

Graduate Discrete Math (21-701) Notes

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Abstract

Lecture notes based on Graduate Discrete Math (21-701)

1 Graphs

Definition 1.1. *Graph is a set of objects (V, E) and $E \subseteq \binom{V}{2}$*

Definition 1.2. *Walk is a sequence of vertices*

Definition 1.3. *A path is a walk without repeated vertices*

Definition 1.4. *A proper K -coloring of a graph is a function $c : V \rightarrow [k]$ such that $\forall u, v \in V, u \sim v \implies c(u) \neq c(v)$*

Theorem 1.5. *A graph is 2 colorable if and only if there is no odd cycles in G*

Proof. (\implies) AFSOC there exist an odd cycle, C in G . Define the vertices of C as v_1, v_2, \dots, v_k where k is odd. Define $c(v) = \begin{cases} \text{red} & d(v, v_1) \text{ is even} \\ \text{blue} & d(v, v_1) \text{ is odd} \end{cases}$

Then $c(v_1)$ and $c(v_k)$ are both red so a contradiction.

(\impliedby) We can assume each component is connected. Choose v_0 and define $c(v) = \begin{cases} \text{red} & d(v, v_0) \text{ is even} \\ \text{blue} & d(v, v_0) \text{ is odd} \end{cases}$

If there exist vertices u, v with uv an edge such that $d(u, v_0) \equiv d(v, v_0) \pmod{2}$ then consider the cycle, C formed by shortest path from $v_0 \rightarrow u$ and $v_0 \rightarrow v$ with uv . Then $|C| = d(u, v_0) + d(v, v_0) + 1$ is odd and we're done. \square

2 Hypergraphs

Definition 2.1. *A collection \mathcal{H} of subsets of a vertex set V .*

Definition 2.2. *\mathcal{H} is k -uniform if $|f| = k, \forall f \in \mathcal{H}$*

Definition 2.3. *A proper k -coloring of \mathcal{H} is an assignment $c : V \rightarrow [k]$ such that $\forall f \in \mathcal{H}, |c(f)| = k$*

Definition 2.4. *A rainbow coloring of \mathcal{H} is an assignment $c : V \rightarrow [k] \forall f \in \mathcal{H}, |c(f)| = |f|$*

Example 2.5. *What is the least number of edges in a k -uniform graph that is not 2-colorable?*

Let this number be $m(k)$ then $m(1) = 0, m(2) = 3, m(3) \geq 7$

Theorem 2.6. *If \mathcal{H} is a 3-uniform hypergraph with less than 6 edges then \mathcal{H} is 2-colorable*

Proof. Using induction on $|V|$

(Base Case) For $n = 6$, consider all balanced 2-colorings of V there are $\binom{6}{3} = 20$. Each hyperedge is incompatible with 2 of those colorings (namely those where the edges are 3 blue or 3 red). Thus, at least $20 - 12 > 0$ of these colorings can be proper.

(Induction Hypothesis) Suppose $n \geq 7$

Claim 2.7. *There are 2 vertices u and v not in any common edge.*

Each edge connects $\binom{3}{2} = 3$ pairs of vertices. There are $\binom{7}{2} = 21$ pairs of vertices overall. So some pair of vertices is not connected as $21 > 18$.

Define \mathcal{H}' by merging u, v into w

Claim 2.8. *\mathcal{H}' is 3-uniform*

Because no edge contains both u and v the merging doesn't create a 2 set and every edge is still has size 3.

Additionally, $||\mathcal{H}'|| \leq ||\mathcal{H}|| \leq 6$ so by induction hypothesis \mathcal{H}' is 2-colorable. Giving both u and v the same color as w and keeping the rest of the colors the same.

If an edge of e of \mathcal{H} avoids $\{u, v\}$ then it is properly colored in \mathcal{H}' . If e contains u or v then after merging it corresponds to an edge of \mathcal{H}' containing w . If e is monochromatic in \mathcal{H} then it would be monochromatic in \mathcal{H}' . This would be a contradiction so edge is monochromatic in \mathcal{H} and thus a proper coloring. \square

Remark 2.9. *Suppose it has 7 edges and vertices. Consider the coloring 4 red and 3 blue. Then there are $\binom{7}{3} = 35$ such colorings. If \mathcal{H} is not 2 colorable then there are $\binom{3}{3} + \binom{4}{3} = 5$ excluded coloring for all distinct edges. There are 4 forbidden configurations for any configurations that are not 2 colorable \mathcal{H} with $|\mathcal{H}| = 7$ on 7 vertices*

3 Probabilistic Method

Theorem 3.1. $m(k) \geq 2^{k-1}$

Proof. Color vertices of \mathcal{H} randomly red or blue. For each edge f , define E_f to be the event that f is monochromatic then $Pr[E_f] = \frac{1}{2^{k-1}}$

$$Pr \left[\bigcup_{f \in \mathcal{H}} E_f \right] \leq \sum_{f \in \mathcal{H}} Pr [E_f] = \frac{|\mathcal{H}|}{2^{k-1}} < 1$$

So there is non-zero probability that there exist a coloring with no monochromatic edges if $|\mathcal{H}| < 2^{k-1}$ \square

Theorem 3.2. Erdős-Selfridge Theorem: Given hypergraph \mathcal{H} , consider a game between a maker and breaker. The maker's goal is to color some edge all blue and breaker's goal is to prevent all blue edges.

If \mathcal{H} is k -uniform and $|\mathcal{H}| < 2^{k-1}$ then the breaker has a winning strategy even as player 2.

Proof. Let $\phi(f) = \begin{cases} 0 & \text{if blocked by breaker} \\ \frac{2^{\#\text{blue}\in f}}{2^n} & \text{otherwise} \end{cases}$

be the "danger function". Define

$$\phi(\mathcal{H}) = \sum_{f \in \mathcal{H}} \phi(f)$$

Observe that if an edge is all blue, then $\phi(\mathcal{H}) \geq 1$

At start of the game $\phi(\mathcal{H}) = \frac{|\mathcal{H}|}{2^n}$. The worst case for when maker moves is increasing by $\frac{|\mathcal{H}|}{2^n}$ when the chosen vertex is in all edges. Then when breaker moves,

$$-\sum_{f \ni v_1} \phi(f).$$

When maker goes after

$$\sum_{f \ni v_2} \phi(f).$$

Notice

$$\sum_{f \ni v_1} \phi(f) > \sum_{f \ni v_2} \phi(f)$$

otherwise breaker played optimally.

So as long as $\frac{|\mathcal{H}|}{2^{n-1}} < 1$ then breaker wins. \square

Definition 3.3. Incidence matrix of a hypergraph \mathcal{H} with $|V| = n$ and $|\mathcal{H}| = m$ is defined as

$$I_{i,j} = \begin{cases} 1 & \text{if } v_n \in f_m \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.4. Hall's Theorem

If G is a bipartite on (A, B) there is a complete matching if and only if

$$\forall S \subseteq A, |\Gamma(S)| \geq |S|$$

where $\Gamma(S) = \{u \in B \mid \exists v \in S, u \sim v\}$.

Theorem 3.5. Consider complete graph $\mathcal{P}(X)$ where $|X| = n$. $\mathcal{P}(X)$ has levels

$$\binom{X}{0}, \binom{X}{1}, \dots, \binom{X}{n}$$

$\forall k < \frac{n}{2}$, there is an injection

$$f_k : \binom{X}{k} \rightarrow \binom{X}{k+1}$$

such that $\forall S \in \binom{X}{k}, S \subseteq f_k(S)$

Proof. Consider bipartite graph $\left(\binom{X}{k}, \binom{X}{k+1} \right)$, if $f \in \binom{X}{k}, g \in \binom{X}{k+1}$ then we define $f \sim g$ if $f \subseteq g$. Then for some $S \subseteq \binom{X}{k}$ then $|\Gamma(S)| \geq \frac{|S|(n-k)}{k+1}$. \square

Definition 3.6. For a sperner system is a hypergraph \mathcal{H} that satisfy if

$$\forall f, g \in \mathcal{H}, f \not\subseteq g$$

Theorem 3.7. If \mathcal{H} is a sperner system of n -vertices then

$$|\mathcal{H}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Theorem 3.8. LYM Inequality on a sperner family \mathcal{H} ,

$$\sum_{f \in \mathcal{H}} \frac{1}{\binom{n}{|f|}} \leq 1$$

Proof. Suppose $F = \{\emptyset, \{1\}, \dots, \{1, 2, \dots, n\}\}$

Note: Any sperner family can share at most one edge with F .

Consider a random permutation $\sigma \in S_n$ and define $\mathcal{H}_\sigma = \{\sigma(f) | f \in \mathcal{H}\}$.

For any $\sigma \in S_n$, $|\mathcal{H}_\sigma \cap F| \leq 1$

Now choose any σ , uniformly at random and define $\mathcal{X} = |\mathcal{H}_\sigma \cap F|$ is a random variable and $\mathcal{X} \leq 1$.

Let $\mathcal{X} = I_f$ where $I_f = \begin{cases} 1 & \sigma(f) \in F \\ 0 & \text{otherwise} \end{cases}$

$$\mathbb{E}[\mathcal{X}] = \sum_{f \in \mathcal{H}} \mathbb{E}[I_f] = \sum_{f \in \mathcal{H}} \frac{1}{\binom{n}{|f|}} \leq 1$$

□

Definition 3.9. Define the "shadow" of $\mathcal{H} \subseteq \binom{X}{r}$ as $\partial\mathcal{H} \subseteq \binom{X}{r-1}$

$$\partial\mathcal{H} = \left\{ S \subseteq \binom{X}{r-1} : \exists T \in \mathcal{H}, S \subseteq T \right\}$$

Theorem 3.10. Let $n = |X|$ then

$$\frac{|\mathcal{H}|}{\binom{n}{r}} \leq \frac{|\partial\mathcal{H}|}{\binom{n}{r-1}}$$

with equality only if \mathcal{H} is empty on $\binom{X}{r}$

Proof. Suppose \mathcal{H} is a sperner system, not all on one level.

Write $\mathcal{H}_i = \mathcal{H} \cap \binom{X}{i}$ then $H = \mathcal{H}_i \cup \mathcal{H}_{i+1} \cup \dots \cup \mathcal{H}_j$ where $i < j$ and \mathcal{H}_i nonempty.

We can instead of \mathcal{H}_j we can write $\partial\mathcal{H}_j$ as $\partial\mathcal{H}_j \subseteq \mathcal{H}_j$

Suppose \mathcal{H} maximizes the sum $\sum_{f \in \mathcal{H}} \frac{1}{\binom{n}{|f|}}$ among all sperner graphs.

Let $S \in \partial\mathcal{H}$ and $T \subseteq \mathcal{H}$. Define a bipartite graph from $S \rightarrow T$ and edges if $S \subseteq T$.

For $T \in \mathcal{H}$, $\deg(T) = r$ for $S \in \partial\mathcal{H}$, $\deg(S) = n - (r - 1)$.

So $|\mathcal{H}| \cdot r = b \leq |\partial\mathcal{H}| \cdot (n - r + 1)$.

Then $|\mathcal{H}| \cdot r \binom{n}{r} \leq |\partial\mathcal{H}| \cdot (n - r + 1) \binom{n}{r-1} \implies \frac{|\mathcal{H}|}{\binom{n}{r}} \leq \frac{|\partial\mathcal{H}|}{\binom{n}{r-1}}$

□

Definition 3.11. An intersecting hypergraph has any 2 hyperedges intersect.

Theorem 3.12. For an intersecting hypergraph on n -vertices and r -uniform,

If $r = \frac{n}{2}$ then we can fix 1 vertex and complete the remaining $\frac{n}{2} - 1$ vertices. So $\binom{n-1}{\frac{n}{2}-1}$

If $r > \frac{n}{2}$ then 2^n .

If $r < \frac{n}{2}$ then $\binom{n-1}{r-1}$.

We'll prove the last statement

Proof. Assume $n = lk$ for some l and for any $\sigma \in S_n$ define $\mathcal{H}_\sigma = \{\sigma(f) | f \in \mathcal{H}\}$. Define \mathcal{F} to be a k -uniform hypergraph with l non-intersecting edges.

If \mathcal{H} is intersecting then $|\mathcal{H}_\sigma \cap \mathcal{F}| \leq 1$.

Let $\mathcal{X} = |\mathcal{H}_\sigma \cap \mathcal{F}|$. Then

$$\mathbb{E}[\mathcal{X}] = \sum_f \mathbb{E}[I_f] = |\mathcal{H}| \mathbb{P}[\sigma(f) \in \mathcal{F}] = |\mathcal{H}| \frac{l}{\binom{n}{k}} = \frac{|\mathcal{H}|}{\binom{n-1}{k-1}}$$

So $|\mathcal{H}| \leq \binom{n-1}{k-1}$

Consider the case when n is not divisible by k . If $n \geq 2k$ then fix a cyclic ordering π of the n vertices. For that ordering consider the n cyclic k -intervals for $i = 1, 2, \dots, n$

$$I_i(\pi) = \{\pi(i), \pi(i+1), \dots, \pi(i+k-1)\}$$

indices taken modulo n .

