The observed relationship between standard deviations of security returns and their average covariances with the returns on other securities takes some of the sting out of earlier comments. Thus, in a diversified portfolio of many securities, the contribution of an individual security to the variance of the portfolio's return depends to a greater extent on the security's covariances with the returns on other securities in the portfolios than on the variance of the security's return. Nevertheless, because of the relationship between security return variances and covariances, the variance of a security's return is likely to be a good indication of how the security will contribute to the variance of the return on a diversified portfolio. Similarly, although the variance of the return on a diversified portfolio depends primarily on the covariances between the returns on the securities in the portfolio, the variances of the component security returns are nevertheless likely to be a good indication of the level of the variance of the return on the portfolio. Generally, the larger the variances of component security returns, the larger the variance of the return on the portfolio.

V. Conclusions

We have considered the two-parameter portfolio model in some detail, both theoretical and numerical. We are ready now to consider the characteristics of equilibrium prices and the relationship between expected return and risk in a market where investors make portfolio decisions according to the two-parameter model.

CHAPTER 8

Capital Market Equilibrium in a Two-Parameter World

I. Introduction

Chapter 7 presents a model for portfolio decisions by investors in a world where probability distributions of returns on portfolios are normal. In this two-parameter model, the investor finds it possible to summarize the distribution of the return on any portfolio in terms of its mean and its standard deviation, and he can rank portfolio return distributions solely in terms of the values of these parameters. Moreover, the investor is assumed to like expected portfolio return, but he is risk-averse in the sense that he dislikes standard deviation of portfolio return. These assumptions about the investor's tastes lead to the fundamental result of the two-parameter model—the investor's optimal portfolio is efficient. To be efficient, a portfolio must have the property that no other portfolio with the same or higher expected return has lower standard deviation of return.

This chapter is concerned with the implications of the portfolio model for capital market equilibrium. That is, if investors make portfolio decisions in accordance with the two-parameter model, how will this affect the process of price formation in the capital market? More specifically, if investors try to hold efficient portfolios, what sort of relationships between expected return and risk can we expect to observe in the capital market? The next chapter then considers whether the expected return-risk relationships that characterize the capital market in a two-parameter world are descriptive of the data generated by the real-world capital market.

The fiist step in the present chapter is to discuss expected return and risk from the viewpoint of an individual investor. We then find that, with a few simplifying assumptions, the type of expected return-risk relationships that apply to individual investors also apply to the market.

II. The Relationship Between Expected Return and Risk in an Efficient Portfolio

The decision problem facing the investor is precisely as in Chapter 7. At time 1 the investor has wealth w_1 that he must allocate to current consumption c_1 and to an investment $w_1 - c_1$ in some portfolio. The market value of his portfolio at time 2 is then his consumption c_2 at time 2. The consumption-investment decision takes place in a capital market assumed to be perfect or frictionless in the sense that an investor can purchase as much or as little of any security as he sees fit (securities are infinitely divisible), there are no transactions costs in purchasing and selling securities, and any investor can buy or sell as much as he likes of any security without affecting its price. Finally, the investor's decision is assumed to be in accordance with the two-parameter model.

A. The Risks of Securities and Portfolios

In the two-parameter model, the risk of a portfolio is measured by the standard deviation or, equivalently, by the variance of its return. The logic is that a risk-averse investor is averse to dispersion of portfolio return. With normal portfolio return distributions, dispersion is completely summarized by variance. The risk of a security in a portfolio is then determined by the contribution of the security to the variance of the return on the portfolio.

In formal terms, the return, expected return, and variance of return on a portfolio p are

$$\tilde{R}_p = \sum_{i=1}^{n} x_{ip} \tilde{R}_i \tag{1}$$

$$E(\tilde{R}_p) = \sum_{i=1}^{n} x_{ip} E(\tilde{R}_i) \tag{2}$$

$$\sigma^2(\tilde{R}_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ip} x_{jp} \sigma_{ij}, \tag{3}$$

where n is the number of securities available for inclusion in portfolios, x_{ip} is the proportion of portfolio funds $w_1 - c_1$ invested in security i in portfolio p, $E(\tilde{R}_i)$ is the expected return on security i, and $\sigma_{ij} = \text{cov}(\tilde{R}_i, \tilde{R}_j)$ is the covariance between the returns on securities i and j.

Rewriting (3) as

$$\sigma^2(\tilde{R}_p) = \sum_{i=1}^{n} x_{ip} \left(\sum_{j=1}^{n} x_{jp} \sigma_{ij} \right) = \sum_{i=1}^{n} x_{ip} \,\text{cov}(\tilde{R}_i, \tilde{R}_p), \tag{4}$$

the contribution of security i to the risk or variance of the return on p is

$$x_{ip} \left(\sum_{j=1}^{n} x_{jp} \sigma_{ij} \right) = x_{ip} \,\text{cov}(\tilde{R}_i, \tilde{R}_p). \tag{5}$$

Thus, one could interpret (5) as the risk of security i in portfolio p. Chapter 7 suggests, however, that it is more convenient to call the weighted average of covariances,

$$\sum_{j=1}^{n} x_{jp} \sigma_{ij} = \text{cov}(\tilde{R}_i, \tilde{R}_p), \tag{6}$$

the risk of security i in p, and we see shortly why this is the more convenient measure of risk. If we interpret (6) as the risk of security i in portfolio p, then from (4) the risk of the portfolio is the weighted average of the risks of individual securities.

We have known since Chapter 2 what the two-parameter model says about the risks of portfolios and the risks of securities in portfolios. We now want to determine what the model says about the relationships between expected return and risk. We find that for any efficient portfolio there is an equation relating the expected return on any security in that portfolio to the risk of the security in the portfolio. More specifically, the mathematical conditions that a portfolio must satisfy to be efficient define the relationship between expected return and risk for individual securities in that portfolio. Much of the rest of this chapter involves developing this point in detail.

B. The Mathematics of Minimum Variance Portfolios

To be efficient, a portfolio must have the property that no other portfolio with the same or higher expected return has lower standard deviation of return. Equivalently, if a portfolio is efficient, then (a) it has the maximum possible expected return given the variance of its return, and (b) it has the smallest possible variance of return given its expected return.

Any portfolio that satisfies condition (b) is called a minimum variance portfolio. Any such minimum variance portfolio can be viewed as the solution to a problem of the form:

$$\underset{\substack{x_{ip} \\ i=1,\dots,n}}{\text{Minimize}} \quad \sigma^2(\tilde{R}_p), \tag{7a}$$

subject to the constraints

$$\sum_{i=1}^{n} x_{ip} E(\tilde{R}_i) = E(\tilde{R}_e) \tag{7b}$$

$$\sum_{i=1}^{n} x_{ip} = 1.0. \tag{7c}$$

Here $E(\tilde{R}_e)$ is some given level of expected return. The problem stated in equations (7a) to (7c) is to choose proportions x_{ip}, $i = 1, \cdots, n$, invested in individual securities that minimize the variance of portfolio return subject to the constraints that expected portfolio return is equal to $E(\tilde{R}_e)$ and that the sum of the proportions invested in individual securities is 1.0.

In geometric terms, suppose the left boundary of the portfolio opportunity set is the solid curve shown in Figure 8.1. (For the moment, the dashed line in the figure is to be ignored.) The minimum variance portfolio with expected return $E(\tilde{R}_e)$ is then a point along this left boundary, say the point e. The solution to (7a) to (7c) is the set of n proportions invested in individual securities that give the minimum variance portfolio with expected return $E(\tilde{R}_e)$. Once these weights are determined, the variance of the portfolio's return is determined, which, in combination with the target value of the expected portfolio return, gives the geometric location of the portfolio.

Every point along the left boundary of the investment opportunity set is a minimum variance portfolio and so can be viewed as the solution to a problem stated in the form of equations (7a-c). One can think of the boundary in Figure 8.1 as determined by the solutions to lots of problems stated in the form of (7a-c). The portfolio e is obtained from the solution to (7a-c) when the target level of expected portfolio return is $E(\tilde{R}_e)$. Other points

FIGURE 8.1

Minimum Variance Portfolios

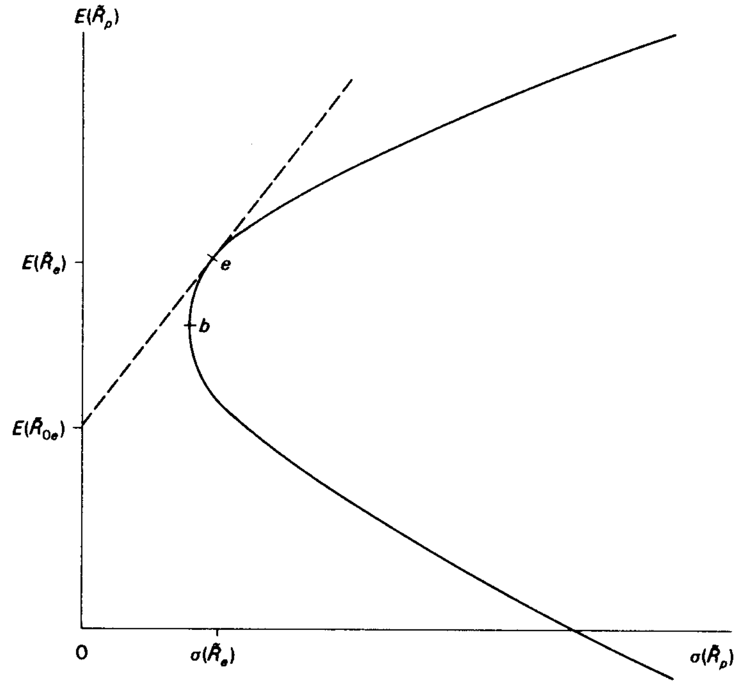
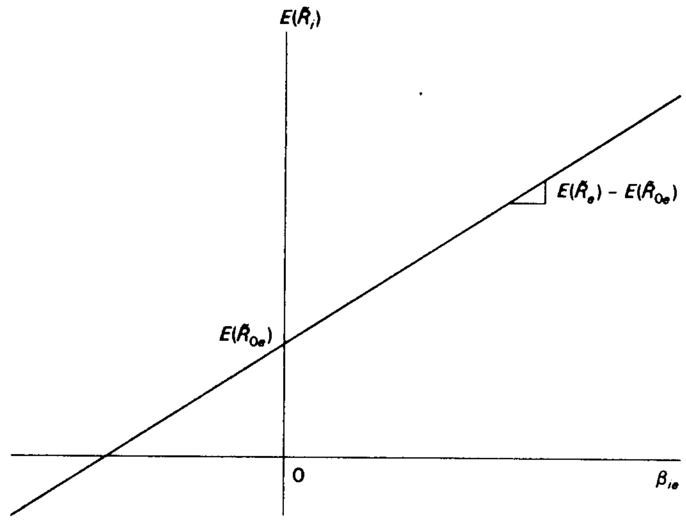


FIGURE 8.2

Relationship Between Expected Return and Risk for Securities in the Minimum Variance Portfolio e



along the boundary are obtained by re-solving (7a) to (7c) for different values of the target expected portfolio return.

Note that in Figure 8.1 efficient portfolios are minimum variance portfolios, but not all minimum variance portfolios are efficient. An efficient portfolio, like a minimum variance portfolio, minimizes variance given its expected return, but an efficient portfolio also maximizes expected return given its variance. Thus, the efficient portfolios in Figure 8.1 are those along the left boundary above the point b, but points below b along the boundary are also minimum variance portfolios.

We now discuss the mathematical details of the solution to the problem stated in equations (7a-c). The outcome is an equation relating the expected return on a security to its risk in the minimum variance portfolio that has expected return $E(\tilde{R}_e)$. In two places, the discussion uses elementary calculus. The nonmathematical reader can either skip down to equation (16) or, better, try to follow the verbal discussions that accompany the mathematics.

To solve the problem stated in equations (7a) to (7c), first form the Lagrangian expression

$$\sigma^2(\tilde{R}_p) + 2\lambda_e\left[E(\tilde{R}_e) - \sum_{i=1}^{n} x_{ip}E(\tilde{R}_i)\right] + 2\phi_e\left[1 - \sum_{i=1}^{n} x_{ip}\right], \qquad (8)$$

where $2\lambda_e$ and $2\phi_e$ are the Lagrange multipliers for the constraints of (7b) and (7c). The convenience of stating the Lagrange multipliers in this way is soon apparent. Minimizing the variance of portfolio return subject to the constraints of (7b) and (7c) involves differentiating (8) with respect to $2\lambda_e$, $2\phi_e$, and x_{ip}, $i = 1, \ldots, n$, and setting these partial derivatives equal to 0.0.* For $2\lambda_e$ and $2\phi_e$, this procedure simply tells us that the proportions invested in individual securities must satisfy (7b) and (7c). For the x_{ip}, $i = 1, \ldots, n$, however, the procedure yields the n new conditions

$$\sum_{j=1}^{n} x_{je}\sigma_{ij} - \lambda_e E(\tilde{R}_i) - \phi_e = 0.0, \quad i = 1, \ldots, n, \qquad (9)$$

where x_{je}, $j = 1, \ldots, n$, are the specific proportions invested in individual securities that define the minimum variance portfolio with expected return $E(\tilde{R}_e)$.

