

# Complex Analysis Notes

Wilson Pan

February 18, 2026

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# 1 Jan 12

## 1.1 Holomorphic Functions and Complex Differentiation

**Definition 1.1.** Let  $\Omega$  be an open set in  $\mathbb{C}$  and  $f$  a complex-valued function on  $\Omega$ . A function  $f$  is holomorphic at a point  $z_0 \in \Omega$  if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}.$$

converges to a limit when  $h \rightarrow 0$ . The limit of the quotient, when it exists, is denoted  $f'(z_0)$ .

**Theorem 1.2.** A function  $F(x, y) = (u(x, y), v(x, y))$  is said to be differentiable at a point  $P_0(x_0, y_0)$  if there exist a linear transformation  $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$\frac{|F(P_0 + H) - F(P_0) - J(H)|}{|H|} \rightarrow 0 \text{ as } |H| \rightarrow 0.$$

## 1.2 Jacobian and Cauchy-Riemann Equations

**Definition 1.3.** The jacobian matrix of  $F$  is

$$J = J_F(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

**Theorem 1.4.** Cauchy-Riemann Equations: If  $f$  is holomorphic then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

*Proof.* If we write  $z = x + iy$ ,  $z_0 = x_0 + iy_0$  and  $f(z) = f(x, y)$  then

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \rightarrow 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0) \end{aligned}$$

Similarly, now take  $h$  purely imaginary, say  $h = ih_2$ , then

$$\begin{aligned} f'(z_0) &= \lim_{h_2 \rightarrow 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} \\ &= \frac{1}{i} \frac{\partial f}{\partial y}(z_0) \end{aligned}$$

Since  $f$  is holomorphic we have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

Writing  $f = u + iv$  we have the stated relation. □

## 2 Jan 14

### 2.1 Holomorphic Equivalence and Wirtinger Derivatives

**Theorem 2.1.**  $f : \Omega \rightarrow \mathbb{C}$  holomorphic on  $\Omega$  equivalently  $F = \begin{pmatrix} u \\ v \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . The partials  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$  exist and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Moreover,  $F$  is  $\mathbb{R}$ -differentiable, there exist a linear map  $\mathcal{L} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $F((x, y) + (h_1, h_2)) - F(x, y) = \mathcal{L}(h_1, h_2) + o(h)$

$$\mathcal{L} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

*Proof.* We have

$$\frac{|F((x, y) + (h_1, h_2)) - F(x, y) - \mathcal{L}(h_1, h_2)|}{|h|} = \frac{\left| \begin{pmatrix} u(\bar{z} + \bar{h}) - u(\bar{z}) - \frac{\partial u}{\partial x}h_1 - \frac{\partial u}{\partial y}h_2 \\ v(\bar{z} + \bar{h}) - v(\bar{z}) - \frac{\partial v}{\partial x}h_1 - \frac{\partial v}{\partial y}h_2 \end{pmatrix} \right|}{|h|} \quad (1)$$

$$= \frac{|f(z + h) - f(z) - f'(z)h|}{|h|} \rightarrow 0 \quad (2)$$

Where  $f(z + h) = u(z + h) + iv(z + h)$ ,  $f(z) = u(z) + iv(z)$  and  $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ .  $\square$

**Corollary 2.2.**  $f$  holomorphic implies  $F$  is  $\mathbb{R}$ -differentiable. Let  $a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$DF_z = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad (4)$$

$$= \sqrt{a^2 + b^2} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad (5)$$

$$= |f'(z)| \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad (6)$$

**Theorem 2.3.** If  $F$  is  $\mathbb{R}$ -differentiable with  $u, v$  satisfying Cauchy Riemann Equations then  $f$  is holomorphic.

*Proof.* Let  $a = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ . We just need to check that  $f(z + h) - f(z) - ah = o(h)$   $\square$

**Definition 2.4.** The Wirtinger Derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right).$$

**Remark 2.5.** The cauchy-riemann equations if and only if  $\frac{\partial f}{\partial \bar{z}} = 0$

We initially had the map  $z \mapsto \bar{z}$  which is not holomorphic. Let  $f(z) = \bar{z}$  and  $\frac{\partial f}{\partial \bar{z}} = 1$  then we've captured this with  $\frac{\partial}{\partial \bar{z}}$ .

If we have  $z, \bar{z}$  as independent variables with  $z = x + iy$ . Consider the map

$$(z, \bar{z}) \mapsto (x(z, \bar{z}), y(z, \bar{z})) \mapsto f(x, y).$$

We have

$$\partial_z f = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right).$$

**Theorem 2.6.**

$$\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} = \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{1}{4} \Delta.$$

### 3 Jan 16

#### 3.1 Power Series

**Definition 3.1.** Given a  $\mathbb{C}$  valued seq  $(a_n)_{n=0}^{\infty}$ , form

$$\sum_{n=0}^{\infty} a_n z^n.$$

A power series

1. Converges if  $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n z^n$  exists
2. Converges absolutely if  $\sum_{n=0}^{\infty} |a_n| |z|^n$  converges

**Remark 3.2.** By translation, all we say will apply to  $\sum a_n(z - z_0)^n$

**Theorem 3.3.** Given  $\sum_{n=0}^{\infty} a_n z^n$ ,  $\exists R \in [0, +\infty]$  such that

1. It converges, if  $D_R = \{|z| < R\}$
2. It diverges at every  $z$  such that  $|z| > R$ .

Moreover,

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}}}. \quad (\text{Hadamard's Formula})$$

**Remark 3.4.** On  $\partial D_R$  it can often be delicate.

*Proof.* Fix  $|z| < R$ , then  $\frac{1}{R}|z| < 1$  and fix  $\epsilon > 0$  such that

$$\left( \frac{1}{R} + \epsilon \right) |z| < 1.$$

Then, eventually

$$|a_n|^{\frac{1}{n}} < \frac{1}{R} + \epsilon.$$

Then

$$|a_n| |z|^n \leq \left( |a_n|^{\frac{1}{n}} |z| \right)^n \leq \left( \left( \frac{1}{R} + \epsilon \right) |z| \right)^n \leq q^n.$$

So by the comparison test with  $\sum q^n < \infty$  □

**Example 3.5.**

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z).$$

This has  $R = +\infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos(z).$$

This has  $R = +\infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin(z).$$

This has  $R = +\infty$

Euler's Formula is

$$e^{iz} = \cos(z) + i \sin(z).$$

**Theorem 3.6.** *The function  $f(z) = \sum a_n z^n$  is holomorphic on  $D_R$ ,  $f'$  is a power series*

$$f'(z) = \sum n a_n z^{n-1}.$$

*Whose disk of convergence is  $R$ .*

*Proof.* Let  $g(z) = \sum n a_n z^{n-1}$ ,  $z \in D_R$ . We want to show that  $f'$  exist and  $f' = g$ .

Fix  $z_0 \in D_R$ , let  $r < R$  such that  $|z_0| < r < R$ .

Fix  $N \geq 1$  (to be chosen later)

$$f(z) = \sum_{n=0}^N a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n = S_N(z) + E_N(z).$$

From here we want to show that

$$\frac{f(z_0 + h) - f(z_0)}{h} \rightarrow g(z_0).$$

For all  $h$  such that  $|z_0 + h| < r$

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) &= \left( \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right) + (S'_N(z_0) - g(z_0)) + \left( \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right) \\ &= A + B + C \end{aligned}$$

Fix  $\epsilon > 0$  we want to show that  $\exists \delta$  such that  $\forall |h| < \delta, |A + B + C| < \epsilon$ .

$$\begin{aligned} |C| &\leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(z_0 + h)^n - z_0^n}{h} \right| \\ &= \sum_{n=N+1}^{\infty} |a_n| \left| (z_0 + h)^{n-1} + (z_0 + h)^{n-2} z_0 + \cdots + z_0^n \right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| |nr^{n-1}| < \frac{\epsilon}{3} \quad (\text{for all } N \geq N_1(\epsilon)) \end{aligned}$$

Note the last summation is the tail of  $\sum a_n nr^{n-1}$  which converges by  $r < R$ .

