

Metodos matemáticos 2

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1. Función Gamma

Límite al infinito(Euler)

$$\begin{aligned}\Gamma(z) &\equiv \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2) \cdots (z+n)} n^z, \quad z \in \mathbb{Z}^+ \text{ o } z \in \mathbb{C} \\ \Gamma(z+1) &= \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{(z+1)(z+2) \cdots (z+n+1)} n^{z+1} \\ &= z \cdot \Gamma(z) = \lim_{n \rightarrow \infty} \frac{n^z}{z+n+1} \Gamma(z) = \Gamma(z+1) = z \cdot \Gamma(z)\end{aligned}$$

Aplicando lo anterior a $z=1,2,3,\dots,n$

$$\begin{aligned}\Gamma(1) &= \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{(1)(2) \cdots (n+1)} n^1 = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \\ \Gamma(2) &= z \cdot \Gamma(z) = \Gamma(z+1) = \Gamma(1+1) = 1 \\ \Gamma(3) &= 2 \cdot \Gamma(1) = \Gamma(2+1) = 2 \cdot 1 \\ \Gamma(4) &= 3 \cdot \Gamma(3) = \Gamma(3+1) = 3 \cdot 2 \cdot 1 \\ &\vdots \\ &\vdots \\ &\vdots \\ \Gamma(n) &= (n-1)!\end{aligned}$$

Integral Definida(Integral de Euler)

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \operatorname{Re}(z) > 0$$

Ej.

Como aparecen en física

$$\Gamma(z) = 2 \int_0^\infty e^{-t^2} t^{z+1} dt \quad \text{o} \quad \Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right) \right]^{z-1} dt$$

Si $z = \frac{1}{2} \Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$ es integral error de Gauss

$$F(z, n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt, \quad \operatorname{Re}(z) > 0 \text{ con } n \text{ entero positivo } \ni$$

$$e^{-t} = \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n}\right)^n$$

$$\Rightarrow F(z, n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \int_0^1 (1-u)^n (un)^{z-1} n du \ni u = \frac{t}{n}$$

$$\frac{F(z, n)}{n^z} = \int_0^1 (1-u)^n (u)^{z-1} du$$

Ahora por integración por partes usando $u = (1-u)^n$, $du = n(1-u)^{n-1} du$, $v = \frac{u^z}{z}$, $dv = u^{z-1} du$ tenemos

$$\begin{aligned}\frac{F(z, n)}{n^z} &= (1-u)^n \frac{u^z}{z} \Big|_0^1 - \int_0^1 \frac{u^z}{z} n(1-u)^{n-1} du \\ F(z, n) &= n^z \cdot \frac{n(n-1) \cdots 1}{z(z+1) \cdots (z+n-1)} = \int_0^1 u^{z+n-1} du \\ &= \frac{1 \cdot 2 \cdots n}{z(z+1) \cdots (z+n)} n^z \Rightarrow \lim_{n \rightarrow \infty} F(z, n) = \Gamma(z)\end{aligned}$$

Producto infinito(Weierstrass)

$$\frac{1}{\Gamma(z)} = ze^{\delta z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

$$\delta := \text{constante de Euler-Mascheron}$$

$$\delta := 0.5772156619$$

Ecuación estadística de Maxwell-Boltzmann

$$e^{-E/KT}$$

K es la constante de Boltzmann
T es la temperatura absoluta
E(energía): estado de energía ocupada

probabilidad de estar en estado de energía es $Y_{kt} = \beta$

$$\begin{array}{ll} P(E) = Ce^{-\beta E} & \text{Para un gas idel sin estructura} \\ n(E)dE & n(E)^{1/2} \\ 1 = C \int n(E)e^{-\beta E} dE & E=\text{energía cinética} \end{array}$$

$$\begin{aligned} 1 &= c \int_0^{\infty} E^{1/2} e^{-\beta E} dE = \frac{C\Gamma(3/2)}{\beta^{3/2}}, \quad \beta E = T \Rightarrow dE = \frac{dT}{\beta} \\ 1 &= C \int_0^{\infty} e^{-t} \left(\frac{t}{\beta}\right)^{1/2} \frac{1}{\beta} dt \\ &= C \int_0^{\infty} e^{-t} t^{3/2-1} dt \cdot \frac{1}{\beta^{3/2}} \\ &= \frac{C\Gamma(3/2)}{\beta^{3/2}} \\ &= c \cdot \frac{\sqrt{\pi}}{2 \cdot \beta^{3/2}} \end{aligned}$$

$$\therefore C = \frac{2 \cdot \beta^{3/2}}{\sqrt{\pi}}$$

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Relaciones de funciones

$$\Gamma(z+1) = z\Gamma(z)$$

Fomula de reflexión

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\operatorname{sen}(z\pi)}$$

Fórmulade duplicación de Legendre

$$\Gamma(1+z)\Gamma\left(z+\frac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\Gamma(2z+1)$$

Notación doble factorial:

$$\begin{aligned}(2n+1)!! &= 1 \cdot 3 \cdot 5 \cdots (2n+1) = \frac{(2n+1)!}{2^n n!} \\ (2n)!! &= 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n! \\ (-1)!! &= 1\end{aligned}$$

PROPIEDADES ANALÍTICAS

$[\Gamma(z)]^{-1}$ tiene singularidades en $z = 0, -1, -2, \dots$ y no tiene cuando $z = -1, 2, 3, \dots$ y no tiene ceros en el plano complejo finito positivo.

