Extremal Combinatorics Notes

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Definition 1.1. *Turan Numbers (Forbidden Subgraph Problems): F is a graph and G graph is F-free if G contains no copy of F as a subgraph.*

We want to maximize the size of G subject to G being F-free. Where size=$e(G)$ = # edges in G

Definition 1.2. $ex(n, F) = \max\{e(G)|G$ *is F-free and G is n-vertex graph*$\}$

Theorem 1.3. *Crude Turan's Theorem: If G is n-vertex graph,*

$$e(G) = \epsilon\binom{n}{2}.$$

then G contains an independent set of size greater than or equal $\frac{1}{2}\frac{1}{\epsilon} + \frac{1}{2}$
In particular, if $e(G) \geq (1-\epsilon)\binom{n}{2}$ edges, then G contains a clique on greater than or equal $\frac{1}{2\epsilon} + \frac{1}{2}$ vertices.

$$ex(n, K_t) \leq \left(1 - \frac{1}{2t-1}\right)\binom{n}{2}.$$

Proof. Pick k vertices in G at random without replacement. Say we pick $v_1, ..., v_k$
For each edge between v_i and v_j, delete v_i.
Let X = # deleted vertices
Note: $X \leq e(G[v_1, ..., v_k]) =: Y$

$$\mathbb{E}[Y] = \sum_{1\leq i\leq j\leq k} \mathbb{P}[v_i v_j \in e(G)] = \binom{k}{2}\frac{\epsilon\binom{n}{2}}{\binom{n}{2}}s = \epsilon\binom{k}{2}.$$

Thus, there is a choice of $v_1, ..., v_k$ such that $Y \leq \epsilon\binom{k}{2}$. Fix it. Let $\alpha(G)$ = size of largest independent set in G satisfies $\alpha(G) \geq k - Y \geq k - \epsilon\binom{k}{2}$

$$\left(k - \epsilon\binom{k}{2}\right)' = 1 + \frac{\epsilon}{2} - \epsilon k.$$

Set $k = \frac{1+\frac{\epsilon}{2}}{2} + \delta$ for some $|\delta| \leq \frac{1}{2}$.

$$\begin{aligned}
\alpha(G) &\geq \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon\left(\left(\frac{1}{\epsilon} + \frac{1}{2} + \delta\right)\left(\frac{1}{\epsilon} - \frac{1}{2} + \delta\right)\right)/2 \\
&= \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon\frac{(\frac{1}{\epsilon} + \delta)^2 - \frac{1}{4}}{2} \\
&= \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon\frac{(1/\epsilon)^2 + 2\frac{1}{\epsilon}\delta + \delta^2 - \frac{1}{4}}{2} \\
&= \frac{1}{2}\frac{1}{\epsilon} + \frac{1}{2} + \frac{\frac{1}{4} - \delta^2}{2} \geq \frac{1}{2}\frac{1}{\epsilon} + \frac{1}{2}
\end{aligned}$$

□

Remark 1.4. *Recall: k-uniform hypergraph on vertex set V is a subset of*

$$\binom{V}{k} := \{e \subseteq V : |e| = k\}.$$

Example 1.5. *In k-uniform hypergraph on vertex with $\epsilon\binom{n}{k}$ edges there is an independent set of size $\geq c_k \frac{1}{\epsilon}^{\frac{1}{k-1}}$*

Theorem 1.6. *Mantel's Theorem: $ex(n, K_3) = \lfloor\frac{n^2}{4}\rfloor$*
More generally, define Turan's graph $T_{n,r}$ to be the n-vertex complete r-partite graph with part sizes as equal as possible.
Note: $K_{r+1} \not\subseteq T_{n,r}$ because among $r+1$ vertices some pair is in same part.

Theorem 1.7. *Turan's Theorem: $ex(n, K_{r+1}) = e(T_{n,r})$ and furthermore $T_{n,r}$ is the only extremizer.*

$$e(T_{n,r}) = \left(1 - \frac{1}{r}\right)\binom{n}{2} + O(n).$$

Proof. Let G be a graph on n vertices that is K_{r+1}-free and has the maximum possible number of edges (an extremizer). We proceed in two steps.

Suppose G is not a complete multipartite graph. Then non-adjacency is not an equivalence relation, implying there exist two non-adjacent vertices x, y that do not have identical neighborhoods.

Assume $\deg(x) \leq \deg(y)$. Delete x and then clone y to obtain y'. The new graph G' has

$$e(G') = e(G) - \deg(x) + \deg(y) \geq e(G).$$

Also, G' is K_{r+1}-free because any copy of K_{r+1} has at most one of y or y' (since they are non-adjacent). If a clique contains y', it can be swapped for y (since $N(y') \subseteq N(y)$), which would imply $K_{r+1} \subseteq G$, a contradiction.

Repeating this process transforms G into a complete multipartite graph without decreasing edges. Thus, the extremizer must be a complete multipartite graph.
Since G is K_{r+1}-free, it must be k-partite with $k \leq r$. To maximize edges, we set $k = r$. Let the parts have sizes $n_1, \ldots, n_r$.

$$e(G) = \binom{n}{2} - \sum_{i=1}^{r}\binom{n_i}{2}$$

To maximize $e(G)$, we must minimize the subtracted sum of pairs inside the parts. This occurs when n_i are as equal as possible ($|n_i - n_j| \leq 1$). If $n_i \geq n_j + 2$, moving a vertex from part i to part j increases the edge count by $(n_i - 1) - n_j > 0$.

Thus, the unique extremizer is the balanced complete r-partite graph, $T_{n,r}$. $\square$

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Proof. Second proof of Turan's Theorem with induction on r.
(Induction Step) Let x be a vertex of largest degree, let $A = N(x)$ be its neighborhood.

$$B := V(G) \setminus A.$$

We have $e(G) = e(A) + e(A, B) + e(B)$ and A is K_{r-1}-free so $e(A) \leq e(T_{|A|,r})$
Each vertex in B has deg $\leq \deg(x) = |A|$
So,

$$e(A, B) + e(B) \leq \sum_{v \in B} \deg(v) \leq |B||A|.$$

So,

$$e(G) \leq e(T_{|A|,r}) + |A||B| \leq e(T_{n,r}).$$

The right hand side is equal to number of edges in complete r-partite graph with one part of size $|B|$ and other being as equal as possible.
Equality holds iff B is independent and A must be a copy of $e(T_{|A|,r-1})$. Hence G is $T_{n,r}$ for equality to hold. $\square$

Theorem 2.1. *Observe that if F is a graph then*

$$\frac{ex(n, F)}{\binom{n}{2}} \geq \frac{ex(n+1, F)}{\binom{n+1}{2}}.$$

Proof. Pick n vertices in an F-free $(n+1)$ vertex graph G at random without replacement; let G' be the induced graph on these n vertices.

$$e(G') \leq ex(n, F).$$

$$\mathbb{E}(e(G')) = \binom{n}{2} \frac{e(G)}{\binom{n+1}{2}} \implies \frac{e(G)}{\binom{n+1}{2}} \leq \frac{ex(n, F)}{\binom{n}{2}}.$$

$\square$

Definition 2.2.

$$\pi(F) := \lim_{n \to \infty} \frac{ex(n, F)}{\binom{n}{2}}.$$

The Turan density of F is

$$\pi(K_{r+1}) = 1 - \frac{1}{r}.$$

Theorem 2.3. *Let F be any graph and let $\epsilon > 0$. Let G be an n-vertex graph. There is a $\delta > 0$ such that*

$$e(G) \geq (\pi(F) + \epsilon) \binom{n}{2} \text{ and } n \geq n_0(F, \epsilon).$$

then there are at least $\delta n^{|V(F)|}$ copies of F.

