

Complex Analysis Notes

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1 Jan 12

Definition 1.1. Let Ω be an open set in \mathbb{C} and f a complex-valued function on Ω . A function f is holomorphic at a point $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}.$$

converges to a limit when $h \rightarrow 0$. The limit of the quotient, when it exists, is denoted $f'(z_0)$.

Theorem 1.2. A function $F(x, y) = (u(x, y), v(x, y))$ is said to be differentiable at a point $P_0(x_0, y_0)$ if there exist a linear transformation $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\frac{|F(P_0 + H) - F(P_0) - J(H)|}{|H|} \rightarrow 0 \text{ as } |H| \rightarrow 0.$$

Definition 1.3. The jacobian matrix of F is

$$J = J_F(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Theorem 1.4. Cauchy-Riemann Equations: If f is holomorphic then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof. If we write $z = x + iy$, $z_0 = x_0 + iy_0$ and $f(z) = f(x, y)$ then

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \rightarrow 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0) \end{aligned}$$

Similarly, now take h purely imaginary, say $h = ih_2$, then

$$\begin{aligned} f'(z_0) &= \lim_{h_2 \rightarrow 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} \\ &= \frac{1}{i} \frac{\partial f}{\partial y}(z_0) \end{aligned}$$

Since f is holomorphic we have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

Writing $f = u + iv$ we have the stated relation. \square

2 Jan 14

Theorem 2.1. $f : \Omega \rightarrow \mathbb{C}$ holomorphic on Ω equivalently $F = \begin{pmatrix} u \\ v \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. The partials $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Moreover, F is \mathbb{R} -differentiable, there exist a linear map $\mathcal{L} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F((x, y) + (h_1, h_2)) - F(x, y) = \mathcal{L}(h_1, h_2) + o(h)$

$$\mathcal{L} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Proof. We have

$$\frac{|F((x, y) + (h_1, h_2)) - F(x, y) - \mathcal{L}(h_1, h_2)|}{|h|} = \frac{\left| \begin{pmatrix} u(\bar{z} + \bar{h}) - u(\bar{z}) - \frac{\partial u}{\partial x}h_1 - \frac{\partial u}{\partial y}h_2 \\ v(\bar{z} + \bar{h}) - v(\bar{z}) - \frac{\partial v}{\partial x}h_1 - \frac{\partial v}{\partial y}h_2 \end{pmatrix} \right|}{|h|} \quad (1)$$

$$= \frac{|f(z + h) - f(z) - f'(z)h|}{|h|} \rightarrow 0 \quad (2)$$

Where $f(z + h) = u(z + h) + iv(z + h)$, $f(z) = u(z) + iv(z)$ and $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. \square

Corollary 2.2. f holomorphic implies F is \mathbb{R} -differentiable. Let $a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$DF_z = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad (4)$$

$$= \sqrt{a^2 + b^2} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad (5)$$

$$= |f'(z)| \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad (6)$$

Theorem 2.3. If F is \mathbb{R} -differentiable with u, v satisfying Cauchy Riemann Equations then f is holomorphic.

Proof. Let $a = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. We just need to check that $f(z + h) - f(z) - ah = o(h)$. \square

Definition 2.4. The Wirtinger Derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Remark 2.5. The cauchy-riemann equations if and only if $\frac{\partial f}{\partial \bar{z}} = 0$

We initially had the map $z \mapsto \bar{z}$ which is not holomorphic. Let $f(z) = \bar{z}$ and $\frac{\partial f}{\partial \bar{z}} = 1$ then we've captured this with $\frac{\partial}{\partial \bar{z}}$.

If we have z, \bar{z} as independent variables with $z = x + iy$. Consider the map

$$(z, \bar{z}) \mapsto (x(z, \bar{z}), y(z, \bar{z})) \mapsto f(x, y).$$

We have

$$\partial_z f = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right).$$

Theorem 2.6.

$$\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{1}{4} \Delta.$$

3 Jan 16

Definition 3.1. Given a \mathbb{C} valued seq $(a_n)_{n=0}^{\infty}$, form

$$\sum_{n=0}^{\infty} a_n z^n.$$

A power series

1. Converges if $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n z^n$ exists
2. Converges absolutely if $\sum_{n=0}^{\infty} |a_n| |z|^n$ converges

Remark 3.2. By translation, all we say will apply to $\sum a_n(z - z_0)^n$

Theorem 3.3. Given $\sum_{n=0}^{\infty} a_n z^n$, $\exists R \in [0, +\infty]$ such that

1. It converges, if $D_R = \{|z| < R\}$
2. It diverges at every z such that $|z| > R$.

Moreover,

$$R = \frac{1}{\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}}. \quad (\text{Hadamard's Formula})$$

Remark 3.4. On ∂D_R it can often be delicate.

Proof. Fix $|z| < R$, then $\frac{1}{R}|z| < 1$ and fix $\epsilon > 0$ such that

$$\left(\frac{1}{R} + \epsilon\right) |z| < 1.$$

Then, eventually

$$|a_n|^{\frac{1}{n}} < \frac{1}{R} + \epsilon.$$

Then

$$|a_n| |z|^n \leq \left(|a_n|^{\frac{1}{n}} |z|\right)^n \leq \left(\left(\frac{1}{R} + \epsilon\right) |z|\right)^n \leq q^n.$$

So by the comparison test with $\sum q^n < \infty$ □

Example 3.5.

