M6803 Assignment3

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1.1 Ex.1.

(a). Perfrom LDLT decomposition on the matrix A. Since the matrix is symmetrical, we only need to calculate L.

$$L_{21} = \frac{20}{8} = 2.5 \tag{1}$$
$$L_{31} = \frac{15}{8} = 1.875 \tag{2}$$
$$U_{22} = 30 \tag{3}$$
$$U_{13} = 15 \tag{4}$$
$$L_{32} = \frac{A_{32} - L_{31}U_{12}}{U_{22}} = 0.4167 \tag{5}$$
$$U_{23} = 12.5 \tag{6}$$
$$U_{33} = A_{33} - L_{31}U_{13} - L_{32}U_{23} = 26.667 \tag{7}$$

Therefore, we have following LDLT decomposition:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2.5 & 1 & 0 \\ 1.875 & 0.4167 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 26.667 \end{bmatrix} \begin{bmatrix} 1 & 2.5 & 1.875 \\ 0 & 1 & 0.4167 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Then, transform the result to Cholesky decomposition

$$A = \begin{bmatrix} 2.8284 & 0 & 0 \\ 7.0711 & 5.4772 & 0 \\ 5.3033 & 2.2822 & 5.1640 \end{bmatrix} \begin{bmatrix} 2.8284 & 7.0711 & 5.3033 \\ 0 & 5.4772 & 2.2822 \\ 0 & 0 & 5.1640 \end{bmatrix} \tag{9}$$

The Equation can be solved as $Ax = L(L^T x) = b$. By using substitute procedure, we can get

$$L^T x = \begin{bmatrix} 17.677 \\ 22.822 \\ -8.876 \end{bmatrix} \tag{10}$$

Then, back substitute, we obtain

$$x = \begin{bmatrix} -2.734 \\ 4.883 \\ -1.719 \end{bmatrix} \tag{11}$$

(b). Because the matrix is not diagonally dominant matrix, Jacobi method can not converge. What is worse, two largest elements in row one and row two are in the same column. Therefore, we cannot adjust rows or columns to make Jacobi method converge. Following is the Jacobi iteration for original matrix:

$$x^{(n+1)} = \begin{bmatrix} 6.25 \\ 31.25 \\ 12.5 \end{bmatrix} - \begin{bmatrix} 0 & 2.5 & 1.875 \\ 0.25 & 0 & 0.625 \\ 0.25 & 0.83333333 & 0 \end{bmatrix} x^{(n)} \tag{12}$$

Therefore,

$$x^{(1)} = \begin{bmatrix} 6.25 \\ 31.25 \\ 12.5 \end{bmatrix} - \begin{bmatrix} 0 & 2.5 & 1.875 \\ 0.25 & 0 & 0.625 \\ 0.25 & 0.83333333 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3.125 \\ 32.125 \\ 11.0833 \end{bmatrix} \tag{13}$$

$$x^{(2)} = \begin{bmatrix} 6.25 \\ 31.25 \\ 12.5 \end{bmatrix} - \begin{bmatrix} 0 & 2.5 & 1.875 \\ 0.25 & 0 & 0.625 \\ 0.25 & 0.83333333 & 0 \end{bmatrix} \begin{bmatrix} 3.125 \\ 32.125 \\ 11.0833 \end{bmatrix} = \begin{bmatrix} -94.843 \\ 23.541 \\ -15.052 \end{bmatrix} \tag{14}$$

$$x^{(3)} = \begin{bmatrix} 6.25 \\ 31.25 \\ 12.5 \end{bmatrix} - \begin{bmatrix} 0 & 2.5 & 1.875 \\ 0.25 & 0 & 0.625 \\ 0.25 & 0.83333333 & 0 \end{bmatrix} \begin{bmatrix} -94.843 \\ 23.541 \\ -15.052 \end{bmatrix} = \begin{bmatrix} -24.3815 \\ 64.3684 \\ 16.5928 \end{bmatrix} \tag{15}$$

$$x^{(4)} = \begin{bmatrix} 6.25 \\ 31.25 \\ 12.5 \end{bmatrix} - \begin{bmatrix} 0 & 2.5 & 1.875 \\ 0.25 & 0 & 0.625 \\ 0.25 & 0.83333333 & 0 \end{bmatrix} \begin{bmatrix} -24.3815 \\ 64.3684 \\ 16.5928 \end{bmatrix} = \begin{bmatrix} -185.782 \\ 26.974 \\ -35.045 \end{bmatrix} \tag{16}$$

1.2 Ex.2

We know the Taylor expansion of functin $f(x+h)$ is

$$f(x+h) = f(x) + h\frac{d}{dx}f(x) + \frac{h^2}{2}\frac{d^2}{dx^2}f(x) + \frac{h^3}{6}\frac{d^3}{dx^3}f(x) + \frac{h^4}{24}\frac{d^4}{dx^4}f(x) + \mathcal{O}\left(h^4\right) \tag{17}$$

