

Convex Optimization Notes

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Contents

1 Jan 13: Introduction to Optimization

1.1 Basic Definitions

Definition 1.1. An *optimization problem* is defined as

$$\min_{x \in \mathcal{C}} f_0(x) \quad \text{where} \quad \forall i \in [m], f_i(x) \leq b_i$$

where \mathcal{C} is the constraint set and $f_0: \mathcal{C} \rightarrow \mathbb{R}$.

Definition 1.2. The *standard form* of an optimization problem is

$$\min_{x \in \mathcal{D}} f_0(x) \quad \text{subject to} \quad \forall i \in [m], f_i(x) \leq 0; \quad \forall j \in [p], h_j(x) = 0$$

where $x \in \text{dom}(f_i) \cap \text{dom}(h_j) =: \mathcal{D}$.

f_0 is termed the objective function, and x is the optimization variable.

f_i and h_j are constraint functions.

Definition 1.3. A *feasible solution* is any $x \in \mathcal{D}$ that satisfies all the constraints.

Definition 1.4. The *optimal solution* is $x^* \in \arg \min_{x \in \mathcal{D}} f_0(x)$.

Remark 1.5. If no feasible solution exists, we define $\min_{x \in \mathcal{C}} f(x) = +\infty$ and, similarly, $\max_{x \in \mathcal{C}} f(x) = -\infty$.

2 Jan 15: Convex Sets and Functions

2.1 Convexity

Definition 2.1. A set C is **convex** if $\forall x_1, x_2 \in C$ and $\forall \theta \in [0, 1]$, we have $\theta x_1 + (1 - \theta)x_2 \in C$.

Definition 2.2. A function $f: \mathcal{X} \rightarrow \mathbb{R}$ is **convex** if for all $x_1, x_2 \in \mathcal{X}$ and $\theta \in [0, 1]$,

$$f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2) \quad (*)$$

Remark 2.3. Convex functions are differentiable almost everywhere.

Theorem 2.4. If f is both convex and concave, then f is affine (i.e., linear plus constant).

Definition 2.5. f is **strictly convex** if $(*)$ holds with strict inequality $<$ whenever $x_1 \neq x_2$ and $\theta \in (0, 1)$.

Definition 2.6. A **convex optimization problem** is one where f_0 , f_i , and \mathcal{C} are convex, and h_j are affine/linear. That is,

$$\min_{x \in \mathcal{C}} f_0(x) \quad \text{such that} \quad f_i(x) \leq 0 \quad \forall i \in [m], \quad h_j(x) = 0 \quad \forall j \in [p]$$

Theorem 2.7. The set of feasible solutions in Definition 2.6 is convex.

2.2 Local vs. Global Optima

Theorem 2.8. x^* is a **local minimizer** if there exists $\epsilon > 0$ such that for all y with $\|x^* - y\| \leq \epsilon$,

$$f(x^*) \leq f(y)$$

Theorem 2.9. x^* is a **local maximizer** if there exists $\epsilon > 0$ such that for all y with $\|x^* - y\| \leq \epsilon$,

$$f(x^*) \geq f(y)$$

Theorem 2.10. For any convex optimization problem, every local minimum is a global minimum.

Proof. Suppose \hat{x} is a local minimizer not equal to global minimizer x^* . Take ϵ as any witness to \hat{x} being a local minimum. Let

$$y = \frac{\epsilon}{\|\hat{x} - x^*\|} x^* + \left(1 - \frac{\epsilon}{\|\hat{x} - x^*\|}\right) \hat{x}$$

Note: $\|\hat{x} - x^*\| \leq \epsilon$, otherwise \hat{x} is not a local minimizer in that neighborhood.

$$y - \hat{x} = \frac{\epsilon(x^* - \hat{x})}{\|x^* - \hat{x}\|} \quad ; \quad \|y - \hat{x}\| = \epsilon$$

Since f is convex,

$$f(y) \leq \frac{\epsilon}{\|\hat{x} - x^*\|} f(x^*) + \left(1 - \frac{\epsilon}{\|\hat{x} - x^*\|}\right) f(\hat{x}) < f(\hat{x})$$

Thus, a contradiction. □

3 Jan 20: Convex Combinations, Hulls, and Cones

3.1 Types of Combinations

Definition 3.1. A **convex combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 = 1$.

Definition 3.2. An **affine combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1 + \theta_2 = 1$.

Definition 3.3. A **linear combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1, \theta_2 \in \mathbb{R}$.

Definition 3.4. A **conic combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1, \theta_2 \geq 0$.

Definition 3.5. Given a set C , the **convex hull** of C is

$$\text{conv}(C) = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in C, \theta_i \in [0, 1], \sum_{i=1}^k \theta_i = 1 \right\}$$

Remark 3.6. The following are true:

1. $C \subseteq \text{conv}(C)$
2. $\text{conv}(C)$ is convex
3. It is the smallest convex set containing C
4. If a convex set $S \supseteq C$ then $S \supseteq \text{conv}(C)$

Theorem 3.7. Any closed convex set can be written as $\overline{\text{conv}}(C)$ for some set C .

Definition 3.8. The **relative interior** of C is defined as

$$\text{relint}(C) = \{x \in C : \exists \epsilon > 0, B(x, \epsilon) \cap \text{Aff}(C) \subseteq C\}$$

Definition 3.9. C is a **cone** if $\alpha x \in C$ whenever $x \in C$ and $\alpha \geq 0$.

