Complex Analysis Notes

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1.1 Holomorphic Functions and Complex Differentiation

Definition 1.1. *Let Ω be an open set in $\mathbb{C}$ and f a complex-valued function on Ω. A function f is holomorphic at a point $z_0 \in \Omega$ if the quotient*

$$\frac{f(z_0+h)-f(z_0)}{h}.$$

converges to a limit when $h \to 0$. The limit of the quotient, when it exists, is denoted $f'(z_0)$.

Theorem 1.2. *A function $F(x, y) = (u(x, y), v(x, y))$ is said to be differentiable at a point $P_0(x_0, y_0)$ if there exist a linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\frac{|F(P_0+H)-F(P_0)-J(H)|}{|H|} \to 0 \text{ as } |H| \to 0.$$

1.2 Jacobian and Cauchy-Riemann Equations

Definition 1.3. *The jacobian matrix of F is*

$$J = J_F(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Theorem 1.4. *Cauchy-Riemann Equations: If f is holomorphic then*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof. If we write $z = x + iy, z_0 = x_0 + iy_0$ and $f(z) = f(x, y)$ then

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \to 0} \frac{f(x_0+h_1, y_0)-f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0) \end{aligned}$$

Similarly, now take h purely imaginary, say $h = ih_2$, then

$$\begin{aligned} f'(z_0) &= \lim_{h_2 \to 0} \frac{f(x_0, y_0+h_2)-f(x_0, y_0)}{ih_2} \\ &= \frac{1}{i}\frac{\partial f}{\partial y}(z_0) \end{aligned}$$

Since f is holomorphic we have

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Writing $f = u + iv$ we have the stated relation. $\square$

2 Jan 14

2.1 Holomorphic Equivalence and Wirtinger Derivatives

Theorem 2.1. *$f : \Omega \to \mathbb{C}$ holomorphic on Ω equivalently $F = \begin{pmatrix} u \\ v \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$. The partials $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and satisfy the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Moreover, F is $\mathbb{R}$-differentiable, there exist a linear map $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F((x, y)+(h_1, h_2))-F(x, y) = \mathcal{L}(h_1, h_2) + o(h)$

$$\mathcal{L} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Proof. We have

$$\frac{|F((x,y)+(h_1,h_2)) - F(x,y) - \mathcal{L}(h_1,h_2)|}{|h|} = \frac{\left|\begin{pmatrix} u(\overline{z}+\overline{h}) - u(\overline{z}) - \frac{\partial u}{\partial x}h_1 - \frac{\partial u}{\partial y}h_2 \\ v(\overline{z}+\overline{h}) - v(\overline{z}) - \frac{\partial v}{\partial x}h_1 - \frac{\partial v}{\partial y}h_2 \end{pmatrix}\right|}{|h|} \tag{1}$$

$$= \frac{|f(z+h) - f(z) - f'(z)h|}{|h|} \to 0 \tag{2}$$

Where $f(z+h) = u(z+h) + iv(z+h)$, $f(z) = u(z) + iv(z)$ and $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. □

Corollary 2.2. *f holomorphic implies F is $\mathbb{R}$-differentiable. Let $a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$*

$$DF_z = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{4}$$

$$= \sqrt{a^2+b^2} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{5}$$

$$= |f'(z)| \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{6}$$

Theorem 2.3. *If F is $\mathbb{R}$-differentiable with u, v satisfying Cauchy Riemann Equations then f is holomorphic.*

Proof. Let $a = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. We just need to check that $f(z+h) - f(z) - ah = o(h)$ □

Definition 2.4. *The Wirtinger Derivatives*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Remark 2.5. *The cauchy-riemann equations if and only if $\frac{\partial f}{\partial \overline{z}} = 0$*

We initially had the map $z \mapsto \overline{z}$ which is not holomorphic. Let $f(z) = \overline{z}$ and $\frac{\partial f}{\partial \overline{z}} = 1$ then we've captured this with $\frac{\partial}{\partial \overline{z}}$.
If we have $z, \overline{z}$ as independent variables with $z = x + iy$. Consider the map

$$(z, \overline{z}) \mapsto (x(z, \overline{z}), y(z, \overline{z})) \mapsto f(x, y).$$

We have

$$\partial_z f = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Theorem 2.6.

$$\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \overline{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{4}\triangle.$$

3 Jan 16

3.1 Power Series

Definition 3.1. *Given a $\mathbb{C}$ valued seq $(a_n)_{n=0}^{\infty}$, form*

$$\sum_{n=0}^{\infty} a_n z^n.$$

A power series

1. *Converges if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n$ exists*
2. *Converges absolutely if $\sum_{n=0}^{\infty} |a_n||z|^n$ converges*

Remark 3.2. *By translation, all we say will apply to $\sum a_n(z-z_0)^n$*

Theorem 3.3. *Given $\sum_{n=0}^{\infty} a_n z^n$, $\exists R \in [0, +\infty]$ such that*

1. *It converges, if $D_R = \{|z| < R\}$*
2. *It diverges at every at z such that $|z| > R$.*

Moreover,

$$R = \frac{1}{\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}}. \tag{Hadamard's Formula}$$

Remark 3.4. *On ∂D_R it can often be delicate.*

Proof. Fix $|z| < R$, then $\frac{1}{R}|z| < 1$ and fix $\epsilon > 0$ such that

$$\left(\frac{1}{R} + \epsilon\right)|z| < 1.$$

Then, eventually

$$|a_n|^{\frac{1}{n}} < \frac{1}{R} + \epsilon.$$

Then

$$|a_n||z|^n \le \left(|a_n|^{\frac{1}{n}}|z|\right)^n \le \left(\left(\frac{1}{R} + \epsilon\right)|z|\right)^n \le q^n.$$

So by the comparison test with $\sum q^n < \infty$ $\square$

Example 3.5.

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z).$$

This has $R = +\infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos(z).$$

This has $R = +\infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin(z).$$

This has $R = +\infty$
Euler's Formula is

$$e^{iz} = \cos(z) + i\sin(z).$$

Theorem 3.6. *The function $f(z) = \sum a_n z^n$ is holomorphic on D_R, f' is a power series*

$$f'(z) = \sum n a_n z^{n-1}.$$

Whose disk of convergence is R.

Proof. Let $g(z) = \sum n a_n z^{n-1}$, $z \in D_R$. We want to show that f' exist and $f' = g$.
Fix $z_0 \in D_R$, let $r < R$ such that $|z_0| < r < R$.
Fix $N \geq 1$ (to be chosen later)

$$f(z) = \sum_{n=0}^{N} a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n = S_N(z) + E_N(z).$$

From here we want to show that

$$\frac{f(z_0+h) - f(z_0)}{h} \to g(z_0).$$

For all h such that $|z_0 + h| < r$

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) &= \left(\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right) + (S_N'(z_0) - g(z_0)) + \left(\frac{E_N(z_0+h)-E_N(z_0)}{h}\right) \\ &= A + B + C\end{aligned}$$

Fix $\epsilon > 0$ we want to show that $\exists \delta$ such that $\forall |h| < \delta, |A+B+C| < \epsilon$.

$$\begin{aligned}|C| &\leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \\ &= \sum_{n=N+1}^{\infty} |a_n| \left|(z_0+h)^{n-1} + (z_0+h)^{n-2} z_0 + \cdots + z_0^n\right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n||n r^{n-1}| < \frac{\epsilon}{3} && \text{(for all } N \geq N_1(\epsilon))\end{aligned}$$

Note the last summation is the tail of $\sum a_n n r^{n-1}$ which converges by $r < R$.
For B:

$$S_N'(z_0) = \sum_{n=0}^{N} n a_n z_0^{n-1} \to g(z_0). \qquad \text{(by def of } g)$$

So $\forall N \geq N_2$, $|B| < \frac{\epsilon}{3}$.
$|A| < \frac{\epsilon}{3}$ for all h sufficiently small by the definition of the complex derivative and the fact that S_N is plainly holomorphic. □

3.2 Analytic Functions

Definition 3.7. *Let Ω be open and define $f : \Omega \to \mathbb{C}$ is analytic at $z_0 \in \Omega$, if there exists a power series $\sum a_n (z - z_0)^n$ with positive radius of convergence such that $f(z) = \sum a_n (z-z_0)^n$ is a neighborhood of z_0.*
f is analytic on Ω, if analytic at every point of $z_0 \in \Omega$.

