Macroeconomics 2-Assignment 2

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1 Define a competitive equilibrium for this economy

A competitive equilibrium for this economy comprises of price $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations for the firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and the household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that:

- Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ solves:

$$\begin{aligned} \max_{\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} p_t(y_t - r_t k_t^d - w_t l_t^d) \\ \text{s.t.} \quad & y_t = z(k_t^d)^{\alpha}(l_t^d)^{1-\alpha}, \quad \forall t \geq 0, \\ & y_t, k_t^d, l_t^d \geq 0 \; \forall t \geq 0 \end{aligned} \tag{1}$$

- Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves:

$$\begin{aligned} \max_{\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta_t\left(\frac{(c_t)^{1-\sigma}}{1-\sigma} - \chi\frac{(l_t^s)^{1+\eta}}{1+\eta}\right) \\ \text{s.t.} \quad & \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t^s + w_t l_t^s) + \pi, \quad \forall t \geq 0, \\ & x_{t+1} = (1-\delta)x_t + i_t, \quad \forall t \geq 0, \\ & 0 \leq l_t^s \leq 1, \; 0 \leq k_t^s \leq x_t \quad \forall t \geq 0, \\ & c_t, x_{t+1} \geq 0 \; \forall t \geq 0, \\ & k_0 \quad \textit{given} \end{aligned} \tag{2}$$

- Markets clearing conditions
  - $y_t = c_t + i_t$ (Goods Market)
  - $l_t^d = l_t^s = l_t$ (Labour Market)
  - $k_t^d = k_t^s = k_t$ (Capital Services Market)

2 Find the steady state value for {c, l, k, y, r, w}

$$\begin{aligned} \max_{\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} p_t(y_t - r_t k_t^d - w_t l_t^d) \\ \text{s.t.} \quad & y_t = z(k_t^d)^{\alpha}(l_t^d)^{1-\alpha}, \quad \forall t \geq 0, \\ & y_t, k_t^d, l_t^d \geq 0 \ \forall t \geq 0 \end{aligned} \tag{3}$$

The first order conditions of the firm's problem with respect to k_t and l_t are the following:

$$\begin{aligned} r_t &= z\alpha(k_t)^{\alpha-1}(l_t)^{1-\alpha} \\ w_t &= z(1-\alpha)(k_t)^{\alpha}(l_t)^{-\alpha} \\ \pi &= \sum_{t=0}^{\infty} p_t(z(k_t)^{\alpha}(l_t)^{1-\alpha} - r_t k_t - w_t l_t) = 0 \end{aligned} \tag{4}$$

$$\begin{aligned} \max_{\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta_t\left(\frac{(c_t)^{1-\sigma}}{1-\sigma} - \chi\frac{(l_t^s)^{1+\eta}}{1+\eta}\right) \\ \text{s.t.} \quad & \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t^s + w_t l_t^s) + \pi, \quad \forall t \geq 0, \\ & x_{t+1} = (1-\delta)x_t + i_t, \quad \forall t \geq 0, \\ & 0 \leq l_t^s \leq 1, \ 0 \leq k_t^s \leq x_t \quad \forall t \geq 0, \\ & c_t, x_{t+1} \geq 0 \ \forall t \geq 0, \\ & k_0 \quad \textit{given} \end{aligned} \tag{5}$$

$$\begin{aligned} \mathcal{L}(\{c_t, k_t, l_t\}_{t=0}^{\infty}, \lambda_t) = & \sum_{t=0}^{\infty} \beta^t\left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi\frac{l_t^{1+\eta}}{1+\eta}\right) \\ & + \lambda_t\left(\sum_{t=0}^{\infty} p_t(z k_t^{\alpha} l_t^{1-\alpha} - c_t - k_{t+1} + (1-\delta)k_t\right) \end{aligned} \tag{6}$$

