Wandering around a fibrous network[1]

when all the paths look very much alike

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https://wilfridskendall.github.io/talks/Short-Fibre-SIRSN-talk-2024/Short-Fibre-SIRSN-talk-handout.pdf

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Development of work on line process models for transportation networks (Aldous & WSK, 2008; WSK, 2011, 2014),
particularly concerning line process models for online maps:
(Aldous & Ganesan, 2013; Aldous, 2014; Kahn, 2016; WSK, 2017, 2020).

Current objective: model online maps, but replace infinitely long straight lines by line segments ("sticks") or moderately curved paths ("stiff fibres").

Structure:

1. Modelling efficient spatial transport systems by an axiomatic approach to scale-invariant random spatial networks (SIRSN);
2. Practicalities (why the two current SIRSN models are unsatisfactory);
3. Sticks and fibres (brief notes on some key results and proofs).

(I) Models for efficient spatial transport systems

How to model real-life online maps? a task for stochastic geometry.

1. Mechanism to connect generic nodes x and y;
2. "Scale -invariant" networks (SIRSN) as potential models;
3. What do "best routes" look like?
4. What can be said about flows? – not for today.

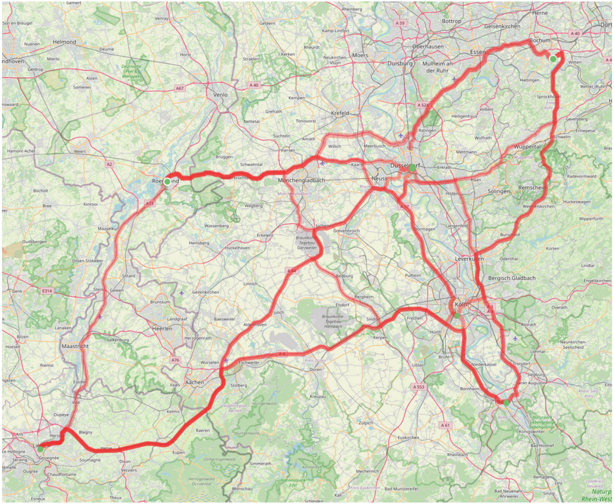
Related to work on mathematics for cities, for example:

- Work on city shape (Bender *et al.*, 2004; Courtat, 2012);
- Route-finding (Bongiorno *et al.*, 2021);
- "Transit nodes" (Bast *et al.*, 2007).

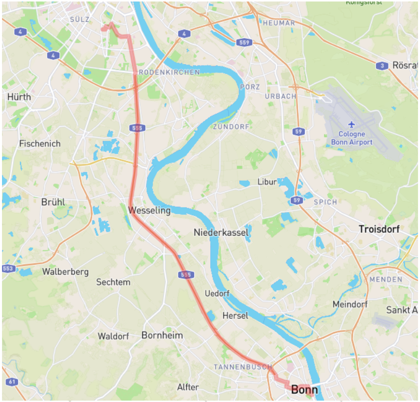
Work so far: models based on *eg* Poisson line processes (Plp) (Aldous & WSK, 2008; WSK, 2011, 2017, 2020; Kahn, 2016; Blanc, 2023).

We'd strongly prefer: models based on *localizable* object processes (*eg* Poisson stick processes, Poisson fibre processes).

Online maps look as if they might be scale-invariant!



An online map as a mechanism for generating networks



Given $x_1, \ldots, x_n$, the online map should generate a *network*

$$\mathcal{N}(x_1, \ldots, x_n) \quad = \quad \bigcup \left\{ \mathcal{R}(x_i, x_j) \; ; \; 1 \leq i < j \leq n \right\} .$$

NB: disallow one-way streets: $\mathcal{R}(x_i, x_j) = \mathcal{R}(x_j, x_i)$.

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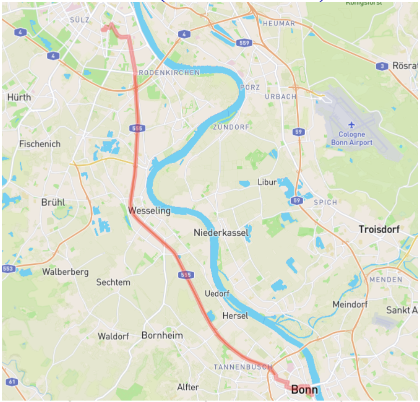
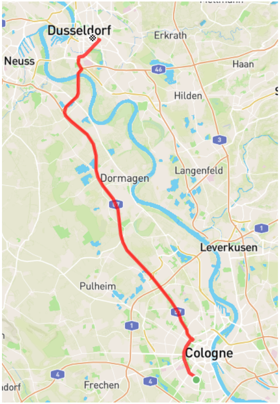
Axiom 1: (Statistical) scale-invariance of networks