Corollary 3.8. *If f analytic on $\Omega \implies f$ is holomorphic on Ω.*
In fact, $\impliedby$.

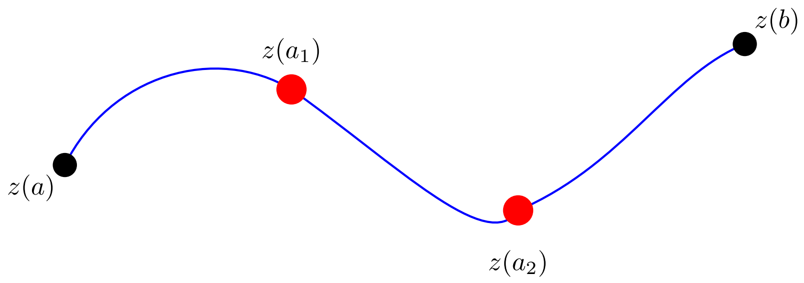
4 Jan 21

4.1 Parametrized Curves

Definition 4.1. *A parametrized curve (in $\mathbb{C}$) is a function $z(t) : [a, b] \to \mathbb{C}$*

1. *Smooth if z' is continuous and $\forall t \in [a, b]$, $z'(t) \neq 0$*
2. *Piecewise smooth if z is continuous so there exists $a_0 = a < a_1 < ... < a_k = b$ such that $z\mathrm{res}_{[a_j, a_{j+1}]}$ for all $j = 0, ..., k-1$ is smooth.*
3. *Let $\tilde{z} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ then two parameterization are equivalent if there exist c^1-bijection $s \mapsto t(s) : [\tilde{a}, \tilde{b}] \to [a, b]$ and $t'(s) > 0$ for all s such that $\tilde{z}(s) = z(t(s))$.*

Example 4.2. *A piecewise smooth curve can be visualized as follows:*



The curve consists of smooth segments joined at corner points (marked in red) where the derivative is discontinuous.

Definition 4.3. *The family of all equivalent parameterization determines a curve (oriented) $\gamma \subset \mathbb{C}, \gamma = z([a, b])$.*
$z(a)$ is the starting point and $z(b)$ is the endpoint of γ. $\gamma^{-1} = \gamma$ with reversed orientation

Definition 4.4. *γ is closed if $z(a) = z(b)$ and simple if $z(t_1) \neq z(t_2)$ for all $t_1 \neq t_2$.*

Example 4.5. $z(T) = re^{-it}, t \in [0, 2\pi]$

Definition 4.6. *A curve has positive orientation if interior is on the left or counterclockwise.*

4.2 Contour Integration

Definition 4.7. *Give a smooth curve $\gamma \subset \mathbb{C}$ (with parameterization $z : [a, b] \to \mathbb{C}$) with $f : \gamma \to \mathbb{C}$ continuous.*
We set

$$\int_\gamma f = \int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

Since $f(z(t))z'(t) \in \mathbb{C}$ we split the integral into

$$\int_a^b \Re(...)dt + i \int_a^b \Im(...)dt.$$

Remark 4.8. *This is well defined as the right hand side does not depend on the choice of the parametrization.*

Definition 4.9. *For a piece-wise smooth curve, we define*

$$\int_\gamma f = \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} f(z(t))z'(t)dt.$$

Definition 4.10. *We define*

$$|\gamma| = \mathit{length}(\gamma) = \int_a^b |z'(t)|dt.$$

Theorem 4.11. *Some very basic properties:*

1. *Linearity*
$$\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g.$$
2. $$\int_{\gamma^{-1}} f = -\int_\gamma f.$$
3. $$\left|\int_\gamma f\right| \leq \left(\sup_\gamma |f|\right) |\gamma|.$$

4.3 Primitives and the Fundamental Theorem

Definition 4.12. *The primitive of $f : \Omega \to \mathbb{C}$ with open $\Omega \subset \mathbb{C}$ is any $F : \Omega \to \mathbb{C}$ holomorphic such that $F' = f$ on Ω*

Theorem 4.13. *The Fundamental Theorem of Calculus*
For f with primitive F on Ω, $\gamma \subset \mathbb{C}$ curve from $w_1 \to w_2$ we have

$$\int_\gamma f = F(w_2) - F(w_1).$$

Proof. We can write

$$\int_\gamma f = \int_a^b f(z(t))z'(t)dt = \int_a^b \frac{d}{dt}(F(z(t)))dt = F(z(b)) - F(z(a)).$$

□

Corollary 4.14. *If γ is closed then*

$$\int_\gamma f = 0.$$

Corollary 4.15. *If f is holomorphic on Ω with $f' = 0$ on Ω then f is constant.*

Proof. Fix $z_0 \in \Omega$, for $z \in \Omega$, take $\gamma : z_0 \to z$ then

$$0 = \int_\gamma f' = f(z) - f(z_0).$$

□

Example 4.16. *Let $z(t) = e^{it}$*

$$\int_{\partial D} \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}dt = 2\pi i \neq 0.$$

So $\frac{1}{z}$ does not admit a primitive.

5 Jan 23

5.1 Glossary of Elementary Functions

Glossary of Elementary Functions

1. Polynomials: $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ for $a_i \in \mathbb{C}$ and is entirely holomorphic in $\mathbb{C}$, $a_n \neq 0$ and $n = \deg P$
2. Rational: $R(z) = \frac{P(z)}{Q(z)}$ where P, Q are polynomials with no common factors. The zeros of Q are called poles and their order is called the order of a given pole.
3. Exponential: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

5.2 Fundamental Theorem of Algebra

Lemma 5.1. *Let $P(z)$ be a complex polynomial of positive degree n, if $|P(z)|$ has a local minimum at z_0 then $P(z_0) = 0$.*

Proof. WLOG $z_0 = 0$ as we can translate $Q(z) = P(z - z_0)$.
AFSOC $P(0) \neq 0$ and WLOG $P(0) = 1$ by $\frac{P}{P(0)}$. We can write $P(z) = 1 + a_k z^k + \cdots + a_z^n$ with $a_k, a_n \neq 0$.
Consider $z_\delta = \delta e^{i\theta}$. For all $\delta > 0$ sufficiently small:

$$\left|a_{k+1} z_\delta + \cdots + a_n z_\delta^{n-k}\right| < \frac{|a_k|}{2}.$$

Then

$$|P(z)| \leq \left|1 + a_k z_\delta^k\right| + \left|a_{k+1} z_\delta^{k+1} + \cdots + a_n z_\delta^n\right| < \left|1 + a_k z_\delta^k\right| + \left|z_\delta^k\right| \frac{|a_k|}{2}.$$

Now we can choose δ such that $a_k z_\delta^k < 0$ and real.

$$\left|1 + a_k z_\delta^k\right| + \left|z_\delta^k\right| \frac{|a_k|}{2} = 1 + a_k z_\delta^k - \frac{1}{2} a_k z_\delta^k = 1 + \frac{1}{2} a_k z_\delta^k < 1.$$

□

Theorem 5.2. *Fundamental Theorem of Algebra:*

Proof. $|P(z)| \to +\infty$ as $|z| \to +\infty$ therefore $\exists R$ such that $\forall |z| > R$, $|P(z)| > |P(0)|$ thus $\inf_{z\in\mathbb{C}} |P(z)| = \inf_{z \in D_R} |P(z)|$ and such a value is attained somewhere at z_0 then by lemma 5.1 we're done. □

Lemma 5.3. *$P(z) = a_n(z - z_1) \cdots (z - z_n)$ where $z_1, ..., z_n$ are roots of P.*
The order of a zero z_j is its multiplicity, the number of times it appears in the sequence. If z_j is of the order k_j,

$$P(z) = \underbrace{Q(z)}_{\text{non-vanishing}}(z - z_j)^{k_j}.$$

A zero z_0 is of order k_0 if and only if $P(z_0) = P'(z_0) = ... = P^{(k_0-1)}(z_0) = 0$ and $P^{(k_0)}(z_0) \neq 0$

