

Wandering around a fibrous network¹

when all the paths look very much alike

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<https://wilfridskendall.github.io/talks/Aarhus-2024/Short-Fibre-SIRSN-talk-handout.pdf>

Development of work on line process models for transportation networks
(Aldous & WSK, 2008; WSK, 2011, 2014),
particularly concerning line process models for online maps:
(Aldous & Ganesan, 2013; Aldous, 2014; Kahn, 2016; WSK, 2017, 2020).



Objective of this work: model online maps, but replace infinitely long straight lines by line segments (“sticks”) or moderately curved paths (“stiff fibres”).

Structure of talk:

- ① Modelling efficient spatial transport systems (illustrating and setting up axiomatic approach to SIRSN);
- ② Practicalities (why the two current SIRSN models are unsatisfactory);
- ③ Sticks and fibres (brief notes on some key results and proofs).

(I) Models for efficient spatial transport systems

How to model real-life online maps? a task for stochastic geometry.

- ① Mechanism to connect generic nodes x and y ;
- ② “Scale -invariant” networks ([SIRSN](#)) as potential models;
- ③ What do “best routes” look like?
- ④ What can be said about flows? – not for today.

Related to work on mathematics for cities, for example:

- Work on city shape ([Bender et al., 2004; Courtat, 2012](#));
- Route-finding ([Bongiorno et al., 2021](#));
- “Transit nodes” ([Bast et al., 2007](#)).

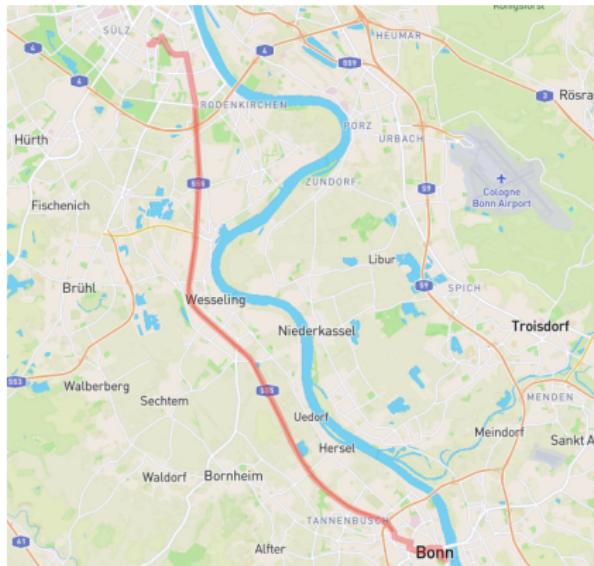
Work so far: models based on eg Poisson line processes ([Plp](#)) ([Aldous & WSK, 2008; WSK, 2011, 2017, 2020; Kahn, 2016; Blanc, 2023](#)).

We'd strongly prefer: models based on *localizable* object processes (eg Poisson stick processes, Poisson fibre processes).

Online maps look as if they might be scale-invariant!



An online map as a mechanism for generating networks

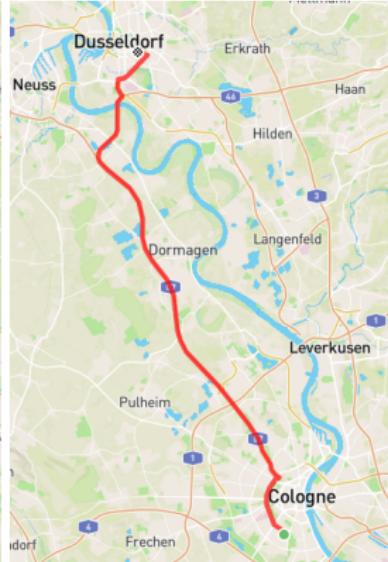
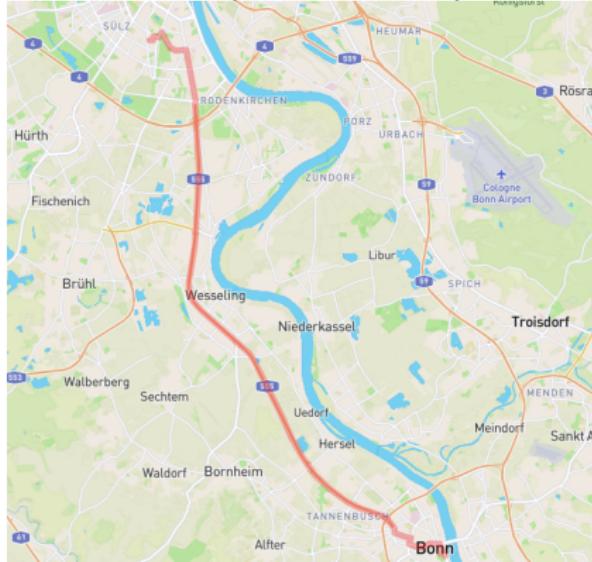


