A Guide to Shear Force and Bending Moment Diagrams

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Introduction

In this guide we're going to take a look at shear and moment diagrams in detail. Determining shear and moment diagrams is an essential skill for any engineer. Unfortunately it's probably the one structural analysis skill students struggle with most.

This is a problem. Without understanding the shear forces and bending moments developed in a structure you can't complete a design. Shear force and bending moment diagrams tell us about the underlying state of stress in the structure. So naturally they're the starting point in any design process.

Another reason every graduating engineer needs to have a solid grasp of shear forces and bending moments is because they're absolutely going to be tested in almost every graduate interview. The quickest way to tell a great CV writer from a great graduate engineer is to ask them to sketch a qualitative bending moment diagram for a given structure and load combination!

So in this guide we'll give you a thorough introduction to shear forces, bending moments and how to draw shear and moment diagrams. We won't be able to cover everything in this one guide but hopefully you'll reach the end knowing more than when you started! If you want a deeper dive into this, check out my course on Mastering Shear Force and Bending Moment Diagrams.

Contents

1 What is a Bending Moment?

Let's start with a basic question; what is a bending moment? To answer this we need to consider what's happening internally in a structure under load. Consider a simply supported beam subject to a uniformly distorted load, Fig. 1.

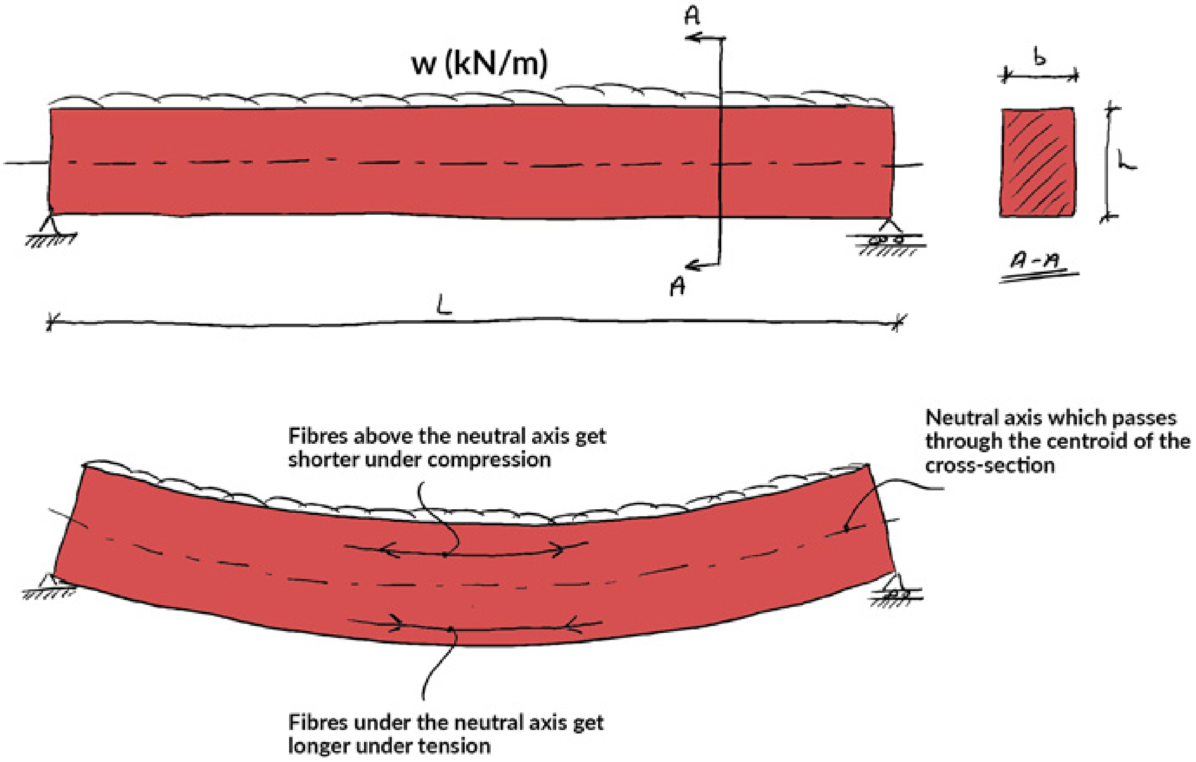

Figure 1: Simply supported beam under the influence of a uniformly distributed load.

The beam will deflect under the load. In order for the beam to deflect as shown, the fibres in the top of the beam must contract or get shorter. The fibres in the bottom of the beam must get longer.

We can say the top of the beam is in compression while the bottom is in tension (notice the direction of the arrows on the fibres in the deflected beam). Now, at some position in the depth of the beam, compression must turn into tension. There is a plane in the beam where this transition between tension and compression occurs. This plane is called the neutral plane or sometimes the neutral axis.

Imagine taking a vertical cut through the beam at some distance x along the beam. We can represent the strain and stress variation throughout the depth of the beam with strain and stress distribution diagrams, Fig. 2.

Remember, strain is just the change in length or a fibre or element divided by the original length. In this case we're considering the longitudinal strain or strain perpendicular (normal) to the cut face.

Compression strains above the neutral axis exist because the longitudinal fibres in the beam

