

# Metodos matemáticos 2

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## 1. Función Gamma

L'imate al infinito(Euler)

$$\begin{aligned}\Gamma(z) &\equiv \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2) \cdots (z+n)} n^z, \quad z \in \mathbb{Z}^+ \text{ o } z \in \mathbb{C} \\ \Gamma(z+1) &= \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{(z+1)(z+2) \cdots (z+n+1)} n^{z+1} \\ &= z \cdot \Gamma(z) = \lim_{n \rightarrow \infty} \frac{nz}{z+n+1} \Gamma(z) = \Gamma(z+1) = z \cdot \Gamma(z)\end{aligned}$$

Aplicando lo anterior a  $z=1,2,3,\dots,n$

$$\begin{aligned}\Gamma(1) &= \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{(1)(2) \cdots (n+1)} n^1 = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \\ \Gamma(2) &= z \cdot \Gamma(z) = \Gamma(z+1) = \Gamma(1+1) = 1 \\ \Gamma(3) &= 2 \cdot \Gamma(1) = \Gamma(2+1) = 2 \cdot 1 \\ \Gamma(4) &= 3 \cdot \Gamma(3) = \Gamma(3+1) = 3 \cdot 2 \cdot 1 \\ &\vdots \\ &\vdots \\ &\vdots \\ \Gamma(n) &= (n-1)!\end{aligned}$$

Integral Definida(Integral de Euler)

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \operatorname{Re}(z) > 0$$

Ej. como aparecen en f'isica

$$\Gamma(z) = 2 \int_0^\infty e^{-t^2} t^{z+1} dt \quad \text{o} \quad \Gamma(z) = \int_0^1 \left[ \ln\left(\frac{1}{t}\right) \right]^{z-1} dt$$

Si  $z = \frac{1}{2} \Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$  es integral error de Gauss

$$F(z, n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt, \quad \operatorname{Re}(z) > 0 \text{ con } n \text{ entero positivo} \ni$$

$$e^{-t} = \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n}\right)^n$$

$$\Rightarrow F(z, n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \int_0^1 (1-u)^n (un)^{z-1} n du \ni u = \frac{t}{n}$$

$$\frac{F(z, n)}{n^z} = \int_0^1 (1-u)^n (u)^{z-1} du$$

Ahora por integraci' on por partes usando  $u = (1 - u)^n$ ,  $du = n(1 - u)^{n-1}du$ ,  $v = \frac{u^z}{z}$ ,  $dv = u^{z-1}du$  tenemos

$$\begin{aligned}\frac{F(z, n)}{n^z} &= (1 - u)^n \frac{u^z}{z} \Big|_0^1 - \int_0^1 \frac{u^z}{z} n(1 - u)^{n-1} du \\ F(z, n) &= n^z \cdot \frac{n(n-1) \cdots 1}{z(z+1) \cdots (z+n-1)} = \int_0^1 u^{z+n-1} du \\ &= \frac{1 \cdot 2 \cdots n}{z(z+1) \cdots (z+n)} n^z \Rightarrow \lim_{n \rightarrow \infty} F(z, n) = \Gamma(z)\end{aligned}$$

**Producto infinito(Weierstrass)**

$$\frac{1}{\Gamma(z)} = ze^{\delta z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

$\delta :=$  constante de Euler-Mascheron

$$\delta := 0.5772156619$$

**Ecuaci' on estad' istica de Maxwell-Boltzmann**

$$\begin{aligned}e^{-E/KT} \quad & K \text{ es la constante de Boltzmann} \\ & T \text{ es la temperatura absoluta} \\ & E(\text{energ' ia}): \text{estado de energ' ia ocupada}\end{aligned}$$

probabilidad de estar en estado de energ' ia es  $Y_{kt} = \beta$

$$\begin{aligned}P(E) &= Ce^{-\beta E} && \text{Para un gas idel sin estructura} \\ n(E)dE & && n(E)^{1/2} \\ 1 &= C \int n(E)e^{-\beta E} dE && E=\text{energ' ia cin' etica}\end{aligned}$$

