

# Perfect Epidemics

## Seminar at University College Dublin

W S Kendall    S B Connor

Warwick, York

20 October 2025



# Introduction

Homage to Dublin  
(Book of Kells, 9th century)



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Work on perfect simulation ([CFTP](#)) for epidemics, now being written up.  
WSK acknowledges the support of UK EPSRC grant EP/R022100.



Handout is on the web: use the QR-code or visit  
[wilfridskendall.github.io/talks/PerfectEpidemics](https://wilfridskendall.github.io/talks/PerfectEpidemics).

# Plan of talk

*Gregory:* Is there any other point to which you would wish to draw my attention?

*Holmes:* To the curious incident of the dog in the night-time.

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- (e) Example with real data.

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- ➍ Simplest possible example: *random-walk-CFTP*  
(can boost to use Ising model to do simple image reconstruction).

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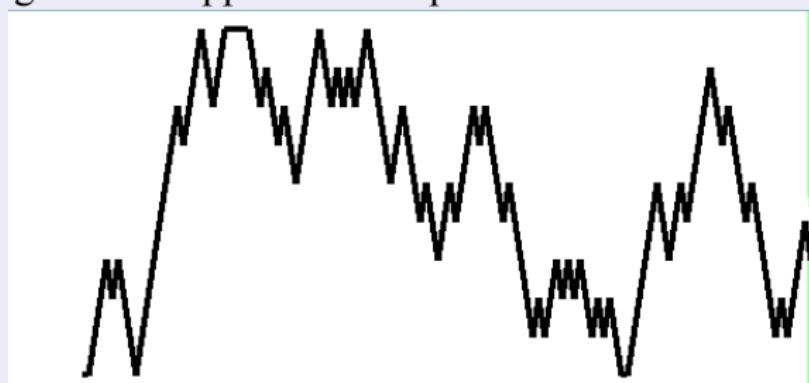
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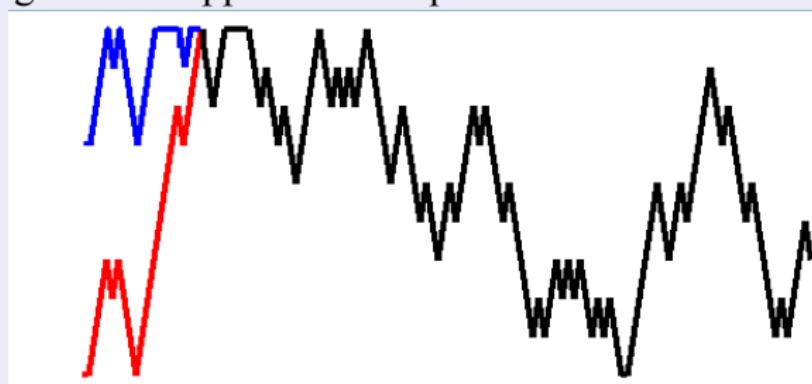
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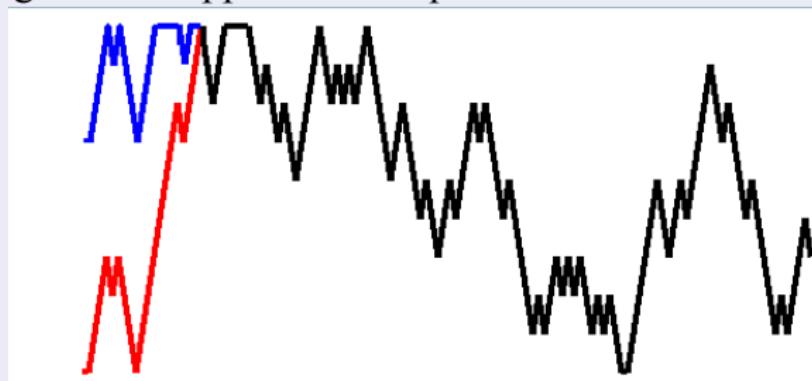
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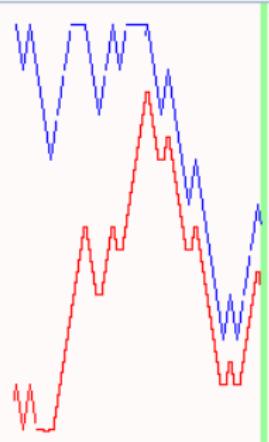
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- ④ Generally **not true** that location *at coupling* is a draw from equilibrium.