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z).$$

This has $R = +\infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos(z).$$

This has $R = +\infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin(z).$$

This has $R = +\infty$

Euler's Formula is

$$e^{iz} = \cos(z) + i \sin(z).$$

Theorem 3.6. The function $f(z) = \sum a_n z^n$ is holomorphic on D_R , f' is a power series

$$f'(z) = \sum n a_n z^{n-1}.$$

Whose disk of convergence is R .

Proof. Let $g(z) = \sum n a_n z^{n-1}$, $z \in D_R$. We want to show that f' exist and $f' = g$.
 Fix $z_0 \in D_R$, let $r < R$ such that $|z_0| < r < R$.
 Fix $N \geq 1$ (to be chosen later)

$$f(z) = \sum_{n=0}^N a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n = S_N(z) + E_N(z).$$

From here we want to show that

$$\frac{f(z_0 + h) - f(z_0)}{h} \rightarrow g(z_0).$$

For all h such that $|z_0 + h| < r$

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) &= \left(\frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right) + (S'_N(z_0) - g(z_0)) + \left(\frac{E_N(z_0 + h) - E_N(z_0)}{h} \right) \\ &= A + B + C \end{aligned}$$

Fix $\epsilon > 0$ we want to show that $\exists \delta$ such that $\forall |h| < \delta, |A + B + C| < \epsilon$.

$$\begin{aligned} |C| &\leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(z_0 + h)^n - z_0^n}{h} \right| \\ &= \sum_{n=N+1}^{\infty} |a_n| \left| (z_0 + h)^{n-1} + (z_0 + h)^{n-2} z_0 + \dots + z_0^n \right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| |nr^{n-1}| < \frac{\epsilon}{3} \quad (\text{for all } N \geq N_1(\epsilon)) \end{aligned}$$

Note the last summation is the tail of $\sum a_n nr^{n-1}$ which converges by $r < R$.

For B :

$$S'_N(z_0) = \sum_{n=0}^N n a_n z_0^{n-1} \rightarrow g(z_0). \quad (\text{by def of } g)$$

So $\forall N \geq N_2, |B| < \frac{\epsilon}{3}$.

$|A| < \frac{\epsilon}{3}$ for all h sufficiently small by the definition of the complex derivative and the fact that S_N is plainly holomorphic. \square

Definition 3.7. Let Ω be open and define $f : \Omega \rightarrow \mathbb{C}$ is analytic at $z_0 \in \Omega$, if there exists a power series $\sum a_n (z - z_0)^n$ with positive radius of convergence such that $f(z) = \sum a_n (z - z_0)^n$ is a neighborhood of z_0 .

f is analytic on Ω , if analytic at every point of $z_0 \in \Omega$.

Corollary 3.8. If f analytic on $\Omega \implies f$ is holomorphic on Ω .

In fact, \Leftarrow .

4 Jan 21

Definition 4.1. A parametrized curve (in \mathbb{C}) is a function $z(t) : [a, b] \rightarrow \mathbb{C}$

1. Smooth if z' is continuous and $\forall t \in [a, b], z'(t) \neq 0$
2. Piecewise smooth if z is continuous so there exists $a_0 = a < a_1 < \dots < a_k = b$ such that $z|_{[a_j, a_{j+1}]}$ for all $j = 0, \dots, k - 1$ is smooth.
3. Let $\tilde{z} : [\tilde{a}, \tilde{b}] \rightarrow \mathbb{C}$ then two parameterizations are equivalent if there exist c^1 -bijection $s \mapsto t(s) : [\tilde{a}, \tilde{b}] \rightarrow [a, b]$ and $t'(s) > 0$ for all s such that $\tilde{z}(s) = z(t(s))$.

Example 4.2. A piecewise smooth curve can be visualized as follows:



The curve consists of smooth segments joined at corner points (marked in red) where the derivative is discontinuous.

Definition 4.3. The family of all equivalent parameterizations determines a curve (oriented) $\gamma \subset \mathbb{C}, \gamma = z([a, b])$.

$z(a)$ is the starting point and $z(b)$ is the endpoint of γ . $\gamma^{-1} = \gamma$ with reversed orientation

Definition 4.4. γ is closed if $z(a) = z(b)$ and simple if $z(t_1) \neq z(t_2)$ for all $t_1 \neq t_2$.

Example 4.5. $z(T) = re^{-it}, t \in [0, 2\pi]$

Definition 4.6. A curve has positive orientation if interior is on the left or counterclockwise.

Definition 4.7. Give a smooth curve $\gamma \subset \mathbb{C}$ (with parameterization $z : [a, b] \rightarrow \mathbb{C}$) with $f : \gamma \rightarrow \mathbb{C}$ continuous.

We set

$$\int_{\gamma} f = \int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt.$$

Since $f(z(t)) z'(t) \in \mathbb{C}$ we split the integral into

$$\int_a^b \Re(\dots) dt + i \int_a^b \Im(\dots) dt.$$

Remark 4.8. This is well defined as the right hand side does not depend on the choice of the parameterization.

