

# Perfect Epidemics

2nd Workshop, UK Research Network in Stochastics  
University of Liverpool

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Warwick, York

27 June 2025



# Introduction

“Once we came to accept the photographic image as reality, the way to its future simulation was open.”  
[Lev Manovich]



Handout is on the web: use the QR-code or visit  
[wilfridskendall.github.io/talks/PerfectEpidemics](https://wilfridskendall.github.io/talks/PerfectEpidemics).

Work on perfect simulation ([CFTP](#)) for epidemics, now being written up.  
WSK acknowledges the support of UK EPSRC grant EP/R022100.

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- ➍ Simplest possible example: *random-walk-CFTP*  
(can boost to use Ising model to do simple image reconstruction).

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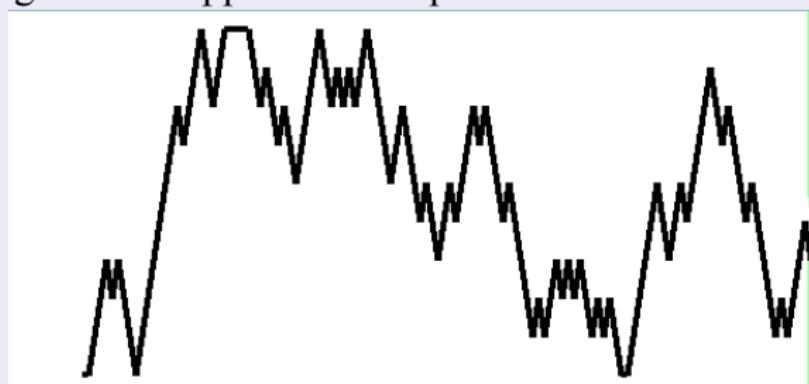
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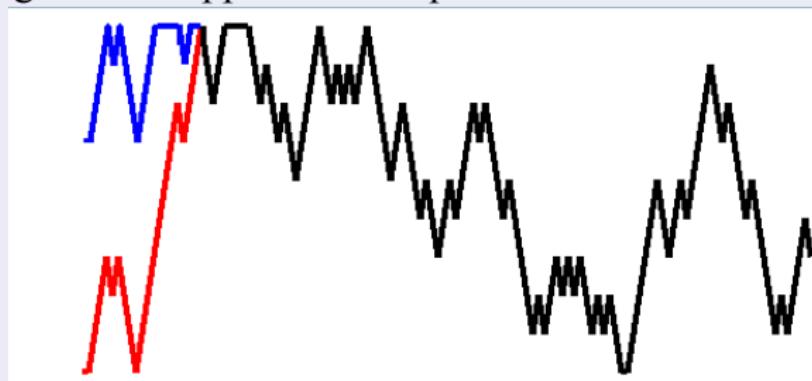
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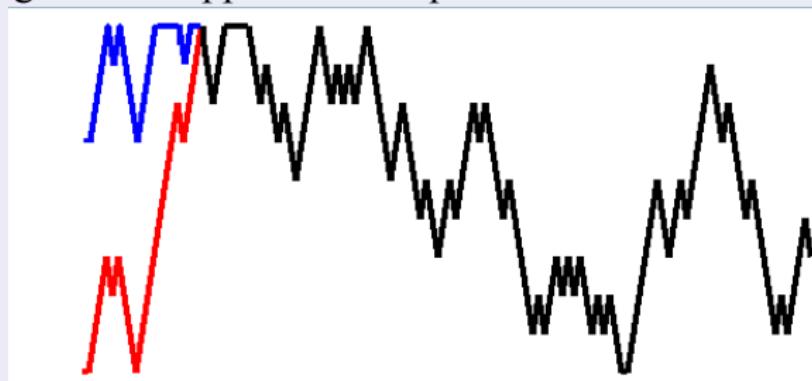
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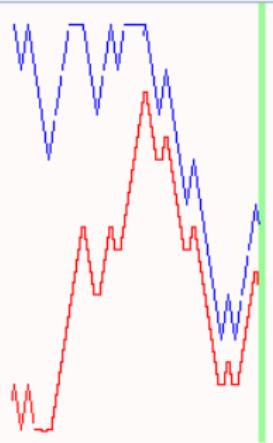
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- ④ Generally **not true** that location *at coupling* is a draw from equilibrium.

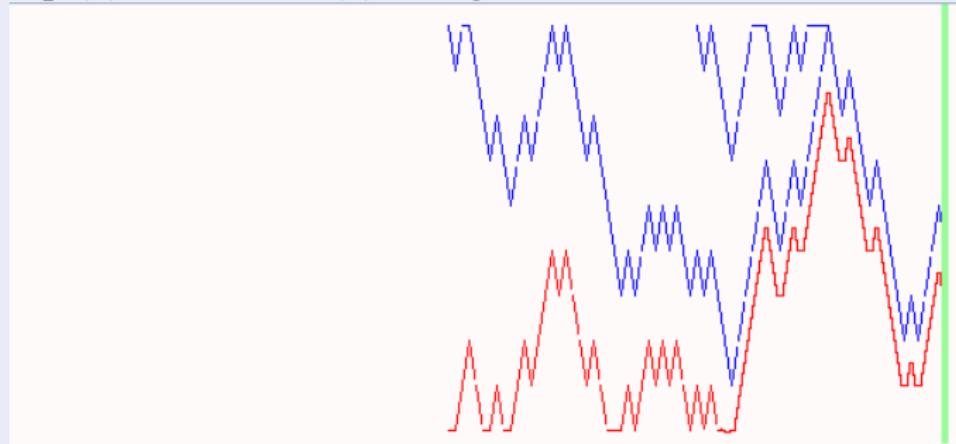
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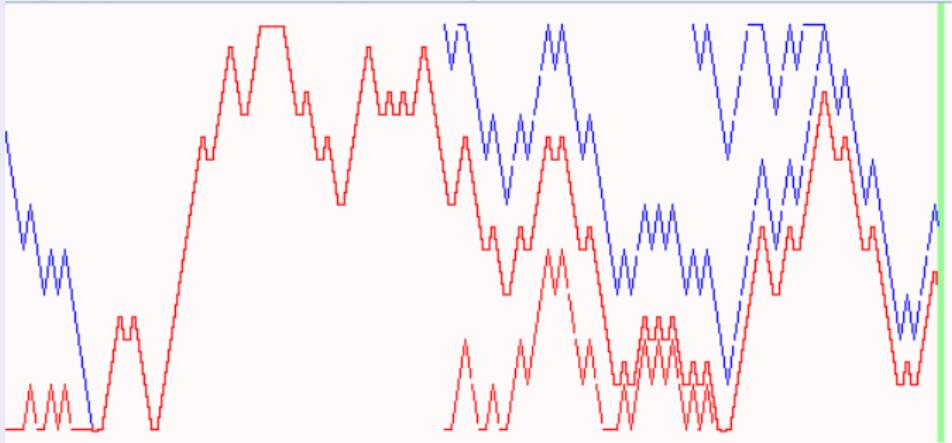


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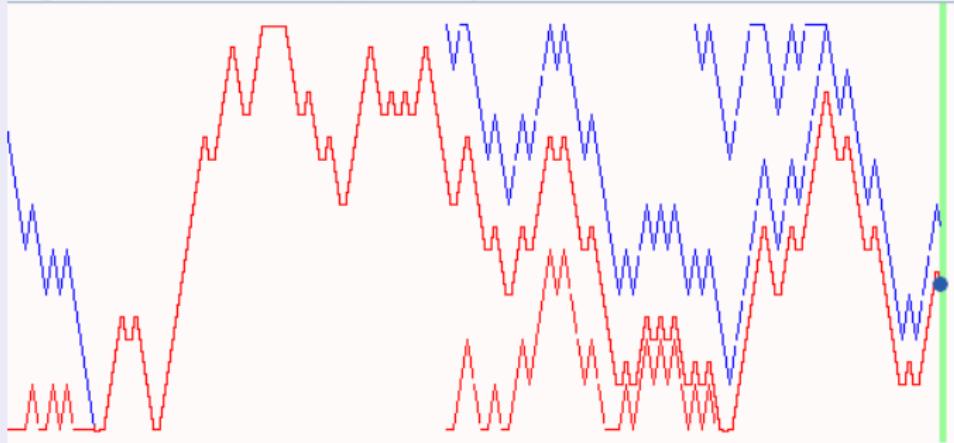
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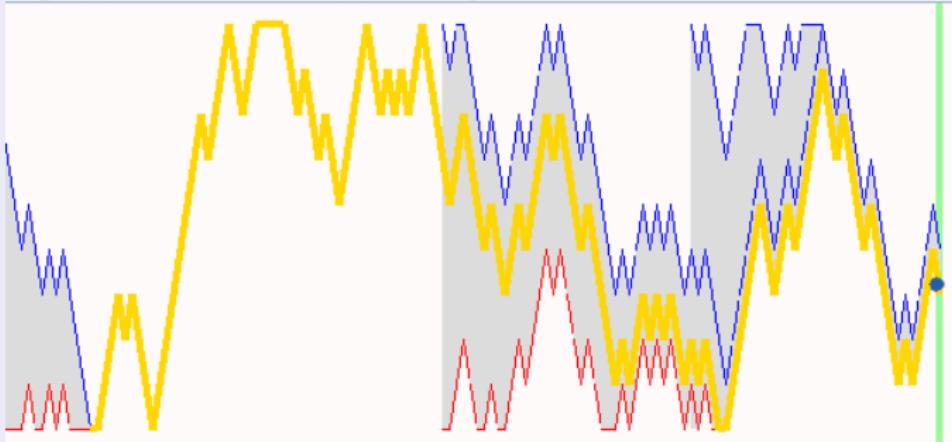
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- ⑤ The common value (golden thread) is an exact draw from equilibrium!

