

Convex Optimization Notes

Wilson Pan

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1 Jan 13

Definition 1.1. An *optimization problem* is defined as

$$\min_{x \in \mathcal{C}} f_0(x) \quad \text{where} \quad \forall i \in [m], f_i(x) \leq b_i$$

where \mathcal{C} is the constraint set and $f_0: \mathcal{C} \rightarrow \mathbb{R}$.

Definition 1.2. The *standard form* of an optimization problem is

$$\min_{x \in \mathcal{D}} f_0(x) \quad \text{subject to} \quad \forall i \in [m], f_i(x) \leq 0; \quad \forall j \in [p], h_j(x) = 0$$

where $x \in \text{dom}(f_i) \cap \text{dom}(h_j) =: \mathcal{D}$.

f_0 is termed the objective function, and x is the optimization variable.

f_i and h_j are constraint functions.

Definition 1.3. A *feasible solution* is any $x \in \mathcal{D}$ that satisfies all the constraints.

Definition 1.4. The *optimal solution* is $x^* \in \arg \min_{x \in \mathcal{D}} f_0(x)$.

Remark 1.5. If no feasible solution exists, we define $\min_{x \in \mathcal{C}} f(x) = +\infty$ and, similarly, $\max_{x \in \mathcal{C}} f(x) = -\infty$.

2 Jan 15

Definition 2.1. A set C is **convex** if $\forall x_1, x_2 \in C$ and $\forall \theta \in [0, 1]$, we have $\theta x_1 + (1 - \theta)x_2 \in C$.

Definition 2.2. A function $f: \mathcal{X} \rightarrow \mathbb{R}$ is **convex** if for all $x_1, x_2 \in \mathcal{X}$ and $\theta \in [0, 1]$,

$$f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2) \quad (*)$$

Remark 2.3. Convex functions are differentiable almost everywhere.

Theorem 2.4. If f is both convex and concave, then f is affine (i.e., linear plus constant).

Definition 2.5. f is **strictly convex** if $(*)$ holds with strict inequality $<$ whenever $x_1 \neq x_2$ and $\theta \in (0, 1)$.

Definition 2.6. A **convex optimization problem** is one where f_0 , f_i , and C are convex, and h_j are affine/linear. That is,

$$\min_{x \in C} f_0(x) \quad \text{such that} \quad f_i(x) \leq 0 \quad \forall i \in [m], \quad h_j(x) = 0 \quad \forall j \in [p]$$

Theorem 2.7. The set of feasible solutions in Definition 2.6 is convex.

Theorem 2.8. x^* is a **local minimizer** if there exists $\epsilon > 0$ such that for all y with $\|x^* - y\| \leq \epsilon$,

$$f(x^*) \leq f(y)$$

Theorem 2.9. x^* is a **local maximizer** if there exists $\epsilon > 0$ such that for all y with $\|x^* - y\| \leq \epsilon$,

$$f(x^*) \geq f(y)$$

Theorem 2.10. For any convex optimization problem, every local minimum is a global minimum.

Proof. Suppose \hat{x} is a local minimizer not equal to global minimizer x^* . Take ϵ as any witness to \hat{x} being a local minimum. Let

$$y = \frac{\epsilon}{\|\hat{x} - x^*\|} x^* + \left(1 - \frac{\epsilon}{\|\hat{x} - x^*\|}\right) \hat{x}$$

Note: $\|\hat{x} - x^*\| \leq \epsilon$, otherwise \hat{x} is not a local minimizer in that neighborhood.

$$y - \hat{x} = \frac{\epsilon(x^* - \hat{x})}{\|x^* - \hat{x}\|} \quad ; \quad \|y - \hat{x}\| = \epsilon$$

Since f is convex,

$$f(y) \leq \frac{\epsilon}{\|\hat{x} - x^*\|} f(x^*) + \left(1 - \frac{\epsilon}{\|\hat{x} - x^*\|}\right) f(\hat{x}) < f(\hat{x})$$

Thus, a contradiction. □

3 Jan 20

Definition 3.1. A **convex combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 = 1$.

Definition 3.2. An **affine combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1 + \theta_2 = 1$.

