

Complex Analysis Notes

Wilson Pan

February 16, 2026

Contents

1	Jan 12	3
1.1	Holomorphic Functions and Complex Differentiation	3
1.2	Jacobian and Cauchy-Riemann Equations	3
2	Jan 14	4
2.1	Holomorphic Equivalence and Wirtinger Derivatives	4
3	Jan 16	5
3.1	Power Series	5
3.2	Analytic Functions	6
4	Jan 21	7
4.1	Parametrized Curves	7
4.2	Contour Integration	7
4.3	Primitives and the Fundamental Theorem	8
5	Jan 23	9
5.1	Glossary of Elementary Functions	9
5.2	Fundamental Theorem of Algebra	9
6	Jan 26	10
6.1	Cauchy's and Goursat's Theorems	10
6.2	Cauchy's Theorem in a Disk	11
7	Jan 28	14
7.1	Applications of Cauchy's Theorem	14
8	Jan 30	17
8.1	Cauchy's Integral Formula	17
8.2	Regularity and Liouville's Theorem	17
9	Feb 2	19
9.1	Rigidity and Analytic Continuation	19
9.2	Morera's Theorem and Uniform Convergence	19
10	Feb 4	20
10.1	Parameter Integrals and Gluing	20
10.2	Schwarz Reflection Principle	21
11	Feb 9	22
11.1	Meromorphic Functions and Poles	22
11.2	Laurent Series and Residues	22
11.3	The Residue Formula	23
12	Feb 11	25
12.1	Integral Evaluation via Residues	25
12.2	Removable Singularities	25

13 Feb 13	27
13.1 Classification of Singularities	27
13.2 Meromorphic Functions and Behavior at Infinity	27
14 Feb 16	28
14.1 Rational Functions via Meromorphy	28

1 Jan 12

1.1 Holomorphic Functions and Complex Differentiation

Definition 1.1. Let Ω be an open set in \mathbb{C} and f a complex-valued function on Ω . A function f is holomorphic at a point $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}.$$

converges to a limit when $h \rightarrow 0$. The limit of the quotient, when it exists, is denoted $f'(z_0)$.

Theorem 1.2. A function $F(x, y) = (u(x, y), v(x, y))$ is said to be differentiable at a point $P_0(x_0, y_0)$ if there exist a linear transformation $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\frac{|F(P_0 + H) - F(P_0) - J(H)|}{|H|} \rightarrow 0 \text{ as } |H| \rightarrow 0.$$

1.2 Jacobian and Cauchy-Riemann Equations

Definition 1.3. The jacobian matrix of F is

$$J = J_F(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Theorem 1.4. Cauchy-Riemann Equations: If f is holomorphic then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof. If we write $z = x + iy$, $z_0 = x_0 + iy_0$ and $f(z) = f(x, y)$ then

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \rightarrow 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0) \end{aligned}$$

Similarly, now take h purely imaginary, say $h = ih_2$, then

$$\begin{aligned} f'(z_0) &= \lim_{h_2 \rightarrow 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} \\ &= \frac{1}{i} \frac{\partial f}{\partial y}(z_0) \end{aligned}$$

Since f is holomorphic we have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

Writing $f = u + iv$ we have the stated relation. □

2 Jan 14

2.1 Holomorphic Equivalence and Wirtinger Derivatives

Theorem 2.1. $f : \Omega \rightarrow \mathbb{C}$ holomorphic on Ω equivalently $F = \begin{pmatrix} u \\ v \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. The partials $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Moreover, F is \mathbb{R} -differentiable, there exist a linear map $\mathcal{L} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F((x, y) + (h_1, h_2)) - F(x, y) = \mathcal{L}(h_1, h_2) + o(h)$

$$\mathcal{L} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Proof. We have

$$\frac{|F((x, y) + (h_1, h_2)) - F(x, y) - \mathcal{L}(h_1, h_2)|}{|h|} = \frac{\left| \begin{pmatrix} u(\bar{z} + \bar{h}) - u(\bar{z}) - \frac{\partial u}{\partial x} h_1 - \frac{\partial u}{\partial y} h_2 \\ v(\bar{z} + \bar{h}) - v(\bar{z}) - \frac{\partial v}{\partial x} h_1 - \frac{\partial v}{\partial y} h_2 \end{pmatrix} \right|}{|h|} \quad (1)$$

$$= \frac{|f(z + h) - f(z) - f'(z)h|}{|h|} \rightarrow 0 \quad (2)$$

Where $f(z + h) = u(z + h) + iv(z + h)$, $f(z) = u(z) + iv(z)$ and $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$. \square

Corollary 2.2. f holomorphic implies F is \mathbb{R} -differentiable. Let $a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$DF_z = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad (4)$$

$$= \sqrt{a^2 + b^2} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad (5)$$

$$= |f'(z)| \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad (6)$$

Theorem 2.3. If F is \mathbb{R} -differentiable with u, v satisfying Cauchy Riemann Equations then f is holomorphic.

Proof. Let $a = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$. We just need to check that $f(z + h) - f(z) - ah = o(h)$ \square

Definition 2.4. The Wirtinger Derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Remark 2.5. The cauchy-riemann equations if and only if $\frac{\partial f}{\partial \bar{z}} = 0$

We initially had the map $z \mapsto \bar{z}$ which is not holomorphic. Let $f(z) = \bar{z}$ and $\frac{\partial f}{\partial \bar{z}} = 1$ then we've captured this with $\frac{\partial}{\partial \bar{z}}$.

If we have z, \bar{z} as independent variables with $z = x + iy$. Consider the map

$$(z, \bar{z}) \mapsto (x(z, \bar{z}), y(z, \bar{z})) \mapsto f(x, y).$$

We have

$$\partial_z f = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right).$$

Theorem 2.6.

$$\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{1}{4} \Delta.$$

3 Jan 16

3.1 Power Series

Definition 3.1. Given a \mathbb{C} valued seq $(a_n)_{n=0}^\infty$, form

$$\sum_{n=0}^{\infty} a_n z^n.$$

A power series

1. Converges if $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n z^n$ exists
2. Converges absolutely if $\sum_{n=0}^{\infty} |a_n| |z|^n$ converges

Remark 3.2. By translation, all we say will apply to $\sum a_n (z - z_0)^n$

Theorem 3.3. Given $\sum_{n=0}^{\infty} a_n z^n$, $\exists R \in [0, +\infty]$ such that

1. It converges, if $D_R = \{|z| < R\}$
2. It diverges at every z such that $|z| > R$.

Moreover,

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}}}. \quad (\text{Hadamard's Formula})$$

Remark 3.4. On ∂D_R it can often be delicate.

Proof. Fix $|z| < R$, then $\frac{1}{R}|z| < 1$ and fix $\epsilon > 0$ such that

$$\left(\frac{1}{R} + \epsilon\right) |z| < 1.$$

Then, eventually

$$|a_n|^{\frac{1}{n}} < \frac{1}{R} + \epsilon.$$

Then

$$|a_n| |z|^n \leq \left(|a_n|^{\frac{1}{n}} |z|\right)^n \leq \left(\left(\frac{1}{R} + \epsilon\right) |z|\right)^n \leq q^n.$$

So by the comparison test with $\sum q^n < \infty$

□

Example 3.5.

