

Algebra I: Class Notes

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1 Lecture 1: Group Actions & The Orbit-Stabilizer Theorem (Jan 12)

1.1 Permutations and Automorphisms

Definition 1.1 (Symmetric Group). Let X be a set. The set of all permutations (bijections) of X is denoted by Sym_X (or sometimes Σ_X). Under function composition, Sym_X forms a group.

Definition 1.2 (Automorphism Group). Let (G, \cdot) be a group. An automorphism of G is a bijection $\phi : G \rightarrow G$ that is also a homomorphism, i.e.,

$$\phi(g_1g_2) = \phi(g_1)\phi(g_2) \quad \forall g_1, g_2 \in G.$$

The set of all such automorphisms is denoted by $\text{Aut}(G)$. It forms a group under composition.

1.2 Group Actions

Definition 1.3 (Group Action). Let G be a group and X be a set. An **action** of G on X is a homomorphism $\phi : G \rightarrow \text{Sym}_X$.

We typically write the action as $g \cdot x := \phi(g)(x)$. This notation satisfies two axioms (equivalent to the homomorphism property):

1. **Identity:** $1 \cdot x = x$ for all $x \in X$.
2. **Compatibility:** $g_1 \cdot (g_2 \cdot x) = (g_1g_2) \cdot x$ for all $g_1, g_2 \in G, x \in X$.

Theorem 1.4 (Equivalence Relation on X). Let G act on X . Define a relation \sim on X by:

$$x_1 \sim x_2 \iff \exists g \in G \text{ such that } g \cdot x_1 = x_2.$$

Then \sim is an equivalence relation.

Proof. We check the properties:

1. **Reflexive:** $1 \cdot x = x \implies x \sim x$.
2. **Symmetric:** If $g \cdot x_1 = x_2$, acting by g^{-1} gives $g^{-1} \cdot (g \cdot x_1) = g^{-1} \cdot x_2 \implies (g^{-1}g) \cdot x_1 = g^{-1} \cdot x_2 \implies x_1 = g^{-1} \cdot x_2$. Thus $x_2 \sim x_1$.
3. **Transitive:** If $g \cdot x_1 = x_2$ and $h \cdot x_2 = x_3$, then $h \cdot (g \cdot x_1) = x_3 \implies (hg) \cdot x_1 = x_3$. Thus $x_1 \sim x_3$.

□

1.3 Orbits and Stabilizers

The equivalence classes under the relation \sim partition the set X . These classes are called **orbits**.

Definition 1.5 (Orbit). For $x \in X$, the orbit of x is the set of all places x can be moved to by G :

$$\mathcal{O}_x = Orb(x) = \{g \cdot x : g \in G\}.$$

Definition 1.6 (Stabilizer). For $x \in X$, the stabilizer of x is the set of elements in G that fix x :

$$G_x = \text{Stab}(x) = \{g \in G : g \cdot x = x\}.$$

Note that G_x is a subgroup of G (denoted $G_x \leq G$).

Theorem 1.7 (Orbit-Stabilizer Theorem). Let G act on X . For any $x \in X$, there is a bijection between the orbit \mathcal{O}_x and the set of left cosets G/G_x . Consequently:

$$|\mathcal{O}_x| = [G : G_x].$$

If G is finite, $|G| = |\mathcal{O}_x| \cdot |G_x|$.

Proof. Define a map $\psi : G/G_x \rightarrow \mathcal{O}_x$ by $\psi(gG_x) = g \cdot x$.

1. **Well-defined:** Suppose $g_1G_x = g_2G_x$. Then $g_1^{-1}g_2 \in G_x$, so $(g_1^{-1}g_2) \cdot x = x$, implying $g_2 \cdot x = g_1 \cdot x$.
2. **Injectivity:** If $\psi(g_1G_x) = \psi(g_2G_x)$, then $g_1 \cdot x = g_2 \cdot x$. Multiplying by g_1^{-1} , we get $x = g_1^{-1}g_2 \cdot x$, so $g_1^{-1}g_2 \in G_x$, implying $g_1G_x = g_2G_x$.
3. **Surjectivity:** By definition, any $y \in \mathcal{O}_x$ is of the form $g \cdot x$ for some g , which is exactly $\psi(gG_x)$.

Thus, ψ is a bijection. □

2 Lecture 2: Cauchy's Theorem & Conjugation (Jan 14)

2.1 Cauchy's Theorem

Lemma 2.1. If X is a finite set and G is a p -group (a group of order p^k) acting on X , then:

$$|X| \equiv |X^G| \pmod{p},$$

where $X^G = \{x \in X : g \cdot x = x, \forall g \in G\}$ is the set of fixed points.

Theorem 2.2 (Cauchy's Theorem). If G is a finite group and p is a prime dividing $|G|$, then G has an element of order p .

Proof (McKay's Proof). Let X be the set of p -tuples of elements of G whose product is the identity:

$$X = \{(g_1, \dots, g_p) \in G^p : g_1 g_2 \cdots g_p = 1\}.$$

Notice that g_p is uniquely determined by the first $p - 1$ elements ($g_p = (g_1 \cdots g_{p-1})^{-1}$), so $|X| = |G|^{p-1}$. Since $p \mid |G|$, we have $p \mid |X|$.

Let \mathbb{Z}_p (cyclic group of order p) act on X by cyclic shift:

$$\sigma \cdot (g_1, g_2, \dots, g_p) = (g_2, g_3, \dots, g_p, g_1).$$

This is a valid action because if $g_1 \cdots g_p = 1$, then $g_1(g_2 \cdots g_p) = 1 \implies g_2 \cdots g_p = g_1^{-1}$, so $(g_2 \cdots g_p)g_1 = g_1^{-1}g_1 = 1$.

By the Lemma, $|X| \equiv |X^{\mathbb{Z}_p}| \pmod{p}$. Fixed points are tuples (a, a, \dots, a) such that $a^p = 1$. Since $(1, \dots, 1) \in X^{\mathbb{Z}_p}$, the set of fixed points is non-empty. Since p divides $|X|$ and p divides the congruence $|X| - |X^{\mathbb{Z}_p}|$, p must divide $|X^{\mathbb{Z}_p}|$. Therefore, there are at least p fixed points. There must exist some $a \neq 1$ such that $a^p = 1$. \square

2.2 Conjugation

Conjugation is a specific action of G on itself.

Definition 2.3. For $g, h \in G$, the **conjugate** of g by h is $g^h := hgh^{-1}$. The map $\phi_h : G \rightarrow G$ defined by $g \mapsto g^h$ is an automorphism (an inner automorphism).

Definition 2.4 (Classes and Centralizers). Let G act on itself by conjugation ($h \cdot g = hgh^{-1}$).

- The orbit of g is the **Conjugacy Class** of g : $Cl(g) = \{hgh^{-1} : h \in G\}$.
- The stabilizer of g is the **Centralizer** of g : $C_G(g) = \{h \in G : hg = gh\}$.

By Orbit-Stabilizer: $|Cl(g)| = [G : C_G(g)]$.

