# Extremal Combinatorics Notes

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**Definition 1.1.** *Turan Numbers (Forbidden Subgraph Problems): $F$ is a graph and $G$ graph is $F$-free if $G$ contains no copy of $F$ as a subgraph.*

*We want to maximize the size of $G$ subject to $G$ being $F$-free. Where size=$e(G)$ = # edges in $G$*

**Definition 1.2.** $ex(n, F) = \max\{e(G)|G \text{ is } F\text{-free and } G \text{ is } n\text{-vertex graph}\}$

**Theorem 1.3.** *Crude Turan's Theorem: If $G$ is $n$-vertex graph,*

$$e(G) = \epsilon \binom{n}{2}.$$

*then $G$ contains an independent set of size greater than or equal $\frac{1}{2}\frac{1}{\epsilon} + \frac{1}{2}$*
*In particular, if $e(G) \geq (1-\epsilon)\binom{n}{2}$ edges, then $G$ contains a clique on greater than or equal $\frac{1}{2\epsilon} + \frac{1}{2}$ vertices.*

$$ex(n, K_t) \leq \left(1 - \frac{1}{2t-1}\right)\binom{n}{2}.$$

*Proof.* Pick $k$ vertices in $G$ at random without replacement. Say we pick $v_1, ..., v_k$
For each edge between $v_i$ and $v_j$, delete $v_i$.
Let $X = $ # deleted vertices
Note: $X \leq e(G[v_1, ..., v_k]) =: Y$

$$\mathbb{E}[Y] = \sum_{1 \leq i \leq j \leq k} \mathbb{P}[v_i v_j \in e(G)] = \binom{k}{2}\frac{\epsilon\binom{n}{2}}{\binom{n}{2}}s = \epsilon\binom{k}{2}.$$

Thus, there is a choice of $v_1, ..., v_k$ such that $Y \leq \epsilon\binom{k}{2}$. Fix it. Let $\alpha(G) = $ size of largest independent set in $G$ satisfies $\alpha(G) \geq k - Y \geq k - \epsilon\binom{k}{2}$

$$\left(k - \epsilon\binom{k}{2}\right)' = 1 + \frac{\epsilon}{2} - \epsilon k.$$

Set $k = \frac{1+\frac{\epsilon}{2}}{2} + \delta$ for some $|\delta| \leq \frac{1}{2}$.

$$\alpha(G) \geq \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon\left(\left(\frac{1}{\epsilon} + \frac{1}{2} + \delta\right)\left(\frac{1}{\epsilon} - \frac{1}{2} + \delta\right)\right)/2$$

$$= \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon\frac{(\frac{1}{\epsilon}+\delta)^2 - \frac{1}{4}}{2}$$

$$= \frac{1}{\epsilon} + \frac{1}{2} + \delta - \epsilon\frac{(1/\epsilon)^2 + 2\frac{1}{\epsilon}\delta + \delta^2 - \frac{1}{4}}{2}$$

$$= \frac{1}{2}\frac{1}{\epsilon} + \frac{1}{2} + \frac{\frac{1}{4} - \delta^2}{2} \geq \frac{1}{2}\frac{1}{\epsilon} + \frac{1}{2}$$

$\square$

**Remark 1.4.** *Recall: $k$-uniform hypergraph on vertex set $V$ is a subset of*

$$\binom{V}{k} := \{e \subseteq V : |e| = k\}.$$

**Example 1.5.** *In $k$-uniform hypergraph on vertex with $\epsilon\binom{n}{k}$ edges there is an independent set of size $\geq c_k \frac{1}{\epsilon}^{\frac{1}{k-1}}$*

**Theorem 1.6.** *Mantel's Theorem: $ex(n, K_3) = \lfloor\frac{n^2}{4}\rfloor$*
*More generally, define Turan's graph $T_{n,r}$ to be the $n$-vertex complete $r$-partite graph with part sizes as equal as possible.*
*Note: $K_{r+1} \nsubseteq T_{n,r}$ because among $r+1$ vertices some pair is in same part.*

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**Theorem 1.7.** *Turan's Theorem: $ex(n, K_{r+1}) = e(T_{n,r})$ and furthermore $T_{n,r}$ is the only extremizer.*

$$e(T_{n,r}) = \left(1 - \frac{1}{r}\right)\binom{n}{2} + O(n).$$

*Proof.* Let $G$ be a graph on $n$ vertices that is $K_{r+1}$-free and has the maximum possible number of edges (an extremizer). We proceed in two steps.

Suppose $G$ is not a complete multipartite graph. Then non-adjacency is not an equivalence relation, implying there exist two non-adjacent vertices $x, y$ that do not have identical neighborhoods.

Assume $\deg(x) \leq \deg(y)$. Delete $x$ and then clone $y$ to obtain $y'$. The new graph $G'$ has

$$e(G') = e(G) - \deg(x) + \deg(y) \geq e(G).$$

Also, $G'$ is $K_{r+1}$-free because any copy of $K_{r+1}$ has at most one of $y$ or $y'$ (since they are non-adjacent). If a clique contains $y'$, it can be swapped for $y$ (since $N(y') \subseteq N(y)$), which would imply $K_{r+1} \subseteq G$, a contradiction.

Repeating this process transforms $G$ into a complete multipartite graph without decreasing edges. Thus, the extremizer must be a complete multipartite graph.
Since $G$ is $K_{r+1}$-free, it must be $k$-partite with $k \leq r$. To maximize edges, we set $k = r$. Let the parts have sizes $n_1, \ldots, n_r$.

$$e(G) = \binom{n}{2} - \sum_{i=1}^{r} \binom{n_i}{2}$$

To maximize $e(G)$, we must minimize the subtracted sum of pairs inside the parts. This occurs when $n_i$ are as equal as possible ($|n_i - n_j| \leq 1$). If $n_i \geq n_j + 2$, moving a vertex from part $i$ to part $j$ increases the edge count by $(n_i - 1) - n_j > 0$.

Thus, the unique extremizer is the balanced complete $r$-partite graph, $T_{n,r}$. $\square$

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*Proof.* Second proof of Turan's Theorem with induction on $r$.

(Induction Step) Let $x$ be a vertex of largest degree, let $A = N(x)$ be its neighborhood.

$$B := V(G) \setminus A.$$

We have $e(G) = e(A) + e(A, B) + e(B)$ and $A$ is $K_{r-1}$-free so $e(A) \le e(T_{|A|,r})$

Each vertex in $B$ has $\deg \le \deg(x) = |A|$

So,

$$e(A, B) + e(B) \le \sum_{v \in B} \deg(v) \le |B||A|.$$

So,

$$e(G) \le e(T_{|A|,r}) + |A||B| \le e(T_{n,r}).$$

The right hand side is equal to number of edges in complete r-partite graph with one part of size $|B|$ and other being as equal as possible.

Equality holds iff $B$ is independent and $A$ must be a copy of $e(T_{|A|,r-1})$. Hence $G$ is $T_{n,r}$ for equality to hold. $\square$

**Theorem 2.1.** *Observe that if $F$ is a graph then*

$$\frac{ex(n, F)}{\binom{n}{2}} \ge \frac{ex(n+1, F)}{\binom{n+1}{2}}.$$

*Proof.* Pick $n$ vertices in an $F$-free $(n+1)$ vertex graph $G$ at random without replacement; let $G'$ be the induced graph on these $n$ vertices.

$$e(G') \le ex(n, F).$$

$$\mathbb{E}(e(G')) = \binom{n}{2} \frac{e(G)}{\binom{n+1}{2}} \implies \frac{e(G)}{\binom{n+1}{2}} \le \frac{ex(n, F)}{\binom{n}{2}}.$$

$\square$

**Definition 2.2.**

$$\pi(F) := \lim_{n \to \infty} \frac{ex(n, F)}{\binom{n}{2}}.$$

*The Turan density of $F$ is*

$$\pi(K_{r+1}) = 1 - \frac{1}{r}.$$

**Theorem 2.3.** *Let $F$ be any graph and let $\epsilon > 0$. Let $G$ be an $n$-vertex graph. There is a $\delta > 0$ such that*

$$e(G) \ge (\pi(F) + \epsilon) \binom{n}{2} \text{ and } n \ge n_0(F, \epsilon).$$