Definition 3.10. Given a set C , the **conic hull** of C is

$$\text{conic}(C) = \left\{ \sum_{i=1}^k \theta_i x_i : x_i \in C, \theta_i \geq 0 \right\}$$

Theorem 3.11. The conic hull of C is the smallest convex cone containing C .

3.2 Norms and Convex Sets

Definition 3.12. The ℓ_p **norm** is defined as

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{1/p}$$

Remark 3.13. *The following are true:*

1. For $p \in (0, 1)$: $\|x\|_p$ is not a convex function
2. For $p \geq 1$: $\|x\|_p$ is convex
3. For $p > 1$: $\|x\|_p$ is strictly convex

Example 3.14. *Examples of convex sets:*

1. Hyperplane: $\{x : a^T x = b\}$
2. Halfspace: $\{x : a^T x \leq b\}$
3. Polyhedron: $\{x \in \mathbb{R}^d : Ax \leq b, Cx = d\}$
4. Polytope: a bounded polyhedron.

3.3 Optimality and the Normal Cone

Theorem 3.15. *A set S is **strictly convex** if for all $x_1 \neq x_2$ and $\theta \in (0, 1)$, $\theta x_1 + (1 - \theta)x_2 \in \text{int}(S)$.*

Definition 3.16. *The **normal cone** is defined to be*

$$N_C(x) = \{g : g^T(y - x) \leq 0, \forall y \in C\}$$

Remark 3.17. *If $x \in \text{int}(C)$ then $N_C(x) = \{0\}$.*

Theorem 3.18. *If f is differentiable, then x^* is optimal if and only if $-\nabla f(x^*) \in N_C(x^*)$.*

4 Jan 22: Optimality Conditions and Characterizations of Convexity

4.1 Optimality and Operations on Convex Sets

Theorem 4.1. *If f is convex and differentiable, and C is a convex set, then any optimal solution x^* to $\min_{x \in C} f(x)$ must satisfy $-\nabla f(x^*) \in N_C(x^*)$.*

Theorem 4.2. *The set of optimal solutions to a convex optimization problem is a convex set.*

Definition 4.3. *If C is convex then*

1. **Translation:** $C + a = \{x : x - a \in C\}$
2. **Scaling:** $\alpha C = \{x : \frac{x}{\alpha} \in C\}$
3. **Intersection:** *If $\{C_\alpha\}_{\alpha \in A}$ is a collection of convex sets, then $\bigcap_{\alpha \in A} C_\alpha$ is convex.*

Theorem 4.4. *The following are true:*

1. *If $C \subseteq \mathbb{R}^n$ is convex, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then $AC + b = \{Ax + b : x \in C\}$ is a convex set.*
2. *If $f(x) = Ax + b$, then $f^{-1}(C)$ is convex.*
3. *$C_1 + C_2 = \{x + y : x \in C_1, y \in C_2\}$ is convex.*
4. *If $C_1 \subseteq \mathbb{R}^m$, $C_2 \subseteq \mathbb{R}^n$ then $C_1 \times C_2 = \{(x, y) \in \mathbb{R}^{m+n} : x \in C_1, y \in C_2\}$ is convex.*
5. *For any $C \subseteq \mathbb{R}^n \times \mathbb{R}_{>0}^m$, define*

$$P(C) = \left\{ \left(\frac{x_1}{t}, \dots, \frac{x_n}{t} \right) : (x_1, \dots, x_n, t) \in C \right\}$$

If C is convex, so are $P(C)$ and $P^{-1}(C)$.

4.2 First and Second Order Conditions

Definition 4.5.

$$\text{epi}(f) = \{(x, t) : x \in \text{dom}(f), t \geq f(x)\}$$

Definition 4.6. First Order Definition of Convexity: If f is differentiable, then f is convex if and only if for all $x, y \in \text{dom}(f)$,

$$f(y) \geq f(x) + \nabla f(x)^T(y - x)$$

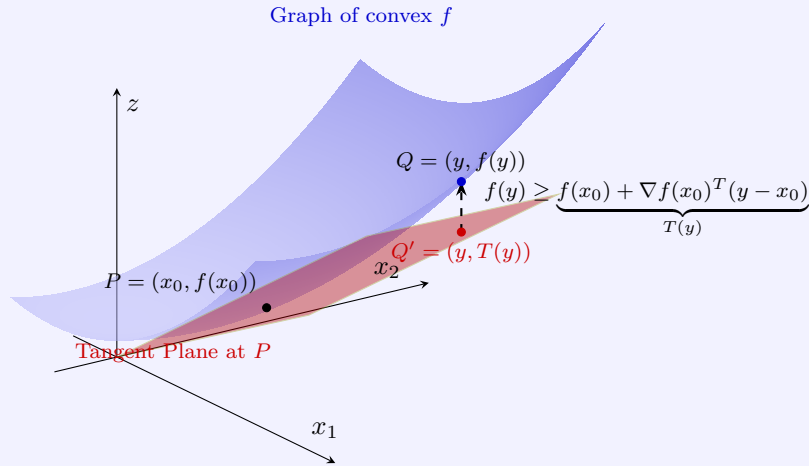


Figure 1: First Order Definition of Convexity

Second Order Definition of Convexity: If f is twice differentiable, then f is convex if and only if $\nabla^2 f(x) \succeq 0$.

Definition 4.7. We say $A \succeq B$ if $A - B$ is positive semi-definite. This is equivalent to

$$a^T \nabla^2 f(x) a \geq 0 \quad \text{for all } a \in \mathbb{R}^d$$

4.3 Subdifferentials

Definition 4.8. The **subdifferential** is

$$\partial f(x) = \{g : f(y) \geq f(x) + g^T(y - x), \forall y \in C\}$$

Any such $g \in \partial f(x)$ is called a **subgradient**.