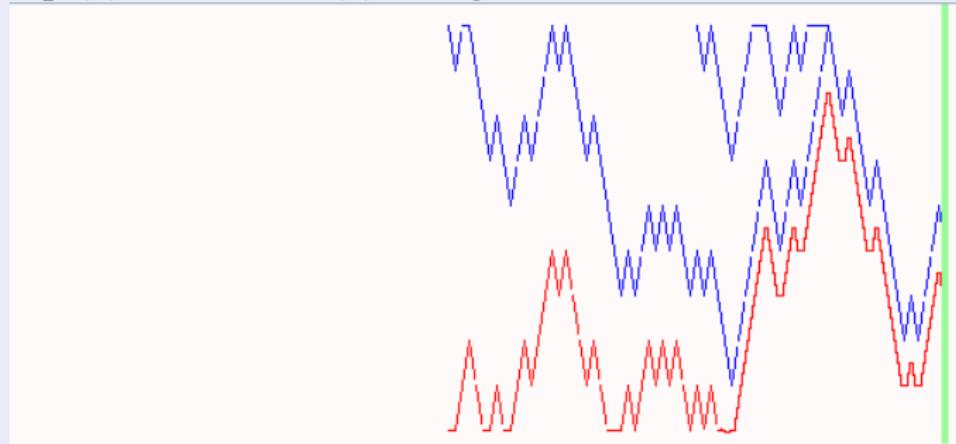
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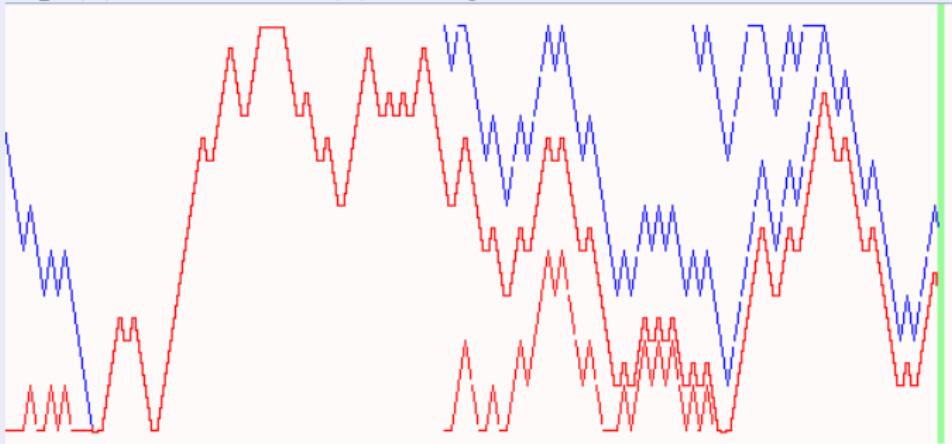


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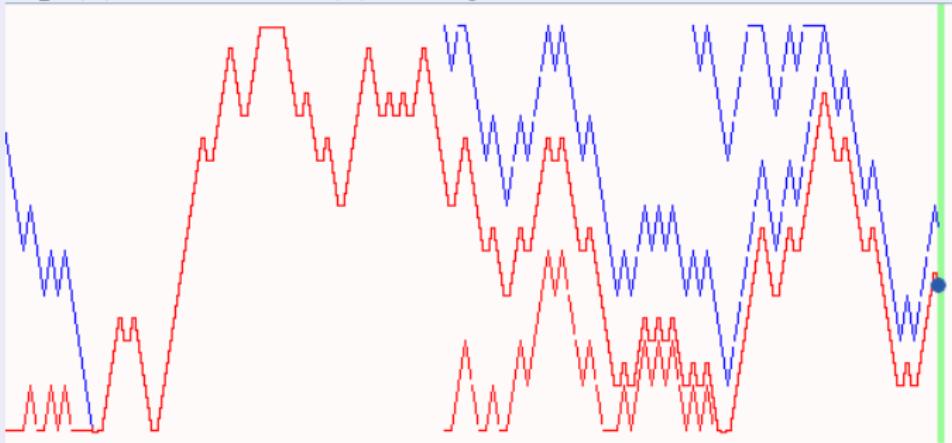
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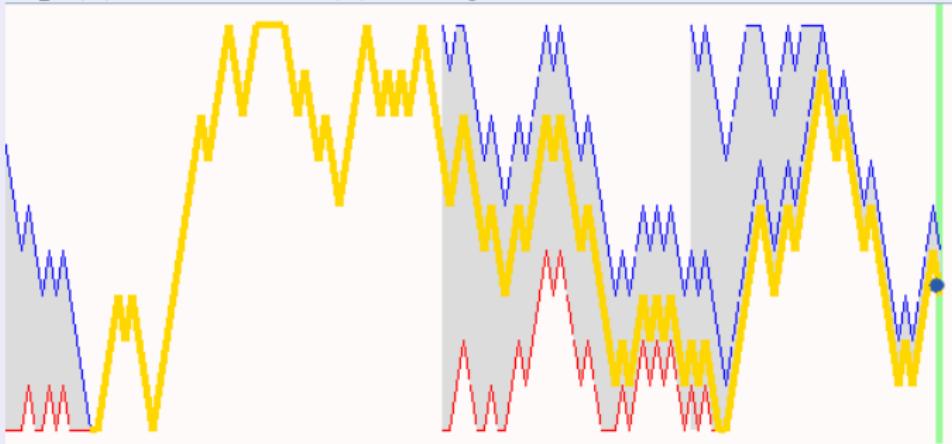
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- ④ When coupled, top and bottom yield a common value at time 0.
- ⑤ The common value (golden thread) is an exact draw from equilibrium!

