

# Convex Optimization Notes

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# 1 Jan 13: Introduction to Optimization

## 1.1 Basic Definitions

**Definition 1.1.** An *optimization problem* is defined as

$$\min_{x \in \mathcal{C}} f_0(x) \quad \text{where } \forall i \in [m], f_i(x) \leq b_i$$

where  $\mathcal{C}$  is the constraint set and  $f_0: \mathcal{C} \rightarrow \mathbb{R}$ .

**Definition 1.2.** The *standard form* of an optimization problem is

$$\min_{x \in \mathcal{D}} f_0(x) \quad \text{subject to } \forall i \in [m], f_i(x) \leq 0; \quad \forall j \in [p], h_j(x) = 0$$

where  $x \in \text{dom}(f_i) \cap \text{dom}(h_j) =: \mathcal{D}$ .

$f_0$  is termed the objective function, and  $x$  is the optimization variable.

$f_i$  and  $h_j$  are constraint functions.

**Definition 1.3.** A *feasible solution* is any  $x \in \mathcal{D}$  that satisfies all the constraints.

**Definition 1.4.** The *optimal solution* is  $x^* \in \arg \min_{x \in \mathcal{D}} f_0(x)$ .

**Remark 1.5.** If no feasible solution exists, we define  $\min_{x \in \mathcal{C}} f(x) = +\infty$  and, similarly,  $\max_{x \in \mathcal{C}} f(x) = -\infty$ .

## 2 Jan 15: Convex Sets and Functions

### 2.1 Convexity

**Definition 2.1.** A set  $C$  is **convex** if  $\forall x_1, x_2 \in C$  and  $\forall \theta \in [0, 1]$ , we have  $\theta x_1 + (1 - \theta)x_2 \in C$ .

**Definition 2.2.** A function  $f: \mathcal{X} \rightarrow \mathbb{R}$  is **convex** if for all  $x_1, x_2 \in \mathcal{X}$  and  $\theta \in [0, 1]$ ,

$$f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2) \quad (*)$$

**Remark 2.3.** Convex functions are differentiable almost everywhere.

**Theorem 2.4.** If  $f$  is both convex and concave, then  $f$  is affine (i.e., linear plus constant).

**Definition 2.5.**  $f$  is **strictly convex** if  $(*)$  holds with strict inequality  $<$  whenever  $x_1 \neq x_2$  and  $\theta \in (0, 1)$ .

**Definition 2.6.** A **convex optimization problem** is one where  $f_0, f_i$ , and  $\mathcal{C}$  are convex, and  $h_j$  are affine/linear. That is,

$$\min_{x \in \mathcal{C}} f_0(x) \quad \text{such that} \quad f_i(x) \leq 0 \quad \forall i \in [m], \quad h_j(x) = 0 \quad \forall j \in [p]$$

**Theorem 2.7.** The set of feasible solutions in Definition 2.6 is convex.

### 2.2 Local vs. Global Optima

**Theorem 2.8.**  $x^*$  is a **local minimizer** if there exists  $\epsilon > 0$  such that for all  $y$  with  $\|x^* - y\| \leq \epsilon$ ,

$$f(x^*) \leq f(y)$$

**Theorem 2.9.**  $x^*$  is a **local maximizer** if there exists  $\epsilon > 0$  such that for all  $y$  with  $\|x^* - y\| \leq \epsilon$ ,

$$f(x^*) \geq f(y)$$

**Theorem 2.10.** For any convex optimization problem, every local minimum is a global minimum.

*Proof.* Suppose  $\hat{x}$  is a local minimizer not equal to global minimizer  $x^*$ . Take  $\epsilon$  as any witness to  $\hat{x}$  being a local minimum. Let

$$y = \frac{\epsilon}{\|\hat{x} - x^*\|} x^* + \left(1 - \frac{\epsilon}{\|\hat{x} - x^*\|}\right) \hat{x}$$

Note:  $\|\hat{x} - x^*\| \leq \epsilon$ , otherwise  $\hat{x}$  is not a local minimizer in that neighborhood.

$$y - \hat{x} = \frac{\epsilon(x^* - \hat{x})}{\|x^* - \hat{x}\|} \quad ; \quad \|y - \hat{x}\| = \epsilon$$

Since  $f$  is convex,

$$f(y) \leq \frac{\epsilon}{\|\hat{x} - x^*\|} f(x^*) + \left(1 - \frac{\epsilon}{\|\hat{x} - x^*\|}\right) f(\hat{x}) < f(\hat{x})$$

Thus, a contradiction.  $\square$

### 3 Jan 20: Convex Combinations, Hulls, and Cones

#### 3.1 Types of Combinations

**Definition 3.1.** A *convex combination* of  $x_1, x_2$  is  $\theta_1x_1 + \theta_2x_2$  where  $\theta_1, \theta_2 \geq 0$  and  $\theta_1 + \theta_2 = 1$ .

**Definition 3.2.** An *affine combination* of  $x_1, x_2$  is  $\theta_1x_1 + \theta_2x_2$  where  $\theta_1 + \theta_2 = 1$ .

**Definition 3.3.** A *linear combination* of  $x_1, x_2$  is  $\theta_1x_1 + \theta_2x_2$  where  $\theta_1, \theta_2 \in \mathbb{R}$ .

