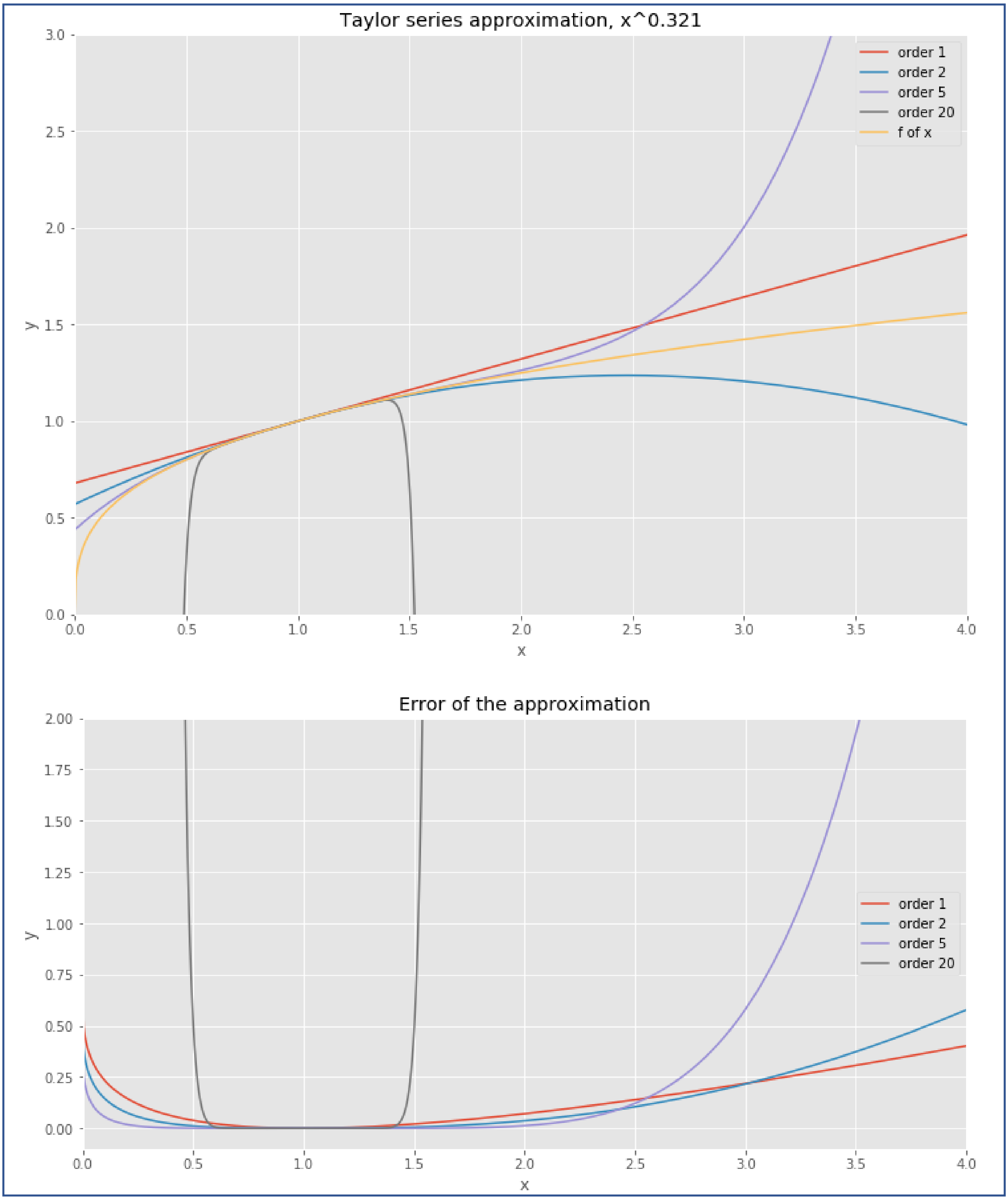
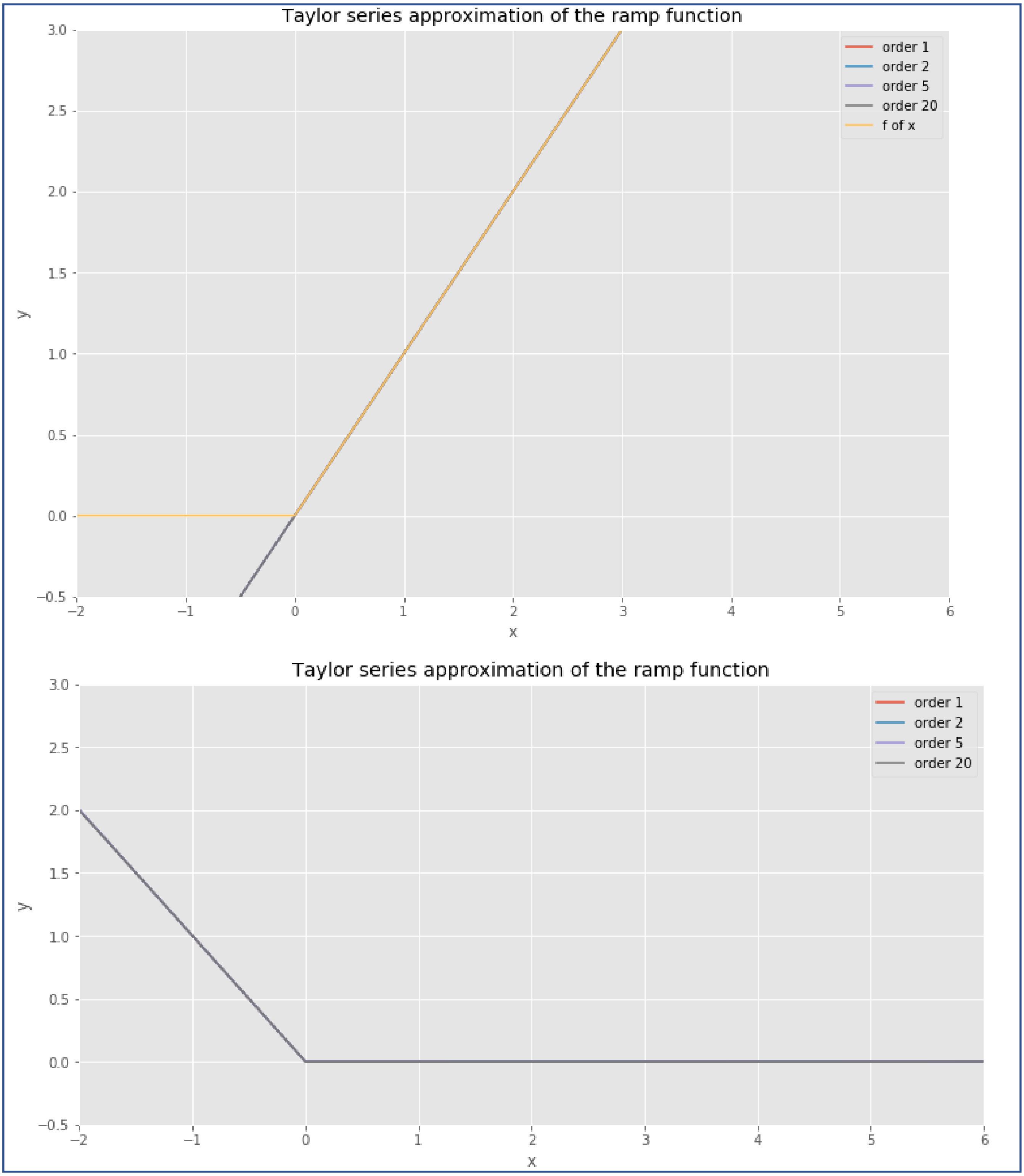
Homework 1 – Adam Wilczyński

Question 1.1



Comment: Error of the approximation of the 20th order is very small in the neighbourhood of the $x_0=1$. However, it grows rapidly when the distance between x and x_0 crosses certain threshold (about 0.45). Approximation of the lower orders perform well as the distance is growing. For x>3, Taylor approximation of 2nd order has the lowest error.

Question 1.2



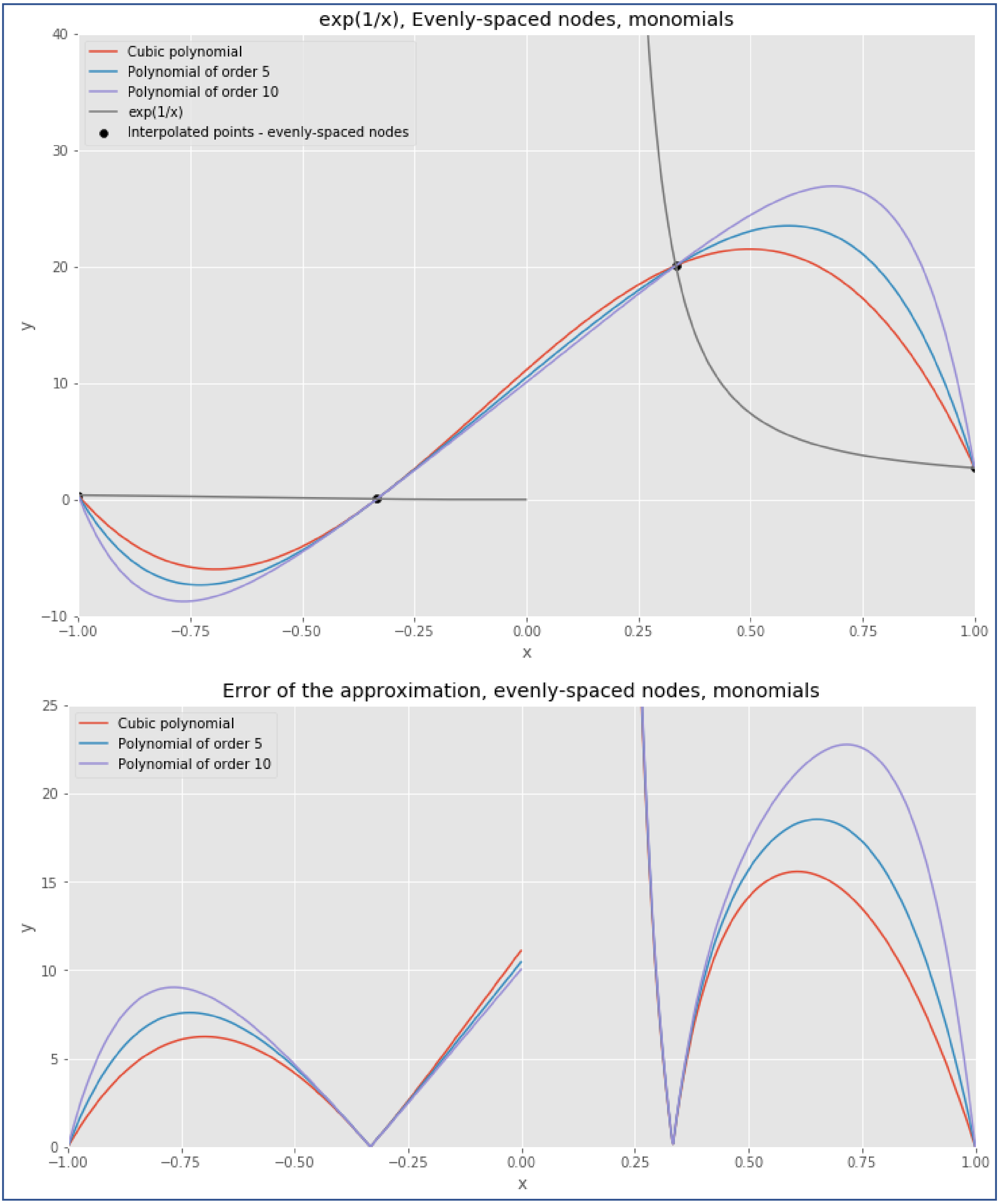
Comment: Approximations of all orders are almost the same. The only difference is due to numerical constraint of the computer.

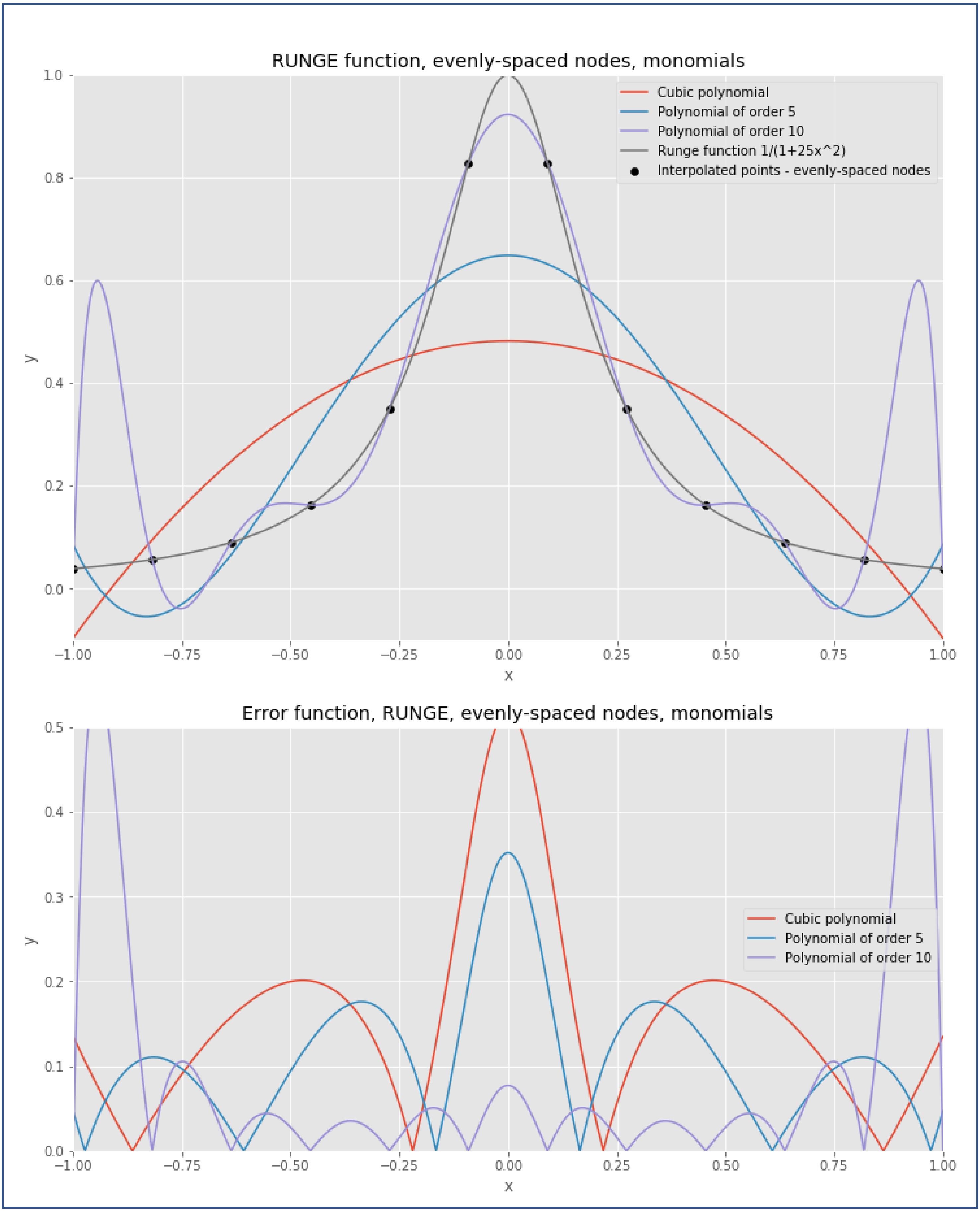
```
Taylor expansion at n=1 1.0*x
Taylor expansion at n=2 1.0*x
Taylor expansion at n=5 1.0*x
Taylor expansion at n=20 1.0*x + 3.6828445904877e-15*(x - 2)**20 - 8.41793049254331e-15*(x - 2)**19 - 7.49721934492139e-15*(x - 2)**18 - 1.40572862717276e-14*(x - 2)**17 - 1.59277015593476e-14*(x - 2)**12 - 4.61965562805297e-14*(x - 2)**8
```

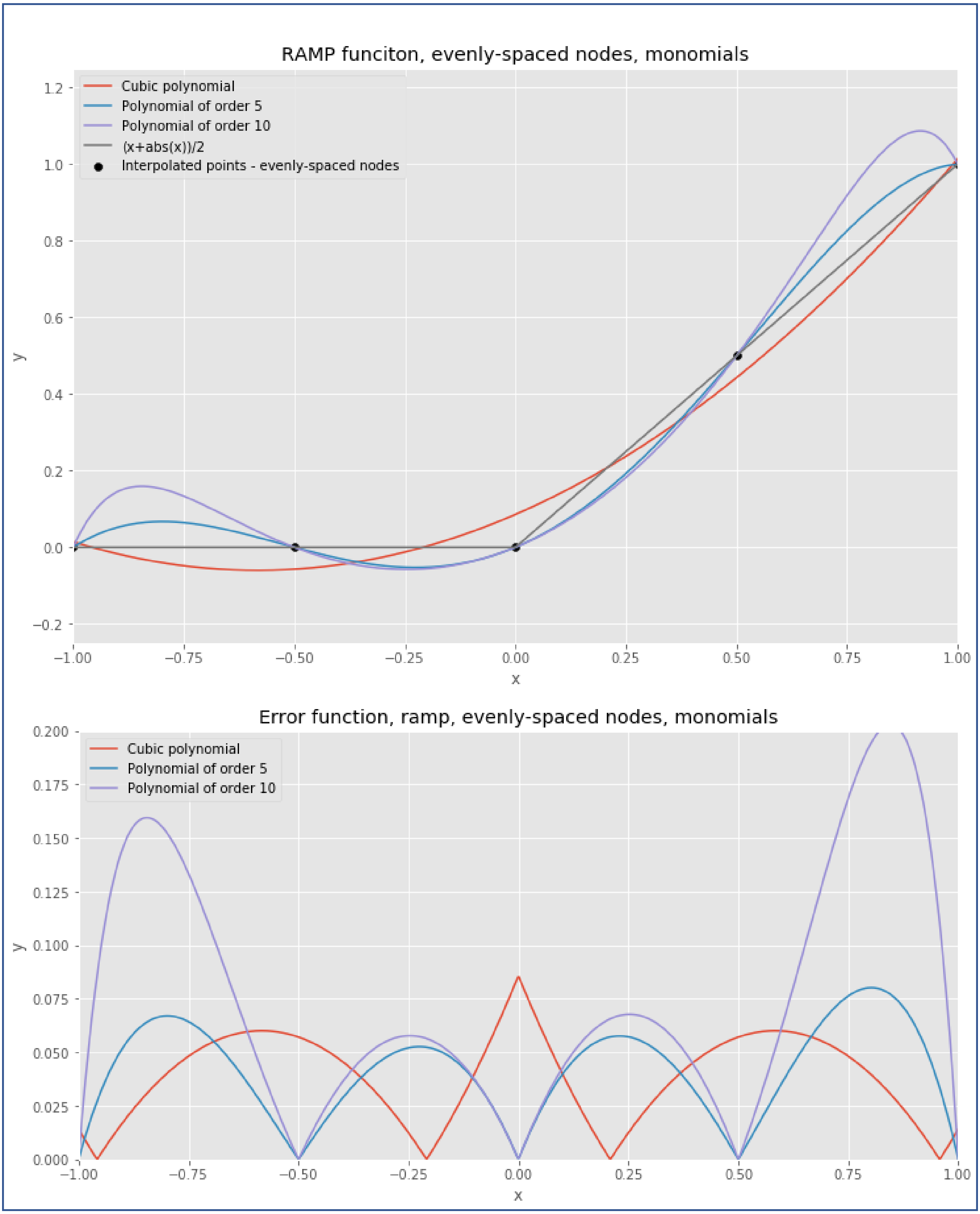
One can notice that computer estimated some very small coefficients in 20th order approximation. This small difference is unobservable on the graph.