For  $B$ :

$$S'_N(z_0) = \sum_{n=0}^N n a_n z_0^{n-1} \rightarrow g(z_0). \quad (\text{by def of } g)$$

So  $\forall N \geq N_2$ ,  $|B| < \frac{\epsilon}{3}$ .

$|A| < \frac{\epsilon}{3}$  for all  $h$  sufficiently small by the definition of the complex derivative and the fact that  $S_N$  is plainly holomorphic.  $\square$

## 3.2 Analytic Functions

**Definition 3.7.** *Let  $\Omega$  be open and define  $f : \Omega \rightarrow \mathbb{C}$  is analytic at  $z_0 \in \Omega$ , if there exists a power series  $\sum a_n(z - z_0)^n$  with positive radius of convergence such that  $f(z) = \sum a_n(z - z_0)^n$  is a neighborhood of  $z_0$ .*

*f is analytic on  $\Omega$ , if analytic at every point of  $z_0 \in \Omega$ .*

**Corollary 3.8.** *If  $f$  analytic on  $\Omega \implies f$  is holomorphic on  $\Omega$ .*

*In fact,  $\iff$ .*

## 4 Jan 21

### 4.1 Parametrized Curves

**Definition 4.1.** A parametrized curve (in  $\mathbb{C}$ ) is a function  $z(t) : [a, b] \rightarrow \mathbb{C}$

1. Smooth if  $z'$  is continuous and  $\forall t \in [a, b], z'(t) \neq 0$
2. Piecewise smooth if  $z$  is continuous so there exists  $a_0 = a < a_1 < \dots < a_k = b$  such that  $z|_{[a_j, a_{j+1}]} \in C^1$  for all  $j = 0, \dots, k-1$  is smooth.
3. Let  $\tilde{z} : [\tilde{a}, \tilde{b}] \rightarrow \mathbb{C}$  then two parameterizations are equivalent if there exist  $C^1$ -bijection  $s \mapsto t(s) : [\tilde{a}, \tilde{b}] \rightarrow [a, b]$  and  $t'(s) > 0$  for all  $s$  such that  $\tilde{z}(s) = z(t(s))$ .

**Example 4.2.** A piecewise smooth curve can be visualized as follows:



The curve consists of smooth segments joined at corner points (marked in red) where the derivative is discontinuous.

**Definition 4.3.** The family of all equivalent parameterizations determines a curve (oriented)  $\gamma \subset \mathbb{C}, \gamma = z([a, b])$ .  
 $z(a)$  is the starting point and  $z(b)$  is the endpoint of  $\gamma$ .  $\gamma^{-1} = \gamma$  with reversed orientation

**Definition 4.4.**  $\gamma$  is closed if  $z(a) = z(b)$  and simple if  $z(t_1) \neq z(t_2)$  for all  $t_1 \neq t_2$ .

**Example 4.5.**  $z(T) = re^{-it}, t \in [0, 2\pi]$

**Definition 4.6.** A curve has positive orientation if interior is on the left or counterclockwise.

### 4.2 Contour Integration

**Definition 4.7.** Give a smooth curve  $\gamma \subset \mathbb{C}$  (with parameterization  $z : [a, b] \rightarrow \mathbb{C}$ ) with  $f : \gamma \rightarrow \mathbb{C}$  continuous.

We set

$$\int_{\gamma} f = \int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt.$$

Since  $f(z(t)) z'(t) \in \mathbb{C}$  we split the integral into

$$\int_a^b \Re(\dots) dt + i \int_a^b \Im(\dots) dt.$$

**Remark 4.8.** This is well defined as the right hand side does not depend on the choice of the parameterization.

**Definition 4.9.** For a piece-wise smooth curve, we define

$$\int_{\gamma} f = \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} f(z(t)) z'(t) dt.$$

**Definition 4.10.** We define

$$|\gamma| = \text{length}(\gamma) = \int_a^b |z'(t)| dt.$$

**Theorem 4.11.** Some very basic properties:

1. Linearity

$$\int_{\gamma} (\alpha f + \beta g) = \alpha \int_{\gamma} f + \beta \int_{\gamma} g.$$

2.

$$\int_{\gamma^{-1}} f = - \int_{\gamma} f.$$

3.

$$\left| \int_{\gamma} f \right| \leq \left( \sup_{\gamma} |f| \right) |\gamma|.$$

### 4.3 Primitives and the Fundamental Theorem

**Definition 4.12.** The primitive of  $f : \Omega \rightarrow \mathbb{C}$  with open  $\Omega \subset \mathbb{C}$  is any  $F : \Omega \rightarrow \mathbb{C}$  holomorphic such that  $F' = f$  on  $\Omega$

**Theorem 4.13.** The Fundamental Theorem of Calculus

For  $f$  with primitive  $F$  on  $\Omega$ ,  $\gamma \subset \mathbb{C}$  curve from  $w_1 \rightarrow w_2$  we have

$$\int_{\gamma} f = F(w_2) - F(w_1).$$

*Proof.* We can write

$$\int_{\gamma} f = \int_a^b f(z(t)) z'(t) dt = \int_a^b \frac{d}{dt} (F(z(t))) dt = F(z(b)) - F(z(a)).$$

□

**Corollary 4.14.** If  $\gamma$  is closed then

$$\int_{\gamma} f = 0.$$

**Corollary 4.15.** If  $f$  is holomorphic on  $\Omega$  with  $f' = 0$  on  $\Omega$  then  $f$  is constant.

*Proof.* Fix  $z_0 \in \Omega$ , for  $z \in \Omega$ , take  $\gamma : z_0 \rightarrow z$  then

$$0 = \int_{\gamma} f' = f(z) - f(z_0).$$

□

**Example 4.16.** Let  $z(t) = e^{it}$

$$\int_{\partial D} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt = 2\pi i \neq 0.$$

*So  $\frac{1}{z}$  does not admit a primitive.*

## 5 Jan 23

### 5.1 Glossary of Elementary Functions

#### Glossary of Elementary Functions

1. Polynomials:  $P(z) = a_0 + a_1z + \dots + a_nz^n$  for  $a_i \in \mathbb{C}$  and is entirely holomorphic in  $\mathbb{C}$ ,  $a_n \neq 0$  and  $n = \deg P$
2. Rational:  $R(z) = \frac{P(z)}{Q(z)}$  where  $P, Q$  are polynomials with no common factors. The zeros of  $Q$  are called poles and their order is called the order of a given pole.
3. Exponential:  $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

### 5.2 Fundamental Theorem of Algebra

**Lemma 5.1.** Let  $P(z)$  be a complex polynomial of positive degree  $n$ , if  $|P(z)|$  has a local minimum at  $z_0$  then  $P(z_0) = 0$ .

*Proof.* WLOG  $z_0 = 0$  as we can translate  $Q(z) = P(z - z_0)$ .

AFSOC  $P(0) \neq 0$  and WLOG  $P(0) = 1$  by  $\frac{P}{P(0)}$ . We can write  $P(z) = 1 + a_kz^k + \dots + a_nz^n$  with  $a_k, a_n \neq 0$ . Consider  $z_\delta = \delta e^{i\theta}$ . For all  $\delta > 0$  sufficiently small:

$$|a_{k+1}z_\delta + \dots + a_nz_\delta^{n-k}| < \frac{|a_k|}{2}.$$

Then

$$|P(z)| \leq |1 + a_kz_\delta^k| + |a_{k+1}z_\delta^{k+1} + \dots + a_nz_\delta^n| < |1 + a_kz_\delta^k| + |z_\delta^k| \frac{|a_k|}{2}.$$

Now we can choose  $\delta$  such that  $a_kz_\delta^k < 0$  and real.

$$|1 + a_kz_\delta^k| + |z_\delta^k| \frac{|a_k|}{2} = 1 + a_kz_\delta^k - \frac{1}{2}a_kz_\delta^k = 1 + \frac{1}{2}a_kz_\delta^k < 1.$$

□

**Theorem 5.2.** Fundamental Theorem of Algebra:

*Proof.*  $|P(z)| \rightarrow +\infty$  as  $|z| \rightarrow +\infty$  therefore  $\exists R$  such that  $\forall |z| > R$ ,  $|P(z)| > |P(0)|$  thus  $\inf_{z \in \mathbb{C}} |P(z)| = \inf_{z \in D_R} |P(z)|$  and such a value is attained somewhere at  $z_0$  then by lemma 5.1 we're done. □

**Lemma 5.3.**  $P(z) = a_n(z - z_1) \cdots (z - z_n)$  where  $z_1, \dots, z_n$  are roots of  $P$ .

The order of a zero  $z_j$  is its multiplicity, the number of times it appears in the sequence. If  $z_j$  is of the order  $k_j$ ,

$$P(z) = \underbrace{Q(z)}_{\text{non-vanishing}} (z - z_j)^{k_j}.$$

A zero  $z_0$  is of order  $k_0$  if and only if  $P(z_0) = P'(z_0) = \dots = P^{(k_0-1)}(z_0) = 0$  and  $P^{(k_0)}(z_0) \neq 0$

**Theorem 5.4.** (Gauss-Lucas) The roots of  $P'$  lies in the convex hull of the roots of  $P$ . In particular, if  $P$  is real rooted then so is  $P'$ .

