Perfect Epidemics

Seminar at University College Dublin

W S Kendall   S B Connor

Warwick, York

20 October 2025

Warwick Statistics

Introduction

Homage to Dublin
(Book of Kells, 9th century)

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Work on perfect simulation (CFTP) for epidemics, now being written up.
WSK acknowledges the support of UK EPSRC grant EP/R022100.

Handout is on the web: use the `QR-code` or visit `wilfridskendall.github.io/talks/PerfectEpidemics`.

Plan of talk

Gregory:	Is there any other point to which you would wish to draw my attention?
Holmes:	To the curious incident of the dog in the night-time.
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5. Example with real data.

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   - seek coalescence.
4. Simplest possible example: *random-walk-CFTP*
   (can boost to use Ising model to do simple image reconstruction).

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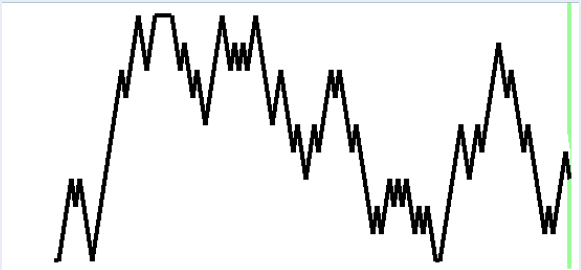

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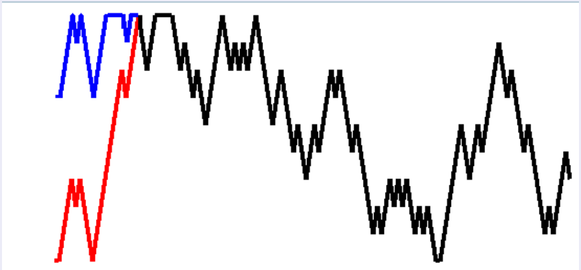

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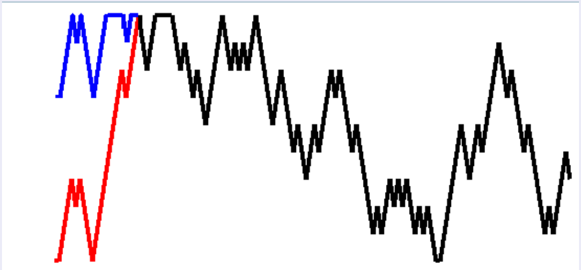

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4. Generally not true that location *at* coupling is a draw from equilibrium.

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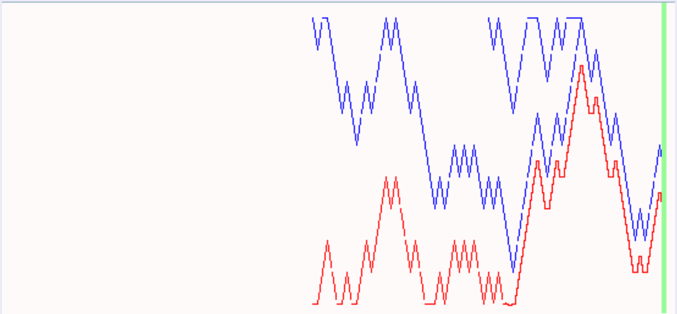

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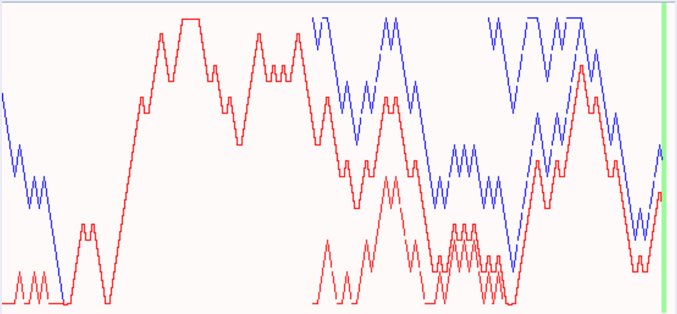

