ALGEBRA I 21-610 SPRING 2026 HOMEWORK III

Homework must be typeset (preferably in TeX) and submitted online in Canvas as a PDF file.

Due by class time on Wed Feb 4.

All groups mentioned may be infinite unless I tell you explicitly that they are finite.

(1) Prove that if G is cyclic of order n then $\operatorname{Aut}(G)$ is isomorphic to the group formed by $\{m : 0 < m < n \text{ and } \gcd(m,n) = 1\}$ with the operation of multiplication mod n. Prove that if n is prime then $\operatorname{Aut}(G)$ is cyclic of order $n-1$.

Let $G = \langle g \rangle$ and $M = \{m : 0 < m < n \text{ and } \gcd(m,n) = 1\}$ To show the first statement, consider $\phi : M \to \operatorname{Aut}(G)$ defined by $\phi(m) = f_m$ where $f_m(k) = k^m$ for $k \in G$. $f_m \in \operatorname{Aut}(G)$ as

$$f_m(g^p \cdot g^q) = g^{m(p+q)} = g^{mp} \cdot g^{mq} = f_m(g^p) \cdot f_m(g^q). \tag{for $p, q \leq n$}$$

To show ϕ is a homomorphism, for all $p, q \in M$, need to show

$$\phi(pq) = \phi(p) \circ \phi(q).$$

We can show this by showing they agree on all values of $k \in G$. We have

$$\phi(pq)(k) = f_{pq}(k) = k^{pq} = f_p(k^q) = f_p(f_q(k)) = \phi(p)(f_q(k)) = \phi(p)(\phi(q)(k)).$$

So ϕ is a homomorphism. To show ϕ is injective, we need to show that if $\phi(p) = \phi(q)$ then $p = q$. We have

$$\phi_p(g^m) = \phi_q(g^m) \iff g^{mp} = g^{mq} \iff mp \equiv mq \pmod{n} \iff p \equiv q \pmod{n}.$$

Last line from $p, q \in M$ as it's relatively prime to n so $p = q$ as $p, q < n$. So ϕ is injective. Since ϕ is injective on finite sets, it's also surjective and thus ϕ is an isomorphism.

To show the second statement, since n is prime, for all $0 < m < n$, $\gcd(m,n) = 1$ so $|M| = n-1$ and $\operatorname{Aut}(G)$ is cyclic of order $n-1$. From previous part, $\operatorname{Aut}(G)$ is isomorphic to M so $\operatorname{Aut}(G)$ is cyclic of order $n-1$ as there must exist an element of order $n-1$ in $\operatorname{Aut}(G)$.

(2) Let H and N be groups and let $\alpha : H \to \mathrm{Aut}(N)$ be a HM. Define a binary operation $\cdot$ on $\{(h, n) : h \in H, n \in N\}$ as follows: $(h_1, n_1) \cdot (h_2, n_2) = (h_1 *_H h_2, \alpha(h_2^{-1})(n_1) *_N n_2)$.

(a) Prove that $\cdot$ makes the given set of ordered pairs into a group which we will denote by $H \ltimes_\alpha N$. Identify 1 and $(h, n)^{-1}$.

(b) Let $H' = H \times 1$ and $N' = 1 \times N$. Prove that $H' \simeq H$, $N' \simeq N$, $H' \leq H \ltimes_\alpha N$, $N' \lhd H \ltimes_\alpha N$, $H' \cap N' = 1$ and $H \ltimes_\alpha N = H'N'$.

(c) Prove that $H \ltimes_\alpha N/N' \simeq H$.

Solutions.

(a) Let H and N be groups and let $\alpha : H \to \mathrm{Aut}(N)$ be a HM. Define a binary operation $\cdot$ on $\{(h, n) : h \in H, n \in N\}$ as follows: $(h_1, n_1) \cdot (h_2, n_2) = (h_1h_2, \alpha(h_2^{-1})(n_1)n_2)$.

(i) Prove that $\cdot$ makes the given set of ordered pairs into a group which we will denote by $H \ltimes_\alpha N$. Identify 1 and $(h, n)^{-1}$.

Proof. **Closure:** Since H and N are groups and $\alpha(h_2^{-1}) \in \mathrm{Aut}(N)$, the product of any two elements results in components within H and N.

