Perfect Epidemics

Applied Probability Seminar
Department of Statistics, University of Warwick

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Warwick, York

17 January 2025

Introduction

IMAGE

QUOTE

Handout available on the web: either use the `QR-code`

or visit https://wilfridskendall.github.io/talks/PerfectEpidemics/.

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- Simplest possible example: *random-walk-CFTP*
  (can boost to use Ising model to do simple image reconstruction).

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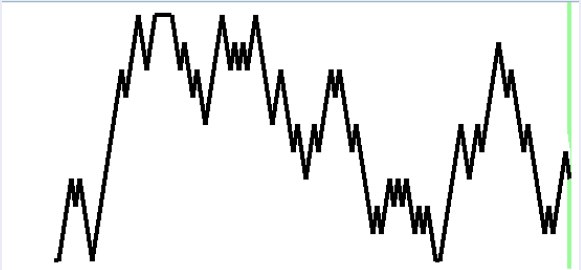
- Conventional MCMC picks a starting point, then runs the simple random walk for long time till approximate equilibrium.



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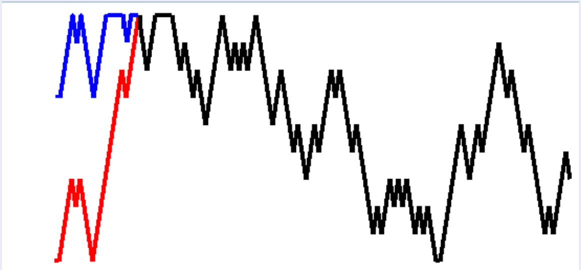

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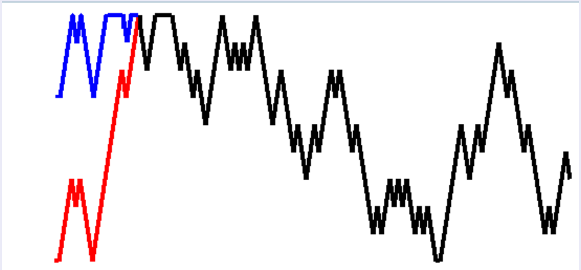
- Generally not true that location at coupling is a draw from equilibrium.



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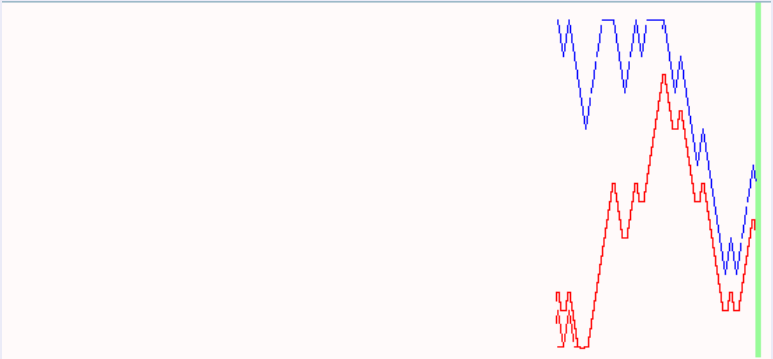

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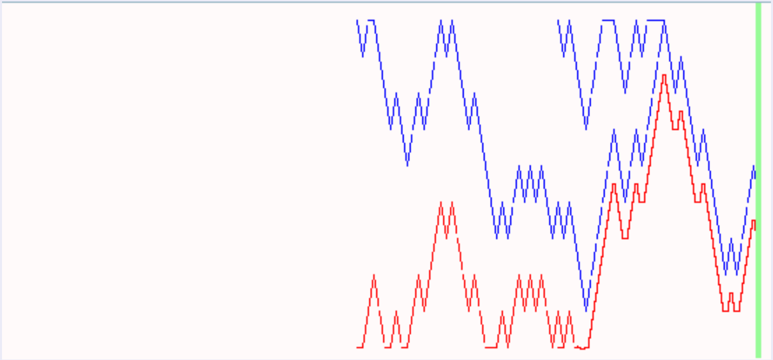

