Simulation of a single and double pendulum

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Abstract:

The suitability of four different numerical methods was tested (Implicit Euler, Explicit Euler, Runge Kutta 4, and Leapfrog). The methods were investigated by solving differential equations related to the displacement and velocity of a single pendulum system. Based on the results of the single pendulum analysis, the most suitable differential equation was chosen to solve a more complex double pendulum system.

Introduction:

The aim of the first part of the experiment is to find a numerical method suitable for a general oscillatory problem with damping. The aim of the second part is to use the suitable numerical method to investigate the dynamics of a double pendulum. The Implicit Euler, Explicit Euler, Runge Kutta 4 and Leapfrog numerical methods can be used to solve differential equations in the form of:

$$\frac{\boldsymbol{dy}}{\boldsymbol{dn}} + \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{n}) = \boldsymbol{0}$$

The numerical methods have finite precision, therefore each calculation introduces errors due to rounding or approximation. Methods that produce unphysical result or methods that are too computationally expensive are considered unsuitable. The step size at which a method ceases to be accurate or stable is important due to the constraints on computational resources. The ideal method will have a relatively high tolerance for altering the step size and will not break any fundamental physical conservation laws. The methods will be considered for damped (D>0) and un-damped (D=0) cases.

Methods:

The equation of motion governing the single pendulum is [1]:

$$ml\frac{d^2\theta}{dt^2} = -mgsin\theta - D\frac{d\theta}{dt}$$

The equation can be simplified by rescaling and making the small angle approximation $sin\theta \sim \theta$:

$$\frac{d^2\theta}{d\boldsymbol{t}^2} + \beta\frac{d\theta}{d\boldsymbol{t}} - \theta = 0$$

Where $\beta = \frac{D}{m\sqrt{g*l}}$ and $\boldsymbol{t} = t\sqrt{\frac{g}{l}}$

A further substitution of $v = \frac{d\theta}{d\boldsymbol{t}}$ yields a suitable form to solve using our numerical methods:

$$\frac{dv}{d\boldsymbol{t}} - \beta v + \theta = 0$$

Stability analysis

The conservation of energy of the pendulum system is measured to determine the critical time steps for which the method becomes unstable. The total energy of the single pendulum system is given by:

$$T.E. = K.E + P.E. = \frac{1}{2}mgl^2(\frac{d\theta}{dt})^2 + mgl(1 - cos(\theta))\ [2]$$

Making the small angle approximation for $cos(x) \sim 1 - \frac{x^2}{2}$ leads to the following expression:

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$$T.E. = \frac{1}{2}ml^2(\frac{d\theta}{dt})^2 + \frac{1}{2}mgl\,\theta^2$$

The stability test (stab_test) takes the initial and final results for angle and angular velocity a specified method. It then computes the total initial and final energies and compares them with a leniency of 5%. The leniency is necessary because the numerical methods are not expected to fully obey the conservation of energy after each iteration. However, large deviations from this leniency would render a method unsuitable for examining realistic physical phenomena. Therefore, if the leniency is unsatisfied the method is considered unstable. The test for the critical time step involves iterating the stability test over a range of step sizes until the method becomes unstable.

Table 1 Stability analysis results, critical time displayed for each method, step size starts iterating at 0.01 and increments by 0.01

Damping	Explicit Euler	Leapfrog	RK4	Implicit Euler
D=0.2	0.201	0.068	2.951	Unconditionally stable
D=0	0.007	0.276	2.829	Unconditionally stable

The stability analysis results suggest that the Leapfrog method is relatively stable for the un-damped pendulum, but unstable for the damped pendulum. The miniscule value of 0.068 is displayed because of the 5% leniency of the stability test. The Explicit Euler method is relatively stable for the damped Pendulum and unstable for the un-damped pendulum (again the miniscule value of 0.007 is due to the built in leniency). Therefore, the results suggest that Explicit Euler method could only be used for damped systems and the Leapfrog method could only be used on frictionless oscillators. The leapfrog method is therefore unsuitable for a general oscillatory problem with damping. Figures 3 and 4 further demonstrate the instability of the leapfrog method for a damped system. RK4 has the largest tolerance for step sizes among the explicit methods. Therefore, larger step sizes can be used which contributes the efficiency of the program. RK4 also exhibits realistic behaviour in terms of energy conservation for both zero damping and non-zero damping scenarios. Therefore, the RK4 method is the recommended method for a general oscillatory problem with damping.