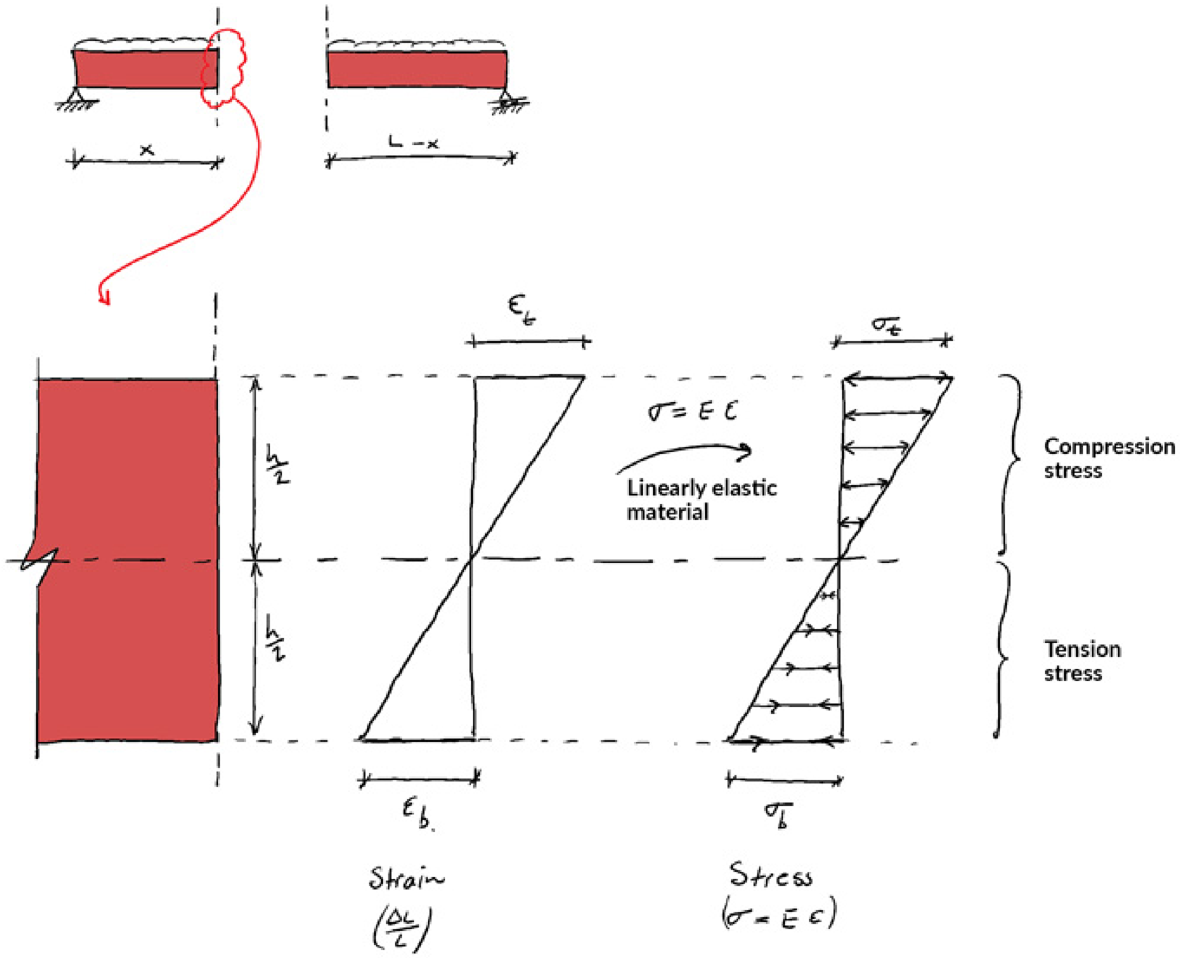

Figure 2: Strain and stress distribution revealed by making an imaginary cut through the beam at a distance x along the beam.

are getting shorter. Tensile strains occur in the bottom because the fibres are extending or getting longer.

If we can assume this beam is made of a linearly elastic material the stresses are linearly proportional to the strains. This simply means we need to multiple the strain at some point in the beam by the Young's modulus (modulus of elasticity) to get the corresponding stress at that point in the beam.

We know that if we multiply a stress by the area over which it acts, we get the resultant force on that area. The same is true for the stress acting on the cut face of the beam. The compression stresses can be represented by a compression force (stress resultant) while the tensile stresses can be replaced by an equivalent tensile force, Fig. 3.

$$F_c = \underbrace{\left[\sigma_t \times \frac{1}{2}\right]}_{\text{average stress}} \underbrace{\left[b \times \frac{h}{2}\right]}_{\text{area}} \tag{1.1}$$

As a result of the external loading on the structure and the deflection that this induces, we

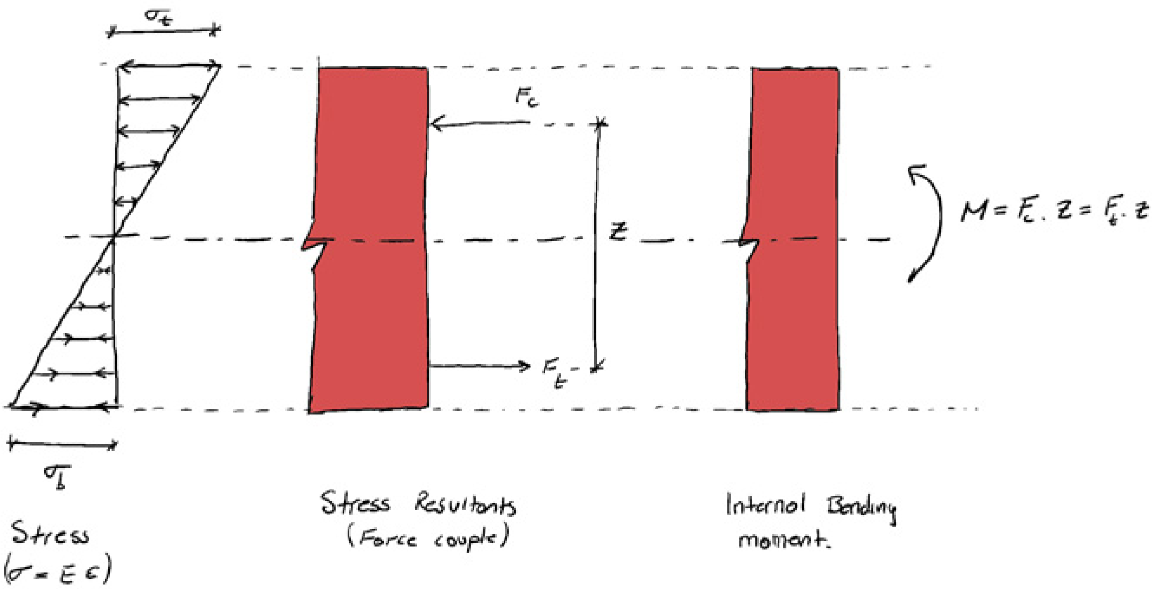

Figure 3: Normal stresses due to bending the their stress resulting forming a couple.

end up with two forces acting on the cut cross-section. These forces are:

- equal in magnitude (must be to maintain force equilibrium)
- parallel to each other (and perpendicular to the cut face)
- acting in opposite directions
- separated by a distance or lever arm, z

You might recognise this pair of forces as forming a couple or moment M.

$$M = F_c \times z = F_t \times z \tag{1.2}$$

The internal bending moment M, is the bending moment we represent in a bending moment diagram. The bending moment diagram shows how M (and therefore stress) varies across a structure.

Since the bending moment is linked directly to the corresponding stresses, our bending moment diagram is a proxy for how the stresses vary throughout the structure.

Now, if we know the state of longitudinal (normal) stress due to bending at a given section in a structure we can work out the corresponding bending moment.

However, more often it's the case that we know the value of the bending moment at a point and use this to work out the maximum values of normal stress at that location.

