

Perfect Epidemics

Applied Probability Seminar

Department of Statistics, University of Warwick

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Warwick, York

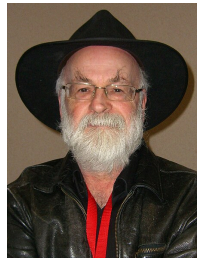
17 January 2025



Introduction

“Maybe the only significant difference between a really smart simulation and a human being was the noise they made when you punched them.”

([The Long Earth](#), Pratchett & Baxter, 2012)



Handout is on the web: use the QR-code or visit
wilfridskendall.github.io/talks/PerfectEpidemics.

WSK acknowledges the support of UK EPSRC grant EP/R022100.

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- Simplest possible example: *random-walk-CFTP*
(can boost to use Ising model to do simple image reconstruction).

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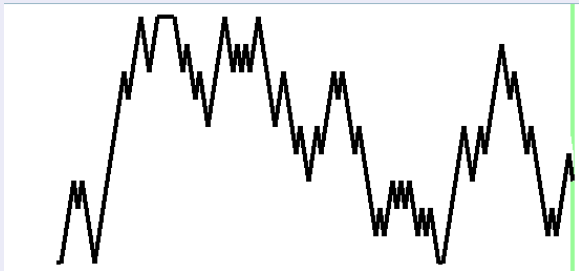
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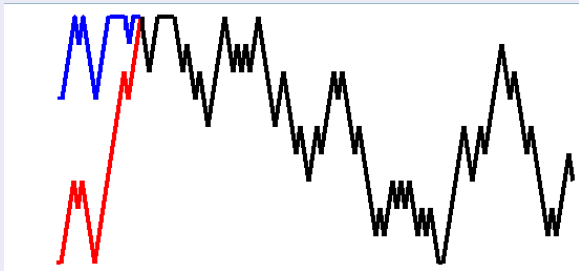
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- Conventional MCMC picks a starting point, then runs the simple random walk for long time till approximate equilibrium.
- How long? One way to *estimate* this is to run two (or several?) coupled copies till they meet. If probability of meeting by time T is high, then deviation of X_T from equilibrium is statistically small;



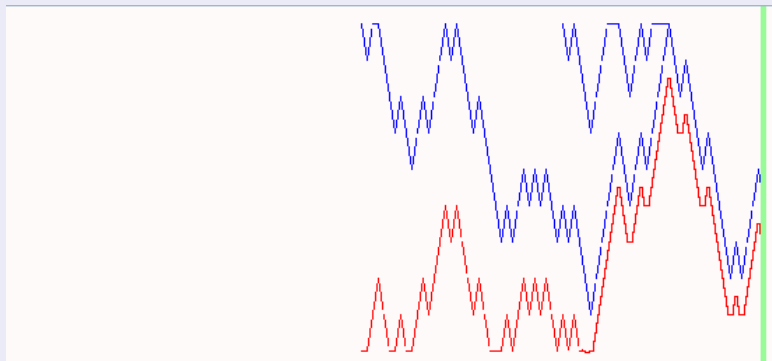
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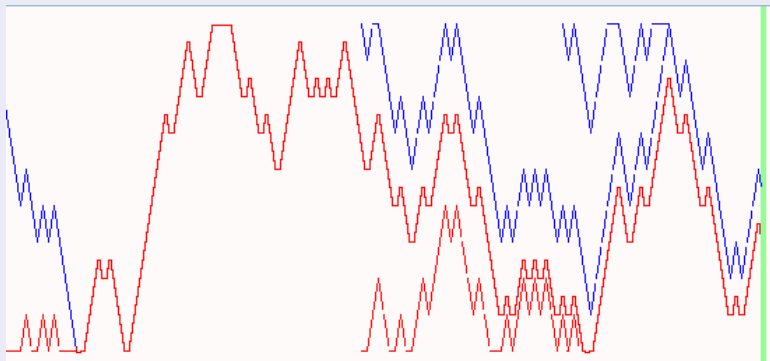
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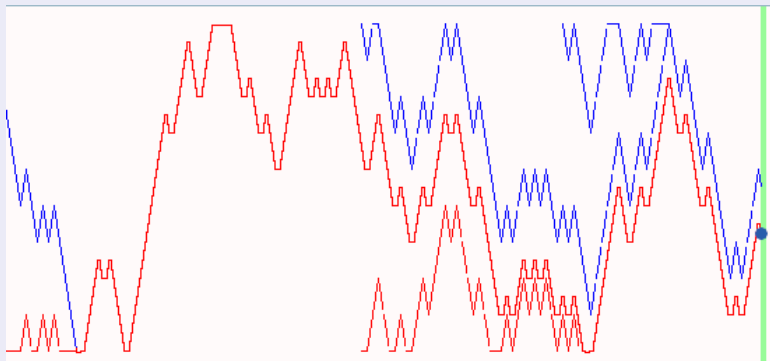
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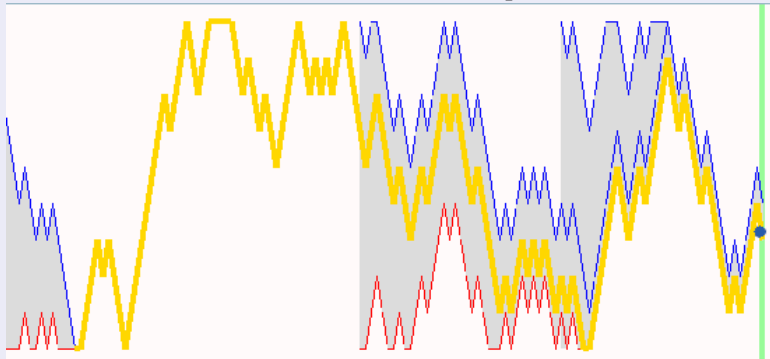
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- The common value is an exact draw from equilibrium!