Given x_1, \dots, x_n , the online map should generate a *network*

$$\mathcal{N}(x_1, \dots, x_n) = \bigcup \{\mathcal{R}(x_i, x_j) ; 1 \leq i < j \leq n\} .$$

NB: we disallow one-way streets: $\mathcal{R}(x_i, x_j) = \mathcal{R}(x_j, x_i)$.

Axiom 1: (Statistical) scale-invariance of networks

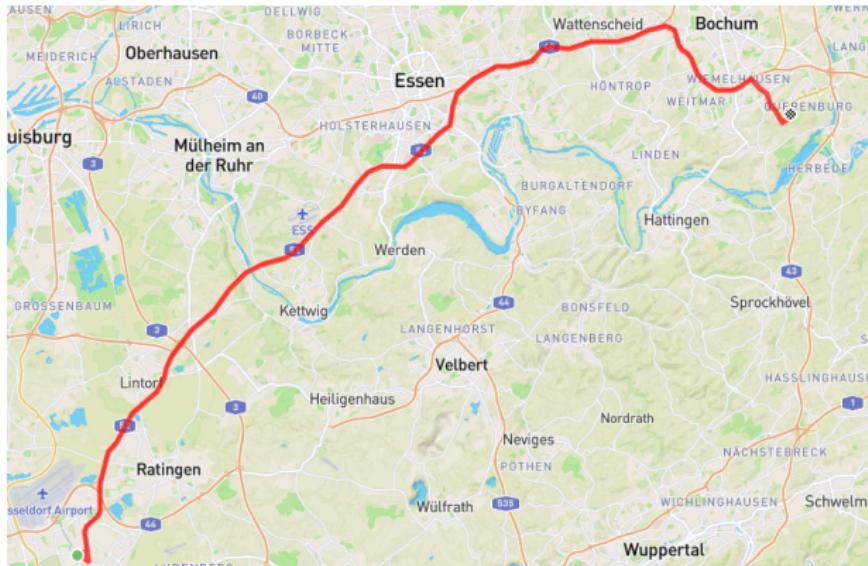


We require *statistical scale invariance*: for all similarities \mathfrak{S} of \mathbb{R}^m :

$$\mathcal{L}(\mathfrak{S}\mathcal{N}(x_1, \dots, x_n)) = \mathcal{L}(\mathcal{N}(\mathfrak{S}x_1, \dots, \mathfrak{S}x_n)).$$

NB: we model $\mathcal{R}(x_i, x_j)$ hence $\mathcal{N}(x_1, \dots, x_n)$ as random.

Axiom 2: Finite mean length of routes



We insist that all routes have finite mean length.

Axiom 3: SIRSN property



NB: SIRSN = “Scale-invariant Random Spatial Network”.

We require massive re-use of route components (strong SIRSN property).

Weak SIRSN: Source/terminus nodes spread evenly everywhere on plane.
“Route-re-use” means mean-network-length-per-unit-area must be finite.