Associativity: Let $(h_1, n_1), (h_2, n_2), (h_3, n_3) \in H \ltimes_\alpha N$.

$$\begin{aligned}((h_1, n_1)(h_2, n_2))(h_3, n_3) &= (h_1h_2, \alpha(h_2^{-1})(n_1)n_2)(h_3, n_3)\\ &= (h_1h_2h_3, \alpha(h_3^{-1})[\alpha(h_2^{-1})(n_1)n_2]n_3)\\ &= (h_1h_2h_3, \alpha(h_3^{-1})(\alpha(h_2^{-1})(n_1)) \cdot \alpha(h_3^{-1})(n_2) \cdot n_3)\\ &= (h_1h_2h_3, \alpha((h_2h_3)^{-1})(n_1) \cdot \alpha(h_3^{-1})(n_2) \cdot n_3)\end{aligned}$$

Calculating the other grouping:

$$\begin{aligned}(h_1, n_1)((h_2, n_2)(h_3, n_3)) &= (h_1, n_1)(h_2h_3, \alpha(h_3^{-1})(n_2)n_3)\\ &= (h_1h_2h_3, \alpha((h_2h_3)^{-1})(n_1) \cdot [\alpha(h_3^{-1})(n_2)n_3])\end{aligned}$$

The terms are identical, so associativity holds.

Identity: The identity is $(1_H, 1_N)$.

$$(h, n)(1, 1) = (h \cdot 1, \alpha(1^{-1})(n) \cdot 1) = (h, n)$$

$$(1, 1)(h, n) = (1 \cdot h, \alpha(h^{-1})(1) \cdot n) = (h, 1 \cdot n) = (h, n)$$

Inverses: Let (x, y) be the inverse of (h, n). We require $(h, n)(x, y) = (1, 1)$.

$$(hx, \alpha(x^{-1})(n)y) = (1, 1)$$

From the first component, $hx = 1 \implies x = h^{-1}$. Substituting x into the second component:

$$\alpha((h^{-1})^{-1})(n)y = 1 \implies \alpha(h)(n)y = 1$$

$$y = [\alpha(h)(n)]^{-1} = \alpha(h)(n^{-1})$$

Thus, $(h, n)^{-1} = (h^{-1}, \alpha(h)(n^{-1}))$.

To verify, we have $(h, n)(h^{-1}, \alpha(h)(n^{-1})) = (hh^{-1}, \alpha(h)(n)\alpha(h)(n^{-1})) = (1, \alpha(h)(nn^{-1})) = (1, 1)$. $\square$

(ii) Let $H' = H \times 1$ and $N' = 1 \times N$. Prove that $H' \simeq H$, $N' \simeq N$, $H' \leq H \ltimes_\alpha N$, $N' \lhd H \ltimes_\alpha N$, $H' \cap N' = 1$ and $H \ltimes_\alpha N = H'N'$.

Proof. **Isomorphisms:** The maps $\phi : H \to H'$ via $h \mapsto (h, 1)$ and $\psi : N \to N'$ via $n \mapsto (1, n)$ are clearly bijective homomorphisms (verification omitted for brevity). Thus $H' \simeq H$ and $N' \simeq N$.

Subgroups: H' is a subgroup because $(h_1, 1)(h_2, 1)^{-1} = (h_1, 1)(h_2^{-1}, 1) = (h_1h_2^{-1}, \alpha(h_2)(1) \cdot 1) = (h_1h_2^{-1}, 1) \in H'$. N' is a subgroup because $(1, n_1)(1, n_2)^{-1} = (1, n_1)(1, n_2^{-1}) = (1, \alpha(1)(n_1)n_2^{-1}) = (1, n_1n_2^{-1}) \in N'$.

Normality of N': Let $g = (h, n) \in G$ and $k = (1, m) \in N'$. We compute gkg^{-1}. First, compute gk:

$$(h, n)(1, m) = (h \cdot 1, \alpha(1^{-1})(n)m) = (h, nm)$$

Now multiply by $g^{-1} = (h^{-1}, \alpha(h)(n^{-1}))$:

$$\begin{aligned}(h, nm)(h^{-1}, \alpha(h)(n^{-1})) &= (hh^{-1}, \alpha((h^{-1})^{-1})(nm) \cdot \alpha(h)(n^{-1})) \\ &= (1, \alpha(h)(nm) \cdot \alpha(h)(n^{-1})) \\ &= (1, \alpha(h)(nmn^{-1}))\end{aligned}$$

Since $nmn^{-1} \in N$ and $\alpha(h) \in \mathrm{Aut}(N)$, the second component is in N. Thus $gkg^{-1} \in N'$, so $N' \lhd G$.