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z).$$

This has $R = +\infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos(z).$$

This has $R = +\infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin(z).$$

This has $R = +\infty$

Euler's Formula is

$$e^{iz} = \cos(z) + i \sin(z).$$

Theorem 3.6. The function $f(z) = \sum a_n z^n$ is holomorphic on D_R , f' is a power series

$$f'(z) = \sum n a_n z^{n-1}.$$

Whose disk of convergence is R .

Proof. Let $g(z) = \sum na_n z^{n-1}$, $z \in D_R$. We want to show that f' exist and $f' = g$.
 Fix $z_0 \in D_R$, let $r < R$ such that $|z_0| < r < R$.
 Fix $N \geq 1$ (to be chosen later)

$$f(z) = \sum_{n=0}^N a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n = S_N(z) + E_N(z).$$

From here we want to show that

$$\frac{f(z_0 + h) - f(z_0)}{h} \rightarrow g(z_0).$$

For all h such that $|z_0 + h| < r$

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) &= \left(\frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right) + (S'_N(z_0) - g(z_0)) + \left(\frac{E_N(z_0 + h) - E_N(z_0)}{h} \right) \\ &= A + B + C \end{aligned}$$

Fix $\epsilon > 0$ we want to show that $\exists \delta$ such that $\forall |h| < \delta, |A + B + C| < \epsilon$.

$$\begin{aligned} |C| &\leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(z_0 + h)^n - z_0^n}{h} \right| \\ &= \sum_{n=N+1}^{\infty} |a_n| |(z_0 + h)^{n-1} + (z_0 + h)^{n-2} z_0 + \dots + z_0^{n-1}| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1} < \frac{\epsilon}{3} \end{aligned} \quad (\text{for all } N \geq N_1(\epsilon))$$

Note the last summation is the tail of $\sum a_n n r^{n-1}$ which converges by $r < R$.

For B :

$$S'_N(z_0) = \sum_{n=0}^N n a_n z_0^{n-1} \rightarrow g(z_0). \quad (\text{by def of } g)$$

So $\forall N \geq N_2, |B| < \frac{\epsilon}{3}$.

$|A| < \frac{\epsilon}{3}$ for all h sufficiently small by the definition of the complex derivative and the fact that S_N is plainly holomorphic. \square

3.2 Analytic Functions

Definition 3.7. Let Ω be open and define $f : \Omega \rightarrow \mathbb{C}$ is analytic at $z_0 \in \Omega$, if there exists a power series $\sum a_n (z - z_0)^n$ with positive radius of convergence such that $f(z) = \sum a_n (z - z_0)^n$ is a neighborhood of z_0 .

f is analytic on Ω , if analytic at every point of $z_0 \in \Omega$.

Corollary 3.8. If f analytic on $\Omega \implies f$ is holomorphic on Ω .

In fact, \Leftarrow .

4 Jan 21

4.1 Parametrized Curves

Definition 4.1. A parametrized curve (in \mathbb{C}) is a function $z(t) : [a, b] \rightarrow \mathbb{C}$

1. Smooth if z' is continuous and $\forall t \in [a, b], z'(t) \neq 0$
2. Piecewise smooth if z is continuous so there exists $a_0 = a < a_1 < \dots < a_k = b$ such that $z|_{[a_j, a_{j+1}]}$ for all $j = 0, \dots, k-1$ is smooth.
3. Let $\tilde{z} : [\tilde{a}, \tilde{b}] \rightarrow \mathbb{C}$ then two parameterization are equivalent if there exist c^1 -bijection $s \mapsto t(s) : [\tilde{a}, \tilde{b}] \rightarrow [a, b]$ and $t'(s) > 0$ for all s such that $\tilde{z}(s) = z(t(s))$.

Example 4.2. A piecewise smooth curve can be visualized as follows:



The curve consists of smooth segments joined at corner points (marked in red) where the derivative is discontinuous.

Definition 4.3. The family of all equivalent parameterization determines a curve (oriented) $\gamma \subset \mathbb{C}, \gamma = z([a, b])$.

$z(a)$ is the starting point and $z(b)$ is the endpoint of γ . $\gamma^{-1} = \gamma$ with reversed orientation

Definition 4.4. γ is closed if $z(a) = z(b)$ and simple if $z(t_1) \neq z(t_2)$ for all $t_1 \neq t_2$.

Example 4.5. $z(t) = re^{-it}, t \in [0, 2\pi]$

Definition 4.6. A curve has positive orientation if interior is on the left or counterclockwise.

4.2 Contour Integration

Definition 4.7. Give a smooth curve $\gamma \subset \mathbb{C}$ (with parameterization $z : [a, b] \rightarrow \mathbb{C}$) with $f : \gamma \rightarrow \mathbb{C}$ continuous.

We set

$$\int_{\gamma} f = \int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt.$$

Since $f(z(t)) z'(t) \in \mathbb{C}$ we split the integral into

$$\int_a^b \Re(\dots) dt + i \int_a^b \Im(\dots) dt.$$

Remark 4.8. This is well defined as the right hand side does not depend on the choice of the parametrization.

Definition 4.9. For a piece-wise smooth curve, we define

$$\int_{\gamma} f = \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} f(z(t)) z'(t) dt.$$

Definition 4.10. We define

$$|\gamma| = \text{length}(\gamma) = \int_a^b |z'(t)| dt.$$

Theorem 4.11. *Some very basic properties:*

1. *Linearity*

$$\int_{\gamma} (\alpha f + \beta g) = \alpha \int_{\gamma} f + \beta \int_{\gamma} g.$$

2.

$$\int_{\gamma^{-1}} f = - \int_{\gamma} f.$$

3.

$$\left| \int_{\gamma} f \right| \leq \left(\sup_{\gamma} |f| \right) |\gamma|.$$

4.3 Primitives and the Fundamental Theorem

Definition 4.12. *The primitive of $f : \Omega \rightarrow \mathbb{C}$ with open $\Omega \subset \mathbb{C}$ is any $F : \Omega \rightarrow \mathbb{C}$ holomorphic such that $F' = f$ on Ω*

Theorem 4.13. *The Fundamental Theorem of Calculus*

For f with primitive F on Ω , $\gamma \subset \mathbb{C}$ curve from $w_1 \rightarrow w_2$ we have

$$\int_{\gamma} f = F(w_2) - F(w_1).$$

Proof. We can write

$$\int_{\gamma} f = \int_a^b f(z(t))z'(t)dt = \int_a^b \frac{d}{dt}(F(z(t)))dt = F(z(b)) - F(z(a)).$$

□

Corollary 4.14. *If γ is closed then*

$$\int_{\gamma} f = 0.$$

Corollary 4.15. *If f is holomorphic on Ω with $f' = 0$ on Ω then f is constant.*

Proof. Fix $z_0 \in \Omega$, for $z \in \Omega$, take $\gamma : z_0 \rightarrow z$ then

$$0 = \int_{\gamma} f' = f(z) - f(z_0).$$

□

Example 4.16. *Let $z(t) = e^{it}$*

$$\int_{\partial D} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} i e^{it} dt = 2\pi i \neq 0.$$

So $\frac{1}{z}$ does not admit a primitive.