$$\begin{aligned}1 &= c \int_0^{\infty} E^{1/2} e^{-\beta E} dE = \frac{C\Gamma(3/2)}{\beta^{3/2}}, \quad \beta E = T \Rightarrow dE = \frac{dT}{\beta} \\ 1 &= C \int_0^{\infty} e^{-t} \left(\frac{t}{\beta}\right)^{1/2} \frac{1}{\beta} dt \\ &= C \int_0^{\infty} e^{-t} t^{3/2-1} dt \cdot \frac{1}{\beta^{3/2}} \\ &= \frac{C\Gamma(3/2)}{\beta^{3/2}} \\ &= c \cdot \frac{\sqrt{\pi}}{2 \cdot \beta^{3/2}}\end{aligned}$$

$$\therefore C = \frac{2 \cdot \beta^{3/2}}{\sqrt{\pi}}$$

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Relaciones de funciones

$$\Gamma(z+1) = z\Gamma(z)$$

Fomula de reflexi'on

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\operatorname{sen}(z\pi)}$$

F'ormuláde duplicaci'on de Legendre

$$\Gamma(1+z)\Gamma\left(z+\frac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\Gamma(2z+1)$$

Notac'i'on doble factorial:

$$\begin{aligned}(2n+1)!! &= 1 \cdot 3 \cdot 5 \cdots (2n+1) = \frac{(2n+1)!}{2^n n!} \\ (2n)!! &= 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n! \\ (-1)!! &= 1\end{aligned}$$

**PROPIEDADES ANAL'ITICAS**

$[\Gamma(z)]^{-1}$  tiene singularidades en  $z = 0, -1, -2, \dots$  y no tiene cuando  $z = -1, 2, 3, \dots$  y no tiene ceros en el plano complejo finito positivo.

El residuo  $R_n$  cuando  $z = -n$  donde  $n$  es un entero mayor o igual a cero.

$$\begin{aligned}R_n &= \lim_{n \rightarrow 0} (\epsilon \Gamma(-n + \epsilon)) \\ &= \lim_{n \rightarrow 0} \frac{\epsilon \Gamma(-n+1+\epsilon)}{-n+\epsilon} \\ ) &= \lim_{n \rightarrow 0} \frac{\epsilon \Gamma(-n+2+\epsilon)}{(+\epsilon)(+1+\epsilon)} \\ &= \lim_{n \rightarrow 0} \frac{\epsilon \Gamma(1+\epsilon)}{(-n+\epsilon) \cdots (\epsilon)} \\ &= \frac{(-1)^n}{n!}, \text{ el residuo alterna signos en } z = -n\end{aligned}$$

Integral de Schaeffli

$$\int_C e^{-t} t^v dt = (e^2) \Gamma(v+1)$$

donde C es el contorno INSERTAR IMAGEN Esta integral es 'util cuando V no es entero.

Notaci'on factorial:

$$\prod(z) = z! = \Gamma(z+1)$$

Funci'on Digamma

$$\begin{aligned}\ln(\Gamma(z+1)) &= z \cdot \Gamma(z) \\ &= \lim_{n \rightarrow \infty} \frac{z}{n} \ln \left[ \frac{n! \cdot n^z}{(z+1)(z+2) \cdots (z+n)} \right] \\ \frac{d}{dz} \ln(\Gamma(z+1)) &= \frac{d}{dz} \lim_{n \rightarrow \infty} [\ln(n!) + z \cdot \ln(n) - \ln(z+1) - \ln(z+2) - \dots - \ln(z+n)] \\ \frac{d}{dz} \ln(\Gamma(z+1)) &\equiv \underbrace{\varphi(z+1)}_{\text{Funci'on digamma}} = \lim_{n \rightarrow \infty} \left( \ln(n) - \frac{1}{z+1} - \frac{1}{z+2} - \dots - \frac{1}{z+n} \right)\end{aligned}$$

$$\varphi(z+1) = \frac{[\Gamma(z+1)]'}{\Gamma(z+1)}$$

Si sumamos y restamos un n'umero arm'onico

$$\begin{aligned}H_n &= \sum_{m=1}^n \frac{1}{m} \\ \varphi(z+1) &= \lim_{n \rightarrow \infty} \underbrace{[(\ln(n) - H_n)]}_{=-\delta \text{ cuando } n \rightarrow \infty} - \sum \left( \frac{1}{z+m} - \frac{1}{m} \right) \\ \Rightarrow \varphi(z+1) &= -\delta + \sum_{m=1}^{\infty} \frac{z}{m(m+z)}, \quad \text{si } z=0 \Rightarrow \varphi(1) = -\delta\end{aligned}$$

para  $n > 0$ ,  $n$  entero.