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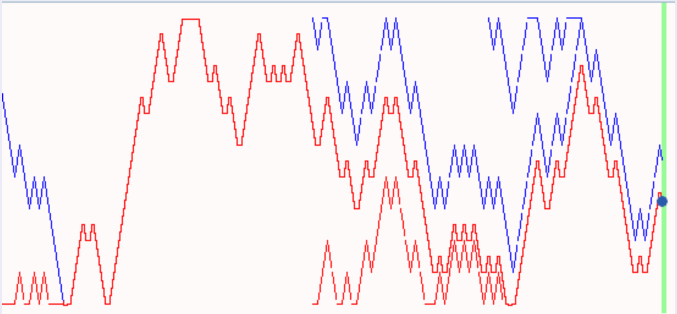

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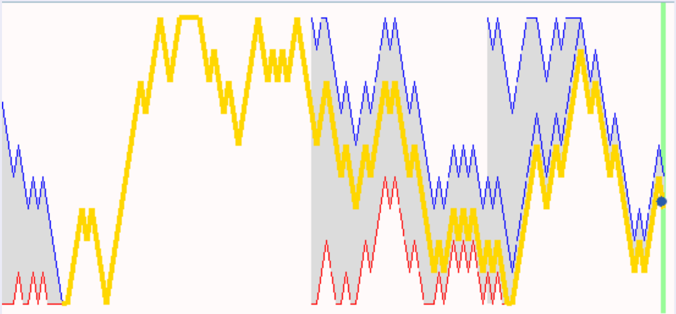

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5. The common value (golden thread) is an exact draw from equilibrium!

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4. Detailed expositions: WSK (2005), Huber (2015). (Want to implement CFTP in R? see WSK, 2015.)

2. Perfect Epidemics: a challenge problem for CFTP

S-I-R deterministic epidemic: differential equation system for (s, i, r)

$$\begin{aligned}
\textbf{Susceptible:} \quad & s' &= \; & -\alpha\, s\, i\,, \\
\textbf{Infected:} \quad & i' &= \; & (\alpha\, s - \beta\,)\; i\,, \\
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Infection:	$S \to S - 1\,,$	$I \to I + 1$	at rate	$\alpha\, S\, I\,,$
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Both models share an unrealistic assumption: homogeneous mixing.

In contrast, Fraser *et al* (2023) deploy a UK model with $N{=}10^6$ agents!

There are *many* important inferential questions (Cori & Kucharski, 2024).

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Wikipedia: "The British-registered *Diamond Princess* was the first cruise ship to have a major `[COVID-19]` outbreak on board, with the ship quarantined at Yokohama from 4 February 2020 for about a month. Of 3711 passengers and crew, around 700 people became infected and 9 people died."
Evidently $\alpha\ s_0/\beta \gg 1$ – as was sadly later confirmed, a sorrow for us all.

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5. Can we use perfect simulation?

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The simplest possible variant of contact tracing:
“When did the infections occur, supposing we only observe removals?”
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5. Result: *trajectory-valued chain*, unconditioned S-I-R as equilibrium.

From incidents to unconditioned epidemic trajectories (1/3)

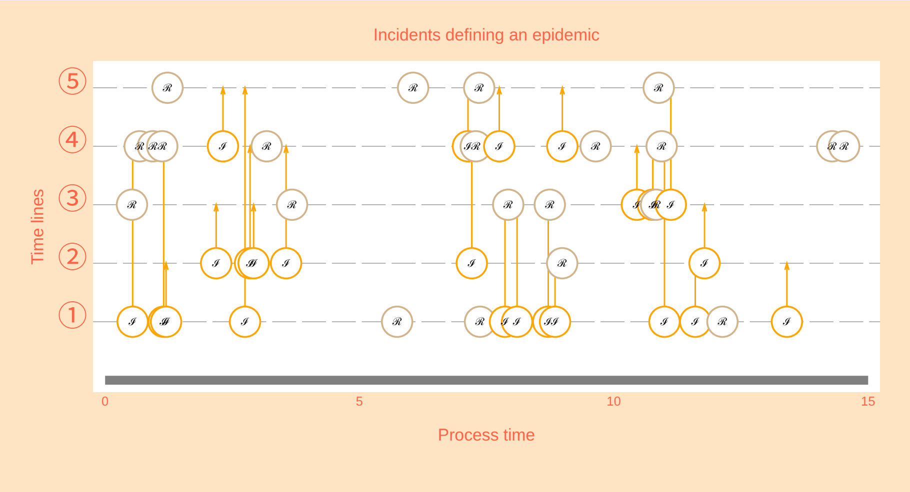

Figure 1: Light-orange circles denote potential infections (arrows point upwards to targets); light-brown circles denote potential removals.