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- ④ Detailed expositions: WSK (2005), Huber (2015).  
(Want to implement CFTP in R? see WSK, 2015.)

### 3. Perfect Epidemics: a challenge problem for CFTP

S-I-R deterministic epidemic:

based on susceptibles  $s$ , infectives  $i$ , removals  $r$ :

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Both make an unrealistic assumption: homogeneous mixing.

In contrast, Fraser & Others (2023) use a UK model with  $10^6$  agents!

There are many important inferential questions (Cori & Kucharski, 2024).



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*Wikipedia*: “The British-registered *Diamond Princess* was the first cruise ship to have a major [COVID-19] outbreak on board, with the ship quarantined at Yokohama from 4 February 2020 for about a month. Of 3711 passengers and crew, around 700 people became infected and 9 people died.”

Evidently  $\alpha s_0 / \beta \gg 1$  – as was sadly later confirmed, a sorrow for us all.



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- ➎ Can we use **perfect simulation**?

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- ④ First step: evolve whole S-I-R trajectory in *algorithmic time* (alter potential infections and removals using immigration-death in discrete algorithmic time).
- ⑤ Result: *trajectory-valued chain*, unconditioned S-I-R as equilibrium.

# From incidents to unconditioned epidemic trajectories (1/3)

Incidents defining an epidemic

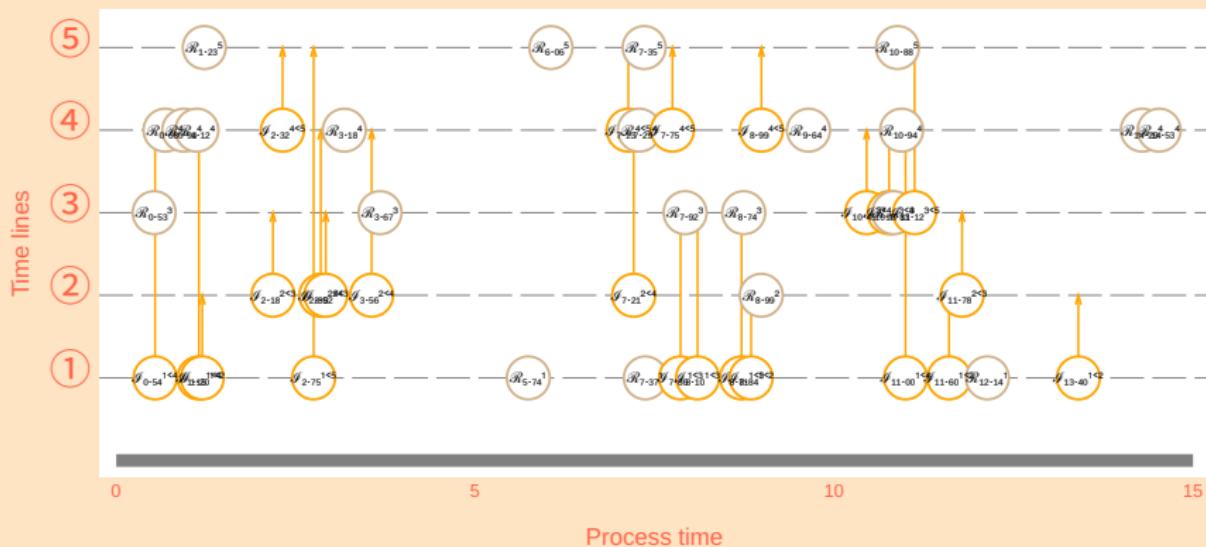


Figure 1: Light-orange circles denote potential infections (arrows point upwards to targets); light-brown circles denote potential removals.

## From incidents to unconditioned epidemic trajectories (2/3)

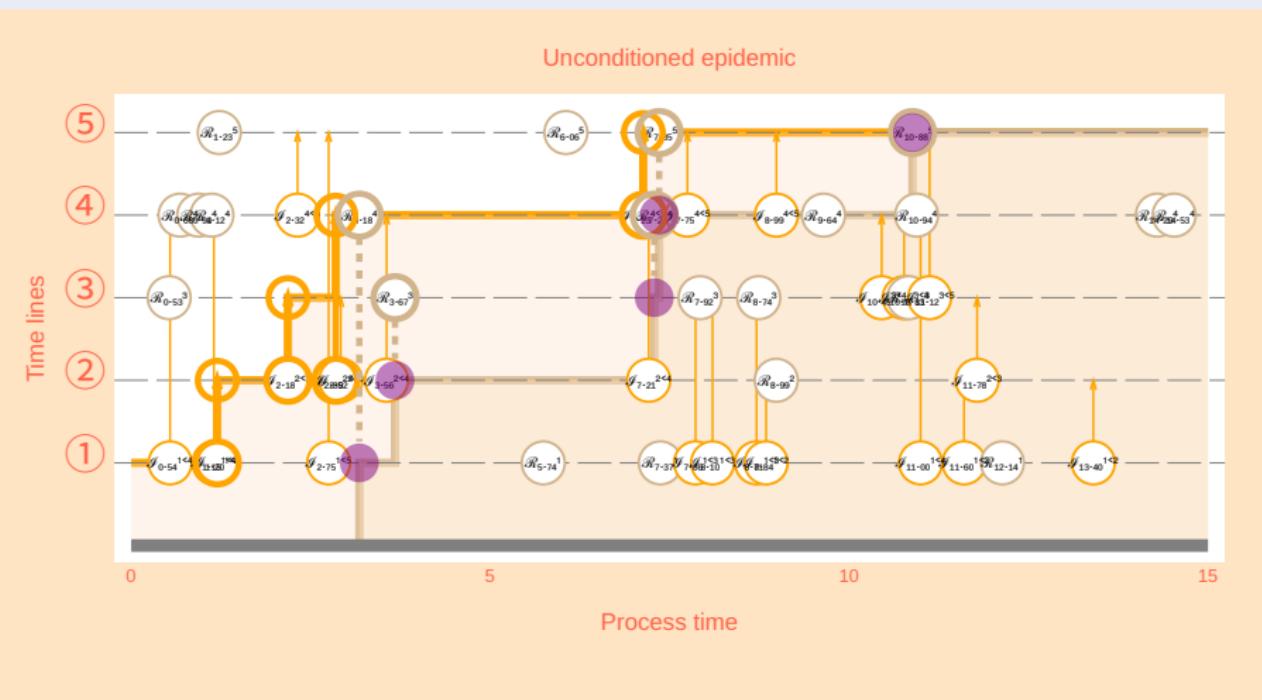


Figure 2: (a) *Infection* activates if target lies on the lowest uninfected timeline; and (b) *removal* activates if on infected timeline; remove lowest infected (purple disk).

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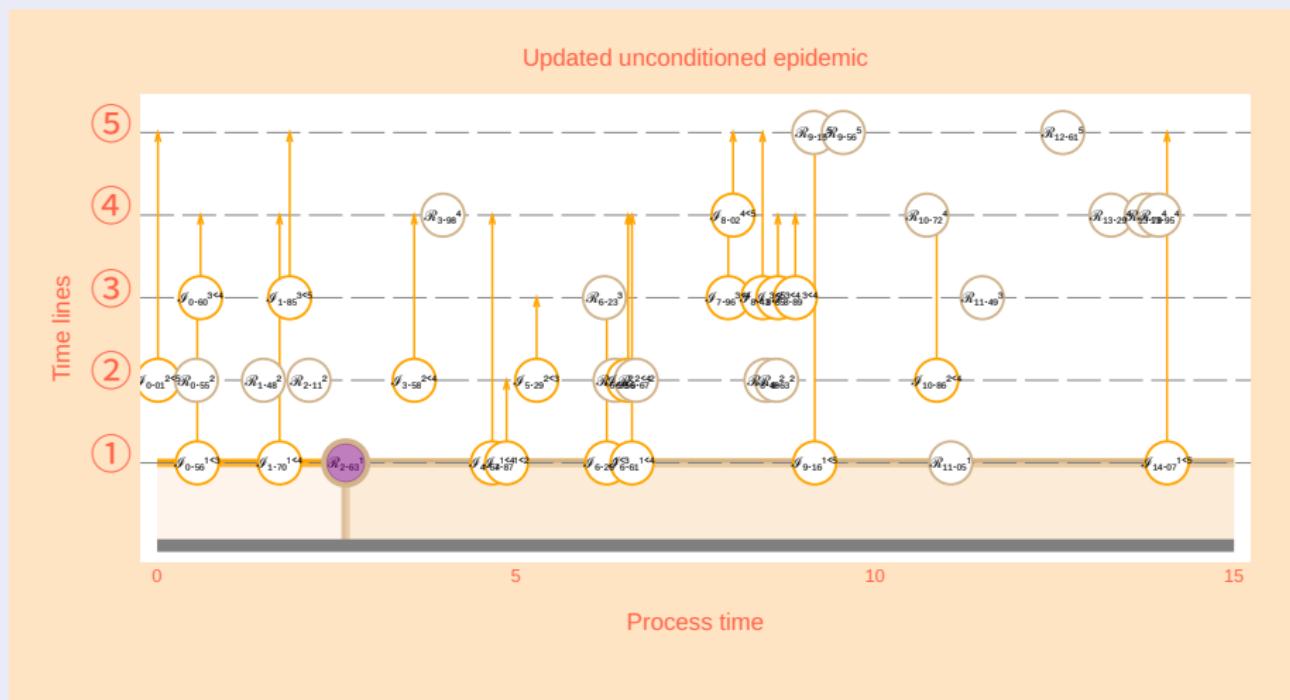


Figure 3: A step in algorithmic time for the unconditioned epidemic simply involves replacing all original incidents by an entirely new set of incidents.