**Theorem 5.5.** If  $R$  a rational function then  $R'$  has the same roots as  $R$  with order greater by 1.

**Theorem 5.6.** The following are true

1.  $(e^z)' = e^z$
2.  $e^{w+z} = e^w e^z$
3.  $\overline{e^z} = e^{\bar{z}}$
4.  $|e^{iy}| = 1$  and  $|e^{x+iy}| = e^x$
5.  $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$  and  $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

## 6 Jan 26

### 6.1 Cauchy's and Goursat's Theorems

**Theorem 6.1.** *Cauchy Theorem: If  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic on  $\Omega$  and  $\gamma$  is a closed curve in  $\Omega$  then*

$$\int_{\gamma} f = 0.$$

**Theorem 6.2.** *Goursat's Theorem: If  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic and  $\Omega$  is open, then  $\forall \Delta \subset \Omega$  we have*

$$\int_{\partial\Delta} f = 0.$$



The integral around the positively oriented (counterclockwise) boundary of any triangle  $\Delta \subset \Omega$  vanishes.

*Proof.* Let  $\Delta = \Delta^{(0)}$  and partition into four triangles by connecting the midpoints of the sides:

$$\Delta^{(0)} = \Delta_1^{(1)} \cup \Delta_2^{(1)} \cup \Delta_3^{(1)} \cup \Delta_4^{(1)}.$$



When we integrate along each sub-triangle with positive orientation, the interior edges cancel (traversed in opposite directions, shown with opposing arrows):

$$\int_{\partial\Delta^{(0)}} f = \sum_{j=1}^4 \int_{\partial\Delta_j^{(1)}} f.$$

By the triangle inequality

$$\left| \int_{\partial\Delta^{(0)}} f \right| \leq \sum_{j=1}^4 \left| \int_{\partial\Delta_j^{(1)}} f \right| \leq 4 \max_{j=1,2,3,4} \left| \int_{\partial\Delta_j^{(1)}} f \right|.$$

Thus there exists  $j_0 \in \{1, 2, 3, 4\}$  such that

$$\left| \int_{\partial\Delta^{(0)}} f \right| \leq 4 \left| \int_{\partial\Delta_{j_0}^{(1)}} f \right|.$$

Set  $\Delta^{(1)} = \Delta_{j_0}^{(1)}$ . Repeating this subdivision process, we obtain a nested sequence of triangles

$$\Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \dots$$

such that

$$\left| \int_{\partial\Delta^{(0)}} f \right| \leq 4^n \left| \int_{\partial\Delta^{(n)}} f \right|.$$

Note that  $\text{diam}(\Delta^{(n)}) = 2^{-n} \text{diam}(\Delta^{(0)})$  and  $|\partial\Delta^{(n)}| = 2^{-n} |\partial\Delta^{(0)}|$ .

Since the triangles are nested compact sets with diameters shrinking to zero, by Cantor's intersection theorem:

$$\bigcap_{n=0}^{\infty} \Delta^{(n)} = \{z_0\}$$

for some  $z_0 \in \Delta^{(0)} \subset \Omega$ . Since  $f$  is holomorphic at  $z_0$ , we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

where  $\psi(z) \rightarrow 0$  as  $z \rightarrow z_0$ .

Since  $f(z_0)$  and  $f'(z_0)(z - z_0)$  have primitives ( $f(z_0)z$  and  $\frac{1}{2}f'(z_0)(z - z_0)^2$  respectively), their integrals over the closed curve  $\partial\Delta^{(n)}$  vanish. Thus:

$$\int_{\partial\Delta^{(n)}} f(z) dz = \int_{\partial\Delta^{(n)}} \psi(z)(z - z_0) dz.$$

For  $z \in \partial\Delta^{(n)}$ , we have  $|z - z_0| \leq \text{diam}(\Delta^{(n)}) = 2^{-n} \text{diam}(\Delta^{(0)})$ .

Let  $\epsilon_n = \sup_{z \in \partial\Delta^{(n)}} |\psi(z)|$ . Since  $\psi(z) \rightarrow 0$  as  $z \rightarrow z_0$  and  $\Delta^{(n)} \rightarrow \{z_0\}$ , we have  $\epsilon_n \rightarrow 0$ . Therefore:

$$\begin{aligned} \left| \int_{\partial\Delta^{(n)}} f(z) dz \right| &\leq \sup_{z \in \partial\Delta^{(n)}} |\psi(z)(z - z_0)| \cdot |\partial\Delta^{(n)}| \\ &\leq \epsilon_n \cdot 2^{-n} \text{diam}(\Delta^{(0)}) \cdot 2^{-n} |\partial\Delta^{(0)}| \\ &= \epsilon_n \cdot 4^{-n} \text{diam}(\Delta^{(0)}) |\partial\Delta^{(0)}| \end{aligned}$$

Combining with our earlier inequality:

$$\left| \int_{\partial\Delta^{(0)}} f \right| \leq 4^n \left| \int_{\partial\Delta^{(n)}} f \right| \leq 4^n \cdot \epsilon_n \cdot 4^{-n} \text{diam}(\Delta^{(0)}) |\partial\Delta^{(0)}| = \epsilon_n \cdot \text{diam}(\Delta^{(0)}) |\partial\Delta^{(0)}|$$

Since  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and the right-hand side is independent of  $n$  except for  $\epsilon_n$ , we conclude:

$$\int_{\partial\Delta} f = 0.$$

□

## 6.2 Cauchy's Theorem in a Disk

**Theorem 6.3.** *Cauchy's Theorem in a disk:* Let  $D \subset \mathbb{C}$  be an open disk and  $f$  holomorphic on  $D$ . Then for any closed curve  $\gamma$  in  $D$  we have

$$\int_{\gamma} f = 0.$$

*Proof.* WLOG let  $D$  be a disk centered at 0. Define  $F : D \rightarrow \mathbb{C}$  by

$$F(z) = \int_{\gamma_z} f$$

where  $\gamma_z$  is the path  $0 \rightarrow \Re(z) \rightarrow z$  (horizontal then vertical).



We claim  $F$  is a primitive for  $f$ , i.e.,  $F'(z) = f(z)$ .