Proof. Let n_0 be large enough such that $\frac{ex(n_0,F)}{\binom{n_0}{2}} \leq \pi(F) + \frac{\epsilon}{2}$. Given n-vertex G as in theorem. Pick n_0 vertices at random without replacement. Let G' be the n_0-vertex graph induced on these chosen vertices.

$$\mathbb{E}[e(G')] = \frac{e(G)}{\binom{n}{2}} \binom{n_0}{2} \geq (\pi(F) + \epsilon) \binom{n_0}{2}.$$

Say G' is good if

$$e(G') \geq \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2}.$$

$$\begin{aligned}
\mathbb{E}(e(G')) &\leq \mathbb{P}(G' \text{ is bad}) \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2} + \mathbb{P}(G' \text{ is good}) \binom{n_0}{2} \\
&\leq \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2} + \mathbb{P}(G' \text{ is good}) \binom{n_0}{2}
\end{aligned}$$

Then $\frac{\epsilon}{2}\binom{n_0}{2} \le \mathbb{P}(G' \text{ is good})\binom{n_0}{2}$ so $P(G' \text{ is good}) \ge \frac{\epsilon}{2}$
Pick $|V(F)|$ many vertices without replacement inside G', call the subgraph they induce G''. So

$$G \supset G' \supset G''.$$

$$\mathbb{P}(G'' \text{ contains } F) \ge \mathbb{P}(G' \text{ is good}) \cdot \mathbb{P}(G'' \text{ contains } F | G' \text{ is good}) \ge \frac{\epsilon}{2}\frac{1}{\binom{n_0}{|V(F)|}}.$$

Note that G'' is uniformly randomly chosen $|V(F)|$-vertex subset of G. This implies the # copies of F in $G \ge \mathbb{P}(G''\text{contains F})\binom{n}{|V(F)|} \ge P(...)\frac{n^{|V(F)|}(1-o(1))}{|V(F)|!}$. We can choose $\delta = \frac{1}{\binom{n_0}{|V(F)|}}$ □

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Theorem 3.1 (Erdős-Stone-Simonovits). *Let $T_{n,r}$ denote the Turán graph. We know that:*

$$ex(n, T_{s,(r+1)}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

This implies that if H is any graph with chromatic number $\chi(H) = r + 1$, then

$$ex(n, H) \le \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

Conversely, if $\chi(H) > r$ (i.e., H is not r-partite), then $T_{n,r}$ is H-free. Hence,

$$ex(n, H) \ge e(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

Combining these, for any non-bipartite graph H:

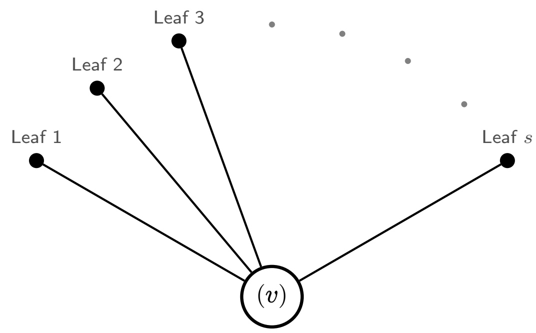
$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}.$$

It is important to note that this theorem holds for bipartite graphs as well, where $\chi(H) = 2$. However, substituting $r = \chi(H) - 1 = 1$ gives a coefficient of $1 - 1/1 = 0$. This implies that $ex(n, H) = o(n^2)$, but it does not provide the precise asymptotic behavior (e.g., $n^{2-\epsilon}$). Therefore, while valid, the Erdős-Stone-Simonovits theorem is not the primary tool for determining extremal numbers of bipartite graphs; we rely on theorems like Kővári-Sós-Turán for those bounds.

Theorem 3.2 (Kővári-Sós-Turán 1954).

$$ex(n, K_{s,t}) \le \frac{1}{2}(t-1)^{\frac{1}{s}} \cdot n^{2-\frac{1}{s}} + O(n).$$

Proof. Let G be a $K_{s,t}$-free graph with n vertices. We will count X, the number of copies of $K_{1,s}$ (stars with s leaves).



We count this in two ways:

1. Since there is no $K_{s,t}$, any set of s vertices can have at most $t - 1$ common neighbors. Thus:

$$X \le \binom{n}{s}(t-1).$$

2. Let d_v be the degree of vertex v. Then:

$$X = \sum_{v \in V(G)} \binom{d_v}{s}.$$

A simple-minded application of Jensen's inequality here fails because the binomial coefficient $\binom{x}{s}$ is not convex for all x. To fix this, we define the function:

$$f_s(x) = \begin{cases} \binom{x}{s} & \text{if } x \ge s - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f_s(x)$ is convex for all x because its derivative:

$$\frac{d}{dx} f_s(x) = \begin{cases} \left(\frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x-s+1}\right)\binom{x}{s} & \text{if } x > s-1 \\ 0 & \text{otherwise} \end{cases}$$

is non-decreasing. Since d_v is an integer, $\binom{d_v}{s} = f_s(d_v)$. We can now validly apply Jensen's inequality:

$$X = \sum_{v \in V} f_s(d_v) \geq n f_s(d_{\text{avg}}) = n f_s\left(\frac{2m}{n}\right),$$

where $m = e(G)$.

Combining (1) and (2):

$$n f_s\left(\frac{2m}{n}\right) \leq \binom{n}{s}(t-1).$$

If $m \leq \frac{1}{2}(s-1)n$, the bound holds trivially. Otherwise, we approximate $\binom{x}{s} \approx \frac{x^s}{s!}$:

$$n\frac{\left(\frac{2m}{n} - s\right)^s}{s!} \leq (t-1)\frac{n^s}{s!}.$$

Rearranging yields:

$$\frac{2m}{n} \leq (t-1)^{\frac{1}{s}} n^{1-\frac{1}{s}} + s \implies m \leq \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + O(n).$$

□

Example 3.3 (Unit Distance Problem)**.** *How can we place n points in $\mathbb{R}^2$ to maximize the number of pairs (p, q) such that $||p - q|| = 1$?*

A strong construction is a $\sqrt{n} \times \sqrt{n}$ grid. Rather than scaling the grid to unit step size, we scale it such that the distance 1 corresponds to a distance d in the integer grid that occurs most frequently.

The possible distances in an unscaled integer grid are $\sqrt{a^2 + b^2}$ for $0 \leq a, b < \sqrt{n}$. By number theory (Landau-Ramanujan), integers representable as a sum of two squares are those where primes $p \equiv 3 \pmod 4$ appear with even exponents.

To get a better bound, we look for "highly composite" integers. Specifically, if we choose a number d composed of a product of many distinct primes $p \equiv 1 \pmod 4$, the number of ways to write d as a sum of two squares is very large. Erdős used this to show there exists a distance occurring:

$$n^{1+\frac{c}{\log \log n}}.$$

This is significantly larger than the $n\sqrt{\log n}$ bound derived from the Pigeonhole Principle on generic integers.

Theorem 3.4 (Upper Bound for Unit Distances)**.** *The number of unit distances is $O(n^{3/2})$.*

Proof. Given n points in $\mathbb{R}^2$, construct a graph where vertices are points and $v \sim u$ if $||v - u|| = 1$. Observe the graph is $K_{2,3}$-free. Geometrically, this is because two circles with radius 1 intersect at most 2 times; thus, no 2 vertices can share 3 common neighbors.
By KST with $s = 2, t = 3$:

$$e(G) \leq \frac{1}{2}\sqrt{3-1} \cdot n^{2-\frac{1}{2}} + o(n^2) = \frac{1}{\sqrt{2}} n^{\frac{3}{2}} + o(n^2).$$

□

Definition 3.5. *Define $K^{(3)}_{s_1,s_2,t}$ to be the complete 3-uniform, 3-partite hypergraph with parts of size s_1, s_2, t.*

Theorem 3.6 (Generalization of KST to Hypergraphs)**.** *For the* 3*-uniform case:*

$$ex(n, K^{(3)}_{s_1,s_2,t}) \le C_{s_1,s_2,t} n^{3-\frac{1}{s_1 s_2}} + O(n^2).$$

Proof. We will count the number of stars $K_{1,1,s_1}$ (sets of edges sharing a common pair of vertices). Let X be this number. For any pair $\{v_1, v_2\}$, let d_{v_1,v_2} be the co-degree (number of edges containing $\{v_1, v_2\}$).

