REVIEW

DISCRETE STATE,
DISCRETE TIME

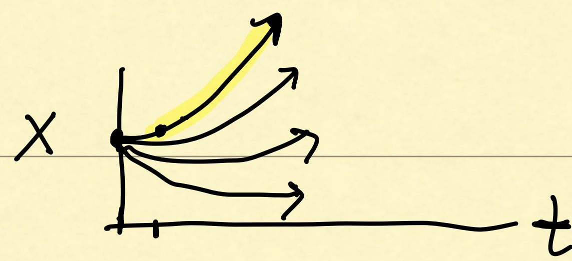
MARKOV CHAINS

DISCRETE STATE
CONTINUOUS TIME

- POISSON PROCESS
- CONTINUOUS-TIME MARKOV CHAIN

CONTINUOUS STATE,
CONTINUOUS TIME

$\rightarrow$



STOCHASTIC DYNAMICS

STOCHASTIC DIFFERENTIAL EQUATION

HETEROGENEITY,
PARAMETRIC NOISE,
PARAMETRIC STOCHASTICITY

$$dx = f(x)dt + dW_t$$

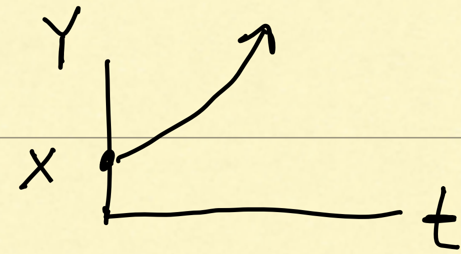
NOISE TERM

HETEROGENEITY

EX. $$\frac{dy}{dt} = Ay$$

$$y(0) = X$$

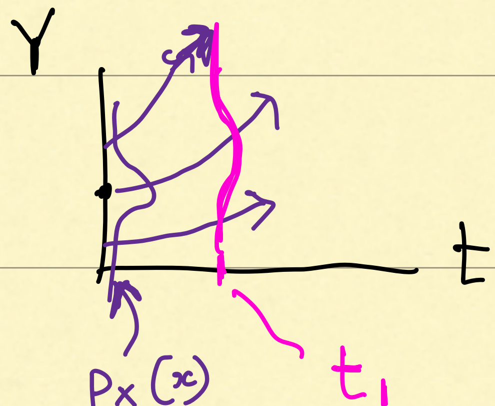
SOLUTION



$$y(t) = Xe^{At}$$

SUPPOSE $X \sim p_x(x)$

$\Rightarrow$ y IS A RANDOM VARIABLE



AT $t = t_1$, WHAT IS

$p_y(y)$?

$\nearrow$ $p_y(y; t)$

$$\int p_y(y)\, dy = 1$$

SUPPOSE $X \sim p_X(x)$

SUPPOSE $Y \sim g(X)$

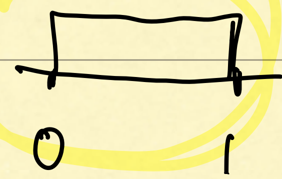
SOME FUNCTION.

WHAT IS $p_Y(y)$?