If f is differentiable at x then $\partial f(x) = \{\nabla f(x)\}$.

Theorem 4.9. f is convex if and only if ∂f is non-empty.

5 Jan 27: Matrix Theory, Lipschitz Continuity, and Smoothness

5.1 Gradient Monotonicity and SVD

Definition 5.1. *Gradient Monotonicity: If f is differentiable, then f is convex if and only if $\nabla f(x)$ is monotone. So we can conclude*

$$(\nabla f(x) - \nabla f(y))^T(x - y) \geq 0 \iff f \text{ is convex.}$$

Theorem 5.2. *For a $A \in \mathbb{R}^{m \times n}$ we can decompose*

$$A = U \Sigma V^T.$$

where $U \in \mathbb{R}^{m \times k}$ with orthonormal column, $\Sigma \in \mathbb{R}^{k \times k}$ is a diagonal matrix with non-negative entries, and $V \in \mathbb{R}^{n \times k}$ with orthonormal column.

Definition 5.3. *(u_i, v_i, σ_i) form a s.v. triplet if $Av_i = \sigma_i u_i$ and $A^T u_i = \sigma_i v_i$
 $\|u_i\| = \|v_i\| = 1$ and $\sigma_i \geq 0$*

Theorem 5.4. *For any $A \in \mathbb{R}^{n \times n}$ then $A^T A$ is always positive semi definite*

Proof. We can write $A = V \Sigma U^T$ then $A^T A = V \Sigma^T U^T U \Sigma V^T = V \Sigma^2 V^T$ □

Definition 5.5. *Spectral Radius: $\max_i \{|\lambda_i| : \lambda_i \in \Lambda(A)\} = \rho(A)$*

Definition 5.6. *The norm must satisfy the following properties:*

1. $\|A\| \geq 0$
2. $\|\alpha A\| = |\alpha| \|A\|$
3. $\|A\| = 0$ if and only if $A = 0$
4. $\|A + A'\| \leq \|A\| + \|A'\|$

5.2 Matrix Norms

Definition 5.7. *Operator/Spectral Norm:*

$$\|A\|_{op} = \|A\|_2 = \max_{\|x\|=1} \|Ax\|_2.$$

Definition 5.8. *Frobenius Norm:*

$$\|A\|_F = \sqrt{\sum_{i,j} A_{ij}^2}.$$

5.3 Lipschitz Continuity and Smoothness

Definition 5.9. *f is L -Lipschitz if*

$$|f(x) - f(y)| \leq L \|x - y\|_2 \quad \forall x, y \in \text{dom}(f).$$

If f is differentiable then f is L -Lipschitz if and only if $|\nabla f| \leq L$

Definition 5.10. *Differentiable f is β -smooth if*

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq \beta \|x - y\|_2 \quad \forall x, y \in \text{dom}(f).$$

If f is twice differentiable then f is β -smooth if and only if $\nabla^2 f(x) \preceq \beta I$.

Theorem 5.11. *If f is twice differentiable and β -smooth then*

$$f(y) \leq f(x) + \nabla f(x)^T(y - x) + \frac{\beta}{2} \|y - x\|_2^2 \quad \forall x, y \in \text{dom}(f).$$

6 Feb 3: Gradient Descent and Convergence

6.1 Descent Directions and Step Sizes

Definition 6.1. "Descent Direction" any h for which $f(x + \eta h) \leq f(x)$.

Theorem 6.2. Assume f differentiable and β -smooth with $x_{t+1} \leftarrow x_t - \eta \nabla f(x_t)$.

1. If $\eta \leq \frac{2}{\beta}$ then $f(x_{t+1}) \leq f(x_t)$
2. If $\eta \leq \frac{1}{\beta}$ then $f(x_{t+1}) \leq f(x_t) - \frac{\eta}{2} \|\nabla f(x_t)\|_2^2$

Proof. Recall

$$f(y) \leq f(x) + \nabla f(x)^T(y - x) + \frac{\beta}{2} \|y - x\|_2^2$$

and

$$\|\nabla f(y) - \nabla f(x)\|_2 \leq \beta \|y - x\|_2$$

Let $y \leftarrow x + h$ then $f(x + h) \leq f(x) + \nabla f(x)^T h + \frac{\beta}{2} \|h\|_2^2$

1. If $h = -\frac{2}{\beta} \nabla f(x)$ then $\nabla f(x)^T h = -\frac{2}{\beta} \|\nabla f(x)\|_2^2$ and $\frac{\beta}{2} \|h\|_2^2 = \frac{2}{\beta} \|\nabla f(x)\|_2^2$ so $f(x + h) \leq f(x)$
2. If $h = -\frac{1}{\beta} \nabla f(x)$ then $\nabla f(x)^T h = -\frac{1}{\beta} \|\nabla f(x)\|_2^2$ and $\frac{\beta}{2} \|h\|_2^2 = \frac{1}{2\beta} \|\nabla f(x)\|_2^2$.

□

Theorem 6.3. If f is differentiable, β smooth, and x^* be any optimizer. For any $k > 0$

$$\min_{i=0, \dots, k} \|\nabla f(x_i)\|_2 \leq \sqrt{\frac{2\beta}{k} \left(\underbrace{f(x_0) - f(x^*)}_{\Delta_0} \right)}$$

Proof. AFSOC $\forall i \in \{0, \dots, k\}$, $\|\nabla f(x_i)\|_2 > \sqrt{\frac{2\beta}{k} \Delta_0}$.