*then there are at least $\delta n^{|V(F)|}$ copies of $F$.*

*Proof.* Let $n_0$ be large enough such that $\frac{ex(n_0, F)}{\binom{n_0}{2}} \le \pi(F) + \frac{\epsilon}{2}$. Given $n$-vertex $G$ as in theorem. Pick $n_0$ vertices at random without replacement. Let $G'$ be the $n_0$-vertex graph induced on these chosen vertices.

$$\mathbb{E}[e(G')] = \frac{e(G)}{\binom{n}{2}} \binom{n_0}{2} \ge (\pi(F) + \epsilon) \binom{n_0}{2}.$$

Say $G'$ is good if

$$e(G') \ge \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2}.$$

$$\mathbb{E}(e(G')) \le \mathbb{P}(G' \text{ is bad}) \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2} + \mathbb{P}(G' \text{ is good}) \binom{n_0}{2}$$

$$\le \left(\pi(F) + \frac{\epsilon}{2}\right) \binom{n_0}{2} + \mathbb{P}(G' \text{ is good}) \binom{n_0}{2}$$

Then $\frac{\epsilon}{2}\binom{n_0}{2} \leq \mathbb{P}(G' \text{ is good})\binom{n_0}{2}$ so $P(G' \text{ is good}) \geq \frac{\epsilon}{2}$

Pick $|V(F)|$ many vertices without replacement inside $G'$, call the subgraph they induce $G''$. So

$$G \supset G' \supset G''.$$

$$\mathbb{P}(G'' \text{ contains } F) \geq \mathbb{P}(G' \text{ is good}) \cdot \mathbb{P}(G'' \text{ contains } F | G' \text{ is good}) \geq \frac{\epsilon}{2}\frac{1}{\binom{n_0}{|V(F)|}}.$$

Note that $G''$ is uniformly randomly chosen $|V(F)|$-vertex subset of $G$. This implies the # copies of $F$ in $G \geq \mathbb{P}(G'' \text{contains F})\binom{n}{|V(F)|} \geq P(...)\frac{n^{|V(F)|}(1-o(1))}{|V(F)|!}$. We can choose $\delta = \frac{1}{\binom{n_0}{|V(F)|}}$  $\square$

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**Theorem 3.1** (Erdős-Stone-Simonovits). *Let $T_{n,r}$ denote the Turán graph. We know that:*

$$ex(n, T_{s,(r+1)}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

*This implies that if $H$ is any graph with chromatic number $\chi(H) = r + 1$, then*

$$ex(n, H) \leq \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

*Conversely, if $\chi(H) > r$ (i.e., $H$ is not $r$-partite), then $T_{n,r}$ is $H$-free. Hence,*

$$ex(n, H) \geq e(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

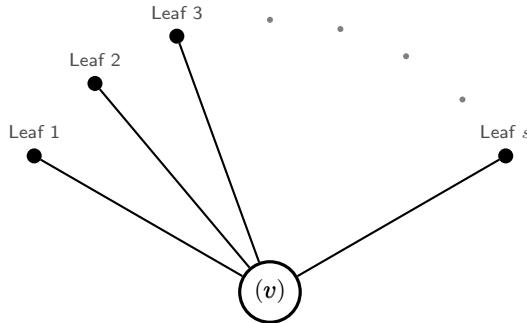*Combining these, for any non-bipartite graph $H$:*

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}.$$

*It is important to note that this theorem holds for bipartite graphs as well, where $\chi(H) = 2$. However, substituting $r = \chi(H) - 1 = 1$ gives a coefficient of $1 - 1/1 = 0$. This implies that $ex(n, H) = o(n^2)$, but it does not provide the precise asymptotic behavior (e.g., $n^{2-\epsilon}$). Therefore, while valid, the Erdős-Stone-Simonovits theorem is not the primary tool for determining extremal numbers of bipartite graphs; we rely on theorems like Kővári-Sós-Turán for those bounds.*

**Theorem 3.2** (Kővári-Sós-Turán 1954).

$$ex(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{\frac{1}{s}} \cdot n^{2-\frac{1}{s}} + O(n).$$

*Proof.* Let $G$ be a $K_{s,t}$-free graph with $n$ vertices. We will count $X$, the number of copies of $K_{1,s}$ (stars with $s$ leaves).



We count this in two ways:

1. Since there is no $K_{s,t}$, any set of $s$ vertices can have at most $t - 1$ common neighbors. Thus:

$$X \leq \binom{n}{s}(t-1).$$

2. Let $d_v$ be the degree of vertex $v$. Then:

$$X = \sum_{v \in V(G)} \binom{d_v}{s}.$$

A simple-minded application of Jensen's inequality here fails because the binomial coefficient $\binom{x}{s}$ is not convex for all $x$. To fix this, we define the function:

$$f_s(x) = \begin{cases} \binom{x}{s} & \text{if } x \geq s - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f_s(x)$ is convex for all $x$ because its derivative:

$$\frac{d}{dx}f_s(x) = \begin{cases} \left(\frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x-s+1}\right)\binom{x}{s} & \text{if } x > s-1 \\ 0 & \text{otherwise} \end{cases}$$

is non-decreasing. Since $d_v$ is an integer, $\binom{d_v}{s} = f_s(d_v)$. We can now validly apply Jensen's inequality:

$$X = \sum_{v \in V} f_s(d_v) \geq nf_s(d_{\text{avg}}) = nf_s\left(\frac{2m}{n}\right),$$

where $m = e(G)$.

Combining (1) and (2):

$$nf_s\left(\frac{2m}{n}\right) \leq \binom{n}{s}(t-1).$$

If $m \leq \frac{1}{2}(s-1)n$, the bound holds trivially. Otherwise, we approximate $\binom{x}{s} \approx \frac{x^s}{s!}$:

$$n\frac{\left(\frac{2m}{n} - s\right)^s}{s!} \leq (t-1)\frac{n^s}{s!}.$$

Rearranging yields:

$$\frac{2m}{n} \leq (t-1)^{\frac{1}{s}}n^{1-\frac{1}{s}} + s \implies m \leq \frac{1}{2}(t-1)^{\frac{1}{s}}n^{2-\frac{1}{s}} + O(n).$$

$\square$

**Example 3.3** (Unit Distance Problem). *How can we place $n$ points in $\mathbb{R}^2$ to maximize the number of pairs $(p,q)$ such that $||p - q|| = 1$?*

*A strong construction is a $\sqrt{n} \times \sqrt{n}$ grid. Rather than scaling the grid to unit step size, we scale it such that the distance $1$ corresponds to a distance $d$ in the integer grid that occurs most frequently.*

*The possible distances in an unscaled integer grid are $\sqrt{a^2 + b^2}$ for $0 \leq a, b < \sqrt{n}$. By number theory (Landau-Ramanujan), integers representable as a sum of two squares are those where primes $p \equiv 3$ (mod 4) appear with even exponents.*

*To get a better bound, we look for "highly composite" integers. Specifically, if we choose a number $d$ composed of a product of many distinct primes $p \equiv 1$ (mod 4), the number of ways to write $d$ as a sum of two squares is very large. Erdős used this to show there exists a distance occurring:*

$$n^{1+\frac{c}{\log\log n}}.$$

*This is significantly larger than the $n\sqrt{\log n}$ bound derived from the Pigeonhole Principle on generic integers.*

**Theorem 3.4** (Upper Bound for Unit Distances). *The number of unit distances is $O(n^{3/2})$.*

*Proof.* Given $n$ points in $\mathbb{R}^2$, construct a graph where vertices are points and $v \sim u$ if $||v - u|| = 1$. Observe the graph is $K_{2,3}$-free. Geometrically, this is because two circles with radius 1 intersect at most 2 times; thus, no 2 vertices can share 3 common neighbors.
By KST with $s = 2, t = 3$:

$$e(G) \leq \frac{1}{2}\sqrt{3-1} \cdot n^{2-\frac{1}{2}} + o(n^2) = \frac{1}{\sqrt{2}}n^{\frac{3}{2}} + o(n^2).$$

$\square$

**Definition 3.5.** *Define $K^{(3)}_{s_1,s_2,t}$ to be the complete 3-uniform, 3-partite hypergraph with parts of size $s_1, s_2, t$.*

**Theorem 3.6** (Generalization of KST to Hypergraphs). *For the 3-uniform case:*

$$ex(n, K^{(3)}_{s_1,s_2,t}) \leq C_{s_1,s_2,t} n^{3-\frac{1}{s_1 s_2}} + O(n^2).$$

*Proof.* We will count the number of stars $K_{1,1,s_1}$ (sets of edges sharing a common pair of vertices). Let $X$ be this number. For any pair $\{v_1, v_2\}$, let $d_{v_1,v_2}$ be the co-degree (number of edges containing $\{v_1, v_2\}$).

