Perfect Epidemics

Applied Probability Seminar
Department of Statistics, University of Warwick

W S Kendall   S B Connor
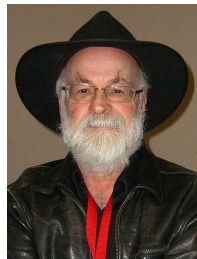
Warwick, York

17 January 2025

Introduction

"Maybe the only significant difference between a really smart simulation and a human being was the noise they made when you punched them." (The Long Earth, Pratchett & Baxter, 2012)

Handout is on the web: use the `QR-code` or visit `wilfridskendall.github.io/talks/PerfectEpidemics`.

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- Simplest possible example: *random-walk-CFTP* (can boost to use Ising model to do simple image reconstruction).

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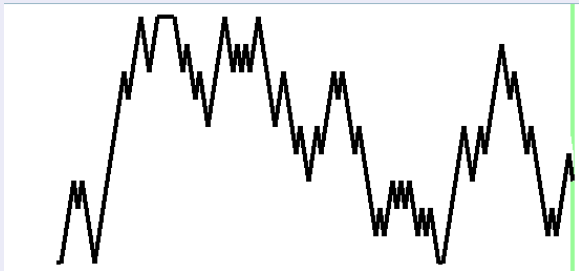
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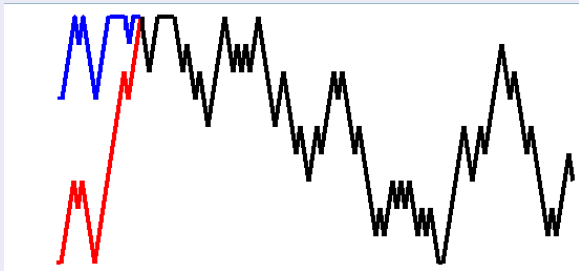 - at state 0 replace -1 jump by staying still.

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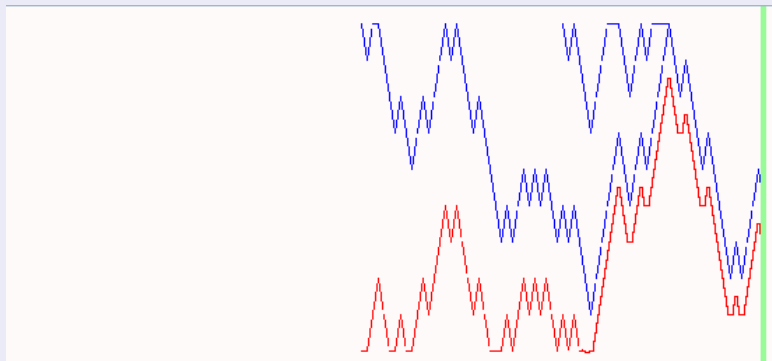- Generally not true that location *at* coupling is a draw from equilibrium.

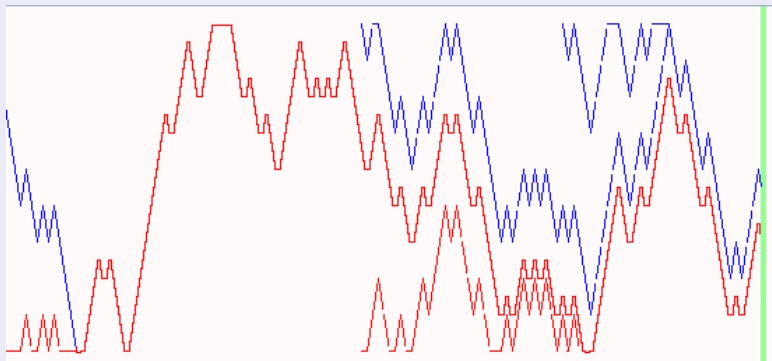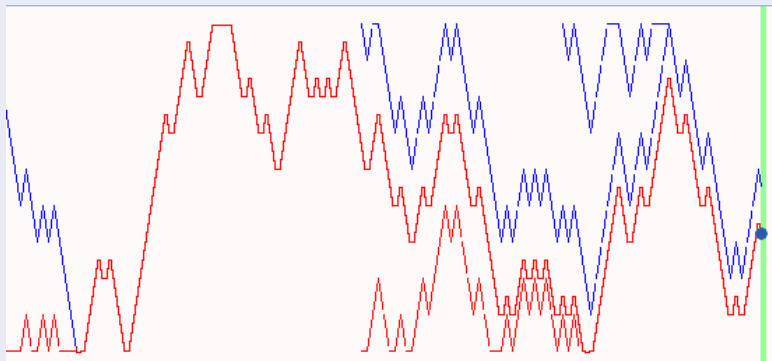- So now start at top (9) and bottom (0) at negative time $-T$, run to time 0.