**Definition 3.4.** A *conic combination* of  $x_1, x_2$  is  $\theta_1x_1 + \theta_2x_2$  where  $\theta_1, \theta_2 \geq 0$ .

**Definition 3.5.** Given a set  $C$ , the *convex hull* of  $C$  is

$$\text{conv}(C) = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in C, \theta_i \in [0, 1], \sum_{i=1}^k \theta_i = 1 \right\}$$

**Remark 3.6.** The following are true:

1.  $C \subseteq \text{conv}(C)$
2.  $\text{conv}(C)$  is convex
3. It is the smallest convex set containing  $C$
4. If a convex set  $S \supseteq C$  then  $S \supseteq \text{conv}(C)$

**Theorem 3.7.** Any closed convex set can be written as  $\overline{\text{conv}}(C)$  for some set  $C$ .

**Definition 3.8.** The *relative interior* of  $C$  is defined as

$$\text{relint}(C) = \{x \in C : \exists \epsilon > 0, B(x, \epsilon) \cap \text{Aff}(C) \subseteq C\}$$

**Definition 3.9.**  $C$  is a *cone* if  $\alpha x \in C$  whenever  $x \in C$  and  $\alpha \geq 0$ .

**Definition 3.10.** Given a set  $C$ , the *conic hull* of  $C$  is

$$\text{conic}(C) = \left\{ \sum_{i=1}^k \theta_i x_i : x_i \in C, \theta_i \geq 0 \right\}$$

**Theorem 3.11.** The conic hull of  $C$  is the smallest convex cone containing  $C$ .

#### 3.2 Norms and Convex Sets

**Definition 3.12.** The  $\ell_p$  *norm* is defined as

$$\|x\|_p = \left( \sum_{i=1}^d |x_i|^p \right)^{1/p}$$

**Remark 3.13.** The following are true:

1. For  $p \in (0, 1)$ :  $\|x\|_p$  is not a convex function
2. For  $p \geq 1$ :  $\|x\|_p$  is convex
3. For  $p > 1$ :  $\|x\|_p$  is strictly convex

**Example 3.14.** Examples of convex sets:

1. Hyperplane:  $\{x : a^T x = b\}$
2. Halfspace:  $\{x : a^T x \leq b\}$
3. Polyhedron:  $\{x \in \mathbb{R}^d : Ax \leq b, Cx = d\}$
4. Polytope: a bounded polyhedron.

### 3.3 Optimality and the Normal Cone

**Theorem 3.15.** A set  $S$  is **strictly convex** if for all  $x_1 \neq x_2$  and  $\theta \in (0, 1)$ ,  $\theta x_1 + (1-\theta)x_2 \in \text{int}(S)$ .

**Definition 3.16.** The **normal cone** is defined to be

$$N_C(x) = \{g : g^T(y - x) \leq 0, \forall y \in C\}$$

**Remark 3.17.** If  $x \in \text{int}(C)$  then  $N_C(x) = \{0\}$ .

**Theorem 3.18.** If  $f$  is differentiable, then  $x^*$  is optimal if and only if  $-\nabla f(x^*) \in N_C(x^*)$ .

## 4 Jan 22: Optimality Conditions and Characterizations of Convexity

### 4.1 Optimality and Operations on Convex Sets

**Theorem 4.1.** If  $f$  is convex and differentiable, and  $C$  is a convex set, then any optimal solution  $x^*$  to  $\min_{x \in C} f(x)$  must satisfy  $-\nabla f(x^*) \in N_C(x^*)$ .

**Theorem 4.2.** The set of optimal solutions to a convex optimization problem is a convex set.

**Definition 4.3.** If  $C$  is convex then

1. **Translation:**  $C + a = \{x : x - a \in C\}$
2. **Scaling:**  $\alpha C = \{x : \frac{x}{\alpha} \in C\}$
3. **Intersection:** If  $\{C_\alpha\}_{\alpha \in A}$  is a collection of convex sets, then  $\bigcap_{\alpha \in A} C_\alpha$  is convex.

**Theorem 4.4.** The following are true:

1. If  $C \subseteq \mathbb{R}^n$  is convex,  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , then  $AC + b = \{Ax + b : x \in C\}$  is a convex set.
2. If  $f(x) = Ax + b$ , then  $f^{-1}(C)$  is convex.
3.  $C_1 + C_2 = \{x + y : x \in C_1, y \in C_2\}$  is convex.
4. If  $C_1 \subseteq \mathbb{R}^m$ ,  $C_2 \subseteq \mathbb{R}^n$  then  $C_1 \times C_2 = \{(x, y) \in \mathbb{R}^{m+n} : x \in C_1, y \in C_2\}$  is convex.
5. For any  $C \subseteq \mathbb{R}^n \times \mathbb{R}_{>0}^m$ , define

$$P(C) = \left\{ \left( \frac{x_1}{t}, \dots, \frac{x_n}{t} \right) : (x_1, \dots, x_n, t) \in C \right\}$$

If  $C$  is convex, so are  $P(C)$  and  $P^{-1}(C)$ .