Definition 2.5 (Center of G). *The **Center**, $Z(G)$, is the set of elements that commute with everything:*

$$Z(G) = \{z \in G : zg = gz, \forall g \in G\} = \bigcap_{g \in G} C_G(g).$$

$Z(G)$ is the kernel of the conjugation homomorphism $G \rightarrow \text{Aut}(G)$. Thus $Z(G) \trianglelefteq G$.

Definition 2.6 (Normalizer). *Let $H \leq G$. The **Normalizer** of H in G is:*

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

$N_G(H)$ is the largest subgroup of G in which H is normal.

3 Lecture 3: Sylow Preliminaries & Isomorphism Theorems (Jan 16)

3.1 Product of Subgroups

Definition 3.1. Let $K, N \leq G$. Define $KN = \{kn : k \in K, n \in N\}$.

- In general, KN is not a subgroup.
- If $N \trianglelefteq G$ (or just $N \subseteq N_G(K)$), then KN is a subgroup.
- Size formula: $|KN| = \frac{|K||N|}{|K \cap N|}$.

Theorem 3.2 (Second Isomorphism Theorem). Let $K \leq G$ and $N \trianglelefteq G$. Then $K \cap N \trianglelefteq K$, and

$$\frac{KN}{N} \cong \frac{K}{K \cap N}.$$

Proof. Consider the natural projection $\pi : G \rightarrow G/N$. Restrict it to K , i.e., $\phi = \pi_{\text{res}_K} : K \rightarrow G/N$. The image of ϕ is $\{kN : k \in K\} = KN/N$. The kernel of ϕ is $\{k \in K : kN = N\} = \{k \in K : k \in N\} = K \cap N$. By the First Isomorphism Theorem, $K/\ker(\phi) \cong \text{Im}(\phi)$. \square

3.2 Sylow Definitions

Definition 3.3 (p -group). Let p be a prime. A group G is a p -group if every element has order a power of p . For finite groups, this is equivalent to $|G| = p^k$.

Definition 3.4 (Sylow p -subgroup). Let $|G| = p^n m$ where $p \nmid m$. A subgroup $P \leq G$ is called a **Sylow p -subgroup** if $|P| = p^n$.

$$Syl_p(G) = \{P \leq G : P \text{ is a Sylow } p\text{-subgroup}\}.$$

4 Lecture 4: The Sylow Theorems (Jan 21)

Theorem 4.1 (The Sylow Theorems). Let G be a finite group of order $p^n m$ where $p \nmid m$.

1. **Existence:** $Syl_p(G) \neq \emptyset$. (There exists a subgroup of order p^n).
2. **Conjugacy:** Any two Sylow p -subgroups are conjugate in G . That is, if $P, Q \in Syl_p(G)$, $\exists g \in G$ such that $gPg^{-1} = Q$.
3. **Number:** Let $n_p = |Syl_p(G)|$. Then:
 - $n_p \equiv 1 \pmod{p}$.
 - $n_p \mid m$ (equivalently $n_p \mid |G|$).

Proof. Let p be a prime such that $p \mid |G|$. Define the set of all p -subgroups:

$$\Sigma = \{H : H \leq G, |H| = p^n \text{ for some } n > 0\}.$$

Define Ω as the set of maximal elements in Σ under inclusion:

$$\Omega = \{H : H \in \Sigma, \text{ there is no } K \in \Sigma \text{ such that } H \subsetneq K\}.$$

We let G act on the set of subgroups $\{H : H \leq G\}$ by conjugation. Since conjugation is an isomorphism ($H^g \simeq H$), it preserves the order of subgroups. Thus:

$$H \in \Sigma \iff H^g \in \Sigma \quad \text{and} \quad H \in \Omega \iff H^g \in \Omega.$$

Therefore, G acts on Ω by $g \cdot H = H^g$.

Claim 1: Let $H \in \Omega$. Consider the action of H on Ω by conjugation. Then H is the **unique fixed point of this action**.

Proof of Claim:

- **Existence:** It is trivial to see that H is a fixed point, as $H^h = hHh^{-1} = H$ for any $h \in H$.
- **Uniqueness:** Let $K \in \Omega$ be a fixed point of the action of H .

$$K \text{ is fixed by } H \iff K^h = K \quad \forall h \in H \iff H \leq N_G(K).$$

Since $K \trianglelefteq N_G(K)$, and $H \leq N_G(K)$, the product HK is a subgroup of $N_G(K)$, and thus $HK \leq G$. Using our counting lemma:

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Since $H, K \in \Omega$, their orders are powers of p . Thus $|HK|$ is a power of p , meaning $HK \in \Sigma$.

We have $H \leq HK$. Since H is maximal in Σ (by definition of Ω) and $HK \in \Sigma$, it must be that $H = HK$. Similarly, $K \leq HK$ implies $K = HK$. Therefore, $H = K$.

Claim 2: Size of Ω Modulo p **Statement:** $|\Omega| \equiv 1 \pmod{p}$.*Proof of Claim:* Let $H \in \Omega$. We decompose Ω into disjoint orbits under the action of H .

$$\Omega = \{H\} \cup \bigcup_{K \neq H} \text{Orb}_H(K).$$

By Claim 1, H is the only fixed point (an orbit of size 1). For any $K \in \Omega$ with $K \neq H$, the stabilizer of K in H is a proper subgroup, so by the Orbit-Stabilizer theorem for p -groups, $|\text{Orb}_H(K)|$ is divisible by p .

$$|\Omega| = 1 + \sum(\text{multiples of } p) \implies |\Omega| \equiv 1 \pmod{p}.$$

Claim 3: Transitivity (Conjugacy of Elements in Ω)**Statement:** Any two elements of Ω are conjugate.*Proof of Claim:* Suppose for contradiction that $H, K \in \Omega$ are not conjugates. Let $\theta_1 = \text{Orb}_G(H)$ and $\theta_2 = \text{Orb}_G(K)$. Since they are distinct orbits, $\theta_1 \cap \theta_2 = \emptyset$.Consider the action of H on these sets:

- On θ_1 : $H \in \theta_1$. By Claim 1, H is the unique fixed point of H in Ω . Since $\theta_1 \subseteq \Omega$, H is the unique fixed point in θ_1 . Thus, $|\theta_1| \equiv 1 \pmod{p}$.
- On θ_2 : Since $H \in \theta_1$ and $\theta_1 \cap \theta_2 = \emptyset$, we have $H \notin \theta_2$. Therefore, θ_2 contains no fixed points under the action of H (because the only fixed point in all of Ω is H). Thus, every orbit of H inside θ_2 has size divisible by p . This implies $|\theta_2| \equiv 0 \pmod{p}$.

By symmetry, we can swap the roles of H and K . Running the same argument with K acting on the sets implies $|\theta_2| \equiv 1 \pmod{p}$ and $|\theta_1| \equiv 0 \pmod{p}$.

This results in a contradiction (e.g., $|\theta_1| \equiv 1$ and $|\theta_1| \equiv 0$). Thus, $\theta_1 = \theta_2$, so H and K are conjugate.

