Wandering around a fibrous network[1]

when all the paths look very much alike

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Abstract: I will give an overview of work I have been interested in for the last decade or so, concerned with generating models based on Poisson processes which supply insight into route-finding and traffic flows. Initial constructions (Aldous, 2014; Kahn, 2016; WSK, 2017) showed existence of such models satisfying the SIRSN axioms formulated by Aldous (2014). But these constructions all involve infinitely long linear paths, and in particular do not easily admit local influences. Can one do better? I shall discuss recent work showing that SIRSN can indeed arise under far less stringent conditions, based on line segments or even suitably stiff fibres.

https://wilfridskendall.github.io/talks/Short-Fibre-SIRSN-talk-2024/Short-Fibre-SIRSN-talk-handout.pdf

Development of work on line process models for transportation networks (Aldous & WSK, 2008; WSK, 2011, 2014),
particularly concerning line process models for online maps:
(Aldous & Ganesan, 2013; Aldous, 2014; Kahn, 2016; WSK, 2017, 2020).

Current objective: model online maps, but replace infinitely long straight lines by line segments (“sticks”) or moderately curved paths (“stiff fibres”).

Structure of talk:

1. Modelling efficient spatial transport systems by an axiomatic approach to scale-invariant random spatial networks (SIRSN);
2. Practicalities (why the two current SIRSN models are unsatisfactory);
3. Sticks and fibres (brief notes on some key results and proofs).

(I) Models for efficient spatial transport systems

How to model real-life online maps? a task for stochastic geometry.

1. Mechanism to connect generic nodes x and y;
2. "Scale -invariant" networks (SIRSN) as potential models;
3. What do "best routes" look like?
4. What can be said about flows? – not for today.

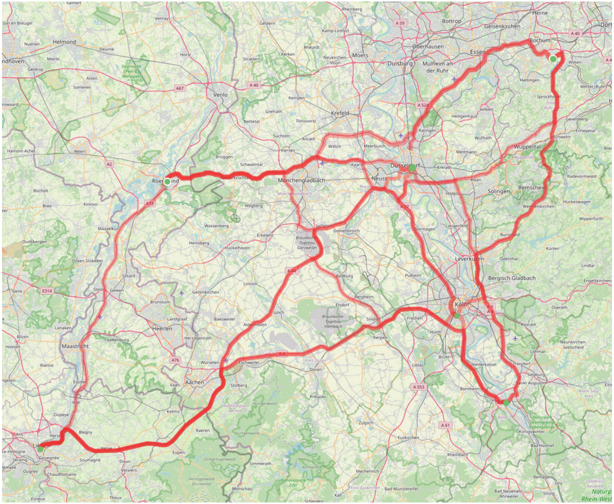
Related to work on mathematics for cities, for example:

- Work on city shape (Bender *et al.*, 2004; Courtat, 2012);
- Route-finding (Bongiorno *et al.*, 2021);
- "Transit nodes" (Bast *et al.*, 2007).

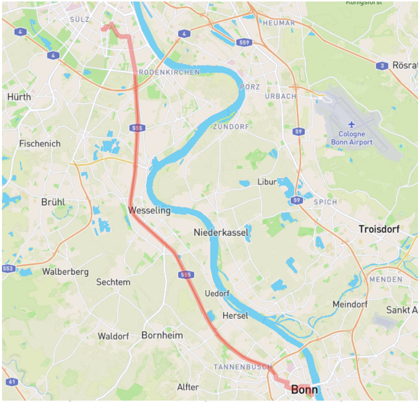
Work so far: models based on *eg* Poisson line processes (Plp) (Aldous & WSK, 2008; WSK, 2011, 2017, 2020; Kahn, 2016; Blanc, 2023).

We'd strongly prefer: models based on *localizable* object processes (*eg* Poisson stick processes, Poisson fibre processes).

Online maps look as if they might be scale-invariant!



An online map as a mechanism for generating networks

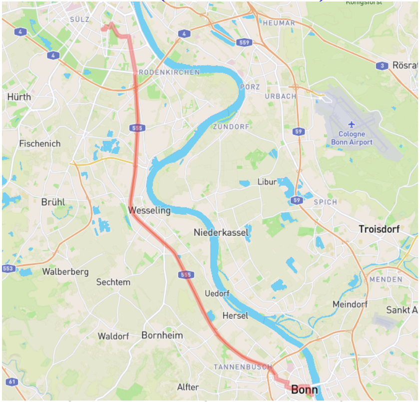


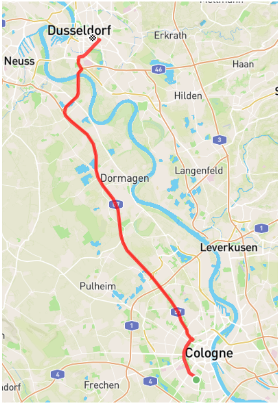
Given $x_1, \ldots, x_n$, the online map should generate a *network*

$$\mathcal{N}(x_1, \ldots, x_n) \quad = \quad \bigcup \{ \mathcal{R}(x_i, x_j) \, ; \, 1 \leq i < j \leq n \} \, .$$

NB: disallow one-way streets: $\mathcal{R}(x_i, x_j) = \mathcal{R}(x_j, x_i)$.