ASIDE

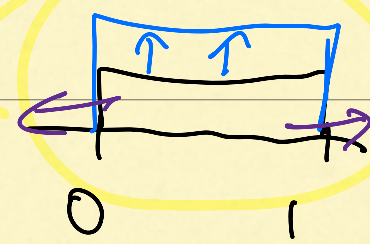
SUPPOSE $Y = cX$

$X \sim \text{UNIF}(0,1)$



IT IS <u>NOT</u> TRUE THAT

$$p_Y(y) = c\,p_X(x)$$



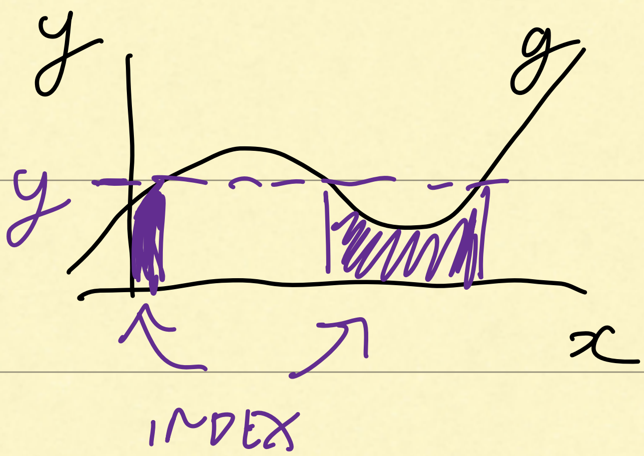
FOR A GENERAL FUNCTION g

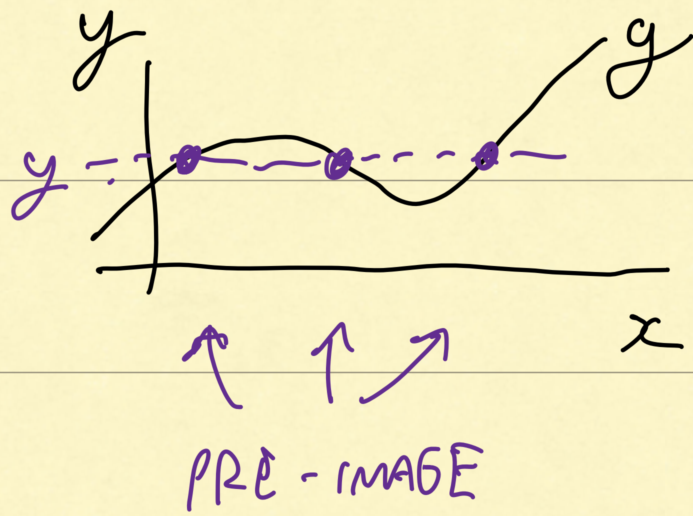
DEFINE $$I_y = \{x : g(x) \le y\}$$

INDEX SET OF y

$$\{x_k\} = \{x : g(x) = y\}$$

PRE-IMAGE OF y





CUMULATIVE OF Y

$$F_Y(y) = \mathbb{P}(Y \le y)$$

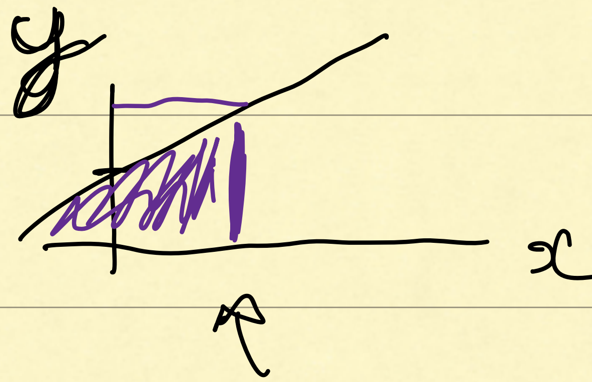
$$= \mathbb{P}(x \in I_y)$$

$$= \int_{x \in I_y} p_X(x)\, dx$$

EX $\quad y = ax + b \qquad Y = aX + b$

$a > 0$

$g(x) = ax + b$



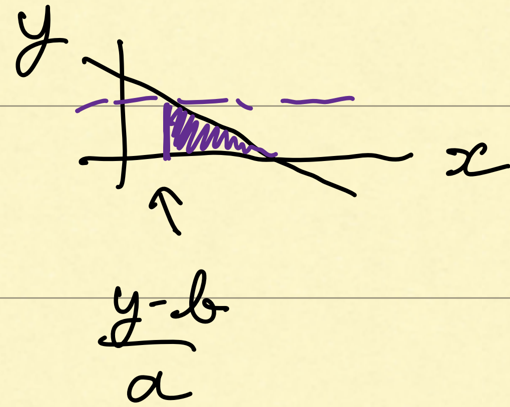
$$I_y = \left\{ x \le \frac{y-b}{a} \right\} \qquad x = \frac{y-b}{a}$$

