Convex Optimization Notes

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Definition 1.1. *An* ***optimization problem*** *is defined as*

$$\min_{x \in \mathcal{C}} f_0(x) \quad \textit{where} \quad \forall i \in [m],\ f_i(x) \le b_i$$

where $\mathcal{C}$ *is the constraint set and* $f_0 \colon \mathcal{C} \to \mathbb{R}$.

Definition 1.2. *The* ***standard form*** *of an optimization problem is*

$$\min_{x \in \mathcal{D}} f_0(x) \quad \textit{subject to} \quad \forall i \in [m],\ f_i(x) \le 0; \quad \forall j \in [p],\ h_j(x) = 0$$

where $x \in \mathrm{dom}(f_i) \cap \mathrm{dom}(h_j) =: \mathcal{D}$.
f_0 *is termed the* objective function, *and* x *is the optimization variable.*
f_i *and* h_j *are constraint functions.*

Definition 1.3. *A* ***feasible solution*** *is any* $x \in \mathcal{D}$ *that satisfies all the constraints.*

Definition 1.4. *The* ***optimal solution*** *is* $x^* \in \arg\min_{x \in \mathcal{D}} f_0(x)$.

Remark 1.5. *If no feasible solution exists, we define* $\min_{x \in \mathcal{C}} f(x) = +\infty$ *and, similarly,* $\max_{x \in \mathcal{C}} f(x) = -\infty$.

2 Jan 15

Definition 2.1. *A set C is **convex** if $\forall x_1, x_2 \in C$ and $\forall \theta \in [0,1]$, we have $\theta x_1 + (1-\theta)x_2 \in C$.*

Definition 2.2. *A function $f\colon \mathcal{X} \to \mathbb{R}$ is **convex** if for all $x_1, x_2 \in \mathcal{X}$ and $\theta \in [0,1]$,*

$$f(\theta x_1 + (1-\theta)x_2) \le \theta f(x_1) + (1-\theta) f(x_2) \tag{*}$$

Remark 2.3. *Convex functions are differentiable almost everywhere.*

Theorem 2.4. *If f is both convex and concave, then f is affine (i.e., linear plus constant).*

Definition 2.5. *f is **strictly convex** if $(*)$ holds with strict inequality $<$ whenever $x_1 \neq x_2$ and $\theta \in (0,1)$.*

Definition 2.6. *A **convex optimization problem** is one where f_0, f_i, and $\mathcal{C}$ are convex, and h_j are affine/linear. That is,*

$$\min_{x \in \mathcal{C}} f_0(x) \quad \textit{such that} \quad f_i(x) \le 0 \;\; \forall i \in [m], \quad h_j(x) = 0 \;\; \forall j \in [p]$$

Theorem 2.7. *The set of feasible solutions in Definition 2.6 is convex.*

Theorem 2.8. *x^* is a **local minimizer** if there exists $\epsilon > 0$ such that for all y with $\|x^* - y\| \le \epsilon$,*

$$f(x^*) \le f(y)$$

Theorem 2.9. *x^* is a **local maximizer** if there exists $\epsilon > 0$ such that for all y with $\|x^* - y\| \le \epsilon$,*

$$f(x^*) \ge f(y)$$

Theorem 2.10. *For any convex optimization problem, every local minimum is a global minimum.*

Proof. Suppose $\hat{x}$ is a local minimizer not equal to global minimizer x^*. Take ϵ as any witness to $\hat{x}$ being a local minimum. Let

$$y = \frac{\epsilon}{\|\hat{x} - x^*\|} x^* + \left(1 - \frac{\epsilon}{\|\hat{x} - x^*\|}\right) \hat{x}$$

Note: $\|\hat{x} - x^*\| \le \epsilon$, otherwise $\hat{x}$ is not a local minimizer in that neighborhood.

$$y - \hat{x} = \frac{\epsilon(x^* - \hat{x})}{\|x^* - \hat{x}\|} \quad ; \quad \|y - \hat{x}\| = \epsilon$$