### 4.2 First and Second Order Conditions

**Definition 4.5.**

$$\text{epi}(f) = \{(x, t) : x \in \text{dom}(f), t \geq f(x)\}$$

**Definition 4.6. First Order Definition of Convexity:** If  $f$  is differentiable, then  $f$  is convex if and only if for all  $x, y \in \text{dom}(f)$ ,

$$f(y) \geq f(x) + \nabla f(x)^T(y - x)$$

**Second Order Definition of Convexity:** If  $f$  is twice differentiable, then  $f$  is convex if and only if  $\nabla^2 f(x) \succeq 0$ .

**Definition 4.7.** We say  $A \succeq B$  if  $A - B$  is positive semi-definite. This is equivalent to

$$a^T \nabla^2 f(x) a \geq 0 \quad \text{for all } a \in \mathbb{R}^d$$

### 4.3 Subdifferentials

**Definition 4.8.** *The subdifferential is*

$$\partial f(x) = \{g : f(y) \geq f(x) + g^T(y - x), \forall y \in C\}$$

*Any such  $g \in \partial f(x)$  is called a **subgradient**.*

*If  $f$  is differentiable at  $x$  then  $\partial f(x) = \{\nabla f(x)\}$ .*

**Theorem 4.9.**  *$f$  is convex if and only if  $\partial f$  is non-empty.*

## 5 Jan 27: Matrix Theory, Lipschitz Continuity, and Smoothness

### 5.1 Gradient Monotonicity and SVD

**Definition 5.1.** *Gradient Monotonicity: If  $f$  is differentiable, then  $f$  is convex if and only if  $\nabla f(x)$  is monotone. So we can conclude*

$$(\nabla f(x) - \nabla f(y))^T(x - y) \geq 0 \iff f \text{ is convex.}$$

**Theorem 5.2.** *For a  $A \in \mathbb{R}^{m \times n}$  we can decompose*

$$A = U \sum V^t.$$

where  $U \in \mathbb{R}^{m \times k}$  with orthonormal column,  $\sigma \in \mathbb{R}^{k \times k}$  is a diagonal matrix with non-negative entries, and  $V \in \mathbb{R}^{n \times k}$  with orthonormal column.

**Definition 5.3.**  $(u_i, v_i, \sigma_i)$  form a s.v. triplet if  $Av_i = \sigma_i u_i$  and  $A^T u_i = \sigma_i v_i$   
 $\|u_i\| = \|v_i\| = 1$  and  $\sigma_i \geq 0$

**Theorem 5.4.** *For any  $A \in \mathbb{R}^{n \times n}$  then  $A^T A$  is always positive semi definite*

*Proof.* We can write  $A = V \sum U$  then  $A^T A = V \sum U^T U \sum V^T = V \sum^2 V^T$  □

**Definition 5.5.** *Spectral Radius:  $\max_i \{|\lambda_i| : \lambda_i \in \Lambda(A)\} = \rho(A)$*

**Definition 5.6.** *The norm must satisfy the following properties:*

1.  $\|A\| \geq 0$
2.  $\|\alpha A\| = |\alpha| \|A\|$
3.  $\|A\| = 0$  if and only if  $A = 0$
4.  $\|A + A'\| \leq \|A\| + \|A'\|$

### 5.2 Matrix Norms

**Definition 5.7.** *Operator/Spectral Norm:*

$$\|A\|_{op} = \|A\|_2 = \max_{\|x\|=1} \|Ax\|_2.$$

**Definition 5.8.** *Frobenius Norm:*

$$\|A\|_F = \sqrt{\sum_{i,j} A_{ij}^2}.$$

### 5.3 Lipschitz Continuity and Smoothness

**Definition 5.9.**  *$f$  is  $L$ -Lipschitz if*

$$|f(x) - f(y)| \leq L \|x - y\|_2 \quad \forall x, y \in \text{dom}(f).$$

*If  $f$  is differentiable then  $f$  is  $L$ -Lipschitz if and only if  $|\nabla f| \leq L$*

**Definition 5.10.** Differentiable  $f$  is  $\beta$ -smooth if

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq \beta \|x - y\|_2 \quad \forall x, y \in \text{dom}(f).$$

If  $f$  is twice differentiable then  $f$  is  $\beta$ -smooth if and only if  $\nabla^2 f(x) \preceq \beta I$ .

**Theorem 5.11.** If  $f$  is twice differentiable and  $\beta$ -smooth then

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{\beta}{2} \|y - x\|_2^2 \quad \forall x, y \in \text{dom}(f).$$

## 6 Feb 3: Gradient Descent and Convergence

### 6.1 Descent Directions and Step Sizes

**Definition 6.1.** "Descent Direction" any  $h$  for which  $f(x + \eta h) \leq f(x)$ .

**Theorem 6.2.** Assume  $f$  differentiable and  $\beta$ -smooth with  $x_{t+1} \leftarrow x_t - \eta \nabla f(x_t)$ .

1. If  $\eta \leq \frac{2}{\beta}$  then  $f(x_{t+1}) \leq f(x_t)$
2. If  $\eta \leq \frac{1}{\beta}$  then  $f(x_{t+1}) \leq f(x_t) - \frac{\eta}{2} \|\nabla f(x_t)\|_2^2$