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- The common value is an exact draw from equilibrium!
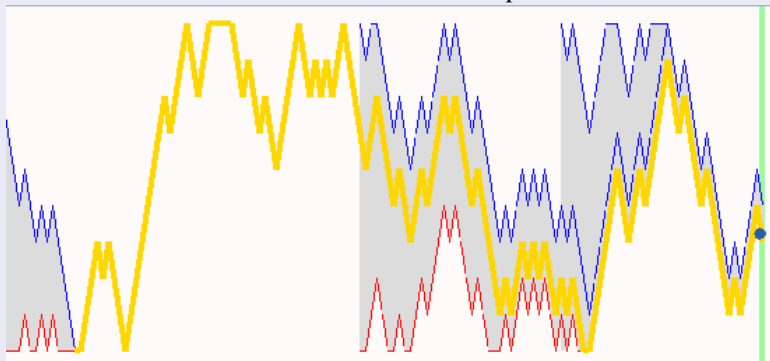
2: A short section on some theory about CFTP

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- Detailed expositions: WSK (2005), Huber (2015).
  (Want to implement CFTP in R? see WSK, 2015.)

3: Perfect Epidemics: a challenge problem for CFTP

Many important inferential questions (Cori & Kucharski, 2024).

Simplest models (versus UK model with 10^6 agents!, Fraser & Others, 2023):

S-I-R deterministic epidemic: susceptibles s, infectives i, removals r (constant total population $s + i + r = n$):

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Infection:	$S \to S - 1\,,$	$I \to I + 1\,,$	at rate	$\alpha\, S I\,,$
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These models make an unrealistic assumption: homogeneous mixing.

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Wikipedia: "The British-registered *Diamond Princess* was the first cruise ship to have a major [COVID-19] outbreak on board, with the ship quarantined at Yokohama from 4 February 2020 for about a month. Of 3711 passengers and crew, around 700 people became infected and 9 people died."

Evidently $\alpha\, s_0/\beta \gg 1$ – as was sadly later confirmed, a sorrow for us all.

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5. Can we use perfect simulation?

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"When did the infections occur, supposing we only observe removals?"
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5. Result: *trajectory-valued chain*, unconditioned S-I-R as equilibrium.