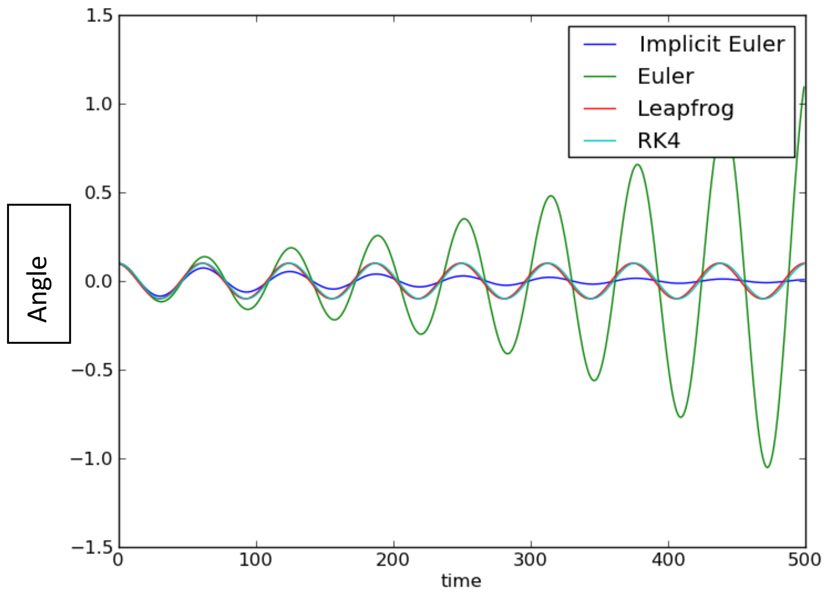

Figure 1 Angle against time, h=0.1, D=0, steps=500

Fig. 1 demonstrates the instability of the Euler method for a non-damped case. The Leapfrog method and RK4 method basically overlap, whereas the implicit Euler method gives an underestimate. Instead of observing a constant amplitude, the implicit Euler method incorrectly plots a decreasing amplitude. This is a non-physical solution.

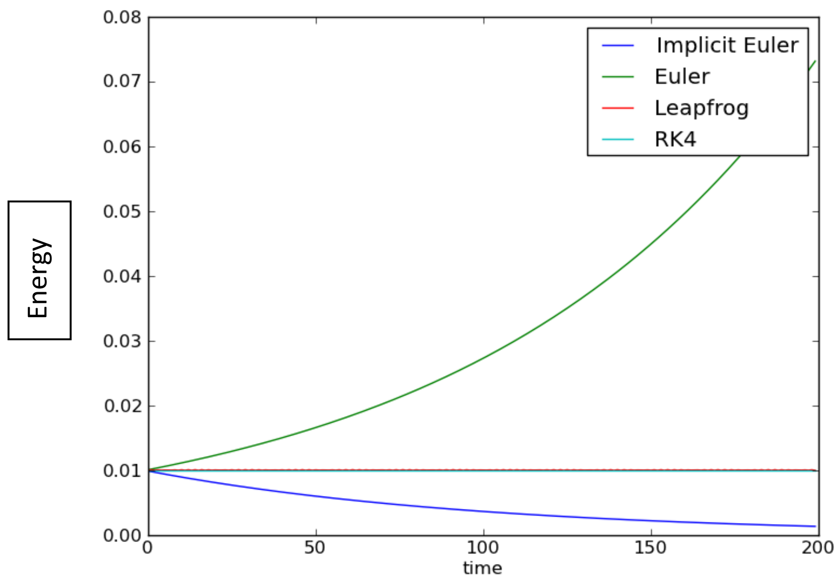

Figure 2 Energy of methods against time, h=0.1, D=0, steps=200

Fig. 2 further demonstrates the instability of the Euler method as it breaks the conservation of energy. The implicit Euler method also breaks the conservation of energy, because energy is disappearing from a closed system. Both the Leapfrog and RK4 method give an expected constant value for the energy.