We do this using the Moment-Curvature equation a.k.a. the Engineer's Bending Equation...

$$\sigma = \frac{-M\,y}{I} \tag{1.3}$$

...which relates the stress, σ at a distance y from the neutral axis, to the moment, M. Where I is the second moment of area for the cross-section.

Hopefully now you can clearly see how bending moments arise;

- external forces induce deflections
- strains develop (which we see at a larger scale as structural deflections)
- where we have strains, we must have stresses (remember Young's modulus)
- these stresses, can be represented with their force resultants that ultimately form a couple or internal bending moment, M

2 What is a Shear Force?

We can now turn our attention to shear forces and start with a simple definition;

> A shear force is any force acting perpendicularly to the longitudinal axis of the structure. We're typically interesting in internal shear forces that are the resultant of internal shear stresses developed in the structure.

Building on our discussion of bending moments, the shear force represented in the shear force diagram is also the resultant of shear stresses acting at a given point in the structure. Consider the cut face of the beam discussed above, Fig. 4.

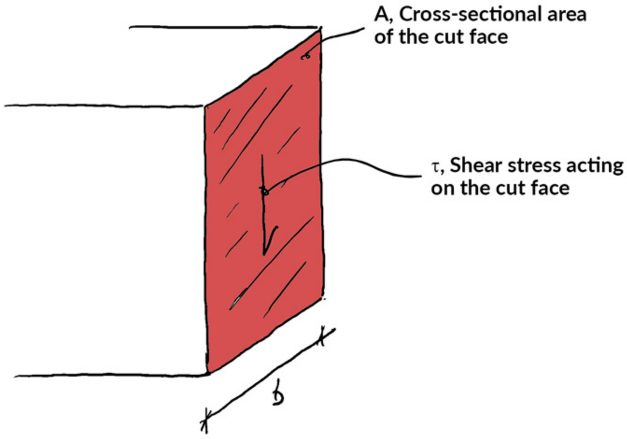

Figure 4: Shear stress acting parallel to the vertical cut face of the beam

The shear stress, τ acting on this cut face is evenly distributed across the width of the face and acts parallel to the cut face. The average value of the shear stress, τ_{avg} is simply the shear force at this point in the structure, V, divided by the cross-sectional area over which it acts, A,

$$\tau_{avg} = \frac{V}{A} \tag{2.1}$$

However, this is just the average value of the shear stress acting on the face. The shear stress actually varies parabolically through the depth of the section according to the following equation,

$$\tau = \frac{VQ}{Ib} \tag{2.2}$$

where, Q is the first moment of area of the area above the level at which the shear stress is being determined, I is the second moment of area of the cross-section and b is the width of the section, Fig. 5.

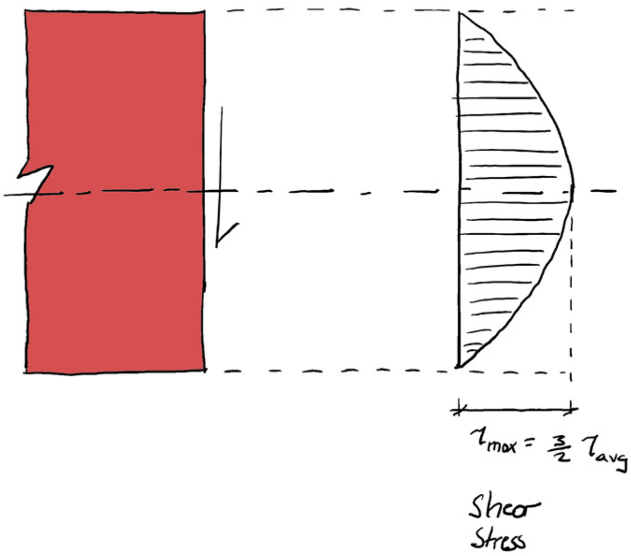

Figure 5: Variation of shear stress through the depth of the beam section

So, just as with bending moments discussed previously, shear stresses arise due to the application of external forces onto the beam. These forces induce shear stress within the beam that can be represented by the (shear) force resultant. It is this shear force that we represent on a shear force diagram which indicates how the shear force varies across the structure.

3 Calculating Internal Shear Forces and Bending Moments

Up to this point we've considered the link between the normal (bending) stress and associated bending moment and the shear stress and associated shear force. Based on this you should be comfortable with the idea that knowing the value of bending moment and shear force at a point is important for understanding the stresses in the structure at that point.

Now we're going to consider the problem of calculating shear forces and bending moments not from the point of view of internal stresses but by considering equilibrium of the structure.

In reality, this is practically how we determine the shear force and bending moment at a point in the structure. Again, let's consider the simply supported beam from above, subject to a uniformly distributed load, w kN/m, Fig. 6.

Simple statics tell us that if the beam is in a state of static equilibrium, the left and right side support reactions are,

$$V_a = V_b = \frac{wL}{2} \tag{3.1}$$

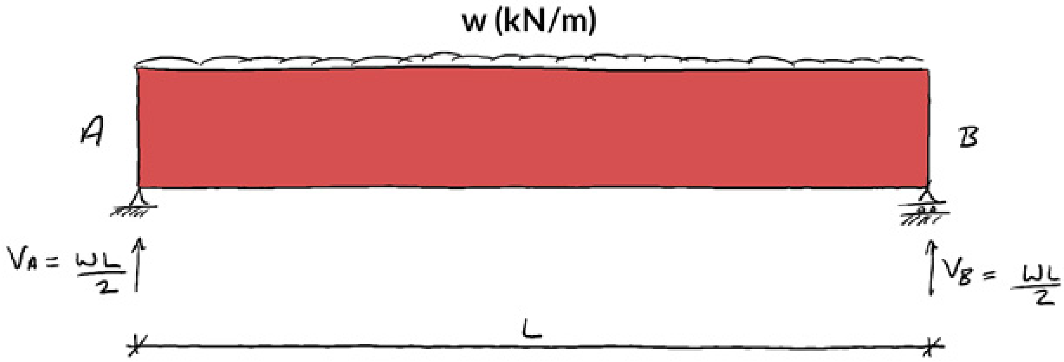

Figure 6: Simply supported beam subject to a uniformly distributed load with intensity w kN/m

If the structure is in a state of static equilibrium (which it is), then any sub-structure or part of the structure must also be in a state of static equilibrium under the stabilising action of the internal stress resultants.

This is a key point! Imagine taking a cut through the structure and separating it into 2 sub-structures. When we cut the structure, we 'reveal' the internal stress resultants (bending moment and shear force), Fig. 7.

