Metodos matemáticos 2

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1. Función Gamma

L'imite al infinito(Euler)

$$\begin{aligned}
\Gamma(z) \equiv & \lim_{n\to\infty} \tfrac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z, \quad z \in \mathbb{Z}^+ \ o \ z \in \mathbb{C} \\
\Gamma(z+1) = & \lim_{n\to\infty} \tfrac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)\cdots(z+n+1)} n^{z+1} \\
= & z\cdot\Gamma(z) = \lim_{n\to\infty} \tfrac{nz}{z+n+1}\Gamma(z) = \Gamma(z+1) = z\cdot\Gamma(z)
\end{aligned}$$

Aplicando lo anterior a z=1,2,3...n

$$\begin{aligned}
\Gamma(1) = & \lim_{n\to\infty} \tfrac{1\cdot 2\cdot 3\cdots n}{(1)(2)\cdots(n+1)} n^z = \lim_{n\to\infty} \tfrac{n}{n+1} = 1 \\
\Gamma(2) = & z\cdot\Gamma(z) = \Gamma(z+1) = \Gamma(1+1) = 1 \\
\Gamma(3) = & 2\cdot\Gamma(1) = \Gamma(2+1) = 2\cdot 1 \\
\Gamma(4) = & 3\cdot\Gamma(3) = \Gamma(3+1) = 3\cdot 2\cdot 1 \\
& . \\
& . \\
& . \\
\Gamma(n) = & (n-1)!
\end{aligned}$$

Itegral Definida(Integral de Euler)

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \quad Re(z) > 0$$

Ej. como aparecen en f'isica

$$\Gamma(z) = 2\int_0^\infty e^{-t^2}t^{z+1}dt \quad o \quad \Gamma(z) = \int_0^1 \left[ln(\frac{1}{t})\right]^{z-1} dt$$

Si $z = \frac{1}{2} \Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$ es integral error de Gauss

$$F(z,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}dt, \quad Re(z) > 0 \ con\ n\ entero\ positivo \ni$$

$$e^{-t} = \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n$$

$$\Rightarrow F(z,n) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt = \int_0^1 (1-u)^n(un)^{z-1}ndu \ni u = \frac{t}{n}$$

$$\frac{F(z,n)}{n^z} = \int_0^1 (1-u)^n(u)^{z-1}du$$

Ahora por integraci'on por partes usando $u = (1-u)^n$, $du = n(1-u)^{n-1}du$, $v = \frac{u^z}{z}$, $dv = u^{z-1}du$ tenemos

$$\begin{aligned} \frac{F(z,n)}{n^z} = & \quad (1-u)^n \left. \frac{u^z}{z} \right|_0^1 - \int_0^1 \frac{u^z}{z} n(1-u)^{n-1} du \\ F(z,n) = & \quad n^z \cdot \frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)} = \int_0^1 u^{z+n-1} du \\ = & \quad \frac{1\cdot 2\cdots n}{z(z+1)\cdots(z+n)} n^z \Rightarrow \lim_{n\to\infty} F(z,n) = \Gamma(z) \end{aligned}$$

Producto infinito(Weierstrass)

$$\frac{1}{\Gamma(z)} = ze^{\delta z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

$\delta :=$ constante de Euler-Mascheron
$\delta := 0.5772156619$

Ecuaci'on estad'istica de Maxwell-Boltzmann

$e^{-E/KT}$

K es la constante de Boltzmann
T es la temperatura absoluta
E(energ'ia): estado de energ'ia ocupada

probabilidad de estar en estado de energ'ia es $Y_{kt} = \beta$

$P(E) = Ce^{-\beta E}$	*Para un gas idel sin estructura*
$n(E)dE$	$n(E)^{1/2}$
$1 = C\int n(E)e^{-\beta E}dE$	*E=energ'ia cin'etica*

$$\begin{aligned} 1 = & \quad c\int_0^{\infty} E^{1/2} e^{-\beta E} dE = \frac{C\Gamma(3/2)}{\beta^{3/2}}, \quad \beta E = T \Rightarrow dE = \frac{dt}{\beta} \\ 1 = & \quad C\int_0^{\infty} e^{-t} \left(\frac{t}{\beta}\right)^{1/2} \frac{1}{\beta} dt \\ & \quad = C\int_0^{\infty} e^{-t} t^{3/2-1} dt \cdot \frac{1}{\beta^{3/2}} \\ = & \quad \frac{C\Gamma(3/2)}{\beta^{3/2}} \\ = & \quad c \cdot \frac{\sqrt{\pi}}{2\cdot\beta^{3/2}} \end{aligned}$$