**Case 1:**  $h \in \mathbb{R}$  (horizontal increment).

The path  $\gamma_{z+h}$  goes  $0 \rightarrow \Re(z) + h \rightarrow z + h$ . Observe that:

$$F(z + h) - F(z) = \int_{\gamma_{z+h}} f - \int_{\gamma_z} f.$$



The difference of paths can be decomposed using Goursat's theorem. The integral over the rectangle with vertices  $\Re(z), \Re(z) + h, z + h, z$  vanishes, so:

$$F(z + h) - F(z) = \int_{\Re(z)}^{\Re(z)+h} f(\zeta) d\zeta + \int_z^{z+h} f(\zeta) d\zeta = \int_z^{z+h} f(\zeta) d\zeta$$

where the first integral is along the real axis and the second is horizontal at height  $\Im(z)$ . By the rectangle lemma, these combine to give just the horizontal segment from  $z$  to  $z + h$ :

$$\frac{F(z + h) - F(z)}{h} = \frac{1}{h} \int_z^{z+h} f(\zeta) d\zeta \rightarrow f(z) \text{ as } h \rightarrow 0.$$

**Case 2:**  $h = ik$  with  $k \in \mathbb{R}$  (vertical increment).

Similarly,  $\gamma_{z+ik}$  goes  $0 \rightarrow \Re(z) \rightarrow z + ik$ . The paths  $\gamma_z$  and  $\gamma_{z+ik}$  share the horizontal segment  $0 \rightarrow \Re(z)$ , so:

$$F(z + ik) - F(z) = \int_z^{z+ik} f(\zeta) d\zeta.$$



Thus:

$$\frac{F(z + ik) - F(z)}{ik} = \frac{1}{ik} \int_z^{z+ik} f(\zeta) d\zeta \rightarrow f(z) \text{ as } k \rightarrow 0.$$

**General  $h$ :** For arbitrary  $h = h_1 + ih_2$ , we write:

$$\frac{F(z + h) - F(z)}{h} = \frac{F(z + h) - F(z + h_1)}{h} + \frac{F(z + h_1) - F(z)}{h}.$$

Using the above cases and the continuity of  $f$ , both terms converge appropriately, giving  $F'(z) = f(z)$ .

Since  $f$  has a primitive  $F$  on  $D$ , by the Fundamental Theorem of Calculus, for any closed curve  $\gamma$  in  $D$ :

$$\int_{\gamma} f = F(z(b)) - F(z(a)) = 0.$$

□

## 7 Jan 28

### 7.1 Applications of Cauchy's Theorem

**Example 7.1.**

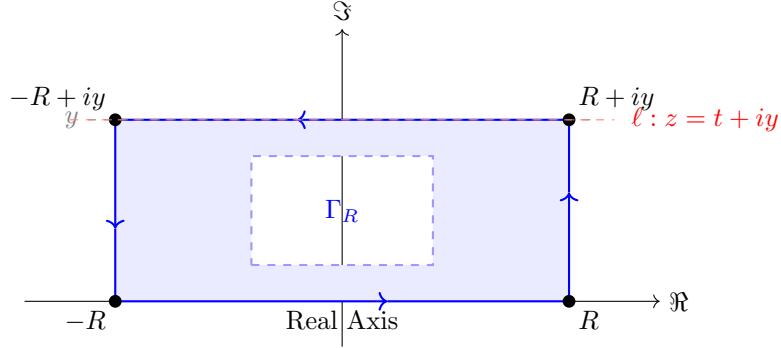
$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1 \iff \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

*Proof.* First, note that by substitution  $u = \sqrt{\pi}x$ , we have:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du.$$

So the equivalence follows if we can show  $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$ .

Consider the function  $f(z) = e^{-\pi z^2}$ , which is entire (holomorphic everywhere). For fixed  $y \in \mathbb{R}$ , consider the rectangular contour  $\Gamma_R$  with vertices at  $-R, R, R + iy, -R + iy$  (traversed counterclockwise).



By Cauchy's theorem:

$$\int_{\Gamma_R} e^{-\pi z^2} dz = 0.$$

Parameterizing the four sides:

$$\begin{aligned} \int_{\Gamma_R} e^{-\pi z^2} dz &= \int_{-R}^R e^{-\pi t^2} dt + \int_0^y e^{-\pi(R+it)^2} i dt \\ &\quad + \int_R^{-R} e^{-\pi(t+iy)^2} dt + \int_y^0 e^{-\pi(-R+it)^2} i dt \end{aligned}$$

We now bound the integrals over the vertical segments. For the right vertical segment, parameterize  $z = R + it$  with  $t \in [0, y]$  (assuming  $y > 0$ ; the case  $y < 0$  is similar). We have:

$$(R + it)^2 = R^2 + 2iRt - t^2 = (R^2 - t^2) + 2iRt$$

so

$$e^{-\pi(R+it)^2} = e^{-\pi(R^2-t^2)} \cdot e^{-2\pi i R t}.$$

Since  $|e^{-2\pi i R t}| = 1$  for all real  $R$  and  $t$ , we have:

$$|e^{-\pi(R+it)^2}| = e^{-\pi(R^2-t^2)} \cdot |e^{-2\pi i R t}| = e^{-\pi(R^2-t^2)} \cdot 1 = e^{-\pi(R^2-t^2)}.$$

For  $t \in [0, y]$ , we have  $t^2 \leq y^2$ , so  $R^2 - t^2 \geq R^2 - y^2$ . Therefore:

$$|e^{-\pi(R+it)^2}| \leq e^{-\pi(R^2-y^2)} = e^{-\pi R^2} e^{\pi y^2}.$$

By the ML-inequality, the length of the path is  $|y|$ , so:

$$\left| \int_0^y e^{-\pi(R+it)^2} i dt \right| \leq |y| \cdot e^{-\pi R^2} e^{\pi y^2} = |y| e^{\pi y^2} e^{-\pi R^2}.$$

Similarly, for the left vertical segment with  $z = -R + it$  where  $t \in [y, 0]$ :

$$(-R + it)^2 = R^2 - 2iRt - t^2 = (R^2 - t^2) - 2iRt$$

so

$$e^{-\pi(-R+it)^2} = e^{-\pi(R^2-t^2)} \cdot e^{2\pi i R t}.$$

Since  $|e^{2\pi i R t}| = 1$  for all real  $R$  and  $t$ , we have:

$$|e^{-\pi(-R+it)^2}| = e^{-\pi(R^2-t^2)} \cdot |e^{2\pi i R t}| = e^{-\pi(R^2-t^2)} \cdot 1 = e^{-\pi(R^2-t^2)}.$$

For  $t \in [y, 0]$  (or equivalently  $t \in [0, y]$  if we reverse the parameterization), we have  $t^2 \leq y^2$ , so:

$$|e^{-\pi(-R+it)^2}| \leq e^{-\pi(R^2-y^2)} = e^{-\pi R^2} e^{\pi y^2}.$$

Therefore:

$$\left| \int_y^0 e^{-\pi(-R+it)^2} i dt \right| \leq |y| \cdot e^{-\pi R^2} e^{\pi y^2} = |y| e^{\pi y^2} e^{-\pi R^2}.$$

Both vertical integrals are bounded by  $C e^{-\pi R^2}$  for some constant  $C$  depending on  $y$  but independent of  $R$ . Since  $e^{-\pi R^2} \rightarrow 0$  as  $R \rightarrow \infty$ , we conclude that:

$$\lim_{R \rightarrow \infty} \int_0^y e^{-\pi(R+it)^2} i dt = 0 \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_y^0 e^{-\pi(-R+it)^2} i dt = 0.$$

Therefore, taking the limit as  $R \rightarrow \infty$ :

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = \int_{-\infty}^{\infty} e^{-\pi(t+iy)^2} dt.$$

Expanding  $e^{-\pi(t+iy)^2} = e^{-\pi(t^2+2ity-y^2)} = e^{-\pi t^2} e^{-2\pi i t y} e^{\pi y^2}$ , we get:

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = e^{\pi y^2} \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-2\pi i t y} dt.$$

In particular, taking  $y = 0$  gives the standard Gaussian integral. The standard result is  $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$ , which completes the proof.  $\square$

### Example 7.2.

$$\int_0^{\infty} \frac{1 - \cos(x)}{x^2} dx = \frac{\pi}{2}.$$

*Proof.* Consider the holomorphic function  $f(z) = \frac{1-e^{iz}}{z^2}$  on  $\mathbb{C} \setminus \{0\}$ . Fix  $\epsilon, R$  and consider the contour  $\Gamma$  which is disk with radius  $R$  and a small disk with radius  $\epsilon$  centered at the origin.