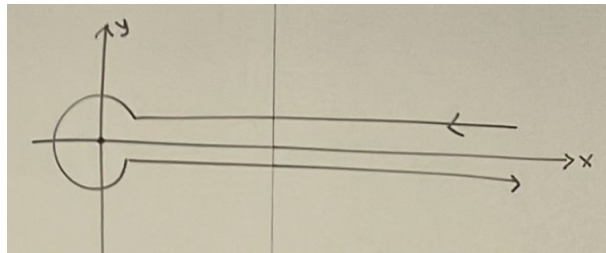
El residuo R_n cuando $z = -n$ donde n es un entero mayor o igual a cero.

$$\begin{aligned}R_n &= \lim_{\epsilon \rightarrow 0} (\epsilon \Gamma(-n + \epsilon)) \\ &= \lim_{\epsilon \rightarrow 0} \frac{\epsilon \Gamma(-n+1+\epsilon)}{-n+\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\epsilon \Gamma(-n+2+\epsilon)}{(-n+\epsilon)(-n+1+\epsilon)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\epsilon \Gamma(1+\epsilon)}{(-n+\epsilon) \cdots (\epsilon)} \\ &= \frac{(-1)^n}{n!}, \text{ el residuo alterna signos en } z = -n\end{aligned}$$

Integral de Schaeffli

$$\int_C e^{-t} t^v dt = (e^{2\pi i v}) \Gamma(v+1)$$

donde C es el contorno



Esta integral es útil cuando V no es entero.

Notación factorial:

$$\prod(z) = z! = \Gamma(z+1)$$

Función Digamma

$$\begin{aligned}\ln(\Gamma(z+1)) &= z \cdot \Gamma(z) \\ &= \lim_{n \rightarrow \infty} \frac{z}{n} \ln \left[\frac{n! \cdot n^z}{(z+1)(z+2) \cdots (z+n)} \right]\end{aligned}$$

$$\frac{d}{dz} \ln(\Gamma(z+1)) = \frac{d}{dz} \lim_{n \rightarrow \infty} [\ln(n!) + z \cdot \ln(n) - \ln(z+1) - \ln(z+2) - \dots - \ln(z+n)]$$

$$\frac{d}{dz} \ln(\Gamma(z+1)) \equiv \underbrace{\varphi(z+1)}_{\text{Función digamma}} = \lim_{n \rightarrow \infty} \left(\ln(n) - \frac{1}{z+1} - \frac{1}{z+2} - \dots - \frac{1}{z+n} \right)$$

$$\varphi(z+1) = \frac{[\Gamma(z+1)]'}{\Gamma(z+1)}$$

Si sumamos y restamos un número armónico

$$H_n = \sum_{m=1}^n \frac{1}{m}$$

$$\varphi(z+1) = \lim_{n \rightarrow \infty} \left[\underbrace{(\ln(n) - H_n)}_{-\delta \text{ cuando } n \rightarrow \infty} - \sum \left(\frac{1}{z+m} - \frac{1}{m} \right) \right]$$

$$\Rightarrow \varphi(z+1) = -\delta + \sum_{m=1}^{\infty} \frac{z}{m(m+z)}, \quad \text{si } z=0 \Rightarrow \varphi(1) = -\delta$$

para $n > 0$, n entero.

$$\begin{aligned} \varphi(n+1) &= -\delta + \sum_{m=1}^n \frac{1}{m} \\ &= -\delta + H_n \end{aligned}$$

Función Poligamma

Cuando se deriva muchas veces la función digamma

$$\begin{aligned} \varphi^{(m)}(z+1) &\equiv \frac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z+1) \\ \varphi^{(m)}(z+1) &= (-1)^{m+1} m! \cdot \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}}, \quad m = 1, 2, 3, \dots \end{aligned}$$

Si $z=0$ la función se conoce como zeta de Riemann

$$\begin{aligned} \zeta(m) &= \sum_{n=1}^{\infty} \frac{1}{n^m} \\ \Rightarrow \varphi_{(1)}^{(m)} &= (-1)^{m+1} \cdot m! \cdot \zeta(m+1), \quad m = 1, 2, 3, \dots \end{aligned}$$

Expansión de Maclaurin

$$\begin{aligned} \ln \Gamma(z+1) &= \sum_{n=1}^{\infty} \frac{z^n}{n!} \varphi_{(1)}^{(n-1)} \\ &= -\delta \cdot z + \sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n} \zeta(n) \end{aligned}$$

converge en $|z| < 1$ para $z = x \Rightarrow$ se puede calcular $\Gamma(z+1)$ para números reales o complejos.

Suma de series

Transformar la serie por medio de fracciones parciales y expresar la serie infinita como sumas finitas de funciones gamma y poligamma.