PROBLEM II.B

1. Show that differentiating (8) with respect to x_{ip} and setting the derivative equal to 0.0 yields (9).

*That this process leads to a minimum rather than a maximum is primarily a consequence of the convexity of $\sigma^2(\tilde{R}_p)$ as a function of x_{ip}, $i = 1, \ldots, n$, a fact that we state without proof.

ANSWER

1. As a first step, differentiating (8) with respect to x_{ip} and setting the derivative equal to 0.0 leads to

$$\frac{\partial\sigma^2(\tilde{R}_e)}{\partial x_{ie}} - 2\lambda_e E(\tilde{R}_i) - 2\phi_e = 0.0,$$

where $\partial\sigma^2(\tilde{R}_e)/\partial x_{ie}$ is $\partial\sigma^2(\tilde{R}_p)/\partial x_{ip}$ evaluated at the values $x_{ip} = x_{ie}$, $i = 1, \ldots, n$, that represent the solution to the problem stated in (7a) to (7c). Thus, all we must show is that

$$\frac{\partial\sigma^2(\tilde{R}_p)}{\partial x_{ip}} = 2\sum_{j=1}^{n} x_{jp}\sigma_{ij}. \qquad (10)$$

The most transparent way to establish (10) is to first write $\sigma^2(\tilde{R}_p)$ as

$$\sigma^2(\tilde{R}_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ip}x_{jp}\sigma_{ij}$$

$$= \begin{bmatrix} x_{1p}^2\sigma_{11} + x_{1p}x_{2p}\sigma_{12} + x_{1p}x_{3p}\sigma_{13} + \cdots + x_{1p}x_{np}\sigma_{1n} \\ + x_{2p}x_{1p}\sigma_{21} + x_{2p}^2\sigma_{22} + x_{2p}x_{3p}\sigma_{23} + \cdots + x_{2p}x_{np}\sigma_{2n} \\ + x_{3p}x_{1p}\sigma_{31} + x_{3p}x_{2p}\sigma_{32} + x_{3p}^2\sigma_{33} + \cdots + x_{3p}x_{np}\sigma_{3n} \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ + x_{np}x_{1p}\sigma_{n1} + x_{np}x_{2p}\sigma_{n2} + x_{np}x_{3p}\sigma_{n3} + \cdots + x_{np}^2\sigma_{nn} \end{bmatrix}.$$

For any given i, the terms involving x_{ip} are those in the ith row and those in the ith column of this block:

$$\begin{bmatrix} + x_{1p}x_{ip}\sigma_{1i} \\ + x_{2p}x_{ip}\sigma_{2i} \\ \vdots \\ + x_{ip}x_{1p}\sigma_{i1} + x_{ip}x_{2p}\sigma_{i2} + \cdots + x_{ip}^2\sigma_{ii} + \cdots + x_{ip}x_{np}\sigma_{in} \\ \vdots \\ + x_{np}x_{ip}\sigma_{ni} \end{bmatrix}.$$

It is then easy to see that

$$\frac{\partial \sigma^2(\tilde{R}_p)}{\partial x_{ip}} = \sum_{\substack{j=1\\ j\neq i}}^{n} x_{jp}\sigma_{ij} + \sum_{\substack{j=1\\ j\neq i}}^{n} x_{jp}\sigma_{ji} + 2x_{ip}\sigma_{ii} = 2\sum_{j=1}^{n} x_{jp}\sigma_{ij},$$

since $\sigma_{ij} = \sigma_{ji}$. It is now also easy to see why it is convenient to state the Lagrange multipliers in (8) as $2\lambda_e$ and $2\phi_e$ rather than as λ_e and ϕ_e.

The n equations described by (9), along with (7b) and (7c), determine the values of the Lagrange multipliers $2\lambda_e$ and $2\phi_e$ and the proportions invested in individual securities that yield the minimum variance portfolio with expected return $E(\tilde{R}_e)$. Thus, the equations of (9) are conditions on the proportions invested in individual securities that must be met by a minimum variance portfolio. We now show that (9) implies the relationship between the expected return on a security and its risk in the minimum variance portfolio e.

Since (9) holds for every security, it holds for security k

$$\sum_{j=1}^{n} x_{je}\sigma_{kj} - \lambda_e E(\tilde{R}_k) - \phi_e = 0.0, \tag{11}$$

and (9) and (11) together imply

$$\sum_{j=1}^{n} x_{je}\sigma_{kj} - \lambda_e E(\tilde{R}_k) = \sum_{j=1}^{n} x_{je}\sigma_{ij} - \lambda_e E(\tilde{R}_i). \tag{12}$$

Multiplying both sides of (12) by x_{ke} and then summing over k, we get

$$\sigma^2(\tilde{R}_e) - \lambda_e E(\tilde{R}_e) = \sum_{j=1}^{n} x_{je}\sigma_{ij} - \lambda_e E(\tilde{R}_i). \tag{13}$$

PROBLEM II.B

2. Write out the details involved in going from (12) to (13).

ANSWER

2. First multiply both sides of (12) by x_{ke} to get

$$x_{ke}\sum_{j=1}^{n} x_{je}\sigma_{kj} - \lambda_e x_{ke}E(\tilde{R}_k) = x_{ke}\left(\sum_{j=1}^{n} x_{je}\sigma_{ij} - \lambda_e E(\tilde{R}_i)\right). \tag{14}$$

Applying equations (2) and (4) to portfolio e yields, with a change in subscripting,

$$E(\tilde{R}_e) = \sum_{k=1}^{n} x_{ke}E(\tilde{R}_k)$$

$$\sigma^2(\tilde{R}_e) = \sum_{k=1}^{n} x_{ke}\sum_{j=1}^{n} x_{je}\sigma_{kj}.$$

Since none of the terms within the parentheses on the right-hand side of (14) involve security k, and since $\sum_{k=1}^{n} x_{ke} = 1.0$, summing (14) over k yields (13).

We can now rearrange (13) to get

$$E(\tilde{R}_i) - E(\tilde{R}_e) = \frac{1}{\lambda_e}\left(\sum_{j=1}^{n} x_{je}\sigma_{ij} - \sigma^2(\tilde{R}_e)\right), \quad i = 1, \ldots, n. \tag{15}$$

Since equation (15) is a direct implication of (9), we can interpret (15) as a condition on the weights x_{je}, $j = 1, \ldots, n$, that must be met if these weights describe the minimum variance portfolio that has expected return $E(\tilde{R}_e)$. In determining the weights x_{je}, $j = 1, \ldots, n$, that cause (15) to be satisfied for every security i, the risk of portfolio e, $\sigma^2(\tilde{R}_e)$, and the risk of each security in portfolio e, $\sum_{j=1}^{n} x_{je}\sigma_{ij}$, $i = 1, \ldots, n$, are also being determined. Once the weights are known, however, we can interpret (15) as the relationship between the expected return on any security i and the risk of the security in the minimum variance portfolio e. The equation says that the difference between the expected return on any security i and the expected return on e is proportional to the difference between the risk of i in e and the risk of e, and where the proportionality factor is $1/\lambda_e$.

To complete the interpretation of the relationship between expected return and risk in the minimum variance portfolio e, we must interpret the quantity $1/\lambda_e$. The Lagrange multiplier $2\lambda_e$ in (8) is the rate of change of the minimum value of $\sigma^2(\tilde{R}_p)$ in (7a) with respect to a small increase in the target value of the expected portfolio return;

$$2\lambda_e = \frac{d\sigma^2(\tilde{R}_e)}{dE(\tilde{R}_e)}.$$

This derivative is related to the slope of the boundary of the opportunity set at the point e in Figure 8.1. If S_e denotes the slope, then

$$S_e = \frac{dE(\tilde{R}_e)}{d\sigma(\tilde{R}_e)};$$

that is, the slope of the boundary at the point e is the rate of change of expected return with respect to a change in the minimum value of the portfolio standard deviation. To show the relationship between λ_e and S_e, we use the chain rule for differentiation to determine that

$$\frac{1}{S_e} = \frac{d\sigma(\tilde{R}_e)}{dE(\tilde{R}_e)} = \frac{d\sigma(\tilde{R}_e)}{d\sigma^2(\tilde{R}_e)} \frac{d\sigma^2(\tilde{R}_e)}{dE(\tilde{R}_e)}$$

$$= \frac{1}{2\sigma(\tilde{R}_e)} \frac{d\sigma^2(\tilde{R}_e)}{dE(\tilde{R}_e)} = \frac{\lambda_e}{\sigma(\tilde{R}_e)},$$

so that

$$\frac{1}{\lambda_e} = \frac{S_e}{\sigma(\tilde{R}_e)}. \tag{16}$$

In words, the proportionality factor $1/\lambda_e$ in the expected return-risk relationship (15) is the slope of the left boundary of the investment opportunity set at the point e in Figure 8.1 divided by the standard deviation of the return on the minimum variance portfolio e.

We can now transform (15) into an expression that has a somewhat more intuitive interpretation and is also more in keeping with the form of the equation used later in empirical tests. Substituting (16) and (6) into (15) and rearranging yields the equation

$$E(\tilde{R}_i) = [E(\tilde{R}_e) - S_e\sigma(\tilde{R}_e)] + \frac{S_e}{\sigma(\tilde{R}_e)} \operatorname{cov}(\tilde{R}_i, \tilde{R}_e). \tag{17}$$

The square brackets are to indicate that the quantity

$$E(\tilde{R}_e) - S_e\sigma(\tilde{R}_e) \equiv E(\tilde{R}_{0e}) \tag{18}$$

has a special interpretation. It is the expected return on any security whose return is uncorrelated and thus has zero covariance with the return on the portfolio e. Using the somewhat mnemonic notation $E(\tilde{R}_{0e})$ for this quantity, the slope of the left boundary of the investment opportunity set at the point corresponding to the portfolio e is

$$S_e = \frac{E(\tilde{R}_e) - E(\tilde{R}_{0e})}{\sigma(\tilde{R}_e)}. \tag{19}$$

PROBLEM II.B

3. What is the geometric interpretation of (18) and (19)?

ANSWER

3. Since S_e is the slope of the boundary of the investment opportunity set at the point e in Figure 8.1, we can see from inspection of the figure that $E(\tilde{R}_e) - S_e\sigma(\tilde{R}_e)$ is the intersection on the $E(\tilde{R}_p)$ axis of the (dashed) line tangent to the boundary at e. If $E(\tilde{R}_e) - S_e\sigma(\tilde{R}_e)$ is called $E(\tilde{R}_{0e})$ to indicate that it is also the expected return on any security whose return is uncorrelated with the return on e, then it is clear from Figure 8.1 that the slope of the investment opportunity boundary at the point corresponding to the portfolio e is as given by (19).

With (18) and (19), the expected return-risk relationship of (17) becomes

$$E(\tilde{R}_i) = E(\tilde{R}_{0e}) + [E(\tilde{R}_e) - E(\tilde{R}_{0e})]\,\beta_{ie}, \quad i = 1, \ldots, n, \tag{20}$$

where

$$\beta_{ie} = \frac{\operatorname{cov}(\tilde{R}_i, \tilde{R}_e)}{\sigma^2(\tilde{R}_e)} \tag{21}$$

is the risk of security i in the portfolio e measured relative to the risk of the portfolio.

C. *Interpretation of the Results*

If the portfolio e is efficient—that is, if as in Figure 8.1 it is along the positively sloped segment of the left boundary of the opportunity set—then (20) has an intuitive interpretation. If e is efficient, then $S_e > 0$, and it follows from (18) that $E(\tilde{R}_e)$ is greater than $E(\tilde{R}_{0e})$. Thus, the term $[E(\tilde{R}_e) - E(\tilde{R}_{0e})]\,\beta_{ie}$ in (20) can be interpreted as the risk premium in the relationship between the expected return on security i and its risk in the portfolio e. Moreover, any security whose return is uncorrelated with the return on e, and so has $\beta_{ie} = 0.0$, is riskless as far as e is concerned, since such a security contributes nothing to $\sigma^2(\tilde{R}_e)$. In these terms, equation (20) says that the expected return on any security i is equal to the expected return on a security that is riskless in e plus a risk premium that is the difference between the expected return on e and $E(\tilde{R}_{0e})$, multiplied by β_{ie}, the risk of security i in e measured relative to the risk of e.

