Perfect Epidemics

Applied Probability Seminar
Department of Statistics, University of Warwick

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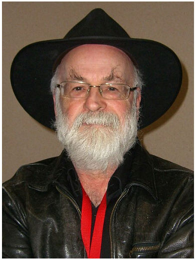
Warwick, York

17 January 2025

Introduction



"Maybe the only significant difference between a really smart simulation and a human being was the noise they made when you punched them."
(The Long Earth, Pratchett & Baxter, 2012)

Handout is on the web: use the `QR-code` or visit `wilfridskendall.github.io/talks/PerfectEpidemics`.

This is initial work on using perfect simulation (CFTP) for epidemics.
WSK acknowledges the support of UK EPSRC grant EP/R022100.

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- Simplest possible example: *random-walk-CFTP*
  (can boost to use Ising model to do simple image reconstruction).

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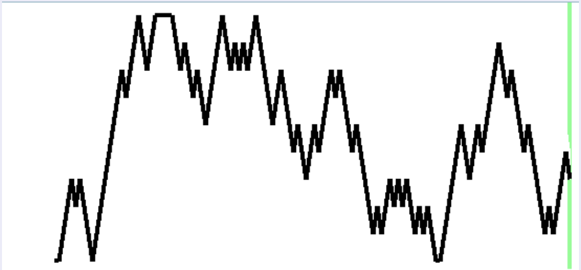

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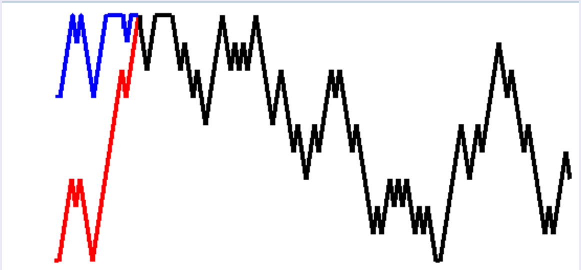

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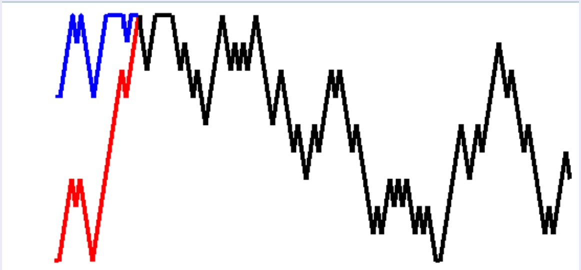
- Generally not true that location *at* coupling is a draw from equilibrium.



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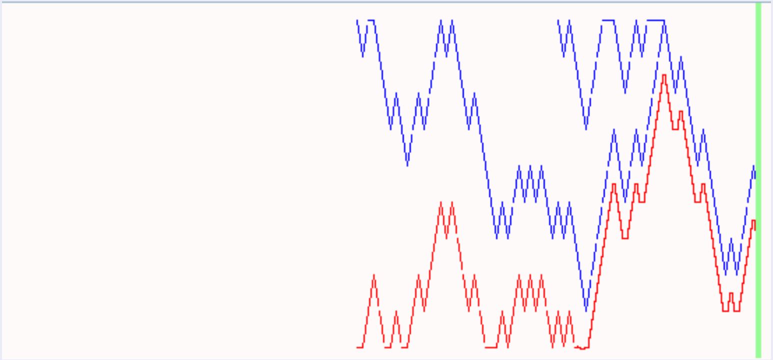

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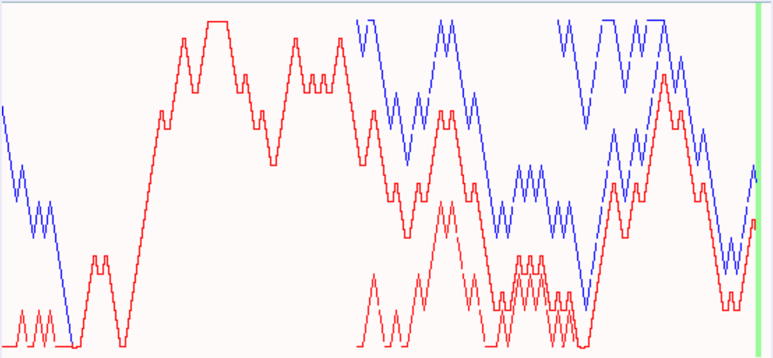

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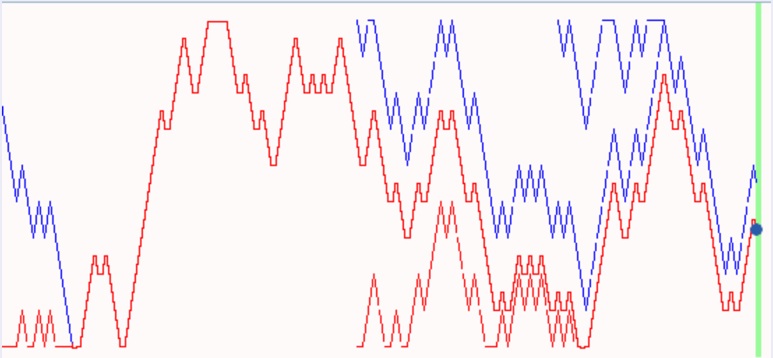