$$\therefore C = \frac{2\cdot\beta^{3/2}}{\sqrt{\pi}}$$

lunes 10 de julio
Relaciones de funciones

$$\Gamma(z+1) = z\Gamma(z)$$

Fomula de reflexi'on

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sen(z\pi)}$$

F'ormuláde duplicaci'on de Legendre

$$\Gamma(1+z)\Gamma(z+\frac{1}{2}) = 2^{-2z}\sqrt{\pi}\Gamma(2z+1)$$

Notac'i'on doble factorial:

$$\begin{aligned} (2n+1)!! = &\quad 1\cdot 3\cdot 5\cdots(2n+1) = \tfrac{(2n+1)!}{2^n n} \\ (2n)!! = &\quad 2\cdot 4\cdot 6\cdots(2n) = 2^n n! \\ (-1)!! = &\quad 1 \end{aligned}$$

PROPIEDADES ANAL'ITICAS

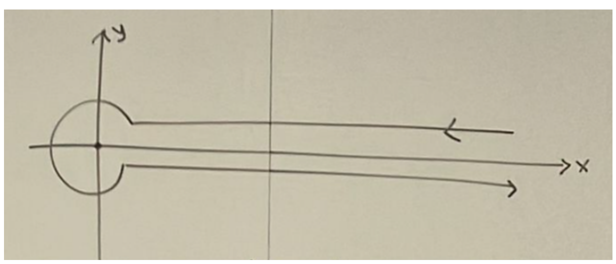
$[\Gamma(z)]^{-1}$ tiene singularidades en $z = 0, -1, -2...$ y no tiene cuando $z = -1, 2, 3...$ y no tiene ceros en el plano complejo finito positivo.
El residuo R_n cuando $z = -n$ donde n es un entero mayor o igual a cero.

$$\begin{aligned} R_n = &\quad \lim_{n\to 0}(\epsilon\Gamma(-n+\epsilon)) \\ = &\quad \lim_{n\to 0}\tfrac{\epsilon\Gamma(-n+1+\epsilon)}{-n+\epsilon} \\ = &\quad \lim_{n\to 0}\tfrac{\epsilon\Gamma(-n+2+\epsilon)}{(+\epsilon)(+1+\epsilon)} \\ = &\quad \lim_{n\to 0}\tfrac{\epsilon\Gamma(1+\epsilon)}{(-n+\epsilon)\cdots(\epsilon))} \\ = &\quad \tfrac{(-1)^n}{n!}, \textit{el residuo alterna signos en } z = -n \end{aligned}$$

Integral de Schaefli

$$\int_C e^{-t}t^v dt = (e^{2\pi iv})\Gamma(v+1)$$

donde C es el contorno



Esta integral es 'util cuando V no es entero.
Notaci'on factorial:

$$\prod(z) = z! = \Gamma(z+1)$$

Funci'on Digamma

$$\begin{aligned} \ln(\Gamma(z+1)) = &\quad z\cdot\Gamma(z) \\ = &\quad \lim_{n\to\infty}\tfrac{z}{z}\ln\left[\tfrac{n!\cdot n^z}{(z+1)(z_2)\cdots(z+n)}\right] \\ \tfrac{d}{dz}\ln(\Gamma(z+1)) = &\quad \tfrac{d}{dz}\lim_{n\to\infty}[\ln(n!) + z\cdot\ln(n) - \ln(z+1) - \ln(z+2) - ...\ln(z+n)] \\ \tfrac{d}{dz}\ln(\Gamma(z+1)) \equiv \underbrace{\varphi(z+1)}_{Funci'on\ digamma} = &\quad \lim_{n\to\infty}\left(\ln(n) - \tfrac{1}{z+1} - \tfrac{1}{z+2}\cdots - \tfrac{1}{z+n}\right) \end{aligned}$$

$$\varphi(z+1) = \frac{[\Gamma(z+1)]'}{\Gamma(z+1)}$$

Si sumamos y restamos un n'umero arm'onico

$$H_n = \sum_{m=1}^{n} \frac{1}{m}$$

$$\varphi(z+1) = \lim_{n\to\infty} \underbrace{[(\ln(n) - H_n)}_{=-\delta\ cuando\ n\to\infty} - \sum \left(\frac{1}{z+m} - \frac{1}{m} \right)$$

$$\Rightarrow \varphi(z+1) = -\delta + \sum_{m=1}^{\infty} \frac{z}{m(m+z)}, \quad si\ z{=}0 \Rightarrow \varphi(1) = -\delta$$

para $n > 0$, n entero.