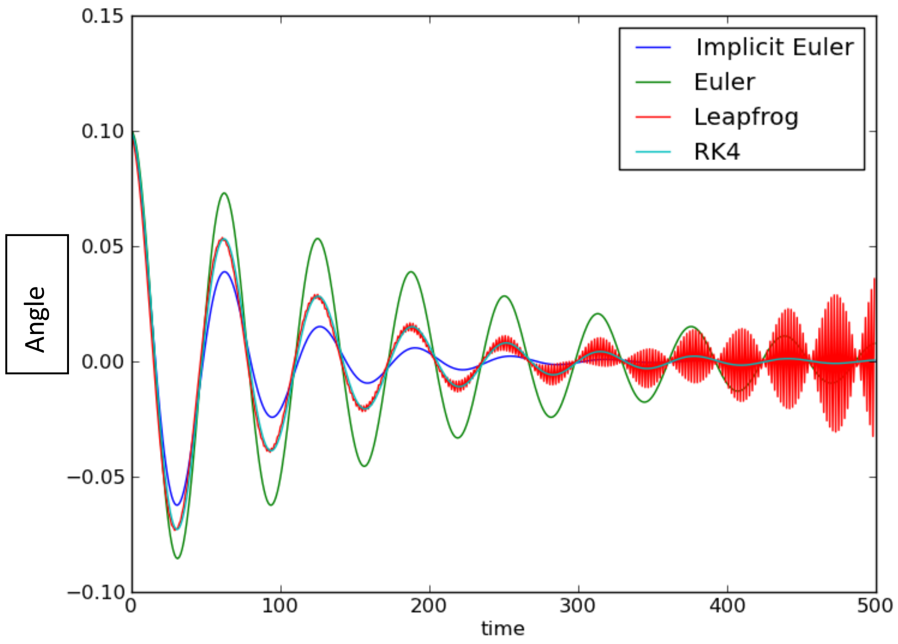

Figure 3 Angle against time, h=0.1, D=0.2, steps=500

For the damped scenarios all the methods except the Leapfrog method converge. Initially, the leapfrog method and RK4 methods overlap, however between time=200 and 300 the Leapfrog method starts exhibiting an increasing beating behaviour. Explicit Euler gives an overestimate for the angle and the Implicit Euler gives an underestimate for the angle.

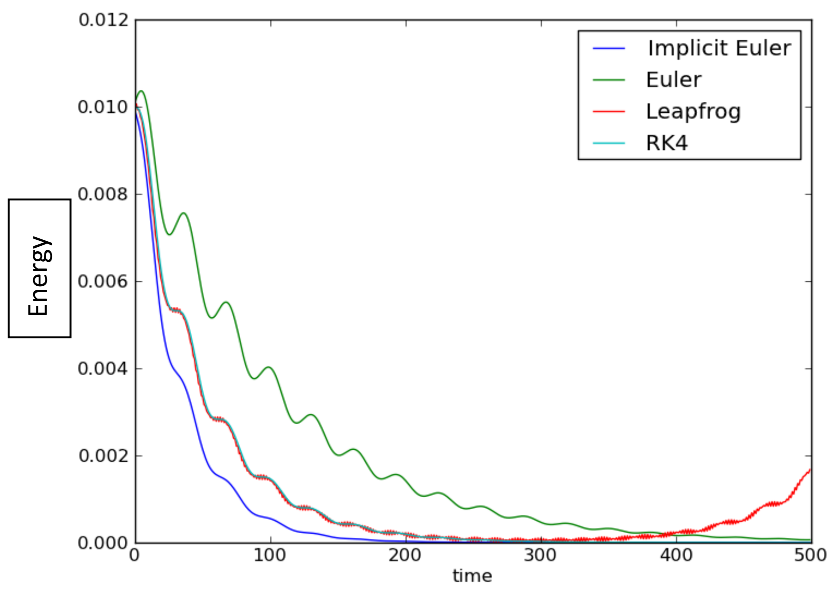

Figure 4 Energy of methods against time, h=0.1, D=0.2, steps=500

Fig. 4 further demonstrates the instability of the Leapfrog method for large values of time. All the other methods tend to 0 as expected of a damped pendulum. The Explicit Euler method overestimates the energy and the Implicit Euler underestimates the energy of the system.