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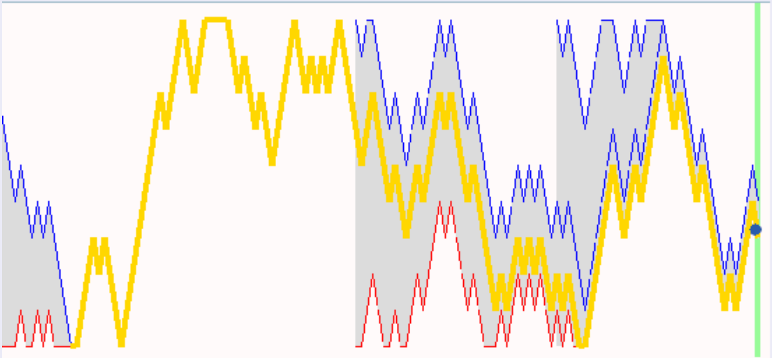
- The common value is an exact draw from equilibrium!



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- Detailed expositions: WSK (2005), Huber (2015).
  (Want to implement CFTP in R? see WSK, 2015.)

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3: Perfect Epidemics: a challenge problem for CFTP

Many important inferential questions (Cori & Kucharski, 2024).

Simplest models (versus UK model with 10^6 agents!, Fraser & Others, 2023):

S-I-R deterministic epidemic: susceptibles s, infectives i, removals r (constant total population $s + i + r = n$):

$$\begin{aligned} s' &= -\alpha\, s\, i\,, \\ i' &= (\alpha\, s - \beta\,)\; i\,, \\ r' &= \beta\, i\,. \end{aligned}$$

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Infection:	$S \to S - 1\,,$	$I \to I + 1$	at rate	$\alpha\, S\, I\,,$
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Both models make an unrealistic assumption: homogeneous mixing.

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Wikipedia: "The British-registered *Diamond Princess* was the first cruise ship to have a major `[COVID-19]` outbreak on board, with the ship quarantined at Yokohama from 4 February 2020 for about a month. Of 3711 passengers and crew, around 700 people became infected and 9 people died."
Evidently $\alpha\ s_0/\beta \gg 1$ – as was sadly later confirmed, a sorrow for us all.

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5. Can we use perfect simulation?

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An absurdly simple variant of contact tracing:

"When did the infections occur, supposing we only observe removals?"

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5. Result: *trajectory-valued chain*, unconditioned S-I-R as equilibrium.

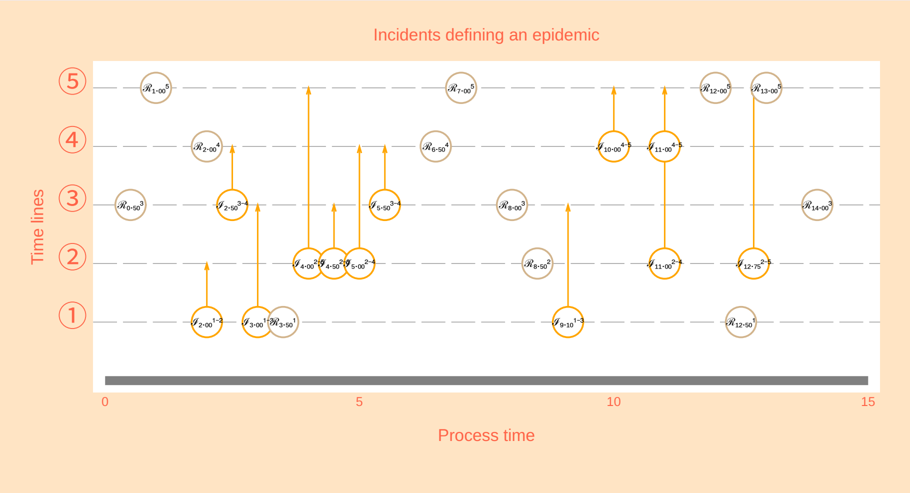

Figure 1: Light-orange circles denote potential infections (arrows point upwards to targets); light-brown circles denote potential removals.