$$\begin{aligned}\varphi(n+1) &= \delta + \sum_{m=1}^n \frac{1}{m} \\ &= -\delta + H_n\end{aligned}$$

### Funci'ón Poligamma

Cuando se deriva muchas veces la funci'ón digamma

$$\begin{aligned}\varphi^{(m)}(z+1) &\equiv \frac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z+1) \\ \varphi^{(m)}(z+1) &= (-1)^{m+1} m! \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}}, \quad m = 1, 2, 3, \dots\end{aligned}$$

Si  $z=0$  la funci'ón se conoce como zeta de Riemann

$$\zeta(m) = \sum_{n=1}^{\infty} \frac{1}{n^m}$$

$$\Rightarrow \varphi_{(1)}^{(m)} = (-1)^{m+1} \cdot m! \cdot \zeta(m+1), \quad m = 1, 2, 3, \dots$$

### Expansi'ón de Maclaurin

$$\begin{aligned}\ln \Gamma(z+1) &= \sum_{n=1}^{\infty} \frac{z^n}{n!} \varphi_{(1)}^{(n-1)} \\ &= -\delta \cdot z + \sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n} \zeta(n)\end{aligned}$$

converge en  $|z| < 1$  para  $z = x \Rightarrow$  se puede calcular  $\Gamma(z+1)$  para n'umeros reales o complejos.

### Suma de series

Transformar la serie por medio de fracciones parciales y expresar la serie infinita como sumas finitas de funciones gamma y poligamma.

### EJEMPLO:

$$\begin{aligned}\varphi^{(m)}(z+2) &= \varphi^{(m)}(z+1) + (-1)^m \frac{m!}{(z+1)^{m+1}} \\ \varphi^{(m)}(z+2) - \varphi^{(m)}(z+1) &= \frac{(-1)^m m!}{(z+1)^{m+1}}\end{aligned}$$

$$\frac{d^{(m)}}{dz^{(m)}} [\varphi(z+2) - \varphi(z+1)]$$

$$\begin{aligned}\varphi(z+2) - \varphi(z+1) &= \sum_{m=1}^{\infty} \frac{z+1}{m(m+z+1)} - \sum_{m=1}^{\infty} \frac{z}{m(m+z)} \\ &= \sum_{m=1}^{\infty} \left[ \frac{1}{m} - \frac{1}{m+z+1} - \frac{1}{m} + \frac{1}{m+z} \right] \\ &= \frac{1}{z+1}\end{aligned}$$

$$\frac{d^m}{dz^m} [\varphi(z+2) - \varphi(z+1)] = \frac{d^m}{dz^m} \left[ \frac{1}{z+1} \right] = \frac{(-1)^m m!}{(z+1)^{m+1}}$$

Demuestre que:

$$\frac{1}{2} \ln \left[ \frac{\pi}{\operatorname{sen}(\pi \cdot z)} \right] = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n} z^{2n}, \quad |z| < 1$$

$$\blacksquare \quad \frac{1}{2} \ln \left[ \underbrace{\frac{\Gamma(z)}{z\Gamma(z)=\Gamma(z+1)}} \quad \Gamma(z-1) \right] = \frac{1}{2} \ln \left[ \frac{\pi}{\operatorname{sen}(z\pi)} \right]$$

$$\blacksquare \quad \ln \Gamma(z+1) = \lim_{n \rightarrow \infty} [\ln(n!) + z \ln(n) - \ln(z+1) - \dots - \ln(z+n)]$$

$$\Gamma(z)\Gamma(1-z) \rightsquigarrow \Gamma(1+z)\Gamma(1-z)$$

$$\begin{aligned}\ln [\Gamma(1+z)\Gamma(1-z)] &= -\delta \cdot z + \sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n} \zeta(n) \\ &\quad + (-\delta(-z)) + \sum_{n=2}^{\infty} (-1)^n \frac{(-z)^n}{n} \zeta(n) \\ &= \sum_{n=2}^{\infty} \zeta(n) \underbrace{\left[ \frac{z^n}{n} + \frac{(-z)^n}{n} \right]}_{z^{2n}} \\ &= 2 \sum_{n=1}^{\infty} \cancel{(-1)^{2n}} \overset{z^{2n}}{1} \zeta(2n) \frac{z^{2n}}{2n}\end{aligned}$$