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- Detailed expositions: WSK (2005), Huber (2015).
(Want to implement CFTP in R? see WSK, 2015.)

3: Perfect Epidemics: a challenge problem for CFTP

Many important inferential questions (Cori & Kucharski, 2024).

Simplest models (versus UK model with 10^6 agents!, Fraser & Others, 2023):

S-I-R deterministic epidemic: susceptibles s , infectives i , removals r
(constant total population $s + i + r = n$):

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Both models make an unrealistic assumption: homogeneous mixing.

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Wikipedia: “The British-registered *Diamond Princess* was the first cruise ship to have a major [COVID-19] outbreak on board, with the ship quarantined at Yokohama from 4 February 2020 for about a month. Of 3711 passengers and crew, around 700 people became infected and 9 people died.”

Evidently $\alpha s_0/\beta \gg 1$ – as was sadly later confirmed, a sorrow for us all.



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- 5 Can we use **perfect simulation**?

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- ➍ First step: evolve whole S-I-R trajectory in *algorithmic time* (alter potential infections and removals using immigration-death in discrete algorithmic time).
- ➎ Result: *trajectory-valued chain*, unconditioned S-I-R as equilibrium.

From incidents to unconditioned epidemic trajectories (1/3)

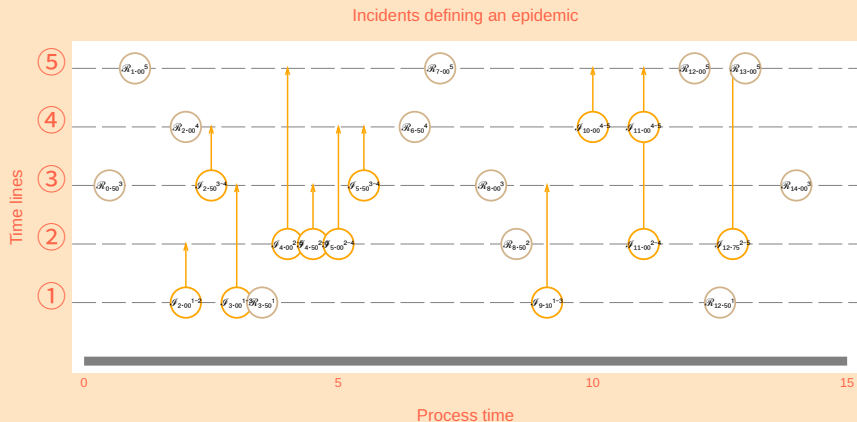


Figure 1: Light-orange circles denote potential infections (arrows point upwards to targets); light-brown circles denote potential removals.

