Graduate Discrete Math (21-701) Notes

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Abstract

Lecture notes based on Graduate Discrete Math (21-701)

1 Graphs

Definition 1.1. *Graph is a set of objects* (V, E) *and* $E \subseteq \binom{V}{2}$

Definition 1.2. *Walk is a sequence of vertices*

Definition 1.3. *A path is a walk without repeated vertices*

Definition 1.4. *A proper* K*-coloring of a graph is a function* $c : V \to [k]$ *such that* $\forall u, v \in V, u \sim v \implies c(u) \neq c(v)$

Theorem 1.5. *A graph is 2 colorable if and only if there is no odd cycles in* G

Proof. ($\implies$) AFSOC there exist an odd cycle, C in G. Define the vertices of C as $v_1, v_2, ..., v_k$ where k is odd. Define $c(v) = \begin{cases} \text{red} & d(v, v_1) \text{ is even} \\ \text{blue} & d(v, v_1) \text{ is odd} \end{cases}$
Then $c(v_1)$ and $c(v_k)$ are both red so a contradiction.

($\impliedby$) We can assume each component is connected. Choose v_0 and define $c(v) = \begin{cases} \text{red} & d(v, v_0) \text{ is even} \\ \text{blue} & d(v, v_0) \text{ is odd} \end{cases}$
If there exist vertices u, v with uv an edge such that $d(u, v_0) \equiv d(v, v_0) \pmod 2$ then consider the cycle, C formed by shortest path from $v_0 \to u$ and $v_0 \to v$ with uv. Then $|C| = d(u, v_0) + d(v, v_0) + 1$ is odd and we're done. □

2 Hypergraphs

Definition 2.1. *A collection* $\mathcal{H}$ *of subsets of a vertex set* V*.*

Definition 2.2. $\mathcal{H}$ *is* k*-uniform if* $|f| = k, \forall f \in \mathcal{H}$

Definition 2.3. *A proper* k*-coloring of* $\mathcal{H}$ *is an assignment* $c : V \to [k]$ *such that* $\forall f \in \mathcal{H}, |c(f)| = k$

Definition 2.4. *A rainbow coloring of* $\mathcal{H}$ *is an assignment* $c : V \to [k]$ $\forall f \in \mathcal{H}, |c(f)| = |f|$

Example 2.5. *What is the least number of edges in a* k*-uniform graph that is not* 2*-colorable?*

Let this number be $m(k)$ then $m(1) = 0, m(2) = 3, m(3) \geq 7$

Theorem 2.6. *If $\mathcal{H}$ is a 3-uniform hypergraph with less than 6 edges then $\mathcal{H}$ is 2-colorable*

Proof. Using induction on $|V|$
(Base Case) For $n = 6$, consider all balanced 2-colorings of V there are $\binom{6}{3} = 20$. Each hyperedge is incompatible with 2 of those colorings (namely those were the edges are 3 blue or 3 red). Thus, at least $20 - 12 > 0$ of these colorings can be proper.
(Induction Hypothesis) Suppose $n \geq 7$

Claim 2.7. *There are 2 vertices u and v not in any common edge.*

Each edge connects $\binom{3}{2} = 3$ pairs of vertices. There are $\binom{7}{2} = 21$ pairs of vertices overall. So some pair of vertices is not connected as $21 > 18$.

Define $\mathcal{H}'$ by merging u, v into w

Claim 2.8. *$\mathcal{H}'$ is 3-uniform*

Because no edge contains both u and v the merging doesn't create a 2 set and every edge is still has size 3.

Additionally, $||\mathcal{H}'|| \leq ||\mathcal{H}|| \leq 6$ so by induction hypothesis $\mathcal{H}'$ is 2-colorable. Giving both u and v the same color as w and keeping the rest of the colors the same.

If an edge of e of $\mathcal{H}$ avoids $\{u, v\}$ then it is properly colored in $\mathcal{H}'$. If e contains u or v then after merging it corresponds to an edge of $\mathcal{H}'$ containing w. If e is monochromatic in $\mathcal{H}$ then it would be monochromatic in $\mathcal{H}'$. This would be a contradiction so edge is monochromatic in $\mathcal{H}$ and thus a proper coloring. □

Remark 2.9. *Suppose it has 7 edges and vertices. Consider the coloring 4 red and 3 blue. Then there are $\binom{7}{3} = 35$ such colorings. If $\mathcal{H}$ is not 2 colorable then there are $\binom{3}{3} + \binom{4}{3} = 5$ excluded coloring for all distinct edges. There are 4 forbidden configurations for any configurations that are not 2 colorable $\mathcal{H}$ with $|\mathcal{H}| = 7$ on 7 vertices*

3 Probabilistic Method

Theorem 3.1. $m(k) \geq 2^{k-1}$

Proof. Color vertices of $\mathcal{H}$ randomly red or blue. For each edge f, define E_f to be the event that f is monochromatic then $Pr[E_f] = \frac{1}{2^{k-1}}$

$$Pr\left[\bigcup_{f\in\mathcal{H}} E_f\right] \leq \sum_{f\in\mathcal{H}} Pr\left[E_f\right] = \frac{|\mathcal{H}|}{2^{k-1}} < 1$$

So there is non-zero probability that there exist a coloring with no monochromatic edges if $|\mathcal{H}| < 2^{k-1}$ □

Theorem 3.2. *Erdős-Selfridge Theorem: Given hypergraph $\mathcal{H}$, consider a game between a maker and breaker. The maker's goal is to color some edge all blue and breaker's goal is to prevent all blue edges.*

If $\mathcal{H}$ is k-uniform and $|\mathcal{H}| < 2^{k-1}$ then the breaker has a winning strategy even as player 2.

Proof. Let $\phi(f) = \begin{cases} 0 & \text{if blocked by breaker} \\ \frac{2^{\#\text{blue}\in f}}{2^n} & \text{otherwise} \end{cases}$
be the "danger function". Define

$$\phi(\mathcal{H}) = \sum_{f\in\mathcal{H}} \phi(f)$$

Observe that if an edge is all blue, then $\phi(\mathcal{H}) \geq 1$
At start of the game $\phi(\mathcal{H}) = \frac{|\mathcal{H}|}{2^n}$. The worst case for when maker moves is increasing by $\frac{|\mathcal{H}|}{2^n}$ when the chosen vertex is in all edges. Then when breaker moves,

$$-\sum_{f\ni v_1} \phi(f).$$

When maker goes after

$$\sum_{f\ni v_2} \phi(f).$$

Notice

$$\sum_{f\ni v_1} \phi(f) > \sum_{f\ni v_2} \phi(f)$$

otherwise breaker played optimally.
So as long as $\frac{|\mathcal{H}|}{2^{n-1}} < 1$ then breaker wins. □

Definition 3.3. *Incidence matrix of a hypergraph $\mathcal{H}$ with $|V| = n$ and $|\mathcal{H}| = m$ is defined as*

$$I_{i,j} = \begin{cases} 1 & \text{if } v_n \in f_m \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.4. *Hall's Theorem*
If G is a bipartite on (A, B) there is a complete matching if and only if

$$\forall S \subseteq A, |\Gamma(S)| \geq |S|$$

where $\Gamma(S) = \{u \in B | \exists v \in S, u \sim v\}$.

Theorem 3.5. *Consider complete graph $\mathcal{P}(X)$ where $|X| = n$.*
$\mathcal{P}(X)$ has levels

$$\binom{X}{0}, \binom{X}{1}, \dots, \binom{X}{n}$$

$\forall k < \frac{n}{2}$, there is an injection

$$f_k : \binom{X}{k} \to \binom{X}{k+1}$$

such that $\forall S \in \binom{X}{k}, S \subseteq f_k(S)$

Proof. Consider bipartite graph $\left(\binom{X}{k}, \binom{X}{k+1}\right)$, if $f \in \binom{X}{k}, g \in \binom{X}{k+1}$ then we define $f \sim g$ if $f \subseteq g$. Then for some $S \subseteq \binom{X}{k}$ then $|\Gamma(S)| \geq \frac{|S|(n-k)}{k+1}$. □

Definition 3.6. *For a sperner system is a hypergraph $\mathcal{H}$ that satisfy if*

$$\forall f, g \in \mathcal{H}, f \not\subseteq g$$

Theorem 3.7. *If $\mathcal{H}$ is a sperner system of n-vertices then*

$$|\mathcal{H}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Theorem 3.8. *LYM Inequality on a sperner family $\mathcal{H}$,*

$$\sum_{f\in\mathcal{H}} \frac{1}{\binom{n}{|f|}} \leq 1$$

Proof. Suppose $F = \{\emptyset, \{1\}, ..., \{1, 2, ..., n\}\}$
Note: Any sperner family can share at most one edge with F.
Consider a random permutation $\sigma \in S_n$ and define $\mathcal{H}_\sigma = \{\sigma(f)|f \in \mathcal{H}\}$.
For any $\sigma \in S_n, |\mathcal{H}_\sigma \cap F| \leq 1$
Now choose any σ, uniformly at random and define $\mathcal{X} = |\mathcal{H}_\sigma \cap F|$ is a random variable and $\mathcal{X} \leq 1$.
Let $\mathcal{X} = I_f$ where $I_f = \begin{cases} 1 & \sigma(f) \in F \\ 0 & \text{otherwise} \end{cases}$

$$\mathbb{E}[\mathcal{X}] = \sum_{f\in\mathcal{H}} \mathbb{E}[I_f] = \sum_{f\in\mathcal{H}} \frac{1}{\binom{n}{|f|}} \leq 1$$

□

Definition 3.9. *Define the "shadow" of $\mathcal{H} \subseteq \binom{X}{r}$ as $\partial\mathcal{H} \subseteq \binom{X}{r-1}$*

$$\partial\mathcal{H} = \left\{ S \subseteq \binom{X}{r-1} : \exists T \in \mathcal{H}, S \subseteq T \right\}$$

Theorem 3.10. *Let $n = |X|$ then*

$$\frac{|\mathcal{H}|}{\binom{n}{r}} \leq \frac{|\partial\mathcal{H}|}{\binom{n}{r-1}}$$

with equality only if $\mathcal{H}$ is empty on $\binom{X}{r}$

Proof. Suppose $\mathcal{H}$ is a sperner system, not all on one level.
Write $\mathcal{H}_i = \mathcal{H} \cap \binom{X}{i}$ then $H = \mathcal{H}_i \cup \mathcal{H}_{i+1} \cup \cdots \cup \mathcal{H}_j$ where $i < j$ and $\mathcal{H}_i$ nonempty.
We can instead of $\mathcal{H}_j$ we can write $\partial\mathcal{H}_j$ as $\partial\mathcal{H}_j \subseteq \mathcal{H}_j$
Suppose $\mathcal{H}$ maximizes the sum $\sum_{f\in\mathcal{H}} \frac{1}{\binom{n}{|f|}}$ among all sperner graphs.
Let $S \in \partial\mathcal{H}$ and $T \subseteq \mathcal{H}$. Define a bipartite graph from $S \to T$ and edges if $S \subseteq T$.
For $T \in \mathcal{H}, deg(T) = r$ for $S \in \partial\mathcal{H}, deg(S) = n - (r - 1)$.
So $|\mathcal{H}| \cdot r = b \leq |\partial\mathcal{H}| \cdot (n - r + 1)$.
Then $|\mathcal{H}| \cdot r\binom{n}{r} \leq |\partial\mathcal{H}| \cdot (n - r + 1)\binom{n}{r} \implies \frac{|\mathcal{H}|}{\binom{n}{r}} \leq \frac{|\partial\mathcal{H}}{\binom{n}{r-1}}$ □

Definition 3.11. *An intersecting hypergraph has any 2 hyperedges intersect.*

Theorem 3.12. *For an intersecting hypergraph on n-vertices and r-uniform,*

If $r = \frac{n}{2}$ then we can fix 1 vertex and complete the remaining $\frac{n}{2} - 1$ vertices. So $\binom{n-1}{\frac{n}{2}-1}$

If $r > \frac{n}{2}$ then 2^n.

If $r < \frac{n}{2}$ then $\binom{n-1}{r-1}$.

We'll prove the last statement

Proof. Assume $n = lk$ for some l and for any $\sigma \in S_n$ define $\mathcal{H}_\sigma = \{\sigma(f)|f \in \mathcal{H}\}$. Define $\mathcal{F}$ to be a k-uniform hypergraph with l non-intersecting edges.
If $\mathcal{H}$ is intersecting then $|\mathcal{H}_\sigma \cap \mathcal{F}| \leq 1$.
Let $\mathcal{X} = |\mathcal{H}_\sigma \cap \mathcal{F}|$. Then

$$\mathbb{E}[\mathcal{X}] = \sum_f \mathbb{E}[I_f] = |\mathcal{H}|\mathbb{P}[\sigma(f) \in \mathcal{F}] = |\mathcal{H}|\frac{l}{\binom{n}{k}} = \frac{|\mathcal{H}|}{\binom{n-1}{k-1}}$$

So $|\mathcal{H}| \leq \binom{n-1}{k-1}$

Consider the case when n is not divisible by k. If $n \geq 2k$ then fix a cyclic ordering π of the n vertices. For that ordering consider the n cyclic k-intervals for $i = 1, 2, ..., n$

$$I_i(\pi) = \{\pi(i), \pi(i+1), ..., \pi(i+k-1)\}$$

indices taken modulo n.
For a given π define

$$\mathcal{X}_\pi := \#\{f \in \mathcal{H} | f \text{ is one of the intervals } I_i(\pi)\}$$

Any two sets counted in $\mathcal{X}_\pi$ must intersect since $\mathcal{H}$ is intersecting. Among the n cyclic k-intervals at most k of them can be pairwise intersecting since we can fix one vertex however $k+1$ intervals will force two of them to be disjoint. So for every $\pi, \mathcal{X}_\pi \leq k$.
So

$$\mathbb{E}[\mathcal{X}_\pi] = |\mathcal{H}| \frac{k!(n-k)!}{(n-1)!} = |\mathcal{H}| \frac{n}{\binom{n}{k}} \leq k \implies |\mathcal{H}| \leq \binom{n-1}{k-1}$$

□

We'll try constructing such a configuration.