5 Jan 23

5.1 Glossary of Elementary Functions

Glossary of Elementary Functions

1. Polynomials: $P(z) = a_0 + a_1z + \cdots + a_nz^n$ for $a_i \in \mathbb{C}$ and is entirely holomorphic in \mathbb{C} , $a_n \neq 0$ and $n = \deg P$
2. Rational: $R(z) = \frac{P(z)}{Q(z)}$ where P, Q are polynomials with no common factors. The zeros of Q are called poles and their order is called the order of a given pole.
3. Exponential: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

5.2 Fundamental Theorem of Algebra

Lemma 5.1. Let $P(z)$ be a complex polynomial of positive degree n , if $|P(z)|$ has a local minimum at z_0 then $P(z_0) = 0$.

Proof. WLOG $z_0 = 0$ as we can translate $Q(z) = P(z - z_0)$.

AFSOC $P(0) \neq 0$ and WLOG $P(0) = 1$ by $\frac{P}{P(0)}$. We can write $P(z) = 1 + a_kz^k + \cdots + a_nz^n$ with $a_k, a_n \neq 0$.

Consider $z_\delta = \delta e^{i\theta}$. For all $\delta > 0$ sufficiently small:

$$|a_{k+1}z_\delta + \cdots + a_nz_\delta^{n-k}| < \frac{|a_k|}{2}.$$

Then

$$|P(z)| \leq |1 + a_kz_\delta^k| + |a_{k+1}z_\delta^{k+1} + \cdots + a_nz_\delta^n| < |1 + a_kz_\delta^k| + |z_\delta^k| \frac{|a_k|}{2}.$$

Now we can choose δ such that $a_kz_\delta^k < 0$ and real.

$$|1 + a_kz_\delta^k| + |z_\delta^k| \frac{|a_k|}{2} = 1 + a_kz_\delta^k - \frac{1}{2}a_kz_\delta^k = 1 + \frac{1}{2}a_kz_\delta^k < 1.$$

□

Theorem 5.2. Fundamental Theorem of Algebra:

Proof. $|P(z)| \rightarrow +\infty$ as $|z| \rightarrow +\infty$ therefore $\exists R$ such that $\forall |z| > R$, $|P(z)| > |P(0)|$ thus $\inf_{z \in \mathbb{C}} |P(z)| = \inf_{z \in D_R} |P(z)|$ and such a value is attained somewhere at z_0 then by lemma 5.1 we're done. □

Lemma 5.3. $P(z) = a_n(z - z_1) \cdots (z - z_n)$ where z_1, \dots, z_n are roots of P .

The order of a zero z_j is its multiplicity, the number of times it appears in the sequence. If z_j is of the order k_j ,

$$P(z) = \underbrace{Q(z)}_{\text{non-vanishing}} (z - z_j)^{k_j}.$$

A zero z_0 is of order k_0 if and only if $P(z_0) = P'(z_0) = \cdots = P^{(k_0-1)}(z_0) = 0$ and $P^{(k_0)}(z_0) \neq 0$

Theorem 5.4. (Gauss-Lucas) The roots of P' lies in the convex hull of the roots of P . In particular, if P is real rooted then so is P' .

Theorem 5.5. If R a rational function then R' has the same roots as R with order greater by 1.

Theorem 5.6. The following are true

1. $(e^z)' = e^z$
2. $e^{w+z} = e^w e^z$
3. $\overline{e^z} = e^{\bar{z}}$
4. $|e^{iy}| = 1$ and $|e^{x+iy}| = e^x$
5. $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

6 Jan 26

6.1 Cauchy's and Goursat's Theorems

Theorem 6.1. *Cauchy Theorem: If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic on Ω and γ is a closed curve in Ω then*

$$\int_{\gamma} f = 0.$$

Theorem 6.2. *Goursat's Theorem: If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic and Ω is open, then $\forall \Delta \subset \Omega$ we have*

$$\int_{\partial\Delta} f = 0.$$



The integral around the positively oriented (counterclockwise) boundary of any triangle $\Delta \subset \Omega$ vanishes.

Proof. Let $\Delta = \Delta^{(0)}$ and partition into four triangles by connecting the midpoints of the sides:

$$\Delta^{(0)} = \Delta_1^{(1)} \cup \Delta_2^{(1)} \cup \Delta_3^{(1)} \cup \Delta_4^{(1)}.$$



When we integrate along each sub-triangle with positive orientation, the interior edges cancel (traversed in opposite directions, shown with opposing arrows):

$$\int_{\partial\Delta^{(0)}} f = \sum_{j=1}^4 \int_{\partial\Delta_j^{(1)}} f.$$

By the triangle inequality

$$\left| \int_{\partial\Delta^{(0)}} f \right| \leq \sum_{j=1}^4 \left| \int_{\partial\Delta_j^{(1)}} f \right| \leq 4 \max_{j=1,2,3,4} \left| \int_{\partial\Delta_j^{(1)}} f \right|.$$

Thus there exists $j_0 \in \{1, 2, 3, 4\}$ such that

$$\left| \int_{\partial\Delta^{(0)}} f \right| \leq 4 \left| \int_{\partial\Delta_{j_0}^{(1)}} f \right|.$$

Set $\Delta^{(1)} = \Delta_{j_0}^{(1)}$. Repeating this subdivision process, we obtain a nested sequence of triangles

$$\Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \dots$$

such that

$$\left| \int_{\partial \Delta^{(0)}} f \right| \leq 4^n \left| \int_{\partial \Delta^{(n)}} f \right|.$$

Note that $\text{diam}(\Delta^{(n)}) = 2^{-n} \text{diam}(\Delta^{(0)})$ and $|\partial \Delta^{(n)}| = 2^{-n} |\partial \Delta^{(0)}|$.

Since the triangles are nested compact sets with diameters shrinking to zero, by Cantor's intersection theorem:

$$\bigcap_{n=0}^{\infty} \Delta^{(n)} = \{z_0\}$$

for some $z_0 \in \Delta^{(0)} \subset \Omega$. Since f is holomorphic at z_0 , we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

where $\psi(z) \rightarrow 0$ as $z \rightarrow z_0$.

Since $f(z_0)$ and $f'(z_0)(z - z_0)$ have primitives ($f(z_0)z$ and $\frac{1}{2}f'(z_0)(z - z_0)^2$ respectively), their integrals over the closed curve $\partial \Delta^{(n)}$ vanish. Thus:

$$\int_{\partial \Delta^{(n)}} f(z) dz = \int_{\partial \Delta^{(n)}} \psi(z)(z - z_0) dz.$$

For $z \in \partial \Delta^{(n)}$, we have $|z - z_0| \leq \text{diam}(\Delta^{(n)}) = 2^{-n} \text{diam}(\Delta^{(0)})$.