The approximations failed to align to the function after the kink at $x=0$. For $x<0$, the error is significantly positive and linearly increasing with distance from the point $x=0$

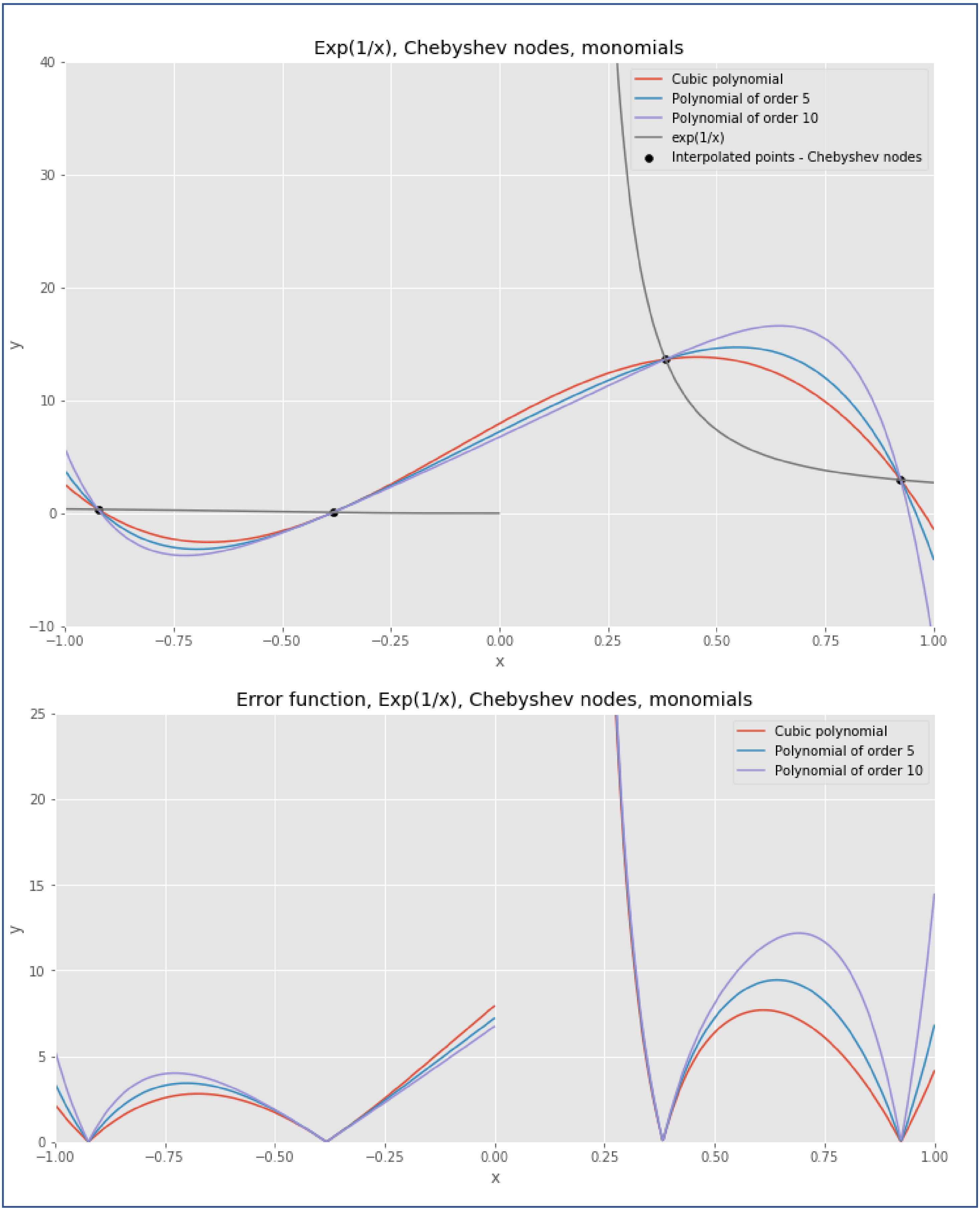
Question 1.3.1



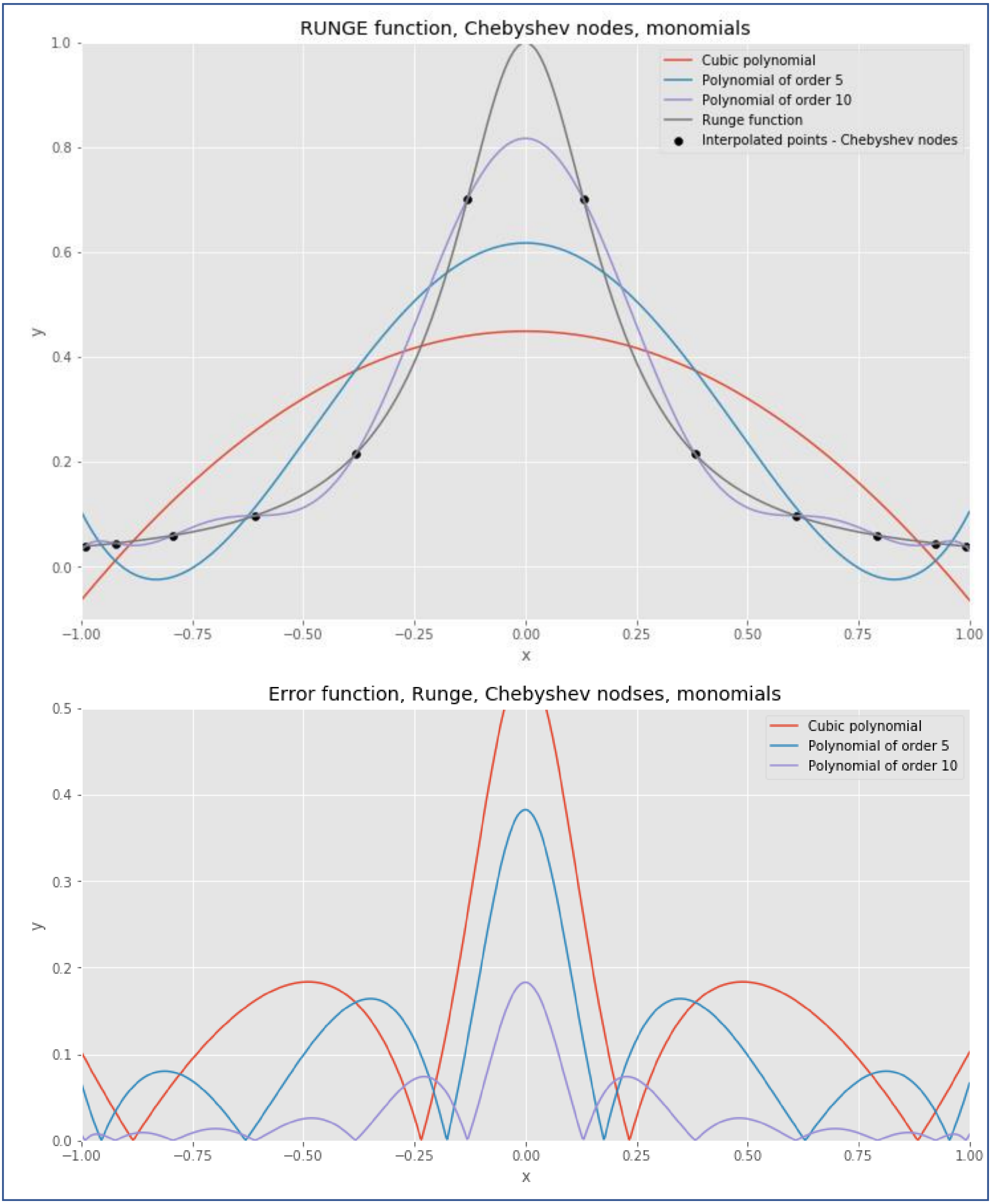

RUNGE function, evenly-spaced nodes, monomials
Cubic polynomial
Polynomial of order 5
Polynomial of order 10
Runge function 1/(1+25x^2)
Interpolated points - evenly-spaced nodes
y
x
1.0
0.8
0.6
0.4
0.2
0.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
Error function, RUNGE, evenly-spaced nodes, monomials
Cubic polynomial
Polynomial of order 5
Polynomial of order 10
y
x
0.5
0.4
0.3
0.2
0.1
0.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00



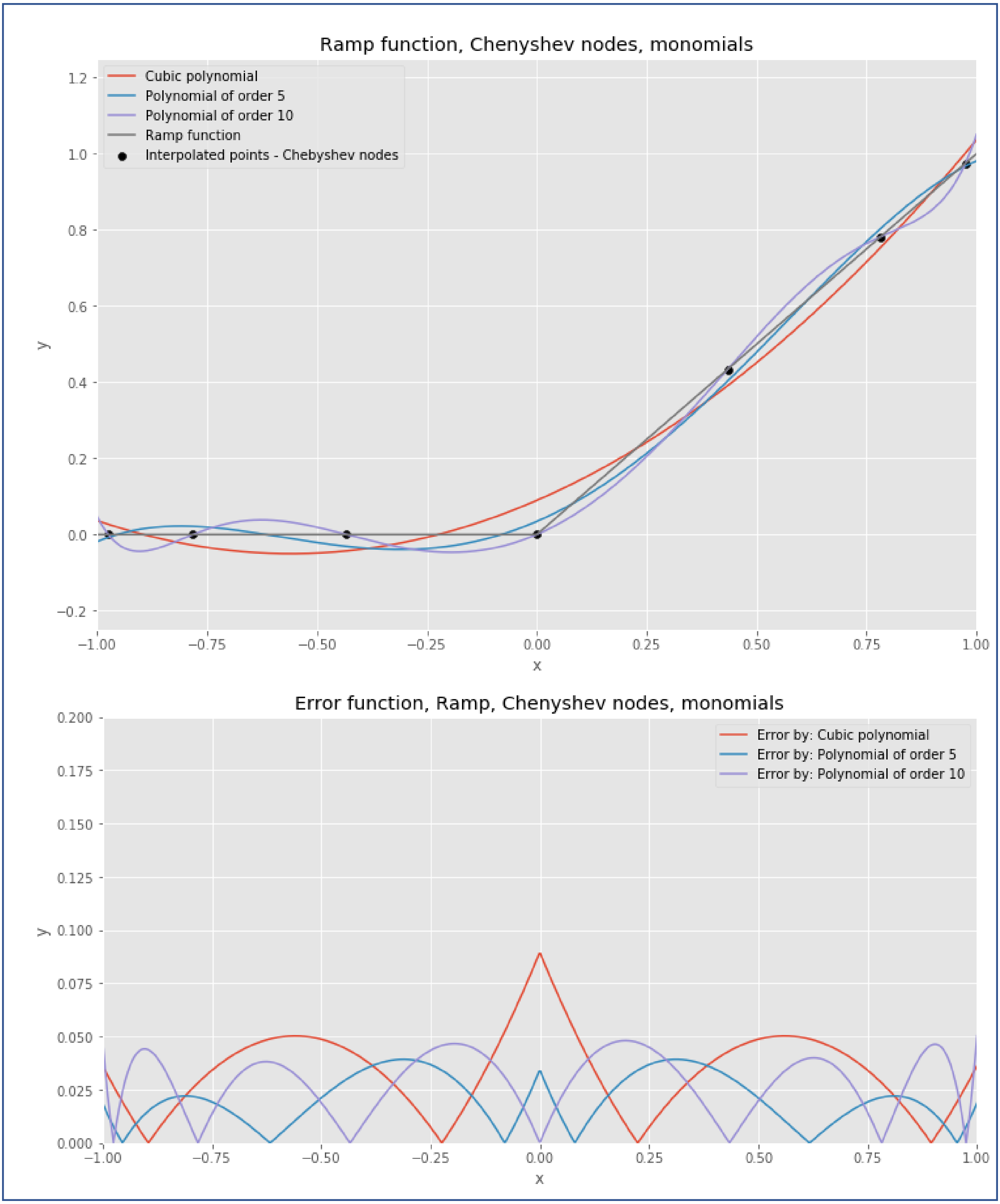
RAMP funciton, evenly-spaced nodes, monomials
Cubic polynomial
Polynomial of order 5
Polynomial of order 10
(x+abs(x))/2
Interpolated points - evenly-spaced nodes
y
x
Error function, ramp, evenly-spaced nodes, monomials
Cubic polynomial
Polynomial of order 5
Polynomial of order 10
y
x



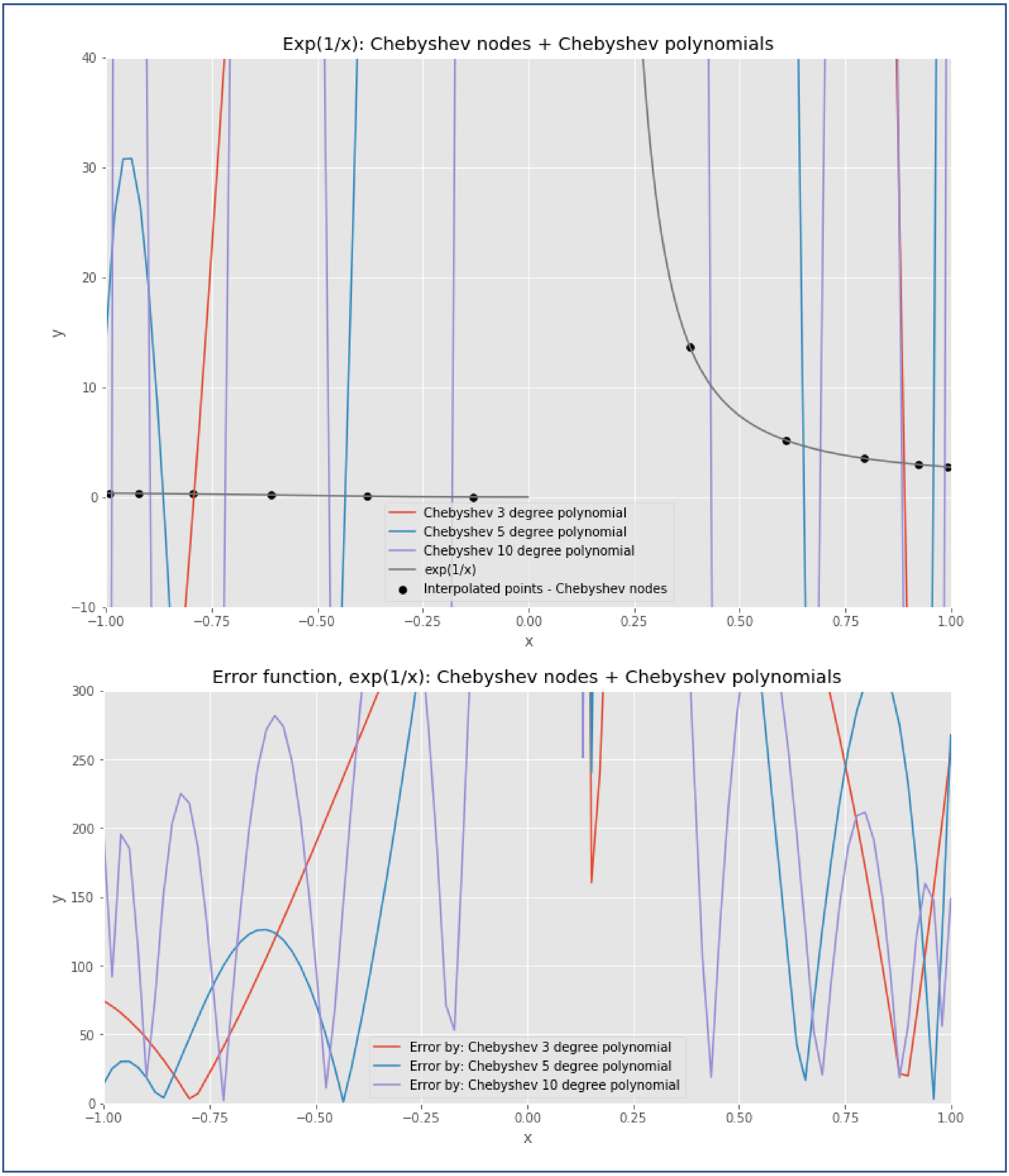
Exp(1/x), Chebyshev nodes, monomials
Cubic polynomial
Polynomial of order 5
Polynomial of order 10
exp(1/x)
Interpolated points - Chebyshev nodes
y
x
Error function, Exp(1/x), Chebyshev nodes, monomials
Cubic polynomial
Polynomial of order 5
Polynomial of order 10
y
x



