6

Limits to Infinity

6.1 Introduction

When we studied limits of functions in Chapter 3, we often considered expressions of the form "$\lim_{x \to c} f(x) = L$," where the symbols c and L both denoted real numbers. However, it is also possible to consider limits involving not only real numbers, but limits to "infinity" and "negative infinity," which are written $\lim_{x \to \infty} f(x) = L$, and $\lim_{x \to -\infty} f(x) = L$, and $\lim_{x \to c} f(x) = \infty$, and $\lim_{x \to c} f(x) = -\infty$. It is also possible to combine these two types of limits, for example $\lim_{x \to \infty} f(x) = \infty$. In all of the types of limits that involve ∞ and $-\infty$, it is important to recognize that the symbols "∞" and "$-\infty$" are not real numbers, but are rather a shorthand way of indicating that something is growing without bound either in the positive direction or in the negative direction.

We will define the different types of limits to infinity in Section 6.2. In Sections 6.3 and 6.4 we discuss two very useful topics involving limits to infinity, both of which appear in calculus courses, namely, l'Hôpital's Rule and improper integrals. Limits to infinity have many applications; for example, we will use such limits, and improper integrals, in our discussion of trigonometric functions and π in Sections 7.3 and 7.4. Other useful applications include Laplace transforms, which in turn are used for solving differential equations, and continuous probability; such topics are beyond the scope of this book. See [BD09, Chapter 6] for Laplace transforms as used for differential equations, and see [Ros10, Chapter 5] for continuous probability.

As was the case in previous chapters, here too we assume that the reader is informally familiar with the standard elementary functions and their basic properties, such as continuity and differentiability, in order to have sufficiently many functions to see interesting examples of the material in this chapter. We will see a rigorous treatment of the elementary functions in Chapter 7; the proofs in that chapter, though making use of some of the general ideas in the present chapter, will not make use of the examples in the present chapter, and there is no circular reasoning.

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6.2 Limits to Infinity

We define two types of limits to infinity. The first type, which we will refer to as Type 1 limits to infinity, and which is denoted $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, has x go to infinity or negative infinity, but has the value of $f(x)$ go to a real number L. The second type, which we will refer to as Type 2 limits to infinity, and which is denoted $\lim_{x\to c} f(x) = \infty$ or $\lim_{x\to c} f(x) = -\infty$, has x go to a real number c, but has the value of $f(x)$ go to infinity or negative infinity. Type 1 limits to infinity correspond to horizontal asymptotes of graphs of functions, as seen in Figure 6.2.1 (i), and Type 2 limits correspond to vertical asymptotes, as seen in the Part (ii) of the figure.

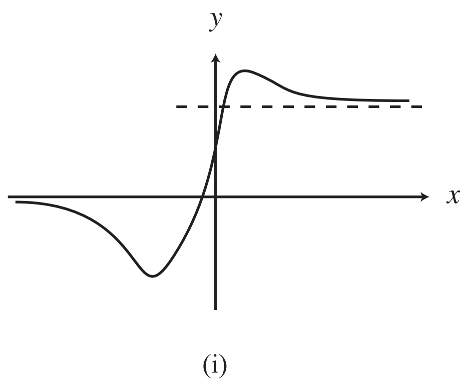

Fig. 6.2.1.

We start our discussion with Type 1 limits to infinity. In the ordinary type of limit, denoted $\lim_{x\to c} f(x) = L$, we mean intuitively that $f(x)$ gets closer and closer to a number L as the value of x gets closer and closer to a number c. By contrast, in a limit of the form $\lim_{x\to\infty} f(x) = L$, the idea is that $f(x)$ gets closer and closer to a number L as the value of x gets larger and larger, which is symbolically denoted by "$x \to \infty$," though there is no real number "∞" that the number x is getting closer and closer to. Hence, in our definition of this type of limit, we replace the expression "$|x - c| < \delta$," which is thought of as x being near c, with the expression "$x > M$," which is thought of as x being large.

For the following definition, recall the definition of right unbounded interval and left unbounded interval given in Definition 2.3.6.

Definition 6.2.1.

1. Let $I \subseteq \mathbb{R}$ be a right unbounded interval, let $f\colon I \to \mathbb{R}$ be a function and let $L \in \mathbb{R}$. The number L is the **limit** of f as x goes to infinity, written

$$\lim_{x\to\infty} f(x) = L,$$