21-623 Complex Analysis **TT**

Homework 1 (due 21th Jan 2026)

Do N O T distribute

1. Let $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = \sqrt{|\mathrm{Re}(z)\mathrm{Im}(z)|}$.

(i) Show that f is *not* holomorphic at 0. (ii) Show that f satisfies the Cauchy-Riemann equations at 0.

Why does this not contradict the theorem from class?

2. Suppose a series of complex numbers $\sum_{n=0}^{\infty} a_n$ converges. Show that then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is not smaller than 1. Moreover, show that for every region in the unit disc of the form $S_M = \{z \in \mathbb{C},\ |z| < 1,\ |1-z| \le M(1-|z|)\}$, $M > 0$, we have

$$\lim_{S_M \ni z \to 1} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n$$

(Abel's theorem).

3. We have said that the convergence of power series on the boundary of their discs of convergence can be sometimes subtle. Here are some basic examples – prove the following statements.

(a) The power series $\sum_{n=0}^{\infty} n z^n$ does not converge on any point of the unit circle.

(b) The power series $\sum_{n=1}^{\infty} \frac{1}{n^2} z^n$ converges at every point of the unit circle.

(c) The power series $\sum_{n=1}^{\infty} \frac{1}{n} z^n$ converges at every point of the unit circle except $z = 1$.

4. A subset S of the set of positive integers is called an arithmetic progression of step r if $S = \{a, a+r, a+2r, a+3r, \dots\}$ for some positive integers a and r. Show that the set of positive integers cannot be partitioned into a finite number of subsets which are arithmetic progressions of distinct steps (excluding the trivial partition into just one set with $a = r = 1$).

Hints. There is an intimate connection between integers and analysis: generating functions! First convince yourself that if such a partition was possible, we would have $\frac{z}{1-z} = \sum_{j=1}^{k} \frac{z^{a_j}}{1-z^{d_j}}$ with, say $d_1 < d_2 < \cdots < d_k$ (for all $|z| < 1$). Then consider this identity around the point $z_0 = e^{2\pi i/d_k}$.