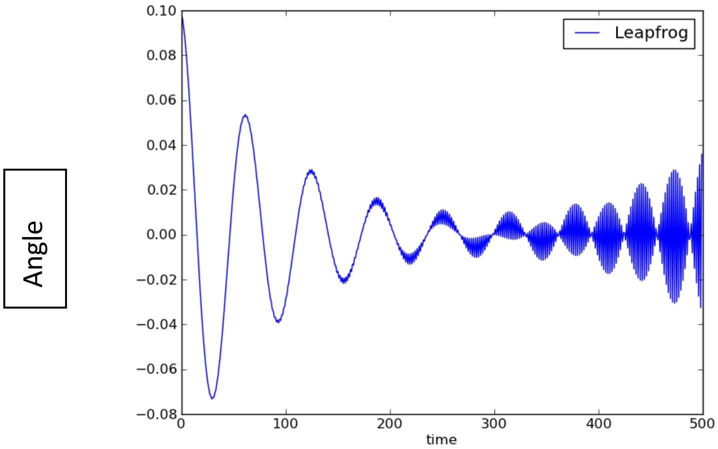

Figure 5 demonstrates the emergent beating pattern h=0.1, D=0.2, steps=500

Upon closer inspection the leapfrog method has a general beating pattern (fig 5) that overtakes the correct pattern as time tends to infinity. This beating pattern is unexpected and unphysical. Perhaps this pattern is visible because each value in the Leapfrog method is derived from the gradient of the previous value the errors are exacerbated as the algorithm is repeated.

Part 2 - Double Pendulum

The system of two pendulums can also be described by a system of first order differential equations [1]. The RK4 method was used, since it produced the most stable solutions for the single pendulum. The stability analysis for the two pendulum system is similar to the one carried out for the single pendulum system. The stability is verified by checking the solutions obey energy conservation with a leniency of 5%. We check the solution is conservative by computing the Langrangian for the system.

The total energy variation of a double pendulum is given by [2]:

$$L = (K1 + K2) - (P1 + P2)$$

$$Energy\ variation = Mlg(\frac{1}{2}l^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}Rl^2\left(\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\varphi}{dt}\right)^2 + 2\frac{d\theta}{dt}\frac{d\varphi}{dt}\cos(\theta-\varphi)\right) - \cos(\theta) - R(\cos(\theta) + \cos(\varphi))$$

G is proportional to the damping of the system. R is the ratio between the mass of the second and first pendulums.

$$R = \frac{M}{m} \quad , \quad G = \frac{D}{m\sqrt{gl}}$$

Table 2, critical time displayed for RK4 for various conditions, step size starts iterating at 0.01 and increments by 0.01