From incidents to unconditioned epidemic trajectories (2/3)

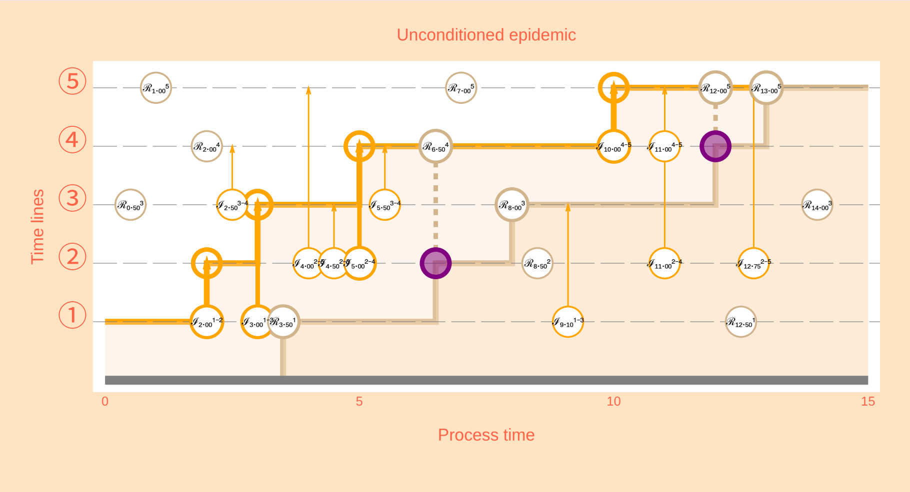

Figure 2: Activate (a) *infection* if target on lowest uninfected timeline; (b) *removal* if in infected region, then remove lowest infected (purple disk if different timeline).

From incidents to unconditioned epidemic trajectories (3/3)

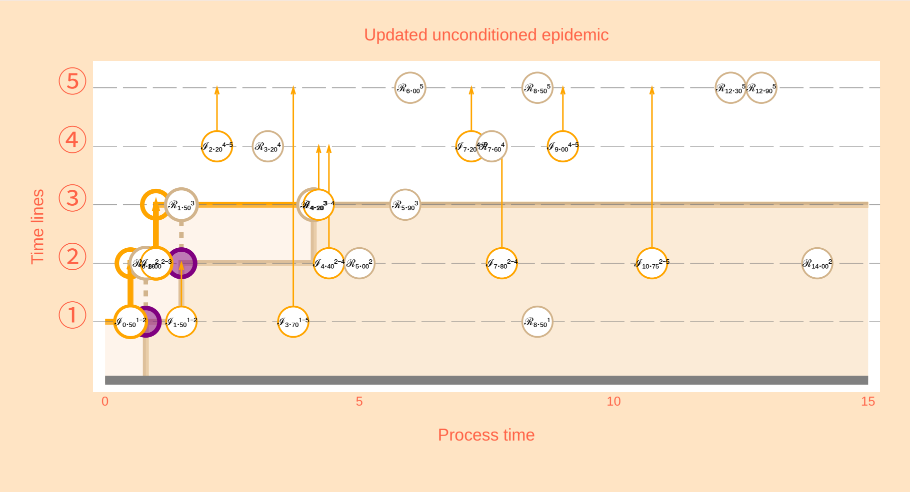

Figure 3: A step in algorithmic time for the unconditioned epidemic simply involves replacing the original incidents by a new set of incidents.

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- Re-express using continuously varying τ. Process time runs over $[0, T]$.

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- For later purposes it is convenient to stage the replacement as follows:
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- Re-express using continuously varying τ. Process time runs over $[0, T]$.
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- The connection "restriction=conditioning" is thereby preserved.
- Crucially, step 2 ensures composition action is irreducible over S! (So equilibrium under conditioning is unique.)

Illustration of technical point (1/8)

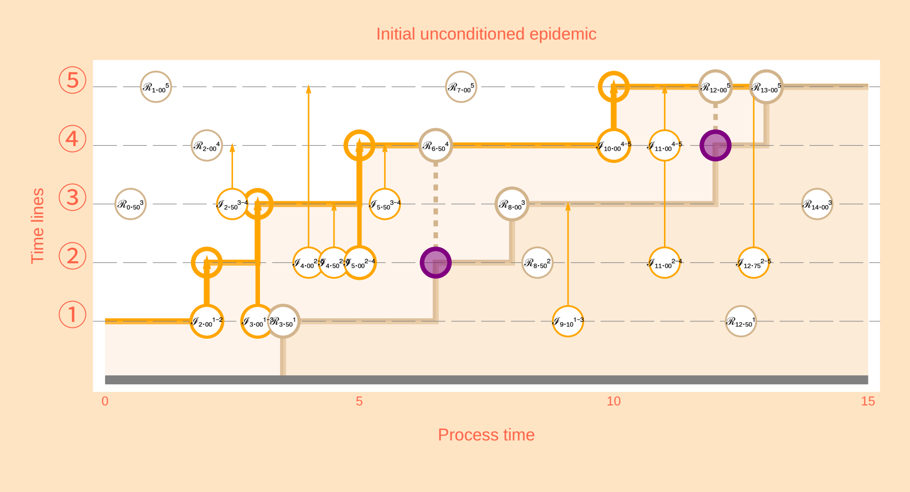

Figure 4: No change yet to removals or infections;