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- Thus the original update is expressible as a (continuous) composition of updates, each of which satisfies detailed balance in equilibrium.
- The connection “restriction=conditioning” holds (**needs proof**).
- Crucially, re-sampling step 2 ensures composite evolution is irreducible over  $S$ ! (So equilibrium under conditioning is unique.)

# Free evolution evolving in continuous algorithmic time

GIF MP4

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- **Housekeeping details** used to establish that monotonicity still works: *laziest feasible epidemic (LFE)* and *no-fly zone (NFZ)*.

# Initial conditioned epidemic

The initial conditioned epidemic

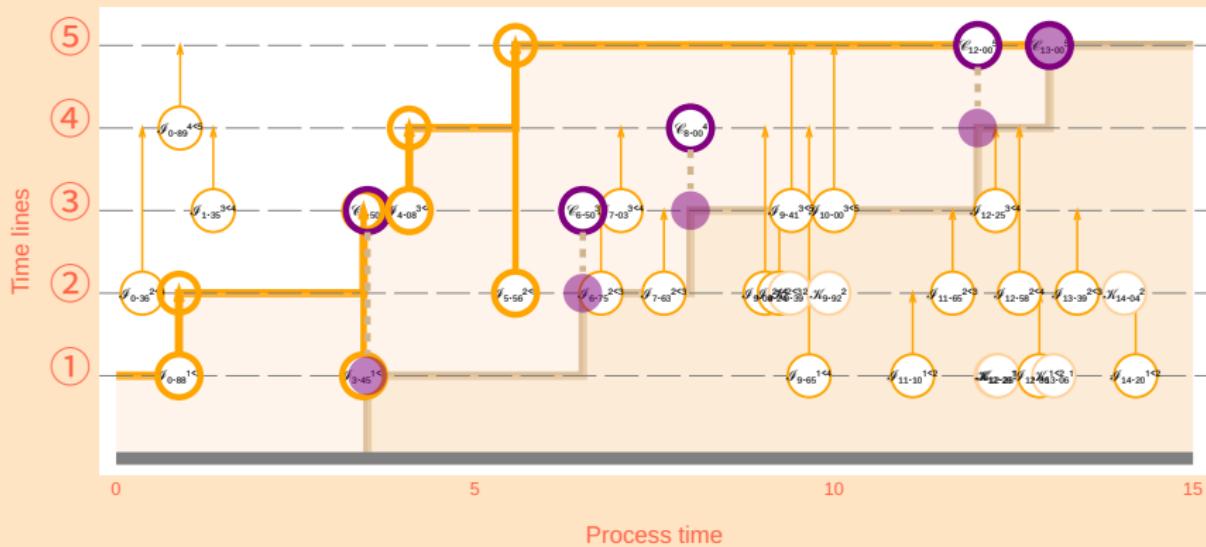


Figure 4: Initial conditioned epidemic, with conditioned removals indicated using purple circles (and purple disks when non-target timelines are infected).

# Conditional epidemic update

Fully updated conditioned epidemic

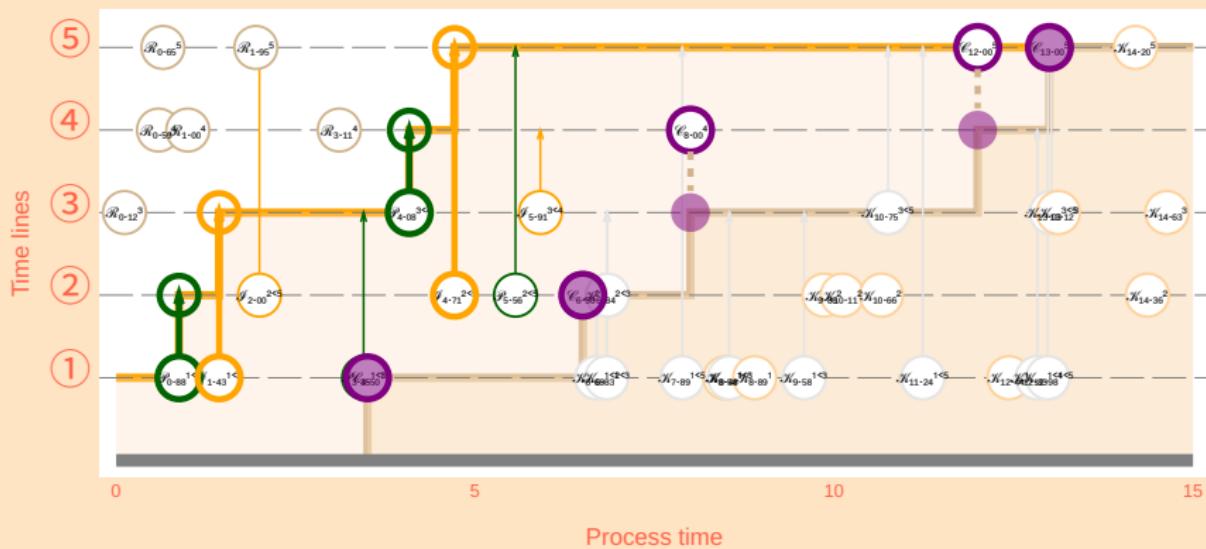


Figure 5: Epidemic updated under restriction: all conditioned removals remain activated, no new removals are activated. Green infections have been “perpetuated”.

# Laziest feasible epidemic (LFE)

Fully updated conditioned epidemic with LFE

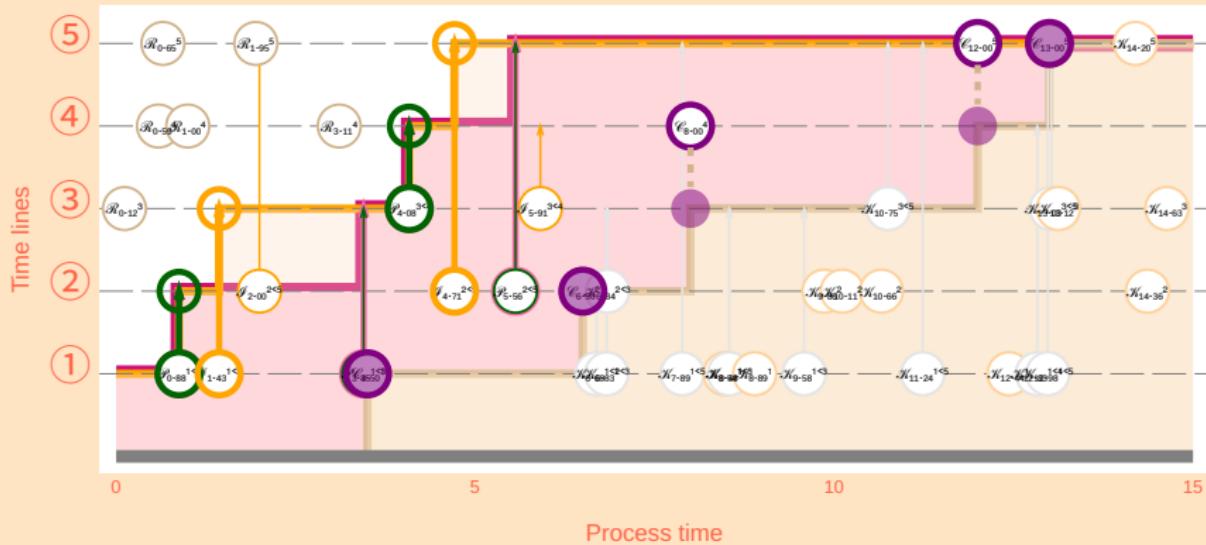


Figure 6: LFE computed recursively working right-to-left: the slowest sequence of infections deals with all infected timelines in order (includes perpetuated infections).

# No-fly zone (NFZ)

Fully updated conditioned epidemic with NFZ

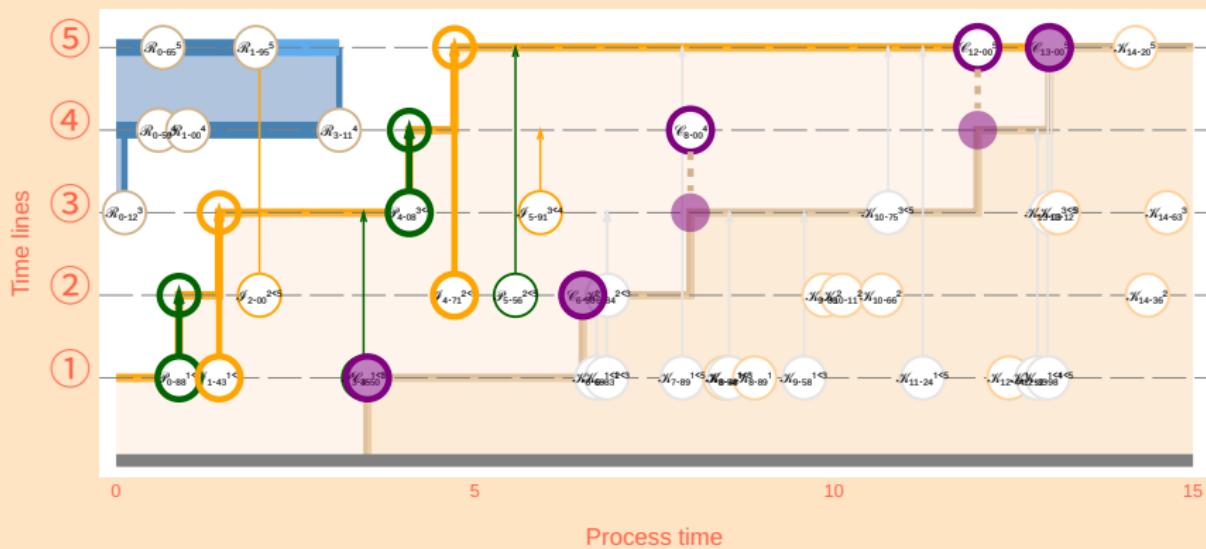


Figure 7: NFZ computed recursively working right-to-left: it traces the region of timelines that must not be infected if one is not to activate unobserved removals.