Figure 1: Light-orange circles denote potential infections (arrows point upwards to targets); light-brown circles denote potential removals.
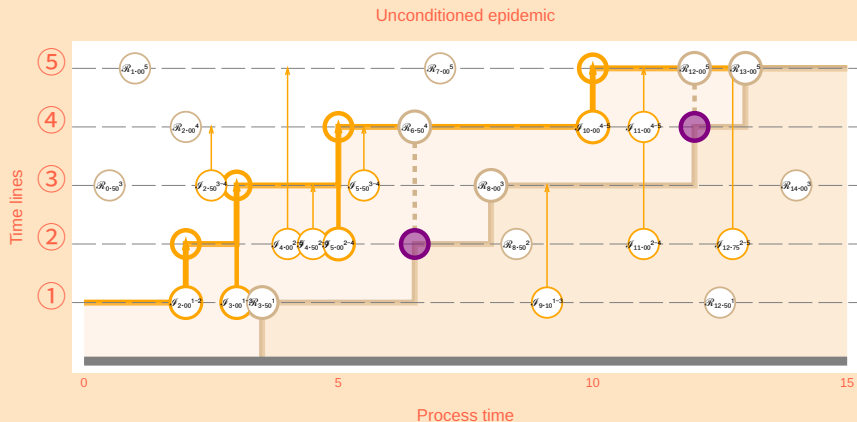
Figure 2: Activate *infection* if target on lowest uninfected timeline. *removal* if in infected region and remove lowest infected (purple disk if different timeline).
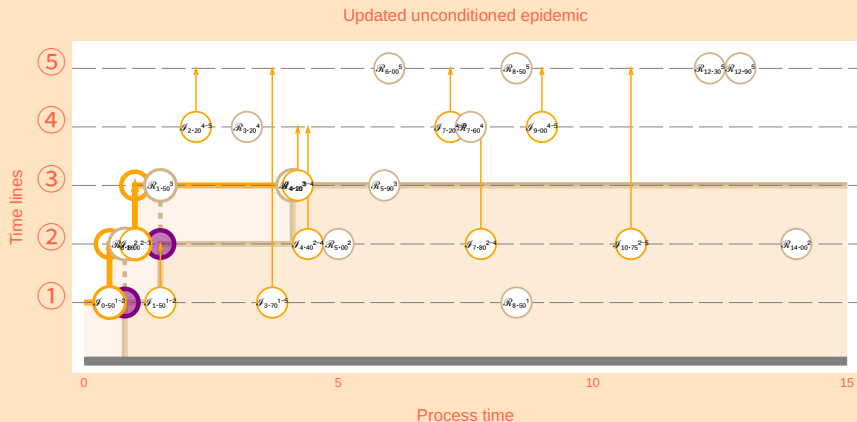
Figure 3: A step in algorithmic time for the unconditioned epidemic simply involves replacing the original incidents by a new set of incidents.