For a given π define

$$\mathcal{X}_\pi := \#\{f \in \mathcal{H} | f \text{ is one of the intervals } I_i(\pi)\}$$

Any two sets counted in \mathcal{X}_π must intersect since \mathcal{H} is intersecting. Among the n cyclic k -intervals at most k of them can be pairwise intersecting since we can fix one vertex however $k+1$ intervals will force two of them to be disjoint. So for every π , $\mathcal{X}_\pi \leq k$.

So

$$\mathbb{E}[\mathcal{X}_\pi] = |\mathcal{H}| \frac{k!(n-k)!}{(n-1)!} = |\mathcal{H}| \frac{n}{\binom{n}{k}} \leq k \implies |\mathcal{H}| \leq \binom{n-1}{k-1}$$

□

We'll try constructing such a configuration.

If $|\mathcal{H}| = \binom{n-1}{k-1}$ then $|\mathcal{H}_\sigma \cap \mathcal{F}| = k$ for each $\sigma \in S_n$. Then there is an i such that $\mathcal{I} = \left\{ \begin{array}{l} \{i-k+1, i-k+2, \dots, i\} \\ \{i-k+2, i-k+3, \dots, i+1\} \\ \vdots \\ \{i, i+1, \dots, i+k+1\} \end{array} \right\}$

Suppose $a_1, \dots, a_{k-1} \in [n]$ with no $a_j = i-k, i-k-1, \dots, i, \dots, i-k+1$

Consider a permutation σ sending $a_1 \rightarrow i+1, a_2 \rightarrow i+2, \dots, a_{k-1} \rightarrow i+k-1$ and fixing $i-k, \dots, i$.

We know, $|\mathcal{H}_\sigma \cap \mathcal{F}|$ includes all edges of \mathcal{I}

Now let σ be any permutation such that $\mathcal{H} \cap \mathcal{F}$ includes i of I . It suffices to show $\mathcal{H}_\sigma \cap \mathcal{F}$ includes all of \mathcal{I} for any transposition.

Lemma: Adjacent transposition generates S_n .

(Case 1) If $j, j+1 \in \{i-k+1, i, \dots, i+k-1\}$, neither is i so they're both on same side of i .

Letting $f_0 = \{i-k+1, \dots, i\}$ and $f_1 = \{i, \dots, i+k-1\}$ then $\tau(f_0) = f_0$ and $\tau(f_1) = f_1$

(Case 2) If $j = i+k-1$ and $j+1 = i+k$ then $\tau(f_0) = f_0$ and $\tau(\{i-k, \dots, i-1\}) = \{i-k, \dots, i-1\}$.

Theorem 3.13. Let $\alpha_1, \dots, \alpha_n \sim Ber(p)$, choosing numbers β_1, \dots, β_n with $\sum \beta_i = 1$ then $\mathbb{P}[\sum \beta_i \alpha_i \geq \frac{1}{2}] \geq p$

Proof. Define \mathcal{H} on $[n]$ by $f \in \mathcal{H}$ if $\sum_{i \in f} \alpha_i \geq \frac{1}{2}$. For simplicity, assume no sum is $\frac{1}{2}$. Then

$$\mathbb{P} \left[\sum \beta_i \alpha_i \geq \frac{1}{2} \right] = \sum_{f \in \mathcal{H}} p^{|f|} (1-p)^{n-|f|}$$

Define $h_k = |\mathcal{H} \cap \binom{X}{k}|$

$$\begin{aligned}\mathbb{P}\left[\sum \beta_i \alpha_i \geq \frac{1}{2}\right] &= \sum_k h_k p^k (1-p)^{n-k} \\ &= \sum_{k \leq \frac{n}{2}} h_k p^k (1-p)^{n-k} + h_{n-k} p^{n-k} (1-p)^k \\ &= \sum_{k \leq \frac{n}{2}} (h_k + h_{n-k}) p^{n-k} (1-p)^k - h_k (p^{n-k} (1-p)^k - p^k (1-p)^{n-k})\end{aligned}$$

Note: $h_k + h_{n-k} \geq \binom{n}{k}$ since it or its complement has to be in \mathcal{H}

$$\begin{aligned}&\geq \sum_{k \leq \frac{n}{2}} \binom{n}{k} p^{n-k} (1-p)^k - h_k (p^{n-k} (1-p)^k - p^k (1-p)^{n-k}) \\ &= \sum_{k \leq \frac{n}{2}} \binom{n}{k} p^{n-k} (1-p)^k + \binom{n-1}{k-1} p^k (1-p)^{n-k} \\ &= p \sum_{k \leq \frac{n}{2}} \binom{n-1}{k-1} p^k (1-p)^{n-k} \\ &= p \sum_{k \leq \frac{n}{2}} \binom{n-1}{k-1} p^k (1-p)^{n-k} \\ &= p \sum_{k \leq \frac{n}{2}} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-k)-(k-1)} \\ &= p\end{aligned}$$

□

Theorem 3.14. If there are 10 points in the plane then they can be covered by 10 non-intersecting unit circles.

Proof. Given any collection $X \subseteq \mathbb{R}^2$, $|x| = 10$. Consider a random translation of the hexagonal circle pattern.

Let $\mathcal{Z} = \#$ points in X covered then

$$\mathbb{E}[\mathcal{Z}] = \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_{10}] = 10 \cdot \frac{\pi}{\frac{6}{\sqrt{3}}} \approx 9.07$$

So there exist a translation such that $\mathcal{Z} = 10$

□

Theorem 3.15. Given a graph, G on n vertices and $\frac{nd}{2}$ edges, $d \geq 1$. Then $\alpha(G) \geq \frac{n}{2d}$.

Proof. Let $S \subseteq V$ be a random subset defined by $\mathbb{P}[v \in S] = p$, p to be determined. Let $X = |S|$ and $Y = \mathbb{E}[G|_S]$. For each $e = \{i, j\} \in E$, let Y_e be indicator random variable for the event $i, j \in S$ so that

$$Y = \sum_{e \in E} Y_e$$

For any such e ,

$$\mathbb{E}[Y_e] = \mathbb{P}[i, j \in S] = p^2$$

$$\mathbb{E}[Y] = \frac{nd}{2} p^2$$

Clearly, $\mathbb{E}[X] = np$ so $\mathbb{E}[X - Y] = np - \frac{nd}{2} p^2$.

Setting $p = \frac{1}{d}$ then $\mathbb{E}[X - Y] = \frac{n}{2d}$.

So there exist a S such that the number of vertices minus the number of edges is at least $\frac{n}{2d}$.

Create S^* from S by deleting one vertex from each edge in S and delete it and this leaves S^* with at least $\frac{n}{2d}$ vertices. With all edges destroyed we leave S^* an independent set. \square

Theorem 3.16. Erdos Chromatic Number Girth Theorem

$\forall k \in \mathbb{Z}^+, \exists$ graph of girth greater than or equal to k and chromatic number k .

Proof. Idea: Choose random graph $G \sim G(n, p)$. To show a graph satisfies both properties we need the the number of short cycles (length less than k) to be 0 and there are no independent set of size no more than $\frac{n}{k}$.

For the first statement let $X = \#\text{cycles with length } \leq k$ then

$$\mathbb{E}[X] = \sum_C \mathbb{E}[I_c] = \sum_{j=3}^k \sum_{|C|=j} \mathbb{E}[I_C] = \sum_{j=3}^k \binom{n}{j} \frac{(j-1)!}{2} p^j \leq \sum_{j=3}^k n^j p^j \leq (np)^{k+1}$$

To have $\mathbb{E}[X] = O(1)$, we need $p = O\left(\frac{1}{n}\right)$

We want no independent set of size $a \approx \frac{n}{k}$ so

$$\begin{aligned} \mathbb{P}[\alpha(G_{n,p}) \geq a] &\leq \binom{n}{a} (1-p)^{\binom{a}{2}} \\ &\leq n^a (1-p)^{a(a-1)/2} \\ &\leq n^a e^{-pa(a-1)/2} \\ &= \left(ne^{-p(a-1)/2}\right)^a \\ &= \left(e^{\ln(n)-p(a-1)/2}\right)^a \end{aligned}$$

Not possible since we need $p \geq 5 \ln(n)/a$ but $p = O\left(\frac{1}{n}\right)$ from previous condition.

To fix this issue consider an alteration. If $p = \frac{n^\epsilon}{n}$ and $0 < \epsilon < \frac{1}{k}$ then we have $\mathbb{P}(\alpha(G_{n,p}) \geq \frac{n}{2k}) \rightarrow 0$ since $\frac{n^\epsilon}{n} \gg \frac{5 \ln(n) - 2k}{n}$.

To fix the short cycle issue,

$$\mathbb{E}[X] = \sum_{j=3}^k (np)^j \leq (k-3)(np)^k \leq kn^{\epsilon k}$$

By Markov's inequality

$$\mathbb{P}\left(X \geq \frac{n}{2}\right) \leq \frac{kn^{\epsilon k}}{n/2}$$

Choose n large enough such that both probabilities are greater than $\frac{1}{2}$. Then there exists a graph on n vertices with no independent set of size $\frac{n}{2k}$ and less than $\frac{n}{2}$ short cycles. Delete an vertex from each

short cycle to make a graph G' with $\frac{n}{2}$ vertices, no short cycles and no independent set of size $\frac{n}{2k}$. So

$$\chi(G') = \frac{n'}{\alpha(G')} \geq \frac{\frac{n}{2}}{\frac{n}{2k}} \geq k.$$

□

When does $G_{n,p}$ have triangles?

If $X = \#\text{triangles}$ then

$$\mathbb{E}[X] = \sum_{\text{Triangle } T \in K_n} E[I_T] = \binom{n}{3} p^3 \sim n^3 p^3 / 6 = O(1)$$

if $p = O\left(\frac{1}{n}\right)$

Is $G_{n,p}$ connected for $p = \frac{c}{n}$?

Let $X = \#\text{spanning trees}$ of $G_{n,p}$ then

$$\mathbb{E}[X] = \sum_{T \in G_{n,p}} E(I_T) = n^{n-2} p^{n-1} = n^{n-2} \frac{c^{n-1}}{n^{n-1}} = \frac{c^{n-1}}{n} \rightarrow \infty.$$

Let $Y = \#\text{isolated vertices}$ then

$$\mathbb{E}[Y] = \sum_{v \in V} \mathbb{P}(v \text{ isolated}) = n(1-p)^{n-1} \approx ne^{-p(n-1)} \approx ne^{-c}$$

For $p = O\left(\frac{1}{n}\right)$, let $X = \#\text{triangles}$. Then

$$\mathbb{E}[X] = \binom{n}{3} p^3 \rightarrow 0.$$

So $\mathbb{P}(G_{n,p} \text{ having triangles}) \rightarrow 0$.

Theorem 3.17. Threshold of \mathcal{H} in $G_{n,p}$.

Consider $G_{n,p}$ with $p = p(n)$ and \mathcal{H} is fixed graph with k vertices and l edges.

Define $\epsilon = \epsilon(\mathcal{H}) = \frac{l}{k}$ and $\epsilon' = \epsilon'(\mathcal{H}) = \max_{J \subseteq \mathcal{H}} \epsilon(J)$. If $p^\epsilon \cdot n \rightarrow 0$ then $\mathbb{E}[\#\mathcal{H} \text{ in } G_{n,p}] \rightarrow 0$.

Proof. $\mathbb{E}[\#\mathcal{H} \in G_{n,p}] \leq \binom{n}{k} h p^l \leq C(n p^\epsilon)^k$ where $h = \frac{k!}{\text{Aut}(\mathcal{H})}$

If $p^{\epsilon'} n \rightarrow 0$, there is some argument for densest subgraph J . \square

Now to show the other side, if $p^{\epsilon'} n \rightarrow \infty$ (if $p = \omega\left(\frac{1}{n^{1/\epsilon'}}\right)$) then $G_{n,p}$ has \mathcal{H} as a subgraph with high probability.