# Conditioned evolution evolving in continuous algorithmic time

GIF MP4

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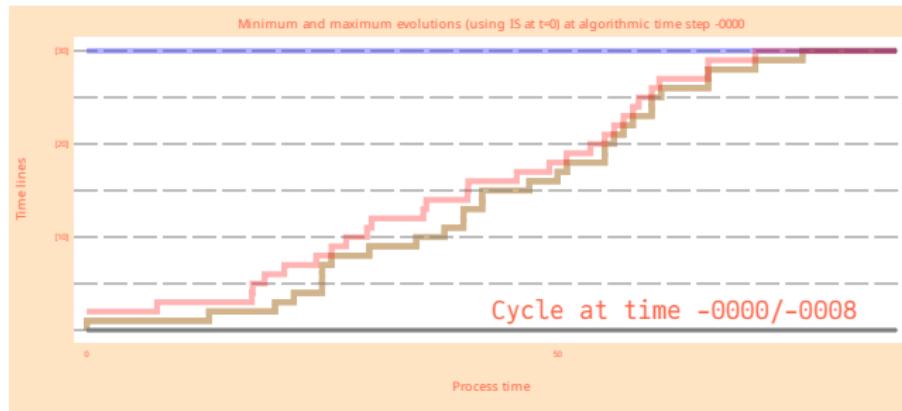
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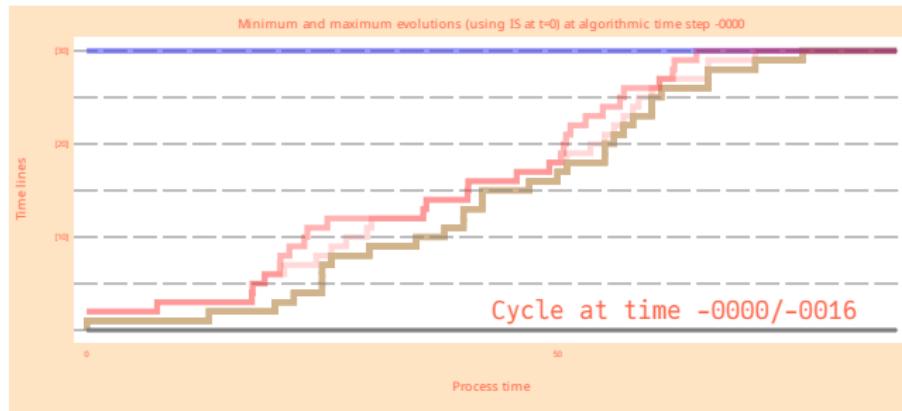
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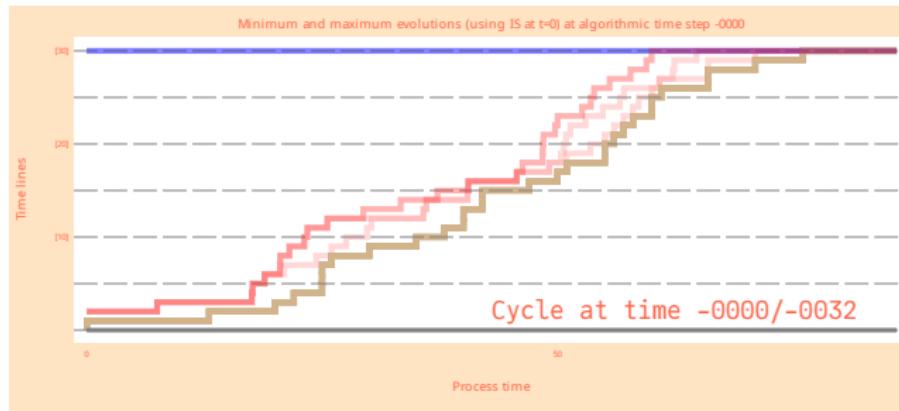
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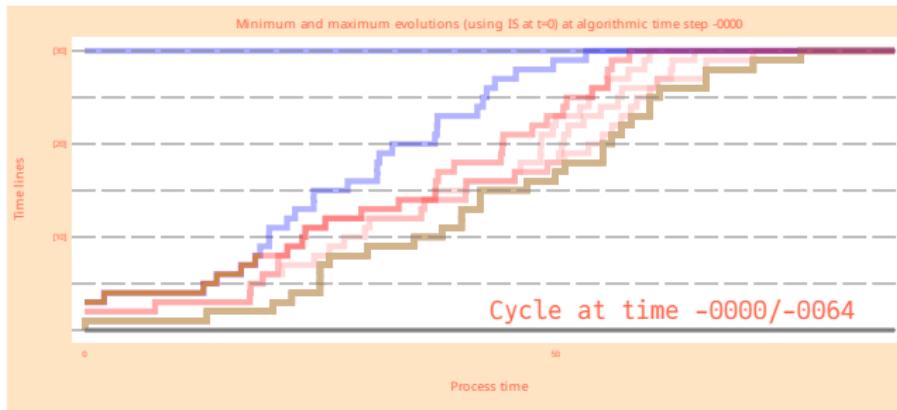
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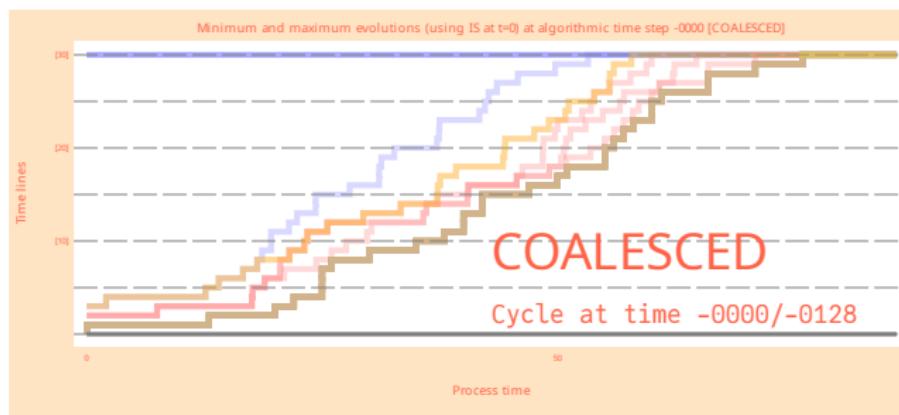
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- Thank you for your attention! **QUESTIONS?**



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## Image information

<i>Image</i>	<i>Attribution</i>	
<i>Terry Pratchett</i> Classic CFTP for a simple random walk	Luigi Novi Result of code written by WSK	<i>CC BY 3.0</i>
<i>Diamond Princess</i> Epidemic CFTP images and animation	Alpsdake Result of code written by WSK	<i>CC BY-SA 4.0</i>

## Previous instances of this talk

<i>Date</i>	<i>Title</i>		<i>Location</i>	
19/04/24	Perfect Epidemics	Short Research Talk	12mn	Warwick
15/05/24	McMC+Perfect Simulation	Graduate Seminar, Aristotle Univ.	50mn	Thessaloniki
17/01/25	Perfect Epidemics	Applied Probability Seminar	50mn	Warwick
27/06/25	Perfect Epidemics	UK Research Network Stochastics	45mn	Liverpool

# Appendix A: A “near-maximal” configuration

A small set for a conditioned epidemic.

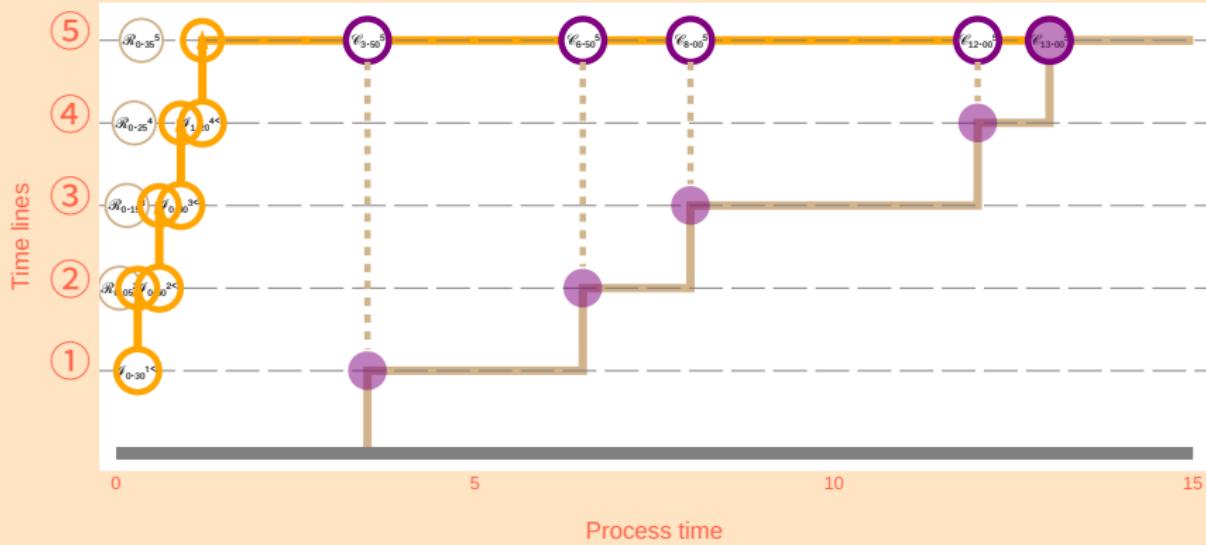


Figure 8: A conditioned epidemic in which all activated infections occur before time 3.0, also before smallest observed removal time.