M_L and M_R are the internal bending moments on either side of the imaginary cut while V_L and V_R are the internal shear forces on either side of the imaginary cut.

> M_R and V_R represent the influence of the left hand side of the structure (sub-structure 1) on the right hand side of the structure (sub-structure 2) and vice versa.

We've just said that each one of these sub-structures is stabilised by the influence of the internal bending moment and shear force revealed by the imaginary cuts.

This means, if we want to find the value of internal bending moment or shear force at any point in a structure, we simply cut the structure at that point to expose the internal stress resultants (M and V). Then calculate what values they must have to ensure the sub-structure

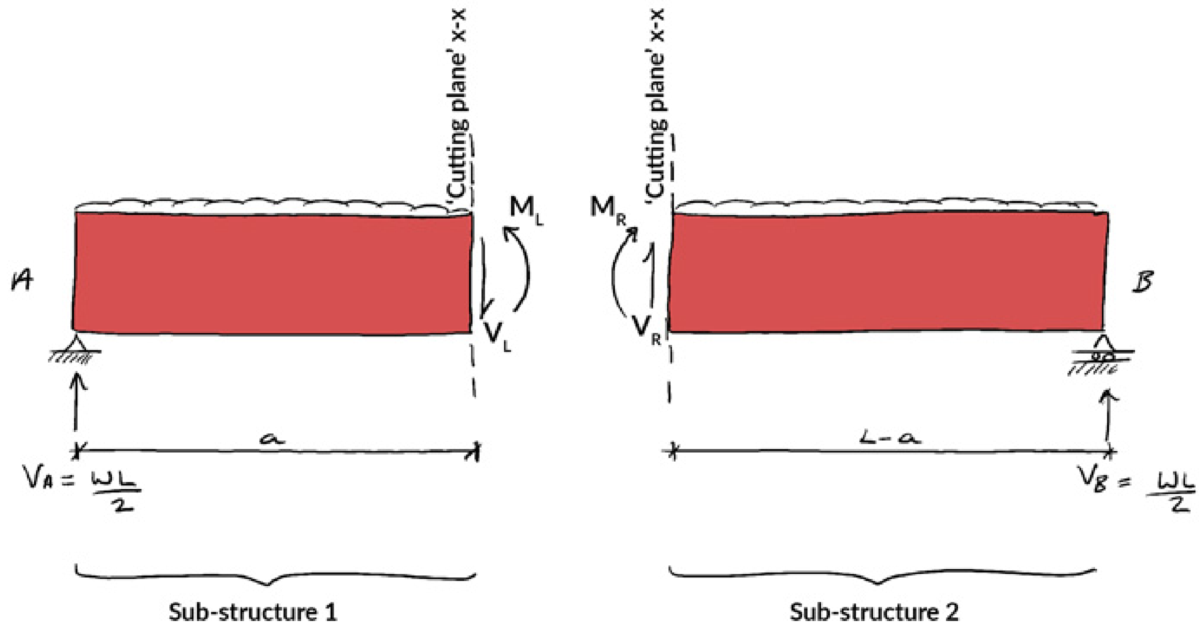

Figure 7: Imaginary cut creating two sub-structures and revealing the internal stress resultants

remains in equilibrium! For example the sub-structure below, Fig. 8, must remain in equilibrium under the combined influence of:

- the external distributed load acting on the sub-structure
- the left hand reaction, V_A (note it has not been reduced just because we're considering the sub-structure. It retains at its original value)
- the internal bending moment at the cut, M_L
- the internal shear force at the cut, V_L

This starts to make more sense when we plug some numbers into an example. For the beam above, let's imagine it has a span $L = 6$ m, applied loading of $w = 10$ kN/m and imagine we cut the beam at $a = 2$ m from the left hand support.

The left hand reaction, V_A is,

$$V_A = \frac{wL}{2} = \frac{10 \times 6}{2} = 30 \text{ kN} \tag{3.2}$$

Now taking the sum of the moments about the cut and assuming clockwise moments are positive,

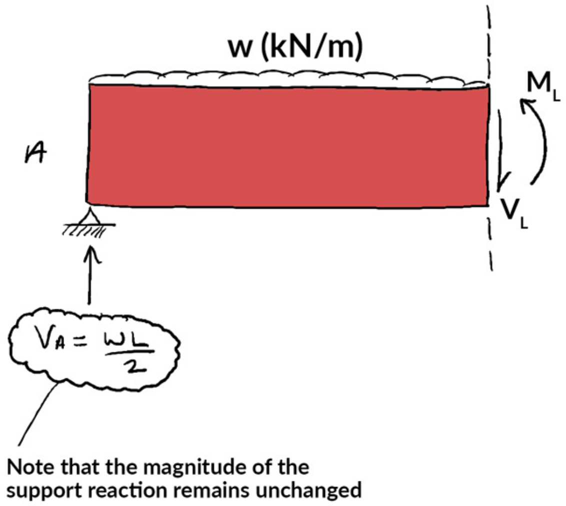

Figure 8: Sub-structure 1 stabilised by the internal shear force and bending moment revealed by the cut.

$$\sum M_{\text{cut}} = 0 \tag{3.3}$$

$$(30\text{ kN} \times 2\text{ m}) - (10\text{ kN/m} \times 2\text{m} \times 1\text{m}) - M_L = 0 \tag{3.4}$$

$$\therefore\ M_L = 40\text{ kNm} \tag{3.5}$$

So, the internal bending moment required to maintain moment equilibrium of the sub-structure is 40 kNm. Similarly, if we take the sum of the vertical forces acting on the sub-structure, this would yield $V_L = 10$ kN.

4 Building Shear and Moment Diagrams

In the last section we worked out how to evaluate the internal shear force and bending moment at a discrete location using imaginary cuts. But to draw a shear force and bending moment diagram, we need to know how these values change across the structure.

What we really want is an equation that tells us the value of the shear force and bending moment as a function of x. Where x is the position along the beam. Consider making an imaginary cut, just like above, except now we can make the cut at a distance x along the beam, Fig. 9.

Now the internal shear force and bending moment revealed by the cut are functions of x, the cut position. Here, we'll determine an expression for $M(x)$. But the procedure is exactly the same to determine $V(x)$.