1. **Upper Bound on X:** Fix a set $U$ of size $s_1$. Define an auxiliary graph $G_U$ on $V(G)$ where $v_1 \sim v_2$ in $G_U$ if $\{v_1, v_2, u\} \in E(G)$ for all $u \in U$. If $G_U$ contains a copy of $K_{s_2,t}$, then combined with $U$, we form a $K^{(3)}_{s_1,s_2,t}$. Thus, $G_U$ must be $K_{s_2,t}$-free.

$$X \leq \binom{n}{s_1} ex(n, K_{s_2,t}) \leq C\binom{n}{s_1} n^{2-\frac{1}{s_2}}.$$

2. **Lower Bound on X:** Summing over pairs:

$$X = \sum_{\{v_1,v_2\} \in \binom{V}{2}} \binom{d_{v_1,v_2}}{s_1} \geq \binom{n}{2} f_{s_1}\left(\frac{3m}{\binom{n}{2}}\right).$$

Combining bounds (assuming $m$ is large enough):

$$\binom{n}{2} \frac{\left(\frac{3m}{n^2/2}\right)^{s_1}}{s_1!} \lesssim \frac{n^{s_1}}{s_1!} n^{2-\frac{1}{s_2}}.$$

Simplifying:

$$n^2 \left(\frac{m}{n^2}\right)^{s_1} \lesssim n^{s_1+2-\frac{1}{s_2}} \implies m^{s_1} \lesssim n^{3s_1-\frac{1}{s_2}}.$$

Taking the $s_1$-th root:

$$m \leq Cn^{3-\frac{1}{s_1 s_2}}.$$

$\square$

# 4 Jan 21: Supersaturation and Generalized Turán Problems

## 4.1 Supersaturation

The classical Turán problem asks for the maximum number of edges in a graph avoiding a subgraph $F$. Supersaturation asks: what happens if we have just *slightly* more edges than this threshold? It turns out we don't just get one copy of $F$, but exponentially many.

**Lemma 4.1** (Supersaturation Lemma). *If $G$ is a graph on $n$ vertices with $e(G) \geq (\pi(F) + \epsilon)\binom{n}{2}$ edges, then $G$ contains at least $\delta(\epsilon)n^{|V(F)|}$ copies of $F$, where $\delta(\epsilon) > 0$ is a constant independent of $n$.*

**Corollary 4.2.** *For the complete graph $K_{r+1}$, we know $\pi(K_{r+1}) = 1 - \frac{1}{r}$. Thus:*

$$e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right)\binom{n}{2} \implies G \text{ contains } \delta(\epsilon)n^{r+1} \text{ copies of } K_{r+1}.$$

## 4.2 Hypergraph Kővári-Sós-Turán

We generalize the KST theorem from bipartite graphs to $r$-uniform hypergraphs. We are looking for the Turán number of the complete $r$-partite hypergraph $K_{s_1,\ldots,s_r}^{(r)}$. Note that the theorem statement often simplifies the last parameter to $t$.

**Theorem 4.3** (Generalized Kővári-Sós-Turán). *Let $K = K_{s_1,s_2,\ldots,s_{r-1},t}^{(r)}$ be the complete $r$-partite $r$-uniform hypergraph with part sizes $s_1,\ldots,s_{r-1},t$. Then:*

$$ex(n, K) = O_{s,t}\left(n^{r - \frac{1}{s_1 s_2 \cdots s_{r-1}}}\right).$$

*Proof.* We proceed by induction on the uniformity $r$. The base case $r = 2$ is the standard graph KST theorem. Assume the bound holds for $(r-1)$-uniform hypergraphs.

Let $G$ be an $r$-uniform hypergraph with $n$ vertices and $m$ edges. We define a "hypergraph star" $K_{1,1,\ldots,s_1}^{(r)}$ as a set of $s_1$ edges that all share a common core of $r - 1$ vertices.

Let $X$ be the number of such stars (copies of $K_{1,1,\ldots,1,s_1}^{(r)}$) in $G$. We double count $X$.

**1. Lower Bound (Convexity):** For every subset $S \in \binom{V(G)}{r-1}$ of size $r - 1$, let the *co-degree* $d_S$ be the number of edges in $G$ containing $S$. A set $S$ contributes $\binom{d_S}{s_1}$ stars to the count.

$$X = \sum_{S \in \binom{V(G)}{r-1}} \binom{d_S}{s_1}.$$

Using Jensen's Inequality (and the convexity of $\binom{x}{s_1}$), we bound this sum:

$$X \geq \binom{n}{r-1}\binom{\bar{d}}{s_1}, \quad \text{where } \bar{d} = \frac{\sum d_S}{\binom{n}{r-1}} = \frac{rm}{\binom{n}{r-1}}.$$

Approximating for large $n$: $\bar{d} \approx \frac{r!m}{n^{r-1}}$ and $\binom{x}{k} \approx \frac{x^k}{k!}$, we get:

$$X \gtrsim n^{r-1} \cdot \left(\frac{m}{n^{r-1}}\right)^{s_1}. \tag{1}$$

**2. Upper Bound (Induction):** Fix a set of vertices $V_1 = \{v_1, \ldots, v_{s_1}\}$. We ask: how many ways can we complete this into a star?

Define an auxiliary $(r-1)$-uniform hypergraph $G_{V_1}$ on $V(G)$. A set $e'$ of size $r - 1$ is an edge in $G_{V_1}$ if $e' \cup \{v_i\} \in E(G)$ for all $i = 1, \ldots, s_1$. In other words, $e'$ is in the common neighborhood of all $v_i$.

If $G_{V_1}$ contains a copy of $K_{s_2,\ldots,s_{r-1},t}^{(r-1)}$, then combined with $V_1$, we would form a copy of the forbidden $K_{s_1,\ldots,t}^{(r)}$ in $G$. Thus, $G_{V_1}$ must be free of $K_{s_2,\ldots,t}^{(r-1)}$. By the inductive hypothesis:

$$e(G_{V_1}) \leq C \cdot n^{(r-1) - \frac{1}{s_2 \cdots s_{r-1}}}.$$

Summing over all choices of $V_1$ (there are $\binom{n}{s_1}$ such sets):

$$X \leq \binom{n}{s_1} \cdot \max_{V_1} e(G_{V_1}) \lesssim n^{s_1} \cdot n^{r-1 - \frac{1}{s_2 \cdots s_{r-1}}}. \tag{2}$$

**3. Conclusion:** Comparing (1) and (2):

$$n^{r-1}\left(\frac{m}{n^{r-1}}\right)^{s_1} \lesssim n^{s_1+r-1-\frac{1}{s_2\cdots s_{r-1}}}.$$

Rearranging for $m$:

$$m^{s_1} \lesssim n^{(r-1)s_1} \cdot n^{s_1-\frac{1}{s_2\cdots s_{r-1}}} = n^{rs_1-\frac{1}{s_2\cdots s_{r-1}}}.$$