## Appendix B: Notes towards a monotonicity proof

### Summary of plan of proof:

- ① Let  $EPI_{\tau=0}^{\pm}$  represent two epidemic trajectories ( $\pm$ ) at algorithmic time  $\tau=0$ , viewed as subsets of “timeline-space”  $\{(k, [0, T)) : k = 1, 2, \dots\}$ .

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- ② Suppose at algorithmic time  $\tau=0$  the *fast*  $\text{EPI}_{\tau=0}^+$  is never later than the *slow*  $\text{EPI}_{\tau=0}^-$  so  $\text{EPI}_{\tau=0}^+ \supseteq \text{EPI}_{\tau=0}^-$ ; additionally suppose monotonicity holds for conditional removal marks: if  $\mathcal{C}_{\tau=0}^{\pm}$  are conditional removals at fixed process time  $t$  then  $\mathcal{C}_{\tau=0}^+ \text{ timeline} \geq \mathcal{C}_{\tau=0}^- \text{ timeline}$ .

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- ③ Then a related monotonicity holds for the laziest feasible epidemics:  
 $\text{LFE}_{\tau=1}^+ \leq \text{LFE}_{\tau=1}^-$  at algorithmic time  $\tau=1$ .

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- ④ Likewise a similar monotonicity (but reversing the set-theoretic inclusion!) holds for no-fly zones:  $\text{NFZ}_{\tau=1}^+ \subseteq \text{NFZ}_{\tau=1}^-$ .

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- ④ Likewise a similar monotonicity (but reversing the set-theoretic inclusion!) holds for no-fly zones:  $\text{NFZ}_{\tau=1}^+ \subseteq \text{NFZ}_{\tau=1}^-$ .
- ⑤ Now prove  $\text{EPI}_{\tau=1}^+ \supseteq \text{EPI}_{\tau=1}^-$  moreover if  $\mathcal{C}_{\tau=1}^+$  matches  $\mathcal{C}_{\tau=1}^-$  at process time  $t$  then  $\mathcal{C}_{\tau=1}^+ \text{ timeline} \geq \mathcal{C}_{\tau=1}^- \text{ timeline}$ .

## $\text{LFE}_{\tau=1}$ : recursive construction

Let  $\text{LFE}_{\tau=1}^{\pm}(k)$  be the (process) time of the latest infection of timeline  $k$  needed if all  $\mathcal{C}^{\pm}$ s of  $\text{EPI}_{\tau=1}^{\pm}$  are to be infected.

- For the top timeline  $n$ ,  $\text{LFE}_{\tau=1}^{\pm}(n)$  must precede any  $\mathcal{C}^{\pm}$  on timeline  $n$ ; set  $\text{LFE}_{\tau=1}^{\pm}(n) = T$  if no such  $\mathcal{C}^{\pm}$ .

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- ② For  $k < n$  with LFE <sub>$\tau=1$</sub> <sup>±</sup>( $k+1$ ) =  $T$ , again LFE <sub>$\tau=1$</sub> <sup>±</sup>( $k$ ) must precede any  $\mathcal{C}^\pm$  on timeline  $k$ ; set LFE <sub>$\tau=1$</sub> <sup>±</sup>( $k$ ) =  $T$  if no such  $\mathcal{C}^\pm$ .

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- ③ Suppose  $n_0$  is largest  $k$  with  $\text{LFE}_{\tau=1}^{\pm}(k) < T$ . Working downwards through  $\ell = n_0 - 1, \dots, 1$ ,  $\text{LFE}_{\tau=1}^{\pm}(\ell)$  is the time of the latest infection targeting  $\ell + 1$  and based in the infected region such that

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- ③ Now work inductively. Suppose monotonicity holds for  $k+1, \dots, n$ . Then  $LFE_{\tau=1}^+(k) \leq LFE_{\tau=1}^+(k+1) \leq LFE_{\tau=1}^-(k+1)$ . But the set of “times of  $\mathcal{C}^-$  on timelines  $\ell, \ell+1, \dots, n$ ” is a subset of the set of “times of  $\mathcal{C}^+$  on timelines  $\ell, \ell+1, \dots, n$ ”. So if  $b_k^\pm$  is the resulting right-constraint on  $LFE_{\tau=1}^\pm(k)$  then  $b_k^+ \leq b_k^-$ .

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  - (a) no perpetuation occurs (use fact, all infections are shared);

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- ② In particular, if  $LFE_{\tau=1}^+(k) = T$  then  $LFE_{\tau=1}^-(k) = T$  also.
- ③ Now work inductively. Suppose monotonicity holds for  $k+1, \dots, n$ . Then  $LFE_{\tau=1}^+(k) \leq LFE_{\tau=1}^+(k+1) \leq LFE_{\tau=1}^-(k+1)$ . But the set of “times of  $\mathcal{C}^-$  on timelines  $\ell, \ell+1, \dots, n$ ” is a subset of the set of “times of  $\mathcal{C}^+$  on timelines  $\ell, \ell+1, \dots, n$ ”. So if  $b_k^\pm$  is the resulting right-constraint on  $LFE_{\tau=1}^\pm(k)$  then  $b_k^+ \leq b_k^-$ .
- ④ Suppose  $EPI_{\tau=0}^\pm$  infects timeline  $k+1$  at time  $a_k^\pm$ :  $a_k^+ \leq a_k^- \leq b_k^-$  by monotonicity for  $EPI_{\tau=0}^\pm$ . If no  $\tau=1$  infections infect timeline  $k+1$  in  $[a_k^\pm, b_k^\pm]$ , then  $LFE_{\tau=1}^\pm(k)$  perpetuates  $a_k^\pm$  using  $EPI_{\tau=0}^\pm(k)$ . Then argue case-by-case:
  - (a) no perpetuation occurs (use fact, all infections are shared);
  - (b)  $LFE_{\tau=1}^-(k)$  is perpetuated (so no useful infections after perpetuation);

## LFE <sub>$\tau=1$</sub> : monotonicity

- ① Re-sample  $\mathcal{C}^\pm$  timelines by accept-reject: same proposals for both  $\pm$ . As  $EPI_{\tau=0}^+ \supseteq EPI_{\tau=0}^-$ , so  $\mathcal{C}$  timelines for  $EPI_{\tau=1}^+$  no lower than for  $EPI_{\tau=1}^-$ .
- ② In particular, if  $LFE_{\tau=1}^+(k) = T$  then  $LFE_{\tau=1}^-(k) = T$  also.
- ③ Now work inductively. Suppose monotonicity holds for  $k+1, \dots, n$ . Then  $LFE_{\tau=1}^+(k) \leq LFE_{\tau=1}^+(k+1) \leq LFE_{\tau=1}^-(k+1)$ . But the set of “times of  $\mathcal{C}^-$  on timelines  $\ell, \ell+1, \dots, n$ ” is a subset of the set of “times of  $\mathcal{C}^+$  on timelines  $\ell, \ell+1, \dots, n$ ”. So if  $b_k^\pm$  is the resulting right-constraint on  $LFE_{\tau=1}^\pm(k)$  then  $b_k^+ \leq b_k^-$ .
- ④ Suppose  $EPI_{\tau=0}^\pm$  infects timeline  $k+1$  at time  $a_k^\pm$ :  $a_k^+ \leq a_k^- \leq b_k^-$  by monotonicity for  $EPI_{\tau=0}^\pm$ . If no  $\tau=1$  infections infect timeline  $k+1$  in  $[a_k^\pm, b_k^\pm]$ , then  $LFE_{\tau=1}^\pm(k)$  perpetuates  $a_k^\pm$  using  $EPI_{\tau=0}^\pm(k)$ . Then argue case-by-case:
  - (a) no perpetuation occurs (use fact, all infections are shared);
  - (b)  $LFE_{\tau=1}^-(k)$  is perpetuated (so no useful infections after perpetuation);
  - (c) only  $LFE_{\tau=1}^+(k)$  is perpetuated (then use  $\tau=0$  monotonicity).

## NFZ <sub>$\tau=1$</sub> : iterative construction

- ① Set  $\text{NFZ}_{\tau=1}^{\pm,*}$  to be union of regions  $(k, [0, t])$  for all  $\mathcal{R}$ s of  $\text{EPI}_{\tau=1}^{\pm}$ , for timeline  $k$  and time  $t$  of  $\mathcal{R}$ . Set  $\text{NFZ}_{\tau=1}^{\pm,*} = \{(k, [0, t_k^*]) : t_k^* > 0\}$ .

**NB:** ignore  $\mathcal{J}$  proposals targeting  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ : either these are rejected ((b) above) or  $\text{NFZ}_{\tau=1}^{\pm,j}$  is modified ((a) above) so they aren't relevant! *Relevant* accepted  $\mathcal{J}$ s are exactly those *not* targeting the final  $\text{NFZ}_{\tau=1}^{\pm}$ .