Taking the sum of the moments about the cut and again assuming clockwise moments are positive,

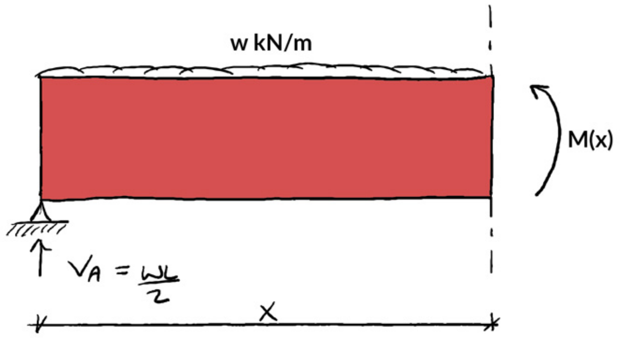

Figure 9: Sub-structure 1 stabilised by the internal shear force and bending moment revealed by the cut.

$$\sum M_{\text{cut}} = 0 \tag{4.1}$$

$$\frac{wL}{2}x - wx\frac{x}{2} - M(x) = 0 \tag{4.2}$$

$$M(x) = \frac{wLx}{2} - \frac{wx^2}{2} \tag{4.3}$$

$$M(x) = \frac{wx}{2}(L - x) \tag{4.4}$$

Now we can use equation (4.4) to determine the value of the internal bending moment for any value of x along the beam. Plotting the bending moment diagram is simply a matter of plotting the equation.

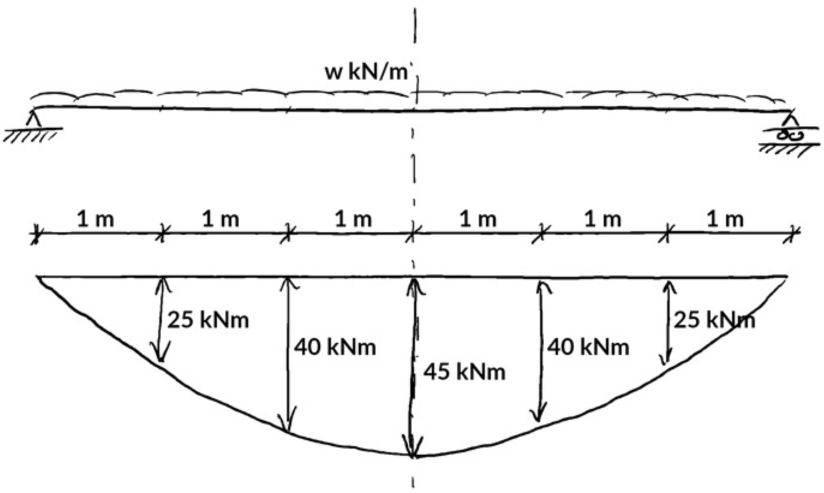

Figure 10: Bending moment evaluated at 1 m intervals along the beam.

4.1 Finding the location of the maximum bending moment

In the example above, the structure and loading is symmetrical so it's pretty easy to recognise the location of the maximum moment and then subsequently to evaluate it.

However this may not always be the case. So it's helpful to have a technique to identify the location of the maximum moment without needing to plot the full bending moment diagram.

In this example, the bending moment for the whole structure is described by a single equation, equation (4.4). You might remember from basic calculus that to identify the location of the local maximum point in a function we simply differentiate the function to get the equation for the slope. Then it's just a matter of setting this function equal to zero and solving for x, Fig. 11.

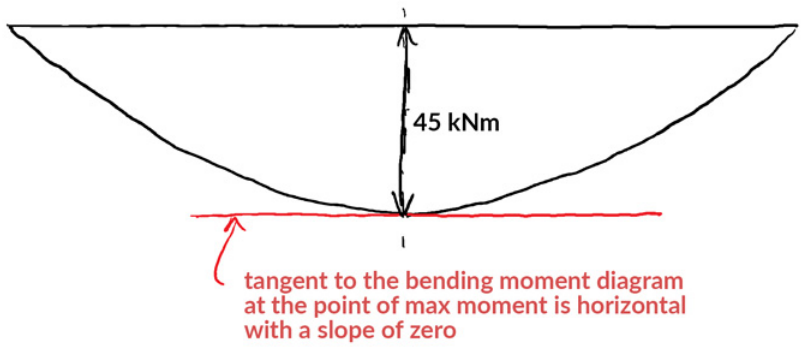

Figure 11: Horizontal tangent identifying location of maximum moment.

In other words, at the location of the maximum bending moment, the slope of the bending moment diagram is zero. So we just need to solve for this location. Once we have the location we can evaluate the bending moment using equation (4.4).

So, to demonstrate let's first evaluate the differential of equation (4.4),

$$\frac{\mathrm{d}}{\mathrm{d}x}M(x) = \frac{wL}{2} - wx \tag{4.5}$$

Remember, equation (4.5) represents the slope of the bending moment diagram. So we now let it equal to zero and solve for x.

$$\frac{wL}{2} - wx = 0 \tag{4.6}$$

$$\therefore x = \frac{L}{2} \tag{4.7}$$

As we would expect, the bending moment is a maximum at the mid-span, $x = L/2$. Now we can evaluate equation (4.4) at $x = L/2 = 3$ m.

$$M(x = L/2) = \frac{wL^2}{4} - \frac{WL^2}{8} \tag{4.8}$$

$$\therefore M_{\text{max}} = \frac{wL^2}{8} = \frac{10 \times 6^2}{8} = 45 \text{ kNm} \tag{4.9}$$

There we have it; the location and magnitude of the maximum bending moment in this simply supported beam, all with some basic calculus.

5 Drawing Shear Force and Bending Moment Diagrams - An Example

Now that we have a grasp of the fundamentals, let's see how it all ties together with a bigger more complex worked example. This example is an extract from this course. Just a quick heads up, if you're new to shear force and bending moment diagrams, this question might be a bit of a challenge. If you get a bit lost with this example, it might be worth your time taking a look at this DegreeTutors course. It's aimed at bringing you from scratch all the way up to being comfortable determining complex shear and moment diagrams.

Ok, let's get on with it. We want to determine the shear force and bending moment diagrams for the following simply supported beam.