Illustration of technical point (2/8)

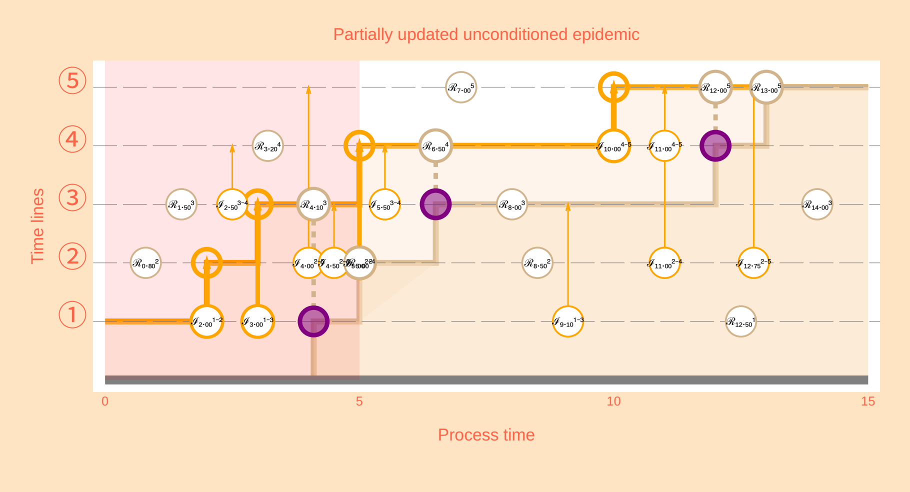

Figure 5: Replace first third of removals, infections unchanged;

Illustration of technical point (3/8)

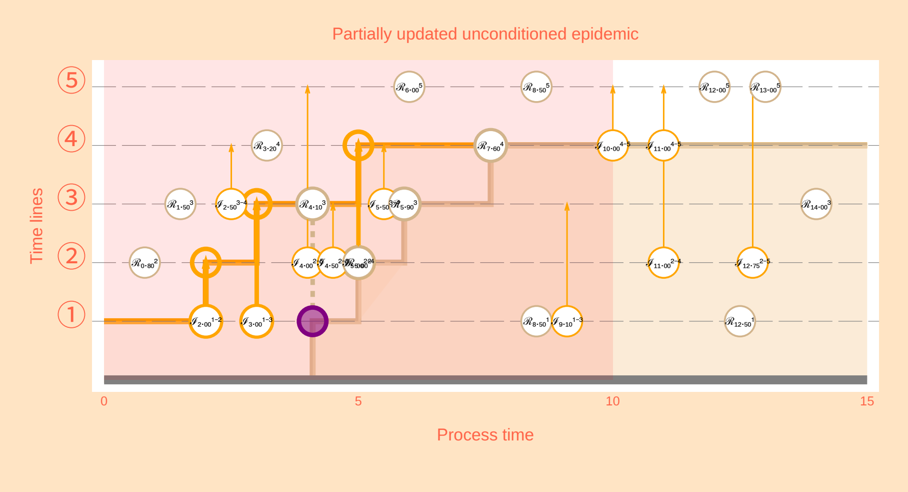

Figure 6: Replace first two-thirds of removals, infections unchanged;

Illustration of technical point (4/8)

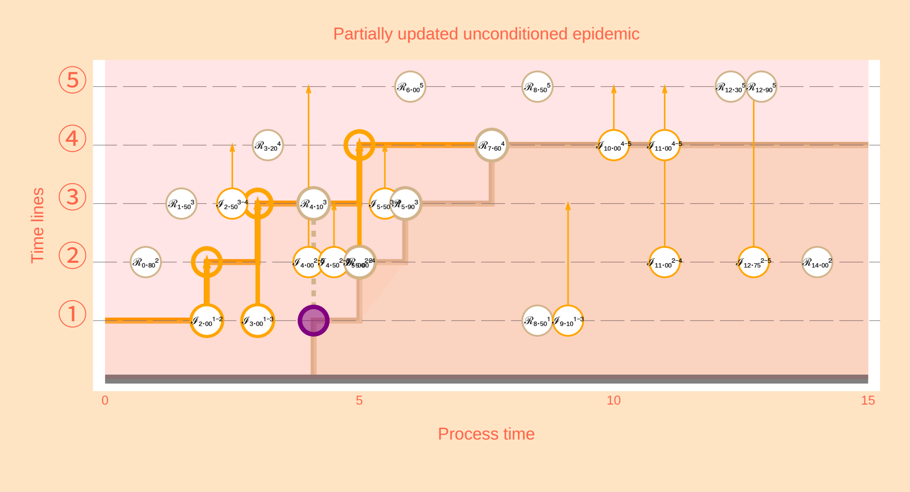

Figure 7: All removals resampled, infections as yet unchanged;

Illustration of technical point (5/8)