Definition 4.9. For a piece-wise smooth curve, we define

$$\int_{\gamma} f = \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} f(z(t)) z'(t) dt.$$

Definition 4.10. We define

$$|\gamma| = \text{length}(\gamma) = \int_a^b |z'(t)| dt.$$

Theorem 4.11. Some very basic properties:

1. Linearity

$$\int_{\gamma} (\alpha f + \beta g) = \alpha \int_{\gamma} f + \beta \int_{\gamma} g.$$

2.

$$\int_{\gamma^{-1}} f = - \int_{\gamma} f.$$

3.

$$\left| \int_{\gamma} f \right| \leq \left(\sup_{\gamma} |f| \right) |\gamma|.$$

Definition 4.12. The primitive of $f : \Omega \rightarrow \mathbb{C}$ with open $\Omega \subset \mathbb{C}$ is any $F : \Omega \rightarrow \mathbb{C}$ holomorphic such that $F' = f$ on Ω

Theorem 4.13. The Fundamental Theorem of Calculus

For f with primitive F on Ω , $\gamma \subset \mathbb{C}$ curve from $w_1 \rightarrow w_2$ we have

$$\int_{\gamma} f = F(w_2) - F(w_1).$$

Proof. We can write

$$\int_{\gamma} f = \int_a^b f(z(t))z'(t)dt = \int_a^b \frac{d}{dt}(F(z(t)))dt = F(z(b)) - F(z(a)).$$

□

Corollary 4.14. If γ is closed then

$$\int_{\gamma} f = 0.$$

Corollary 4.15. If f is holomorphic on Ω with $f' = 0$ on Ω then f is constant.

Proof. Fix $z_0 \in \Omega$, for $z \in \Omega$, take $\gamma : z_0 \rightarrow z$ then

$$0 = \int_{\gamma} f' = f(z) - f(z_0).$$

□

Example 4.16. Let $z(t) = e^{it}$

$$\int_{\partial D} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt = 2\pi i \neq 0.$$

So $\frac{1}{z}$ does not admit a primitive.

5 Jan 23

Glossary of Elementary Functions

1. Polynomials: $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ for $a_i \in \mathbb{C}$ and is entirely holomorphic in \mathbb{C} , $a_n \neq 0$ and $n = \deg P$
2. Rational: $R(z) = \frac{P(z)}{Q(z)}$ where P, Q are polynomials with no common factors. The zeros of Q are called poles and their order is called the order of a given pole.
3. Exponential: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Lemma 5.1. Let $P(z)$ be a complex polynomial of positive degree n , if $|P(z)|$ has a local minimum at z_0 then $P(z_0) = 0$.

Proof. WLOG $z_0 = 0$ as we can translate $Q(z) = P(z - z_0)$.

AFSOC $P(0) \neq 0$ and WLOG $P(0) = 1$ by $\frac{P}{P(0)}$. We can write $P(z) = 1 + a_k z^k + \cdots + a_n z^n$ with $a_k, a_n \neq 0$. Consider $z_\delta = \delta e^{i\theta}$. For all $\delta > 0$ sufficiently small:

$$|a_{k+1} z_\delta + \cdots + a_n z_\delta^{n-k}| < \frac{|a_k|}{2}.$$

Then

$$|P(z)| \leq |1 + a_k z_\delta^k| + |a_{k+1} z_\delta^{k+1} + \cdots + a_n z_\delta^n| < |1 + a_k z_\delta^k| + |z_\delta^k| \frac{|a_k|}{2}.$$

Now we can choose δ such that $a_k z_\delta^k < 0$ and real.

$$|1 + a_k z_\delta^k| + |z_\delta^k| \frac{|a_k|}{2} = 1 + a_k z_\delta^k - \frac{1}{2} a_k z_\delta^k = 1 + \frac{1}{2} a_k z_\delta^k < 1.$$

□

Theorem 5.2. Fundamental Theorem of Algebra:

Proof. $|P(z)| \rightarrow +\infty$ as $|z| \rightarrow +\infty$ therefore $\exists R$ such that $\forall |z| > R$, $|P(z)| > |P(0)|$ thus $\inf_{z \in \mathbb{C}} |P(z)| = \inf_{z \in D_R} |P(z)|$ and such a value is attained somewhere at z_0 then by lemma 5.1 we're done. □

Lemma 5.3. $P(z) = a_n(z - z_1) \cdots (z - z_n)$ where z_1, \dots, z_n are roots of P .

The order of a zero z_j is its multiplicity, the number of times it appears in the sequence. If z_j is of the order k_j ,

$$P(z) = \underbrace{Q(z)}_{\text{non-vanishing}} (z - z_j)^{k_j}.$$

A zero z_0 is of order k_0 if and only if $P(z_0) = P'(z_0) = \dots = P^{(k_0-1)}(z_0) = 0$ and $P^{(k_0)}(z_0) \neq 0$

Theorem 5.4. (Gauss-Lucas) The roots of P' lies in the convex hull of the roots of P . In particular, if P is real rooted then so is P' .

Theorem 5.5. If R a rational function then R' has the same roots as R with order greater by 1.