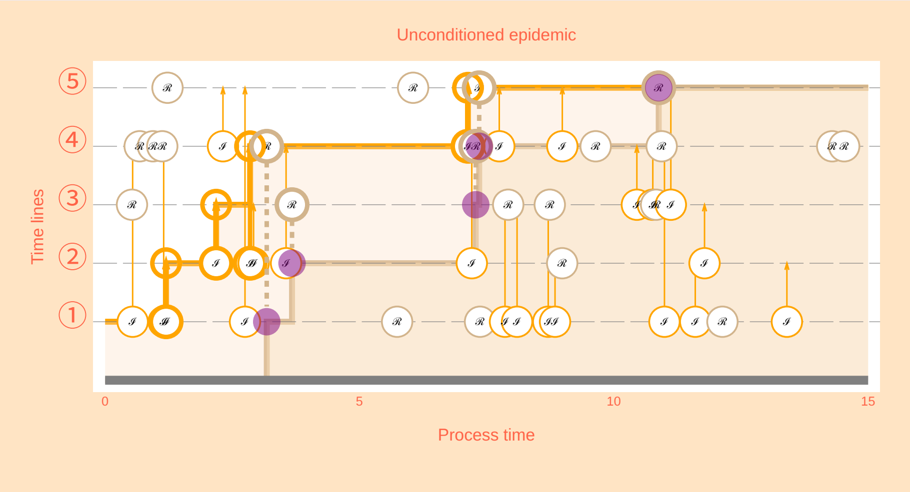

Figure 2: (a) *Infections* activate if on infected timeline and pointing to lowest uninfected timeline; (b) *Removals* activate if on infected timeline; remove lowest infected (purple disk).

From incidents to unconditioned epidemic trajectories (3/3)

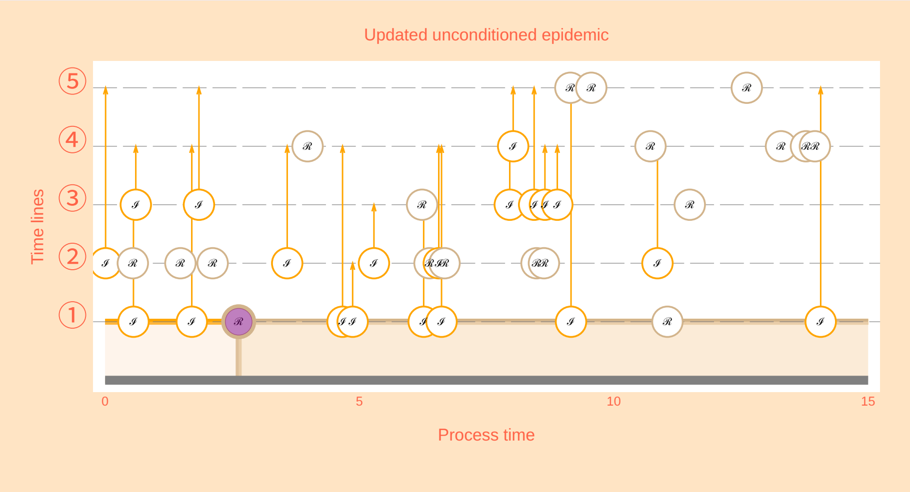

Figure 3: A step in algorithmic time for the unconditioned epidemic simply involves replacing all original incidents by an entirely new set of incidents.