From incidents to unconditioned epidemic trajectories (2/3)

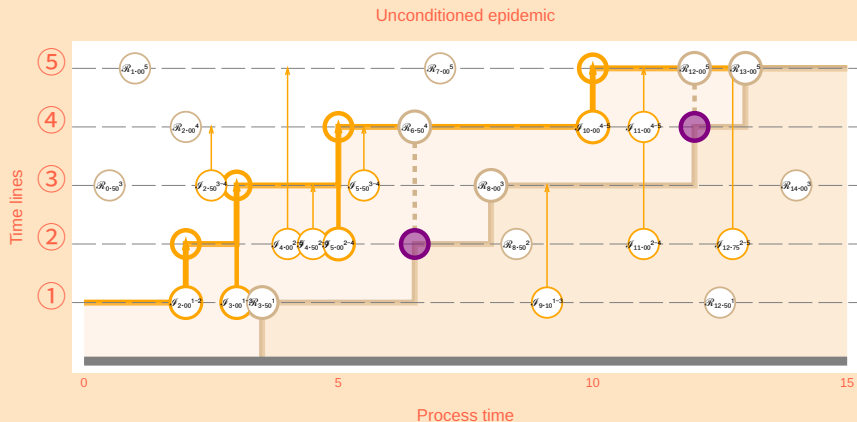


Figure 2: Activate (a) *infection* if target on lowest uninfected timeline; (b) *removal* if in infected region, then remove lowest infected (purple disk if different timeline).

From incidents to unconditioned epidemic trajectories (3/3)

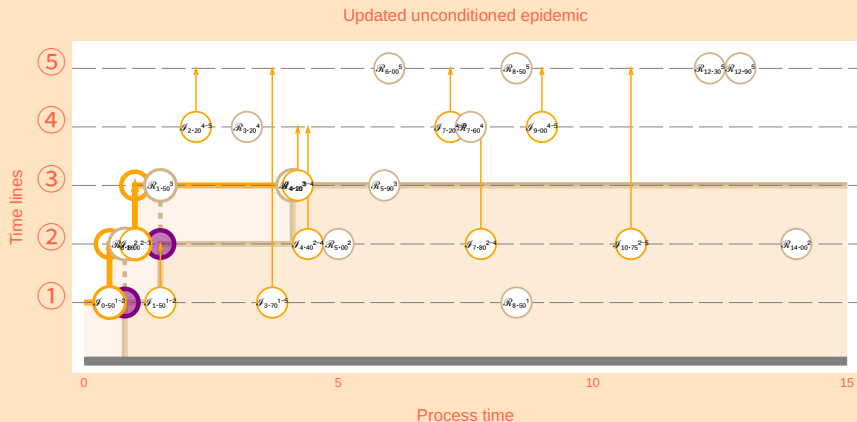


Figure 3: A step in algorithmic time for the unconditioned epidemic simply involves replacing the original incidents by a new set of incidents.

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 - 3 For $(2n+1)T < \tau < (2n+2)T$, update old I s in $((2n+2)T - \tau, T)$.

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- Updates in algorithmic time τ are then (algorithmic-)*time-reversible*: so restriction to subset S of state-space (of *activated* removals occurring precisely at specified set of times) implies a new equilibrium which is the old equilibrium conditioned to lie in S .
- For later purposes it is convenient to stage the replacement as follows:
 - ➊ Replace removals (R s);
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 - ➌ Replace infections (I s).
- Re-express using continuously varying τ . Process time runs over $[0, T]$.
 - ➊ For $2nT < \tau < (2n+1)T$, update old R s with times in $(0, \tau - 2nT)$;
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- Thus the original update is expressible as a (continuous) composition of updates, each of which satisfies detailed balance in equilibrium.
- The connection “restriction=conditioning” is thereby preserved.
- Crucially, step 2 ensures composition action is irreducible over S !
(So equilibrium under conditioning is unique.)

Illustration of technical point (1/8)

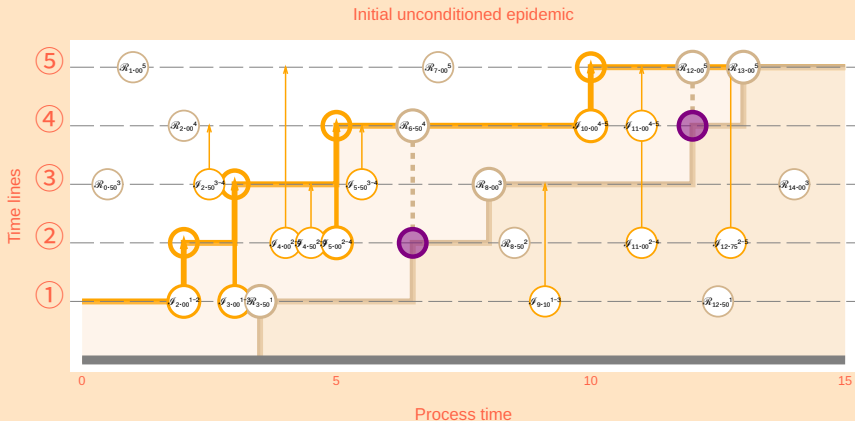


Figure 4: No change yet to removals or infections;