Crucial technical point

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Crucial technical point

- Updates in algorithmic time τ are then (algorithmic-)*time-reversible*: so restriction to subset S of state-space (in our case, *activated* removals occurring precisely at specified set of times) implies a new equilibrium which is the old equilibrium conditioned to lie in S.
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- The connection "restriction=conditioning" is thereby preserved.
- Crucially, step 2 ensures composition action is irreducible over S! (So equilibrium under conditioning is unique.)
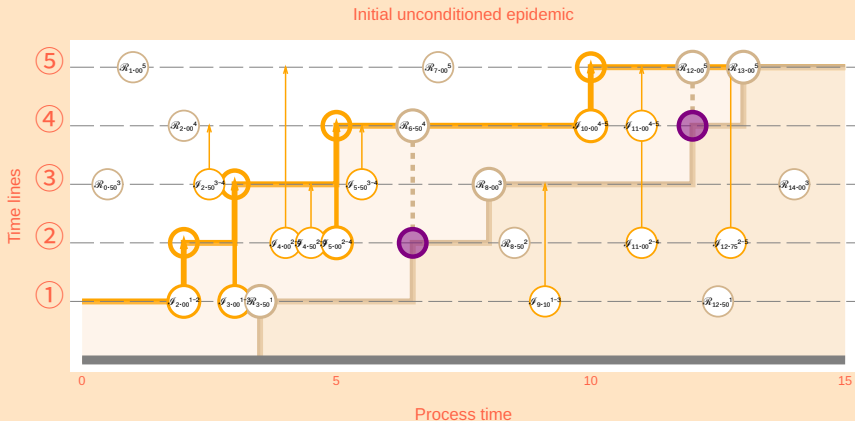
Figure 4: No change to removals or infections
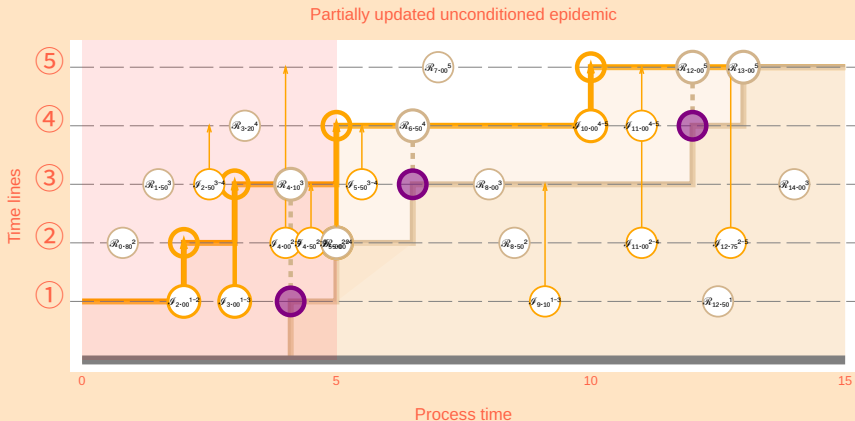
Figure 5: Replace first third of removals, infections unchanged
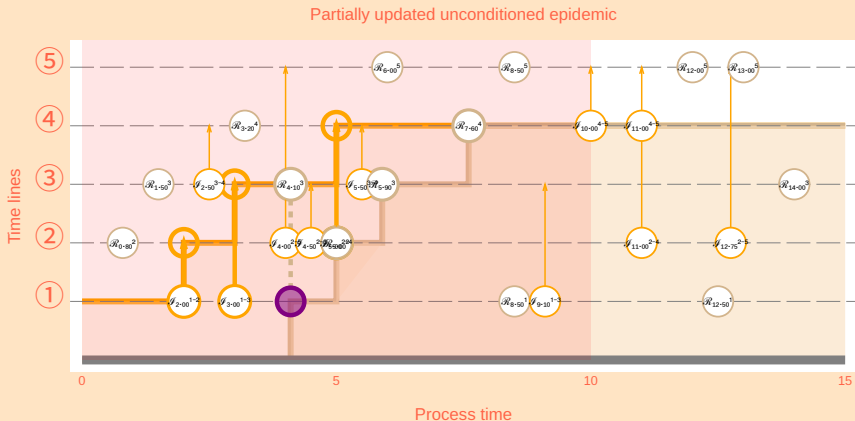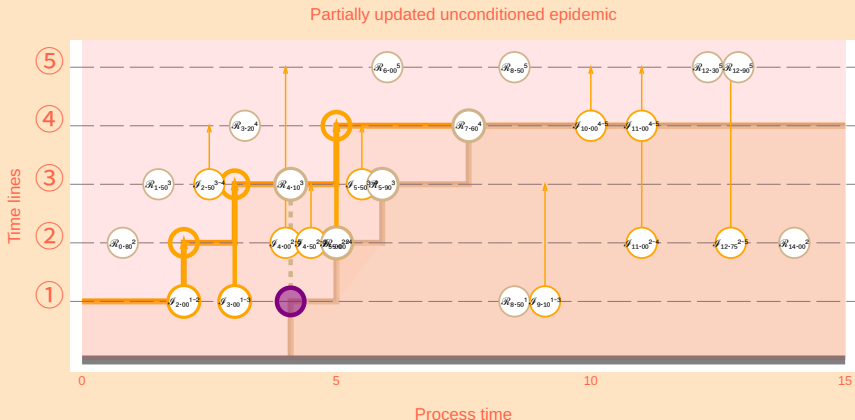
Figure 6: Replace first two-thirds of removals, infections unchanged