Crucial technical point

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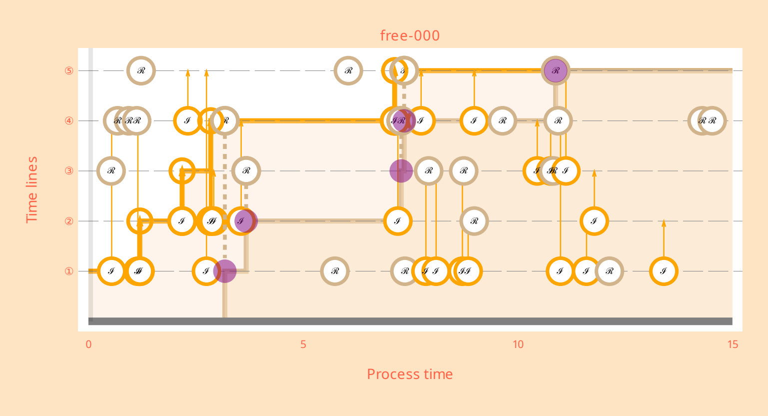
- Thus the original update is expressible as a (continuous) composition of updates, each of which satisfies detailed balance in equilibrium.
- The connection "restriction=conditioning" still holds.
- Crucially, re-sampling step 2 ensures composite evolution is irreducible over S! (So equilibrium under conditioning is unique.)

Free evolution evolving in continuous algorithmic time



GIF MP4

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- Housekeeping details used to establish that monotonicity still works: *laziest feasible epidemic* (LFE) and *no-fly zone* (NFZ).

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Initial conditioned epidemic

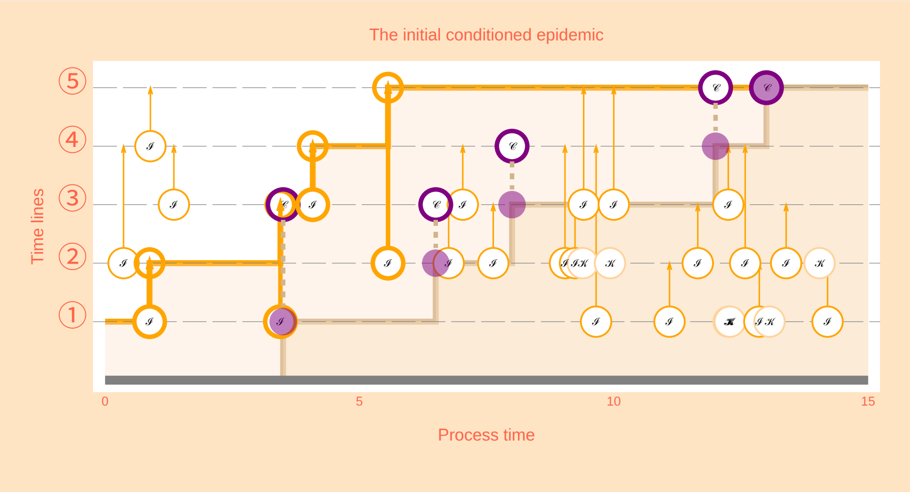


Figure 4: Initial conditioned epidemic, with conditioned removals indicated using purple circles (and purple disks when non-target timelines are infected).

Conditional epidemic update

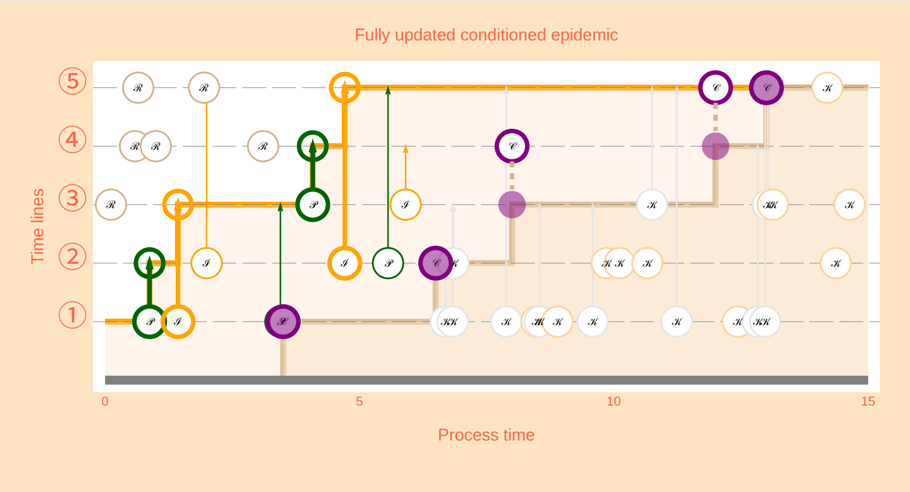


Figure 5: Epidemic updated under restriction: all conditioned removals remain activated, no new removals are activated. Green infections have been "perpetuated".

