Macroeconomics 2-Assignment 1

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Consider the Neo-Classical growth model. Time is discrete and goes on forever. There is a representative agent that derives utility only from consumption and discounts future utility at a rate b. The agent owns k0 units of capital and has an endowment of time that can be used for labor or leisure every period. The time endowment is normalized to 1. There is a representative firm that hires labor and rents capital to produce using a constant returns to scale technology. Capital rental rate is r and the wage is w. Capital depreciates fully after use.

1 Define a competitive equilibrium for this economy

A competitive equilibrium for this economy comprise of price $\{p_t, w_t, r_t\}$, and allocations for the firm $\{k_t^d, n_t^d, y_t\}$, and the household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}$ such that:

- Given $\{p_t, w_t, r_t\}$, the allocation of the representative

  firm $\{k_t^d, n_t^d, y_t\}$ solves:

$$\begin{aligned} \max_{\{y_t, k_t, n_t\}} \quad & \Sigma_{t=0}^{\infty} p_t (y_t - r_t k_t - n_t w_t) \\ \text{s.t.} \quad & y_t = F(k_t, n_t), \quad \textit{for all } t \geq 0, \\ & y_t, k_t, n_t \geq 0 \end{aligned} \tag{1}$$

- Given $\{p_t, w_t, r_t\}$, the allocation of the representative

  household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}$ solves:

$$
\begin{aligned}
\max_{\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}} \quad & \Sigma_{t=0}^{\infty}\beta^t U(c_t) \\
\text{s.t.} \quad & \Sigma_{t=0}^{\infty} p_t(c_t+i_t) = \Sigma_{t=0}^{\infty} p_t(r_t k_t + n_t w_t)) + \pi, \quad for\ all\ t \geq 0, \\
& x_{t+1} = (1-\delta)x_t + i_t, \quad for\ all\ t \geq 0, \\
& 0 \leq n_t \leq 1,\ 0 \leq k_t \leq x_t,, \\
& c_t, x_{t+1} \geq 0\ for\ all\ t \geq 0, \\
& x_0\ \ given
\end{aligned} \tag{2}
$$

- Markets clearing
  - $y_t = c_t + i_t$ (Goods Markets)
  - $n_t^d = n_t^s$ (Labour Markets)
  - $k_t^d = k_t^s$ (Capital Services Markets)

2 Define the social planner's problem for this economy.

The social planner's problem for this economy is defined as follows:

$$
\begin{aligned}
\max_{\{c_t, k_t, n_t\}} \quad & \Sigma_{t=0}^{\infty}\beta^t U(c_t) \\
\text{s.t.} \quad & F(k_t, n_t) = c_t + k_t - (1-\delta)k_t, \quad for\ all\ t \geq 0, \\
& y_t, k_t, n_t \geq 0, \\
& c_t \geq 0,\ k_t \geq 0,\ 0 \leq n_t \leq 1,, \\
& k_0 \leq \bar{k_0}
\end{aligned} \tag{3}
$$

The value function is the total lifetime utility of the representative household if the social planner chooses $\{y_t, k_t, n_t\}$ optimally and the initial capital stock in the economy is $\bar{k_0}$.

3 Show that the equilibrium allocation of consumption, capital, and labor coincides with those of the planner's.

By the first welfare theorem, equilibrium allocation of consumption, capital, and labor coincides with those of the planner's. In fact, household's preference satisfies local non-satiation condition and the first welfare theorem states that any competitive equilibrium leads to a Pareto efficient allocation of resources under local non-satiation of household preference.
By local non-satiation we mean that, for any consumption allocation x and any $\epsilon > 0$, there's some allocation x' such that $\|x - x'\| \leq \epsilon$ and $U(x') > U(x)$.

4 Pose the planner's dynamic programming problem. Write down the appropriate Bellman equation.

Let's assume U to be continuously differentiable, strictly increasing, strictly concave and bounded. In addition, let's assume that U satisfies Inada conditions. The discount factor β belongs to $(0, 1)$. Furthermore, let's assume that F is continuously differentiable and homogenous of degree 1; strictly increasing and strictly concave. $F(0, n) = F(k, 0) = 0$ for all $k, n > 0$. Let's assume also that F satisfies Inada conditions.
From these assumptions, it turns out that $n_t = 1$ and $k_0 = \bar{k_0}$ for all t since households do not value leisure in their utility function and production function is strictly increasing in capital.
To simplify notation we define $f(k) = F(k, 1) + (1 - \delta)k$ for all k. Exploiting the implications of the assumptions, and substituting for $c_t = f(k_t) - k_{t+1}$, we can rewrite the previous social planner's problem as:

$$\begin{aligned} \max_{\{c_t, k_t, n_t\}} \quad & \Sigma_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\ \text{s.t.} \quad & 0 \leq k_{t+1} \leq f(k_t), \\ & k_0 = \bar{k_0} > 0 \end{aligned} \tag{4}$$