If $|\mathcal{H}| = \binom{n-1}{k-1}$ then $|\mathcal{H}_\sigma \cap \mathcal{F}| = k$ for each $\sigma \in S_n$. Then there is an i such that $\mathcal{I} = \begin{cases} \{i-k+1, i-k+2, ..., i\} \\ \{i-k+2, i-k+3, ..., i+1\} \\ \vdots \\ \{i, i+1, ..., i+k+1\} \end{cases}$

Suppose $a_1, ..., a_{k-1} \in [n]$ with no $a_j = i-k, i-k-1, ..., i, ..., i-k+1$
Consider a permutation σ sending $a_1 \to i+1, a_2 \to i+2, ..., a_{k-1} \to i+k-1$ and fixing $i-k, ..., i$.
We know, $|\mathcal{H}_\sigma \cap \mathcal{F}|$ includes all edges of $\mathcal{I}$
Now let σ be any permutation such that $\mathcal{H} \cap \mathcal{F}$ includes i of I. It suffices to show $\mathcal{H}_\sigma \cap \mathcal{F}$ includes all of $\mathcal{I}$ for any transposition.

Lemma 3.13. *Adjacent transposition generates S_n.*

Proof. (Case 1) If $j, j+1 \in \{i-k+1, 1, ..., i+k-1\}$, neither is i so they're both on same side of i.
Letting $f_0 = \{i-k+1, ..., i\}$ and $f_1 = \{i, ..., i+k-1\}$ then $\tau(f_0) = f_0$ and $\tau(f_1) = f_1$
(Case 2) If $j = i+k-1$ and $j+1 = i+k$ then $\tau(f_0) = f_0$ and $\tau(\{i-k, ..., i-1\}) = \{i-k, ..., i-1\}$. □

Theorem 3.14. *Let $\alpha_1, ..., \alpha_n \sim Ber(p)$, choosing numbers $\beta_1, ..., \beta_n$ with $\sum \beta_i = 1$ then $\mathbb{P}[\sum \beta_i \alpha_i \geq \frac{1}{2}] \geq p$*

Proof. Define $\mathcal{H}$ on $[n]$ by $f \in \mathcal{H}$ if $\sum_{i \in f} a_i \geq \frac{1}{2}$ For simplicity, assume no sum is $\frac{1}{2}$. Then

$$\mathbb{P}\left[\sum \beta_i \alpha_i \geq \frac{1}{2}\right] = \sum_{f \in \mathcal{H}} p^{|f|}(1-p)^{n-|f|}$$

Define $h_k = \left|\mathcal{H} \cap \binom{X}{k}\right|$

$$
\begin{aligned}
\mathbb{P}\left[\sum \beta_i \alpha_i \geq \frac{1}{2}\right] &= \sum_k h_k p^k (1-p)^{n-k} \\
&= \sum_{k \leq \frac{n}{2}} h_k p^k (1-p)^{n-k} + h_{n-k} p^{n-k} (1-p)^k \\
&= \sum_{k \leq \frac{n}{2}} (h_k + h_n) p^{n-k} (1-p)^k - h_k \left(p^{n-k}(1-p)^k - p^k (1-p)^{n-k}\right) \\
&\text{Note: } h_k + h_{n-k} \geq \binom{n}{k} \text{ since it or its complement has to be in } \mathcal{H} \\
&\geq \sum_{k \leq \frac{n}{2}} \binom{n}{k} p^{n-k} (1-p)^k - h_k \left(p^{n-k}(1-p)^k - p^k (1-p)^{n-k}\right) \\
&= \sum_{k \leq \frac{n}{2}} \binom{n}{k} p^{n-k} (1-p)^k + \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= p \sum_{k \leq \frac{n}{2}} \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= p \sum_{k \leq \frac{n}{2}} \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= p \sum_{k \leq \frac{n}{2}} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-k)-(k-1)} \\
&= p
\end{aligned}
$$

□

Theorem 3.15. *If there are 10 points in the plane then they can be covered by 10 non-intersecting unit circles.*

Proof. Given any collection $X \subseteq \mathbb{R}^2, |x| = 10$. Consider a random translation of the hexagonal circle pattern.
Let $\mathcal{Z} = \#$ points in X covered then

$$\mathbb{E}[\mathcal{Z}] = \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_{10}] = 10 \cdot \frac{\frac{\pi}{6}}{\sqrt{3}} \approx 9.07$$

So there exist a translation such that $\mathcal{Z} = 10$ □

Theorem 3.16. *Given a graph, G on n vertices and $\frac{nd}{2}$ edges, $d \geq 1$. Then $\alpha(G) \geq \frac{n}{2d}$.*

Proof. Let $S \subseteq V$ be a random subset defined by $\mathbb{P}[v \in S] = p$, p to be determined. Let $X = |S|$ and $Y = \mathbb{E}[G_{|S}]$. For each $e = \{i,j\} \in E$, let Y_e be indicator random variable for the event $i,j \in S$ so that

$$Y = \sum_{e \in E} Y_e$$

For any such e,

$$\mathbb{E}[Y_e] = \mathbb{P}[i,j \in S] = p^2$$

$$\mathbb{E}[Y] = \frac{nd}{2} p^2$$

Clearly, $\mathbb{E}[X] = np$ so $\mathbb{E}[X-Y] = np - \frac{nd}{2}p^2$.
Setting $p = \frac{1}{d}$ then $\mathbb{E}[X-Y] = \frac{n}{2d}$.
So there exist a S such that the number of vertices minus the number of edges is at least $\frac{n}{2d}$.
Create S^* from S by deleting one vertex from each edge in S and delete it and this leaves S^* with at least $\frac{n}{2d}$ vertices. With all edges destroyed we leave S^* an independent set. □

Theorem 3.17. *Erdos Chromatic Number Girth Theorem*
$\forall k \in \mathbb{Z}^+, \exists$ *graph of girth greater than or equal to* k *and chromatic number* k.

Proof. Idea: Choose random graph $G \sim G(n,p)$. To show a graph satisfies both properties we need the the number of short cycles (length less than k) to be 0 and there are no independent set of size no more than $\frac{n}{k}$.
For the first statement let $X = $ #cycles with length $\leq k$ then

$$\mathbb{E}[X] = \sum_C \mathbb{E}[I_c] = \sum_{j=3}^{k} \sum_{|C|=j} \mathbb{E}[I_C] = \sum_{j=3}^{k} \binom{n}{j} \frac{(j-1)!}{2} p^j \leq \sum_{j=3}^{k} n^j p^j \leq (np)^{k+1}$$

To have $\mathbb{E}[X] = O(1)$, we need $p = O\left(\frac{1}{n}\right)$

We want no independent set of size $a \approx \frac{n}{k}$ so

$$\begin{aligned}\mathbb{P}[\alpha(G_{n,p}) \geq a] &\leq \binom{n}{a}(1-p)^{\binom{a}{2}} \\ &\leq n^a(1-p)^{a(a-1)/2} \\ &\leq n^a e^{-pa(a-1)/2} \\ &= \left(ne^{-p(a-1)/2}\right)^a \\ &= \left(e^{\ln(n)-p(a-1)/2}\right)^a\end{aligned}$$

Not possible since we need $p \geq 5\ln(n)/a$ but $p = O\left(\frac{1}{n}\right)$ from previous condition.

To fix this issue consider an alteration. If $p = \frac{n^\epsilon}{n}$ and $0 < \epsilon < \frac{1}{k}$ then we have $\mathbb{P}(\alpha(G_{n,p}) \geq \frac{n}{2k}) \to 0$ since $\frac{n^\epsilon}{n} >> \frac{5\ln(n)-2k}{n}$.
To fix the short cycle issue,

$$\mathbb{E}[X] = \sum_{j=3}^{k} (np)^j \leq (k-3)(np)^k \leq kn^{\epsilon k}$$

By Markov's inequality

$$\mathbb{P}\left(X \geq \frac{n}{2}\right) \leq \frac{kn^{\epsilon k}}{n/2}$$

Choose n large enough such that both probabilities are greater than $\frac{1}{2}$. Then there exists a graph on n vertices with no independent set of size $\frac{n}{2k}$ and less than $\frac{n}{2}$ short cycles. Delete an vertex from each short cycle to make a graph G' with $\frac{n}{2}$ vertices, no short cycles and no independent set of size $\frac{n}{2k}$. So

$$\mathcal{X}(G') = \frac{n'}{\alpha(G')} \geq \frac{\frac{n}{2}}{\frac{n}{2k}} \geq k.$$

□

When does $G_{n,p}$ have triangles?
If $X = $ #triangles then

$$\mathbb{E}[X] = \sum_{\text{Triangle } T \in K_n} E[I_T] = \binom{n}{3} p^3 \sim n^3p^3/6 = O(1)$$

if $p = O\left(\frac{1}{n}\right)$
Is $G_{n,p}$ connected for $p = \frac{c}{n}$?
Let $X = $ #spanning trees of $G_{n,p}$ then

$$\mathbb{E}[X] = \sum_{T \in G_{n,p}} E(I_T) = n^{n-2}p^{n-1} = n^{n-2}\frac{c^{n-1}}{n^{n-1}} = \frac{c^{n-1}}{n} \to \infty.$$

Let $Y = \#\text{isolated vertices}$ then

$$\mathbb{E}[Y] = \sum_{v \in V} \mathbb{P}(v \text{ isolated}) = n(1-p)^{n-1} \approx ne^{-p(n-1)} \approx ne^{-c}$$

For $p = O\left(\frac{1}{n}\right)$, let $X = \#\text{triangles}$. Then

$$\mathbb{E}[X] = \binom{n}{3} p^3 \to 0.$$

So $\mathbb{P}(G_{n,p} \text{ having triangles}) \to 0$.

Theorem 3.18. *Threshold of $\mathcal{H}$ in $G_{n,p}$.*
Consider $G_{n,p}$ with $p = p(n)$ and $\mathcal{H}$ is fixed graph with k vertices and l edges.
Define $\epsilon = \epsilon(\mathcal{H}) = \frac{l}{k}$ and $\epsilon' = \epsilon'(\mathcal{H}) = \max_{J \subseteq \mathcal{H}} \epsilon(J)$. If $p^{\epsilon} \cdot n \to 0$ then $\mathbb{E}[\#\mathcal{H}$ in $G_{n,p}] \to 0$.

Proof. $\mathbb{E}[\#\mathcal{H} \in G_{n,p}] \leq \binom{n}{k} hp^l \leq C(np^{\epsilon})^k$ where $h = \frac{k!}{\text{Aut}(\mathcal{H})}$
If $p^{\epsilon'} n \to 0$, there is some argument for densest subgraph J. □

Now to show the other side, if $p^{\epsilon'} n \to \infty$ (if $p = \omega\left(\frac{1}{n^{1/\epsilon'}}\right)$ then $G_{n,p}$ has $\mathcal{H}$ as a subgraph with high probability.