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- ④ Detailed expositions: WSK (2005), Huber (2015).  
(Want to implement CFTP in R? see WSK, 2015.)

## 2. Perfect Epidemics: a challenge problem for CFTP

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Both models share an unrealistic assumption: homogeneous mixing.

In contrast, Fraser *et al* (2023) deploy a UK model with  $N=10^6$  agents!

There are *many* important inferential questions (Cori & Kucharski, 2024).



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*Wikipedia*: “The British-registered *Diamond Princess* was the first cruise ship to have a major [COVID-19] outbreak on board, with the ship quarantined at Yokohama from 4 February 2020 for about a month. Of 3711 passengers and crew, around 700 people became infected and 9 people died.”

Evidently  $\alpha s_0 / \beta \gg 1$  – as was sadly later confirmed, a sorrow for us all.



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- ➎ Can we use **perfect simulation**?

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The simplest possible variant of contact tracing:

“When did the infections occur, supposing we only observe removals?”

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- ④ First step: evolve whole **S-I-R** trajectory in *algorithmic time* (alter potential infections and removals using immigration-death in discrete algorithmic time).

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- ④ First step: evolve whole **S-I-R** trajectory in *algorithmic time* (alter potential infections and removals using immigration-death in discrete algorithmic time).
- ⑤ Result: *trajectory-valued chain*, unconditioned **S-I-R** as equilibrium.

# From incidents to unconditioned epidemic trajectories (1/3)



Figure 1: Light-orange circles denote potential infections (arrows point upwards to targets); light-brown circles denote potential removals.

## From incidents to unconditioned epidemic trajectories (2/3)

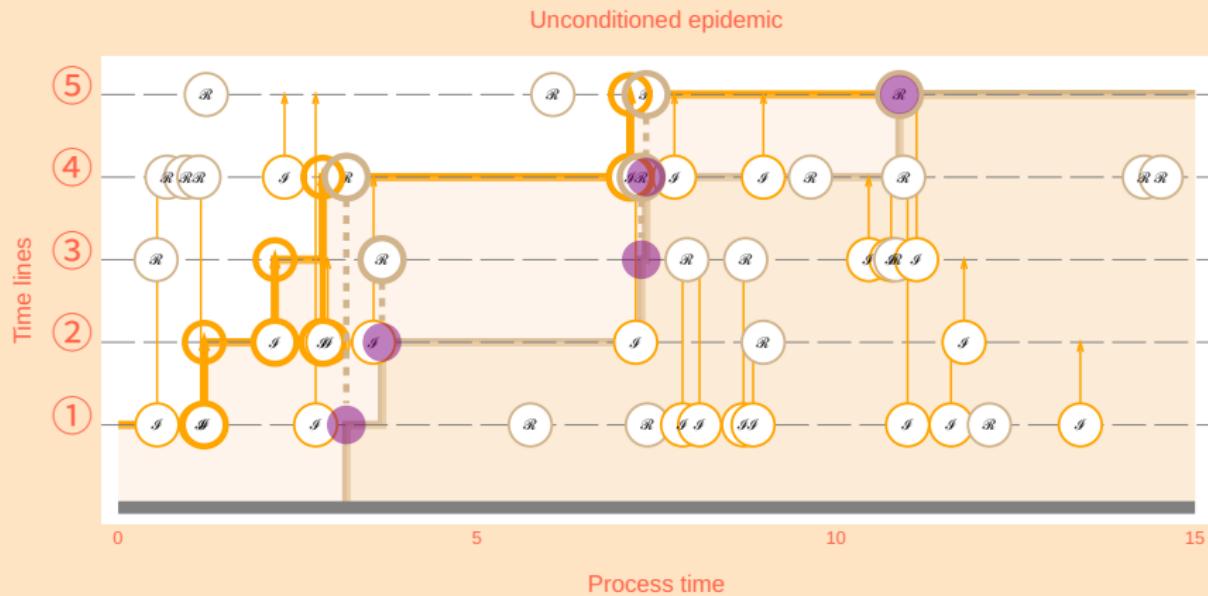


Figure 2: (a) *Infections* activate if on infected timeline and pointing to lowest uninfected timeline; (b) *Removals* activate if on infected timeline; remove lowest infected (purple disk).