Ej.

$$\begin{aligned} \varphi^{(m)}(z+2) &= \varphi^{(m)}(z+1) + (-1)^m \frac{m!}{(z+1)^{m+1}} \\ \varphi^{(m)}(z+2) - \varphi^{(m)}(z+1) &= \frac{(-1)^m m!}{(z+1)^{m+1}} \\ \frac{d^{(m)}}{dz^{(m)}} [\varphi(z+2) - \varphi(z+1)] &= \\ \varphi(z+2) - \varphi(z+1) &= \sum_{m=1}^{\infty} \frac{z+1}{m(m+z+1)} - \sum_{m=1}^{\infty} \frac{z}{m(m+z)} \\ &= \sum_{m=1}^{\infty} \left[\frac{1}{m} - \frac{1}{m+z+1} - \frac{1}{m} + \frac{1}{m+z} \right] \\ &= \frac{1}{z+1} \\ \frac{d^m}{dz^m} [\varphi(z+2) - \varphi(z+1)] &= \frac{d^m}{dz^m} \left[\frac{1}{z+1} \right] = \frac{(-1)^m m!}{(z+1)^{m+1}} \end{aligned}$$

Demuestre que:

$$\frac{1}{2} \ln \left[\frac{\pi \cdot z}{\operatorname{sen}(\pi \cdot z)} \right] = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n} z^{2n}, \quad |z| < 1$$

$$\blacksquare \frac{1}{2} \ln \left[\underbrace{\Gamma(z)}_{z\Gamma(z)=\Gamma(z+1)} \Gamma(z-1) \right] = \frac{1}{2} \ln \left[\frac{\pi}{\operatorname{sen}(z\pi)} \right]$$

$$\blacksquare \ln \Gamma(z+1) = \lim_{n \rightarrow \infty} [\ln(n!) + z \ln(n) - \ln(z+1) - \dots - \ln(z+n)]$$

$$\Gamma(z)\Gamma(1-z) \rightsquigarrow \Gamma(1+z)\Gamma(1-z)$$

$$\ln [\Gamma(1+z)\Gamma(1-z)] = -\delta \cdot z + \sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n} \zeta(n)$$

$$+(-\delta(-z)) + \sum_{n=2}^{\infty} (-1)^n \frac{(-z)^n}{n} \zeta(n)$$

$$= \sum_{n=2}^{\infty} \zeta(n) \underbrace{\left[\frac{z^n}{n} + \frac{(-z)^n}{n} \right]}_{z^{2n}}$$

$$= 2 \sum_{n=1}^{\infty} (-1)^{2n-1} \zeta(2n) \frac{z^{2n}}{2n}$$

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La Función Beta

$$\Gamma(p+q) \propto \Gamma(p)\Gamma(q)$$

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t} t^n dt, \quad \text{Variante de gamma para definir la función beta}$$

$$\Gamma(m)\Gamma(n) = \int_0^\infty e^{-u} u^m du \int_0^\infty e^{-v} v^n dv$$

$$\begin{aligned} u &= x^2 & v &= y^2 \\ du &= 2x dx & dv &= 2y dy \end{aligned}$$

$$\begin{aligned} \Gamma(m)\Gamma(n) &= \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} x^{2m} (2x) y^{2n} (2y) dx dy \\ &= 4 \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} x^{2m+1} y^{2n+1} dx dy \end{aligned}$$

otro cambio de variables

$$x = r \cos(\theta) \quad y = r \sin(\theta)$$

Usando el jacobiano

$$\begin{aligned} &\left| \begin{array}{cc} \frac{\partial x}{\partial r} = \cos\theta & \frac{\partial x}{\partial \theta} = -r \sin\theta \\ \frac{\partial y}{\partial r} = \sin\theta & \frac{\partial y}{\partial \theta} = r \cos\theta \end{array} \right| = r dr d\theta \\ &= \int_0^\infty \int_0^{\pi/2} e^{-r^2 \cos^2 \theta} e^{-r^2 \sin^2 \theta} r^{2m+2n+3} \cos^{2m+1} \theta \sin^{2n+1} \theta dr d\theta \end{aligned}$$

$$\Gamma(m)\Gamma(n) = 2\Gamma(m+n+1) \int_0^{\pi/2} \cos^{2m+1} \theta \sin^{2n+1} \theta d\theta$$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt; \quad \text{Re}(z) > 0$$

$$\Gamma(q)\Gamma(p) = \int_0^\infty e^{-s} s^{2q-1} ds \int_0^\infty e^{-t} t^{2p-1} dt$$

$$\begin{aligned} s &= r \cos\theta & t &= r \sin\theta \\ r^2 &= s^2 + t^2 & ds dt &= r dr d\theta \end{aligned}$$

$$\Gamma(q)\Gamma(p) = 4 \int \int e^{-r^2 \cos^2 \theta} e^{-r^2 \sin^2 \theta} r^{2q-1} \cos^{2q-1} r^{2p-1} \sin^{2p-1} r dr d\theta$$

$$= 2 \cdot 2 \int_0^\infty e^{-r^2} r^{2q+2p-1} \cos^{2q-1} \sin^{2p-1} dr d\theta$$

$$\Gamma(p)\Gamma(q) = 2 \int \cos^{2q-1} \sin^{2p-1} d\theta$$

$$\frac{m!n!}{(m+n+1)!} = 2 \int_0^{\pi/2} \cos^{2m+1} \theta \sin^{2n+1} \theta d\theta$$

$$\left. \boxed{b(p, q) = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}} \right\} \text{definición de función beta}$$

$$b(p, q) = b(q, p)$$

Si se sustituye $t = \cos^2 \theta$ y $dt = -2 \cos \theta \sin \theta d\theta$

$$\begin{aligned} \cos^{2m} \theta + \sin^{2n} \theta &= 1 \\ \sin^{2n} \theta &= 1 - t \end{aligned} \quad \star = 2 \int_0^1 \cos^{2m} \theta \cos^{2n} \theta (1-t)^n \sin^{2n} \theta dt$$

$$\begin{aligned} \sin^{2n+1} \theta &= \sin^{2n} \theta \sin \theta \\ &= (1-t)^n \sin \theta = \int_0^1 t^m (1-t)^n dt \Rightarrow \text{otra forma de escribir } B(m+1, n+1) \end{aligned}$$

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Ej.