	R=0.01	R=1	R=100
G=0	2.699	1.539	0.029
G=1	0.03	1.563	0.022

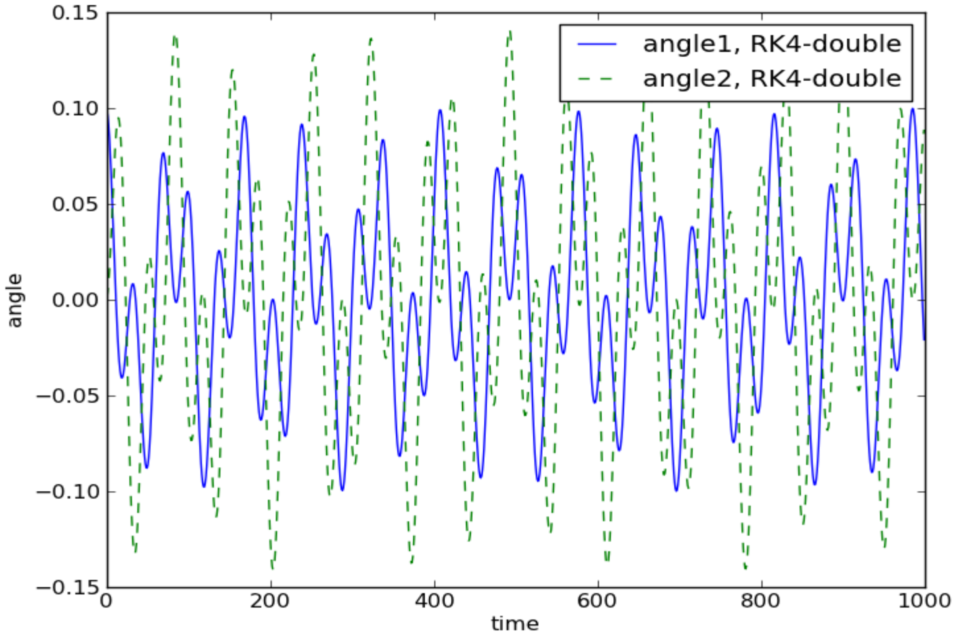

Figure 6 Double pendulum, h=0.1, G=0, R=1, steps=1000

Figure 6 shows the un-damped periodic motion for bobs of equal mass using the RK4 method. Initially the angle of the second bob is zero. As angular momentum is transferred from the first bob, the second bob increases its oscillations. This is the second most stable solution (t=1.539).

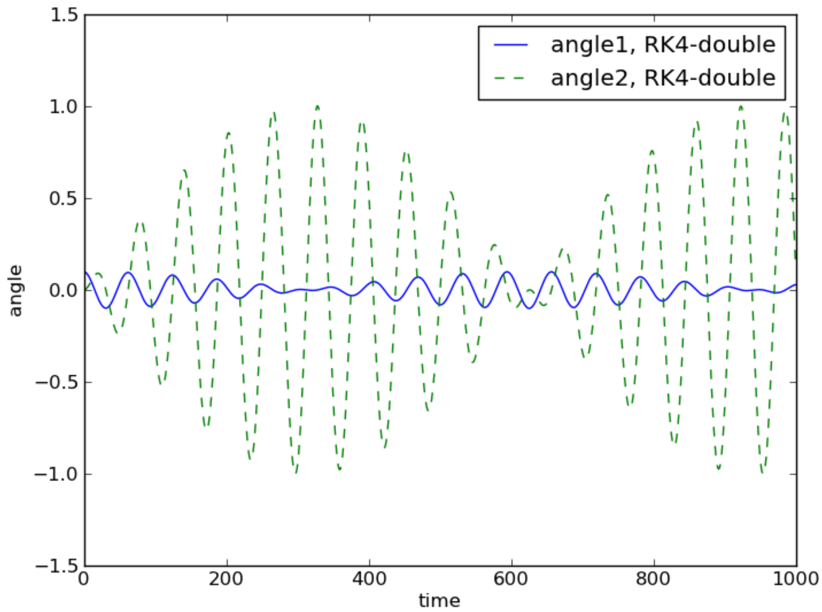

Figure 7 Double pendulum, h=0.1, G=0, R=0.01, steps=1000

Figure 7 illustrates the how each pendulum oscillates with a different frequency. A small change in oscillation of the first bob makes a large difference (about one order of magnitude) to the oscillation of the second bob. This is because at R=0.01 the first bob is 100 times more massive than the second bob.

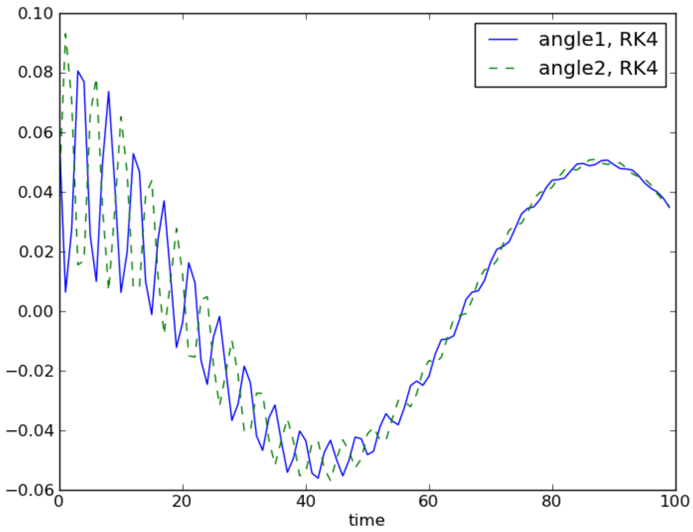

Figure 8 Double pendulum, h=0.1, G=0, R=100, steps=100

For R=100 table 2 states that the critical time step is 0.029. Since our time step in figure 9 exceeds the critical time step, we can see the system is unstable. The first bob oscillates with a large frequency due to the big difference in mass. The time step is too large to capture the rapid movement of the first bob.