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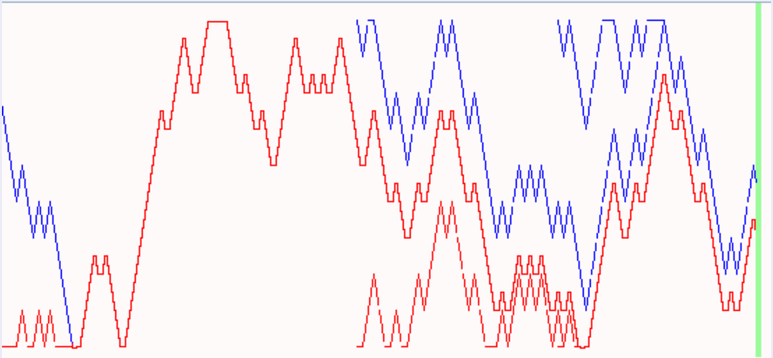

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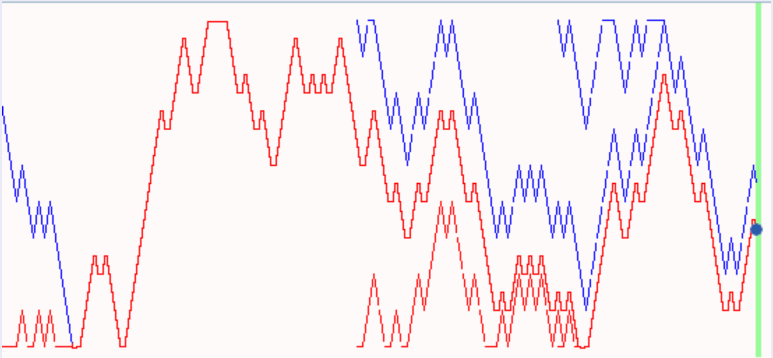

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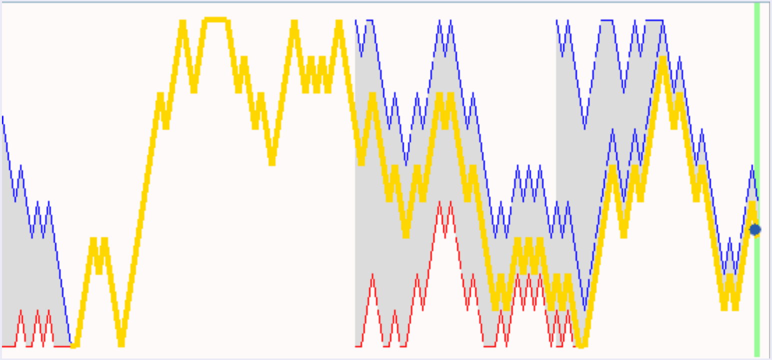
- The common value is an exact draw from equilibrium!



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- Detailed expositions: WSK (2005), Huber (2015). (Want to implement CFTP in R? see WSK, 2015.)

3: Perfect Epidemics: a challenge problem for CFTP

Many important inferential questions (Cori & Kucharski, 2024).

Simplest models (versus UK model with 10^6 agents!, Fraser & Others, 2023):

S-I-R deterministic epidemic: susceptibles s, infectives i, removals r (constant total population $s + i + r = n$):

$$\begin{aligned} s' &= -\alpha\, s\, i\,, \\ i' &= (\alpha\, s - \beta\,)\; i\,, \\ r' &= \beta\, i\,. \end{aligned}$$

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S-I-R stochastic epidemic: a Markov chain (S, I, R) with transitions

Infection:	$S \to S - 1\,,$	$I \to I + 1$	at rate	$\alpha\, S\, I\,,$
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Massive assumption for these simple models: **homogeneous mixing**.

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Wikipedia: "The British-registered *Diamond Princess* was the first cruise ship to have a major `[COVID-19]` outbreak on board, with the ship quarantined at Yokohama from 4 February 2020 for about a month. Of 3711 passengers and crew, around 700 people became infected and 9 people died."

Evidently $\alpha\ s_0/\beta \gg 1$ – as was sadly later confirmed, a sorrow for us all.

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5. Can we use perfect simulation?

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An absurdly simple variant on contact tracing:

"When did the infections occur. if we only observe removals?"

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4. Evolve whole S-I-R trajectory in *algorithmic time* (alter potential infections and removals using immigration-death in discrete algorithmic time).
5. Result: *trajectory-valued chain*, unconditioned S-I-R as equilibrium.