Proof. Let $X = \#\mathcal{H}$ subgraph in $G_{n,p}$ then by Chebyshev,

$$\mathbb{P}[X \leq 0] \leq \frac{\text{Var}[X]}{\mathbb{E}[X]^2}.$$

We can compute

$$\mathbb{E}[X^2] = \sum_{H_1, H_2 \in \mathcal{H}} \mathbb{P}[H_1, H_2 \subseteq G_{n,p}] = \sum_{t=0}^k \sum_{|H_1 \cap H_2|=t} \mathbb{P}(H_1, H_2 \subseteq G_{n,p})$$

For $t = 0$ we have

$$\sum_{|H_1 \cap H_2|=0} \mathbb{P}(H_1 \subseteq G_{n,p}) \mathbb{P}(H_2 \subseteq G_{n,p}) \leq \mathbb{E}[X]^2$$

So

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 \leq \sum_{t=1}^k \sum_{|H_1 \cap H_2|=t} \mathbb{P}(H_1, H_2 \subseteq G) = \sum_{t=1}^k \sum_{|H_1 \cap H_2|=t} \binom{k}{t} \binom{n-k}{k-t} h p^{e(H_1 \cup H_2)}$$

By PIE, $e(H_1 \cup H_2) \geq 2l - \epsilon't$ since $e(H_1 \cap H_2) \leq \epsilon't$ as $H_1 \cap H_2$ is a subgraph of H_1

$$\begin{aligned} \sum_{t=1}^k \sum_{|H_1 \cap H_2|=t} \binom{k}{t} \binom{n-k}{k-t} h p^{e(H_1 \cup H_2)} &\leq \sum_{t=1}^k \sum_{|H_1 \cap H_2|=t} \binom{k}{t} \binom{n-k}{k-t} h p^{2l - \epsilon't} \\ &= \sum_{t=1}^k \sum_{|H_1 \cap H_2|=t} \binom{n}{k} h \cdot h \binom{n-k}{k-t} p^{l - \epsilon't} \\ &\quad (\sum_{H \in \mathcal{H}} p^l = \mathbb{E}[X] = \binom{n}{k} \cdot h \cdot p^l) \\ &\leq \mathbb{E}[X] \sum_{t=1}^k h \binom{k}{t} \binom{n-k}{k-t} p^{l - \epsilon't} \\ &\leq \mathbb{E}[X] \sum_{t=1}^k h \cdot C \cdot n^k \cdot \frac{1}{n^t} \cdot p^{l - \epsilon't} \\ &\leq \mathbb{E}[X] \sum_{t=1}^k C' \cdot h \cdot \binom{n}{k} \cdot p^l \cdot \frac{1}{(np^{\epsilon'})^t} \\ &= \mathbb{E}[X]^2 \sum_{t=1}^k C' \left(\frac{1}{np^{\epsilon'}} \right)^t \rightarrow 0 \end{aligned}$$

\square

Theorem 3.18. Chernoff Bound

Suppose you had independent random values ζ_1, \dots, ζ_n with $\zeta_i \in \{-1, 1\} \forall i$ and $\mathbb{P}(\zeta_i = 1) = \mathbb{P}(\zeta_i = -1) = \frac{1}{2}$. Let $X = \sum_{i=1}^n \zeta_i$

$$\begin{aligned}
 \mathbb{P}(X > a) &= \mathbb{P}(e^{tX} > e^{ta}) \\
 &\leq \frac{\mathbb{E}(e^{tX})}{e^{ta}} \\
 &= \frac{\mathbb{E}(e^{t\sum \zeta_i})}{e^{ta}} \\
 &= \frac{\prod_{i=1}^n \mathbb{E}[e^{t\zeta_i}]}{e^{ta}} \\
 &= \frac{\left(\frac{e^t + e^{-t}}{2}\right)^n}{e^{ta}} \\
 &\leq e^{nt^2/2 - ta} \\
 &= e^{a^2/2n - a^2/n} \quad (\text{for } t = \frac{a}{n}) \\
 &= e^{-\frac{a^2}{2n}}
 \end{aligned}$$

Question: How many vectors can I have in \mathbb{R}^d all at a common pairwise angle.

Equivalently, $\exists \alpha, v_i \cdot v_j = \begin{cases} \alpha & \text{if } i \neq j \\ 1 & \text{otherwise} \end{cases}$

$$\begin{aligned}
 \text{Let } V &= \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix} \\
 \text{Then } V^T V &= G = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_m \\ \vdots & v_2 \cdot v_2 & \cdots & \\ \vdots & \vdots & \ddots & \end{bmatrix}
 \end{aligned}$$

Claim: $\text{Rank}(G) \leq d$.

For each $\vec{x} \in \text{Ker}(V) \implies V\vec{x} = 0 \implies V^T V \vec{x} = 0 \implies G\vec{x} = 0$

We know $\text{Rank}(V) \leq d$ so $\text{Rank}(G) = \text{Rank}(V^T V) \leq d$.

We can write $G = \alpha J_m + (1-\alpha)I_m$ then J_m has eigenvalues m with multiplicity at least 1 and eigenvectors with multiplicity $\geq m-1$.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ \vdots \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

If \vec{x} is an eigenvector of J_m with eigenvalue λ then $G\vec{x} = \alpha\lambda\vec{x} + (1-\alpha)\vec{x} = (\alpha\lambda + 1 - \alpha)\vec{x}$.

G has eigenvalues $1 - \alpha$ with multiplicity at least $m-1$.

So $\text{Rank}(G) \geq n-1$ and consequently $d \geq n-1$.

Question: How many unit vectors v_1, \dots, v_n can I have in \mathbb{R}^d with all pairs approx equal angle.

Equivalently, $\exists \alpha$ such that $(*) v_i \cdot v_j \in (\alpha - \epsilon, \alpha + \epsilon)$ if $i \neq j$ and 1 if $i = j$.

Theorem 3.19. $\forall \epsilon \in (0, 1)$, there is a $c > 0$ such that for all sufficiently large d there is a collection $v_1, \dots, v_m \in \mathbb{R}^d$ of vectors satisfying $(*)$ with $m \geq 2^{cd}$

Consider the easier problem when $\alpha = 0$. Choose m vectors $v_i \in \{1, -1\}^d$ at random with $\mathbb{P}(v_i^j = +1) = \mathbb{P}(v_i^j = -1) = \frac{1}{2}$. We know $v_i \cdot v_i = n$ then let $u_i = \frac{v_i}{\sqrt{n}}$. Then $\mathbb{E}(u_i \cdot u_v) = 0$ as $\zeta_{i,j,k} =$

$\begin{cases} 1 & \text{if } v_i^k \cdot v_j^k = 1 \\ -1 & \text{if } v_i^k \cdot v_j^k = -1 \end{cases}$ By Chernoff,

$$\mathbb{P}(v_i \cdot v_j \notin (-\epsilon n, \epsilon n)) \leq 2e^{-\epsilon^2 n/2}$$

$$\mathbb{P}(\exists i, j \text{ s.t. } u_i \cdot u_j \notin (-\epsilon n, \epsilon n)) \leq \binom{m}{2} 2e^{-\epsilon^2 n/2} \leq m^2 e^{-\epsilon^2 n/2} = m^2 \beta^{-n}$$

Theorem 3.20. If $\zeta_1, \zeta_2, \dots, \zeta_n$ are independent random variables with $\mathbb{E}(\zeta_i) = 0, |\zeta_i| \leq 1, X = \sum \zeta_i$ then

$$\mathbb{P}(X \geq a) \leq e^{-\frac{a^2}{2n}}$$

Question: I flip biased coins with head prob = $\frac{1}{3}$ n times. Bound the probability # head $\geq \frac{n}{2}$.

Define $\zeta_i = \begin{cases} 1 & \text{if head} \\ \frac{-p}{1-p} & \text{otherwise} \end{cases}$ (Subtract expected value and dividing by $1-p$) Then $\mathbb{E}[\zeta_i] = 0, |\zeta_i| \leq 1$

for $p < \frac{1}{2}$.

Let $X = h + (n-h)\frac{-p}{1-p}$

Theorem 3.21. For random variables X, Y

$$\begin{aligned} \mathbb{E}[XY] &= \sum_{x,y} xy \mathbb{P}(X=x, Y=y) \\ &= \sum_{x,y} xy \mathbb{P}(Y=y) \mathbb{P}(X=x|Y=y) \\ &= \sum_y y \mathbb{P}(Y=y) \sum_x x \mathbb{P}(X=x|Y=y) \\ &= \mathbb{E}(Y \mathbb{E}(X|Y)) \end{aligned}$$

Theorem 3.22. Let X be a random variable and A an event then

$$\mathbb{E}(X|A) = \frac{1}{\mathbb{P}(A)} \sum_{w \in A} p(w) \mathcal{X}(w)$$

Theorem 3.23. Let ζ_1, \dots, ζ_n be random variable with $\mathbb{E}(\zeta_i) = 0$ and $|\zeta_i| \leq 1$.

$$\mathbb{E}(e^{t \sum \zeta_i}) \leq \left(\frac{e^t + e^{-t}}{2} \right)^n$$

Proof.

$$\mathbb{E}(e^{t \sum \zeta_i}) = \mathbb{E} \left(\prod_{i=1}^n e^{t \zeta_i} \right) = \mathbb{E} \left(\prod_{i=1}^{n-1} e^{t \zeta_i} \mathbb{E} \left(e^{t \zeta_n} \mid \prod_{i=1}^{n-1} e^{t \zeta_i} \right) \right)$$

We know want to upper bound $\mathbb{E} \left(e^{t \zeta_n} \mid \prod_{i=1}^{n-1} e^{t \zeta_i} \right)$.

By convexity, $e^{t \zeta_i} \leq h(\zeta_i) = \frac{1}{2} [(1-\zeta_i)e^{-t} + (1+\zeta_i)e^t] \implies \mathbb{E}[e^{t \zeta_i}] \leq \frac{e^t + e^{-t}}{2}$

$$\mathbb{E} \left(e^{t \zeta_n} \mid \prod_{i=1}^{n-1} e^{t \zeta_i} \right) \leq \left(\frac{e^t + e^{-t}}{2} \right) \mathbb{E} \left(\prod_{i=1}^{n-1} e^{t \zeta_i} \right) \leq \left(\frac{e^t + e^{-t}}{2} \right)^n$$

□

Definition 3.24. Random variables X_0, X_1, \dots is a martingale if it satisfies the following properties

1. $\mathbb{E}[|X_i|] < \infty$
2. $\mathbb{E}[X_{i+1}|X_1, \dots, X_i] = X_i$

Theorem 3.25. Azuma's Theorem: If X_0, X_1, \dots is martingale and $|X_{i+1} - X_i| \leq 1$ then

$$\mathbb{P}(X_n - X_0 \geq \lambda\sqrt{n}) \leq e^{-\lambda^2/2}$$

Proof. By Markov's Inequality we have

$$\mathbb{P}(X_n - X_0 \geq \lambda\sqrt{n}) \leq \frac{\mathbb{E}[e^{t(X_n - X_0)}]}{e^{t\lambda\sqrt{n}}}$$

We can telescope the numerator as

$$\begin{aligned} \mathbb{E}[e^{t(X_n - X_0)}] &= \mathbb{E}\left[\prod_{k=1}^n e^{t(X_k - X_{k-1})}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{t(X_n - X_{n-1})}|X_0, X_1, X_2, \dots, X_{n-1}\right] \prod_{k=1}^{n-1} e^{t(X_k - X_{k-1})}\right] \end{aligned}$$

By Theorem 3.23 we have $\mathbb{E}[e^{t(X_n - X_{n-1})}|X_1, X_2, \dots, X_{n-1}] \leq e^{t^2/2}$ so

$$\mathbb{E}\left[\mathbb{E}\left[e^{t(X_n - X_{n-1})}|X_0, X_1, X_2, \dots, X_{n-1}\right] \prod_{k=1}^n e^{t(X_k - X_{k-1})}\right] \leq e^{nt^2/2}$$

So

$$\mathbb{P}(X_n - X_0 \geq \lambda\sqrt{n}) \leq e^{nt^2/2 - t\lambda\sqrt{n}}.$$

The RHS achieves it's max at $t = \frac{\lambda}{\sqrt{n}}$. Thus

$$\mathbb{P}(X_n - X_0 \geq \lambda\sqrt{n}) \leq e^{-\lambda^2/2}$$

□

Definition 3.26. Doob's Martingale: Let X and Y_1, Y_2, \dots be random variables with $\mathbb{E}[|Y_i|] < \infty$. Define $X_i := \mathbb{E}(X|Y_1, \dots, Y_i)$ for $i \geq 1$. with $X_0 = \mathbb{E}[X]$

Theorem 3.27. McDiarmid's Inequality: Let $f : \mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_n \rightarrow \mathbb{R}$ that is 1-lipschitz. If Y_1, Y_2, \dots, Y_n are independent random variables where $Y_i \in \mathcal{Y}_i$. For $X := f(Y_1, \dots, Y_n)$ satisfies

$$\mathbb{P}(X - \mathbb{E}(X) \geq \lambda\sqrt{n}) \leq e^{-2\lambda^2}.$$

Example 3.28. For m balls into n bins, let $X = \#$ empty bins then

$$\mathbb{E}[X] = n \left(1 - \frac{1}{n}\right)^m \approx ne^{-\alpha}$$

where $m = \alpha n$. If we let $Y_i = \text{pos of ball } i$ then $f(Y_1, \dots, Y_m) = \#$ empty bins. So $\mathbb{P}(X \geq \mathbb{E}[X] + \lambda\sqrt{n}) \leq e^{-2\lambda^2}$

Theorem 3.29.

$$\mathbb{P}(A|B \cap C) = \frac{\mathbb{P}(A \cap B|C)}{\mathbb{P}(B|C)}$$

Theorem 3.30. Lovasz Local Lemma: Given a collection $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ of bad events. If B_1, \dots, B_n have a dependency graph of max degree d and $4pd \leq 1$ where $\mathbb{P}(B_i) \leq p$ then

$$\mathbb{P}\left(\bigcap \overline{B_i}\right) > 0.$$

Proof. We'll prove the statement by induction on $k = |S|$ for $S \subseteq [n]$. Assume $4pd \leq 1$ then we want to show $\forall i, \mathbb{P}\left(B_i \mid \bigcap_{j \in S} \overline{B_j}\right) \leq 2p$.