This interpretation of (20) makes sense, however, only when the portfolio e is efficient as well as of minimum variance. If e is minimum variance but inefficient (that is, if it is along the negatively sloped portion of the left boundary of the opportunity set), then $S_e < 0$, and, from (18) and (19), $E(\tilde{R}_e)$ is less than $E(\tilde{R}_{0e})$, so that (20) must be interpreted in terms of risk discounts rather than premiums. Since the investor is only concerned with efficient portfolios, no harm is done if we use the risk premium interpretation of (20).

Figure 8.2 gives a geometric interpretation of the expected return-risk relationship of (20). The figure emphasizes that the relationship between $E(\tilde{R}_i)$ and β_{ie} is linear. It has intercept $E(\tilde{R}_{0e})$ on the $E(\tilde{R}_i)$ axis and slope $E(\tilde{R}_e) - E(\tilde{R}_{0e})$. In geometric terms, the fact that e is a minimum variance portfolio implies that the weights x_{ie}, $i = 1, \ldots, n$ are chosen in such a way that all securities end up with combinations of $E(\tilde{R}_i)$ and β_{ie} that lie along the line in Figure 8.2. Thus, the n available securities would be arrayed at different points along the line.

It is instructive to look at Figures 8.1 and 8.2 from the viewpoint of the investor faced with a two-parameter world. If the boundary of the investment opportunity set is as represented by the solid curve in Figure 8.1, then the efficient portion of the boundary, the segment above the point b, shows the relationship between expected return and portfolio risk that is relevant when the investor is considering which portfolio to choose. Once he chooses some efficient portfolio, say the portfolio e, then equation (20) and Figure 8.2 show the relationship between expected security return and security risk within the portfolio e. Thus, whereas Figure 8.1 shows the trade-offs of expected return for risk among efficient portfolios, Figure 8.2 shows how the expected returns on individual securities are related to their risks in a specific efficient portfolio.

There is a different version of Figure 8.2 and equation (20) for each minimum variance portfolio. Thus, using equations (3) and (6) we can rewrite (21) and (20) as

$$\beta_{ie} \equiv \frac{\text{cov}(\tilde{R}_i, \tilde{R}_e)}{\sigma^2(\tilde{R}_e)} = \frac{\sum_{j=1}^{n} x_{je}\sigma_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ie}x_{je}\sigma_{ij}} \tag{22}$$

$$E(\tilde{R}_i) = E(\tilde{R}_{0e}) + [E(\tilde{R}_e) - E(\tilde{R}_{0e})] \frac{\sum_{j=1}^{n} x_{je}\sigma_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ie}x_{je}\sigma_{ij}}. \tag{23}$$

The expected security returns $E(\tilde{R}_i)$, $i = 1, \ldots, n$, and the pairwise security return covariances σ_{ij} $(i, j = 1, \ldots, n)$ are parameters of the joint distribution of security returns and so are the same from one efficient portfolio to another. But the weights x_{je}, $j = 1, \ldots, n$, change from one minimum variance portfolio to another; consequently, the risk of a security is different in different portfolios. Since expected security returns do not change from one minimum variance portfolio to another, the fact that the risks of individual securities change means that the intercept $E(\tilde{R}_{0e})$ and the slope $[E(\tilde{R}_e) - E(\tilde{R}_{0e})]$ in (20) must also change from one minimum variance portfolio to another.

PROBLEMS II.C

1. Discuss in general terms how the intercept $E(\tilde{R}_{0e})$ and the slope $[E(\tilde{R}_e) - E(\tilde{R}_{0e})]$ in (20) change as one considers efficient portfolios with higher levels of expected return.

2. Show that equation (20) applies to portfolios as well as to individual securities. That is, show that for any minimum variance portfolio e and any portfolio p

$$E(\tilde{R}_p) = E(\tilde{R}_{0e}) + [E(\tilde{R}_e) - E(\tilde{R}_{0e})]\,\beta_{pe}, \tag{24}$$

where

$$\beta_{pe} = \frac{\text{cov}(\tilde{R}_p, \tilde{R}_e)}{\sigma^2(\tilde{R}_e)}. \tag{25}$$

3. We interpret the intercept $E(\tilde{R}_{0e})$ in (20) as the expected return on any security whose return is uncorrelated with the return on the portfolio e. It is well to note that $E(\tilde{R}_{0e})$ is indeed the expected return on such a security if one exists, but (20) does not require the existence of such a security. Equation (20) simply says that within an efficient portfolio there is a linear relationship between expected security returns and security risks in that portfolio.

With the result of the preceding problem, however, the reader can easily show that when short-selling of securities is allowed, it is always possible to construct a portfolio whose return is uncorrelated with the return on e, and the expected return on any such portfolio is $E(\tilde{R}_{0e})$. (Show it!) Thus, when short-selling of securities is possible, $E(\tilde{R}_{0e})$ can be interpreted as the expected return on any security or portfolio whose return is uncorrelated with the return on e. The importance of this result becomes clear later.

ANSWERS

1. One can apply equations (17) to (20) to any efficient portfolio. Thus, thinking of e as an arbitrary efficient portfolio, the intercept

$$E(\tilde{R}_{0e}) = E(\tilde{R}_e) - S_e\sigma(\tilde{R}_e)$$

in (20) is always the intercept on the $E(\tilde{R}_p)$ axis of a line tangent to the efficient boundary at the point corresponding to the portfolio e. Since the efficient boundary is positively sloped and concave, this means that $E(\tilde{R}_{0e})$ is higher for efficient portfolios with higher levels of expected return and that $E(\tilde{R}_{0e})$ increases faster than $E(\tilde{R}_e)$, so that the slope $[E(\tilde{R}_e) - E(\tilde{R}_{0e})]$ in (20) declines as one considers portfolios further up along the efficient boundary.

2. First, note that

$$\beta_{pe} = \frac{\operatorname{cov}(\tilde{R}_p, \tilde{R}_e)}{\sigma^2(\tilde{R}_e)} = \frac{\operatorname{cov}\left(\sum_{i=1}^{n} x_{ip}\tilde{R}_i, \tilde{R}_e\right)}{\sigma^2(\tilde{R}_e)}$$

$$= \sum_{i=1}^{n} x_{ip} \frac{\operatorname{cov}(\tilde{R}_i, \tilde{R}_e)}{\sigma^2(\tilde{R}_e)} = \sum_{i=1}^{n} x_{ip}\beta_{ie}. \tag{26}$$

If we now multiply through both sides of (20) by x_{ip} and then sum over i, (24) follows directly.

Thus, equation (20), the relationship between expected security returns and security risks within a minimum variance portfolio, turns out to apply to portfolios as well as to securities. This result is important in the empirical tests of the two-parameter model in the next chapter.

3. It suffices to show that when short-selling is allowed, it is always possible to use any two securities to form a portfolio whose return is uncorrelated with the return on e. The answer to the preceding problem then implies that the expected return on any such portfolio is $E(\tilde{R}_{0e})$.

Let i and j be any two securities and let

$$\tilde{R}_p = x\tilde{R}_i + (1 - x)\tilde{R}_j.$$

From the answer to the preceding problem,

$$\beta_{pe} = x\beta_{ie} + (1 - x)\beta_{je}.$$

It is clear that if the value of x is unrestricted, it is always possible to choose x so that $\beta_{pe} = 0.0$.

Finally, equation (20) is derived from the solution to the variance minimization problem of equations (7a) to (7c). Since the constraints do not include statements about the signs of the proportions of portfolio funds invested in individual securities, unrestricted short-selling of securities is assumed. If short-selling is ruled out and if the proportions are constrained to be nonnegative, the mathematics of the analysis of expected return-risk relationships is more complex. The results, however, are similar. In particular, equation (20), with precisely the same interpretation as above, is the relationship between the expected return on a security and its risk in the portfolio e. As one might expect, however, the relationship only applies to securities that appear in e with nonzero weights. The reader can also determine that if (20) only applies to securities that appear in the minimum variance portfolio e at a nonzero level, then (24), the "portfolio" version of (20), only applies to portfolios of such securities.

III. Market Relationships Between Expected Return and Risk When There Is Risk-free Borrowing and Lending

The preceding can be viewed as a discussion of the relationships between expected return and risk that are relevant for an individual investor in a two-parameter world. When the investor is choosing a portfolio, the relevant relationship between expected return and risk is that described by the curve of efficient portfolios. Once he chooses an efficient portfolio, there is a relationship between expected security returns and their risks in that portfolio which is a direct consequence of the fact that the portfolio is efficient.

It is time, however, to step beyond the analysis of risk and expected return as seen by the individual investor and to consider what a two-parameter world implies about the process of price formation in the capital market. That is, if investors make portfolio decisions at time 1 according to the two-parameter model, what does this imply about the prices of securities that are set in the market at time 1? More specifically, we now know how to talk about expected return and risk from the viewpoint of an investor. What remains to be determined is whether the portfolio decisions of individual investors, considered together, cause securities to be priced in such a way that there are similar expected return-risk relationships that apply to the market.

Note the change in perspective in going from the investor to the market. From the viewpoint of the investor, security prices at time 1 are taken as given. The investor is assumed to be small relative to the market, so that security prices are given parameters in his decision problem. When we look at the two-parameter world from the viewpoint of the market, however, we must recognize that security prices are determined by the decisions of investors. The effect of any investor on prices is negligible, but the portfolio decisions of all investors, considered together, determine prices.

A. *Complete Agreement*

The task of the market at time 1 is to determine a market-clearing or market equilibrium set of prices, that is, prices where supply equals demand for each security. Once equilibrium prices are determined, the picture of the efficient set facing an investor is determined. The investor considers this picture as showing the relationship between expected portfolio return and portfolio risk. For the purposes of his own investment decision, this view is entirely correct. Given the prices of securities set at time 1, however, each investor's view of the efficient set depends on an assessment of the joint distribution of the time 2 prices or values of securities that may be unique to him.* If disagreement among investors about the joint distribution of security prices at time 2 is substantial, there may be no meaningful sense in which one can talk about expected return and risk from the viewpoint of the market.

One can reasonably argue, however, that investors would only make portfolio decisions according to the two-parameter model if their assessments of the joint distribution of future security prices were descriptively valid. Although we present the two-parameter model in a one-period framework, we have in mind a multiperiod world where, period after period, the investor makes portfolio decisions in accordance with the two-parameter model.† It does not make sense for the investor to behave in this way unless his assessments of portfolio opportunities are accurate. If he is consistently inaccurate, he will come to feel that the whole decision-making framework is of little value.‡ In short, if one assumes that investors make portfolio decision according to the two-parameter model, then one must assume that they can obtain reasonably accurate assessments of the parameters that the model requires as inputs to a portfolio decision. This in turn implies that there is considerable consensus among investors in their assessments of the joint distribution of future values of securities and thus considerable consensus in how they view the efficient set of portfolios.§

Although "considerable consensus" is the general notion we have in mind, such a concept is too vague for a formal model. To make life simple, we assume that the degree of agreement among investors is complete rather than just considerable; that is, at time 1 there is complete agreement among investors with respect to the joint distribution of security values at time 2. Given the equilibrium prices set at time 1, this means that every investor has the same view of the set of efficient portfolios available at time 1. The common

*When we talk about the prices or values of securities at time 2, we mean to include any dividends or interest paid on the securities at time 2.

†The conditions under which the two-parameter model applies period after period are discussed in Fama (1970) and Merton (1973). These papers are rather difficult mathematically, and so the topic is not considered in this book, which is meant to be more of an introduction to theory and empirical work. Suffice it to say that the empirical evidence of Fama and MacBeth (1974) seems to be consistent with the conditions required for period-by-period application of the one-period model.

‡The argument is not special to the two-parameter model. Any framework for rational decision-making must assume that accurate assessments of relevant parameters are available. Otherwise, the formal decision-making apparatus has little value.

§It is, of course, reasonable to ask how such a consensus might arise. In a multiperiod framework, the simplest case is when, period after period, the equilibrium current prices of securities and the joint distribution of next period's prices are such that the efficient-set curve facing the investor is the same every period. The empirical relevance of this situation is discussed by Fama and MacBeth (1974).

picture of the efficient set shows the relationship between expected portfolio return and portfolio risk for every investor and thus for the market.

