0 \dots v_{i-1}$ is also a path of same length. Thus, the neighbors of v_i are inside the path. So $\{v_0, v_1, \dots, v_r\}$ is a connected component of G and thus $r \geq k + 1$ else $e(G) \leq \frac{(r-1)n}{2}$. Otherwise, if $v_0 \not\sim v_r$ then there exist i such that $v_0 \sim v_{i+1}$ and $v_r \sim v_i$ then we can produce a cycle $v_0 v_{i+1} v_{i+2} \dots v_r v_i v_{i-1} \dots v_0$ and this is of length r , so it becomes the previous case. By pigeonhole, such a i exist as $\{i : v_0 v_{i+1} \in E(G)\}$ and $\{i : v_r v_i \in E(G)\}$ must intersect as $r \leq k$ by $v_0 v_r \notin E(G)$ □

6 Jan 26: Projective Geometry

Definition 6.1. Projective Plane over a Field. Let \mathbb{F} be a field.

1. *Points:* The set of lines through 0 in \mathbb{F}^3 ; equivalently, the 1-dimensional subspaces.
2. *Lines:* The set of planes through 0 in \mathbb{F}^3 ; equivalently, the 2-dimensional subspaces.
3. *Incidence:* A point P lies on a line ℓ if $P \subset \ell$.

Observe:

1. Any two points determine a unique line.
2. Any two lines intersect in a unique point.
3. There exist 4 points no three of which are collinear.

Inside a projective plane over \mathbb{F} , there is a copy of the affine plane \mathbb{F}^2 embedded as $\{z = 1\}$.

We denote this by $\mathbb{P}^2(\mathbb{F})$.

We say $p \sim \ell$ if $p \in \ell$ for p a point and ℓ a line of $\mathbb{P}^2(\mathbb{F}_q)$.

Remark 6.2. There are $q^3 - 1$ nonzero vectors in $\mathbb{F}_q^3 \setminus \{0\}$. We identify points of $\mathbb{P}^2(\mathbb{F}_q)$ with equivalence classes of nonzero vectors under

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \quad \text{for } \lambda \in \mathbb{F}_q^\times.$$

There are $q - 1$ nonzero choices for λ , so the number of points in $\mathbb{P}^2(\mathbb{F}_q)$ is

$$\frac{q^3 - 1}{q - 1} = q^2 + q + 1.$$

By a similar argument, the number of lines is also $q^2 + q + 1$.

Each line contains exactly $q + 1$ points and dually, each point lies on exactly $q + 1$ lines. There is a natural bijection (called a polarity) between points and lines, preserving incidence: given a point p , its corresponding line is the orthogonal complement p^\perp .

Theorem 6.3. Construction of a C_4 -free graph:

We construct an improved C_4 -free graph as follows:

- The vertices are the points of $\mathbb{P}^2(\mathbb{F}_q)$.
- Points p_1 and p_2 are adjacent if and only if $p_1 \in p_2^\perp$ (i.e., p_1 lies on the line associated to p_2 's orthogonal complement).

For any pair p_1, p_2 , their common neighborhood is

$$p_1^\perp \cap p_2^\perp,$$

which is a single point. Thus, the graph is C_4 -free (contains no 4-cycles). This gives $q^2 + q + 1$ vertices with average degree $\approx q$.

Note: There exist p such that $p \subseteq p^\perp$.

Definition 6.4. General Projective Planes. If P (points) and L (lines) are sets with an incidence relation such that:

1. For any two points, there is a unique line through them.
2. Any two lines meet at a unique point.
3. There exist four points, no three collinear.



Example: The Fano plane (order 2) with 7 points and 7 lines. Each line contains 3 points, and each point lies on 3 lines.

Definition 6.5. Order of a Projective Plane. *The order of a projective plane is the number of points per line minus one (so q if each line has $q + 1$ points).*

Remark 6.6. *Are there planes of order which are not prime powers?*

Theorem 6.7. Construction of C_{2k} -free graphs for $k = 2, 3, 5$:

Let q be a prime power and let \mathbb{F}_q be the finite field with q elements. Define a graph G on the vertex set \mathbb{F}_q^k (the set of k -tuples over \mathbb{F}_q) as follows:

For $\mathbf{a} = (a_0, a_1, \dots, a_{k-1})$ and $\mathbf{b} = (b_0, b_1, \dots, b_{k-1})$ in \mathbb{F}_q^k , declare $\mathbf{a} \sim \mathbf{b}$ if and only if for all $j = 0, 1, \dots, k-2$,

$$b_j = a_j + a_{j+1} \cdot b_{k-1}$$

where indices are taken modulo k (so $a_k = a_0$).