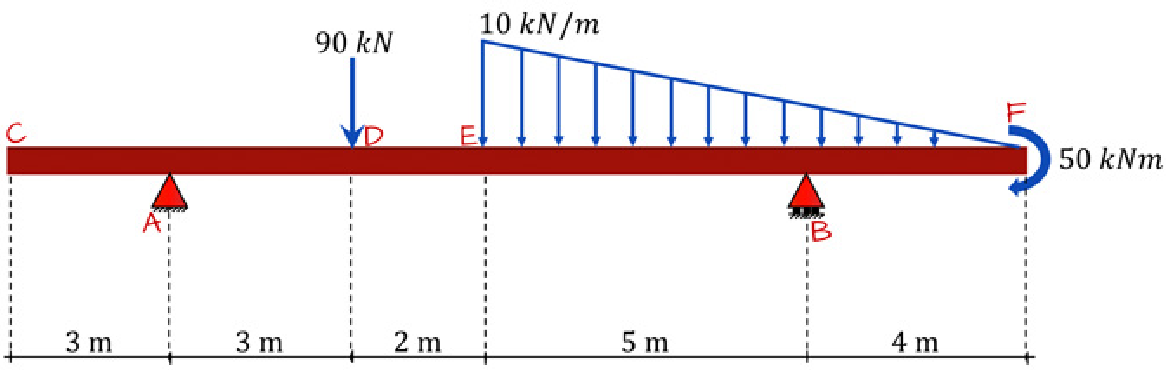

Figure 12: Simply supported beam example question.

You can continue reading through the solution below, or if you prefer video, you can watch me walk through the solution here.

5.1 Calculating support reactions

The first step in analysing any statically determinate structure is working out the support reactions. We can kick-off by taking the sum of the moments about point A, to determine the unknown vertical reaction at B, V_B,

$$\sum M_{\mathsf{A}} = 0 \text{ (assume clockwise positive)} \tag{5.1}$$

$$(90\text{kN} \times 3\text{m}) + 50\text{kNm} + (10\text{kN/m} \times 9\text{m} \times 8\text{m} \times 0.5) - 10\text{m} \times V_B = 0 \tag{5.2}$$

$$\therefore V_B = 68\text{kN} \tag{5.3}$$

Now with only one unknown force, we can consider the sum of the forces in the vertical direction to calculate the unknown reaction at A, V_A,

$$V_A + 68\text{kN} - 90\text{kN} - 10\text{kN/m} \times 9\text{m} \times 0.5 = 0 \tag{5.4}$$

$$\therefore V_A = 67\text{kN} \tag{5.5}$$

5.2 Drawing the shear force diagram

Our approach to drawing the shear force diagram is actually very straightforward. We're going to 'trace the impact of the loads' across the beam from left to right.

The first load on the structure is $V_A = 67$kN acting upwards, this raises the shear force diagram from zero to $+67$kN at point A. The shear force then remains constant as we move from left to right until we hit the external load of 90kN acting down at D. This will cause the shear force diagram to 'drop' down by 90kN at D to a value of $+67 - 90 = -23$kN.

This process of following or tracing the loads across the structure continues across the full beam until you've completely traced out the shear force diagram.

When we reach the linearly varying load at E, we make use of the relationship between load intensity, q and shear force V that tells us that the slope of the shear force diagram is equal to the negative of the load intensity at a point,

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q \tag{5.6}$$

This is telling us that the linearly varying distributed load between E and F will produce a curved shear force diagram described by a polynomial equation. In other words, the shear force diagram starts curving at E with a linearly reducing slope as we move towards F, ultimately finishing at F with a slope of zero (horizontal). When the full loading for the beam is traced out, we end up with the following, Fig. 13,

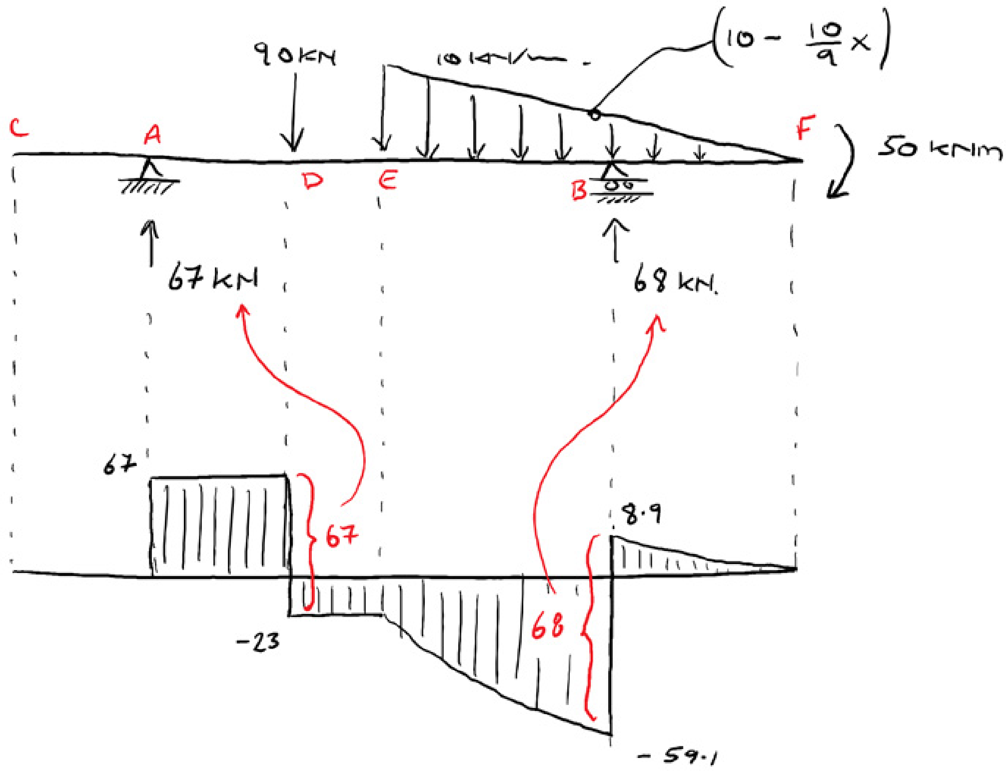

Figure 13: Shear force diagram obtained by 'tracing the loads across the structure'.

It's worth pausing for a moment to explain how the shear force to the left of B, $V_{B,L} = -59.1$kN was calculated. This is obtained by subtracting the total vertical load between E and B from the shear force of -23kN at E.

$$V_{B,L} = -23\text{kN} - (4.44\text{kN/m} \times 5\text{m}) - (10 - 4.44)\text{kN/m} \times 5\text{m} \times 0.5 \tag{5.7}$$

$$V_{B,L} = -59.1\text{kN} \tag{5.8}$$

5.3 Drawing the bending moment diagram

Once we've completed the shear force diagram, the bending moment diagram becomes much easier to determine. This is because we can make use of the following relationship between the shear force V and the slope of the bending moment diagram,

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V \tag{5.9}$$

Similarly to equation (5.6), this expression allows us to infer a qualitative shape for the bending moment diagram, based on the shear force diagram we've already calculated.