Illustration of technical point (2/8)

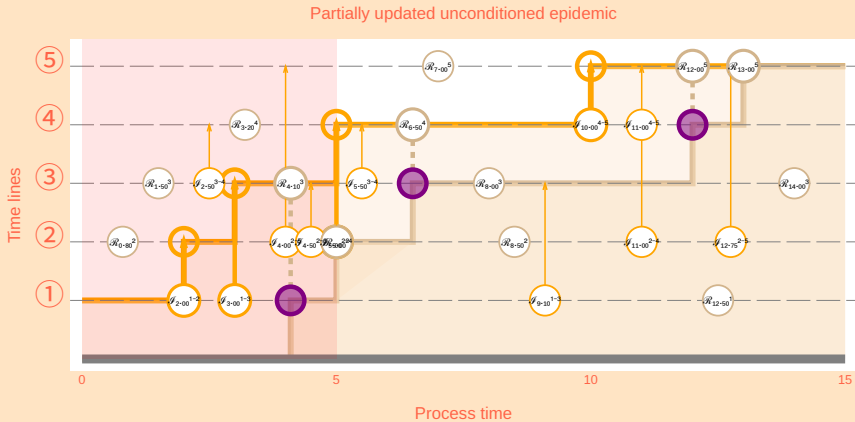


Figure 5: Replace first third of removals, infections unchanged;

Illustration of technical point (3/8)

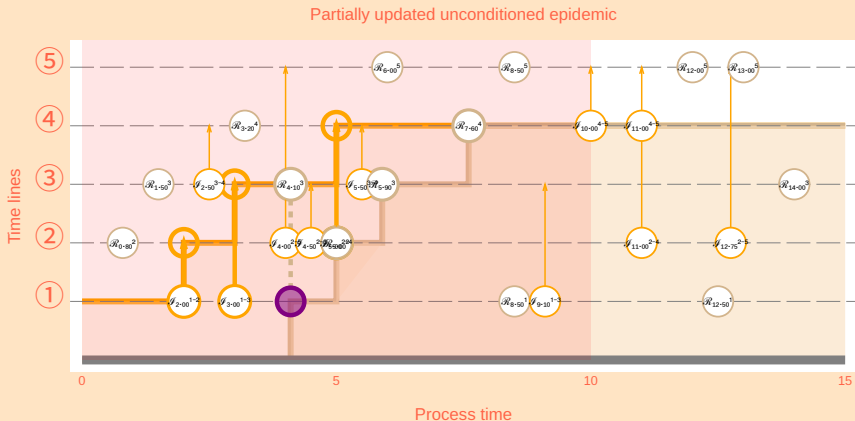


Figure 6: Replace first two-thirds of removals, infections unchanged;

Illustration of technical point (4/8)

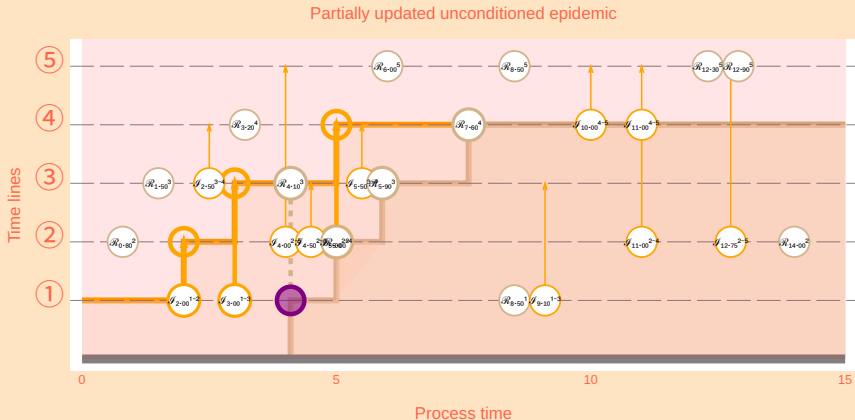


Figure 7: All removals resampled, infections as yet unchanged;