Since f is convex,

$$f(y) \le \frac{\epsilon}{\|\hat{x} - x^*\|} f(x^*) + \left(1 - \frac{\epsilon}{\|\hat{x} - x^*\|}\right) f(\hat{x}) < f(\hat{x})$$

Thus, a contradiction. $\square$

3 Jan 20

Definition 3.1. *A **convex combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 = 1$.*

Definition 3.2. *An **affine combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1 + \theta_2 = 1$.*

Definition 3.3. *A **linear combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1, \theta_2 \in \mathbb{R}$.*

Definition 3.4. *A **conic combination** of x_1, x_2 is $\theta_1 x_1 + \theta_2 x_2$ where $\theta_1, \theta_2 \geq 0$.*

Definition 3.5. *Given a set C, the **convex hull** of C is*

$$\operatorname{conv}(C) = \left\{ \sum_{i=1}^{k} \theta_i x_i \;\middle|\; x_i \in C,\, \theta_i \in [0,1],\, \sum_{i=1}^{k} \theta_i = 1 \right\}$$

Remark 3.6. *The following are true:*

1. $C \subseteq \operatorname{conv}(C)$
2. $\operatorname{conv}(C)$ *is convex*
3. *It is the smallest convex set containing C*
4. *If a convex set $S \supseteq C$ then $S \supseteq \operatorname{conv}(C)$*

Theorem 3.7. *Any closed convex set can be written as $\overline{\operatorname{conv}}(C)$ for some set C.*

Definition 3.8. *The **relative interior** of C is defined as*

$$\operatorname{relint}(C) = \{x \in C : \exists \epsilon > 0,\ B(x,\epsilon) \cap \operatorname{Aff}(C) \subseteq C\}$$

Definition 3.9. *C is a **cone** if $\alpha x \in C$ whenever $x \in C$ and $\alpha \geq 0$.*

Definition 3.10. *Given a set C, the **conic hull** of C is*

$$\operatorname{conic}(C) = \left\{ \sum_{i=1}^{k} \theta_i x_i : x_i \in C,\ \theta_i \geq 0 \right\}$$

Theorem 3.11. *The conic hull of C is the smallest convex cone containing C.*

Definition 3.12. *The ℓ_p **norm** is defined as*

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p}$$

Remark 3.13. *The following are true:*

1. *For $p \in (0,1)$: $\|x\|_p$ is not a convex function*
2. *For $p \geq 1$: $\|x\|_p$ is convex*
3. *For $p > 1$: $\|x\|_p$ is strictly convex*

Example 3.14. *Examples of convex sets:*

1. *Hyperplane: $\{x : a^T x = b\}$*
2. *Halfspace: $\{x : a^T x \leq b\}$*
3. *Polyhedron: $\{x \in \mathbb{R}^d : Ax \leq b,\ Cx = d\}$*
4. *Polytope: a bounded polyhedron.*

Theorem 3.15. *A set S is **strictly convex** if for all $x_1 \neq x_2$ and $\theta \in (0,1)$, $\theta x_1 + (1-\theta) x_2 \in \operatorname{int}(S)$.*

Definition 3.16. *The **normal cone** is defined to be*

$$N_C(x) = \{g : g^T(y - x) \leq 0,\ \forall y \in C\}$$

Remark 3.17. *If $x \in \operatorname{int}(C)$ then $N_C(x) = \{0\}$.*

Theorem 3.18. *If f is differentiable, then x^* is optimal if and only if $-\nabla f(x^*) \in N_C(x^*)$.*

4 Jan 22

Theorem 4.1. *If f is convex and differentiable, and C is a convex set, then any optimal solution x^* to $\min_{x \in C} f(x)$ must satisfy $-\nabla f(x^*) \in N_C(x^*)$.*

Theorem 4.2. *The set of optimal solutions to a convex optimization problem is a convex set.*