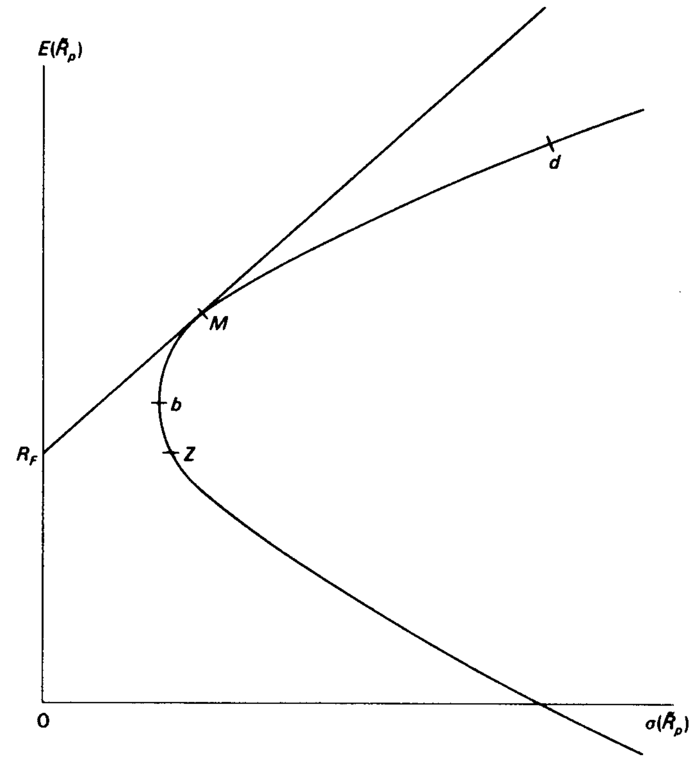
The complete agreement assumption also helps us to determine which of the various portfolios available are efficient. There are two basic approaches. We consider first the simpler approach, which is based on the assumed existence of risk-free borrowing and lending.

B. *The Efficient Set When There Is Risk-free Borrowing and Lending*

Suppose that at time 1 investors can borrow and lend at a risk-free rate of interest R_F. Like the prices of other securities, the value of the risk-free rate is determined as part of the market-clearing process at time 1. As in the case of other securities, an equilibrium or market-clearing value of R_F implies a value such that supply equals demand; the total quantity that investors want to borrow is equal to the quantity that others want to lend.

FIGURE 8.3

Market Equilibrium with Unrestricted Risk-free Borrowing and Lending



We know from Chapter 7 that with risk-free borrowing and lending the efficient set has a simple form. For example, suppose that the curve from b through d in Figure 8.3 shows the portfolios that would be efficient in the absence of risk-free borrowing and lending. With risk-free borrowing and lending, however, efficient portfolios are along the line from R_F that just sits on or is tangent to the curve, that is, the line from R_F through M in Figure 8.3. Portfolios along this line are combinations of risk-free borrowing or lending with the portfolio M, where the proportion x of portfolio funds is invested in F and $(1 - x)$ is in M. The returns, expected returns, and standard deviations of returns on such portfolios of F and M are

$$\tilde{R}_p = xR_F + (1 - x)\tilde{R}_M \tag{27}$$

$$E(\tilde{R}_p) = xR_F + (1 - x)E(\tilde{R}_M) \tag{28}$$

$$\sigma(\tilde{R}_p) = (1 - x)\sigma(\tilde{R}_M), \quad x \leq 1. \tag{29}$$

When $x = 1.0$, all portfolio funds are invested in F; when $x = 0.0$, all funds are in M. Portfolios with $0.0 < x \leq 1.0$ are lending portfolios, since such values of x imply that a positive fraction of portfolio funds is lent at the rate R_F. Portfolios with $x < 0.0$ are borrowing portfolios; the investor borrows $-x(w_1 - c_1)$ and puts this plus the portfolio funds $w_1 - c_1$ into M. Lending portfolios give combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ that are on the straight line between R_F and M in Figure 8.3. Borrowing portfolios give combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ that are along the extension of the line through M. Efficient portfolios differ in terms of how portfolio funds are split between F and M, but all efficient portfolios are just combinations of F and M. M is the only efficient portfolio of all positive variance securities, that is, securities with strictly positive return variances.

C. *Market Equilibrium When There Is Risk-free Borrowing and Lending*

Consider now what this result implies about the characteristics of a market equilibrium at time 1. Since there is assumed to be complete agreement among investors with respect to the joint distribution of security values at time 2, given a set of security prices and a value for the risk-free rate at time 1, there is a tangency portfolio like M in Figure 8.3 that all investors try to combine with F. Some investors want to combine the tangency portfolio with borrowing at the risk-free rate, while others want to combine it with lending, but the tangency portfolio is the only portfolio of only positive variance securities for which investors enter demands.

Remember, though, that a market equilibrium requires a market-clearing

set of prices; a market equilibrium requires that, in aggregate, investors demand all securities and demand them in the proportions in which they are outstanding. Given the nature of the efficient set when there is risk-free borrowing and lending, this market-clearing condition means that a market equilibrium is not attained until the one tangency portfolio that all investors try to combine with risk-free borrowing or lending is a portfolio of all the positive variance securities in the market, where each security is weighted by the ratio of the total market value at time 1 of all its outstanding units to the total market value of all outstanding units of all securities. In short, a market equilibrium is not reached until the tangency portfolio M in Figure 8.3 is the value-weighted version of the market portfolio. In addition, the value of R_F must be such that the aggregates of demands and supplies of loans are equal.

In slightly different terms, given the common assessment by investors of the joint distribution of security values at time 2, a set of prices for securities at time 1 and a value of R_F imply a representation of investment opportunities like that in Figure 8.3. A different risk-free rate and a different set of security prices imply a different picture of investment opportunities, but one that is always similar to Figure 8.3 in the sense that all efficient portfolios are combinations of borrowing or lending with one tangency portfolio of positive variance securities. The tangency portfolio is, however, different for different sets of security prices at time 1 and different values of R_F. A market equilibrium—a set of security prices that clears the securities market and a value of R_F that clears the borrowing-lending market—requires that the tangency portfolio be the value-weighted version of the market portfolio.

Since the market portfolio M is efficient, equation (20), which shows the relationship between expected security returns and their risks in an arbitrary minimum variance portfolio e, can be applied to M. We have

$$E(\tilde{R}_i) = E(\tilde{R}_{0M}) + [E(\tilde{R}_M) - E(\tilde{R}_{0M})]\,\beta_{iM}, \quad i = 1, \ldots, n, \tag{30}$$

where

$$\beta_{iM} = \frac{\text{cov}\,(\tilde{R}_i, \tilde{R}_M)}{\sigma^2(\tilde{R}_M)} \tag{31}$$

is the risk of security i in M measured relative to the risk of M, and $E(\tilde{R}_{0M})$ is the expected return on any security or portfolio whose return is uncorrelated with the return on M.

In the present view of the world, there indeed exists a security, F, whose return is uncorrelated with $\tilde{R}_M$. Substituting R_F for $E(\tilde{R}_{0M})$ in (30) yields

$$E(\tilde{R}_i) = R_F + [E(\tilde{R}_M) - R_F]\,\beta_{iM}, \quad i = 1, \ldots, n. \tag{32}$$

In words, the expected return on any security i is the risk-free rate R_F plus a risk premium which is the risk measure β_{iM} multiplied by the difference between the expected return on M and R_F.

Equation (32) is just (20) applied to M, but the interpretation of (32) goes far beyond that of (20). With the assumption that investors agree on the joint distribution of security values at time 2, and with the assumed existence of risk-free borrowing and lending, it is not happenstance that the value-weighted market portfolio M is efficient. The efficiency of M is a necessary condition for a market equilibrium. With complete agreement and risk-free borrowing and lending, a market equilibrium requires that R_F and the prices of securities at time 1 are set so that M is efficient, which means that R_F and the prices of securities must be set in such a way that (32) holds for every security. Thus, with complete agreement and risk-free borrowing and lending, the expected return-risk relationship (32) is the implication of the two-parameter model for the process of price formation in the capital market at time 1.

PROBLEM III.C

1. In the present model, what is the formal expression for the relationship between the expected return $E(\tilde{R}_e)$ on an efficient portfolio and its risk $\sigma(\tilde{R}_e)$?

ANSWER

1. From inspection of Figure 8.3, the relationship is

$$E(\tilde{R}_e) = R_F + \left[\frac{E(\tilde{R}_M) - R_F}{\sigma(\tilde{R}_M)}\right] \sigma(\tilde{R}_e), \tag{33}$$

where M is the value-weighted market portfolio, the subscript e indicates an arbitrary efficient portfolio, and

$$\frac{E(\tilde{R}_M) - R_F}{\sigma(\tilde{R}_M)} = S_M \tag{34}$$

is the slope of the efficient-set at the point corresponding to M and at every other point.

Be clear on the difference between (33) and (32). Equation (33) is the relationship between the expected returns and risks of efficient portfolios, while (32) is the relationship between expected security returns and security risks within the particular efficient portfolio M.

D. Criticisms of the Model

The two-parameter model of market equilibrium discussed above is credited to Sharpe (1964) and Lintner (1965a), and it is usually called the Sharpe-Lintner model. The model is also often called the capital asset pricing model, although we do not use the term here. There are two common criticisms of the model. First, the model says that all investors hold the market portfolio in combination with different amounts of borrowing or lending, and one does not observe that all investors hold the market portfolio. Second, borrowing or lending is never completely risk-free. Any funds borrowed at time 1 must be repaid from the market value of the portfolio at time 2. Since we assume that return distributions are normal and since normal distributions are unbounded, there is always some chance that the borrower cannot deliver in full to the lender. Moreover, in the real-world capital market, the promises of the borrower are usually stated in terms of money. The investor values money, however, only for the "real" consumption (that is, goods and services) that it will buy. Thus, if the purchasing power of money at time 2 is uncertain, at time 1 a contract that pays a perfectly certain amount of money at time 2 is not perfectly certain in "real" terms and so is not considered as risk-free by the investor.*

There are several responses to these criticisms of the Sharpe-Lintner model. I offer just one, and by now it is familiar. Throughout the book I have emphasized that any model proposes a simplified view of the world but that this is not sufficient basis for its rejection. Thus, even though it is not realistic in all of its details, we may be willing to go along with the Sharpe-Lintner model of market equilibrium as long as its implications for equilibrium prices of securities are empirically descriptive. The primary purpose of the Sharpe-Lintner model is to develop testable implications of the two-parameter portfolio model for the process of price formation in the capital market. Despite the fact that in many respects it is oversimplified, the model is vindicated if its implication about equilibrium security prices—in particular, the expected return-risk relationship of (32)—seems to be a good description of real-world data.

The Sharpe-Lintner model is, however, just one view of the implications of the two-parameter portfolio model for the process of price formation in the capital market. There are other two-parameter models of market equilibrium that likewise yield testable propositions concerning how the attempts of

*The implication of these comments is that the portfolio model itself should be developed in "real" terms rather than in terms of dollars of consumption at time 1 and time 2. When we speak of "dollars" in the context of the model, we have in mind the general notion of a unit of purchasing power.

investors to hold efficient portfolios affect equilibrium prices of securities, and the testable propositions of these models differ somewhat from those of the Sharpe-Lintner model. When we confront the two-parameter model with real-world data in Chapter 9, we want to be armed with the implications of as many two-parameter models of market equilibrium as possible. In this way we minimize the chance of falsely rejecting the proposition that prices and returns reflect the attempts of investors to hold efficient portfolios when, in fact, the failure of the tests is due to a bad specific model of market equilibrium.

In short, testing the implications of the two-parameter portfolio model for the process of price formation in the capital market requires a two-parameter model of market equilibrium, and such models of market equilibrium require more restrictive assumptions than the portfolio model. If our basic interest is to test whether the process of price formation is dominated by investors concerned with portfolio efficiency, then we want to be aware of the various two-parameter models of market equilibrium that are consistent with this basic proposition.

IV. Market Relationships Between Expected Return and Risk When Short-Selling of Positive Variance Securities Is Unrestricted

The implications of the Sharpe-Lintner model for the nature of a market equilibrium at time 1 follow from the fact that with complete agreement among investors with respect to the joint distribution of security values at time 2 and with unrestricted risk-free borrowing and lending, all efficient portfolios are combinations of borrowing or lending with one efficient portfolio of positive variance securities. The requirement that all securities be cleared from the market then implies that in a market equilibrium the one efficient portfolio of only positive variance securities is the market portfolio. This line of reasoning does not require that positive variance securities can be sold short. The major alternative to the Sharpe-Lintner model of market equilibrium is less restrictive than the Sharpe-Lintner model in the sense that it does not assume the existence of a risk-free security; but in another sense it is more restrictive, since it assumes that short-selling of positive variance securities is unrestricted. The alternative model is credited primarily to Black (1972).