Let $S = T \cup U$ where $T = \{j \in S \mid j \sim i\}$.

$$\begin{aligned} \mathbb{P}\left(B_i \mid \bigcap_{j \in T} \overline{B_j} \cap \bigcap_{j \in U} \overline{B_j}\right) &= \frac{\mathbb{P}\left(B_i \cap \bigcap_{j \in T} \overline{B_j} \mid \bigcap_{j \in U} \overline{B_j}\right)}{\mathbb{P}\left(\bigcap_{j \in T} \overline{B_j} \mid \bigcap_{j \in U} \overline{B_j}\right)} \\ &\leq \frac{p}{1 - \mathbb{P}\left(\bigcup_{j \in T} B_j \mid \bigcap_{j \in U} \overline{B_j}\right)} \\ &\leq \frac{p}{1 - \sum_{j \in T} \mathbb{P}(B_j \mid \bigcap_{k \in U} \overline{B_k})} \\ &\leq \frac{p}{1 - 2dp} \quad (\text{Induction Hypothesis}) \\ &\leq 2p \quad (\text{Assumption}) \end{aligned}$$

We're done with the induction.

Then

$$\mathbb{P}\left(\bigcap_{i=1}^n \overline{B_i}\right) = \mathbb{P}\left(\overline{B_n} \mid \bigcap_{i=1}^{n-1} \overline{B_i}\right) \cdot \mathbb{P}\left(\bigcap_{i=1}^{n-1} \overline{B_i}\right) \geq \frac{(1-2p)^n}{p} > 0$$

□

Theorem 3.31. *Local Lemma (General Version)* Suppose \mathcal{B} is a collection of "bad" events with some dependency graph. Suppose we can assign real number $0 < X_A < 1$ to each $A \in \mathcal{B}$ such that

$$\mathbb{P}(A) \leq X_A \prod_{B \sim A} (1 - X_B).$$

Then

$$\mathbb{P}\left(\bigcap_{B \in \mathcal{B}} \overline{B}\right) \geq \prod_{B \in \mathcal{B}} (1 - X_B) > 0$$

Proof. We can prove it by induction on $|S|$ that if $B_1, \dots, B_t, B_{t+1}, \dots, B_S \in \mathcal{B}$ such that $A \sim B_1, \dots, B_t$ and $A \not\sim B_{t+1}, \dots, B_S$ to show

$$\mathbb{P}\left(A \mid \bigcap_{i=1}^S \overline{B_i}\right) \leq X_A$$

We have

$$\begin{aligned} \mathbb{P}\left(A \mid \bigcap_{i=1}^t \overline{B_i} \cap \bigcap_{i=t+1}^S \overline{B_i}\right) &= \frac{\mathbb{P}\left(A \cap \bigcap_{i=1}^t \overline{B_i} \mid \bigcap_{i=t+1}^S \overline{B_i}\right)}{\mathbb{P}\left(\bigcap_{i=1}^t \overline{B_i} \mid \bigcap_{i=t+1}^S \overline{B_i}\right)} \\ &\leq \frac{\mathbb{P}\left(A \mid \bigcap_{i=t+1}^S \overline{B_i}\right)}{\mathbb{P}\left(\bigcap_{i=1}^t \overline{B_i} \mid \bigcap_{i=t+1}^S \overline{B_i}\right)} \\ &\leq \frac{X_A \prod_{B \sim A} (1 - X_B)}{\prod_{i=1}^t (1 - X_B)} \\ &\leq X_A \end{aligned}$$

Thus we're done with induction. Using the statement

$$\mathbb{P}\left(\bigcap_{i=1}^S \overline{B_i}\right) = \mathbb{P}(\overline{B_1} \mid \overline{B_2}) \times \cdots \times \mathbb{P}\left(\overline{B_n} \mid \bigcap_{i=1}^{n-1} \overline{B_i}\right) \geq \prod_{i=1}^n (1 - X_{B_i}) > 0$$

□

Theorem 3.32. *Axel's Theorem:*

$\forall \epsilon > N_\epsilon$ and an infinite binary sequence such that $\forall n > N_\epsilon$, any 2 consecutive block of length n differ in $\geq (\frac{1}{2} - \epsilon) n$ places.

Proof. Let the bad events be $B_{i,n}$ where for each i , intervals $[i, \dots, i+n], [i+n+1, \dots, i+2n]$ differ by less than $(\frac{1}{2} - \epsilon) n$

Let $X = \#$ places where they differ then $\mathbb{E}[X] = \frac{n}{2}$.

By Chernoff-Hoeffding's Lemma we have

$$\mathbb{P}(X - \mathbb{E}[X] \geq -\epsilon n) = \mathbb{P}(X \geq n/2 - \epsilon n) \leq e^{-2\epsilon^2 n} \leq e^{-\epsilon^2 n/10}$$

Let $X_{B_{i,n}} = e^{-\epsilon^2 n/20} \cdot \frac{1}{n^3}$ then fix B_{i_0, n_0} then we have

$$X_{B_{i_0, n_0}} = e^{-\epsilon^2 n_0/20} \cdot \frac{1}{n_0^3}$$

$$e^{-\epsilon^2 n_0/20} \cdot \frac{1}{n_0^3} \prod_{n=N_\epsilon}^T \prod_{i=i_0-2n}^{i_0+2n} \left(1 - e^{-\epsilon^2 n/20} \cdot \frac{1}{n^3}\right)^{2n+2n_0} \geq$$

□

From Homework #3

Theorem 3.33. $\forall \epsilon > 0, \exists N_\epsilon, \forall T, \exists$ a binary sequence of length T such that
(*) $\forall n > N_\epsilon$ identical blocks of length n are separated by distance $\geq (2 - \epsilon)^n$

Theorem 3.34. Konig's Infinity Lemma: Let G be a connected, locally finite, infinite graph then G contains an infinite path.

Theorem 3.35. $\forall \epsilon > 0, \exists N_\epsilon, \forall T, \exists$ an infinite binary sequence such that vertices of my tree are finite binary sequences with property (*).

Let \mathcal{T} be a complete tree of all binary sequences with all vertices and join $S \rightarrow S'$ if S can be obtained from S' by removing last digit of S' .

We want to show \mathcal{T} is locally finite and infinite. It is locally finite as each node has at most 2 children. It is infinite because for any string that satisfy (*), any of its prefix has to satisfy (*). By Theorem 3.34, there must be an infinite path with property (*)

4 Topology

Definition 4.1. A topology is a set X and a collection \mathcal{O} of open sets satisfying

1. $\emptyset \in \mathcal{O}, X \in \mathcal{O}$
2. \mathcal{O} is closed under finite intersection
3. \mathcal{O} is closed under arbitrary union

A collection of basic open sets are closed under finite intersections.

Definition 4.2. X is compact if every cover has a finite subcover

Definition 4.3. Product topogy is

$$\prod_{\alpha \in A} \mathcal{O}_\alpha$$

where $\mathcal{O}_\alpha \subseteq X_\alpha$ is open and $\mathcal{O}_\alpha = X_\alpha$ except for finitely many.

Theorem 4.4. If X_α where $\alpha \in I$ are compact topological spaces then $\prod_{\alpha \in I} X_\alpha$ is compact.

5 Ramsey Numbers

Definition 5.1. Ramsey number $R(k, l) = \min_{n \geq 1} \{K_n \text{ contains a red } K_k \text{ or blue } K_l\}$
We can see $R(3, 3) = 6$

Theorem 5.2. $R(k, l) \leq R(k - 1, l) + R(k, l - 1)$

Proof. Let $n \geq R(k - 1, l) + R(k, l - 1)$ and consider a red/blue coloring of K_n . Fix v_0 . Since v_0 has $\geq R(k - 1, l) + R(k, l - 1) - 1$ edges,

(Case 1) If v_0 has $\geq R(k - 1, l)$ red edges then the induced subgraph of the neighbors, G' must have red K_{k-1} or blue K_l . If red K_{k-1} then $G' \cup v_0$ is a K_k , otherwise we have blue K_l .

(Case 2) If v_0 has $\geq R(k, l - 1)$ blue neighbors then same argument as case 1.

Thus we're done □

Theorem 5.3. $R(k, k) > (1 - O(1)) \frac{k}{e\sqrt{2}} 2^{k/2}$

Proof. Flip fair coins to color a K_n red or blue. Let $X = \#$ monotonic K_k then

$$\mathbb{E}[X] = \binom{n}{k} 2^{1-\binom{k}{2}} \leq \left(\frac{en}{k}\right)^k \cdot 2 \cdot 2^{-k(k-1)/2}$$

If $\mathbb{E}[X] < 1$ then $R(k, k) > n$.

$$2^{1/k} \left(\frac{en}{k}\right) 2^{(k-1)/2} < 1$$

Thus

$$n < (1 - O(1)) \frac{k}{e\sqrt{2}} 2^{k/2}$$

Consequently, $R(k, k) > (1 - O(1)) \frac{k}{e\sqrt{2}} 2^{k/2}$ \square

Theorem 5.4. Alterations: Color edges of K_n randomly red or blue. Delete an vertex from each monochromatic K_k . Let $X = n - \#$ monochromatic cliques.

$$\begin{aligned} \mathbb{E}[X] &= n - \binom{n}{k} 2^{1-\binom{k}{2}} \\ &\geq n - \left(\frac{en}{k}\right)^k \cdot 2 \cdot 2^{-k(k+1)/2} \\ &= n - 2 \left(\frac{en}{k} \cdot 2^{\frac{-k-1}{2}}\right) \end{aligned}$$

Let $n = \frac{k}{e} \cdot 2^{k/2}$ then

$$\frac{k}{e} \cdot 2^{k/2} - 2^{k/2} = (1 - o(1)) \cdot \frac{k}{e} \cdot 2^{k/2}$$

Theorem 5.5. Using Lovasz Local Lemma: Given k , fix n , randomly red/blue color edges.

Proof. Bad events: B_k for $k \in \mathcal{K}$ where \mathcal{K} is the collection of k -clique.

Then $\mathbb{P}(B_k) = 2^{1-\binom{n}{k}}$.

If K_1, K_2 share any edges, set $B_{K_1} \sim B_{K_2}$ in dependency graph. Then

$$D \leq \binom{k}{2} \binom{n}{k-2}$$

Consequently

$$\begin{aligned} epD &\leq e \cdot 2^{1-\binom{n}{k}} \left(2 \binom{k}{2} \binom{n}{k-2}\right) < 1 \\ 4e \left(\left(\frac{en}{k-2}\right)^{k-2} \binom{k}{2}\right) &< 2^{\binom{k}{2}} \\ \left(2e \binom{k}{2}\right)^{\frac{1}{k-2}} \cdot \frac{en}{k-2} &< 2^{\binom{k}{2}-\frac{1}{k-2}} = 2^{\frac{k+1}{2}} \\ (1 + o(1)) \frac{en}{k-2} &< 2^{\frac{k+1}{2}} \end{aligned}$$

So

$$n < (1 - o(1)) \frac{k\sqrt{2}}{e} 2^{k/2}$$

Thus $R(k) > (1 - o(1)) \frac{k\sqrt{2}}{e} 2^{k/2}$ \square

Definition 5.6. Define K_k^j as the complete j uniform hypergraph on n vertices with k vertices

Definition 5.7. Define $R_j(k) = \min n$ such that any red/blue coloring of $\binom{[n]}{j}$ has a monochromatic K_k^j

Theorem 5.8. $R_r(k, l) \leq R_{r-1}(R_r(k-1, l), R_r(k, l-1))$

Proof. Let $N = R_{r-1}(R_r(k-1, l), R_r(k, l-1)) + 1$ and fix v . There are $N+1$ other vertices, Y . Each edge containing v includes an $r-1$ edge in Y . Let it inherit the color of the r edges.

(Case 1) We have $R_r(k-1, l)$ vertices in Y such that all $r-1$ subsets are red. (Case 1A) \square

Let $C(k) = \min n$ such that $\forall X \subseteq \mathbb{R}^2$ such that $|X| = n$ and X has a subset S where $|S| = k$ and S is in convex position.

Then $C(1) = 1, C(2) = 2, C(3) = 3, C(4) = 5, C(5) = 9, \dots$

Theorem 5.9. $C(k) \leq R_4(5, k)$

Lemma: If $S \subseteq \mathbb{R}^2$ is k -points in general position such that any 4 of them are in convex position, then they all are. (Easy to see by triangulation)

Given a set F of four points color F red if not in convex position and blue otherwise.

Note: K_k^n is a complete k -uniform hypergraph on n vertices.

A red K_4^5 is impossible as $C(4) \leq 5$. Since we can find a blue K_4^k and we win by lemma.

Color 3-tuples according to whether "sorted slopes" are increasing or decreasing. If $n \geq R_3(k, k)$, I can find k vertices all of whose 3 tuples are caps or all 3 tuples are cups.

Definition 5.10. Let $CC(k, l) = \min n$ such that any n pts in general position, no two have same x coordinate, have a k -cup or an l -cap.

Theorem 5.11. Erdos Szekeres: $CC(k, l) = \binom{k+l-4}{k-2} + 1$

Proof. I have k -cup or a l -cup. We'll show it by induction on $k+l$. Assume no l -cap. I do have a $(k-1)$ -cup or l cup by induction. If I delete the last point of each $(k-1)$ -cup. Then only $\binom{k+1-5}{k-3}$ points remain. So I deleted $\binom{k_l-4}{k-2} + 1 - \binom{k+l-5}{k-2} + 1$. \square

Theorem 5.12. For all positive integers k and r , there exists N such that any r -coloring of the numbers $1, 2, \dots, N$ has a monochromatic k -term arithmetic progression.