Therefore, we write down Taylor expansion for

$$f(x-2h) = f(x) - 2h\frac{d}{dx}f(x) + \frac{4h^2}{2}\frac{d^2}{dx^2}f(x) - \frac{8h^3}{6}\frac{d^3}{dx^3}f(x) + \frac{16h^4}{24}\frac{d^4}{dx^4}f(x) + \mathcal{O}\left(h^5\right) \tag{18}$$

$$f(x-h) = f(x) - h\frac{d}{dx}f(x) + \frac{h^2}{2}\frac{d^2}{dx^2}f(x) - \frac{h^3}{6}\frac{d^3}{dx^3}f(x) + \frac{h^4}{24}\frac{d^4}{dx^4}f(x) + \mathcal{O}\left(h^5\right) \tag{19}$$

$$f(x+h) = f(x) + h\frac{d}{dx}f(x) + \frac{h^2}{2}\frac{d^2}{dx^2}f(x) + \frac{h^3}{6}\frac{d^3}{dx^3}f(x) + \frac{h^4}{24}\frac{d^4}{dx^4}f(x) + \mathcal{O}\left(h^5\right) \tag{20}$$

$$f(x+2h) = f(x) + 2h\frac{d}{dx}f(x) + \frac{4h^2}{2}\frac{d^2}{dx^2}f(x) + \frac{8h^3}{6}\frac{d^3}{dx^3}f(x) + \frac{16h^4}{24}\frac{d^4}{dx^4}f(x) + \mathcal{O}\left(h^5\right) \tag{21}$$

It can be easily observed that

$$\begin{aligned} &f(x+2h) - f(x-2h) - 2\left(f(x+h) + f(x-h)\right) \\ &= 4h\frac{d}{dx}f(x) + \frac{16h^3}{6}\frac{d^3}{dx^3}f(x) - 2\left(2h\frac{d}{dx}f(x) + \frac{2h^3}{6}\frac{d^3}{dx^3}f(x)\right) + \mathcal{O}\left(h^5\right) \\ &= 2h^3\frac{d^3}{dx^3}f(x) + \mathcal{O}\left(h^5\right) \end{aligned} \tag{22}$$

Therefore, we obtain the centered finite-difference approximation to the third derivate that is second-order accurate:

$$\frac{d^3}{dx^3}f(x) = \frac{f(x+2h) - f(x-2h) - 2\left(f(x+h) + f(x-h)\right)}{2} + \mathcal{O}\left(h^2\right) \tag{23}$$

1.3 Ex.3

Firstly, we derive the governing equation of the given system.

$$V = \pi D^4 h/4 \tag{24}$$

$$\frac{dV}{dt} = -Q_{out} = -CA\sqrt{2gh} \tag{25}$$

Substitute the given parameters into the equation, we obtain:

$$\frac{dh}{dt} = -\frac{0.55\sqrt{2gh}}{10^4} \tag{26}$$

$$h(0) = 2.75 \tag{27}$$

Inserting 4th order RK method in to Excel for this problem, we have following plot for $h(t)$

Meanwhile, iteration of RK methods with step $t = 1000$s is attached:

Table 1: RK iteration table

t	h	f	k1	k2	k3	k4
0	2.75	-0.0004	-0.4038	-0.38869	-0.38927	-0.37413
1000	2.361024	-0.00037	-0.37415	-0.35902	-0.35965	-0.34448
2000	2.001695	-0.00034	-0.34451	-0.32935	-0.33003	-0.31483
3000	1.672012	-0.00031	-0.31486	-0.29967	-0.30042	-0.28517
4000	1.371975	-0.00029	-0.28521	-0.26999	-0.27082	-0.25552
5000	1.101584	-0.00026	-0.25557	-0.24029	-0.24123	-0.22586
6000	0.86084	-0.00023	-0.22592	-0.21058	-0.21166	-0.19619
7000	0.649742	-0.0002	-0.19628	-0.18085	-0.18211	-0.16651
8000	0.468292	-0.00017	-0.16663	-0.15108	-0.1526	-0.13681
9000	0.31649	-0.00014	-0.13699	-0.12126	-0.12317	-0.10706
10000	0.194338	-0.00011	-0.10734	-0.09133	-0.09389	-0.07717
11000	0.101847	-7.8E-05	-0.07771	-0.06111	-0.06501	-0.04673
12000	0.039064	-4.8E-05	-0.04813	-0.02982	-0.03784	-0.00851
13000	0.007069	-2E-05	-0.02047	#NUM!	#NUM!	#NUM!

Therefore, approximately after 13500 seconds, the tank will be empty. The online version of this homework (including the Excel worksheet) is upload to `http://goo.gl/P9hNVq`.