Then we have

$$f(x_{i+1}) < f(x_i) - \frac{1}{2\beta} \left(\frac{2\beta \Delta_0}{k} \right) \quad (\forall i \in \{0, \dots, k\})$$

Continuing down the chain we have

$$f(x_{k+1}) < f(x_0) - \Delta_0 < f(x^*)$$

Thus a contradiction. □

6.2 Convergence Rates

Theorem 6.4. If f is β -smooth and convex, $\eta = \frac{1}{\beta}$ and x^* any minimum then

$$f(x_k) - f(x^*) \leq \frac{\beta \|x_0 - x^*\|_2^2}{2k}$$

Proof. Observe that

$$\|x_{t+1} - x^*\|^2 = \|x_t - x^* - \eta \nabla f(x_t)\|^2 = \|x_t - x^*\|^2 - \eta^2 \|\nabla f(x_t)\|^2 - 2\eta \nabla f(x_t)^T (x_t - x^*)$$

Rearranging and by convexity we have

$$\begin{aligned}
 f(x_t) - f(x^*) &\leq \nabla f(x_t)^T(x_t - x^*) = \frac{1}{2\eta} \left[\underbrace{\|x_t - x^*\|^2}_{\delta_t} - \underbrace{\|x_{t+1} - x^*\|^2}_{\delta_{t+1}} + \frac{\eta}{2} \|\nabla f(x_t)\|^2 \right] \\
 &\leq \frac{1}{2\eta} [\delta_t - \delta_{t+1} + 2\eta(f(x_t) - f(x_{t+1}))] \\
 &= \frac{1}{2\eta} (\delta_t - \delta_{t+1}) + f(x_t) - f(x_{t+1})
 \end{aligned}$$

So we have

$$f(x_{t+1}) - f(x^*) \leq \frac{\beta}{2} (\delta_t - \delta_{t+1})$$

Adding across all terms

$$\sum_{i=0}^{k-1} f(x_{i+1}) - f(x^*) \leq \frac{\beta}{2} (\|x_0 - x^*\|^2)$$

So we have the relation

$$k(f(x_k) - f(x^*)) \leq \sum_{i=0}^{k-1} f(x_{i+1}) - f(x^*) \leq \frac{\beta}{2} \|x_0 - x^*\|^2$$

□

6.3 Strong Convexity

Definition 6.5. f is α -strongly convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\alpha}{2} t(1-t) \|x - y\|_2^2.$$

Additionally, if f is differentiable then

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{\alpha}{2} \|y - x\|_2^2.$$

If f is twice differentiable then f is α -strongly convex if and only if $\nabla^2 f(x) \succeq \alpha I$.

Theorem 6.6. Assume f is β -smooth, α -strongly convex, $\eta = \frac{1}{\beta}$ then

$$\|x_k - x^*\|^2 \leq \left(1 - \frac{1}{\gamma}\right)^k \|x_0 - x^*\|^2.$$

We define the conditional number as $\gamma = \frac{\beta}{\alpha}$

Proof.

$$\frac{\alpha}{2} \|x_k - x^*\|^2 \leq \frac{\beta}{2} (\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2) \leq \left(1 - \frac{\alpha}{\beta}\right) \|x_t - x^*\|^2 \leq \left(1 - \frac{1}{\gamma}\right) \|x_k - x^*\|^2$$

□

7 Feb 5: Subgradients and the Subgradient Method

7.1 Review and Subgradient Calculus

Recall 7.1. From previous class

1. $f(x + \eta h) \approx f(x) + \eta h^T \nabla f(x)$ and when we set $h = -\nabla f(x)$ we have

$$f(x - \eta \nabla f(x)) \approx f(x) - \eta \|\nabla f(x)\|_2^2.$$

Backtracking Line Search: Pick $\gamma_1 \in (0, 1)$ and $\gamma_2 \in (0, 1)$, start with initial step size η . While $f(x - \eta \nabla f(x)) > f(x) - \gamma_1 \eta \|\nabla f(x)\|_2^2$ (Armijo condition not satisfied), set $\eta \leftarrow \gamma_2 \eta$ and retry. Once the condition is satisfied, use step size η to update $x \leftarrow x - \eta \nabla f(x)$.

Recall 7.2. g is a subgradient of f at x if

$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \text{dom}(f).$$

Lemma 7.3. The following are true:

1. $\partial f(x)$ makes sense for non convex functions too, could be empty
2. If f is convex, then for $x \in \text{RelInt}(\text{dom}(f))$ we have that $\partial f(x)$ is non-empty
3. $\partial f(x)$ is a convex
4. If f is convex and differentiable at x then $\partial f(x) = \{\nabla f(x)\}$
5. If $\partial f(x)$ is non-empty everywhere, f is convex.

Theorem 7.4. The following are true:

1. $\partial(af) = a\partial f(x)$
2. $\partial(f + g) = \partial f(x) + \partial g(x)$
3. If $g(x) = f(Ax + b)$ then $\partial g(x) = A^T \partial f(Ax + b)$

Example 7.5. For $f(x) = \max_{i=1, \dots, n} f_i(x)$ we have $\partial f(x) = \text{Conv} \left(\bigcup_{i=1, \dots, n} \partial f_i(x) \right)$

Example 7.6. For $f(x) = |x|$ we have $\partial f(x) = \text{sign}(x) = \max\{-x, x\}$ so $\partial f(0) = [-1, 1]$.

