

Three centuries of random lines: from Buffon's needle to scale-invariant networks

Theo Cacoullos Memorial Lecture 2024

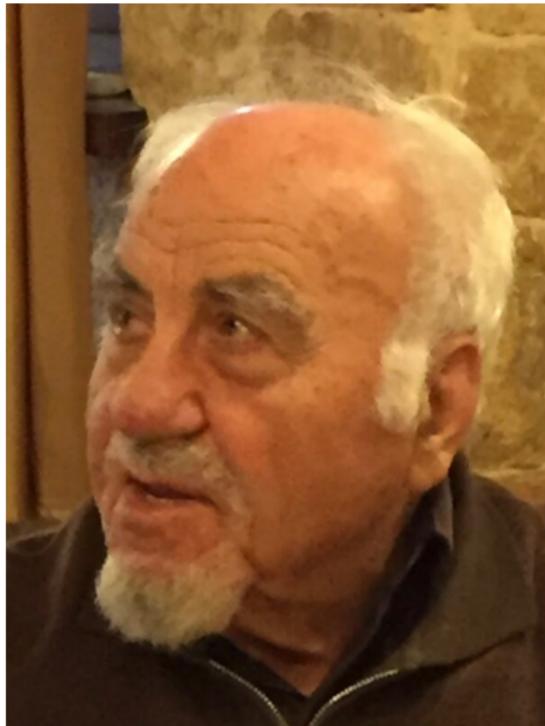
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Introduction



Cacoullos:

“An excellent teacher who never used notes
but did use spontaneous humor”

(IMS obituary November 16, 2021)

I count it a real honour to be invited to present the second Theo Cacoullos Memorial lecture. These remarks on his presentational style make it clear that Theo Cacoullos Memorial lecturers have much to live up to! I will do my very best.

Theophilos Cacoullos 1932-2020

Recent Origins: Particle Tracks in Apatite Crystals



An apatite crystal

- Nearly half a century ago, I assisted in a study ([Laslett *et al.*, 1982](#)) of fission track lengths in apatite crystals (tracks from fission of ^{238}U nuclei; crystal is sliced then etched to reveal intercepting tracks);
- *In principle* the length histogram indicates thermal hence geological history; but direct observation is massively **biased** by
 - (a) length selection bias,
 - (b) truncation (a kind of *Wicksell Problem*),
 - (c) projection effects (if non-horizontal);
- How to model a pattern of segments? (*Poisson segment process*: a kind of *Boolean model*);
- How to get good length histogram? (select *horizontal* enclosed tracks hitting tracks that intersect slice).

Historical Origins: pre-Revolutionary France



Georges-Louis Leclerc,
Comte de Buffon, 1707 – 1788

Buffon rose from undistinguished origins in the France of the *Ancien Régime*, to become a major and celebrated philosopher and scientist. He embodied a remarkable mixture of empiricism and rationalism.

“Writing well consists of thinking, feeling and expressing well, of clarity of mind, soul and taste ... The style is the man himself”

Throwing baguettes over one's shoulder (I)



Baguette as used by the Count.

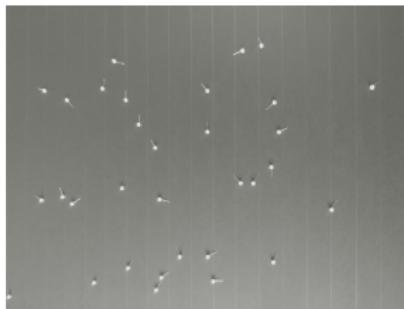


Table used by UCL experiment

- Comte de Buffon was very interested in probability (read him in Hey *et al.*, 2010);
- Probability from geometry not gambling;
- There ensued a whole sub-culture of experimental determinations of π (see Lazzarini, 1901; Stigler, 1991),

- Most recently at the inaugural UCL Bloomsbury Probability Colloquium March 2024;
- Strips at 3cm, pins at 1.5cm, with 24 intersections out of 45 throws.
- This estimates $\pi \approx 45/21 = 2.14 \dots$: the true value of π lies just inside the 95% confidence interval!