Theorem 5.13. If \mathbb{N} is partitioned into 2 sets, one contains arbitrary long arithmetic progression.

Statement 1: $\forall k, \exists N$ such that any 2 coloring of $[N]$ has a monochromatic k -term arithmetic progression. If such a statement is false for k_0 , then for all n there is a coloring of $[n]$ with no k_0 arithmetic progression. With Konig's Lemma there exists a coloring of \mathbb{N} with no k_0 -term arithmetic progression.

Statement 1 implies

Statement 2 $\forall r, \forall k, \exists W(k, r)$ such that any r -coloring of $[N]$ for $N \geq W(k, r)$ admits a k -term monochromatic A.P.

Some values of $W(k, r)$ are $W(k; 1) = k$, $W(2, r) = r + 1$, $W(2, 2) = 3$ and $W(3, 2) = 9$.

Theorem 5.14. $W(3, 2) \leq 325$

Note: The technique used here can be used for the general case.

Proof. Consider 65 blocks of 5 spots each. Within the first 33 blocks, there must be 2 blocks of the same coloring. Let the blocks be $b_1, b_2 \in [33]$. Of the first block consider the first 3 spots then if it's same color then we're done, WLOG for $a_1, a_2 \in [3]$ with $a_1 < a_2$ say $5b_1 + a_1, 5b_1 + a_2$ be red. Let $a_3 = 2a_2 - a_1 \in [5]$. If $7b_1 + a_3$ is red then we're done as $7b_1 + a_1, 7b_1 + a_2, 7b_1 + a_3$ is a mono A.P. So say $7b_1 + a_3$ is blue.

Since b_2 is the same coloring then let $b_3 = 2b_2 - b_1 \in [65]$. If $7b_3 + a_3$ is red then we have $7b_1 + a_1, 7b_2 + a_2, 7b_3 + a_3$. Otherwise if blue we have $7b_1 + a_3, 7b_2 + a_3, 7b_3 + a_3$.

Thus we're done and $W(3, 2) \geq 65 \cdot 5 = 325$ \square

Definition 5.15. $WF(k, l, r) = \text{minimum } N \text{ such that any } r\text{-coloring of } [N] \text{ admits } l \text{ color focused } k\text{-term A.P or a } k+1 \text{ term A.P.}$

Theorem 5.16.

$$WF(2, 2, r) \leq (2r^{2r+1} + 1)(2r + 1)$$

$$WF(2, 3, r) \leq (2r^{2r^{2r+1}+1} + 1)(2r^{2r+1} + 1)(2r + 1)$$

Definition 5.17. Hales-Jewett: $\forall r, \forall n, \exists d$ such that in any r -coloring of $[n]^d$ hypercube, there is a monochromatic line.

Definition 5.18. A combinatorial line is a set of points represented by a string in $([n] \cup \{x\})^d \setminus [n]^d$. The points of the line are obtained by substituting $x = 1, 2, \dots, n$.

Definition 5.19. A geometric line $([n] \cup \{x, \bar{x}\})^d \setminus [n]^d$ obtained by substituting in $x = 1, \dots, n$ and $\bar{x} = n - x + 1$.

Given an A.P-free coloring of $[N]$ want to give a line free coloring of $[N]^d$. Define $\phi : [n]^d \rightarrow (n-1)d$ by $\phi(a_0, a_1, \dots, a_{d-1}) = a_0 + a_1 + \dots + a_{d-1}$. Then we have

$$HJ(2, r) \leq d \iff 2^d < r \iff HJ(2, r) \leq \log_2 r$$

$HJ^c(2, r) = r$ as if we take any of $(0, \dots, 0), (1, 0, \dots, 0), \dots$ there are $d+1 > r \implies$ a monochromatic combinatorial line.

For $HJ(3, 2)$, take $p \in [3]^d$

Additive Combinatorics

Definition 5.20.

$$A + A = \{a + a' | a, a' \in A\}$$

$$A \cdot A = \{a \cdot a' | a, a' \in A\}$$

Theorem 5.21. $\max(|A + A|, |A \cdot A|) \geq |A|^{1+\epsilon}$

Suppose we have a set A , $X = A + A, Y = A \cdot A$. Let $\mathcal{P} = X \times Y = (A + A) \times (A \cdot A)$. Let $\mathcal{L} = \{\{y | y = a(x - a')\} | a, a' \in A\}$. Then $|\mathcal{L}| = |A|^2$.

Define $i(\mathcal{L}, \mathcal{P})$ to be the number of incidences between the points and lines in \mathcal{P} and \mathcal{L} . For any line containing $a, a' \in A$, the equation is $y = a(x - a')$. For a point $p = (a' + a'', a \cdot a'')$ we have $a' + a'' \in A + A$ and $a \cdot a'' \in A \cdot A$.

$$i(\mathcal{L}, \mathcal{P}) \geq |\mathcal{L}| \cdot |A| = |A|^3$$

Then

$$i(\mathcal{L}, \mathcal{P}) = O(|\mathcal{L}|^{2/3}|\mathcal{P}|^{2/3} + |\mathcal{L}| + |\mathcal{P}|).$$

So $|A|^3 \leq i(\mathcal{L}, \mathcal{P}) \leq C(|A|^{4/3}(|A+A| \cdot |A \cdot A|^{2/3}))$ as $|\mathcal{L}| + |\mathcal{P}| = O(|\mathcal{L}|^{2/3}|\mathcal{P}|^{2/3})$. We also have $|A|^2 \leq |\mathcal{P}| \leq |A|^4$ and $|\mathcal{P}|^{1/2} \leq |\mathcal{L}| \leq |\mathcal{P}|$. So $C|A|^{5/2} \leq |A \cdot A||A+A|$ and consequently

$$\max(|A+A|, |A \cdot A|) \geq \epsilon|A|^{5/4}$$

Planar Graphs

Theorem 5.22. *Euler's Formula for Planar Graphs:*

$$|V| - |E| + |F| = 2$$

Theorem 5.23. *Suppose G is a connected planar graph with $m \geq 3$. Then*

$$m \leq 3n - 6$$

Proof. Consider the bipartite graph of $E(G)$ and $|F(G)|$. For each edge there is at most 2 faces and each face is closed by at least 3 edges. So

$$2|E| \leq \sum \deg(e) = \sum \deg(f) \geq |F| \cdot 3$$

$$\text{So } n - m + f = 2 \implies n - m + \frac{2}{3}m \geq 2 \implies n - \frac{1}{3}m \geq 2$$

□

Definition 5.24. *Let $Cr(G)$ is the minimum number of crossing in any drawing.*

Given G , if $e(G) \geq 3n$, G is not planar $Cr(G) \geq m - 3n$ since at least $m - 3n$ edges must be removed to make G planar.

Consider G with G_p = graph where each vertex stays with probability p . Then $\mathbb{E}(np) = pn$ and $\mathbb{E}(mp) = p^2m$. Then

$$p^4Cr(G) \geq \mathbb{E}(Cr(G_p)) \geq \mathbb{E}(m_p - 3n_p) = \mathbb{E}(m_p) - 3\mathbb{E}(n_p) \geq p^2m - 3pn$$

So $Cr(G) \geq \frac{m}{p^2} - \frac{3n}{p^3}$ and is maximized when $p = \frac{4n}{m}$ only when $4n \leq m$. Then $\frac{m}{p^2} - \frac{3n}{p^3} = \frac{m^3}{64n^2}$.

Theorem 5.25. *For any collection \mathcal{L} of lines in \mathbb{R}^3 , there are at most $O(|\mathcal{L}|^{3/2})$ joints.*

We just have to show the following lemma to imply the theorem.

Lemma 5.26. *In any collection of lines with $|J|$ joints, there exist some line in $\leq 3|J|^{1/3}$ joints.*

Lemma 5.26 \implies theorem 5.25 as we define $J(L) = \text{most joints in } |L| \text{ lines}$.

$$J(L) \leq J(L-1) + 3J^{1/3} \leq J(L-2) + 3(J-1)^{1/3} + 3J^{1/3} \leq \dots$$

$$\text{So } J \leq 3J^{1/3}L \iff J^{2/3} \leq 3L \iff J \leq \sqrt{27}L^{3/2}$$

Given an arbitrary field, $\text{Poly}_D(\mathbb{F}^n)$ and $S = \{a_1, \dots, a_k\}$ for $a_i \in \mathbb{F}^n$. We want to find a nonzero polynomial that vanishes at J .

Let $T : \text{Poly}_D(\mathbb{F}^n) \rightarrow \mathbb{F}^k$ defined as $T(p) = \begin{bmatrix} p(a_1) \\ \vdots \\ p(a_k) \end{bmatrix}$

By rank nullity theorem,

$$\dim(Im(T)) \leq k \implies \dim(\ker(T)) \geq \dim(\text{Poly}_D(\mathbb{F}^n)) - k$$

If $\dim(\text{Poly}_D(\mathbb{F}^n)) > k$, $\exists p \in \text{Poly}_D(\mathbb{F}^n)$ vanishes at S , $|S| = k$.

Let

$$\mathcal{D} = \{x_1^{d_1}, x_2^{d_2}, \dots, x_n^{d_n} \mid \sum d_i \leq D\}$$

This is a basis for $\text{Poly}_D(\mathbb{F}^n)$. By stars and bars we have $|\mathcal{D}| = \binom{D+n}{n} \geq \frac{D^n}{n!} > k$. We need $\frac{D^3}{3!} > J$ so $D > 3J^{1/3}$.

AFSOC each line has more than $D > 3J^{1/3}$ joints.

If $p \in \text{Poly}_D(\mathbb{F})$, $\forall a \in \mathbb{F}, \exists c \in \mathbb{F}$ such that $p(x) = (x - a)q(x) + c$. If a is a root, $p(x) = (x - a)q(x)$.

A line is a function $\gamma(t) = a + bt$ for $a, b \in \mathbb{F}^n$ then $q(t) := p(\gamma(t))$ is a polynomial in $\text{Poly}_D(\mathbb{F})$. $\deg(q)$ has to have at most the degree of p so $\deg(q) < D$. By our assumption the line has more than D joints so $\deg(q) > D$. The only way $q(t)$ can have more than D roots is if q is the zero polynomial. So we can conclude our polynomial p must be identically 0 on the union of all lines in \mathcal{L}

Each joint is the intersection of 3 lines and p is zero on all lines. So the direction derivative along each of the lines is 0 and as they are linearly independent we have $\nabla p = 0$ at every joint. Consider $p_1 = \frac{\partial p}{\partial x}$, $\deg(p_1) \leq D - 1$ and p_1 vanishes at every joint in J so we contradict the minimality of D so the assumption that all lines have $> D$ joints is false.

Lemma 5.27. If $P(x_1, \dots, x_n)$ is a non-zero polynomial over \mathbb{F}_q , with total degree $D \leq q - 1$ then $P(x)$ cannot be zero for all $x \in \mathbb{F}_q^n$.

Proof. We can write

$$P(x) = \sum_{k=0}^D q_k(x_1, \dots, x_{n-1}) x_n^k$$

We'll show the statement by induction on n .

(**Base Case** $n = 1$) AFSOC $P(x_1)$ is nonzero and vanishes on all of \mathbb{F}_q . Since $\deg(P) \leq q - 1$ but P has q distinct roots, P must be the zero polynomial. This is a contradiction.

(**Inductive Step**) AFSOC $P(x) = 0$ on all $x \in \mathbb{F}_q^n$. We write

$$P(x) = \sum_{k=0}^D q_k(x_1, \dots, x_{n-1}) x_n^k$$

Since $P(x)$ is a nonzero polynomial, there must exist at least one k for which $q_k(x_1, \dots, x_{n-1})$ is a non-zero polynomial.

Fix the first $n - 1$ variables. Let (a_1, \dots, a_{n-1}) be an arbitrary point in \mathbb{F}_q^{n-1} . Define

$$Q(t) := P(a_1, \dots, a_{n-1}, t)$$

We can express $Q(t) = \sum_{k=0}^D q_k(a_1, \dots, a_{n-1}) t^k$. For this fixed (a_1, \dots, a_{n-1}) , each $q_k(a_1, \dots, a_{n-1})$ is a constant in \mathbb{F}_q . This implies $\deg(Q) \leq D \leq q - 1$.

By assumption, $P(x) = 0$ everywhere, so $Q(t) = P(a_1, \dots, a_{n-1}, t) = 0$ for all $t \in \mathbb{F}_q$. From our base case, a single-variable polynomial of degree $\leq q - 1$ that has q roots must be the zero polynomial. This means all coefficients of $Q(t)$ must be zero.

Therefore $q_k(a_1, \dots, a_{n-1}) = 0$ for all k . Since (a_1, \dots, a_{n-1}) was arbitrarily chosen, this holds for all points in \mathbb{F}_q^{n-1} .

This means each q_k is a polynomial in $n - 1$ variables that vanishes on all of \mathbb{F}_q^{n-1} . The total degree of P is $D = \max_k(\deg(q_k) + k)$, which implies $\deg(q_k) \leq D \leq q - 1$. By our inductive hypothesis, a polynomial in $n - 1$ variables of degree $\leq q - 1$ that vanishes everywhere must be the zero polynomial.