Theorem 5.6. The following are true

1. $(e^z)' = e^z$
2. $e^{w+z} = e^w e^z$
3. $\overline{e^z} = e^{\bar{z}}$
4. $|e^{iy}| = 1$ and $|e^{x+iy}| = e^x$
5. $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

6 Jan 26

Theorem 6.1. *Cauchy Theorem: If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic on Ω and γ is a closed curve in Ω then*

$$\int_{\gamma} f = 0.$$

Theorem 6.2. *Goursat's Theorem: If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic and Ω is open, then $\forall \Delta \subset \Omega$ we have*

$$\int_{\partial\Delta} f = 0.$$



The integral around the positively oriented (counterclockwise) boundary of any triangle $\Delta \subset \Omega$ vanishes.

Proof. Let $\Delta = \Delta^{(0)}$ and partition into four triangles by connecting the midpoints of the sides:

$$\Delta^{(0)} = \Delta_1^{(1)} \cup \Delta_2^{(1)} \cup \Delta_3^{(1)} \cup \Delta_4^{(1)}.$$



When we integrate along each sub-triangle with positive orientation, the interior edges cancel (traversed in opposite directions, shown with opposing arrows):

$$\int_{\partial\Delta^{(0)}} f = \sum_{j=1}^4 \int_{\partial\Delta_j^{(1)}} f.$$

By the triangle inequality

$$\left| \int_{\partial\Delta^{(0)}} f \right| \leq \sum_{j=1}^4 \left| \int_{\partial\Delta_j^{(1)}} f \right| \leq 4 \max_{j=1,2,3,4} \left| \int_{\partial\Delta_j^{(1)}} f \right|.$$

Thus there exists $j_0 \in \{1, 2, 3, 4\}$ such that

$$\left| \int_{\partial\Delta^{(0)}} f \right| \leq 4 \left| \int_{\partial\Delta_{j_0}^{(1)}} f \right|.$$

Set $\Delta^{(1)} = \Delta_{j_0}^{(1)}$. Repeating this subdivision process, we obtain a nested sequence of triangles

$$\Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \dots$$

such that

$$\left| \int_{\partial \Delta^{(0)}} f \right| \leq 4^n \left| \int_{\partial \Delta^{(n)}} f \right|.$$

Note that $\text{diam}(\Delta^{(n)}) = 2^{-n} \text{diam}(\Delta^{(0)})$ and $|\partial \Delta^{(n)}| = 2^{-n} |\partial \Delta^{(0)}|$.

Since the triangles are nested compact sets with diameters shrinking to zero, by Cantor's intersection theorem:

$$\bigcap_{n=0}^{\infty} \Delta^{(n)} = \{z_0\}$$

for some $z_0 \in \Delta^{(0)} \subset \Omega$. Since f is holomorphic at z_0 , we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

where $\psi(z) \rightarrow 0$ as $z \rightarrow z_0$.

Since $f(z_0)$ and $f'(z_0)(z - z_0)$ have primitives ($f(z_0)z$ and $\frac{1}{2}f'(z_0)(z - z_0)^2$ respectively), their integrals over the closed curve $\partial \Delta^{(n)}$ vanish. Thus:

$$\int_{\partial \Delta^{(n)}} f(z) dz = \int_{\partial \Delta^{(n)}} \psi(z)(z - z_0) dz.$$

For $z \in \partial \Delta^{(n)}$, we have $|z - z_0| \leq \text{diam}(\Delta^{(n)}) = 2^{-n} \text{diam}(\Delta^{(0)})$.

Let $\epsilon_n = \sup_{z \in \partial \Delta^{(n)}} |\psi(z)|$. Since $\psi(z) \rightarrow 0$ as $z \rightarrow z_0$ and $\Delta^{(n)} \rightarrow \{z_0\}$, we have $\epsilon_n \rightarrow 0$. Therefore:

$$\begin{aligned} \left| \int_{\partial \Delta^{(n)}} f(z) dz \right| &\leq \sup_{z \in \partial \Delta^{(n)}} |\psi(z)(z - z_0)| \cdot |\partial \Delta^{(n)}| \\ &\leq \epsilon_n \cdot 2^{-n} \text{diam}(\Delta^{(0)}) \cdot 2^{-n} |\partial \Delta^{(0)}| \\ &= \epsilon_n \cdot 4^{-n} \text{diam}(\Delta^{(0)}) |\partial \Delta^{(0)}| \end{aligned}$$

Combining with our earlier inequality:

$$\left| \int_{\partial \Delta^{(0)}} f \right| \leq 4^n \left| \int_{\partial \Delta^{(n)}} f \right| \leq 4^n \cdot \epsilon_n \cdot 4^{-n} \text{diam}(\Delta^{(0)}) |\partial \Delta^{(0)}| = \epsilon_n \cdot \text{diam}(\Delta^{(0)}) |\partial \Delta^{(0)}|$$

Since $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and the right-hand side is independent of n except for ϵ_n , we conclude:

$$\int_{\partial \Delta} f = 0.$$

□

Theorem 6.3. *Cauchy's Theorem in a disk: Let $D \subset \mathbb{C}$ be an open disk and f holomorphic on D . Then for any closed curve γ in D we have*

$$\int_{\gamma} f = 0.$$

Proof. WLOG let D be a disk centered at 0. Define $F : D \rightarrow \mathbb{C}$ by

$$F(z) = \int_{\gamma_z} f$$

where γ_z is the path $0 \rightarrow \Re(z) \rightarrow z$ (horizontal then vertical).