Taking the $s_1$-th root:

$$m \lesssim n^{r-\frac{1}{s_1 s_2\cdots s_{r-1}}}. \qquad \square$$

## 4.3 The Erdős-Stone Theorem

**Definition 4.4** (Blow-up). *A **blow-up** of a graph $H$, denoted $H(s)$, is obtained by replacing each vertex $v \in V(H)$ with an independent set $I_v$ of size $s$, and replacing each edge $(u,v) \in E(H)$ with a complete bipartite graph between $I_u$ and $I_v$.*
*The Turán graph $T_{n,r}$ is essentially a blow-up of $K_r$ with parts of size $n/r$.*

**Theorem 4.5** (Erdős-Stone). *If $e(G) \geq \left(1 - \frac{1}{r} + \epsilon\right)\binom{n}{2}$ and $n$ is sufficiently large, then $G$ contains $T_{s,(r+1)}$ (a blow-up of $K_{r+1}$ with parts of size $s$).*
*Consequently, $G$ contains any graph $H$ with $\chi(H) = r + 1$ (since any such $H$ is a subgraph of a large enough blow-up of $K_{r+1}$).*

*Proof.* **Step 1: Find many cliques.** By the Supersaturation Corollary, since the edge density is greater than the Turán threshold for $K_{r+1}$, $G$ contains many copies of $K_{r+1}$:

$$\#K_{r+1} \geq \delta n^{r+1}.$$

**Step 2: Construct Auxiliary Hypergraph.** Define an $(r + 1)$-uniform hypergraph $\mathcal{H}$ with vertex set $V(G)$. Let a set of $r + 1$ vertices form an edge in $\mathcal{H}$ if and only if they form a $K_{r+1}$ in the original graph $G$.
From Step 1, we know $e(\mathcal{H}) \geq \delta n^{r+1}$. Since $\delta$ is constant, this is a dense hypergraph.
**Step 3: Find a dense structure in the hypergraph.** We apply the **Hypergraph KST Theorem** to $\mathcal{H}$. Since $\mathcal{H}$ is dense (order $n^{r+1}$), it exceeds the KST threshold (which is roughly $n^{(r+1)-\epsilon}$). Therefore, $\mathcal{H}$ must contain a copy of the complete $(r + 1)$-partite hypergraph $K_{s,s,\ldots,s}^{(r+1)}$.
**Step 4: Map back to $G$.** Let the parts of this hypergraph copy be $U_1, \ldots, U_{r+1}$, each of size $s$. The definition of the complete hypergraph implies that for *any* choice of vertices $u_1 \in U_1, \ldots, u_{r+1} \in U_{r+1}$, the set $\{u_1, \ldots, u_{r+1}\}$ is an edge in $\mathcal{H}$.
By definition of $\mathcal{H}$, this means $\{u_1, \ldots, u_{r+1}\}$ forms a clique $K_{r+1}$ in $G$. If every possible tuple forms a clique, then every pair of vertices in distinct parts $U_i, U_j$ must be connected in $G$.
Thus, $G[U_1 \cup \cdots \cup U_{r+1}]$ contains the complete multipartite graph with parts of size $s$, which is exactly $T_{s(r+1)}$. $\qquad \square$

## 4.4 Degenerate Turán Numbers (Bipartite Graphs)

When $H$ is bipartite ($r = 1$), the Erdős-Stone bound gives $1 - 1/1 = 0$, which is trivial. We rely on KST bounds $ex(n, K_{s,t}) = O(n^{2-1/s})$. We examine the sharpness of these bounds.

**Theorem 4.6** (Tightness of KST).

1. ***Case $s = 1$ (Trees/Forests):***
$$ex(n, K_{1,t}) \leq \frac{1}{2}(t - 1)n.$$

   *This is tight for a graph consisting of disjoint copies of $K_t$ (or a matching if $t = 1$).*

2. ***Case $s = 2$ ($C_4$-free graphs):*** *We know $ex(n, K_{2,2}) = O(n^{3/2})$. This is tight.*

   ***Construction (Finite Geometry):*** *Let $q$ be a prime power. Construct a bipartite graph $G$ with parts $A$ and $B$, where $|A| = |B| = q^2$.*

   - *Identify vertices in $A$ and $B$ with points in the affine plane $\mathbb{F}_q^2$.*
   - *Define edge $(a,b) \sim (x,y)$ if $ax + by = 1$, where $(a,b) \in A$ and $(x,y) \in B$.*

For a fixed vertex $v = (a, b) \in A$, its neighborhood is the set of solutions $(x, y)$ to the linear equation $ax + by = 1$. This describes a line in $\mathbb{F}_q^2$, which contains $q$ points (unless $(a, b) = (0, 0)$, which we can discard).

$$e(G) \approx q^2 \cdot q = q^3.$$

Since $n = 2q^2$, we have $q \approx \sqrt{n/2}$, so $e(G) \approx (n/2)^{3/2} = \Theta(n^{3/2})$.

**Why is it $C_4$-free?** A $C_4$ corresponds to two vertices in $A$ sharing two common neighbors in $B$. Geometrically, this means two distinct lines intersect at two distinct points. In a plane, two distinct lines intersect at most at one point. Thus, no $C_4$ exists.

3. **Case $s = 3$ ($K_{3,3}$-free):** Tightness $ex(n, K_{3,3}) = \Theta(n^{5/3})$ is known.

   **Construction Sketch:** Vertices are points in $\mathbb{F}_q^3$. Adjacency is defined by $(x, y, z) \sim (a, b, c)$ if $(x - a)^2 + (y - b)^2 + (z - c)^2 = 1$. Geometrically, the neighborhood of a point is a unit sphere. If $G$ contained $K_{3,3}$, three vertices in one part would share three common neighbors. This would imply that three unit spheres intersect at three points. In Euclidean geometry, three spheres intersect at most at 2 points (a circle intersects a sphere at 2 points). This logic carries over to $\mathbb{F}_q$ provided $-1$ is not a square.

4. **Case $s = 4$ ($K_{4,4}$-free):** It is an **open problem** whether the KST bound $O(n^{2-1/4})$ is tight for $K_{4,4}$.

## 4.5   Extremal Numbers for Trees

**Lemma 4.7.** *If a graph $G$ has average degree $d$, it contains a subgraph $H$ with minimum degree $\delta(H) \geq d/2$.*

*Proof.* Let $n = |V(G)|$. We have $e(G) = \frac{dn}{2}$. Iteratively delete any vertex with degree strictly less than $d/2$. Let $S$ be the set of removed vertices. The number of edges removed is strictly less than $|S| \cdot \frac{d}{2}$. Even if we remove $n - 1$ vertices, the number of edges removed is $< (n-1)\frac{d}{2} < \frac{dn}{2}$. Thus, the edge set cannot be emptied. The process must stop at a non-empty subgraph where every remaining vertex has degree $\geq d/2$. $\square$

**Theorem 4.8.** *If $T$ is a tree with $r$ edges (so $r + 1$ vertices), then:*

$$ex(n, T) \leq rn.$$

*Proof.* Suppose $e(G) > rn$. Then the average degree of $G$ is $d(G) = \frac{2e(G)}{n} > 2r$. By the Lemma, there exists a subgraph $H \subseteq G$ with minimum degree $\delta(H) > r$. Since degrees are integers, $\delta(H) \geq r$.

We embed $T$ into $H$ greedily:

1. Order the vertices of $T$ as $v_1, v_2, \ldots, v_{r+1}$ such that each $v_i$ (for $i > 1$) has exactly one neighbor among $\{v_1, \ldots, v_{i-1}\}$. (This is always possible by doing a BFS/DFS from a root).

2. Map $v_1$ to any vertex $u_1 \in V(H)$.

3. Suppose we have embedded $v_1, \ldots, v_{i-1}$ as $u_1, \ldots, u_{i-1}$. Let $v_j$ be the parent of $v_i$ in the tree ($j < i$).

4. We need to map $v_i$ to a neighbor of $u_j$ in $H$.

5. The vertex $u_j$ has at least $r$ neighbors in $H$. We have used at most $i - 1$ vertices so far. Since $i \leq r + 1$, we have used at most $r$ vertices.