Theorem 5.4. *(Gauss-Lucas) The roots of P' lies in the convex hull of the roots of P. In particular, if P is real rooted then so is P'.*

Theorem 5.5. *If R a rational function then R' has the same roots as R with order greater by 1.*

Theorem 5.6. *The following are true*

1. $(e^z)' = e^z$
2. $e^{w+z} = e^w e^z$
3. $\overline{e^z} = e^{\overline{z}}$
4. $|e^{iy}| = 1$ *and* $|e^{x+iy}| = e^x$
5. $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$ *and* $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$

6 Jan 26

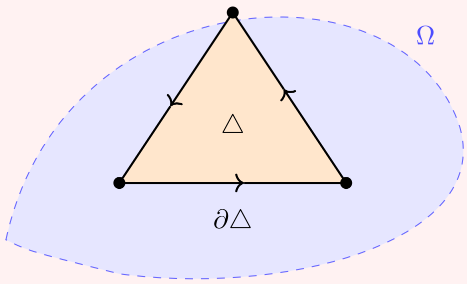
6.1 Cauchy's and Goursat's Theorems

Theorem 6.1. *Cauchy Theorem: If* $f : \Omega \to \mathbb{C}$ *is holomorphic on* Ω *and* γ *is a closed curve in* Ω *then*

$$\int_\gamma f = 0.$$

Theorem 6.2. *Goursat's Theorem: If* $f : \Omega \to \mathbb{C}$ *is holomorphic and* Ω *is open, then* $\forall \triangle \subset \Omega$ *we have*

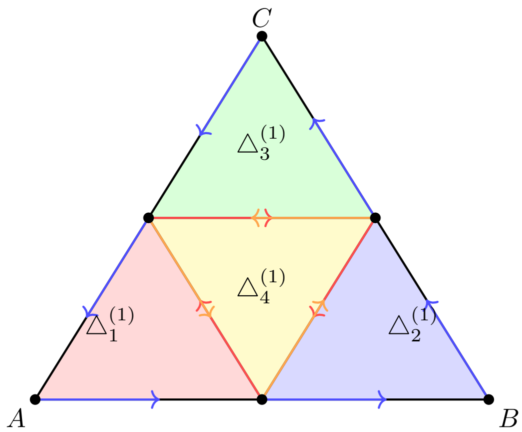
$$\int_{\partial\triangle} f = 0.$$



The integral around the positively oriented (counterclockwise) boundary of any triangle $\triangle \subset \Omega$ *vanishes.*

Proof. Let $\triangle = \triangle^{(0)}$ and partition into four triangles by connecting the midpoints of the sides:

$$\triangle^{(0)} = \triangle_1^{(1)} \cup \triangle_2^{(1)} \cup \triangle_3^{(1)} \cup \triangle_4^{(1)}.$$



When we integrate along each sub-triangle with positive orientation, the interior edges cancel (traversed in opposite directions, shown with opposing arrows):

$$\int_{\partial\triangle^{(0)}} f = \sum_{j=1}^{4} \int_{\partial\triangle_j^{(1)}} f.$$

By the triangle inequality

$$\left|\int_{\partial\triangle^{(0)}} f\right| \leq \sum_{j=1}^{4} \left|\int_{\partial\triangle_j^{(1)}} f\right| \leq 4 \max_{j=1,2,3,4} \left|\int_{\partial\triangle_j^{(1)}} f\right|.$$

Thus there exists $j_0 \in \{1,2,3,4\}$ such that

$$\left|\int_{\partial\triangle^{(0)}} f\right| \leq 4 \left|\int_{\partial\triangle_{j_0}^{(1)}} f\right|.$$

Set $\triangle^{(1)} = \triangle_{j_0}^{(1)}$. Repeating this subdivision process, we obtain a nested sequence of triangles

$$\triangle^{(0)} \supset \triangle^{(1)} \supset \triangle^{(2)} \supset \cdots$$

such that

$$\left|\int_{\partial\triangle^{(0)}} f\right| \leq 4^n \left|\int_{\partial\triangle^{(n)}} f\right|.$$

Note that $\mathrm{diam}(\triangle^{(n)}) = 2^{-n}\mathrm{diam}(\triangle^{(0)})$ and $|\partial\triangle^{(n)}| = 2^{-n}|\partial\triangle^{(0)}|$.
Since the triangles are nested compact sets with diameters shrinking to zero, by Cantor's intersection theorem:

$$\bigcap_{n=0}^{\infty} \triangle^{(n)} = \{z_0\}$$

for some $z_0 \in \triangle^{(0)} \subset \Omega$. Since f is holomorphic at z_0, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

where $\psi(z) \to 0$ as $z \to z_0$.
Since $f(z_0)$ and $f'(z_0)(z - z_0)$ have primitives ($f(z_0)z$ and $\frac{1}{2}f'(z_0)(z - z_0)^2$ respectively), their integrals over the closed curve $\partial\triangle^{(n)}$ vanish. Thus:

$$\int_{\partial\triangle^{(n)}} f(z)dz = \int_{\partial\triangle^{(n)}} \psi(z)(z - z_0)dz.$$

For $z \in \partial\triangle^{(n)}$, we have $|z - z_0| \leq \mathrm{diam}(\triangle^{(n)}) = 2^{-n}\mathrm{diam}(\triangle^{(0)})$.
Let $\epsilon_n = \sup_{z\in\partial\triangle^{(n)}} |\psi(z)|$. Since $\psi(z) \to 0$ as $z \to z_0$ and $\triangle^{(n)} \to \{z_0\}$, we have $\epsilon_n \to 0$.
Therefore:

$$\begin{aligned}\left|\int_{\partial\triangle^{(n)}} f(z)dz\right| &\leq \sup_{z\in\partial\triangle^{(n)}} |\psi(z)(z - z_0)| \cdot |\partial\triangle^{(n)}| \\ &\leq \epsilon_n \cdot 2^{-n}\mathrm{diam}(\triangle^{(0)}) \cdot 2^{-n}|\partial\triangle^{(0)}| \\ &= \epsilon_n \cdot 4^{-n}\mathrm{diam}(\triangle^{(0)})|\partial\triangle^{(0)}|\end{aligned}$$

Combining with our earlier inequality:

$$\left|\int_{\partial\triangle^{(0)}} f\right| \leq 4^n \left|\int_{\partial\triangle^{(n)}} f\right| \leq 4^n \cdot \epsilon_n \cdot 4^{-n}\mathrm{diam}(\triangle^{(0)})|\partial\triangle^{(0)}| = \epsilon_n \cdot \mathrm{diam}(\triangle^{(0)})|\partial\triangle^{(0)}|$$

Since $\epsilon_n \to 0$ as $n \to \infty$ and the right-hand side is independent of n except for ϵ_n, we conclude:

$$\int_{\partial\triangle} f = 0.$$

$\square$

6.2 Cauchy's Theorem in a Disk

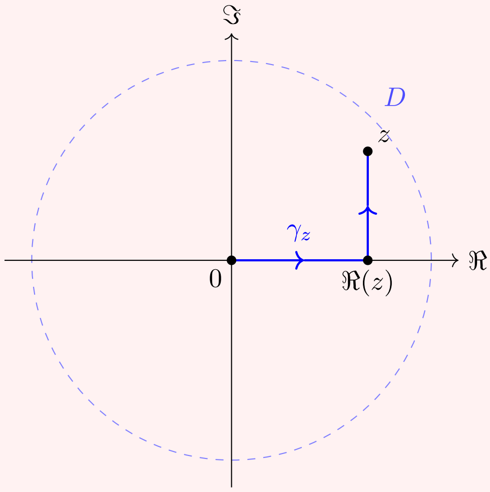
Theorem 6.3. *Cauchy's Theorem in a disk: Let $D \subset \mathbb{C}$ be an open disk and f holomorphic on D. Then for any closed curve γ in D we have*

$$\int_\gamma f = 0.$$

Proof. WLOG let D be a disk centered at 0. Define $F : D \to \mathbb{C}$ by

$$F(z) = \int_{\gamma_z} f$$

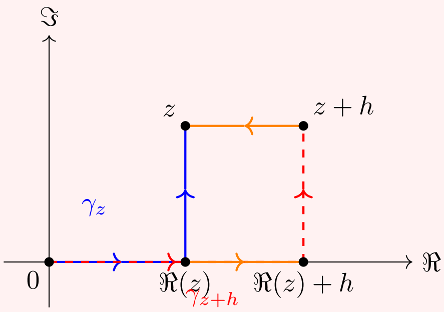
where γ_z is the path $0 \to \Re(z) \to z$ (horizontal then vertical).