The first order conditions are the following:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t c_t^{-\sigma} - \lambda_t p_t = 0 \\
\frac{\partial \mathcal{L}}{\partial l_t} &= -\beta^t \chi l_t^{\eta} + \lambda_t p_t z(1-\alpha) k_t^{\alpha} l_t^{-\alpha} = 0 \\
\frac{\partial \mathcal{L}}{\partial k_t} &= \lambda_t p_t (z\alpha k_t^{\alpha-1} l_t^{1-\alpha} + 1 - \delta) - \lambda_{t-1} p_{t-1} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &= z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta) k_t - c_t = 0
\end{aligned}
\tag{7}
$$

In steady state, $c_t = c, k_t = k, l_t = l$. Normalizing $p_0 = 1$, we get:

$$
\begin{aligned}
\lambda_t &= c \\
p_t &= \beta^t \\
c^{\sigma} l^{\eta} &= z(1-\alpha) k^{\alpha} l^{-\alpha} / \chi \\
z\alpha k^{\alpha-1} l^{1-\alpha} &= 1/\beta - 1 + \delta \\
c &= z k^{\alpha} l^{1-\alpha} - \delta k
\end{aligned}
\tag{8}
$$

Next, we will express the steady state variables in term of $U = \frac{k}{l}$ and $V = \frac{c}{l}$.

$$
\begin{aligned}
U &= \frac{k}{l} = \left(\frac{z\alpha\beta}{1 - \beta + \beta\delta} \right)^{\frac{1}{1-\alpha}} \\
V &= \frac{c}{l} = z(\frac{k}{l})^{\alpha} - \delta \frac{k}{l} = zU^{\alpha} - \delta U \\
c^{\sigma} l^{\eta} &= (lV)^{\sigma} l^{\eta} \\
&= z(1-\alpha)(\frac{k}{l})^{\alpha} / \chi \\
&\Rightarrow l^{\sigma+\eta} = \frac{z(1-\alpha)U^{\alpha}}{\chi V^{\sigma}} \\
&\Rightarrow l = \left(\frac{z(1-\alpha)U^{\alpha}}{\chi V^{\sigma}} \right)^{\frac{1}{\sigma+\eta}} \qquad (*)
\end{aligned}
\tag{9}
$$

Having l in steady state, we can obtain other steady state variables:

$$\begin{aligned}
k &= Ul \quad (*) \\
c &= Vl \quad (*) \\
y &= zk^{\alpha}l^{1-\alpha} = zU^{\alpha}l \quad (*) \\
r &= \alpha z k^{\alpha-1}l^{1-\alpha} = \alpha z U^{\alpha-1} \quad (*) \\
w &= (1-\alpha)zk^{\alpha}l^{-\alpha} = (1-\alpha)zU^{\alpha} \quad (*)
\end{aligned} \tag{10}$$

3 Pose the planner's dynamic programming problem. Write down the appropriate Bellman equation.

The social planner's problem is as follows:

$$\begin{aligned}
\max_{\{k_t,c_t,l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \Big(\frac{(c_t)^{1-\sigma}}{1-\sigma} - \chi \frac{(l_t)^{1+\eta}}{1+\eta}\Big) \\
s.t. \quad z(k_t)^{\alpha}(l_t)^{1-\alpha} = c_t + k_{t+1} - (1-\delta)k_t \quad \forall t \geq 0 \\
c_t \geq 0, k_t \geq 0, 0 \leq l_t \leq 1 \quad \forall t \geq 0 \\
k_0 \; given
\end{aligned} \tag{11}$$

The associated Bellman equation is defined as follows:

$$v(k_0) = \max_{\substack{0 \leq k' \leq z(k)^{\alpha}(l)^{1-\alpha}+(1-\delta)k \\ 0 \leq l \leq 1}} \Big\{\frac{(z(k)^{\alpha}(l)^{1-\alpha} + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} - \chi \frac{(l)^{1+\eta}}{1+\eta} + \beta v(k')\Big\} \tag{12}$$

