

keeping advantage of the localization feature of vortex methods is to homogenize the grid by moving randomly each particle within a square of size h before resampling (see Ref. 37 for the convergence proof of a method in this spirit in which resampling is done with a completely random choice).

All these methods, except the last one based on random rezoning, are second order in space. Their time accuracy is limited only by the splitting of the Navier–Stokes equations. This means that a high-order splitting method, like the one analyzed in Ref. 24, which alternates half steps of advection and diffusion, virtually yields second-order methods.

5.4. The Method of Particle Strength Exchange

In this section we derive a general framework for the construction of high-order viscous algorithms for vortex methods introduced by Mas-Gallic in 1987 [147].

The starting point is the Lagrangian description of the Navier–Stokes equations as approximated on the particle locations:

$$\frac{d\omega_p}{dt} = v \Delta\omega_p. \quad (5.4.1)$$

The goal of the method is to derive formulas for consistent evaluations of $\Delta\omega_p$ on the particle locations.

5.4.1. Design

The key idea is to replace the diffusion operator by an integral one. The links between integral and diffusion operators have long been exploited in the field of kinetic equations, but in general in the other direction, namely, to derive a diffusion approximation of integral operators that model collisions of particles. The motivation here is that integrals are much better suited for particle methods than second-order differential operators: As already observed on many occasions, inserting particles inside an integral is indeed equivalent to writing a numerical quadrature of this integral.

Let us now describe in detail the resulting methods. Let η be a kernel satisfying the following moment properties:

$$\begin{aligned} \int x_i x_j \eta(\mathbf{x}) d\mathbf{x} &= 2 \delta_{ij} \quad \text{for } i, j = 1, 2, \\ \int x_1^{i_1} x_2^{i_2} \eta(\mathbf{x}) d\mathbf{x} &= 0 \quad \text{if } i_1 + i_2 = 1 \text{ or } 3 \leq i_1 + i_2 \leq r + 1, \\ \int |\mathbf{x}|^{r+2} |\eta(\mathbf{x})| d\mathbf{x} &< \infty. \end{aligned} \quad (5.4.2)$$

For $\varepsilon > 0$ we set $\eta_\varepsilon(\mathbf{x}) = \varepsilon^{-2} \eta(\mathbf{x}/\varepsilon)$ (recall that we consider the two-dimensional case; here and subsequently appropriate modifications need to be done for three dimensions). We then consider the integral operator Δ_ε defined by

$$\Delta_\varepsilon \omega(\mathbf{x}) = \varepsilon^{-2} \int [\omega(\mathbf{y}) - \omega(\mathbf{x})] \eta_\varepsilon(\mathbf{y} - \mathbf{x}) d\mathbf{y}. \quad (5.4.3)$$

This provides an integral approximation of the Laplace operator according to the following result.

Proposition 5.4.1. *Under assumptions (5.4.2), if $\omega \in W^{r+2,p}(\mathbf{R}^2)$ for some $p \in [1, +\infty]$, then the following estimate holds (with the notation defined in Appendix A):*

$$\|\Delta_\varepsilon \omega - \Delta \omega\|_{0,p} \leq C \varepsilon^r \|\omega\|_{r+2,p}. \quad (5.4.4)$$

Proof. Let us check this assertion only for $p = +\infty$. The Taylor expansion of ω at the point \mathbf{x} to the order $r + 2$ yields

$$\begin{aligned} \omega(\mathbf{y}) &= \omega(\mathbf{x}) + (\mathbf{y} - \mathbf{x}, \nabla) \omega(\mathbf{x}) + \cdots + \frac{1}{(r+1)!} (\mathbf{y} - \mathbf{x}, \nabla)^{(r+1)} \omega(\mathbf{x}) \\ &\quad + O(|\mathbf{y} - \mathbf{x}|^{r+2} \|\omega\|_{r+2,\infty}). \end{aligned}$$