Corollary: Since there is only one orbit, $\Omega = \theta_1$. Thus $|\Omega| = |\theta_1| \equiv 1 \pmod{p}$.**Claim 4: Divisibility****Statement:** $|\Omega| \mid |G|$.*Proof of Claim:* Let $H \in \Omega$. Since Ω is exactly the orbit of H under conjugation (from Claim 3), the size of the orbit is the index of the stabilizer:

$$|\Omega| = [G : \text{Stab}_G(H)] = [G : N_G(H)] = \frac{|G|}{|N_G(H)|}.$$

Thus $|\Omega|$ divides $|G|$.**Claim 5: Identification with Sylow Subgroups****Statement:** $\Omega = \text{Syl}_p(G)$.*Proof of Claim:* We know $\text{Syl}_p(G) \subseteq \Omega$ because Sylow subgroups are maximal p -subgroups by definition. We must show the reverse: every $H \in \Omega$ is a Sylow p -subgroup.

Let $H \in \Omega$. Assume for contradiction that $H \notin Syl_p(G)$. Let $|G| = p^s m$ where $p \nmid m$. Since H is a p -group but not Sylow, $|H| = p^t$ with $t < s$.

Consider the normalizer $N_G(H)$. From Claim 2, we know $|\Omega| = [G : N_G(H)] \equiv 1 \pmod{p}$. Therefore, p does *not* divide $[G : N_G(H)]$. Since $|G| = [G : N_G(H)] \cdot |N_G(H)|$, all factors of p in $|G|$ must reside in $|N_G(H)|$. Thus, $|N_G(H)|$ is divisible by p^s .

Consequently, the index $[N_G(H) : H] = \frac{|N_G(H)|}{|H|} = \frac{\text{(multiple of } p^s\text{)}}{p^t}$ is divisible by p (since $s > t$).

By Cauchy's Theorem applied to the quotient group $N_G(H)/H$, there exists a subgroup of order p , say $\bar{K} \leq N_G(H)/H$. By the Correspondence Theorem, there exists a subgroup K such that $H \leq K \leq N_G(H)$ with $|K| = p \cdot |H| = p^{t+1}$.

Since K is a p -group, $K \in \Sigma$. However, $H \subsetneq K$, which contradicts the definition of H as a maximal element in Ω .

Therefore, H must already be of order p^s . Thus $\Omega = Syl_p(G)$. □

5 Lecture 5: Normal Series & Solvability (Jan 23)

5.1 Normal and Subnormal Series

Definition 5.1. A **subnormal series** of a group G is a chain of subgroups:

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_k = G$$

such that $H_i \trianglelefteq H_{i+1}$ for all i . The quotient groups H_{i+1}/H_i are called the **factors** of the series.

Definition 5.2 (Composition Series). A subnormal series is a **composition series** if all factors H_{i+1}/H_i are **simple** groups (non-trivial groups with no normal subgroups other than $\{1\}$ and themselves).

Theorem 5.3 (Jordan-Hölder). Any two composition series of a finite group G are equivalent. That is, they have the same length, and their factors are isomorphic (up to reordering).

5.2 Solvable Groups

Definition 5.4. A group G is **solvable** if it has a subnormal series where all factors H_{i+1}/H_i are **abelian**.

Definition 5.5 (Commutator). The commutator of g, h is $[g, h] = ghg^{-1}h^{-1}$. The **Derived Subgroup** (or Commutator Subgroup) G' or $[G, G]$ is the subgroup generated by all commutators.

Proposition 5.6. G/N is abelian if and only if $[G, G] \leq N$. Thus, $[G, G]$ is the smallest normal subgroup such that the quotient is abelian.

Remark 5.7 (Derived Series). Define $G^{(0)} = G$, $G^{(1)} = [G, G]$, and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. G is solvable if and only if the derived series terminates at $\{1\}$ (i.e., $G^{(n)} = \{1\}$ for some n).

6 Lecture 6: Solvable and Derived Series (Jan 26)

Lemma 6.1. Let $\alpha \in \text{Aut}(G)$ then $\alpha([g, h]) = [\alpha(g), \alpha(h)]$.

Lemma 6.2. For $N \trianglelefteq G$ then G/N is abelian iff $[G, G] \leq N$.

Theorem 6.3. If $N \trianglelefteq G$, $g, h \in G$ and $[gN, hN] = [g, h]N$. Additionally, gN and hN commutes iff $[g, h]N = N$ iff $[g, h] \subseteq N$.

Definition 6.4. G is solvable iff there is a subnormal series $\langle H_i : 0 \leq i \leq t \rangle$ such that $[H_{i+1}, H_{i+1}] \leq H_i$ for all $0 \leq i \leq t-1$.

Theorem 6.5. The following are true

1. If G solvable and $K \leq G$ then K is solvable
2. If G is solvable and $N \trianglelefteq G$ then G/N is solvable.
3. For $N \trianglelefteq G$, G solvable iff both N and G/N are solvable

Proof. We prove the previous theorems

1. Let $\langle H_i : 0 \leq i \leq t \rangle$ be a series in G such that $[H_{i+1}, H_{i+1}] \leq H_i$ for all $0 \leq i \leq t-1$. We can let $H'_i = K \cap H_i$ for $0 \leq i \leq t$ and we need to verify $\langle H'_i : 0 \leq i \leq t \rangle$ is a solvable series in K .
2. Let $\langle H_i : 0 \leq i \leq t \rangle$ be a series in G such that $[H_{i+1}, H_{i+1}] \leq H_i$ for all $0 \leq i \leq t-1$. Let $H'_i = \phi_N[H_i] = H_i N / N$. Verify it is solvable.
3. The forward direction is trivial by (1) and (2). For the reverse direction, N solvable and G/N solvable $\langle N_i : 0 \leq i \leq s \rangle$ subnormal in N and N_{i+1}/N_i abelian and similar for $\langle H_j N / N : 0 \leq j \leq t \rangle$ subnormal in G/N and $(H_{j+1} N / N)_{j+1} / (H_j N / N)$ abelian. We know $H_{j+1} \trianglelefteq H_j$ and $[H_{j+1}, H_{j+1}] \leq H_j$ for all $0 \leq j \leq t-1$. So $N_0 = 1 \trianglelefteq N_1 \cdots \trianglelefteq N_s = N = H_0 \trianglelefteq H_1 \cdots \trianglelefteq H_t = G$.

□

Definition 6.6. Given G , the derived series of G is given by $G'_0 = G$ and $G'_{i+1} = [G'_i, G'_i]$ for $i \geq 0$.

Theorem 6.7. G is solvable iff there is n such that $G'_n = \{1\}$.

Proof. For the backwards direction, by construction G'_i / G'_{i+1} is abelian. Let $H_j = G_{n-j}$. For the forward direction, let $\langle H_j : 0 \leq j \leq n \rangle$ be a subnormal series in G , H_{j+1} / H_j abelian. Show by induction that $G'_i \leq H_{n-i}$ for all i . □

Theorem 6.8. Suppose G abelian and simple, let $a \in G$ and $a \neq 1$ then $G = \langle a \rangle$ and $|a|$ is prime.

Theorem 6.9. *Let G be simple and let $\langle H_i : 0 \leq i \leq t \rangle$ be subnormal series in G . Then there is $n < t$ such that $H_i = 1$ for $0 \leq i \leq n$ and $H_i = G$ for $n < i \leq t$.*

7 Lecture 7: Nilpotent Groups and Free Groups (Jan 28)

7.1 Nilpotent Groups

Definition 7.1. For $H, K \leq G$, $[H, K] = \langle [h, k] : h \in H, k \in K \rangle$

Remark 7.2. G'_i is characteristic in G for all i , in particular $G'_i \trianglelefteq G$.