Illustration of technical point (5/8)

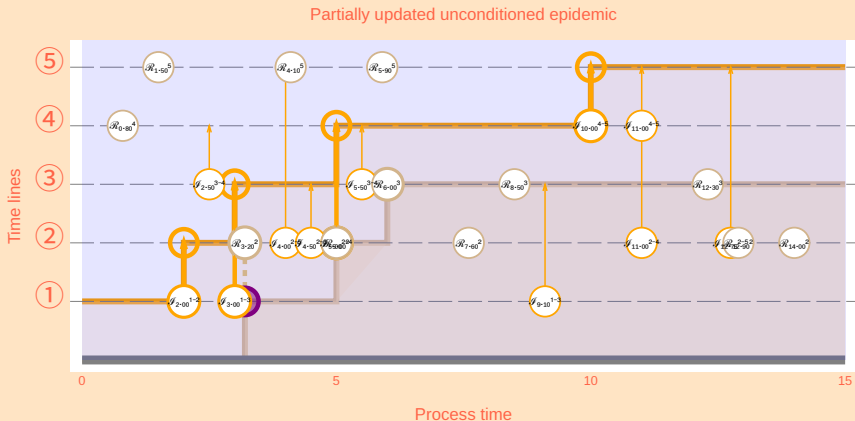


Figure 8: Re-sample all removal timelines, infections as yet unchanged;

Illustration of technical point (6/8)

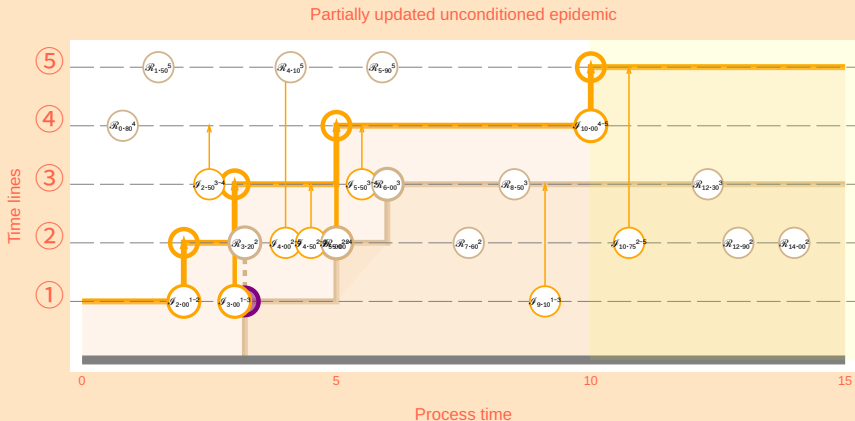


Figure 9: Re-sample last third of infections;

Illustration of technical point (7/8)

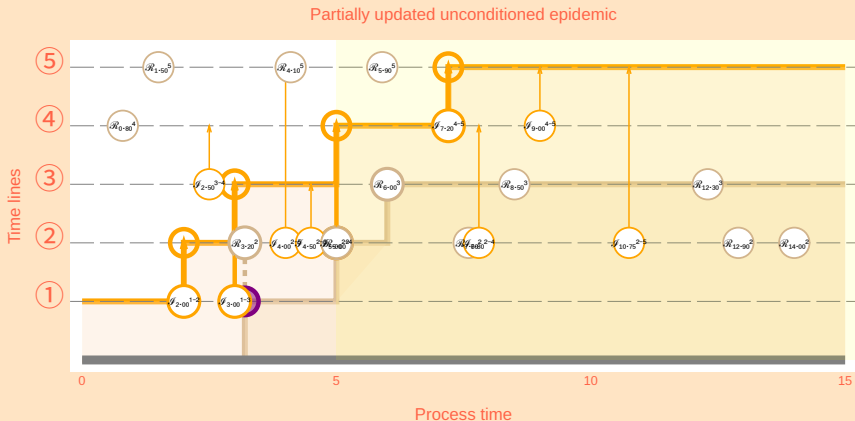


Figure 10: Re-sample last two-thirds of infections;

Illustration of technical point (8/8)