6. We only care that $u_j$ has neighbors not already used in the current tree embedding. The degree of $u_j$ is $\geq r$. The number of currently used vertices is $i - 1 \leq r$.

7. The bound requires strict inequality or careful counting. If $\delta(H) \geq r$, $u_j$ has $r$ neighbors. We have used $i - 1$ vertices total. If $i - 1 < r$, there is space. If $i - 1 = r$, we are placing the last vertex. The neighbors of $u_j$ could be occupied by the other $r - 1$ vertices of the tree.

8. However, the conjecture is actually $\frac{1}{2}(r)n$. The bound $rn$ is loose enough. If $e(G) > rn$, avg deg $> 2r$, min deg $\geq r + 1$. Then $u_j$ has $r + 1$ neighbors, but the tree only has $r + 1$ vertices total. At step $i$, only $i - 1 \leq r$ vertices are occupied. Thus there is always at least one free neighbor for $u_j$.

$\square$

# 5    Jan 23

## 5.1    Turan Number of Cycles and Girth Problem

**Definition 5.1.** *If $\mathcal{F}$ is a family of graphs, $ex(n, \mathcal{F}) =$ maximum number of edges in an $n$ vertex graph $G$ that contains no $F \in \mathcal{F}$.*

**Lemma 5.2.** $ex(n, \mathcal{F}) = \left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}$ *if* $r = \min_{F \in \mathcal{F}} \mathcal{X}(F)$

*Proof.*

$$ex(n, \mathcal{F}) \leq \min_{F \in \mathcal{F}} ex(n, F).$$

By Erdos Stone, we're done. $\qquad\square$

**Theorem 5.3.** *Let $\mathcal{F} = \{C_3, ..., C_{2k}\}$ then $ex(n, \mathcal{F}) \leq n^{1+\frac{1}{k}}$*

**Lemma 5.4.** *$G$ is $\mathcal{F}$ free iff $G$ has girth $\geq 2k + 1$*

**Lemma 5.5.** *Every graph $G$ contains a subgraph $G'$ that is bipartite with $e(G') \geq \frac{1}{2}e(G)$.*

*Proof.* Color vertices red or blue uniformly and independent. Then delete the edges that are monochromatic By independence, for any $e \in E(G)$

$$\Pr[e \in E(G')] = \frac{1}{2} \implies E[e(G')] = \frac{1}{2}e(G).$$

$\qquad\square$

*We can also prove this with the greedy algorithm and will achieve a slightly better bound.*

*Proof.* To prove theorem 5.3:
Suppose $G$ has at least $n^{1+\frac{1}{k}}$, by lemma 5.2 we may replace $G$ by a sub-graph of $\min \deg d \geq n^{1+\frac{1}{k}}$.
We can do a depth first search by picking a root vertex $v$. Let $V_i$ be the set of vertices at distance $i$ from $v$. So $V_0 = \{v\}$ Observe that $G[V_0, V_1, ..., V_k]$ forms a tree with root $v$ and $V_i$ being the descendent at at level $i$ else we get a cycle of length at most $2k$. So we have

$$|V_0| + |V_1| + \cdots + |V_k| \geq 1 + d + (d-1)d + d(d-1)^2 + \cdots + d(d-1)^{k-1} = 1 + d^k > n.$$

So, we have a contradiction. $\qquad\square$

**Theorem 5.6.** $ex(n, C_{2k}) \leq c_k n^{1+\frac{1}{k}}$

**Theorem 5.7.** $ex(n, \{C_4, C_6, ..., C_{2k}\}) \leq 2n^{1+\frac{1}{k}}$

**Remark 5.8.** *The exponent $1 + \frac{1}{k}$ is sharp if $k = 2, 3, 5$*

## 5.2    Turan Number for Paths

**Theorem 5.9.** *Let $P_k$ be a path with $k$ edges then*

$$ex(n, P_k) \leq \frac{k}{2}n.$$

*Proof.* By induction on $n$ it is enough to work with connected graph as we can just split into connected components. Delete vertices of degree $\leq \frac{k}{2}$. If $e(G) > \frac{k}{2}n$ then this also holds after deletions. So WLOG $G$ has minimum degree at least $\frac{k}{2}$.
Let $v_0, v_1, ..., v_r$ be the longest path in $G$. The neighbors of $v_0, v_r$ are in the path otherwise you can extend the path. If $v_0 \sim v_r$ then $v_0 v_1 ... v_r$ is a cycle and so $v_i v_{i+1} ... v_r v_0 ... v_{i-1}$ is also a path of same length. Thus, the neighbors of $v_i$ are inside the path. So $\{v_0, v_1, ..., v_r\}$ is a connected conponent of $G$ and thus $v \geq k + 1$ else $e(G) \leq \frac{(r-1)n}{2}$. Otherwise, if $v_0 \not\sim v_r$ then there exist $i$ such that $v_0 \sim v_{i+1}$ and $v_r \sim v_i$ then we can produce a cycle $v_0 v_{i+1} v_{i+2} ... v_r v_i v_{i-1} ... v_0$ and this is of length $r$, so it becomes the previous case. By pigeonhole, such a $i$ exist as $\{i : v_0 v_i + 1 \in E(G)\}$ and $\{i : v_r v_i \in E(G)\}$ must intersect as $r \leq k$ by $v_0 v_r \notin E(G)$ $\qquad\square$

# 6 Jan 26: Projective Geometry

**Definition 6.1. *Projective Plane over a Field.*** *Let $\mathbb{F}$ be a field.*

1. *Points: The set of lines through $0$ in $\mathbb{F}^3$; equivalently, the $1$-dimensional subspaces.*

2. *Lines: The set of planes through $0$ in $\mathbb{F}^3$; equivalently, the $2$-dimensional subspaces.*

3. *Incidence: A point $P$ lies on a line $\ell$ if $P \subset \ell$.*

*Observe:*

1. *Any two points determine a unique line.*

2. *Any two lines intersect in a unique point.*

3. *There exist 4 points no three of which are collinear.*

*Inside a projective plane over $\mathbb{F}$, there is a copy of the affine plane $\mathbb{F}^2$ embedded as $\{z = 1\}$.*
*We denote this by $\mathbb{P}^2(\mathbb{F})$.*
*We say $p \sim \ell$ if $p \in \ell$ for $p$ a point and $\ell$ a line of $\mathbb{P}^2(\mathbb{F}_q)$.*

**Remark 6.2.** *There are $q^3 - 1$ nonzero vectors in $\mathbb{F}_q^3 \setminus \{0\}$. We identify points of $\mathbb{P}^2(\mathbb{F}_q)$ with equivalence classes of nonzero vectors under*

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \quad \text{for } \lambda \in \mathbb{F}_q^\times.$$

*There are $q - 1$ nonzero choices for $\lambda$, so the number of points in $\mathbb{P}^2(\mathbb{F}_q)$ is*

$$\frac{q^3 - 1}{q - 1} = q^2 + q + 1.$$

*By a similar argument, the number of lines is also $q^2 + q + 1$.*
*Each line contains exactly $q + 1$ points and dually, each point lies on exactly $q + 1$ lines. There is a natural bijection (called a* polarity*) between points and lines, preserving incidence: given a point $p$, its corresponding line is the orthogonal complement $p^\perp$.*

**Theorem 6.3. *Construction of a $C_4$-free graph:***
*We construct an improved $C_4$-free graph as follows:*

- *The vertices are the points of $\mathbb{P}^2(\mathbb{F}_q)$.*

- *Points $p_1$ and $p_2$ are adjacent if and only if $p_1 \in p_2^\perp$ (i.e., $p_1$ lies on the line associated to $p_2$'s orthogonal complement).*

*For any pair $p_1, p_2$, their common neighborhood is*

$$p_1^\perp \cap p_2^\perp,$$

*which is a single point. Thus, the graph is $C_4$-free (contains no 4-cycles). This gives $q^2 + q + 1$ vertices with average degree $\approx q$.*
*Note: There exist $p$ such that $p \subseteq p^\perp$.*