*Proof.* Recall

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{\beta}{2} \|y - x\|_2^2.$$

and

$$\|\nabla f(y) - \nabla f(x)\|_2 \leq \beta \|y - x\|_2.$$

Let  $y \leftarrow x + h$  then  $f(x + h) \leq f(x) + \nabla f(x)^T h + \frac{\beta}{2} \|h\|_2^2$

1. If  $h = -\frac{2}{\beta} \nabla f(x)$  then  $\nabla f(x)^T h = -\frac{2}{\beta} \|\nabla f(x)\|_2^2$  and  $\frac{\beta}{2} \|h\|_2^2 = \frac{2}{\beta} \|\nabla f(x)\|_2^2$  so  $f(x + h) \leq f(x)$
2. If  $h = -\frac{1}{\beta} \nabla f(x)$  then  $\nabla f(x)^T h = -\frac{1}{\beta} \|\nabla f(x)\|_2^2$  and  $\frac{\beta}{2} \|h\|_2^2 = \frac{1}{2\beta} \|\nabla f(x)\|_2^2$ .

□

**Theorem 6.3.** If  $f$  is differentiable,  $\beta$  smooth, and  $x^*$  be any optimizer. For any  $k > 0$

$$\min_{i=0, \dots, k} \|\nabla(f(x_i))\|_2 \leq \sqrt{\frac{2\beta}{k} \left( \underbrace{f(x_0) - f(x^*)}_{\Delta_0} \right)}.$$

*Proof.* AFSOC  $\forall i \in \{0, \dots, k\}, \|\nabla f(x_i)\| > \sqrt{\frac{2\beta}{k} \Delta_0}$  Then we have

$$f(x_{i+1}) < f(x_i) - \frac{1}{2\beta} \left( \frac{2\beta \Delta_0}{k} \right). \quad (\forall i \in \{0, \dots, k\})$$

Continuing down the chain we have

$$f(x_{k+1}) < f(x_0) - \Delta_0 < f(x^*).$$

Thus a contradiction. □

## 6.2 Convergence Rates

**Theorem 6.4.** If  $f$  is  $\beta$ -smooth and convex,  $\eta = \frac{1}{\beta}$  and  $x^*$  any minimum then

$$f(x_k) - f(x^*) \leq \frac{\beta \|x_0 - x^*\|_2^2}{2k}.$$

*Proof.* Observe that

$$\|x_{t+1} - x^*\|^2 = \|x_t - x^* - \eta \nabla f(x_t)\|^2 = \|x_t - x^*\|^2 - \eta^2 \|\nabla f(x_t)\|^2 - 2\eta \nabla f(x_t)^T (x_t - x^*).$$

Rearranging and by convexity we have

$$\begin{aligned} f(x_t) - f(x^*) &\leq \nabla f(x_t)^T (x_t - x^*) = \frac{1}{2\eta} \left[ \underbrace{\|x_t - x^*\|^2}_{\delta_t} - \underbrace{\|x_{t+1} - x^*\|^2}_{\delta_{t+1}} + \frac{\eta}{2} \|\nabla f(x_t)\|^2 \right] \\ &\leq \frac{1}{2\eta} [\delta_t - \delta_{t+1} + 2\eta(f(x_t) - f(x_{t+1}))] \\ &= \frac{1}{2\eta} (\delta_t - \delta_{t+1}) + f(x_t) - f(x_{t+1}) \end{aligned}$$

So we have

$$f(x_{t+1}) - f(x^*) \leq \frac{\beta}{2} (\delta_t - \delta_{t+1}).$$

Adding across all terms

$$\sum_{i=0}^{k-1} f(x_{i+1}) - f(x^*) \leq \frac{\beta}{2} (\|x_0 - x^*\|^2).$$

So we have the relation

$$k(f(x_k) - f(x^*)) \leq \sum_{i=0}^{k-1} f(x_{i+1}) - f(x^*) \leq \frac{\beta}{2} \|x_0 - x^*\|^2.$$

□

## 6.3 Strong Convexity

**Definition 6.5.**  $f$  is  $\alpha$ -strongly convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\alpha}{2} t(1-t) \|x - y\|_2^2.$$

Additionally, if  $f$  is differentiable then

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\alpha}{2} \|y - x\|_2^2.$$

If  $f$  is twice differentiable then  $f$  is  $\alpha$ -strongly convex if and only if  $\nabla^2 f(x) \succeq \alpha I$ .

**Theorem 6.6.** Assume  $f$  is  $\beta$ -smooth,  $\alpha$ -strongly convex,  $\eta = \frac{1}{\beta}$  then  $\|x_k - x^*\|^2 \leq \left(1 - \frac{1}{\gamma}\right)^k \|x_0 - x^*\|^2$ .