4 Find χ such that $l_{ss} = 0.4$

From $l_{ss}^{\sigma+\eta} = \frac{z(1-\alpha)U^{\alpha}}{\chi V^{\sigma}}$, we can derive χ as: $\chi = \frac{z(1-\alpha)U^{\alpha}}{l_{ss}^{\sigma+\eta}V^{\sigma}}$

Plug in $\alpha = 1/3, z = 1, \delta = 2, \eta = 1, l_{ss} = 0.4$, $\beta = 0.9$ we obtain

$$\chi = 38.81 \tag{13}$$

5 Solve the planner' problem numerically using value function iteration. You must do it using:

5.1 Plain VFI

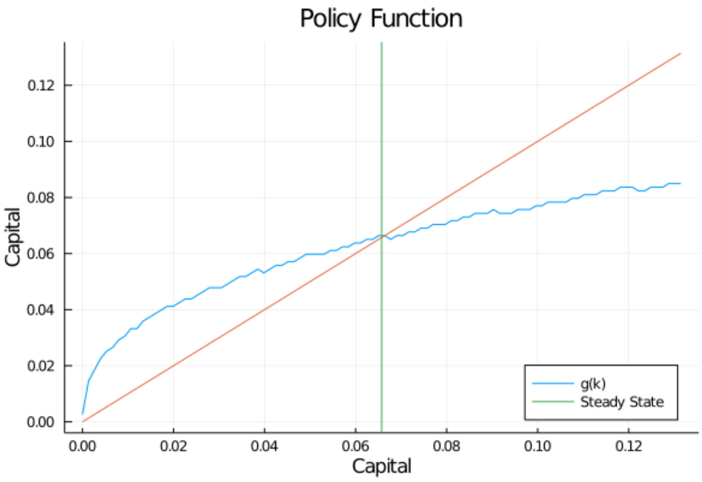

Figure 1: Approximated Value Function

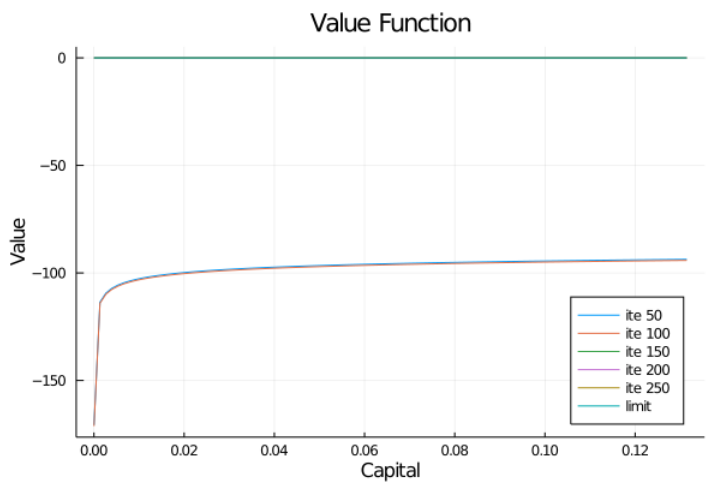

Figure 2: Approximated Policy Function

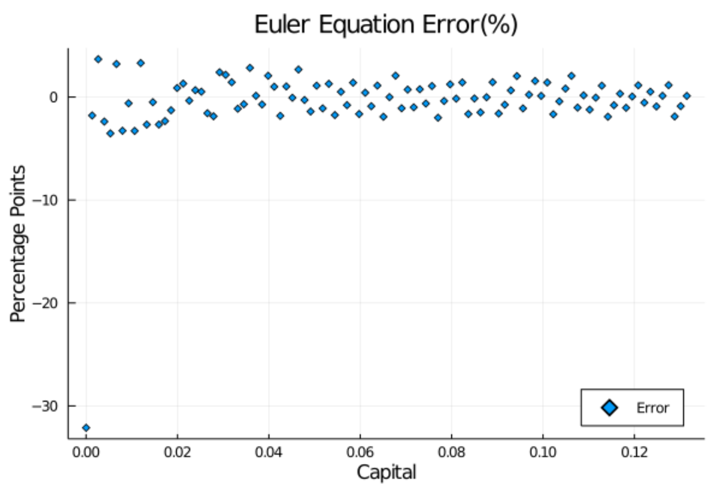

Figure 3: Approximated Error(%)