**Definition 6.4. *General Projective Planes.*** *If $P$ (points) and $L$ (lines) are sets with an incidence relation such that:*

1. *For any two points, there is a unique line through them.*

2. *Any two lines meet at a unique point.*

3. *There exist four points, no three collinear.*

*Example: The Fano plane (order 2) with 7 points and 7 lines. Each line contains 3 points, and each point lies on 3 lines.*

**Definition 6.5. *Order of a Projective Plane.*** *The order of a projective plane is the number of points per line minus one (so $q$ if each line has $q + 1$ points).*

**Remark 6.6.** *Are there planes of order which are not prime powers?*

**Theorem 6.7. *Construction of $C_{2k}$-free graphs for $k = 2, 3, 5$:***
*Let $q$ be a prime power and let $\mathbb{F}_q$ be the finite field with $q$ elements. Define a graph $G$ on the vertex set $\mathbb{F}_q^k$ (the set of $k$-tuples over $\mathbb{F}_q$) as follows:*
*For $\mathbf{a} = (a_0, a_1, \ldots, a_{k-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{k-1})$ in $\mathbb{F}_q^k$, declare $\mathbf{a} \sim \mathbf{b}$ if and only if for all $j = 0, 1, \ldots, k - 2$,*

$$b_j = a_j + a_{j+1} \cdot b_{k-1}$$

*where indices are taken modulo $k$ (so $a_k = a_0$).*
*Then $G$ is $C_{2k}$-free, i.e., it contains no cycle of length $2k$.*

*Proof.* Suppose for contradiction that $G$ contains a cycle of length $2k$:

$$\mathbf{a}^1, \mathbf{b}^1, \mathbf{a}^2, \mathbf{b}^2, \ldots, \mathbf{a}^k, \mathbf{b}^k,$$

with $\mathbf{a}^i \sim \mathbf{b}^i \sim \mathbf{a}^{i+1}$ for all $i$, indices modulo $k$.

**Step 1: System of equations.** For each $i = 1, \ldots, k$ and $j = 0, \ldots, k - 2$, adjacency gives:

$$b_j^i = a_j^i + a_{j+1}^i \cdot b_{k-1}^i \qquad \text{(from } \mathbf{a}^i \sim \mathbf{b}^i\text{)}$$
$$b_j^i = a_j^{i+1} + a_{j+1}^{i+1} \cdot b_{k-1}^i \qquad \text{(from } \mathbf{b}^i \sim \mathbf{a}^{i+1}\text{)}$$

Subtract:

$$(a_j^i - a_j^{i+1}) + (a_{j+1}^i - a_{j+1}^{i+1}) \cdot b_{k-1}^i = 0$$

or

$$(a_j^{i+1} - a_j^i) + (a_{j+1}^{i+1} - a_{j+1}^i) \cdot b_{k-1}^i = 0$$

Summing over $i$:

$$\sum_{i=1}^{k} (a_j^{i+1} - a_j^i) + \sum_{i=1}^{k} (a_{j+1}^{i+1} - a_{j+1}^i) b_{k-1}^i = 0$$

The first sum telescopes to zero (since indices are cyclic), so for each $j = 0, \ldots, k - 2$,

$$\sum_{i=1}^{k} (a_{j+1}^{i+1} - a_{j+1}^i) \, b_{k-1}^i = 0 \qquad (3)$$

**Step 2: Specialize** $j = k - 2$. Let $\triangle_i = a_{k-1}^{i+1} - a_{k-1}^i$ and $x_i = b_{k-1}^i$. Then:

$$\sum_{i=1}^{k} \triangle_i x_i = 0$$

and

$$\sum_{i=1}^{k} \triangle_i = 0$$

by telescoping.

**Step 3: Linear algebra.** The system (3) for $j = 0, \ldots, k - 2$ can be written as a $(k-1) \times k$ Vandermonde-type matrix $V$ applied to the vector $(\triangle_1, \ldots, \triangle_k)$:

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ \vdots & \vdots & & \vdots \\ x_1^{k-2} & x_2^{k-2} & \cdots & x_k^{k-2} \end{pmatrix}$$

The equations become $V \cdot (\triangle_1, \ldots, \triangle_k)^\top = 0$. Since $\sum_{i=1}^k \triangle_i = 0$, the vector is in the kernel of the row $(1, 1, \ldots, 1)$.

**Step 4: Key property of the $x_i$.**

**Claim 6.8.** *For each $i$, $x_i \neq x_{i+1}$ (indices modulo $k$).*

*Proof.* If $x_i = x_{i+1}$, then $b_{k-1}^i = b_{k-1}^{i+1}$. By the adjacency relations, this forces $\mathbf{b}^i = \mathbf{b}^{i+1}$, so $\mathbf{a}^{i+1} = \mathbf{a}^{i+2}$, contradicting that the cycle is proper. $\square$

**Step 5: Contradiction for $k = 2, 3, 5$.** Since not all $\triangle_i$ are zero (otherwise the cycle is trivial), and $\sum_{i=1}^k \triangle_i = 0$, equation $\sum_{i=1}^k \triangle_i x_i = 0$ forces a linear dependence among the $x_i$ (not all distinct). Thus, for these small $k$ we can classify possibilities:

**Claim 6.9.** *For $k = 2, 3, 5$, if $x_i = x_j$ for some $i \neq j$, then $x_i = x_{i+1}$ for some $i$, contradicting the above claim.*

- **Case $k = 2$:** Only possible if $x_1 = x_2$, violating $x_1 \neq x_2$.

- **Case $k = 3$:** If $x_1 = x_2$, $x_2 = x_3$, or $x_3 = x_1$, we violate the claim.

- **Case $k = 5$:** If $x_i = x_j$ for $i \neq j$, by relabeling, could assume $x_1 = x_j$ for $j = 2, 3, 4, 5$:

    - $j = 2$ or $5$ gives $x_1 = x_2$ or $x_5 = x_1$, directly a violation.
    - $j = 3$: $x_1 = x_3$. The system $\sum_{i=1}^5 \triangle_i = 0$, $\sum_{i=1}^5 \triangle_i x_i = 0$ with $x_1 = x_3$ and $x_i \neq x_{i+1}$ for any $i$ only allows further coincidences that eventually force $x_i = x_{i+1}$.
    - $j = 4$: similarly, $x_1 = x_4$ eventually forces a pair $x_i = x_{i+1}$.

- **Case $k = 4$:** The pattern $x_1 = x_3 \neq x_2 = x_4$ is possible with no consecutive equal $x_i$, which is why this construction fails for $k = 4$.

Therefore, for $k = 2, 3, 5$, no proper $2k$-cycle exists in $G$. $\square$

# 7 Jan 28

Recall: $ex(n, K_{s,t}) \leq c_{s,t} n^{2-\frac{1}{s}} + o(n)$. We will see that $ex(n, K_{s,t}) = \theta\left(n^{2-\frac{1}{s}}\right)$ for $t \geq t_0(s)$

**Theorem 7.1.** *Construction of the extremal graph for $ex(n, K_{s,t})$.*

*Proof.* We will first consider a random graph by picking edges with probability $p = n^{-\frac{1}{s}}$. Let $x^{(1)}, ..., x^{(s)}$ be $s$ vertices chosen uniformly at random without replacement on the left. Let $N(x^{(1),...,x^{(s)}})$ that are the common neighbors with $d(x^{(1)}, ..., x^{(s)}) = |N(...)|$. Let our random variable be $d(x^{(1)}, ..., x^{(s)})$ and is distributed according to

$$Binomial(n, p^s) = Binomial\left(n, \frac{1}{n}\right) \sim Poisson(1).$$

So $\mathbb{E}[d(x^{(1)}, ..., x^{(s)})] = 1$ and $\mathbb{E}[d(...)^k] = o_k(1)$. With probability $\frac{1}{n^\epsilon}$ there is at least $\epsilon \log n$ neighbors. Yet there are $\binom{n}{s}$ such $s$ tuples $(x^{(1)}, ..., x^{(s)})$. and then $\frac{n}{s}$ disjoint. This does not work.