$$\begin{aligned} \varphi(n+1) = &\quad -\delta + \textstyle\sum_{m=1}^{n} \frac{1}{m} \\ = &\quad -\delta + H_n \end{aligned}$$

Funci'on Poligamma

Cuando se deriva muchas veces la funci'on digamma

$$\begin{aligned} \varphi^{(m)}(z+1) \equiv &\quad \tfrac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z+1) \\ \varphi^{(m)}(z+1) = &\quad (-1)^{m+1} m! \textstyle\sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}}, \quad m = 1, 2, 3... \end{aligned}$$

Si z=0 la funci'on se conoce como zeta de Riemann

$$\zeta(m) = \sum_{n-1}^{\infty} \frac{1}{n^m}$$

$$\Rightarrow \varphi_{(1)}^{(m)} = (-1)^{m+1} \cdot m! \cdot \zeta(m+1), \quad m = 1, 2, 3...$$

Expansi'on de Maclaurin

$$\begin{aligned} \ln \Gamma(z+1) = &\quad \textstyle\sum_{n=1}^{\infty} \frac{z^n}{n!} \varphi_{(1)}^{(n-1)} \\ = &\quad -\delta \cdot z + \textstyle\sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n} \zeta(n) \end{aligned}$$

converge en $|z| < 1$ para $z = x \Rightarrow$ se puede calcular $\Gamma(z+1)$ para n'umeros reales o complejos.

Suma de series

Transformar la serie por medio de fracciones parciales y expresar la serie infinita como sumas finitas de funciones gamma y poligamma.

EJEMPLO:

$$\begin{aligned} \varphi^{(m)}(z+2) = &\quad \varphi^{(m)}(z+1) + (-1)^m \tfrac{m!}{(z+1)^{m+1}} \\ \varphi^{(m)}(z+2) - \varphi^{(m)}(z+1) = &\quad \tfrac{(-1)^m m!}{(z+1)^{m+1}} \end{aligned}$$

$$\tfrac{d^{(m)}}{dz^{(m)}} [\varphi(z+2) - \varphi(z+1)]$$

$$\begin{aligned} \varphi(z+2) - \varphi(z+1) = &\quad \textstyle\sum_{m=1}^{\infty} \frac{z+1}{m(m+z+1)} - \sum_{m=1}^{\infty} \frac{z}{m(m+z)} \\ = &\quad \textstyle\sum_{m=1}^{\infty} \left[\frac{1}{m} - \frac{1}{m+z+1} - \frac{1}{m} + \frac{1}{m+z} \right] \\ = &\quad \tfrac{1}{z+1} \end{aligned}$$

$$\frac{d^m}{dz^m} [\varphi(z+2) - \varphi(z+1)] = \frac{d^m}{dz^m} \left[\frac{1}{z+1} \right] = \frac{(-1)^m m!}{(z+1)^{m+1}}$$

Demuestre que:

$$\frac{1}{2} \ln \left[\frac{\pi \cdot z}{sen(\pi \cdot z)} \right] = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n} z^{2n}, \quad |z| < 1$$

- $\frac{1}{2} \ln \left[\underbrace{\Gamma(z)}_{z\Gamma(z)=\Gamma(z+1)} \quad \Gamma(z-1) \right] = \frac{1}{2} \ln \left[\frac{\pi}{sen(z\pi)} \right]$
- $\ln \Gamma(z+1) = \lim_{n\to\infty} [\ln(n!) + z\ln(n) - \ln(z+1) - ... - \ln(z+n)]$

$$\Gamma(z)\Gamma(1-z) \rightsquigarrow \Gamma(1+z)\Gamma(1-z)$$

$$\begin{aligned}
\ln\left[\Gamma(1+z)\Gamma(1-z)\right] = \quad & -\delta\cdot z + \textstyle\sum_{n=2}^{\infty}(-1)^n\frac{z^n}{n}\zeta(n) \\
& +(-\delta(-z)) + \textstyle\sum_{n=2}^{\infty}(-1)^n\frac{(-z)^n}{n}\zeta(n) \\
= \quad & \textstyle\sum_{n=2}^{\infty}\zeta(n)\underbrace{\left[\dfrac{z^n}{n}+\dfrac{(-z)^n}{n}\right]}_{z^{2n}} \\
= \quad & 2\textstyle\sum_{n=1}^{\infty}\cancel{(-1)^{2n}}\xrightarrow{} 1\,\zeta(2n)\frac{z^{2n}}{2n}
\end{aligned}$$