Then G is C_{2k} -free, i.e., it contains no cycle of length $2k$.

Proof. Suppose for contradiction that G contains a cycle of length $2k$:

$$\mathbf{a}^1, \mathbf{b}^1, \mathbf{a}^2, \mathbf{b}^2, \dots, \mathbf{a}^k, \mathbf{b}^k,$$

with $\mathbf{a}^i \sim \mathbf{b}^i \sim \mathbf{a}^{i+1}$ for all i , indices modulo k .

Step 1: System of equations. For each $i = 1, \dots, k$ and $j = 0, \dots, k-2$, adjacency gives:

$$b_j^i = a_j^i + a_{j+1}^i \cdot b_{k-1}^i \quad (\text{from } \mathbf{a}^i \sim \mathbf{b}^i)$$

$$b_j^i = a_j^{i+1} + a_{j+1}^{i+1} \cdot b_{k-1}^i \quad (\text{from } \mathbf{b}^i \sim \mathbf{a}^{i+1})$$

Subtract:

$$(a_j^i - a_j^{i+1}) + (a_{j+1}^i - a_{j+1}^{i+1}) \cdot b_{k-1}^i = 0$$

or

$$(a_j^{i+1} - a_j^i) + (a_{j+1}^{i+1} - a_{j+1}^i) \cdot b_{k-1}^i = 0$$

Summing over i :

$$\sum_{i=1}^k (a_j^{i+1} - a_j^i) + \sum_{i=1}^k (a_{j+1}^{i+1} - a_{j+1}^i) b_{k-1}^i = 0$$

The first sum telescopes to zero (since indices are cyclic), so for each $j = 0, \dots, k-2$,

$$\sum_{i=1}^k (a_{j+1}^{i+1} - a_{j+1}^i) b_{k-1}^i = 0 \quad (3)$$

Step 2: Specialize $j = k - 2$. Let $\Delta_i = a_{k-1}^{i+1} - a_{k-1}^i$ and $x_i = b_{k-1}^i$. Then:

$$\sum_{i=1}^k \Delta_i x_i = 0$$

and

$$\sum_{i=1}^k \Delta_i = 0$$

by telescoping.

Step 3: Linear algebra. The system (3) for $j = 0, \dots, k - 2$ can be written as a $(k - 1) \times k$ Vandermonde-type matrix V applied to the vector $(\Delta_1, \dots, \Delta_k)$:

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ \vdots & \vdots & & \vdots \\ x_1^{k-2} & x_2^{k-2} & \cdots & x_k^{k-2} \end{pmatrix}$$

The equations become $V \cdot (\Delta_1, \dots, \Delta_k)^\top = 0$. Since $\sum_{i=1}^k \Delta_i = 0$, the vector is in the kernel of the row $(1, 1, \dots, 1)$.

Step 4: Key property of the x_i .

Claim 6.8. For each i , $x_i \neq x_{i+1}$ (indices modulo k).

Proof. If $x_i = x_{i+1}$, then $b_{k-1}^i = b_{k-1}^{i+1}$. By the adjacency relations, this forces $\mathbf{b}^i = \mathbf{b}^{i+1}$, so $\mathbf{a}^{i+1} = \mathbf{a}^{i+2}$, contradicting that the cycle is proper. \square

Step 5: Contradiction for $k = 2, 3, 5$. Since not all Δ_i are zero (otherwise the cycle is trivial), and $\sum_{i=1}^k \Delta_i = 0$, equation $\sum_{i=1}^k \Delta_i x_i = 0$ forces a linear dependence among the x_i (not all distinct). Thus, for these small k we can classify possibilities:

Claim 6.9. For $k = 2, 3, 5$, if $x_i = x_j$ for some $i \neq j$, then $x_i = x_{i+1}$ for some i , contradicting the above claim.