# From incidents to unconditioned epidemic trajectories (3/3)

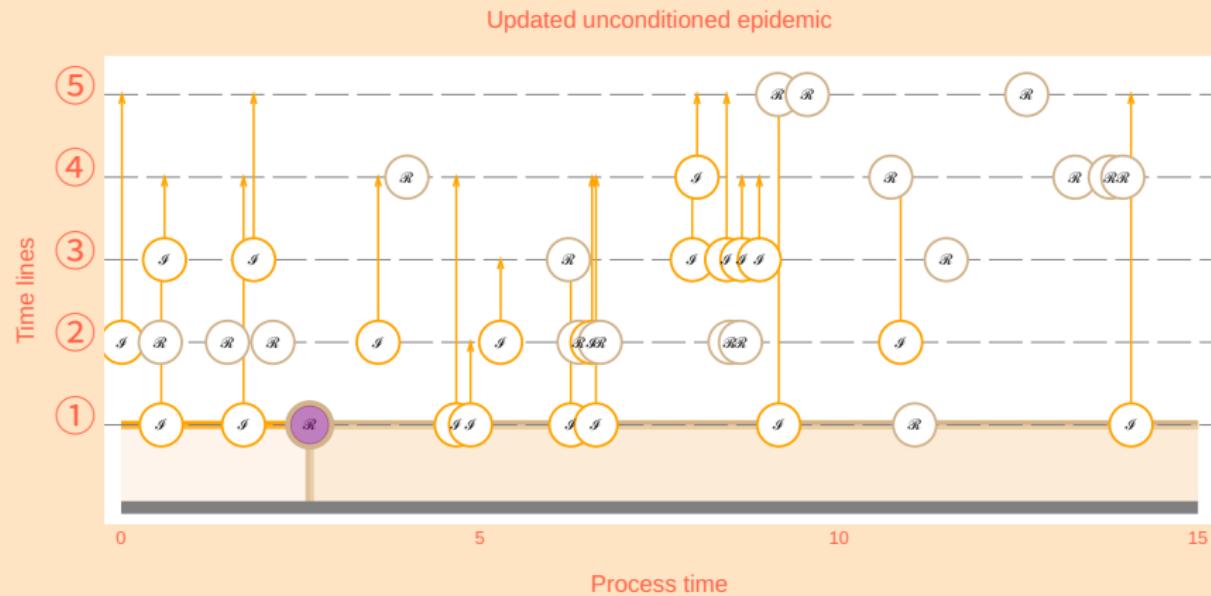


Figure 3: A step in algorithmic time for the unconditioned epidemic simply involves replacing all original incidents by an entirely new set of incidents.

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- Thus the original update is expressible as a (continuous) composition of updates, each of which satisfies detailed balance in equilibrium.
- The connection “restriction=conditioning” still holds.
- Crucially, re-sampling step 2 ensures composite evolution is irreducible over  $S$ ! (So equilibrium under conditioning is unique.)

# Free evolution evolving in continuous algorithmic time

GIF MP4



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- Does this produce a *feasible* and suitably *monotonic* algorithm?
- **Housekeeping details** used to establish that monotonicity still works: *laziest feasible epidemic (LFE)* and *no-fly zone (NFZ)*.