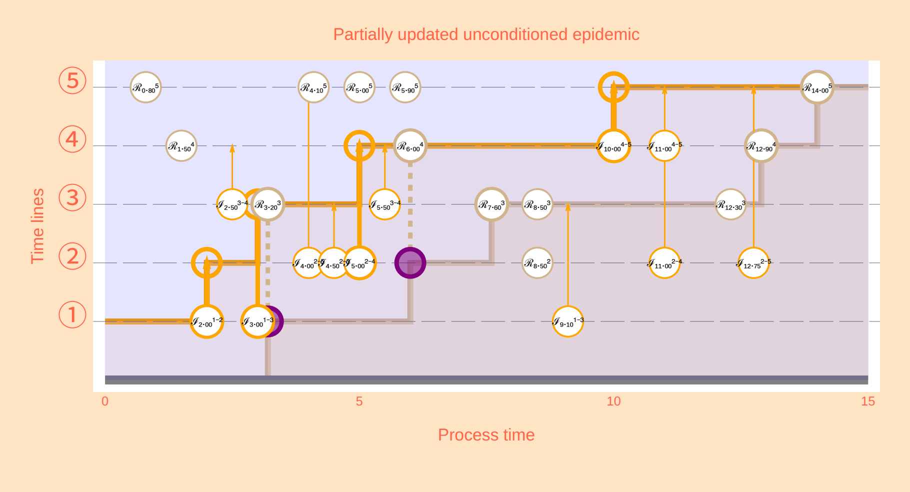

Figure 8: Re-sample all removal timelines, infections as yet unchanged;

Illustration of technical point (6/8)

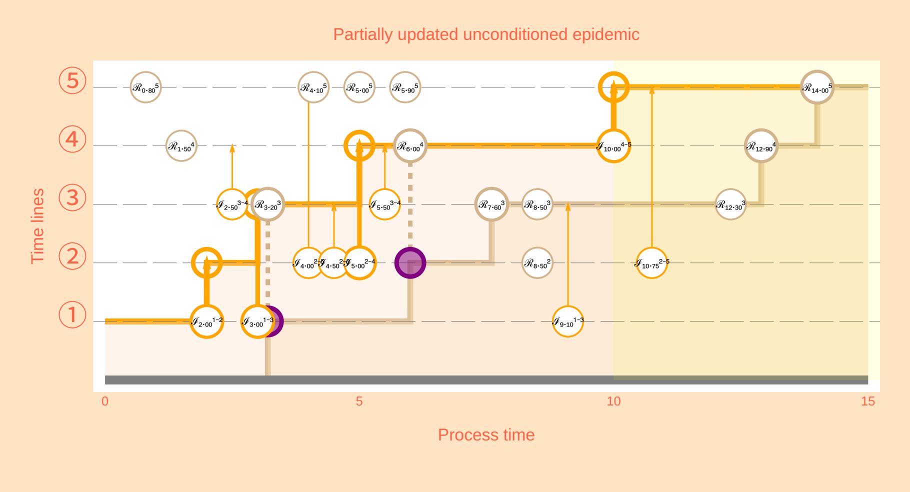

Figure 9: Re-sample last third of infections;

Illustration of technical point (7/8)

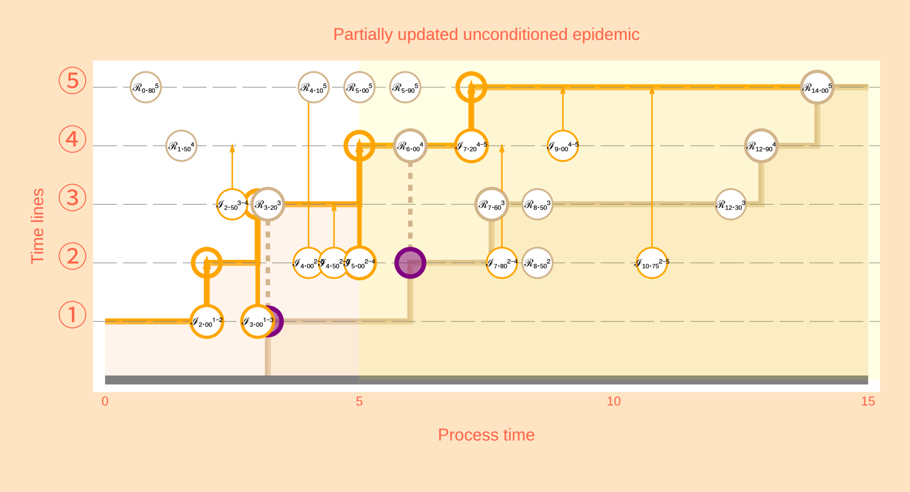

Figure 10: Re-sample last two-thirds of infections;

Illustration of technical point (8/8)