In the Sharpe-Lintner model, a market equilibrium at time 1 requires the efficiency of the market portfolio. Thus, equation (20), when applied to M, can be interpreted as a condition on equilibrium prices of securities at time 1 rather than as just the expected return-risk relationship that holds for securities within an efficient portfolio. Equivalently, to show that a market equilibrium requires the efficiency of M is to transform (20) into a market equilibrium expected return-risk relationship. The implications of the Black model for equilibrium security prices are likewise based on the efficiency of the market portfolio in a market equilibrium. We now see, however, that the path to showing that a market equilibrium requires the efficiency of M is more tortuous than in the Sharpe-Lintner model.

A. *The Efficiency of the Market Portfolio*

OVERVIEW

Since the formal arguments quickly get rather involved, we begin with a brief informal discussion of the basis of the efficiency of the market portfolio in the Black model. Suppose that when a market equilibrium is established at time 1, the set of minimum variance portfolios is the solid curve in Figure 8.1. Since the Black model, like the Sharpe-Lintner model, assumes complete agreement among investors with respect to the joint distribution of security prices at time 2, when a market equilibrium is attained at time 1, each investor perceives that Figure 8.1 is the relevant picture of portfolio opportunities. Depending on tastes, each investor then chooses some efficient portfolio, so we can think of investors as choosing different points from along the efficient segment of the curve of minimum variance portfolios, the segment above point b.

A market equilibrium requires that total investor demand for each security be equal to total supply. Equivalently, in a market equilibrium, the portfolio of the efficient portfolios chosen by investors, where each investor's portfolio is weighted by the ratio of his invested wealth to the total invested wealth of all investors, must be the market portfolio M. It follows that to show that a market equilibrium requires that M be efficient, it is sufficient to show that any portfolio of efficient portfolios, with component efficient portfolios receiving positive weights, is itself efficient. This is what we now do.

PROPERTIES OF THE MINIMUM VARIANCE BOUNDARY

Go back to equations (7) to (9). Recall that the n conditions of (9), along with (7b) and (7c), determine the n proportions of portfolio funds invested in individual securities that define the minimum variance portfolio with expected return $E(\tilde{R}_e)$. These $N + 2$ equations also determine the values of the Lagrange multipliers $2\lambda_e$ and $2\phi_e$ for this portfolio. For the moment, however, let us assume that someone gives us the values of λ_e and ϕ_e. Then we can use the n equations of (9) to solve for the values of $x_{ie}, i = 1, \ldots, n$.*

Stating (9) in matrix form, we get

$$\underset{(n\times n)}{A}\ \underset{(n\times 1)}{X_e} = \lambda_e \underset{(n\times 1)}{E(\tilde{R})} + \phi_e \underset{(n\times 1)}{[1]}\ , \tag{35}$$

where A is the $n \times n$ matrix of the pairwise security return covariances σ_{ij} $(i, j = 1, \ldots, n)$; X_e is the $n \times 1$ vector of the proportions of portfolio funds invested in individual securities; $E(\tilde{R})$ is the $n \times 1$ vector of expected security returns; and $[1]$ is an $n \times 1$ vector of ones. If D is the inverse of A with typical element d_{ij}, we can solve (35) for X_e to get

$$X_e = \lambda_e DE(\tilde{R}) + \phi_e D[1], \tag{36}$$

$$x_{ie} = \lambda_e \left[\sum_{j=1}^{n} d_{ij}E(\tilde{R}_j)\right] + \phi_e \left[\sum_{j=1}^{n} d_{ij}\right], \quad i = 1, \ldots, n. \tag{37}$$

Thus, if someone gives us the values of λ_e and ϕ_e, the n equations of (37) give the values of the proportions invested in individual securities that define the minimum variance portfolio with expected return $E(\tilde{R}_e)$.

There is a better way to look at equation (37). If we only consider values of λ_e and ϕ_e that are consistent with solutions to the problem stated in equations (7a–c) for different values of the target expected portfolio return $E(\tilde{R}_e)$, then by varying λ_e and ϕ_e in (37) we generate the different values of the proportion of portfolio funds invested in security i $(i = 1, \ldots, n)$ in different minimum variance portfolios. By varying λ_e and ϕ_e through all the feasible values of these Lagrange multipliers, we generate the proportions invested in individual securities in each of the portfolios along the minimum variance boundary. In effect, this is how the boundary is determined, and a better understanding of the process allows us to attain easily the goal of establishing that in a market equilibrium the market portfolio M is efficient.

The problem comes down to identifying the feasible combinations of λ_e and ϕ_e. For λ_e, the answer follows from equation (16) and the discussion in Chapter 7 of the nature of the minimum variance boundary when unlimited short-selling is possible. In particular, the boundary of minimum variance portfolios in Figure 8.1 is a hyperbola that extends indefinitely upward and to the right and indefinitely downward and to the right. As one moves from the point b upward on the boundary, the slope S_e of the curve goes from ∞ toward a finite positive asymptote; when one moves downward from b, S_e goes from $-\infty$ toward a finite negative asymptote. As one moves either up or down the boundary away from b, $\sigma(\tilde{R}_e)$ increases continuously. With these observations, we can then infer from (16) that $\lambda_e = \sigma(\tilde{R}_e)/S_e$ can take any positive or negative value.

*The next paragraph requires a little matrix algebra. The reader should follow along so that the line of reasoning and the notation become familiar, even if the mathematics is not completely comprehensible.

Thus, the set of minimum variance portfolios is generated by varying λ_e in (37) between $-\infty$ and ∞, combining each value of λ_e with the appropriate value of ϕ_e. To determine the appropriate value of ϕ_e, we simply note from (8) that $2\phi_e$ is the Lagrange multiplier for the constraint (7c). For any given λ_e, the appropriate ϕ_e is the value that makes the sum of the n values of x_{ie} equal to 1.0.

With a little new notation, we can now obtain a simple description of how the set of minimum variance portfolios is generated when short-selling of securities is unrestricted. First, define two new portfolios u and v as

$$\tilde{R}_u = \sum_{i=1}^{n} x_{iu}\tilde{R}_i \text{ and } \tilde{R}_v = \sum_{i=1}^{n} x_{iv}\tilde{R}_i, \tag{38}$$

$$x_{iu} = \sum_{j=1}^{n} d_{ij}E(\tilde{R}_j) \Big/ \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}E(\tilde{R}_j), \quad i = 1, \ldots, n, \tag{39}$$

$$x_{iv} = \sum_{j=1}^{n} d_{ij} \Big/ \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}, \quad i = 1, \ldots, n. \tag{40}$$

From inspection of these two expressions we can see that

$$\sum_{i=1}^{n} x_{iu} = 1.0 \text{ and } \sum_{i=1}^{n} x_{iv} = 1.0, \tag{41}$$

so that u and v are standard portfolios. If we next define

$$y_{eu} = \lambda_e \left[\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}E(\tilde{R}_j)\right] \tag{42}$$

$$y_{ev} = \phi_e \left[\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\right], \tag{43}$$

then (37) can be rewritten as

$$x_{ie} = y_{eu}x_{iu} + y_{ev}x_{iv}, \quad i = 1, \ldots, n. \tag{44}$$

Since the double sum in (42) is a constant and so is independent of λ_e, the fact that λ_e can take any value between $-\infty$ and ∞ means that y_{eu} in (44) can take any value between $-\infty$ and ∞. Since the double sum in (43) is a constant and so is independent of ϕ_e, for any given value of y_{eu} in (42) the appropriate value of y_{ev} in (43) involves choosing ϕ_e so that

$$\sum_{i=1}^{n} x_{ie} = 1.0. \tag{45}$$

But from (41) and (44)

$$\sum_{i=1}^{n} x_{ie} = \sum_{i=1}^{n} (y_{eu}x_{iu} + y_{ev}x_{iv}) = y_{eu} + y_{ev}. \tag{46}$$

Thus, to satisfy (45) for any given value of y_{eu} in (44), y_{ev} must be chosen to satisfy

$$y_{eu} + y_{ev} = 1.0. \tag{47}$$

In words, to determine the proportion of portfolio funds invested in security i ($i = 1, \ldots, n$) in different minimum variance portfolios, we simply vary y_{eu} in (44), combining each value of y_{eu} with the value of y_{ev} that satisfies (47). Equivalently, with (44) the return on any minimum variance, portfolio e can be written as

$$\begin{aligned}\tilde{R}_e &= \sum_{i=1}^{n} x_{ie}\tilde{R}_i \\ &= \sum_{i=1}^{n} (y_{eu}x_{iu} + y_{ev}x_{iv})\tilde{R}_i \\ &= y_{eu}\left(\sum_{i=1}^{n} x_{iu}\tilde{R}_i\right) + y_{ev}\left(\sum_{i=1}^{n} x_{iv}\tilde{R}_i\right) \\ \tilde{R}_e &= y_{eu}\tilde{R}_u + y_{ev}\tilde{R}_v.\end{aligned} \tag{48}$$

Thus, any minimum variance portfolio e is a combination of the portfolios u and v, where the proportion y_{eu} of portfolio funds is invested in u and $y_{ev} = 1.0 - y_{eu}$ is invested in v. Any such combination of u and v is a minimum variance portfolio and the set of minimum variance portfolios includes all combinations of u and v that satisfy (47).

PROBLEMS IV.A

1. Show that u and v are themselves minimum variance portfolios.

2. Show that v is in fact the minimum variance portfolio b in Figure 8.1.

ANSWERS

1. The portfolio u is a portfolio of u and v with $y_{eu} = 1.0$ and $y_{ev} = 0.0$, and v is a portfolio of u and v with $y_{eu} = 0.0$ and $y_{ev} = 1.0$. Since any combinations of u and v that satisfy (47) are minimum variance portfolios, u and v are minimum variance portfolios.

2. At the point b in Figure 8.1, $\lambda_b = \sigma(\tilde{R}_b)/S_b = 0$, so that, from (42),

$$x_{ib} = \phi_b \sum_{j=1}^{n} d_{ij}.$$

Since ϕ_b must be chosen so that $\Sigma_{i=1}^{n} x_{ib} = 1$, we can conclude from (40) that $x_{ib} = x_{iv}$. Thus, the portfolio v defined by (40) is the minimum variance portfolio that has the smallest possible return variance.

With these results it is now easy to show that any portfolio of minimum variance portfolios is a minimum variance portfolio. The return on any portfolio p of minimum variance portfolios can be written as

$$\tilde{R}_p = \sum_e x_e\tilde{R}_e \tag{49}$$

$$\sum_e x_e = 1.0, \tag{50}$$

where x_e is the proportion of portfolio funds invested in minimum variance portfolio e and where the notation Σ_e is meant to indicate that we are taking a sum over some finite number of minimum variance portfolios. Substituting (48) into (49) yields

$$\tilde{R}_p = \sum_e x_e(y_{eu}\tilde{R}_u + y_{ev}\tilde{R}_v) = \left(\sum_e x_e y_{eu}\right)\tilde{R}_u + \left(\sum_e x_e y_{ev}\right)\tilde{R}_v.$$

Any combination of u and v with proportions invested in u and v that sum to 1.0 is a minimum variance portfolio. Thus p is a minimum variance portfolio if

$$\sum_e x_e y_{eu} + \sum_e x_e y_{ev} = 1.0.$$

But as stated in (48), each of the component portfolios e in (49) is a portfolio of u and v that satisfies (47). Thus, from (50) and (47),

$$\sum_e x_e y_{eu} + \sum_e x_e y_{ev} = \sum_e x_e(y_{eu} + y_{ev}) = \sum_e x_e = 1.0.$$

If any portfolio of minimum variance portfolios is a minimum variance portfolio, then any portfolio of efficient portfolios is a minimum variance portfolio, since efficient portfolios are minimum variance portfolios. Moreover, any portfolio of efficient portfolios where the proportions invested in component efficient portfolios are all nonnegative is an efficient portfolio, since such a combination of efficient portfolios is a minimum variance portfolio and it necessarily has an expected return within the range of expected returns covered by the efficient segment of the minimum variance boundary.

MARKET EQUILIBRIUM AND THE MARKET PORTFOLIO

It follows almost directly that with complete agreement among investors concerning the joint distribution of security values at time 2, and with no restrictions on short-selling, a market equilibrium at time 1 implies that the market portfolio M is efficient. A market equilibrium at time 1 requires a set of security prices such that the aggregate demand for each security by investors is equal to the supply of the security. Equivalently, a market equilibrium requires that when one combines the portfolios chosen by investors, weighting each investor's portfolio by the ratio of his invested wealth to the sum of the invested wealths of all investors, then the resulting portfolio is the market portfolio. Since investors choose efficient portfolios, and since the invested wealth of each investor is assumed to be nonnegative, from the above analysis we can conclude that in a market equilibrium the market portfolio is efficient.