- **Case $k = 2$:** Only possible if $x_1 = x_2$, violating $x_1 \neq x_2$.
- **Case $k = 3$:** If $x_1 = x_2$, $x_2 = x_3$, or $x_3 = x_1$, we violate the claim.
- **Case $k = 5$:** If $x_i = x_j$ for $i \neq j$, by relabeling, could assume $x_1 = x_j$ for $j = 2, 3, 4, 5$:
 - $j = 2$ or 5 gives $x_1 = x_2$ or $x_5 = x_1$, directly a violation.
 - $j = 3$: $x_1 = x_3$. The system $\sum_{i=1}^5 \Delta_i = 0$, $\sum_{i=1}^5 \Delta_i x_i = 0$ with $x_1 = x_3$ and $x_i \neq x_{i+1}$ for any i only allows further coincidences that eventually force $x_i = x_{i+1}$.
 - $j = 4$: similarly, $x_1 = x_4$ eventually forces a pair $x_i = x_{i+1}$.
- **Case $k = 4$:** The pattern $x_1 = x_3 \neq x_2 = x_4$ is possible with no consecutive equal x_i , which is why this construction fails for $k = 4$.

Therefore, for $k = 2, 3, 5$, no proper $2k$ -cycle exists in G . \square

7 Jan 28

Recall: $ex(n, K_{s,t}) \leq c_{s,t} n^{2-\frac{1}{s}} + o(n)$. We will see that $ex(n, K_{s,t}) = \theta \left(n^{2-\frac{1}{s}} \right)$ for $t \geq t_0(s)$

Theorem 7.1. *Construction of the extremal graph for $ex(n, K_{s,t})$.*

Proof. We will first consider a random graph by picking edges with probability $p = n^{-\frac{1}{s}}$. Let $x^{(1)}, \dots, x^{(s)}$ be s vertices chosen uniformly at random without replacement on the left. Let $N(x^{(1)}, \dots, x^{(s)})$ that are the common neighbors with $d(x^{(1)}, \dots, x^{(s)}) = |N(\dots)|$. Let our random variable be $d(x^{(1)}, \dots, x^{(s)})$ and is distributed according to

$$\text{Binomial}(n, p^s) = \text{Binomial}\left(n, \frac{1}{n}\right) \sim \text{Poisson}(1).$$

So $\mathbb{E}[d(x^{(1)}, \dots, x^{(s)})] = 1$ and $\mathbb{E}[d(\dots)^k] = o_k(1)$. With probability $\frac{1}{n^\epsilon}$ there is at least $\epsilon \log n$ neighbors. Yet there are $\binom{n}{s}$ such s tuples $(x^{(1)}, \dots, x^{(s)})$. and then $\frac{n}{s}$ disjoint. This does not work.

Consider another construction of $K_{s,t}$ -free graph with $\Omega(n^{2-\frac{1}{s}})$ edges.

For a bipartite graph with \mathbb{F}_p^s where $n = 2p^s$ on each side we say $x \sim y$ if $f(x, y) = 0$. Pick equation f uniformly at random among all polynomials in s, t of degree at most d in each of x and y . It would be

$$f = \sum_{|\alpha|, |\beta| \leq d} c_{\alpha\beta} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} y_1^{\beta_1} y_2^{\beta_2} \dots y_s^{\beta_s}. \quad (c_{\alpha\beta} \in \mathbb{F}_p, |\alpha| = \alpha_1 + \dots + \alpha_d)$$

Part I: This random graph is locally (neighborhood look "similar") similar to uniform random graph.

Part II: Globally very different/more discrete. \square

Lemma 7.2. *For any fixed $x, y \in \mathbb{F}_p^s$*

$$\Pr[x \sim y] = \frac{1}{p}.$$

Proof. We have

$$f = c_{0,0} + \sum_{\alpha, \beta \neq 0} c_{\alpha\beta} x^\alpha y^\beta.$$

For any choice of $c_{\alpha, \beta}$ for $(\alpha, \beta) \neq (0, 0)$ there is a unique choice of $c_{0,0}$ such that $f(x, y) = 0$.

$$\mathbb{E}[e(G)] = \frac{1}{p} n^2 = c_s n^{2-\frac{1}{s}}.$$

\square

Let $x^{(1)}, \dots, x^{(s)}$ be s vertices on say left, $N(x^{(1)}, \dots, x^{(s)})$ be the common neighbors with of $x^{(1)}, \dots, x^{(s)}$ and this is same as $\{y \in \mathbb{F}_p^s : f(x^{(1)}, y) = 0, \dots, f(x^{(s)}, y) = 0\}$.