Proof. Let $X = \#\mathcal{H}$ subgraph in $G_{n,p}$ then by Chebyshev,

$$\mathbb{P}[X \leq 0] \leq \frac{\text{Var}[X]}{\mathbb{E}[X]^2}.$$

We can compute

$$\mathbb{E}[X^2] = \sum_{H_1, H_2 \in \mathcal{H}} \mathbb{P}[H_1, H_2 \subseteq G_{n,p}] = \sum_{t=0}^{k} \sum_{|H_1 \cap H_2| = t} \mathbb{P}(H_1, H_2 \subseteq G_{n,p})$$

For $t = 0$ we have

$$\sum_{|H_1 \cap H_2| = 0} \mathbb{P}(H_1 \subseteq G_{n,p}) \mathbb{P}(H_2 \subseteq G_{n,p}) \leq \mathbb{E}[X]^2$$

So

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 \leq \sum_{t=1}^{k} \sum_{|H_1 \cap H_2| = t} \mathbb{P}(H_1, H_2 \subseteq G) = \sum_{t=1}^{k} \sum_{|H_1 \cap H_2| = t} \binom{k}{t} \binom{n-k}{k-t} hp^{e(H_1 \cup H_2)}$$

By PIE, $e(H_1 \cup H_2) \geq 2l - \epsilon' t$ since $e(H_1 \cap H_2) \leq \epsilon' t$ as $H_1 \cap H_2$ is a subgraph of H_1

$$\begin{aligned}
\sum_{t=1}^{k} \sum_{|H_1 \cap H_2| = t} \binom{k}{t} \binom{n-k}{k-t} hp^{e(H_1 \cup H_2)} &\leq \sum_{t=1}^{k} \sum_{|H_1 \cap H_2| t} \binom{k}{t} \binom{n-k}{k-t} hp^{2l - \epsilon' t} \\
&= \sum_{t=1}^{k} \sum_{|H_1 \cap H_2| t} \binom{n}{k} h \cdot h \binom{n-k}{k-t} p^{l - \epsilon' t} \\
&\qquad \left(\textstyle\sum_{H \in \mathcal{H}} p^l = \mathbb{E}[X] = \binom{n}{k} \cdot h \cdot p^l\right) \\
&\leq \mathbb{E}[X] \sum_{t=1}^{k} h \binom{k}{t} \binom{n-k}{k-t} p^{l - \epsilon' t} \\
&\leq \mathbb{E}[X] \sum_{t=1}^{k} h \cdot C \cdot n^k \cdot \frac{1}{n^t} \cdot p^{l - \epsilon' t} \\
&\leq \mathbb{E}[X] \sum_{t=1}^{k} C' \cdot h \cdot \binom{n}{k} \cdot p^l \cdot \frac{1}{(np^{\epsilon'})^t} \\
&= \mathbb{E}[X]^2 \sum_{t=1}^{k} C' \left(\frac{1}{np^{\epsilon'}}\right)^t \to 0
\end{aligned}$$

□

Theorem 3.19. *Chernoff Bound*
Suppose you had independent random values $\zeta_1, ..., \zeta_n$ with $\zeta_i \in \{-1, 1\}$ $\forall i$ and $\mathbb{P}(\zeta_i = 1) = \mathbb{P}(\zeta = -1) = \frac{1}{2}$.
Let $X = \sum_{i=1}^{n} \zeta_i$

$$\begin{aligned} \mathbb{P}(X > a) &= \mathbb{P}(e^{tX} > e^{ta}) \\ &\leq \frac{\mathbb{E}(e^{tX})}{e^{ta}} \\ &= \frac{\mathbb{E}(e^{t\sum \zeta_i})}{e^{ta}} \\ &= \frac{\prod_{i=1}^{n} \mathbb{E}[e^{t\zeta_i}]}{e^{ta}} \\ &= \frac{\left(\frac{e^t+e^{-t}}{2}\right)^n}{e^{ta}} \\ &\leq e^{nt^2/2-ta} \\ &= e^{a^2/2n-a^2/n} && (\text{for } t = \tfrac{a}{n}) \\ &= e^{-\frac{a^2}{2n}} \end{aligned}$$

Question: How many vectors can I have in $\mathbb{R}^d$ all at a common pairwise angle.

Equivalently, $\exists \alpha, v_i \cdot v_j = \begin{cases} \alpha & \text{if } i \neq j \\ 1 & \text{otherwise} \end{cases}$

Let $V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix}$

Then $V^TV = G = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_m \\ \vdots & v_2 \cdot v_2 & \cdots & \\ \vdots & \vdots & \ddots & \end{bmatrix}$

Claim: $Rank(G) \leq d$.
For each $\vec{x} \in \text{Ker}(V) \implies V\vec{x} = 0 \implies V^TV\vec{x} = 0 \implies G\vec{x} = 0$
We know $Rank(V) \leq d$ so $Rank(G) = Rank(V^TV) \leq d$.

We can write $G = \alpha J_m + (1-\alpha)I_m$ then J_m has eigenvalues m with multiplicity at least 1 and eigenvectors with multiplicity $\geq m - 1$.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ \vdots \end{bmatrix} \cdots \begin{bmatrix} 1 \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

If $\vec{x}$ is an eigenvector of J_m with eigenvalue λ then $G\vec{x} = \alpha\lambda x + (1-\alpha)x = (\alpha\lambda + 1 - \alpha)x$.
G has eigenvalues $1 - \alpha$ with multiplicity at least $m - 1$.
So $Rank(G) \geq n - 1$ and consequently $d \geq n - 1$.

Question: How many unit vectors $v_1, .., v_n$ can I have in $\mathbb{R}^d$ with all pairs approx equal angle.
Equivalently, $\exists \alpha$ such that $(*)$ $v_i \cdot v_j \in (\alpha - \epsilon, \alpha + \epsilon$ if $i \neq j$ and 1 if $i = j$.

Theorem 3.20. $\forall \epsilon \in (0, 1)$, *there is a $c > 0$ such that for all sufficiently large d there is a collection $v_1, ..., v_m \in \mathbb{R}^d$ of vectors satisfying $(*)$ with $m \geq 2^{cd}$*

Consider the easier problem when $\alpha = 0$. Choose m vectors $v_i \in \{1, -1\}^d$ at random with $\mathbb{P}(v_i^j = +1) = \mathbb{P}(v_i^j = -1) = \frac{1}{2}$. We know $v_i \cdot v_i = n$ then let $u_i = \frac{v_i}{\sqrt{n}}$. Then $\mathbb{E}(u_i \cdot u_v) = 0$ as $\zeta_{i,j,k} =$

$\begin{cases} 1 & \text{if } v_i^k \cdot v_j^k = 1 \\ -1 & \text{if } v_i^k \cdot v_j^k = -1 \end{cases}$ By Chernoff,

$$\mathbb{P}(v_i \cdot v_j \notin (-\epsilon n, \epsilon n)) \leq 2e^{-\epsilon^2 n/2}$$

$$\mathbb{P}(\exists i, j \text{ s.t. } u_i \cdot u_j \notin (-\epsilon n, \epsilon n)) \leq \binom{m}{2} 2e^{-\epsilon^2 n/2} \leq m^2 e^{-\epsilon^2 n/2} = m^2 \beta^{-n}$$

Theorem 3.21. *If $\zeta_1, \zeta_2, ..., \zeta_n$ are independent random variables with $\mathbb{E}(\zeta_i) = 0, |\zeta_i| \leq 1, X = \sum \zeta_i$ then*

$$\mathbb{P}(X \geq a) \leq e^{-\frac{a^2}{2n}}$$

Question: I flip biased coins with head prob $= \frac{1}{3}$ n times. Bound the probability # head $\geq \frac{n}{2}$.
Define $\zeta_i = \begin{cases} 1 & \text{if head} \\ \frac{-p}{1-p} & \text{otherwise} \end{cases}$ (Subtract expected value and dividing by $1-p$) Then $\mathbb{E}[\zeta_i] = 0, |\zeta_i| \leq 1$ for $p < \frac{1}{2}$.
Let $X = h + (n-h)\frac{-p}{1-p}$

Theorem 3.22. *For random variables X, Y*

$$\begin{aligned} \mathbb{E}[XY] &= \sum_{x,y} xy\mathbb{P}(X = x, Y = y) \\ &= \sum_{x,y} xy\mathbb{P}(Y = y)\mathbb{P}(X = x|Y = y) \\ &= \sum_{y} y\mathbb{P}(Y = y) \sum_{x} x\mathbb{P}(X = x|Y = y) \\ &= \mathbb{E}(Y\mathbb{E}(X|Y)) \end{aligned}$$

Theorem 3.23. *Let X be a random variable and A an event ten*

$$\mathbb{E}(X|A) = \frac{1}{\mathbb{P}(A)} \sum_{w \in A} p(w)\mathcal{X}(w)$$

Theorem 3.24. *Let $\zeta_1, ..., \zeta_n$ be random variable with $\mathbb{E}(\zeta_i) = 0$ and $|\zeta_i| \leq 1$.*

$$\mathbb{E}(e^{t\sum \zeta_i}) \leq \left(\frac{e^t + e^{-t}}{2}\right)^n$$

Proof.

$$\mathbb{E}(e^{t\sum \zeta_i}) = \mathbb{E}\left(\prod_{i=1}^{n} e^{t\zeta_i}\right) = \mathbb{E}\left(\prod_{i=1}^{n-1} e^{t\zeta_i}\mathbb{E}\left(e^{t\zeta_n}|\prod_{i=1}^{n-1} e^{t\zeta_i}\right)\right)$$

We know want to upper bound $\mathbb{E}\left(e^{t\zeta_n}|\prod_{i=1}^{n-1} e^{t\zeta_i}\right)$.
By convexity, $e^{t\zeta_i} \leq h(\zeta_i) = \frac{1}{2}\left[(1-\zeta_i)e^{-t} + (1+\zeta_i)e^t\right] \implies \mathbb{E}\left[e^{t\zeta_i}\right] \leq \frac{e^t + e^{-t}}{2}$

$$\mathbb{E}\left(e^{t\zeta_n}|\prod_{i=1}^{n-1} e^{t\zeta_i}\right) \leq \left(\frac{e^t + e^{-t}}{2}\right)\mathbb{E}\left(\prod_{i=1}^{n-1} e^{t\zeta_i}\right) \leq \left(\frac{e^t + e^{-t}}{2}\right)^n$$

□

Definition 3.25. *Random variables $X_0, X_1, ...$ is a martingale if it satisfies the following properties*

1. $\mathbb{E}[|X_i|] < \infty$

2. $\mathbb{E}[X_{i+1}|X_1, ..., X_i] = X_i$

Theorem 3.26. *Azuma's Theorem: If $X_0, X_1, ...$ is martingale and $|X_{i+1} - X_i| \leq 1$ then*

$$\mathbb{P}(X_n - X_0 \geq \lambda\sqrt{n}) \leq e^{-\lambda^2/2}$$

Proof. By Markov's Inequality we have

$$\mathbb{P}(X_n - X_0 \geq \lambda\sqrt{n}) \leq \frac{\mathbb{E}\left[e^{t(X_n - X_0)}\right]}{e^{t\lambda\sqrt{n}}}$$

We can telescope the numerator as

$$\begin{aligned}\mathbb{E}\left[e^{t(X_n-X_0)}\right] &= \mathbb{E}\left[\prod_{k=1}^{n} e^{t(X_k - X_{k-1})}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{t(X_n - X_{n-1})}|X_0, X_1, X_2, ..., X_{n-1}\right]\prod_{k=1}^{n-1} e^{t(X_k - X_{k-1})}\right]\end{aligned}$$

By Theorem 3.23 we have $\mathbb{E}\left[e^{t(X_n - X_{n-1})}|X_1, X_2, ...X_{n-1}\right] \leq e^{t^2/2}$ so

$$\mathbb{E}\left[\mathbb{E}\left[e^{t(X_n - X_{n-1})}|X_0, X_1, X_2, ..., X_{n-1}\right]\prod_{k=1}^{n} e^{t(X_k - X_{k-1})}\right] \leq e^{nt^2/2}$$

So

$$\mathbb{P}(X_n - X_0 \geq \lambda\sqrt{n}) \leq e^{nt^2/2 - t\lambda\sqrt{n}}.$$

The RHS achieves it's max at $t = \frac{\lambda}{\sqrt{n}}$. Thus

$$\mathbb{P}(X_n - X_0 \geq \lambda\sqrt{n}) \leq e^{-\lambda^2/2}$$

□

Definition 3.27. *Doob's Martingale: Let X and $Y_1, Y_2, ...$ be random variables with $\mathbb{E}[|Y_i|] < \infty$. Define $X_i := \mathbb{E}(X|Y_1, ..., Y_i)$ for $i \geq 1$. with $X_0 = \mathbb{E}[X]$*

Theorem 3.28. *McDiarmid's Inequality: Let $f : \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_n \to \mathbb{R}$ that is 1-lipschitz.*
If $Y_1, Y_2, ..., Y_n$ are independent random variables where $Y_i \in \mathcal{Y}_i$.
For $X := f(Y_1, ..., Y_n)$ satisfies

$$\mathbb{P}(X - \mathbb{E}(X) \geq \lambda\sqrt{n}) \leq e^{-2\lambda^2}.$$

Example 3.29. *For m balls into n bins, let $X = \#$ empty bins then*

$$\mathbb{E}[X] = n\left(1 - \frac{1}{n}\right)^m \approx ne^{-\alpha}$$

where $m = \alpha n$. If we let $Y_i =$ pos of ball i then $f(Y_1, ..., Y_m) = \#$ empty bins. So $\mathbb{P}(X \geq \mathbb{E}[X] + \lambda\sqrt{n}) \leq e^{-2\lambda^2}$

Theorem 3.30.

$$\mathbb{P}(A|B \cap C) = \frac{\mathbb{P}(A \cap B|C)}{\mathbb{P}(B|C)}$$

Theorem 3.31. *Lovasz Local Lemma: Given a collection $\mathcal{B} = \{B_1, B_2, ..., B_n\}$ of bad events. If $B_1, ..., B_n$ have a dependency graph of max degree d and $4pd \leq 1$ where $\mathbb{P}(B_i) \leq p$ then*

$$\mathbb{P}\left(\bigcap \overline{B_i}\right) > 0.$$

Proof. We'll prove the statement by induction on $k = |S|$ for $S \subseteq [n]$. Assume $4pd \leq 1$ then we want to show $\forall i, \mathbb{P}\left(B_i | \bigcap_{j\in S} \overline{B_j}\right) \leq 2p$.