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La Funci'on Beta

$$\Gamma(p+q) \propto \Gamma(p)\Gamma(q)$$

$$n! = \Gamma(n+1)\& \int_0^\infty e^{-t}t^2dt, \quad \textit{Variante de gamma para definir la funi'on beta}$$

$$\Gamma(m)\Gamma(n) = \quad \int_0^\infty e^{-u}u^m fu \int_0^\infty e^{-v}v^n dv$$

$$\begin{array}{cc} u = x^2 & v = y^2 \\ du = 2xdx & dv = 2ydy \end{array}$$

$$\begin{aligned} \Gamma(m)\Gamma(n) = & \quad \int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}x^{2m}(2x)y^{2m}(2y)dxdy \\ = & \quad 4\int_0^\infty e^{-x^2}e^{-x^2}e^{-y^2}x^{2m+1}y^{2m+1}dxdy \end{aligned}$$

otro cambio de variables

$$x = rcos(\theta) \quad y = rsen(\theta)$$

Usando el jacobiano

$$\begin{vmatrix} \frac{\partial x}{\partial r} = cos\theta dr & \frac{\partial x}{\partial \theta} = -rsen\theta d\theta \\ & \\ \frac{\partial y}{\partial r} = sen\theta dr & \frac{\partial y}{\partial \theta} rcos\theta d\theta \end{vmatrix} = rdrd\theta$$

$$= \quad \int_0^\infty \int_0^{\pi/2} \cancelto{e^{-r^2cos^2\theta}}{e^{-r^2}} \quad r^{2m+2n+3}cos^{2m+1}\theta sen^{2n+1}drd\theta$$

$$Gamma(m)\Gamma(n) = \quad 2\Gamma(m+n+1)\int_0^{\pi/2} cos^{2m+1}\theta sen^{2n+1}\theta d\theta$$

$$\begin{aligned} \Gamma(z) = & \quad 2\int_0^\infty e^{-t^2}t^{2z-1}dt; \; Re(z>0) \\ \Gamma(q)\Gamma(p) = & \quad 4\int_0^\infty e^{-s^2}s^{2q-1}ds \int_0^\infty e^{-t^2}t^{2p-1}dt \end{aligned}$$

$$\begin{array}{cc} s = rcos\theta & t = rsen\theta \\ r^2 = s^2+t^2 & dsdt = rdrd\theta \end{array}$$

$$\begin{aligned} \Gamma(q)\Gamma(p) = & \quad 4\int\int e^{-r^2cos^2\theta}e^{-r^2sen^2\theta}r^{2q-1}cos^{2q-1}r^{2p-1}sen^{2p-1}rdrd\theta \\ = & \quad 2\cdot 2\int_0^\infty e^{-r^2}r^{2q+2p-1}cos^{2q-1}sen^{2p-1}drd\theta \\ \Gamma(p)\Gamma(q) = & \quad 2\int cos^{2q-1}sen^{2p-1}d\theta \end{aligned}$$

$$\frac{m!n!}{(m+n+1)!}\star = 2\int_0^{pi/2} cos^{2m+1}\theta sen^{2n+1}\theta d\theta$$

$$\boxed{b(p,q) = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}}$$ definici'on de funci'on beta

$$b(p,q) = b(q,p)$$

Si se sustituye $t = cos^2\theta$ y $dt = -2cos\theta sen\theta d\theta$

$$\begin{aligned} \cancelto{t}{cos^2\theta} + sen^2\theta = & \quad 1 \\ sen^2\theta = & \quad 1-t \\ sen^{2n+1}\theta = & \quad sen^{2n}\theta sen\theta \\ = & \quad (1-t)^n sen\theta \end{aligned} \qquad \begin{aligned} \star = & \quad 2\int_0^1 cos^{2m}\theta\cancel{cos\theta}(1-t)^n\cancel{sen\theta}dt \\ = & \quad \int_0^1 t^m(1-t)^n dt \Rightarrow \; \textit{otra forma de escribir} B(m+1,n+1) \end{aligned}$$