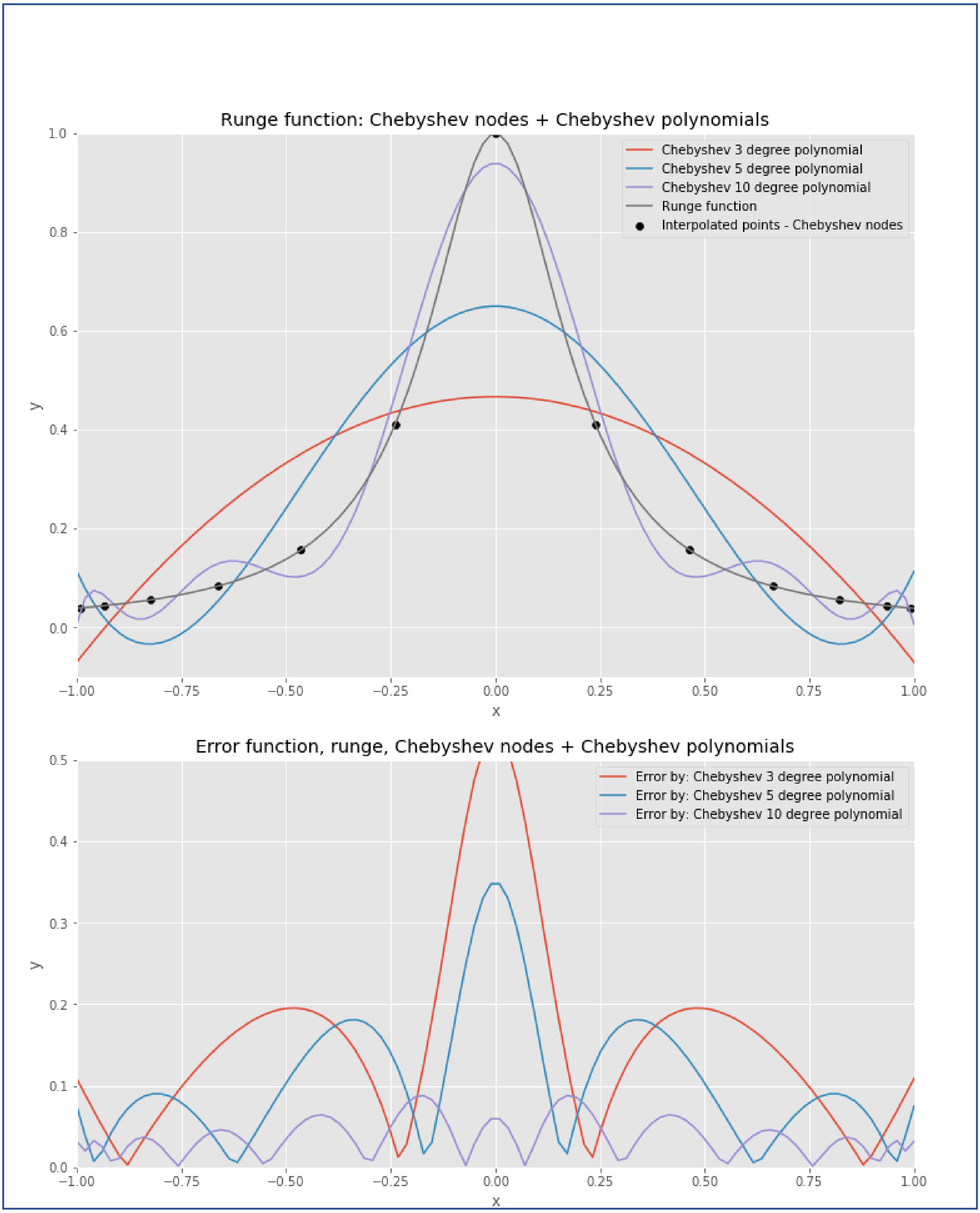
RUNGE function, Chebyshev nodes, monomials
Cubic polynomial
Polynomial of order 5
Polynomial of order 10
Runge function
Interpolated points - Chebyshev nodes
y
x
1.0
0.8
0.6
0.4
0.2
0.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
Error function, Runge, Chebyshev nodses, monomials
Cubic polynomial
Polynomial of order 5
Polynomial of order 10
y
x
0.5
0.4
0.3
0.2
0.1
0.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00



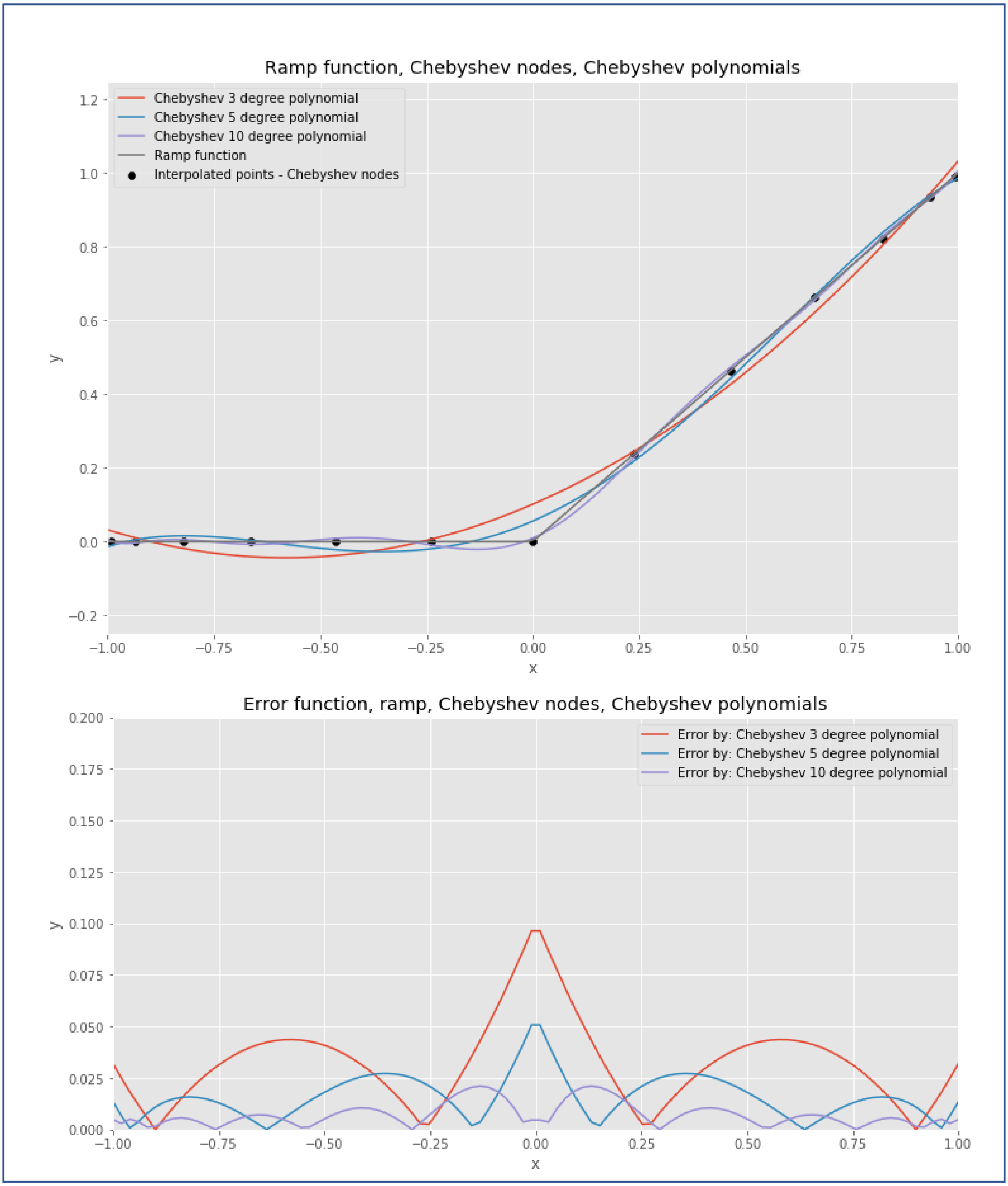
Ramp function, Chenyshev nodes, monomials
Cubic polynomial
Polynomial of order 5
Polynomial of order 10
Ramp function
Interpolated points - Chebyshev nodes
y
x
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Error function, Ramp, Chenyshev nodes, monomials
Error by: Cubic polynomial
Error by: Polynomial of order 5
Error by: Polynomial of order 10
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200



Comment: The problem here is that in order to use Chebyshev polynomial of 10 degree it is necessary to use more than 10 nodes. However, when more than 10 nodes are used, some of them are closed to X=0, where the function exp(1/x) may have very sharp slope and approaches infinity from right side. Then, Chebyshev polynomial becomes extremely volatile and generates huge errors.

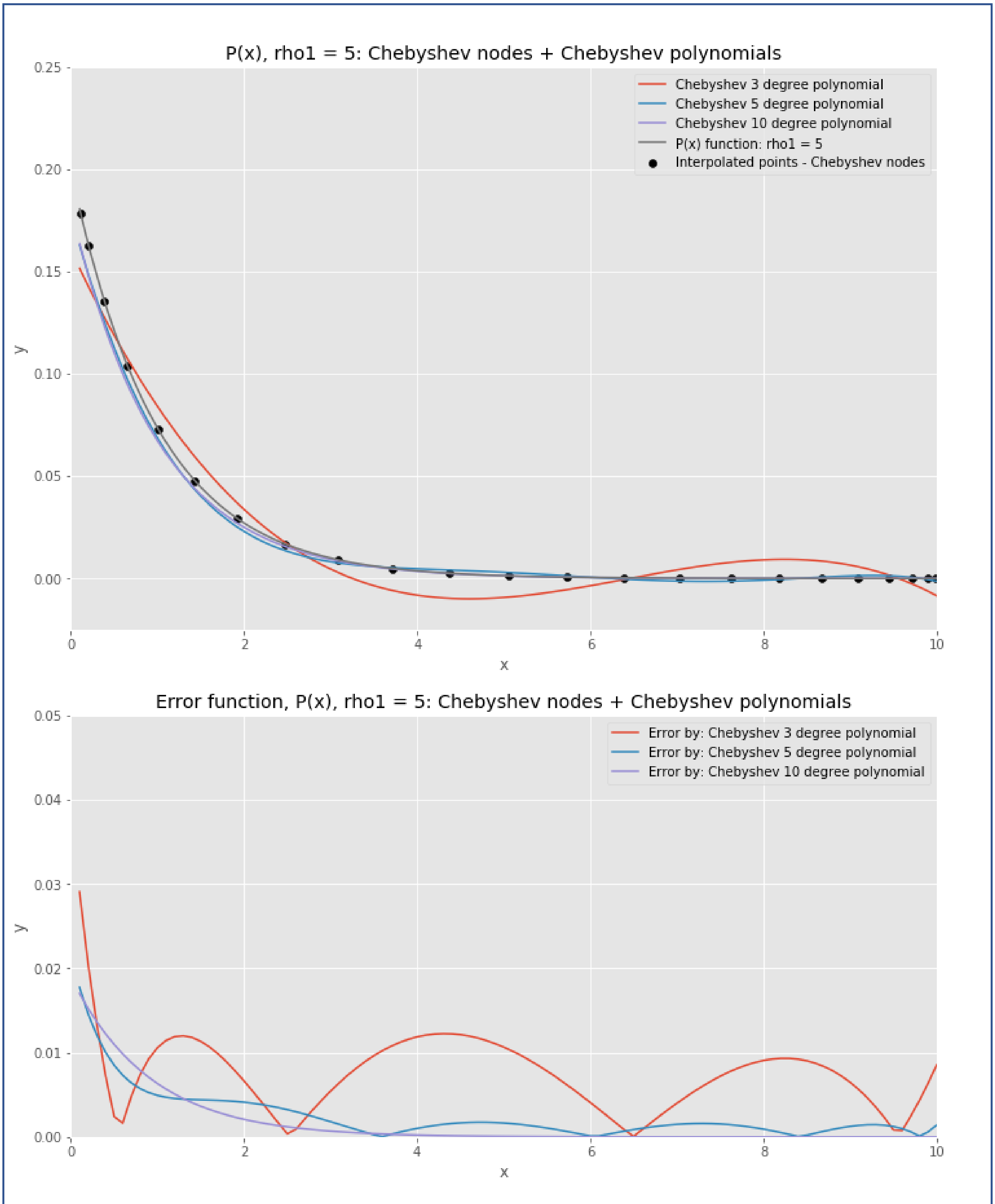


Comment: In this case, errors are similar to errors reported by method of monomials.

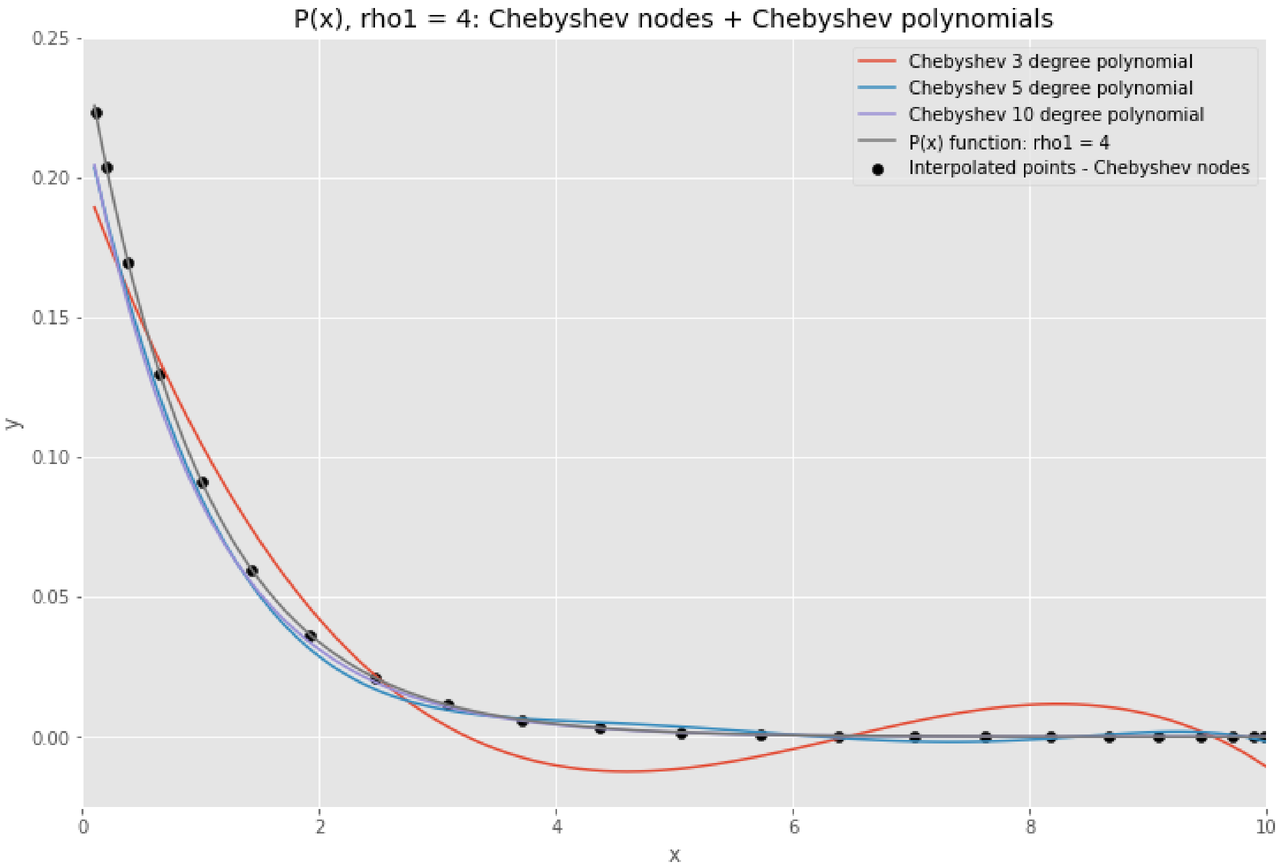


Comment: In this case, method of Cheb. Node & Cheb. Poly. Turned out to be significantly better than Evenly-spaced nodes & monomials or Cheb. Nodes and monomials. Especially, Chebyshev polynomials of the higher degrees perform very well. The error by: Chebyshev 10^{th} degree polynomial does not exceeds 0.025. In comparison, monomial of 10^{th} degree reported error which was close to 0.05 at few points.