Una partícula de masa m se mueve en un potencial simétrico esta descrito por $v(x) = A|x|^n$ y energía total como $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + v(x) = E$

Resolviendo para $\frac{dx}{dt}$ e integrando encontramos que el movimiento periódico es:

$$\zeta = \sqrt{2m} \int_0^{X_{max}} \frac{dx}{(E - Ax)^{1/2}}$$

donde X_{max} es el punto de inflexión clásico dado por $Ax_{max}^n = E$

Muestre que:

$$\tau = \frac{2}{n} \sqrt{\frac{e\pi m}{E}} \left(\frac{E}{A}\right)^{1/2} \left[\frac{\Gamma(1/n)}{\Gamma(\frac{1}{n} + \frac{1}{2})} \right]$$

$$B(p+1, q+1) = B(1/n, 1/2) = \int_0^1 t^p (1-t)^q dt$$

$$\frac{1}{n} = p+1 \Rightarrow p = \frac{1}{n} - 1$$

$$\frac{1}{2} = q+1 \Rightarrow q = \frac{1}{2} - 1$$

$$\int_0^1 t^{\frac{1}{n}-1} (1-t)^{-1/2} dt$$

$$t = \frac{Ax^n}{E} \Rightarrow \left(\frac{Et}{A}\right)^{1/n}$$

$$dt = \frac{A}{E} n(x^{n-1}) dx$$

$$dx = \underbrace{\frac{E}{A}}_{\frac{1}{n} \frac{x}{t}} \frac{1}{n} \frac{x}{x^n} dt$$

$$\begin{aligned} \tau &= 2 \frac{\sqrt{2m}}{\sqrt{E}} \int_0^1 \frac{\frac{x}{n} dt}{(1-t)^{1/2}} \\ &= \left(\frac{E}{A}\right)^{1/n} \frac{2}{n} \sqrt{\frac{2m}{E}} \int_0^1 \frac{\frac{(Et)^{1/n}}{A^{1/n}} dt}{(1-t)^{1/2}} \\ &= \frac{2}{n} \sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{1/n} \underbrace{\int_0^1 t^{\frac{1}{n}-1} (1-t)^{-1/2} dt}_{\frac{\Gamma(\frac{1}{n}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n} + \frac{1}{2})} \sqrt{\pi}} \end{aligned}$$

$$\tau = \frac{2}{n} \sqrt{\frac{2\pi m}{E}} \left(\frac{E}{A}\right)^{1/n} \frac{\Gamma(\frac{1}{n})}{\Gamma(\frac{1}{n} + \frac{1}{2})}$$

2. SERIES DE STIRLING

$$\ln(n!) \rightsquigarrow \ln \Gamma(z)$$

para z no entero, $|z|$ muy grande, no Series de Maclaurin

$$\Gamma(z+1) = z\Gamma(z)$$

Se usa una expansión asintótica para $\ln(\Gamma(z))$ esta expansión es la que se conoce como **Serie de Stirling** o **Formula de Stirling**
Series asintóticas → **Series semi-convergentes** → **Series de Poincaré**

Se consideran 2 tipos de integrales

$$I_1(x) = \int_x^\infty e^{-u} f(u) du$$

$$I_2(x) = \int_0^\infty e^{-u} f\left(\frac{u}{x}\right) du$$

Qué podemos hacer con las series asintóticas:

1. Si multiplicamos 2 series asintóticas vamos a tener otra serie asintótica.
2. Se pueden integrar término a término u el resultado será, otra serie asintótica de la forma

$$\int_x^\infty f(x) dx$$

3. La diferenciación es válido únicamente bajo ciertas condiciones.

Método para generar una serie asintótica:

$$\left. \begin{array}{l} \text{Método de Steepest Descents} \\ \rightarrow \text{Saddle point} \rightarrow \text{punto silla} \\ \rightarrow \text{Método de integrandos Holomorfos} \end{array} \right\} \begin{array}{l} \text{a lo largo del camino} \\ \text{máximo decrecimiento} \end{array}$$

Derivación de la fórmula de integración de Euler-Maclaurin

→ uno de los usos de los polinomios de Bernoulli ←

Números de Bernoulli → Polinomios de Bernoulli

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}$$

Como $\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}$ es una serie de Taylor se pueden identificar a B_n como una sucesión de derivadas de la función generadora valuadas en cero.