Intersection and Product: $H' \cap N' = \{(h, 1)\} \cap \{(1, n)\} = \{(1, 1)\}$. $H'N' = \{(h, 1)(1, n) \mid h \in H, n \in N\} = \{(h, \alpha(1)(1)n)\} = \{(h, n)\} = G$. □

(iii) Prove that $H \ltimes_\alpha N/N' \simeq H$.

Proof. Define $\pi : H \ltimes_\alpha N \to H$ by $\pi(h, n) = h$. This is a homomorphism because:

$$\pi((h_1, n_1)(h_2, n_2)) = \pi(h_1h_2, \dots) = h_1h_2 = \pi(h_1, n_1)\pi(h_2, n_2)$$

The map is surjective (for any h, $\pi(h, 1) = h$) and $\ker(\pi) = \{(h, n) \mid h = 1\} = \{(1, n)\} = N'$. By the First Isomorphism Theorem, $G/N' \simeq H$. □

(3) Let $H \leq G$ and $N \lhd G$ with $H \cap N = 1$ and $HN = G$. Prove that $G \simeq H \ltimes_\alpha N$ for a suitable choice of HM $\alpha : H \to \mathrm{Aut}(N)$.

Solution. Define $\alpha_h(n) = hnh^{-1}$ and this would indeed map from $N \to N$ as $N \lhd G$. This is indeed a homomorphism as

$$a_{h_1 h_2}(n) = h_1 h_2 n h_2^{-1} h_1^{-1} = h_1(h_2 n h_2^{-1})h_1^{-1} = \alpha_{h_1}(\alpha_{h_2}(n)).$$

Consider $\phi : H \ltimes_\alpha N \to G$ defined as $\phi(h, n) = hn$. This is a homomorphism as

$$\begin{aligned}
\phi((h_1, n_1) \cdot (h_2, n_2)) &= \phi(h_1 * h_2, \alpha_{h_2^{-1}}(n_1) * n_2) \\
&= \phi(h_1 * h_2, h_2^{-1} n_1 h_2 n_2) \\
&= h_1 h_2 h_2^{-1} n_1 h_2 n_2 \\
&= h_1 n_1 h_2 n_2 \\
&= \phi(h_1, n_1)\phi(h_2, n_2)
\end{aligned}$$

This is surjective as by assumption $HN = G$ so for any $g \in G$ we can write it as $g = hn$ for some $h \in H$ and $n \in N$.
We also have

$$\ker(\phi) = \{(h, n) \in H \ltimes_\alpha N : \phi(h, n) = e_G\} = \{(h, n) \in H \ltimes_\alpha N : hn = e_G\} = \{e_G\}.$$

The last line is from $H \cap N = 1$ since $n = h^{-1}$ but $h, h^{-1} \in H$ then h has to be identity as otherwise $|H \cap N| \geq 2$. So by first isomorphism theorem we have

$$H \ltimes_\alpha N \simeq G.$$

(4) Let p be an odd prime and let $|G| = 2p$.
  (a) Prove that there exist subgroups H and N with $|H| = 2$ and $|N| = p$.
  (b) Prove that $N \lhd G$, $H \cap N = 1$ and $G = HN$.
  (c) Prove that there are two possible structures for G: G is cyclic of order $2p$ or G is isomorphic to the group of symmetries of a regular p-gon.

Part A. By Sylow's Theorem, we know $\mathrm{Syl}_2(G)$ and $\mathrm{Syl}_p(G)$ are not empty. For any subgroup in $\mathrm{Syl}_2(G)$, the order is 2 and similarly for $\mathrm{Syl}_p(G)$ the order is p.

Part B.