Figure 7: All removals resampled, infections unchanged
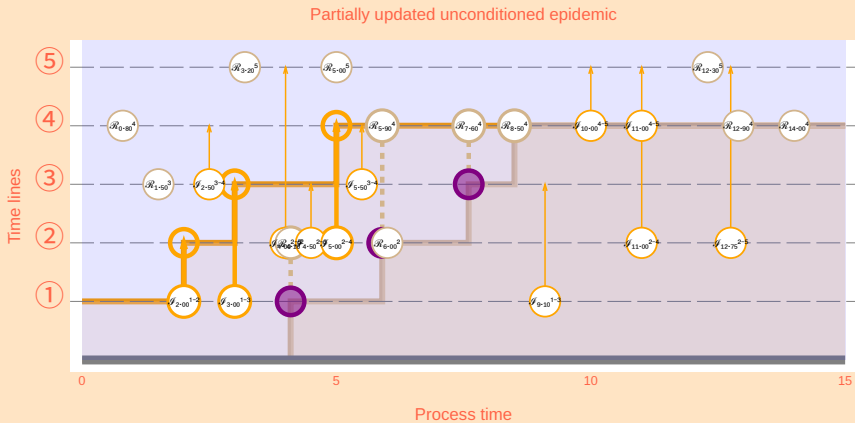
Figure 8: Re-sample all removal timelines, infections unchanged
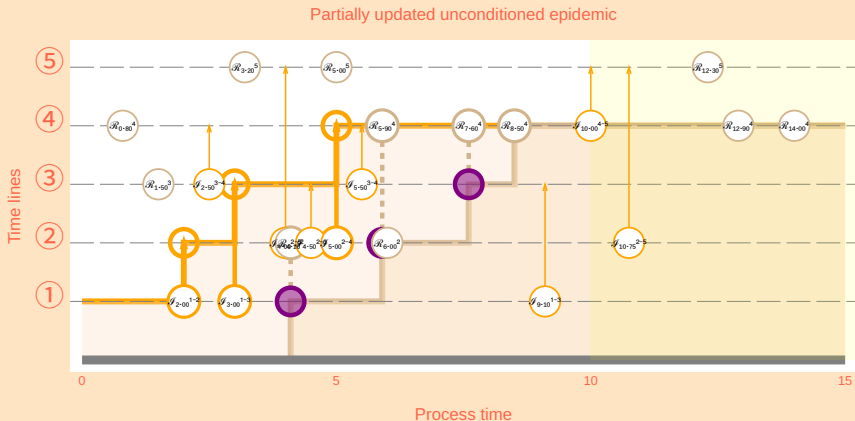
Figure 9: Re-sample last third of infections
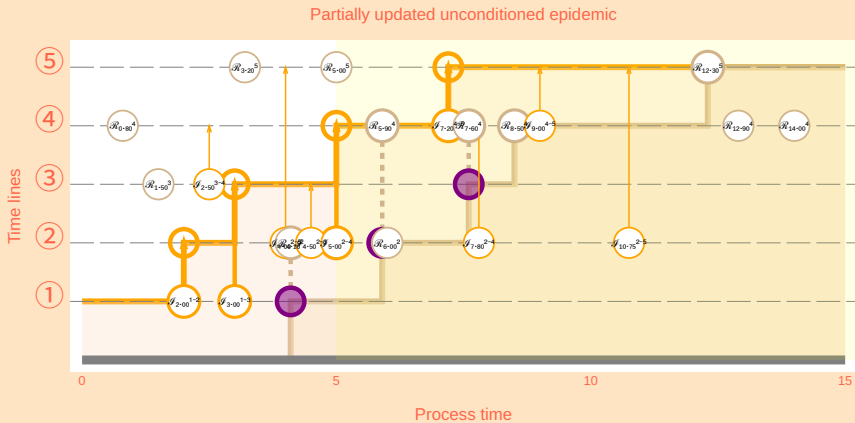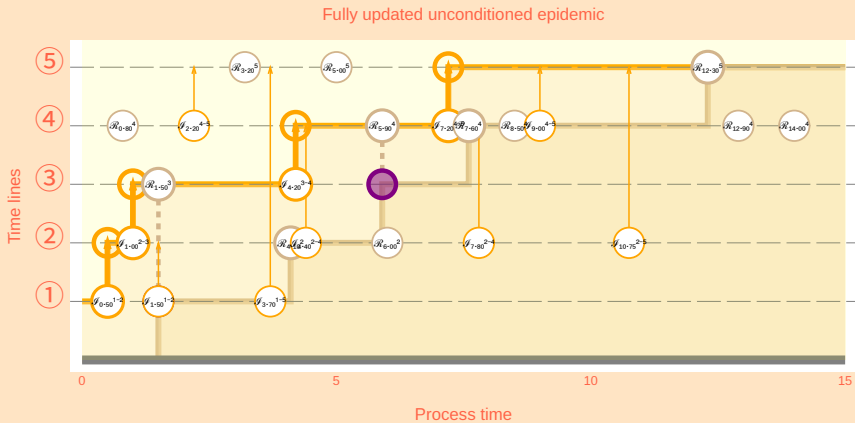
Figure 10: Re-sample last two-thirds of infections

Figure 11: All infections now re-sampled.