Lemma 7.3. If $N \trianglelefteq G$, $N \leq H \leq G$ then

$$\frac{H}{N} \leq Z\left(\frac{G}{N}\right) \iff [G, H] \leq N.$$

Definition 7.4. A normal series in G , $\langle H_i : 0 \leq i \leq t \rangle$ is a central series if $\frac{H_{i+1}}{H_i} \leq Z\left(\frac{G}{H_i}\right)$ for all $0 \leq i \leq t - 1$. Equivalently, $[G, H_{i+1}] \leq H_i$ for all $0 \leq i \leq t - 1$.

Definition 7.5. G is nilpotent iff G has a central series.

Remark 7.6. Nilpotent groups are solvable.

Definition 7.7. Let G be a group

1. The descending central series is the sequence of subgroup given by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$ for $i \geq 1$.
2. The ascending central series is the sequence of subgroup given by $Z_0(G) = 1$ and

$$\frac{Z_{n+1}(G)}{Z_n(G)} = Z\left(\frac{G}{Z_n(G)}\right).$$

Remark 7.8. By induction, $Z_n(G)$ and $\gamma_n(G)$ are characteristic in G . and $Z_n(G) \trianglelefteq G$ and $\gamma_n(G) \trianglelefteq G$ for all $n \geq 0$.

Remark 7.9. For any G and $N \trianglelefteq G$, $[G, N] \leq N$ because for any $g \in G, n \in N$ we have $[g, n] = gng^{-1}n^{-1}$ and $n^g \in N$ since $N \trianglelefteq G$. So $[G, N] \leq N$.

Theorem 7.10. G is nilpotent iff there is a $N \geq 0$ such that $Z_n(G) = G$ iff there is $n \geq 0$ such that $\gamma_{n+1}(G) = \{1\}$. Moreover, the least n such that $Z_n(G) = G$ is the least n such that $\gamma_{n+1}(G) = \{1\}$.

Proof. Take any central series then the decreasing and ascending central series will grow at least as fast as the arbitrary central series. \square

7.2 Free Groups

Example 7.11. Consider $\mathbb{Z} = (\mathbb{Z}, +)$, 1 is a generator $\mathbb{Z} = \langle 1 \rangle$. The universal property of \mathbb{Z} , 1. For any group G and any $g \in G$, there is a unique homomorphism $\phi : \mathbb{Z} \rightarrow G$ and $\phi(1) = g$.

Proof. If ϕ exist let $\phi(n) = g^n$ then ϕ is a homomorphism. \square

g is free because whatever 1 is mapped to in the homomorphism it won't mess you up compared to a finite group.

Definition 7.12. F is a free group on 2 element if

1. There exist $a, b \in F$, $F = \langle a, b \rangle$
2. For all G , all $g, h \in G$ there is a unique homomorphism $\phi : F \rightarrow G$ such that $\phi(a) = g$ and $\phi(b) = h$.

8 Lecture 8: Free Groups

Theorem 8.1. For any set X , there are a group F and an injective function $i : X \rightarrow F$ such that

- For any group G and any function $f : X \rightarrow G$ there is a unique homomorphism $\phi : F \rightarrow G$ such that $\phi(i(x)) = f(x)$ for all $x \in X$.

$$\begin{array}{ccc} X & \xhookrightarrow{i} & F \\ & \searrow f & \downarrow \phi \\ & & G \end{array}$$

The diagram commutes: $\phi \circ i = f$. The homomorphism ϕ is unique.

- F is generated by $i[X]$

Proof. A word (on alphabet X) is a (possibly empty) finite sequence from $X \times \mathbb{Z}$. We will write $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ for the word x_1, x_2, \dots, x_k for $\langle (x_1, n_1), (x_2, n_2), \dots, (x_k, n_k) \rangle$.

Example 8.2. Let $X = \{x, y, z\}$ then $w = x^0 x^{15} y^{-23} z^{27}$ is a word.

Let w be the set of words. Given a word, we can possibly perform a reduction step

- If word contains an entry $\langle x, 0 \rangle$, remove it.
- If the word contains successive entries $\langle x, n \rangle$ and $\langle x, m \rangle$ then replace them with $\langle x, n + m \rangle$.

Example 8.3. $y^{15} x^3 x^{-2} x^{-1} y^2 z^5 \rightarrow y^{15} x^1 x^{-1} y^2 z^5 \rightarrow y^{15} x^0 y^2 z^5 \rightarrow y^{15} y^2 z^5 \rightarrow y^{17} z^5$.

Let R be set of all reduced words, where no more reductions can be done.

For any word w , there is a unique reduced word w' such that w can be transformed into w' by a finite set of reductions.

We will not be able to show this property using group properties but instead we'll produce a group F with the property that every word is equivalent to a unique reduced word.

Given $x \in X$, I will replace $f_x : F \rightarrow F$

- w does not begin with some x^n then $f_x(w) = xw$
- w is of the form $x^{-1}v$ then $f_x(w) = v$
- w is of the form $x^n v$ for $n \neq -1$ then $f_x(w) = x^{n+1}v$

I also define $g_x : R \rightarrow R$ by

- w does not begin with some x^n then $g_x(w) = x^{-1}w$
- w is of the form $x^1 v$ then $g_x(w) = v$
- w is of the form $x^n v$ for $n \neq 1$ then $g_x(w) = x^{n-1}v$

We can observe that $f_x \circ g_x = g_x \circ f_x = 1_R$.

For each $x \in X$, $f_x \in \sum_R$ (the group of permutations of R) and $g_x = f_x^{-1}$. Let F be the subgroup of \sum_R generated by $\{f_x : x \in X\}$ or $\{f_{x_1}^{n_1}, \dots, f_{x_t}^{n_t} : x_i \in X, n_i \in \mathbb{Z}\}$.

Let $i : X \rightarrow F$ with $i(x) = f_x$. For any word $w = x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$. Let $f_w = f_{x_1}^{n_1} f_{x_2}^{n_2} \cdots f_{x_t}^{n_t} \in F$.

Key Facts:

1. If w is obtained from \bar{w} by a single reduction step, $f_w = f_{\bar{w}}$.
2. If w is obtained from \bar{w} by any sequence of reduction steps, then $f_w = f_{\bar{w}}$.
3. If $w \in R$ then $f_w(\langle \rangle) = w$
4. For any word w , $f_w(\langle \rangle)$ the unique reduced word to which w can be transformed by reduction step.
5. If w_1, w_2 are reduced words then $v =$ unique reduction of $w_1 w_2$, $f_v = f_{w_1} f_{w_2}$

Let F' be the group whose underlying set is R whose operation is "concatenate and reduce". Then $F \simeq F'$ by $f_v \mapsto v$. To finish, let G be a group, $h : X \rightarrow G$ function. We want to find there is a unique homomorphism ϕ with $F' \rightarrow G$ and $\phi \circ i = h$.