Strong SIRSN: Source/terminus nodes spread **densely** everywhere on plane.
“Route-re-use” means mean-network-length-per-unit-area is finite *for
“long-distance” network*.

Scale-Invariant Random Spatial Networks (SIRSN)

Axioms (following Aldous & Ganesan, 2013; Aldous, 2014).

Given: a (random) network $\mathcal{N}(x_1, \dots, x_n)$ connecting nodes. Require:

- ① **Scale-invariance**: for each Euclidean similarity \mathfrak{S} ,
$$\mathcal{L}(\mathfrak{S}N(x_1, \dots, x_n)) = \mathcal{L}(\mathfrak{S}\mathcal{N}(x_1, \dots, x_n)).$$
- ② If D_1 is length of fastest route between two points at unit distance apart
then mean length is finite: $\mathbb{E}[D_1] < \infty$.
- ③ **Weak SIRSN property**: network connecting points of (independent) unit intensity Poisson point process has finite average length density;
or
(Strong) SIRSN property: finite length intensity of “long-range” routes
for points of *dense* Poisson point process.

Do SIRSN exist at all?

Yes (first version): Hierarchical SIRSN (Aldous, 2014),
a *randomized* dense dyadic speed-marked planar *lattice*.

Do SIRSN exist at all? a more natural answer

Yes (second version): Poisson line process SIRSN: planar *and also* higher dimensions too (Aldous, 2014; Kahn, 2016; WSK, 2017).

Construction uses speed-marked (improper) Plp Π .

Π is governed by an intensity measure ϖ_{Plp} on $(0, \infty) \times \mathbb{R} \times [0, \pi]$, scaling speed $v \rightarrow \lambda v$ versus space $x \rightarrow \lambda^{\gamma-1}x$:

$$d\varpi_{\text{Plp}} = \frac{1}{2}(\gamma-1)v^{-\gamma} dv dr d\theta \quad (\text{make sure } \gamma > 1!).$$

NB: $\frac{1}{2} dr d\theta$: intensity measure for standard Poisson line process.

NB ($\gamma > 1$): Lines of speed ≥ 1 form standard Poisson line process $\Pi^{(\geq 1)}$.

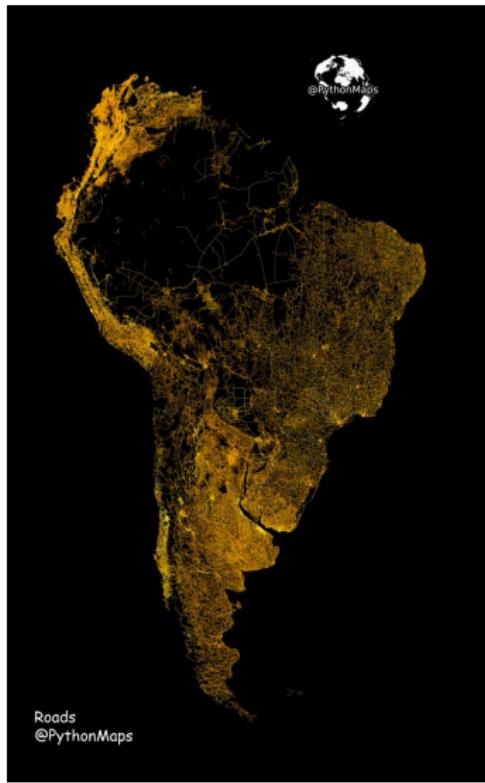
Use fastest Π -paths for routes. Require $\gamma > 2$ so

- Π -paths cannot reach ∞ in finite time;
- Π -paths can reach any prescribed point in finite time.

Proving Π forms a SIRSN is non-trivial!