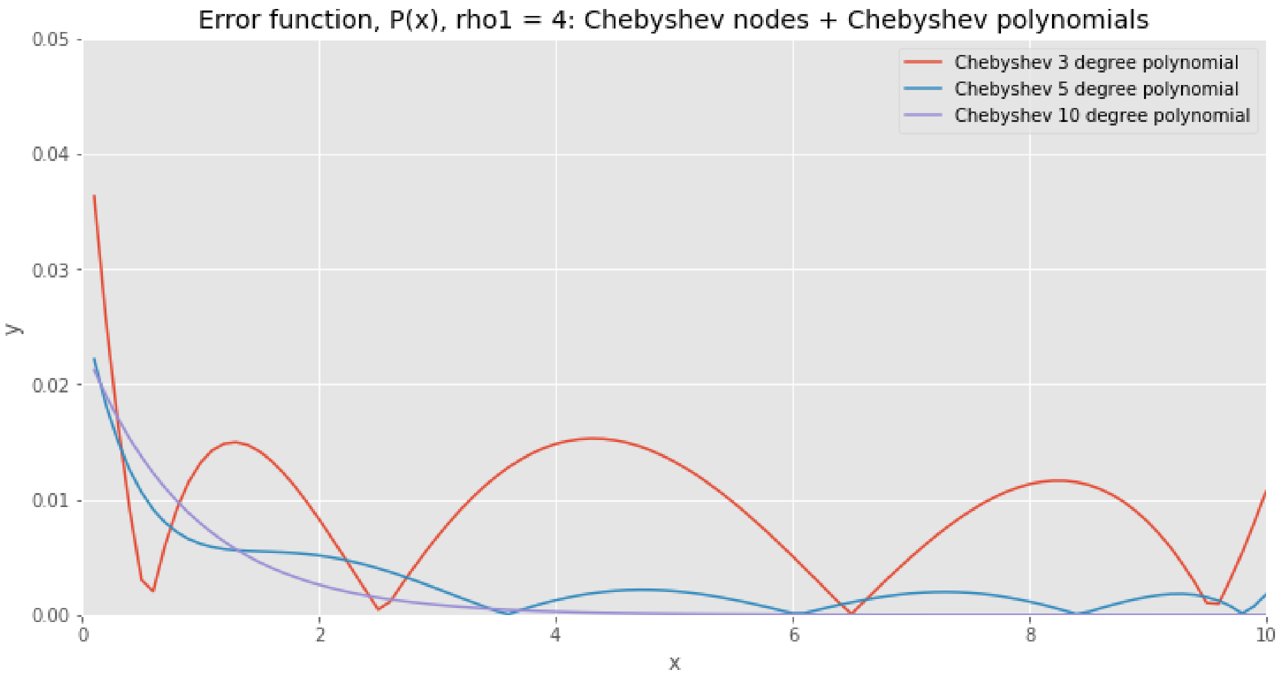
Question 1.4



P(x), rho1 = 5: Chebyshev nodes + Chebyshev polynomials
Chebyshev 3 degree polynomial
Chebyshev 5 degree polynomial
Chebyshev 10 degree polynomial
P(x) function: rho1 = 5
Interpolated points - Chebyshev nodes
y
x
0.25
0.20
0.15
0.10
0.05
0.00
0
2
4
6
8
10
Error function, P(x), rho1 = 5: Chebyshev nodes + Chebyshev polynomials
Error by: Chebyshev 3 degree polynomial
Error by: Chebyshev 5 degree polynomial
Error by: Chebyshev 10 degree polynomial
y
x
0.05
0.04
0.03
0.02
0.01
0.00
0
2
4
6
8
10



P(x), rho1 = 4: Chebyshev nodes + Chebyshev polynomials
Chebyshev 3 degree polynomial
Chebyshev 5 degree polynomial
Chebyshev 10 degree polynomial
P(x) function: rho1 = 4
Interpolated points - Chebyshev nodes
y
x



Error function, P(x), rho1 = 4: Chebyshev nodes + Chebyshev polynomials
Chebyshev 3 degree polynomial
Chebyshev 5 degree polynomial
Chebyshev 10 degree polynomial
y
x

QUESTION 2

CES: $f(k,h) = \left((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$

MPL: $\frac{\partial f(k,h)}{\partial h} = \frac{\sigma}{\sigma-1}\left((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \alpha \frac{\sigma-1}{\sigma} h^{\frac{-1}{\sigma}}$

MPK: $\frac{\partial f(k,h)}{\partial k} = \frac{\sigma}{\sigma-1}\left((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} (1-\alpha)\frac{\sigma-1}{\sigma} k^{\frac{-1}{\sigma}}$

MRS: $\frac{MPK}{MPL} = \frac{1-\alpha}{\alpha}\left(\frac{h}{k}\right)^{\frac{1}{\sigma}}$

Marginal Rate of Substitution

ES $= \frac{d\ln\left(\frac{h}{k}\right)}{d\ln(MRS)} = \frac{d\ln\left(\frac{h}{k}\right)}{d\ln\left(\frac{1-\alpha}{\alpha}\left(\frac{h}{k}\right)^{\frac{1}{\sigma}}\right)}$

Elasticity of Subst.

$= \frac{d\ln\left(\frac{h}{k}\right)}{\frac{\alpha}{1-\alpha}\left(\frac{h}{k}\right)^{-\frac{1}{\sigma}} \frac{1-\alpha}{\alpha}\frac{1}{\sigma}\left(\frac{h}{k}\right)^{\frac{1-\sigma}{\sigma}} d\left(\frac{h}{k}\right)}$

$= \frac{\frac{k}{h}\, d\left(\frac{h}{k}\right)}{\left(\frac{h}{k}\right)^{-1} d\left(\frac{h}{k}\right)\frac{1}{\sigma}} = \sigma$

Labor share: $\frac{wh}{Y}$

① $\frac{MPL}{MPK} = \frac{w}{r}$ by competitive market

assume: $r = 1$

② $wh + rk = Y$

from ① $$\frac{\frac{\sigma}{\sigma-1}\left((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \alpha \frac{\sigma-1}{\sigma} h^{\frac{-1}{\sigma}}}{\frac{\sigma}{\sigma-1}\left((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}(1-\alpha)\frac{\sigma-1}{\sigma} k^{\frac{-1}{\sigma}}} = \frac{\alpha}{1-\alpha}\left(\frac{k}{h}\right)^{\frac{1}{\sigma}}$$

so $$\left(\frac{1-\alpha}{\alpha} w\right)^{\sigma} h = k$$

plugging into ②

$$wh + \left(\frac{1-\alpha}{\alpha} w\right)^{\sigma} h = Y$$

$$h\left(w + \left(\frac{1-\alpha}{\alpha} w\right)^{\sigma}\right) = Y$$

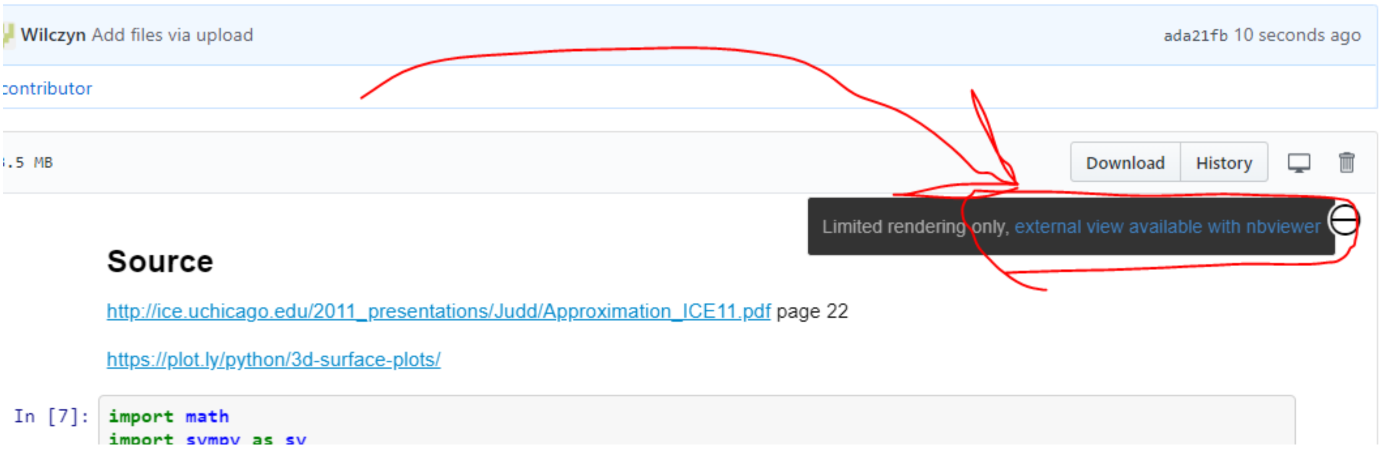
so $$\frac{wh}{Y} = \frac{wh}{wh + \left(\frac{1-\alpha}{\alpha}\right)^{\sigma} w^{\sigma} h} = \frac{wh}{wh + \left(\frac{1-\alpha}{\alpha}\right)^{\sigma} w^{\sigma-1} wh}$$

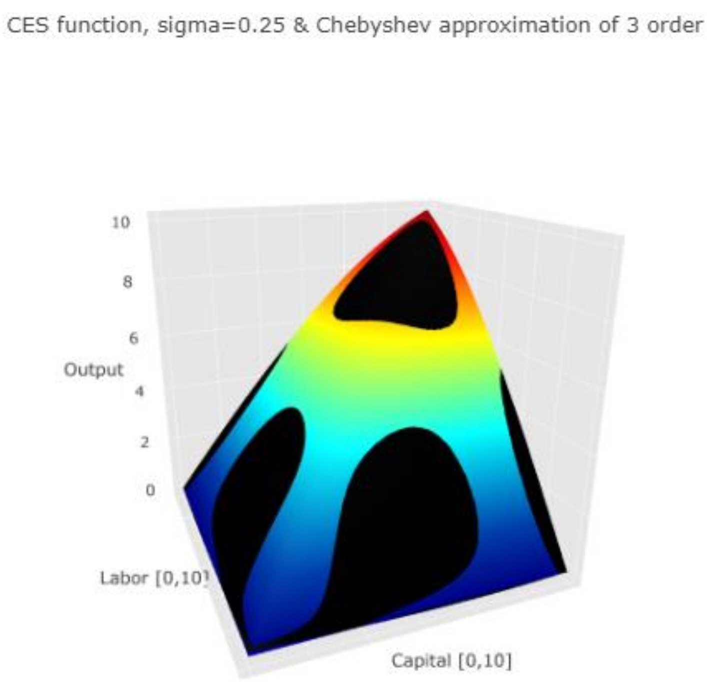
$$= \frac{1}{1 + (1-\alpha)^{\sigma} \alpha^{-\sigma} w^{\sigma} w^{-1}} = \frac{w\alpha^{\sigma}}{1 + (1-\alpha)^{\sigma} w^{\sigma}}$$

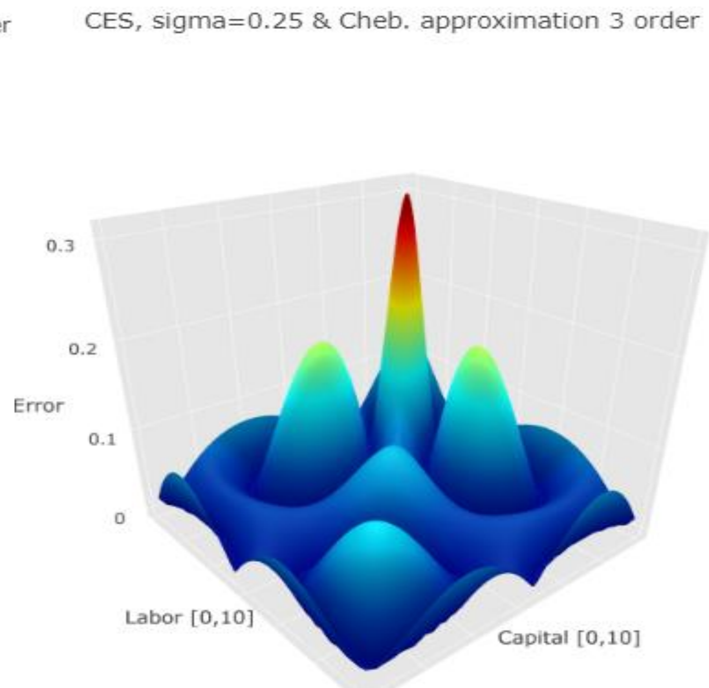
Labor share $= \frac{w\alpha^{\sigma}}{1+(1-\alpha)^{\sigma} w^{\sigma}}$ where $w \in \mathbb{R}_+$ as it is a level of wages

Remark: Chebyshev approximation is a black surface. Original CES function is a colorful surface.

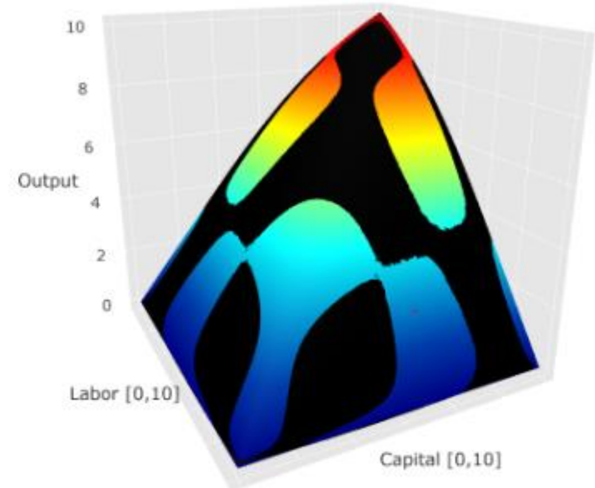
To see 3D interactive charts using GitHub, please navigate to nbviewet:



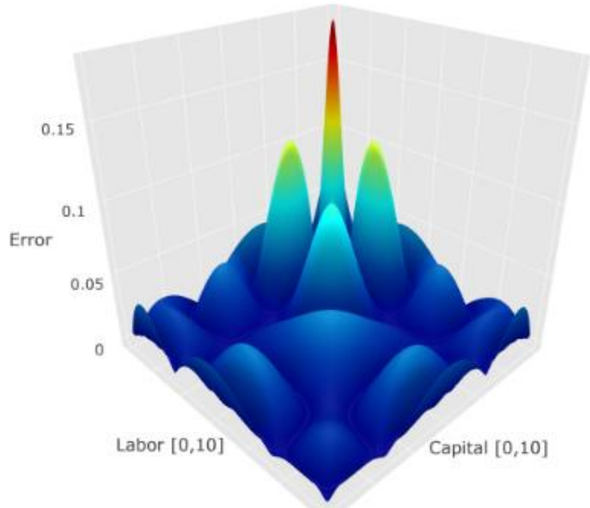




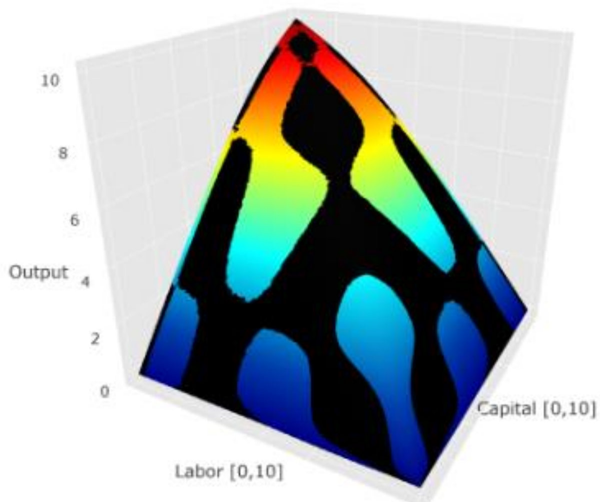
CES function, sigma=0.25 & Chebyshev approximation of 4 order



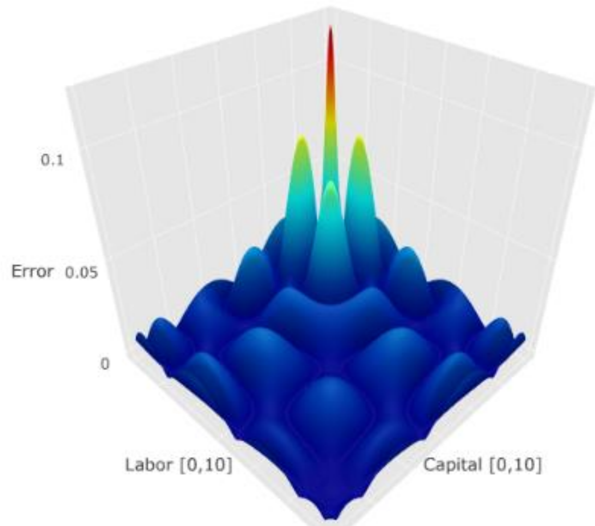
CES, sigma=0.25 & Cheb. approximation 4 order



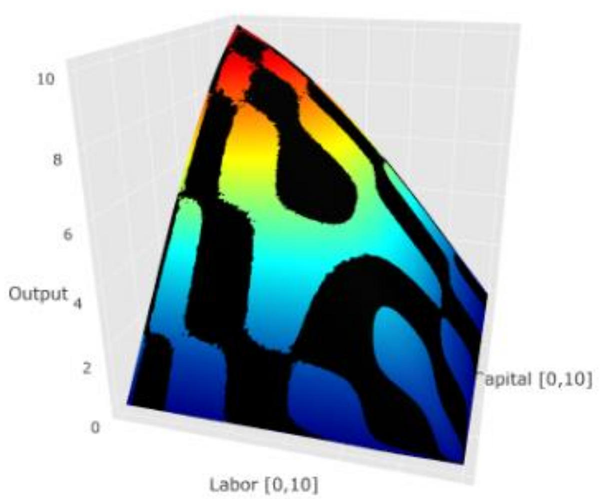
CES function, sigma=0.25 & Chebyshev approximation of 5 order



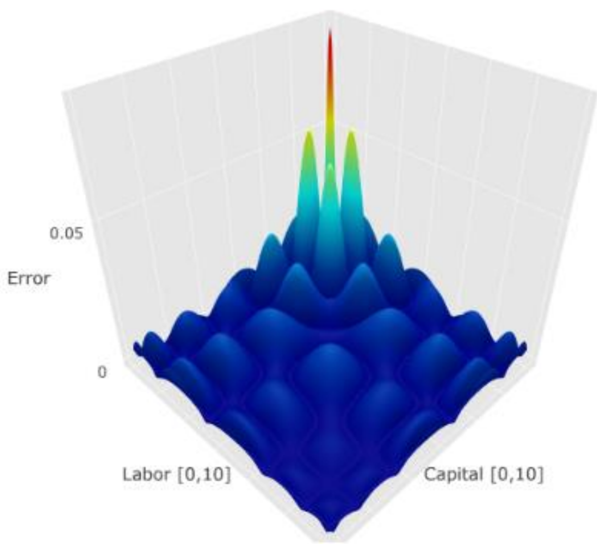
CES, sigma=0.25 & Cheb. approximation 5 order