Let $\epsilon_n = \sup_{z \in \partial \Delta^{(n)}} |\psi(z)|$. Since $\psi(z) \rightarrow 0$ as $z \rightarrow z_0$ and $\Delta^{(n)} \rightarrow \{z_0\}$, we have $\epsilon_n \rightarrow 0$. Therefore:

$$\begin{aligned} \left| \int_{\partial \Delta^{(n)}} f(z) dz \right| &\leq \sup_{z \in \partial \Delta^{(n)}} |\psi(z)(z - z_0)| \cdot |\partial \Delta^{(n)}| \\ &\leq \epsilon_n \cdot 2^{-n} \text{diam}(\Delta^{(0)}) \cdot 2^{-n} |\partial \Delta^{(0)}| \\ &= \epsilon_n \cdot 4^{-n} \text{diam}(\Delta^{(0)}) |\partial \Delta^{(0)}| \end{aligned}$$

Combining with our earlier inequality:

$$\left| \int_{\partial \Delta^{(0)}} f \right| \leq 4^n \left| \int_{\partial \Delta^{(n)}} f \right| \leq 4^n \cdot \epsilon_n \cdot 4^{-n} \text{diam}(\Delta^{(0)}) |\partial \Delta^{(0)}| = \epsilon_n \cdot \text{diam}(\Delta^{(0)}) |\partial \Delta^{(0)}|$$

Since $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and the right-hand side is independent of n except for ϵ_n , we conclude:

$$\int_{\partial \Delta} f = 0.$$

□

6.2 Cauchy's Theorem in a Disk

Theorem 6.3. *Cauchy's Theorem in a disk: Let $D \subset \mathbb{C}$ be an open disk and f holomorphic on D . Then for any closed curve γ in D we have*

$$\int_{\gamma} f = 0.$$

Proof. WLOG let D be a disk centered at 0. Define $F : D \rightarrow \mathbb{C}$ by

$$F(z) = \int_{\gamma_z} f$$

where γ_z is the path $0 \rightarrow \Re(z) \rightarrow z$ (horizontal then vertical).



We claim F is a primitive for f , i.e., $F'(z) = f(z)$.

Case 1: $h \in \mathbb{R}$ (horizontal increment).

The path γ_{z+h} goes $0 \rightarrow \Re(z) + h \rightarrow z + h$. Observe that:

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f - \int_{\gamma_z} f.$$



The difference of paths can be decomposed using Goursat's theorem. The integral over the rectangle with vertices $\Re(z)$, $\Re(z) + h$, $z + h$, z vanishes, so:

$$F(z+h) - F(z) = \int_{\Re(z)}^{\Re(z)+h} f(\zeta) d\zeta + \int_z^{z+h} f(\zeta) d\zeta = \int_z^{z+h} f(\zeta) d\zeta$$

where the first integral is along the real axis and the second is horizontal at height $\Im(z)$. By the rectangle lemma, these combine to give just the horizontal segment from z to $z+h$:

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_z^{z+h} f(\zeta) d\zeta \rightarrow f(z) \text{ as } h \rightarrow 0.$$

Case 2: $h = ik$ with $k \in \mathbb{R}$ (vertical increment).

Similarly, γ_{z+ik} goes $0 \rightarrow \Re(z) \rightarrow z + ik$. The paths γ_z and γ_{z+ik} share the horizontal segment $0 \rightarrow \Re(z)$, so:

$$F(z+ik) - F(z) = \int_z^{z+ik} f(\zeta) d\zeta.$$



Thus:

$$\frac{F(z + ik) - F(z)}{ik} = \frac{1}{ik} \int_z^{z+ik} f(\zeta) d\zeta \rightarrow f(z) \text{ as } k \rightarrow 0.$$

General h : For arbitrary $h = h_1 + ih_2$, we write:

$$\frac{F(z + h) - F(z)}{h} = \frac{F(z + h) - F(z + h_1)}{h} + \frac{F(z + h_1) - F(z)}{h}.$$

Using the above cases and the continuity of f , both terms converge appropriately, giving $F'(z) = f(z)$.

Since f has a primitive F on D , by the Fundamental Theorem of Calculus, for any closed curve γ in D :

$$\int_{\gamma} f = F(z(b)) - F(z(a)) = 0.$$

□

7 Jan 28

7.1 Applications of Cauchy's Theorem

Example 7.1.

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1 \iff \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Proof. First, note that by substitution $u = \sqrt{\pi}x$, we have:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du.$$

So the equivalence follows if we can show $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$.

Consider the function $f(z) = e^{-\pi z^2}$, which is entire (holomorphic everywhere). For fixed $y \in \mathbb{R}$, consider the rectangular contour Γ_R with vertices at $-R, R, R + iy, -R + iy$ (traversed counterclockwise).



By Cauchy's theorem:

$$\int_{\Gamma_R} e^{-\pi z^2} dz = 0.$$

Parameterizing the four sides:

$$\begin{aligned} \int_{\Gamma_R} e^{-\pi z^2} dz &= \int_{-R}^R e^{-\pi t^2} dt + \int_0^y e^{-\pi(R+it)^2} i dt \\ &\quad + \int_R^{-R} e^{-\pi(t+iy)^2} dt + \int_y^0 e^{-\pi(-R+it)^2} i dt \end{aligned}$$

We now bound the integrals over the vertical segments. For the right vertical segment, parameterize $z = R + it$ with $t \in [0, y]$ (assuming $y > 0$; the case $y < 0$ is similar). We have:

$$(R + it)^2 = R^2 + 2iRt - t^2 = (R^2 - t^2) + 2iRt$$

so

$$e^{-\pi(R+it)^2} = e^{-\pi(R^2-t^2)} \cdot e^{-2\pi iRt}.$$

Since $|e^{-2\pi iRt}| = 1$ for all real R and t , we have:

$$|e^{-\pi(R+it)^2}| = e^{-\pi(R^2-t^2)} \cdot |e^{-2\pi iRt}| = e^{-\pi(R^2-t^2)} \cdot 1 = e^{-\pi(R^2-t^2)}.$$

For $t \in [0, y]$, we have $t^2 \leq y^2$, so $R^2 - t^2 \geq R^2 - y^2$. Therefore:

$$|e^{-\pi(R+it)^2}| \leq e^{-\pi(R^2-y^2)} = e^{-\pi R^2} e^{\pi y^2}.$$

By the ML-inequality, the length of the path is $|y|$, so:

$$\left| \int_0^y e^{-\pi(R+it)^2} i dt \right| \leq |y| \cdot e^{-\pi R^2} e^{\pi y^2} = |y| e^{\pi y^2} e^{-\pi R^2}.$$

Similarly, for the left vertical segment with $z = -R + it$ where $t \in [y, 0]$:

$$(-R + it)^2 = R^2 - 2iRt - t^2 = (R^2 - t^2) - 2iRt$$

so

$$e^{-\pi(-R+it)^2} = e^{-\pi(R^2-t^2)} \cdot e^{2\pi iRt}.$$

Since $|e^{2\pi iRt}| = 1$ for all real R and t , we have:

$$|e^{-\pi(-R+it)^2}| = e^{-\pi(R^2-t^2)} \cdot |e^{2\pi iRt}| = e^{-\pi(R^2-t^2)} \cdot 1 = e^{-\pi(R^2-t^2)}.$$