The planner faces between letting the consumer eat today versus investing in the capital stock.
The associated Bellman equation is defined as:

$V(k) = Max\{U(f(k) - k') + \beta V(k')\}, \quad \text{with } 0 \leq k' \leq f(k).$

5 Solve the planner's dynamic programming problem (find the value and policy functions).

Pose $u(c) = \ln c$ and $f(k,l) = zk^{\alpha}l^{1-\alpha}$

The Bellman becomes: $V(k) = Max\{\ln(f(k) - k') + \beta \ln(k')\}$ $0 \leq k' \leq zk^{\alpha}$.

I am going to guess a particular functional form of a solution and then verify that the solution has in fact this form.

Let's pose V(k)=A+B lnk

Let's find the value of k' that maximises $ln(zk^{\alpha} - k') + \beta(A + Bln(k'))$. From the first order condition we find that: $\frac{1}{zk^{\alpha}-k'} = \frac{\beta B}{k'}$, Hence, $k' = \frac{zk^{\alpha}\beta B}{1+\beta B}$.

$A + Blnk = ln(zk^{\alpha} - k') + \beta(A + B\ln(k'))$

Plugging the expression of k' in the above equality and proceed by identification, we find $B = \frac{\alpha}{1-\alpha\beta}$.

Plugging back the expression of B in the above equality, we obtain.

$A(1-\beta) = \frac{\alpha\beta}{1-\alpha\beta}ln(\alpha\beta) + ln(1-\alpha\beta) + \alpha\frac{lnz}{1-\alpha\beta}$ Hence, $A = \frac{1}{1-\beta}\left(\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta) + \alpha\frac{lnz}{1-\alpha\beta}\right)$

Therefore, the value function is $V^*(k) = A + B\ln(k)$ with A, B as determined above. The associated optimal policy function is $k' = \frac{zk^{\alpha}\beta B}{1+\beta B} = g(k)$. Plugging in the expression of B we obtain $g(k) = z\alpha\beta k^{\alpha}$

6 Use the solution to the planner's problem to obtain the steady state value of {c, k, r, w, y}

Solving the Social Planner's problem, we obtain the following first order condition:

$\frac{-\beta^t}{zk_t^{\alpha}-k_{t+1}} - \frac{\beta^{t+1}\alpha zk_{t+1}^{\alpha}}{zk_{t+1}^{\alpha}-k_{t+2}} = 0$ and so the Euler equation is $c_{t+1} = \alpha\beta zk_{t+1}^{\alpha-1}c_t$.

At the steady state, $c_{t+1} = c_t = c^*$ and $k_{t+1} = k_t = k^*$. Hence, $k^* = (\alpha\beta z)^{\frac{1}{1-\alpha}}$

Using the resource constraint at the steady state, we obtain $c^* = f(k^*) - k^* = zk^{*^{\alpha}} - k^*$

So $c^* = z(\alpha\beta z)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta z)^{\frac{1}{1-\alpha}}$, $r* = f'(k^*) = \alpha zk^{*^{\alpha-1}}$, and

$w* = z(1-\alpha)k^{*^{\alpha}}l^{1-\alpha}$; $y* = r^*k^* - w^*l$

7 **Assume that $\alpha = \frac{1}{3}$, $z = 1$. Use the solution to the planner's problem to obtain the path of {c, k, r, w, y} starting from the steady state after the following changes**

$k^* = (\frac{1}{3}\beta)^{\frac{3}{2}}$;

$c^* = (\frac{1}{3}\beta)^{\frac{1}{2}} - (\frac{1}{3}\beta)^{\frac{3}{2}} = (\frac{1}{3}\beta)^{\frac{1}{2}}\left(1 - \frac{1}{3}\beta\right)$;

$r^* = f'(k^*) = \frac{1}{3}\beta k^{*\frac{-2}{3}} = \beta^{-1}$;

$w^* = \frac{2}{3}k^{*\frac{1}{3}}$

$y^* = r^*k^* - w^*$