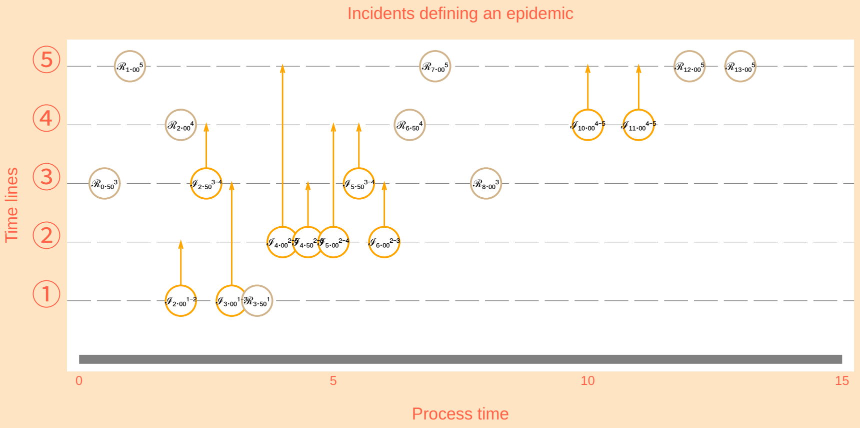

Figure 1: Light-orange circles denote potential infections (arrows point upwards to targets); light-brown circles denote potential removals.

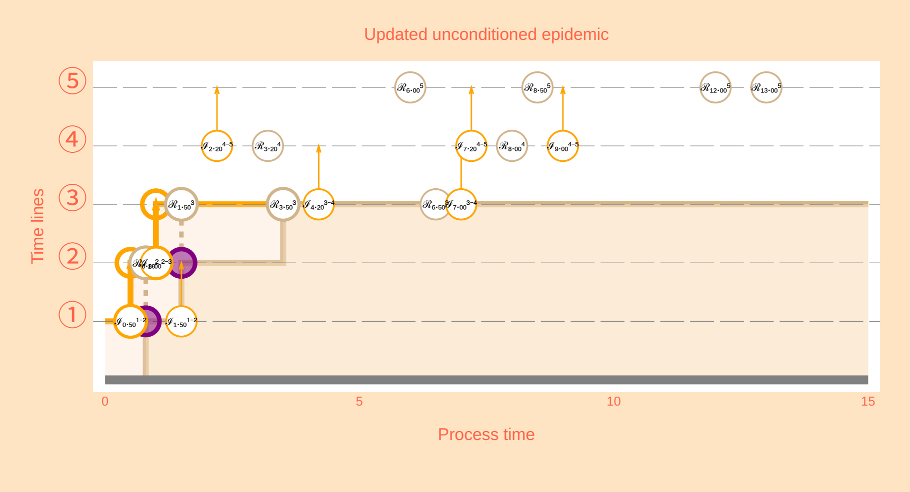

Figure 3: A step in algorithmic time for the unconditioned epidemic simply involves replacing all incidents by a new set of incidents.

Crucial technical point

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- Updates in algorithmic time τ are then (algorithmic-)*time-reversible*: so restriction to subset S of state-space (in our case, *activated* removals occurring precisely at specified set of times) implies a new equilibrium which is the old equilibrium conditioned to lie in S.
- For later purposes it is convenient to stage the replacement as follows:
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- It is convenient to re-express this for continuously varying τ:
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- Crucially, step 2 ensures composition action is irreducible over S! (So equilibrium under conditioning is unique.)

Illustration of technical point (1/8)

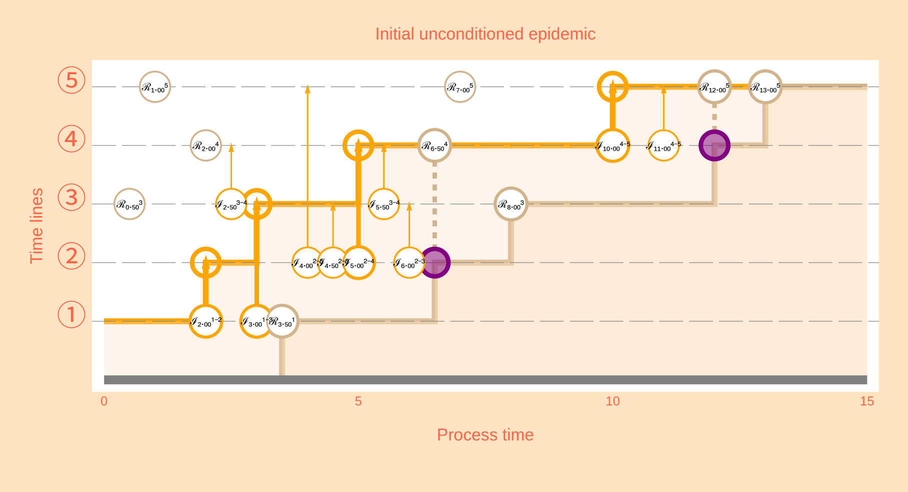

Figure 4: No change to removals or infections

Illustration of technical point (2/8)