Consider another construction of $K_{s,t}$-free graph with $\Omega(n^{2-\frac{1}{s}})$ edges.
For a bipartite graph with $\mathbb{F}_p^s$ where $n = 2p^s$ on each side we say $x \sim y$ if $f(x, y) = 0$. Pick equation $f$ uniformly at random among all polynomials in $s, t$ of degree at most $d =$ in each of $x$ and $y$. It would be

$$f = \sum_{|\alpha|, |\beta| \leq d} c_{\alpha\beta} x_1^{\alpha_1} x_2^{\alpha_2} ... x_s^{\alpha_s} y_1^{\beta_1} y_2^{\beta_2} ... y_s^{\beta_s}. \qquad (c_{\alpha\beta} \in \mathbb{F}_p, |\alpha| = \alpha_1 + \cdots + \alpha_d)$$

Part I: This random graph is locally (neighborhood look "similar") similar to uniform random graph.
Part II: Globally very different/more discrete. $\qquad \square$

**Lemma 7.2.** *For any fixed $x, y \in \mathbb{F}_p^s$*

$$\Pr[x \sim y] = \frac{1}{p}.$$

*Proof.* We have

$$f = c_{0,0} + \sum_{\alpha, \beta \neq 0} c_{\alpha\beta} x^\alpha b^\beta.$$

For any choice of $c_{\alpha,\beta}$ fir $(\alpha, \beta) \neq (0, 0)$ there is a unique choice of $c_{0,0}$ such that $f(x, y) = 0$.

$$\mathbb{E}[e(G)] = \frac{1}{p} n^2 = c_s n^{2-\frac{1}{s}}.$$

$\qquad \square$

Let $x^{(1)}, ..., x^{(s)}$ be $s$ vertices on say left, $N(x^{(1)}, ..., x^{(s)})$ be the common neighbors with of $x^{(1)}, ..., x^{(s)}$ and this is same as $\{y \in \mathbb{F}_p^s : f(x^{(1)}, y) = 0, ..., f(x^{(s)}, y) = 0\}$.

**Lemma 7.3.** *If $g$ is a random polynomial of degree of $D$ in $m$ variables and $z^{(1)}, ..., z^{(t)} \in \mathbb{F}_p^m$ then the random variables $g(z^{(1)}), ..., g(z^t)$ are independent if $D > t$ and $p > t^2$.*

*Warm up: When $m = 1$, write $g(x) = \underbrace{a_0 + a_1 x + \cdots + a_{t-1} x^{t-1}}_{g_{small}(x)} + \underbrace{\cdots + a_D x^D}_{g_{large}(x)}$.*
*For any choice of $g_{large}(x)$ and any values $b_1, ..., b_t$ there is a unique polynomial $g_{small}$ such that $g(z^{(i)}) = g_{small}(z^{(i)}) + g_{large}(z^{(i)}) = b_i$ for $i = 1, ..., t$ if and only if $g_{small}(z^{(i)}) = b_i - g_{large}(z^{(i)})$ by lagrange interpolation.*

*For the general case, if $z_1^{(1)}, ..., z_1^{(t)}, ..., z_m^{(1)}, ..., z_m^{(t)}$ are distinct, then $g(x_1, ..., x_m) = \overline{g}(x_1) +$ other terms and use $m = 1 =$ case. Pick a invertible linear transform $f : \mathbb{F}_p^s \to \mathbb{F}_p^s$ such that $Tz^{(1)}, ..., Tz^{(t)}$ are distinct. To pick such a $T$, let $T(x_1, ..., x_s) = (x_1 + S(x_2, ..., x_s), x_2, ..., x_s)$. For $z^{(i)}, z^{(j)}$ distinct, the space of $S$ for which $T(z^{(i)}) = T(z^{(j)})$ is of co-dimension 1, so a random $S$ has probability $\frac{1}{p}$ of satisfying this. So $\Pr[\exists i, j \text{ such that } T(z^{(i)}) = T(z^{(j)})] \leq \frac{1}{p}\binom{t}{2} < 1$.*

*We can now conclude from*

$$\Pr[f(x^{(1)}, y) = 0, ..., f(x^{(s)}, y) = 0] = \frac{1}{p^s}.$$

*If $d > s$ and $p > s^2$ then*

$$\mathbb{E}[N(x^{(1)}, ..., x^{(s)})] = 1.$$

*Furthermore, if $\alpha$ is a positive integer and $p \cdot s < d$ then*

$$\mathbb{E}\left[\left|N(x^{(1)}, ..., x^{(s)})\right|^p\right] = \mathbb{E}\left[\left(\sum_{y \in \mathbb{F}_p^s} R(y)\right)^\alpha\right] = \mathbb{E}\left[\sum_{y^{(1)}, ..., y^{(\alpha)}} R(y^{(1)})...R(y^{(\alpha)})\right].$$

*Where* $R(y) = \begin{cases} 1 & \text{if } y \in N(x^{(1)}, ..., x^{(s)}) \\ 0 & \text{otherwise} \end{cases}$

$$= \sum_{y^{(1)}, ..., y^{(\alpha)}} p^{-s\#\text{distinct } y's} \leq \sum_k \sum_{y^{(1)}, ..., y^{(k)} \in \mathbb{F}_p^s} p^{-sk} = \sum_k p^{sk} p^{-sk} \cdot \#\text{ways to partition a objection into } k \text{ bins.}$$

*Last equality from having to choose $y^{(i)}$ such that $k$ are distinct. This is in $O(1)$. We want to assign values to $y^{(i)}$ such that exactly $k$ are distinct. Equivalently, choose $k$ distinct values in $\mathbb{F}_p^s$. Then choose which of the $y^{(i)}$ go to which bin.*

$$\Pr[|N(x^{(1)}, ..., x^{(s)})| \geq T] \leq \frac{\mathbb{E}\left[\left|N(x^{(1)}, ..., x^{(s)})\right|^\alpha\right]}{T^\alpha} = \frac{C_{\alpha,s}}{T^\alpha}.$$

*The set $N(x^{(1)}, ..., x^{(s)})$ is contained in $\{y \in \mathbb{F}_p^s : f(x^{(1)}, y) = 0, ..., f(x^{(s)}, y) = 0\}$.*
*A vector subspace $V$ of $\mathbb{F}_p^s$ of dimension $D$ has $p^D$ elements.*

**Theorem 7.4.** *Lang-Weil Theorem: If $g_1, \ldots, g_r$ are polynomials of degree $\leq D$ in $s$ variables over $\mathbb{F}_p$, then the set $\{y \in \mathbb{F}_p^s : g_1(y) = \cdots = g_r(y) = 0\}$ has either $O_{D,r}(1)$ points, or more than $\frac{p}{2}$ points.*

*So $\Pr[\text{some } s \text{ vertices have neighborhood of size} \geq O_{s,\alpha}(1)] \leq C_{s,\alpha}\left(\frac{p}{2}\right)^{-\alpha}\binom{p^s}{s}$. Choose $\alpha = s^2 + 1$. To ensure this probability goes to $0$, we need to choose $d = s^2 + 1$.*

# 8   Regularity Lemmas: Feb 4

**Definition 8.1.** *Binary word is sequence $w_1, w_2, ..., w_n \in \{0,1\}$ and write $w = w_1 w_2 ..., w_n$. The density of $w$ is*

$$d(w) = \frac{\#1\text{'s in } w}{n}.$$

**Definition 8.2.** *Word $w$ is $\epsilon$-regular if for any subword $w'$ of $w$ of length $\geq \epsilon \cdot len(w)$ satisfies*

$$|d(w) - d(w')| \leq \epsilon.$$

**Lemma 8.3.** *By Chernoff bound,*

$$\Pr[|d(w') - d(w)| \geq \epsilon] \leq e^{-c_\epsilon n}.$$

# 9  Feb 6

**Definition 9.1.** *Partition $\mathcal{P}'$ is a refinement of partition $\mathcal{P}$ if each $w' \in \mathcal{P}'$ is a sub-word of some word in $\mathcal{P}$*