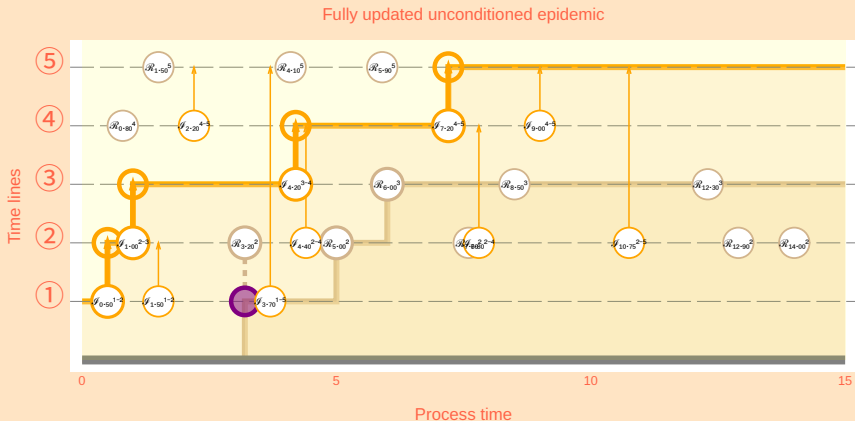


Figure 11: All infections now re-sampled.

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- ▶ an infection can be introduced only if no new observed removals result.
- Does this produce a *feasible* and suitably monotonic algorithm?
- **Housekeeping details** required to establish that monotonicity still works.
Key notions: *last feasible epidemic* (LFE) and *no-fly zone* (NFZ).

Initial conditional epidemic

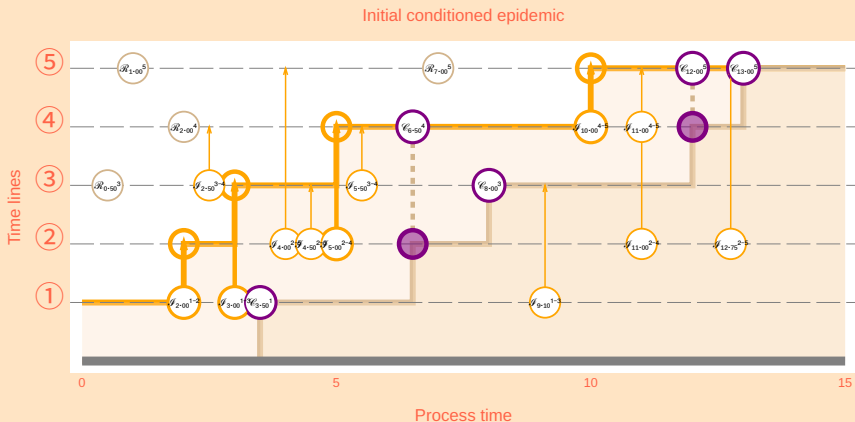


Figure 12: Initial epidemic with conditioned removals indicated using purple circles (and purple disks when different timelines are infected).

Conditional epidemic update

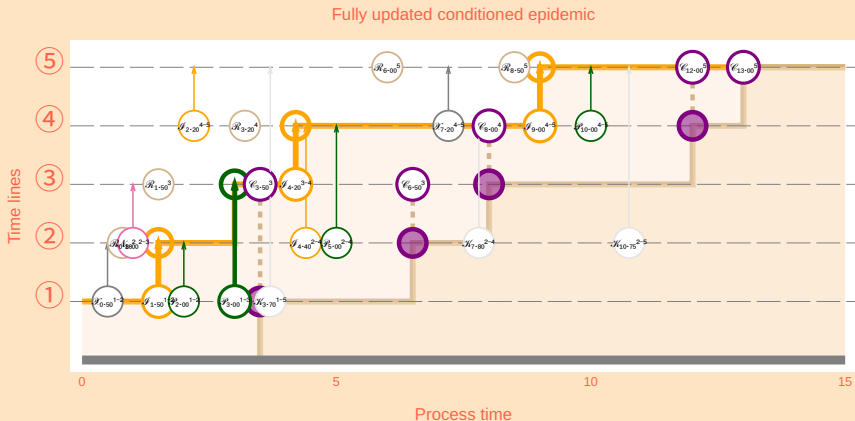


Figure 13: Epidemic updated under restriction: all conditioned removals remain activated, no new removals are activated. Green infections have been “perpetuated”.