Let $S = T \cup U$ where $T = \{j \in S | j \sim i\}$.

$$\begin{aligned}
\mathbb{P}\left(B_i | \bigcap_{j\in T} \overline{B_j} \cap \bigcap_{j\in U} \overline{B_j}\right) &= \frac{\mathbb{P}\left(B_i \cap \bigcap_{j\in T} \overline{B_j} | \bigcap_{j\in U} \overline{B_j}\right)}{\mathbb{P}\left(\bigcap_{j\in T} \overline{B_j} | \cap_{j\in U} \overline{B_j}\right)} \\
&\leq \frac{p}{1 - \mathbb{P}\left(\bigcup_{j\in T} B_j | \cap_{j\in U} \overline{B_j}\right)} \\
&\leq \frac{p}{1 - \sum_{j\in T} \mathbb{P}(B_j | \bigcap_{k\in U} \overline{B_k})} \\
&\leq \frac{p}{1-2dp} && \text{(Induction Hypothesis)} \\
&\leq 2p && \text{(Assumption)}
\end{aligned}$$

We're done with the induction.
Then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{B_i}\right) = \mathbb{P}\left(\overline{B_n} | \bigcap_{i=1}^{n-1} \overline{B_i}\right) \cdot \mathbb{P}\left(\bigcap_{i=1}^{n-1} \overline{B_i}\right) \geq \frac{(1-2p)^n}{p} > 0$$

□

Theorem 3.32. *Local Lemma (General Version) Suppose $\mathcal{B}$ is a collection of ”bad” events with some dependency graph. Suppose we can assign real number $0 < X_A < 1$ to each $A \in B$ such that*

$$\mathbb{P}(A) \leq X_A \prod_{B\sim A} (1 - X_B).$$

Then

$$\mathbb{P}\left(\bigcap_{B\in\mathcal{B}} \overline{B}\right) \geq \prod_{B\in\mathcal{B}} (1 - X_B) > 0$$

Proof. We can prove it by induction on $|S|$ that if $B_1, ..., B_t, B_{t+1}, ..., B_S \in \mathcal{B}$ such that $A \sim B_1, ..., B_t$ and $A \not\sim B_{t+1}, ..., B_S$ to show

$$\mathbb{P}\left(A | \bigcap_{i=1}^{S} \overline{B_i}\right) \leq X_A$$

We have

$$\begin{aligned}
\mathbb{P}\left(A | \bigcap_{i=1}^{t} \overline{B_i} \cap \bigcap_{i=t+1}^{T} \overline{B_i}\right) &= \frac{\mathbb{P}\left(A \cap \bigcap_{i=1}^{T} \overline{B_i} | \bigcap_{i=t+1}^{S} \overline{B_i}\right)}{\mathbb{P}(\bigcap_{i=1}^{t} \overline{B_i} | \bigcap_{i=t+1}^{S} \overline{B_i})} \\
&\leq \frac{\mathbb{P}\left(A | \bigcap_{i=t+1}^{S} \overline{B_i}\right)}{\mathbb{P}(\bigcap_{i=1}^{t} \overline{B_i} | \bigcap_{i=t+1}^{S} \overline{B_i})} \\
&\leq \frac{X_A \prod_{B\sim A}(1 - X_B)}{\prod_{i=1}^{t}(1 - X_B)} \\
&\leq X_A
\end{aligned}$$

Thus we're done with induction. Using the statement

$$\mathbb{P}\left(\bigcap_{i=1}^{S} \overline{B_i}\right) = \mathbb{P}\left(\overline{B_1} | \overline{B_2}\right) \times \cdots \times \mathbb{P}\left(\overline{B_n} | \bigcap_{i=1}^{S} \overline{B_i}\right) \geq \prod_{i=1}^{n} (1 - X_{B_i}) > 0$$

□

Theorem 3.33. *Axel's Theorem:*
$\forall \epsilon > N_\epsilon$ and an infinite binary sequence such that $\forall n > N_\epsilon$, any 2 consecutive block of length n differ in $\geq \left(\frac{1}{2} - \epsilon\right) n$ places.

Proof. Let the bad events be $B_{i,n}$ where for each i, intervals $[i, ..., i+n], [i+n+1, ..., i+2n]$ differ by less than $\left(\frac{1}{2} - \epsilon\right) n$
Let $X = \#$ places where they differ then $\mathbb{E}[X] = \frac{n}{2}$.
By Chernoff-Hoeffding's Lemma we have

$$\mathbb{P}(X - \mathbb{E}[X] \geq -\epsilon n) = \mathbb{P}(X \geq n/2 - \epsilon n) \leq e^{-2\epsilon^2 n} \leq e^{-\epsilon^2 n/10}$$

Let $X_{B_{i,n}} = e^{-\epsilon^2 n/20} \cdot \frac{1}{n^3}$ then fix B_{i_0,n_0} then we have

$$X_{B_{i_0,n_0}} = e^{-\epsilon^2 n_0/20} \cdot \frac{1}{n_0^3}$$

$$e^{-\epsilon^2 n_0/20} \cdot \frac{1}{n_0^3} \prod_{n=N_\epsilon}^{T} \prod_{i=i_0-2n}^{i_0+2n} \left(1 - e^{-\epsilon^2 n/20} \cdot \frac{1}{n^3}\right)^{2n+2n_0} \geq$$

□

From Homework #3

Theorem 3.34. $\forall \epsilon > 0, \exists N_\epsilon, \forall T, \exists$ *a binary sequence of length T such that*
$(*)$ $\forall n > N_\epsilon$ identical blocks of length n are separated by distance $\geq (2-\epsilon)^n$

Theorem 3.35. *Konig's Infinity Lemma: Let G be a connected, locally finite, infinite graph then G contains an infinite path.*

Theorem 3.36. $\forall \epsilon > 0, \exists N_\epsilon, \forall T, \exists$ *an infinite binary sequence such that vertices of my tree are finite binary sequences with property $(*)$.*
Let $\mathcal{T}$ be a complete tree of all binary sequences with all vertices and join $S \to S'$ if S can be obtained from S' by removing last digit of S'.
We want to show $\mathcal{T}$ is locally finite and infinite. It is locally finite as each node has at most 2 children. It is infinite because for any string that satisfy $(*)$, any of its prefix has to satisfy $(*)$. By Theorem 3.34, there must be an infinite path with property $(*)$

4 Topology

Definition 4.1. *A topology is a set X and a collection $\mathcal{O}$ of open sets satisfying*

1. *$\emptyset \in \mathcal{O}, X \in \mathcal{O}$*
2. *$\mathcal{O}$ is closed under finite intersection*
3. *$\mathcal{O}$ is closed under arbitrary union*

A collection of basic open sets are closed under finite intersections.

Definition 4.2. *X is compact if every cover has a finite subcover*

Definition 4.3. *Product topogy is*

$$\prod_{\alpha \in A} \mathcal{O}_\alpha$$

where $\mathcal{O}_\alpha \subseteq X_\alpha$ is open and $\mathcal{O}_\alpha = X_\alpha$ except for finitely many.

Theorem 4.4. *If X_α where $\alpha \in I$ are compact topological spaces then $\prod_{\alpha\in I} X_\alpha$ is compact.*

5 Ramsey Numbers

Definition 5.1. *Ramsey number $R(k,l) = \min_{n\geq 1}\{K_n$ contains a red K_k or blue $K_l\}$*
We can see $R(3,3) = 6$

Theorem 5.2. $R(k,l) \leq R(k-1,l) + R(k,l-1)$

Proof. Let $n \geq R(k-1,l) + R(k,l-1)$ and consider a red/blue coloring of K_n. Fix v_0. Since v_0 has $\geq R(k-1,l) + R(k,l-1) - 1$ edges,
(Case 1) If v_0 has $\geq R(k-1,l)$ red edges then the induced subgraph of the neighbors, G' must have red K_{k-1} or blue K_l. If red K_{k-1} then $G' \cup v_0$ is a K_k, otherwise we have blue K_l.
(Case 2) If v_0 has $\geq R(k,l-1)$ blue neighbors then same argument as case 1.
Thus we're done □

Theorem 5.3. $R(k,k) > (1-O(1))\frac{k}{e\sqrt{2}}2^{k/2}$

Proof. Flip fair coins to color a K_n red or blue. Let $X = \#$ monotonic K_k then

$$\mathbb{E}[X] = \binom{n}{k}2^{1-\binom{k}{2}} \leq \left(\frac{en}{k}\right)^k \cdot 2 \cdot 2^{-k(k-1)/2}$$

If $\mathbb{E}[X] < 1$ then $R(k,k) > n$.

$$2^{1/k}\left(\frac{en}{k}\right)2^{(k-1)/2} < 1$$

Thus

$$n < (1 - O(1)\frac{k}{e\sqrt{2}}2^{k/2}$$

Consequently, $R(k,k) > (1-O(1)\frac{k}{e\sqrt{2}}2^{k/2}$ □

Theorem 5.4. *Alterations: Color edges of K_n randomly red or blue. Delete an vertex from each monochromatic K_k. Let $X = n-$ # monochromatic cliques.*

$$\begin{aligned}\mathbb{E}[X] &= n - \binom{n}{k}2^{1-\binom{k}{2}} \\ &\geq n - \left(\frac{en}{k}\right)^k \cdot 2 \cdot 2^{-k(k+1)/2} \\ &= n - 2\left(\frac{en}{k}\cdot 2^{\frac{-k-1}{2}}\right)\end{aligned}$$

Let $n = \frac{k}{e}\cdot 2^{k/2}$ then

$$\frac{k}{e}\cdot 2^{k/2} - 2^{k/2} = (1-o(1))\cdot\frac{k}{e}\cdot 2^{k/2}$$

Theorem 5.5. *Using Lovasc Local Lemma: Given k, fix n, randomly red/blue color edges.*

Proof. Bad events: B_k for $k \in \mathcal{K}$ where $\mathcal{K}$ is the collection of k-clique.
Then $\mathbb{P}(B_k) = 2^{1-\binom{n}{k}}$.
If K_1, K_2 share any edges, set $B_{K_1} \sim B_{K_2}$ in dependency graph. Then

$$D \leq \binom{k}{2}\binom{n}{k-2}$$

Consequently

$$epD \leq e \cdot 2^{1-\binom{n}{k}} \left(2\binom{k}{2}\binom{n}{k-2}\right) < 1$$

$$4e\left(\left(\frac{en}{k-2}\right)^{k-2}\binom{k}{2}\right) < 2^{\binom{k}{2}}$$

$$\left(2e\binom{k}{2}\right)^{\frac{1}{k-2}} \cdot \frac{en}{k-2} < 2^{\binom{k}{2}-\frac{1}{k-2}} = 2^{\frac{k+1}{2}}$$

$$(1+o(1))\frac{en}{k-2} < 2^{\frac{k+1}{2}}$$

So

$$n < (1-o(1))\frac{k\sqrt{2}}{e}2^{k/2}$$

Thus $R(k) > (1-o(1))\frac{k\sqrt{2}}{e}2^{k/2}$ □

Definition 5.6. *Define K_k^j as the complete j uniform hypergraph on n vertices with k vertices*

Definition 5.7. *Define $R_j(k)$ = minimum n such that any red/blue coloring of $\binom{[n]}{j}$ has a monochromatic K_k^j*

Theorem 5.8. $R_r(k,l) \leq R_{r-1}(R_r(k-1,l), R_r(k,l-1))$

Proof. Let $N = R_{r-1}(R_r(k-1,l), R_r(k,l-1)) + 1$ and fix v. There are $N+1$ other vertices, Y. Each edge containing v includes an $r-1$ edge in Y. Let it inherit the color of the r edges.
(Case 1) We have $R_r(k-1,l)$ vertices in Y such that all $r-1$ subsets are red. (Case 1A) □

Let $C(k)$ = minimum n such that $\forall X \subseteq \mathbb{R}^2$ such that $|X| = n$ and X has a subset S where $|S| = k$ and S is in convex position.
Then $C(1) = 1, C(2) = 2, C(3) = 3, C(4) = 5, C(5) = 9, \ldots$

Theorem 5.9. $C(k) \leq R_4(5,k)$

Lemma: If $S \subseteq \mathbb{R}^2$ is k-points in general position such that any 4 of them are in convex position, then they all are. (Easy to see by triangulation)

Given a set F of four points color F color F red if not in convex position and blue otherwise.
Note: K_k^n is a complete k-uniform hypergraph on n vertices.
A red K_4^5 is impossible as $C(4) \leq 5$. Since we can find a blue K_4^k and we win by lemma.
Color 3-tuples according to whether "sorted slopes" are increasing or decreasing. If $n \geq R_3(k,k)$, I can find k vertices all of whose 3 tuples are caps or all 3 tuples are cups.