Consider the shear force between A and D for example, 14; it's constant, which means the slope of the bending moment diagram is also constant (an inclined straight line). Between D and E, the shear force is still constant but has changed sign. This tells us the slope of the bending moment diagram has also changed sign, i.e. the bending moment diagram has a local peak at D.

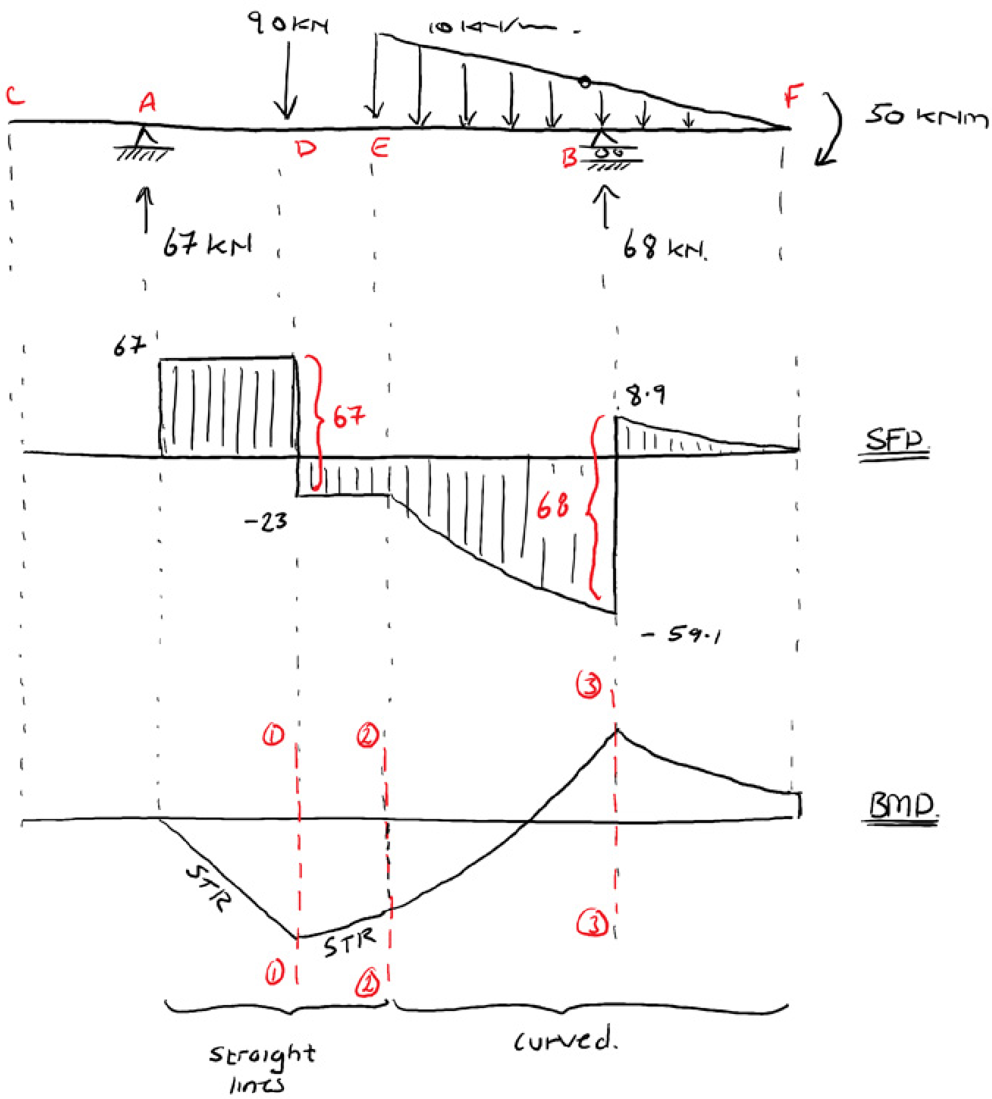

Figure 14: Shear force diagram and qualitative bending moment diagram.

The fact that the shear force is a polynomial (curve) between E and F also tells us the bending moment's slope is continuously changing, i.e. it's also a curve. But the fact that the shear force changes sign at B, means the bending moment diagram has a peak at that point.

Finally, the externally applied moment at F tells us that the bending moment diagram at this location has a value of 50kNm. We can combine all this information together to sketch out a qualitative bending moment diagram, based purely on the information encoded in the shear force diagram.

Now we simply have to cut the structure at discrete locations (indicated with red dashed lines above) to establish the various key values required to quantitatively define the bending moment diagram. In this case three cuts are sufficient:

- at D to determine the local peak - Cut 1-1
- at E to determine the value on the boundary between the straight and curved sections of the bending moment diagram - Cut 2-2
- at B to determine the local peak - Cut 3-3

Cut 1-1

As we've seen above, to determine the internal bending moment at D, M_D, we cut the structure to reveal the internal bending moment at this point, 15. Then by considering moment equilibrium of the sub-structure we can solve for the value of M_D.

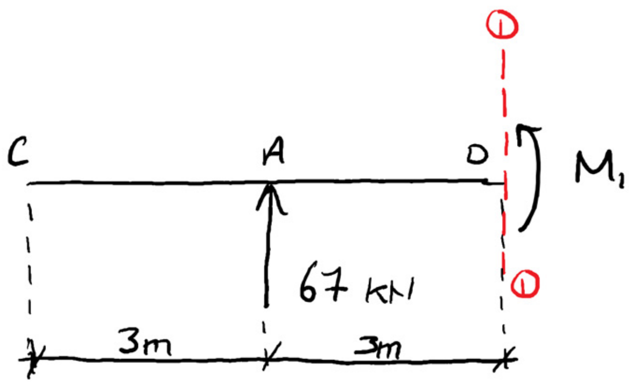

Figure 15: Free-body diagram for cut 1-1

Taking the sum of the moments about the cut,

$$\sum M_{\text{cut 1}} = 0 \text{ (assume clockwise positive)} \tag{5.10}$$

$$-M_1 + (67\text{kN} \times 3\text{m}) = 0 \tag{5.11}$$

$$M_1 = 201\text{kNm} \tag{5.12}$$

Cut 2-2

Repeating this process for cut 2-2, 16,

$$\sum M_{\text{cut 2}} = 0 \text{ (assume clockwise positive)} \tag{5.13}$$

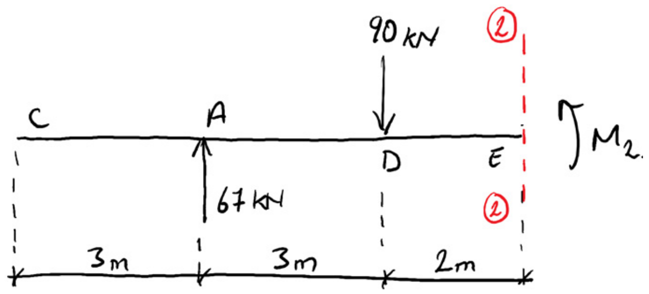

Figure 16: Free-body diagram for cut 2-2

$$-M_2 + (67\text{kN} \times 5\text{m}) - (90\text{kN/m} \times 2\text{m}) = 0 \tag{5.14}$$

$$M_2 = 155\text{kNm} \tag{5.15}$$

Cut 3-3

And finally for cut 3-3, 17 this time considering equilibrium of the sub-structure to the right-hand side of the cut