For $t \in [y, 0]$ (or equivalently $t \in [0, y]$ if we reverse the parameterization), we have $t^2 \leq y^2$, so:

$$|e^{-\pi(-R+it)^2}| \leq e^{-\pi(R^2-y^2)} = e^{-\pi R^2} e^{\pi y^2}.$$

Therefore:

$$\left| \int_y^0 e^{-\pi(-R+it)^2} i dt \right| \leq |y| \cdot e^{-\pi R^2} e^{\pi y^2} = |y| e^{\pi y^2} e^{-\pi R^2}.$$

Both vertical integrals are bounded by $C e^{-\pi R^2}$ for some constant C depending on y but independent of R . Since $e^{-\pi R^2} \rightarrow 0$ as $R \rightarrow \infty$, we conclude that:

$$\lim_{R \rightarrow \infty} \int_0^y e^{-\pi(R+it)^2} i dt = 0 \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_y^0 e^{-\pi(-R+it)^2} i dt = 0.$$

Therefore, taking the limit as $R \rightarrow \infty$:

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = \int_{-\infty}^{\infty} e^{-\pi(t+iy)^2} dt.$$

Expanding $e^{-\pi(t+iy)^2} = e^{-\pi(t^2+2ity-y^2)} = e^{-\pi t^2} e^{-2\pi ity} e^{\pi y^2}$, we get:

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = e^{\pi y^2} \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-2\pi ity} dt.$$

In particular, taking $y = 0$ gives the standard Gaussian integral. The standard result is $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$, which completes the proof. \square

Example 7.2.

$$\int_0^{\infty} \frac{1 - \cos(x)}{x^2} dx = \frac{\pi}{2}.$$

Proof. Consider the holomorphic function $f(z) = \frac{1-e^{iz}}{z^2}$ on $\mathbb{C} \setminus \{0\}$. Fix ϵ, R and consider the contour Γ which is disk with radius R and a small disk with radius ϵ centered at the origin.



We can express $e^{iz} \approx 1 + iz + g(z)$ where $g(z) = e^{iz} - 1 - iz$ so

$$\begin{aligned} \int_{\gamma_\epsilon} f(z) dz &= \int_{\gamma_\epsilon} \frac{1 - (1 + iz + g(z))}{z^2} dz \\ &= \int_{\gamma_\epsilon} \frac{-g(z)}{z^2} dz + \int_{\gamma_\epsilon} \frac{-i}{z} dz \end{aligned}$$

We can evaluate each part as

$$\int_{\gamma_\epsilon} \frac{dz}{z} = - \int_0^\pi \frac{ie^{it}}{e^{it}} dt = -\pi i.$$

$$\int_{\gamma_\epsilon} \frac{g(z)}{z^2} dz \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

□

8 Jan 30

8.1 Cauchy's Integral Formula

Theorem 8.1. *Cauchy's Integral Formula: Let Ω be open $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Let \bar{D} be open in Ω be a disk then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof. Fix $z \in D_1$, let $\gamma_{\epsilon, \delta}$ be a key-hole contour but we omit z . Then

$$\int_{\gamma_{\epsilon, \delta}} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$



Fix $\epsilon > 0$ and $\delta \rightarrow 0$ we want to show that

$$\int_{\gamma_{\epsilon, \delta}} \frac{f(\zeta)}{\zeta - z} d\zeta \rightarrow \int_{\partial D} - \int_{C_\epsilon}.$$

For

$$\int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_\epsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + f(z) \int_{C_\epsilon} \frac{d\zeta}{\zeta - z}.$$

The first term is bounded as holomorphic so as $\epsilon \rightarrow 0$, it goes to 0. The second part integrates to $f(z) \cdot 2\pi i$. \square

8.2 Regularity and Liouville's Theorem

Theorem 8.2. *Regularity: Holomorphic implies Analytic.*

Let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic, $\bar{D} = \bar{D}_r(z_0) \subset \Omega$ then f is analytic at z_0 i.e

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Proof. We know

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We can write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) + (z_0 - z)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n.$$

Note: $\frac{z-z_0}{\zeta-z_0} < 1$.

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} \underbrace{\sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \frac{f(\zeta)}{\zeta - z}}_{\text{conv. uniformly on } \partial D} d\zeta \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\
&= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\left(\frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right)}_{a_n}
\end{aligned}$$

So $a_n = \frac{f^{(n)}(z_0)}{n!}$. □

Corollary 8.3. *Cauchy's Formula:*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Corollary 8.4. *Cauchy's Inequality:*

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \sup_{\zeta \in \partial D} \left| \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \right| = \frac{n!}{r^n} \|f\|_{\infty}.$$

Theorem 8.5. *Liouville's Theorem: If f entire and bounded then f is constant.*

Proof.

$$|f'(z_0)| \leq \frac{\|f\|_{\infty}}{r}$$

Since f is bounded and r can be taken arbitrarily large (as f is entire), sending $r \rightarrow \infty$ gives $|f'(z_0)| = 0$. Thus, f is constant. □

9 Feb 2

9.1 Rigidity and Analytic Continuation

Theorem 9.1. (*Rigidity Theorem*): Let Ω be a region and $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. If $z_1, z_2, \dots \in \Omega$ distinct sequence with a limit in Ω . If $f(z_n) = 0, \forall n$ then $f = 0$ on Ω .

Proof. Say $z_n \rightarrow \omega \in \Omega$ and $D = D_r(\omega) \subset \Omega$ be a disk centered at ω with radius r . If $f = 0$ on D then by regularity implies

$$f(z) = \sum_{n \geq 0} a_n(z - \omega)^n \text{ on } D.$$

Suppose $f \neq 0$ on D then $\exists m$ such that $a_m \neq 0$, let's take the smallest such m . Then $f(z) = a_m(z - \omega)^m(1 + g(z))$ where $g(z) \rightarrow 0$ as $z \rightarrow \omega$. For all sufficiently large $k, z_k \in D$ then

$$0 = f(z_k) = a_m(z_k - \omega)^m(1 + g(z_k)). \quad (a_m \neq 0 \text{ and } z_k - \omega \neq 0)$$

A contradiction as the RHS is nonzero. To finish $f = 0$ on Ω , we use $\mathcal{U} = \text{int}\{z \in \Omega : f(z) = 0\}$. We have \mathcal{U} is open, $\omega \in \mathcal{U}$ so \mathcal{U} is non-empty. Additionally, \mathcal{U} is closed. We have $\Omega \setminus \mathcal{U}$ is open and non-empty so $\exists z_k \in \Omega \setminus \mathcal{U}$ such that $z_k \rightarrow \omega' \in \Omega \setminus \mathcal{U}$. Then $f(z_k) \neq 0$ for all k so $f \neq 0$ on $D_r(\omega')$. A contradiction as $D_r(\omega') \subset D$. Thus, $f = 0$ on Ω . \square

Corollary 9.2. $f, g : \underset{\text{region}}{\Omega} \rightarrow \mathbb{C}$ holomorphic. If $f(z_n) = g(z_n), \forall z_n \in \Omega$ distinct sequence with a limit in Ω then $f = g$ on Ω .