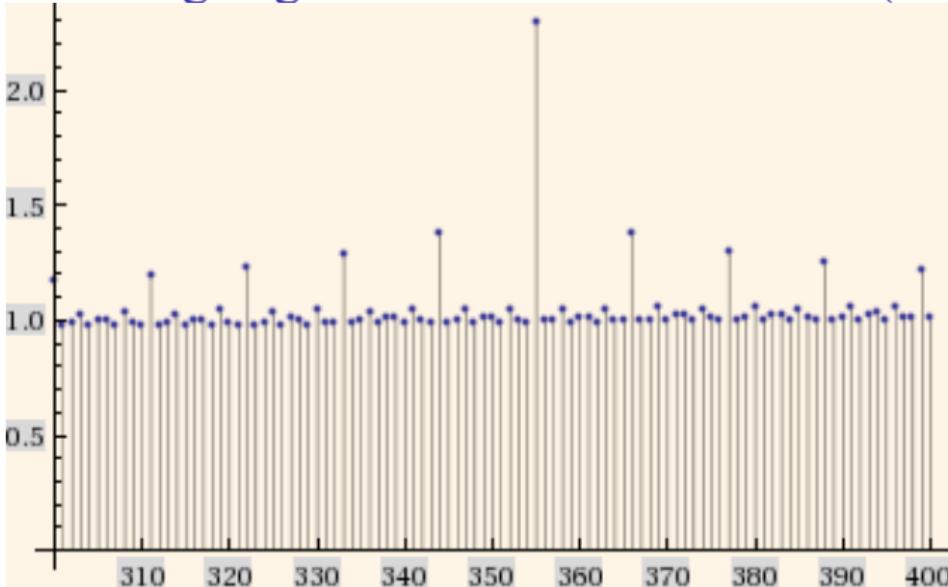
Throwing baguettes over one's shoulder (II)

Suppose you have to design such an experiment.

- Unit-length needle, unit-ruled floor. You must choose n the number of trials.
- The number of successes X is $\text{Binomial}(n, 2/\pi)$.
- Measure success by $1/\sqrt{|2/\pi - X/n|}$.
- Use your skill and judgement (and your balance between accuracy and patience!) to decide on total number of trials $n \in [300, 400]$

In the next slide, we graph the mean measure of success against n to see how much the choice of n actually makes any difference!

Throwing baguettes over one's shoulder (III)



Log of mean of $1/\sqrt{2/\pi - X/n}$ for X with distribution Binomial($n, 2/\pi$).

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{...}}}}}} \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + 1/1}} = \cfrac{355}{113} = 3.1415929203539825 .$$

Poisson line processes

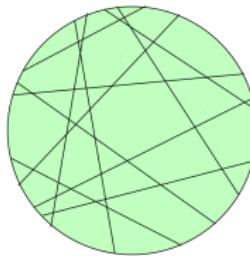


Figure 1: Poisson line process

- Parametrization (actually coordinatization of once-punctured projective plane) (**Chiu et al., 2013**):
 - ▶ invariant measure $\frac{1}{2}dr d\theta$;
 - ▶ Higher-dimensional analogues eg of lines in 3-space;
- Alternate parametrization:
 - ▶ invariant measure now $\frac{1}{2} \sin \phi dr d\phi$;
- Random line pattern using *Poisson point process*:
 - ▶ mention “void probability” approach;
- Simple (standard) line-process calculations:
 - ▶ mean number of lines hitting segment equals segment length;
 - ▶ mean number of lines hitting two sides a and b of a triangle but missing