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- Does this produce a *feasible* and suitably monotonic algorithm?
- Housekeeping details required to establish that monotonicity still works. Key notions: *last feasible epidemic* (LFE) and *no-fly zone* (NFZ).
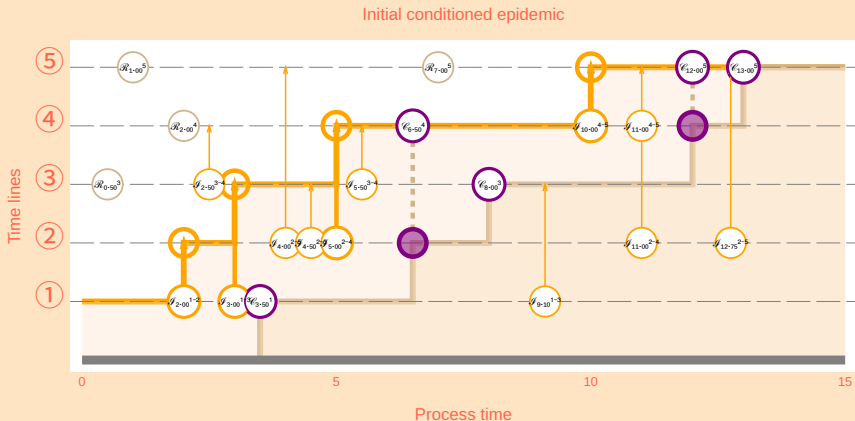
Initial conditional epidemic

Figure 12: Initial epidemic with conditioned removals indicated using purple circles (and purple disks when different timelines are infected).
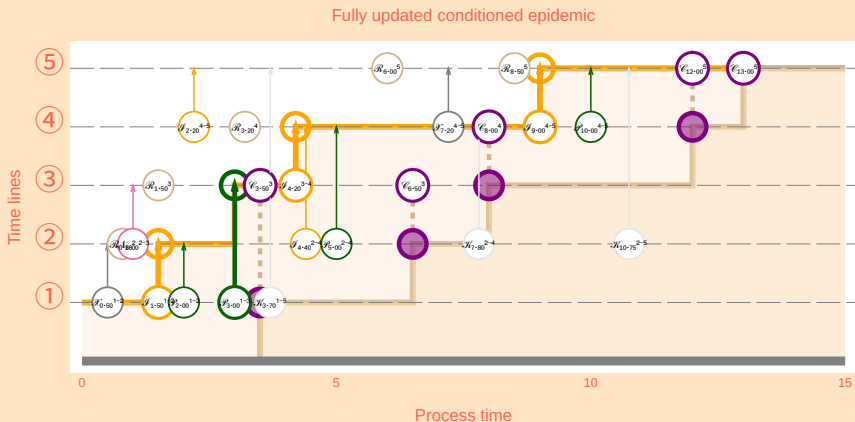
Conditional epidemic update

Figure 13: Epidemic updated under restriction: all conditioned removals remain activated, no new removals are activated. Green infections have been "perpetuated".
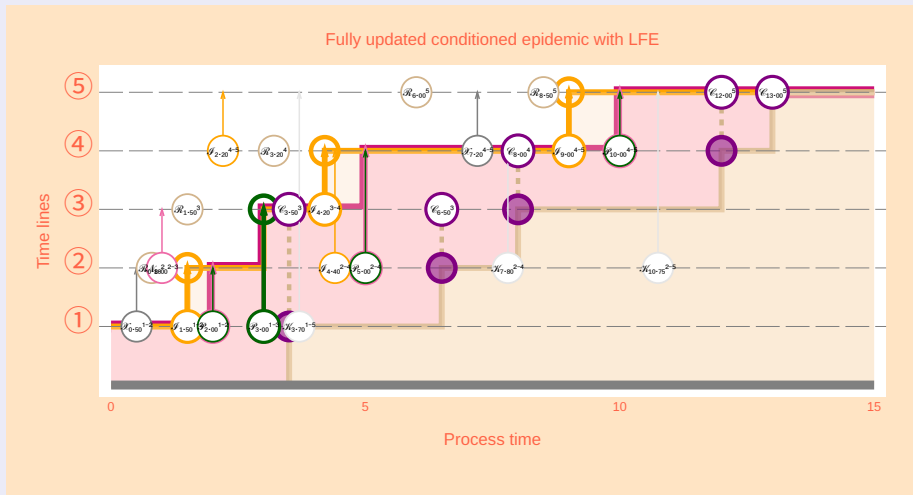
Last feasible epidemic (LFE)