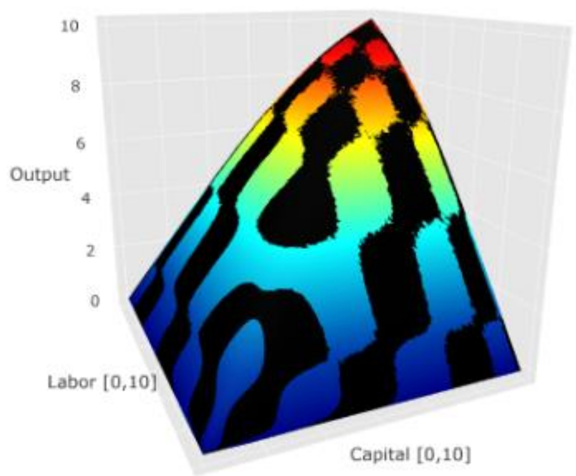
CES function, sigma=0.25 & Chebyshev approximation of 6 order



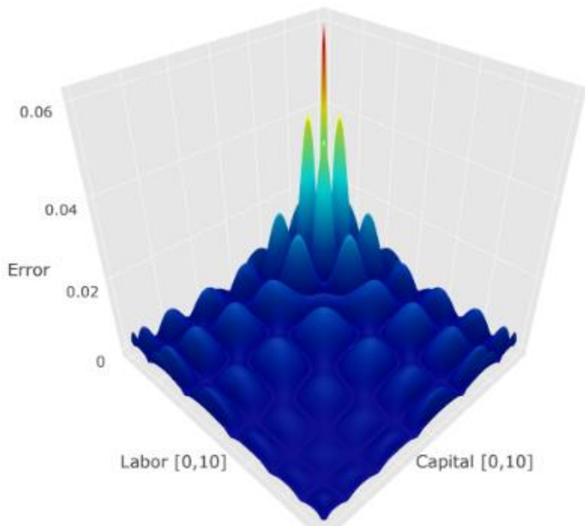
CES, sigma=0.25 & Cheb. approximation 6 order



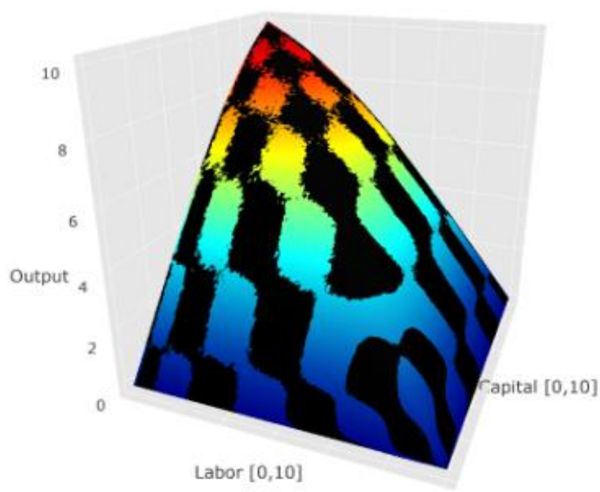
CES function, sigma=0.25 & Chebyshev approximation of 7 order



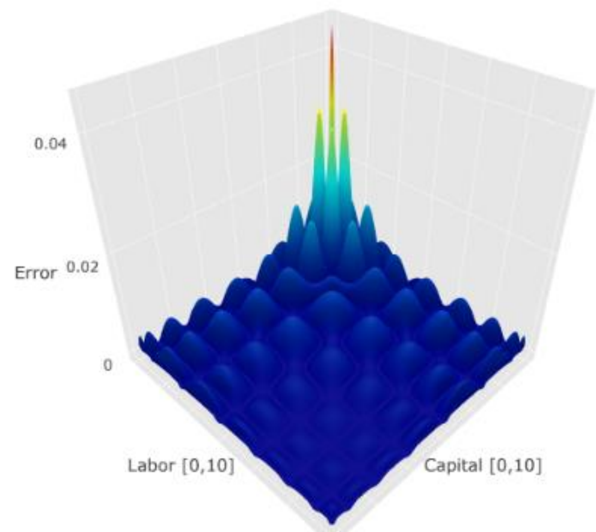
CES, sigma=0.25 & Cheb. approximation 7 order



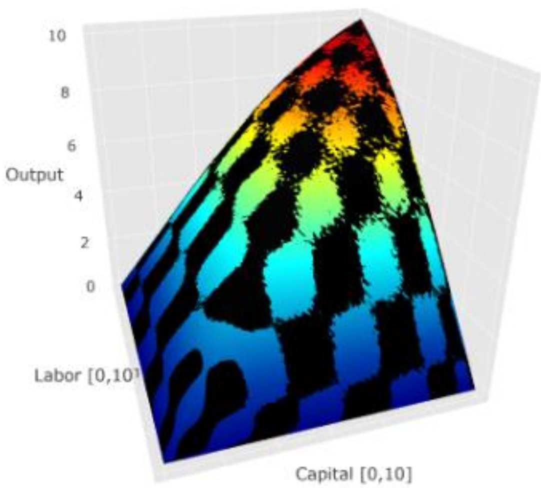
CES function, sigma=0.25 & Chebyshev approximation of 8 order



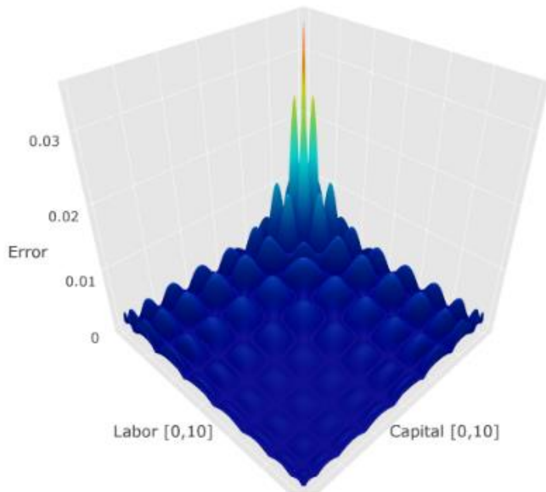
CES, sigma=0.25 & Cheb. approximation 8 order

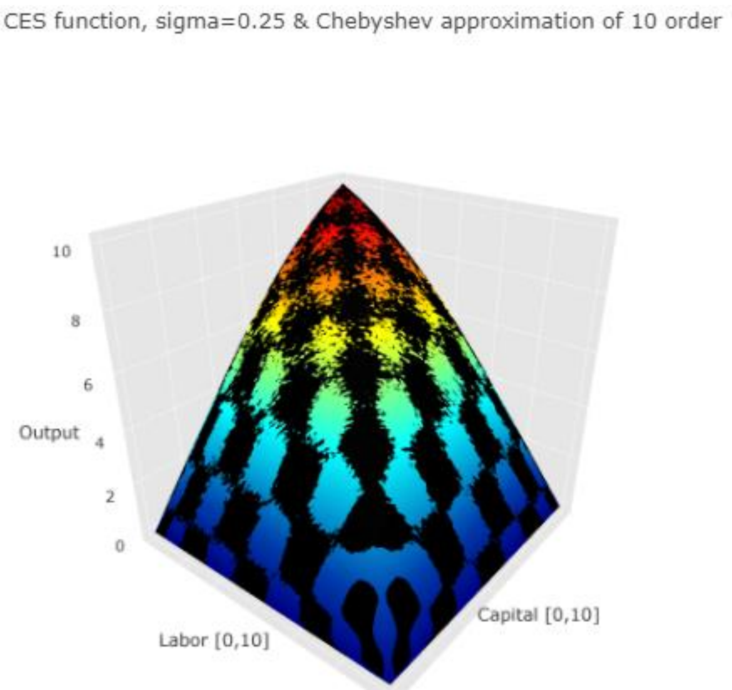


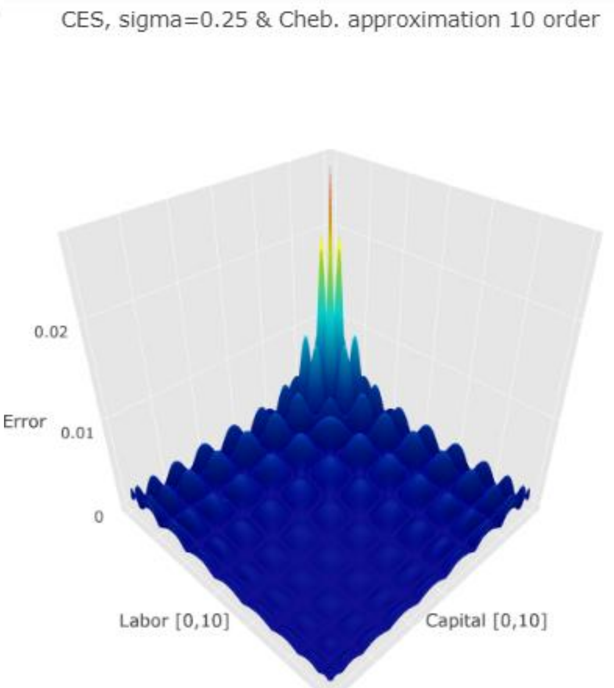
CES function, sigma=0.25 & Chebyshev approximation of 9 order



CES, sigma=0.25 & Cheb. approximation 9 order







Computer encountered a floating-point calculation error and failed to create a graph for the approximation of 11th order