(a) To show $N \lhd G$ we'll show $n_p := |\mathrm{Syl}_p(G)| = 1$. By the third Sylow's theorem, we know $n_p \equiv 1 \pmod p$ and $n_p \mid 2$ so $n_p = 1$. We know that any two Sylow p-subgroups are conjugate and since there is only one Sylow p-subgroup, it must be normal as $\forall g \in G, gNg^{-1} = N$.

(b) To show $H \cap N = 1$, we know $H = \{e, h\}$ for some $h \in H$ and $|h| = 2$, by lagrange's theorem, the order of it's element must divide the order of the group which is p but p is odd prime so $2 \nmid p$ so $H \cap N = 1$.

(c) We can use

$$|HN| = \frac{|H||N|}{|H \cap N|} = \frac{2p}{1} = 2p.$$

So $G = HN$.

Part C. Since $|N| = p$ then N is cyclic so let $N = \langle n \rangle$ then for any $h \in H$ we have $hnh^{-1} \in N$ so $hnh^{-1} = n^k$ for some $k \in \{1, 2, ..., p-1\}$. We know $h^2 = 1$ so $n = h^2nh^{-2}$ and

$$h(hnh^{-1})h^{-1} = h(n^k)h^{-1} = (hnh^{-1})^k = n^{k^2}.$$

So we can conclude $n = h^2nh^{-2} = n^{k^2}$. So $k^2 \equiv 1 \pmod p$, this occurs only when $k = 1$ or $k = -1$.

(a) If $k = 1$ then $hnh^{-1} = n \implies hn = nh$ so G is abelian as all elements can be expressed as $g = hn$ by $G = HN$. Additionally, as G contains an element of order 2 and p then there exist an element of order $2p$ as $\gcd(2, p) = 1$ so G is cyclic. Thus isomorphic to $\mathbb{Z}_{2p}$

(b) If $k = -1$ then $hnh^{-1} = n^{-1} \implies hn = n^{-1}h$ with $|h| = 2$, $|n| = p$ so G is isomorphic to the group of symmetries of a regular p-gon.

(5) Let Q be the group of invertible complex 2×2 matrices generated by A and B where

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(a) Find $|Q|$.
(b) Compute $Z(Q)$ and $[Q, Q]$.
(c) Find an increasing chain $(N_i)_{0 \leq i \leq 3}$ of normal subgroups of Q such that $|N_i| = 2^i$.

(a) We have $|A| = 4$ as

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I, A^3 = A^2(-I) = -A, A^4 = (A^2)^2 = I.$$

Similarly $|B| = 4$,

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I, B^3 = B^2(-I) = -B, B^4 = (B^2)^2 = I.$$

Analyzing the product we have

$$AB = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \text{ and } BA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

So we can conclude $AB = -BA$ and lastly we look at $(AB)^2$

$$(AB)^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

Because of the relation $BA = -AB$, we can take any string of A's and B's (e.g., $BABA^2B$) and move all A's to the left and all B's to the right. Every time we swap a B past an A, we just change the sign. Thus there are 8 elements in Q namely, $I, -I, A, -A, B, -B, AB, -AB$.

(b) Clearly, $I, -I \in Z(Q)$ and $A, -A, B, -B$ not in $Z(Q)$ as $AB = -BA$ so $AB \neq BA$. For AB we have $(AB)A = (-BA)A = B$ but $A(AB) = A(-BA) = -B$. So AB and $-AB$ are not in $Z(Q)$. So $Z(G) = \{I, -I\}$.

We know $|Q/\{I, -I\}| = \frac{8}{2} = 4$ so $Q/\{I, -I\}$ is abelian as it's order is a prime squared. We can conclude $[Q, Q] \leq \{I, -I\}$. Take $A, B \in Q$ then we have $[A, B] = ABA^{-1}B^{-1} = AB(-A)(-B) = (AB)^2 = -I$ and $[I, A] = IAI^{-1}A^{-1} = I$ so $[Q, Q] = \{I, -I\}$.