Laziest feasible epidemic (LFE)

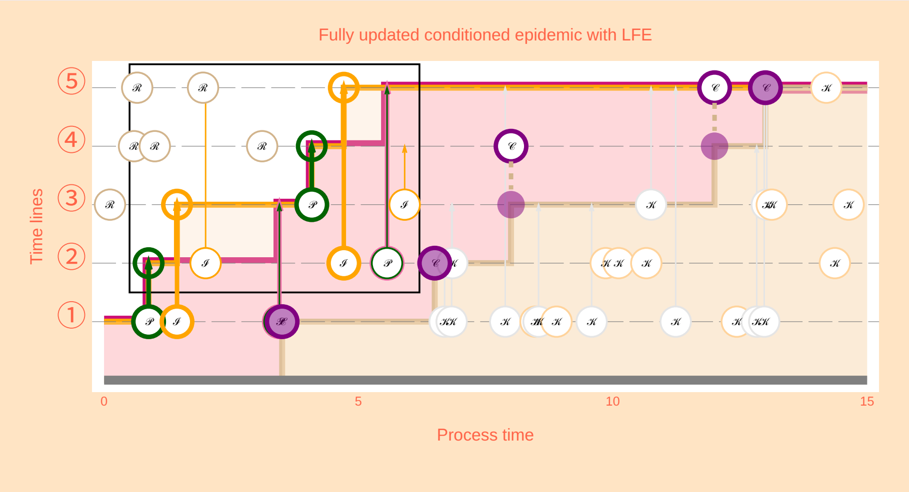
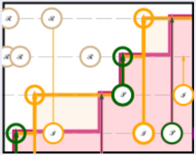


Figure 6: LFE computed recursively working right-to-left: slowest sequence of infections (and perpetuated infections) generating all conditioned removals. Can be used to identify perpetuated infections.

LFE: construction details



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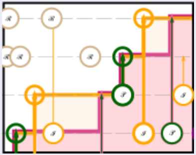

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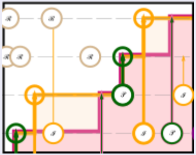

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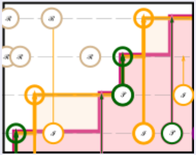

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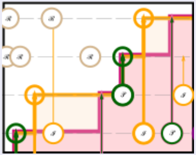

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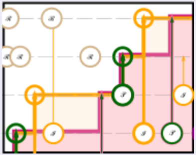
LFE: construction details



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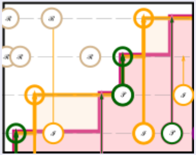

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No-fly zone (NFZ)

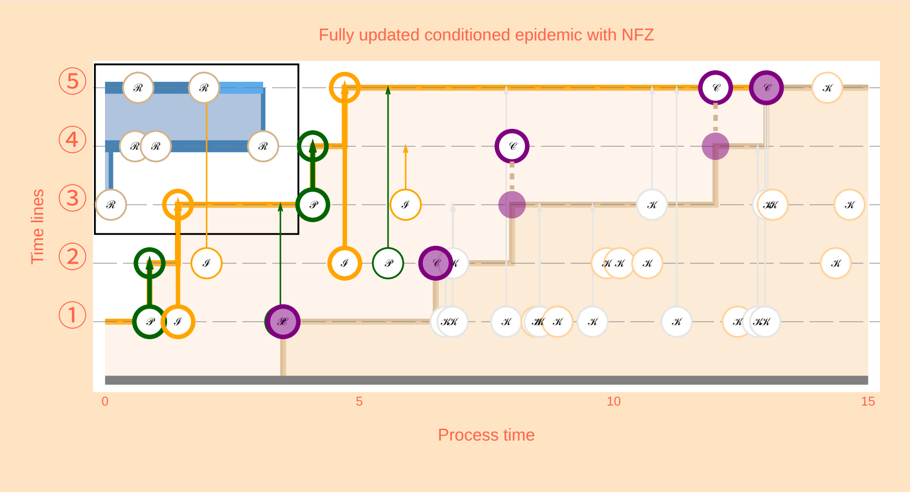
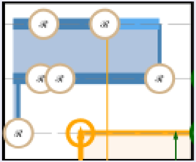


Figure 7: NFZ computed recursively working right-to-left: it traces a region of timelines such that unobserved removals are not activated if region not infected.