Definition 9.3. If $\Omega_1 \subset \Omega_2$, are two regions, $f_i : \Omega_i \rightarrow \mathbb{C}$ holomorphic for $i = 1, 2$ such that $f_1 = f_2$ on Ω_1 then we say f_2 is the analytic continuation of f_1 into Ω_2 .

9.2 Morera's Theorem and Uniform Convergence

Theorem 9.4. (*Morera's Theorem*): If $f : \mathcal{D} \rightarrow \mathbb{C}$ continuous such that $\forall \bar{\Delta} \subset \mathcal{D}$ and $\int_{\partial \Delta} f = 0$ then f is holomorphic.

Proof. Repeating the proof of Cauchy's Theorem in a disk, gives that f has a primitive $F = \int_{\gamma_z} f$ in \mathcal{D} . Since F is holomorphic, F' exists but so does F'', F''', \dots so f' exists. \square

Theorem 9.5. If $f_n : \underset{\text{open}}{\Omega} \rightarrow \mathbb{C}$ holomorphic with f_n converging uniformly to f on every compact $K \subset \Omega$ then f is holomorphic.

Proof. Use Morera's Theorem. \square

Moreover, if f'_n converges uniformly to f' , using $\Omega_\delta = \{z \in \Omega : \bar{D}_\delta(z) \subset \Omega\}$.

Claim: $\forall F$ holomorphic in Ω , $\|F'\|_{\infty, \Omega_\delta} \leq \frac{1}{\delta} \|F\|_{\infty, \Omega_\delta}$.

By Cauchy's Formula:

$$|F'(z)| = \left| \frac{1}{2\pi i} \int_{\partial \mathcal{D}_\delta(z)} \frac{F(w)}{(w - z)^2} dw \right| \leq \frac{1}{2\pi} \int_{\partial \mathcal{D}_\delta(z)} \frac{1}{|w - z|^2} \|F\|_{\infty, \Omega} \leq \frac{1}{2\pi} \frac{2\pi\delta}{\delta^2} \|F\|_{\infty, \Omega} \leq \frac{1}{\delta} \|F\|_{\infty, \Omega}.$$

10 Feb 4

10.1 Parameter Integrals and Gluing

Theorem 10.1. *Let $F : \Omega \times [0, 1] \rightarrow \mathbb{C}$ satisfy*

1. $\forall \Delta \in [0, 1], z \mapsto F(z, \Delta)$ *is holomorphic*
2. F *is continuous on $\Omega \times [0, 1]$*

Then $z \mapsto \int_0^1 F(z, \Delta) d\Delta$ is holomorphic on Ω .

Proof. Consider $F_n(z) = \frac{1}{n} \sum_{k=1}^n F(z, \frac{k}{n})$ and by (1) each F_n is holomorphic on Ω .

$$\begin{aligned} \left| F_n(z) - \int_0^1 F(z, \Delta) d\Delta \right| &= \left| F_n(z) - \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} F(z, \Delta) d\Delta \right| \\ &= \left| \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \underbrace{\left[F(z, \frac{k}{n}) - F(z, \Delta) \right]}_{\leq \epsilon} d\Delta \right| \\ &\leq \epsilon \end{aligned}$$

□

Definition 10.2. $\Omega \subset \mathbb{C}$ *be open, symmetric with respect to real axis, $\Omega^+ = \Omega \cap \{z \in \mathbb{C}, \Im(z) > 0\}, \Omega^- = \Omega \cap \{z \in \mathbb{C}, \Im(z) < 0\}$ and $I = \Omega \cap \{\Im(z) = 0\}$*

Theorem 10.3. *Let each $f^\pm : \Omega^\pm \rightarrow \mathbb{C}$ be holomorphic and extend continuous to I such that $f^+ = f^-$ on I . Then*

$$f = \begin{cases} f^+(z) & \text{if } z \in \Omega^+ \\ f^+(z) = f^-(z) & \text{if } z \in I \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}.$$

is holomorphic on Ω .

Proof. Sufficient to handle $z \in I$. Fix such z take $D \subset \Omega$ centered at z . Take $\overline{\Delta} \subset D$ we want to show $\int_{\partial \Delta} f = 0$.

Case 1: If $\overline{\Delta} \subset \Omega^+$ (or Ω^-) then we're done.



Case 2: If a vertex or side of Δ is on I , $\int_{\partial \Delta_\epsilon} f = 0$ for when $\epsilon \rightarrow 0, \partial \Delta_\epsilon \rightarrow \partial \Delta$.



Case 3: If I cuts the triangle then we can split the triangle into smaller triangles and we're done.



The triangle is split into smaller triangles, each either entirely in Ω^+ or Ω^- , or with edges on I , so we can apply Cases 1 and 2. \square

10.2 Schwarz Reflection Principle

Theorem 10.4. *Schwarz Reflection Principle: Let $f : \Omega^+ \rightarrow \mathbb{C}$ be holomorphic and extend continuous to I with $f^+/I \in \mathbb{R}$ then f^+ extends analytically to Ω*

Proof. Let $f^- : \Omega^- \rightarrow \mathbb{C}$ be defined by $f^-(z) = \overline{f^+(\bar{z})}$. We claim f^- is holomorphic on Ω^- , then the previous theorem applies because $f^+ = f^-$ on I .

Fix $z_0 \in \Omega^-$ then $\bar{z}_0 \in \Omega^+$, we know $f^+(\bar{z}) = \sum_{n=0}^{\infty} a_n(\bar{z} - \bar{z}_0)^n$ for all $\bar{z} \in D(\bar{z}_0)$. Thus

$$\overline{f^+(\bar{z})} = \sum_{n=0}^{\infty} \overline{a_n}(\bar{z} - \bar{z}_0)^n.$$

We know $\overline{\bar{z} - \bar{z}_0} = z - z_0$ so

$$f^-(z) = \overline{f^+(\bar{z})} = \sum \overline{a_n}(z - z_0)^n.$$

\square

11 Feb 9

11.1 Meromorphic Functions and Poles

Definition 11.1. Meromorphic functions are "determined" by zeros and singularities.

A point singularity of f is a point $z_0 \in \mathbb{C}$ such that f is defined in a neighborhood of z_0 but not at z_0 .

$$\mathcal{D}_\delta(z_0) \setminus \{z_0\}.$$

Example 11.2. $f(z) = \frac{1}{z}$ has a singularity at $z_0 = 0$.

Remark 11.3. Zeros of a holomorphic f are isolated, unless $f \equiv 0$.

Theorem 11.4. (Local Description Near Zeros) Let $f : \underset{\text{open}}{\Omega} \rightarrow \mathbb{C}$ holomorphic and $f(z_0) = 0$, $z_0 \in \Omega$, $f \not\equiv 0$ then $\exists \mathcal{U} \ni z_0$, $g : \mathcal{U} \rightarrow \mathbb{C}$ holomorphic and vanishing ($\forall z \in \mathcal{U}, g(z) \neq 0$) $\exists! n > 0$ such that $f(z) = (z - z_0)^n g(z)$.