1. If ϕ exists then

$$\begin{aligned}\phi(x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}) &= \phi(i(x_1))^{n_1} \phi(i(x_2))^{n_2} \cdots \phi(i(x_t))^{n_t} \\ &= h(x_1)^{n_1} h(x_2)^{n_2} \cdots h(x_t)^{n_t}\end{aligned}$$

2. Verify ϕ is a homomorphism.

□

Example 8.4. Let $X = \langle x, y \rangle$, let G be any group with 2 generators say $G = \langle a, b \rangle$. Let F be a free group on X . Let $\phi : F \rightarrow G$ be unique homomorphism such that $\phi(x) = a$ and $\phi(y) = b$. ϕ is surjective because G is generated by a and b . Thus, $G \simeq \frac{F}{\ker \phi}$.

9 Lecture 9: Generators and Relations (Feb 2)

Definition 9.1. let X be a set, let F be a free group on X . Let $A \subseteq F$ and $N = \text{least normal subgroup of } F \text{ containing } A$.

Theorem 9.2. Let $\bar{F} = F/N$ and $j : X \rightarrow \bar{F}$ by $j(x) = xN$.

$$\begin{array}{ccc} F & \xrightarrow{\phi_N} & \bar{F} \\ i \uparrow & j \nearrow & \\ X & & \end{array}$$

The following are true:

1. \bar{F} is generated by $\{j(x) : x \in X\}$. Let $\bar{x} = j(x)$
2. If $a = x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \in A$ then $a \in N$ and $\bar{x}_1^{n_1} \bar{x}_2^{n_2} \cdots \bar{x}_t^{n_t} = \phi_N(a) = 1_{\bar{F}}$

"Universal Property of \bar{F} ": Let G be a group and let $h : X \rightarrow G$ such that $a = x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \in A$ and $h(x_1)^{n_1} h(x_2)^{n_2} \cdots h(x_t)^{n_t} = 1_G$. There is an unique homomorphism $\tau : \bar{F} \rightarrow G$ such that

$$\begin{array}{ccc} X & \xrightarrow{j} & \bar{F} \\ & h \searrow & \downarrow \tau \\ & & G \end{array}$$

So $h = \tau \circ j$

Proof. Since $\{\bar{x} : x \in X\}$ generates \bar{F} there is at most one τ . By universal property of F , there is unique homomorphism $\phi : F \rightarrow G$ such that

$$\phi(x^1) = h(x). \quad (\text{For all } x \in X)$$

Use property of F

$$\begin{array}{ccc} X & \xrightarrow{i} & F \\ & h \searrow & \downarrow \phi \\ & & G \end{array}$$

Claim: By hypothesis on h , $A \subseteq \ker(\phi)$.

Let $a = x_1^{t_1} x_2^{t_2} \cdots x_k^{t_k} \in A$ then $\phi(a) = \phi(x_1)^{t_1} \phi(x_2)^{t_2} \cdots \phi(x_k)^{t_k} = h(x_1)^{t_1} h(x_2)^{t_2} \cdots h(x_k)^{t_k} = 1_G$ since $h(x_1)^{t_1} h(x_2)^{t_2} \cdots h(x_k)^{t_k} = 1_G$. Thus, $a \in \ker(\phi)$.

By choice of N , $N \leq \ker(\phi)$, we'll attempt to define $\tau : \bar{F} \rightarrow G$ by $\tau(wN) = \phi(w)$ for $w \in F$.

τ is well defined because $w_1N = w_2N \implies w_1^{-1}w_2 \in N \implies w_1^{-1}w_2 \in \ker(\phi) \implies \phi(w_1)^{-1}\phi(w_2) = 1 \implies \phi(w_1) = \phi(w_2)$. Showing τ is a homomorphism is straightforward.

$$\begin{array}{ccc} X & \xrightarrow{i} & F \\ & \searrow h & \downarrow \phi_N \\ & & \overline{F} \end{array}$$

$$\begin{array}{ccc} & & \swarrow \phi \\ & & G \end{array}$$

□

10 Lecture 10: Presentation of Groups (Feb 4)

Example 10.1. *Dihedral Group: $D_n = \text{symmetry group of a regular } n\text{-gon}$.*

Let $D_n = \langle a, b \rangle$ where $a = \text{rotation by } \frac{2\pi}{n}$ and $b = \text{reflection about the vertical axis}$.

Then D_n is generated by a and b with the relations $a^n = 1$ and $b^2 = 1$.

We have $bab^{-1}a = 1$

Let $X = \{x, y\}$ and F be free group and $A = \{x^n, y^2, yxy^{-1}x\}$ and $N = \text{least normal subgroup } \supseteq A$. Let $\bar{x} = xN$ and $\bar{y} = yN$. Then $F/N = \langle \bar{x}, \bar{y} \rangle$ and

$$\overline{yxy^{-1}x} = \overline{yxy^{-1}}\overline{x}^{-1} = 1_{\overline{F}}.$$

There is a unique homomorphism $\tau : F/N \rightarrow D_n$ such that $\tau(\bar{x}) = a$ and $\tau(\bar{y}) = b$ as $D_n = \langle a, b \rangle$ so τ is surjective. So $|F/N| \geq 2n$.

Claim: $|F/N| \leq 2n$ (so τ is isomorphism)

Proof: Using relations from before, we can simply write any expression in \bar{x}, \bar{y} down to $\bar{x}^s\bar{y}^t$ for $0 \leq s \leq n$ and $0 \leq t \leq 2$.

11 Lecture 11: Category Theory (Feb 6)

11.1 Categories

- Example 11.1.**
1. Category of sets, if we have two sets A and B then $A \rightarrow B$ is a function $f : A \rightarrow B$
 2. Category of groups, if we have two groups G and H then $G \rightarrow H$ is a homomorphism $\phi : G \rightarrow H$
 3. Category of topological spaces, if we have two topological spaces X and Y then $X \rightarrow Y$ is a continuous function $f : X \rightarrow Y$

Definition 11.2. A category \mathcal{C} consists of the following data:

1. A collection of objects $Ob(\mathcal{C})$
2. A collection of morphisms $Mor(\mathcal{C})$
3. For each morphism f , there are objects $dom(f)$ and $cod(f)$ such that $dom(f) \rightarrow cod(f)$ is a morphism in $Mor(\mathcal{C})$
4. A composition operation $Mor(\mathcal{C}) \times Mor(\mathcal{C}) \rightarrow Mor(\mathcal{C})$
5. An identity morphism $1_A : A \rightarrow A$ for each object $A \in Ob(\mathcal{C})$

Lemma 11.3. The following are true:

1. If $a \xrightarrow{f} b$ then $f1_a = f$ and $1_b f = f$.
2. If $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ then $h(gf) = (hg)f$

Example 11.4. Posets: \mathbb{P} is a poset with \leq binary relation such that

1. $a \leq a$ for all $a \in \mathbb{P}$
2. $a \leq b$ and $b \leq a \implies a = b$
3. $a \leq b$ and $b \leq c \implies a \leq c$

Given poset \mathbb{P} , make "poset category" with objects the elements of \mathbb{P} . There is only one morphism $a \rightarrow b$ if $a \leq b$ and no morphism if $a \not\leq b$.

Definition 11.5. Let \mathcal{C} be a category. An object is an initial if for all object b there is exactly one arrow $a \rightarrow b$.