Example 7.7. Let C be a convex set and $I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$ is convex. Additionally, for $x \in C$,

$$\partial I_C(x) = N_C(x) = \{g : g^T(y - x) \leq 0 \forall y \in C\}$$

Proof. For $x \in C$, $I_C(y) \geq I_C(x) + g^T(y - x)$ for all $y \in C$ then we have

$$0 \geq 0 + g^T(y - x).$$

□

7.2 Subgradient Method

Definition 7.8. Subgradient Method: $x_{t+1} \leftarrow x_t - \eta g_x$ for some $g_x \in \partial f(x)$

1. Subgradients are in general not descent directions
2. The min norm subgradient is a descent direction

Example 7.9. $f(x) = \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ is convex, then for $x \neq 0$ we have $\partial f(x) = \frac{x}{\|x\|_2}$.

$$\partial f(0) = \{g : \|g\|_2 \leq 1\}.$$

8 Feb 10: Subgradient Method Analysis and Projections

8.1 Subgradient Method Rates

Definition 8.1. For the subgradient method for $\min_{x \in \mathbb{R}^n} f(x)$ and f convex where

$$x_{t+1} = x_t - \eta_t g_t \text{ where } g_t \in \partial f(x_t).$$

We define the best iterate as

$$x_T^{(\text{best})} = \arg \min_{i=0, \dots, T} f(x_i).$$

Theorem 8.2. Assume f is G -lipschitz and convex. Let $\|x_0 - x^*\| \leq R$ then pick

$$\eta_t = \frac{R}{G\sqrt{T}} \text{ guarantees that } f(x_T^{(\text{best})}) - f(x^*) \leq \frac{GR}{\sqrt{T}}.$$

Theorem 8.3. A convex function f is G -lipschitz iff

$$\|g_x\| \leq G \quad \forall x \in \text{dom}(f) \text{ and } \forall g_x \in \partial f(x).$$

Theorem 8.4. For nonconvex, differentiable f , f being G -lipschitz iff

$$\|\nabla f(x)\| \leq G \quad \forall x \in \text{dom}(f).$$

Theorem 8.5. Assume f is convex and G -Lipschitz, and that an optimal solution x^* exists with $\|x_0 - x^*\| \leq R$ for some $R > 0$.

Pick $\eta_t \rightarrow 0$ such that $\sum_{t=1}^{\infty} \eta_t = \infty$ and $\sum_{t=1}^T \eta_t^2 < \infty$ then

$$f(x_T^{(\text{best})}) \rightarrow f(x^*) \text{ as } T \rightarrow \infty.$$

Theorem 8.6. For the subgradient method with step sizes $\{\eta_t\}_{t=1}^T$ on a convex, G -Lipschitz function f , we have

$$f(x_T^{(\text{best})}) - f(x^*) \leq \frac{\|x_0 - x^*\|^2 + G^2 \sum_{t=1}^T \eta_t^2}{2 \sum_{t=1}^T \eta_t}.$$

Verification of Theorem 8.2. Set $\eta_t = \frac{R}{G\sqrt{T}}$ for all t . With $\|x_0 - x^*\| \leq R$:

$$\begin{aligned} \sum_{t=1}^T \eta_t &= T \cdot \frac{R}{G\sqrt{T}} = \frac{R\sqrt{T}}{G}, \\ \sum_{t=1}^T \eta_t^2 &= T \cdot \frac{R^2}{G^2 T} = \frac{R^2}{G^2}. \end{aligned}$$

Substituting into the bound (using $\|x_0 - x^*\|^2 \leq R^2$):

$$f(x_T^{(\text{best})}) - f(x^*) \leq \frac{R^2 + G^2 \cdot \frac{R^2}{G^2}}{2 \cdot \frac{R\sqrt{T}}{G}} = \frac{2R^2}{2R\sqrt{T}/G} = \frac{2R^2 \cdot G}{2R\sqrt{T}} = \frac{GR}{\sqrt{T}}.$$

Hence Theorem 8.2 holds with the $O(1/\sqrt{T})$ rate. □

Proof.

$$\begin{aligned} \|x_{t+1} - x^*\|^2 &= \|x_t - \eta_t g_t - x^*\|^2 \\ &= \|x_t - x^*\|^2 - 2\eta_t g_t^T (x_t - x^*) + \eta_t^2 \|g_t\|^2 \\ &\leq \|x_t - x^*\|^2 + 2\eta_t (f(x^*) - f(x_t)) + \eta_t^2 G^2 \end{aligned} \tag{*}$$

Last step follows from $f(x^*) \geq f(x_t) + g_t^T(x^* - x_t) \iff -g_t^T(x_t - x^*) \geq f(x^*) - f(x_t)$.
 Adding (*) up from $0, \dots, T-1$ we have

$$\|x_T - x^*\|^2 \leq \|x_0 - x^*\|^2 + G^2 \sum_{t=0}^{T-1} \eta_t^2 + 2 \sum_{t=0}^{T-1} \eta_t (f(x^*) - f(x_t)).$$

So

$$2 \sum_{t=0}^{T-1} \eta_t (f(x_T^{(\text{best})}) - f(x^*)) \leq 2 \sum_{t=0}^{T-1} \eta_t (f(x_t) - f(x^*)) \leq \|x_0 - x^*\|^2 + G^2 \sum_{t=0}^{T-1} \eta_t^2.$$

□

8.2 Polyak Stepsize and Feasibility

Theorem 8.7. *Polyak's Stepsize: If $f(x^*)$ is known,*

$$\eta_t = \frac{f(x_t) - f(x^*)}{\|g_t\|^2}.$$

Theorem 8.8. *Given C_1, \dots, C_k convex sets find $x^* \in \bigcap_{i=1}^k C_i$.*

Proof. Define $f_i(x) = \min_{y \in C_i} \|x - y\|^2 = \text{dist}(x, C_i)$ and $f(x) = \max_{i=1, \dots, k} f_i(x)$.