$$\Rightarrow B_n = \frac{d^n}{dt^n} \left(\frac{t}{e^t - 1} \right) \Big|_{t=0}$$

$$B_0 = \lim_{n \rightarrow 0} \frac{t}{e^t - 1} = 1$$

$$B_1 = \lim_{n \rightarrow 0} \frac{d}{dt} \left[\frac{t}{e^t - 1} \right] \Big|_{t=0} = -\frac{1}{2}$$

$$B_1 = \lim_{n \rightarrow 0} \left(\frac{1}{e^t - 1} - \frac{te^t}{(e^t - 1)^2} \right)$$

$$B_1 = \lim_{n \rightarrow 0} \left(\frac{e^t - 1 - te^t}{(e^t - 1)^2} \right) = \lim_{n \rightarrow 0} \frac{e^t - t - 1}{2(e^t - 1)}$$

$$= -\frac{1}{2}$$

La fórmula Euler-Maclaurin para evaluar integrales definidas en $(0, \infty)$

$$\begin{aligned} \int_0^\infty &= \frac{1}{2} f(0) + f(1) + f(2) + f(3...) \\ &+ \frac{B_2}{2!} f'(0) + \frac{B_4}{4!} f^{(3)}(0) + \frac{B_6}{6!} f^{(5)}(0) \end{aligned}$$

Donde B_n son los números de Bernoulli

$$B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, \dots$$

$$\int_0^\infty \frac{dx}{(z+x)^2} = \frac{1}{z}$$

z no está en el eje real negativo

$$f(1) + f(2) + f(3) + \dots = \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} =$$

$$\varphi^{(m)}(z+1) = \frac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z+1)$$

$$_{m=1,2,3\dots} = (-1)^{m+1} m! \sum_{n=2}^{\infty} \frac{1}{(z+n)^{m+1}}$$

m=1

$$(-1)^2 \cdot 1! \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} = \varphi^{(1)}(z+1)$$

$$f_{(0)}^{(2n+1)} = \left(\frac{d}{dx} \right)^{2n-1} \frac{1}{(z+x)^2} \Big|_{x=0} = -\frac{(2n)!}{z^{2n+1}}$$

$$\int_0^{\infty} f(x) dx = \int_0^{\infty} \frac{dx}{(z+x)^2} = \frac{1}{z} = \frac{1}{2} \frac{1}{z^2} + \underbrace{\varphi_{(z+1)}^{(1)}}_{\text{poligamma con m=1}} - \frac{B_2}{z^3} - \frac{B_4}{z^5} - \dots$$

$$\varphi_{(z+1)}^{(1)} = \frac{d}{dz} \underbrace{\varphi(z+1)}_{\text{digamma}}$$

$$\varphi_{(z+1)}^{(1)} = \frac{1}{z} - \frac{1}{2z^2} + \frac{B_2}{z^3} + \frac{B_4}{z^5} + \dots$$

$$= \frac{1}{z} - \frac{1}{2z^2} + \underbrace{\sum_{n=1}^{\infty} \frac{B_{2n}}{z^{2n+1}}}_{\text{Es una serie asintótica que es útil si se utilizan pocos términos}}$$

Al integrar $\varphi_{(z+1)}^{(1)}$ obtenemos $\varphi_{(z+1)}$ [Digamma]

$$\varphi_{(z+1)} = C_1 + \ln z + \frac{1}{2z} - \frac{B_2}{2z^2} - \frac{B_4}{4z^4} - \dots$$

$$= C_1 + \ln z + \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{z^{2n}}$$

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$$\varphi_{(z+1)} = C_1 + \ln z + \frac{1}{2z} - \frac{B_2}{2z^2} - \frac{B_4}{4z^4} - \dots, \quad B_{2n} \text{ Números de Bernoulli}$$

$$= C_1 + \ln z + \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}}$$

$$\varphi(z+1) \equiv \frac{d}{dz} \ln \Gamma(z+1)$$

$$\ln \Gamma(z+1) = C_2 + \left(z + \frac{1}{2}\right) \ln z + (c_1 - 1)z + \frac{B_2}{2z} + \dots + \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \dots$$

$$\Rightarrow C_1 = 0 \text{ y } c_2 = \frac{1}{2} \ln 2\pi$$

$$\ln \Gamma(z+1) = \frac{1}{2} \ln 2\pi + \left(z + \frac{1}{2}\right) \ln z - z + \frac{1}{12z} + \frac{1}{360z^3} + \frac{1}{1260z^5} + \dots$$

$$\lim_{n \rightarrow \infty} \frac{z!}{\sqrt{zz^ze^{-z}}} = \sqrt{2\pi}$$

$$z! \approx \sqrt{2\pi} \sqrt{zz^ze^{-z}}$$

$$\Gamma(z+1) \approx \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z}$$

$$\ln \Gamma(z+1) \approx \frac{1}{2} \ln 2\pi + \left(z + \frac{1}{2}\right) (\ln z) - z$$

$$\ln n! \approx n \ln n - n$$

Ej.

n es natural.

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$\lim_{n \rightarrow \infty} \sqrt[n^3]{n} = 1$$

$$\lim_{n \rightarrow \infty} \sqrt[n^3]{n^7} = 1$$

$$\lim_{n \rightarrow \infty} \sqrt[p(n)]{q(n)} = 1$$

pero...

$$\lim_{n \rightarrow \infty} \sqrt[n]{n!} = \infty$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sqrt[n]{n!} = \lim_{n \rightarrow \infty} \sqrt[n]{\sqrt{2\pi} \sqrt{nn^n} e^{-n}}$$

$$= \lim_{n \rightarrow \infty} \sqrt[n]{\sqrt{2\pi n}} \cdot n \cdot e^{-1} = \lim_{n \rightarrow \infty} \sqrt[n]{2\pi n} \cdot n \cdot e^{-1}$$

$$= \infty$$

$$\lim_{n \rightarrow \infty} \frac{2n!}{4^n (n+1)!} = 0$$

$$(n+1)! = (n+1)n!(\sqrt{2\pi n} n^n e^{-n})$$

$$2n! = \sqrt{2\pi} \sqrt{2n} (2n)^{2n} e^{-2n}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{2\pi} \sqrt{2n} (2n)^{2n} (e)^{-2n}}{4^n (n+1) (\sqrt{2\pi n} \cdot n^{2n} e^{-2n})^2}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{\pi n}}{(n+1)(\pi n)} = 0$$

Ej.