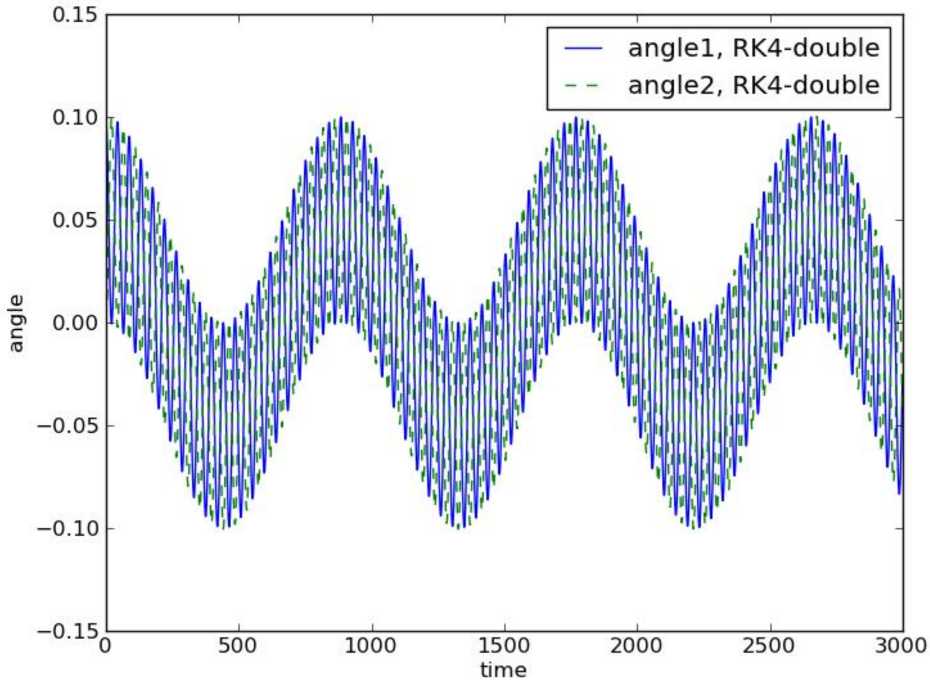

Figure 9 Double pendulum, h=0.01, G=0, R=100, steps=3000

Figure 10 shows an accurate result for the same initial conditions as figure 9, because the time step is below the critical time step (0.01<0.029). For a larger R we need to decrease the time step to get an accurate solution.

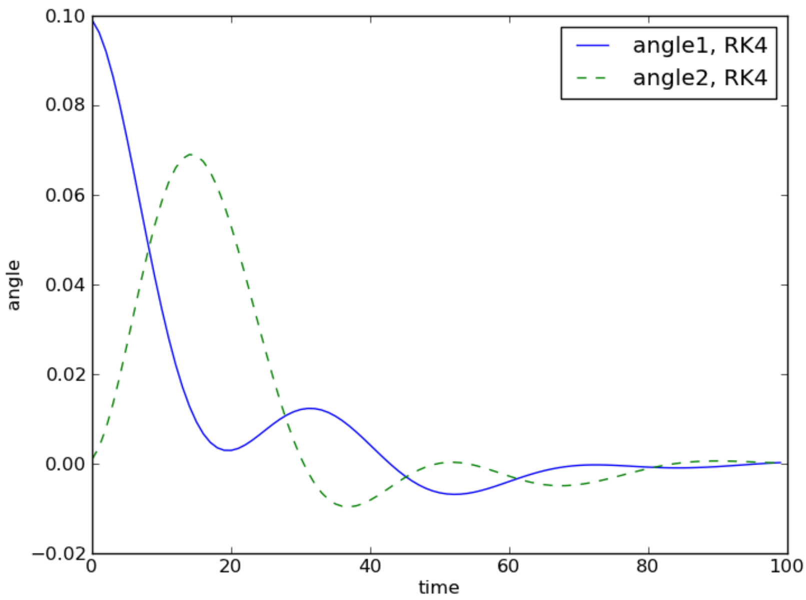

Figure 10 G=1 R=1 h=0.1 steps =1000

Figure 11 shows a damped system of two pendulums of equal mass. From table 2, the conditions in figure 11 are shown to be the most stable solution (t=1.563).