We claim F is a primitive for f, i.e., $F'(z) = f(z)$.

Case 1: $h \in \mathbb{R}$ (horizontal increment).
The path γ_{z+h} goes $0 \to \Re(z) + h \to z + h$. Observe that:

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f - \int_{\gamma_z} f.$$



The difference of paths can be decomposed using Goursat's theorem. The integral over the rectangle with vertices $\Re(z), \Re(z) + h, z + h, z$ vanishes, so:

$$F(z+h) - F(z) = \int_{\Re(z)}^{\Re(z)+h} f(\zeta)d\zeta + \int_z^{z+h} f(\zeta)d\zeta = \int_z^{z+h} f(\zeta)d\zeta$$

where the first integral is along the real axis and the second is horizontal at height $\Im(z)$. By the rectangle lemma, these combine to give just the horizontal segment from z to $z + h$:

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_z^{z+h} f(\zeta)d\zeta \to f(z) \text{ as } h \to 0.$$

Case 2: $h = ik$ with $k \in \mathbb{R}$ (vertical increment).
Similarly, γ_{z+ik} goes $0 \to \Re(z) \to z + ik$. The paths γ_z and γ_{z+ik} share the horizontal segment $0 \to \Re(z)$, so:

$$F(z+ik) - F(z) = \int_z^{z+ik} f(\zeta)d\zeta.$$

7 Jan 28

7.1 Applications of Cauchy's Theorem

Example 7.1.

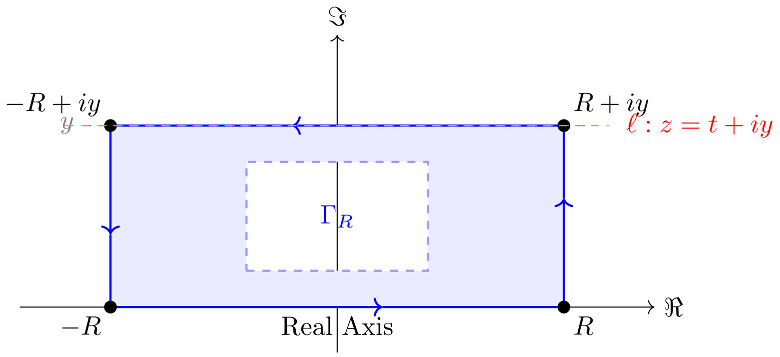
$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}\,dx = 1 \iff \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1.$$

Proof. First, note that by substitution $u = \sqrt{\pi}x$, we have:

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\,du.$$

So the equivalence follows if we can show $\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$.

Consider the function $f(z) = e^{-\pi z^2}$, which is entire (holomorphic everywhere). For fixed $y \in \mathbb{R}$, consider the rectangular contour Γ_R with vertices at $-R, R, R+iy, -R+iy$ (traversed counterclockwise).



By Cauchy's theorem:

$$\int_{\Gamma_R} e^{-\pi z^2}\,dz = 0.$$

Parameterizing the four sides:

$$\int_{\Gamma_R} e^{-\pi z^2}\,dz = \int_{-R}^{R} e^{-\pi t^2}\,dt + \int_{0}^{y} e^{-\pi(R+it)^2}\,i\,dt + \int_{R}^{-R} e^{-\pi(t+iy)^2}\,dt + \int_{y}^{0} e^{-\pi(-R+it)^2}\,i\,dt$$

We now bound the integrals over the vertical segments. For the right vertical segment, parameterize $z = R + it$ with $t \in [0, y]$ (assuming $y > 0$; the case $y < 0$ is similar). We have:

$$(R+it)^2 = R^2 + 2iRt - t^2 = (R^2 - t^2) + 2iRt$$

so

$$e^{-\pi(R+it)^2} = e^{-\pi(R^2-t^2)} \cdot e^{-2\pi iRt}.$$

Since $|e^{-2\pi iRt}| = 1$ for all real R and t, we have:

$$|e^{-\pi(R+it)^2}| = e^{-\pi(R^2-t^2)} \cdot |e^{-2\pi iRt}| = e^{-\pi(R^2-t^2)} \cdot 1 = e^{-\pi(R^2-t^2)}.$$

For $t \in [0, y]$, we have $t^2 \le y^2$, so $R^2 - t^2 \ge R^2 - y^2$. Therefore:

$$|e^{-\pi(R+it)^2}| \le e^{-\pi(R^2-y^2)} = e^{-\pi R^2} e^{\pi y^2}.$$

By the ML-inequality, the length of the path is $|y|$, so:

$$\left|\int_{0}^{y} e^{-\pi(R+it)^2}\,i\,dt\right| \le |y| \cdot e^{-\pi R^2} e^{\pi y^2} = |y| e^{\pi y^2} e^{-\pi R^2}.$$

Similarly, for the left vertical segment with $z = -R + it$ where $t \in [y, 0]$:

$$(-R+it)^2 = R^2 - 2iRt - t^2 = (R^2 - t^2) - 2iRt$$

so

$$e^{-\pi(-R+it)^2} = e^{-\pi(R^2-t^2)} \cdot e^{2\pi iRt}.$$

Since $|e^{2\pi iRt}| = 1$ for all real R and t, we have:

$$|e^{-\pi(-R+it)^2}| = e^{-\pi(R^2-t^2)} \cdot |e^{2\pi iRt}| = e^{-\pi(R^2-t^2)} \cdot 1 = e^{-\pi(R^2-t^2)}.$$

For $t \in [y, 0]$ (or equivalently $t \in [0, y]$ if we reverse the parameterization), we have $t^2 \leq y^2$, so:

$$|e^{-\pi(-R+it)^2}| \leq e^{-\pi(R^2-y^2)} = e^{-\pi R^2} e^{\pi y^2}.$$

Therefore:

$$\left| \int_y^0 e^{-\pi(-R+it)^2} i\, dt \right| \leq |y| \cdot e^{-\pi R^2} e^{\pi y^2} = |y| e^{\pi y^2} e^{-\pi R^2}.$$

Both vertical integrals are bounded by $Ce^{-\pi R^2}$ for some constant C depending on y but independent of R. Since $e^{-\pi R^2} \to 0$ as $R \to \infty$, we conclude that:

$$\lim_{R\to\infty} \int_0^y e^{-\pi(R+it)^2} i\, dt = 0 \quad \text{and} \quad \lim_{R\to\infty} \int_y^0 e^{-\pi(-R+it)^2} i\, dt = 0.$$

Therefore, taking the limit as $R \to \infty$:

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = \int_{-\infty}^{\infty} e^{-\pi(t+iy)^2} dt.$$

Expanding $e^{-\pi(t+iy)^2} = e^{-\pi(t^2+2ity-y^2)} = e^{-\pi t^2} e^{-2\pi ity} e^{\pi y^2}$, we get:

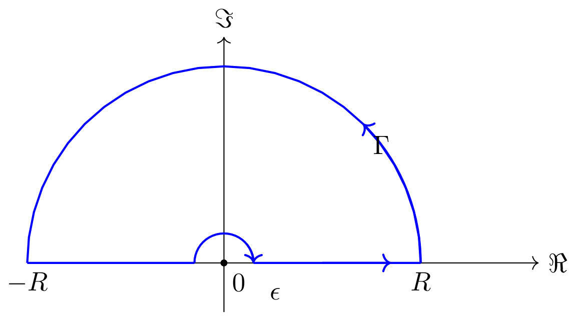
$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = e^{\pi y^2} \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-2\pi ity} dt.$$

In particular, taking $y = 0$ gives the standard Gaussian integral. The standard result is $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$, which completes the proof. $\square$

Example 7.2.

$$\int_0^{\infty} \frac{1 - \cos(x)}{x^2} dx = \frac{\pi}{2}.$$

Proof. Consider the holomorphic function $f(z) = \frac{1-e^{iz}}{z^2}$ on $\mathbb{C} \setminus \{0\}$. Fix ϵ, R and consider the contour Γ which is disk with radius R and a small disk with radius ϵ centered at the origin.