Lemma 7.3. *If g is a random polynomial of degree of D in m variables and $z^{(1)}, \dots, z^{(t)} \in \mathbb{F}_p^m$ then the random variables $g(z^{(1)}), \dots, g(z^{(t)})$ are independent if $D > t$ and $p > t^2$.*

Warm up: When $m = 1$, write $g(x) = \underbrace{a_0 + a_1 x + \dots + a_{t-1} x^{t-1}}_{g_{\text{small}}(x)} + \underbrace{\dots + a_D x^D}_{g_{\text{large}}(x)}$.

For any choice of $g_{\text{large}}(x)$ and any values b_1, \dots, b_t there is a unique polynomial g_{small} such that $g(z^{(i)}) = g_{\text{small}}(z^{(i)}) + g_{\text{large}}(z^{(i)}) = b_i$ for $i = 1, \dots, t$ if and only if $g_{\text{small}}(z^{(i)}) = b_i - g_{\text{large}}(z^{(i)})$ by lagrange interpolation.

For the general case, if $z_1^{(1)}, \dots, z_1^{(t)}, \dots, z_m^{(1)}, \dots, z_m^{(t)}$ are distinct, then $g(x_1, \dots, x_m) = \bar{g}(x_1) + \text{other terms}$ and use $m = 1$ case. Pick an invertible linear transform $f : \mathbb{F}_p^s \rightarrow \mathbb{F}_p^s$ such that $Tz^{(1)}, \dots, Tz^{(t)}$ are distinct. To pick such a T , let $T(x_1, \dots, x_s) = (x_1 + S(x_2, \dots, x_s), x_2, \dots, x_s)$. For $z^{(i)}, z^{(j)}$ distinct, the space of S for which $T(z^{(i)}) = T(z^{(j)})$ is of co-dimension 1, so a random S has probability $\frac{1}{p}$ of satisfying this. So $\Pr[\exists i, j \text{ such that } T(z^{(i)}) = T(z^{(j)})] \leq \frac{1}{p} \binom{t}{2} < \frac{1}{p}$.

We can now conclude from

$$\Pr[f(x^{(1)}, y) = 0, \dots, f(x^{(s)}, y) = 0] = \frac{1}{p^s}.$$

If $d > s$ and $p > s^2$ then

$$\mathbb{E}[N(x^{(1)}, \dots, x^{(s)})] = 1.$$

Furthermore, if α is a positive integer and $p \cdot s < d$ then

$$\mathbb{E} \left[\left| N(x^{(1)}, \dots, x^{(s)}) \right|^p \right] = \mathbb{E} \left[\left(\sum_{y \in \mathbb{F}_p^s} R(y) \right)^\alpha \right] = \mathbb{E} \left[\sum_{y^{(1)}, \dots, y^{(\alpha)}} R(y^{(1)}) \dots R(y^{(\alpha)}) \right].$$

Where $R(y) = \begin{cases} 1 & \text{if } y \in N(x^{(1)}, \dots, x^{(s)}) \\ 0 & \text{otherwise} \end{cases}$

$$= \sum_{y^{(1)}, \dots, y^{(\alpha)}} p^{-s \cdot \# \text{distinct } y\text{'s}} \leq \sum_k \sum_{y^{(1)}, \dots, y^{(k)} \in \mathbb{F}_p^s} p^{-sk} = \sum_k p^{sk} p^{-sk} \cdot \# \text{ways to partition a objection into } k \text{ bins.}$$

Last equality from having to choose $y^{(i)}$ such that k are distinct. This is in $O(1)$. We want to assign values to $y^{(i)}$ such that exactly k are distinct. Equivalently, choose k distinct values in \mathbb{F}_p^s . Then choose which of the $y^{(i)}$ go to which bin.

$$\Pr[|N(x^{(1)}, \dots, x^{(s)})| \geq T] \leq \frac{\mathbb{E} \left[\left| N(x^{(1)}, \dots, x^{(s)}) \right|^\alpha \right]}{T^\alpha} = \frac{C_{\alpha, s}}{T^\alpha}.$$

The set $N(x^{(1)}, \dots, x^{(s)})$ is contained in $\{y \in \mathbb{F}_p^s : f(x^{(1)}, y) = 0, \dots, f(x^{(s)}, y) = 0\}$. A vector subspace V of \mathbb{F}_p^s of dimension D has p^D elements.