(c) Trivially, $N_0 = \{I\}$, $N_1 = Z(Q) = \{I, -I\}$, and $N_3 = Q$. We can choose a subgroup say $K = \{I, -I, A, -A\}$ as we've shown earlier in part a this is a subgroup ($A^i \in K$). As $|Q/K| = 2$ then $K \triangleleft Q$ so we're done by letting $N_2 = K$

(6) Recall from class that:
- A *central series* in a group G is a normal series $(H_i)_{0\leq i\leq t}$ such that $H_{i+1}/H_i \leq Z(G/H_i)$ for all $i < t$, equivalently $[G, H_{i+1}] \leq H_i$ for all $i < t$.
- G is *nilpotent* if and only if it has a central series.
- The *ascending central series* is defined by the recursion $Z_0(G) = 1$, $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$. It's an increasing sequence of characteristic subgroups.
- The *descending central series* is defined by the recursion $\gamma_1(G) = G$, $\gamma_{n+1}(G) = [G, \gamma_n(G)]$. It's a decreasing sequence of characteristic subgroups.

Let G be a nilpotent group with a central series $(H_i)_{0\leq i\leq t}$. Prove that

(a) $H_i \subseteq Z_i(G)$ for all i, so that $Z_t(G) = G$.
(b) $\gamma_i(G) \subseteq H_{t-i+1}$ for all i, so that $\gamma_{t+1}(G) = 1$.
(c) The least n such that $Z_n(G) = G$, the least n such that $\gamma_{n+1}(G) = 1$, and the least n such that there is some central series $(K_j)_{0\leq j\leq n}$ are all equal.

Solutions.

(a) We prove $H_i \subseteq Z_i(G)$ by induction on i.
Base Case ($i = 0$): $H_0 = 1$ and $Z_0(G) = 1$, so $H_0 \subseteq Z_0(G)$ holds trivially.
Inductive Step: Assume $H_i \subseteq Z_i(G)$. By the definition of a central series, $[G, H_{i+1}] \subseteq H_i$. By the induction hypothesis, $H_i \subseteq Z_i(G)$, so $[G, H_{i+1}] \subseteq Z_i(G)$.
Recall the definition of the upper central series: $Z_{i+1}(G)$ is the subgroup such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. Equivalently, $x \in Z_{i+1}(G) \iff [G, x] \subseteq Z_i(G)$.
Since $[G, H_{i+1}] \subseteq Z_i(G)$, it follows by definition that $H_{i+1} \subseteq Z_{i+1}(G)$.
Thus, by induction, $H_i \subseteq Z_i(G)$ for all i. For $i = t$, since $H_t = G$, we have $G \subseteq Z_t(G)$, which implies $Z_t(G) = G$.

(b) We prove $\gamma_i(G) \subseteq H_{t-i+1}$ by induction on i.
Base Case ($i = 1$): $\gamma_1(G) = G$ and $H_{t-1+1} = H_t = G$. Thus $\gamma_1(G) \subseteq H_t$ holds.
Inductive Step: Assume $\gamma_i(G) \subseteq H_{t-i+1}$. By definition, $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. Using the induction hypothesis:

$$\gamma_{i+1}(G) = [G, \gamma_i(G)] \subseteq [G, H_{t-i+1}]$$

Since (H_k) is a central series, we know $[G, H_k] \subseteq H_{k-1}$. Letting $k = t - i + 1$, we get:

$$[G, H_{t-i+1}] \subseteq H_{t-i} = H_{t-(i+1)+1}$$

Therefore, $\gamma_{i+1}(G) \subseteq H_{t-(i+1)+1}$.
By induction, the inclusion holds for all i. In particular, for $i = t + 1$, we have $\gamma_{t+1}(G) \subseteq H_{t-(t+1)+1} = H_0 = 1$. Thus $\gamma_{t+1}(G) = 1$.

(c) Let n_Z be the least integer such that $Z_{n_Z}(G) = G$. Let n_γ be the least integer such that $\gamma_{n_\gamma+1}(G) = 1$. Let n_{min} be the minimal length of any central series of G.
Since the Upper Central Series (Z_i) is itself a central series of length