We can express  $e^{iz} \approx 1 + iz + g(z)$  where  $g(z) = e^{iz} - 1 - iz$  so

$$\begin{aligned} \int_{\gamma_\epsilon} f(z) dz &= \int_{\gamma_\epsilon} \frac{1 - (1 + iz + g(z))}{z^2} dz \\ &= \int_{\gamma_\epsilon} \frac{-g(z)}{z^2} dz + \int_{\gamma_\epsilon} \frac{-i}{z} dz \end{aligned}$$

We can evaluate each part as

$$\int_{\gamma_\epsilon} \frac{dz}{z} = - \int_0^\pi \frac{ie^{it}}{e^{it}} dt = -\pi i.$$

$$\int_{\gamma_\epsilon} \frac{g(z)}{z^2} dz \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

□

## 8 Jan 30

### 8.1 Cauchy's Integral Formula

**Theorem 8.1.** *Cauchy's Integral Formula: Let  $\Omega$  be open  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic. Let  $\overline{D}$  be open in  $\Omega$  be a disk then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

*Proof.* Fix  $z \in D_1$ , let  $\gamma_{\epsilon, \delta}$  be a key-hole contour but we omit  $z$ . Then

$$\int_{\gamma_{\epsilon, \delta}} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$



Fix  $\epsilon > 0$  and  $\delta \rightarrow 0$  we want to show that

$$\int_{\gamma_{\epsilon, \delta}} \frac{f(\zeta)}{\zeta - z} d\zeta \rightarrow \int_{\partial D} - \int_{C_\epsilon}.$$

For

$$\int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_\epsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + f(z) \int_{C_\epsilon} \frac{d\zeta}{\zeta - z}.$$

The first term is bounded as holomorphic so as  $\epsilon \rightarrow 0$ , it goes to 0. The second part integrates to  $f(z) \cdot 2\pi i$ .  $\square$

### 8.2 Regularity and Liouville's Theorem

**Theorem 8.2.** *Regularity: Holomorphic implies Analytic.*

*Let  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic,  $\overline{D} = \overline{D}_r(z_0) \subset \Omega$  then  $f$  is analytic at  $z_0$  i.e*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

*Proof.* We know

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We can write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) + (z_0 - z)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z-z_0}{\zeta-z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n.$$

Note:  $\frac{z-z_0}{\zeta-z_0} < 1$ .

$$\begin{aligned}
 f(z) &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} \underbrace{\sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n \frac{f(\zeta)}{\zeta - z}}_{\text{conv. uniformly on } \partial D} d\zeta \\
 &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} \left( \frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\
 &= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\left( \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right)}_{a_n}
 \end{aligned}$$

So  $a_n = \frac{f^{(n)}(z_0)}{n!}$ . □

**Corollary 8.3.** *Cauchy's Formula:*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

**Corollary 8.4.** *Cauchy's Inequality:*

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \sup_{\zeta \in \partial D} \left| \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \right| = \frac{n!}{r^n} \|f\|_{\infty} \partial D.$$

**Theorem 8.5.** *Liouville's Theorem:* If  $f$  entire and bounded then  $f$  is constant.

*Proof.*

$$|f'(z_0)| \leq \frac{\|f\|_{\infty}}{r}$$

Since  $f$  is bounded and  $r$  can be taken arbitrarily large (as  $f$  is entire), sending  $r \rightarrow \infty$  gives  $|f'(z_0)| = 0$ . Thus,  $f$  is constant. □

## 9 Feb 2

### 9.1 Rigidity and Analytic Continuation

**Theorem 9.1.** (*Rigidity Theorem*): Let  $\Omega$  be a region and  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic. If  $z_1, z_2, \dots \in \Omega$  distinct sequence with a limit in  $\Omega$ . If  $f(z_n) = 0, \forall n$  then  $f = 0$  on  $\Omega$ .

*Proof.* Say  $z_n \rightarrow \omega \in \Omega$  and  $D = D_r(\omega) \subset \Omega$  be a disk centered at  $\omega$  with radius  $r$ . If  $f = 0$  on  $D$  then by regularity implies

$$f(z) = \sum_{n \geq 0} a_n(z - \omega)^n \text{ on } D.$$

Suppose  $f \neq 0$  on  $D$  then  $\exists m$  such that  $a_m \neq 0$ , let's take the smallest such  $m$ . Then  $f(z) = a_m(z - w)^m(1 + g(z))$  where  $g(z) \rightarrow 0$  as  $z \rightarrow w$ . For all sufficiency large  $k, z_k \in D$  then

$$0 = f(z_k) = a_m(z_k - w)^m(1 + g(z_k)). \quad (a_m \neq 0 \text{ and } z_k - w \neq 0)$$

A contradiction as the RHS is nonzero. To finish  $f = 0$  on  $\Omega$ , we use  $\mathcal{U} = \text{int}\{z \in \Omega : f(z) = 0\}$ . We have  $\mathcal{U}$  is open,  $w \in \mathcal{U}$  so  $\mathcal{U}$  is non-empty. Additionally,  $\mathcal{U}$  is closed. We have  $\Omega \setminus \mathcal{U}$  is open and non-empty so  $\exists z_k \in \Omega \setminus \mathcal{U}$  such that  $z_k \rightarrow \omega' \in \Omega \setminus \mathcal{U}$ . Then  $f(z_k) \neq 0$  for all  $k$  so  $f \neq 0$  on  $D_r(\omega')$ . A contradiction as  $D_r(\omega') \subset D$ . Thus,  $f = 0$  on  $\Omega$ .  $\square$

**Corollary 9.2.**  $f, g : \underset{\text{region}}{\Omega} \rightarrow \mathbb{C}$  holomorphic. If  $f(z_n) = g(z_n), \forall z_n \in \Omega$  distinct sequence with a limit in  $\Omega$  then  $f = g$  on  $\Omega$ .

**Definition 9.3.** If  $\Omega_1 \subset \Omega_2$ , are two regions,  $f_i : \Omega_i \rightarrow \mathbb{C}$  holomorphic for  $i = 1, 2$  such that  $f_1 = f_2$  on  $\Omega_1$  then we say  $f_2$  is the analytic continuation of  $f_1$  into  $\Omega_2$ .

### 9.2 Morera's Theorem and Uniform Convergence

**Theorem 9.4.** (*Morera's Theorem*): If  $f : \mathcal{D} \rightarrow \mathbb{C}$  continuous such that  $\forall \bar{\Delta} \subset \mathcal{D}$  and  $\int_{\partial \Delta} f = 0$  then  $f$  is holomorphic.

*Proof.* Repeating the proof of Cauchy's Theorem in a disk, gives that  $f$  has a primitive  $F = \int_{\gamma_z} f$  in  $\mathcal{D}$ . Since  $F$  is holomorphic,  $F'$  exists but so does  $F'', F''', \dots$  so  $f'$  exists.  $\square$

**Theorem 9.5.** If  $f_n : \underset{\text{open}}{\Omega} \rightarrow \mathbb{C}$  holomorphic with  $f_n$  converging uniformly to  $f$  on every compact  $K \subset \Omega$  then  $f$  is holomorphic.

*Proof.* Use Morera's Theorem.  $\square$

Moreover, if  $f'_n$  converges uniformly to  $f'$ , using  $\Omega_s = \{z \in \Omega : \bar{\mathcal{D}}_\delta(z) \subset \Omega\}$ .

Claim:  $\forall F$  holomorphic in  $\Omega$ ,  $\|F'\|_{\infty, \Omega_\delta} \leq \frac{1}{\delta} \|F\|_{\infty, \Omega_\delta}$ .

By Cauchy's Formula:

$$|F'(z)| = \left| \frac{1}{2\pi i} \int_{\partial \mathcal{D}_\delta(z)} \frac{F(w)}{(w - z)^2} dw \right| \leq \frac{1}{2\pi} \int_{\partial \mathcal{D}_\delta(z)} \frac{1}{|w - z|^2} \|F\|_{\infty, \Omega} \leq \frac{1}{2\pi} \frac{2\pi\delta}{\delta^2} \|F\|_{\infty, \Omega} \leq \frac{1}{\delta} \|F\|_{\infty, \Omega}.$$

# 10 Feb 4

## 10.1 Parameter Integrals and Gluing

**Theorem 10.1.** Let  $F : \Omega \times [0, 1] \rightarrow \mathbb{C}$  satisfy

1.  $\forall \Delta \in [0, 1], z \mapsto F(z, \Delta)$  is holomorphic
2.  $F$  is continuous on  $\Omega \times [0, 1]$

Then  $z \mapsto \int_0^1 F(z, \Delta) d\Delta$  is holomorphic on  $\Omega$ .