Theorem 7.4. *Lang-Weil Theorem: If g_1, \dots, g_r are polynomials of degree $\leq D$ in s variables over \mathbb{F}_p , then the set $\{y \in \mathbb{F}_p^s : g_1(y) = \dots = g_r(y) = 0\}$ has either $O_{D, r}(1)$ points, or more than $\frac{p}{2}$ points.*

So $\Pr[\text{some } s \text{ vertices have neighborhood of size } \geq O_{s, \alpha}(1)] \leq C_{s, \alpha} \left(\frac{p}{2}\right)^{-\alpha} \binom{p^s}{s}$. Choose $\alpha = s^2 + 1$. To ensure this probability goes to 0, we need to choose $d = s^2 + 1$.

8 Regularity Lemmas: Feb 4

Definition 8.1. Binary word is sequence $w_1, w_2, \dots, w_n \in \{0, 1\}$ and write $w = w_1 w_2 \dots, w_n$.
The density of w is

$$d(w) = \frac{\#1's \text{ in } w}{n}.$$

Definition 8.2. Word w is ϵ -regular if for any subword w' of w of length $\geq \epsilon \cdot \text{len}(w)$ satisfies

$$|d(w) - d(w')| \leq \epsilon.$$

Lemma 8.3. By Chernoff bound,

$$\Pr[|d(w') - d(w)| \geq \epsilon] \leq e^{-c_\epsilon n}.$$

9 Feb 6

Definition 9.1. Partition \mathcal{P}' is a refinement of partition \mathcal{P} if each $w' \in \mathcal{P}'$ is a sub-word of some word in \mathcal{P}

Theorem 9.2. (Regularity Lemma for Binary Words)

For all $\epsilon > 0$, every word $w \in \{0, 1\}^n$ can be partitioned into $\leq M(\epsilon)$ many sub-words $w^{(1)}, \dots, w^{(m)}$ (i.e. $w = w^{(1)} \dots w^{(m)}$). So that the total length of $w^{(i)}$'s that are ϵ -irregular is at most ϵn .

Proof. For a partition \mathcal{P} of w into sub-words $\mathcal{P} = \{w^{(1)}, \dots, w^{(m)}\}$ we define $f(\mathcal{P}) = \sum_{w' \in \mathcal{P}} d(w)^2 |w'|/|w|$. Alternatively, we pick a symbol of w at random, say $i \in [n]$ uniformly, then pick another symbol at random from the same part as i 's, say j is picked.

$$\Pr[w_i = w_j] = \sum_{w' \in \mathcal{P}} \frac{|w'|}{|w|} (d(w')^2 + (1 - d(w'))^2) = 2f(\mathcal{P}) + \sum_{w'} \frac{|w'|}{|w|} - 2 \sum_{w'} d(w') \frac{|w'|}{|w|} = 2f(\mathcal{P}) + 1 - 2d(w).$$

Lemma 9.3. If \mathcal{P}' is a refinement of \mathcal{P} then

$$f(\mathcal{P}') = f(\mathcal{P}) + \sum_{w' \in \mathcal{P}'} (d(w') - d(\text{parent}(w')))^2 \frac{|w'|}{|w|}.$$

Proof. Expanding the second term we have

$$\sum_{w' \in \mathcal{P}'} d(w') \frac{|w'|}{|w|} - 2d(w')d(\text{parent}(w')) + d(\text{parent}(w'))^2 \frac{|w'|}{|w|} = \sum_1 + \sum_2 + \sum_3.$$

We have $\sum_1 = f(\mathcal{P}')$ and

$$\begin{aligned} \sum_2 &= \sum_{\tilde{w} \in \mathcal{P}} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} \left(-2d(w')d(\tilde{w}) \frac{|w'|}{|w|} \right) \\ &= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w}) \frac{1}{|w|} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} d(w')|w'| \\ &= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w}) \frac{1}{|w|} \cdot d(\tilde{w})|\tilde{w}| \\ &= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w})^2 \frac{|\tilde{w}|}{|w|} \\ &= -2f(\mathcal{P}) \end{aligned}$$

Then for the third sum

$$\begin{aligned} \sum_3 &= \sum_{\tilde{w} \in \mathcal{P}} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} d(\tilde{w})^2 \frac{|w'|}{|w|} \\ &= \sum_{\tilde{w} \in \mathcal{P}} d(\tilde{w})^2 \frac{|\tilde{w}|}{|w|} = f(\mathcal{P}) \end{aligned}$$

So we have $\sum_1 + \sum_2 + \sum_3 = f(\mathcal{P}') - f(\mathcal{P})$ □

Start with the trivial partition (i.e. $\mathcal{P} = \{w\}$) and repeat while $\sum_{w' \in \mathcal{P}} |w'|$ is ϵ -irregular $|w'| \leq \epsilon |w|$.