# Initial conditioned epidemic

The initial conditioned epidemic

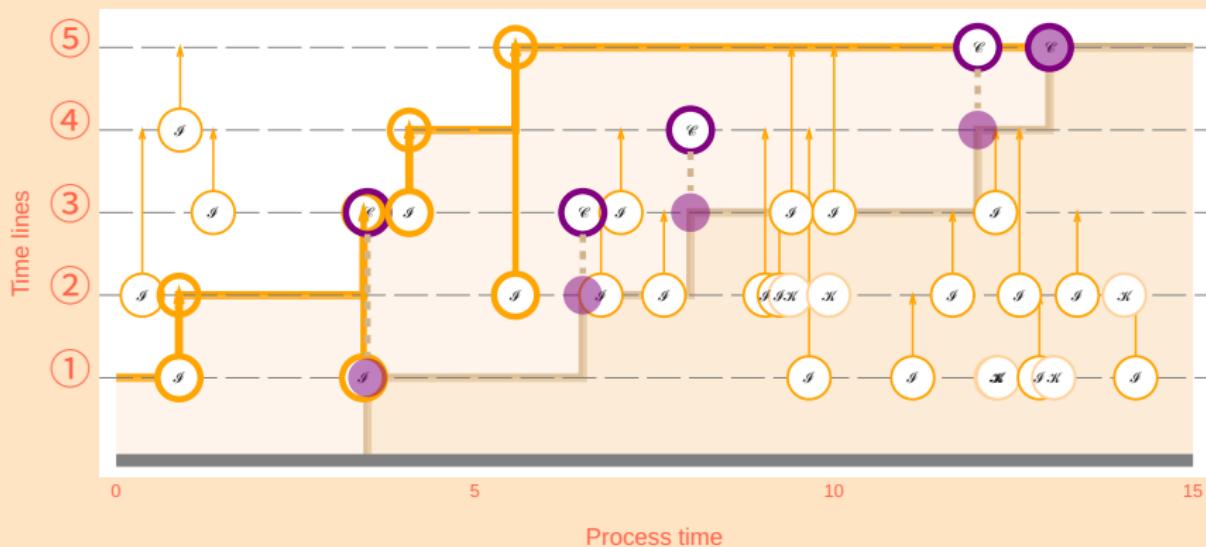


Figure 4: Initial conditioned epidemic, with conditioned removals indicated using purple circles (and purple disks when non-target timelines are infected).

# Conditional epidemic update

Fully updated conditioned epidemic

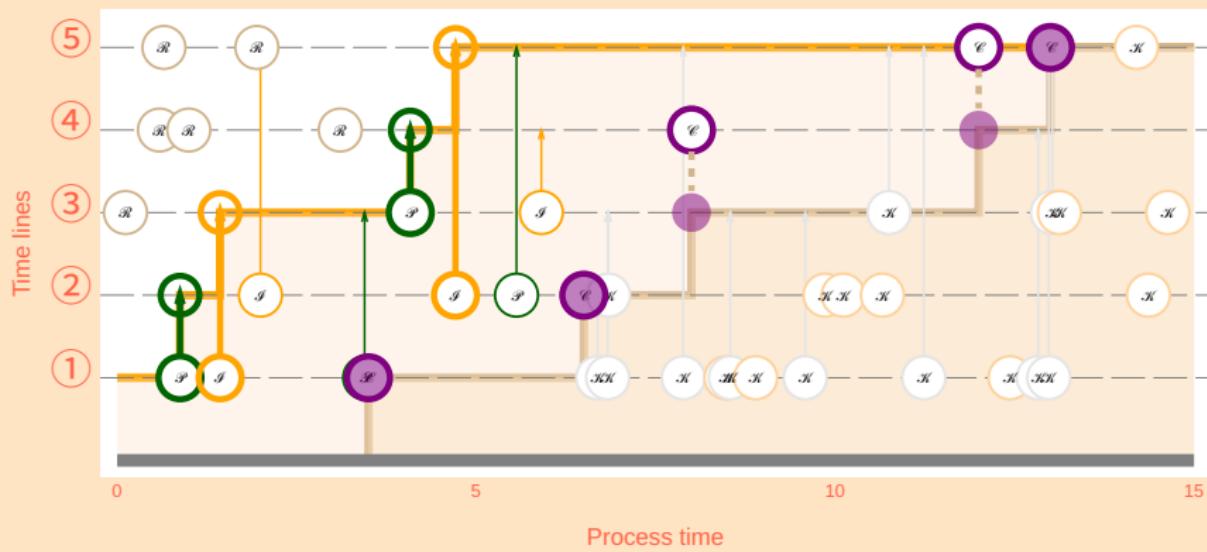


Figure 5: Epidemic updated under restriction: all conditioned removals remain activated, no new removals are activated. Green infections have been “perpetuated”.

# Laziest feasible epidemic (LFE)

Fully updated conditioned epidemic with LFE

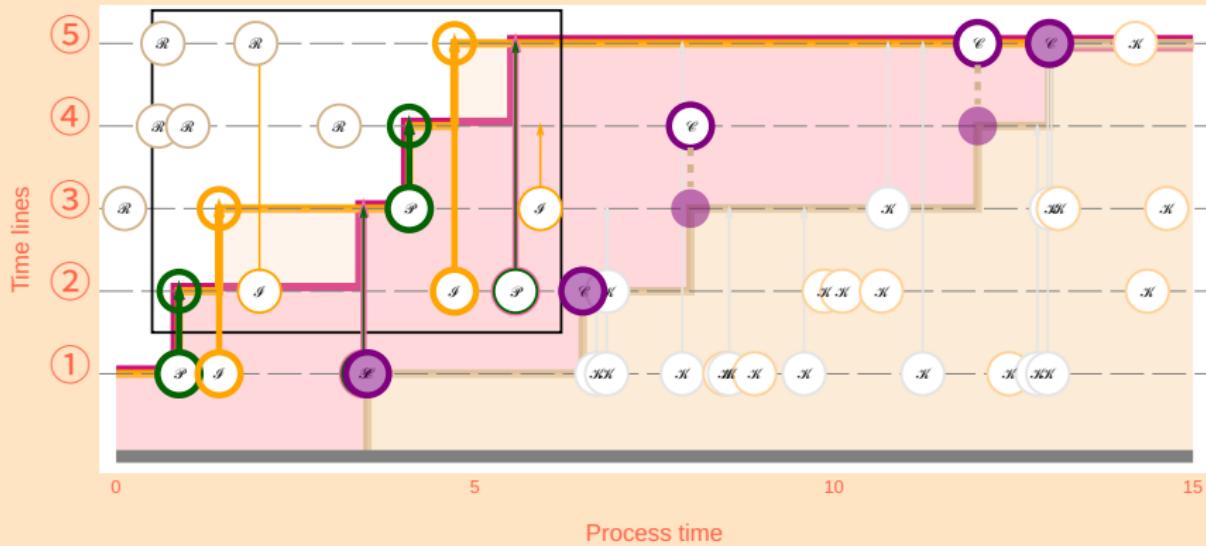
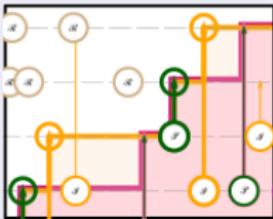