plot.ly »

```
C:\Users\adamw\Anaconda3\lib\site-packages\ipykernel_launcher.py:71: RuntimeWarning:

invalid value encountered in double_scalars

---------------------------------------------------------------------------
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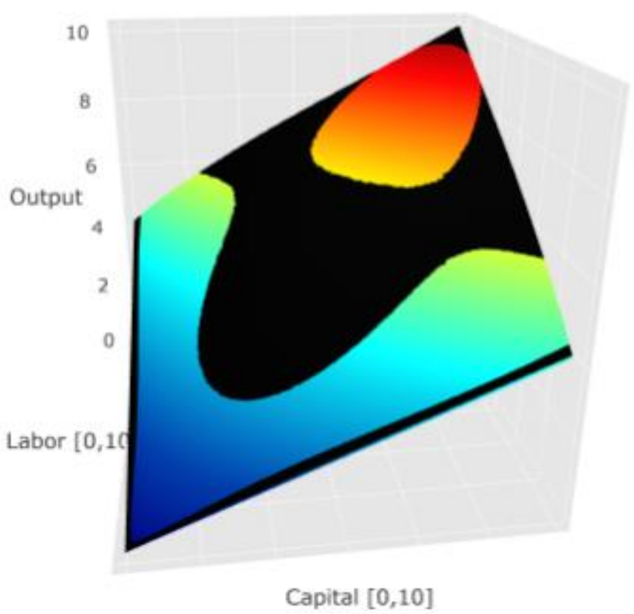
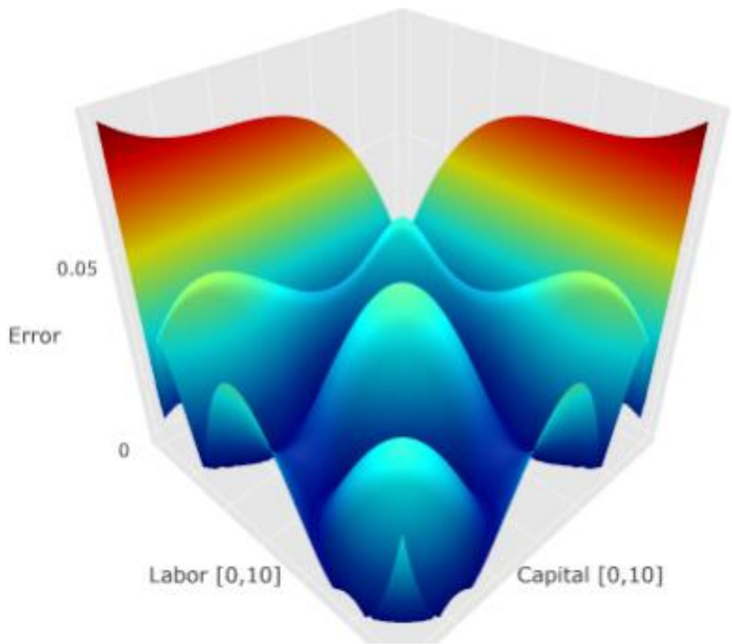
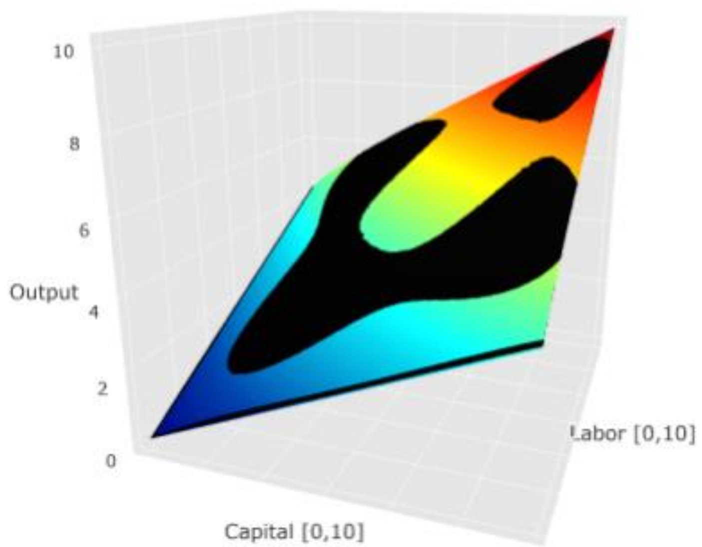
CES, sigma=5.0 & Cheb. approximation 3 order



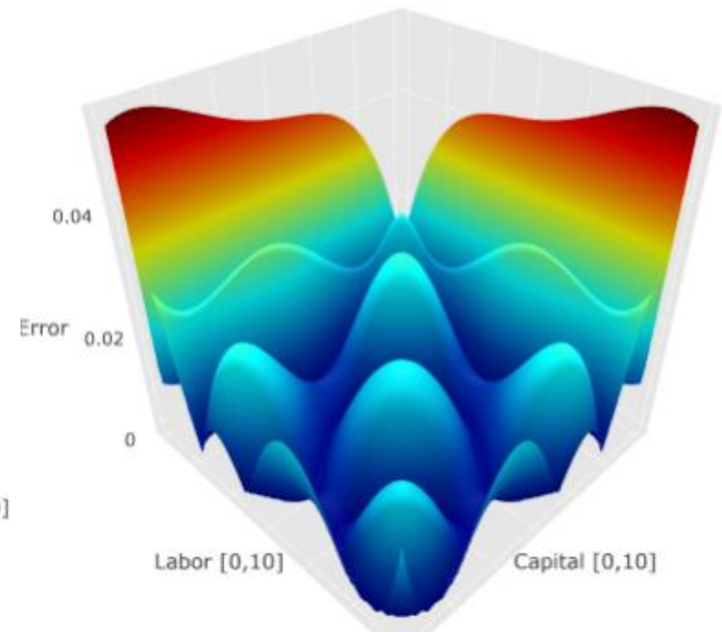
CES, sigma=5.0 & Cheb. approximation 3 order



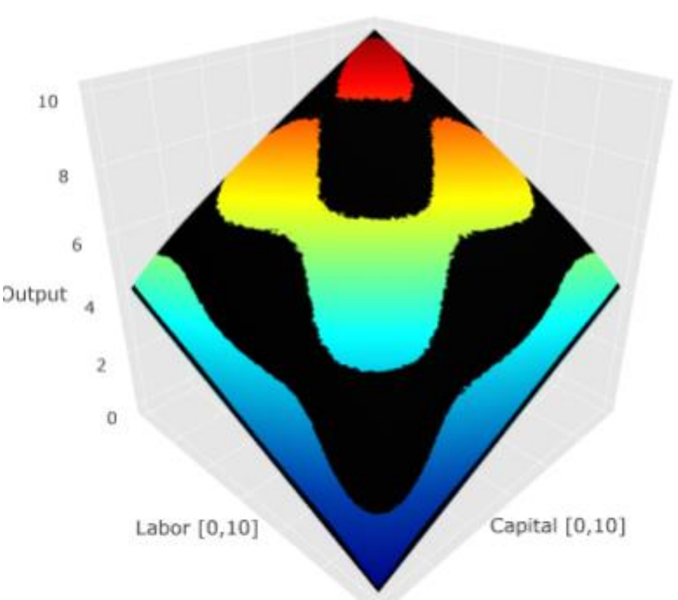
CES, sigma=5.0 & Cheb. approximation 4 order



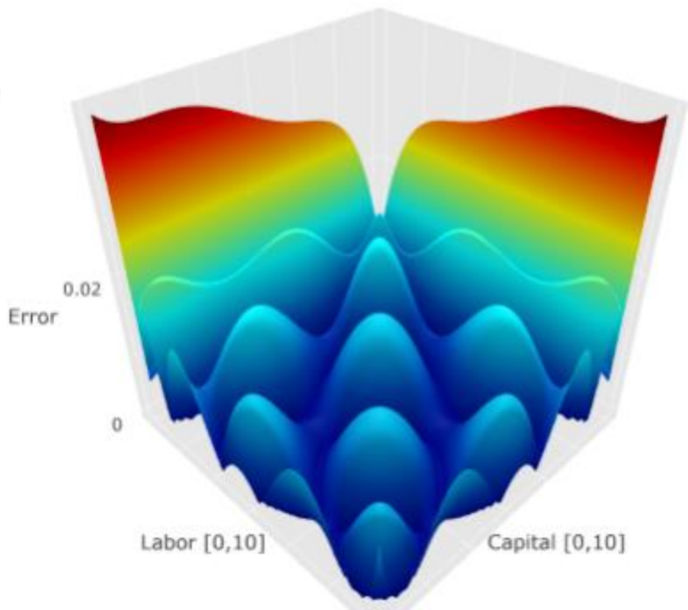
CES, sigma=5.0 & Cheb. approximation 4 order



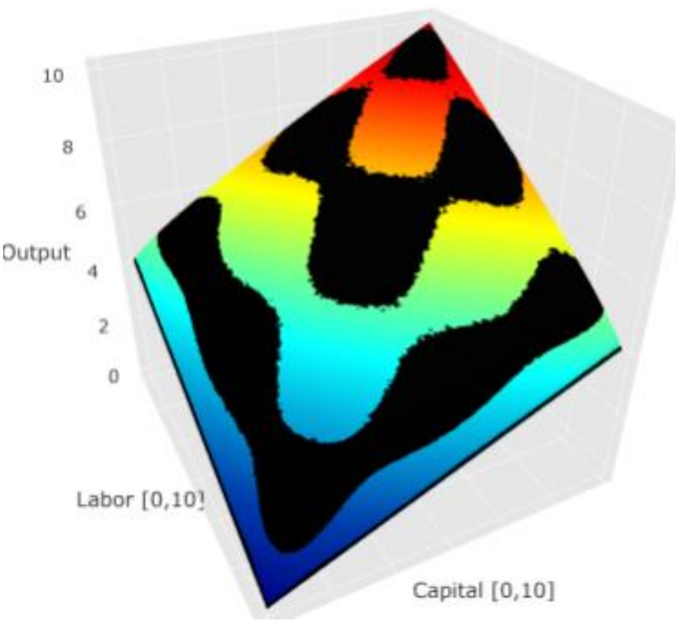
CES, sigma=5.0 & Cheb. approximation 5 order



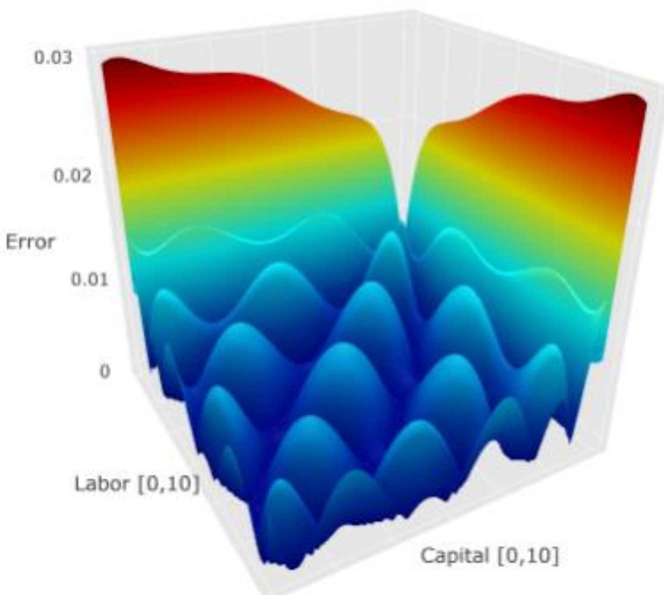
CES, sigma=5.0 & Cheb. approximation 5 order



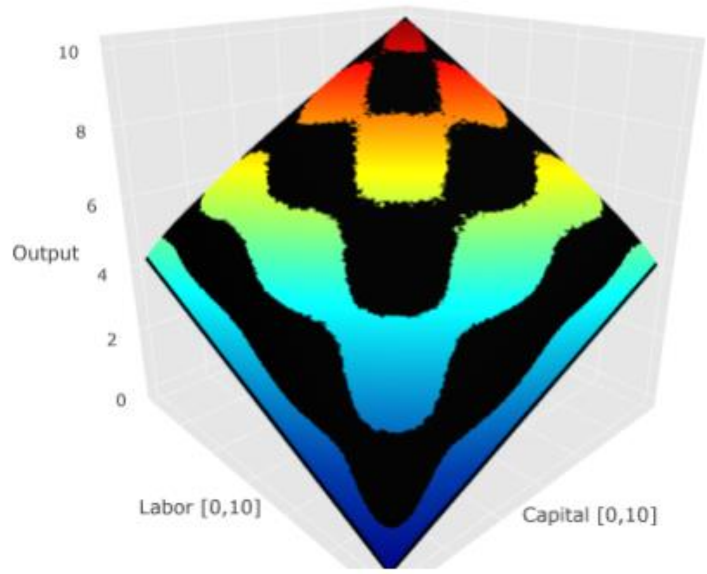
CES, sigma=5.0 & Cheb. approximation 6 order



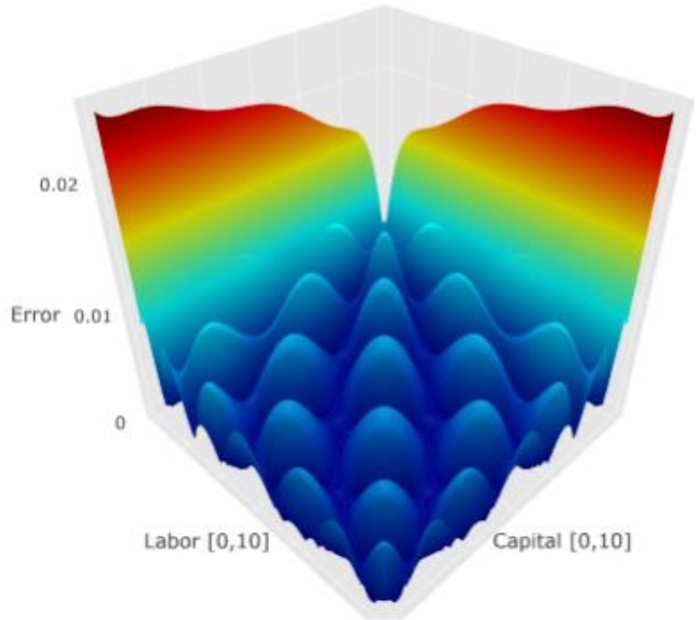
CES, sigma=5.0 & Cheb. approximation 6 order



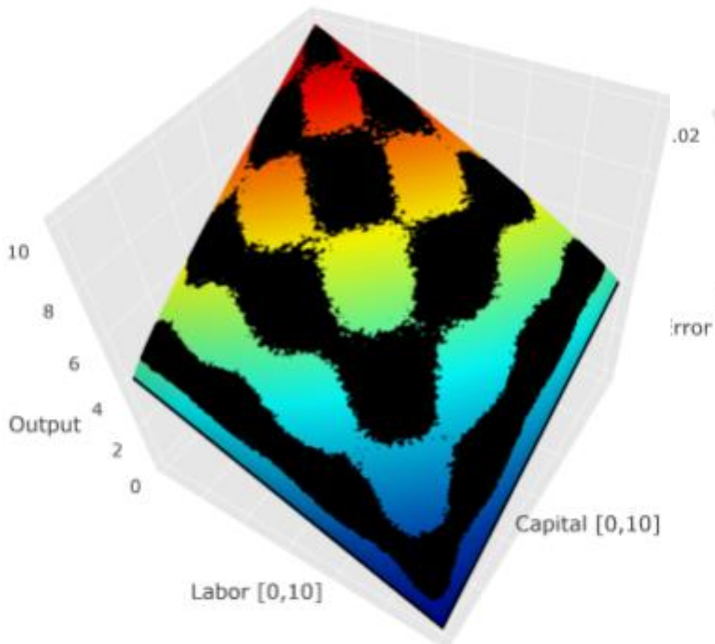
CES, sigma=5.0 & Cheb. approximation 7 order



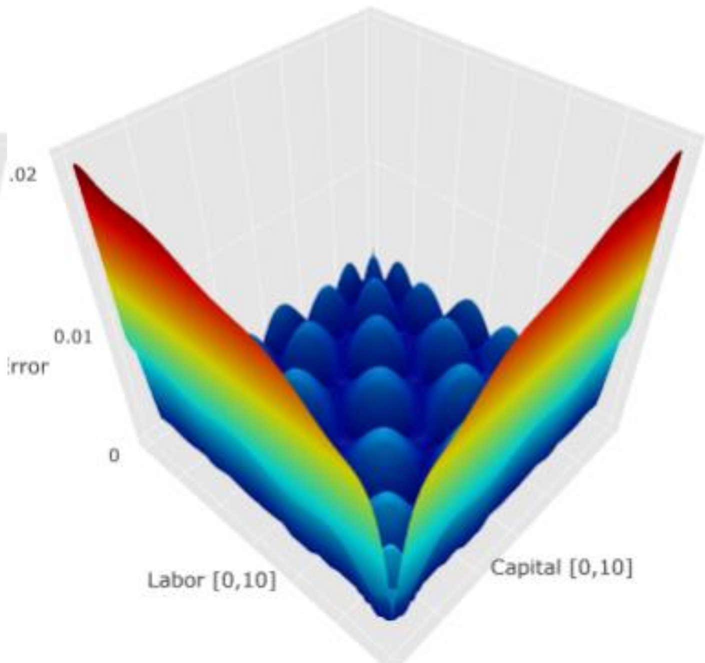
CES, sigma=5.0 & Cheb. approximation 7 order



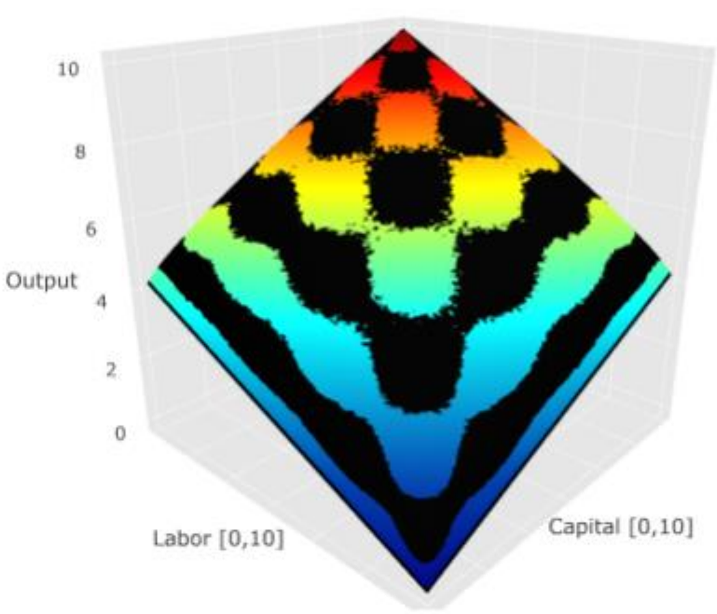
CES, sigma=5.0 & Cheb. approximation 8 order



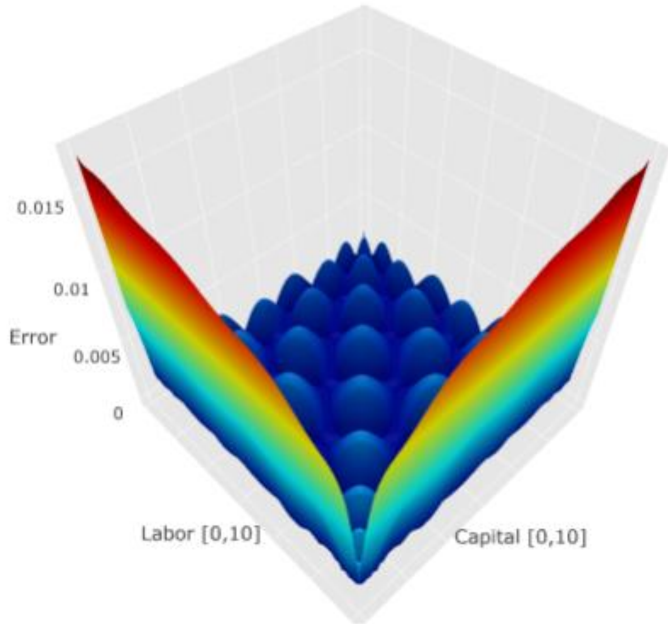
CES, sigma=5.0 & Cheb. approximation 8 order