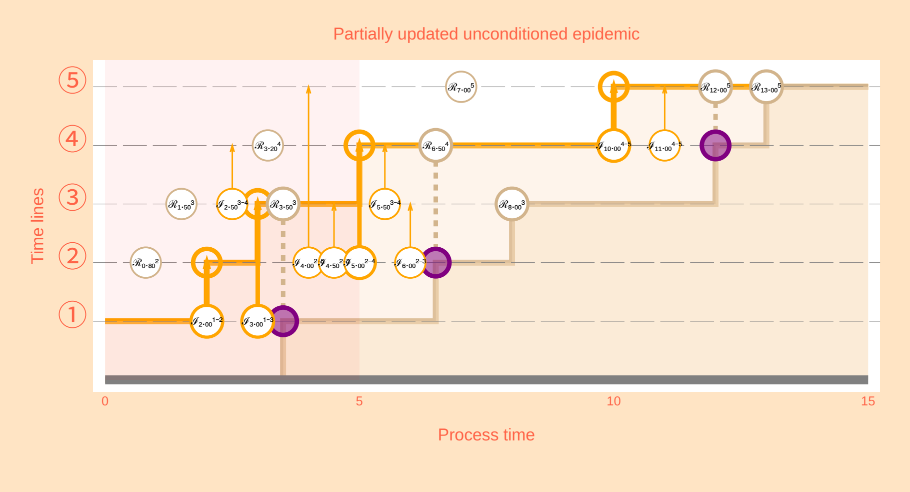

Figure 5: Replace first third of removals, infections unchanged

Illustration of technical point (3/8)

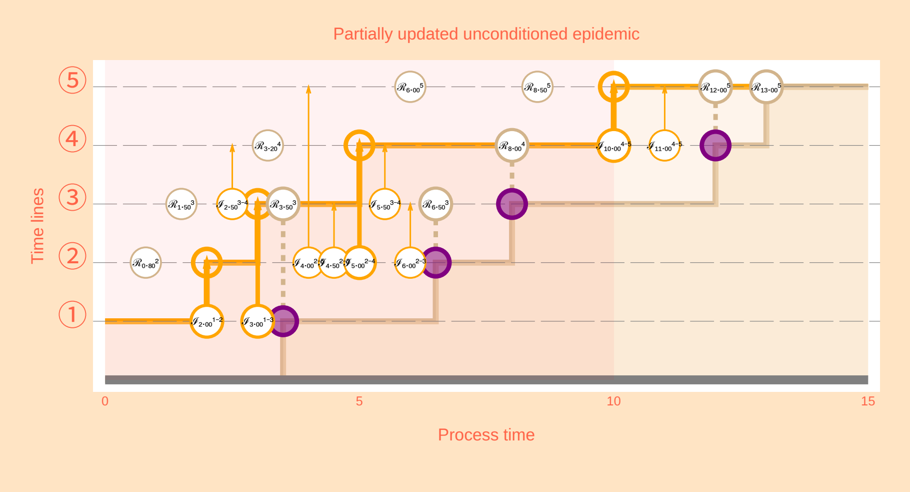

Figure 6: Replace first two-thirds of removals, infections unchanged

Illustration of technical point (4/8)