Definition 5.10. *Let $CC(k,l)$ = min n such that any n pts in general position, no two have same x coordinate, have a k-cup or an l-cap.*

Theorem 5.11. *Erdos Szekeres:* $CC(k,l) = \binom{k+l-4}{k-2} + 1$

Proof. I have k-cup or a l-cup. We'll show it by induction on $k+l$. Assume no l-cap. I do have a $(k-1)$-cup or l cup by induction. If I delete the last point of each $(k-1)$-cup. Then only $\binom{k+1-5}{k-3}$ points remain. So I deleted $\binom{k_l-4}{k-2} + 1 - \binom{k+l-5}{k-2} + 1$. □

Theorem 5.12. *For all positive integers k and r, there exists N such that any r-coloring of the numbers $1, 2, ..., N$ has a monochromatic k-term arithmetic progression.*

Theorem 5.13. *If $\mathbb{N}$ is partitioned into 2 sets , one contains arbitrary long arithmetic progression.* ***Statement 1:*** *$\forall k, \exists N$ such that any 2 coloring of $[N]$ has a monochromatic k-term arithmetic progression. If such a statement is false for k_0, then for all n there is a coloring of $[n]$ with no k_0 arithmetic progression. With Konig's Lemma there exists a coloring of $\mathbb{N}$ with no k_0-term arithmetic progression.*

Statement 1 implies
Statement 2 *$\forall r, \forall k, \exists W(k,r)$ such that any r-coloring of $[N]$ for $N \geq W(k,r)$ admits a k-term monochromatic A.P.*

Some values of $W(k,r)$ are $W(k;1) = k$, $W(2,r) = r+1$, $W(2,2) = 3$ and $W(3,2) = 9$.

Theorem 5.14. $W(3,2) \leq 325$
Note: The technique used here can be used for the general case.

Proof. Consider 65 blocks of 5 spots each. Within the first 33 blocks, there must be 2 blocks of the same coloring. Let the blocks be $b_1, b_2 \in [33]$. Of the first block consider the first 3 spots then if it's same color then we're done, WLOG for $a_1, a_2 \in [3]$ with $a_1 < a_2$ say $5b_1 + a_1, 5b_1 + a_2$ be red. Let $a_3 = 2a_2 - a_1 \in [5]$. If $7b_1 + a_3$ is red then we're done as $7b_1 + a_1, 7b_1 + a_2, 7b_1 + a_3$ is a mono A.P. So say $7b_1 + a_3$ is blue.
Since b_2 is the same coloring then let $b_3 = 2b_2 - b_1 \in [65]$. If $7b_3 + a_3$ is red then we have $7b_1 + a_1, 7b_2 + a_2, 7b_3 + a_3$. Otherwise if blue we have $7b_1 + a_3, 7b_2 + a_3, 7b_3 + a_3$.
Thus we're done and $W(3,2) \geq 65 \cdot 5 = 325$ □

Definition 5.15. *$WF(k,l,r)$ = minimum N such that any r-coloring of $[N]$ admits l color focused k-term A.P or a $k+1$ term A.P.*

Theorem 5.16.

$$WF(2,2,r) \leq (2r^{2r+1} + 1)(2r+1)$$

$$WF(2,3,r) \leq (2r^{2r^{2r+1}+1} + 1)(2r^{2r+1} + 1)(2r+1)$$

Definition 5.17. *Hales-Jewett: $\forall r, \forall n, \exists d$ such that in any r-coloring of $[n]^d$ hypercube, there is a monochromatic line.*

Definition 5.18. *A combinatorial line is a set of points represented by a string in $([n] \cup \{x\})^d \setminus [n]^d$. The points of the line are obtained by substituting $x = 1, 2, ..., n$.*

Definition 5.19. *A geometric line $([n] \cup \{x, \overline{x}\})^d \setminus [n]^d$ obtained by substituting in $x = 1, ..., n$ and $\overline{x} = n - x + 1$.*

Given an A.P-free coloring of $[N]$ want to give a line free coloring of $[N]^d$. Define $\phi : [n]^d \to (n-1)d$ by $\phi(a_0, a_1, ..., a_{d-1}) = a_0 + a_1 + \cdots + a_{d-1}$. Then we have

$$HJ(2,r) \leq d \iff 2^d < r \iff HJ(2,r) \leq \log_2 r$$

$HJ^c(2,r) = r$ as if we take any of $(0, ..., 0), (1, 0, ..., 0), ...$ there are $d+1 > r \implies$ a monochromatic combinatorial line.

For $HJ(3,2)$, take $p \in [3]^d$

Additive Combinatorics

Definition 5.20.

$$A + A = \{a + a' | a, a' \in A\} \qquad A \cdot A = \{a \cdot a' | a, a' \in A\}$$

Theorem 5.21. $\max(|A+A|, |A \cdot A|) \geq |A|^{1+\epsilon}$

Suppose we have a set A, $X = A + A, Y = A \cdot A$. Let $\mathcal{P} = X \times Y = (A + A) \times (A \cdot A)$. Let $\mathcal{L} = \{\{y | y = a(x - a')\} | a, a' \in A\}$. Then $|\mathcal{L}| = |A|^2$.

Define $i(\mathcal{L}, \mathcal{P})$ to be the number of incidences between the points and lines in $\mathcal{P}$ and $\mathcal{L}$.
For any line containing $a, a' \in A$, the equation is $y = a(x - a')$. For a point $p = (a' + a'', a \cdot a'')$ we have $a' + a'' \in A + A$ and $a \cdot a'' \in A \cdot A$.

$$i(\mathcal{L}, \mathcal{P}) \geq |\mathcal{L}| \cdot |A| = |A|^3$$

Then

$$i(\mathcal{L}, \mathcal{P}) = O(|\mathcal{L}|^{2/3} |\mathcal{P}|^{2/3} + |\mathcal{L}| + |\mathcal{P}|).$$

So $|A|^3 \leq i(\mathcal{L}, \mathcal{P}) \leq C(|A|^{4/3}(|A + A| \cdot |A \cdot A|^{2/3})$ as $|\mathcal{L}| + |\mathcal{P}| = O(|\mathcal{L}|^{2/3}|\mathcal{P}|^{2/3})$. We also have $|A|^2 \leq |\mathcal{P}| \leq |A|^4$ and $|\mathcal{P}|^{1/2} \leq |\mathcal{L}| \leq |\mathcal{P}|$. So $C|A|^{5/2} \leq |A \cdot A||A + A|$ and consequently

$$\max(|A + A|, |A \cdot A|) \geq \epsilon |A|^{5/4}$$

Planar Graphs

Theorem 5.22. *Euler's Formula for Planar Graphs:*

$$|V| - |E| + |F| = 2$$

Theorem 5.23. *Suppose G is a connected planar graph with $m \geq 3$. Then*

$$m \leq 3n - 6$$

Proof. Consider the bipartite graph of $E(G)$ and $|F(G)|$. For each edge there is at most 2 faces and each face is closed by at least 3 edges. So

$$2|E| \leq \sum \deg(e) = \sum \deg(f) \geq |F| \cdot 3$$

So $n - m + f = 2 \implies n - m + \frac{2}{3}m \geq 2 \implies n - \frac{1}{3}m \geq 2$ □

Definition 5.24. *Let $Cr(G)$ is the minimum number of crossing in any drawing.*

Given G, if $e(G) \geq 3n$, G is not planar $Cr(G) \geq m - 3n$ since at least $m - 3n$ edges must be removed to make G planar.
Consider G with G_p =graph where each vertex stays with probability p. Then $\mathbb{E}(np) = pn$ and $\mathbb{E}(mp) = p^2 m$. Then

$$p^4 Cr(G) \geq \mathbb{E}(Cr(G_p)) \geq \mathbb{E}(m_p - 3n_p) = \mathbb{E}(m_p) - 3\mathbb{E}(n_p) \geq p^2 m - 3pn$$

So $Cr(G) \geq \frac{m}{p^2} - \frac{3n}{p^3}$ and is maximized when $p = \frac{4n}{m}$ only when $4n \leq m$. Then $\frac{m}{p^2} - \frac{3n}{p^3} = \frac{m^3}{64n^2}$.

Theorem 5.25. *For any collection $\mathcal{L}$ of lines in $\mathbb{R}^3$, there are at most $O(|\mathcal{L}|^{3/2})$ joints.*

We just have to show the following lemma to imply the theorem.

Lemma 5.26. *In any collection of lines with $|J|$ joints, there exist some line in $\leq 3|J|^{1/3}$ joints.*

Lemma 5.26 $\implies$ theorem 5.25 as we define $J(L) =$ most joints in $|L|$ lines.

$$J(L) \leq J(L-1) + 3J^{1/3} \leq J(L-2) + 3(J-1)^{1/3} + 3J^{1/3} \leq \cdots$$

So $J \leq 3J^{1/3}L \iff J^{2/3} \leq 3L \iff J \leq \sqrt{27}L^{3/2}$
Given an arbitrary field, $\text{Poly}_D(\mathbb{F}^n)$ and $S = \{a_1, ..., a_k\}$ for $a_i \in \mathbb{F}^n$. We want to find a nonzero polynomial that vanishes at J.

Let $T : \text{Poly}_D(\mathbb{F}^n) \to \mathbb{F}^k$ defined as $T(p) = \begin{bmatrix} p(a_1) \\ \vdots \\ p(a_k) \end{bmatrix}$
By rank nullity theorem,

$$\dim(Im(T)) \leq k \implies \dim(\ker(T)) \geq \dim(\text{Poly}_D(\mathbb{F}^n)) - k$$

If $\dim(\text{Poly}_D(\mathbb{F}^n) > k, \exists p \in \text{Poly}_D(\mathbb{F}^n)$ vanishes at S, $|S| = k$.
Let

$$\mathcal{D} = \{x_1^{d_1}, x_2^{d_2}, ..., x_n^{d_n} | \sum d_i \leq D\}$$

This is a basis for $\text{Poly}_D(\mathbb{F}^n)$. By stars and bars we have $|\mathcal{D}| = \binom{D+n}{n} \geq \frac{D^n}{n!} > k$. We need $\frac{D^3}{3!} > J$ so $D > 3J^{1/3}$.
AFSOC each line has more than $D > 3J^{1/3}$ joints.
If $p \in \text{Poly}_D(\mathbb{F}), \forall a \in \mathbb{F}, \exists c \in \mathbb{F}$ such that $p(x) = (x-a)q(x) + c$. If a is a root, $p(x) = (x-a)q(x)$.
A line is a function $\gamma(t) = a + bt$ for $a, b \in \mathbb{F}^n$ then $q(t) := p(\gamma(t))$ is a polynomial in $\text{Poly}_D(\mathbb{F})$. $\deg(q)$ has to have at most the degree of p so $\deg(q) < D$. By our assumption the line has more than D joints so $\deg(q) > D$. The only way $q(t)$ can have more than D roots is if q is the zero polynomial. So we can conclude our polynomial p must be identically 0 on the union of all lines in $\mathcal{L}$

Each joint is the intersection of 3 lines and p is zero on all lines. So the direction derivative along each of the lines is 0 and as they are linearly independent we have $\nabla p = 0$ at every joint. Consider $p_1 = \frac{\partial p}{dx}$, $\deg(p_1) \leq D - 1$ and p_1 vanishes at every joint in J so we contradict the minimality of D so the assumption that all lines have $> D$ joints is false.

Lemma 5.27. *If $P(x_1, ..., x_n)$ is a non-zero polynomial over $\mathbb{F}_q$, with total degree $D \le q - 1$ then $P(x)$ cannot be zero for all $x \in \mathbb{F}_q^n$.*

Proof. We can write

$$P(x) = \sum_{k=0}^{D} q_k(x_1, ..., x_{n-1}) x_n^k$$

We'll show the statement by induction on n.

(Base Case $n = 1$**)** AFSOC $P(x_1)$ is nonzero and vanishes on all of $\mathbb{F}_q$. Since $\deg(P) \le q - 1$ but P has q distinct roots, P must be the zero polynomial. This is a contradiction.

(Inductive Step) AFSOC $P(x) = 0$ on all $x \in \mathbb{F}_q^n$. We write

$$P(x) = \sum_{k=0}^{D} q_k(x_1, ..., x_{n-1}) x_n^k$$

Since $P(x)$ is a nonzero polynomial, there must exist at least one k for which $q_k(x_1, ..., x_{n-1})$ is a non-zero polynomial.

Fix the first $n - 1$ variables. Let $(a_1, ..., a_{n-1})$ be an arbitrary point in $\mathbb{F}_q^{n-1}$. Define

$$Q(t) := P(a_1, ..., a_{n-1}, t)$$

We can express $Q(t) = \sum_{k=0}^{D} q_k(a_1, ..., a_{n-1}) t^k$. For this fixed $(a_1, ..., a_{n-1})$, each $q_k(a_1, ..., a_{n-1})$ is a constant in $\mathbb{F}_q$. This implies $\deg(Q) \le D \le q - 1$.

By assumption, $P(x) = 0$ everywhere, so $Q(t) = P(a_1, ..., a_{n-1}, t) = 0$ for all $t \in \mathbb{F}_q$. From our base case, a single-variable polynomial of degree $\le q - 1$ that has q roots must be the zero polynomial. This means all coefficients of $Q(t)$ must be zero.

Therefore $q_k(a_1, ..., a_{n-1}) = 0$ for all k. Since $(a_1, ..., a_{n-1})$ was arbitrarily chosen, this holds for all points in $\mathbb{F}_q^{n-1}$.

This means each q_k is a polynomial in $n - 1$ variables that vanishes on all of $\mathbb{F}_q^{n-1}$. The total degree of P is $D = \max_k(\deg(q_k) + k)$, which implies $\deg(q_k) \le D \le q - 1$. By our inductive hypothesis, a polynomial in $n - 1$ variables of degree $\le q - 1$ that vanishes everywhere must be the zero polynomial.