For $w' \in \mathcal{P}$ that is ϵ -irregular, find a way to write as $w' = w^{(1)}w^{(2)}w^{(3)}$ where $|d(w^{(2)}) - d(w^{(1)})| \geq \epsilon$ and $|w^{(2)}| \geq \epsilon |w'|$. Replace w' with the three words $w^{(1)}, w^{(2)}, w^{(3)}$ to obtain a new partition of w , call it \mathcal{P}' . Repeat with \mathcal{P} replaced by \mathcal{P}' .

To analyze this algorithm, if we drop some words we have

$$f(\mathcal{P}') \geq f(\mathcal{P}) + \sum_{\substack{w' \in \mathcal{P}' \\ w' \text{ is } \epsilon\text{-irregular}}} \sum_{w^{(2)}} (d(w^{(2)}) - d(w')) \frac{|w^{(2)}|}{|w|} \geq f(\mathcal{P}) + \sum_{\substack{w' \in \mathcal{P}' \\ w' \text{ is } \epsilon\text{-irregular}}} \epsilon^2 \frac{|w'|}{|w|} = f(\mathcal{P}) + \epsilon^4.$$

Note: $f(\mathcal{P}) \leq 1$ for any partition \mathcal{P} .

So the number of steps in the algorithm is $\leq \frac{1}{\epsilon^4}$. The theorem holds with $M(\epsilon) = 3^{1/\epsilon^4}$ \square

Example 9.4. Define a twin in a word $w \in \{0,1\}^n$ is a pair of subsequences $x, y \in w$ that are disjoint (no symbol in both x and y) and are equal as word.

We define

$$t(w) = \text{the length of longest pair twins in } w \text{ and } t(n) = \min_{w \in \{0,1\}^n} t(w).$$

We have the trivial bounds $\frac{n}{4} \leq t(n) \leq \frac{n}{2}$.

Theorem 9.5.

$$t(n) \geq \left(\frac{1}{2} - o(1) \right) n.$$

Proof. Pick small $\epsilon > 0$ and cut w into ϵ -regular words $t \leq \epsilon n$ junk symbols. Enough $t(w) \geq (\frac{1}{2} - c\epsilon) |w|$ for ϵ -regular w .

Cut w into $\frac{1}{\epsilon}$ equally long sub-words $w^{(1)}, \dots, w^{(m)}$ where $m = \frac{1}{\epsilon}$.

The first twin is 0's from $w^{(1)}$ + 1's from $w^{(2)}$ + 0's from $w^{(3)}$...

The second twin is 1's from $w^{(1)}$ + 0's from $w^{(2)}$ + 1's from $w^{(3)}$...

Choose twins then

$$|\# \text{ 0's in } w^{(i)}| - |\# \text{ 1's in } w^{(2)}| \leq \epsilon |w^{(1)}|.$$

So, # symbols not in either of the twins is less than or equal to

$$\epsilon |w| (1's \text{ in } w^{(1)}) + \epsilon |w| (1's \text{ in } w^{(2)}) + \sum_i 4\epsilon |w^{(i)}|.$$

\square

10 Feb 13

Definition 10.1. For disjoint sets $U, V \subset V(G)$, the density between U and V is

$$d(U, V) = \frac{e(U, V)}{|U||V|}.$$

Definition 10.2. A pair (U, V) of disjoint subsets of $V(G)$ is ϵ -regular if $\forall U' \subseteq U, V' \subseteq V$ such that $|U'| \geq \epsilon|U|$ and $|V'| \geq \epsilon|V|$ we have

$$|d(U, V) - d(U', V')| \leq \epsilon.$$

Definition 10.3. We say a partition $V(G) = V_1 \cup \dots \cup V_k \cup J$ is ϵ -regular partition when

1. $|J| \leq \epsilon|V(G)|$
2. $|V_1| = |V_2| = \dots = |V_k|$
3. All except $\leq \epsilon k^2$ pairs (V_i, V_j) are ϵ -regular

Theorem 10.4. (Szemerédi's Regularity Lemma) $\forall \epsilon > 0, m$ there exist a constant $M = M(\epsilon, m)$ such that every graph admits an ϵ -regular partition $V(G) = V_1 \cup \dots \cup V_k \cup J$ where the number of parts $m \leq k \leq M$.