## NFZ $_{\tau=1}$ : iterative construction

- ① Set  $\text{NFZ}_{\tau=1}^{\pm,*}$  to be union of regions  $(k, [0, t])$  for all  $\mathcal{R}$ s of  $\text{EPI}_{\tau=1}^{\pm}$ , for timeline  $k$  and time  $t$  of  $\mathcal{R}$ . Set  $\text{NFZ}_{\tau=1}^{\pm,*} = \{(k, [0, t_k^*]) : t_k^* > 0\}$ .
- ② Set  $\text{NFZ}_{\tau=1}^{\pm,0} = \{(k, [0, t_k]) : t_k > 0\}$  as monotonic envelope of  $\text{NFZ}_{\tau=1}^{\pm,*}$ :  $\{t_k\}$  is smallest non-decreasing sequence majorizing  $\{t_k^*\}$ .

**NB:** ignore  $\mathcal{J}$  proposals targeting  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ : either these are rejected ((b) above) or  $\text{NFZ}_{\tau=1}^{\pm,j}$  is modified ((a) above) so they aren't relevant! *Relevant* accepted  $\mathcal{J}$ s are exactly those *not* targeting the final  $\text{NFZ}_{\tau=1}^{\pm}$ .

## NFZ $_{\tau=1}$ : iterative construction

- ① Set  $\text{NFZ}_{\tau=1}^{\pm,*}$  to be union of regions  $(k, [0, t])$  for all  $\mathcal{R}$ s of  $\text{EPI}_{\tau=1}^{\pm}$ , for timeline  $k$  and time  $t$  of  $\mathcal{R}$ . Set  $\text{NFZ}_{\tau=1}^{\pm,*} = \{(k, [0, t_k^*]) : t_k^* > 0\}$ .
- ② Set  $\text{NFZ}_{\tau=1}^{\pm,0} = \{(k, [0, t_k]) : t_k > 0\}$  as monotonic envelope of  $\text{NFZ}_{\tau=1}^{\pm,*}$ :  $\{t_k\}$  is smallest non-decreasing sequence majorizing  $\{t_k^*\}$ .
- ③ Work backwards through new “*non-removed*”  $\mathcal{J}$ s. At step  $j$ , time  $t$ ,

**NB:** ignore  $\mathcal{J}$  proposals targeting  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ : either these are rejected ((b) above) or  $\text{NFZ}_{\tau=1}^{\pm,j}$  is modified ((a) above) so they aren’t relevant! *Relevant* accepted  $\mathcal{J}$ s are exactly those *not* targeting the final  $\text{NFZ}_{\tau=1}^{\pm}$ .

## NFZ <sub>$\tau=1$</sub> : iterative construction

- ① Set  $\text{NFZ}_{\tau=1}^{\pm,*}$  to be union of regions  $(k, [0, t])$  for all  $\mathcal{R}$ s of  $\text{EPI}_{\tau=1}^{\pm}$ , for timeline  $k$  and time  $t$  of  $\mathcal{R}$ . Set  $\text{NFZ}_{\tau=1}^{\pm,*} = \{(k, [0, t_k^*]) : t_k^* > 0\}$ .
- ② Set  $\text{NFZ}_{\tau=1}^{\pm,0} = \{(k, [0, t_k]) : t_k > 0\}$  as monotonic envelope of  $\text{NFZ}_{\tau=1}^{\pm,*}$ :  $\{t_k\}$  is smallest non-decreasing sequence majorizing  $\{t_k^*\}$ .
- ③ Work backwards through new “*non-removed*”  $\mathcal{J}$ s. At step  $j$ , time  $t$ ,
  - ⓐ accept  $\mathcal{J}$  if it targets  $\text{NFZ}_{\tau=1}^{\pm,j-1}$  at timeline  $k$  but infection fails: set  $\text{NFZ}_{\tau=1}^{\pm,j} = \text{NFZ}_{\tau=1}^{\pm,j-1} \cup \{(k-1, [0, t])\}$ ;  
otherwise set  $\text{NFZ}_{\tau=1}^{\pm,j} = \text{NFZ}_{\tau=1}^{\pm,j-1}$  and

**NB:** ignore  $\mathcal{J}$  proposals targeting  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ : either these are rejected ((b) above) or  $\text{NFZ}_{\tau=1}^{\pm,j}$  is modified ((a) above) so they aren’t relevant! *Relevant* accepted  $\mathcal{J}$ s are exactly those *not* targeting the final  $\text{NFZ}_{\tau=1}^{\pm}$ .

## NFZ $_{\tau=1}$ : iterative construction

- ① Set  $\text{NFZ}_{\tau=1}^{\pm,*}$  to be union of regions  $(k, [0, t])$  for all  $\mathcal{R}$ s of  $\text{EPI}_{\tau=1}^{\pm}$ , for timeline  $k$  and time  $t$  of  $\mathcal{R}$ . Set  $\text{NFZ}_{\tau=1}^{\pm,*} = \{(k, [0, t_k^*]) : t_k^* > 0\}$ .
- ② Set  $\text{NFZ}_{\tau=1}^{\pm,0} = \{(k, [0, t_k]) : t_k > 0\}$  as monotonic envelope of  $\text{NFZ}_{\tau=1}^{\pm,*}$ :  $\{t_k\}$  is smallest non-decreasing sequence majorizing  $\{t_k^*\}$ .
- ③ Work backwards through new “*non-removed*”  $\mathcal{J}$ s. At step  $j$ , time  $t$ ,
  - ④a) **accept**  $\mathcal{J}$  if it targets  $\text{NFZ}_{\tau=1}^{\pm,j-1}$  at timeline  $k$  but infection fails: set  $\text{NFZ}_{\tau=1}^{\pm,j} = \text{NFZ}_{\tau=1}^{\pm,j-1} \cup \{(k-1, [0, t])\}$ ;  
otherwise set  $\text{NFZ}_{\tau=1}^{\pm,j} = \text{NFZ}_{\tau=1}^{\pm,j-1}$  and
  - ④b) **reject**  $\mathcal{J}$  if it would infect part of  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ ;

**NB:** ignore  $\mathcal{J}$  proposals targeting  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ : either these are rejected ((b) above) or  $\text{NFZ}_{\tau=1}^{\pm,j}$  is modified ((a) above) so they aren’t relevant! *Relevant* accepted  $\mathcal{J}$ s are exactly those *not* targeting the final  $\text{NFZ}_{\tau=1}^{\pm}$ .

## NFZ <sub>$\tau=1$</sub> : iterative construction

- ① Set  $\text{NFZ}_{\tau=1}^{\pm,*}$  to be union of regions  $(k, [0, t])$  for all  $\mathcal{R}$ s of  $\text{EPI}_{\tau=1}^{\pm}$ , for timeline  $k$  and time  $t$  of  $\mathcal{R}$ . Set  $\text{NFZ}_{\tau=1}^{\pm,*} = \{(k, [0, t_k^*]) : t_k^* > 0\}$ .
- ② Set  $\text{NFZ}_{\tau=1}^{\pm,0} = \{(k, [0, t_k]) : t_k > 0\}$  as monotonic envelope of  $\text{NFZ}_{\tau=1}^{\pm,*}$ :  $\{t_k\}$  is smallest non-decreasing sequence majorizing  $\{t_k^*\}$ .
- ③ Work backwards through new “*non-removed*”  $\mathcal{J}$ s. At step  $j$ , time  $t$ ,
  - ④(a) **accept**  $\mathcal{J}$  if it targets  $\text{NFZ}_{\tau=1}^{\pm,j-1}$  at timeline  $k$  but infection fails: set  $\text{NFZ}_{\tau=1}^{\pm,j} = \text{NFZ}_{\tau=1}^{\pm,j-1} \cup \{(k-1, [0, t])\}$ ;  
otherwise set  $\text{NFZ}_{\tau=1}^{\pm,j} = \text{NFZ}_{\tau=1}^{\pm,j-1}$  and
  - ④(b) **reject**  $\mathcal{J}$  if it would infect part of  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ ;
  - ④(c) **accept**  $\mathcal{J}$  if it doesn’t target  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ .

**NB:** ignore  $\mathcal{J}$  proposals targeting  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ : either these are rejected ((b) above) or  $\text{NFZ}_{\tau=1}^{\pm,j}$  is modified ((a) above) so they aren’t relevant! *Relevant* accepted  $\mathcal{J}$ s are exactly those *not* targeting the final  $\text{NFZ}_{\tau=1}^{\pm}$ .

## NFZ $_{\tau=1}$ : iterative construction

- ① Set  $\text{NFZ}_{\tau=1}^{\pm,*}$  to be union of regions  $(k, [0, t])$  for all  $\mathcal{R}$ s of  $\text{EPI}_{\tau=1}^{\pm}$ , for timeline  $k$  and time  $t$  of  $\mathcal{R}$ . Set  $\text{NFZ}_{\tau=1}^{\pm,*} = \{(k, [0, t_k^*]) : t_k^* > 0\}$ .
- ② Set  $\text{NFZ}_{\tau=1}^{\pm,0} = \{(k, [0, t_k]) : t_k > 0\}$  as monotonic envelope of  $\text{NFZ}_{\tau=1}^{\pm,*}$ :  $\{t_k\}$  is smallest non-decreasing sequence majorizing  $\{t_k^*\}$ .
- ③ Work backwards through new “non-removed”  $\mathcal{J}$ s. At step  $j$ , time  $t$ ,
  - ④ accept  $\mathcal{J}$  if it targets  $\text{NFZ}_{\tau=1}^{\pm,j-1}$  at timeline  $k$  but infection fails: set  $\text{NFZ}_{\tau=1}^{\pm,j} = \text{NFZ}_{\tau=1}^{\pm,j-1} \cup \{(k-1, [0, t])\}$ ; otherwise set  $\text{NFZ}_{\tau=1}^{\pm,j} = \text{NFZ}_{\tau=1}^{\pm,j-1}$  and
  - ⑤ reject  $\mathcal{J}$  if it would infect part of  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ ;
  - ⑥ accept  $\mathcal{J}$  if it doesn’t target  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ .
- ④ Set  $\text{NFZ}_{\tau=1}^{\pm} = \text{NFZ}_{\tau=1}^{\pm,j}$  if a total of  $j$  new  $\mathcal{J}$ s are proposed for  $\text{EPI}_{\tau=1}^{\pm}$ , so no more  $\mathcal{J}$ s remain!