NFZ: construction details



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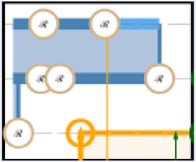

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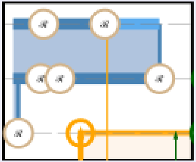

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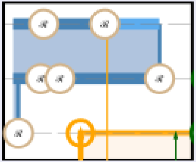

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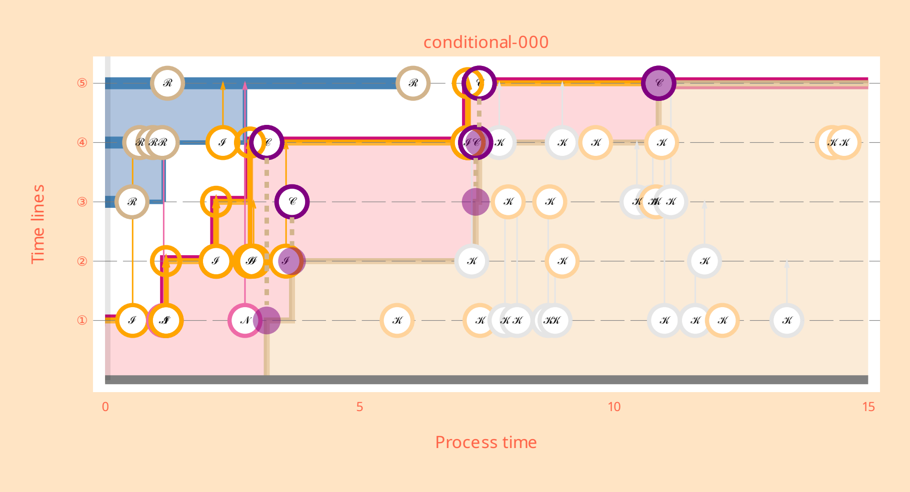
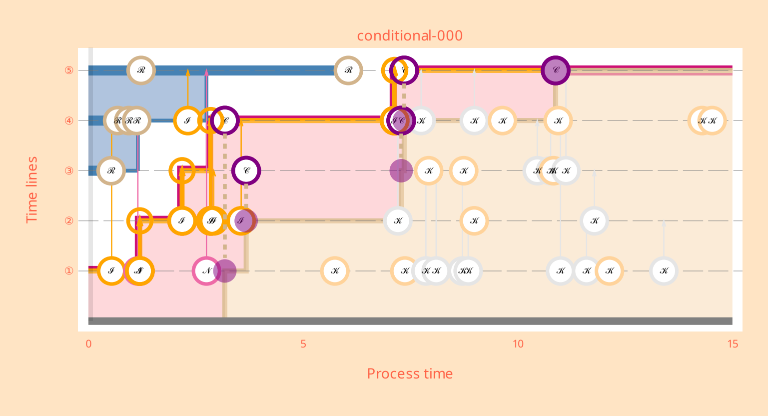
2. Use the above expression to prove that the shard, and thus the entire NFZ, depends monotonically on the old epidemic history.

4. CFTP monotonicity



Full monotonicity (hence CFTP) follows by showing *new* epidemic history depends monotonically on LFE and NFZ.

Conditioned evolution evolving in continuous algorithmic time



GIF MP4

If a new $\mathcal{J}_t^{i<j}$ has i, j in infected zone then LFE is relevant;
if i in infected zone and j in susceptible zone then NFZ is relevant.

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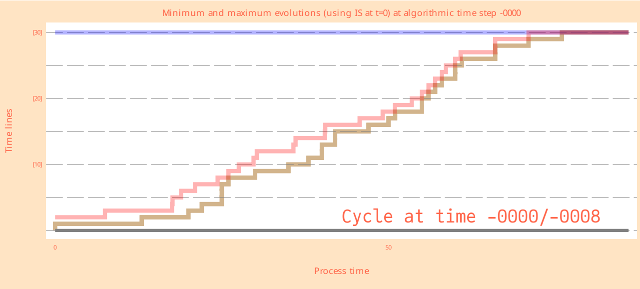