Thus, each q_k is the zero polynomial. This implies $P(x)$ is the zero polynomial, which contradicts our initial assumption that $P(x)$ is a non-zero polynomial. □

Theorem 5.28. *If $N \subseteq \mathbb{F}_q^n$ is a set with the property that for all $x \in \mathbb{F}_q^n$, there is a line L_x such that $L_x \setminus \{x\} \subseteq N$, then $|N| \ge \epsilon_n q^n$ where $\epsilon_n > 0$ depends only on n. (The proof shows $\epsilon_n = (10n)^{-n}$).*

Proof. Assume for the sake of contradiction that $|N| < \left(\frac{q}{10n}\right)^n$.

We know from the polynomial method that there exists a non-zero polynomial $p \in \mathrm{Poly}_D(\mathbb{F}_q^n)$ that vanishes on N, with degree $D \le 2n|N|^{1/n}$.

Using our AFSOC, we can bound this degree D:

$$D \le 2n|N|^{1/n} < 2n\left(\frac{q}{10n}\right) = \frac{q}{5}$$

So, we have found a non-zero polynomial p with total degree $D < q/5$.

Now, consider any arbitrary $x \in \mathbb{F}_q^n$. By the theorem's premise, there is a line L_x through x such that $L_x \setminus \{x\} \subseteq N$. We can parametrize this line as $\gamma(t) = x + d \cdot t$ for $t \in \mathbb{F}_q$, where $d \in \mathbb{F}_q^n \setminus \{0\}$ is a direction vector. Note that $\gamma(0) = x$, and $L_x \setminus \{x\} = \{\gamma(t) \mid t \in \mathbb{F}_q \setminus \{0\}\}$.

Define a new, single-variable polynomial $R(t) := p(\gamma(t))$. The degree of $R(t)$ is at most the total degree of p, so $\deg(R) \le D < q/5$.

Since p vanishes on N, p must vanish on $L_x \setminus \{x\}$. This means $R(t) = p(\gamma(t)) = 0$ for all $t \in \mathbb{F}_q \setminus \{0\}$. The set $\mathbb{F}_q \setminus \{0\}$ has $q - 1$ elements, so $R(t)$ has $q - 1$ distinct roots.

We have a polynomial $R(t)$ with $\deg(R) \le D < q/5$. For any $q \ge 3$, we have $q/5 \le q - 2$ (since $10 \le 4q$).

Thus, $R(t)$ is a polynomial with degree strictly less than $q-1$, but it has $q-1$ roots. A non-zero polynomial cannot have more roots than its degree. Therefore, $R(t)$ must be the zero polynomial.

If $R(t)$ is the zero polynomial, it must be zero for all t, including $t=0$.

$$R(0) = p(\gamma(0)) = p(x) = 0$$

Since $x \in \mathbb{F}_q^n$ was arbitrary, we have shown $p(x) = 0$ for all $x \in \mathbb{F}_q^n$. We also know $\deg(p) = D < q/5$, which implies $\deg(p) \leq q-1$.

By Lemma 5.27, any polynomial with degree $\leq q-1$ that vanishes on all of $\mathbb{F}_q^n$ must be the zero polynomial. This contradicts our choice of p as a non-zero polynomial.

Therefore, our initial assumption was false, and we must have $|N| \geq \left(\frac{q}{10n}\right)^n$. □

Lemma 5.29. *If $p \in poly_{a-1}(\mathbb{F}_q^n)$ is nonzero, $|zero(P)| < q^n$.*
Maximized when $x_1^{q-1} - 1$

Theorem 5.30. *Schartz-Zippel: If nonzero $p \in poly_D(\mathbb{F}^n)$ and $S \subseteq \mathbb{F}$ a finite subset. For random $s_1, ..., s_n \in S$*

$$\mathbb{P}_{s_1,...,s_n}(p(s_1, ..., s_n) = 0) \leq \frac{D}{|S|}$$

In other words, $|zero(p) \cap S^n| \leq D|S|^{n-1}$

Proof. Let $p \in poly_D(\mathbb{F}^n)$ be nonzero. We're done if $n=1$. Do induction on n.

$$p(x_1, ..., x_n) = \sum_{k=0}^{n} q_k(x_1, ..., x_{n-1})x_n^k.$$

Choose $k_0$to be largest such that $q_{k_0} \neq 0$. By induction

$$\mathbb{P}_{s_1,...,s_{n-1}}(q_{k_0}(s_1, ..., s_{n-1}) = 0) \leq \frac{D-k_0}{|S|}$$

$\mathbb{P}_{s_1,...,s_n}(p(s_1, ..., s_n) = 0) \leq \mathbb{P}_{s_1,...,s_n}(q_{k_0}(s_1, ..., s_{n-1}) = 0) + \mathbb{P}_{s_1,...,s_n}(p(s_1, ..., s_n) = 0 | q_{k_0}(s_1, ...s_{n-1}) \neq 0)$

Note: This is just $\mathbb{P}(B) \leq \mathbb{P}(C) + \mathbb{P}(B|\neg C) \cdot \mathbb{P}(\neg C)$
$q_{k_0}(s_1, ..., s_{n-1}) \neq 0 \implies p(s_1, ..., s_{n-1}, x_n)$ has degree k_0.

$$\frac{\sum_{s_1,...,s_{n-1}} \prod_{q(s_1,...,s_{n-1})\neq 0} \frac{1}{|S|^{n-1}} \mathbb{P}(p(s_1, ..., s_n) = 0 | q_{k_0}(s_1, ..., s_{n-1}) \neq 0)}{\sum_{s_1,...,s_{n-1}} \prod_{q(s_1,...,s_{n-1})\neq 0} \frac{1}{|S|^{n-1}}}$$

$$\mathbb{P}_{s_1,...,s_n}(q_{k_0}(s_1, ..., s_{n-1}) + \mathbb{P}_{s_1,...,s_n}(p(s_1, ..., s_n) = 0 | q_{k_0}(s_1, ...s_{n-1}) \neq 0) \leq \frac{D-k_0}{|S|} + \frac{k_0}{|S|} = \frac{D}{|S|}$$

□

Theorem 5.31. *Extremal Schwartz-Zippel. If p nonzero of degree d, $p = 0$ on S^n then $|S| \leq d$.*

Example 5.32.

$$p = \prod_{s \in S}(x_i - s)$$

is 0 for all $x \in S^n$. This holds for $S \times \{1\} \times \{1\} \times \cdots$.

Example 5.33.

$$q = \prod_{a_1 \in S_1}(x_1 - a_1) \prod_{a_2 \in S_2}(x_2 - a_2) \prod_{a_3 \in S_3}(x_3 - a_3)$$

Say S_1, S_2, S_3 has size $4, 3, 2$, respectively. If in a $5 \times 4 \times 3$ box then q is definitely not zero polynomial.

Theorem 5.34. *Combinatorial Nullstellensatz: Suppose p is a nonzero polynomial in $Poly_d(\mathbb{F}^n)$ of degree d and the monomial $x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ for $j_1+\cdots+j_n = d$ has nonzero coefficients then $\forall S_1,...,S_n \subseteq \mathbb{F}$ with $|S_i| \geq j_i+1$ for all i, $p(x)\neq 0$ for some $x \in S_1\times S_2\times\cdots\times S_n$.*

Proof. We'll show it by induction on d.
Suppose $f \equiv 0$ on $S_1\times S_2\times\cdots\times S_n$ where $S_i \subseteq \mathbb{F}$.
WLOG $j_i \geq 1, \forall s \in S_1$ we have $f(x) = (x_1-s)q_s(x)+r(x)$ then $\deg(q_s) = d-1$, moreover coefficients of $x_1^{j_1-1}x_2^{j_2}\cdots x_n^{j_n}$. For any $s \in S_1, s_2 \in S_2, ...$ we have $f(s,s_2,...,s_n) = 0 \implies r_s(s,s_2,...,s_n) = 0\forall s_2,...,s_n \implies r_s(s_1,...,s_n) = 0\forall s_1 \in S_1, s_2 \in S_2,...$
By our assumption we have $0 = f(s_1,...,s_n) = (s_1-s)q_s(s_1,...,s_n)$ for all $s, s_1 \in S_1,...,s_n \in S_n$. For $s \neq s_1$ we learn $q_s(s_1,...,s_n) = 0$. Since q_s is zero on $S_1\setminus\{s\}\times S_2\times\cdots\times S_n \implies$ we must have $|S_1\setminus\{s\}| \leq j_1-1$ or for some i, $|S_i| \leq j_i$. by induction hypothesis. □

Example 5.35. *$\mathcal{X}'(G)$ list chromatic number of G. We want to find $\mathcal{X}'(C_n)$.
Consider an assignment c_i to each vertex $i, c_i \in \mathbb{N}$. $f(c_1,...,c_n) = (c_2-c_1)(c_3-c_2)\cdots(c_n-c_{n-1})(c_1-c_n)$. The leading term of f is $2c_1c_2\cdots c_n$. For all sets $S_1,...,S_n \subseteq \mathbb{N}$, chromatic number implies $\exists c_1 \in S_1, c_2 \in S_2,...,c_n \in S_n$ such that $f(c_1,...,c_n) \neq 0$.*

Example 5.36. *Cauchy Davenport: Let p prime, $A, B \subseteq \mathbb{Z}_p$*

$$|A+B| \geq \min(p, |A|+|B|-1)$$

Proof. Case 1: If $|A|+|B|-1 \geq p$
Consider $x \in \mathbb{Z}_p$ then $|x-A| = |A|$ so $|A-x|+|B| \geq p+1$ and $\exists y \in A_x \cap B \implies \exists a \in A, b \in B$ such that $y = b, y = x-a$ so $x = a+b$.

Case 2: If $|A|+|B|-1 < p$
Consider any set $C \subseteq \mathbb{Z}$ of size $|C| = |A|+|B|-2$. We want to show $\exists x \in A+B, x \notin C$.
Define
$$f(a,b) = \prod_{c\in C}(a+b-c)$$
We have $\deg(f) = |C| = |A|+|B|-2$, consider the monomial $a^{|A|-1}b^{|B|-1}$ then the coefficient is $\binom{|A|+|B|-2}{|A|-1} \neq 0$ in $\mathbb{Z}_p$. So for $|A|, |B|$ we have a choice of a, b such that $a+b \notin C$. □

Definition 5.37. *Finite Kakeya: In $\mathbb{F}_a^n, \forall a, \exists b$ such that $\{at+b|t \in \mathbb{F}_a\} \subseteq K$. Then K is a kakeya set.*

Theorem 5.38. *Chevalley–Warning theorem:
Let $a = p^l$ for $f_1,...,f_k \in \mathbb{F}_a[x_1,...,x_n]$.
If $\sum_i \deg(f_i) < n$ then the number of common zeros is a multiple of p. In particular: if there's 1 common zero then there is more.*

Example 5.39. *Given any n numbers $a_1,...,a_n$ there is a nonempty subset that sums to 0 (mod n).*

Proof. Let $S_0 = \{\}, S_1 = \{a_1\},...,S_n = \{a_1,...,a_n\}$ then there exists i,j such that $S_i = S_j$ so □

Theorem 5.40. *Erdos-Ginzburg-Ziv Theorem:*
How large a collection of numbers do I require to ensure that some n-subset sum to a multiple of n?
$2n-1$ is enough

Proof. (Main Case) $n = p$ is a prime
Given numbers $a_1, ..., a_{2p-1}$, we'll give two polynomials in $2p-1$ variables $x_1, ..., x_{p-1}$
We want a polynomial such that x_i behaves like indicators for $a_i \in S$. So $x_i^{p-1} \equiv 1 \bmod p$ by FLT.

$$f(x_1, .., x_{2p-1}) = \sum_{x_i} x_i^{p-1} = \#\{i | x_i \neq 0\}$$

$$g(x_1, ..., x_{2p-1}) = \sum_{x_i} a_i x_i^{p-1} = \sum_{x_i \neq 0} a_i$$

We have $2p-2 < 2p-1$ and the trivial solution exist so a non-trivial solution exist by Chevalley-Warning
(General Case) Induction on n, $a_1, ..., a_{2n-1}$
If n not prime, let p be a prime factor of n, $m = \frac{n}{p}$. Find a set I_i, $|I_i| = p$ such that $\sum_{j \in I_i} a_j = 0 \pmod{p}$ for $i \in [2m-1]$
Say $\sum_{i \in I_j} a_i = b_i \equiv 0 \pmod{p}$
Let $c_i = \frac{b_i}{p}$, we can find $c_{i_1}, c_{i_2}, ..., c_{i_m}$ such that

$$\sum_{j=1}^{m} c_{i_j} = \sum_{j=1}^{m} \sum_{t \in I_{i_j}} t = \left(\sum_{j=1}^{m} c_{i_j} \right) p \equiv 0$$

□

Theorem 5.41. *There exist an order of at least d^2 2-distance set in $\mathbb{R}^d$?*

Proof. Suppose $S = \{p_1, ..., p_m\}$ has just 2 distances α and β. Consider the polymomial, $f \in \mathbb{R}[x_1, ..., x_d]$ defined as

$$f_i(X) = (||X - p_i||^2 - \alpha^2)(||X - p_i||^2 - \beta^2)$$

$$f_i(X) = \begin{cases} \alpha^2\beta^2 & \text{if } X = p_i \\ 0 & \text{otherwise} \end{cases}$$