Definition 11.6. An arrow $a \xrightarrow{f} b$ in \mathcal{C} is an isomorphism iff there is another $b \xrightarrow{g} a$ such that $gf = 1_b$ and $fg = 1_a$.

Lemma 11.7. *If a, b are initial objects in \mathcal{C} they are uniquely isomorphic.*

Proof. Let $a \xrightarrow{f} b$ and $b \xrightarrow{g} a$ be unique arrows from a to b and b to a respectively. Then we have $a \xrightarrow{gf} a$ and $b \xrightarrow{fg} b$ so $gf = 1_a$ and $fg = 1_b$. \square

Definition 11.8. *Object b is terminal iff for all a there is a unique $a \rightarrow b$ iff b is initial in \mathcal{C}^{op}*

Lemma 11.9. *Two terminal objects are uniquely isomorphic.*

Definition 11.10. *Let \mathcal{C} and \mathcal{D} be categories. A functor from \mathcal{C} to \mathcal{D} is $\mathcal{C} \xrightarrow{F} \mathcal{D}$. Assign to each object a of \mathcal{C} an object $F(a)$ of \mathcal{D} . For each arrow $a \xrightarrow{f} b$ there is an arrow $F(a) \xrightarrow{F(f)} F(b)$ with $f(1_a) = 1_{F(a)}$. Additionally, $F(gf) = F(g)F(f)$*

12 Lecture 12: Ring Theory (Feb 8)

Definition 12.1. A ring is a set R with two binary operations $+$ and \times such that

1. $(R, +)$ is an abelian group
2. (R, \times) is associative
3. $a \times (b + c) = a \times b + a \times c$ and $(b + c) \times a = b \times a + c \times a$ for all $a, b, c \in R$

Example 12.2. The following are rings

1. $\mathbb{Z}, 2\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
2. $M_2(\mathbb{Z})$
3. $\{f : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous}\}$
4. $\{f : \mathbb{R} \rightarrow \mathbb{R} \text{ differentiable}\}$

Definition 12.3. R is commutative iff $rs = sr$ for all $rs \in R$

Definition 12.4. R is unital iff there exists $1_R \in R$ such that $1_R r = r 1_R = r$ for all $r \in R$

Definition 12.5. Let R, S be rings. A ring homomorphism from R to S if $\phi : R \rightarrow S$ satisfy:

1. $\phi(r + s) = \phi(r) + \phi(s)$ for all $r, s \in R$
2. $\phi(rs) = \phi(r)\phi(s)$ for all $r, s \in R$
3. $\phi(1_R) = 1_S$

Definition 12.6. Let S be a subring of R if S is a ring and the inclusion map $i : S \rightarrow R$ is a ring homomorphism.

Definition 12.7. Let $\phi : R \rightarrow S$ be a ring homomorphism then

1. $\ker(\phi) = \{r \in R : \phi(r) = 0\}$ is a subring of R
2. $\phi(R) = \{s \in S : \exists r \in R, \phi(r) = s\}$ is a subring of S

Remark 12.8. In general, not the case that $\ker(\phi)$ is a subring of R .

Definition 12.9. Let R be a ring. An ideal of R is a subring I of R such that

1. $I \leq (R, +)$
2. For all $a \in R$ and $b \in I$, $ab \in I$.

Definition 12.10. Let I be an ideal of R then we define $r + I = \{r + i : i \in I\}$

13 Lecture 13: Ideals and Quotient Rings (Feb 10)

Theorem 13.1. *First Isomorphism Theorem of Rings:* Let R, S be rings and $\phi : R \rightarrow S$ be a homomorphism with $\ker(\phi)$ an ideal of R and $\text{im}(\phi)$ a subring of S . Then we get isomorphism

$$\tau : \frac{R}{\ker(\phi)} \xrightarrow{\cong} \text{im}(\phi) \text{ and } \tau : r + \ker(\phi) \mapsto \phi(r).$$

Theorem 13.2. The following are true for any ideals I, J of R :

1. $I + J = \text{smallest ideal containing } I \text{ and } J$
2. $I \cap J = \text{largest ideal contained in } I \text{ and } J$
3. $IJ = \{\sum_{i=1}^n r_i s_i \mid n \in \mathbb{N}, r_i \in I, s_i \in J\} \subseteq I \cap J$
4. Ideals of R/I are in bijection with ideals of R that contain I

Definition 13.3. R is a zero ring if $R = \langle 0_R \rangle$

Note: If $1_R = 0_R$ then $r \cdot 1 = r \cdot 0 = 0$ and R is a zero ring.

Definition 13.4. R is an integral domain iff

1. $1_R \neq 0_R$ (R is not a zero ring)
2. For all $a, b \in R$

$$ab = 0 \implies a = 0 \text{ or } b = 0.$$

Definition 13.5. R is a field iff

1. $1_R \neq 0_R$
2. Every element of $R \setminus \{0_R\}$ has a multiplicative inverse

Exercise 13.6. If R is a field then R is an integral domain.

Definition 13.7. Let R be a ring. Then $u \in R$ is a unit iff u has a multiplicative inverse.

We let

$$U(R) = \{u \in R : u \text{ is a unit}\}.$$

$U(R)$ is a group under multiplication.

Definition 13.8. An ideal I of R is principal iff $I = aR$

Lemma 13.9. Let R be any ring then R is the largest ideal and $\{0_R\}$ is the smallest ideal.

Remark 13.10. We have $R/R = \{0_R\}$ and $R/\{0_R\} \simeq R$

Theorem 13.11. Let R be a field and let I be an ideal then $I = 0$ or $I = R$

Proof. Assume $I \neq 0$, let $a \in I$ and $a \neq 0$. There exist $b \in R$ such that $ab = 1 \in I$. In any ring $1 \in I$ iff $I = R$.

□

Theorem 13.12. If $R \neq 0$ are the only two ideals of R then R is a field.

Proof. Let $a \in R, a \neq 0$ then $(a) \neq 0$ as $a \in (a)$ so $(a) = R$ so $1 \in (a)$ and $1 = ab$ for some b .

□

Definition 13.13. An ideal I of R is maximal iff

1. $I \neq R$
2. For all ideals $J \supseteq I$, $J = I$ or $J = R$

I is maximal in poset $\{J : J \text{ ideal}, J \neq R\}$ ordered by \subseteq

Lemma 13.14. If I is an ideal of R ,

$$I \text{ maximal} \iff R/I \text{ is a field.}$$

Proof. $R \neq I \iff 1_{R/I} \neq 0_{R/I}$. By I maximal iff there are only two ideals of R/I which are 0 and R/I .

□

Definition 13.15. Ideal I is prime iff

1. $I \neq R$
2. For all $a, b \in R$, $ab \in I \implies a \in I$ or $b \in I$

Lemma 13.16. R/I is an integral domain iff I is a prime ideal.

Definition 13.17. $\text{Spec}(R) = \{I : I \text{ is a prime ideal of } R\}$

Theorem 13.18. Let R be a ring. I is an ideal of R , $I \neq R$. Then there is a maximal ideal J such that $I \subseteq J$.