We define the conditional number as  $\gamma = \frac{\beta}{\alpha}$

*Proof.*

$$\frac{\alpha}{2} \|x_k - x^*\|^2 \leq \frac{\beta}{2} (\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2) \leq \left(1 - \frac{\alpha}{\beta}\right) \|x_t - x^*\|^2 \leq \left(1 - \frac{1}{\gamma}\right) \|x_k - x^*\|^2.$$

□

## 7 Feb 5: Subgradients and the Subgradient Method

### 7.1 Review and Subgradient Calculus

**Recall 7.1.** From previous class

1.  $f(x + \eta h) \approx f(x) + \eta h^T \nabla f(x)$  and when we set  $h = -\nabla f(x)$  we have

$$f(x - \eta \nabla f(x)) \approx f(x) - \eta \|\nabla f(x)\|_2^2.$$

**Backtracking Line Search:** Pick  $\gamma_1 \in (0, 1)$  and  $\gamma_2 \in (0, 1)$ , start with initial step size  $\eta$ . While  $f(x - \eta \nabla f(x)) > f(x) - \gamma_1 \eta \|\nabla f(x)\|_2^2$  (Armijo condition not satisfied), set  $\eta \leftarrow \gamma_2 \eta$  and retry. Once the condition is satisfied, use step size  $\eta$  to update  $x \leftarrow x - \eta \nabla f(x)$ .

**Recall 7.2.**  $g$  is a subgradient of  $f$  at  $x$  if

$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \text{dom}(f).$$

**Lemma 7.3.** The following are true:

1.  $\partial f(x)$  makes sense for non convex functions too, could be empty
2. If  $f$  is convex, then for  $x \in \text{RelInt}(\text{dom}(f))$  we have that  $\partial f(x)$  is non-empty
3.  $\partial f(x)$  is a convex
4. If  $f$  is convex and differentiable at  $x$  then  $\partial f(x) = \{\nabla f(x)\}$
5. If  $\partial f(x)$  is non-empty everywhere,  $f$  is convex.

**Theorem 7.4.** The following are true:

1.  $\partial(af) = a\partial f(x)$
2.  $\partial(f + g) = \partial f(x) + \partial g(x)$
3. If  $g(x) = f(Ax + b)$  then  $\partial g(x) = A^T \partial f(Ax + b)$

**Example 7.5.** For  $f(x) = \max_{i=1,\dots,n} f_i(x)$  we have  $\partial f(x) = \text{Conv} \left( \bigcup_{i=1,\dots,n} \partial f_i(x) \right)$

**Example 7.6.** For  $f(x) = |x|$  we have  $\partial f(x) = \text{sign}(x) = \max -x, x$  so  $\partial f(0) = [-1, 1]$ .

**Example 7.7.** Let  $C$  be a convex set and  $I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$  is convex. Additionally, for  $x \in C$ ,  $\partial I_C(x) = N_C(x) = \{g : g^T(y - x) \leq 0 \forall y \in C\}$

*Proof.* For  $x \in C$ ,  $I_C(y) \geq I_C(x) + g^T(y - x)$  for all  $y \in C$  then we have

$$0 \geq 0 + g^T(y - x).$$

□

## 7.2 Subgradient Method

**Definition 7.8.** *Subgradient Method:*  $x_{t+1} \leftarrow x_t - \eta g_x$  for some  $g_x \in \partial f(x)$

1. Subgradients are in general not descent directions
2. The min norm subgradient is a descent direction

**Example 7.9.**  $f(x) = \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$  is convex, then for  $x \neq 0$  we have  $\partial f(x) = \frac{x}{\|x\|_2}$ .

$$\partial f(0) = \{g : \|g\|_2 \leq 1\}.$$

## 8 Feb 10: Subgradient Method Analysis and Projections

### 8.1 Subgradient Method Rates

**Definition 8.1.** For the subgradient method for  $\min_{x \in \mathbb{R}^n} f(x)$  and  $f$  convex where

$$x_{t+1} = x_t - \eta_t g_t \text{ where } g_t \in \partial f(x_t).$$

We define the best iterate as

$$x_T^{(\text{best})} = \arg \min_{i=0, \dots, T} f(x_i).$$

**Theorem 8.2.** Assume  $f$  is  $G$ -lipschitz and convex. Let  $\|x_0 - x^*\| \leq R$  then pick

$$\eta_t = \frac{R}{G\sqrt{T}} \text{ guarantees that } f(x_T^{(\text{best})}) - f(x^*) \leq \frac{GR}{\sqrt{T}}.$$

**Theorem 8.3.** A convex function  $f$  is  $G$ -lipschitz iff

$$\|g_x\| \leq G \quad \forall x \in \text{dom}(f) \text{ and } \forall g_x \in \partial f(x).$$

**Theorem 8.4.** For nonconvex, differentiable  $f$ ,  $f$  being  $G$ -lipschitz iff

$$\|\nabla f(x)\| \leq G \quad \forall x \in \text{dom}(f).$$

**Theorem 8.5.** Assume  $f$  is convex and  $G$ -Lipschitz, and that an optimal solution  $x^*$  exists with  $\|x_0 - x^*\| \leq R$  for some  $R > 0$ .