Un conjunto de N partículas distinguibles es asignado a los estados $\psi_i \ni i = 1, 2, \dots, M$. Si el número de partículas en los diversos estados son $n_1, n_2, n_3, \dots, n_M$ (con $M \ll N$). El número de formas que se pueden hacer es:

$$w = \frac{N!}{n_1! n_2! \dots n_M!}$$

La entropía asociada es $S = k \ln w$ donde k es la constante de Boltzmann.

En el límite cuando $N \rightarrow \infty$, con $n_i = P_i N$, (P_i es la fracción del as partículas en le estado i). Encuentre S en función de N y P_i .

$$\text{Pista: } \sum_i P_i = 1$$

Sol.

$$\begin{aligned} \ln N! &\approx N \ln N - N \quad \text{usando la condición } \sum_i n_i = N \\ S = k \ln W &= k [\ln N! - \sum_i \ln n_i!] \\ &= k \left[(N \ln N - N) - \left(\sum_{i=1}^M n_i + n \ln n - n_i \right) \right] \end{aligned}$$

lunes 24 de julio

$$\begin{aligned}\zeta(z) &= \sum_{n=1}^{\infty} n^{-z} && \text{por definici3n} \\ \zeta(z) &= \int_0^{\infty} \frac{u^{z-1}}{e^u-1} du\end{aligned}$$

$$\begin{aligned}\zeta(1) &= \infty && \zeta(2) = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \\ \zeta(3) &= 1,202056 && \zeta(4) = \frac{\pi^4}{90} \\ \zeta(5) &= 1,0369 && \zeta(6) = \frac{\pi^6}{945}\end{aligned}$$

$$Re(z) > 1$$

$$\begin{aligned}\zeta(-1) &= \frac{1}{1-1} + \frac{1}{2-1} + \dots \\ &= 1 + 2 + 3 + 4 + \dots \\ &= -\frac{1}{12}\end{aligned}$$

$$\begin{aligned}\zeta(-2n) &= 0 \\ \text{Ceros "triviales"}\end{aligned}$$

$$\begin{aligned}z &= 2 + i && n^z = \frac{1}{2}^{2+i} = \frac{1}{2}^2 \frac{1}{2}^i \\ n &= \frac{1}{2}\end{aligned}$$

Otra forma de definir Zeta de Riemann

$$\zeta(s) = \prod_{\substack{\text{primos} \\ p=2}}^{\infty} (1 - p^{-s})^{-1}$$

franja cr3tica:

$$\begin{aligned}\zeta(z) &= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t-1} dt \frac{e^{-t}}{e^{-t}} = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1} \cdot e^{-t}}{1-e^{-t}} \\ &= \int_0^{\infty} t^{z-1} e^{-t} \sum_{m=1}^{\infty} e^{-mt} dt && \left(\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m \right) \\ &= \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} \sum_{m=1}^{\infty} e^{-mt} dt && = \frac{1}{\Gamma(z)} \int_0^{\infty} \sum_{m=1}^{\infty} \left(\frac{x}{m} \right)^{z-1} e^{-x} \frac{dx}{m} \\ & && = \frac{1}{\Gamma(z)} \left(\sum_{m=1}^{\infty} \frac{1}{m^z} \right) \int_0^{\infty} x^{z-1} e^{-x} dx\end{aligned}$$

Ej.

$$\begin{aligned}\int_0^{\infty} \frac{x^n \cdot e^x}{(e^x-1)^2} dx &= n! \zeta(n) && \begin{aligned} u &= x^n && v = -\frac{1}{e^x-1} \\ du &= nx^{n-1} dx && dv = \frac{e^x}{(e^x-1)^2} dx \end{aligned} \\ &= -\frac{x^n}{e^x-1} \Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{e^x-1} nx^{n-1} dx && \zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t-1} dt \\ &= \int_0^{\infty} \frac{x^{n-1}}{e^x-1} dx = n(n+1)! \zeta(z)\end{aligned}$$

Funciones Incompletas

Gamma incompleta

$$\delta(a, x) = \int_0^x e^{-t} t^{a-1} dt \quad R(a) > 0$$

$$\Gamma(a, x) = \int_x^{\infty} e^{-t} t^{a-1} dt$$

$$\delta(a, x) + \Gamma(a, x) = \text{Gamma}(a) = \Gamma(a)$$

Cuando a es entero:

$$\delta(n, x) = (n-1)! \left(1 - e^{-x} \sum_{s=0}^{n-1} \frac{x^s}{s!} \right)$$

$$\Gamma(n, x) = (n-1)! e^{-x} \sum_{s=0}^{n-1} \frac{x^s}{s!}$$

Cuando a no es entero:

$$\begin{aligned}\delta(a, x) &\rightarrow \text{serie de potencias para } x \text{ pequeno} &\Rightarrow \delta(a, x) &= x^a \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!(a+n)} \\ \Gamma(a, x) &\rightarrow \text{serie asintótica} &\Rightarrow \Gamma(a, x) &\sim x^{a-1} e^{-x} \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a-n)} \cdot \frac{1}{x^n}\end{aligned}$$