When this expansion is used on the right-hand side of Eq. (5.4.3), by assumptions (5.4.2), the contributions of all the terms of the form $\int (\mathbf{y} - \mathbf{x})^\alpha \eta_\varepsilon(\mathbf{y} - \mathbf{x}) d\mathbf{y}$ vanish except for the terms

$$\begin{aligned} &\frac{\partial^2 \omega}{\partial x_1^2}(\mathbf{x}) \int (y_1 - x_1)^2 \eta_\varepsilon(\mathbf{y} - \mathbf{x}) d\mathbf{y} + \frac{\partial^2 \omega}{\partial x_2^2}(\mathbf{x}) \int (y_2 - x_2)^2 \eta_\varepsilon(\mathbf{y} - \mathbf{x}) d\mathbf{y} \\ &+ \|\omega\|_{r+2,\infty} O \left[\int |\mathbf{y} - \mathbf{x}|^{r+2} \eta_\varepsilon(\mathbf{y} - \mathbf{x}) d\mathbf{y} \right], \end{aligned}$$

which, after the change of variables $\mathbf{z} = [(\mathbf{y} - \mathbf{x})/\varepsilon]$ in the integrals, yields

$$\Delta_\varepsilon \omega(\mathbf{x}) = \Delta \omega(\mathbf{x}) + O(\varepsilon^r).$$

□

Particle approximation of the diffusion can then be defined from numerical integrations of Δ_ε . The resulting scheme is written as

$$\omega^h(\mathbf{x}, t) = \sum_p v_p \omega_p^h(t) \delta(\mathbf{x} - \mathbf{x}_p^h), \quad (5.4.5)$$

where the trajectories of the particles \mathbf{x}_p^h are computed with the velocity field

$\mathbf{u}^h = \mathbf{K}_\varepsilon \star \omega^h$ and $\omega_p^h(t)$ obey the following system of ODEs

$$\frac{d\omega_p^h}{dt} = v\varepsilon^{-2} \sum_q (v_q \omega_q^h - v_q \omega_p^h) \eta_\varepsilon(\mathbf{x}_q^h - \mathbf{x}_p^h). \quad (5.4.6)$$

Concerning moment conditions (5.4.2), let us first observe that, for symmetry reasons, any function η with radial invariance would satisfy

$$\begin{aligned} \int x_i \eta(\mathbf{x}) d\mathbf{x} &= \int x_1 x_2 \eta(\mathbf{x}) d\mathbf{x} = 0, \\ \int x_1^2 \eta(\mathbf{x}) d\mathbf{x} &= \int x_2^2 \eta(\mathbf{x}) d\mathbf{x} \end{aligned}$$

and thus would provide, on proper rescaling, a kernel satisfying conditions (5.4.2) for $r = 2$. Several constructions of the kernel η are proposed in Ref. 73. They in particular relate moment conditions (5.4.2) to the cutoff conditions determining the accuracy of vortex blobs in inviscid calculations (see Section 2.4). One possibility is to use

$$\eta = \Delta\zeta.$$

If ζ is a cutoff of the order of r , then it is straightforward to check that η satisfies conditions (5.4.2) with the same value of r . This choice was later rediscovered by Fishelov. However, Fishelov's scheme, which is derived in Ref. 83 from the approximation $\Delta\omega \simeq \omega \star \Delta\zeta_\varepsilon$, is given by

$$\frac{d\omega_p^h}{dt} = v \sum_q (v_q \omega_q^h) \Delta\zeta_\varepsilon(\mathbf{x}_q^h - \mathbf{x}_p^h).$$

Therefore it does not incorporate the correction term $-\omega_p \sum_q v_q \Delta\zeta_\varepsilon(\mathbf{x}_p^h - \mathbf{x}_q^h)$, which in Eq. (5.4.5) implicitly results from the integral approximation approach. Like the resampling scheme of Eq. (5.3.2), it therefore fails to be conservative.

Another systematic construction of arbitrarily high-order kernels consists of using the formula

$$\eta(\mathbf{x}) = -2 \frac{\nabla\zeta(\mathbf{x}) \cdot \mathbf{x}}{|\mathbf{x}|^2}. \quad (5.4.7)$$

If ζ satisfies the cutoff moment properties at order r (in the sense of Section 2.3), then it is readily seen that η satisfies Eq. (5.4.2) at the same order. For a cutoff

with radial symmetry, as is most often the case, this formula gives

$$\eta(\mathbf{x}) = -2 \frac{\zeta'(|\mathbf{x}|)}{|\mathbf{x}|}.$$

It is noteworthy that, if one starts from the Gaussian kernel $\mathcal{G}(\mathbf{x}) = (4\pi)^{-1} \exp(-|\mathbf{x}|^2/4)$, the above construction gives $\eta = \mathcal{G}$. For a more general presentation of these methods and their extension to anisotropic diffusion, we refer to Ref. 73.