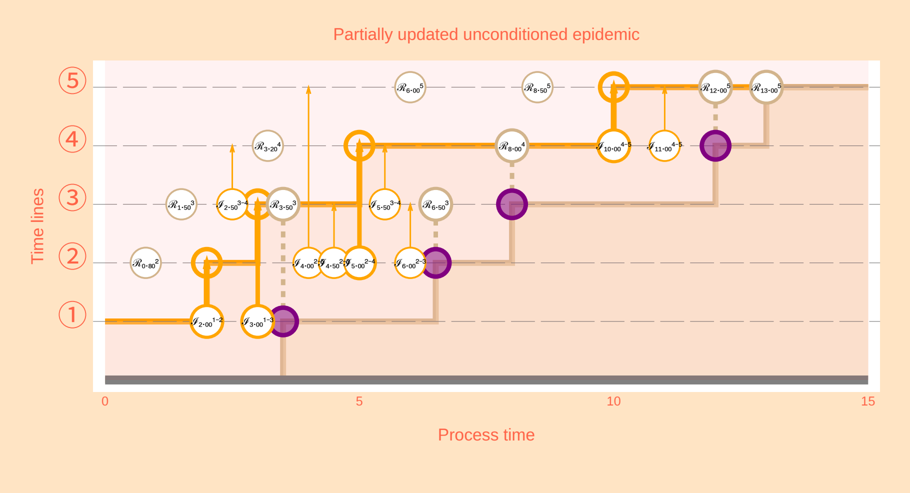

Figure 7: Replace all removals, infections unchanged

Illustration of technical point (5/8)

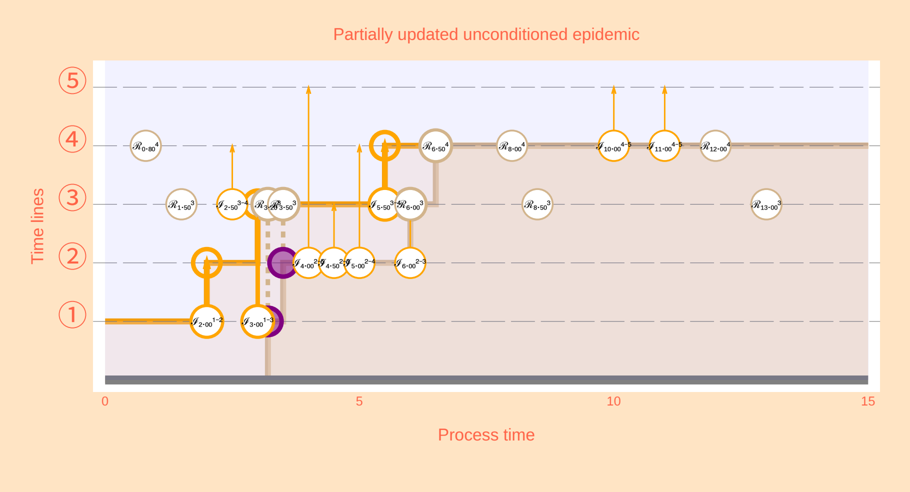

Figure 8: Re-sample all removal timelines, infections unchanged

Illustration of technical point (6/8)

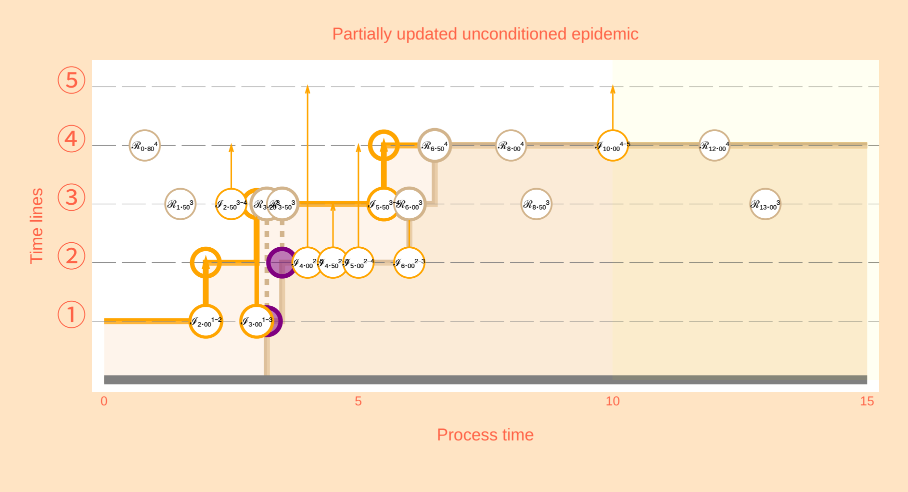

Figure 9: Re-sample last third of infections

Illustration of technical point (7/8)