(II) From lines to (non-infinite) fibres

But we want to be able to model networks like this:



Sticks and stiff fibres: foundations

Use *object processes* (random objects located at points of a Poisson process):

- **Scale-invariant**, so speed-marked (“object soup”);
- Sticks or C^1 fibres, so notion of length;
- *Stiff* fibres (each fibre within ε of its intrinsic overall direction).
- Define using intensity measure on **configuration space**:

$$\begin{array}{ccccccc} \mathbb{R}^m & \times & \mathrm{SO}_m & \times & (0, \infty) & \times & \text{Shape} \\ x & & \omega & & r & & \text{shape} \\ (\text{location}) & & (\text{rotation}) & & (\text{length}) & & (v) \end{array}$$

- Parameters:
 - ▶ fibre **shape** distribution;
 - ▶ **rescaled length** distribution (joint with shape);
 - ▶ **specific length intensity** λ for set of fibres of more than unit speed;
 - ▶ **scaling index** γ .

Sticks and stiff fibres: techniques

- ① Integrate the measurable field of orientations using Lipschitz paths;
- ② Sobolev compactness theorem for Lipschitz paths: hence existence of “geodesics” (fastest connecting paths);
- ③ For routes, select geodesics in scale-invariant way using auxiliary scale-invariant randomness;
- ④ Validation of SIRSN axioms depends on comparison of (fractional power) of connection time with Exponential distribution.

Finesse away percolation issues by supposing fibre soup is *thick*: scaled length distribution has finite first moment but infinite second moment (quantified by tail condition)

and do the sums for

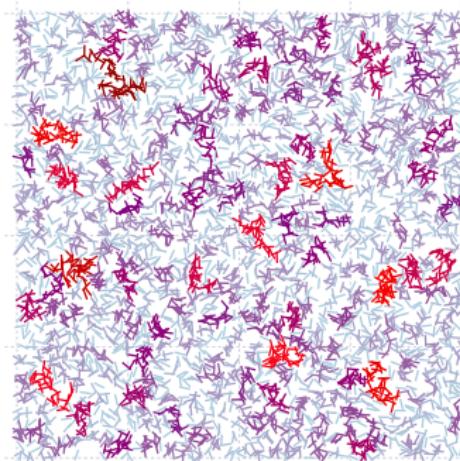
- probability of “timely” connection of two close points by a single fibre;
- concatenating point-pairs for “timely” connection of two far points;
- deriving exponential moment of fractional power(connection time);
- deriving estimate on “time-diameter” for compact set.

(III) Beyond thickness: percolation

The SIRSN property should not need “thickness” (single-fibre connection).

Conjecture: Super-critical percolation suffices

I conjecture that a Poisson soup of stiff fibres can form a SIRSN if the family of fibres of greater than unit speed exhibits super-critical percolation.



A suitable first-passage percolation property may suffice (routes can use “instantaneous stops”).

Conclusion

- Strong positive evidence that “SIRSN are everywhere!” (not just when infinite lines are involved).
- There are very simple scale-invariant stick soups which are thick: so this theory is not vacuous!
- Future prospects:
 - ▶ traffic, benchmark models for traffic flow (cf Gameros Leal, 2017);
 - ▶ adding spatial inhomogeneity;
 - ▶ moving beyond Poisson, *eg* adding simple interactions.



Thank you for your attention! **QUESTIONS?**



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Previous versions of this talk

In earlier forms:

- Colloquium lecture *Scale-invariant random spatial networks (SIRSN) from line patterns*: Auckland 9 August 2016; Bern 10 June 2016; Aarhus 6 June 2016; Manchester Probability Seminar 13 April 2016; Journée de Rham lecture at Lausanne 22 April 2015.
- Four-lecture course *Poisson line processes and spatial transportation networks*: Edinburgh 21-22 June 2018.
- Talks on more general varieties of SIRSN: Dunkerque 17-19 November 2021, Singapore 21 June 2022, Bern 15 September 2022, Cologne 29 March 2023.
- Shorter form talks (25-30 minutes): “Warwick Stats 50” 8 September 2022, Baxendale birthday workshop Warwick 9 June 2023, OEBN memorial workshop Aarhus 29 May 2024.

These notes were produced from `Short-Fibre-SIRSN-talk.qmd`:

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