**NB:** ignore  $\mathcal{J}$  proposals targeting  $\text{NFZ}_{\tau=1}^{\pm,j-1}$ : either these are rejected ((b) above) or  $\text{NFZ}_{\tau=1}^{\pm,j}$  is modified ((a) above) so they aren’t relevant! *Relevant* accepted  $\mathcal{J}$ s are exactly those *not* targeting the final  $\text{NFZ}_{\tau=1}^{\pm}$ .

## NFZ <sub>$\tau=1$</sub> : monotonicity

Establish monotonicity for  $\text{NFZ}_{\tau=1}^{\pm,*}$ ,  $\text{NFZ}_{\tau=1}^{\pm,0}$ ,  $\text{NFZ}_{\tau=1}^{\pm,1}$ ,  $\text{NFZ}_{\tau=1}^{\pm,2}$ , ... in turn:

- ① Since  $\text{EPI}_{\tau=0}^+ \supseteq \text{EPI}_{\tau=0}^-$  and the set of  $\mathcal{R}$ s for  $\text{EPI}_{\tau=1}^\pm$  are formed by intersecting the same  $\mathcal{R}$  pattern with the complements of  $\text{EPI}_{\tau=0}^\pm$ , it follows that  $\text{NFZ}_{\tau=1}^{+,*} \subseteq \text{NFZ}_{\tau=1}^{-,*}$ .

## NFZ <sub>$\tau=1$</sub> : monotonicity

Establish monotonicity for  $\text{NFZ}_{\tau=1}^{\pm,*}$ ,  $\text{NFZ}_{\tau=1}^{\pm,0}$ ,  $\text{NFZ}_{\tau=1}^{\pm,1}$ ,  $\text{NFZ}_{\tau=1}^{\pm,2}$ , ... in turn:

- ① Since  $\text{EPI}_{\tau=0}^+ \supseteq \text{EPI}_{\tau=0}^-$  and the set of  $\mathcal{R}$ s for  $\text{EPI}_{\tau=1}^\pm$  are formed by intersecting the same  $\mathcal{R}$  pattern with the complements of  $\text{EPI}_{\tau=0}^\pm$ , it follows that  $\text{NFZ}_{\tau=1}^{+,*} \subseteq \text{NFZ}_{\tau=1}^{-,*}$ .
- ② Monotonicity for  $\text{NFZ}_{\tau=1}^{\pm,0}$  is a direct consequence.

## NFZ <sub>$\tau=1$</sub> : monotonicity

Establish monotonicity for  $\text{NFZ}_{\tau=1}^{\pm,*}$ ,  $\text{NFZ}_{\tau=1}^{\pm,0}$ ,  $\text{NFZ}_{\tau=1}^{\pm,1}$ ,  $\text{NFZ}_{\tau=1}^{\pm,2}$ , ... in turn:

- ① Since  $\text{EPI}_{\tau=0}^+ \supseteq \text{EPI}_{\tau=0}^-$  and the set of  $\mathcal{R}$ s for  $\text{EPI}_{\tau=1}^\pm$  are formed by intersecting the same  $\mathcal{R}$  pattern with the complements of  $\text{EPI}_{\tau=0}^\pm$ , it follows that  $\text{NFZ}_{\tau=1}^{+,*} \subseteq \text{NFZ}_{\tau=1}^{-,*}$ .
- ② Monotonicity for  $\text{NFZ}_{\tau=1}^{\pm,0}$  is a direct consequence.
- ③ Given  $\text{NFZ}_{\tau=1}^{+,j-1} \subseteq \text{NFZ}_{\tau=1}^{-,j-1}$ , create  $\text{NFZ}_{\tau=1}^{\pm,j}$  by proposing  $\mathcal{J}$  at time  $t$  targeting timeline  $k$ , based in  $\text{EPI}_{\tau=0}^+$  infected region. Then  $\text{NFZ}_{\tau=1}^{+,j} = \text{NFZ}_{\tau=1}^{+,j-1} \cup \{(k-1, [0, t])\}$  exactly when  $\mathcal{J}$  fails to infect in  $\text{EPI}_{\tau=1}^+$ . Then

## NFZ <sub>$\tau=1$</sub> : monotonicity

Establish monotonicity for NFZ <sub>$\tau=1$</sub> <sup>±,\*</sup>, NFZ <sub>$\tau=1$</sub> <sup>±,0</sup>, NFZ <sub>$\tau=1$</sub> <sup>±,1</sup>, NFZ <sub>$\tau=1$</sub> <sup>±,2</sup>, ... in turn:

- ① Since EPI <sub>$\tau=0$</sub> <sup>+</sup>  $\supseteq$  EPI <sub>$\tau=0$</sub> <sup>-</sup> and the set of  $\mathcal{R}$ s for EPI <sub>$\tau=1$</sub> <sup>±</sup> are formed by intersecting the same  $\mathcal{R}$  pattern with the complements of EPI <sub>$\tau=0$</sub> <sup>±</sup>, it follows that NFZ <sub>$\tau=1$</sub> <sup>+,\*</sup>  $\subseteq$  NFZ <sub>$\tau=1$</sub> <sup>-,\*</sup>.
- ② Monotonicity for NFZ <sub>$\tau=1$</sub> <sup>±,0</sup> is a direct consequence.
- ③ Given NFZ <sub>$\tau=1$</sub> <sup>+,j-1</sup>  $\subseteq$  NFZ <sub>$\tau=1$</sub> <sup>-,j-1</sup>, create NFZ <sub>$\tau=1$</sub> <sup>±,j</sup> by proposing  $\mathcal{J}$  at time  $t$  targeting timeline  $k$ , based in EPI <sub>$\tau=0$</sub> <sup>+</sup> infected region. Then  $\text{NFZ}_{\tau=1}^{+,j} = \text{NFZ}_{\tau=1}^{+,j-1} \cup \{(k-1, [0, t])\}$  exactly when  $\mathcal{J}$  fails to infect in EPI <sub>$\tau=1$</sub> <sup>+</sup>. Then
  - ⓐ we know  $k$  timeline at  $t$  is in NFZ <sub>$\tau=1$</sub> <sup>-,j-1</sup>  $\supseteq$  NFZ <sub>$\tau=1$</sub> <sup>+,j-1</sup>;

## NFZ <sub>$\tau=1$</sub> : monotonicity

Establish monotonicity for NFZ <sub>$\tau=1$</sub> <sup>±,\*</sup>, NFZ <sub>$\tau=1$</sub> <sup>±,0</sup>, NFZ <sub>$\tau=1$</sub> <sup>±,1</sup>, NFZ <sub>$\tau=1$</sub> <sup>±,2</sup>, ... in turn:

- ① Since EPI <sub>$\tau=0$</sub> <sup>+</sup>  $\supseteq$  EPI <sub>$\tau=0$</sub> <sup>-</sup> and the set of  $\mathcal{R}$ s for EPI <sub>$\tau=1$</sub> <sup>±</sup> are formed by intersecting the same  $\mathcal{R}$  pattern with the complements of EPI <sub>$\tau=0$</sub> <sup>±</sup>, it follows that NFZ <sub>$\tau=1$</sub> <sup>+,\*</sup>  $\subseteq$  NFZ <sub>$\tau=1$</sub> <sup>-,\*</sup>.
- ② Monotonicity for NFZ <sub>$\tau=1$</sub> <sup>±,0</sup> is a direct consequence.
- ③ Given NFZ <sub>$\tau=1$</sub> <sup>+,j-1</sup>  $\subseteq$  NFZ <sub>$\tau=1$</sub> <sup>-,j-1</sup>, create NFZ <sub>$\tau=1$</sub> <sup>±,j</sup> by proposing  $\mathcal{J}$  at time  $t$  targeting timeline  $k$ , based in EPI <sub>$\tau=0$</sub> <sup>+</sup> infected region. Then  
 $NFZ_{\tau=1}^{+,j} = NFZ_{\tau=1}^{+,j-1} \cup \{(k-1, [0, t])\}$  exactly when  $\mathcal{J}$  fails to infect in EPI <sub>$\tau=1$</sub> <sup>+</sup>. Then
  - (a) we know  $k$  timeline at  $t$  is in NFZ <sub>$\tau=1$</sub> <sup>-,j-1</sup>  $\supseteq$  NFZ <sub>$\tau=1$</sub> <sup>+,j-1</sup>;
  - (b) infection fails for EPI <sub>$\tau=1$</sub> <sup>+</sup> because timeline  $k-1$  is not infected at  $t$  in EPI <sub>$\tau=0$</sub> <sup>+</sup>. But we know EPI <sub>$\tau=0$</sub> <sup>+</sup>  $\supseteq$  EPI <sub>$\tau=0$</sub> <sup>-</sup>, so timeline  $k-1$  is not infected at  $t$  in EPI <sub>$\tau=0$</sub> <sup>-</sup> either. So infection in EPI <sub>$\tau=1$</sub> <sup>-</sup> also fails.