Definition 4.3. *If C is convex then*

1. ***Translation:*** $C + a = \{x : x - a \in C\}$
2. ***Scaling:*** $\alpha C = \{x : \frac{x}{\alpha} \in C\}$
3. ***Intersection:*** *If $\{C_\alpha\}_{\alpha \in A}$ is a collection of convex sets, then $\bigcap_{\alpha \in A} C_\alpha$ is convex.*

Theorem 4.4. *The following are true:*

1. *If $C \subseteq \mathbb{R}^n$ is convex, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then $AC + b = \{Ax + b : x \in C\}$ is a convex set.*
2. *If $f(x) = Ax + b$, then $f^{-1}(C)$ is convex.*
3. $C_1 + C_2 = \{x + y : x \in C_1,\, y \in C_2\}$ *is convex.*
4. *If $C_1 \subseteq \mathbb{R}^m$, $C_2 \subseteq \mathbb{R}^n$ then $C_1 \times C_2 = \{(x, y) \in \mathbb{R}^{m+n} : x \in C_1,\, y \in C_2\}$ is convex.*
5. *For any $C \subseteq \mathbb{R}^n \times \mathbb{R}^m_{>0}$, define*

$$P(C) = \left\{ \left(\frac{x_1}{t}, \dots, \frac{x_n}{t} \right) : (x_1, \dots, x_n, t) \in C \right\}$$

   If C is convex, so are $P(C)$ and $P^{-1}(C)$.

Definition 4.5.

$$\operatorname{epi}(f) = \{(x, t) : x \in \operatorname{dom}(f),\, t \geq f(x)\}$$

Definition 4.6. ***First Order Definition of Convexity:*** *If f is differentiable, then f is convex if and only if for all $x, y \in \operatorname{dom}(f)$,*

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

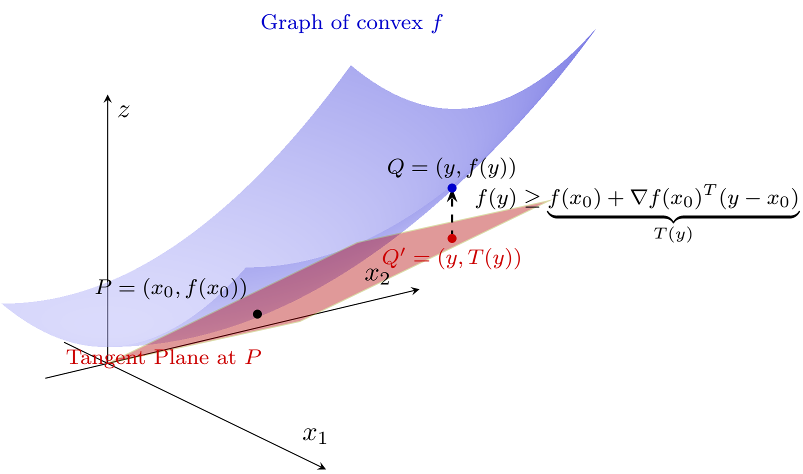

Figure 1: First Order Definition of Convexity

Second Order Definition of Convexity: *If f is twice differentiable, then f is convex if and only if $\nabla^2 f(x) \succeq 0$.*

Definition 4.7. *We say $A \succeq B$ if $A - B$ is positive semi-definite. This is equivalent to*

$$a^T \nabla^2 f(x) a \geq 0 \quad \text{for all } a \in \mathbb{R}^d$$

Definition 4.8. *The* ***subdifferential*** *is*

$$\partial f(x) = \left\{g : f(y) \geq f(x) + g^T(y - x),\ \forall y \in C\right\}$$

Any such $g \in \partial f(x)$ *is called a* ***subgradient****.*
If f *is differentiable at* x *then* $\partial f(x) = \{\nabla f(x)\}$.