Figure 14: LFE computed recursively working right-to-left: the slowest sequence of infections deals with all infected timelines in order (includes perpetuated infections).
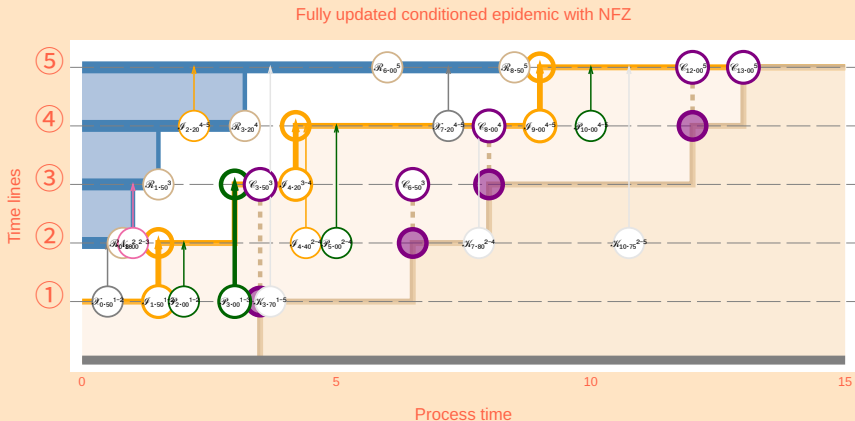
Figure 15: NFZ computed recursively working right-to-left: trace region of timelines that must not be infected if one is not to activate unobserved removals.

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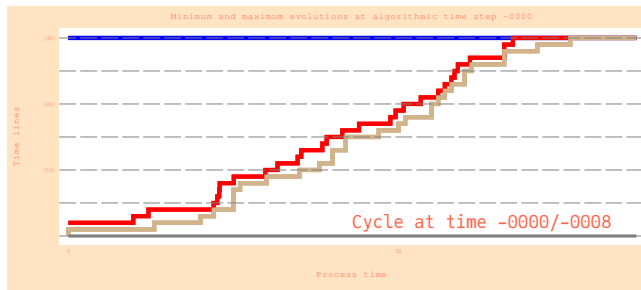 - ▶ *all* removals are recorded;

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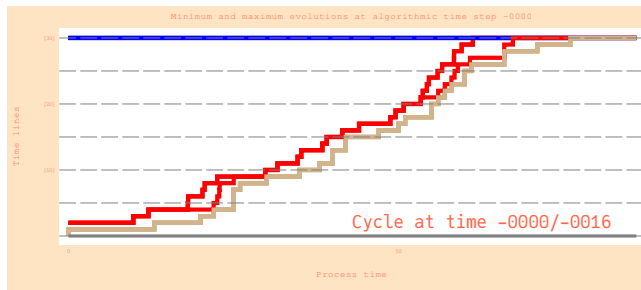 - ▶ no further removals after last observed removal (makes life easier).
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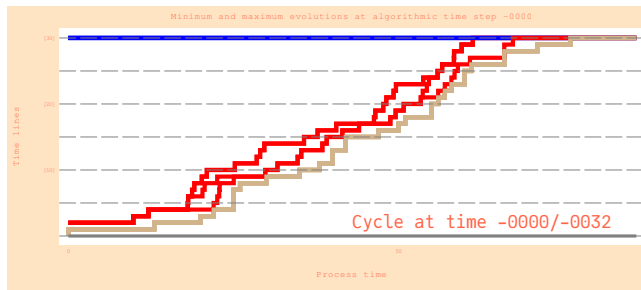
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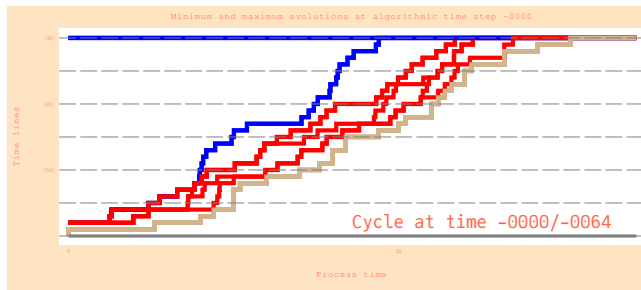
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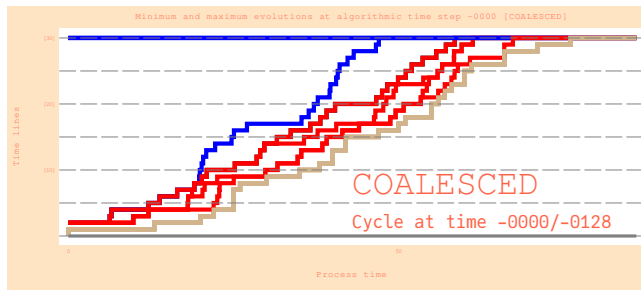Minimum and maximum evolutions at algorithmic time step -0000