We claim F is a primitive for f , i.e., $F'(z) = f(z)$.

Case 1: $h \in \mathbb{R}$ (horizontal increment).

The path γ_{z+h} goes $0 \rightarrow \Re(z) + h \rightarrow z + h$. Observe that:

$$F(z + h) - F(z) = \int_{\gamma_{z+h}} f - \int_{\gamma_z} f.$$



The difference of paths can be decomposed using Goursat's theorem. The integral over the rectangle with vertices $\Re(z), \Re(z) + h, z + h, z$ vanishes, so:

$$F(z + h) - F(z) = \int_{\Re(z)}^{\Re(z)+h} f(\zeta) d\zeta + \int_z^{z+h} f(\zeta) d\zeta = \int_z^{z+h} f(\zeta) d\zeta$$

where the first integral is along the real axis and the second is horizontal at height $\Im(z)$. By the rectangle lemma, these combine to give just the horizontal segment from z to $z + h$:

$$\frac{F(z + h) - F(z)}{h} = \frac{1}{h} \int_z^{z+h} f(\zeta) d\zeta \rightarrow f(z) \text{ as } h \rightarrow 0.$$

Case 2: $h = ik$ with $k \in \mathbb{R}$ (vertical increment).

Similarly, γ_{z+ik} goes $0 \rightarrow \Re(z) \rightarrow z + ik$. The paths γ_z and γ_{z+ik} share the horizontal segment $0 \rightarrow \Re(z)$, so:

$$F(z + ik) - F(z) = \int_z^{z+ik} f(\zeta) d\zeta.$$



Thus:

$$\frac{F(z + ik) - F(z)}{ik} = \frac{1}{ik} \int_z^{z+ik} f(\zeta) d\zeta \rightarrow f(z) \text{ as } k \rightarrow 0.$$

General h : For arbitrary $h = h_1 + ih_2$, we write:

$$\frac{F(z + h) - F(z)}{h} = \frac{F(z + h) - F(z + h_1)}{h} + \frac{F(z + h_1) - F(z)}{h}.$$

Using the above cases and the continuity of f , both terms converge appropriately, giving $F'(z) = f(z)$.

Since f has a primitive F on D , by the Fundamental Theorem of Calculus, for any closed curve γ in D :

$$\int_{\gamma} f = F(z(b)) - F(z(a)) = 0.$$

□

7 Jan 28

Example 7.1.

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1 \iff \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Proof. First, note that by substitution $u = \sqrt{\pi}x$, we have:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du.$$

So the equivalence follows if we can show $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$.

Consider the function $f(z) = e^{-\pi z^2}$, which is entire (holomorphic everywhere). For fixed $y \in \mathbb{R}$, consider the rectangular contour Γ_R with vertices at $-R, R, R + iy, -R + iy$ (traversed counterclockwise).



By Cauchy's theorem:

$$\int_{\Gamma_R} e^{-\pi z^2} dz = 0.$$

Parameterizing the four sides:

$$\begin{aligned} \int_{\Gamma_R} e^{-\pi z^2} dz &= \int_{-R}^R e^{-\pi t^2} dt + \int_0^y e^{-\pi(R+it)^2} i dt \\ &\quad + \int_R^{-R} e^{-\pi(t+iy)^2} dt + \int_y^0 e^{-\pi(-R+it)^2} i dt \end{aligned}$$

We now bound the integrals over the vertical segments. For the right vertical segment, parameterize $z = R + it$ with $t \in [0, y]$ (assuming $y > 0$; the case $y < 0$ is similar). We have:

$$(R + it)^2 = R^2 + 2iRt - t^2 = (R^2 - t^2) + 2iRt$$

so

$$e^{-\pi(R+it)^2} = e^{-\pi(R^2-t^2)} \cdot e^{-2\pi i R t}.$$

Since $|e^{-2\pi i R t}| = 1$ for all real R and t , we have:

$$|e^{-\pi(R+it)^2}| = e^{-\pi(R^2-t^2)} \cdot |e^{-2\pi i R t}| = e^{-\pi(R^2-t^2)} \cdot 1 = e^{-\pi(R^2-t^2)}.$$

For $t \in [0, y]$, we have $t^2 \leq y^2$, so $R^2 - t^2 \geq R^2 - y^2$. Therefore:

$$|e^{-\pi(R+it)^2}| \leq e^{-\pi(R^2-y^2)} = e^{-\pi R^2} e^{\pi y^2}.$$

By the ML-inequality, the length of the path is $|y|$, so:

$$\left| \int_0^y e^{-\pi(R+it)^2} i dt \right| \leq |y| \cdot e^{-\pi R^2} e^{\pi y^2} = |y| e^{\pi y^2} e^{-\pi R^2}.$$

Similarly, for the left vertical segment with $z = -R + it$ where $t \in [y, 0]$:

$$(-R + it)^2 = R^2 - 2iRt - t^2 = (R^2 - t^2) - 2iRt$$

so

$$e^{-\pi(-R+it)^2} = e^{-\pi(R^2-t^2)} \cdot e^{2\pi i R t}.$$

Since $|e^{2\pi i R t}| = 1$ for all real R and t , we have:

$$|e^{-\pi(-R+it)^2}| = e^{-\pi(R^2-t^2)} \cdot |e^{2\pi i R t}| = e^{-\pi(R^2-t^2)} \cdot 1 = e^{-\pi(R^2-t^2)}.$$