Definition 3.3. A **linear combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1, \theta_2 \in \mathbb{R}$.

Definition 3.4. A **conic combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1, \theta_2 \geq 0$.

Definition 3.5. Given a set C , the **convex hull** of C is

$$\text{conv}(C) = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in C, \theta_i \in [0, 1], \sum_{i=1}^k \theta_i = 1 \right\}$$

Remark 3.6. The following are true:

1. $C \subseteq \text{conv}(C)$
2. $\text{conv}(C)$ is convex
3. It is the smallest convex set containing C
4. If a convex set $S \supseteq C$ then $S \supseteq \text{conv}(C)$

Theorem 3.7. Any closed convex set can be written as $\overline{\text{conv}}(C)$ for some set C .

Definition 3.8. The **relative interior** of C is defined as

$$\text{relint}(C) = \{x \in C : \exists \epsilon > 0, B(x, \epsilon) \cap \text{Aff}(C) \subseteq C\}$$

Definition 3.9. C is a **cone** if $\alpha x \in C$ whenever $x \in C$ and $\alpha \geq 0$.

Definition 3.10. Given a set C , the **conic hull** of C is

$$\text{conic}(C) = \left\{ \sum_{i=1}^k \theta_i x_i : x_i \in C, \theta_i \geq 0 \right\}$$

Theorem 3.11. The conic hull of C is the smallest convex cone containing C .

Definition 3.12. The ℓ_p **norm** is defined as

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{1/p}$$

Remark 3.13. The following are true:

1. For $p \in (0, 1)$: $\|x\|_p$ is not a convex function
2. For $p \geq 1$: $\|x\|_p$ is convex
3. For $p > 1$: $\|x\|_p$ is strictly convex

Example 3.14. Examples of convex sets:

1. Hyperplane: $\{x : a^T x = b\}$
2. Halfspace: $\{x : a^T x \leq b\}$
3. Polyhedron: $\{x \in \mathbb{R}^d : Ax \leq b, Cx = d\}$
4. Polytope: a bounded polyhedron.

Theorem 3.15. A set S is **strictly convex** if for all $x_1 \neq x_2$ and $\theta \in (0, 1)$, $\theta x_1 + (1 - \theta)x_2 \in \text{int}(S)$.

Definition 3.16. The **normal cone** is defined to be

$$N_C(x) = \{g : g^T(y - x) \leq 0, \forall y \in C\}$$

Remark 3.17. If $x \in \text{int}(C)$ then $N_C(x) = \{0\}$.

Theorem 3.18. If f is differentiable, then x^* is optimal if and only if $-\nabla f(x^*) \in N_C(x^*)$.

4 Jan 22

Theorem 4.1. If f is convex and differentiable, and C is a convex set, then any optimal solution x^* to $\min_{x \in C} f(x)$ must satisfy $-\nabla f(x^*) \in N_C(x^*)$.

Theorem 4.2. The set of optimal solutions to a convex optimization problem is a convex set.

Definition 4.3. If C is convex then

1. **Translation:** $C + a = \{x : x - a \in C\}$
2. **Scaling:** $\alpha C = \{x : \frac{x}{\alpha} \in C\}$
3. **Intersection:** If $\{C_\alpha\}_{\alpha \in A}$ is a collection of convex sets, then $\bigcap_{\alpha \in A} C_\alpha$ is convex.

Theorem 4.4. The following are true:

1. If $C \subseteq \mathbb{R}^n$ is convex, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then $AC + b = \{Ax + b : x \in C\}$ is a convex set.
2. If $f(x) = Ax + b$, then $f^{-1}(C)$ is convex.
3. $C_1 + C_2 = \{x + y : x \in C_1, y \in C_2\}$ is convex.
4. If $C_1 \subseteq \mathbb{R}^m$, $C_2 \subseteq \mathbb{R}^n$ then $C_1 \times C_2 = \{(x, y) \in \mathbb{R}^{m+n} : x \in C_1, y \in C_2\}$ is convex.
5. For any $C \subseteq \mathbb{R}^n \times \mathbb{R}_{>0}^m$, define

$$P(C) = \left\{ \left(\frac{x_1}{t}, \dots, \frac{x_n}{t} \right) : (x_1, \dots, x_n, t) \in C \right\}$$

If C is convex, so are $P(C)$ and $P^{-1}(C)$.