14 Lecture 14: Zorn's Lemma and Modules (Feb 13)

Lemma 14.1. *Zorn's Lemma: If \mathbb{P} is a poset such that every chain in \mathbb{P} has an upper bound, then for every element p of \mathbb{P} there is a $q \geq p$ with q maximal.*

Example 14.2. *Claim: \mathbb{Q} satisfies the hypothesis of Zorn's Lemma.*

Proof. Let $(I_a : a \in A)$ be a chain in \mathbb{Q} . That is, I_a is ideal with $I_a \neq R$ and for $a, b \in A, I_a \subseteq I_b$ or $I_b \subseteq I_a$. We need upper bound, let

$$J = \bigcup_{a \in A} I_a = \{r : \text{there is } a \in A, r \in I_a\}.$$

Verify J is an ideal and for $r, s \in J$ then $r \in I_a$ and $s \in I_b$. WLOG let $I_a \subseteq I_b$ so $r, s \in I_b$ so $r + s \in I_b \subseteq J$.

Verify $J \neq R$, easy as $1 \notin I_a$ all a so $1 \notin J$. \square

14.1 Modules

Definition 14.3. *Let R be a ring. A R -module is an abelian group $(M, +)$ equipped with a function $R \times M \rightarrow M$ by $(r, m) \mapsto rm$ such that*

1. $0_R m = 0_M, 1m = m$ for all $m \in M$
2. $(r_1 + r_2)m = r_1m + r_2m$ and $m(r_1 + r_2) = mr_1 + mr_2$
3. $r(sm) = (rs)m$ for $r, s \in R$ and $m \in M$

Definition 14.4. *If M is a R -module, we can form “linear combinations”*

$$\sum_{i=1}^n r_i m_i \text{ for } r_i \in R \text{ and } m_i \in M.$$

Example 14.5. *Let $R = \mathbb{Z}$ and $(G, +)$ be an abelian group. Then G is a \mathbb{Z} -module.*

Definition 14.6. *If M is a R -module, a submodule of M is N such that $N \leq (M, +)$, $rn \in N$ for $r \in R$ and $n \in N$.*

Remark 14.7. *If R is a ring, we can view R as a R -module. The ideals of R are the submodules of R (viewed as a R -module).*

Definition 14.8. *Let M, N be R -modules. A R -module homomorphism is a function $\phi : M \rightarrow N$ such that*

1. ϕ is a group homomorphism for $(M, +)$ to $(N, +)$
2. $\phi(rm) = r\phi(m)$ for $r \in R$ and $m \in M$

Note: This is like linear transformation in linear algebra.

15 Lecture 15: Modules (Feb 16)

Definition 15.1. Let M and N be R -modules then we say $\phi : M \rightarrow N$ is R -linear if

$$\phi \left(\sum_{i=1}^n r_i m_i \right) = \sum_{i=1}^n r_i \phi(m_i) \text{ for } r_i \in R \text{ and } m_i \in M.$$

Definition 15.2. Let $M \leq N$ (M is submodule of N), $N/M = \{n + M : n \in N\}$. We define

1. $(n_1 + M) + (n_2 + M) = (n_1 + n_2) + M$
2. $r(n + M) = (rn) + M$

We can define the R -linear quotient map $\phi_M : N \rightarrow N/M$ by $\phi : n \mapsto n + M$ with $\ker(\phi_M) = M$.

Theorem 15.3. First Isomorphism Theorem for Modules: Let $\phi : M_1 \rightarrow M_2$ linear, then

$$\text{im}(\phi) \simeq \frac{M_1}{\ker(\phi)}.$$

via $m + \ker(\phi) \mapsto \phi(m)$

Definition 15.4. Given R -module M and set $X \subseteq M$, the least submodule containing X is

$$\text{Span}(X) = \left\{ \sum_{i=1}^n r_i x_i \mid n \in \mathbb{N}, r_i \in R, x_i \in X \right\}.$$

We specifically define $\text{Span}(\emptyset) = \{0_M\}$

Definition 15.5. If R is a ring, $X \subseteq R$ we write $\langle X \rangle$ for the least ideal containing X

$$\langle X \rangle = \left\{ \sum_{i=1}^n r_i x_i \mid n \in \mathbb{N}, r_i \in R, x_i \in X \right\}.$$

Definition 15.6. A ring R is a principal ideal ring iff all ideals of R are principal (a principal ideal is of form Ra for some $a \in R$).

Definition 15.7. R is a principal ideal domain iff R is integral domain and principal ideal ring.

Definition 15.8. A ring R is noetherian iff every ideal of R is finitely generated. (An ideal is finitely generated if $I = Ra_1 + \cdots + Ra_n$ for some $a_1, \dots, a_n \in R$.)

Theorem 15.9. If I is an ideal of R the ideals R/I are in bijection with $\{J : J \text{ ideal of } R, J \supseteq I\}$.

Also prime ideals of R/I correspond to prime ideals of R that contain I .

Example 15.10. Common PID's

1. \mathbb{Z}
2. $k[x]$ for k a field

Definition 15.11. R is a Euclidean Domain iff

1. R is integral domain
2. There is a function $\phi : R \setminus \{0\} \rightarrow \mathbb{N}$ such that for all $a, b \in R$, for $b \neq 0$, there exist q, r such that $a = bq + r$ either $r = 0$ or $\phi(r) < \phi(b)$

Example 15.12. If $R = \mathbb{Z}$ then $\phi(n) = |n|$ is a Euclidean Domain.

Example 15.13. If $R = k[x]$ for k a field then $\phi(f) = \deg(f)$ is a Euclidean function.

Theorem 15.14. If R is a Euclidean Domain then R is a PID.

Proof. Let R be euclidean domain and let I be the ideal of R . If $I = 0$ then we're done. If $I \neq 0$ consider $\{\phi(r) | r \in I, r \neq 0\} \subseteq \mathbb{N}$. Let n be the minimal element and let $a \in I \setminus \{0\}$ such that $\phi(a) = n$. As $a \in I$, $Ra \subseteq I$. Let $b \in I$, find q, r such that $b = aq + r$ either $r = 0$ or $\phi(r) < n$. We have $r = b - aq \in I$. So $r = 0$ and $b \in Ra$. \square

Lemma 15.15. Facts about Integral Domains: Let R be an integral domain.

1. If $a \neq 0$ and $ab = ac$ then $a(b - c) = 0$
2. $Ra = Rb$ iff a and b are associates

Lemma 15.16. Let R be any ring, the following are true about units

1. u unit iff u has a multiplicative inverse
2. Units of R form a group under multiplication
3. $a, b \in R$ are called associates iff there is an unit u such that $au = b$.

Example 15.17. $\mathbb{Z}[x]$ is not a PID. Consider the ideal $\langle 2, x \rangle = \{f : \text{constant term of } f \text{ is even}\}$.

Theorem 15.18. R Noetherian iff $R[x]$ is Noetherian.

16 Lecture 16: Integral Domains

Definition 16.1. Let $p \in R$

1. p is prime if and only if p is a nonzero non-unit and for all $r, s \in R$, $p|rs$ implies $p|r$ or $p|s$.
2. p is irreducible if and only if p is a nonzero non-unit and for all $r, s \in R$ and $p = rs$ then one of r or s is a unit.