5.2 Modified Howard's Policy Iteration (you must choose the number of policy iterations)

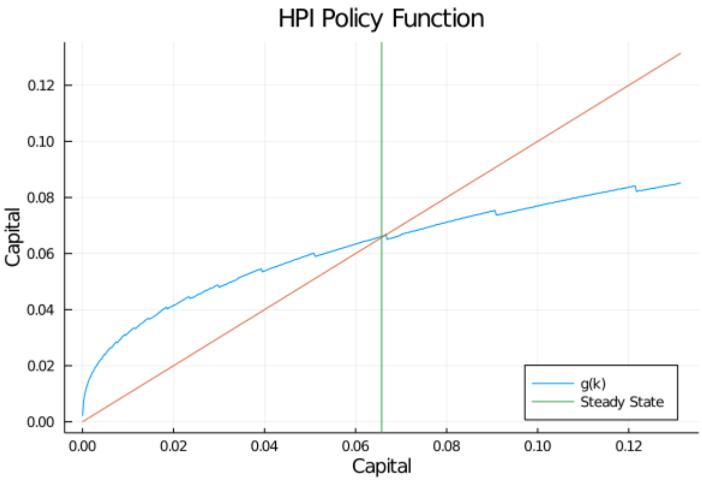

Figure 4: Approximated Value Function

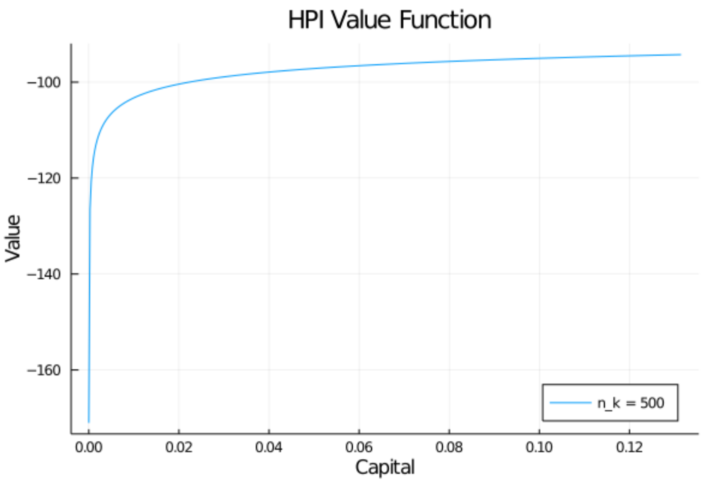

Figure 5: Approximated Policy Function

Figure 6: Approximated Error(%)