One advantage of the PSE schemes just described over the resampling methods introduced in Section 5.3 is that their definition does not require any viscous splitting of the Navier–Stokes equations. The time discretization of Eq. (5.4.6) can be done through classical time-stepping schemes, leading to a method that can be virtually of any order in space and time, provided the proper choice of η is made. It is, however, interesting to note that, if system (5.4.6) is discretized by the explicit forward Euler scheme with a time step δt , one advances from time $t_n = n\delta t$ to time t_{n+1} by

$$\omega_p^{n+1} = \omega_p^n + \frac{v\delta t}{\varepsilon^2} \sum_q v_q (\omega_q^n - \omega_p^n) \eta_\varepsilon(\mathbf{x}_q^n - \mathbf{x}_p^n), \quad (5.4.8)$$

which is the resampling scheme of Eq. (5.3.3), with η instead of \mathcal{G} and $\varepsilon^2 = v\delta t$. As stated in Section 5.3, resampling schemes can thus be seen as a particular low-order time discretization of PSE schemes. Finally, as we will see in Subsection 5.4.3, the PSE schemes can also be formulated for spatially varying ε .

5.4.2. Numerical Analysis

Before sketching the analysis of these methods in the linear case, let us point out the particular role played by nonnegative even kernels η . For this class of kernels, one can easily check that the algorithm is unconditionally enstrophy decreasing, which is what can be expected for the Navier–Stokes equations in the absence of boundaries.

If we indeed define by $\mathcal{E}^h(t) = \sum_p v_p |\omega_p^h(t)|^2$ a discrete version of the enstrophy $\mathcal{E}(t) = \int |\boldsymbol{\omega}(\mathbf{x}, t)|^2 d\mathbf{x}$, we get from Eq. (5.4.6)

$$\frac{1}{2} \dot{\mathcal{E}}^h = \sum_p v_p \dot{\omega}_p^h \omega_p^h = v \varepsilon^{-2} \sum_{p,q} v_p v_q (\omega_q^h - \omega_p^h) \omega_p^h \eta_\varepsilon(\mathbf{x}_q^h - \mathbf{x}_p^h).$$

Writing then $\omega_p = \frac{1}{2}(\omega_p + \omega_q) + \frac{1}{2}(\omega_p - \omega_q)$ and observing that, by symmetry, since η is even, the contribution of the first term in the sum is zero, we obtain

$$\dot{\mathcal{E}}^h = -v \varepsilon^{-2} \sum_{p,q} v_p v_q (\omega_q^h - \omega_p^h)^2 \eta_\varepsilon(\mathbf{x}_q^h - \mathbf{x}_p^h) \leq 0,$$

provided that $\eta > 0$. Furthermore it can be shown (see Ref. 62) that, on reinitialization of the particle distribution to guarantee the overlapping of particles for all time, the discrete enstrophy has the same decay property as for the continuous equation, that is,

$$\mathcal{E}^h(t) \leq \frac{C}{vt},$$

where C depends on only the initial vorticity.

The drawback of nonnegative kernels is that they are second order only, as the moments of order 4 cannot vanish.

We give now the main steps of the convergence analysis in the linear case for general diffusion approximations (we refer to Ref. 73 for details). We focus again on the linear convection-diffusion equations (5.2.4) and (5.2.5), and we assume that the initial particle distribution consists of particles located at the nodes of a uniform grid of mesh size h .

We first give some notation: $S^v(t)$, and $S_\varepsilon^v(t)$ will respectively denote the linear operators that give the solutions, at time t , of Eq. (5.2.4) and of the same equation with Δ replaced by Δ_ε (susequently referred to as the Δ_ε equation). We first observe that

$$\Delta_\varepsilon \omega = \varepsilon^{-2}(\omega \star \eta_\varepsilon - \lambda \omega),$$

where $\lambda = \int \eta(\mathbf{x}) d\mathbf{x}$. This implies that, for all $p \in [1, \infty]$,

$$\|\Delta_\varepsilon \omega\|_{L^p} \leq C \varepsilon^{-2} \|\omega\|_{L^p}, \quad (5.4.9)$$

and the Δ_ε equation immediately yields

$$\frac{d}{dt} \|S_\varepsilon^v(t)\omega_0\|_{L^2} \leq C v \varepsilon^{-2} \|S_\varepsilon^v(t)\omega_0\|_{L^2}.$$

We thus obtain a finite bound for the enstrophy (for finite times) only if $v\varepsilon^{-2} = O(1)$. As a result, the Δ_ε model and the particle method that has been derived from it, must be seen as a vanishing viscosity model (however, if $\eta \geq 0$, it does provide a valid approximation for fixed v and $\varepsilon \rightarrow 0$).