1. **Upper Bound on X:** Fix a set U of size s_1. Define an auxiliary graph G_U on $V(G)$ where $v_1 \sim v_2$ in G_U if $\{v_1, v_2, u\} \in E(G)$ for all $u \in U$. If G_U contains a copy of $K_{s_2,t}$, then combined with U, we form a $K^{(3)}_{s_1,s_2,t}$. Thus, G_U must be $K_{s_2,t}$-free.

$$X \le \binom{n}{s_1} ex(n, K_{s_2,t}) \le C \binom{n}{s_1} n^{2-\frac{1}{s_2}}.$$

2. **Lower Bound on X:** Summing over pairs:

$$X = \sum_{\{v_1,v_2\} \in \binom{V}{2}} \binom{d_{v_1,v_2}}{s_1} \ge \binom{n}{2} f_{s_1}\left(\frac{3m}{\binom{n}{2}}\right).$$

Combining bounds (assuming m is large enough):

$$\binom{n}{2} \frac{\left(\frac{3m}{n^2/2}\right)^{s_1}}{s_1!} \lesssim \frac{n^{s_1}}{s_1!} n^{2-\frac{1}{s_2}}.$$

Simplifying:

$$n^2 \left(\frac{m}{n^2}\right)^{s_1} \lesssim n^{s_1+2-\frac{1}{s_2}} \implies m^{s_1} \lesssim n^{3s_1-\frac{1}{s_2}}.$$

Taking the s_1-th root:

$$m \le C n^{3-\frac{1}{s_1 s_2}}.$$

$\square$

4 Jan 21: Supersaturation and Generalized Turán Problems

4.1 Supersaturation

The classical Turán problem asks for the maximum number of edges in a graph avoiding a subgraph F. Supersaturation asks: what happens if we have just *slightly* more edges than this threshold? It turns out we don't just get one copy of F, but exponentially many.

Lemma 4.1 (Supersaturation Lemma). *If G is a graph on n vertices with $e(G) \geq (\pi(F)+\epsilon)\binom{n}{2}$ edges, then G contains at least $\delta(\epsilon)n^{|V(F)|}$ copies of F, where $\delta(\epsilon) > 0$ is a constant independent of n.*

Corollary 4.2. *For the complete graph K_{r+1}, we know $\pi(K_{r+1}) = 1 - \frac{1}{r}$. Thus:*

$$e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right)\binom{n}{2} \implies G \text{ contains } \delta(\epsilon)n^{r+1} \text{ copies of } K_{r+1}.$$

4.2 Hypergraph Kővári-Sós-Turán

We generalize the KST theorem from bipartite graphs to r-uniform hypergraphs. We are looking for the Turán number of the complete r-partite hypergraph $K^{(r)}_{s_1,\ldots,s_r}$. Note that the theorem statement often simplifies the last parameter to t.

Theorem 4.3 (Generalized Kővári-Sós-Turán). *Let $K = K^{(r)}_{s_1,s_2,\ldots,s_{r-1},t}$ be the complete r-partite r-uniform hypergraph with part sizes $s_1,\ldots,s_{r-1},t$. Then:*

$$ex(n,K) = O_{s,t}\left(n^{r-\frac{1}{s_1 s_2 \cdots s_{r-1}}}\right).$$

Proof. We proceed by induction on the uniformity r. The base case $r = 2$ is the standard graph KST theorem. Assume the bound holds for $(r-1)$-uniform hypergraphs.
Let G be an r-uniform hypergraph with n vertices and m edges. We define a "hypergraph star" $K^{(r)}_{1,1,\ldots,s_1}$ as a set of s_1 edges that all share a common core of $r-1$ vertices.
Let X be the number of such stars (copies of $K^{(r)}_{1,1,\ldots,1,s_1}$) in G. We double count X.

1. Lower Bound (Convexity): For every subset $S \in \binom{V(G)}{r-1}$ of size $r-1$, let the *co-degree* d_S be the number of edges in G containing S. A set S contributes $\binom{d_S}{s_1}$ stars to the count.

$$X = \sum_{S \in \binom{V(G)}{r-1}} \binom{d_S}{s_1}.$$

Using Jensen's Inequality (and the convexity of $\binom{x}{s_1}$), we bound this sum:

$$X \geq \binom{n}{r-1}\binom{\bar{d}}{s_1}, \quad \text{where } \bar{d} = \frac{\sum d_S}{\binom{n}{r-1}} = \frac{rm}{\binom{n}{r-1}}.$$

Approximating for large n: $\bar{d} \approx \frac{r!m}{n^{r-1}}$ and $\binom{x}{k} \approx \frac{x^k}{k!}$, we get:

$$X \gtrsim n^{r-1} \cdot \left(\frac{m}{n^{r-1}}\right)^{s_1}. \tag{1}$$

2. Upper Bound (Induction): Fix a set of vertices $V_1 = \{v_1,\ldots,v_{s_1}\}$. We ask: how many ways can we complete this into a star?
Define an auxiliary $(r-1)$-uniform hypergraph G_{V_1} on $V(G)$. A set e' of size $r-1$ is an edge in G_{V_1} if $e' \cup \{v_i\} \in E(G)$ for all $i = 1,\ldots,s_1$. In other words, e' is in the common neighborhood of all v_i.
If G_{V_1} contains a copy of $K^{(r-1)}_{s_2,\ldots,s_{r-1},t}$, then combined with V_1, we would form a copy of the forbidden $K^{(r)}_{s_1,\ldots,t}$ in G. Thus, G_{V_1} must be free of $K^{(r-1)}_{s_2,\ldots,t}$. By the inductive hypothesis:

$$e(G_{V_1}) \leq C \cdot n^{(r-1)-\frac{1}{s_2 \cdots s_{r-1}}}.$$

Summing over all choices of V_1 (there are $\binom{n}{s_1}$ such sets):

$$X \leq \binom{n}{s_1} \cdot \max_{V_1} e(G_{V_1}) \lesssim n^{s_1} \cdot n^{r-1-\frac{1}{s_2 \cdots s_{r-1}}}. \tag{2}$$

3. Conclusion: Comparing (1) and (2):

$$n^{r-1}\left(\frac{m}{n^{r-1}}\right)^{s_1} \lesssim n^{s_1+r-1-\frac{1}{s_2\cdots s_{r-1}}}.$$

Rearranging for m:

$$m^{s_1} \lesssim n^{(r-1)s_1} \cdot n^{s_1-\frac{1}{s_2\cdots s_{r-1}}} = n^{rs_1-\frac{1}{s_2\cdots s_{r-1}}}.$$

Taking the s_1-th root:

$$m \lesssim n^{r-\frac{1}{s_1 s_2\cdots s_{r-1}}}.$$

□

4.3 The Erdős-Stone Theorem

Definition 4.4 (Blow-up)**.** *A **blow-up** of a graph H, denoted $H(s)$, is obtained by replacing each vertex $v \in V(H)$ with an independent set I_v of size s, and replacing each edge $(u,v) \in E(H)$ with a complete bipartite graph between I_u and I_v.*
The Turán graph $T_{n,r}$ is essentially a blow-up of K_r with parts of size n/r.

Theorem 4.5 (Erdős-Stone)**.** *If $e(G) \geq \left(1-\frac{1}{r}+\epsilon\right)\binom{n}{2}$ and n is sufficiently large, then G contains $T_{s,(r+1)}$ (a blow-up of K_{r+1} with parts of size s).*
Consequently, G contains any graph H with $\chi(H) = r+1$ (since any such H is a subgraph of a large enough blow-up of K_{r+1}).