Lemma 16.2. The following are true

1. As R is an ID, for any $a \in R$, a nonzero non-unit iff $(a) \neq \{0, R\}$
2. If p is prime, then (p) is a prime ideal
3. p prime implies p is irreducible.
4. If R PID then prime = irreducible.
5. In any ID, (0) is a prime ideal.

Definition 16.3. R is a unique factorization domain (UFD) iff

1. R is an integral domain
2. Every non-zero nonunit in R is a product u_1, \dots, u_n , $n > 0$, u_i irreducible.
3. If $m, n > 0$ then $u_1 \dots u_n = v_1 \dots v_m$ and u_i, v_j irreducible, then $m = n$ and u_i are associates and permutations of v_j

Recall 16.4. An R -module M is finitely generated (fg) if there are $m_1, \dots, m_n \in M$ such that $M = \text{Span}(m_1, \dots, m_n)$.

Definition 16.5. An R -module N is Noetherian iff every submodule of N is finitely generated.

Equivalently, Ring R is Noetherian iff R is Noetherian as an R -module that is all ideals are finitely generated.

Example 16.6. Non-Noetherian Rings:

$$R = \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \dots, x_n].$$

Consider the ideal

$$I = \langle x_1, x_2, \dots \rangle = \{f \in R : f \text{ has no constant term}\} = \{f : f(0, \dots) = 0\}.$$

Proof. If I is finitely generated there is $h_1, \dots, h_n \in I$ such that

$$I = Rh_1 + \dots + Rh_n = (h_1) + \dots + (h_n).$$

Each h_i has finite x_j 's in it so find N so large that x_N does not appear in any h_i . As $x_N \in I$ there are $f_1, \dots, f_n \in R$ such that $x_N = \sum_{i=1}^n f_i h_i$. Substitute

$$x_j = \begin{cases} 0 & \text{if } j < N \\ 1 & \text{if } j \geq N \end{cases}.$$

Then we conclude $1 = 0$ which is a contradiction. \square

Theorem 16.7. *Let M be a R -module then the following are equivalent*

1. *M is Noetherian*
2. *For any increasing chain of submodules $N_0 \subseteq N_1 \subseteq N_2 \subseteq \dots$ of M , the chain is eventually constant.*
3. *For any nonempty set X of submodules of M , there is some $N \in X$ which is maximal under inclusion.*

Proof. (1) \implies (2): Consider $N_\infty = \bigcup_{i=0}^{\infty} N_i = \{m : \exists i \in \mathbb{N}, m \in N_i\}$.

Verify that N_∞ is a submodule of M , as M is noetherian, N_∞ is finitely generated say $N_\infty = \text{Span}(X)$ with $X \subseteq N_\infty$ finite. Find i such that $X \subseteq N_i$ then $N_\infty = \text{Span}(X) \subseteq N_i \subseteq N_j \subseteq N_\infty$ for $j \geq i$.

(2) \implies (1): AFSOC there is $N \leq M$ with N not finitely generated. Take $n_0 \in N$ and $n_0 \neq 0$. Choose $n_1 \in N \setminus Rn_0$ and generally

$$n_k \subseteq N \setminus \left(\sum_{i < k} Rn_i \right).$$

In the end, let $N_i = \sum_{j < i} Rn_j$ then $N_0 < N_1 < N_2 < \dots$, a contradiction. \square

17 Lecture 17: PID (Feb 19)

Lemma 17.1. If R PID then

1. R is Noetherian
2. 0 is a prime ideal and 0 is maximal iff R is a field
3. For $a \neq 0$, (a) prime iff a prime iff (a) maximal iff a irreducible

Theorem 17.2. If R is a PID then R UFD

Claim 17.3. Every nonzero non-unit $a \in R$ is a product of irreducibles.

Proof. If not, choose a that is not a finite product of irreducibles such that (a) is maximal under inclusion, a not finite product of irreducibles. a is not irreducible.

So $a = bc$ where b, c not units. So (a) $\subsetneq (b)$ and (a) $\subsetneq (c)$ so b, c are finite products of irreducibles. Contradiction so a is a finite product of irreducibles. \square

Claim 17.4. If $u_1 \dots u_n = v_1 \dots v_m$ with u_i, v_j irreducible then $m = n$ and u_i, v_j are the same up to permutations and associates.

Proof. u_1 divides $v_1 \dots v_m$. As R PID, u_1 is prime so $u_1 | v_j$ for some j . As v_j is irreducible, u_i either unit or an associate of v_j . As u_i irreducible, u_i not a unit. So u_1 and v_j are associates. Say $u_1 = x \cdot v_j$ for x unit. Then

$$xu_2 \dots u_n = v_1 \dots v_{j-1} v_{j+1} \dots v_m.$$

Finish by induction. \square

Corollary 17.5. If k field, $k[x]$ is a UFD.

Lemma 17.6. If R PID and $p \in R$ irreducible then $R/(p)$ is a field.

Theorem 17.7. (Hilbert's Basissatz Theorem) If R Noetherian ring then $R[x]$ is Noetherian ring.

Proof. Let I be ideal of $R[x]$. For each n let

$$J_n = \left\{ r \in R : \exists c_0, \dots, c_{n-1} \in R, rx^n + \sum_{i=0}^{n-1} c_i x^i \in I \right\}.$$

J_n is an ideal of R .

Claim: $J_n \subseteq J_{n+1}$

Let $r \in J_n$ say $rx^n + \sum_{i=0}^{n-1} c_i x^i \in I$, as I ideal, $x(rx^n + \dots) \in I$ so $r \in J_{n+1}$.

As R is noetherian, there is N such that $J_n = J_N$ for all $n \geq N$.

For each $n \leq N$, as R noetherian, J_n is finitely generated ideal of R . For each $n \leq N$, choose a finite set $F_n \subseteq I$ such that if $\deg(f) = n$ then $f \in F_n$ and $\{\text{leading coeff of } f : f \in F_n\}$

$f \in F_n\}$ generates J_n .

Let $F = \bigcup_{n \leq N} F_n$ and claim that F generates I (as an ideal of $R[x]$). Show by induction on $\deg(h)$ that every $h \in I$ is an $R[x]$ -linear combination of elements of F .

(Base Case) $h = 0$, $\deg(h) \leq N$ and $h = rx^n + \sum_{i < n} c_i x^i$ where $r \in J_n$. SO find $\lambda_1, \dots, \lambda_t \in R$ and $g_1, \dots, g_t \in F_n$ such that $r = \text{leading coefficient of } \sum_{i=1}^t \lambda_i g_i$. Look at $h - \sum_{i=1}^t \lambda_i g_i \in I$ has lower degree.

(Induction Step) Let $h \in I$ with $\deg(h) = n > N$, $h = rx^n + \sum_{i < n} c_i x^i$ with $r \in J_n = J_N$. Find $g'_1, \dots, g'_s \in F_N, \lambda_1, \dots, \lambda_s \in R$ such that $r = \text{leading coefficients of } \sum_{j=1}^s \lambda_j g'_j$ with leading term in rx^N . So now look at $h - \sum_{j=1}^s \lambda_j x^{n-N} g'_j$. This has degree $< n$, so done by induction. \square