n_Z, we must have $n_{min} \leq n_Z$.
Let $(H_i)_{0\leq i\leq t}$ be any central series. From part (a), we know $H_t \subseteq Z_t(G)$. Since $H_t = G$, $Z_t(G) = G$. This implies $n_Z \leq t$. Since this holds for *any* central series of length t, it holds for the minimal one, so $n_Z \leq n_{min}$.
Combining the above, $n_Z = n_{min}$.
From part (b), given any central series of length t, $\gamma_{t+1}(G) = 1$. This implies $n_\gamma \leq t$. Taking the minimal central series (where $t = n_{min}$), we get $n_\gamma \leq n_{min}$.
Conversely, consider the series defined by the lower central series: set $K_j = \gamma_{n_\gamma - j+1}(G)$ for $0 \leq j \leq n_\gamma$. Then $K_0 = \gamma_{n_\gamma+1}(G) = 1$ and $K_{n_\gamma} = \gamma_1(G) = G$. Check the central condition: $[G, K_{j+1}] = [G, \gamma_{n_\gamma - j}(G)] = \gamma_{n_\gamma - j+1}(G) = K_j$. Thus, this is a valid central series of length n_γ. Since n_{min} is the minimal length of any central series, $n_{min} \leq n_\gamma$.
Combining these inequalities, $n_\gamma = n_{min}$.
Conclusion: $n_Z = n_{min} = n_\gamma$.

(7) Let U_n be the group of $n \times n$ real invertible upper triangular matrices with 1's on the diagonal, and let B_n be the group of all $n \times n$ real invertible upper triangular matrices. Prove that:
  (a) $U_n \lhd B_n$.
  (b) U_n is solvable.
  (c) B_n/U_n is abelian.
  (d) B_n is solvable.

Solutions.

(a) Define a map $\phi : B_n \to (\mathbb{R}^\times)^n$ (where $(\mathbb{R}^\times)^n$ is the group of diagonal matrices under multiplication) by $\phi(A) = (A_{11}, A_{22}, \dots, A_{nn})$.
This is a homomorphism because for upper triangular matrices A and B, the diagonal entries of the product are the products of the diagonal entries: $(AB)_{ii} = A_{ii}B_{ii}$.
The kernel of this map is the set of matrices where every diagonal entry is 1, which is exactly U_n. Since the kernel of any homomorphism is a normal subgroup, $U_n \lhd B_n$.

(b) We prove U_n is solvable by induction on n.
Base Case ($n = 2$): $U_2 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$. This group is isomorphic to the additive group of real numbers $(\mathbb{R}, +)$ via the map $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto x$. Since $(\mathbb{R}, +)$ is abelian, U_2 is solvable.
Inductive Step: Assume U_n is solvable. Consider the homomorphism $\psi : U_{n+1} \to U_n$ defined by removing the last row and the last column (mapping the $(n+1) \times (n+1)$ matrix to its top-left $n \times n$ block).
Let $A = \begin{pmatrix} A' & v \\ 0 & 1 \end{pmatrix} \in U_{n+1}$, where $A' \in U_n$ and $v \in \mathbb{R}^n$. Then $\psi(A) = A'$. This map is surjective. The kernel is:

$$K = \ker(\psi) = \left\{ \begin{pmatrix} I_n & v \\ 0 & 1 \end{pmatrix} : v \in \mathbb{R}^n \right\}.$$

We can observe that $K \cong (\mathbb{R}^n, +)$, which is an abelian group. Thus K is solvable.
By the First Isomorphism Theorem, $U_{n+1}/K \cong U_n$. Since both K (the kernel) and U_{n+1}/K (the quotient, isomorphic to U_n) are solvable, the group U_{n+1} is solvable.
Thus, by induction, U_n is solvable for all $n \geq 1$.

(c) To show B_n/U_n is abelian, it suffices to show that the derived subgroup $[B_n, B_n]$ is contained in U_n.
Let $A, B \in B_n$. Since diagonal entries of upper triangular matrices commute, for any $1 \leq i \leq n$:

$$(ABA^{-1}B^{-1})_{ii} = A_{ii}B_{ii}A_{ii}^{-1}B_{ii}^{-1} = A_{ii}A_{ii}^{-1}B_{ii}B_{ii}^{-1} = 1.$$

Since all diagonal entries of the commutator $ABA^{-1}B^{-1}$ are 1, the commutator belongs to U_n. Therefore, $[B_n, B_n] \leq U_n$, which implies B_n/U_n is abelian.

(d) We recall the property that if $N \lhd G$ and both N and G/N are solvable, then G is solvable.

From part (a), $U_n \lhd B_n$. From part (b), U_n is solvable. From part (c), B_n/U_n is abelian, and all abelian groups are solvable.
Therefore, B_n is solvable.