Proof. **Step 1: Find many cliques.** By the Supersaturation Corollary, since the edge density is greater than the Turán threshold for K_{r+1}, G contains many copies of K_{r+1}:

$$\#K_{r+1} \geq \delta n^{r+1}.$$

Step 2: Construct Auxiliary Hypergraph. Define an $(r+1)$-uniform hypergraph $\mathcal{H}$ with vertex set $V(G)$. Let a set of $r+1$ vertices form an edge in $\mathcal{H}$ if and only if they form a K_{r+1} in the original graph G.
From Step 1, we know $e(\mathcal{H}) \geq \delta n^{r+1}$. Since δ is constant, this is a dense hypergraph.
Step 3: Find a dense structure in the hypergraph. We apply the **Hypergraph KST Theorem** to $\mathcal{H}$. Since $\mathcal{H}$ is dense (order n^{r+1}), it exceeds the KST threshold (which is roughly $n^{(r+1)-\epsilon}$). Therefore, $\mathcal{H}$ must contain a copy of the complete $(r+1)$-partite hypergraph $K^{(r+1)}_{s,s,\ldots,s}$.
Step 4: Map back to G. Let the parts of this hypergraph copy be $U_1,\ldots,U_{r+1}$, each of size s. The definition of the complete hypergraph implies that for *any* choice of vertices $u_1 \in U_1,\ldots,u_{r+1} \in U_{r+1}$, the set $\{u_1,\ldots,u_{r+1}\}$ is an edge in $\mathcal{H}$.
By definition of $\mathcal{H}$, this means $\{u_1,\ldots,u_{r+1}\}$ forms a clique K_{r+1} in G. If every possible tuple forms a clique, then every pair of vertices in distinct parts U_i, U_j must be connected in G.
Thus, $G[U_1 \cup \cdots \cup U_{r+1}]$ contains the complete multipartite graph with parts of size s, which is exactly $T_{s(r+1)}$. □

4.4 Degenerate Turán Numbers (Bipartite Graphs)

When H is bipartite ($r=1$), the Erdős-Stone bound gives $1-1/1=0$, which is trivial. We rely on KST bounds $ex(n,K_{s,t}) = O(n^{2-1/s})$. We examine the sharpness of these bounds.

Theorem 4.6 (Tightness of KST)**.**

1. ***Case** $s=1$ **(Trees/Forests):***

   $$ex(n,K_{1,t}) \leq \frac{1}{2}(t-1)n.$$

   This is tight for a graph consisting of disjoint copies of K_t (or a matching if $t=1$).

2. ***Case** $s=2$ **(C_4-free graphs):** We know $ex(n,K_{2,2}) = O(n^{3/2})$. This is tight.*

   ***Construction (Finite Geometry):** Let q be a prime power. Construct a bipartite graph G with parts A and B, where $|A| = |B| = q^2$.*

   - *Identify vertices in A and B with points in the affine plane $\mathbb{F}_q^2$.*
   - *Define edge $(a,b) \sim (x,y)$ if $ax+by=1$, where $(a,b) \in A$ and $(x,y) \in B$.*

For a fixed vertex $v = (a, b) \in A$, its neighborhood is the set of solutions (x, y) to the linear equation $ax + by = 1$. This describes a line in $\mathbb{F}_q^2$, which contains q points (unless $(a, b) = (0, 0)$, which we can discard).

$$e(G) \approx q^2 \cdot q = q^3.$$

Since $n = 2q^2$, we have $q \approx \sqrt{n/2}$, so $e(G) \approx (n/2)^{3/2} = \Theta(n^{3/2})$.

Why is it C_4-free? *A C_4 corresponds to two vertices in A sharing two common neighbors in B. Geometrically, this means two distinct lines intersect at two distinct points. In a plane, two distinct lines intersect at most at one point. Thus, no C_4 exists.*

3. ***Case*** $s = 3$ ***($K_{3,3}$-free):*** *Tightness $ex(n, K_{3,3}) = \Theta(n^{5/3})$ is known.*

   Construction Sketch: *Vertices are points in $\mathbb{F}_q^3$. Adjacency is defined by $(x, y, z) \sim (a, b, c)$ if $(x-a)^2 + (y-b)^2 + (z-c)^2 = 1$. Geometrically, the neighborhood of a point is a unit sphere. If G contained $K_{3,3}$, three vertices in one part would share three common neighbors. This would imply that three unit spheres intersect at three points. In Euclidean geometry, three spheres intersect at most at 2 points (a circle intersects a sphere at 2 points). This logic carries over to $\mathbb{F}_q$ provided -1 is not a square.*

4. ***Case*** $s = 4$ ***($K_{4,4}$-free):*** *It is an* **open problem** *whether the KST bound $O(n^{2-1/4})$ is tight for $K_{4,4}$.*

4.5 Extremal Numbers for Trees

Lemma 4.7. *If a graph G has average degree d, it contains a subgraph H with minimum degree $\delta(H) \geq d/2$.*

Proof. Let $n = |V(G)|$. We have $e(G) = \frac{dn}{2}$. Iteratively delete any vertex with degree strictly less than $d/2$. Let S be the set of removed vertices. The number of edges removed is strictly less than $|S| \cdot \frac{d}{2}$. Even if we remove $n-1$ vertices, the number of edges removed is $< (n-1)\frac{d}{2} < \frac{dn}{2}$. Thus, the edge set cannot be emptied. The process must stop at a non-empty subgraph where every remaining vertex has degree $\geq d/2$. □

Theorem 4.8. *If T is a tree with r edges (so $r + 1$ vertices), then:*

$$ex(n, T) \leq rn.$$

Proof. Suppose $e(G) > rn$. Then the average degree of G is $d(G) = \frac{2e(G)}{n} > 2r$. By the Lemma, there exists a subgraph $H \subseteq G$ with minimum degree $\delta(H) > r$. Since degrees are integers, $\delta(H) \geq r$.

We embed T into H greedily:

1. Order the vertices of T as $v_1, v_2, \ldots, v_{r+1}$ such that each v_i (for $i > 1$) has exactly one neighbor among $\{v_1, \ldots, v_{i-1}\}$. (This is always possible by doing a BFS/DFS from a root).
2. Map v_1 to any vertex $u_1 \in V(H)$.
3. Suppose we have embedded $v_1, \ldots, v_{i-1}$ as $u_1, \ldots, u_{i-1}$. Let v_j be the parent of v_i in the tree $(j < i)$.
4. We need to map v_i to a neighbor of u_j in H.
5. The vertex u_j has at least r neighbors in H. We have used at most $i - 1$ vertices so far. Since $i \leq r + 1$, we have used at most r vertices.
6. We only care that u_j has neighbors not already used in the current tree embedding. The degree of u_j is $\geq r$. The number of currently used vertices is $i - 1 \leq r$.
7. The bound requires strict inequality or careful counting. If $\delta(H) \geq r$, u_j has r neighbors. We have used $i - 1$ vertices total. If $i - 1 < r$, there is space. If $i - 1 = r$, we are placing the last vertex. The neighbors of u_j could be occupied by the other $r - 1$ vertices of the tree.
8. However, the conjecture is actually $\frac{1}{2}(r)n$. The bound rn is loose enough. If $e(G) > rn$, avg deg $> 2r$, min deg $\geq r + 1$. Then u_j has $r + 1$ neighbors, but the tree only has $r + 1$ vertices total. At step i, only $i - 1 \leq r$ vertices are occupied. Thus there is always at least one free neighbor for u_j.