Figure 6: LFE computed recursively working right-to-left: slowest sequence of infections (and perpetuated infections) generating all conditioned removals. Can be used to identify perpetuated infections.

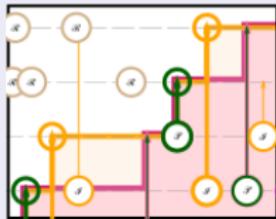
## LFE: construction details



① Recursive definition of LFE: working over  $[0, T)$  with  $s_{N+1} = T$  and

Choose (maximally)  $s_i$  with  $\mathcal{P}_{s_i}^{j < i+1} / \mathcal{I}_{s_i}^{j < i+1}$   
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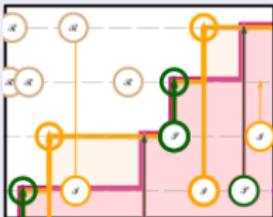
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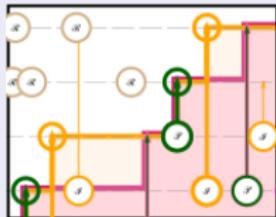
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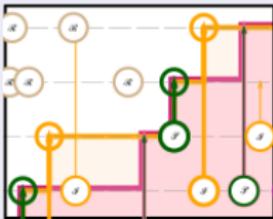
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- ➌ Comparisons based on intrinsic definition show monotonic dependence  
of LFE on old epidemic history.

Fully updated conditioned epidemic with NFZ

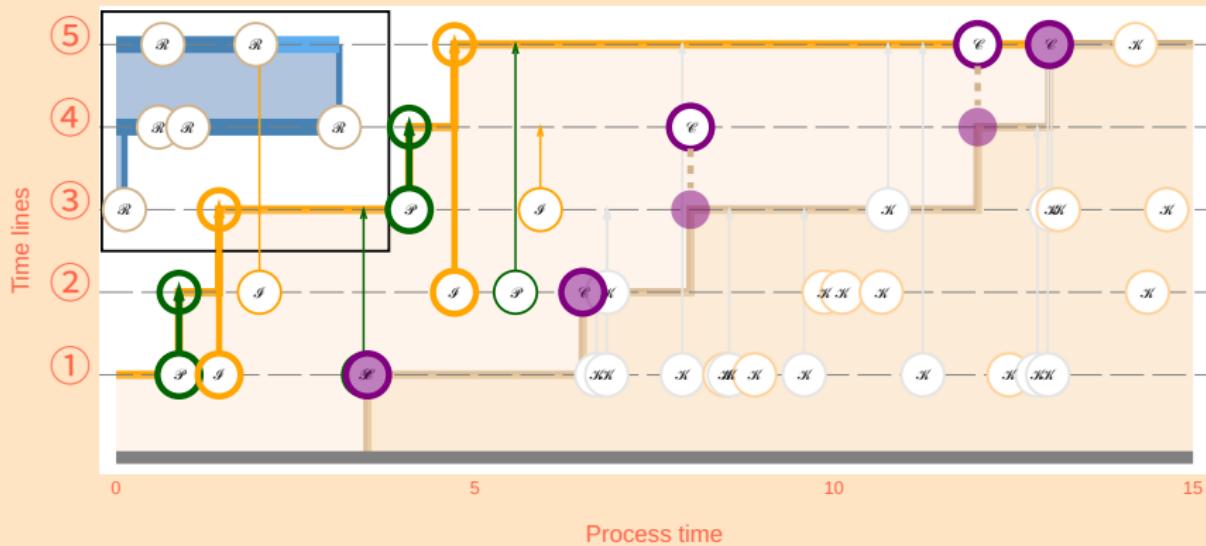
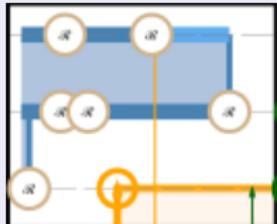


Figure 7: NFZ computed recursively working right-to-left: it traces a region of timelines such that unobserved removals are not activated if region not infected.