We can express $e^{iz} \approx 1 + iz + g(z)$ where $g(z) = e^{iz} - 1 - iz$ so

$$\begin{aligned}\int_{\gamma_\epsilon} f(z)dz &= \int_{\gamma_\epsilon} \frac{1 - (1 + iz + g(z))}{z^2} dz \\ &= \int_{\gamma_\epsilon} \frac{-g(z)}{z^2} dz + \int_{\gamma_\epsilon} \frac{-i}{z} dz\end{aligned}$$

We can evaluate each part as

$$\int_{\gamma_\epsilon} \frac{dz}{z} = -\int_0^\pi \frac{ie^{it}}{e^{it}} dt = -\pi i.$$

$$\int_{\gamma_\epsilon} \frac{g(z)}{z^2} dz \to 0 \text{ as } \epsilon \to 0.$$

□

8 Jan 30

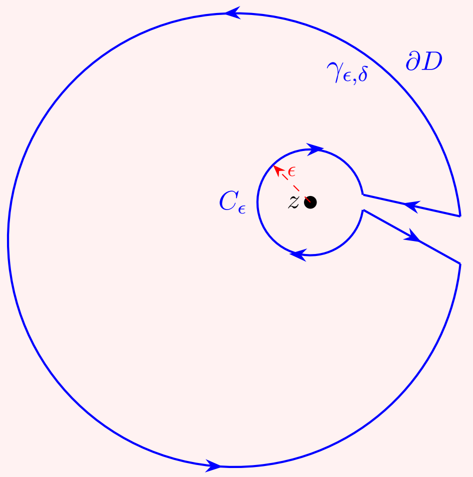
8.1 Cauchy's Integral Formula

Theorem 8.1. *Cauchy's Integral Formula: Let* Ω *be open* $f : \Omega \to \mathbb{C}$ *be holomorphic. Let* $\overline{D}$ *be open in* Ω *be a disk then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof. Fix $z \in D_1$, let $\gamma_{\epsilon,\delta}$ be a key-hole contour but we omit z. Then

$$\int_{\gamma_{\epsilon,\delta}} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$



Fix $\epsilon > 0$ and $\delta \to 0$ we want to show that

$$\int_{\gamma_{\epsilon,\delta}} \frac{f(\zeta)}{\zeta - z} d\zeta \to \int_{\partial D} - \int_{C_\epsilon}.$$

For

$$\int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_\epsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + f(z) \int_{C_\epsilon} \frac{d\zeta}{\zeta - z}.$$

The first term is bounded as holomorphic so as $\epsilon \to 0$, it goes to 0. The second part integrates to $f(z) \cdot 2\pi i$. $\square$

8.2 Regularity and Liouville's Theorem

Theorem 8.2. *Regularity: Holomorphic implies Analytic.*
Let $f : \Omega \to \mathbb{C}$ be holomorphic, $\overline{D} = \overline{D}_r(z_0) \subset \Omega$ then f is analytic at z_0 i.e

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Proof. We know

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We can write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) + (z_0 - z)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n.$$

Note: $\frac{z-z_0}{\zeta - z_0} < 1$.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} \underbrace{\sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \frac{f(\zeta)}{\zeta - z}}_{\text{conv. uniformly on } \partial D} d\zeta \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\left(\frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right)}_{a_n} \end{aligned}$$

So $a_n = \frac{f^{(n)}(z_0)}{n!}$. □

Corollary 8.3. *Cauchy's Formula:*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Corollary 8.4. *Cauchy's Inequality:*

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \sup_{\zeta \in \partial D} \left| \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \right| = \frac{n!}{r^n} ||f||_\infty \partial D.$$

Theorem 8.5. *Liouville's Theorem: If f entire and bounded then f is constant.*

Proof.

$$|f'(z_0)| \leq \frac{\|f\|_\infty}{r}$$

Since f is bounded and r can be taken arbitrarily large (as f is entire), sending $r \to \infty$ gives $|f'(z_0)| = 0$. Thus, f is constant. □

9 Feb 2

9.1 Rigidity and Analytic Continuation

Theorem 9.1. *(Rigidity Theorem): Let Ω be a region and $f : \Omega \to \mathbb{C}$ be holomorphic. If $z_1, z_2, \ldots \in \Omega$ distinct sequence with a limit in Ω. If $f(z_n) = 0, \forall n$ then $f = 0$ on Ω.*

Proof. Say $z_n \to \omega \in \Omega$ and $D = D_r(\omega) \subset \Omega$ be a disk centered at ω with radius r. If $f = 0$ on D then by regularity implies

$$f(z) = \sum_{n \geq 0} a_n (z - \omega)^n \text{ on } D.$$

Suppose $f \neq 0$ on D then $\exists m$ such that $a_m \neq 0$, let's take the smallest such m. Then $f(z) = a_m(z - w)^m(1 + g(z))$ where $g(z) \to 0$ as $z \to w$. For all sufficiency large k, $z_k \in D$ then

$$0 = f(z_k) = a_m(z_k - w)^m(1 + g(z_k)). \qquad (a_m \neq 0 \text{ and } z_k - w \neq 0)$$

A contradiction as the RHS is nonzero. To finish $f = 0$ on Ω, we use $\mathcal{U} = \operatorname{int}\{z \in \Omega : f(z) = 0\}$ We have $\mathcal{U}$ is open, $w \in \mathcal{U}$ so $\mathcal{U}$ is non-empty. Additionally, $\mathcal{U}$ is closed. We have $\Omega \setminus \mathcal{U}$ is open and non-empty so $\exists z_k \in \Omega \setminus \mathcal{U}$ such that $z_k \to \omega' \in \Omega \setminus \mathcal{U}$. Then $f(z_k) \neq 0$ for all k so $f \neq 0$ on $D_r(\omega')$. A contradiction as $D_r(\omega') \subset D$. Thus, $f = 0$ on Ω. □

Corollary 9.2. *$f, g : \underset{region}{\Omega} \to \mathbb{C}$ holomorphic. If $f(z_n) = g(z_n), \forall z_n \in \Omega$ distinct sequence with a limit in Ω then $f = g$ on Ω.*

Definition 9.3. *If $\Omega_1 \subset \Omega_2$, are two regions, $f_i : \Omega_i \to \mathbb{C}$ holomorphic for $i = 1, 2$ such that $f_1 = f_2$ on Ω_1 then we say f_2 is the analytic continuation of f_1 into Ω_2.*

9.2 Morera's Theorem and Uniform Convergence

Theorem 9.4. *(Morera's Theorem): If $f : \mathcal{D} \to \mathbb{C}$ continuous such that $\forall \triangle \subset \mathcal{D}$ and $\int_{\partial \triangle} f = 0$ then f is holomorphic.*

Proof. Repeating the proof of Cauchy's Theorem in a disk, gives that f has a primitive $F = \int_{\gamma_z} f$ in $\mathcal{D}$. Since F is holomorphic, F' exists but so does $F'', F''', \ldots$ so f' exists. □

Theorem 9.5. *If $f_n : \underset{open}{\Omega} \to \mathbb{C}$ holomorphic with f_n converging uniformly to f on every compact $K \subset \Omega$ then f is holomorphic.*