*Proof.* Consider  $F_n(z) = \frac{1}{n} \sum_{k=1}^n F(z, \frac{k}{n})$  and by (1) each  $F_n$  is holomorphic on  $\Omega$ .

$$\begin{aligned} \left| F_n(z) - \int_0^1 F(z, \Delta) d\Delta \right| &= \left| F_n(z) - \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} F(z, \Delta) d\Delta \right| \\ &= \left| \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left[ \underbrace{F(z, \frac{k}{n}) - F(z, \Delta)}_{\leq \epsilon} \right] d\Delta \right| \\ &\leq \epsilon \end{aligned}$$

□

**Definition 10.2.**  $\Omega \subset \mathbb{C}$  be open, symmetric with respect to real axis,  $\Omega^+ = \Omega \cap \{z \in \mathbb{C}, \Im(z) > 0\}$ ,  $\Omega^- = \Omega \cap \{z \in \mathbb{C}, \Im(z) < 0\}$  and  $I = \Omega \cap \{\Re(z) = 0\}$

**Theorem 10.3.** Let each  $f^\pm : \Omega^\pm \rightarrow \mathbb{C}$  be holomorphic and extend continuous to  $I$  such that  $f^+ = f^-$  on  $I$ . Then

$$f = \begin{cases} f^+(z) & \text{if } z \in \Omega^+ \\ f^+(z) = f^-(z) & \text{if } z \in I \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}.$$

is holomorphic on  $\Omega$ .

*Proof.* Sufficient to handle  $z \in I$ . Fix such  $z$  take  $D \subset \Omega$  centered at  $z$ . Take  $\overline{\Delta} \subset D$  we want to show  $\int_{\partial\Delta} f = 0$ .

**Case 1:** If  $\overline{\Delta} \subset \Omega^+$  (or  $\Omega^-$ ) then we're done.



**Case 2:** If a vertex or side of  $\Delta$  is on  $I$ ,  $\int_{\partial\Delta_\epsilon} f = 0$  for when  $\epsilon \rightarrow 0, \partial\Delta_\epsilon \rightarrow \partial\Delta$ .



**Case 3:** If  $I$  cuts the triangle then we can split the triangle into smaller triangles and we're done.



The triangle is split into smaller triangles, each either entirely in  $\Omega^+$  or  $\Omega^-$ , or with edges on  $I$ , so we can apply Cases 1 and 2.  $\square$

## 10.2 Schwarz Reflection Principle

**Theorem 10.4.** *Schwarz Reflection Principle: Let  $f : \Omega^+ \rightarrow \mathbb{C}$  be holomorphic and extend continuously to  $I$  with  $f^+/I \in \mathbb{R}$  then  $f^+$  extends analytically to  $\Omega$*

*Proof.* Let  $f^- : \Omega^- \rightarrow \mathbb{C}$  be defined by  $f^-(z) = \overline{f^+(\bar{z})}$ . We claim  $f^-$  is holomorphic on  $\Omega^-$ , then the previous theorem applies because  $f^+ = f^-$  on  $I$ .

Fix  $z_0 \in \Omega^-$  then  $\bar{z}_0 \in \Omega^+$ , we know  $f^+(\bar{z}) = \sum_{n=0}^{\infty} a_n (\bar{z} - \bar{z}_0)^n$  for all  $\bar{z} \in D(\bar{z}_0)$ . Thus

$$\overline{f^+(\bar{z})} = \sum_{n=0}^{\infty} \overline{a_n} (\bar{z} - \bar{z}_0)^n.$$

We know  $\overline{\bar{z} - \bar{z}_0} = z - z_0$  so

$$f^-(z) = \overline{f^+(\bar{z})} = \sum \overline{a_n} (z - z_0)^n.$$

$\square$

# 11 Feb 9

## 11.1 Meromorphic Functions and Poles

**Definition 11.1.** Meromorphic functions are "determined" by zeros and singularities.

A point singularity of  $f$  is a point  $z_0 \in \mathbb{C}$  such that  $f$  is defined in a neighborhood of  $z_0$  but not at  $z_0$ .

$$\mathcal{D}_\delta(z_0) \setminus \{z_0\}.$$

**Example 11.2.**  $f(z) = \frac{1}{z}$  has a singularity at  $z_0 = 0$ .

**Remark 11.3.** Zeros of a holomorphic  $f$  are isolated, unless  $f \equiv 0$ .

**Theorem 11.4.** (Local Description Near Zeros) Let  $f : \Omega_{open} \rightarrow \mathbb{C}$  holomorphic and  $f(z_0) = 0$ ,  $z_0 \in \Omega$ ,  $f \not\equiv 0$  then  $\exists \mathcal{U} \ni z_0$ ,  $g : \mathcal{U} \rightarrow \mathbb{C}$  holomorphic and vanishing ( $\forall z \in \mathcal{U}, g(z) \neq 0$ )  $\exists! n > 0$  such that  $f(z) = (z - z_0)^n g(z)$ .

*Proof.* We know

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

$f \not\equiv 0$  implies that  $n = \text{smallest } k \text{ such that } a_k \neq 0$ . We can write  $f(z) = (z - z_0)^n g(z)$ .  $\square$

**Definition 11.5.** Let  $n = \text{the multiplicity/order of } z_0$ . When  $n = 1$ ,  $z_0$  is called a simple zero. We say  $f : \mathcal{D}_\delta(z_0) \setminus \{z_0\} \rightarrow \mathbb{C}$  has a pole at  $z_0$  if  $\frac{1}{f}$  extended by 0 at  $z_0$  is holomorphic in  $\mathcal{D}_\epsilon(z_0)$ , for some  $0 < \epsilon < \delta$ .

**Theorem 11.6.** (Local Discontinuity near Poles) Let  $f : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$  has a pole at  $z_0$  in  $\Omega$  then  $\exists \mathcal{U} \ni z_0$ ,  $\exists h : \mathcal{U} \xrightarrow{holo} \mathbb{C}$  nonvanishing then  $\exists! n > 0$  such that  $f(z) = (z - z_0)^{-n} h(z)$ .

*Proof.* Apply previous theorem to  $\frac{1}{f}(z) = (z - z_0)^n g(z)$ .  $\square$

## 11.2 Laurent Series and Residues

**Theorem 11.7.** (Laurent Series Expansion) If  $f$  has a pole of order  $n$  at  $z_0$  then locally

$$f(z) = \underbrace{\frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{\overbrace{a_{-1}}^{\text{residue of } f \text{ at } z_0}}{z - z_0}}_{\text{principal part}} + \underbrace{G(z)}_{\text{holo part of } f}.$$

where  $G(z)$  is holomorphic and nonvanishing near  $z_0$ .