## NFZ: construction details

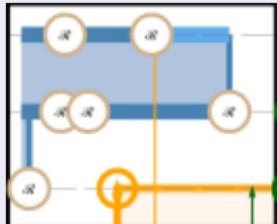


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Full monotonicity (hence CFTP) follows by showing new epidemic history depends monotonically on LFE and NFZ.

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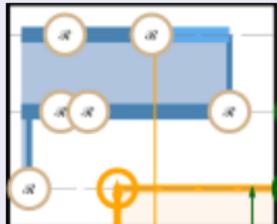


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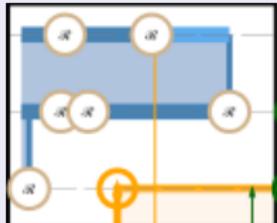


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- ➋ Can then show monotonic dependence of NFZ on old epidemic history.

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## Conditioned evolution evolving in continuous algorithmic time

GIF MP4

If a new  $\mathcal{I}_t^{i < j}$  has  $i, j$  in infected zone then LFE is relevant;  
if  $i$  in infected zone and  $j$  in susceptible zone then NFZ is relevant.

## 4. Example

- Smallpox outbreak in a closed community of 120 individuals in Abakaliki, Nigeria (much studied! see page 125 of **Bailey, 1975**).

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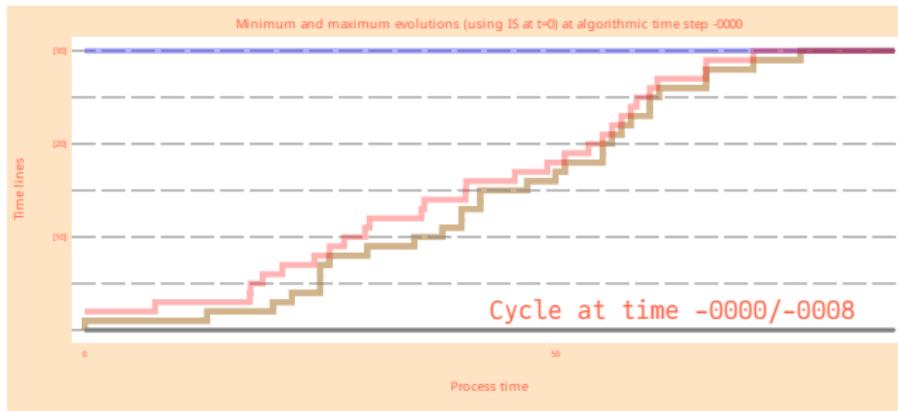
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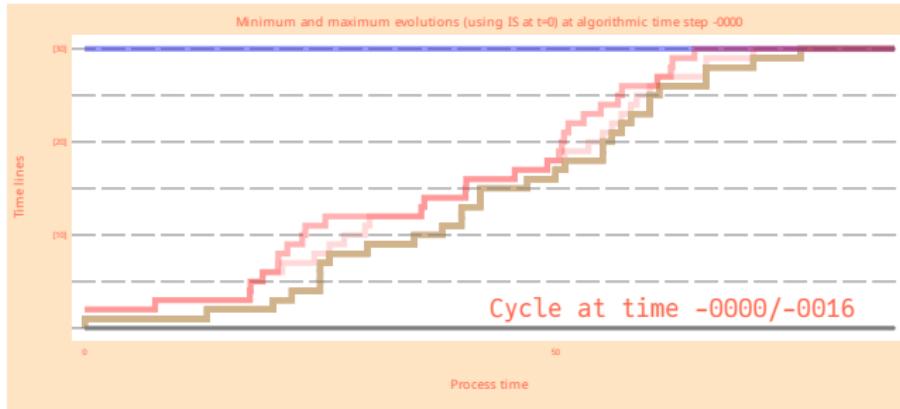
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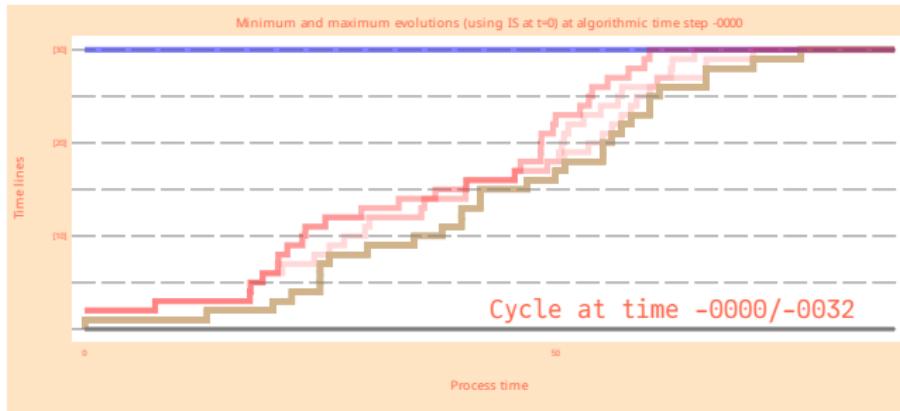
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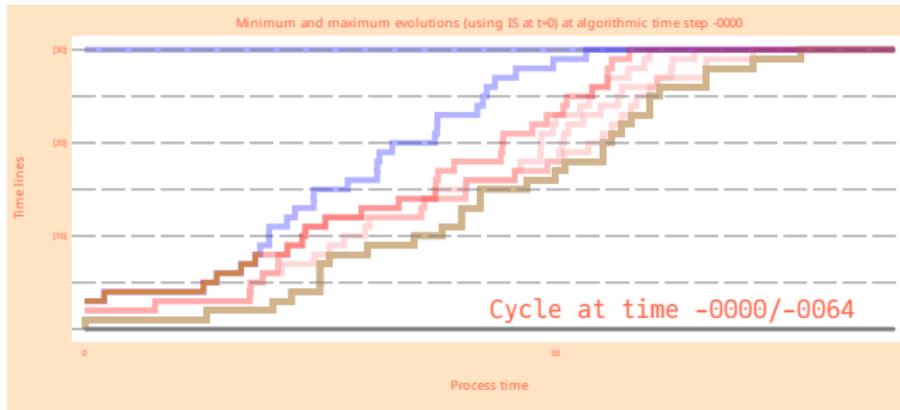
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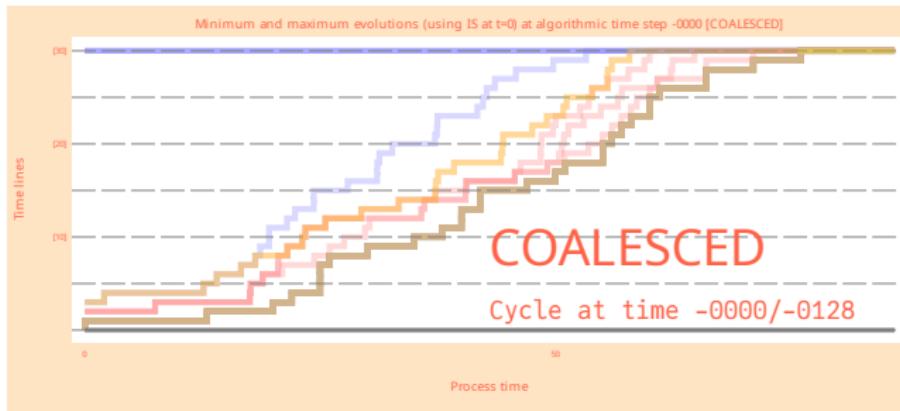
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- **Finally:** generalize to other suitable compartment models?