For $t \in [y, 0]$ (or equivalently $t \in [0, y]$ if we reverse the parameterization), we have $t^2 \leq y^2$, so:

$$|e^{-\pi(-R+it)^2}| \leq e^{-\pi(R^2-y^2)} = e^{-\pi R^2} e^{\pi y^2}.$$

Therefore:

$$\left| \int_y^0 e^{-\pi(-R+it)^2} i dt \right| \leq |y| \cdot e^{-\pi R^2} e^{\pi y^2} = |y| e^{\pi y^2} e^{-\pi R^2}.$$

Both vertical integrals are bounded by $C e^{-\pi R^2}$ for some constant C depending on y but independent of R . Since $e^{-\pi R^2} \rightarrow 0$ as $R \rightarrow \infty$, we conclude that:

$$\lim_{R \rightarrow \infty} \int_0^y e^{-\pi(R+it)^2} i dt = 0 \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_y^0 e^{-\pi(-R+it)^2} i dt = 0.$$

Therefore, taking the limit as $R \rightarrow \infty$:

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = \int_{-\infty}^{\infty} e^{-\pi(t+iy)^2} dt.$$

Expanding $e^{-\pi(t+iy)^2} = e^{-\pi(t^2+2ity-y^2)} = e^{-\pi t^2} e^{-2\pi i t y} e^{\pi y^2}$, we get:

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = e^{\pi y^2} \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-2\pi i t y} dt.$$

In particular, taking $y = 0$ gives the standard Gaussian integral. The standard result is $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$, which completes the proof. \square

Example 7.2.

$$\int_0^{\infty} \frac{1 - \cos(x)}{x^2} dx = \frac{\pi}{2}.$$

Proof. Consider the holomorphic function $f(z) = \frac{1-e^{iz}}{z^2}$ on $\mathbb{C} \setminus \{0\}$. Fix ϵ, R and consider the contour Γ which is disk with radius R and a small disk with radius ϵ centered at the origin.



We can express $e^{iz} \approx 1 + iz + g(z)$ where $g(z) = e^{iz} - 1 - iz$ so

$$\begin{aligned} \int_{\gamma_\epsilon} f(z) dz &= \int_{\gamma_\epsilon} \frac{1 - (1 + iz + g(z))}{z^2} dz \\ &= \int_{\gamma_\epsilon} \frac{-g(z)}{z^2} dz + \int_{\gamma_\epsilon} \frac{-i}{z} dz \end{aligned}$$

We can evaluate each part as

$$\begin{aligned} \int_{\gamma_\epsilon} \frac{dz}{z} &= - \int_0^\pi \frac{ie^{it}}{e^{it}} dt = -\pi i. \\ \int_{\gamma_\epsilon} \frac{g(z)}{z^2} dz &\rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

\square

8 Jan 30

Theorem 8.1. *Cauchy's Integral Formula:* Let Ω be open $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Let \overline{D} be open in Ω be a disk then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof. Fix $z \in D_1$, let $\gamma_{\epsilon, \delta}$ be a key-hole contour but we omit z . Then

$$\int_{\gamma_{\epsilon, \delta}} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Fix $\epsilon > 0$ and $\delta \rightarrow 0$ we want to show that

$$\int_{\gamma_{\epsilon, \delta}} \frac{f(\zeta)}{\zeta - z} d\zeta \rightarrow \int_{\partial D} - \int_{C_\epsilon} .$$

For

$$\int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_\epsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + f(z) \int_{C_\epsilon} \frac{d\zeta}{\zeta - z}.$$

The first term is bounded as holomorphic so as $\epsilon \rightarrow 0$, it goes to 0. The second part integrates to $f(z) \cdot 2\pi i$. \square

Theorem 8.2. *Regularity: Holomorphic implies Analytic.*

Let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic, $\overline{D} = \overline{D}_r(z_0) \subset \Omega$ then f is analytic at z_0 i.e

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Proof. We know

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We can write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) + (z_0 - z)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n.$$

Note: $\frac{z - z_0}{\zeta - z_0} < 1$.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} \underbrace{\sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n}_{\text{conv. uniformly on } \partial D} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\left(\frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right)}_{a_n} \end{aligned}$$

So $a_n = \frac{f^{(n)}(z_0)}{n!}$. \square

Corollary 8.3. *Cauchy's Formula:*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Corollary 8.4. *Cauchy's Inequality:*

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \sup_{\zeta \in \partial D} \left| \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \right| = \frac{n!}{r^n} \|f\|_{\infty} \partial D.$$

Theorem 8.5. *Liouville's Theorem: If f entire and bounded then f is constant.*

Proof.

$$|f'(z_0)| \leq \frac{\|f\|_\infty}{r}$$

Since f is bounded and r can be taken arbitrarily large (as f is entire), sending $r \rightarrow \infty$ gives $|f'(z_0)| = 0$.
Thus, f is constant. \square

9 Feb 2

Theorem 9.1. (*Rigidity Theorem*): Let Ω be a region and $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. If $z_1, z_2, \dots \in \Omega$ distinct sequence with a limit in Ω . If $f(z_n) = 0, \forall n$ then $f = 0$ on Ω .