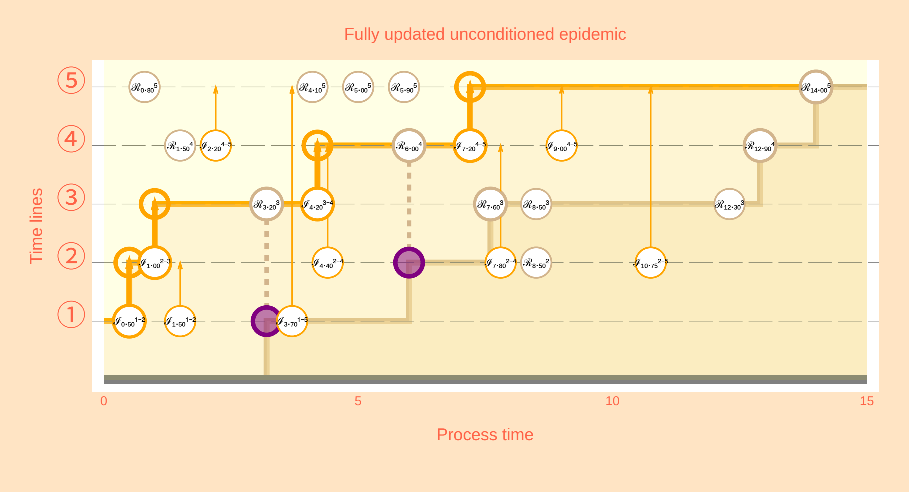

Figure 11: All infections now re-sampled.

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- Housekeeping details required to establish that monotonicity still works. Key notions: *last feasible epidemic* (LFE) and *no-fly zone* (NFZ).

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Initial conditional epidemic

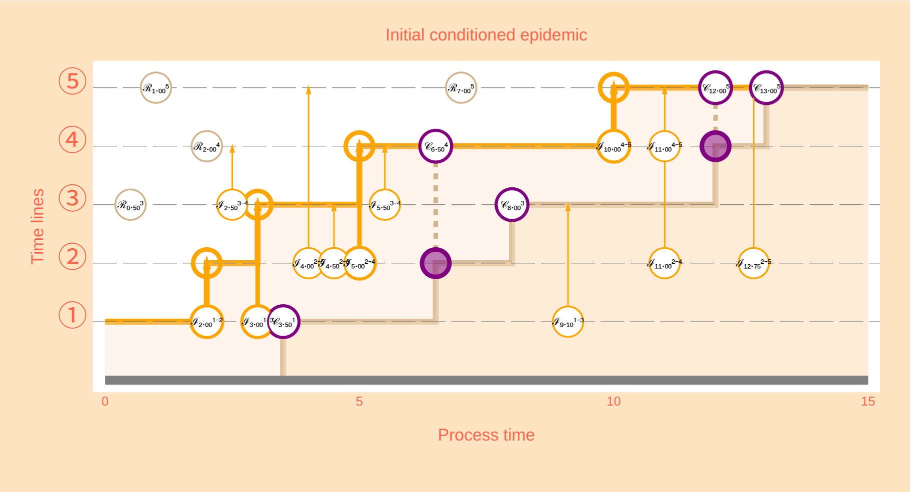

Figure 12: Initial epidemic with conditioned removals indicated using purple circles (and purple disks when different timelines are infected).

Conditional epidemic update

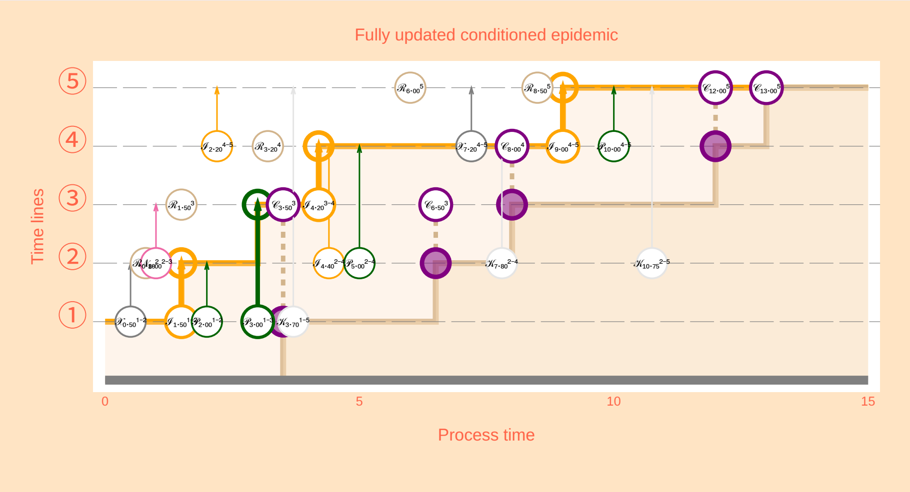

Figure 13: Epidemic updated under restriction: all conditioned removals remain activated, no new removals are activated. Green infections have been "perpetuated".