## NFZ <sub>$\tau=1$</sub> : monotonicity

Establish monotonicity for NFZ <sub>$\tau=1$</sub> <sup>±,\*</sup>, NFZ <sub>$\tau=1$</sub> <sup>±,0</sup>, NFZ <sub>$\tau=1$</sub> <sup>±,1</sup>, NFZ <sub>$\tau=1$</sub> <sup>±,2</sup>, ... in turn:

- ① Since EPI <sub>$\tau=0$</sub> <sup>+</sup>  $\supseteq$  EPI <sub>$\tau=0$</sub> <sup>-</sup> and the set of  $\mathcal{R}$ s for EPI <sub>$\tau=1$</sub> <sup>±</sup> are formed by intersecting the same  $\mathcal{R}$  pattern with the complements of EPI <sub>$\tau=0$</sub> <sup>±</sup>, it follows that NFZ <sub>$\tau=1$</sub> <sup>+,\*</sup>  $\subseteq$  NFZ <sub>$\tau=1$</sub> <sup>-,\*</sup>.
- ② Monotonicity for NFZ <sub>$\tau=1$</sub> <sup>±,0</sup> is a direct consequence.
- ③ Given NFZ <sub>$\tau=1$</sub> <sup>+,j-1</sup>  $\subseteq$  NFZ <sub>$\tau=1$</sub> <sup>-,j-1</sup>, create NFZ <sub>$\tau=1$</sub> <sup>±,j</sup> by proposing  $\mathcal{J}$  at time  $t$  targeting timeline  $k$ , based in EPI <sub>$\tau=0$</sub> <sup>+</sup> infected region. Then  
 $NFZ_{\tau=1}^{+,j} = NFZ_{\tau=1}^{+,j-1} \cup \{(k-1, [0, t])\}$  exactly when  $\mathcal{J}$  fails to infect in EPI <sub>$\tau=1$</sub> <sup>+</sup>. Then

- (a) we know  $k$  timeline at  $t$  is in NFZ <sub>$\tau=1$</sub> <sup>-,j-1</sup>  $\supseteq$  NFZ <sub>$\tau=1$</sub> <sup>+,j-1</sup>;
- (b) infection fails for EPI <sub>$\tau=1$</sub> <sup>+</sup> because timeline  $k-1$  is not infected at  $t$  in EPI <sub>$\tau=0$</sub> <sup>+</sup>. But we know EPI <sub>$\tau=0$</sub> <sup>+</sup>  $\supseteq$  EPI <sub>$\tau=0$</sub> <sup>-</sup>, so timeline  $k-1$  is not infected at  $t$  in EPI <sub>$\tau=0$</sub> <sup>-</sup> either. So infection in EPI <sub>$\tau=1$</sub> <sup>-</sup> also fails.

Thus NFZ <sub>$\tau=1$</sub> <sup>+,j</sup> = NFZ <sub>$\tau=1$</sub> <sup>+,j-1</sup>  $\cup$   $\{(k-1, [0, t])\}$  implies

NFZ <sub>$\tau=1$</sub> <sup>-,j</sup> = NFZ <sub>$\tau=1$</sub> <sup>-,j-1</sup>  $\cup$   $\{(k-1, [0, t])\}$  and so NFZ <sub>$\tau=1$</sub> <sup>+,j</sup>  $\subseteq$  NFZ <sub>$\tau=1$</sub> <sup>-,j</sup>.

## EPI <sub>$\tau=1$</sub> : monotonicity (I)

Consider:

- there is epidemic monotonicity at algorithmic time  $\tau=0$   
(EPI <sub>$\tau=0$</sub> <sup>+</sup>  $\supseteq$  EPI <sub>$\tau=0$</sub> <sup>-</sup>, also  $\mathcal{C}$ s in EPI <sub>$\tau=0$</sub> <sup>+</sup> are never lower than in EPI <sub>$\tau=0$</sub> <sup>-</sup>);

## $\text{EPI}_{\tau=1}$ : monotonicity (I)

Consider:

- there is epidemic monotonicity at algorithmic time  $\tau=0$  ( $\text{EPI}_{\tau=0}^+ \supseteq \text{EPI}_{\tau=0}^-$ , also  $\mathcal{C}$ s in  $\text{EPI}_{\tau=0}^+$  are never lower than in  $\text{EPI}_{\tau=0}^-$ );
- at algorithmic time  $\tau=1$  there is monotonicity of laziest feasible epidemic ( $\text{LFE}_{\tau=1}^+(k) \leq \text{LFE}_{\tau=1}^-(k)$  for all timelines  $k$ );

## $EPI_{\tau=1}$ : monotonicity (I)

Consider:

- there is epidemic monotonicity at algorithmic time  $\tau=0$  ( $EPI_{\tau=0}^+ \supseteq EPI_{\tau=0}^-$ , also  $\mathcal{C}$ s in  $EPI_{\tau=0}^+$  are never lower than in  $EPI_{\tau=0}^-$ );
- at algorithmic time  $\tau=1$  there is monotonicity of laziest feasible epidemic ( $LFE_{\tau=1}^+(k) \leq LFE_{\tau=1}^-(k)$  for all timelines  $k$ );
- and there is monotonicity of no-fly zone ( $NFZ_{\tau=1}^+ \subseteq NFZ_{\tau=1}^-$ ).

## $EPI_{\tau=1}$ : monotonicity (I)

Consider:

- there is epidemic monotonicity at algorithmic time  $\tau=0$  ( $EPI_{\tau=0}^+ \supseteq EPI_{\tau=0}^-$ , also  $\mathcal{C}$ s in  $EPI_{\tau=0}^+$  are never lower than in  $EPI_{\tau=0}^-$ );
- at algorithmic time  $\tau=1$  there is monotonicity of laziest feasible epidemic ( $LFE_{\tau=1}^+(k) \leq LFE_{\tau=1}^-(k)$  for all timelines  $k$ );
- and there is monotonicity of no-fly zone ( $NFZ_{\tau=1}^+ \subseteq NFZ_{\tau=1}^-$ ).

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Two cases to consider. Use  $\partial NFZ$  to represent right-most boundary for a  $NFZ$ , and similarly use  $\partial EPI$  to represent *left*-most boundary for a  $EPI$ ):

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In the first case there is nothing to be done: simply use the remark

$$\partial NFZ_{\tau=1}^\pm \leq \partial EPI_{\tau=1}^\pm \leq LFE_{\tau=1}^\pm.$$

## $EPI_{\tau=1}$ : monotonicity (II)

In the second case argue as follows. Localize to particular timeline  $k$ :

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This completes the proof of monotonicity for  $EPI_{\tau=1}$ .

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- This would apply, for example, in the case of the *Diamond Princess* if  $\alpha$ ,  $\beta$  depended on age and location of cabin on the ship.

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- So each timeline is divided into a *susceptible interval* (empty if it is initially infected), an *infected interval* (empty if it is never infected), and a *removed interval* (empty if it has no conditioned removal).

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- ② List in *time-reverse order* original infections together with sampled *new* candidate  $\tilde{\mathcal{I}}$ s in complements of the removed intervals.
- ③ Work iteratively through this list. Would discarding original  $\mathcal{I}_{i,j}(t)$  result in failure to infect a conditioned removal?. If so, **retain**  $\mathcal{I}_{i,j}(t)$  as *perpetuated infection*  $\mathcal{P}_{i,j}(t)$ , otherwise **discard**.

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As in S-I-R case, the conditioned epidemic is the unique equilibrium.

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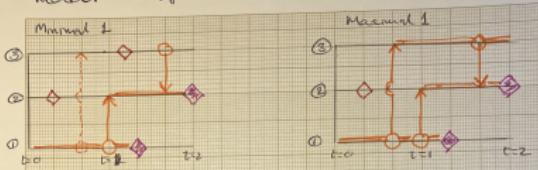
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Then **CFTP** would make sense, and it would only be necessary to show that accessibility of a set of near-maximal configurations guarantees eventual coalescence.

# Counterexample to monotonicity

"Observe" generalization to compartmental model can fail to be monotone!

1/2/25



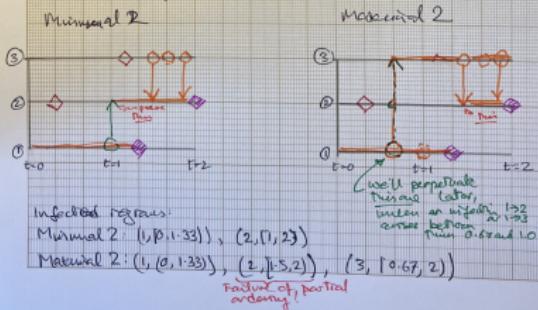
◆ Conditioned removal

◆ Infection

◆ Inactivated removal

This is after deleting all old inactivated removals and replacing by new inactivated removals.

Now work from right-to-left deleting all infections except where so doing would leave a conditioned removal uninfected. (and inactivated removals.)  
At  $t=1$  we get to:



# Other technical information

## Software used in computations

<i>Software</i>	<i>Version</i>	<i>Branch</i>	<i>Last commit</i>
quarto	1.6.39	—	
Running under julia	1.11.5	—	
EpidemicsCFTP	2.2.514	develop	Fri Mar 21 10:43:55 2025
EpidemicsUtilities	0.1.2.174	main	Tue Mar 4 16:32:10 2025
This quarto script	0.2.2.713	develop	Wed Mar 12 14:27:50 2025

## Project information

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**Version:** 0.2.2.715 (develop)

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**Author:** Wilfrid Kendall <[W.S.Kendall@warwick.ac.uk](mailto:W.S.Kendall@warwick.ac.uk)>

**Date:** Fri Jun 20 15:42:30 2025 +0100

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### Comment:

Preparation for Liverpool seminar