Proof. Say $z_n \rightarrow \omega \in \Omega$ and $D = D_r(\omega) \subset \Omega$ be a disk centered at ω with radius r . If $f = 0$ on D then by regularity implies

$$f(z) = \sum_{n \geq 0} a_n(z - \omega)^n \text{ on } D.$$

Suppose $f \neq 0$ on D then $\exists m$ such that $a_m \neq 0$, let's take the smallest such m . Then $f(z) = a_m(z - w)^m(1 + g(z))$ where $g(z) \rightarrow 0$ as $z \rightarrow w$. For all sufficiency large $k, z_k \in D$ then

$$0 = f(z_k) = a_m(z_k - w)^m(1 + g(z_k)). \quad (a_m \neq 0 \text{ and } z_k - w \neq 0)$$

A contradiction as the RHS is nonzero. To finish $f = 0$ on Ω , we use $\mathcal{U} = \text{int}\{z \in \Omega : f(z) = 0\}$. We have \mathcal{U} is open, $w \in \mathcal{U}$ so \mathcal{U} is non-empty. Additionally, \mathcal{U} is closed. We have $\Omega \setminus \mathcal{U}$ is open and non-empty so $\exists z_k \in \Omega \setminus \mathcal{U}$ such that $z_k \rightarrow \omega' \in \Omega \setminus \mathcal{U}$. Then $f(z_k) \neq 0$ for all k so $f \neq 0$ on $D_r(\omega')$. A contradiction as $D_r(\omega') \subset D$. Thus, $f = 0$ on Ω . \square

Corollary 9.2. $f, g : \underset{\text{region}}{\Omega} \rightarrow \mathbb{C}$ holomorphic. If $f(z_n) = g(z_n), \forall z_n \in \Omega$ distinct sequence with a limit in Ω then $f = g$ on Ω .

Definition 9.3. If $\Omega_1 \subset \Omega_2$, are two regions, $f_i : \Omega_i \rightarrow \mathbb{C}$ holomorphic for $i = 1, 2$ such that $f_1 = f_2$ on Ω_1 then we say f_2 is the analytic continuation of f_1 into Ω_2 .

Theorem 9.4. (*Morera's Theorem*): If $f : \mathcal{D} \rightarrow \mathbb{C}$ continuous such that $\forall \bar{\Delta} \subset \mathcal{D}$ and $\int_{\partial \Delta} f = 0$ then f is holomorphic.

Proof. Repeating the proof of Cauchy's Theorem in a disk, gives that f has a primitive $F = \int_{\gamma_z} f$ in \mathcal{D} . Since F is holomorphic, F' exists but so does F'', F''', \dots so f' exists. \square

Theorem 9.5. If $f_n : \underset{\text{open}}{\Omega} \rightarrow \mathbb{C}$ holomorphic with f_n converging uniformly to f on every compact $K \subset \Omega$ then f is holomorphic.

Proof. Use Morera's Theorem. \square

Moreover, if f'_n converges uniformly to f' , using $\Omega_s = \{z \in \Omega : \bar{\mathcal{D}}_\delta(z) \subset \Omega\}$.

Claim: $\forall F$ holomorphic in Ω , $\|F'\|_{\infty, \Omega_\delta} \leq \frac{1}{\delta} \|F\|_{\infty, \Omega_\delta}$.

By Cauchy's Formula:

$$|F'(z)| = \left| \frac{1}{2\pi i} \int_{\partial \mathcal{D}_\delta(z)} \frac{F(w)}{(w - z)^2} dw \right| \leq \frac{1}{2\pi} \int_{\partial \mathcal{D}_\delta(z)} \frac{1}{|w - z|^2} \|F\|_{\infty, \Omega} \leq \frac{1}{2\pi} \frac{2\pi\delta}{\delta^2} \|F\|_{\infty, \Omega} \leq \frac{1}{\delta} \|F\|_{\infty, \Omega}.$$

10 Feb 4

Theorem 10.1. Let $F : \Omega \times [0, 1] \rightarrow \mathbb{C}$ satisfy

1. $\forall \Delta \in [0, 1], z \mapsto F(z, \Delta)$ is holomorphic
2. F is continuous on $\Omega \times [0, 1]$

Then $z \mapsto \int_0^1 F(z, \Delta) d\Delta$ is holomorphic on Ω .

Proof. Consider $F_n(z) = \frac{1}{n} \sum_{k=1}^n F(z, \frac{k}{n})$ and by (1) each F_n is holomorphic on Ω .

$$\begin{aligned} \left| F_n(z) - \int_0^1 F(z, \Delta) d\Delta \right| &= \left| F_n(z) - \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} F(z, \Delta) d\Delta \right| \\ &= \left| \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \underbrace{\left[F(z, \frac{k}{n}) - F(z, \Delta) \right]}_{\leq \epsilon} d\Delta \right| \\ &\leq \epsilon \end{aligned}$$

□

Definition 10.2. $\Omega \subset \mathbb{C}$ be open, symmetric with respect to real axis, $\Omega^+ = \Omega \cap \{z \in \mathbb{C}, \Im(z) > 0\}$, $\Omega^- = \Omega \cap \{z \in \mathbb{C}, \Im(z) < 0\}$ and $I = \Omega \cap \{\Im(z) = 0\}$

Theorem 10.3. Let each $f^\pm : \Omega^\pm \rightarrow \mathbb{C}$ be holomorphic and extend continuous to I such that $f^+ = f^-$ on I . Then

$$f = \begin{cases} f^+(z) & \text{if } z \in \Omega^+ \\ f^+(z) = f^-(z) & \text{if } z \in I \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}.$$

is holomorphic on Ω .