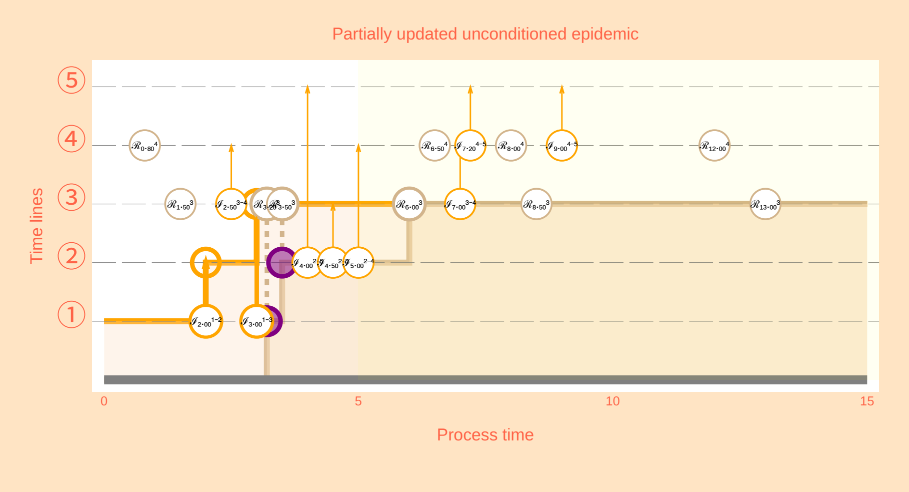

Figure 10: Re-sample last two-thirds of infections

Illustration of technical point (8/8)

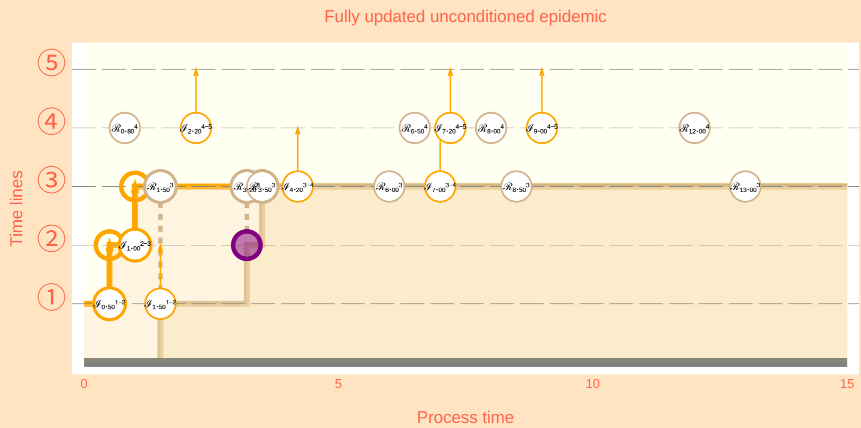

Figure 11: Re-sample all infections

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- Does this produce a *feasible* and suitably monotonic algorithm?
- Housekeeping details required to establish that monotonicity still works. Key notions: *last feasible epidemic* (LFE) and *no-fly zone* (NFZ).

Initial conditional epidemic

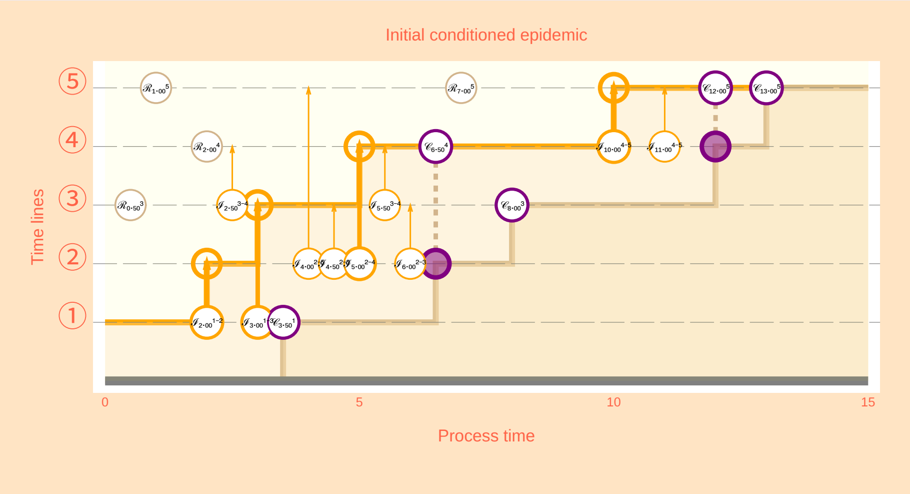

Figure 12: Initial epidemic with conditioned removals indicated using purple circles (and purple disks when different timelines are infected).