5.3 MacQueen-Porteus Bounds

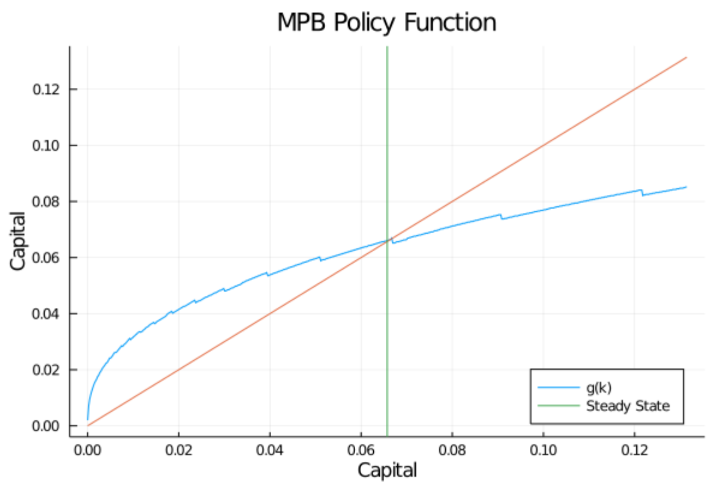

Figure 7: Approximated Value Function

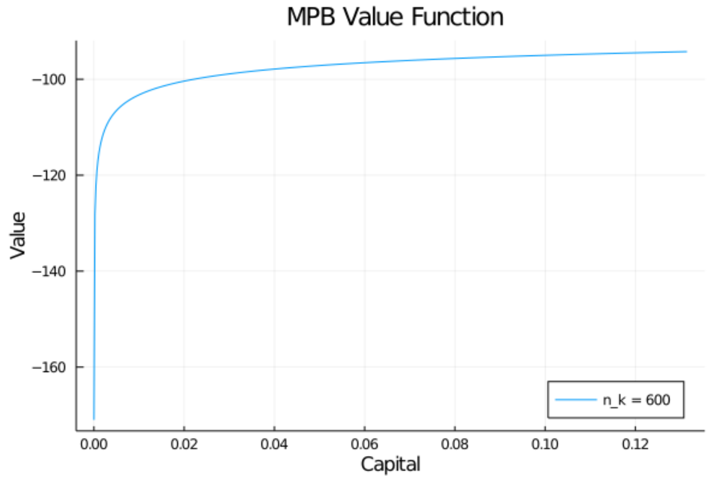

Figure 8: Approximated Policy Function

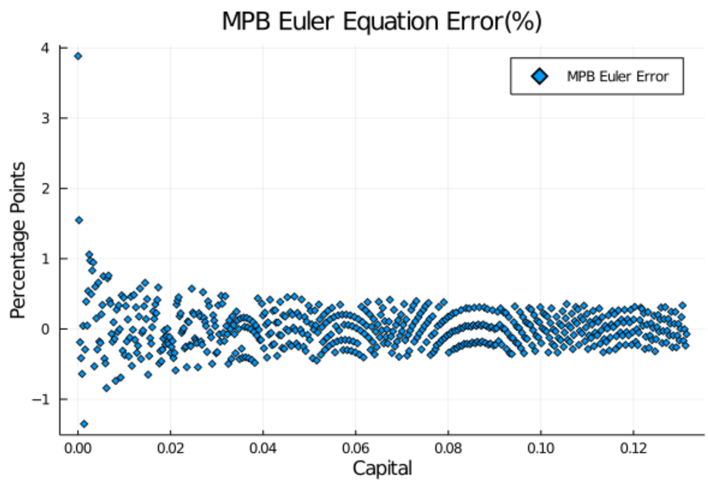

Figure 9: Approximated Error(%)