Definition 4.5.

$$\text{epi}(f) = \{(x, t) : x \in \text{dom}(f), t \geq f(x)\}$$

Definition 4.6. First Order Definition of Convexity: If f is differentiable, then f is convex if and only if for all $x, y \in \text{dom}(f)$,

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$



Figure 1: First Order Definition of Convexity

Second Order Definition of Convexity: If f is twice differentiable, then f is convex if and only if $\nabla^2 f(x) \succeq 0$.

Definition 4.7. We say $A \succeq B$ if $A - B$ is positive semi-definite. This is equivalent to

$$a^T \nabla^2 f(x) a \geq 0 \quad \text{for all } a \in \mathbb{R}^d$$

Definition 4.8. The *subdifferential* is

$$\partial f(x) = \{g : f(y) \geq f(x) + g^T(y - x), \forall y \in C\}$$

Any such $g \in \partial f(x)$ is called a **subgradient**.

If f is differentiable at x then $\partial f(x) = \{\nabla f(x)\}$.

Theorem 4.9. f is convex if and only if ∂f is non-empty.

5 Jan 27

Definition 5.1. *Gradient Monotonicity:* If f is differentiable, then f is convex if and only if $\nabla f(x)$ is monotone. So we can conclude

$$(\nabla f(x) - \nabla f(y))^T(x - y) \geq 0 \iff f \text{ is convex.}$$

Theorem 5.2. For a $A \in \mathbb{R}^{m \times n}$ we can decompose

$$A = U \sum V^t.$$

where $U \in \mathbb{R}^{m \times k}$ with orthonormal column, $\sigma \in \mathbb{R}^{k \times k}$ is a diagonal matrix with non-negative entries, and $V \in \mathbb{R}^{n \times k}$ with orthonormal column.

Definition 5.3. (u_i, v_i, σ_i) form a s.v. triplet if $Av_i = \sigma_i u_i$ and $A^T u_i = \sigma_i v_i$ $\|u_i\| = \|v_i\| = 1$ and $\sigma_i \geq 0$

Theorem 5.4. For any $A \in \mathbb{R}^{n \times n}$ then $A^T A$ is always positive semi definite

Proof. We can write $A = V \sum U$ then $A^T A = V \sum U^T U \sum V^T = V \sum^2 V^T$ □

Definition 5.5. *Spectral Radius:* $\max_i \{|\lambda_i| : \lambda_i \in \Lambda(A)\} = \rho(A)$

Definition 5.6. The norm must satisfy the following properties:

1. $\|A\| \geq 0$
2. $\|\alpha A\| = |\alpha| \|A\|$
3. $\|A\| = 0$ if and only if $A = 0$
4. $\|A + A'\| \leq \|A\| + \|A'\|$

Definition 5.7. *Operator/Spectral Norm:*

$$\|A\|_{op} = \|A\|_2 = \max_{\|x\|=1} \|Ax\|_2.$$

Definition 5.8. *Frobenius Norm:*

$$\|A\|_F = \sqrt{\sum_{i,j} A_{ij}^2}.$$

Definition 5.9. f is L -Lipschitz if

$$|f(x) - f(y)| \leq L \|x - y\|_2 \quad \forall x, y \in \text{dom}(f).$$

If f is differentiable then f is L -Lipschitz if and only if $|\nabla f| \leq L$

Definition 5.10. Differentiable f is β -smooth if

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq \beta \|x - y\|_2 \quad \forall x, y \in \text{dom}(f).$$

If f is twice differentiable then f is β -smooth if and only if $\nabla^2 f(x) \preceq \beta I$.

Theorem 5.11. If f is twice differentiable and β -smooth then

$$f(y) \leq f(x) + \nabla f(x)^T(y - x) + \frac{\beta}{2} \|y - x\|_2^2 \quad \forall x, y \in \text{dom}(f).$$

6 Feb 3

Definition 6.1. "Descent Direction" any h for which $f(x + \eta h) \leq f(x)$.