If $x^* \in C_1 \cap \dots \cap C_k$ then $f(x^*) = 0$.

Recall: $\partial f(x) = \text{Conv} \left(\bigcup_{i=1, \dots, k} \partial f_i(x) \right)$.

Let $P_C(x) = \arg \min_{y \in C} \|x - y\|$

Lemma 8.9. *u is the projection of x onto C iff $\langle x - u, y - u \rangle \leq 0 \ \forall y \in C$*

We have $f_i(x) = \|x - P_{C_i}(x)\|^2$ so $\partial f_i(x) = \frac{x - P_{C_i}(x)}{\|x - P_{C_i}(x)\|}$ if $x \neq 0$.

$$x_{t+1} = x_t - f(x_t) \cdot \frac{x_t - P_{C_i}(x_t)}{\|x_t - P_{C_i}(x_t)\|} = P_{C_i}(x_t).$$

□

9 Feb 12: Projected Subgradient Method

9.1 Projections and Contractions

Definition 9.1. For $\min_{x \in C} f(x)$ we define the projected subgradient method to be

$$\begin{cases} y_{t+1} = x_t - \eta_t g_t \\ x_{t+1} = P_C(y_t) \end{cases}.$$

Theorem 9.2. $P_C(x) = \arg \min_{y \in C} \|x - y\|^2$

Proof. We have $z = P_C(x)$ iff $\forall y \in C, \langle x - z, y - z \rangle \leq 0$. Then for

$$\underbrace{-\nabla f(z)}_{x-z} \in \underbrace{N_C(z)}_{\{g: g^T(y-z) \leq 0 \ \forall y \in C\}}.$$

so we have $(x - z)^T(y - z) \leq 0$ □

Lemma 9.3. Projections are contractions:

$$\| \underbrace{P_C(x_1)}_{z_1} - \underbrace{P_C(x_2)}_{z_2} \|_2 \leq \|x_1 - x_2\|_2.$$

Proof. We have

1. $\forall z_2 \in C, \langle x_1 - z_1, z_2 - z_1 \rangle \leq 0$
2. $\forall z_1 \in C, \langle x_2 - z_2, z_1 - z_2 \rangle \leq 0$
3. Adding these two inequalities we have

$$\langle x_1 - z_1, z_2 - z_1 \rangle + \langle x_2 - z_2, z_1 - z_2 \rangle = \langle x_1 - x_2 + z_2 - z_1, z_2 - z_1 \rangle \leq 0.$$

So

$$\langle x_1 - x_2, z_2 - z_1 \rangle + \|z_2 - z_1\|_2^2 \leq 0 \iff \|z_2 - z_1\|_2^2 \leq \langle x_1 - x_2, z_2 - z_1 \rangle \leq \|x_1 - x_2\| \|z_2 - z_1\|.$$

So we have our result

$$\|z_1 - z_2\| \leq \|x_1 - x_2\|.$$
□

9.2 Convergence and Examples

Theorem 9.4. Rates for PGD are identical to GD and similarly PSGM is identical to SGM.

Proof.

$$\|x_{t+1} - x^*\|^2 = \|P_C(x_t - \eta_t g_t) - x^*\|^2 = \|P_C(x_t - \eta_t g_t) - P_C(x^*)\|^2 \leq \|x_t - \eta_t g_t - x^*\|^2.$$
□

Example 9.5. Consider

$$\min_{x \in \mathbb{R}^d} \frac{1}{2} \|Ax - b\|_2^2 \text{ where } n \geq d \text{ and } A \text{ full rank.}$$

When we solve this by gradient descent then the rate would be

$$\left(1 - \frac{\lambda_{\min}}{\lambda_{\max}}\right)^k.$$

Proof. We have $\nabla f(x) = A^T(Ax - b)$ and $\nabla^2 f(x) = A^T A$. So f is quadratic with Hessian $A^T A$. The eigenvalues of $A^T A$ satisfy $\lambda_{\min} = \lambda_{\min}(A^T A)$ and $\lambda_{\max} = \lambda_{\max}(A^T A)$. Hence f is α -strongly convex with $\alpha = \lambda_{\min}$ and β -smooth with $\beta = \lambda_{\max}$. Taking $\gamma = \beta/\alpha = \lambda_{\max}/\lambda_{\min}$ and $\eta = 1/\beta$, the strongly convex convergence theorem gives

$$\|x_k - x^*\|^2 \leq \left(1 - \frac{1}{\gamma}\right)^k \|x_0 - x^*\|^2 = \left(1 - \frac{\lambda_{\min}}{\lambda_{\max}}\right)^k \|x_0 - x^*\|^2.$$

□

Example 9.6. Consider a similar problem

$$\min_{x \geq 0} \frac{1}{2} \|Ax - b\|_2^2 \text{ where } n \geq d \text{ and } A \text{ full rank.}$$

We can let $y_{t+1} \leftarrow x_t - \eta_t(Ax_t - b)$ and $x_{t+1} \leftarrow \max(0, y_{t+1})$.