Pick  $\eta_t \rightarrow 0$  such that  $\sum_{t=1}^{\infty} \eta_t = \infty$  and  $\sum_{t=1}^T \eta_t^2 < \infty$  then

$$f(x_T^{(\text{best})}) \rightarrow f(x^*) \text{ as } T \rightarrow \infty.$$

**Theorem 8.6.** For the subgradient method with step sizes  $\{\eta_t\}_{t=1}^T$  on a convex,  $G$ -Lipschitz function  $f$ , we have

$$f(x_T^{(\text{best})}) - f(x^*) \leq \frac{\|x_0 - x^*\|^2 + G^2 \sum_{t=1}^T \eta_t^2}{2 \sum_{t=1}^T \eta_t}.$$

Verification of Theorem 8.2. Set  $\eta_t = \frac{R}{G\sqrt{T}}$  for all  $t$ . With  $\|x_0 - x^*\| \leq R$ :

$$\begin{aligned} \sum_{t=1}^T \eta_t &= T \cdot \frac{R}{G\sqrt{T}} = \frac{R\sqrt{T}}{G}, \\ \sum_{t=1}^T \eta_t^2 &= T \cdot \frac{R^2}{G^2 T} = \frac{R^2}{G^2}. \end{aligned}$$

Substituting into the bound (using  $\|x_0 - x^*\|^2 \leq R^2$ ):

$$f(x_T^{(\text{best})}) - f(x^*) \leq \frac{R^2 + G^2 \cdot \frac{R^2}{G^2}}{2 \cdot \frac{R\sqrt{T}}{G}} = \frac{2R^2}{2R\sqrt{T}/G} = \frac{2R^2 \cdot G}{2R\sqrt{T}} = \frac{GR}{\sqrt{T}}.$$

Hence Theorem 8.2 holds with the  $O(1/\sqrt{T})$  rate.  $\square$

*Proof.*

$$\begin{aligned} \|x_{t+1} - x^*\|^2 &= \|x_t - \eta_t g_t - x^*\|^2 \\ &= \|x_t - x^*\|^2 - 2\eta_t g_t^T (x_t - x^*) + \eta_t^2 \|g_t\|^2 \\ &\leq \|x_t - x^*\|^2 + 2\eta_t (f(x^*) - f(x_t)) + \eta_t^2 G^2 \end{aligned} \tag{*}$$

Last step follows from  $f(x^*) \geq f(x_t) + g_t^T(x^* - x_t) \iff -g_t^T(x_t - x^*) \geq f(x^*) - f(x_t)$ .  
Adding (\*) up from 0, ...,  $T - 1$  we have

$$\|x_T - x^*\|^2 \leq \|x_0 - x^*\|^2 + G^2 \sum_{t=0}^{T-1} \eta_t^2 + 2 \sum_{t=0}^{T-1} \eta_t (f(x^*) - f(x_t)).$$

So

$$2 \sum_{t=0}^{T-1} \eta_t (f(x_T^{(\text{best})}) - f(x^*)) \leq 2 \sum_{t=0}^{T-1} \eta_t (f(x_t) - f(x^*)) \leq \|x_0 - x^*\|^2 + G^2 \sum_{t=0}^{T-1} \eta_t^2.$$

□

## 8.2 Polyak Stepsize and Feasibility

**Theorem 8.7.** *Polyak's Stepsize: If  $f(x^*)$  is known,*

$$\eta_t = \frac{f(x_t) - f(x^*)}{\|g_t\|^2}.$$

**Theorem 8.8.** *Given  $C_1, \dots, C_k$  convex sets find  $x^* \in \bigcap_{i=1}^k C_i$ .*

*Proof.* Define  $f_i(x) = \min_{y \in C_i} \|x - y\|^2 = \text{dist}(x, C_i)$  and  $f(x) = \max_{i=1, \dots, k} f_i(x)$ .  
If  $x^* \in C_1 \cap \dots \cap C_k$  then  $f(x^*) = 0$ .

Recall:  $\partial f(x) = \text{Conv}(\bigcup_{i=1, \dots, k} \partial f_i(x))$ .

Let  $P_C(x) = \arg \min_{y \in C} \|x - y\|$

**Lemma 8.9.**  *$u$  is the projection of  $x$  onto  $C$  iff  $\langle x - u, y - u \rangle \leq 0 \forall y \in C$*

We have  $f_i(x) = \|x - P_{C_i}(x)\|^2$  so  $\partial f_i(x) = \frac{x - P_{C_i}(x)}{\|x - P_{C_i}(x)\|}$  if  $x \neq 0$ .

$$x_{t+1} = x_t - f(x_t) \cdot \frac{x_t - P_{C_i}(x_t)}{\|x_t - P_{C_i}(x_t)\|} = P_{C_i}(x_t).$$

□

## 9 Feb 12: Projected Subgradient Method

### 9.1 Projections and Contractions

**Definition 9.1.** For  $\min_{x \in C} f(x)$  we define the projected subgradient method to be

$$\begin{cases} y_{t+1} = x_t - \eta_t g_t \\ x_{t+1} = P_C(y_t) \end{cases}.$$

**Theorem 9.2.**  $P_C(x) = \arg \min_{y \in C} \|x - y\|^2$

*Proof.* We have  $z = P_C(x)$  iff  $\forall y \in C$ ,  $\langle x - z, y - z \rangle \leq 0$ . Then for

$$\underbrace{-\nabla f(z)}_{x-z} \in \underbrace{N_C(z)}_{\{g: g^T(y-z) \leq 0 \forall y \in C\}}.$$

so we have  $(x - z)^T(y - z) \leq 0$  □

**Lemma 9.3.** Projections are contractions:

$$\|\underbrace{P_C(x_1)}_{z_1} - \underbrace{P_C(x_2)}_{z_2}\|_2 \leq \|x_1 - x_2\|_2.$$