Axiom 1: (Statistical) scale-invariance of networks



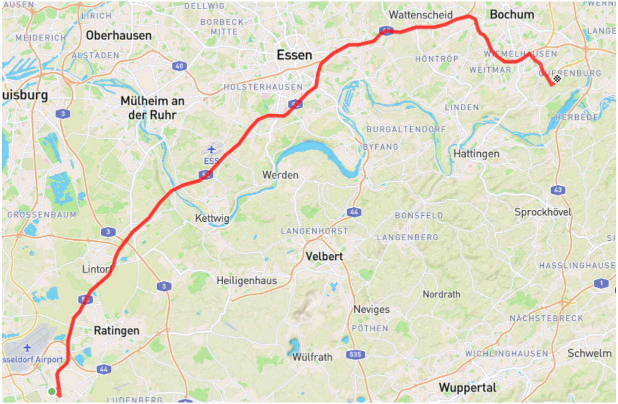


We require *statistical scale invariance*: for all similarities $\mathfrak{S}$ of $\mathbb{R}^m$:

$$\mathcal{L}\left(\mathfrak{S}\mathcal{N}(x_1, \ldots, x_n)\right) \quad = \quad \mathcal{L}\left(\mathcal{N}(\mathfrak{S}x_1, \ldots, \mathfrak{S}x_n)\right) .$$

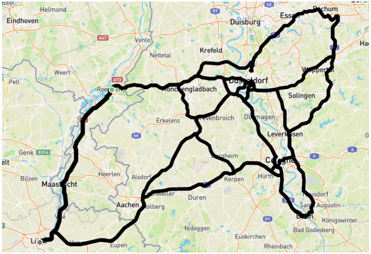
NB: model $\mathcal{R}(x_i, x_j)$ hence $\mathcal{N}(x_1, \ldots, x_n)$ as random.

Axiom 2: Finite mean length of routes



We insist that all routes have finite mean length.

Axiom 3: SIRSN property



NB: SIRSN = “Scale-invariant Random Spatial Network”.

We require massive re-use of route components (strong SIRSN property).

Weak SIRSN: Source/terminus nodes spread evenly everywhere on plane. “Route-re-use” means mean-network-length-per-unit-area must be finite.

Strong SIRSN: Source/terminus nodes spread densely everywhere on plane. “Route-re-use” means mean-network-length-per-unit-area is finite *for the “long-distance” network*.

Scale-Invariant Random Spatial Networks (SIRSN)

Axioms (following Aldous & Ganesan, 2013; Aldous, 2014).

Given: a (random) network $\mathcal{N}(x_1, \ldots, x_n)$ connecting nodes. Require:

1. Scale-invariance: for each Euclidean similarity $\mathfrak{S}$, $\mathcal{L}\left(N(\mathfrak{S}x_1, \ldots, \mathfrak{S}x_n)\right) = \mathcal{L}\left(\mathfrak{S}N(x_1, \ldots, x_n)\right)$.
2. If D_1 is length of fastest route between two points at unit distance apart then mean length is finite: $\mathbb{E}\left[D_1\right] < \infty$.
3. Weak SIRSN property: network connecting points of (independent) unit intensity Poisson point process has finite average length density;
   or
   (Strong) SIRSN property: finite length intensity of "long-range" routes for points of *dense* Poisson point process.

Do SIRSN exist at all?

Yes (first version): Hierarchical SIRSN (Aldous, 2014), a *randomized* dense dyadic speed-marked planar *lattice*.

Do SIRSN exist at all? a more natural answer

Yes (second version): Poisson line process SIRSN: planar *and also* higher dimensions too (Aldous, 2014; Kahn, 2016; WSK, 2017).

Construction uses speed-marked (improper) Plp Π.

Π is governed by an intensity measure ϖ_{Plp} on $(0,\infty)\times\mathbb{R}\times[0,\pi)$, scaling speed $v\to\lambda v$ versus space $x\to\lambda^{\gamma-1}x$:

$$\mathrm{d}\varpi_{\text{Plp}} \quad = \quad \tfrac{1}{2}(\gamma-1)v^{-\gamma}\,\mathrm{d}v\;\;\mathrm{d}r\,\mathrm{d}\theta \qquad (\text{make sure } \gamma>1!)\,.$$

NB: $\frac{1}{2}\,\mathrm{d}r\,\mathrm{d}\theta$: intensity measure for standard Poisson line process.
NB ($\gamma>1$): Lines of speed ≥ 1 form standard Poisson line process $\Pi^{(\geq 1)}$.