Conditional epidemic update

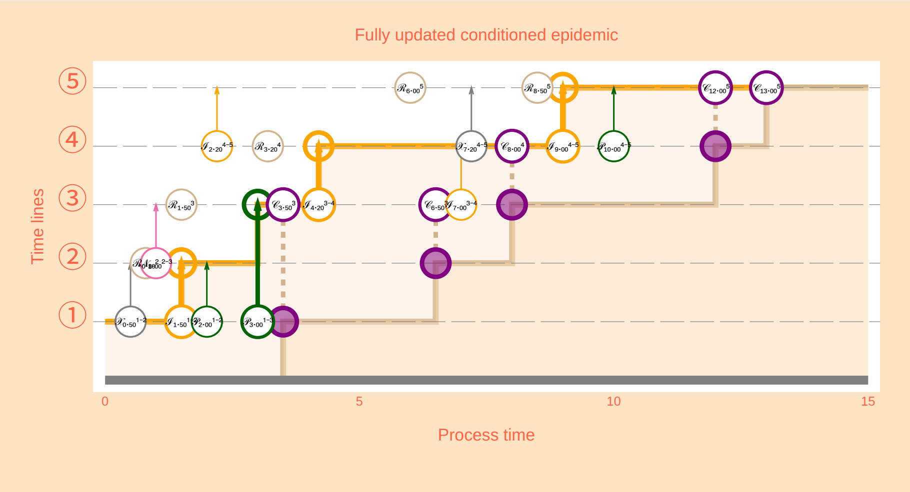

Figure 13: Epidemic updated under restriction: all conditioned removals remain activated, no new removals are activated. Green infections have been “perpetuated”.

Last feasible epidemic (LFE)

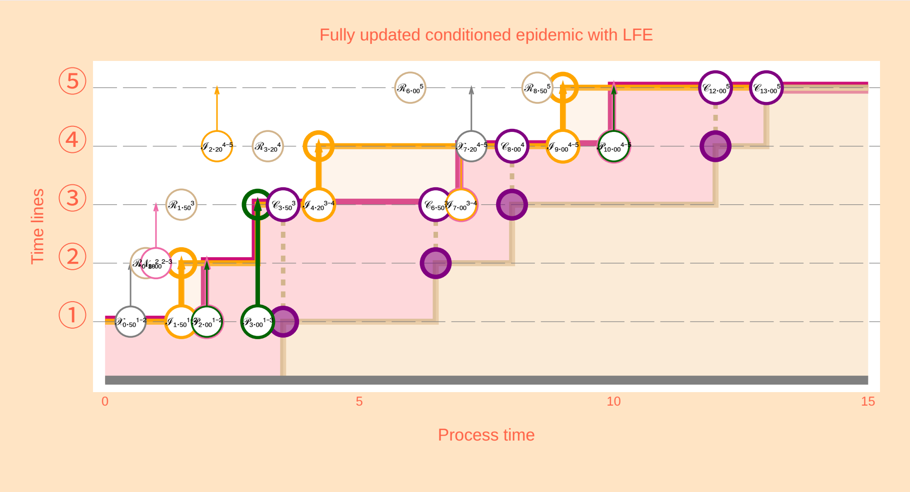

Figure 14: LFE computed recursively working right-to-left: the slowest sequence of infections deals with all infected timelines in order (includes perpetuated infections).