Proof. Sufficient to handle $z \in I$. Fix such z take $D \subset \Omega$ centered at z . Take $\overline{\Delta} \subset D$ we want to show $\int_{\partial\Delta} f = 0$.

Case 1: If $\overline{\Delta} \subset \Omega^+$ (or Ω^-) then we're done.



Case 2: If a vertex or side of Δ is on I , $\int_{\partial\Delta_\epsilon} f = 0$ for when $\epsilon \rightarrow 0, \partial\Delta_\epsilon \rightarrow \partial\Delta$.



Case 3: If I cuts the triangle then we can split the triangle into smaller triangles and we're done.



The triangle is split into smaller triangles, each either entirely in Ω^+ or Ω^- , or with edges on I , so we can apply Cases 1 and 2. \square

Theorem 10.4. Schwarz Reflection Principle: Let $f : \Omega^+ \rightarrow \mathbb{C}$ be holomorphic and extend continuously to I with $f^+/I \in \mathbb{R}$ then f^+ extends analytically to Ω

Proof. Let $f^- : \Omega^- \rightarrow \mathbb{C}$ be defined by $f^-(z) = \overline{f^+(\bar{z})}$. We claim f^- is holomorphic on Ω^- , then the previous theorem applies because $f^+ = f^-$ on I .

Fix $z_0 \in \Omega^-$ then $\bar{z}_0 \in \Omega^+$, we know $f^+(\bar{z}) = \sum_{n=0}^{\infty} a_n (\bar{z} - \bar{z}_0)^n$ for all $\bar{z} \in D(\bar{z}_0)$. Thus

$$\overline{f^+(\bar{z})} = \sum_{n=0}^{\infty} \overline{a_n} (\bar{z} - \bar{z}_0)^n.$$

We know $\overline{\bar{z} - \bar{z}_0} = z - z_0$ so

$$f^-(z) = \overline{f^+(\bar{z})} = \sum \overline{a_n} (z - z_0)^n.$$

\square

11 Feb 9

Definition 11.1. Meromorphic functions are "determined" by zeros and singularities.

A point singularity of f is a point $z_0 \in \mathbb{C}$ such that f is defined in a neighborhood of z_0 but not at z_0 .

$$\mathcal{D}_\delta(z_0) \setminus \{z_0\}.$$

Example 11.2. $f(z) = \frac{1}{z}$ has a singularity at $z_0 = 0$.

Remark 11.3. Zeros of a holomorphic f are isolated, unless $f \equiv 0$.

Theorem 11.4. (Local Description Near Zeros) Let $f : \Omega_{open} \rightarrow \mathbb{C}$ holomorphic and $f(z_0) = 0$, $z_0 \in \Omega$, $f \not\equiv 0$ then $\exists \mathcal{U} \ni z_0$, $g : \mathcal{U} \rightarrow \mathbb{C}$ holomorphic and vanishing ($\forall z \in \mathcal{U}, g(z) \neq 0$) $\exists! n > 0$ such that $f(z) = (z - z_0)^n g(z)$.

Proof. We know

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

$f \not\equiv 0$ implies that $n = \text{smallest } k \text{ such that } a_k \neq 0$. We can write $f(z) = (z - z_0)^n g(z)$. \square

Definition 11.5. Let $n = \text{the multiplicity/order of } z_0$. When $n = 1$, z_0 is called a simple zero. We say $f : \mathcal{D}_\delta(z_0) \setminus \{z_0\} \rightarrow \mathbb{C}$ has a pole at z_0 if $\frac{1}{f}$ extended by 0 at z_0 is holomorphic in $\mathcal{D}_\epsilon(z_0)$, for some $0 < \epsilon < \delta$.

Theorem 11.6. (Local Discontinuity near Poles) Let $f : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ has a pole at z_0 in Ω then $\exists \mathcal{U} \ni z_0, \exists h : \mathcal{U} \xrightarrow{holo} \mathbb{C}$ nonvanishing then $\exists! n > 0$ such that $f(z) = (z - z_0)^{-n} h(z)$.

Proof. Apply previous theorem to $\frac{1}{f}(z) = (z - z_0)^n g(z)$. \square

Theorem 11.7. (Laurent Series Expansion) If f has a pole of order n at z_0 then locally

$$f(z) = \underbrace{\frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{\overbrace{a_{-1}}^{residue \ of \ f \ at \ z_0}}{z - z_0}}_{principal \ part} + \underbrace{G(z)}_{holomorphic \ and \ nonvanishing \ near \ z_0}.$$

where $G(z)$ is holomorphic and nonvanishing near z_0 .