Stochastic geometry of online maps

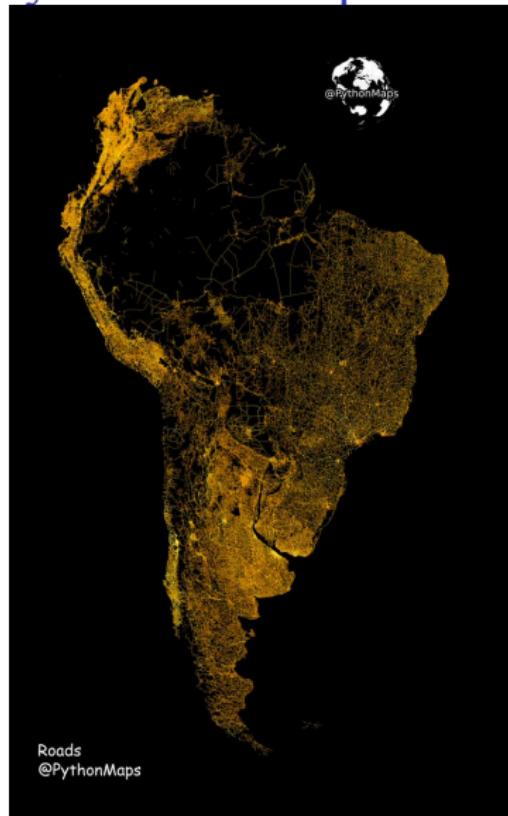
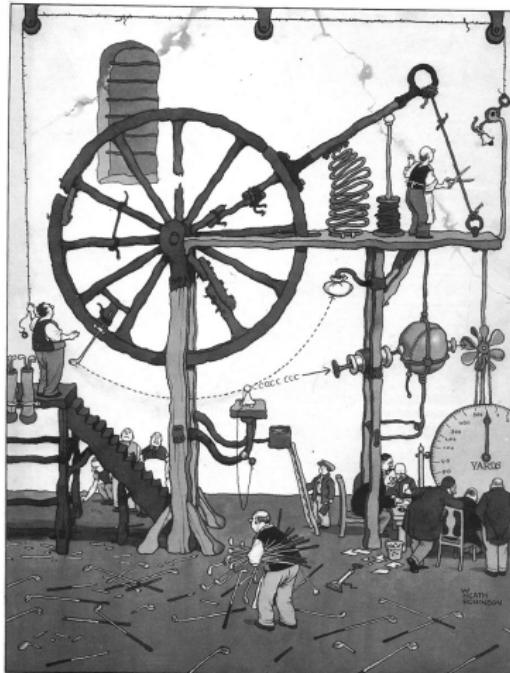


Figure 2: All the roads in South America

- Modelling transportation systems;
- Visual approach to axioms for online maps: - Holland and Northwest Germany;
 - ▶ Generating networks by a (random) mechanism;
 - ▶ Axioms:
 - ① Scale invariance ;
 - ② Finite mean-length;
 - ③ “SIRSN” property (Scale-Invariant Random Spatial Network).
 - ▶ Bertrand Russell: “The axiomatic method has many advantages over honest work.”
 - ▶ Settling the existential question:
 - randomized dyadic speed-marked planar lattice ([Aldous, 2014](#));
 - speed-marked Poisson line process ([Aldous, 2014](#); [Kahn, 2016](#); [WSK, 2017](#))
 - using the line process theory discussed above.
- More realistic models:
 - ▶ Sticks instead of lines;
 - ▶ Thick stick soup;
 - ▶ Thick Pareto stick soup;
 - ▶ Thick stiff fibre soup.

Some further remarks



Testing Golf Drivers

34

Figure 3: Much more work is still to be done

- “Thickness” is a simplifying assumption and still has reality problems:
 - ▶ using finite-mean-length sticks with infinite second-length-moment, means that *any* selected stick will have infinite-mean-length (length-biasing!);
 - ▶ reasonable to suppose that it suffices to have super-critical percolation for the configuration of fibres of greater than unit speed;
 - ▶ But stick percolation is *hard!* Still working to try to develop some relevant theory.
- Everything still works for lines in higher-dimensional space:
 - ▶ Dimension 3 is currently the only other case of interest (could be of interest when studying the human connective!);
 - ▶ Blanc has recently published nice work on an analogue in hyperbolic space.
- A big attraction of the theory in its present state is that one can consider relaxing parts of the scaling symmetry assumption. The “SIRSN effect” (Axiom 3) is probably still going to work if controlled spatial inhomogeneity is allowed. So this approach now provides a flexible framework for statistical and operational research modelling.
- We can introduce traffic, for example by supposing traffic along the connecting route to be uniformly generated between any pair of points. Camerons Loo (2017) worked this out for a simpler model and compared

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Technical information

Image	Original location	
Theo Cacoullos Apatite crystal	IMS Bulletin obituary, November 2021 Apatite_taillée.jpg	By Didier Descouens - Own work, CC BY 3.0
Comte de Buffon Baguette Buffon and Number Theory Buffon Table Poisson line process S. American Roads Heath Robinson	Buffon_1707-1788.jpg baguette-de-pain-.jpg WSK calculation By kind permission of Terry Soo (UCL) WSK calculation By kind permission of @PythonMaps Testing Golf Drivers	Free of copyright Free stock photo - Public Domain

Poisson line process

These notes were produced from Kozani-Cacoullos-2024.qmd:

Author: Wilfrid Kendall W.S.Kendall@warwick.ac.uk
Date: Fri May 10 17:58:53 2024 +0100
Summary: All pictures now logged.
Completed Origins and Baguettes.
Remainder needs full processing!