Theorem 4.9. f *is convex if and only if* ∂f *is non-empty.*

5 Jan 27

Definition 5.1. *Gradient Monotonicity: If f is differentiable, then f is convex if and only if $\nabla f(x)$ is monotone. So we can conclude*

$$(\nabla f(x) - \nabla f(y))^T(x - y) \geq 0 \iff f \text{ is convex.}$$

Theorem 5.2. *For a $A \in \mathbb{R}^{m \times n}$ we can decompose*

$$A = U \sum V^t.$$

where $U \in \mathbb{R}^{m \times k}$ with orthonormal column, $\sigma \in \mathbb{R}^{k \times k}$ is a diagonal matrix with non-negative entries, and $V \in \mathbb{R}^{n \times k}$ with orthonormal column.

Definition 5.3. *(u_i, v_i, σ_i) form a s.v. triplet if $Av_i = \sigma_i u_i$ and $A^T u_i = \sigma_i v_i$*
$||u_i|| = ||v_i|| = 1$ and $\sigma_i \geq 0$

Theorem 5.4. *For any $A \in \mathbb{R}^{n \times n}$ then $A^T A$ is always positive semi definite*

Proof. We can write $A = V \sum U$ then $A^T A = V \sum U^T U \sum V^T = V \sum^2 V^T$ □

Definition 5.5. *Spectral Radius:* $\max_i\{|\lambda_i| : \lambda_i \in \Lambda(A)\} = \rho(A)$

Definition 5.6. *The norm must satisfy the following properties:*

1. $||A|| \geq 0$
2. $||\alpha A|| = |\alpha| ||A||$
3. $||A|| = 0$ *if and only if* $A = 0$
4. $||A + A'|| \leq ||A|| + ||A'||$

Definition 5.7. *Operator/Spectral Norm:*

$$||A||_{op} = ||A||_2 = \max_{||x||=1} ||Ax||_2.$$

Definition 5.8. *Frobenius Norm:*

$$||A||_F = \sqrt{\sum_{i,j} A_{ij}^2}.$$

Definition 5.9. *f is L-Lipschitz if*

$$|f(x) - f(y)| \leq L||x - y||_2 \quad \forall x, y \in \text{dom}(f).$$

If f is differentiable then f is L-Lipschitz if and only if $|\nabla f| \leq L$

Definition 5.10. *Differentiable f is β-smooth if*

$$||\nabla f(x) - \nabla f(y)||_2 \leq \beta||x - y||_2 \quad \forall x, y \in \text{dom}(f).$$

If f is twice differentiable then f is β-smooth if and only if $\nabla^2 f(x) \preceq \beta I$.

Theorem 5.11. *If f is twice differentiable and β-smooth then*

$$f(y) \leq f(x) + \nabla f(x)^T(y - x) + \frac{\beta}{2}||y - x||_2^2 \quad \forall x, y \in \text{dom}(f).$$

6 Feb 3

Definition 6.1. *"Descent Direction" any h for which $f(x+\eta h) \leq f(x)$.*

Theorem 6.2. *Assume f differentiable and β-smooth with $x_{t+1} \leftarrow x_t - \eta \nabla f(x_t)$.*

1. *If $\eta \leq \frac{2}{\beta}$ then $f(x_{t+1}) \leq f(x_t)$*
2. *If $\eta \leq \frac{1}{\beta}$ then $f(x_{t+1}) \leq f(x_t) - \frac{\eta}{2}||\nabla f(x_t)||_2^2$*

Proof. Recall

$$f(y) \leq f(x) + \nabla f(x)^T(y-x) + \frac{\beta}{2}||y-x||_2^2.$$

and

$$||\nabla f(y) - \nabla f(x)||_2 \leq \beta ||y-x||_2.$$

Let $y \leftarrow x+h$ then $f(x+h) \leq f(x) + \nabla f(x)^T h + \frac{\beta}{2}||h||_2^2$

1. If $h = -\frac{2}{\beta}\nabla f(x)$ then $\nabla f(x)^T h = -\frac{2}{\beta}||\nabla f(x)||_2^2$ and $\frac{\beta}{2}||h||_2^2 = \frac{2}{\beta}||\nabla f(x)||_2^2$ so $f(x+h) \leq f(x)$
2. If $h = -\frac{1}{\beta}\nabla f(x)$ then $\nabla f(x)^T h = -\frac{1}{\beta}||\nabla f(x)||_2^2$ and $\frac{\beta}{2}||h||_2^2 = \frac{1}{2\beta}||\nabla f(x)||_2^2$.