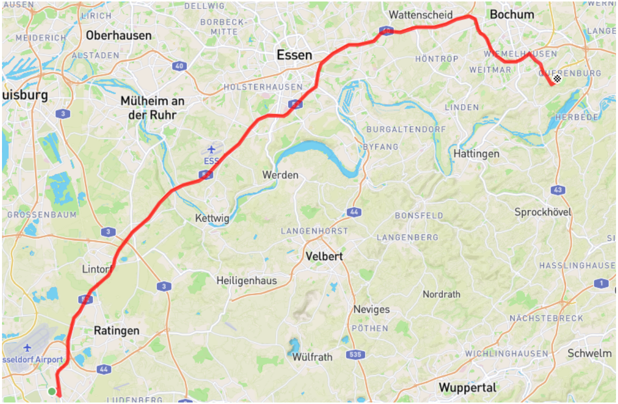


We require *statistical scale invariance*: for all similarities $\mathfrak{S}$ of $\mathbb{R}^m$:

$$\mathcal{L}\left(\mathfrak{S}\mathcal{N}(x_1, \ldots, x_n)\right) \quad = \quad \mathcal{L}\left(\mathcal{N}(\mathfrak{S}x_1, \ldots, \mathfrak{S}x_n)\right) \,.$$

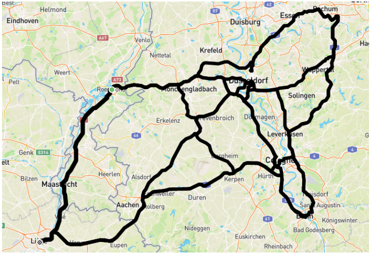
NB: model $\mathcal{R}(x_i, x_j)$ hence $\mathcal{N}(x_1, \ldots, x_n)$ as random.

Axiom 2: Finite mean length of routes



We insist that all routes have finite mean length.

Axiom 3: SIRSN property



NB: SIRSN = “Scale-invariant Random Spatial Network”.

We require massive re-use of route components (strong SIRSN property).

Weak SIRSN: Source/terminus nodes spread evenly everywhere on plane. “Route-re-use” means mean-network-length-per-unit-area must be finite.

Strong SIRSN: Source/terminus nodes spread densely everywhere on plane. “Route-re-use” means mean-network-length-per-unit-area is finite *for the “long-distance” network.*

Scale-Invariant Random Spatial Networks (SIRSN)

Axioms (following Aldous & Ganesan, 2013; Aldous, 2014).

Given: a (random) network $\mathcal{N}(x_1, \ldots, x_n)$ connecting nodes. Require:

1. Scale-invariance: for each Euclidean similarity $\mathfrak{S}$, $\mathcal{L}\left(N(\mathfrak{S}x_1, \ldots, \mathfrak{S}x_n)\right) = \mathcal{L}\left(\mathfrak{S}N(x_1, \ldots, x_n)\right)$.
2. If D_1 is length of fastest route between two points at unit distance apart then mean length is finite: $\mathbb{E}\left[D_1\right] < \infty$.
3. Weak SIRSN property: network connecting points of (independent) unit intensity Poisson point process has finite average length density;
   or
   (Strong) SIRSN property: finite length intensity of “long-range” routes for points of *dense* Poisson point process.

Do SIRSN exist at all?

Yes (first version): Hierarchical SIRSN (Aldous, 2014),
a *randomized* dense dyadic speed-marked planar *lattice*.

Do SIRSN exist at all? a more natural answer

Yes (second version): Poisson line process SIRSN: planar *and also* higher dimensions too (Aldous, 2014; Kahn, 2016; WSK, 2017).

Construction uses speed-marked (improper) Plp Π.

Π is governed by an intensity measure ϖ_{Plp} on $(0,\infty)\times\mathbb{R}\times[0,\pi)$, scaling speed $v \to \lambda v$ versus space $x \to \lambda^{\gamma-1}x$:

$$\mathrm{d}\varpi_{\text{Plp}} \quad = \quad \tfrac{1}{2}(\gamma-1)v^{-\gamma}\,\mathrm{d}v\;\;\mathrm{d}r\,\mathrm{d}\theta \qquad \text{(make sure } \gamma > 1!)\,.$$

NB: $\frac{1}{2}\,\mathrm{d}r\,\mathrm{d}\theta$: intensity measure for standard Poisson line process.
NB ($\gamma > 1$): Lines of speed ≥ 1 form standard Poisson line process $\Pi^{(\geq 1)}$.

Use fastest Π-paths for routes. Require $\gamma > 2$ so

- Π-paths cannot reach ∞ in finite time;
- Π-paths can reach any prescribed point in finite time.