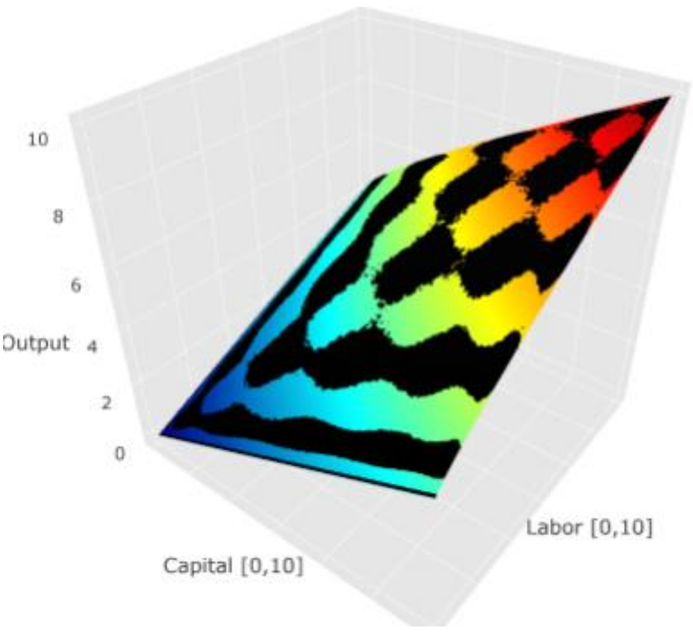
CES, sigma=5.0 & Cheb. approximation 9 order



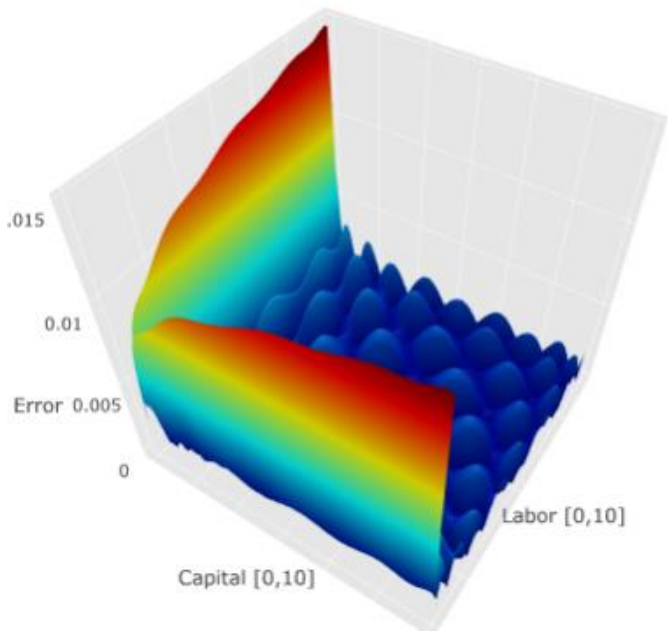
CES, sigma=5.0 & Cheb. approximation 9 order



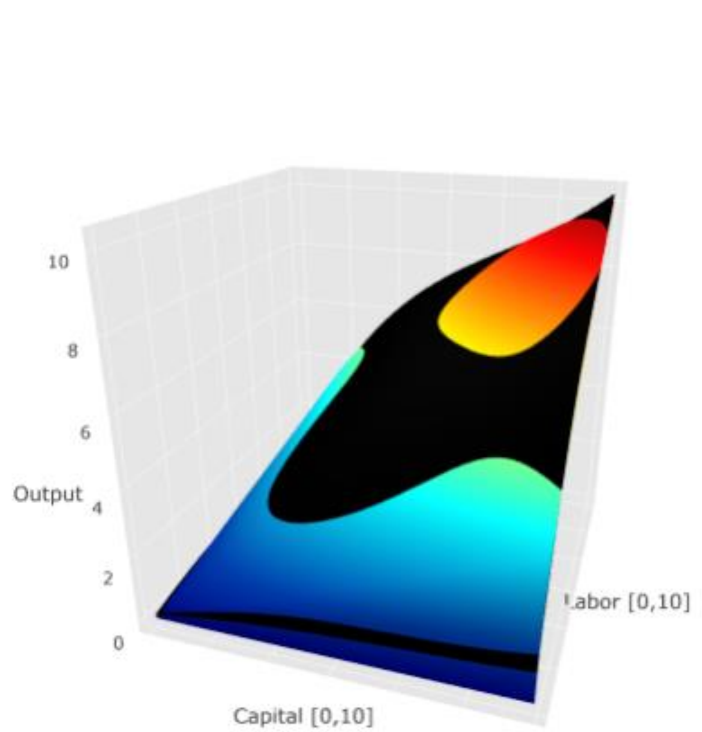
CES, sigma=5.0 & Cheb. approximation 10 order



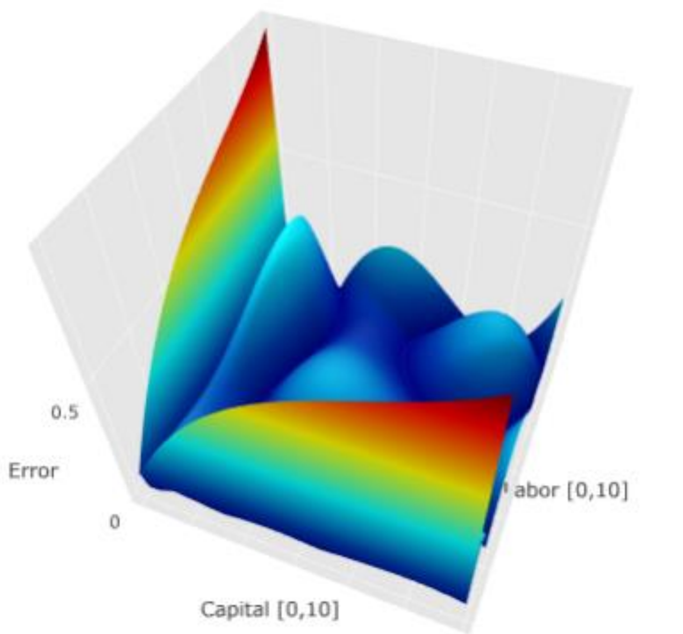
CES, sigma=5.0 & Cheb. approximation 10 order



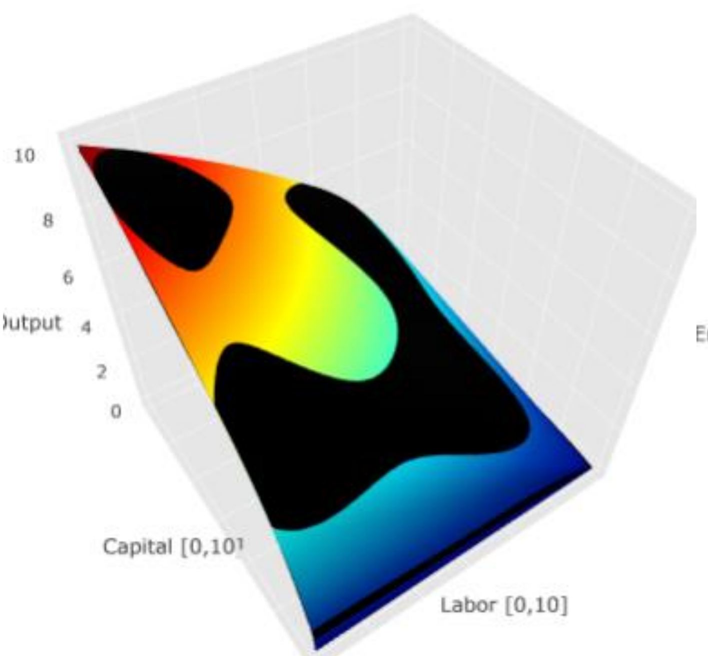
CES, sigma=0.99 & Cheb. approximation 3 order



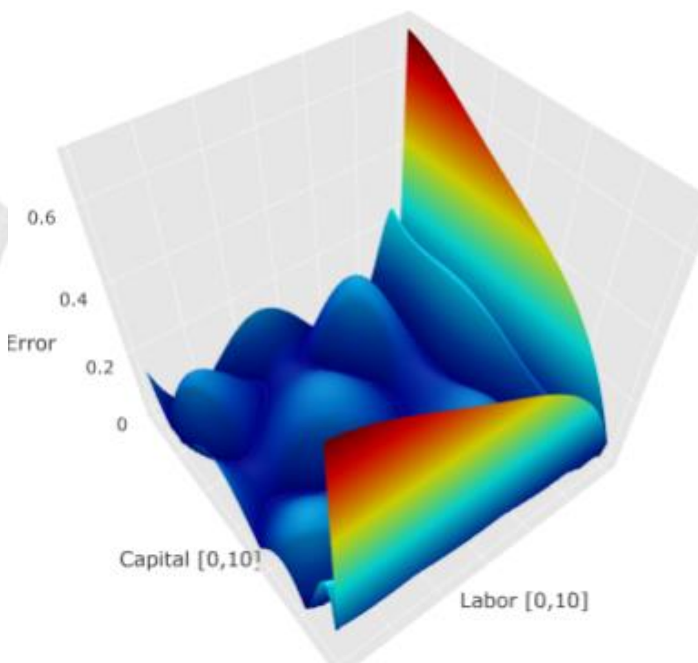
CES, sigma=0.99 & Cheb. approximation 3 order



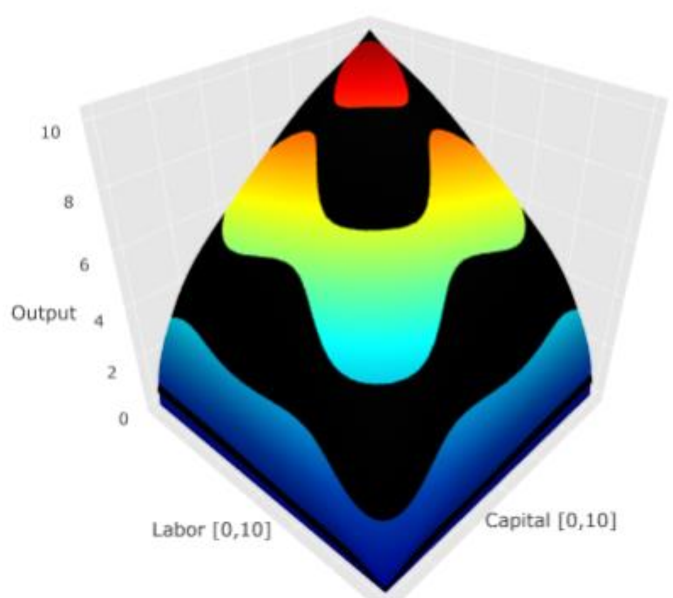
CES, sigma=0.99 & Cheb. approximation 4 order



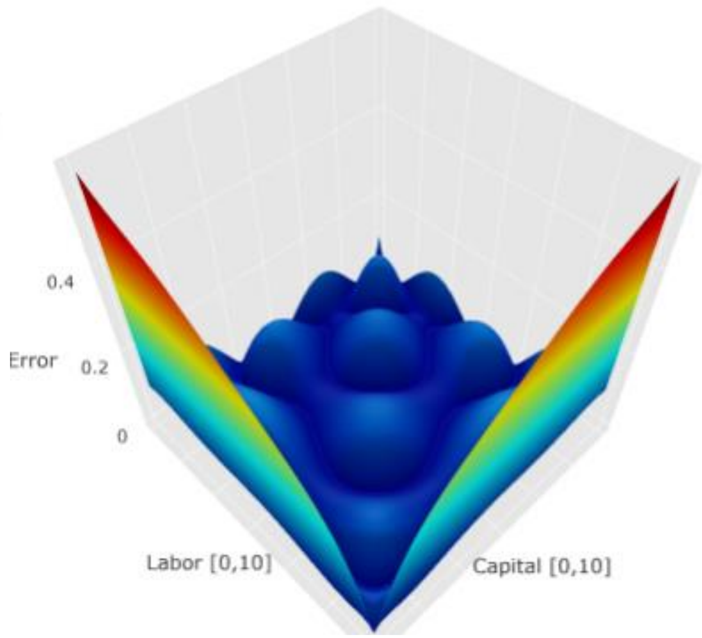
CES, sigma=0.99 & Cheb. approximation 4 order



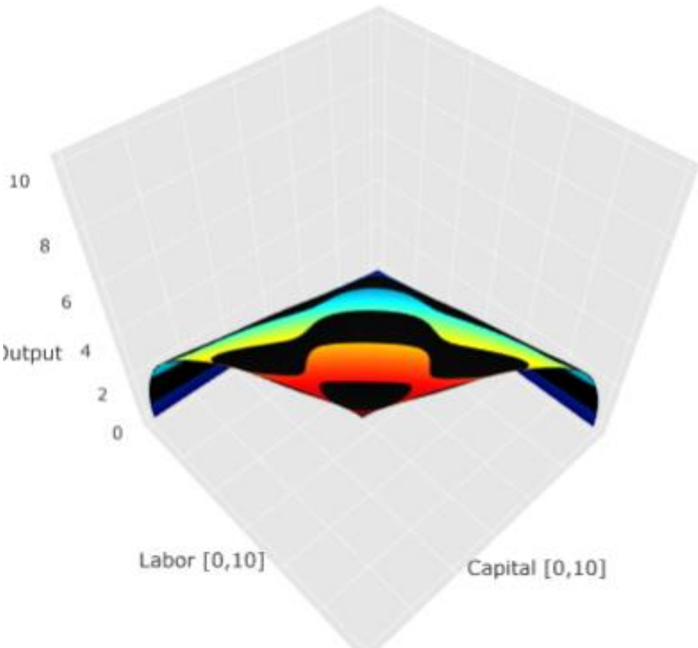
CES, sigma=0.99 & Cheb. approximation 5 order



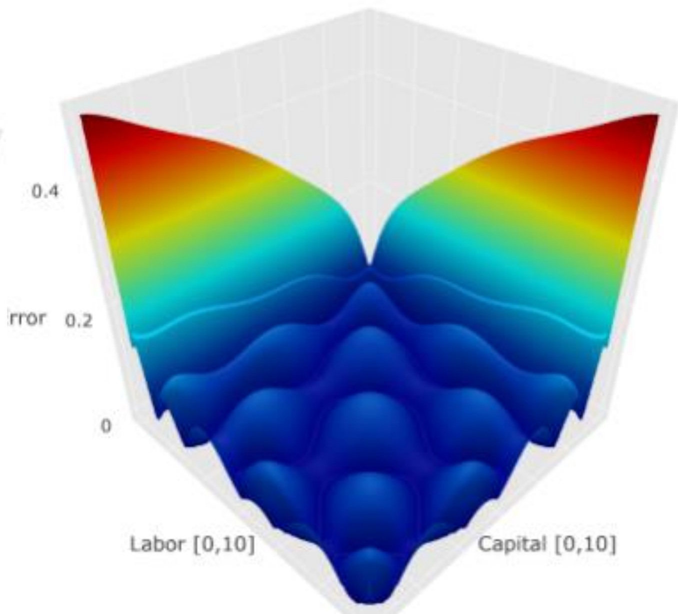
CES, sigma=0.99 & Cheb. approximation 5 order



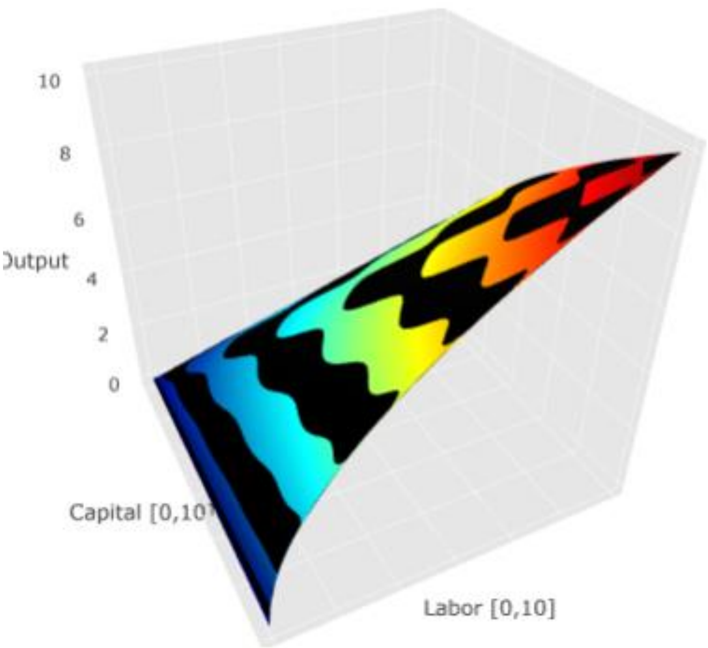
CES, sigma=0.99 & Cheb. approximation 6 order



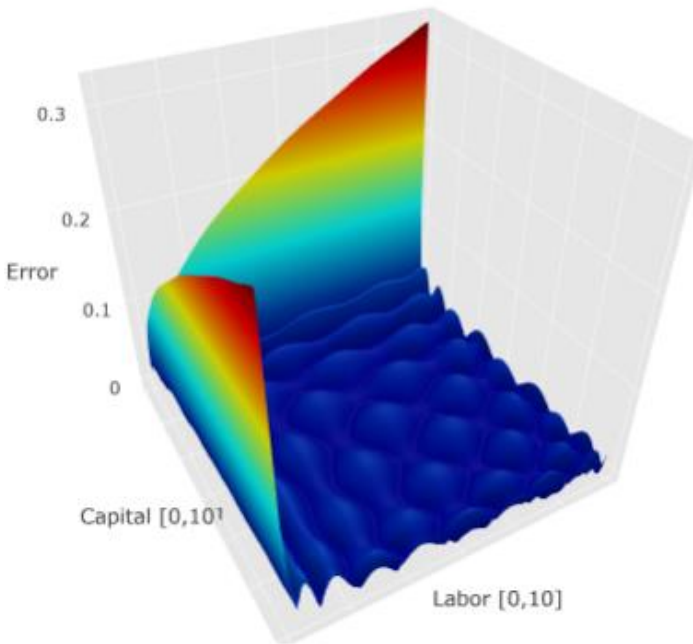
CES, sigma=0.99 & Cheb. approximation 6 order

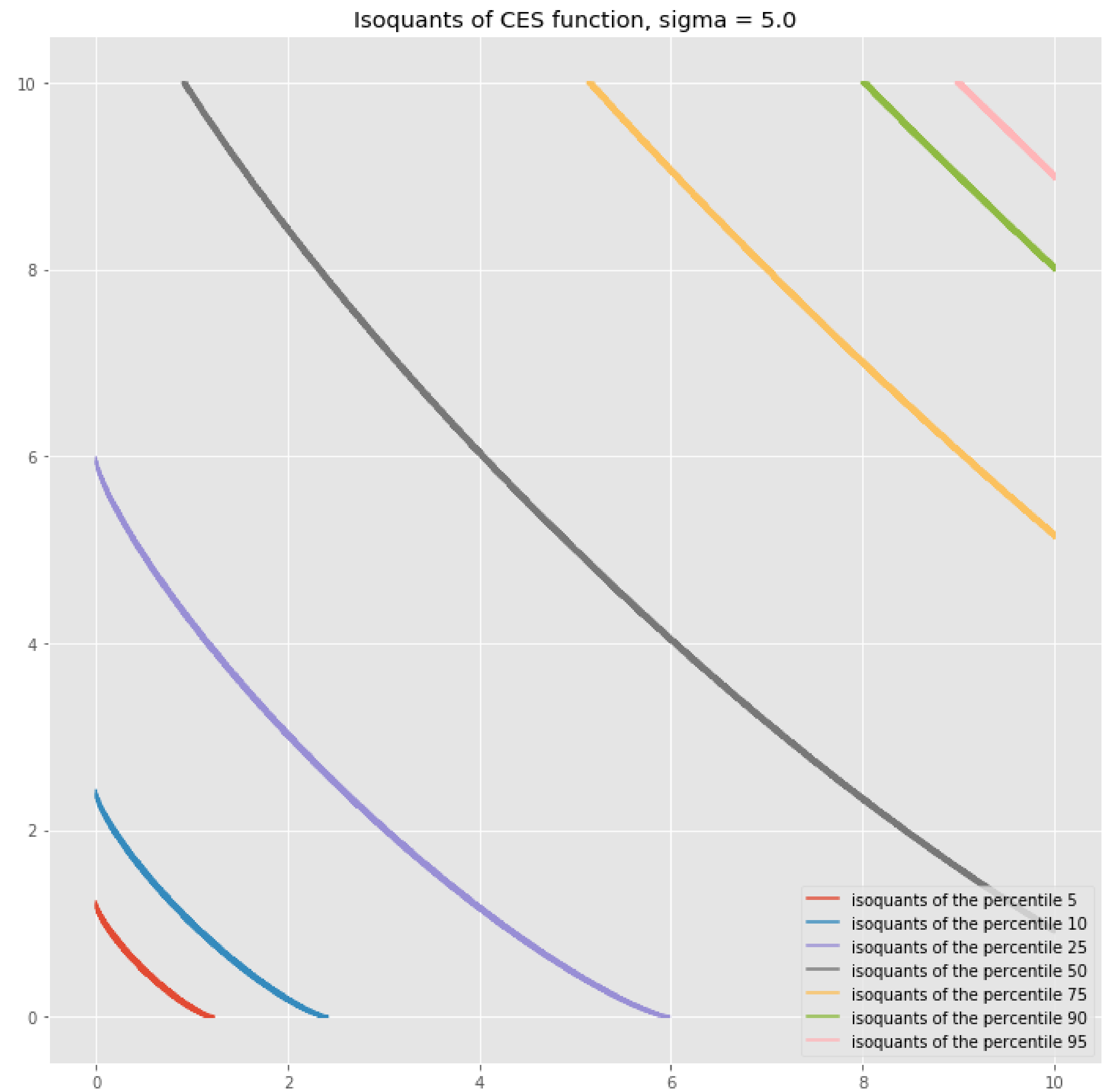


CES, sigma=0.99 & Cheb. approximation 10 order

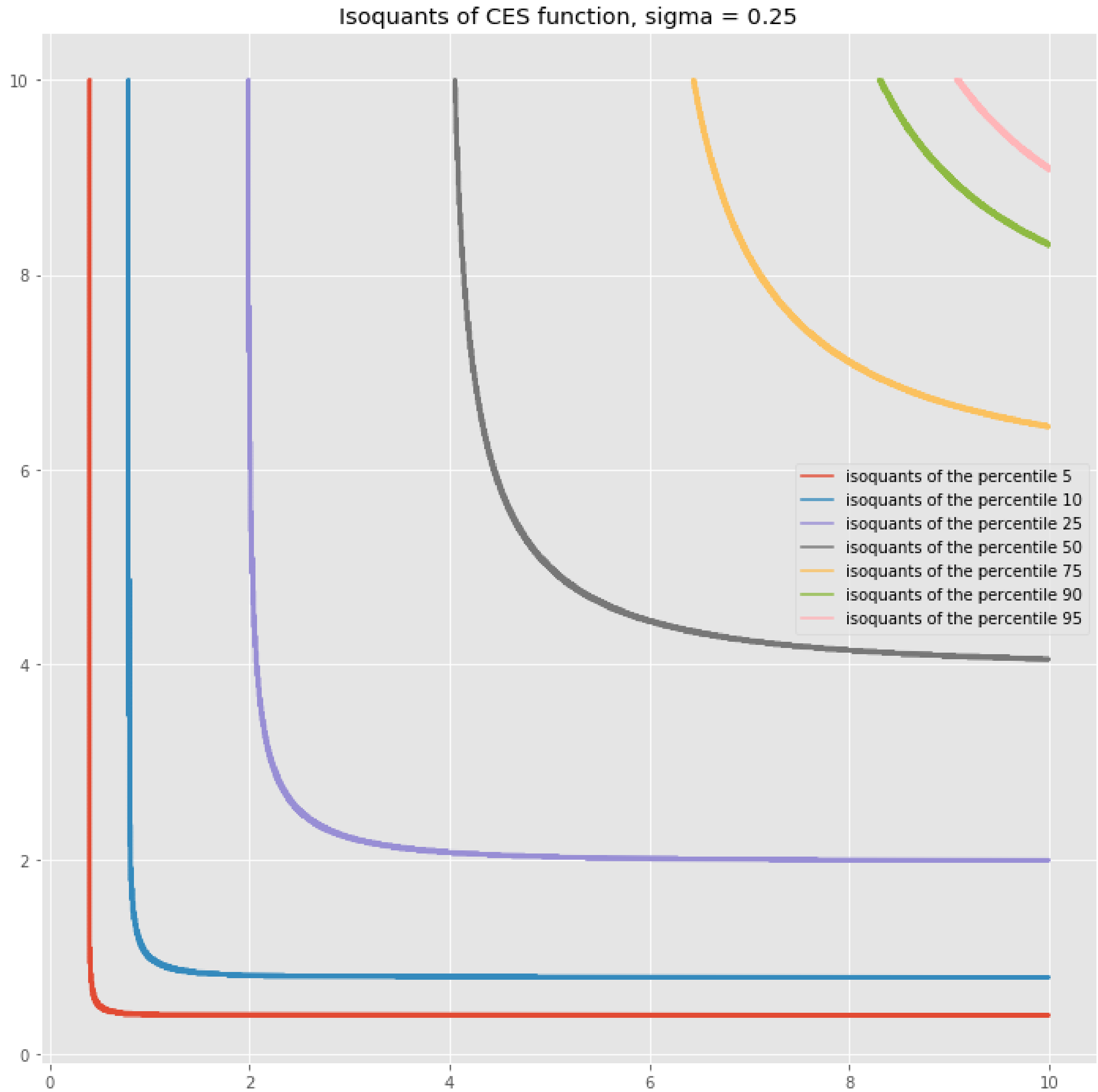


CES, sigma=0.99 & Cheb. approximation 10 order

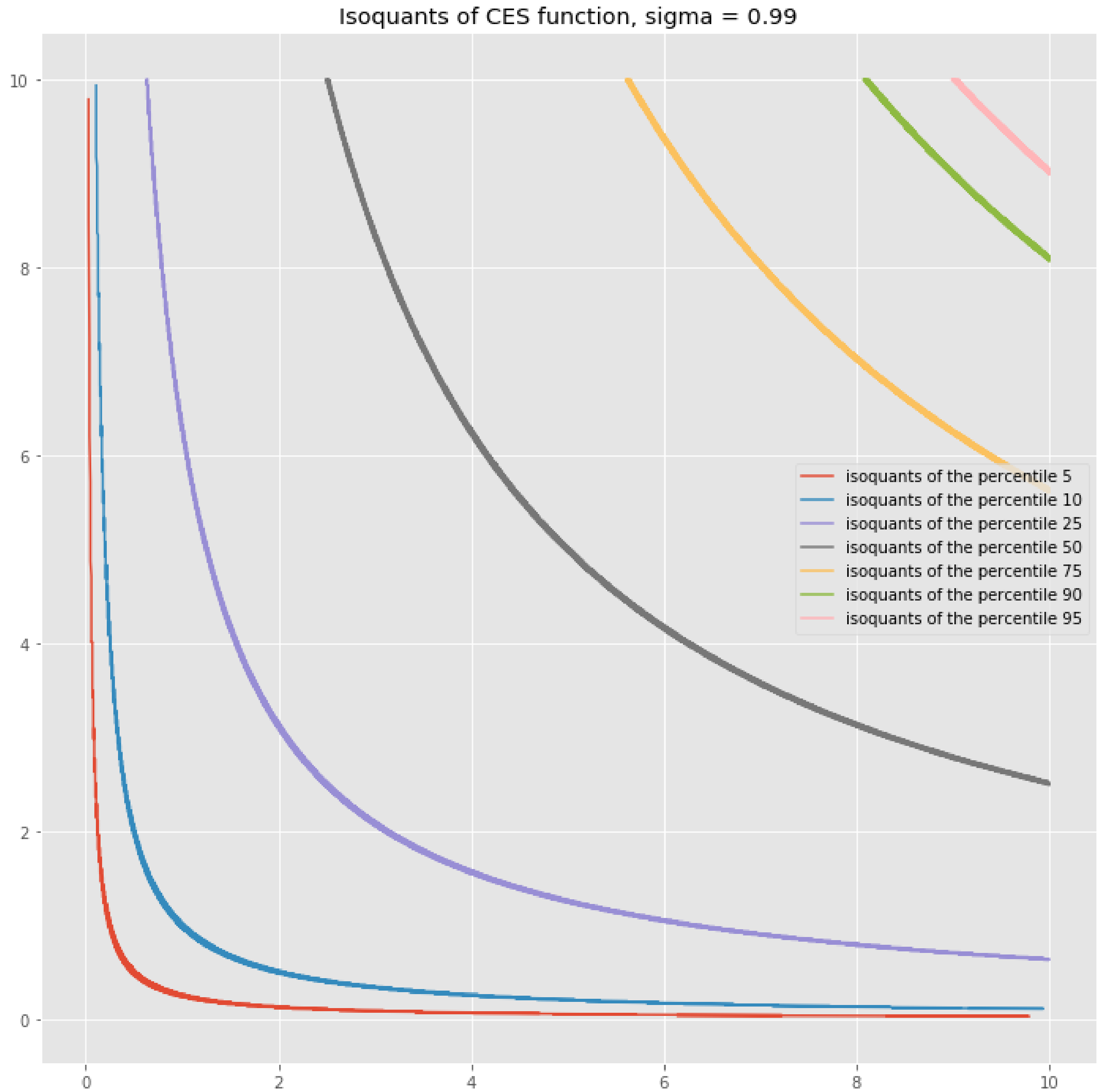


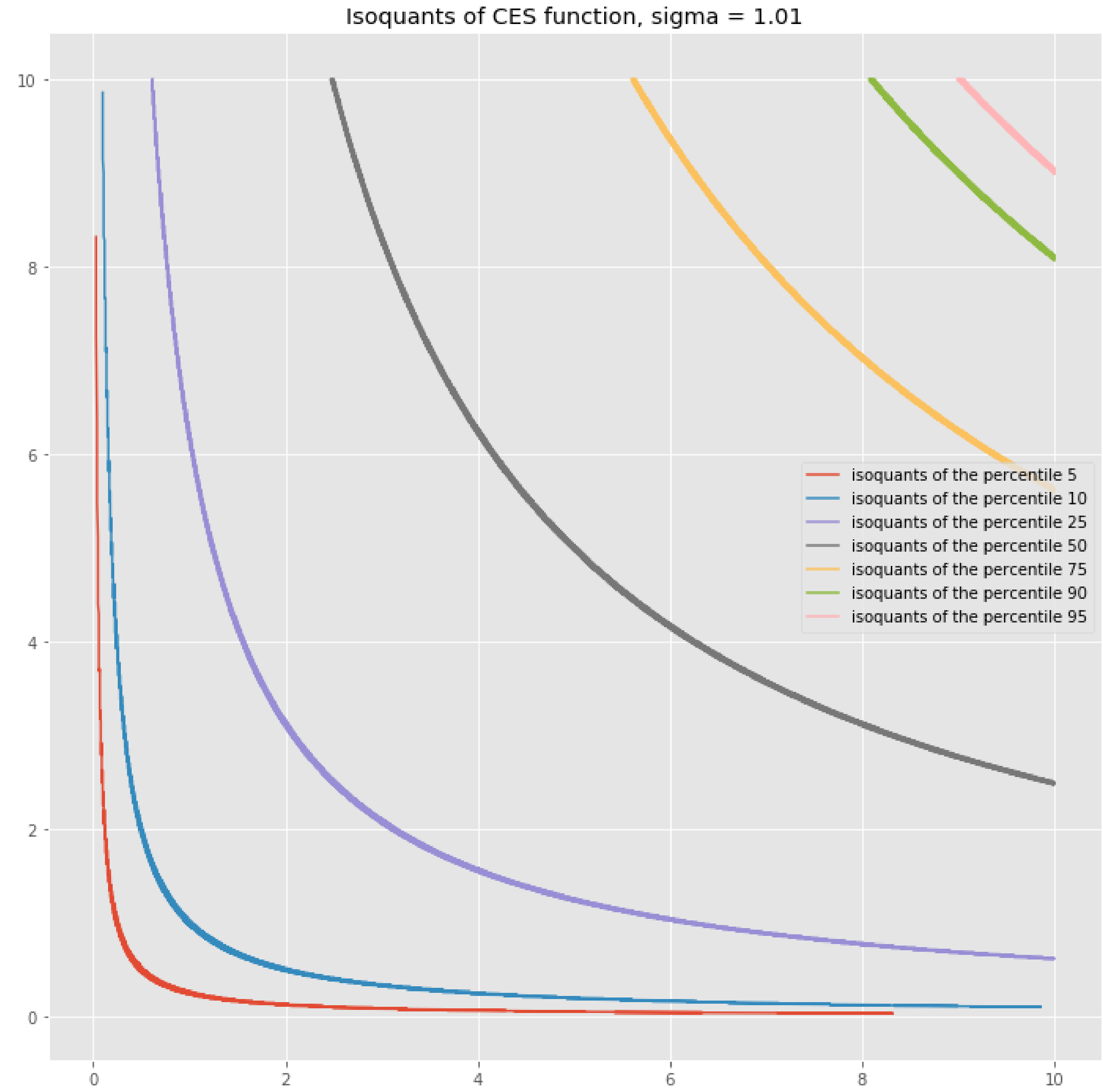


Isoquants of CES function, sigma = 5.0
isoquants of the percentile 5
isoquants of the percentile 10
isoquants of the percentile 25