Moreover,  $\text{res}_{z_0}(f) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}}((z - z_0)^n f(z))$

*Proof.* By theorem 11.6,

$$f(z) = (z - z_0)^{-n} h(z) = (z - z_0)^{-n} \sum_{k=0}^{\infty} b_k (z - z_0)^k = \frac{b_0}{(z - z_0)^n} + \cdots + \frac{b_{n-1}}{(z - z_0)} + \sum_{k=n}^{\infty} b_k (z - z_0)^{k-n}.$$

So we have

$$f(z)(z - z_0)^n = b_0 + b_1(z - z_0) + \cdots + b_{n-1}(z - z_0)^{n-1} + \underbrace{O((z - z_0)^n)}_{\rightarrow 0 \text{ as } \frac{d^{n-1}}{dz^{n-1}}}.$$

$\square$

### 11.3 The Residue Formula

**Theorem 11.8.** (*The Residue Formula*) Suppose  $f : \Omega \setminus \underbrace{\{z_1, z_2, \dots, z_n\}}_{\text{poles}} \rightarrow \mathbb{C}$  is holomorphic. Then for a disk  $\bar{\mathcal{D}} \subset \Omega$  containing  $z_1, \dots, z_n$  we have

$$\int_{\partial\mathcal{D}} f = 2\pi i \sum_{k=1}^n \text{res}_{z_k}(f).$$

*Proof.* Let  $\gamma = \partial\mathcal{D}$  be the boundary of the disk  $\mathcal{D}$ , oriented counterclockwise. For each pole  $z_k$ , choose a small disk  $\mathcal{D}_k$  centered at  $z_k$  with radius  $\varepsilon_k > 0$  small enough so that:

- $\overline{\mathcal{D}_k} \subset \mathcal{D}$  for all  $k$
- $\overline{\mathcal{D}_k} \cap \overline{\mathcal{D}_j} = \emptyset$  for  $k \neq j$
- $f$  is holomorphic on  $\bar{\mathcal{D}} \setminus \bigcup_{k=1}^n \mathcal{D}_k$

Let  $\gamma_k = \partial\mathcal{D}_k$  be the boundary of each small disk, oriented clockwise (negative orientation). Consider the multiply connected region  $\mathcal{D} \setminus \bigcup_{k=1}^n \overline{\mathcal{D}_k}$ .



By Cauchy's theorem for multiply connected regions, the integral of  $f$  around the boundary of this region (taking into account orientations) is zero. The boundary consists of  $\gamma$  (counterclockwise) and each  $\gamma_k$  (clockwise), so:

$$\int_{\gamma} f(z) dz + \sum_{k=1}^n \int_{\gamma_k} f(z) dz = 0.$$

Note that  $\gamma_k$  has clockwise orientation, so reversing it gives:

$$\int_{\gamma} f(z) dz = \sum_{k=1}^n \int_{-\gamma_k} f(z) dz.$$

where  $-\gamma_k$  denotes  $\gamma_k$  with counterclockwise orientation.

Now, for each pole  $z_k$ , consider its Laurent expansion:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j^{(k)} (z - z_k)^j = \frac{a_{-1}^{(k)}}{z - z_k} + \sum_{j \neq -1} a_j^{(k)} (z - z_k)^j.$$

where  $a_{-1}^{(k)} = \text{res}_{z_k}(f)$ .

For  $j \neq -1$ , the function  $(z - z_k)^j$  has an antiderivative on  $\mathbb{C} \setminus \{z_k\}$  (or on all of  $\mathbb{C}$  if  $j \geq 0$ ), so by the fundamental theorem of calculus:

$$\int_{-\gamma_k} (z - z_k)^j dz = 0 \quad \text{for } j \neq -1.$$

For  $j = -1$ , we compute directly. Parameterize  $-\gamma_k$  by  $z(t) = z_k + \varepsilon_k e^{it}$  for  $t \in [0, 2\pi]$ :

$$\int_{-\gamma_k} \frac{1}{z - z_k} dz = \int_0^{2\pi} \frac{1}{\varepsilon_k e^{it}} \cdot i\varepsilon_k e^{it} dt = \int_0^{2\pi} i dt = 2\pi i.$$

Therefore,

$$\int_{-\gamma_k} f(z) dz = a_{-1}^{(k)} \cdot 2\pi i = 2\pi i \cdot \text{res}_{z_k}(f).$$

Combining all terms:

$$\int_{\partial D} f(z) dz = \sum_{k=1}^n \int_{-\gamma_k} f(z) dz = 2\pi i \sum_{k=1}^n \text{res}_{z_k}(f).$$

□

## 12 Feb 11

### 12.1 Integral Evaluation via Residues

**Example 12.1.**

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(a\pi)}.$$

*Proof.* Let  $f(z) = \frac{e^{az}}{1+e^z}$  then we have simple poles at  $z : e^z = 1$  so  $z \in \{..., -\pi i, \pi i, 3\pi i, ...\}$ . We have the residual for

$$\int_{\gamma_1} f + \dots = \int_{\gamma_R} f(z) dz = 2\pi i \operatorname{res}_{\pi i}(f).$$



We have

$$\int_{\gamma_1} f(z) dz \xrightarrow{R \rightarrow +\infty} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = I.$$

On  $\gamma_3$ ,  $z = x + 2\pi i$  with  $x$  from  $R$  to  $-R$ , so

$$\int_{\gamma_3} f(z) dz = - \int_{-R}^R \frac{e^{a(x+2\pi i)}}{1+e^x} dx = -e^{2\pi ai} \cdot I.$$

As  $R \rightarrow \infty$ ,  $\int_{\gamma_2} f \rightarrow 0$  and  $\int_{\gamma_4} f \rightarrow 0$ . Thus

$$(1 - e^{2\pi ai})I = 2\pi i \cdot \operatorname{res}_{\pi i}(f) = 2\pi i \cdot (-e^{a\pi i}) = -2\pi i e^{a\pi i},$$

$$\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}} idy \right| \leq \int_0^{2\pi} \frac{e^{aR}}{e^R + 1} dy \xrightarrow{R \rightarrow \infty} 0$$

Additionally  $\int_{\gamma_4} f \rightarrow 0$ . To compute the residue at  $\pi i$ , we have

$$\lim_{z \rightarrow \pi i} f(z)(z - \pi i) = \lim_{z \rightarrow i\pi} \frac{e^{az}}{\frac{e^z - e^{\pi i}}{z - \pi i}} = \frac{e^{\pi i a}}{e^{\pi i}}.$$

Letting  $R \rightarrow +\infty$  in the residual formula gives

$$I + 0 + 0 - e^{2\pi ai} I = -2\pi i e^{a\pi i}.$$

So

$$I = -\frac{2\pi i e^{\pi i a}}{1 - e^{2\pi ai}} = \frac{\pi}{\sin(a\pi)}.$$

□

### 12.2 Removable Singularities

**Definition 12.2.** Let  $f : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$  be holomorphic, if we can extend  $f$  analytically to  $z_0$  we say that  $f$  has a removable singularity at  $z_0$ .

**Theorem 12.3.** (Riemann's Theorem) Let  $f : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$  be holomorphic, if  $f$  is bounded near  $z_0$  then  $z_0$  is a removable singularity.

*Proof.* Let  $\overline{D} \subset \Omega$  be a disc centered at  $z_0$ . We want to use Cauchy's formula

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw = g(z), \text{ for all } z \in D.$$

It suffices to show that  $f = g$  on  $D \setminus \{z_0\}$  because then  $g$  is the desired extension of  $f$ .

$F(w, z)$  is jointly continuous on  $\partial D \times \overline{D}_{r-\epsilon}(z_0)$  and  $\forall w \in \partial D, z \mapsto F(w, z)$  is holomorphic on  $D$ .

Fix  $z \in D \setminus \{z_0\}$  then

$$\int_{\gamma} \frac{f(w)}{w - z} dw = 0 \implies \int_{\partial D} \frac{f(w)}{w - z} dw = \int_{C_{\epsilon}(z_0)} \frac{f(w)}{w - z} dw + \int_{C_{\epsilon}(z)} \frac{f(w)}{w - z} dw = 0 + 2\pi i f(z).$$

□

## 13 Feb 13

### 13.1 Classification of Singularities

**Theorem 13.1.** (Riemann) Let  $f : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$  has a removable singularity at  $z_0$  if and only if bounded near  $z_0$ .

**Corollary 13.2.** Pole at  $z_0$  if and only if  $|f| \rightarrow +\infty$  as  $z \rightarrow z_0$

*Proof.* ( $\implies$ ) Local description near poles implies  $f(z) = (z - z_0)^{-n}g(z)$  so  $|f| \rightarrow +\infty$   
 $(\impliedby)$  Since  $\left|\frac{1}{f}\right| \rightarrow 0$ , in particular  $\frac{1}{f}$  is bounded near  $z_0$ , Riemann implies  $\frac{1}{f}$  has a removable singularity at  $z_0$ .  $\square$

**Theorem 13.3.** (Casorati-Weierstrass) If  $f : D_r(z_0) \setminus \{z_0\} \rightarrow \mathbb{C}$  is holomorphic with an essential singularity at  $z_0$  then the image of  $f$  is dense in  $\mathbb{C}$ .