Proof. We know

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

$f \not\equiv 0$ implies that $n =$ smallest k such that $a_k \neq 0$. We can write $f(z) = (z - z_0)^n g(z)$. \square

Definition 11.5. Let $n =$ the multiplicity/order of z_0 . When $n = 1$, z_0 is called a simple zero. We say $f : \mathcal{D}_\delta(z_0) \setminus \{z_0\} \rightarrow \mathbb{C}$ has a pole at z_0 if $\frac{1}{f}$ extended by 0 at z_0 is holomorphic in $\mathcal{D}_\epsilon(z_0)$, for some $0 < \epsilon < \delta$.

Theorem 11.6. (Local Discontinuity near Poles) Let $f : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ has a pole at z_0 in Ω then $\exists \mathcal{U} \ni z_0$, $\exists h : \mathcal{U} \xrightarrow{\text{holo}} \mathbb{C}$ nonvanishing then $\exists! n > 0$ such that $f(z) = (z - z_0)^{-n} h(z)$.

Proof. Apply previous theorem to $\frac{1}{f}(z) = (z - z_0)^n g(z)$. \square

11.2 Laurent Series and Residues

Theorem 11.7. (Laurent Series Expansion) If f has a pole of order n at z_0 then locally

$$f(z) = \underbrace{\frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{\overbrace{a_{-1}}^{\text{residue of } f \text{ at } z_0}}{z - z_0}}_{\text{principal part}} + \underbrace{G(z)}_{\text{holo part of } f}.$$

where $G(z)$ is holomorphic and nonvanishing near z_0 .

Moreover, $\text{res}_{z_0}(f) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n f(z))$

Proof. By theorem 11.6,

$$f(z) = (z - z_0)^{-n} h(z) = (z - z_0)^{-n} \sum_{k=0}^{\infty} b_k (z - z_0)^k = \frac{b_0}{(z - z_0)^n} + \cdots + \frac{b_{n-1}}{(z - z_0)} + \sum_{k \geq n} b_k (z - z_0)^{k-n}.$$

So we have

$$f(z)(z - z_0)^n = b_0 + b_1(z - z_0) + \cdots + b_{n-1}(z - z_0)^{n-1} + \underbrace{O((z - z_0)^n)}_{\rightarrow 0 \text{ as } \frac{d^{n-1}}{dz^{n-1}}}$$

\square

11.3 The Residue Formula

Theorem 11.8. (*The Residue Formula*) Suppose $f : \Omega \setminus \underbrace{\{z_1, z_2, \dots, z_n\}}_{\text{poles}} \rightarrow \mathbb{C}$ is holomorphic. Then for

a disk $\overline{\mathcal{D}} \subset \Omega$ containing z_1, \dots, z_n we have

$$\int_{\partial \mathcal{D}} f = 2\pi i \sum_{k=1}^n \text{res}_{z_k}(f).$$

Proof. Let $\gamma = \partial \mathcal{D}$ be the boundary of the disk \mathcal{D} , oriented counterclockwise. For each pole z_k , choose a small disk \mathcal{D}_k centered at z_k with radius $\varepsilon_k > 0$ small enough so that:

- $\overline{\mathcal{D}_k} \subset \mathcal{D}$ for all k
- $\overline{\mathcal{D}_k} \cap \overline{\mathcal{D}_j} = \emptyset$ for $k \neq j$
- f is holomorphic on $\overline{\mathcal{D}} \setminus \bigcup_{k=1}^n \mathcal{D}_k$

Let $\gamma_k = \partial \mathcal{D}_k$ be the boundary of each small disk, oriented clockwise (negative orientation). Consider the multiply connected region $\mathcal{D} \setminus \bigcup_{k=1}^n \overline{\mathcal{D}_k}$.



By Cauchy's theorem for multiply connected regions, the integral of f around the boundary of this region (taking into account orientations) is zero. The boundary consists of γ (counterclockwise) and each γ_k (clockwise), so:

$$\int_{\gamma} f(z) dz + \sum_{k=1}^n \int_{\gamma_k} f(z) dz = 0.$$

Note that γ_k has clockwise orientation, so reversing it gives:

$$\int_{\gamma} f(z) dz = \sum_{k=1}^n \int_{-\gamma_k} f(z) dz.$$

where $-\gamma_k$ denotes γ_k with counterclockwise orientation.

Now, for each pole z_k , consider its Laurent expansion:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j^{(k)} (z - z_k)^j = \frac{a_{-1}^{(k)}}{z - z_k} + \sum_{j \neq -1} a_j^{(k)} (z - z_k)^j.$$

where $a_{-1}^{(k)} = \text{res}_{z_k}(f)$.

For $j \neq -1$, the function $(z - z_k)^j$ has an antiderivative on $\mathbb{C} \setminus \{z_k\}$ (or on all of \mathbb{C} if $j \geq 0$), so by the fundamental theorem of calculus:

$$\int_{-\gamma_k} (z - z_k)^j dz = 0 \quad \text{for } j \neq -1.$$

For $j = -1$, we compute directly. Parameterize $-\gamma_k$ by $z(t) = z_k + \varepsilon_k e^{it}$ for $t \in [0, 2\pi]$:

$$\int_{-\gamma_k} \frac{1}{z - z_k} dz = \int_0^{2\pi} \frac{1}{\varepsilon_k e^{it}} \cdot i\varepsilon_k e^{it} dt = \int_0^{2\pi} i dt = 2\pi i.$$

Therefore,

$$\int_{-\gamma_k} f(z) dz = a_{-1}^{(k)} \cdot 2\pi i = 2\pi i \cdot \text{res}_{z_k}(f).$$

Combining all terms:

$$\int_{\partial\mathcal{D}} f(z) dz = \sum_{k=1}^n \int_{-\gamma_k} f(z) dz = 2\pi i \sum_{k=1}^n \text{res}_{z_k}(f).$$

□

12 Feb 11

12.1 Integral Evaluation via Residues

Example 12.1.