□

Theorem 6.3. *If f is differentiable, β smooth, and x^* be any optimizer. For any $k > 0$*

$$\min_{i=0,\dots,k} ||\nabla(f(x_i))||_2 \leq \sqrt{\frac{2\beta}{k}\left(\underbrace{f(x_0) - f(x^*)}_{\Delta_0}\right)}.$$

Proof. AFSOC $\forall i \in \{0,\dots,k\}$, $||\nabla f(x_i)|| > \sqrt{\frac{2\beta}{k}\Delta_0}$ Then we have

$$f(x_{i+1}) < f(x_i) - \frac{1}{2\beta}\left(\frac{2\beta\Delta_0}{k}\right). \qquad (\forall i \in \{0,\dots,k\})$$

Continuing down the chain we have

$$f(x_{k+1}) < f(x_0) - \Delta_0 < f(x^*).$$

Thus a contradiction. □

Theorem 6.4. *If f is β-smooth and convex, $\eta = \frac{1}{\beta}$ and x^* any minimum then*

$$f(x_k) - f(x^*) \leq \frac{\beta||x_0 - x^*||_2^2}{2k}.$$

Proof. Observe that

$$||x_{t+1} - x^*||^2 = ||x_t - x^* - \eta\nabla f(x_t)||^2 = ||x_t - x^*||^2 - \eta^2||\nabla f(x_t)||^2 - 2\eta\nabla f(x_t)^T(x_t - x^*).$$

Rearranging and by convexity we have

$$\begin{aligned} f(x_t) - f(x^*) \leq \nabla f(x_t)^T(x_t - x^*) &= \frac{1}{2\eta}\left[\underbrace{||x_t - x^*||^2}_{\delta_t} - \underbrace{||x_{t+1} - x^*||^2}_{\delta_{t+1}} + \frac{\eta}{2}||\nabla f(x_t)||^2\right] \\ &\leq \frac{1}{2\eta}\left[\delta_t - \delta_{t+1} + 2\eta(f(x_t) - f(x_{t+1}))\right] \\ &= \frac{1}{2\eta}(\delta_t - \delta_{t+1}) + f(x_t) - f(x_{t+1}) \end{aligned}$$

So we have

$$f(x_{t+1}) - f(x^*) \leq \frac{\beta}{2}(\delta_t - \delta_{t+1}).$$

Adding across all terms

$$\sum_{i=0}^{k-1} f(x_{i+1}) - f(x^*) \leq \frac{\beta}{2}\left(||x_0 - x^*||^2\right).$$

So we have the relation

$$k(f(x_k) - f(x^*)) \leq \sum_{i=0}^{k-1} f(x_{i+1}) - f(x^*) \leq \frac{\beta}{2}||x_0 - x^*||^2.$$

□

Definition 6.5. *f is α-strongly convex if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\alpha}{2}t(1-t)||x - y||_2^2.$$

Additionally, if f is differentiable then

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{\alpha}{2}||y - x||_2^2.$$

If f is twice differentiable then f is α-strongly convex if and only if $\nabla^2 f(x) \succeq \alpha I$.

Theorem 6.6. *Assume f is β-smooth, α-strongly convex, $\eta = \frac{1}{\beta}$ then $||x_k - x^*||^2 \leq \left(1 - \frac{1}{\gamma}\right)^k ||x_0 - x^*||^2$. We define the conditional number as $\gamma = \frac{\beta}{\alpha}$*

Proof.

$$\frac{\alpha}{2}||x_k - x^*||^2 \leq \frac{\beta}{2}\left(||x_t - x^*||^2 - ||x_{t+1} - x^*||^2\right) \leq \left(1 - \frac{\alpha}{\beta}\right)||x_t - x^*||^2 \leq (1 - \frac{1}{\gamma})||x_k - x^*||^2.$$