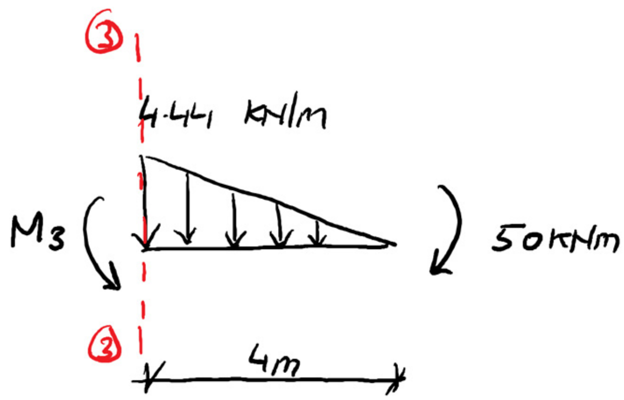

Figure 17: Free-body diagram for cut 3-3

$$\sum M_{\text{cut 3}} = 0 \text{ (assume clockwise positive)} \tag{5.16}$$

$$-M_3 + 50\text{kNm} + (4.44\text{kN/m} \times 4\text{m} \times 0.5 \times (4/3)\text{m}) = 0 \tag{5.17}$$

$$M_3 = 61.84\text{kNm} \tag{5.18}$$

We can now sketch the complete quantitative bending moment diagram for the structure. In fact at this point we can summarise the output of our complete structural analysis, 18.

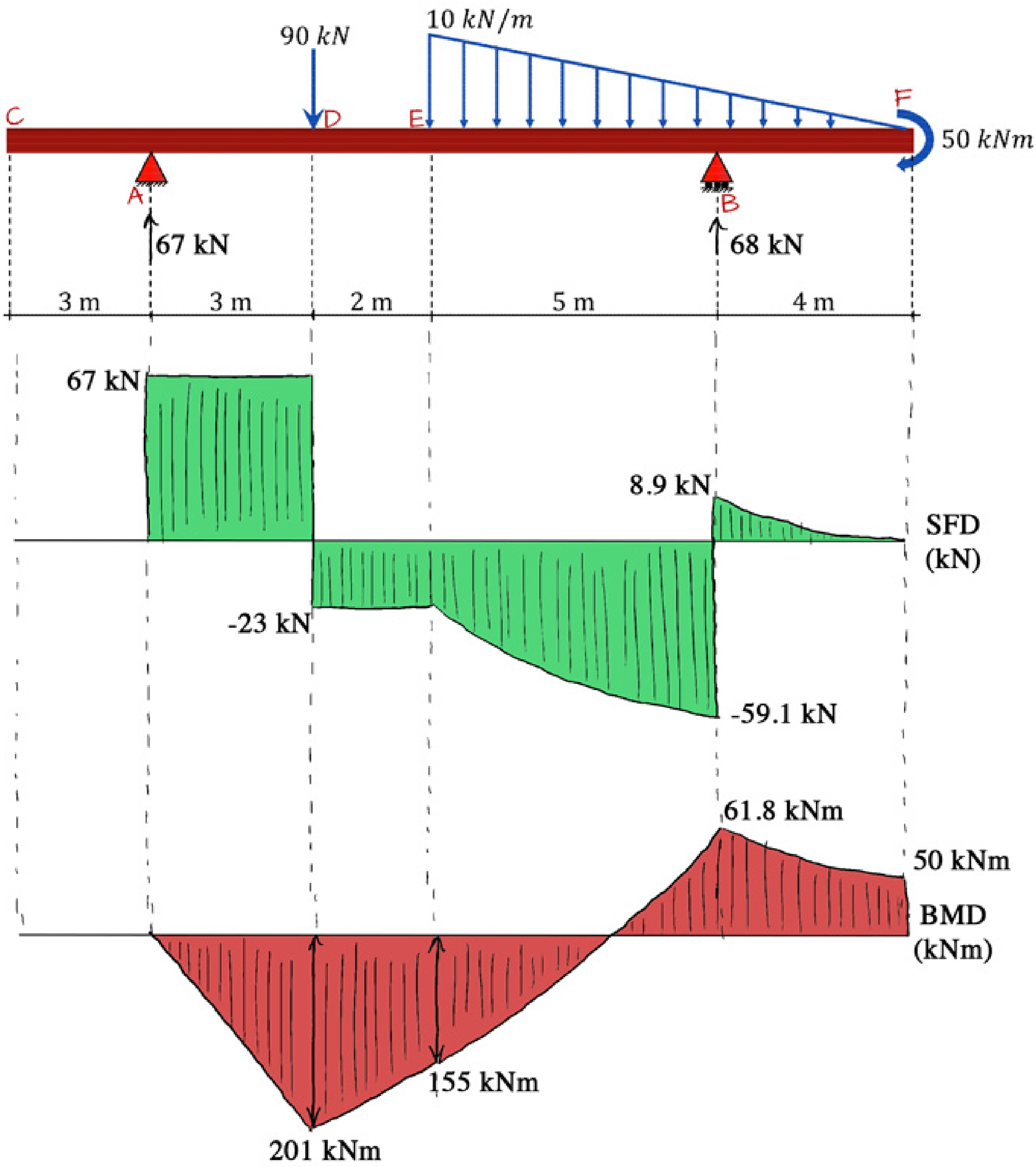

Figure 18

So there you have it. We've linked together the internal normal and shear stresses with the bending moment and shear force diagrams. There is quite a lot more we could say about shear and moment diagrams.

The best way for you to get better at evaluating shear force and bending moment diagrams is through practice. There really are no shortcuts I'm afraid. The good news is, the more you practice, the quicker you get and the stronger your intuition for structural behaviour becomes. That's all for now, I hope you got some value from reading this guide and I'll see you in the next one.