□

5 Jan 23

5.1 Turan Number of Cycles and Girth Problem

Definition 5.1. *If $\mathcal{F}$ is a family of graphs, $ex(n,\mathcal{F})$ = maximum number of edges in an n vertex graph G that contains no $F \in \mathcal{F}$.*

Lemma 5.2. $ex(n,\mathcal{F}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}$ *if* $r = \min_{F\in\mathcal{F}} \mathcal{X}(F)$

Proof.
$$ex(n,\mathcal{F}) \leq \min_{F\in\mathcal{F}} ex(n,F).$$
By Erdos Stone, we're done. □

Theorem 5.3. *Let $\mathcal{F} = \{C_3, ..., C_{2k}\}$ then $ex(n,\mathcal{F}) \leq n^{1+\frac{1}{k}}$*

Lemma 5.4. *G is $\mathcal{F}$ free iff G has girth $\geq 2k+1$*

Lemma 5.5. *Every graph G contains a subgraph G' that is bipartite with $e(G') \geq \frac{1}{2}e(G)$.*

Proof. Color vertices red or blue uniformly and independent. Then delete the edges that are monochromatic By independence, for any $e \in E(G)$
$$\Pr[e \in E(G')] = \frac{1}{2} \implies E[e(G')] = \frac{1}{2}e(G).$$
□

We can also prove this with the greedy algorithm and will achieve a slightly better bound.

Proof. To prove theorem 5.3:
Suppose G has at least $n^{1+\frac{1}{k}}$, by lemma 5.2 we may replace G by a sub-graph of $\min\deg d \geq n^{1+\frac{1}{k}}$.
We can do a depth first search by picking a root vertex v. Let V_i be the set of vertices at distance i from v. So $V_0 = \{v\}$ Observe that $G[V_0, V_1, ..., V_k]$ forms a tree with root v and V_i being the descendent at at level i else we get a cycle of length at most $2k$. So we have
$$|V_0| + |V_1| + \cdots + |V_k| \geq 1 + d + (d-1)d + d(d-1)^2 + \cdots + d(d-1)^{k-1} = 1 + d^k > n.$$
So, we have a contradiction. □

Theorem 5.6. $ex(n, C_{2k}) \leq c_k n^{1+\frac{1}{k}}$

Theorem 5.7. $ex(n, \{C_4, C_6, ..., C_{2k}\}) \leq 2n^{1+\frac{1}{k}}$

Remark 5.8. *The exponent $1+\frac{1}{k}$ is sharp if $k = 2, 3, 5$*

5.2 Turan Number for Paths

Theorem 5.9. *Let P_k be a path with k edges then*
$$ex(n, P_k) \leq \frac{k}{2}n.$$

Proof. By induction on n it is enough to work with connected graph as we can just split into connected components. Delete vertices of degree $\leq \frac{k}{2}$. If $e(G) > \frac{k}{2}n$ then this also holds after deletions. So WLOG G has minimum degree at least $\frac{k}{2}$.
Let $v_0, v_1, ..., v_r$ be the longest path in G. The neighbors of v_0, v_r are in the path otherwise you can extend the path. If $v_0 \sim v_r$ then $v_0v_1...v_r$ is a cycle and so $v_iv_{i+1}...v_rv_0...v_{i-1}$ is also a path of same length. Thus, the neighbors of v_i are inside the path. So $\{v_0, v_1, ..., v_r\}$ is a connected conponent of G and thus $v \geq k+1$ else $e(G) \leq \frac{(r-1)n}{2}$. Otherwise, if $v_0 \not\sim v_r$ then there exist i such that $v_0 \sim v_{i+1}$ and $v_r \sim v_i$ then we can produce a cycle $v_0v_{i+1}v_{i+2}...v_rv_iv_{i-1}...v_0$ and this is of length r, so it becomes the previous case. By pigeonhole, such a i exist as $\{i : v_0v_i + 1 \in E(G)\}$ and $\{i : v_rv_i \in E(G)\}$ must intersect as $r \leq k$ by $v_0v_r \notin E(G)$ □

6 Jan 26: Projective Geometry

Definition 6.1. ***Projective Plane over a Field.*** *Let $\mathbb{F}$ be a field.*

1. *Points: The set of lines through 0 in $\mathbb{F}^3$; equivalently, the 1-dimensional subspaces.*
2. *Lines: The set of planes through 0 in $\mathbb{F}^3$; equivalently, the 2-dimensional subspaces.*
3. *Incidence: A point P lies on a line ℓ if $P \subset \ell$.*

Observe:

1. *Any two points determine a unique line.*
2. *Any two lines intersect in a unique point.*
3. *There exist 4 points no three of which are collinear.*

Inside a projective plane over $\mathbb{F}$, there is a copy of the affine plane $\mathbb{F}^2$ embedded as $\{z = 1\}$.
We denote this by $\mathbb{P}^2(\mathbb{F})$.
We say $p \sim \ell$ if $p \in \ell$ for p a point and ℓ a line of $\mathbb{P}^2(\mathbb{F}_q)$.

Remark 6.2. *There are $q^3 - 1$ nonzero vectors in $\mathbb{F}_q^3 \setminus \{0\}$. We identify points of $\mathbb{P}^2(\mathbb{F}_q)$ with equivalence classes of nonzero vectors under*

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \quad \text{for } \lambda \in \mathbb{F}_q^{\times}.$$

There are $q - 1$ nonzero choices for λ, so the number of points in $\mathbb{P}^2(\mathbb{F}_q)$ is

$$\frac{q^3 - 1}{q - 1} = q^2 + q + 1.$$

By a similar argument, the number of lines is also $q^2 + q + 1$.
Each line contains exactly $q + 1$ points and dually, each point lies on exactly $q + 1$ lines. There is a natural bijection (called a polarity*) between points and lines, preserving incidence: given a point p, its corresponding line is the orthogonal complement $p^{\perp}$.*

Theorem 6.3. ***Construction of a C_4-free graph:***
We construct an improved C_4-free graph as follows:

- *The vertices are the points of $\mathbb{P}^2(\mathbb{F}_q)$.*
- *Points p_1 and p_2 are adjacent if and only if $p_1 \in p_2^{\perp}$ (i.e., p_1 lies on the line associated to p_2's orthogonal complement).*

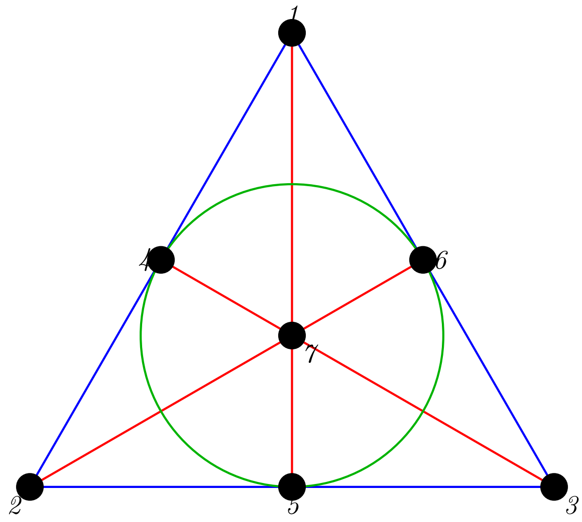
For any pair p_1, p_2, their common neighborhood is

$$p_1^{\perp} \cap p_2^{\perp},$$

which is a single point. Thus, the graph is C_4-free (contains no 4-cycles). This gives $q^2 + q + 1$ vertices with average degree $\approx q$.
Note: There exist p such that $p \subseteq p^{\perp}$.

Definition 6.4. ***General Projective Planes.*** *If P (points) and L (lines) are sets with an incidence relation such that:*

1. *For any two points, there is a unique line through them.*
2. *Any two lines meet at a unique point.*
3. *There exist four points, no three collinear.*



Example: The Fano plane (order 2) with 7 points and 7 lines. Each line contains 3 points, and each point lies on 3 lines.

Definition 6.5. ***Order of a Projective Plane.*** *The order of a projective plane is the number of points per line minus one (so q if each line has $q+1$ points).*

Remark 6.6. *Are there planes of order which are not prime powers?*

Theorem 6.7. ***Construction of C_{2k}-free graphs for $k = 2, 3, 5$:***
Let q be a prime power and let $\mathbb{F}_q$ be the finite field with q elements. Define a graph G on the vertex set $\mathbb{F}_q^k$ (the set of k-tuples over $\mathbb{F}_q$) as follows:
For $\mathbf{a} = (a_0, a_1, \ldots, a_{k-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{k-1})$ in $\mathbb{F}_q^k$, declare $\mathbf{a} \sim \mathbf{b}$ if and only if for all $j = 0, 1, \ldots, k-2$,

$$b_j = a_j + a_{j+1} \cdot b_{k-1}$$

where indices are taken modulo k (so $a_k = a_0$).
Then G is C_{2k}-free, i.e., it contains no cycle of length $2k$.