Theorem 6.2. Assume f differentiable and β -smooth with $x_{t+1} \leftarrow x_t - \eta \nabla f(x_t)$.

1. If $\eta \leq \frac{2}{\beta}$ then $f(x_{t+1}) \leq f(x_t)$
2. If $\eta \leq \frac{1}{\beta}$ then $f(x_{t+1}) \leq f(x_t) - \frac{\eta}{2} \|\nabla f(x_t)\|_2^2$

Proof. Recall

$$f(y) \leq f(x) + \nabla f(x)^T(y - x) + \frac{\beta}{2} \|y - x\|_2^2.$$

and

$$\|\nabla f(y) - \nabla f(x)\|_2 \leq \beta \|y - x\|_2.$$

Let $y \leftarrow x + h$ then $f(x + h) \leq f(x) + \nabla f(x)^T h + \frac{\beta}{2} \|h\|_2^2$

1. If $h = -\frac{2}{\beta} \nabla f(x)$ then $\nabla f(x)^T h = -\frac{2}{\beta} \|\nabla f(x)\|_2^2$ and $\frac{\beta}{2} \|h\|_2^2 = \frac{2}{\beta} \|\nabla f(x)\|_2^2$ so $f(x + h) \leq f(x)$
2. If $h = -\frac{1}{\beta} \nabla f(x)$ then $\nabla f(x)^T h = -\frac{1}{\beta} \|\nabla f(x)\|_2^2$ and $\frac{\beta}{2} \|h\|_2^2 = \frac{1}{2\beta} \|\nabla f(x)\|_2^2$.

□

Theorem 6.3. If f is differentiable, β smooth, and x^* be any optimizer. For any $k > 0$

$$\min_{i=0, \dots, k} \|\nabla f(x_i)\|_2 \leq \sqrt{\frac{2\beta}{k} \underbrace{(f(x_0) - f(x^*))}_{\Delta_0}}.$$

Proof. AFSOC $\forall i \in \{0, \dots, k\}$, $\|\nabla f(x_i)\|_2 > \sqrt{\frac{2\beta}{k} \Delta_0}$ Then we have

$$f(x_{i+1}) < f(x_i) - \frac{1}{2\beta} \left(\frac{2\beta \Delta_0}{k} \right). \quad (\forall i \in \{0, \dots, k\})$$

Continuing down the chain we have

$$f(x_{k+1}) < f(x_0) - \Delta_0 < f(x^*).$$

Thus a contradiction.

□

Theorem 6.4. If f is β -smooth and convex, $\eta = \frac{1}{\beta}$ and x^* any minimum then

$$f(x_k) - f(x^*) \leq \frac{\beta \|x_0 - x^*\|_2^2}{2k}.$$

Proof. Observe that

$$\|x_{t+1} - x^*\|^2 = \|x_t - x^* - \eta \nabla f(x_t)\|^2 = \|x_t - x^*\|^2 - \eta^2 \|\nabla f(x_t)\|^2 - 2\eta \nabla f(x_t)^T (x_t - x^*).$$

Rearranging and by convexity we have

$$\begin{aligned} f(x_t) - f(x^*) &\leq \nabla f(x_t)^T (x_t - x^*) = \frac{1}{2\eta} \left[\underbrace{\|x_t - x^*\|^2}_{\delta_t} - \underbrace{\|x_{t+1} - x^*\|^2}_{\delta_{t+1}} + \frac{\eta}{2} \|\nabla f(x_t)\|^2 \right] \\ &\leq \frac{1}{2\eta} [\delta_t - \delta_{t+1} + 2\eta(f(x_t) - f(x_{t+1}))] \\ &= \frac{1}{2\eta} (\delta_t - \delta_{t+1}) + f(x_t) - f(x_{t+1}) \end{aligned}$$

So we have

$$f(x_{t+1}) - f(x^*) \leq \frac{\beta}{2} (\delta_t - \delta_{t+1}).$$

Adding across all terms

$$\sum_{i=0}^{k-1} f(x_{i+1}) - f(x^*) \leq \frac{\beta}{2} (\|x_0 - x^*\|^2).$$

So we have the relation

$$k(f(x_k) - f(x^*)) \leq \sum_{i=0}^{k-1} f(x_{i+1}) - f(x^*) \leq \frac{\beta}{2} \|x_0 - x^*\|^2.$$

□

Definition 6.5. f is α -strongly convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\alpha}{2}t(1-t)\|x - y\|_2^2.$$

Additionally, if f is differentiable then

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{\alpha}{2}\|y - x\|_2^2.$$

If f is twice differentiable then f is α -strongly convex if and only if $\nabla^2 f(x) \succeq \alpha I$.