Proof. Use Morera's Theorem. □

Moreover, if f_n' converges uniformly to f', using $\Omega_s = \{z \in \Omega : \overline{\mathcal{D}}_\delta(z) \subset \Omega\}$.
Claim: $\forall F$ holomorphic in Ω, $||F'||_{\infty, \Omega_\delta} \leq \frac{1}{\delta}||F||_{\infty, \Omega_\delta}$.
By Cauchy,s Formula:

$$|F'(z)| = \left| \frac{1}{2\pi i} \int_{\partial \mathcal{D}_\delta(z)} \frac{F(w)}{(w - z)^2} dw \right| \leq \frac{1}{2\pi} \int_{\partial \mathcal{D}_\delta(z)} \frac{1}{|w - z|^2} ||F||_{\infty, \Omega} \leq \frac{1}{2\pi} \frac{2\pi\delta}{\delta^2} ||F||_{\infty, \Omega} \leq \frac{1}{\delta} ||F||_{\infty, \Omega}.$$

10 Feb 4

10.1 Parameter Integrals and Gluing

Theorem 10.1. *Let $F : \Omega \times [0,1] \to \mathbb{C}$ satisfy*

1. $\forall \triangle \in [0,1]$, $z \mapsto F(z, \triangle)$ *is holomorphic*
2. F *is continuous on* $\Omega \times [0,1]$

Then $z \mapsto \int_0^1 F(z,\triangle)d\triangle$ *is holomorphic on* Ω.

Proof. Consider $F_n(z) = \frac{1}{n}\sum_{k=1}^n F(z, \frac{k}{n})$ and by (1) each F_n is holomorphic on Ω.

$$\begin{aligned} \left|F_n(z) - \int_0^1 F(z,\triangle)d\triangle\right| &= \left|F_n(z) - \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} F(z,\triangle)d\triangle\right| \\ &= \left|\sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left[\underbrace{F(z,\tfrac{k}{n}) - F(z,\triangle)}_{\leq \epsilon}\right] d\triangle\right| \\ &\leq \epsilon \end{aligned}$$

□

Definition 10.2. $\Omega \subset \mathbb{C}$ *be open, symmetric with respect to real axis,* $\Omega^+ = \Omega \cap \{z \in \mathbb{C}, \Im(z) > 0\}$, $\Omega^- = \Omega \cap \{z \in \mathbb{C}, \Im(z) < 0\}$ *and* $I = \Omega \cap \{\Im(z) = 0\}$

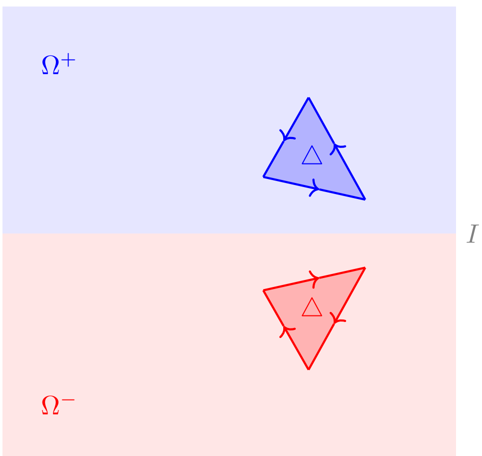
Theorem 10.3. *Let each* $f^\pm : \Omega^\pm \to \mathbb{C}$ *be holomorphic and extend continuous to* I *such that* $f^+ = f^-$ *on* I. *Then*

$$f = \begin{cases} f^+(z) & \text{if } z \in \Omega^+ \\ f^+(z) = f^-(z) & \text{if } z \in I \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}.$$

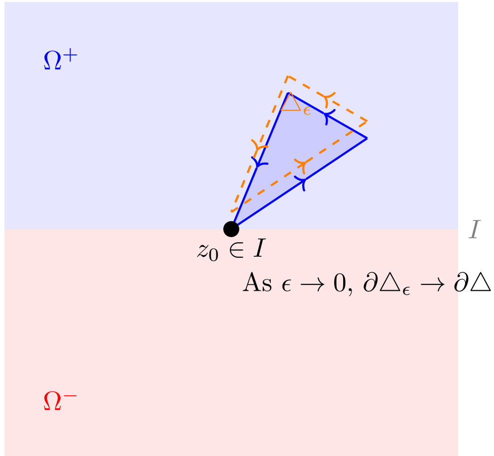
is holomorphic on Ω.

Proof. Sufficient to handle $z \in I$. Fix such z take $D \subset \Omega$ centered at z. Take $\overline{\triangle} \subset D$ we want to show $\int_{\partial\triangle} f = 0$.

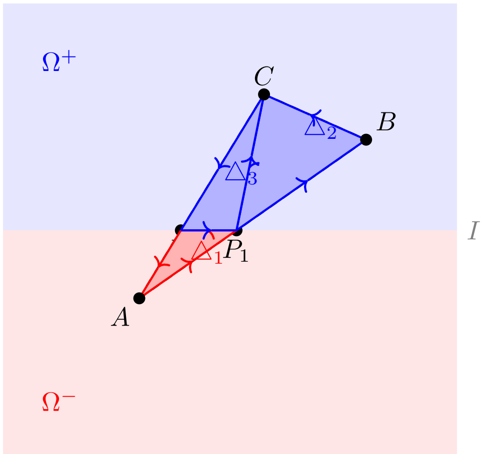
Case 1: If $\overline{\triangle} \subset \Omega^+$ (or Ω^-) then we're done.



Case 2: If a vertex or side of $\triangle$ is on I, $\int_{\partial\triangle_\epsilon} f = 0$ for when $\epsilon \to 0, \partial\triangle_\epsilon \to \partial\triangle$.



Case 3: If I cuts the triangle then we can split the triangle into smaller triangles and we're done.



The triangle is split into smaller triangles, each either entirely in Ω^+ or Ω^-, or with edges on I, so we can apply Cases 1 and 2. $\square$

10.2 Schwarz Reflection Principle

Theorem 10.4. *Schwarz Reflection Principle: Let $f : \Omega^+ \to \mathbb{C}$ be holomorphic and extend continuous to I with $f^+/I \in \mathbb{R}$ then f^+ extends analytically to Ω*

Proof. Let $f^- : \Omega^- \to \mathbb{C}$ be defined by $f^-(z) = \overline{f^+(\overline{z})}$. We claim f^- is holomorphic on Ω^-, then the previous theorem applies because $f^+ = f^-$ on I.
Fix $z_0 \in \Omega^-$ then $\overline{z_0} \in \Omega^+$, we know $f^+(\overline{z}) = \sum_{n=0}^{\infty} a_n(\overline{z} - \overline{z_0})^n$ for all $\overline{z} \in D(\overline{z_0})$. Thus

$$\overline{f^+(\overline{z})} = \sum_{n=0}^{\infty} \overline{a_n}(\overline{z} - \overline{z_0})^n.$$

We know $\overline{\overline{z} - \overline{z_0}} = z - z_0$ so

$$f^-(z) = \overline{f^+(\overline{z})} = \sum \overline{a_n}(z - z_0)^n.$$

$\square$

11 Feb 9

11.1 Meromorphic Functions and Poles

Definition 11.1. *Meromorphic functions are "determined" by zeros and singularities.*
A point singulaity of f is a point $z_0 \in \mathbb{C}$ such that f is defined in a neighborhood of z_0 but not at z_0.

$$\mathcal{D}_\delta(z_0) \setminus \{z_0\}.$$

Example 11.2. $f(z) = \frac{1}{z}$ *has a singularity at* $z_0 = 0$.

Remark 11.3. *Zeros of a holomorphic f are isolated, unless $f \equiv 0$.*

Theorem 11.4. *(Local Description Near Zeros) Let* $f : \underset{open}{\Omega} \to \mathbb{C}$ *holomorphic and* $f(z_0) = 0$, $z_0 \in \Omega, f \not\equiv 0$ *then* $\exists \mathcal{U} \ni z_0$, $g : \mathcal{U} \to \mathbb{C}$ *holomorphic and vanishing* ($\forall z \in \mathcal{U}, g(z) \neq 0$) $\exists! n > 0$ *such that* $f(z) = (z - z_0)^n g(z)$.