GIVEN $F_X(x)$, THEN

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

$$\boxed{p_Y(y) = p_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}}$$

EX SAME BUT $a<0$



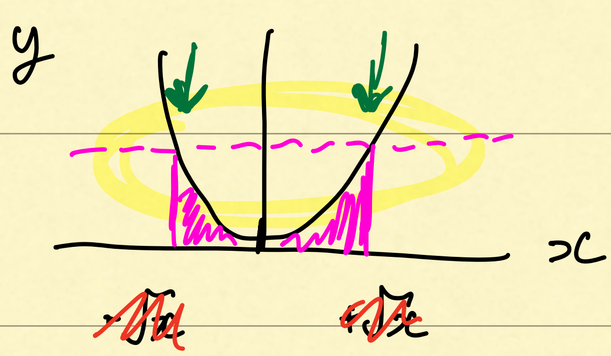
$$F_Y(y) = \mathbb{P}\left(X \geq \frac{y-b}{a}\right)$$
$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

$$p_Y(y) = -1 \cdot p_X\left(\frac{y-b}{a}\right)\frac{1}{a}$$

FOR GENERAL a,

$$p_Y(y) = \left|\frac{1}{a}\right| p_X\left(\frac{y-b}{a}\right)$$

EX $Y = X^2$ $\qquad g(x) = x^2$



$I_y = \{ [-\sqrt{y}, +\sqrt{y}]$ IF $y \geq 0$

EMPTY IF

$-\sqrt{y}$ $+\sqrt{y}$

$y < 0$

$$F_Y(y) = \begin{cases} F_X(+\sqrt{y}) - F_X(-\sqrt{y}) & \text{IF } y \geq 0 \\ 0 & \text{IF } y < 0 \end{cases}$$

$$p_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} p_X(+\sqrt{y}) + \frac{1}{2\sqrt{y}} p_X(+\sqrt{y}) \\ 0 \end{cases}$$

GENERAL FORMULA

LET $X \sim p_X(x)$ AND $Y = g(X)$

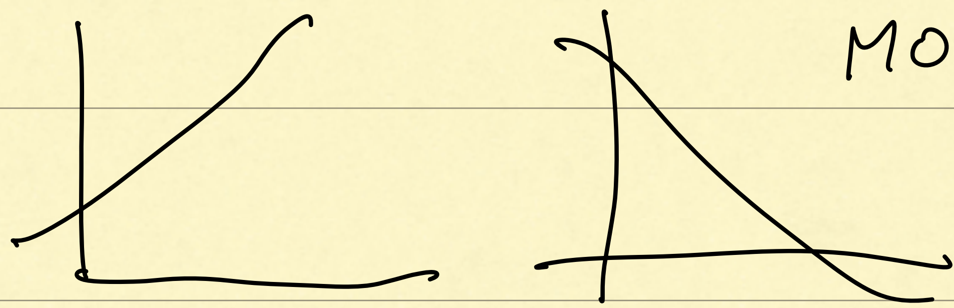
THEN

$$p_Y(y) = \sum_k p_X(x_k(y)) \left\| \left(\frac{dx_k}{dy} \right) \right\|$$

WHERE $\{x_k(y)\}$ IS THE PRE-IMAGE OF y.

IF $\frac{dg}{dx} \neq 0$, THEN

$$p_Y(y) = \sum_k p_X(g_k^{-1}(y)) \left(\left| \frac{dg}{dx} \right| \right)^{-1}$$

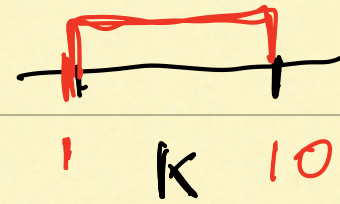


MONOTONIC

IF g IS MONOTONIC THEN

$$p_Y(y) = p_X(g^{-1}(y)) \left(\left| \frac{dg}{dx} \right| \right)^{-1}$$

EX $\quad K \sim \text{UNIF}\,\cancel{(0,1)}\;(1, 10)$



K - RATE CONSTANT
(UNITS OF $\frac{1}{\text{SECOND}}$)

LET $\quad T = \dfrac{1}{K}$ - MEAN TIME
(UNITS OF SECONDS)

WHAT IS $p_T(t)$?