In geometric terms, suppose that when a market equilibrium is attained at time 1, minimum variance portfolios are as shown by the solid curve in Figure 8.1. Then the market portfolio M is along the positively sloped segment of the curve, which describes the set of efficient portfolios. Since M is efficient, equation (30) holds, showing the relationship between expected returns on securities and their risks in M. Moreover, since a market equilibrium requires that M is efficient, (30) can be interpreted as a condition on equilibrium security prices. Equivalently, as in the Sharpe-Lintner model, equation (30) can be interpreted in the Black model as the market equilibrium relationship between expected security returns and their risks in M.

The difference between the Sharpe-Lintner model and the Black model is that in the Sharpe-Lintner model the intercept $E(\tilde{R}_{0M})$ in (30) can be identified as the risk-free rate of interest, whereas in the Black model there is no risk-free security, so that $E(\tilde{R}_{0M})$ is the return on any positive variance security that has $\beta_{0M} = 0.0$, that is, on any positive variance security whose return is uncorrelated with the return on M. Remember that since such a security contributes nothing to the variance of the return on M, it is riskless in M even though it is not risk-free in the sense of having a zero variance of return.

B. Efficient Portfolios as Combinations of the Market Portfolio M *and the Minimum Variance Portfolio* Z

The remaining models of market equilibrium are variants of the Black model in the sense that arguments similar to those above are used to show that a market equilibrium generally requires the market portfolio to be efficient. Before proceeding to these models, however, it is convenient to discuss further the nature of minimum variance and efficient portfolios in a market equilibrium when short-selling of securities is unrestricted. This will improve our perspective both on the Black model and on the variants of the model that are discussed later.

The goal is to show that with unrestricted short-selling, in a market equilibrium any minimum variance portfolio can be expressed as a combination of the market portfolio M and the minimum variance zero-β_{pM} portfolio—that is, the minimum variance portfolio, call it Z, whose return is uncorrelated with the return on M. The first step is to show that the set of minimum variance portfolios can be generated as combinations of any two minimum variance portfolios. Next we show that a minimum variance zero-β_{pM} portfolio always exists. Since M and this zero-β_{pM} portfolio Z are minimum variance portfolios, it then follows that the set of minimum variance portfolios can be generated as combinations of M and Z.

In the preceding section we found that with unrestricted short-selling, any portfolio of minimum variance portfolios is a minimum variance portfolio. Thus, any portfolio of two different minimum variance portfolios is a minimum variance portfolio. Since any two different minimum variance portfolios have different expected returns, with the appropriate weights assigned to these two portfolios we can generate a minimum variance portfolio with any specified level of expected return. It follows that any two minimum variance portfolios can be used to generate the entire set of minimum variance portfolios.

With unrestricted short-selling, in a market equilibrium the market portfolio M is efficient. Thus, one of the two portfolios used to generate minimum variance portfolios can be M. The other can be the minimum variance zero-β_{pM} portfolio Z, as long as we can show that there is indeed a minimum variance portfolio whose return is uncorrelated with the return on M. The existence of such a portfolio is implied by the fact that the range of expected returns covered by minimum variance portfolios is unbounded both from above and from below, so that there is necessarily a minimum variance portfolio with any specified level of expected return. Thus, there is a minimum variance portfolio with expected return equal to $E(\tilde{R}_{0M})$. Since (30) applies to any security or portfolio, any portfolio with expected return equal to $E(\tilde{R}_{0M})$ must have $\beta_{pM} = 0.0$. The existence of the minimum variance zero-β_{pM} portfolio Z is thus established.

In short, the returns on different minimum variance portfolios can be obtained by varying x in

$$\tilde{R}_e = x\tilde{R}_Z + (1 - x)\tilde{R}_M. \tag{51}$$

When minimum variance portfolios are formed in this way, the expected value and variance of the return on a minimum variance portfolio are

$$E(\tilde{R}_e) = xE(\tilde{R}_Z) + (1 - x)E(\tilde{R}_M) \tag{52}$$

$$\sigma^2(\tilde{R}_e) = x^2\sigma^2(\tilde{R}_Z) + (1 - x)^2\sigma^2(\tilde{R}_M), \tag{53}$$

where the absence of the usual covariance term in equation (53) follows from the fact that $\tilde{R}_Z$ and $\tilde{R}_M$ are uncorrelated.

PROBLEMS IV.B

1. Show that the portfolio Z must be below M on the minimum variance boundary; that is, $E(\tilde{R}_Z) < E(\tilde{R}_M)$.

2. Show that the portfolio Z cannot be efficient.

ANSWERS

1. Since M is efficient, $E(\tilde{R}_M) - E(\tilde{R}_{0M})$ in (30) is positive. Since $E(\tilde{R}_Z) = E(\tilde{R}_{0M})$, it follows that Z is below M on the minimum variance boundary.

2. Since $\tilde{R}_Z$ and $\tilde{R}_M$ are uncorrelated, in (51) the portfolio of Z and M that has the smallest possible variance has (from Problem III.A.*3* of Chapter 7),

$$x = \frac{\sigma^2(\tilde{R}_M)}{\sigma^2(\tilde{R}_M) + \sigma^2(\tilde{R}_Z)}.$$

Since this value of x is greater than 0.0 and less than 1.0, it follows that there are portfolios of Z and M formed according to (51) that have smaller variances than either Z or M and that fall between Z and M on the minimum variance boundary. Since $E(\tilde{R}_M) > E(\tilde{R}_Z)$, such portfolios must also have larger expected returns than Z. Thus Z cannot be efficient.

Since $E(\tilde{R}_Z) = E(\tilde{R}_{0M})$, Black (1972) suggests that (30) be rewritten as

$$E(\tilde{R}_i) = E(\tilde{R}_Z) + [E(\tilde{R}_M) - E(\tilde{R}_Z)]\beta_{iM}, \quad i = 1, \ldots, n. \tag{54}$$

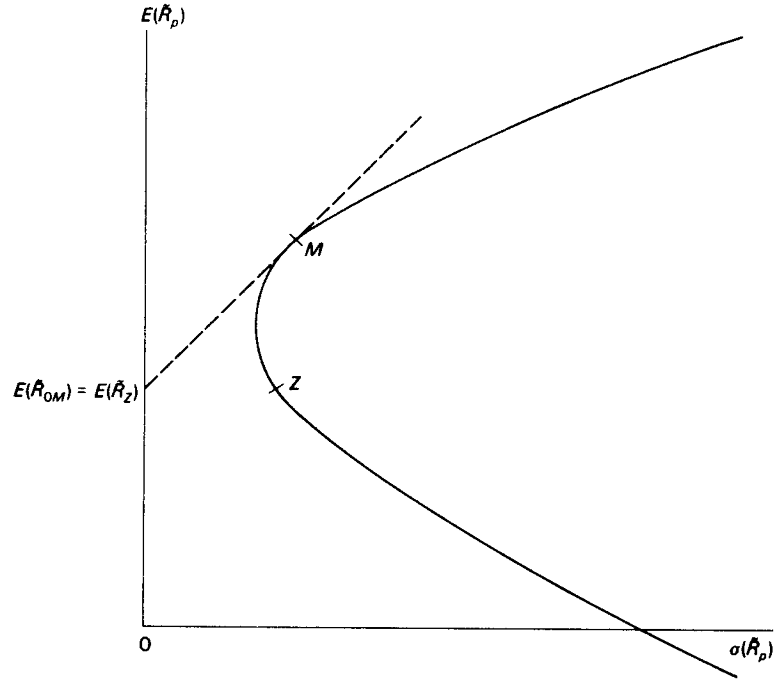
In words, with unrestricted short-selling, in a market equilibrium the prices of securities are set so that the expected return on any security is the ex-

pected return on Z, the minimum variance portfolio whose return is uncorrelated with the return on the market portfolio M, plus a risk premium which is β_{iM}, the risk of security i in M measured relative to the risk of M, times the difference between the expected returns on M and Z.

In geometric terms, the set of minimum variance portfolios available in a market equilibrium might be as shown by the solid curve in Figure 8.4. All

FIGURE 8.4

Market Equilibrium with Unrestricted Short-Selling of Positive Variance Securities



minimum variance portfolios can be obtained as combinations of M and Z. With $x = 1.0$ in (51) we get Z, while with $x = 0.0$ we get M. Portfolios between Z and M on the minimum variance curve have $0 < x < 1$; that is, positive fractions of portfolio funds are invested in both Z and M. Points above M on the curve involve short-selling of Z; that is, $x < 0.0$ in (51). Points below Z involve short-selling of M; that is, $x > 1.0$ and $(1 - x) < 0.0$. The market portfolio is on the efficient segment of the minimum variance curve, but from Problem IV.B.*2* above we know that Z is not. Finally, applying the analysis of equations (17) to (19) to M tells us that the (dashed) line tangent to the minimum variance boundary at the point M in Figure 8.4 must intersect the $E(\tilde{R}_p)$ axis at $E(\tilde{R}_Z) = E(\tilde{R}_{0M})$.

V. Variants of the Model of Market Equilibrium When There Is Unrestricted Short-Selling of Positive Variance Securities

There are several other two-parameter models of market equilibrium that are closely related to the Black model presented above. Like the Black model, they assume that short-selling of positive variance securities (securities whose return distributions have positive variances) is unrestricted and that there is complete agreement among investors with respect to the joint distribution of security values at time 2. The key result in each of these models is that in a market equilibrium the market portfolio M is a minimum variance portfolio, so that (30) or (54) can be interpreted as a market equilibrium relationship between the expected returns on securities and their risks in M. These models differ from the Black model in that they assume the existence of a risk-free security, but they also differ from the Sharpe-Lintner model in that they do not allow unrestricted risk-free borrowing and lending at the same interest rate.

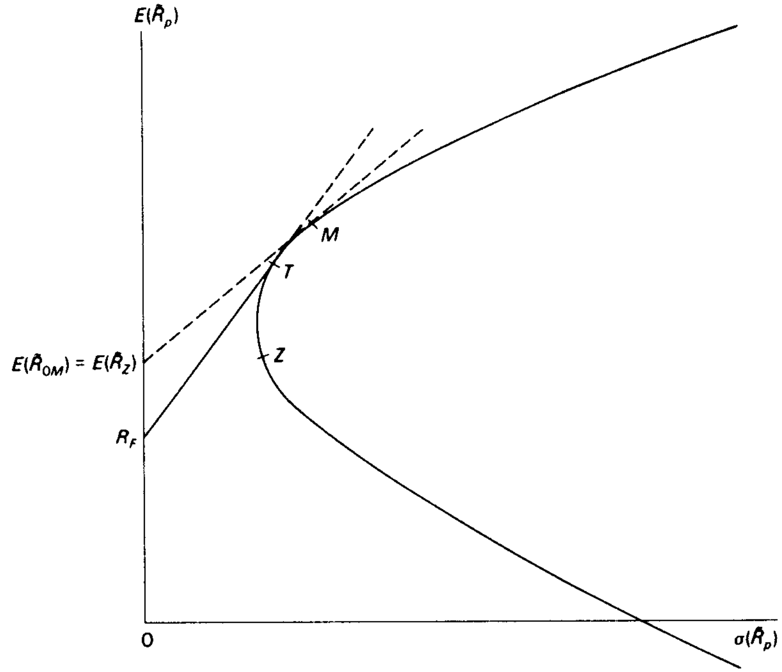
A. *Market Equilibrium When There Is a Risk-free Security But It Cannot Be Sold Short*

For example, Black (1972) discusses a model in which there is a risk-free security F which investors can hold long but cannot sell short. Thus, there is risk-free "lending" but not "borrowing." The risk-free securities might be the securities of firms whose activities have perfectly certain market values at time 2, or they might be bonds issued and guaranteed by the government. In either case, the portfolio opportunities facing investors in a market equilibrium might be as shown in Figure 8.5.

The solid curve in the figure is the set of minimum variance portfolios of only positive variance securities—that is, portfolios which minimize variance at different levels of expected return but are subject to the additional constraint that they not contain any of the risk-free security F. Henceforth, these portfolios are what we refer to when we use the term minimum variance portfolios. These minimum variance portfolios of positive variance securities have the same properties as the minimum variance portfolios of the preceding section. In particular, precisely the same arguments imply that with unrestricted short-selling of positive variance securities, (a) any portfolio of minimum variance portfolios is itself a minimum variance portfolio; (b) any portfolio of positively weighted minimum variance portfolios from along the positively sloped segment of the minimum variance boundary is likewise

FIGURE 8.5

Market Equilibrium with Risk-free Lending But Not Borrowing



along the positively sloped segment of the boundary; and (c) the set of minimum variance portfolios can be generated as portfolios of any two different minimum variance portfolios.