Proof. Suppose for contradiction that G contains a cycle of length $2k$:

$$\mathbf{a}^1, \mathbf{b}^1, \mathbf{a}^2, \mathbf{b}^2, \ldots, \mathbf{a}^k, \mathbf{b}^k,$$

with $\mathbf{a}^i \sim \mathbf{b}^i \sim \mathbf{a}^{i+1}$ for all i, indices modulo k.

Step 1: System of equations. For each $i = 1, \ldots, k$ and $j = 0, \ldots, k-2$, adjacency gives:

$$b_j^i = a_j^i + a_{j+1}^i \cdot b_{k-1}^i \qquad \text{(from } \mathbf{a}^i \sim \mathbf{b}^i\text{)}$$
$$b_j^i = a_j^{i+1} + a_{j+1}^{i+1} \cdot b_{k-1}^i \qquad \text{(from } \mathbf{b}^i \sim \mathbf{a}^{i+1}\text{)}$$

Subtract:

$$(a_j^i - a_j^{i+1}) + (a_{j+1}^i - a_{j+1}^{i+1}) \cdot b_{k-1}^i = 0$$

or

$$(a_j^{i+1} - a_j^i) + (a_{j+1}^{i+1} - a_{j+1}^i) \cdot b_{k-1}^i = 0$$

Summing over i:

$$\sum_{i=1}^{k}(a_j^{i+1} - a_j^i) + \sum_{i=1}^{k}(a_{j+1}^{i+1} - a_{j+1}^i) b_{k-1}^i = 0$$

The first sum telescopes to zero (since indices are cyclic), so for each $j = 0, \ldots, k-2$,

$$\sum_{i=1}^{k}(a_{j+1}^{i+1} - a_{j+1}^i)\, b_{k-1}^i = 0 \tag{3}$$

Step 2: Specialize $j = k-2$. Let $\triangle_i = a^{i+1}_{k-1} - a^i_{k-1}$ and $x_i = b^i_{k-1}$. Then:

$$\sum_{i=1}^{k} \triangle_i x_i = 0$$

and

$$\sum_{i=1}^{k} \triangle_i = 0$$

by telescoping.

Step 3: Linear algebra. The system (3) for $j = 0, \dots, k-2$ can be written as a $(k-1) \times k$ Vandermonde-type matrix V applied to the vector $(\triangle_1, \dots, \triangle_k)$:

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ \vdots & \vdots & & \vdots \\ x_1^{k-2} & x_2^{k-2} & \cdots & x_k^{k-2} \end{pmatrix}$$

The equations become $V \cdot (\triangle_1, \dots, \triangle_k)^\top = 0$. Since $\sum_{i=1}^k \triangle_i = 0$, the vector is in the kernel of the row $(1, 1, \dots, 1)$.

Step 4: Key property of the x_i.

Claim 6.8. *For each i, $x_i \neq x_{i+1}$ (indices modulo k).*

Proof. If $x_i = x_{i+1}$, then $b^i_{k-1} = b^{i+1}_{k-1}$. By the adjacency relations, this forces $\mathbf{b}^i = \mathbf{b}^{i+1}$, so $\mathbf{a}^{i+1} = \mathbf{a}^{i+2}$, contradicting that the cycle is proper. □

Step 5: Contradiction for $k = 2, 3, 5$. Since not all $\triangle_i$ are zero (otherwise the cycle is trivial), and $\sum_{i=1}^k \triangle_i = 0$, equation $\sum_{i=1}^k \triangle_i x_i = 0$ forces a linear dependence among the x_i (not all distinct). Thus, for these small k we can classify possibilities:

Claim 6.9. *For $k = 2, 3, 5$, if $x_i = x_j$ for some $i \neq j$, then $x_i = x_{i+1}$ for some i, contradicting the above claim.*

- **Case $k = 2$:** Only possible if $x_1 = x_2$, violating $x_1 \neq x_2$.
- **Case $k = 3$:** If $x_1 = x_2$, $x_2 = x_3$, or $x_3 = x_1$, we violate the claim.
- **Case $k = 5$:** If $x_i = x_j$ for $i \neq j$, by relabeling, could assume $x_1 = x_j$ for $j = 2, 3, 4, 5$:
  - $j = 2$ or 5 gives $x_1 = x_2$ or $x_5 = x_1$, directly a violation.
  - $j = 3$: $x_1 = x_3$. The system $\sum_{i=1}^5 \triangle_i = 0$, $\sum_{i=1}^5 \triangle_i x_i = 0$ with $x_1 = x_3$ and $x_i \neq x_{i+1}$ for any i only allows further coincidences that eventually force $x_i = x_{i+1}$.
  - $j = 4$: similarly, $x_1 = x_4$ eventually forces a pair $x_i = x_{i+1}$.
- **Case $k = 4$:** The pattern $x_1 = x_3 \neq x_2 = x_4$ is possible with no consecutive equal x_i, which is why this construction fails for $k = 4$.

Therefore, for $k = 2, 3, 5$, no proper $2k$-cycle exists in G. □

7 Jan 28

Recall: $ex(n, K_{s,t}) \le c_{s,t} n^{2-\frac{1}{s}} + o(n)$. We will see that $ex(n, K_{s,t}) = \theta\left(n^{2-\frac{1}{s}}\right)$ for $t \ge t_0(s)$

Theorem 7.1. *Construction of the extremal graph for $ex(n, K_{s,t})$.*

Proof. We will first consider a random graph by picking edges with probability $p = n^{-\frac{1}{s}}$. Let $x^{(1)}, ..., x^{(s)}$ be s vertices chosen uniformly at random without replacement on the left. Let $N(x^{(1), ..., x^{(s)}})$ that are the common neighbors with $d(x^{(1)}, ..., x^{(s)}) = |N(...)|$. Let our random variable be $d(x^{(1)}, ..., x^{(s)})$ and is distributed according to

$$Binomial(n, p^s) = Binomial\left(n, \frac{1}{n}\right) \sim Poisson(1).$$

So $\mathbb{E}[d(x^{(1)}, ..., x^{(s)})] = 1$ and $\mathbb{E}[d(...)^k] = o_k(1)$. With probability $\frac{1}{n^\epsilon}$ there is at least $\epsilon \log n$ neighbors. Yet there are $\binom{n}{s}$ such s tuples $(x^{(1)}, ..., x^{(s)})$. and then $\frac{n}{s}$ disjoint. This does not work.

Consider another construction of $K_{s,t}$-free graph with $\Omega(n^{2-\frac{1}{s}})$ edges.
For a bipartite graph with $\mathbb{F}_p^s$ where $n = 2p^s$ on each side we say $x \sim y$ if $f(x, y) = 0$. Pick equation f uniformly at random among all polynomials in s, t of degree at most $d =$ in each of x and y. It would be

$$f = \sum_{|\alpha|, |\beta| \le d} c_{\alpha\beta} x_1^{\alpha_1} x_2^{\alpha_2} ... x_s^{\alpha_s} y_1^{\beta_1} y_2^{\beta_2} ... y_s^{\beta_s}. \qquad (c_{\alpha\beta} \in \mathbb{F}_p,\ |\alpha| = \alpha_1 + \cdots + \alpha_d)$$

Part I: This random graph is locally (neighborhood look "similar") similar to uniform random graph.
Part II: Globally very different/more discrete. $\square$

Lemma 7.2. *For any fixed $x, y \in \mathbb{F}_p^s$*

$$\Pr[x \sim y] = \frac{1}{p}.$$

Proof. We have

$$f = c_{0,0} + \sum_{\alpha, \beta \ne 0} c_{\alpha\beta} x^\alpha b^\beta.$$

For any choice of $c_{\alpha,\beta}$ fir $(\alpha, \beta) \ne (0, 0)$ there is a unique choice of $c_{0,0}$ such that $f(x, y) = 0$.

$$\mathbb{E}[e(G)] = \frac{1}{p} n^2 = c_s n^{2-\frac{1}{s}}.$$

$\square$

Let $x^{(1)}, ..., x^{(s)}$ be s vertices on say left, $N(x^{(1)}, ..., x^{(s)})$ be the common neighbors with of $x^{(1)}, ..., x^{(s)}$ and this is same as $\{y \in \mathbb{F}_p^s : f(x^{(1)}, y) = 0, ..., f(x^{(s)}, y) = 0\}$.