Thus, each q_k is the zero polynomial. This implies $P(x)$ is the zero polynomial, which contradicts our initial assumption that $P(x)$ is a non-zero polynomial. \square

Theorem 5.28. If $N \subseteq \mathbb{F}_q^n$ is a set with the property that for all $x \in \mathbb{F}_q^n$, there is a line L_x such that $L_x \setminus \{x\} \subseteq N$, then $|N| \geq \epsilon_n q^n$ where $\epsilon_n > 0$ depends only on n . (The proof shows $\epsilon_n = (10n)^{-n}$).

Proof. Assume for the sake of contradiction that $|N| < \left(\frac{q}{10n}\right)^n$.

We know from the polynomial method that there exists a non-zero polynomial $p \in \text{Poly}_D(\mathbb{F}_q^n)$ that vanishes on N , with degree $D \leq 2n|N|^{1/n}$.

Using our AFSOC, we can bound this degree D :

$$D \leq 2n|N|^{1/n} < 2n \left(\frac{q}{10n}\right)^n = \frac{q}{5}$$

So, we have found a non-zero polynomial p with total degree $D < q/5$.

Now, consider any arbitrary $x \in \mathbb{F}_q^n$. By the theorem's premise, there is a line L_x through x such that $L_x \setminus \{x\} \subseteq N$. We can parametrize this line as $\gamma(t) = x + d \cdot t$ for $t \in \mathbb{F}_q$, where $d \in \mathbb{F}_q^n \setminus \{0\}$ is a direction vector. Note that $\gamma(0) = x$, and $L_x \setminus \{x\} = \{\gamma(t) \mid t \in \mathbb{F}_q \setminus \{0\}\}$.

Define a new, single-variable polynomial $R(t) := p(\gamma(t))$. The degree of $R(t)$ is at most the total degree of p , so $\deg(R) \leq D < q/5$.

Since p vanishes on N , p must vanish on $L_x \setminus \{x\}$. This means $R(t) = p(\gamma(t)) = 0$ for all $t \in \mathbb{F}_q \setminus \{0\}$. The set $\mathbb{F}_q \setminus \{0\}$ has $q - 1$ elements, so $R(t)$ has $q - 1$ distinct roots.

We have a polynomial $R(t)$ with $\deg(R) \leq D < q/5$. For any $q \geq 3$, we have $q/5 \leq q - 2$ (since $10 \leq 4q$).

Thus, $R(t)$ is a polynomial with degree strictly less than $q - 1$, but it has $q - 1$ roots. A non-zero polynomial cannot have more roots than its degree. Therefore, $R(t)$ must be the zero polynomial.

If $R(t)$ is the zero polynomial, it must be zero for all t , including $t = 0$.

$$R(0) = p(\gamma(0)) = p(x) = 0$$

Since $x \in \mathbb{F}_q^n$ was arbitrary, we have shown $p(x) = 0$ for all $x \in \mathbb{F}_q^n$. We also know $\deg(p) = D < q/5$, which implies $\deg(p) \leq q - 1$.

By Lemma 5.27, any polynomial with degree $\leq q - 1$ that vanishes on all of \mathbb{F}_q^n must be the zero polynomial. This contradicts our choice of p as a non-zero polynomial.

Therefore, our initial assumption was false, and we must have $|N| \geq (\frac{q}{10n})^n$. \square

Lemma 5.29. *If $p \in \text{poly}_{a-1}(\mathbb{F}_q^n)$ is nonzero, $|\text{zero}(P)| < q^n$.*

Maximized when $x_1^{q-1} - 1$

Theorem 5.30. *Schartz-Zippel: If nonzero $p \in \text{poly}_D(\mathbb{F}^n)$ and $S \subseteq \mathbb{F}$ a finite subset. For random $s_1, \dots, s_n \in S$*

$$\mathbb{P}_{s_1, \dots, s_n}(p(s_1, \dots, s_n) = 0) \leq \frac{D}{|S|}$$

In other words, $|\text{zero}(p) \cap S^n| \leq D|S|^{n-1}$

Proof. Let $p \in \text{poly}_D(\mathbb{F}^n)$ be nonzero. We're done if $n = 1$. Do induction on n .

$$p(x_1, \dots, x_n) = \sum_{k=0}^n q_k(x_1, \dots, x_{n-1}) x_n^k.$$

Choose k_0 to be largest such that $q_{k_0} \neq 0$. By induction

$$\mathbb{P}_{s_1, \dots, s_{n-1}}(q_{k_0}(s_1, \dots, s_{n-1}) = 0) \leq \frac{D - k_0}{|S|}$$

$$\mathbb{P}_{s_1, \dots, s_n}(p(s_1, \dots, s_n) = 0) \leq \mathbb{P}_{s_1, \dots, s_n}(q_{k_0}(s_1, \dots, s_{n-1}) = 0) + \mathbb{P}_{s_1, \dots, s_n}(p(s_1, \dots, s_n) = 0 | q_{k_0}(s_1, \dots, s_{n-1}) \neq 0)$$

Note: This is just $\mathbb{P}(B) \leq \mathbb{P}(C) + \mathbb{P}(B \mid \neg C) \cdot \mathbb{P}(\neg C)$

$q_{k_0}(s_1, \dots, s_{n-1}) \neq 0 \implies p(s_1, \dots, s_{n-1}, x_n)$ has degree k_0 .

$$\frac{\sum_{s_1, \dots, s_{n-1}} \prod_{q(s_1, \dots, s_{n-1}) \neq 0} \frac{1}{|S|^{n-1}} \mathbb{P}(p(s_1, \dots, s_n) = 0 | q_{k_0}(s_1, \dots, s_{n-1}) \neq 0)}{\sum_{s_1, \dots, s_{n-1}} \prod_{q(s_1, \dots, s_{n-1}) \neq 0} \frac{1}{|S|^{n-1}}}$$

$$\mathbb{P}_{s_1, \dots, s_n}(q_{k_0}(s_1, \dots, s_{n-1}) + \mathbb{P}_{s_1, \dots, s_n}(p(s_1, \dots, s_n) = 0 | q_{k_0}(s_1, \dots, s_{n-1}) \neq 0) \leq \frac{D - k_0}{|S|} + \frac{k_0}{|S|} = \frac{D}{|S|}$$

\square

Theorem 5.31. *Extremal Schwartz-Zippel. If p nonzero of degree d , $p = 0$ on S^n then $|S| \leq d$.*

Example 5.32.

$$p = \prod_{s \in S} (x_i - s)$$

is 0 for all $x \in S^n$. This holds for $S \times \{1\} \times \{1\} \times \dots$

Example 5.33.

$$q = \prod_{a_1 \in S_1} (x_1 - a_1) \prod_{a_2 \in S_2} (x_2 - a_2) \prod_{a_3 \in S_3} (x_3 - a_3)$$

Say S_1, S_2, S_3 has size 4, 3, 2, respectively. If in a $5 \times 4 \times 3$ box then q is definitely not zero polynomial.

Theorem 5.34. *Combinatorial Nullstellensatz:* Suppose p is a nonzero polynomial in $\text{Poly}_d(\mathbb{F}^n)$ of degree d and the monomial $x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ for $j_1 + \cdots + j_n = d$ has nonzero coefficients then $\forall S_1, \dots, S_n \subseteq \mathbb{F}$ with $|S_i| \geq j_i + 1$ for all i , $p(x) \neq 0$ for some $x \in S_1 \times S_2 \times \cdots \times S_n$.

Proof. We'll show it by induction on d .

Suppose $f \equiv 0$ on $S_1 \times S_2 \times \cdots \times S_n$ where $S_i \subseteq \mathbb{F}$.

WLOG $j_i \geq 1, \forall s \in S_1$ we have $f(x) = (x_1 - s)q_s(x) + r(x)$ then $\deg(q_s) = d-1$, moreover coefficients of $x_1^{j_1-1}x_2^{j_2}\cdots x_n^{j_n}$. For any $s \in S_1, s_2 \in S_2, \dots$ we have $f(s, s_2, \dots, s_n) = 0 \implies r_s(s, s_2, \dots, s_n) = 0 \forall s_2, \dots, s_n \implies r_s(s_1, \dots, s_n) = 0 \forall s_1 \in S_1, s_2 \in S_2, \dots$

By our assumption we have $0 = f(s_1, \dots, s_n) = (s_1 - s)q_s(s_1, \dots, s_n)$ for all $s, s_1 \in S_1, \dots, s_n \in S_n$. For $s \neq s_1$ we learn $q_s(s_1, \dots, s_n) = 0$. Since q_s is zero on $S_1 \setminus \{s\} \times S_2 \times \cdots \times S_n \implies$ we must have $|S_1 \setminus \{s\}| \leq j_1 - 1$ or for some i , $|S_i| \leq j_i$. by induction hypothesis. \square

Example 5.35. $\mathcal{X}'(G)$ list chromatic number of G . We want to find $\mathcal{X}'(C_n)$.

Consider an assignment c_i to each vertex $i, c_i \in \mathbb{N}$. $f(c_1, \dots, c_n) = (c_2 - c_1)(c_3 - c_2)\cdots(c_n - c_{n-1})(c_1 - c_n)$. The leading term of f is $2c_1c_2\cdots c_n$. For all sets $S_1, \dots, S_n \subseteq \mathbb{N}$, chromatic number implies $\exists c_1 \in S_1, c_2 \in S_2, \dots, c_n \in S_n$ such that $f(c_1, \dots, c_n) \neq 0$.

Example 5.36. Cauchy Davenport: Let p prime, $A, B \subseteq \mathbb{Z}_p$

$$|A + B| \geq \min(p, |A| + |B| - 1)$$

Proof. Case 1: If $|A| + |B| - 1 \geq p$

Consider $x \in \mathbb{Z}_p$ then $|x - A| = |A|$ so $|A - x| + |B| \geq p + 1$ and $\exists y \in A_x \cap B \implies \exists a \in A, b \in B$ such that $y = b, y = x - a$ so $x = a + b$.

Case 2: If $|A| + |B| - 1 < p$

Consider any set $C \subseteq \mathbb{Z}$ of size $|C| = |A| + |B| - 2$. We want to show $\exists x \in A + B, x \notin C$.

Define

$$f(a, b) = \prod_{c \in C} (a + b - c)$$

We have $\deg(f) = |C| = |A| + |B| - 2$, consider the monomial $a^{|A|-1}b^{|B|-1}$ then the coefficient is $\binom{|A|+|B|-2}{|A|-1} \neq 0$ in \mathbb{Z}_p . So for $|A|, |B|$ we have a choice of a, b such that $a + b \notin C$. \square

Definition 5.37. Finite Kakeya: In $\mathbb{F}_a^n, \forall a, \exists b$ such that $\{at + b | t \in \mathbb{F}_a\} \subseteq K$. Then K is a kakeya set.

Theorem 5.38. Chevalley–Warning theorem:

Let $a = p^l$ for $f_1, \dots, f_k \in \mathbb{F}_a[x_1, \dots, x_n]$.

If $\sum_i \deg(f_i) < n$ then the number of common zeros is a multiple of p . In particular: if there's 1 common zero then there is more.

Example 5.39. Given any n numbers a_1, \dots, a_n there is a nonempty subset that sums to 0 (mod n).

Proof. Let $S_0 = \{\}, S_1 = \{a_1\}, \dots, S_n = \{a_1, \dots, a_n\}$ then there exists i, j such that $S_i = S_j$ so \square

Theorem 5.40. Erdos-Ginzburg-Ziv Theorem:

How large a collection of numbers do I require to ensure that some n -subset sum to a multiple of n ?
 $2n - 1$ is enough

Proof. (Main Case) $n = p$ is a prime

Given numbers a_1, \dots, a_{2p-1} , we'll give two polynomials in $2p - 1$ variables x_1, \dots, x_{p-1}

We want a polynomial such that x_i behaves like indicators for $a_i \in S$. So $x_i^{p-1} \equiv 1 \pmod{p}$ by FLT.

$$f(x_1, \dots, x_{2p-1}) = \sum_{x_i} x_i^{p-1} = \#\{i | x_i \neq 0\}$$

$$g(x_1, \dots, x_{2p-1}) = \sum_{x_i} a_i x_i^{p-1} = \sum_{x_i \neq 0} a_i$$

We have $2p - 2 < 2p - 1$ and the trivial solution exist so a non-trivial solution exist by Chevalley-Warning

(General Case) Induction on n , a_1, \dots, a_{2n-1}

If n not prime, let p be a prime factor of n , $m = \frac{n}{p}$. Find a set I_i , $|I_i| = p$ such that $\sum_{j \in I_i} a_j \equiv 0 \pmod{p}$ for $i \in [2m - 1]$

Say $\sum_{i \in I_j} a_i = b_i \equiv 0 \pmod{p}$

Let $c_i = \frac{b_i}{p}$, we can find $c_{i_1}, c_{i_2}, \dots, c_{i_m}$ such that

$$\sum_{j=1}^m c_{i_j} = \sum_{j=1}^m \sum_{t \in I_{i_j}} t = \left(\sum_{j=1}^m c_{i_j} \right) p \equiv 0$$

□

Theorem 5.41. There exist an order of at least d^2 2-distance set in \mathbb{R}^d ?