Remark 10.5. # edges inside parts + # edges adjacent to J + # edges in ϵ -irregular pairs is at most

$$k \binom{n/k}{2} + (\epsilon n) \cdot n + \epsilon k^2 (n/k)^2 \leq \frac{1}{k} n^2 + \epsilon n^2 + \epsilon n^2.$$

Proof. For a partition \mathcal{P} of $V := V(G)$, we define

$$f(\mathcal{P}) = \sum_{U, W \in \mathcal{P}, U \neq W} \underbrace{d(U, W)^2 \frac{|U||W|}{|V|^2}}_{f(U, W)}.$$

Lemma 10.6. Suppose $A, B \subset V$ are disjoint and we have partitions $A = A_1 \cup \dots \cup A_k$ and $B = B_1 \cup \dots \cup B_l$ then

$$\sum_{i, j} f(A_i, B_j) = f(A, B) + \sum_{i, j} (d(A, B) - d(A_i, B_j))^2 \frac{|A_i||B_j|}{|V|^2}.$$

Proof is exactly the same as the proof of the regularity lemma for binary words.

Algorithm: To find an ϵ -regular partition

1. Start with any equipartition of V into m parts.
2. We will start with partitions of the form $V = V_1 \cup V_2 \cup \dots \cup V_k \cup J_1 \cup \dots \cup J_\ell$
3. For each ϵ -irregular pair (V_i, V_j) there exist partitions $V_i = V_{i,1}^{(i,j)} \cup V_{i,2}^{(i,j)}$ and $V_j = V_{j,1}^{(i,j)} \cup V_{j,2}^{(i,j)}$ such that $|d(V_i, V_j) - d(V_{i,1}^{(i,j)}, V_{j,1}^{(i,j)})| \geq \epsilon$ and $|V_{i,1}^{(i,j)}| \geq \epsilon|V_i|$ and $|V_{j,1}^{(i,j)}| \geq \epsilon|V_j|$
Note: We write it as such since there may be many ϵ -irregular pairs (V_i, V_j) and we want to keep track of them.
4. For each i consider all partitions of V_i as $V_i = V_{i,1} \cup V_{i,2}$ and take the common refinement. Two V, V' are in the same part of the refinement iff V, V' are in the same part in each of the partitions.
Observe: Each V_i is cut into $\leq 2^k$ parts.

Let \mathcal{P} be the old partition and \mathcal{P}_{new} be the new partition.

$$\begin{aligned}
f(\mathcal{P}_{\text{new}}) &= \sum_{\substack{A, B \in \mathcal{P}_{\text{new}} \\ A \neq B}} f(A, B) \\
&\geq \sum_i f(J_i, ?) + \sum_{i, j} \sum_{\substack{A, B \\ \text{parent}(A)=V_i \\ \text{parent}(B)=V_j}} f(A, B) \\
&\geq f(\text{Junk}) + \sum_{i \neq j} \left(f(V_i, V_j) + \epsilon^2 \frac{|V_i||V_j|}{|V|^2} \right) \\
&= f(\mathcal{P}) + \epsilon^4 \left(\frac{|V| - |\text{Junk}|}{|V|} \right)^2 \\
&\geq f(\mathcal{P}) + \frac{\epsilon^4}{4}
\end{aligned}$$

5. Cut each of the parts in \mathcal{P}_{new} into equally big parts so total # parts is $\leq k \cdot 4^k$ and some left over where # of leftovers is $\leq k2^k$ and the size of them $\leq k2^k \cdot \frac{|V|}{k4^k} \leq \frac{|V|}{2^k}$

So the total junk size is $\leq \sum_{k \geq m} \frac{|V|}{2^k}$

The total number of iterations is at most $\frac{4}{\epsilon^4}$, and let $g(k) = k \cdot 4^k$. The total number of parts at the end is at most

$$M \leq g^{(\lceil 1/\epsilon^5 \rceil)}(m)$$

where $g^{(t)}$ denotes g composed with itself t times, starting from m ; that is, we iterate g at most $\frac{1}{\epsilon^5}$ times as $\epsilon k^2 \cdot \frac{\epsilon^4}{k^2} = \epsilon^5$.