**Theorem 9.2.** *(Regularity Lemma for Binary Words)*
*For all $\epsilon > 0$, every word $w \in \{0,1\}^n$ can be partitioned into $\leq M(\epsilon)$ many sub-words $w^{(1)}, ..., w^{(m)}$ (i.e. $w = w^{(1)}...w^{(m)}$). So that the total length of $w^{(i)}$'s that are $\epsilon$-irregular is at most $\epsilon n$.*

*Proof.* For a partition $\mathcal{P}$ of $w$ into sub-words $\mathcal{P} = \{w^{(1)}, ..., w^{(m)}\}$ we define $f(\mathcal{P}) = \sum_{w' \in \mathcal{P}} d(w)^2 |w'|/|w|$. Alternatively, we pick a symbol of $w$ at random, say $i \in [n]$ uniformly, then pick another symbol at random from the same part as 1'st, say $j$ is picked.

$$\Pr[w_i = w_j] = \sum_{w' \in \mathcal{P}} \frac{|w'|}{|w|}(d(w')^2 + (1 - d(w'))^2) = 2f(\mathcal{P}) + \sum_{w'} \frac{|w'|}{|w|} - 2\sum_{w'} d(w')\frac{|w'|}{|w|} = 2f(\mathcal{P}) + 1 - 2d(w).$$

**Lemma 9.3.** *If $\mathcal{P}'$ is a refinement of $\mathcal{P}$ then*

$$f(\mathcal{P}') = f(\mathcal{P}) + \sum_{w' \in \mathcal{P}'} (d(w') - d(\text{parent}(w')))^2 \frac{|w'|}{|w|}.$$

*Proof.* Expanding the second term we have

$$\sum_{w' \in \mathcal{P}'} d(w')\frac{|w'|}{|w|} - 2d(w')d(\text{parent}(w')) + d(\text{parent}(w'))^2 \frac{|w'|}{|w|} = \sum_1 + \sum_2 + \sum_3.$$

We have $\sum_1 = f(\mathcal{P}')$ and

$$\sum_2 = \sum_{\tilde{w} \in \mathcal{P}} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} \left(-2d(w')d(\tilde{w})\frac{|w'|}{|w|}\right)$$

$$= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w})\frac{1}{|w|} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} d(w')|w'|$$

$$= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w})\frac{1}{|w|} \cdot d(\tilde{w})|\tilde{w}|$$

$$= \sum_{\tilde{w} \in \mathcal{P}} -2d(\tilde{w})^2 \frac{|\tilde{w}|}{|w|}$$

$$= -2f(\mathcal{P})$$

Then for the third sum

$$\sum_3 = \sum_{\tilde{w} \in \mathcal{P}} \sum_{\substack{w' \in \mathcal{P}' \\ \text{parent}(w') = \tilde{w}}} d(\tilde{w})^2 \frac{|w'|}{|w|}$$

$$= \sum_{\tilde{w} \in \mathcal{P}} d(\tilde{w})^2 \frac{|\tilde{w}|}{|w|} = f(\mathcal{P})$$

So we have $\sum_1 + \sum_2 + \sum_3 = f(\mathcal{P}') - f(\mathcal{P})$ $\qquad\qquad\square$

Start with the trivial partition (i.e. $\mathcal{P} = \{w\}$) and repeat while $\sum_{w' \in \mathcal{P}, w' \text{ is } \epsilon\text{-irregular}} |w'| \leq \epsilon|w|$.
For $w' \in \mathcal{P}$ that is $\epsilon$-irregular, find a way to write as $w' = w^{(1)}w^{(2)}w^{(3)}$ where $|d(w^{(2)}) - d(w^{(1)})| \geq \epsilon$ and $|w^{(2)}| \geq \epsilon|w'|$ Replace $w'$ with the three words $w^{(1)}, w^{(2)}, w^{(3)}$ to obtain a new partition of $w$, call it $\mathcal{P}'$. Repeat with $\mathcal{P}$ replaced by $\mathcal{P}'$.

To analyze this algorithm, if we drop some words we have

$$f(\mathcal{P}') \geq f(\mathcal{P}) + \sum_{\substack{w' \in \mathcal{P}' \\ w' \text{ is } \epsilon\text{-irregular}}} \sum_{w^{(2)}} (d(w^{(2)} - d(w')))\frac{|w^{(2)}|}{|w|} \geq f(\mathcal{P}) + \sum_{\substack{w' \in \mathcal{P}' \\ w' \text{ is } \epsilon\text{-irregular}}} \epsilon^2 \frac{\epsilon|w'|}{|w|} = f(\mathcal{P}) + \epsilon^4.$$

Note: $f(\mathcal{P}) \leq 1$ for any partition $\mathcal{P}$.

So the number of steps in the algorithm is $\leq \frac{1}{\epsilon^4}$. The theorem holds with $M(\epsilon) = 3^{1/\epsilon^4}$ $\qquad \square$

**Example 9.4.** *Define a twin in a word $w \in \{0,1\}^n$ is a pair of subsequences $x, y \in w$ that are disjoint (no symbol in both $x$ and $y$) and are equal as word.*

*We define*
$$t(w) = \text{the length of longest pair twins in } w \text{ and } t(n) = \min_{w \in \{0,1\}^n} t(w).$$

*We have the trivial bounds $\frac{n}{4} \leq t(n) \leq \frac{n}{2}$.*

**Theorem 9.5.**
$$t(n) \geq \left( \frac{1}{2} - o(1) \right) n.$$

*Proof.* Pick small $\epsilon > 0$ and cut $w$ into $\epsilon$-regular words $t \leq \epsilon n$ junk symbols. Enough $t(w) \geq \left( \frac{1}{2} - c\epsilon \right) |w|$ for $\epsilon$-regular $w$.

Cut $w$ into $\frac{1}{\epsilon}$ equally long sub-words $w^{(1)}, ..., w^{(m)}$ where $m = \frac{1}{\epsilon}$.

The first twin is 0's from $w^{(1)}+$ 1's from $w^{(2)}+$ 0's from $w^{(3)}$...

The second twin is 1's from $w^{(1)}+$ 0's from $w^{(2)}+$ 1's from $w^{(3)}$...

Choose twins then
$$|\# \text{ 0's in } w^{(i)}| - |\# \text{ 1's in } w^{(2)}| \leq \epsilon |w^{(1)}|.$$

So, $\#$ symbols not in either of the twins is less than or equal to

$$\epsilon |w|(\text{1's in } w^{(1)}) + \epsilon |w|(\text{1's in } w^{(2)}) + \sum_i 4\epsilon |w^{(i)}|.$$

$\qquad \square$

# 10   Feb 13

**Definition 10.1.** *For disjoint sets $U, V \subset V(G)$, the density between $U$ and $V$ is*

$$d(U, V) = \frac{e(U, V)}{|U||V|}.$$

**Definition 10.2.** *A pair $(U, V)$ of disjoint subsets of $V(G)$ is $\epsilon$-regular if $\forall U' \subseteq U, V' \subseteq V$ such that $|U'| \geq \epsilon |U|$ and $|V'| \geq \epsilon |V|$ we have*

$$|d(U, V) - d(U', V')| \leq \epsilon.$$

**Definition 10.3.** *We say a partition $V(G) = V_1 \cup ... \cup V_k \cup J$ is $\epsilon$-regular partition when*

1. *$|J| \leq \epsilon |V(G)|$*

2. *$|V_1| = |V_2| = ... = |V_k|$*

3. *All except $\leq \epsilon k^2$ pairs $(V_1, V_2)$ are $\epsilon$-regular*

**Theorem 10.4.** *(Szemeredi's Regularity Lemma) $\forall \epsilon > 0, m$ there exist a constant $M = M(\epsilon, m)$ such that every graph admits an $\epsilon$-regular partition $V(G) = V_1 \cup \cdots \cup V_k \cup J$ where the number of parts $m \leq K \leq M$*