*Proof.* We have

1.  $\langle x_1 - z_1, z_2 - z_1 \rangle \leq 0 \forall z_2 \in C$
2.  $\langle x_2 - z_2, z_1 - z_2 \rangle \leq 0 \forall z_1 \in C$
3. Adding these two inequalities we have

$$\langle x_1 - z_1, z_2 - z_1 \rangle + \langle x_2 - z_2, z_1 - z_2 \rangle = \langle x_1 - x_2 + z_2 - z_1, z_2 - z_1 \rangle \leq 0.$$

So

$$\langle x_1 - x_2, z_2 - z_1 \rangle + \|z_2 - z_1\|_2^2 \leq 0 \iff \|z_2 - z_1\|_2^2 \leq \langle x_1 - x_2, z_2 - z_1 \rangle \leq \|x_1 - x_2\| \|z_2 - z_1\|.$$

So we have our result

$$\|z_1 - z_2\| \leq \|x_1 - x_2\|. □$$

### 9.2 Convergence and Examples

**Theorem 9.4.** Rates for PGD are identical to GD and similarly PSGM is identical to SGM.

*Proof.*

$$\|x_{t+1} - x^*\|^2 = \|P_C(x_t - \eta_t g_t) - x^*\|^2 = \|P_C(x_t - \eta_t g_t) - P_C(x^*)\|^2 \leq \|x_t - \eta_t g_t - x^*\|^2.$$

□

**Example 9.5.** Consider

$$\min_{x \in \mathbb{R}^d} \frac{1}{2} \|Ax - b\|_2^2 \text{ where } n \geq d \text{ and } A \text{ full rank}.$$

When we solve this by gradient descent then the rate would be

$$\left(1 - \frac{\lambda_{\min}}{\lambda_{\max}}\right)^k.$$

*Proof.* We have  $\nabla f(x) = A^T(Ax - b)$  and  $\nabla^2 f(x) = A^T A$ . So  $f$  is quadratic with Hessian  $A^T A$ . The eigenvalues of  $A^T A$  satisfy  $\lambda_{\min} = \lambda_{\min}(A^T A)$  and  $\lambda_{\max} = \lambda_{\max}(A^T A)$ . Hence  $f$  is  $\alpha$ -strongly convex with  $\alpha = \lambda_{\min}$  and  $\beta$ -smooth with  $\beta = \lambda_{\max}$ . Taking  $\gamma = \beta/\alpha = \lambda_{\max}/\lambda_{\min}$  and  $\eta = 1/\beta$ , the strongly convex convergence theorem gives

$$\|x_k - x^*\|^2 \leq \left(1 - \frac{1}{\gamma}\right)^k \|x_0 - x^*\|^2 = \left(1 - \frac{\lambda_{\min}}{\lambda_{\max}}\right)^k \|x_0 - x^*\|^2.$$

□

**Example 9.6.** Consider a similar problem

$$\min_{x \geq 0} \frac{1}{2} \|Ax - b\|_2^2 \text{ where } n \geq d \text{ and } A \text{ full rank}.$$

We can let  $y_{t+1} \leftarrow x_t - \eta_t(Ax_t - b)$  and  $x_{t+1} \leftarrow \max(0, y_{t+1})$ .

**Example 9.7.** Example of Fast Projection:

$$C = \{Ay + b | y \in \mathbb{R}^\ell\} \subseteq \mathbb{R}^n \text{ and } P_C(x) = b + A(A^T A)^{-1} A^T(x - b).$$

*Proof.*

$$y^* = \arg \min_{y \in \mathbb{R}^\ell} \frac{1}{2} \|x - (Ay + b)\|_2^2 \implies A^T(Ay^* + b) - x = 0 \implies A^T A y^* = A^T(x - b) \implies y^* = (A^T A)^{-1} A^T(x - b).$$

So  $P_C(x) = Ay^* + b$

□

**Example 9.8.**

$$C = \{y : Ay = b\} \subseteq R^d \text{ and } P_C(x) = x + A^T(AA^T)^{-1}(b - Ax).$$

## 10 Feb 14: Optimality Conditions and Proximal Gradient Descent

**Theorem 10.1.** Consider  $\min_{x \in \mathbb{R}^d} f(x)$ ,  $f$  not assumed convex. Then  $x^*$  is optimal on  $\text{dom}(f)$  iff  $0 \in \partial f(x^*)$ .

*Proof.* ( $\Leftarrow$ ) We have

$$f(y) \geq f(x^*) + \langle g_{x^*}, y - x^* \rangle. \quad (\forall y \in \text{dom}(f) \text{ and } g_x^* \in \partial f(x^*))$$

Since  $0 \in \partial f(x^*)$  we have  $f(y) \geq f(x^*)$  for all  $y \in \text{dom}(f)$ . So  $x^*$  is optimal.