□

7 Feb 5

Recall 7.1. *From previous class*

1. $f(x+\eta h) \approx f(x) + \eta h^T \nabla f(x)$ *and when we set* $h = -\nabla f(x)$ *we have*

$$f(x - \eta \nabla f(x)) \approx f(x) - \eta ||\nabla f(x)||_2^2.$$

Backtracking Line Search: *Pick* $\gamma_1 \in (0,1)$ *and* $\gamma_2 \in (0,1)$*, start with initial step size* η*. While* $f(x - \eta\nabla f(x)) > f(x) - \gamma_1 \eta ||\nabla f(x)||_2^2$ *(Armijo condition not satisfied), set* $\eta \leftarrow \gamma_2 \eta$ *and retry. Once the condition is satisfied, use step size* η *to update* $x \leftarrow x - \eta \nabla f(x)$*.*

Recall 7.2. g *is a subgradient of* f *at* x *if*

$$f(y) \geq f(xlabel) + g^T(y-x) \quad \forall y \in \mathrm{dom}(f).$$

Lemma 7.3. *The following are true:*

1. $\partial f(x)$ *makes sense for non convex functions too, could be empty*
2. *If* f *is convex, then for* $x \in \mathrm{RelInt}(\mathrm{dom}(f))$ *we have that* $\partial f(x)$ *is non-empty*
3. $\partial f(x)$ *is a convex*
4. *If* f *is convex and differentiable at* x *then* $\partial f(x) = \{\nabla f(x)\}$
5. *If* $\partial f(x)$ *is non-empty everywhere,* f *is convex.*

Theorem 7.4. *The following are true:*

1. $\partial(af) = a\partial f(x)$
2. $\partial(f+g) = \partial f(x) + \partial g(x)$
3. *If* $g(x) = f(Ax+b)$ *then* $\partial g(x) = A^T \partial f(Ax+b)$

Example 7.5. *For* $f(x) = \max_{i=1,\ldots,n} f_i(x)$ *we have* $\partial f(x) = \mathrm{Conv}\left(\bigcup_{i=1,\ldots,n} \partial f_i(x)\right)$

Example 7.6. *For* $f(x) = |x|$ *we have* $\partial f(x) = \mathrm{sign}(x) = \max -x, x$ *so* $\partial f(0) = [-1, 1]$.

Example 7.7. *Let* C *be a convex set and* $I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$ *is convex Additionally, for* $x \in C$*,* $\partial I_C(x) = N_C(x) = \{g : g^T(y-x) \leq 0 \forall y \in C\}$

Proof. For $x \in C$, $I_C(y) \geq I_C(x) + g^T(y-x)$ for all $y \in C$ then we have

$$0 \geq 0 + g^T(y-x).$$

□

Definition 7.8. *Subgradient Method:* $x_{t+1} \leftarrow x_t - \eta g_x$ *for some* $g_x \in \partial f(x)$

1. *Subgradients are in general not descent directions*
2. *The min norm subgradient is a descent direction*

Example 7.9. $f(x) = ||x||_2 = \sqrt{\sum_{i=1}^n x_i^2}$ *is convex, then for* $x \neq 0$ *we have* $\partial f(x) = \frac{x}{||x||_2}$.

$$\partial f(0) = \{g : ||g||_2 \leq 1\}.$$

8 Feb 10

Definition 8.1. *For the subgradient method for* $\min_{x\in\mathbb{R}^n} f(x)$ *and* f *convex where*

$$x_{t+1} = x_t - \eta_t g_t \text{ where } g_t \in \partial f(x_t).$$

We define the best iterate as

$$x_T^{(\text{best})} = \arg\min_{i=0,\dots,T} f(x_i).$$

Theorem 8.2. *Assume* f *is* G*-lipschitz and convex. Let* $||x_0 - x^*|| \le R$ *then pick*

$$\eta_t = \frac{R}{G\sqrt{T}} \text{ guarantees that } f\left(x_T^{(\text{best})}\right) - f(x^*) \le \frac{GR}{\sqrt{T}}.$$