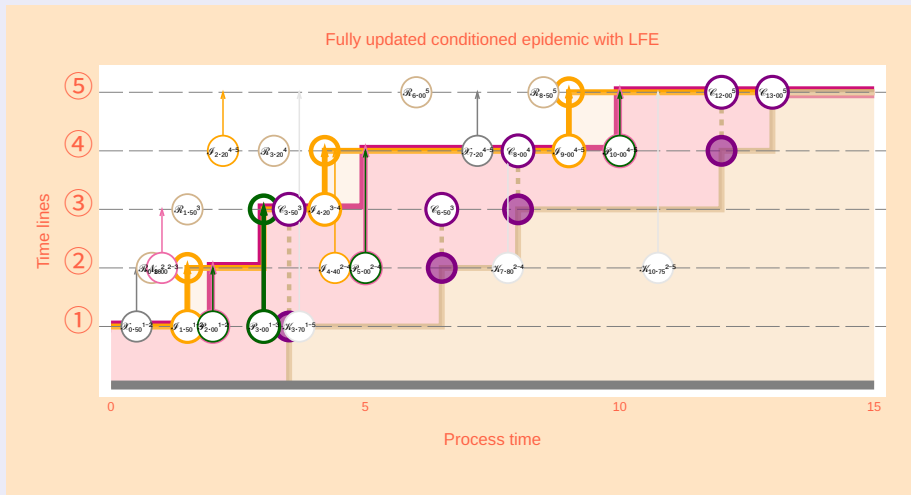


Figure 14: LFE computed recursively working right-to-left: the slowest sequence of infections deals with all infected timelines in order (includes perpetuated infections).

Fully updated conditioned epidemic with NFZ

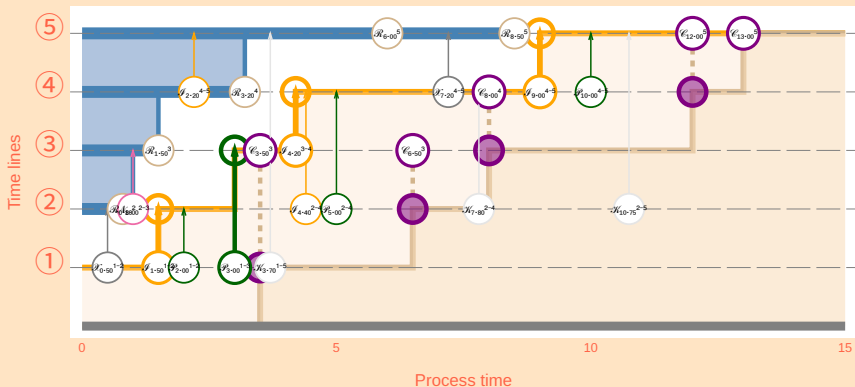


Figure 15: NFZ computed recursively working right-to-left: it traces the region of timelines that must not be infected if one is not to activate unobserved removals.

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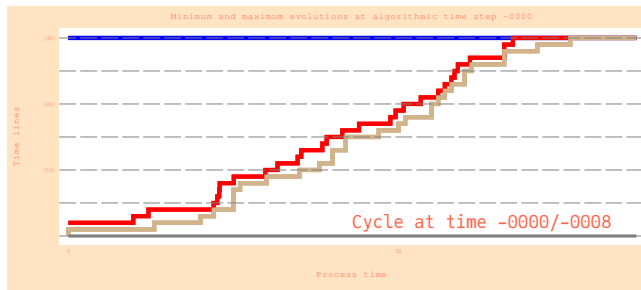
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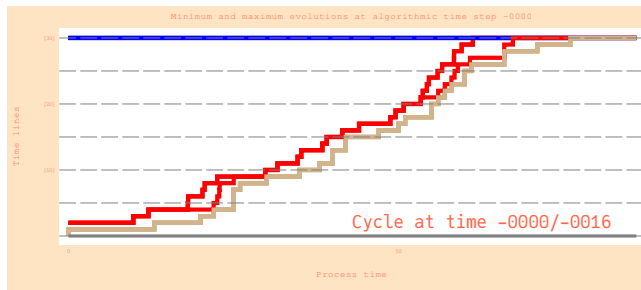
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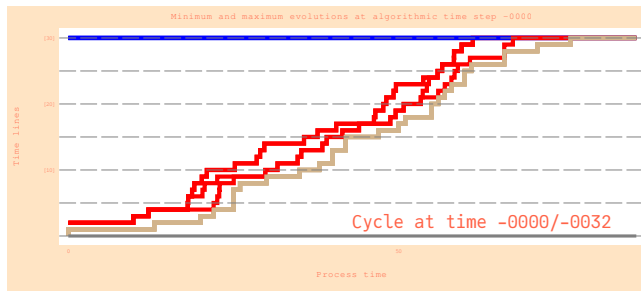
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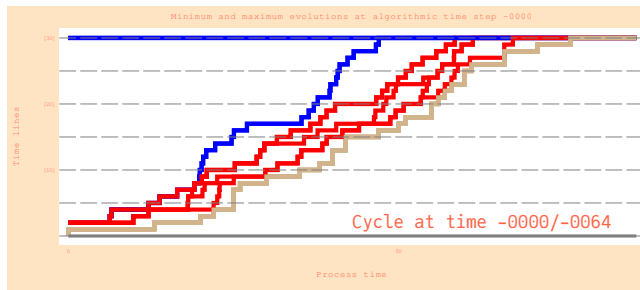
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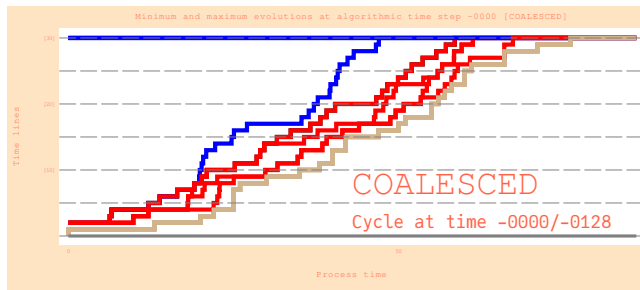
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- Finally: generalize to other suitable compartment models?

