21-610 Homework 2

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Problem 1

Lemma 0.1. *If $H_1, H_2 \leq G$ then their intersection $H_1 \cap H_2 \leq G$*

Proof. $H_1 \cap H_2$ is non-empty as $e \in H_1 \cap H_2$. To show closure under multiplication, if $x, y \in H_1 \cap H_2$ then $x, y \in H_1$ and $x, y \in H_2$ so $xy \in H_1$ and $xy \in H_2$. Thus, $xy \in H_1 \cap H_2$. To show closure under inverses if $x \in H_1 \cap H_2$ then $x \in H_1$ so $x^{-1} \in H_1$ similarly for H_2. So $x^{-1} \in H_1 \cap H_2$. □

Part A

Let $G = p^n m$ where $p \nmid m$ then $|H| = p^n$ and suppose $|N| = p^r q$ where $r \leq n$ and $q|m$ by lagrange theorem. We first show $H \cap N$ is a subgroup. N is trivially a subgroup and H is a subgroup of G by definition of $Syl_p(G)$. So by lemma 0.1 $H \cap N$ is a subgroup.
Since $N \triangleleft G$ then HN is a subgroup. By second isomorphism theorem we have

$$|H \cap N| = \frac{|H||N|}{|HN|}.$$

We know $H \leq HN \leq G$ so $|HN| = p^n s$ where $s \leq m$ as $|H| \Bigg| |HN|$. So we can conclude

$$|H \cap N| = \frac{p^n \cdot p^r q}{p^n s} = p^r \frac{q}{s}.$$

Since $H \cap N \leq H$ then $\frac{q}{s} = 1$ by lagrange as $H \cap N$ is a p-subgroup. Thus $|H \cap N| = p^r$ and $H \cap N \in Syl_p(N)$.

Part B

Consider any $Q \in \mathrm{Syl}_p(N)$ and take any $H \in \mathrm{Syl}_p(G)$ then $H \cap N \in \mathrm{Syl}_p(N)$ by part A. By Sylow's Theorem II there exist $n \in N$ such that $n(H \cap N)n^{-1} = Q$. Expanding we have

$$(nHn^{-1}) \cap (nNn^{-1}) = Q \iff (nHn^{-1}) \cap N = Q.$$

Since $n \in G$ then $nHn^{-1} \in \mathrm{Syl}_p(G)$ as conjugation is a bijection. So we're done.

Problem 2

Part A

Let $|G| = p^n m$ where $p \nmid m$ then $|H| = p^n$ and we can let $|N| = p^r q$ where $r \leq n$ and $q \mid m$.
Then the order of subgroups in $\mathrm{Syl}_p(G/N)$ is p^{n-r} as $|G/N| = \frac{p^n m}{p^r \cdot q} = p^{n-r}\frac{m}{q}$.
Since $N \triangleleft G$, $N \triangleleft HN$. Thus the quotient group HN/N is well-defined. By second isomorphism theorem,

$$\left|\frac{HN}{N}\right| = \frac{|H|}{|H \cap N|}.$$

We've previously shown in problem 1A that $|H \cap N| = p^r$ so $\left|\frac{HN}{N}\right| = \frac{p^n}{p^r} = p^{n-r}$. So $\frac{HN}{N} \in \mathrm{Syl}_p(G/N)$

Part B

Consider any $Q \in \mathrm{Syl}_p(G/N)$ we want to show that $Q = \frac{H'N}{N}$ for some $H' \in \mathrm{Syl}_p(G)$. From part A, we know $P = \frac{HN}{N} \in \mathrm{Syl}_p(G/N)$ for $H \in \mathrm{Syl}_p(G)$. So there exist $gN \in G/N$ such that $Q = gNP(gN)^{-1}$ by second Sylow Theorem. We can rearrange using normality of N

$$Q = (gN)P(gN)^{-1} \iff Q = g\frac{HN}{N}g^{-1} \iff Q = \frac{gHNg^{-1}}{N} \iff Q = \frac{gHg^{-1}N}{N}.$$

We can let $H' = gHg^{-1}$ as we know $gHg^{-1} \in \mathrm{Syl}_p(G)$ by Sylow's second theorem. So $Q = \frac{H'N}{N}$ satisfies the problem.

Problem 3

Let H act on $X := G/H$ by left multiplication. So for some $h \in H$ and $gH \in X$, we have $h(gH) = (hg)H$. Then by fixed point theorem

$$|X| \equiv |X^H| \bmod p.$$

So we want to find $gH \in X$ such that $h(gH) = gH$ for all $h \in H$. Rearranging we have $g^{-1}hgH = H$ and thus $g^{-1}hg \in H$. We want this to hold for all $h \in H$ so $g \in N_G(H)$ and $gH \in N_G(H)/H$. Thus,

$$[G : H] \equiv [N_G(H) : H].$$

Problem 4

We factor $48 = 2^4 \cdot 3$ so $Syl_2(G)$ has subgroups of order 16. We know $|Syl_2(G)|$ is odd and divides 3 by Sylow's Third Theorem so $|Syl_2(G)| \in \{1, 3\}$. Since the problem posits two distinct Sylow 2-subgroups, $|Syl_2(G)| = 3$. Take distinct $P, Q \in Syl_2(G)$ we have for just subsets

$$|PQ| = \frac{|P||Q|}{|P \cap Q|}.$$

Since P, Q are subgroups of G then $|PQ| \leq 48$. Then rearranging

$$|P \cap Q| \geq \frac{|P||Q|}{48} = \frac{16^2}{48} \approx 5.33.$$

Since $P \cap Q \leq P$ then by Lagrange's Theorem $|P \cap Q| \;\Big|\; 16$ and as P and Q are distinct $|P \cap Q| < 16$. So $|P \cap Q| = 8$ as the factors of 16 are $1, 2, 4, 8, 16$.

Problem 5

Lemma 0.2. *If S is a p-subgroup of G and $P \in Syl_p(G)$, then there exists $g \in G$ such that $S \subseteq gPg^{-1}$*

Proof. Let $\Omega = G/P$ then $|\Omega|$ is not divisible by p as P has the same power of p as G. Let S act on Ω by left multiplication. So $s \cdot (gP) = (sg)P.$. Then by fixed point theorem

$$|\Omega| \equiv \left|\Omega^{\text{fix}}\right| \mod p.$$

Since Ω is not divisible by p, there exist at least one fixed point. Let gP be such a fixed point then $s(gP) = gP$ for all $s \in S$. So

$$s(gP) = gP \iff g^{-1}sgP = P \iff g^{-1}sg \in P.$$

Since this holds for all $s \in S, g^{-1}Sg \subseteq P$ and thus $S \subseteq gPg^{-1}$. $\square$

Now to prove the main statement.
Let G be a finite group, $H \leq G$ and $P \in \mathrm{Syl}_p(G)$. Then by Sylow's first theorem, $\mathrm{Syl}_p(H)$ is not empty. So take $S \in \mathrm{Syl}_p(H)$ then by lemma 0.2, there exists $g \in G$ such that $S \subseteq gPg^{-1}$. So $S \leq H \cap P^g$ and $H \cap P^g \leq H$ so $H \cap P^g$ is a p-subgroup in H and as S is a maximal p-subgroup of H so is $H \cap P^g$. Thus $H \cap P^g \in \mathrm{Syl}_p(H)$.