Claim: f_i's are independent.
Suppose $\alpha_1 f_1 + ... + \alpha_m f_m = 0$, $\forall i$, plug in p_i gives $\alpha_i \alpha^2 \beta^2 = 0 \implies \alpha_i = 0$
Claim: $x_i^{d_i} x_j^{d_j}$ with $d_1 + d_2 \leq 4$ covers all possible terms. So there is at most $O(d^2)$ possible choices. □

Example 5.42. *Eventown where each club has even size and even intersection, $\geq 2^{\lceil n + \frac{1}{2} \rceil}$*

Example 5.43. *Oddtown where each club has odd size and even intersection*

Let v_i =indicence vector of club i in $\mathbb{F}_2$. $v_i \cdot v_j = \begin{cases} 0 & \textit{if } i \neq j \\ 1 & \textit{otherwise} \end{cases}$

Suppose $\alpha_1 \vec{v_1} + \cdots + \alpha_n \vec{v_n} = \vec{0}$
$\vec{v_i}$ on both sides so $\alpha_i = 0$

Theorem 5.44. *Every Polygon has a triangulation*

Proof. Choose a convex vertex of the polygon (a vertex that is a vertex of the convex hull) with neighbors q, r. If $\overline{qr} \subset P^\circ$ then we're done. Othewise we can move the point. □

Definition 5.45. *We say polygon $P \sim Q$ by scissor congruency if we can cut up P and reassembled to be Q.*

Lemma 5.46. *(Any rectangle)* $\sim$ *(Any Unit Size Rectangle)*
(Any triangle) $\sim$ *(Unit Side Rectangle)*
(Triangle) $\sim$ *(2 Right Triangle)*
(Right Triangle) $\sim$ *(Rectangle)*

Remark 5.47. *From this lemma we can conclude any polygon is congruent to a rectangle of* $1 \times d$

Theorem 5.48. *Are equal-area polyhedra necessarily plane-dissection equivalent? This is not true.*

Example 5.49. *Unit cube and volume* 1 *reg-tetrahedron are not dissection equivalent.*

Proof. Dihedral angle is an irrational multiple of π
A list of vectors $\vec{v_1}, ...$ is independent if for every finite sum,

$$\sum_{j=1}^{k} \alpha_{i_j} v_{i_j} = 0 \implies \alpha_{i_j} = 0 \forall j$$

Span$(\mathcal{L})$ is the set of vectors representable as finite linear combinations. $\mathcal{L}$ is a basis for V if $\mathcal{L}$ is independent and *Span*$(\mathcal{L}) = V$

Lemma 5.50. *Zorn's Lemma: P is a poset in which every chain has an upper bound then P has a maximal element.*

Doset is a set of independent sequences, ordered by inclusion.

$$\mathcal{L}_1 \subseteq \mathcal{L}_2 \cdots \subseteq$$

Then $\cup \mathcal{L}_i$ is still independent. By Zorn's lemma and Doset we have every vector space has a (possibly infinite) basis.

Define α to be dihedral angle of tetrahedron, we'll use that $\frac{\alpha}{\pi}$ is irrational.
In general, we can define a linear transformation $f : \mathbb{R} \to \mathbb{R}$ such that $f(\pi) = 0$ and $f(\alpha) = 1$.
Since α and π are independent over $\mathbb{Q}$ extended to a basis $\alpha, \pi, v_3, v_4, ...$
Define f using this basis by defining $f(\alpha) = 1, f(\pi) = 0$

If a plane goes through an angle then $l_e = l_{e_1} = l_{e_2}$ and $\theta_e = \theta_{e_1} + \theta_{e_2}$.
If a plane goes through an edge then $l_{e_1} + l_{e_2} = l_e$ and $\theta_e = \theta_{e_1} = \theta_{e_2}$
If a plane goes through another plane and creates a new edge then $l_{e_1} = l_{e_2}$ and $\theta_{e_1} + \theta_{e_2} = \pi$
We can assign a real number to each polytope by

$$\sum_{e \in P} l_e \cdot f(\theta_e)$$

In the first case we would have

$$R(P) = \sum_{e \in P} l_e f(\theta_e)$$

If P is a cube then $R(P) = 0$ as each dihedral angle is 90° so it's a rational multiple of π. However if P is a tetrahedron has irrational multiple of π so $P(R) \neq 0$. □

Linear Algebra

Definition 5.51. *Adjacency Matrices: On the vertex set* $[n]$

$$A = \begin{bmatrix} & \end{bmatrix}$$

$A_{i,j} = \begin{cases} 1 & \textit{if } i \sim j \\ 0 & \textit{otherwise} \end{cases}$

If $f : V \to \mathbb{R}$ then $Af = g$ and if $Af = \lambda f$ then it's an eigen function.

Remark 5.52. *We'll denote the* $n \times n$ *matrix of all ones as* J_n*.*

Theorem 5.53. J_n*'s eigenvalues are* $\lambda_1 = n$ *with multiplicity* 1 *and* $\lambda_2 = 0$ *with multiplicity* $n - 1$*.*

Proof. To show $\lambda = 0$ has multiplicity $n - 1$ the associated eigenvectors are

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ \vdots \end{bmatrix}, \cdots,$$

□

Theorem 5.54. $K_n = J_n - I_n$ *has eigenvalue* $n - 1$ *with multiplicity* 1 *and* -1 *with multiplicity* $n - 1$.

Remark 5.55. *For any regular graph with degree* d*,* d *is an eigenvalue value with multiplicity* 1.

Lemma 5.56. *For any adjacency matrix* A*,* A^2 *has the property that* $A^2_{i,j} = \#$*walks of length* 2 *from* $i \to j$*. This generalizes easily to the general case for* A^k

Proof. This is easily shown from the matrix multiplication

$$A_{i,j} = \sum_{k \in [n]} A(i,k)A(k,j)$$

□

Lemma 5.57. *For a* d*-regular graph with diameter* 2 *graph then the maximum number of vertices is* $1 + d + d(d-1) = d^2 + 1$ *vertices.*
To achieve this bound, I require girth$(G) \geq 5$
Note: Girth is the length of a shortest cycle.
Another way to achieve this bound is the Peterson Graph

Lemma 5.58. *Any Moore graph (regular graph whose girth is at least twice its diameter) has $A^2 + A - (d-1)I = J$.*

Proof. We know $\lambda = d$ an eigenvalue for $f \equiv 1$. By spectral theorem, A has an orthogonal basis of real eigenvectors. Since $f_j \perp f_1$ for $j \neq 1$ then $J \cdot f_j = \vec{0}$ as J is the matrix of all ones. So

$$A^2 f_j + Af_j - (d-1)If_i = 0 \iff \lambda_j^2 + \lambda_j - (d-1) = 0$$

We can conclude $\lambda = \frac{-1 \pm \sqrt{4d-3}}{2}$
We need $1 + m_2 + m_3 = n$ and $d + m_2\lambda_2 + m_3\lambda_3 = 0$
So $2d - (m_2 + m_3) + (m_2 - m_3)\sqrt{4d-3} = 0$
(Case 1) If $\sqrt{4d+3} \notin \mathbb{Q}$ then $m_2 = m_3$ and $2d = d^2 \implies d = 2$
(Case 2) If $\sqrt{4d-3} = s \in \mathbb{Z}$ then $d = \frac{s^2+3}{4}$ then

$$2d - d^2 + (m_2 - m_3)s = 0 \iff 8\left(\frac{s^2+3}{4}\right) - (s^2+3)^2 + 16(m_2 - m_3)s = 0$$

.
Expanding out we have $as^4 + bs^3 + cs^2 + ds + 15 = 0$ then $s|15 \implies s = 1, 3, 5, 15 \implies d = 1, 3, 7, 57$. □

We'll be covering graph where for any 2 vertices u, v there is exactly one common neighbor of u, v

Theorem 5.59. *"There is a politician": $\exists v_0$ such that $\forall u, v_0 \sim u$*
Note: This doesn't hold for infinite vertices by $H_0 =$ 5-cycle and H_{i+1} is H_i with independent path of length 2 added between parts that don't have a common neighbor in H_i.

Step 1 A counterexample must be regular

Step 1A $u \not\sim v \implies \deg(v) \geq \deg(u)$. By symmetry $\deg(v) = \deg(u)$. This is from w_1 being the common neighbor of u, v and w_2 being the common neighbor of w_1, u and z_1 being common neighbor of w_1 and v.

Step 1B Let $\deg(u) = d, \forall v \neq w_i$ we get $\deg = d$ for all $w_2, ..., w_d$, we get $\deg(w_i) = \deg(v) = d$. All but w_1 are known to be degree d. Since w_i not a politican then w_1 must be the politican.

Going back to the graph with diameter 2 graph then $n = 1 + d(d-1) = d^2 - d + 1$. There are exactly 1 path of length 2 between u, v $\deg(u) = d \implies A^2$ is d along diagonal and 1 everywhere else. $A^2 = J + (d-1)I$. J has e.v. n with multiplicity 1 and 0 with multipicity $n-1$. So A^2 has e.v. $n + d - 1$ with multiplicity 1 and $d - 1$ with multiplicity $n - 1$.
A has eignevalues d with multiplcitiy 1, $\sqrt{d-1}$ with multiplicity s and $-\sqrt{d-1}$ with multiplicity t. Also $s + t = n - 1$. The trace of A is 0 so $d + \sqrt{d-1}s - \sqrt{d-1}t = 0 \implies d + (s-t)\sqrt{d-1} = 0$. So $\sqrt{d-1} \in \mathbb{Q} \implies h := \sqrt{d-1} \in \mathbb{N}$.
We have $d = \sqrt{d-1}^2 + 1 = h^2 + 1$. So $d + h(s-t) = 0 \implies h^2 + 1 = h(t-s) \implies h = 1 \implies d = 2$

Theorem 5.60. *Oddtown: Clubs have odd size and intersections are even. The clubs are less than number of people.*

Lemma 5.61 (Fisher's Inequality)**.** *Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of m distinct subsets of a universe X where $|X| = n$. Suppose there exists a constant k such that $|A_i \cap A_j| = k$ for all $i \neq j$. Furthermore, assume that $|A_i| > k$ for all i. Then:*

$$m \leq n$$

Proof. Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be the incidence vectors of the sets $A_1, \ldots, A_m$. That is, the x-th component of vector v_i is 1 if $x \in A_i$ and 0 otherwise.
We aim to show that these vectors are linearly independent. Consider a linear combination of these

vectors equal to the zero vector, with coefficients $\alpha_1, \dots, \alpha_m \in \mathbb{R}$:

$$\sum_{i=1}^{m} \alpha_i v_i = 0 \tag{1}$$

We take the squared Euclidean norm (the dot product with itself) of both sides:

$$\left\langle \sum_{i=1}^{m} \alpha_i v_i, \sum_{j=1}^{m} \alpha_j v_j \right\rangle = 0$$

Expanding using the linearity of the inner product:

$$\sum_{i=1}^{m} \alpha_i^2 \langle v_i, v_i \rangle + \sum_{i \neq j} \alpha_i \alpha_j \langle v_i, v_j \rangle = 0$$

We observe the following properties of the incidence vectors:

- $\langle v_i, v_i \rangle = |A_i|$
- $\langle v_i, v_j \rangle = |A_i \cap A_j| = k$ (for $i \neq j$)

Substituting these values into the equation:

$$\sum_{i=1}^{m} \alpha_i^2 |A_i| + k \sum_{i \neq j} \alpha_i \alpha_j = 0$$

To simplify the second term, we use the identity $(\sum \alpha_i)^2 = \sum \alpha_i^2 + \sum_{i \neq j} \alpha_i \alpha_j$. Rearranging this gives $\sum_{i \neq j} \alpha_i \alpha_j = (\sum \alpha_i)^2 - \sum \alpha_i^2$. We substitute this back into our equation:

$$\sum_{i=1}^{m} \alpha_i^2 |A_i| + k \left[\left(\sum_{i=1}^{m} \alpha_i \right)^2 - \sum_{i=1}^{m} \alpha_i^2 \right] = 0$$

$$\sum_{i=1}^{m} \alpha_i^2 (|A_i| - k) + k \left(\sum_{i=1}^{m} \alpha_i \right)^2 = 0$$

Since we assumed $|A_i| > k$, we have $|A_i| - k > 0$. Also, squares of real numbers are non-negative (assuming $k > 0$ and observing $(\sum \alpha_i)^2 \geq 0$). Therefore, we have a sum of non-negative terms equaling zero. This implies that every individual term must be zero. Specifically:

$$\alpha_i^2 (|A_i| - k) = 0 \quad \forall i$$

Since $|A_i| - k \neq 0$, it must be that $\alpha_i = 0$ for all i.
Thus, the vectors $v_1, \dots, v_m$ are linearly independent. Since they exist in $\mathbb{R}^n$, the dimension of the subspace they span cannot exceed n, implying $m \leq n$. □

Theorem 5.62. *$R(k+1) \geq \binom{k}{3} + 1$. We can group every 3 vertices then color it red*

A quadratic form/homogeneous polynomial say $q(x, y) = x^2 + 2xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. There is an bijection between quadratic form and symmetric matrices.
We can write $A = P^T B P$ where B is a diagonal matrix with eigenvalues

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Definition 5.63. *For a symmetric matrix A,*

1. *Positive definite if $\lambda_i > 0$ for all i*
2. *Negative definite if $\lambda_i < 0$ for all i*

Theorem 5.64. *Given symmetric $A, q_A(X) = X^TAX$. Let A be real, symmetric $n \times n$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigen values of A. We can conclude*

$$\lambda_k = \max_{U \subseteq V} \max_{\dim(U)=k} \min_{X \in U} \frac{X^TAX}{X^TX}$$

Remark 5.65. *This statement doesn't care of the magnitude of X and only the direction. We're looking for the direction of greatest change.*

Proof. We'll first show $\lambda_k \leq \max_U$, For this direction suffices to exhibit one good u. For $v_1, ..., v_n$ orthonormal eigen basis, v_i eigenvalues for λ_i. Let $U_k = \{v_1, ..., v_k\}$ and let $X \in \text{Span}(U_k)$ so $X = \sum_{i=1}^{k} \alpha_i v_i$. WLOG $|X| = 1$ since the magnitude does not change the result. This implies $\sum \alpha_j^2 = 1$.

$$\begin{aligned} X^TAX &= \left(\sum \alpha_j v_j\right)^T A \left(\sum \alpha_j v_j\right) && (2)\\ &= \left(\sum \alpha_j v_j\right)^T \left(\sum \alpha_j \lambda_j v_j\right) && (3)\\ &= \sum \lambda_j \alpha_j^2 && (4) \end{aligned}$$

This is a weighted average of $\lambda_1, ..., \lambda_k$ so this is at least the $\min(\lambda_1, ..., \lambda_k) = \lambda_k$.