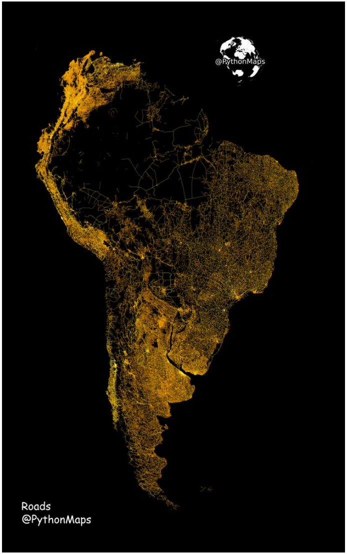
Use fastest Π-paths for routes. Require $\gamma>2$ so

- Π-paths cannot reach ∞ in finite time;
- Π-paths can reach any prescribed point in finite time.

It is non-trivial to prove that Π forms a SIRSN!

(II) From lines to (non-infinite) fibres

But we want to be able to model networks like this:



Sticks and stiff fibres: foundations

Use object processes (random objects located at points of a Poisson process):

- Scale-invariant, so speed-marked ("object soup");
- Sticks or C^1 fibres, so notion of length;
- Stiff fibres (each fibre within ε of its intrinsic overall direction).
- Define using intensity measure on configuration space:

$$\begin{array}{ccccccccc} \mathbb{R}^m & \times & \mathrm{SO}_m & \times & (0,\infty) & \times & \text{Shape} & \times & (0,\infty) \\ x & & \omega & & r & & \text{shape} & & v \\ \text{(location)} & & \text{(rotation)} & & \text{(length)} & & & & \text{(speed)} \end{array}$$

- Parameters (after applying scale-invariance):
  - fibre shape distribution;
  - rescaled length distribution (joint with shape);
  - specific length intensity λ for set of fibres of more than unit speed;
  - scaling index $\gamma = (m-1)(\nu-1) = 1$ for $\nu > 2$.

Sticks and stiff fibres: techniques

1. Integrate the measurable field of orientations using Lipschitz paths;
2. Sobolev compactness theorem for Lipschitz paths: hence existence of "geodesics" (fastest connecting paths);
3. For routes, select geodesics in scale-invariant way using auxiliary scale-invariant randomness;
4. Validation of SIRSN axioms depends on comparison of (fractional power) of connection time with Exponential distribution.

Finesse away percolation issues by supposing fibre soup is *thick*: scaled length distribution has finite first moment but infinite second moment (quantified by tail condition)

and do the sums for

- probability of "timely" connection of two close points by a single fibre;
- concatenating point-pairs for "timely" connection of two far points;
- deriving exponential moment of fractional power(connection time);
- deriving estimate on "time-diameter" for compact set.

(III) Beyond thickness: percolation

The SIRSN property should not need “thickness” (single-fibre connection).

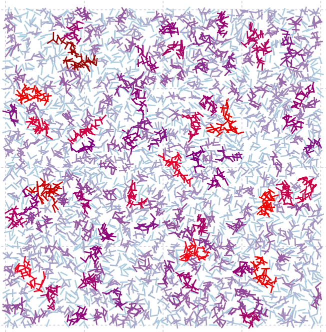
Conjecture: Super-critical percolation suffices.

I conjecture that a Poisson soup of stiff fibres can form a SIRSN if there is super-critical percolation for the family of fibres of greater than unit speed.

Simulation of a caricature:

- Scaled stick-length $= 1$;
- large clusters coloured red,
  small clusters coloured blue;
- using stick-length 1;
- length intensities reduce inversely.

Simulation: percolation is plausible.



Conclusion

- Strong positive evidence that “SIRSN are everywhere!” (not just when infinite lines are involved).
- There are very simple scale-invariant stick soups which are thick: so this theory is not vacuous!
- Future prospects:
  - traffic, benchmark models for traffic flow (cf Gameros Leal, 2017);
  - adding spatial inhomogeneity;
  - moving beyond Poisson, *eg* adding simple interactions.

Thank you for your attention! QUESTIONS?

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Previous versions of this talk

In earlier forms:

- Colloquium lecture *Scale-invariant random spatial networks (SIRSN) from line patterns*: Journée de Rham lecture at Lausanne 22/4/2015; Auckland 9/8/2016; Bern 10/6/2016; Aarhus 6/6/2016; Manchester Probability Seminar 13/4/2016.
- Four-lecture course *Poisson line processes and spatial transportation networks*: Edinburgh 21-22/6/2018.
- Talks on more general varieties of SIRSN: Sandbjerg 16 February 2020; Dunkerque 17-19 November 2021; Singapore 21/6/2022; Bern 15/9/2022; Cologne 29/3/2023; UCD 24/4/2024.
- Shorter form talks (25-30 minutes): "Warwick Stats 50" 8/9/2022; Baxendale birthday Warwick 9/6/2023; OEBN memorial Aarhus 29/5/2024.

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