Proof. Suppose $S = \{p_1, \dots, p_m\}$ has just 2 distances α and β . Consider the polynomial, $f \in \mathbb{R}[x_1, \dots, x_d]$ defined as

$$f_i(X) = (||X - p_i||^2 - \alpha^2)(||X - p_i||^2 - \beta^2)$$

$$f_i(X) = \begin{cases} \alpha^2 \beta^2 & \text{if } X = p_i \\ 0 & \text{otherwise} \end{cases}$$

Claim: f_i 's are independent.

Suppose $\alpha_1 f_1 + \dots + \alpha_m f_m = 0, \forall i$, plug in p_i gives $\alpha_i \alpha^2 \beta^2 = 0 \implies \alpha_i = 0$

Claim: $x_i^{d_i} x_j^{d_j}$ with $d_1 + d_2 \leq 4$ covers all possible terms. So there is at most $O(d^2)$ possible choices. □

Example 5.42. Eventown where each club has even size and even intersection, $\geq 2^{\lceil n+\frac{1}{2} \rceil}$

Example 5.43. Oddtown where each club has odd size and even intersection

$$\text{Let } v_i = \text{indidence vector of club } i \text{ in } \mathbb{F}_2. v_i \cdot v_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{otherwise} \end{cases}$$

Suppose $\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0}$

\vec{v}_i on both sides so $\alpha_i = 0$

Theorem 5.44. Every Polygon has a triangulation

Proof. Choose a convex vertex of the polygon (a vertex that is a vertex of the convex hull) with neighbors q, r . If $\overline{qr} \subset P^\circ$ then we're done. Otherwise we can move the point. □

Definition 5.45. We say polygon $P \sim Q$ by scissor congruency if we can cut up P and reassembled to be Q .

Lemma 5.46. $(\text{Any rectangle}) \sim (\text{Any Unit Size Rectangle})$
 $(\text{Any triangle}) \sim (\text{Unit Side Rectangle})$
 $(\text{Triangle}) \sim (\text{2 Right Triangle})$
 $(\text{Right Triangle}) \sim (\text{Rectangle})$

Remark 5.47. From this lemma we can conclude any polygon is congruent to a rectangle of $1 \times d$

Theorem 5.48. Are equal-area polyhedra necessarily plane-dissection equivalent? This is not true.

Example 5.49. Unit cube and volume 1 reg-tetrahedron are not dissection equivalent.

Proof. Dihedral angle is an irrational multiple of π

A list of vectors v_1, \dots is independent if for every finite sum,

$$\sum_{j=1}^k \alpha_{ij} v_{ij} = 0 \implies \alpha_{ij} = 0 \forall j$$

$\text{Span}(\mathcal{L})$ is the set of vectors representable as finite linear combinations. \mathcal{L} is a basis for V if \mathcal{L} is independent and $\text{Span}(\mathcal{L}) = V$

Lemma 5.50. Zorn's Lemma: P is a poset in which every chain has an upper bound then P has a maximal element.

Doset is a set of independent sequences, ordered by inclusion.

$$\mathcal{L}_1 \subseteq \mathcal{L}_2 \cdots \subseteq$$

Then $\cup \mathcal{L}_i$ is still independent. By Zorn's lemma and Doset we have every vector space has a (possibly infinite) basis.

Define α to be dihedral angle of tetrahedron, we'll use that $\frac{\alpha}{\pi}$ is irrational.

In general, we can define a linear transformation $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(\pi) = 0$ and $f(\alpha) = 1$.

Since α and π are independent over \mathbb{Q} extended to a basis $\alpha, \pi, v_3, v_4, \dots$

Define f using this basis by defining $f(\alpha) = 1, f(\pi) = 0$

If a plane goes through an angle then $l_e = l_{e_1} = l_{e_2}$ and $\theta_e = \theta_{e_1} + \theta_{e_2}$.

If a plane goes through an edge then $l_{e_1} + l_{e_2} = l_e$ and $\theta_e = \theta_{e_1} = \theta_{e_2}$

If a plane goes through another plane and creates a new edge then $l_{e_1} = l_{e_2}$ and $\theta_{e_1} + \theta_{e_2} = \pi$

We can assign a real number to each polytope by

$$\sum_{e \in P} l_e \cdot f(\theta_e)$$

In the first case we would have

$$R(P) = \sum_{e \in P} l_e f(\theta_e)$$

If P is a cube then $R(P) = 0$ as each dihedral angle is 90° so it's a rational multiple of π . However if P is a tetrahedron has irrational multiple of π so $R(P) \neq 0$. \square

Linear Algebra

Definition 5.51. *Adjacency Matrices: On the vertex set $[n]$*

$$A = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$A_{i,j} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

If $f : V \rightarrow \mathbb{R}$ then $Af = g$ and if $Af = \lambda f$ then it's an eigen function.

Remark 5.52. We'll denote the $n \times n$ matrix of all ones as J_n .

Theorem 5.53. J_n 's eigenvalues are $\lambda_1 = n$ with multiplicity 1 and $\lambda_2 = 0$ with multiplicity $n - 1$.

Proof. To show $\lambda = 0$ has multiplicity $n - 1$ the associated eigenvectors are

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ \vdots \end{bmatrix}, \dots,$$

□

Theorem 5.54. $K_n = J_n - I_n$ has eigenvalue $n - 1$ with multiplicity 1 and -1 with multiplicity $n - 1$.

Remark 5.55. For any regular graph with degree d , d is an eigenvalue value with multiplicity 1.

Lemma 5.56. For any adjacency matrix A , A^2 has the property that $A_{i,j}^2 = \#\text{walks of length 2 from } i \rightarrow j$. This generalizes easily to the general case for A^k

Proof. This is easily shown from the matrix multiplication

$$A_{i,j} = \sum_{k \in [n]} A(i,k)A(k,j)$$

□

Lemma 5.57. For a d -regular graph with diameter 2 graph then the maximum number of vertices is $1 + d + d(d - 1) = d^2 + 1$ vertices.

To achieve this bound, I require $\text{girth}(G) \geq 5$

Note: Girth is the length of a shortest cycle.

Another way to achieve this bound is the Peterson Graph

Lemma 5.58. Any Moore graph (regular graph whose girth is at least twice its diameter) has $A^2 + A - (d-1)I = J$.

Proof. We know $\lambda = d$ an eigenvalue for $f \equiv 1$. By spectral theorem, A has an orthogonal basis of real eigenvectors. Since $f_j \perp f_1$ for $j \neq 1$ then $J \cdot f_j = \vec{0}$ as J is the matrix of all ones. So

$$A^2 f_j + Af_j - (d-1)If_j = 0 \iff \lambda_j^2 + \lambda_j - (d-1) = 0$$

We can conclude $\lambda = \frac{-1 \pm \sqrt{4d-3}}{2}$

We need $1 + m_2 + m_3 = n$ and $d + m_2\lambda_2 + m_3\lambda_3 = 0$

So $2d - (m_2 + m_3) + (m_2 - m_3)\sqrt{4d-3} = 0$

(Case 1) If $\sqrt{4d-3} \notin \mathbb{Q}$ then $m_2 = m_3$ and $2d = d^2 \implies d = 2$

(Case 2) If $\sqrt{4d-3} = s \in \mathbb{Z}$ then $d = \frac{s^2+3}{4}$ then

$$2d - d^2 + (m_2 - m_3)s = 0 \iff 8\left(\frac{s^2+3}{4}\right) - (s^2+3)^2 + 16(m_2 - m_3)s = 0$$

Expanding out we have $as^4 + bs^3 + cs^2 + ds + 15 = 0$ then $s|15 \implies s = 1, 3, 5, 15 \implies d = 1, 3, 7, 57$. \square

We'll be covering graph where for any 2 vertices u, v there is exactly one common neighbor of u, v

Theorem 5.59. "There is a politician": $\exists v_0$ such that $\forall u, v_0 \sim u$

Note: This doesn't hold for infinite vertices by $H_0 = 5$ -cycle and H_{i+1} is H_i with independent path of length 2 added between parts that don't have a common neighbor in H_i .

Step 1 A counterexample must be regular

Step 1A $u \not\sim v \implies \deg(v) \geq \deg(u)$. By symmetry $\deg(v) = \deg(u)$. This is from w_1 being the common neighbor of u, v and w_2 being the common neighbor of w_1, u and z_1 being common neighbor of w_1 and v .

Step 1B Let $\deg(u) = d, \forall v \neq w_i$ we get $\deg = d$ for all w_2, \dots, w_d , we get $\deg(w_i) = \deg(v) = d$. All but w_1 are known to be degree d . Since w_1 not a politician then w_1 must be the politician.

Going back to the graph with diameter 2 graph then $n = 1 + d(d-1) = d^2 - d + 1$. There are exactly 1 path of length 2 between u, v $\deg(u) = d \implies A^2$ is d along diagonal and 1 everywhere else. $A^2 = J + (d-1)I$. J has e.v. n with multiplicity 1 and 0 with multiplicity $n-1$. So A^2 has e.v. $n+d-1$ with multiplicity 1 and $d-1$ with multiplicity $n-1$.

A has eigenvalues d with multiplicity 1, $\sqrt{d-1}$ with multiplicity s and $-\sqrt{d-1}$ with multiplicity t . Also $s+t = n-1$. The trace of A is 0 so $d + \sqrt{d-1}s - \sqrt{d-1}t = 0 \implies d + (s-t)\sqrt{d-1} = 0$. So $\sqrt{d-1} \in \mathbb{Q} \implies h := \sqrt{d-1} \in \mathbb{N}$.

We have $d = \sqrt{d-1}^2 + 1 = h^2 + 1$. So $d + h(s-t) = 0 \implies h^2 + 1 = h(t-s) \implies h = 1 \implies d = 2$

Theorem 5.60. Oddtown: Clubs have odd size and intersections are even. The clubs are less than number of people.

Lemma 5.61 (Fisher's Inequality). Let $\mathcal{F} = \{A_1, \dots, A_m\}$ be a family of m distinct subsets of a universe X where $|X| = n$. Suppose there exists a constant k such that $|A_i \cap A_j| = k$ for all $i \neq j$. Furthermore, assume that $|A_i| > k$ for all i . Then:

$$m \leq n$$

Proof. Let $v_1, \dots, v_m \in \mathbb{R}^n$ be the incidence vectors of the sets A_1, \dots, A_m . That is, the x -th component of vector v_i is 1 if $x \in A_i$ and 0 otherwise.

We aim to show that these vectors are linearly independent. Consider a linear combination of these

vectors equal to the zero vector, with coefficients $\alpha_1, \dots, \alpha_m \in \mathbb{R}$:

$$\sum_{i=1}^m \alpha_i v_i = 0 \quad (1)$$

We take the squared Euclidean norm (the dot product with itself) of both sides:

$$\left\langle \sum_{i=1}^m \alpha_i v_i, \sum_{j=1}^m \alpha_j v_j \right\rangle = 0$$

Expanding using the linearity of the inner product:

$$\sum_{i=1}^m \alpha_i^2 \langle v_i, v_i \rangle + \sum_{i \neq j} \alpha_i \alpha_j \langle v_i, v_j \rangle = 0$$

We observe the following properties of the incidence vectors:

- $\langle v_i, v_i \rangle = |A_i|$
- $\langle v_i, v_j \rangle = |A_i \cap A_j| = k$ (for $i \neq j$)

Substituting these values into the equation:

$$\sum_{i=1}^m \alpha_i^2 |A_i| + k \sum_{i \neq j} \alpha_i \alpha_j = 0$$

To simplify the second term, we use the identity $(\sum \alpha_i)^2 = \sum \alpha_i^2 + \sum_{i \neq j} \alpha_i \alpha_j$. Rearranging this gives $\sum_{i \neq j} \alpha_i \alpha_j = (\sum \alpha_i)^2 - \sum \alpha_i^2$. We substitute this back into our equation:

$$\begin{aligned} \sum_{i=1}^m \alpha_i^2 |A_i| + k \left[\left(\sum_{i=1}^m \alpha_i \right)^2 - \sum_{i=1}^m \alpha_i^2 \right] &= 0 \\ \sum_{i=1}^m \alpha_i^2 (|A_i| - k) + k \left(\sum_{i=1}^m \alpha_i \right)^2 &= 0 \end{aligned}$$

Since we assumed $|A_i| > k$, we have $|A_i| - k > 0$. Also, squares of real numbers are non-negative (assuming $k > 0$ and observing $(\sum \alpha_i)^2 \geq 0$). Therefore, we have a sum of non-negative terms equaling zero. This implies that every individual term must be zero. Specifically:

$$\alpha_i^2 (|A_i| - k) = 0 \quad \forall i$$

Since $|A_i| - k \neq 0$, it must be that $\alpha_i = 0$ for all i .

Thus, the vectors v_1, \dots, v_m are linearly independent. Since they exist in \mathbb{R}^n , the dimension of the subspace they span cannot exceed n , implying $m \leq n$. \square

Theorem 5.62. $R(k+1) \geq \binom{k}{3} + 1$. We can group every 3 vertices then color it red

A quadratic form/homogeneous polynomial say $q(x, y) = x^2 + 2xy + 3y^2 = [x \ y] \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. There is an bijection between quadratic form and symmetric matrices.

