# Markov chain Monte Carlo and Perfect simulation

## Lecture at Aristotle University of Thessaloniki

Wilfrid S Kendall

15 May 2024

Handout available on the web:

# Introduction

Figure 1: Αριστοτέλης 384–322 BCE

Aristotle: “The more you know, the more you know you don’t know.”

# Sketch of MCMC (I)

Figure 2: Edward Teller (1908-2003)

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[Fermi once said,] Teller was the only monomaniac he knew who had several manias: see Brown & May (2004).

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- But, physicists always remind us, physicists got there fifty years earlier!

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Question (A) is what this lecture is all about.

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  - Conclusion: effective MCMC requires very careful thought about appropriate length of run — think deeply about the problem!
- Can there ever be a better way?

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- "Perfect simulation" (WSK, 1998): because everyone knows it isn't going to be perfect, whereas people might imagine "exact simulation" would somehow miraculously defeat numerical approximation error :-).

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- Detailed expositions are given by WSK (2005), Huber (2015). WSK (2015) shows how to implement CFTP in R.

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# Applications to Queues and Epidemics

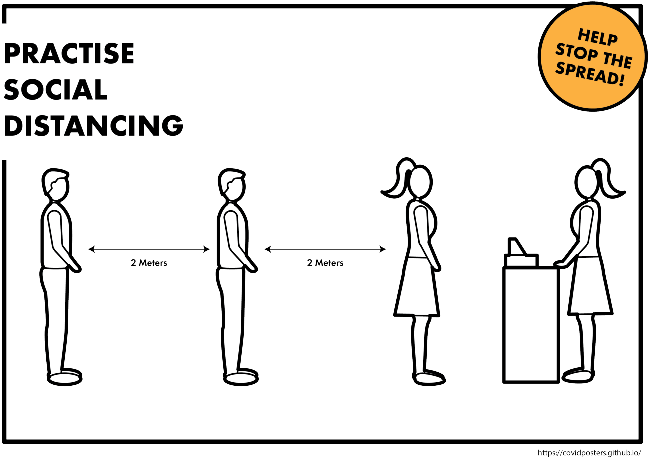


Figure 3: An illustration introducing *both* queues *and* epidemics!

# Perfect Queues

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- Connor & WSK (2015) show how to extend Sigman (2011), showing how Dominated CFTP can be applied to simulate (sub-critical!) queues perfectly (and this has now been generalized by others to the case of non-Poissonian inter-arrival times). (Technical point: pathwise domination requires service times to be assigned in order of commencement of service!)

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- (indicate how the perfect simulation algorithm can be used as a high-dimensional integration device to enable simulation-based Bayesian inference!).

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# Conclusion

- You don’t always have to put up with burn-in issues when doing MCMC;
- CFTP works even for significantly complex and relevant models of real-life phenomena;
- *Of course* really detailed models are still going to resist perfect simulation: but it will always be helpful to compare with a simpler model (using fewer parameters!).
- CFTP is clearly an important tool to be considered by the investigator seeking to do accurate and informative MCMC.

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