The set of efficient portfolios in Figure 8.5 includes, first, the portfolios along the line from R_F to the tangency portfolio T. Such portfolios of F and T are lending portfolios; positive fractions of portfolio funds are invested in both F and T. Portfolios along the dashed extension of the line from R_F through T are not feasible, however, since they would require borrowing or short-selling of the risk-free security F. To get efficient portfolios with expected returns greater than $E(\tilde{R}_T)$, one must move from T up along the boundary of minimum variance portfolios of positive variance securities.

In Figure 8.5 the portfolio M, which is now defined as the market portfolio of only positive variance securities, is efficient. In a market equilibrium, this portfolio must be efficient. As always, a market equilibrium requires a set of security prices such that demand equals supply for each security. Equivalently, if one weights the risky component of each investor's portfolio by the ratio of the funds invested by that investor in positive variance securities to the total investment in positive variance securities by all investors, then in a market equilibrium the portfolio of the risky components of investor portfolios must be the market portfolio of positive variance securities. The risky component of any portfolio along the line from R_F to T in Figure 8.5 is the tangency portfolio T. The risky components of other efficient portfolios chosen by investors are the portfolios themselves, the chosen points along the efficient boundary above T. Thus, in a market equilibrium the market portfolio of positive variance securities is a portfolio of efficient minimum variance portfolios where each component portfolio receives a positive weight. When short-selling of positive variance securities is unrestricted, any such portfolio of efficient minimum variance portfolios is efficient.

PROBLEM V.A

1. Show that in general the market portfolio M is above the tangency portfolio T on the curve of minimum variance portfolios.

ANSWER

1. If all investors choose portfolios along the line from R_F to T in Figure 8.5, then in a market equilibrium T must be the market portfolio M. If some investors choose efficient portfolios above T, then in a market equilibrium M is a portfolio of T and portfolios above T along the minimum variance boundary. In this case, which I call the general case, M has higher expected return than T and so is above T on the minimum variance boundary.

With unrestricted short-selling of positive variance securities, the set of minimum variance portfolios of positive variance securities can be generated as combinations of any two minimum variance portfolios. We have just shown that in a market equilibrium, one of these two minimum variance portfolios can be the market portfolio M. Moreover, the arguments of the preceding section can be used to show that there is a minimum variance (but inefficient) zero-β_{pM} portfolio Z of only positive variance securities, so that the set of minimum variance portfolios can be generated from combinations of M and Z.

In the present model, however, $E(\tilde{R}_Z)$ is generally greater than the risk-free rate R_F. As illustrated in Figure 8.5, $E(\tilde{R}_{0M})$ is the intercept on the $E(\tilde{R}_p)$ axis of the dashed line tangent to the minimum variance boundary at the point corresponding to M. As in the preceding section, $E(\tilde{R}_{0M})$ is the expected return on any risky security or on any portfolio of risky securities whose return is uncorrelated with the return on M. In essence, a market equilibrium

requires that M be a minimum variance portfolio, which means that (30) applies to each of the securities in M and to portfolios of the securities in M. Since M includes all positive variance securities, (30) applies to all positive variance securities and to portfolios of positive variance securities. The minimum variance zero-β_{pM} portfolio Z is just a special case of such a portfolio; and since $\beta_{ZM} = 0.0, E(\tilde{R}_Z) = E(\tilde{R}_{0M})$.

The market portfolio M, however, does not include the risk-free security F. Thus, even though the return on F is uncorrelated with the return on M, there is no reason to expect (30) and (54) to apply to F, and so there is no reason to expect that $R_F = E(\tilde{R}_Z)$. In fact, since the positively sloped segment of the minimum variance boundary is concave, and since M is generally above the tangency portfolio T on the minimum variance boundary, in general $E(\tilde{R}_Z) > R_F$. Only in the special case where M and T coincide do we get $E(\tilde{R}_Z) = R_F$.

We can contrast these results of what might be called the modified Black model with those of the Sharpe-Lintner model. In the Sharpe-Lintner model, there is unrestricted borrowing as well as lending at the risk-free rate R_F. In terms of Figure 8.5, this means that portfolios along the dashed extension of the line from R_F through T are now feasible. Since T is the risky component of every efficient portfolio, in a market equilibrium T must be the market portfolio of positive variance securities M. In this case, although M does not include F, $R_F = E(\tilde{R}_Z) = E(\tilde{R}_{0M})$, since R_F, $E(\tilde{R}_Z)$, and $E(\tilde{R}_{0M})$ all correspond to the intercept on the $E(\tilde{R}_p)$ axis of the line tangent to the minimum variance boundary at the point M. Thus, equation (32) holds for any security or portfolio, and R_F is the expected return on any security or portfolio whose return is uncorrelated with the return on M. In short, in the Sharpe-Lintner model the investment opportunities available in a market equilibrium are as shown in Figure 8.3, whereas in the modified Black model the world is as shown in Figure 8.5.

Finally, we should note one further thing about the modified Black model. In Figure 8.5 any portfolio of positively weighted efficient portfolios, where each component efficient portfolio is T or a portfolio above T on the efficient boundary, is itself an efficient portfolio. Any portfolio of positively weighted efficient portfolios from along the line between F and T is itself efficient. Unlike the Sharpe-Lintner model and the basic Black model, however, in the modified Black model it is not the case that any portfolio of positively weighted efficient portfolios is efficient. For example, portfolios of F and any efficient portfolio above T are inefficient, since such portfolios are always dominated either by portfolios along the line between F and T or by portfolios along the boundary from T through M. In short, the tangency portfolio T is our first instance of what is called a "corner" portfolio. Efficient portfolios below T contain a security, F, which does not appear in portfolios above T. Such corner portfolios appear consistently in the two-parameter models that we consider below.

PROBLEMS V.A

2. In the modified Black model, since there is risk-free lending but not borrowing, there must be a positive amount of the risk-free security F in the market. Show that the market portfolio of both risk-free and positive variance securities is not efficient.

3. In the Sharpe-Lintner model it is usually assumed that the risk-free security is borrowing and lending among investors, so that in a market equilibrium aggregate borrowing is equal to aggregate lending and the net outstanding amount of the risk-free security is zero. How must the model be modified if the values at time 2 of the activities of some firms are perfectly certain at time 1, so that the net outstanding amount of risk-free securities is positive?

ANSWERS

2. In the modified Black model the market portfolio M in Figure 8.5 includes only positive variance securities. The market portfolio of all securities is a portfolio of F and M, where F is weighted by the ratio of the total market value at time 1 of outstanding risk-free securities to the total market value of all outstanding units of all securities, and where M is weighted by the ratio of the total market value of all outstanding units of all positive variance securities to the total market value of all outstanding units of all securities. This combined market portfolio of risk-free and positive variance securities is on the straight line (not shown) between F and M in Figure 8.5, so it is only efficient in the special case where M and T coincide.

3. The Sharpe-Lintner model changes little when the net outstanding amount of risk-free securities is positive rather than zero. In geometric terms, the relevant picture of investment opportunities in a market equilibrium is again Figure 8.3. Now we must say that M is the market portfolio of only positive variance securities, and the solid curve through Z and M is the curve of minimum variance portfolios of positive variance securities. The only new wrinkle in the Sharpe-Lintner model when there is a positive outstanding amount of risk-free securities is that, as in the preceding problem, the market portfolio of risk-free and risky securities is the appropriately weighted combination of F and M, and it would be somewhere on the straight line between R_F and M in Figure 8.3 Unlike the results for the modified Black model in the preceding problem, however, in the Sharpe-Lintner model the market portfolio of all securities, like the market portfolio of only positive variance securities, is efficient.

B. Market Equilibrium When There Is Risk-free Borrowing and Lending But at Different Interest Rates

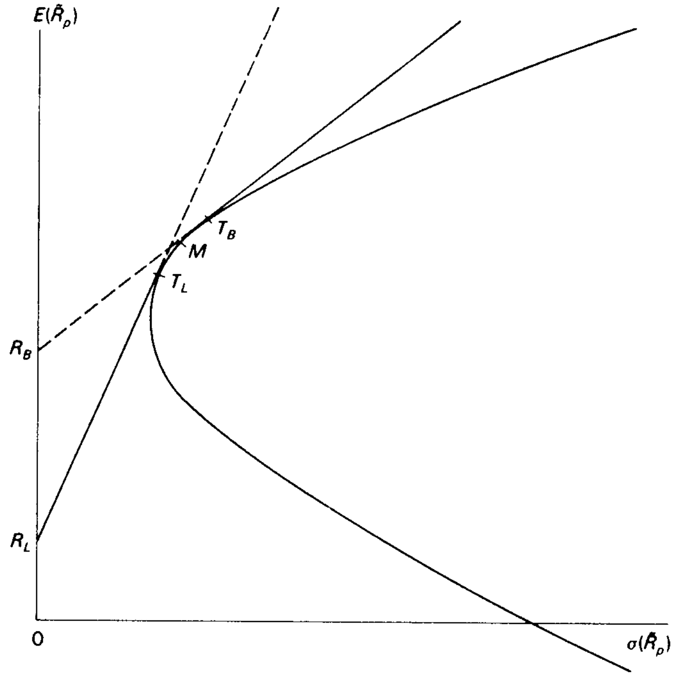
Another variant of the Black model, developed by Brennan (1971), allows for risk-free borrowing and lending, but at different interest rates. The simplest way to set up the Brennan model is to assume that there is risk-free borrowing and lending among investors, so that aggregate borrowing equals lending and there is no net outstanding supply of risk-free securities, but that there is a middleman or broker who exacts a fee from both borrowers and lenders. The same market rate is quoted to borrowers and lenders, but brokerage fees, assumed to be a fixed fraction of the amount borrowed or lent, have the effect of making the rate received by lenders less than the quoted rate, while the rate paid by borrowers is higher than the quoted rate. The assumption of the Brennan model is that there are only brokerage fees in the borrowing-lending market. Positive variance securities can be bought or sold without such transactions costs. Moreover, as in the two variants of the Black model discussed above, short-selling of positive variance securities is unrestricted. Finally, as in all the models of market equilibrium discussed so far, at time 1 there is assumed to be complete agreement among investors with respect to the joint distribution of securities values at time 2.

In the present model, the picture of the investment opportunities available at time 1 might be as shown in Figure 8.6. There are now two different tangency portfolios, T_L and T_B, and three different types of efficient portfolios. The lowest-expected-return efficient portfolios are lending portfolios. These start at the point R_L (the net rate received by lenders) on the vertical axis in Figure 8.6 and go up to the first tangency portfolio T_L on the curve of minimum variance portfolios of positive variance securities. The dashed extension of the line from R_L through T_L does not describe feasible portfolios, since such portfolios imply borrowing at the rate R_L, and borrowers must pay the higher rate R_B. The highest-expected-return efficient portfolios are borrowing portfolios. These are along the solid extension of the line from R_B through the second tangency portfolio T_B. The dashed segment of the line between R_B and T_B does not represent feasible portfolios, since such portfolios imply lending at the rate R_B, and lenders only net the rate R_L. Finally, the remaining efficient portfolios are those between T_B and T_L along the boundary of minimum variance portfolios of positive variance securities.

In Figure 8.6 the market portfolio M is one of the efficient portfolios between T_B and T_L. In a market equilibrium M must indeed be efficient, and the reasoning is similar to that of the modified Black model of the preceding section. In particular, a market equilibrium requires that the portfolio of the

FIGURE 8.6

Market Equilibrium When the Risk-free Borrowing Rate Is Greater Than the Risk-free Lending Rate



risky components of investor portfolios, where the risky component of each investor's portfolio is weighted by the ratio of his total investment in positive variance securities to the total investment in positive variance securities by all investors, be the market portfolio M. The risky component of every efficient portfolio that involves lending is the tangency portfolio T_L. The risky component of every efficient portfolio that involves borrowing is the tangency portfolio T_B. The risky components of other efficient portfolios are simply the portfolios themselves, that is, points along the efficient boundary between T_L and T_B. Thus, in a market equilibrium, the market portfolio can be expressed as a portfolio of positively weighted efficient portfolios between T_L and T_B. Any such portfolio of efficient portfolios of positive variance securities is efficient; as shown in Figure 8.6, the market portfolio must lie somewhere along the efficient boundary between T_L and T_B. Thus, as in the two-parameter models of market equilibrium discussed in preceding sections, we once again have the result that a market equilibrium requires that the market portfolio be efficient, which means that we can interpret (30) or (54) as the market equilibrium relationship between the expected returns on securities and their risks in M.