Example 9.7. Example of Fast Projection:

$$C = \{Ay + b | y \in \mathbb{R}^\ell\} \subseteq \mathbb{R}^n \text{ and } P_C(x) = b + A(A^T A)^{-1} A^T(x - b).$$

Proof.

$$y^* = \arg \min_{y \in \mathbb{R}^\ell} \frac{1}{2} \|x - (Ay + b)\|_2^2 \implies A^T(Ay^* + b) - x = 0 \implies A^T Ay^* = A^T(x - b) \implies y^* = (A^T A)^{-1} A^T(x - b).$$

So $P_C(x) = Ay^* + b$

□

Example 9.8.

$$C = \{y : Ay = b\} \subseteq \mathbb{R}^d \text{ and } P_C(x) = x + A^T(AA^T)^{-1}(b - Ax).$$

10 Feb 14: Optimality Conditions and Proximal Gradient Descent

Theorem 10.1. Consider $\min_{x \in \mathbb{R}^d} f(x)$, f not assumed convex. Then x^* is optimal on $\text{dom}(f)$ iff $0 \in \partial f(x^*)$.

Proof. (\Leftarrow) We have

$$f(y) \geq f(x^*) + \langle g_{x^*}, y - x^* \rangle. \quad (\forall y \in \text{dom}(f) \text{ and } g_{x^*} \in \partial f(x^*))$$

Since $0 \in \partial f(x^*)$ we have $f(y) \geq f(x^*)$ for all $y \in \text{dom}(f)$. So x^* is optimal.

(\Rightarrow)

$$f(y) \geq f(x^*) + \langle 0, y - x^* \rangle \Rightarrow 0 \in \partial f(x^*).$$

□

Theorem 10.2. f is convex and differentiable with C convex then

$$x^* \in \arg \min_{x \in C} f(x) \iff -\nabla f(x^*) \in N_C(x^*).$$

Theorem 10.3. If f is convex and C convex then

$$x^* \text{ is optimal for } f \text{ in } C \iff 0 \in \partial f(x^*) + N_C(x^*).$$

Proof. We can rewrite the objective as

$$x^* \in \arg \min_{x \in C} f(x) = \arg \min_{x \in \mathbb{R}^d} f(x) + I_C(x) \iff 0 \in \partial f(x^*) + N_C(x^*).$$

□

Example 10.4. *Soft Thresholding:* Consider the optimization problem with an L_2 data fidelity term and an L_1 regularization term (Lasso):

$$x^* = \arg \min_{z \in \mathbb{R}^d} \left(\frac{1}{2} \|z - y\|_2^2 + \lambda \|z\|_1 \right).$$

Since the objective function is separable (a sum of terms for each component), we can solve for each coordinate x_j^* independently. The problem reduces to finding scalar x_j^* that minimizes:

$$f(z_j) = \frac{1}{2} (z_j - y_j)^2 + \lambda |z_j|.$$

The optimality condition for a convex function requires that 0 is in the subdifferential of the objective at the solution x_j^* :

$$0 \in \partial \left(\frac{1}{2} (x_j^* - y_j)^2 + \lambda |x_j^*| \right).$$

Taking the derivative of the quadratic term and the subdifferential of the absolute value, we get:

$$0 \in (x_j^* - y_j) + \lambda \partial |x_j^*|.$$

where the subdifferential $\partial |x|$ is defined as:

$$\partial |x| = \begin{cases} \{1\} & \text{if } x > 0 \\ \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \end{cases}$$

We analyze the three possible cases for the solution x_j^* :

1. **Case 1** ($x_j^* > 0$): The subgradient is $\{1\}$.

$$x_j^* - y_j + \lambda = 0 \implies x_j^* = y_j - \lambda.$$

For this to be consistent with our assumption $x_j^* > 0$, we must have $y_j - \lambda > 0$, or $y_j > \lambda$.

2. **Case 2** ($x_j^* < 0$): The subgradient is $\{-1\}$.

$$x_j^* - y_j - \lambda = 0 \implies x_j^* = y_j + \lambda.$$

For this to be consistent with $x_j^* < 0$, we must have $y_j + \lambda < 0$, or $y_j < -\lambda$.

3. **Case 3** ($x_j^* = 0$): The subgradient is the interval $[-1, 1]$. The condition becomes $0 \in -y_j + \lambda[-1, 1]$, which means $y_j \in [-\lambda, \lambda]$. This holds whenever $|y_j| \leq \lambda$.

Combining these cases yields the closed-form **Soft Thresholding Operator**, denoted as $\mathcal{S}_\lambda(y)$:

$$x_j^* = \mathcal{S}_\lambda(y_j) = \begin{cases} y_j - \lambda & \text{if } y_j > \lambda \\ y_j + \lambda & \text{if } y_j < -\lambda \\ 0 & \text{if } |y_j| \leq \lambda \end{cases}$$

Or more compactly using the sign function:

$$x^* = \text{sign}(y) \max(|y| - \lambda, 0).$$

Theorem 10.5. Proximal Gradient Descent: Our objective is

$$\min_{x \in \mathbb{R}^d} \underbrace{g(x)}_{\substack{\text{convex} \\ \text{diff}}} + \underbrace{h(x)}_{\substack{\text{convex} \\ \text{non-diff} \\ \text{simple}}}.$$

Our update rules are

$$\begin{aligned} y^{t+1} &\leftarrow x^t - \eta_t \nabla g(x^t) \\ x^{t+1} &\leftarrow \text{prox}_{\eta_t, h}(y^{t+1}) := \arg \min_{z \in \mathbb{R}^d} \frac{1}{2\eta_t} \|y^t - z\|_2^2 + h(z) \end{aligned}$$

When $g(x) = \|Ax - b\|_2^2$, $h(x) = \lambda \|x\|_1$ then this is Proximal GD called ISTA.