( $\Rightarrow$ )

$$f(y) \geq f(x^*) + \langle 0, y - x^* \rangle \Rightarrow 0 \in \partial f(x^*).$$

□

**Theorem 10.2.**  $f$  is convex and differentiable with  $C$  convex then

$$x^* \in \arg \min_{x \in C} f(x) \iff -\nabla f(x^*) \in N_C(x^*).$$

**Theorem 10.3.** If  $f$  is convex and  $C$  convex then

$$x^* \text{ is optimal for } f \text{ in } C \iff 0 \in \partial f(x^*) + N_C(x^*).$$

*Proof.* We can rewrite the objective as

$$x^* \in \arg \min_{x \in C} f(x) = \arg \min_{x \in \mathbb{R}^d} f(x) + I_C(x) \iff 0 \in \partial f(x^*) + N_C(x^*).$$

□

**Example 10.4.** Soft Thresholding:

$$x^* = \arg \min_{z \in \mathbb{R}^d} \frac{1}{2} \|z - y\|_2^2 + \lambda \|z\|_1.$$

We can conclude that

$$x^* = \begin{cases} y & \text{if } x^* = 0 \\ 0 & \text{if } x^* = \infty \end{cases}.$$

We have  $0 \in x^* - y + \lambda \text{sign}(x^*)$ . So

$$x_j^* = \begin{cases} y_j - \lambda & \text{if } y_j \geq \lambda \\ y_j + \lambda & \text{if } y_j \leq -\lambda \\ 0 & \text{otherwise} \end{cases}.$$

**Theorem 10.5.** Proximal Gradient Descent: Our objective is

$$\min_{x \in \mathbb{R}^d} \underbrace{g(x)}_{\substack{\text{convex} \\ \text{diff}}} + \underbrace{h(x)}_{\substack{\text{convex} \\ \text{non-diff} \\ \text{simple}}}.$$

Our update rules are

$$\begin{aligned} y^{t+1} &\leftarrow x^t - \eta_t \nabla g(x^t) \\ x^{t+1} &\leftarrow \text{prox}_{\eta_t, h}(y^{t+1}) := \arg \min_{z \in \mathbb{R}^d} \frac{1}{2\eta_t} \|y^t - z\|_2^2 + h(z) \end{aligned}$$

When  $g(x) = \|Ax - b\|_2^2$ ,  $h(x) = \lambda \|x\|_1$  then this is Proximal GD called ISTA.

**Remark 10.6.** The intuition is that

$$x_{t+1} \leftarrow \arg \min_z g(x_t) + \nabla g(x_t)^T(z - x_t) + \frac{1}{2\eta_t} \|z - z_t\|_2^2 + h(z) = \frac{\arg \min_z 1}{2\eta_t} \|z - (x_t - \eta_t \nabla g(x_t))\|_2^2 + h(z).$$

**Theorem 10.7.** For  $p(x) = \text{Prox}_{\eta, h}(x) = \frac{\arg \min_z 1}{2\eta} \|x - z\|_2^2 + h(z)$  we have

1.  $\|p(x) - p(y)\|_2^2 \leq \langle x - y, p(x) - p(y) \rangle$
  2.  $\|p(x) - p(y)\| \leq \|x - y\|$
- (1)  $\implies$  (2) by the Cauchy-Schwarz inequality.

*Proof.* We have

$$\frac{1}{\eta}(x - p(x)) \in \partial h(p(x)) \text{ and } \frac{1}{\eta}(y - p(y)) \in \partial h(p(y)).$$

By subgradient monotonicity we have

$$\langle g_x - g_y, x - y \rangle \geq 0 \text{ for all } x, y \text{ and } g_x \in \partial f(x), g_y \in \partial f(y).$$

So

$$\left\langle \frac{1}{\eta}(y - p(y)) - \frac{1}{\eta}(x - p(x)), p(y) - p(x) \right\rangle \geq 0 \iff \langle y - x, p(y) - p(x) \rangle \geq \langle y - x, p(y) - p(x) \rangle \geq \|p(y) - p(x)\|_2^2.$$

□

**Theorem 10.8.** Define  $G_\eta(x) = \frac{x - \text{prox}_{\eta, h}(x - \eta \nabla g(x))}{\eta}$  then the following are true:

1.  $G_\eta(x) = 0 \iff x = \text{prox}_{\eta, h}(x - \eta \nabla g(x))$
2.  $G_\eta(x^*) = 0 \iff 0 \in \nabla g(x^*) + \partial h(x^*) \iff x^* \text{ is optimal} \iff x^* \text{ is a fixed point of prox}$

*Proof.* By definition  $\forall \tilde{x}$  then

$$\tilde{x} - \eta G_\eta(\tilde{x}) = \text{prox}_{\eta, h}\underbrace{(\tilde{x} - \eta \nabla g(\tilde{x}))}_x.$$

$$\begin{aligned} u := \text{prox}_{\eta, h}(x) &\iff \frac{1}{\eta}(x - u) \in \partial h(u) \\ &\iff \tilde{x} - \eta \nabla g(\tilde{x}) - \tilde{x} - \eta G_\eta(\tilde{x}) \in \eta \partial h(\tilde{x} - \eta G_\eta(\tilde{x})) \\ &\iff G_\eta(\tilde{x}) \in \nabla g(\tilde{x}) + \partial h(\tilde{x} - \eta G_\eta(\tilde{x})) \quad \forall \tilde{x} \end{aligned}$$

Thus we imply 1 by choosing  $\tilde{x} = x^*$

□