Theorem 6.6. Assume f is β -smooth, α -strongly convex, $\eta = \frac{1}{\beta}$ then $\|x_k - x^*\|^2 \leq \left(1 - \frac{1}{\gamma}\right)^k \|x_0 - x^*\|^2$.

We define the conditional number as $\gamma = \frac{\beta}{\alpha}$

Proof.

$$\frac{\alpha}{2}\|x_k - x^*\|^2 \leq \frac{\beta}{2} (\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2) \leq \left(1 - \frac{\alpha}{\beta}\right) \|x_t - x^*\|^2 \leq \left(1 - \frac{1}{\gamma}\right) \|x_k - x^*\|^2.$$

□

7 Feb 5

Recall 7.1. From previous class

1. $f(x + \eta h) \approx f(x) + \eta h^T \nabla f(x)$ and when we set $h = -\nabla f(x)$ we have

$$f(x - \eta \nabla f(x)) \approx f(x) - \eta \|\nabla f(x)\|_2^2.$$

Backtracking Line Search: Pick $\gamma_1 \in (0, 1)$ and $\gamma_2 \in (0, 1)$, start with initial step size η . While $f(x - \eta \nabla f(x)) > f(x) - \gamma_1 \eta \|\nabla f(x)\|_2^2$ (Armijo condition not satisfied), set $\eta \leftarrow \gamma_2 \eta$ and retry. Once the condition is satisfied, use step size η to update $x \leftarrow x - \eta \nabla f(x)$.

Recall 7.2. g is a subgradient of f at x if

$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \text{dom}(f).$$

Lemma 7.3. The following are true:

1. $\partial f(x)$ makes sense for non convex functions too, could be empty
2. If f is convex, then for $x \in \text{RelInt}(\text{dom}(f))$ we have that $\partial f(x)$ is non-empty
3. $\partial f(x)$ is a convex
4. If f is convex and differentiable at x then $\partial f(x) = \{\nabla f(x)\}$
5. If $\partial f(x)$ is non-empty everywhere, f is convex.

Theorem 7.4. The following are true:

1. $\partial(af) = a\partial f(x)$
2. $\partial(f + g) = \partial f(x) + \partial g(x)$
3. If $g(x) = f(Ax + b)$ then $\partial g(x) = A^T \partial f(Ax + b)$

Example 7.5. For $f(x) = \max_{i=1, \dots, n} f_i(x)$ we have $\partial f(x) = \text{Conv} \left(\bigcup_{i=1, \dots, n} \partial f_i(x) \right)$

Example 7.6. For $f(x) = |x|$ we have $\partial f(x) = \text{sign}(x) = \max -x, x$ so $\partial f(0) = [-1, 1]$.

Example 7.7. Let C be a convex set and $I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$ is convex. Additionally, for $x \in C$, $\partial I_C(x) = N_C(x) = \{g : g^T(y - x) \leq 0 \forall y \in C\}$

Proof. For $x \in C$, $I_C(y) \geq I_C(x) + g^T(y - x)$ for all $y \in C$ then we have

$$0 \geq 0 + g^T(y - x).$$

□

Definition 7.8. Subgradient Method: $x_{t+1} \leftarrow x_t - \eta g_x$ for some $g_x \in \partial f(x)$

1. Subgradients are in general not descent directions
2. The min norm subgradient is a descent direction

Example 7.9. $f(x) = \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ is convex, then for $x \neq 0$ we have $\partial f(x) = \frac{x}{\|x\|_2}$.