From now on we assume that the parameter ε is linked with v by

$$v/\varepsilon^2 \approx O(1). \quad (5.4.10)$$

If we subtract the Δ_ε equation from Eq. (5.2.4) and use relation (5.4.4), we obtain, under the assumptions of Eq. (5.4.2) and provided that $S^v \omega_0$ is smooth

enough ($\omega_0 \in W^{r+2,\infty}$ is actually a sufficient condition),

$$\frac{\partial(S_\varepsilon^v - S^v)\omega_0}{\partial t} + \operatorname{div}[\mathbf{a}(S_\varepsilon^v - S^v)\omega_0] - v\Delta_\varepsilon[(S_\varepsilon^v - S^v)\omega_0] = O(v^{r/2}).$$

We now use relation (5.4.9) and approximation (5.4.10) together with the stability properties of the advection equation to deduce, for $t \leq T$,

$$\|S_\varepsilon^v(t)\omega_0 - S^v(t)\omega_0\|_{L^\infty} \leq C(T)v^{r/2}. \quad (5.4.11)$$

The Δ_ε equation can thus be viewed as giving the $[r/2]$ first terms in an asymptotic expansion, with respect to v , of the solution to the convection-diffusion equation with small viscosity. This result easily extends to the nonlinear Navier–Stokes equations (one must then use the elliptic regularity properties stated in Section A.3 to deal with the nonlinear terms in the advection equation).

Let us now continue our analysis and in particular investigate the effect of the particle approximation itself. To denote the particle approximation of Δ_ε on the moving particles, let us set

$$\Delta_{\varepsilon,h,t}\omega = \varepsilon^{-2} \sum_p v_p \{\omega[\mathbf{x}_p(t)] - \omega(\mathbf{x})\} \eta_\varepsilon[\mathbf{x}_p(t) - \mathbf{x}],$$

where the subscript h refers to the particle discretization and $\omega = S^v\omega_0$, $\omega_\varepsilon = S_\varepsilon^v\omega_0$. We also define

$$e_p(t) = \omega_p(t) - \omega_\varepsilon[\mathbf{x}_p(t), t].$$

By subtracting Eq. (5.4.6) from Eq. (5.2.4), we obtain

$$\frac{de_p}{dt} = (\Delta_\varepsilon - \Delta_{\varepsilon,h,t})\omega_\varepsilon[\mathbf{x}_p(t), t] + v\varepsilon^{-2} \sum_q v_q (e_q - e_p) \eta_\varepsilon(\mathbf{x}_p - \mathbf{x}_q). \quad (5.4.12)$$

We next observe that the definition of the particle paths \mathbf{X} together with a straightforward application of the quadrature error estimate (A.1.3) allows us to write

$$\begin{aligned} \Delta_{\varepsilon,h,t}\omega_\varepsilon(\mathbf{x}) &= \varepsilon^{-2} \int \{\omega_\varepsilon[\mathbf{X}(t; \mathbf{y}, 0)] - \omega_\varepsilon(\mathbf{x})\} \eta_\varepsilon[\mathbf{X}(t; \mathbf{y}, 0) - \mathbf{x}] d\mathbf{y} \\ &\quad + O(h^m \varepsilon^{-m-2}) = \Delta_\varepsilon \omega_\varepsilon(\mathbf{x}) + O(h^m \varepsilon^{-m-2}). \end{aligned} \quad (5.4.13)$$

This estimate is actually valid on suitable smoothness assumptions on ω_ε and

on the particle paths \mathbf{X} , which in turn respectively follow from the smoothness of ω_0 and the velocity field \mathbf{a} .