isoquants of the percentile 50
isoquants of the percentile 75
isoquants of the percentile 90
isoquants of the percentile 95

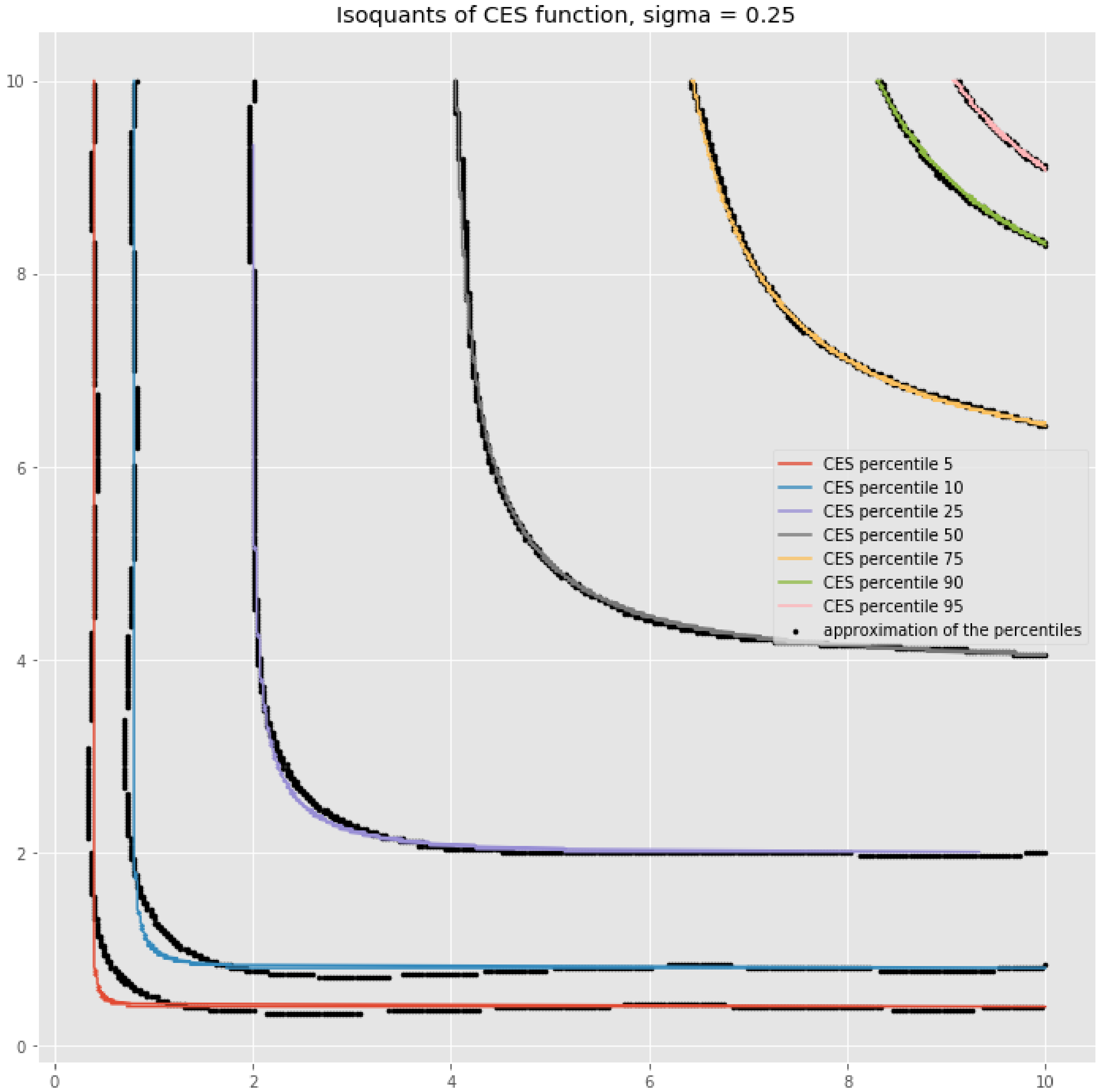


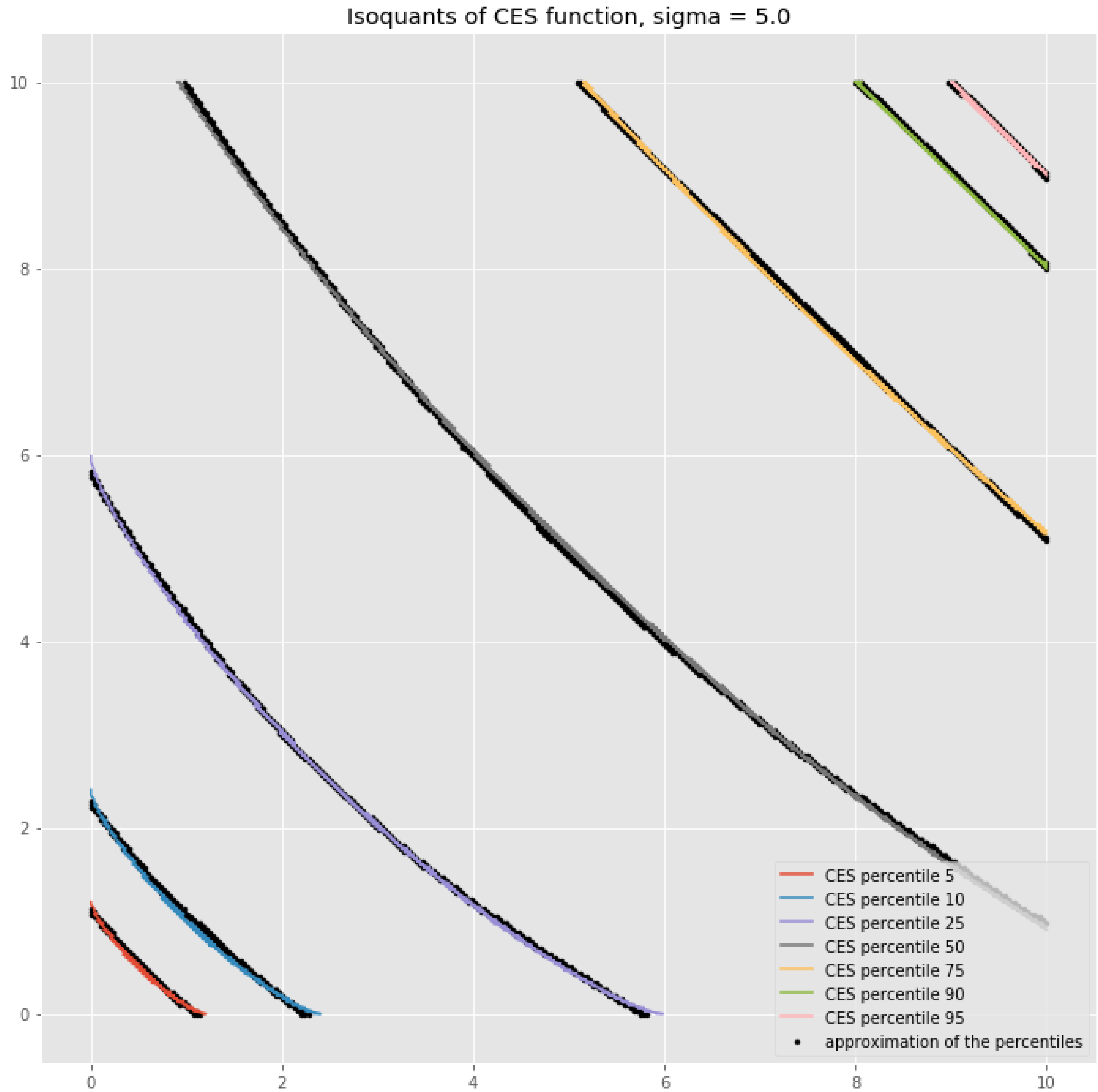
Isoquants of CES function, sigma = 0.25
isoquants of the percentile 5
isoquants of the percentile 10
isoquants of the percentile 25
isoquants of the percentile 50
isoquants of the percentile 75
isoquants of the percentile 90
isoquants of the percentile 95
0
2
4
6
8
10


Isoquants of CES function, sigma = 0.99
isoquants of the percentile 5
isoquants of the percentile 10
isoquants of the percentile 25
isoquants of the percentile 50
isoquants of the percentile 75
isoquants of the percentile 90
isoquants of the percentile 95
0
2
4
6
8
10



Isoquants of CES function, sigma = 1.01
isoquants of the percentile 5
isoquants of the percentile 10
isoquants of the percentile 25
isoquants of the percentile 50
isoquants of the percentile 75
isoquants of the percentile 90
isoquants of the percentile 95
0
2
4
6
8
10


Isoquants of CES function, sigma = 0.25
0
2
4
6
8
10
CES percentile 5
CES percentile 10
CES percentile 25
CES percentile 50
CES percentile 75
CES percentile 90
CES percentile 95
approximation of the percentiles



Isoquants of CES function, sigma = 5.0
CES percentile 5
CES percentile 10
CES percentile 25
CES percentile 50
CES percentile 75
CES percentile 90
CES percentile 95
approximation of the percentiles