15-756 Homework 1

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January 28, 2026

Problem 1

Let x_i, x_j be the amounts in the envelopes. WLOG let $w_i < w_j$ then the algorithm Bob will use is if he sees k in the envelope he draws then he swaps with probability $1 - \frac{k}{n}$ and stays with $\frac{k}{n}$. Then the probability he wins using law of total probability is

$$\begin{aligned}
\Pr(\text{Wins}) &= \Pr(\text{Wins}|\text{Draws } x_i)\Pr(\text{Draws } x_i) + \Pr(\text{Wins}|\text{Draws } x_j)\Pr(\text{Draws } x_j) \\
&= \left(1 - \frac{x_1}{n}\right) \cdot \frac{1}{2} + \left(\frac{x_j}{n}\right) \cdot \frac{1}{2} \\
&= \frac{1}{2}\left(1 + \frac{x_j - x_i}{n}\right) \\
&= \frac{1}{2} + \frac{1}{2} \cdot \left(\frac{x_j - x_i}{n}\right) \\
&= \frac{1}{2} + \Omega\left(\frac{1}{n}\right)
\end{aligned}$$

The last line as $x_j - x_i \geq 1$ so $\frac{x_j - x_i}{n} \geq \frac{1}{n}$.

Problem 2

We can write the difference in times as a sum of two differences.

$$t_3^{(i)} - t_1^{(i)} = t_3^{(i)} - t_2^{(i)} + t_2^{(i)} - t_1^{(i)}.$$

For scenario 1, $t_2^{(1)} - t_1^{(1)} = 0$. Using the fact that the expected value of the next bus arriving in 1 minute then $\mathbb{E}[t_3^{(1)} - t_2^{(1)}] = 1$ so we can conclude that

$$\mathbb{E}\left[t_3^{(1)} - t_1^{(1)}\right] = 1.$$

For scenario 2, by linearity of expectation

$$\mathbb{E}\left[t_3^{(2)} - t_1^{(2)}\right] = \mathbb{E}\left[t_3^{(2)} - t_2^{(2)} + t_2^{(2)} - t_1^{(2)}\right] = \mathbb{E}\left[t_3^{(2)} - t_2^{(2)}\right] + \mathbb{E}\left[t_2^{(2)} - t_1^{(2)}\right].$$

Let $A = t_2^{(2)} - t_1^{(2)}$ and $B = t_3^{(2)} - t_2^{(2)}$.
Letting $N(t)$ be the number of buses that arrived in an interval of length t. Then

$$\Pr(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

We want the case when $k = 0$ so

$$\Pr(N(t) = 0) = e^{-\lambda t}.$$

This is equivalent to $\Pr(A > t)$ as this is the probability no buses arrived in the last t minutes. So

$$\Pr(A > t) = e^{-\lambda t}.$$

So A follows exponential distribution with rate λ where $\lambda = 1$ so $\mathbb{E}[A] = 1$

The second term $\mathbb{E}[B]$ is the expected time since the *previous* bus. Since the bus arrivals are a Poisson process with rate $\lambda = 1$, the inter-arrival times are exponentially distributed. By the memoryless property of the exponential distribution, the time since the last event is also exponentially distributed with mean $1/\lambda = 1$. Thus

$$E[A + B] = 2.$$

Thus the two scenarios are different. The two quantities are not equal as a random point in time is more likely to fall in a longer interval than a shorter one, biasing the expected interval length upwards.

Problem 3

Part 1

Consider dividing the steps into "blocks" of length a so from $[1, a], [a+1, 2a], \ldots$. If Alice ever moves a consecutive steps to the right then she will move a distance of $+a$ which guarantees she reaches the a as otherwise she would've reached 0 already. The probability of stepping right a times in a row is

$$\epsilon = \left(\frac{1}{2}\right)^a.$$

Regardless of where Alice starts in a block, there is always at least ϵ probability she reaches a as she can always touch 0 before reaching a. For Alice to stay inside the interval forever, we must have that she fails to exit every single time so probability of not exiting at any block is at most $1 - \epsilon$. So for her to not exit by block n, it is at most $(1-\epsilon)^n$. So

$$\Pr(\text{Alice never hits } a \text{ or } 0) \leq \lim_{n\to\infty} (1-\epsilon)^n = 0.$$

Consequently,

$$\Pr(\text{Alice eventually hits } a \text{ or } 0) = 1 - \Pr(\text{Alice never hits } a \text{ or } 0) \geq 1.$$

So

$$\Pr(\text{Alice eventually hits } a \text{ or } 0) = 1.$$

Part 2

So we'll continue by induction on i.
(Base Case) When $i = 1$ then there $p_2 = \frac{1}{2}$ so it holds.
(Induction Step) Assume it holds for i then for $i+1$, the probability she reaches $\frac{1}{2^i}$ before 0 by our induction hypothesis was 2^i and from there is equal probability of reaching 2^{i+1} before 0 as she has no preference for direction at any point.
So by law of total probability

$$\begin{aligned}
p_{i+1} &= \Pr(\text{Reaches } 2^{i+1} \text{ before } 0 | \text{ At } 2^i) \cdot \Pr(\text{Reaches } 2^i \text{ before } 0) \\
&\quad + \Pr(\text{Reaches } 2^{i+1} \text{ before } 0 | \text{ At } 0) \Pr(\text{Reaches } 0 \text{ before } 2^i) \\
&= \frac{1}{2} p_i + 0 \cdot (1 - p_i) \\
&= \frac{1}{2^{i+1}}
\end{aligned}$$

So it holds by induction.

Part 3

Define E_i to be the event that reaches 2^i before 0 then for Alice to never hit 0, we must E_i occur for all $i \in \mathbb{N}$.

$$\Pr(E_i) = \frac{1}{2^i}. \qquad \text{(Part 2)}$$

If E_k occurred then E_j occurred for all $j \leq k$. So

$$\Pr(\text{Never reaching } 0) \leq \lim_{i\to\infty} \frac{1}{2^i} = 0.$$

Note: The sequence of events E_i is indexed by integers $i \in \mathbb{N}$, so the limit is perfectly well-defined.
So

$$\Pr(\text{Reach } 0) = 1 - \Pr(\text{Never reaching } 0) = 1.$$

Part 4

Let T be the number of steps until Alice reaches 0 If Alice is currently at 2^i then the minimum number of steps until she gets to 0 is 2^i so

$$\begin{aligned}
\mathbb{E}[T] &\geq \sum_{i=1}^{\infty} \Pr(\text{Alice reaches } 2^i \text{ before 0 and did not reach } 2^{i+1}) \cdot 2^{i+1} \qquad (2^i \text{ to get from and to each}) \\
&= \sum_{i=1}^{\infty} (p_i - p_{i+1})\, 2^{i+1} \\
&= \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} \cdot 2^{i+1} \\
&= \sum_{i=1}^{\infty} 1
\end{aligned}$$

The RHS diverges so $\mathbb{E}[T]$ diverages as well.