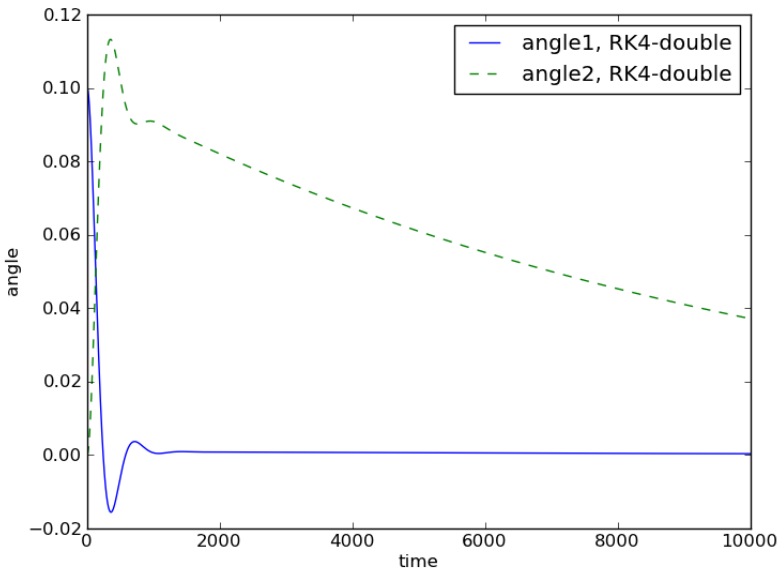

Figure 11 G=1 R=0.01 h=0.01 steps=10000

Figure 12 shows a damped pendulum where the first bob is 100 times heavier than the second one. This system dissipates energy quickly as there are barely any oscillations. It is a stable system.

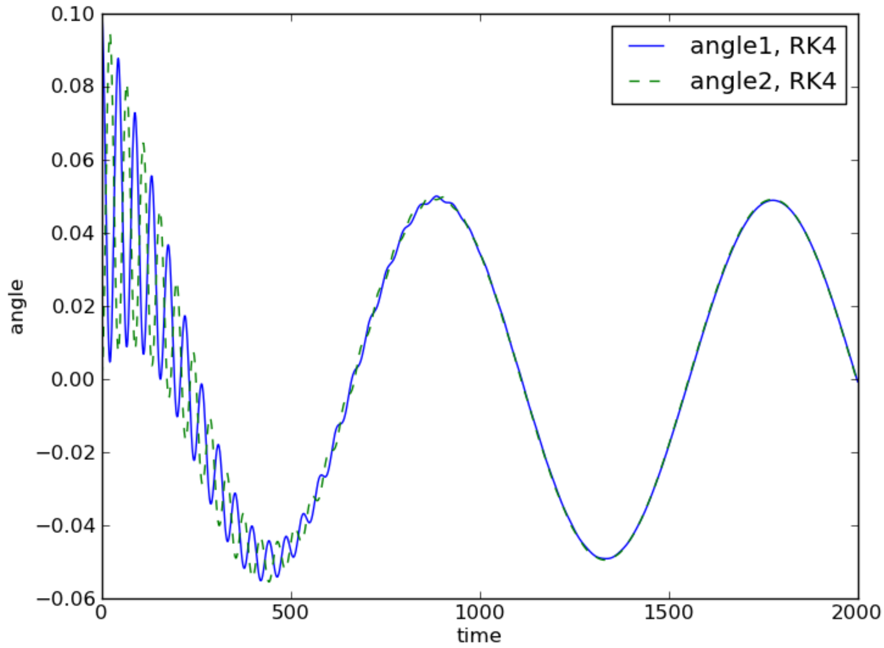

Figure 12 G=1 R=100 h=0.01 steps=2000

Figure 13 shows a damped pendulum system. The oscillations are initially out of phase but converge as time goes to infinity.

Conclusion:

The single and double pendulum systems were used to test the effectiveness of four different numerical methods. The Leapfrog method was shown to be unsuitable for damped systems and the Euler method was shown to be unsuitable for un-damped systems. The Runge-Kutta 4 method was demonstrated to be the most suitable method for a general oscillator with good time-step tolerance and stability properties for both damped and un-damped systems. The RK4 method was further established to be an efficient method when successively applied to the double-pendulum system in part 2 of the project. In part 2 of the project it was shown that the time step has to be increased when R increases to capture the higher frequencies of the oscillators.

Word Count: 1596

Bibliography:

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