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(a\pi)}.$$

Proof. Let $f(z) = \frac{e^{az}}{1+e^z}$ then we have simple poles at $z : e^z = 1$ so $z \in \{\dots, -\pi i, \pi i, 3\pi i, \dots\}$. We have the residual for

$$\int_{\gamma_1} f + \dots = \int_{\gamma_R} f(z) dz = 2\pi i \operatorname{res}_{\pi i}(f).$$



We have

$$\int_{\gamma_1} f(z) dz \xrightarrow{R \rightarrow +\infty} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = I.$$

On γ_3 , $z = x + 2\pi i$ with x from R to $-R$, so

$$\int_{\gamma_3} f(z) dz = - \int_{-R}^R \frac{e^{a(x+2\pi i)}}{1+e^x} dx = -e^{2\pi ai} \cdot I.$$

As $R \rightarrow \infty$, $\int_{\gamma_2} f \rightarrow 0$ and $\int_{\gamma_4} f \rightarrow 0$. Thus

$$(1 - e^{2\pi ai})I = 2\pi i \cdot \operatorname{res}_{\pi i}(f) = 2\pi i \cdot (-e^{a\pi i}) = -2\pi i e^{a\pi i},$$

$$\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}} i dy \right| \leq \int_0^{2\pi} \frac{e^{aR}}{e^R + 1} dy \xrightarrow{R \rightarrow \infty} 0$$

Additionally $\int_{\gamma_4} f \rightarrow 0$. To compute the residue at πi , we have

$$\lim_{z \rightarrow \pi i} f(z)(z - \pi i) = \lim_{z \rightarrow i\pi} \frac{e^{az}}{e^z - e^{\pi i}} = \frac{e^{\pi ia}}{e^{\pi i}}.$$

Letting $R \rightarrow +\infty$ in the residual formula gives

$$I + 0 + 0 - e^{2\pi ai} I = -2\pi i e^{a\pi i}.$$

So

$$I = -\frac{2\pi i e^{\pi ia}}{1 - e^{2\pi ai}} = \frac{\pi}{\sin(a\pi)}.$$

□

12.2 Removable Singularities

Definition 12.2. Let $f : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic, if we can extend f analytically to z_0 we say that f has a removable singularity at z_0 .

Theorem 12.3. (Riemann's Theorem) Let $f : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be holomorphic, if f is bounded near z_0 then z_0 is a removable singularity.

Proof. Let $\overline{D} \subset \Omega$ be a disc centered at z_0 . We want to use Cauchy's formula

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw = g(z), \text{ for all } z \in D.$$

It suffices to show that $f = g$ on $D \setminus \{z_0\}$ because then g is the desired extension of f .

$F(w, z)$ is jointly continuous on $\partial D \times \overline{D}_{r-\epsilon}(z_0)$ and $\forall w \in \partial D, z \mapsto F(w, z)$ is holomorphic on D .

Fix $z \in D \setminus \{z_0\}$ then

$$\int_{\gamma} \frac{f(w)}{w - z} dw = 0 \implies \int_{\partial D} \frac{f(w)}{w - z} dw = \int_{C_{\epsilon}(z_0)} \frac{f(w)}{w - z} dw + \int_{C_{\epsilon}(z)} \frac{f(w)}{w - z} dw = 0 + 2\pi i f(z).$$

□

13 Feb 13

13.1 Classification of Singularities

Theorem 13.1. (Riemann) Let $f : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ has a removable singularity at z_0 if and only if bounded near z_0 .

Corollary 13.2. Pole at z_0 if and only if $|f| \rightarrow +\infty$ as $z \rightarrow z_0$

Proof. (\implies) Local description near poles implies $f(z) = (z - z_0)^{-n}g(z)$ so $|f| \rightarrow +\infty$

(\impliedby) Since $\left|\frac{1}{f}\right| \rightarrow 0$, in particular $\frac{1}{f}$ is bounded near z_0 , Riemann implies $\frac{1}{f}$ has a removable singularity at z_0 . \square

Theorem 13.3. (Casorati-Weierstrass) If $f : D_r(z_0) \setminus \{z_0\} \rightarrow \mathbb{C}$ is holomorphic with an essential singularity at z_0 then the image of f is dense in \mathbb{C} .

Proof. If not, $\exists w \in \mathbb{C}$ such that no values of f near w , $\exists \delta > 0$ s.t. $\forall z, |f(z) - w| > \delta$. Consider

$$g(z) = \frac{1}{f(z) - w} : D_r(z_0) \setminus \{z_0\} \rightarrow \mathbb{C}.$$

Such a g is bounded by $\frac{1}{\delta}$. So g extends holomorphically into $D_r(z_0)$.

(Case 1) $g(z_0) = 0 \implies \frac{1}{g} \rightarrow +\infty$ so $f(z) - w$ has a pole so f has a pole

(Case 2) $g(z_0) \neq 0 \implies \frac{1}{g}$ is well defined and f has a removable singularity. \square

Theorem 13.4. (Picard's Theorem) If $f : D_r(z_0) \setminus \{z_0\} \rightarrow \mathbb{C}$ is holomorphic with an essential singularity at z_0 then the image of f is either \mathbb{C} or \mathbb{C} except for at most one value.

13.2 Meromorphic Functions and Behavior at Infinity

Theorem 13.5. Let $\Omega \subset \mathbb{C}$ be open. A function f is called meromorphic on Ω if $\exists z_1, z_2, \dots$ without a limit point such that f is holomorphic in $\Omega \setminus \{z_1, z_2, \dots\}$ and has poles at z_1, z_2, \dots .

Theorem 13.6. (Behavior at $+\infty$) Suppose f is holomorphic in a neighborhood of ∞ i.e. $\{|z| > R\}$ for some $R > 0$.

Note: $F(z) = f\left(\frac{1}{z}\right)$ is holomorphic in $D_{\frac{1}{R}}(0) \setminus \{0\}$.

We say f has a removable singularity at ∞ if F has a removable singularity at $z = 0$.

Example 13.7. e^z has an essential singularity at $+\infty$.

Remark 13.8. We'll denote $\overline{\mathbb{C}} = \mathbb{C} \cup \{+\infty\}$

Example 13.9. Examples of meromorphic functions on \mathbb{C}

1. Rational functions: $\frac{P}{Q}$
2. The gamma function: $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ (simple poles at $\dots, -2, -1, 0$)
3. The Riemann zeta function: $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ (simple poles at $s = -1, -2, -3, \dots$)

14 Feb 16

14.1 Rational Functions via Meromorphy

Theorem 14.1. *Suppose f is meromorphic on \mathbb{C} with a removable singularity or a pole at $+\infty$ then f is a rational function.*

Proof. Claim: Under these assumptions, f has finitely many poles.

If not, z_1, z_2, \dots are poles and this sequence cannot be bounded otherwise it got a limit point. Say $|z_k| \rightarrow +\infty$, the neighborhoods of ∞ are $\{|z| > R\}$ so f not holomorphic for any $R > 0$ which contradicts that f has a removable singularity at $+\infty$.

Let z_1, \dots, z_N be the poles of f . Near z_k , for each $1 \leq k \leq N$,

$$f(z) = \underbrace{f_k(z)}_{\substack{\text{principal part} \\ \text{poly in } \frac{1}{z-z_k}}} + \underbrace{g_k(z)}_{\text{holomorphic}}.$$

Near ∞ ,

$$f\left(\frac{1}{z}\right) = \underbrace{\tilde{f}_\infty(z)}_{\substack{\text{principal part} \\ \text{poly in } \frac{1}{z}}} + \underbrace{\tilde{g}_\infty(z)}_{\text{holomorphic}}.$$

Consider,

$$H(z) = f(z) - \underbrace{\sum_{j=1}^N f_j(z)}_{\text{rational}} - \underbrace{\tilde{f}_\infty(z)}_{\text{poly}}.$$

Claim: H is holomorphic and bounded.

Near z_k

$$H = f - f_k - \sum_{j \neq k} f_j - \tilde{f}_\infty.$$

□