Cuando a=0

$$\Gamma(0, x) \sim \frac{e^x}{x} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^n}$$

Beta Incompleta

$$B_x(p, q) = \int_0^x t^{p-1} (1-t)^{q-1} dt \quad 0 \leq x \leq 1, p > 0, q > 0$$

Si $x = 0 \Rightarrow B(p, q)$ función Beta

Aparece en:

- Funciones hipergeométricas
- Probabilidades

Viernes 28 de julio

Lunes 31 de julio

Derivaci'on bajo el signo de la integral

Integral de Feynmann

$$\frac{d}{dt} \int f(x, t) dx = \int \frac{\partial f(x, t)}{\partial t} dx$$

Derivaci'on de la funci'on Gamma

$$F(t) = \int_0^{\infty} e^{-tx} dx$$

$$F(t) = \frac{1}{t} \text{ para } t > 0$$

$$\begin{aligned} \frac{dF(t)}{dt} = F'(t) &= \int_0^{\infty} \frac{d}{dt} e^{-tx} dx = \frac{d}{dt} \left[\frac{1}{t} \right] \\ &\Rightarrow \int_0^{\infty} -x e^{-tx} dx = -\frac{1}{t^2} \\ &\Rightarrow \int_0^{\infty} x^n e^{-tx} dx = \frac{n!}{t^{n+1}} \end{aligned}$$

$$n! = \int_0^{\infty} e^{-x} x^n dx$$

$$n! = t^{n+1} \int_0^{\infty} x^n e^{-tx} dx = \int_0^{\infty} x^n e^{-x} dx$$

$$\begin{aligned} u &= tx \\ du &= t dx \end{aligned} \quad \int_0^{\infty} \cancel{t}^{\frac{u}{t}} \cancel{t}^{\frac{u}{t}} e^{-u} \frac{du}{t}$$

$$\int_0^1 \frac{x^2 - 1}{\ln(x)} dx$$

$$\begin{aligned} \frac{d}{dt} I(t) &= \int_0^1 \frac{\partial}{\partial t} \frac{x^t - 1}{\ln(x)} dx \\ &= \int_0^1 \frac{1}{\ln(x)} \frac{\partial}{\partial t} [x^t - 1] dx \\ &= \int_0^1 \frac{1}{\ln(x)} x^t \ln(x) dx \\ &= \int_0^1 x^t dx = \frac{1}{t+1} x^{t+1} \Big|_0^1 \\ &= \frac{1}{t+1} \\ \int \frac{d}{dt} I(t) &= \int \frac{1}{t+1} dt \\ I(t) &= \ln |t+1| + c \end{aligned}$$

si t=0

$$\begin{aligned} I(2) &= \ln |2+1| = \ln |3| \\ I(2) &= \int_0^2 I'(t) dt \\ &= \int_0^2 \frac{1}{t+1} dt = \ln |t+1| \Big|_0^2 \\ &= \ln |3| \end{aligned}$$

$$\int_0^{\infty} \frac{e^{-x^2} \text{sen}(x^2)}{x^2} dx$$

$$\frac{d}{dx} \text{sen}(ax^2) = x^2 \cos(ax^2)$$

$$\frac{d}{da} I(a) = \frac{d}{da} \int_0^\infty \frac{\partial}{\partial a} \frac{e^{-x^2} \text{sen}(x^2)}{x^2} dx$$

$$= \int_0^\infty \frac{e^{-x^2}}{x^2} x^2 \cos(ax^2) dx$$

$$I'(a) = \int_0^\infty e^{-x^2} \cos(ax^2) dx$$

$$= \text{Re} \int_0^\infty e^{-x^2} e^{iax^2} dx$$

$$= \text{Re} \int_0^\infty e^{-x^2(1-ia)} dx$$

$$\int I'(a) da = \int \frac{1}{2} \sqrt{\pi} \text{Re} \left[\frac{1}{\sqrt{1-ia}} \right] da$$

$$I(a) = \frac{\sqrt{\pi}}{2} \text{Re} [2i\sqrt{1-ia}] + c$$

$$a = 0 \Rightarrow \text{sen}(ax^2) = 0 \Rightarrow c = 0$$

$$a=1$$

$$I(1) = \sqrt{\pi} \text{Re} [i\sqrt{1-i}]$$

$$1-i \text{ en forma polar: } re^{i\theta}$$

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1} \left[\frac{-1}{1} \right] = -\frac{\pi}{4}$$

$$1-i = \sqrt{2} e^{-\frac{\pi}{4}i}$$

$$\sqrt{1-i} = \sqrt{\sqrt{2}} e^{-\frac{\pi}{8}i}$$

$$= \sqrt{\sqrt{2}} \left[\cos \frac{\pi}{8} - i \text{sen} \frac{\pi}{8} \right]$$

$$i\sqrt{1-i} = \sqrt{\sqrt{2}} \left[i \cos \frac{\pi}{8} - i^2 \text{sen} \frac{\pi}{8} \right]$$