For the other direction $\lambda_k \geq \max_U$. Given any U_k, we want to show $\exists X \in U_K$ such that $\frac{X^TAX}{X^TX} \leq \lambda_k$. Let $W = \text{Span}(v_k, ..., v_n)$ then $W = n - k + 1$. So there exist a vector $X \neq 0, X \in W \cap U_k$. WLOG, take $|X| = 1$. Since $X \in W, X = \sum_{j=k}^{n} \alpha_j v_j$ and

$$X^TAX = \sum_{j=k}^{n} \lambda_j \alpha_j^2 \leq \max(\lambda_k, ..., \lambda_n) = \lambda_k$$

□

Example 5.66. *Given a d-regular graph G with adjacency matrix A with $\lambda_1 \geq ... \geq \lambda_n \geq -d$. Suppose I had a negative eigenvalue that is less than $-d$ then any vertex when applied the adjacency matrix will be the sum of the neighboring vertices, but we can't have $|\sum| > d^2$.*
Given an independent S of size α. Define a vector $v = nI_S - \alpha I = (n-\alpha)I_S - \alpha I_{\overline{S}}$. Then $v \cdot I = 0$. We know

$$\min_{X \subseteq \mathbb{R}^n} \frac{X^TAX}{X^TX} = \lambda_k$$

So we know $\lambda_k \leq \frac{v^TAv}{v^Tv}$.

$$\begin{aligned} v^Tv &= (nI_S - \alpha I)(nI_S - \alpha I) = \alpha n^2 - 2\alpha^2 n + \alpha^2 n = \alpha n(n-\alpha) && (5)\\ v^TAv &= (nI_S - \alpha I)A(nI_S - \alpha I) && (6)\\ &= nI_S A n I_S - 2\alpha^2 I_S A I + \alpha^2 IAI && (7)\\ &= 0 - 2n\alpha I_S \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} + \alpha^2 nd && (8)\\ &= -n\alpha^2 d && (9) \end{aligned}$$

So we have $\frac{-n\alpha^2 d}{\alpha n(n-\alpha)} = \frac{-\alpha d}{(n-\alpha)} = \frac{d}{1-\frac{n}{\alpha}} \geq \lambda_n$. When we solve for α

Definition 5.67. *Expander Graphs*

Definition 5.68. *Let $G = (V, E)$ and $|V| = n$*
Cheeger Constant

$$h(G) = \min_{S \subseteq V, |S| \leq \frac{n}{2}} \frac{e(S, \overline{S})}{|S|}$$

Remark 5.69. *$h(G) \leq d$ and $h(G) = 0 \iff G$ is disconnected.*

Definition 5.70. *G bipartite on (L, R) with $|L| = |R| = n$ is a (d, α) expander if*

1. *every degree in L is d*
2. *every set S of size $\leq \frac{n}{d}$ in L has $\alpha|S|$ neighbors (in R)*

Theorem 5.71. *Let $d \equiv 4$, choose d edges from each vertex in L independent and 1 and only.*
Claim: With constant probability, result is a $(d, \frac{d}{10})$ bipartite expander.

Proof. Let the bad events be for sets $S \subseteq L, T \subseteq R$, $|S| \leq \frac{n}{d}, |T| < \alpha|S|, E_{S,T} = \{N(S) \subseteq T\}$. Then $\mathbb{P}(E_{S,T}) = \left(\frac{|T|}{n}\right)^{d|S|}$

$$\begin{aligned}
\mathbb{P}(\exists S, T, |S| \leq \frac{n}{d}, |T| = \alpha|S|, E_{S,T}) &\leq \sum_{s=1}^{n/d} \binom{n}{s} \binom{n}{\alpha s} \left(\frac{\alpha s}{n}\right)^d s && (10)\\
&\leq \sum_{s=1}^{n/d} \binom{n}{\alpha s}^2 \left(\frac{\alpha s}{n}\right)^d s && (11)\\
&\leq \sum_{s=1}^{\infty} \left(\frac{en}{\alpha s}\right)^{2\alpha s} \left(\frac{\alpha s}{n}\right)^{ds} && (12)\\
&= \sum_{s=1}^{\infty} \left(\frac{n}{d}\right)^{(2\alpha - d)s} e^{2\alpha s} s^{(d-2\alpha)s} && (13)\\
&= \sum_{s=1}^{\infty} \left(\frac{n}{\alpha s}\right)^{2\alpha - d)s} e^{2\alpha s} && (14)\\
&\leq \sum_{s=1}^{\infty} 10^{(2\alpha - d)s} e^{2\alpha s} && (\alpha \leq d/10)\\
&= \sum_{s=1}^{\infty} \left(10^{2\alpha - d} e^{2\alpha}\right)^s < 1 && (15)
\end{aligned}$$

□

Lemma 5.72. *If p has 1 fermat witnesses, half of the a's relatively prime to p are fermat witnesses.*

Example 5.73. *Prime Algorithm*

1. *Randomly choose a*
2. *Compute $a^p \pmod{p}$*
   If $\not\equiv a$, report not prime else report maybe prime

From the previous lemma if not prime, at least half the a will show it.
We want a deterministic expander graph and on the L

Theorem 5.74. *Similar setup to theroem 5.64,*

$$\lambda_k = \min_{\dim(U)=k-1} \max_{X \perp U} \frac{X^T A X}{X^T X}$$

Theorem 5.75. *We want to relate the spectral gap to the Creeger Constant*

Proof. $\vec{v} = nI_S - sI = (n-s)I_S - sI_{\overline{S}}$ with $s = |S|$
G is a d-regular graph, A is an adjacency matrix, we have

$$\lambda_2 = \max_{x \perp I} \frac{x^T A x}{x^T x} \geq \frac{\vec{v}^T A \vec{v}}{\vec{v}^T v}$$

Note: $\vec{v} \cdot I = ns - ns = 0$ and $\vec{v} \cdot \vec{v} = (n-s)^2 s + s^2(n-s) = s(n-s)n$.
Since the graph is d-regular we have $ds = 2e(S) + e(S, \overline{S})$ and $d(n-s) = 2e(\overline{S}) + e(S, \overline{S})$.

$$\begin{aligned}
\vec{v}^T A \vec{v} = \sum_{(i,j) \in E(G)} v_i v_j &= 2e(S)(n-s)^2 - 2e(S, \overline{S})s(n-s) + 2e(\overline{S})s^2 \\
&= (ds - e(S, \overline{S}))(n-s)^2 - 2e(S, \overline{S})s(n-s) + (d(n-s)e(S, \overline{S}))s^2 \\
&= ds(n-s)n - e(S, \overline{S})[(n-s)^2 + 2s(n-s) + s^2] \\
&= ds(n-s)n - e(S, \overline{S})n^2
\end{aligned}$$

Then substituting back we have

$$\lambda_2 \geq \frac{ds(n-s)n - e(S, \overline{S})n^2}{s(n-s)n} = d - \frac{e(S, \overline{S})n}{s(n-s)}$$

$\forall S \subseteq V, |S| \leq \frac{n}{2}$ we have

$$d - \lambda_2 \leq \frac{2e(S, \overline{S})}{|S|} \implies \lambda_1 - \lambda_2 \leq 2h(G)$$

□

Example 5.76. *Consider $A_{K_n} = J - I$ has eigenvalues $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$.*

Lemma 5.77. *Lower bound on λ_2 with G is d-regular, $A = A_G$, A^2 has eigen values $\lambda_1^2, ..., \lambda_n^2$ $Trace(A^2) = nd = \lambda_1^2 + \cdots + \lambda_n^2$*

$$nd - d^2 = \sum_{k=2}^{n} \lambda_k^2 \leq (n-1)\lambda_*^2 \qquad (\lambda_* = \max_{i \neq 1} |\lambda_i|)$$

So we have $d - o(1) = \frac{nd-d^2}{n-1} \leq \lambda_*^2 \implies \lambda_* \geq \sqrt{d} - o(1)$

Theorem 5.78. *Consider simple random walk on a graph G, G has adjacency matrix A and transition matrix* $P = \begin{bmatrix} \frac{1}{\deg(v_1)} & \cdots & \\ \vdots & \ddots & \\ & & \frac{1}{\deg(v_n)} \end{bmatrix} A$

Remark 5.79. $P(i,j) = \mathbb{P}(\textit{next at } j | \textit{now at } i$

If $v_1, ..., v_n$ are orthonormal eigenbasis to eiganvalues for A, also true for P. Let corresponding eigen values be $\lambda_1 \geq \cdots \geq \lambda_n$
Consider now a stochastic vector x with $\sum_{i=1}^n x_i = 1, x_i \geq 0$, the product $xP = y$ where y is a stochastic vector with the new distribution given we start at distribution x.

$$x^T = \sum \alpha_i \vec{v_i}$$

Since P is symmetric

$$\begin{aligned} xP^t &= P^t(\sum \alpha_i v_i) \tag{16}\\ &= \sum_{i=1}^n (\alpha_i \lambda_i^t v)i \tag{17}\\ &= \alpha_1 v_1 + \sum_{i=2}^n \lambda_i^t \alpha_i v_i \tag{18}\\ &\leq \lambda_*^t \left(\sum \alpha_i v_i\right) \tag{19} \end{aligned}$$

So we can conclude $xP^t \to \frac{1}{n} I$

Theorem 5.80. *A knight makes random knight moves: let τ be the time to return to the bottom left corner. What is $\mathbb{E}[X]$?*

Definition 5.81. *$\rho =$ probability of returning to origin for a random walk on $\mathbb{R}$ then a random walk is recurrent if $\rho = 1$ and transient otherwise.*

Remark 5.82. *A random walk is recurrent on Γ iff $\mathbb{E}[\textit{visits to origin}]$ is ∞*
$\mathbb{P}(\textit{I visit exactly } k \textit{ times}) = p^{k-1}(1-p)$
$\mathbb{E}(\textit{visits}) = \sum_{k=1}^{\infty} kp^{k-1}(1-p) = \frac{1}{1-p}$

For our random walk on $\mathbb{R}$,

$$\mathbb{E}[\text{visit}] = \sum_{n=0}^{\infty} \mathbb{E}[I_n] = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} = \sum_{n=0}^{\infty} a_n \geq \sum_{n \geq N} \frac{1}{\sqrt{\pi n}}$$

Where $a_n \sim \frac{1}{\pi n}$ by Stirling's Formula

The expected time to return to origin is not finite.

Theorem 5.83. *The probability starting from j we reach n before reaching 0 is $p_j = \frac{j}{n}$. $p_j = \frac{1}{2} p_{j-1} + \frac{1}{2} p_{j+1}$*

For a random walk on $\mathbb{R}^2$ let $I_n = 1$ if at 0 at $2n$.

$$\Pr(I_n) = \sum_n \frac{1}{4^{2n}} \sum_k \frac{(2n)!}{n!n!(n-k)!(n-k)!} = \sum_n \frac{1}{4^{2n}} \binom{2n}{n} \sum_k \binom{n}{k}^2 = \sum_n \frac{1}{4^{2n}} \binom{2n}{n}^2$$

For a random walk on $\mathbb{R}^3$ let $I_n = 1$ if I return after $2n$ steps.

$$\mathbb{E}[\text{visits}] = \sum_n \frac{1}{6^{2n}} \sum_{j,k} \frac{(2n)!}{j!^2 k!^2 (n-j-k)!^2} \leq \sum_n \frac{\binom{2n}{n}\binom{n}{n/3, n/3, n/3}}{2^{2n} 3^n} \sum \frac{\binom{n}{j,k,n-j-k}}{3^n}$$

By Stirling's formula, $\frac{n!}{\left(\frac{n}{3}\right)!^3} \approx \frac{(n/e)^n}{\left(\frac{n}{3e}\right)}$

For a random walk on a directed graph, let

$$\pi(y) = \mathbb{E}(\text{visits a random direction from } z \text{ and makes to } y \text{ before visiting } z \text{ again})$$