Conclusion

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- Still to be done: statistical estimation of parameters, generalization to other compartment models.
- Thank you for your attention! **QUESTIONS?**



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Image information

Image	Attribution	
<i>Terry Pratchett</i> Classic CFTP for a simple random walk	Luigi Novi Result of code written by WSK	<i>CC BY 3.0</i>
<i>Diamond Princess</i> Epidemic CFTP images and animation	Alpsdake Result of code written by WSK	<i>CC BY-SA 4.0</i>

A “near-maximal” configuration

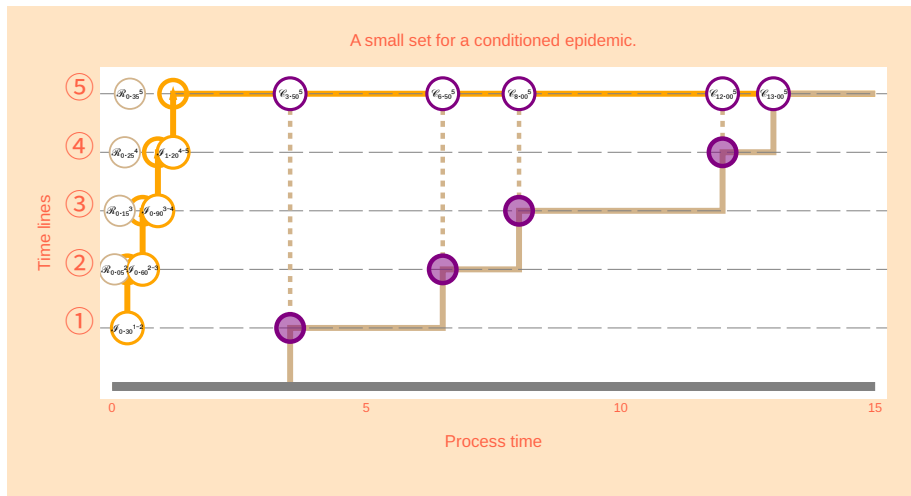


Figure 16: A conditional epidemic in which all activated infections occur before time 3.0, also before smallest observed removal time.

Other technical information

Software versions

Software used in computations:

<i>Software</i>	<i>Version</i>	<i>Branch</i>	<i>Date of last commit</i>
Quarto	1.6.39	—	
Running under julia	1.11.2	—	
Module EpidemicsCFTP	2.2.487	main	Fri Jan 10 20:17:28 2025
Module EpidemicsUtilities	0.1.2.154	main	Wed Jan 15 13:23:36 2025
This Quarto script	2.2.587	master	Thu Jan 16 17:01:55 2025

Revision history

These notes were produced from PerfectEpidemics.qmd:

Author:	Wilfrid Kendall W.S.Kendall@warwick.ac.uk
Date:	Thu Jan 16 17:01:55 2025 +0000
Summary:	Final version for 17 January 2025.