Remark 10.6. The intuition is that

$$x_{t+1} \leftarrow \arg \min_z g(x_t) + \nabla g(x_t)^T (z - x_t) + \frac{1}{2\eta_t} \|z - x_t\|_2^2 + h(z) = \frac{\arg \min_z 1}{2\eta_t} \|z - (x_t - \eta_t \nabla g(x_t))\|_2^2 + h(z).$$

Theorem 10.7. For $p(x) = \text{Prox}_{\eta, h}(x) = \frac{\arg \min_z 1}{2\eta} \|x - z\|_2^2 + h(z)$ we have

1. $\|p(x) - p(y)\|_2^2 \leq \langle x - y, p(x) - p(y) \rangle$
2. $\|p(x) - p(y)\| \leq \|x - y\|$

(1) \implies (2) by the Cauchy-Schwarz inequality.

Proof. We have

$$\frac{1}{\eta}(x - p(x)) \in \partial h(p(x)) \text{ and } \frac{1}{\eta}(y - p(y)) \in \partial h(p(y)).$$

By subgradient monotonicity we have

$$\langle g_x - g_y, x - y \rangle \geq 0 \text{ for all } x, y \text{ and } g_x \in \partial f(x), g_y \in \partial f(y).$$

So

$$\left\langle \frac{1}{\eta}(y - p(y)) - \frac{1}{\eta}(x - p(x)), p(y) - p(x) \right\rangle \geq 0 \iff \langle y - x, p(y) - p(x) \rangle \geq \|p(y) - p(x)\|_2^2.$$

□

Theorem 10.8. Define $G_\eta(x) = \frac{x - \text{prox}_{\eta,h}(x - \eta \nabla g(x))}{\eta}$ then the following are true:

1. $G_\eta(x) = 0 \iff x = \text{prox}_{\eta,h}(x - \eta \nabla g(x))$
2. $G_\eta(x^*) = 0 \iff 0 \in \nabla g(x^*) + \partial h(x^*) \iff x^* \text{ is optimal} \iff x^* \text{ is a fixed point of } \text{prox}$

Proof. By definition $\forall \tilde{x}$ then

$$\tilde{x} - \eta G_\eta(\tilde{x}) = \text{prox}_{\eta,h}(\underbrace{\tilde{x} - \eta \nabla g(\tilde{x})}_x).$$

$$\begin{aligned} u := \text{prox}_{\eta,h}(x) &\iff \frac{1}{\eta}(x - u) \in \partial h(u) \\ &\iff \tilde{x} - \eta \nabla g(\tilde{x}) - \tilde{x} - \eta G_\eta(\tilde{x}) \in \eta \partial h(\tilde{x} - \eta G_\eta(\tilde{x})) \\ &\iff G_\eta(\tilde{x}) \in \nabla g(\tilde{x}) + \partial h(\tilde{x} - \eta G_\eta(\tilde{x})) \quad \forall \tilde{x} \end{aligned}$$

Thus we imply 1 by choosing $\tilde{x} = x^*$

□

11 Feb 19: Stochastic Gradient Descent

Definition 11.1. *Stochastic Gradient Descent:* We have access to $g(x_t, \zeta_t)$ unbiased for $\nabla f(x_t)$ or an element of $\partial f(x_t)$.

$$\mathbb{E}_{\zeta}[g(x, \zeta)] = \nabla f(x) \in \partial f(x).$$

Our algorithm performs update

$$x^{t+1} \leftarrow x^t - \eta_t g(x^t, \zeta_t).$$

Example 11.2. $R_n(\theta) = \frac{1}{n} \sum_{i=1}^n \ell(f_{\theta}(x_i), y_i)$, if n is large, computing ∇R_n is expensive.

$g_i = \nabla \ell(f_{\theta}(x_i), y_i)$ is unbiased for $\nabla R_n(\theta)$ if $i \sim \text{Unif}\{1, \dots, n\}$.

Theorem 11.3. Assume f is convex, $\|x_0 - x^*\| = R$ and $\mathbb{E}[\|g(x_t, \zeta_t)\|_2^2] \leq G^2$ then choosing $\eta = \frac{R}{G\sqrt{k}}$ yields

$$\mathbb{E} \left[f \left(\frac{1}{k} \sum_{i=1}^k x_i \right) - f(x^*) \right] \leq \frac{RG}{\sqrt{k}}.$$

Proof. Define $g_t = \mathbb{E}[\tilde{g}_t]$ then we have

$$\|x_{t+1} - x^*\|^2 = \|x_t - \eta_t \tilde{g}_t - x^*\|^2 = \|x_t - x^*\|^2 - 2\eta_t \tilde{g}_t^T (x_t - x^*) + \eta_t^2 \tilde{g}_t^2.$$

So

$$\begin{aligned} \mathbb{E}[\|x_{t+1} - x^*\|_2^2 \mid x_t] &\leq \|x_t - x^*\|^2 - 2\eta_t g_t^T (x_t - x^*) + \eta_t^2 G^2 \\ &\leq \|x_t - x^*\|^2 - 2\eta_t (f(x_t) - f(x^*)) + \eta_t^2 G^2 \\ \mathbb{E}[\|x_{t+1} - x^*\|_2^2] &\leq \mathbb{E}[\|x_t - x^*\|^2] - 2\eta_t \mathbb{E}[f(x_t) - f(x^*)] + \eta_t^2 G^2 \end{aligned}$$

So we can conclude

$$0 \leq \mathbb{E}[\|x_{t+1} - x^*\|_2^2] \leq R^2 - 2 \sum_{i=1}^k \mathbb{E}[f(x_i) - f(x^*)] + \sum_{t=1}^k \eta_t^2 G^2.$$

Rearranging we have

□