*Proof.* If not,  $\exists w \in \mathbb{C}$  such that no values of  $f$  near  $w$ ,  $\exists \delta > 0$  s.t.  $\forall z, |f(z) - w| > \delta$ . Consider

$$g(z) = \frac{1}{f(z) - w} : D_r(z_0) \setminus \{z_0\} \rightarrow \mathbb{C}.$$

Such a  $g$  is bounded by  $\frac{1}{\delta}$ . So  $g$  extends holomorphically into  $D_r(z_0)$ .

(Case 1)  $g(z_0) = 0 \implies \frac{1}{g} \rightarrow +\infty$  so  $f(z) - w$  has a pole so  $f$  has a pole

(Case 2)  $g(z_0) \neq 0 \implies \frac{1}{g}$  is well defined and  $f$  has a removable singularity.  $\square$

**Theorem 13.4.** (Picard's Theorem) If  $f : D_r(z_0) \setminus \{z_0\} \rightarrow \mathbb{C}$  is holomorphic with an essential singularity at  $z_0$  then the image of  $f$  is either  $\mathbb{C}$  or  $\mathbb{C}$  except for at most one value.

### 13.2 Meromorphic Functions and Behavior at Infinity

**Theorem 13.5.** Let  $\Omega \subset \mathbb{C}$  be open. A function  $f$  is called meromorphic on  $\Omega$  if  $\exists z_1, z_2, \dots$  without a limit point such that  $f$  is holomorphic in  $\Omega \setminus \{z_1, z_2, \dots\}$  and has poles at  $z_1, z_2, \dots$

**Theorem 13.6.** (Behavior at  $+\infty$ ) Suppose  $f$  is holomorphic in a neighborhood of  $\infty$  i.e.  $\{|z| > R\}$  for some  $R > 0$ .

Note:  $F(z) = f(\frac{1}{z})$  is holomorphic in  $D_{\frac{1}{R}}(0) \setminus \{0\}$ .

We say  $f$  has a removable singularity at  $\infty$  if  $F$  has a removable singularity at  $z = 0$ .

**Example 13.7.**  $e^z$  has an essential singularity at  $+\infty$ .

**Remark 13.8.** We'll denote  $\overline{\mathbb{C}} = \mathbb{C} \cup \{+\infty\}$

**Example 13.9.** Examples of meromorphic functions on  $\mathbb{C}$

1. Rational functions:  $\frac{P}{Q}$

2. The gamma function:  $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$  (simple poles at ...,  $-2, -1, 0$ )

3. The Riemann zeta function:  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$  (simple poles at  $s = -1, -2, -3, \dots$ )

## 14 Feb 16

### 14.1 Rational Functions via Meromorphy

**Theorem 14.1.** Suppose  $f$  is meromorphic on  $\mathbb{C}$  with a removable singularity or a pole at  $+\infty$  then  $f$  is a rational function.

*Proof.* Claim: Under these assumptions,  $f$  has finitely many poles.

If not,  $z_1, z_2, \dots$  are poles and this sequence cannot be bounded otherwise it got a limit point. Say  $|z_k| \rightarrow +\infty$ , the neighborhoods of  $\infty$  are  $\{|z| > R\}$  so  $f$  not holomorphic for any  $R > 0$  which contradicts that  $f$  has a removable singularity at  $+\infty$ .

For a function  $F$  which has a pole at  $z = 0$  we have

$$F(z) = \underbrace{\frac{a_n}{z^n} + \dots + \frac{a_{-1}}{z}}_{f(z)} + \underbrace{a_0 + a_1 z + \dots}_{g(z)}.$$

Let  $z_1, \dots, z_N$  be the poles of  $f$ . Near  $z_k$ , for each  $1 \leq k \leq N$ ,

$$f(z) = \underbrace{\frac{f_k(z)}{\text{principal part}}}_{\substack{\text{poly in } \frac{1}{z-z_k}}} + \underbrace{g_k(z)}_{\text{holomorphic}}.$$

Near  $\infty$ ,

$$f\left(\frac{1}{z}\right) = \underbrace{\tilde{f}_\infty(z)}_{\substack{\text{principal part} \\ \text{poly in } \frac{1}{z}}} + \underbrace{\tilde{g}_\infty(z)}_{\text{holomorphic}}.$$

Consider,

$$H(z) = f(z) - \underbrace{\sum_{j=1}^N f_j(z)}_{\text{rational}} - \underbrace{\tilde{f}_\infty(z)}_{\text{poly}}.$$

Claim:  $H$  is holomorphic and bounded.

Near  $z_k$

$$H = \underbrace{f - f_k}_{\text{holo}} - \sum_{j \neq k} f_j - \underbrace{\tilde{f}_\infty}_{\text{holo}}.$$

We can show  $H$  is bounded as

$$H\left(\frac{1}{z}\right) = \underbrace{f\left(\frac{1}{z}\right) - \tilde{f}_\infty(z)}_{\substack{\text{bounded near } z=0 \\ \tilde{g}_\infty(z)}} - \underbrace{\sum_{j=1}^N f_j\left(\frac{1}{z}\right)}_{\substack{\text{poly} \\ \text{bounded near } z=0}}.$$

So  $H\left(\frac{1}{z}\right)$  is bounded in say  $\{|z| < \delta\}$  and  $H(z)$  is bounded in  $\{|z| > 1/\delta\}$ , hence  $H$  is bounded in  $\{|z| \leq 1/\delta\}$ . *dset*  $\square$

## 15 Feb 18

**Theorem 15.1.** (*The Argument Principle*) Let  $f : \Omega \rightarrow \mathbb{C}$  be meromorphic,  $\overline{D} \subset \Omega$  be a disk. If  $f$  never vanishes on  $\partial D$  nor has any poles on  $\partial D$  then

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'}{f} = \# \text{ zeros of } f - \# \text{ poles of } f \text{ in } \overline{D}.$$

*Proof.* If  $f$  has zero of order  $n$  at  $z_0$  then  $f(z) = (z - z_0)^n g(z)$  nonvanishing near  $z_0$  then

$$\frac{f'}{f}(z) = \frac{n}{z - z_0} + \frac{g'}{g}(z).$$

If  $f$  has a pole of order  $n$  at  $z_0$  we can write  $f(z) = (z - z_0)^{-n} h(z)$  then

$$\frac{f'}{f}(z) = -\frac{n}{z - z_0} + \frac{h'}{h}(z).$$

Therefore  $\frac{f'}{f}$  has simple poles at zeros and poles of  $f$  with residue  $\frac{\text{order}}{\text{order}}$ . The residue formula finishes the proof.  $\square$

**Theorem 15.2.** *Rouche's Theorem:* Let  $f, g$  be holomorphic in  $\Omega \supset \overline{D}$  with  $\Omega$  open. If  $|f| > |g|$  on  $\partial D$  then  $f, f + g$  have the same number of zeros in  $D$ .

*Proof.* Let  $f_t = f + t \cdot g$  for  $t \in [0, 1]$  and  $n_t = \#$  of zeros of  $f_t$  in  $D$ . By argument principle we have

$$n_t = \frac{1}{2\pi i} \int_{\partial D} \frac{f'_t}{f_t}.$$

$f_t$  never vanishes on  $\partial D$ . We can conclude

$$|f_t| \geq |f| - t|g| \geq |f| - |g| > 0.$$

We can show continuity of the integrand by

$$\int_0^{2\pi} \underbrace{\frac{f'_t(z_0 + re^{i\theta})}{f_t(z_0 + re^{i\theta})} rie^{i\theta}}_{H(t,\theta)} d\theta.$$

Then  $H(t, \theta)$  is jointly continuous in  $(t, \theta)$ . So  $n_t$  is continuous and discrete so  $n_t$  is constant.  $\square$

*Proof.* Third Proof of fundamental Theorem of Algebra.

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} \underbrace{g(z)}_{f(z)} + \underbrace{a_n z^n}_{f(z)}$$

There exist  $\mathbb{R} > 0$  such that  $|g(z)| < |f(z)|$  on  $|z| = R$   $\square$