$$\partial f(0) = \{g : \|g\|_2 \leq 1\}.$$

8 Feb 10

Definition 8.1. For the subgradient method for $\min_{x \in \mathbb{R}^n} f(x)$ and f convex where

$$X_{t+1} = X_t - \eta_t g_t \text{ where } g_t \in \partial f(X_t).$$

We define the best iterate as

$$X_T^{(\text{best})} = \arg \min_{i=0, \dots, T} f(X_i).$$

Theorem 8.2. Assume f is G -lipschitz and convex. Let $\|x_0 - x^*\| \leq R$ then pick

$$\eta_t = \frac{R}{G\sqrt{T}} \text{ guarantees that } f\left(x_T^{(\text{best})}\right) - f(x^*) \leq \frac{GR}{\sqrt{T}}.$$

Theorem 8.3. A convex function f is G -lipschitz iff

$$\|g_x\| \leq G \quad \forall x \in \text{dom}(f) \text{ and } \forall g_x \in \partial f(x).$$

Theorem 8.4. For nonconvex, differentiable f , f being G -lipschitz iff

$$\|\nabla f(x)\| \leq G \quad \forall x \in \text{dom}(f).$$

Theorem 8.5. Assume f is convex and G -Lipschitz, and that an optimal solution x^* exists with $\|x_0 - x^*\| \leq R$ for some $R > 0$.

Pick $\eta_t \rightarrow 0$ such that $\sum_{t=1}^{\infty} \eta_t = \infty$ and $\sum_{t=1}^T \eta_t^2 < \infty$ then

$$f\left(x_T^{(\text{best})}\right) \rightarrow f(x^*) \text{ as } T \rightarrow \infty.$$

Theorem 8.6. For the subgradient method with step sizes $\{\eta_t\}_{t=1}^T$ on a convex, G -Lipschitz function f , we have

$$f\left(x_T^{(\text{best})}\right) - f(x^*) \leq \frac{\|x_0 - x^*\|^2 + G^2 \sum_{t=1}^T \eta_t^2}{2 \sum_{t=1}^T \eta_t}.$$

This theorem implies Theorem 8.2 since, under the assumptions $\eta_t \rightarrow 0$, $\sum_{t=1}^{\infty} \eta_t = \infty$, and $\sum_{t=1}^{\infty} \eta_t^2 < \infty$, we have

$$\lim_{T \rightarrow \infty} \frac{\|x_0 - x^*\|^2 + G^2 \sum_{t=1}^T \eta_t^2}{2 \sum_{t=1}^T \eta_t} = 0,$$

because the numerator is bounded (the series $\sum_{t=1}^{\infty} \eta_t^2$ converges) while the denominator diverges (the series $\sum_{t=1}^{\infty} \eta_t$ diverges). Hence

$$f\left(x_T^{(\text{best})}\right) - f(x^*) \xrightarrow{T \rightarrow \infty} 0,$$

which is exactly the statement of Theorem 8.2.

Proof.

$$\begin{aligned} \|x_{t+1} - x^*\|^2 &= \|x_t - \eta_t g_t - x^*\|^2 \\ &= \|x_t - x^*\|^2 - 2\eta_t g_t^T(x_t - x^*) + \eta_t^2 \|g_t\|^2 \\ &\leq \|x_t - x^*\|^2 + \eta_t^2 G^2 + 2\eta_t (f(x^*) - f(x_t)) \end{aligned} \quad (*)$$

Last step follows from $f(x^*) \geq f(x_t) + g_t^T(x^* - x_t) \iff -g_t^T(x_t - x^*) \geq f(x^*) - f(x_t)$.

Adding (*) up from $0, \dots, T-1$ we have

$$\|x_T - x^*\|^2 \leq \|x_0 - x^*\|^2 + G^2 \sum_{t=0}^{T-1} \eta_t^2 + 2 \sum_{t=0}^{T-1} \eta_t (f(x^*) - f(x_t)).$$

So

$$2 \sum_{t=0}^{T-1} \eta_t \left(f(x_T^{(\text{best})}) - f(x^*) \right) \leq 2 \sum_{t=0}^{T-1} \eta_t (f(x^*) - f(x_t)) \leq \|x_0 - x^*\|^2 + G^2 \sum_{t=0}^{T-1} \eta_t^2.$$

□