Proof. We know

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

$f \not\equiv 0$ implies that $n =$ smallest k such that $a_k \neq 0$. We can write $f(z) = (z - z_0)^n g(z)$. □

Definition 11.5. *Let $n =$ the multiplicity/order of z_0. When $n = 1$, z_0 is called a simple zero. We say* $f : \mathcal{D}_\delta(z_0) \setminus \{z_0\} \to \mathbb{C}$ *has a pole at z_0 if* $\frac{1}{f}$ *extended by 0 at z_0 is holomorphic in* $\mathcal{D}_\epsilon(z_0)$, *for some* $0 < \epsilon < \delta$.

Theorem 11.6. *(Local Discontinuity near Poles) Let* $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ *has a pole at z_0 in Ω then* $\exists \mathcal{U} \ni z_0$, $\exists h : \mathcal{U} \xrightarrow{holo} \mathbb{C}$ *nonvanishing then* $\exists! n > 0$ *such that* $f(z) = (z - z_0)^{-n} h(z)$.

Proof. Apply previous theorem to $\frac{1}{f}(z) = (z - z_0)^n g(z)$. □

11.2 Laurent Series and Residues

Theorem 11.7. *(Laurent Series Expansion) If f has a pole of order n at z_0 then locally*

$$f(z) = \underbrace{\frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{\overbrace{a_{-1}}^{residue\ of\ f\ at\ z_0}}{z - z_0}}_{principal\ part} + \underbrace{G(z)}_{holo\ part\ of\ f}.$$

where $G(z)$ is holomorphic and nonvanishing near z_0.
Moreover, $res_{z_0}(f) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n f(z))$

Proof. By theorem 11.6,

$$f(z) = (z - z_0)^{-n} h(z) = (z - z_0)^{-n} \sum_{k=0}^{\infty} b_k (z - z_0)^k = \frac{b_0}{(z - z_0)^n} + \cdots + \frac{b_{n-1}}{(z - z_0)} + \sum_{k \geq n}^{\infty} b_k (z - z_0)^{k-n}.$$

So we have

$$f(z)(z - z_0)^n = b_0 + b_1(z - z_0) + \cdots + b_{n-1}(z - z_0)^{n-1} + \underbrace{O((z - z_0)^n)}_{\to 0 \text{ as } \frac{d^{n-1}}{dz^{n-1}}}.$$

□

11.3 The Residue Formula

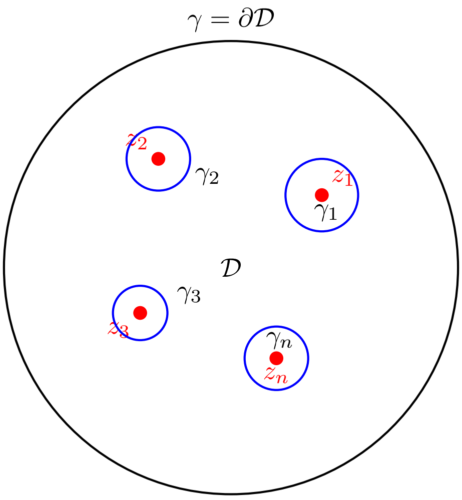
Theorem 11.8. *(The Residue Formula) Suppose* $f : \Omega \setminus \underbrace{\{z_1, z_2, \cdots, z_n\}}_{poles} \to \mathbb{C}$ *is holomorphic. Then for a disk* $\overline{\mathcal{D}} \subset \Omega$ *containing* $z_1, ..., z_n$ *we have*

$$\int_{\partial \mathcal{D}} f = 2\pi i \sum_{k=1}^{n} res_{z_k}(f).$$

Proof. Let $\gamma = \partial \mathcal{D}$ be the boundary of the disk $\mathcal{D}$, oriented counterclockwise. For each pole z_k, choose a small disk $\mathcal{D}_k$ centered at z_k with radius $\varepsilon_k > 0$ small enough so that:

- $\overline{\mathcal{D}_k} \subset \mathcal{D}$ for all k
- $\overline{\mathcal{D}_k} \cap \overline{\mathcal{D}_j} = \emptyset$ for $k \neq j$
- f is holomorphic on $\overline{\mathcal{D}} \setminus \bigcup_{k=1}^{n} \mathcal{D}_k$

Let $\gamma_k = \partial \mathcal{D}_k$ be the boundary of each small disk, oriented clockwise (negative orientation). Consider the multiply connected region $\mathcal{D} \setminus \bigcup_{k=1}^{n} \overline{\mathcal{D}_k}$.



By Cauchy's theorem for multiply connected regions, the integral of f around the boundary of this region (taking into account orientations) is zero. The boundary consists of γ (counterclockwise) and each γ_k (clockwise), so:

$$\int_{\gamma} f(z)dz + \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz = 0.$$

Note that γ_k has clockwise orientation, so reversing it gives:

$$\int_{\gamma} f(z)dz = \sum_{k=1}^{n} \int_{-\gamma_k} f(z)dz.$$

where $-\gamma_k$ denotes γ_k with counterclockwise orientation.
Now, for each pole z_k, consider its Laurent expansion:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j^{(k)} (z - z_k)^j = \frac{a_{-1}^{(k)}}{z - z_k} + \sum_{j \neq -1} a_j^{(k)} (z - z_k)^j.$$

where $a_{-1}^{(k)} = \text{res}_{z_k}(f)$.
For $j \neq -1$, the function $(z - z_k)^j$ has an antiderivative on $\mathbb{C} \setminus \{z_k\}$ (or on all of $\mathbb{C}$ if $j \geq 0$), so by the fundamental theorem of calculus:

$$\int_{-\gamma_k} (z - z_k)^j dz = 0 \quad \text{for } j \neq -1.$$

For $j = -1$, we compute directly. Parameterize $-\gamma_k$ by $z(t) = z_k + \varepsilon_k e^{it}$ for $t \in [0, 2\pi]$:

$$\int_{-\gamma_k} \frac{1}{z - z_k} dz = \int_0^{2\pi} \frac{1}{\varepsilon_k e^{it}} \cdot i\varepsilon_k e^{it} dt = \int_0^{2\pi} i dt = 2\pi i.$$

Therefore,

$$\int_{-\gamma_k} f(z) dz = a_{-1}^{(k)} \cdot 2\pi i = 2\pi i \cdot \operatorname{res}_{z_k}(f).$$

Combining all terms:

$$\int_{\partial \mathcal{D}} f(z) dz = \sum_{k=1}^{n} \int_{-\gamma_k} f(z) dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z_k}(f).$$

$\square$

12 Feb 11

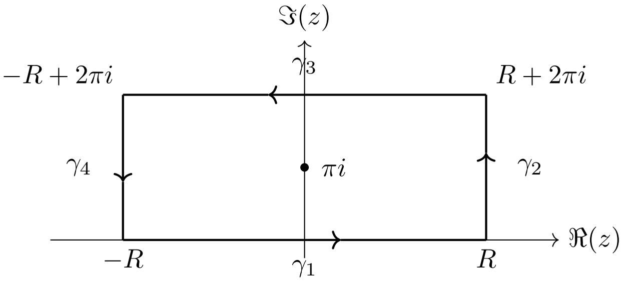
12.1 Integral Evaluation via Residues

Example 12.1.