Lemma 7.3. *If g is a random polynomial of degree of D in m variables and $z^{(1)}, ..., z^{(t)} \in \mathbb{F}_p^m$ then the random variables $g(z^{(1)}), ..., g(z^t)$ are independent if $D > t$ and $p > t^2$.*

Warm up: When $m = 1$, write $g(x) = \underbrace{a_0 + a_1 x + \cdots + a_{t-1} x^{t-1}}_{g_{small}(x)} + \underbrace{\cdots + a_D x^D}_{g_{large}(x)}$.

For any choice of $g_{large}(x)$ and any values $b_1, ..., b_t$ there is a unique polynomial g_{small} such that $g(z^{(i)}) = g_{small}(z^{(i)}) + g_{large}(z^{(i)}) = b_i$ for $i = 1, ..., t$ if and only if $g_{small}(z^{(i)}) = b_i - g_{large}(z^{(i)})$ by lagrange interpolation.

For the general case, if $z_1^{(1)}, ..., z_1^{(t)}, ..., z_m^{(1)}, ..., z_m^{(t)}$ are distinct, then $g(x_1, ..., x_m) = \overline{g}(x_1) +$ other terms and use $m = 1 =$ case. Pick a invertible linear transform $f : \mathbb{F}_p^s \to \mathbb{F}_p^s$ such that $Tz^{(1)}, ..., Tz^{(t)}$ are distinct. To pick such a T, let $T(x_1, ..., x_s) = (x_1 + S(x_2, ..., x_s), x_2, ..., x_s)$. For $z^{(i)}, z^{(j)}$ distinct, the space of S for which $T(z^{(i)}) = T(z^{(j)})$ is of co-dimension 1, so a random S has probability $\frac{1}{p}$ of satisfying this. So $\Pr[\exists i, j$ such that $T(z^{(i)}) = T(z^{(j)})] \le \frac{1}{p}\binom{t}{2} < 1$.

We can now conclude from

$$\Pr[f(x^{(1)}, y) = 0, ..., f(x^{(s)}, y) = 0] = \frac{1}{p^s}.$$

If $d > s$ and $p > s^2$ then

$$\mathbb{E}[N(x^{(1)}, ..., x^{(s)})] = 1.$$

Furthermore, if α is a positive integer and $p \cdot s < d$ then

$$\mathbb{E}\left[\left|N(x^{(1)}, ..., x^{(s)})\right|^p\right] = \mathbb{E}\left[\left(\sum_{y \in \mathbb{F}_p^s} R(y)\right)^\alpha\right] = \mathbb{E}\left[\sum_{y^{(1)}, ..., y^{(\alpha)}} R(y^{(1)})...R(y^{(\alpha)})\right].$$

Where $R(y) = \begin{cases} 1 & \textit{if } y \in N(x^{(1)}, ..., x^{(s)}) \\ 0 & \textit{otherwise} \end{cases}$

$= \sum_{y^{(1)}, ..., y^{(\alpha)}} p^{-s\#\textit{distinct } y\textit{'s}} \leq \sum_k \sum_{y^{(1)}, ..., y^{(k)} \in \mathbb{F}_p^s} p^{-sk} = \sum_k p^{sk} p^{-sk} \cdot \#$*ways to partition a objection into k bins.*

Last equality from having to choose $y^{(i)}$ such that k are distinct. This is in $O(1)$. We want to assign values to $y^{(i)}$ such that exactly k are distinct. Equivalently, choose k distinct values in $\mathbb{F}_p^s$. Then choose which of the $y^{(i)}$ go to which bin.

$$\Pr[|N(x^{(1)}, ..., x^{(s)})| \geq T] \leq \frac{\mathbb{E}\left[\left|N(x^{(1)}, ..., x^{(s)})\right|^\alpha\right]}{T^\alpha} = \frac{C_{\alpha,s}}{T^\alpha}.$$

The set $N(x^{(1)}, ..., x^{(s)})$ is contained in $\{y \in \mathbb{F}_p^s : f(x^{(1)}, y) = 0, ..., f(x^{(s)}, y) = 0\}$.
A vector subspace V of $\mathbb{F}_p^s$ of dimension D has p^D elements.

Theorem 7.4. *Lang-Weil Theorem: If $g_1, \ldots, g_r$ are polynomials of degree $\leq D$ in s variables over $\mathbb{F}_p$, then the set $\{y \in \mathbb{F}_p^s : g_1(y) = \cdots = g_r(y) = 0\}$ has either $O_{D,r}(1)$ points, or more than $\frac{p}{2}$ points.*

So $\Pr[\textit{some } s \textit{ vertices have neighborhood of size} \geq O_{s,\alpha}(1)] \leq C_{s,\alpha} \left(\frac{p}{2}\right)^{-\alpha} \binom{p^s}{s}$. *Choose $\alpha = s^2 + 1$. To ensure this probability goes to 0, we need to choose $d = s^2 + 1$.*

8 Regularity Lemmas: Feb 4

Definition 8.1. *Binary word is sequence* $w_1, w_2, ..., w_n \in \{0, 1\}$ *and write* $w = w_1 w_2 ..., w_n$. *The density of* w *is*

$$d(w) = \frac{\#1\text{'s in } w}{n}.$$

Definition 8.2. *Word* w *is* ϵ*-regular if for any subword* w' *of* w *of length* $\geq \epsilon \cdot len(w)$ *satisfies*

$$|d(w) - d(w')| \leq \epsilon.$$

Lemma 8.3. *By Chernoff bound,*

$$\Pr[|d(w') - d(w)| \geq \epsilon] \leq e^{-c_\epsilon n}.$$

9 Feb 6

Definition 9.1. *Partition $\mathcal{P}'$ is a refinement of partition $\mathcal{P}$ if each $w' \in \mathcal{P}'$ is a sub-word of some word in $\mathcal{P}$*

Theorem 9.2. *(Regularity Lemma for Binary Words)*
For all $\epsilon > 0$, every word $w \in \{0,1\}^n$ can be partitioned into $\leq M(\epsilon)$ many sub-words $w^{(1)}, ..., w^{(m)}$ (i.e. $w = w^{(1)}...w^{(m)}$). So that the total length of $w^{(i)}$'s that are ϵ-irregular is at most ϵn.