Theorem 8.3. *A convex function* f *is* G*-lipschitz iff*

$$||g_x|| \le G \;\forall x \in \mathrm{dom}(f) \text{ and } \forall g_x \in \partial f(x).$$

Theorem 8.4. *For nonconvex, differentiable* f*,* f *being* G*-lipschitz iff*

$$||\nabla f(x)|| \le G \;\forall x \in \mathrm{dom}(f).$$

Theorem 8.5. *Assume* f *is convex and* G*-Lipschitz, and that an optimal solution* x^* *exists with* $||x_0 - x^*|| \le R$ *for some* $R > 0$.
Pick $\eta_t \to 0$ *such that* $\sum_{t=1}^{\infty} \eta_t = \infty$ *and* $\sum_{t=1}^{T} \eta_t^2 < \infty$ *then*

$$f\left(x_T^{(\text{best})}\right) \to f(x^*) \text{ as } T \to \infty.$$

Theorem 8.6. *For the subgradient method with step sizes* $\{\eta_t\}_{t=1}^T$ *on a convex,* G*-Lipschitz function* f*, we have*

$$f\left(x_T^{(\text{best})}\right) - f(x^*) \;\le\; \frac{\|x_0 - x^*\|^2 + G^2 \sum_{t=1}^T \eta_t^2}{2\sum_{t=1}^T \eta_t}.$$

This theorem implies Theorem 8.2 by letting $\eta_t = \frac{R}{G\sqrt{T}}$

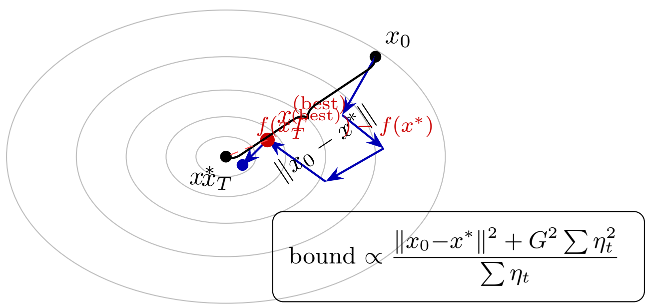

Figure 2: Subgradient method: the path may zigzag (subgradients need not be descent directions). The **best iterate** $x_T^{(\text{best})}$ minimizes f over $\{x_0, \dots, x_T\}$. The bound shrinks when $\sum \eta_t$ is large (more "progress") and when $\|x_0 - x^*\|$ and $\sum \eta_t^2$ are small.

Proof.

$$\begin{aligned} ||x_{t+1} - x^*||^2 &= ||x_t - \eta_t g_t - x^*||^2 \\ &= ||x_t - x^*||^2 - 2\eta_t g_t^T (x_t - x^*) + \eta_t^2 ||g_t||^2 \\ &\le ||x_t - x^*||^2 + 2\eta_t (f(x^*) - f(x_t)) + \eta_t^2 G^2 \qquad (*) \end{aligned}$$

Last step follows from $f(x^*) \ge f(x_t) + g_t^T(x^* - x_t) \iff -g_t^T(x_t - x^*) \ge f(x^*) - f(x_t)$.
Adding $(*)$ up from $0, ..., T-1$ we have

$$||x_T - x^*||^2 \le ||x_0 - x^*||^2 + G^2 \sum_{t=0}^{T-1} \eta_t^2 + 2 \sum_{t=0}^{T-1} \eta_t \left(f(x^*) - f(x_t)\right).$$

So

$$2\sum_{t=0}^{T-1}\eta_t\left(f(x_T^{(\text{best})})-f(x^*)\right)\leq 2\sum_{t=0}^{T-1}\eta_t\left(f(x^*)-f(x_t)\right)\leq ||x_0-x^*||^2+G^2\sum_{t=0}^{T-1}\eta_t^2.$$