We also note that

$$\begin{aligned} & \left| \sum_q v_q (e_q - e_p) \eta_\varepsilon(\mathbf{x}_p - \mathbf{x}_q) \right| \\ & \leq 2 \max_q |e_q| \sum_q v_q |\eta_\varepsilon(\mathbf{x}_q - \mathbf{x}_p)| \leq \max_q |e_q| \\ & \quad \left[\int |\eta(\mathbf{x})| d\mathbf{x} + O(h^m \varepsilon^{-m}) \right]. \end{aligned} \quad (5.4.14)$$

Combining relations (5.4.12), (5.4.13), and (5.4.14) thus leads to

$$\max_p |e_p(t)| \leq C \left[h^m \varepsilon^{-m} + (1 + h^m \varepsilon^{-m}) \int_0^t \max_p |e_p(s)| ds \right].$$

If the usual overlapping condition $h \leq \varepsilon$ is met, this shows, by Gronwall's theorem, that

$$\max_p |e_p(t)| = O(h^m \varepsilon^{-m}).$$

It remains to recall relation (5.4.11) to obtain the error estimate (stated in terms of v for convenience):

$$\max_p |\omega_p(t) - \omega[\mathbf{x}_p(t), t]| = O(v^{r/2} + h^m v^{-m/2}).$$

This result is the expected convergence result for the semidiscrete particle method. Let us now briefly comment on the stability requirements for the fully discrete algorithm, and in particular the already mentioned explicit first-order time discretization (5.4.8) of Eq. (5.4.6). Under the scaling $v = C\varepsilon^2$ one easily deduces from Eq. (5.4.8) that

$$\begin{aligned} \max_p |\omega_p^{n+1}| & \leq \max_p |\omega_p^n| \left\{ 1 + C\delta t \left[\sum_q v_q |\eta_\varepsilon(\mathbf{x}_q - \mathbf{x}_p)| \right] \right\} \\ & \leq \max_p |\omega_p^n| \{1 + C\delta t [\lambda + O(h^m \varepsilon^{-m})]\}. \end{aligned}$$

This shows that the algorithm is stable, provided that $h = O(\varepsilon)$. Now if one

uses nonnegative cutoff, which allows for smaller values of ε , one gets

$$\max_p |\omega_p^{n+1}| \leq \max_p |\omega_p^n|,$$

if

$$v\delta t\varepsilon^{-2} \sum_q v_q \eta_\varepsilon(\mathbf{x}_q - \mathbf{x}_p) \leq 1. \quad (5.4.15)$$

Since

$$\sum_q v_q \eta_\varepsilon(\mathbf{x}_q - \mathbf{x}_p) \simeq \lambda = \int \eta(\mathbf{x}) d\mathbf{x}, \quad (5.4.16)$$

relation (5.4.15) is a stability condition similar to that found in finite-difference methods. However, if one uses relation (5.4.15) to choose the maximal time step allowed, one has to be careful that, in the absence of remeshing, $\sum_q v_q |\eta_\varepsilon(\mathbf{x}_q - \mathbf{x}_p)|$ can significantly differ from λ in parts of the flow where particle spacings get highly distorted, e.g., around stagnation points. Of course at these points the accuracy of the algorithm is questionable (the apparent conflict with the error estimates comes from the fact that, as in the inviscid case, these estimates involve constants that depend, in particular, on the smoothness of the flow map; these constants can grow exponentially in time), and local regridding is certainly necessary. However, it may be desirable to have a method that at least does not blow up in these situations and does not require choosing a time step much smaller than needed in the rest of the flow. One way suggested in Ref. 62 is to modify PSE scheme (5.4.8) into

$$\omega_p^{n+1} = \omega_p^n + v\delta t\varepsilon^{-2} \lambda \frac{\sum_q v_q (\omega_q - \omega_p) \eta_\varepsilon(\mathbf{x}_q - \mathbf{x}_p)}{\sum_q v_q |\eta_\varepsilon(\mathbf{x}_q - \mathbf{x}_p)|}. \quad (5.4.17)$$

In this formulation the algorithm is stable under the condition

$$v\delta t\varepsilon^{-2} \lambda \leq 1.$$

5.4.3. PSE Schemes with Variable-Size Blobs

The PSE methodology has enough flexibility in dealing with viscous effects to allow the treatment of variable viscous scales in the vorticity redistribution scheme. The use of variable-blob size has already been considered in the inviscid case (see Subsection 2.6.3). The underlying idea is that one may wish to