We can write $A = P^T B P$ where B is a diagonal matrix with eigenvalues

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Definition 5.63. For a symmetric matrix A ,

1. Positive definite if $\lambda_i > 0$ for all i
2. Negative definite if $\lambda_i < 0$ for all i

Theorem 5.64. Given symmetric A , $q_A(X) = X^T AX$. Let A be real, symmetric $n \times n$. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigen values of A . We can conclude

$$\lambda_k = \max_{U \subseteq V} \min_{\substack{X \in U \\ \dim(U)=k}} \frac{X^T AX}{X^T X}$$

Remark 5.65. This statement doesn't care of the magnitude of X and only the direction. We're looking for the direction of greatest change.

Proof. We'll first show $\lambda_k \leq \max_U$. For this direction suffices to exhibit one good u . For v_1, \dots, v_n orthonormal eigen basis, v_i eigenvalues for λ_i . Let $U_k = \{v_1, \dots, v_k\}$ and let $X \in \text{Span}(U_k)$ so $X = \sum_{i=1}^k \alpha_i v_i$. WLOG $|X| = 1$ since the magnitude does not change the result. This implies $\sum \alpha_j^2 = 1$.

$$X^T AX = \left(\sum \alpha_j v_j \right)^T A \left(\sum \alpha_j v_j \right) \quad (2)$$

$$= \left(\sum \alpha_j v_j \right)^T \left(\sum \alpha_j \lambda_j v_j \right) \quad (3)$$

$$= \sum \lambda_j \alpha_j^2 \quad (4)$$

This is a weighted average of $\lambda_1, \dots, \lambda_k$ so this is at least the $\min(\lambda_1, \dots, \lambda_k) = \lambda_k$.

For the other direction $\lambda_k \geq \max_U$. Given any U_k , we want to show $\exists X \in U_k$ such that $\frac{X^T AX}{X^T X} \leq \lambda_k$. Let $W = \text{Span}(v_k, \dots, v_n)$ then $W = n - k + 1$. So there exist a vector $X \neq 0, X \in W \cap U_k$. WLOG, take $|X| = 1$. Since $X \in W, X = \sum_{j=k}^n \alpha_j v_j$ and

$$X^T AX = \sum_{j=k}^n \lambda_j \alpha_j^2 \leq \max(\lambda_k, \dots, \lambda_n) = \lambda_k$$

□

Example 5.66. Given a d -regular graph G with adjacency matrix A with $\lambda_1 \geq \dots \geq \lambda_n \geq -d$. Suppose I had a negative eigenvalue that is less than $-d$ then any vertex when applied the adjacency matrix will be the sum of the neighboring vertices, but we can't have $|\sum| > d^2$.

Given an independent S of size α . Define a vector $v = nI_S - \alpha I = (n - \alpha)I_S - \alpha I_{\bar{S}}$. Then $v \cdot I = 0$. We know

$$\min_{X \subseteq \mathbb{R}^n} \frac{X^T AX}{X^T X} = \lambda_k$$

So we know $\lambda_k \leq \frac{v^T Av}{v^T v}$.

$$v^T v = (nI_S - \alpha I)(nI_S - \alpha I) = \alpha n^2 - 2\alpha^2 n + \alpha^2 n = \alpha n(n - \alpha) \quad (5)$$

$$v^T Av = (nI_S - \alpha I)A(nI_S - \alpha I) \quad (6)$$

$$= nI_S A nI_S - 2\alpha^2 I_S A I + \alpha^2 I A I \quad (7)$$

$$= 0 - 2n\alpha I_S \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} + \alpha^2 nd \quad (8)$$

$$= -n\alpha^2 d \quad (9)$$

So we have $\frac{-n\alpha^2 d}{\alpha n(n - \alpha)} = \frac{-\alpha d}{(n - \alpha)} = \frac{d}{1 - \frac{n}{\alpha}} \geq \lambda_n$. When we solve for α

Definition 5.67. Expander Graphs

Definition 5.68. Let $G = (V, E)$ and $|V| = n$
Cheeger Constant

$$h(G) = \min_{S \subseteq V, |S| \leq \frac{n}{2}} \frac{e(S, \bar{S})}{|S|}$$

Remark 5.69. $h(G) \leq d$ and $h(G) = 0 \iff G$ is disconnected.

Definition 5.70. G bipartite on (L, R) with $|L| = |R| = n$ is a (d, α) expander if

1. every degree in L is d
2. every set S of size $\leq \frac{n}{d}$ in L has $\alpha|S|$ neighbors (in R)

Theorem 5.71. Let $d \equiv 4$, choose d edges from each vertex in L independent and 1 and only.

Claim: With constant probability, result is a $(d, \frac{d}{10})$ bipartite expander.

Proof. Let the bad events be for sets $S \subseteq L, T \subseteq R, |S| \leq \frac{n}{d}, |T| < \alpha|S|, E_{S,T} = \{N(S) \subseteq T\}$. Then $\mathbb{P}(E_{S,T}) = \left(\frac{|T|}{n}\right)^{d|S|}$

$$\mathbb{P}(\exists S, T, |S| \leq \frac{n}{d}, |T| = \alpha|S|, E_{S,T}) \leq \sum_{s=1}^{n/d} \binom{n}{s} \binom{n}{\alpha s} \left(\frac{\alpha s}{n}\right)^d s \quad (10)$$

$$\leq \sum_{s=1}^{n/d} \binom{n}{\alpha s}^2 \left(\frac{\alpha s}{n}\right)^d s \quad (11)$$

$$\leq \sum_{s=1}^{\infty} \left(\frac{en}{\alpha s}\right)^{2\alpha s} \left(\frac{\alpha s}{n}\right)^{ds} \quad (12)$$

$$= \sum_{s=1}^{\infty} \left(\frac{n}{d}\right)^{(2\alpha-d)s} e^{2\alpha s} s^{(d-2\alpha)s} \quad (13)$$

$$= \sum_{s=1}^{\infty} \left(\frac{n}{\alpha s}\right)^{2\alpha-d}s e^{2\alpha s} \quad (14)$$

$$\leq \sum_{s=1}^{\infty} 10^{(2\alpha-d)s} e^{2\alpha s} \quad (\alpha \leq d/10)$$

$$= \sum_{s=1}^{\infty} (10^{2\alpha-d} e^{2\alpha})^s < 1 \quad (15)$$

□

Lemma 5.72. If p has 1 fermat witnesses, half of the a 's relatively prime to p are fermat witnesses.

Example 5.73. Prime Algorithm

1. Randomly choose a
2. Compute $a^p \pmod p$
If $\not\equiv a$, report not prime else report maybe prime

From the previous lemma if not prime, at least half the a will show it.
We want a deterministic expander graph and on the L

Theorem 5.74. Similar setup to theorem 5.64,

$$\lambda_k = \min_{\dim(U)=k-1} \max_{X \perp U} \frac{X^T AX}{X^T X}$$

Theorem 5.75. We want to relate the spectral gap to the Cheeger Constant

Proof. $\vec{v} = nI_S - sI = (n-s)I_S - sI_{\bar{S}}$ with $s = |S|$
 G is a d -regular graph, A is an adjacency matrix, we have

$$\lambda_2 = \max_{x \perp I} \frac{x^T Ax}{x^T x} \geq \frac{\vec{v}^T A \vec{v}}{\vec{v}^T v}$$

Note: $\vec{v} \cdot I = ns - ns = 0$ and $\vec{v} \cdot \vec{v} = (n-s)^2 s + s^2(n-s) = s(n-s)n$.

Since the graph is d -regular we have $ds = 2e(S) + e(S, \bar{S})$ and $d(n-s) = 2e(\bar{S}) + e(S, \bar{S})$.

$$\begin{aligned} \vec{v}^T A \vec{v} &= \sum_{(i,j) \in E(G)} v_i v_j = 2e(S)(n-s)^2 - 2e(S, \bar{S})s(n-s) + 2e(\bar{S})s^2 \\ &= (ds - e(S, \bar{S}))(n-s)^2 - 2e(S, \bar{S})s(n-s) + (d(n-s)e(S, \bar{S}))s^2 \\ &= ds(n-s)n - e(S, \bar{S})[(n-s)^2 + 2s(n-s) + s^2] \\ &= ds(n-s)n - e(S, \bar{S})n^2 \end{aligned}$$

Then substituting back we have

$$\lambda_2 \geq \frac{ds(n-s)n - e(S, \bar{S})n^2}{s(n-s)n} = d - \frac{e(S, \bar{S})n}{s(n-s)}$$

$\forall S \subseteq V, |S| \leq \frac{n}{2}$ we have

$$d - \lambda_2 \leq \frac{2e(S, \bar{S})}{|S|} \implies \lambda_1 - \lambda_2 \leq 2h(G)$$

□

Example 5.76. Consider $A_{K_n} = J - I$ has eigenvalues $n - 1$ with multiplicity 1 and -1 with multiplicity $n - 1$.

Lemma 5.77. Lower bound on λ_2 with G is d -regular, $A = A_G$, A^2 has eigen values $\lambda_1^2, \dots, \lambda_n^2$
 $\text{Trace}(A^2) = nd = \lambda_1^2 + \dots + \lambda_n^2$

$$nd - d^2 = \sum_{k=2}^n \lambda_k^2 \leq (n-1)\lambda_*^2 \quad (\lambda_* = \max_{i \neq 1} |\lambda_i|)$$

So we have $d - o(1) = \frac{nd - d^2}{n-1} \leq \lambda_*^2 \implies \lambda_* \geq \sqrt{d} - o(1)$

Theorem 5.78. Consider simple random walk on a graph G , G has adjacency matrix A and transition matrix $P = \begin{bmatrix} \frac{1}{\deg(v_1)} & \cdots \\ \vdots & \ddots \\ & & \frac{1}{\deg(v_n)} \end{bmatrix} A$

Remark 5.79. $P(i, j) = \mathbb{P}(\text{next at } j | \text{now at } i)$

If v_1, \dots, v_n are orthonormal eigenbasis to eigenvalues for A , also true for P . Let corresponding eigenvalues be $\lambda_1 \geq \dots \geq \lambda_n$

Consider now a stochastic vector x with $\sum_{i=1}^n x_i = 1, x_i \geq 0$, the product $xP = y$ where y is a stochastic vector with the new distribution given we start at distribution x .

$$x^T = \sum \alpha_i \vec{v}_i$$

Since P is symmetric

$$xP^t = P^t(\sum \alpha_i v_i) \quad (16)$$

$$= \sum_{i=1}^n (\alpha_i \lambda_i^t v_i) i \quad (17)$$

$$= \alpha_1 v_1 + \sum_{i=2}^n \lambda_i^t \alpha_i v_i \quad (18)$$

$$\leq \lambda_*^t \left(\sum \alpha_i v_i \right) \quad (19)$$

So we can conclude $xP^t \rightarrow \frac{1}{n} I$

Theorem 5.80. A knight makes random knight moves: let τ be the time to return to the bottom left corner. What is $\mathbb{E}[X]?$

Definition 5.81. $\rho = \text{probability of returning to origin for a random walk on } \mathbb{R}$ then a random walk is recurrent if $\rho = 1$ and transient otherwise.

Remark 5.82. A random walk is recurrent on Γ iff $\mathbb{E}[\text{visits to origin}]$ is ∞

$$\mathbb{P}(\text{I visit exactly } k \text{ times}) = p^{k-1}(1-p)$$

$$\mathbb{E}(\text{visits}) = \sum_{k=1}^{\infty} kp^{k-1}(1-p) = \frac{1}{1-p}$$

For our random walk on \mathbb{R} ,

$$\mathbb{E}[\text{visit}] = \sum_{n=0}^{\infty} \mathbb{E}[I_n] = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} = \sum_{n=0}^{\infty} a_n \geq \sum_{n \geq N} \frac{1}{\sqrt{\pi n}}$$

Where $a_n \sim \frac{1}{\pi n}$ by Stirling's Formula

The expected time to return to origin is not finite.

Theorem 5.83. The probability starting from j we reach n before reaching 0 is $p_j = \frac{j}{n}$. $p_j = \frac{1}{2}p_{j-1} + \frac{1}{2}p_{j+1}$

For a random walk on \mathbb{R}^2 let $I_n = 1$ if at 0 at $2n$.

$$\Pr(I_n) = \sum_n \frac{1}{4^{2n}} \sum_k \frac{(2n)!}{n!n!(n-k)!(n-k)!} = \sum_n \frac{1}{4^{2n}} \binom{2n}{n} \sum_k \binom{n}{k}^2 = \sum_n \frac{1}{4^{2n}} \binom{2n}{n}^2$$

For a random walk on \mathbb{R}^3 let $I_n = 1$ if I return after $2n$ steps.

$$\mathbb{E}[\text{visits}] = \sum_n \frac{1}{6^{2n}} \sum_{j,k} \frac{(2n)!}{j!^2 k!^2 (n-j-k)!^2} \leq \sum_n \frac{\binom{2n}{n} \binom{n}{n/3, n/3, n/3}}{2^{2n} 3^n} \sum_n \frac{\binom{n}{j, k, n-j-k}}{3^n}$$

By Stirling's formula, $\frac{n!}{(\frac{n}{3})!^3} \approx \frac{(n/e)^n}{(\frac{n}{3e})^3}$

For a random walk on a directed graph, let

$$\pi(y) = \mathbb{E}(\text{visits a random direction from } z \text{ and makes to } y \text{ before visiting } z \text{ again})$$