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- Thank you for your attention! **QUESTIONS?**



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## Image information

<i>Image</i>	<i>Attribution</i>	
<i>Book of Kells</i>	Huber Gerhard	<i>CC BY 4.0</i>
Classic CFTP for a simple random walk	Result of code written by WSK	
<i>Diamond Princess</i>	Alpsdake	<i>CC BY-SA 4.0</i>
Epidemic CFTP images and animation	Result of code written by WSK	

## Previous instances of this talk

<i>Date</i>	<i>Title</i>		<i>Location</i>
19/04/24	Perfect Epidemics	Short Research Talk	12mn Warwick
15/05/24	McMC+Perfect Simulation	Graduate Seminar, Aristotle Univ.	50mn Thessaloniki
17/01/25	Perfect Epidemics	Applied Probability Seminar	50mn Warwick
27/06/25	Perfect Epidemics	UK Research Network Stochastics	45mn Liverpool
20/10/25	Perfect Epidemics	Seminar	Dublin

# Other technical information

## Software used in computations

<i>Software</i>	<i>Version</i>	<i>Branch</i>	<i>Last commit</i>
quarto	1.6.39	—	
Running under julia	1.12.0	—	
EpidemicsCFTP	2.2.532	develop	Tue Jul 8 17:13:42 2025 +0100
EpidemicsUtilities	0.1.2.177	main	Fri Sep 26 15:35:26 2025 +0100
This quarto script	0.2.2.725	2025-10-09-Dublin-preparation	Tue Oct 14 18:01:39 2025 +0100

## Project information

<b>Version:</b>	0.2.2.729 (2025-10-09-Dublin-preparation)
<b>Author:</b>	Wilfrid Kendall <W.S.Kendall@warwick.ac.uk>
<b>Date:</b>	Fri Oct 17 15:23:14 2025 +0100

### Comment:

Final version of Dublin talk 20 October 2025.