Last feasible epidemic (LFE)

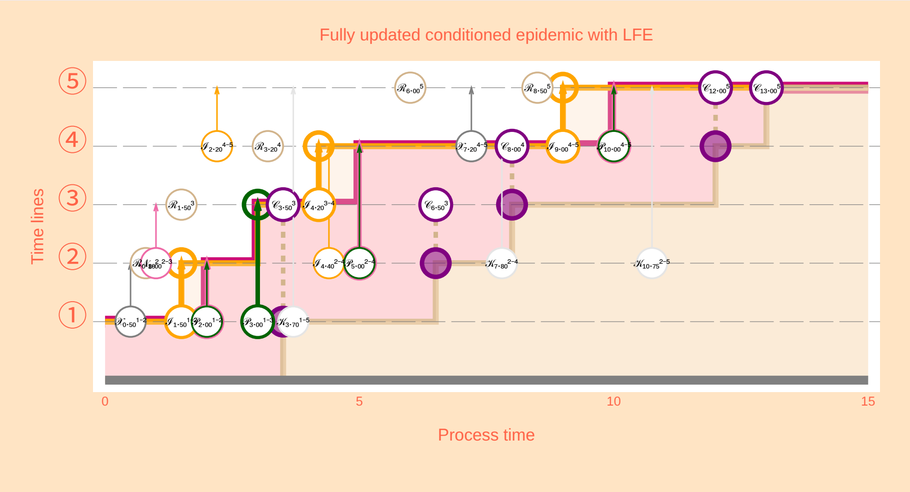

Figure 14: LFE computed recursively working right-to-left: the slowest sequence of infections deals with all infected timelines in order (includes perpetuated infections).

No-fly zone (NFZ)

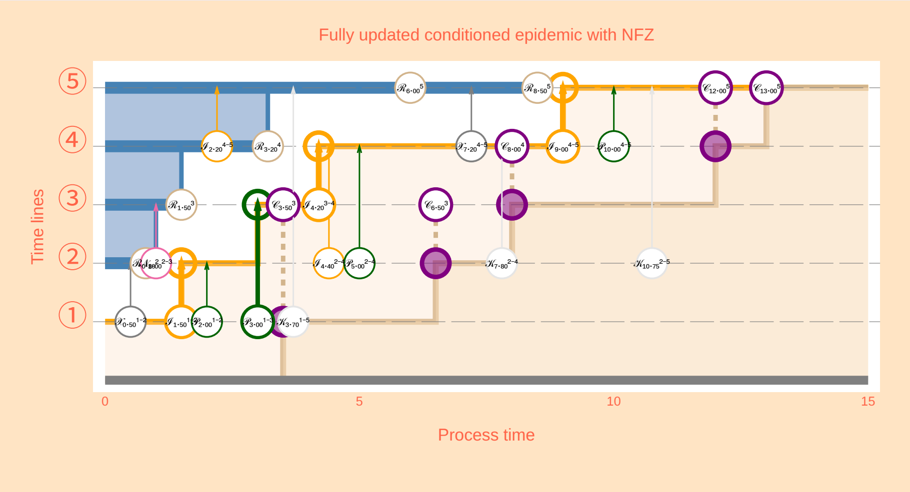

Figure 15: NFZ computed recursively working right-to-left: it traces the region of timelines that must not be infected if one is not to activate unobserved removals.

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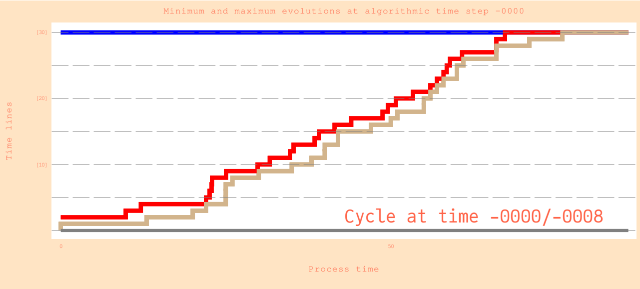

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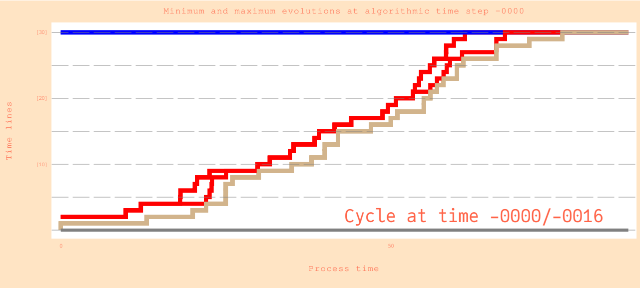