6 Use the solution to the planner's problem to obtain the steady state value of $\{c, k, r, l, w, y\}$

6.1 Capital decreases to 80% of its steady state value

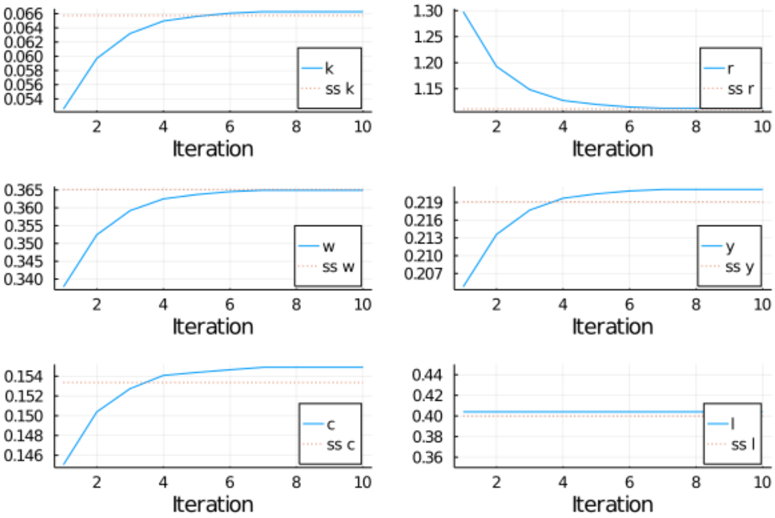

Figure 10: SS capital decreases to 80%

6.2 Productivity increases permanently by 5%

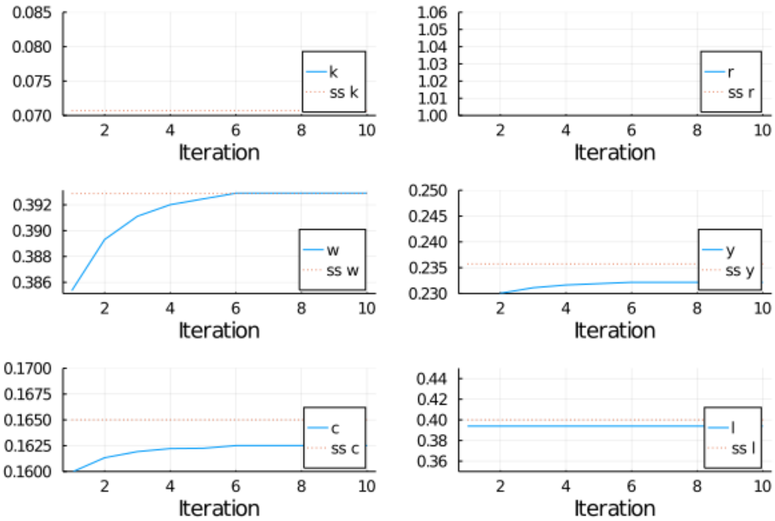

Figure 11: Productivity increases permanently by 5%

7 Prove that the mapping used in Howard's policy iteration algorithm is a contraction.

I will prove this using an alternative version of Blackwell's theorem

$$Tv(k) = \max_{\substack{0 \leq k' \leq z(k)^{\alpha}(l)^{1-\alpha}+(1-\delta)k \\ 0 \leq l \leq 1}} \{\frac{(z(k)^{\alpha}(l)^{1-\alpha} + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} - \chi\frac{(l)^{1+\eta}}{1+\eta} + \beta v(k')\} \tag{14}$$

Let's pose

$$U(f(k) - k', l) = \frac{(f(k) - k')^{1-\sigma}}{1-\sigma} - \chi\frac{l^{1+\eta}}{1+\eta}$$

Let's define $(B[0, \infty), d)$ as our metric space. B being the space of bounded functions on $[0, \infty)$ with d being the sup-norm. I am going to verify that all the hypotheses for Blackwell's theorem are satisfied.

- T maps $(B[0, \infty), d)$ into itself.

  In fact, if we take v to be bounded, since we assumed that U is bounded, so is Tv.

- Monotonicity

  Given v, $w \in B[0, \infty)$, $\beta \in (0, 1)$, and $k \geq 0$ such that $v(k) \leq w(k)$, if $\overline{g}(k)$ denotes the Howard's optimal policy function, hence,

$$\begin{aligned} Tv(k) &= U(\overline{g}(k)) + \beta v(\overline{g}(k)) \\ &\leq U(\overline{g}(k)) + \beta w(\overline{g}(k)) \\ &= Tw(k) \end{aligned} \tag{15}$$

- Discounting

  Given $v \in B[0, \infty)$, $k \geq 0$, $a \geq 0$, and $\beta \in (0, 1)$

$$\begin{aligned} T(v + a)(k) &= U(\overline{g}(k)) + \beta(v(\overline{g}(k)) + a) \\ &= U(\overline{g}(k)) + \beta v(\overline{g}(k)) + \beta a \\ &= Tv(k) + \beta a \end{aligned} \tag{16}$$

The mapping used in Howard's fixed policy function satisfies all the hypotheses for Blackwell's theorem, therefore it's a contraction mapping.