Cycle at time -0000/-0064

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- Finally: generalize to other suitable compartment models?

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- Thank you for your attention! **QUESTIONS?**

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Image information

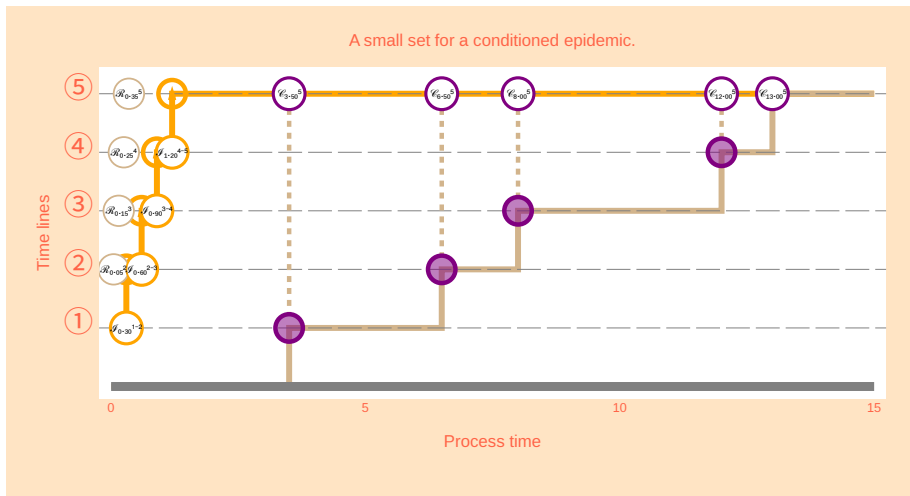| Image	Attribution	
Terry Pratchett	Luigi Novi	*CC BY 3.0*
Classic CFTP for a simple random walk	Result of code written by WSK	
Diamond Princess	Alpsdake	*CC BY-SA 4.0*
Epidemic CFTP images and animation	Result of code written by WSK	

A "near-maximal" configuration

Figure 16: A conditional epidemic in which all activated infections occur before time 3.0, also before smallest observed removal time.

Other technical information

Software versions

Software used in computations:

Software	Version	Branch	Date of last commit
Quarto	1.6.39	—	
Running under `julia`	1.11.2	—	
Module `EpidemicsCFTP`	2.2.487	main	Fri Jan 10 20:17:28 2025
Module `EpidemicsUtilities`	0.1.2.154	main	Wed Jan 15 13:23:36 2025
This Quarto script	2.2.585	master	Wed Jan 15 16:27:48 2025

Revision history

These notes were produced from `PerfectEpidemics.qmd`:

Author:	Wilfrid Kendall W.S.Kendall@warwick.ac.uk
Date:	Wed Jan 15 16:27:48 2025 +0000
Summary:	Editing for style.
	Added appendix on near-maximal configuration.