□

Theorem 8.7. *Polyak's Stepsize: If $f(x^*)$ is known,*

$$\eta_t=\frac{f(x_t)-f(x^*)}{||g_t||^2}.$$

Theorem 8.8. *Given $C_1,...,C_k$ convex sets find $x^*\in\bigcap_{i=1}^k C_i$.*

Proof. Define $f_i(x)=\min_{y\in C_i}||x-y||^2=\text{dist}(x,C_i)$ and $f(x)=\max_{i=1,...,k}f_i(x)$.
If $x^*\in C_1\cap...\cap C_k$ then $f(x^*)=0$.
Recall: $\partial f(x)=\text{Conv}\left(\bigcup_{i=1,...,k}\partial f_i(x)\right)$.
Let $P_C(x)=\arg\min_{y\in C}||x-y||$

Lemma 8.9. *u is the projection of x onto C iff $\langle x-u,y-u\rangle\leq 0\forall y\in C$*

We have $f_i(x)=||x-P_{C_i}(x)||^2$ so $\partial f_i(x)=\frac{x-P_{C_i}(x)}{|x-P_{C_i}(x)||}$ if $x\neq 0$.

$$x_{t+1}=x_t-f(x_t)\cdot\frac{x_t-P_{C_i}(x_t)}{||x_t-P_{C_i}(x_t)||}=P_{C_i}(x_t).$$

□

9 Feb 12

Definition 9.1. *For* $\min_{x\in C} f(x)$ *we define the projected subgradient method to be*

$$\begin{cases} y_{t+1} = x_t - \eta_t g_t \\ x_{t+1} = P_C(y_t) \end{cases}.$$

Theorem 9.2. $P_C(x) = \arg\min_{y\in C} ||x-y||^2$

Proof. We have $z = P_c(x)$ iff $\forall y \in C$, $\langle x-z, y-z\rangle \leq 0$. Then for

$$\underbrace{-\nabla f(z)}_{x-z} \in \underbrace{N_c(z)}_{\{g : g^T(y-z)\leq 0\ \forall y\in C\}}.$$

so we have $(x-z)^T(y-z) \leq 0$ □

Lemma 9.3. *Projections are contractions:*

$$||\underbrace{P_C(x_1)}_{z_1} - \underbrace{P_C(x_2)}_{z_2}||_2 \leq ||x_1 - x_2||_2.$$

Proof. We have

1. $\langle x_1 - z_1, z_2 - z_1\rangle \leq 0 \forall z_2 \in C$
2. $\langle x_2 - z_2, z_1 - z_2\rangle \leq 0 \forall z_1 \in C$
3. Adding these two inequalities we have

$$\langle x_1 - z_1, z_2 - z_1\rangle + \langle x_2 - z_2, z_1 - z_2\rangle = \langle x_1 - x_2 + z_2 - z_1, z_2 - z_1\rangle \leq 0.$$

So

$$\langle x_1 - x_2, z_2 - z_1\rangle + ||z_2 - z_1||_2^2 \leq 0 \iff ||z_2 - z_1||_2^2 \leq \langle x_1 - x_2, z_2 - z_1\rangle \leq ||x_1 - x_2||||z_2 - z_1||.$$

So we have our result

$$\|z_1 - z_2\| \leq ||x_1 - x_2||.$$

□

Theorem 9.4. *Rates for PGD are identical to GD and similarly PSGM is identical to SGM.*

Proof.

$$||x_{t+1} - x^*||^2 = ||P_C(x_t - \eta_t g_t) - x^*||^2 = ||P_C(x_t - \eta_t g_t) - P_C(x^*)|| \leq ||x_t - \eta_t g_t - x^*||^2.$$

□

Example 9.5. *Consider*

$$\min_{x\in\mathbb{R}^d} \frac{1}{2}||Ax - b||_2^2 \textit{ where } n \geq d \textit{ and } A \textit{ full rank }.$$

When we solve this by gradient descent then

Proof. □