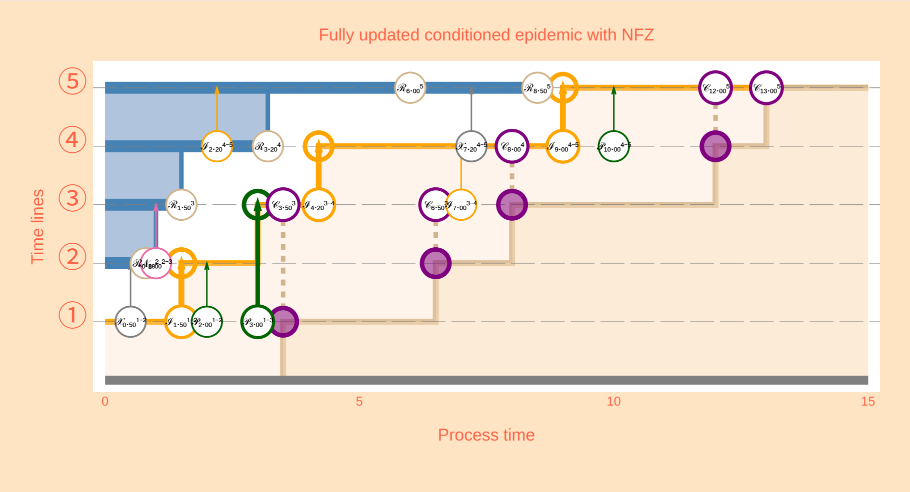

Figure 15: NFZ computed recursively working right-to-left: trace region of timelines that must not be infected if one is not to activate unobserved removals.

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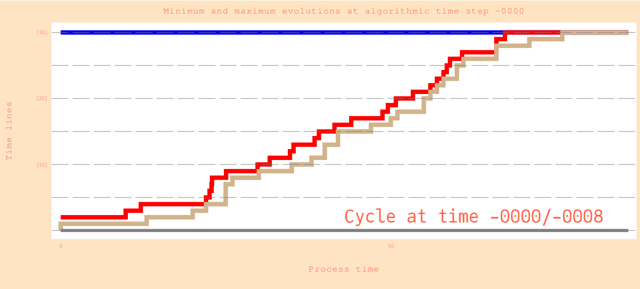

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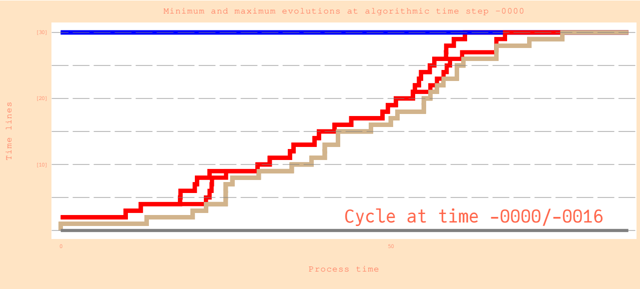

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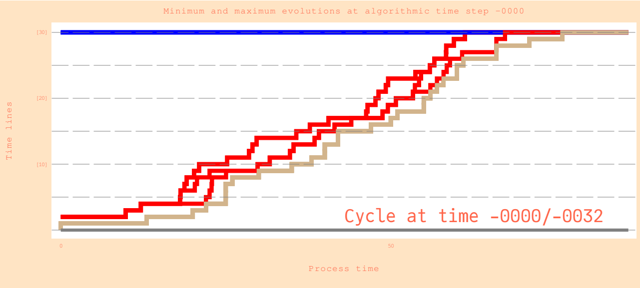

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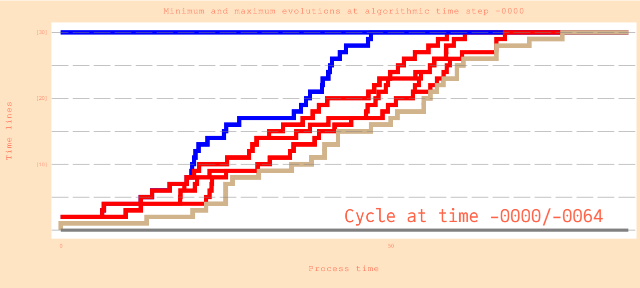

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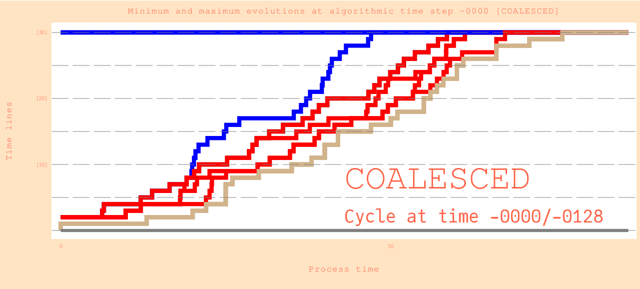

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- Finally: generalize to other suitable compartment models?

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- Thank you for your attention! QUESTIONS?

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Technical information

Image	Attribution	
Classic CFTP for a simple random walk	Result of code written by WSK	
Diamond Princess	Alpsdake	*CC BY-SA 4.0*
Epidemic CFTP images and animation	Result of code written by WSK	

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Summary:	Ameding `figures.qmd`.

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