Proof. For a partition $\mathcal{P}$ of w into sub-words $\mathcal{P} = \{w^{(1)}, ..., w^{(m)}\}$ we define $f(\mathcal{P}) = \sum_{w' \in \mathcal{P}} d(w)^2 |w'|/|w|$. Alternatively, we pick a symbol of w at random, say $i \in [n]$ uniformly, then pick another symbol at random from the same part as 1'st, say j is picked.

$$\Pr[w_i = w_j] = \sum_{w' \in \mathcal{P}} \frac{|w'|}{|w|}(d(w')^2 + (1-d(w'))^2) = 2f(\mathcal{P}) + \sum_{w'} \frac{|w'|}{|w|} - 2\sum_{w'} d(w')\frac{|w'|}{|w|} = 2f(\mathcal{P}) + 1 - 2d(w).$$

Lemma 9.3. *If $\mathcal{P}'$ is a refinement of $\mathcal{P}$ then*

$$f(\mathcal{P}') = f(\mathcal{P}) + \sum_{w' \in \mathcal{P}'} (d(w') - d(\text{parent}(w')))^2 \frac{|w'|}{|w|}.$$

Proof. Expanding the second term we have

$$\sum_{w' \in \mathcal{P}'} d(w')\frac{|w'|}{|w|} - 2d(w')d(\text{parent}(w')) + d(\text{parent}(w'))^2 \frac{|w'|}{|w|} = \sum_1 + \sum_2 + \sum_3 .$$

We have $\sum_1 = f(\mathcal{P}')$ and

$$\begin{aligned}
\sum_2 &= \sum_{\tilde{w} \in \mathcal{P}} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} \left(-2d(w')d(\tilde{w})\frac{|w'|}{|w|}\right) \\
&= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w})\frac{1}{|w|} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} d(w')|w'| \\
&= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w})\frac{1}{|w|} \cdot d(\tilde{w})|\tilde{w}| \\
&= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w})^2 \frac{|\tilde{w}|}{|w|} \\
&= -2f(\mathcal{P})
\end{aligned}$$

Then for the third sum

$$\begin{aligned}
\sum_3 &= \sum_{\tilde{w} \in \mathcal{P}} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} d(\tilde{w})^2 \frac{|w'|}{|w|} \\
&= \sum_{\tilde{w} \in \mathcal{P}} d(\tilde{w})^2 \frac{|\tilde{w}|}{|w|} = f(\mathcal{P})
\end{aligned}$$

So we have $\sum_1 + \sum_2 + \sum_3 = f(\mathcal{P}') - f(\mathcal{P})$ □

Start with the trivial partition (i.e. $\mathcal{P} = \{w\}$) and repeat while $\sum_{w' \in \mathcal{P} w' \text{ is } \epsilon\text{-irregular}} |w'| \leq \epsilon |w|$.
For $w' \in \mathcal{P}$ that is ϵ-irregular, find a way to write as $w' = w^{(1)}w^{(2)}w^{(3)}$ where $|d(w^{(2)}) - d(w^{(1)})| \geq \epsilon$ and $|w^{(2)}| \geq \epsilon|w'|$ Replace w' with the three words $w^{(1)}, w^{(2)}, w^{(3)}$ to obtain a new partition of w, call it $\mathcal{P}'$. Repeat with $\mathcal{P}$ replaced by $\mathcal{P}'$.

To analyze this algorithm, if we drop some words we have

$$f(\mathcal{P}') \geq f(\mathcal{P}) + \sum_{\substack{w' \in \mathcal{P}' \\ w' \text{ is } \epsilon\text{-irregular}}} \sum_{w^{(2)}} (d(w^{(2)} - d(w')))\frac{|w^{(2)}|}{|w|} \geq f(\mathcal{P}) + \sum_{\substack{w' \in \mathcal{P}' \\ w' \text{ is } \epsilon\text{-irregular}}} \epsilon^2 \frac{\epsilon|w'|}{|w|} = f(\mathcal{P}) + \epsilon^4.$$

Note: $f(\mathcal{P}) \leq 1$ for any partition $\mathcal{P}$.
So the number of steps in the algorithm is $\leq \frac{1}{\epsilon^4}$. The theorem holds with $M(\epsilon) = 3^{1/\epsilon^4}$ $\square$

Example 9.4. *Define a twin in a word $w \in \{0,1\}^n$ is a pair of subsequences $x, y \in w$ that are disjoint (no symbol in both x and y) and are equal as word.*
We define

$$t(w) = \text{the length of longest pair twins in } w \text{ and } t(n) = \min_{w \in \{0,1\}^n} t(w).$$

We have the trivial bounds $\frac{n}{4} \leq t(n) \leq \frac{n}{2}$.

Theorem 9.5.

$$t(n) \geq \left(\frac{1}{2} - o(1)\right) n.$$

Proof. Pick small $\epsilon > 0$ and cut w into ϵ-regular words $t \leq \epsilon n$ junk symbols. Enough $t(w) \geq \left(\frac{1}{2} - c\epsilon\right)|w|$ for ϵ-regular w.
Cut w into $\frac{1}{\epsilon}$ equally long sub-words $w^{(1)}, ..., w^{(m)}$ where $m = \frac{1}{\epsilon}$.
The first twin is 0's from $w^{(1)}$+ 1's from $w^{(2)}$+ 0's from $w^{(3)}$...
The second twin is 1's from $w^{(1)}$+ 0's from $w^{(2)}$+ 1's from $w^{(3)}$...
Choose twins then

$$|\#\text{ 0's in } w^{(i)}| - |\#\text{ 1's in } w^{(2)}| \leq \epsilon |w^{(1)}|.$$

So, # symbols not in either of the twins is less than or equal to

$$\epsilon|w|(\text{1's in } w^{(1)}) + \epsilon|w|(\text{1's in } w^{(2)}) + \sum_i 4\epsilon|w^{(i)}|.$$

$\square$

10 Feb 13

Definition 10.1. *For disjoint sets $U, V \subset V(G)$, the density between U and V is*

$$d(U,V) = \frac{e(U,V)}{|U||V|}.$$

Definition 10.2. *A pair (U, V) of disjoint subsets of $V(G)$ is ϵ-regular if $\forall U' \subseteq U, V' \subseteq V$ such that $|U'| \geq \epsilon|U|$ and $|V'| \geq \epsilon|V|$ we have*

$$|d(U,V) - d(U',V')| \leq \epsilon.$$

Definition 10.3. *We say a partition $V(G) = V_1 \cup ... \cup V_k \cup J$ is ϵ-regular partition when*

1. *$|J| \leq \epsilon|V(G)|$*
2. *$|V_1| = |V_2| = ... = |V_k|$*
3. *All except $\leq \epsilon k^2$ pairs (V_1, V_2) are ϵ-regular*

Theorem 10.4. *(Szemeredi's Regularity Lemma) $\forall \epsilon > 0, m$ there exist a constant $M = M(\epsilon, m)$ such that every graph admits an ϵ-regular partition $V(G) = V_1 \cup \cdots \cup V_K \cup J$ where the number of parts $m \leq K \leq M$.*

Remark 10.5. *# edges inside parts + # edges adjacent to J + # edges in ϵ-irregular pairs is at most*

$$k\binom{n/k}{2} + (\epsilon n) \cdot n + \epsilon k^2 (n/k)^2 \leq \frac{1}{k}n^2 + \epsilon n^2 + \epsilon n^2.$$

Proof. For a partition $\mathcal{P}$ of $V := V(G)$, we define

$$f(\mathcal{P}) = \sum_{U,W \in \mathcal{P}, U \neq W} \underbrace{d(U,W)^2 \frac{|U||W|}{|V|^2}}_{f(U,W)}.$$

Lemma 10.6. *Suppose $A, B \subset V$ are disjoint and we have partitions $A = A_1 \cup \cdots \cup A_k$ and $B = B_1 \cup \cdots \cup B_l$ then*

$$\sum_{i,j} f(A_i, B_j) = f(A,B) + \sum_{i,j} (d(A,B) - d(A_i,B_j))^2 \frac{|A_i||B_j|}{|V|^2}.$$

Proof is exactly the same as the proof of the regularity lemma for binary words.

Algorithm: To find an ϵ-regular partition

1. Start with any equipartition of V into m parts.
2. We will start with partitions of the form $V = V_1 \cup V_2 \cup ... \cup V_k \cup J_1 \cup ... \cup J_\ell$
3. For each ϵ-irregular pair (V_i, V_j) there exist partitions $V_i = V_{i,1}^{(i,j)} \cup V_{i,2}^{(i,j)}$ Note: We write is as such since there may be many ϵ-irregular pairs (V_i, V_j) and we want to keep track of them.

$\square$