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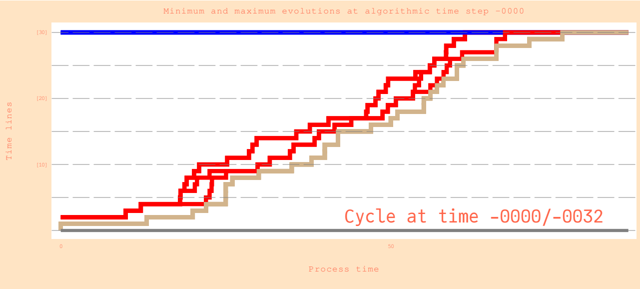

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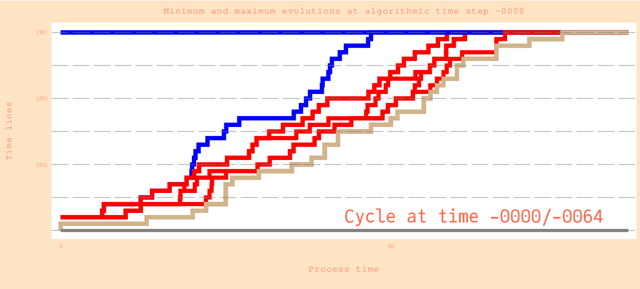

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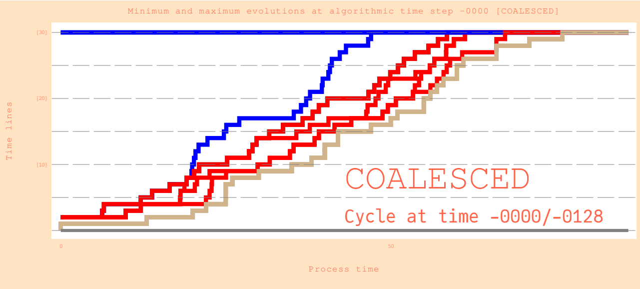

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- Finally: generalize to other suitable compartment models?

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- Thank you for your attention! **QUESTIONS?**

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Image information

Image	Attribution	
Terry Pratchett	Luigi Novi	*CC BY 3.0*
Classic CFTP for a simple random walk	Result of code written by WSK	
Diamond Princess	Alpsdake	*CC BY-SA 4.0*
Epidemic CFTP images and animation	Result of code written by WSK	

Previous instances of this talk

Date	Title		Location
19/04/24	Perfect Epidemics	Short Research Talk (12min)	Warwick
15/05/24	McMC and Perfect Simulation	Graduate Seminar, Aristotle Univ. (50min)	Thessaloniki
17/01/25	Perfect Epidemics	Applied Probability Seminar (50min)	Warwick

A “near-maximal” configuration

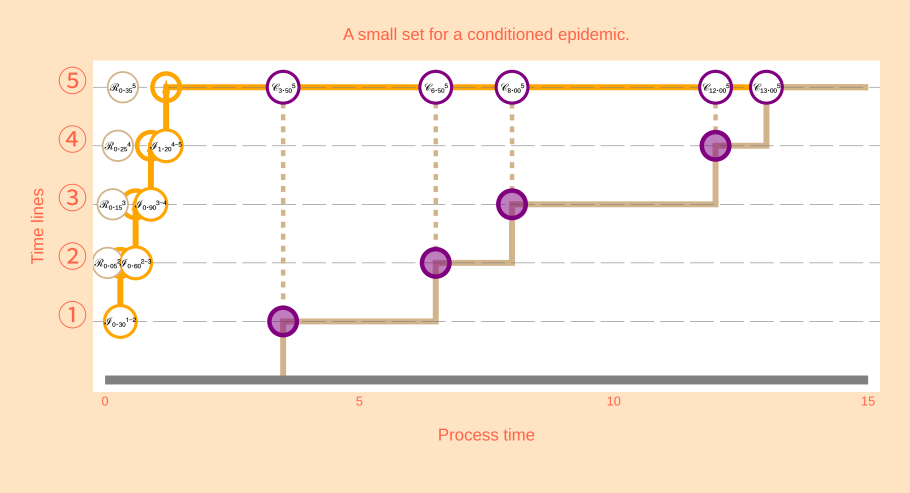

Figure 16: A conditional epidemic in which all activated infections occur before time 3.0, also before smallest observed removal time.

Other technical information

Software versions

Software used in computations:

Software	*Version*	*Branch*	*Date of last commit*
Quarto	`1.6.39`	—	
Running under `julia`	`1.11.2`	—	
Module `EpidemicsCFTP`	`2.2.487`	`main`	`Fri Jan 10 20:17:28 2025`
Module `EpidemicsUtilities`	`0.1.2.154`	`main`	`Wed Jan 15 13:23:36 2025`
This Quarto script	`2.2.589`	`master`	`Fri Jan 17 16:59:04 2025`

Revision history

These notes were produced from `PerfectEpidemics.qmd`:

Author:	Wilfrid Kendall W.S.Kendall@warwick.ac.uk
Date:	Fri Jan 17 16:59:04 2025 +0000
Summary:	This version has edits arising from the 17/1/2025 talk in Warwick. 2.2.589