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(a\pi)}.$$

Proof. Let $f(z) = \frac{e^{az}}{1+e^z}$ then we have simple poles at $z : e^z = 1$ so $z \in \{..., -\pi i, \pi i, 3\pi i, ...\}$. We have the residual for

$$\int_{\gamma_1} f + \cdots = \int_{\gamma_R} f(z)\, dz = 2\pi i \operatorname{res}_{\pi i}(f).$$



We have

$$\int_{\gamma_1} f(z) dz \xrightarrow{R \to +\infty} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = I.$$

On γ_3, $z = x + 2\pi i$ with x from R to $-R$, so

$$\int_{\gamma_3} f(z)\, dz = -\int_{-R}^{R} \frac{e^{a(x+2\pi i)}}{1+e^x} dx = -e^{2\pi a i} \cdot I.$$

As $R \to \infty$, $\int_{\gamma_2} f \to 0$ and $\int_{\gamma_4} f \to 0$. Thus

$$(1 - e^{2\pi a i})I = 2\pi i \cdot \operatorname{res}_{\pi i}(f) = 2\pi i \cdot (-e^{a\pi i}) = -2\pi i e^{a\pi i},$$

$$\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}} i dy \right| \leq \int_0^{2\pi} \frac{e^{aR}}{e^R + 1} dy \xrightarrow{R \to \infty} 0$$

Additionally $\int_{\gamma_4} f \to 0$. To compute the residue at πi, we have

$$\lim_{z \to \pi i} f(z)(z - \pi i) = \lim_{z \to i\pi} \frac{e^{az}}{\frac{e^z - e^{\pi i}}{z - \pi i}} = \frac{e^{\pi i a}}{e^{\pi i}}.$$

Letting $R \to +\infty$ in the residual formula gives

$$I + 0 + 0 - e^{2\pi a i} I = -2\pi i e^{a\pi i}.$$

So

$$I = -\frac{2\pi i e^{\pi i a}}{1 - e^{2\pi a i}} = \frac{\pi}{\sin(a\pi)}.$$

∎

12.2 Removable Singularities

Definition 12.2. *Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic, if we can extend f analytically to z_0 we say that f has a removable singularity at z_0.*

Theorem 12.3. *(Riemann's Theorem) Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic, if f is bounded near z_0 then z_0 is a removable singularity.*

Proof. Let $\overline{D} \subset \Omega$ be a disc centered at z_0. We want to use Cacuhy's formula

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw = g(z), \text{ for all} z \in D.$$

It suffices to show that $f = g$ on $D \setminus \{z_0\}$ because then g is the desired extension of f.
$F(w, z)$ is jointly continuous on $\partial D \times \overline{D}_{r-\epsilon}(z_0)$ and $\forall w \in \partial D$, $z \mapsto F(w, z)$ is holomorphic on D.
Fix $z \in D \setminus \{z_0\}$ then

$$\int_\gamma \frac{f(w)}{w - z} dw = 0 \implies \int_{\partial D} \frac{f(w)}{w - z} dw = \int_{C_\epsilon(z_0)} \frac{f(w)}{w - z} dw + \int_{C_\epsilon(z)} \frac{f(w)}{w - z} dw = 0 + 2\pi i f(z).$$

□

13 Feb 13

13.1 Classification of Singularities

Theorem 13.1. *(Riemann) Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ has a removable singularity at z_0 if and only if bounded near z_0.*

Corollary 13.2. *Pole at z_0 if and only if $|f| \to +\infty$ as $z \to z_0$*

Proof. ($\Longrightarrow$) Local description near poles implies $f(z) = (z - z_0)^{-n} g(z)$ so $|f| \to +\infty$
($\Longleftarrow$) Since $\left|\frac{1}{f}\right| \to 0$, in particular $\frac{1}{f}$ is bounded near z_0, Riemann implies $\frac{1}{f}$ has a removable singularity at z_0. □

Theorem 13.3. *(Casorati-Weierstrass) If $f : D_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic with an essential singularity at z_0 then the image of f is dense in $\mathbb{C}$.*

Proof. If not, $\exists w \in \mathbb{C}$ such that no values of f near w, $\exists \delta > 0$ s.t. $\forall z$, $|f(z) - w| > \delta$.
Consider

$$g(z) = \frac{1}{f(z) - w} : D_r(z_0) \setminus \{z_0\} \to \mathbb{C}.$$

Such a g is bounded by $\frac{1}{\delta}$. So g extends halomorphically into $D_r(z_0)$.

(Case 1) $g(z_0) = 0 \implies \frac{1}{g} \to +\infty$ so $f(z) - w$ has a pole so f has a pole

(Case 2) $g(z_0) \neq 0 \implies \frac{1}{g}$ is well defined and f has a removable singularity.

□

Theorem 13.4. *(Picard's Theorem) If $f : D_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic with an essential singularity at z_0 then the image of f is either $\mathbb{C}$ or $\mathbb{C}$ except for at most one value.*

13.2 Meromorphic Functions and Behavior at Infinity

Theorem 13.5. *Let $\Omega \subset \mathbb{C}$ be open. A function f is called meromorphic on Ω if $\exists z_1, z_2, \ldots$ without a limit point such that f is holomorphic in $\Omega \setminus \{z_1, z_2, \ldots\}$ and has poles at $z_1, z_2, \ldots$.*

Theorem 13.6. *(Behavior at $+\infty$) Suppose f is holomorphic in a neighborhood of $_\infty$ i.e. $\{|z| > R\}$ for some $R > 0$.*
Note: $F(z) = f(\frac{1}{z})$ is holomorphic in $D_{\frac{1}{R}}(0) \setminus \{0\}$.
We say f has a removable singularity at ∞ if F has a removable singularity at $z = 0$.

Example 13.7. *e^z has an essential singularity at $+\infty$.*

Remark 13.8. *We'll denote $\overline{\mathbb{C}} = \mathbb{C} \cup \{+\infty\}$*

Example 13.9. *Examples of meromorphic functions on $\mathbb{C}$*

1. *Rational functions: $\frac{P}{Q}$*
2. *The gamma function: $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ (simple poles at $\ldots, -2, -1, 0$)*
3. *The Riemann zeta function: $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ (simple poles at $s = -1, -2, -3, \ldots$)*

14 Feb 16

14.1 Rational Functions via Meromorphy

Theorem 14.1. *Suppose f is meromorphic on $\mathbb{C}$ with a removable singularity or a pole at $+\infty$ then f is a rational function.*

Proof. Claim: Under these assumptions, f has finitely many poles.
If not, $z_1, z_2, \ldots$ are poles and this sequence cannot be bounded otherwise it got a limit point. Say $|z_k| \to +\infty$, the neighborhoods of ∞ are $\{|z| > R\}$ so f not holomorphic for any $R > 0$ which contradicts that f has a removable singularity at $+\infty$.

For a function F which has a pole at $z = 0$ we have

$$F(z) = \underbrace{\frac{a_n}{z^n} + \ldots + \frac{a_{-1}}{z}}_{f(z)} + \underbrace{a_0 + a_1 z + \cdots}_{g(z)}.$$

Let $z_1, \ldots, z_N$ be the poles of f. Near z_k, for each $1 \le k \le N$,

$$f(z) = \underbrace{f_k(z)}_{\substack{\text{principal part} \\ \text{poly in } \frac{1}{z-z_k}}} + \underbrace{g_k(z)}_{\text{holomorphic}} .$$

Near ∞,

$$f\left(\frac{1}{z}\right) = \underbrace{\tilde{f}_\infty(z)}_{\substack{\text{principal part} \\ \text{poly in } \frac{1}{z}}} + \underbrace{\tilde{g}_\infty(z)}_{\text{holomorphic}} .$$

Consider,

$$H(z) = f(z) - \underbrace{\sum_{j=1}^{N} f_j(z)}_{\text{rational}} - \underbrace{\tilde{f}_\infty(z)}_{\text{poly}}.$$

Claim: H is holomorphic and bounded.
Near z_k

$$H = \underbrace{f - f_k}_{\text{holo}} - \sum_{j \neq k} f_j - \underbrace{f_\infty}_{\text{holo}}.$$

We can show H is bounded as

$$H\left(\frac{1}{z}\right) = \underbrace{f\left(\frac{1}{z}\right) - \tilde{f}_\infty(z)}_{\substack{\tilde{g}_\infty(z) \\ \text{bounded near } z=0}} - \underbrace{\sum_{j=1}^{N} f_j\left(\frac{1}{z}\right)}_{\substack{\text{poly} \\ \text{bounded near } z=0}} .$$

So $H(\frac{1}{z})$ is bounded in say $\{|z| < \delta\}$ and $H(z)$ is bounded in $\{|z| > 1/\delta\}$, hence H is bounded in $\{|z| \le 1/\delta\}$. *dset* ☐

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