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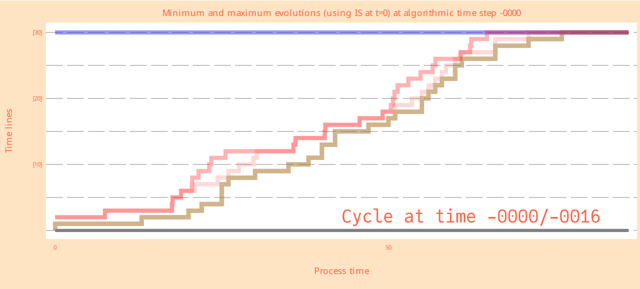

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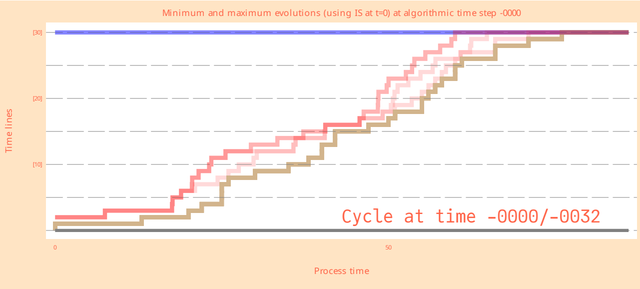

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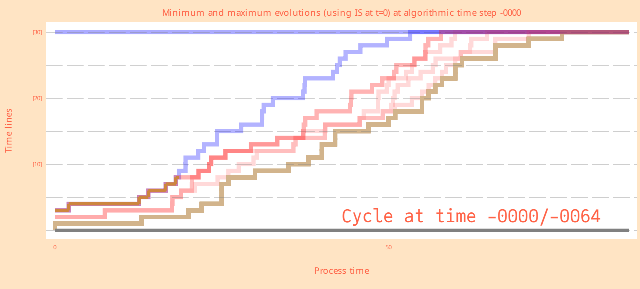

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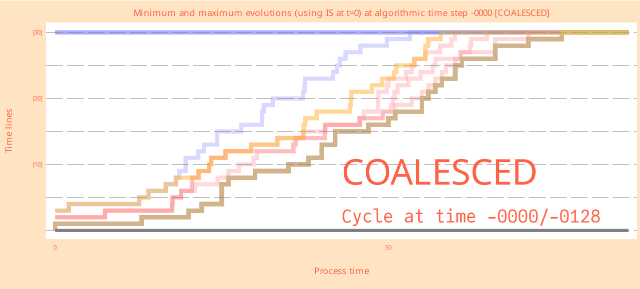

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- **Finally:** generalize to other suitable compartment models?

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- Thank you for your attention! **QUESTIONS?**

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Image information

Image	*Attribution*	
Book of Kells	Huber Gerhard	*CC BY 4.0*
Classic CFTP for a simple random walk	Result of code written by WSK	
Diamond Princess	Alpsdake	*CC BY-SA 4.0*
Epidemic CFTP images and animation	Result of code written by WSK	

Previous instances of this talk

Date	*Title*			*Location*
19/04/24	Perfect Epidemics	Short Research Talk	12mn	Warwick
15/05/24	McMC+Perfect Simulation	Graduate Seminar, Aristotle Univ.	50mn	Thessaloniki
17/01/25	Perfect Epidemics	Applied Probability Seminar	50mn	Warwick
27/06/25	Perfect Epidemics	UK Research Network Stochastics	45mn	Liverpool
20/10/25	Perfect Epidemics	Seminar	50mn	Dublin

Other technical information

Software used in computations

Software	*Version*	*Branch*	*Last commit*
`quarto`	`1.6.39`	—	
Running under `julia`	`1.12.0`	—	
`EpidemicsCFTP`	`2.2.532`	`develop`	`Tue Jul 8 17:13:42 2025 +0100`
`EpidemicsUtilities`	`0.1.2.177`	`main`	`Fri Sep 26 15:35:26 2025 +0100`
This `quarto` script	`0.2.2.725`	`2025-10-09-Dublin-preparation`	`Tue Oct 14 18:01:39 2025 +0100`

Project information

Version:	`0.2.2.733` (`develop`)
Author:	Wilfrid Kendall <`W.S.Kendall@warwick.ac.uk`>
Date:	Fri Oct 24 17:43:38 2025 +0100

Comment:
Minor tweaks.