$$\int_0^\infty \left[\ln(\mathbf{x}) \right] \left[\ln(\tanh(\mathbf{x})) \right] d\mathbf{x}$$

$$I(m) = \int_0^\infty x^m \ln(\tanh(x)) dx$$

a usar:

$$\frac{d}{dm} x^m = x^m \ln(x)$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Viernes 04 de agosto **Ej.**

$$\int_0^{\infty} [\ln(x)] [\ln(\tanh(x))] dx$$

$$I(m) = \int_0^{\infty} x^m [\ln(1 - e^{-2x}) - \ln(1 + e^{-2x})] dx$$

$$= \int_0^{\infty} x^m \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-e)^{-2nx}}{n} \right] dx - \int_0^{\infty} x^m \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (e)^{-2nx}}{n} dx$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d}{dm} x^m = x^m \log(x)$$

$$\tanh(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

Serie de MacLaurin

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

PEDIR COPIA

Funciones Bessel

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

EC. Bessel de orden P.

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

EC. Bessel de orden $\frac{1}{2}$

Método por series de Frobenius

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$xy'' + y' + xy = 0$$

Ec. Bessel de orden 0

$$y_2 = y_1 \ln(x) + x^{r_1+1} \sum_{n=0}^{\infty} b_n x^n$$

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1} + \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$n = 0$$

$$(r)(r-1)C_0 x^{r-1}$$

$$rC_0 x^{r-1}$$

$$n = 1$$

$$(1+r)(r)C_1 x^r$$

$$(1+r)C_1 x^r$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-1}$$

$$\sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$

lunes 07 d e agosto

$$\boxed{xy'' + y' + xy = 0}$$

$$y_1 = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

$$\begin{aligned} y_2 = & y_1 \ln(x) + x^{r_2+1} \sum_{n=1}^{\infty} b_n x^n \\ & \boxed{xy'' \ln(x)} + 2y_1' - \cancel{\frac{1}{x}y_1} + \sum_{n=1}^{\infty} n(n-1)b_n x^{n-1} + \\ & + \boxed{y_1' \ln(x)} + \cancel{\frac{1}{x}y_1} + \sum_{n=1}^{\infty} n b_n x^{n-1} + \\ & + \boxed{xy_1 \ln(x)} + \sum_{n=1}^{\infty} b_n x^{n+1} = 0 \end{aligned}$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n} (n!)^2} + \sum_{n=1}^{\infty} n(n-1) b_n x^{\overset{(n-1)}{\circ}} + \sum_{n=1}^{\infty} n b_n x^{\overset{(n-1)}{\circ}} + \sum_{n=1}^{\infty} b_n x^{\overset{(n-1)}{\circ}} = 0$$

$$\underline{x^0}: \quad n = 1 \quad 1(0)b_1 + 1b_1 = 0 \quad \Rightarrow b_1 = 0$$

$$\underline{x^1}: \quad \frac{2(-1)(2)}{2^2(1!)^2} + 2(2-1)b_2 + 2b_2 = 0 \quad \Rightarrow -1 + 2b_2 + 2b_2 = 0$$

$$\Rightarrow b_2 = \frac{1}{4}$$

$$\underline{x^2}: \quad \text{Ya funciona para todas las series}$$

$$2 \sum_{n=2}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n} (n!)^2} + \sum_{n=3}^{\infty} n(n-1) b_n x^{n-1} + \sum_{n=3}^{\infty} n b_n x^{n-1} + \sum_{n=1}^{\infty} b_n x^{n+1} = 0$$

$$2 \sum_{n=2}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n} (n!)^2} + \sum_{n=2}^{\infty} (n+1)(n) b_{n+1} x^n + \sum_{n=2}^{\infty} (n+1) b_{n+1} x^n + \sum_{n=2}^{\infty} b_{n-1} x^n = 0$$

$$2 \sum_{n=2}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n} (n!)^2} + \sum_{n=2}^{\infty} [n(n+1) b_{n+1} + \underset{(n+1)^2 b_{n+1} \cancel{(n+1)} + b_{n-1}}{(n+1) b_{n+1} + b_{n-1}}] x^n = 0$$

$$2 \sum_{n=2}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n} (n!)^2} + \sum_{n=2}^{\infty} [(n+1)^2 b_{n+1} + b_{n-1}] x^n = 0$$

$$\boxed{n \text{ par} \rightarrow n=2k}$$

$$(2k+1)^2 b_{2k+1} + b_{2k-1} \quad \text{para } k \geq 1$$

$$b_{2k+1} = \frac{-b_{2k-1}}{(2k+1)^2}$$

k=1:

$$b_3 = \frac{-b_1}{3^2} = 0$$

k=2:

$$b_5 = \frac{-b_3}{5^2} = 0$$

$$\boxed{n \text{ impar} \rightarrow n=2k-1}, k \geq 2$$

$$\frac{2(-1)^k 2k}{2^{2k}((k!)^2)} + (2k)^2 b_{2k} + b_{2k-2} = 0$$

$$(2k)^2 b_{2k} + b_{2k-2} = \frac{-(-1)^k 4k}{2^{2k}(k!)^2}$$

suponemos:

$$b_{2k} = \frac{(-1)^{k+1} c_{2k}}{2^{2k}(k!)^2}$$