It is non-trivial to prove that Π forms a SIRSN!

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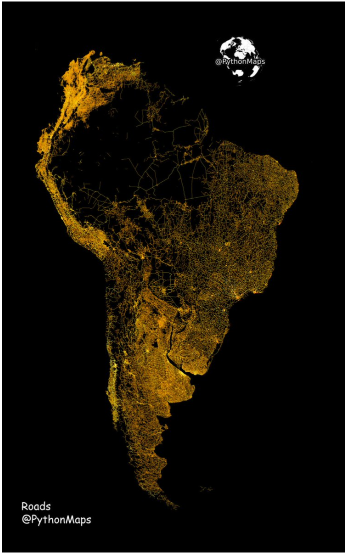
(II) From lines to (non-infinite) fibres

But we want to be able to model networks like this:



Sticks and stiff fibres: foundations

Use object processes (random objects located at points of a Poisson process):

- Scale-invariant, so speed-marked ("object soup");
- Sticks or C^1 fibres, so notion of length;
- Stiff fibres (each fibre within ε of its intrinsic overall direction).
- Define using intensity measure on configuration space:

$$\begin{array}{ccccccccc} \mathbb{R}^m & \times & \mathrm{SO}_m & \times & (0,\infty) & \times & \text{Shape} & \times & (0,\infty) \\ x & & \omega & & r & & \text{shape} & & v \\ \text{(location)} & & \text{(rotation)} & & \text{(length)} & & & & \text{(speed)} \end{array}$$

- Parameters:
  - fibre shape distribution;
  - rescaled length distribution (joint with shape);
  - specific length intensity λ for set of fibres of more than unit speed;
  - scaling index γ.

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Sticks and stiff fibres: techniques

1. Integrate the measurable field of orientations using Lipschitz paths;
2. Sobolev compactness theorem for Lipschitz paths: hence existence of "geodesics" (fastest connecting paths);
3. For routes, select geodesics in scale-invariant way using auxiliary scale-invariant randomness;
4. Validation of SIRSN axioms depends on comparison of (fractional power) of connection time with Exponential distribution.

Finesse away percolation issues by supposing fibre soup is *thick*: scaled length distribution has finite first moment but infinite second moment (quantified by tail condition)

and do the sums for

- probability of "timely" connection of two close points by a single fibre;
- concatenating point-pairs for "timely" connection of two far points;
- deriving exponential moment of fractional power(connection time);
- deriving estimate on "time-diameter" for compact set.

(III) Beyond thickness: percolation

The SIRSN property should not need "thickness" (single-fibre connection).

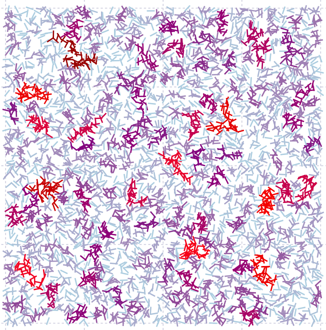
Conjecture: Super-critical percolation suffices.

I conjecture that a Poisson soup of stiff fibres can form a SIRSN if there is super-critical percolation for the family of fibres of greater than unit speed.

Simulation of a caricature:

- Scaled stick-length $= 1$;
- large clusters coloured red, small clusters coloured blue;
- using stick-length 1;
- length intensities reduce inversely.

Simulation: percolation is plausible.



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Conclusion

- Strong positive evidence that “SIRSN are everywhere!”
  (not just when infinite lines are involved).
- There are very simple scale-invariant stick soups which are thick:
  so this theory is not vacuous!
- Future prospects:
  - traffic, benchmark models for traffic flow (cf Gameros Leal, 2017);
  - adding spatial inhomogeneity;
  - moving beyond Poisson, *eg* adding simple interactions.

Thank you for your attention! QUESTIONS?

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Previous versions of this talk

In earlier forms:

- Colloquium lecture *Scale-invariant random spatial networks (SIRSN) from line patterns*: Journée de Rham lecture at Lausanne 22/4/2015; Auckland 9/8/2016; Bern 10/6/2016; Aarhus 6/6/2016; Manchester Probability Seminar 13/4/2016.
- Four-lecture course *Poisson line processes and spatial transportation networks*: Edinburgh 21-22/6/2018.
- Talks on more general varieties of SIRSN: Sandbjerg 16 February 2020; Dunkerque 17-19 November 2021; Singapore 21/6/2022; Bern 15/9/2022; Cologne 29/3/2023; UCD 24/4/2024.
- Shorter form talks (25-30 minutes): "Warwick Stats 50" 8/9/2022; Baxendale birthday Warwick 9/6/2023; OEBN memorial Aarhus 29/5/2024.

These notes were produced from `Short-Fibre-SIRSN-talk.qmd`:

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