So using $g(k) \leq 4^{4^k}$ we have

$$M \leq 4^{4^{4^{\dots 4^m}}} \left\} \frac{2}{\epsilon^5} \text{ times.} \right.$$

□

11 Feb 16

Lemma 11.1 (Triangle Counting Lemma). *Let $X, Y, Z \subset V(G)$ be disjoint with $d(X, Y), d(X, Z), d(Y, Z) \geq 2\epsilon$, and suppose $(X, Y), (X, Z), (Y, Z)$ are ϵ -regular. Then the number of XYZ -triangles is at least*

$$(1 - 2\epsilon)(d(X, Y) - \epsilon)(d(X, Z) - \epsilon)(d(Y, Z) - \epsilon)|X||Y||Z|$$

Proof. Observe the following:

1. $Y_{\text{bad}} = \{x \in X : \deg_Y(x) < (d(X, Y) - \epsilon)|Y|\}$
2. $Z_{\text{bad}} = \{x \in Z : \deg_Z(x) < (d(X, Z) - \epsilon)|Z|\}$

Then $|Y_{\text{bad}}|, |Z_{\text{bad}}| \leq \epsilon|X|$.

Consider Y_{bad} ,

$$e(Y_{\text{bad}}, Y) = \sum_{x \in Y_{\text{bad}}} \deg_Y(x) < |Y_{\text{bad}}|(d(X, Y) - \epsilon)|Y|.$$

By ϵ -regularity we have $d(X, Y) - \epsilon$ is within ϵ of $d(X, Y)$ so a contradiction if $|Y_{\text{bad}}| > \epsilon|X|$.

To prove the Triangle Counting Lemma, let $X' := X \setminus (Y_{\text{bad}} \cup Z_{\text{bad}})$ then $|X'| \geq (1 - 2\epsilon)|X|$ by observation.

Let $x \in X'$ then $N_Y(x) := N(x) \cap Y$ and $N_Z(x) := N(x) \cap Z$.

The number of triangles containing x is

$$e(N_Y(x), N_Z(x)) \geq |N_Y(x)||N_Z(x)|(d(Y, Z) - \epsilon). \quad (*)$$

By ϵ -regularity of (Y, Z) and definition of bad. Additionally

$$(*) \geq (d(X, Y) - \epsilon)(d(X, Z) - \epsilon)(d(Y, Z) - \epsilon)|Y||Z|.$$

So

$$\#\Delta = \sum_{x \in X'} (*) \geq (*) (1 - 2\epsilon)|X|.$$

□

Lemma 11.2. Triangle Removal Lemma: $\forall \epsilon > 0, \exists \delta > 0$ such that every graph with $\leq \delta n^3$ triangles can be made triangle free by deleting only ϵn^2 edges.

Proof. Pick $\epsilon' = \frac{\epsilon}{4}$ and $m = \frac{4}{\epsilon}$ and apply szemerédi's regularity lemma to G to get an ϵ' -regular partition $V(G) = V_1 \cup \dots \cup V_k \cup J$ where $m \leq k \leq M(\epsilon', m)$ and $|J| \leq \epsilon'n$. To clean up the graph

1. Delete all edges incident to J as we do not know anything about J
2. Delete all edges inside ϵ' -irregular parts
3. Delete all edges inside a single V_i
4. Delete edges between V_i and V_j if $d(V_i, V_j) < 2\epsilon'$

If at least one \triangle survives, say xyz with $x \in X, y \in Y, z \in Z$ with $X, Y, Z \in \{V_1, \dots, V_k\}$ then X, Y, Z satisfy the condition of triangle counting lemma. So we have

$$\#\Delta \in G \geq \#XYZ\Delta \geq (1 - 2\epsilon')(d(X, Y) - \epsilon')(d(X, Z) - \epsilon')(d(Y, Z) - \epsilon')|X||Y||Z| \geq (1 - 2\epsilon)\epsilon^3 \left(\frac{|V(G)|(1 - \epsilon')}{M} \right)^3.$$

So the total $\#$ edges deleted in steps (1)-(4) is at most

$$n|J| + \#\epsilon'\text{-irregular pairs} \cdot \left(\frac{n}{k}\right)^2 + k \binom{\frac{n}{k}}{2} + k^2 \cdot \epsilon' \left(\frac{n}{k}\right)^2 \leq \epsilon'n^2 + \epsilon'k^2 \left(\frac{n}{k}\right)^2 + \frac{n^2}{m} + \epsilon'n^2.$$

So the bound we get is $\delta \geq \frac{1}{\text{tower}(\frac{4}{\epsilon^5})}$

□