PROBLEM V.B

1. Show that $R_L < E(\tilde{R}_{0M}) = E(\tilde{R}_Z) < R_B$.

ANSWER

1. The inequality follows from three facts. First, the order of the portfolios T_L, M, and T_B along the efficient boundary must be as shown in Figure 8.6. Second, the efficient boundary between T_L and T_B is strictly concave. Third, R_L, $E(\tilde{R}_{0M})$, and R_B are, respectively, the intercepts on the $E(\tilde{R}_p)$ axis of lines tangent to the boundary at the points T_L, M, and T_B.

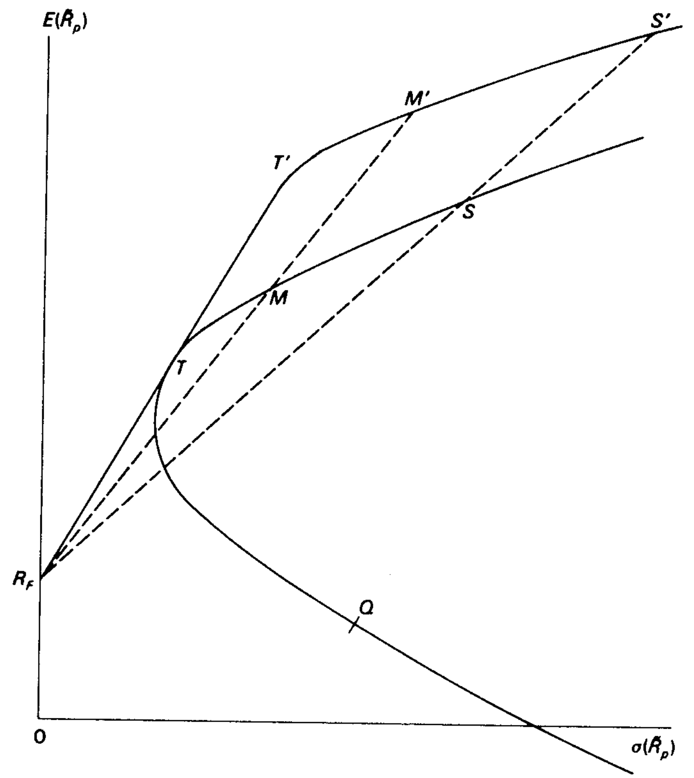
In the Brennan model there are two "corner" portfolios along the efficient boundary, T_L and T_B. Portfolios above T_B on the efficient boundary contain a "security" (borrowing at the rate R_B) which does not appear in any portfolios below T_B on the boundary, while portfolios below T_L on the efficient boundary contain a "security" (lending at the rate R_L) which does not appear in portfolios above T_L on the boundary. Any portfolio of positively weighted portfolios from along the line between R_L and T_L is efficient, and likewise for portfolios of positively weighted portfolios on the curve between T_L and T_B, or for portfolios of efficient portfolios above T_B. But not all portfolios of positively weighted efficient portfolios are efficient. For example, combining risk-free lending at the rate R_L with any minimum variance portfolio above T_L yields an inefficient portfolio.

Finally, note that the borrowing rate R_B and the lending rate R_L do not conform to the expected return-risk equation (54). Since (54) is a relationship between expected return and risk for securities in M, and since M does not include either risk-free borrowing or lending, there is no reason to expect that (54) will apply to these activities.

C. Market Equilibrium When There Is Risk-free Borrowing and Lending But There Are Margin Requirements

The next two-parameter model of market equilibrium is similar to the Black model in that short-selling of positive variance securities is unrestricted, but it is also similar to the Sharpe-Lintner model in that there is risk-free borrowing and lending at a common interest rate. The amount of borrowing, however, is assumed to be restricted to some fixed fraction of an investor's portfolio funds $(w_1 - c_1)$. The fraction is the same for all investors, and it is independent of which portfolio of positive variance securities an investor chooses. Finally, at time 1 there is complete agreement among investors with respect to the joint distribution of security values at time 2.

FIGURE 8.7
Market Equilibrium with Risk-free Lending and Margin Requirements on Risk-free Borrowing



In this model, which we call the restricted borrowing model, the portfolio opportunities facing investors in a market equilibrium might be as shown in Figure 8.7. The solid curve through the points Q and S shows minimum variance portfolios of positive variance securities. Efficient portfolios are first along the line from the risk-free rate R_F through the tangency portfolio T. Portfolios between F and T on the line are combinations of F and T involving different fractions of lending at the rate R_F. Portfolios above T on the line from R_F are borrowing portfolios. The point T' is the portfolio of F and T that involves the maximum allowable fraction of borrowing at the rate R_F. Efficient portfolios along the continuation of the boundary from T' involve combining maximum borrowing with successive minimum variance portfolios of positive variance securities above T on the minimum variance boundary. For example, the efficient portfolio S' involves combining the minimum

variance portfolio S with the maximum allowable fraction of borrowing at the risk-free rate R_F.

In Figure 8.7, the market portfolio M of only risky securities is on the minimum variance boundary. In a market equilibrium this must be the case, and the reasoning is similar to that used in the two preceding models. In particular, in a market equilibrium the portfolio of the risky components of investor portfolios, where the risky component of each investor's portfolio is weighted by the ratio of his investment of his own funds in positive variance securities to the total of such investments by all investors, must be the market portfolio M. The risky components of efficient portfolios are the minimum variance portfolio T and portfolios above T on the curve of minimum variance portfolios of only positive variance securities. Any portfolio of positively weighted minimum variance portfolios where the component portfolios are T and minimum variance portfolios above T is a minimum variance portfolio, and in general it is above T on the minimum variance boundary. Thus, in a market equilibrium, the market portfolio M is a minimum variance portfolio, and in general it is above the tangency portfolio T on the minimum variance boundary.

For the first time, however, the market portfolio of positive variance securities is not efficient. It is the risky component of the efficient portfolio M', but M itself is not efficient. Nevertheless, as long as M is a minimum variance portfolio, (30) or (54) applies. Moreover, since market equilibrium requires that M be a minimum variance portfolio, we can make the by now familiar comments concerning (30) or (54). Since these are conditions that must hold in a market equilibrium, they can be interpreted as conditions on equilibrium prices, or as market equilibrium relationships between the expected returns on positive variance securities and their risks in M. Again, however, since M does not include risk-free securities, (30) and (54) do not apply to F. In fact, recalling that $E(\tilde{R}_{0M}) = E(\tilde{R}_Z)$ is the intercept on the $E(\tilde{R}_p)$ axis of a line tangent to the curve of minimum variance portfolios at the point M, we can see from Figure 8.7 that in the present model, as in the modified Black model (where there is risk-free lending but not borrowing), the risk-free rate R_F is equal to or less than $E(\tilde{R}_Z)$.

PROBLEM V.C

1. In the restricted borrowing model, when will a market equilibrium imply $R_F = E(\tilde{R}_Z)$?

ANSWER

1. From inspection of Figure 8.7 and the definition of $E(\tilde{R}_{0M}) = E(\tilde{R}_Z)$ in (18), we can see that the condition $R_F = E(\tilde{R}_Z)$ requires that the tangency portfolio T be the market portfolio M. This will happen when the only efficient portfolios investors choose to hold are those along the segment of the efficient boundary from R_F to T'. Then T will be the risky component of every efficient portfolio which any investor holds, so that in a market equilibrium T must be the market portfolio M. In this case the present model and the Sharpe-Lintner model are indistinguishable; (32) is the appropriate market equilibrium condition for both models.

VI. Comparison of and Comments on the Various Two-Parameter Models of Market Equilibrium

In all the models of market equilibrium considered so far, a market equilibrium requires that the market portfolio M of positive variance securities be a minimum variance portfolio of positive variance securities, and M is always on the positively sloped segment of the minimum variance boundary. In all models but one, the restricted borrowing model of the preceding section, M is also efficient. The fact that a market equilibrium requires that the market portfolio M be a minimum variance portfolio means that when applied to M, equation (20), which is basically a mathematical condition on the proportions of portfolio funds invested in individual securities that must be met by a minimum variance portfolio, can be interpreted as a condition on equilibrium prices. The fact that a market equilibrium requires that M be a minimum variance portfolio means that in a market equilibrium securities must be priced so that (30), which is (20) applied to M, holds for every positive variance security. Thus, (30) can be interpreted as the implication of the various two-parameter models of market equilibrium for equilibrium prices of securities at time 1. Alternatively, (30) can be interpreted as the market equilibrium relationship between the expected return on any positive variance security and its risk in M.

The differences between the various two-parameter models of market equilibrium, in terms of what they say about prices at time 1, center on additional statements that they make about $E(\tilde{R}_{0M})$ in (30). In all models, $E(\tilde{R}_{0M})$ is the intercept on the $E(\tilde{R}_p)$ axis of the line tangent to the curve of minimum variance portfolios at the point corresponding to the market portfolio M. In all models, $E(\tilde{R}_{0M})$ is the expected return on any positive variance security or on any portfolio of positive variance securities whose return is uncorrelated with the return on M. In the Sharpe-Lintner model, in which there is assumed to be unrestricted risk-free borrowing and lending, $E(\tilde{R}_{0M})$ is equal to R_F, the rate of interest that lenders receive and borrowers pay. Thus, in this model the expected return on positive variance zero-β_{iM} securities and on zero-β_{pM} portfolios of positive variance securities is equal to the return on risk-free securities. In the modified Black model, in which there is risk-free lending but not borrowing, in general $E(\tilde{R}_{0M}) > R_F$, where R_F is now the rate on risk-free lending. The same result, $E(\tilde{R}_{0M}) > R_F$, applies to the restricted borrowing model, in which there is borrowing and lending at the common rate R_F but the amount of borrowing is restricted to a fixed fraction of the investor's portfolio funds $w_1 - c_1$. Thus, in these models, positive variance zero-β_{iM} securities and zero-β_{pM} portfolios of positive variance securities have higher expected returns than risk-free securities. Finally, in the Brennan model, in which there is unrestricted risk-free borrowing and lending but the rate paid by borrowers is higher than the rate required by lenders, $R_L < E(\tilde{R}_{0M}) < R_B$, where R_L and R_B are the lending and borrowing rates.

There are, however, basically just two different types of models. First, there is the Sharpe-Lintner model, in which there is assumed to be unrestricted borrowing and lending at a common risk-free rate R_F but in which it is not necessary that positive variance securities can be sold short. Then there is the Black model and variants thereof, in which unrestricted short-selling of positive variance securities is always assumed. In the basic Black model there are no risk-free securities. In the variants of the model there are risk-free securities, but borrowing is either impossible or restricted, or lending and borrowing are subject to transactions costs.

Given that these models consistently assume that short-selling of positive variance securities is unrestricted and that there are no transactions costs in trading such securities, the assumptions that the variants of the Black model then make about the borrowing-lending market are somewhat contrived. For example, in the Brennan model there are transactions costs for risk-free securities such that the risk-free rate paid by borrowers is greater than the rate received by lenders, but there are no transactions costs in trading positive variance securities. In real-world capital markets there are brokerage fees in trading all securities, and in general they are higher on common stocks than on bonds. Thus, although introducing transactions costs seems to move the model in the direction of greater realism, one can argue that quite the opposite is true if such costs are assumed to exist only for risk-free securities, since this has the effect of assuming the reverse of the actual relationship between the costs of trading risk-free and positive variance securities.

In the other two variants of the Black model, either there is no risk-free borrowing, or risk-free borrowing is restricted to some fixed fraction of portfolio funds. Risk-free lending is unrestricted, and there are no restrictions on either long or short positions in positive variance securities. In the real world, of course, it is impossible for an investor to borrow unlimited amounts at a risk-free rate. Nevertheless, restrictions on short-selling of positive variance securities like common stocks are even more severe. In truth, short-selling of common stocks, in the manner in which we use that term, does not exist. We assume that when an investor sells short a security—that is, borrows the security and sells it on the open market—he gets to use the proceeds from the sale to increase his investments in other securities. In reality, when an investor sells a common stock short, not only does he not get the proceeds from the sale of the security (they reside with the broker), but he must also put up collateral as if he were buying the security long. Given that the funds required are the same as in a long position, a real-world short sale of a common stock is in effect a "long" position in the stock, but one where the short-seller arranges to get the negative of the return on the usual type of purchase.

In any case, the point is that although risk-free borrowing is not unrestricted in the real world, it is less restricted than short-selling of positive variance securities like common stocks. Thus, models that incorporate restrictions on borrowing but allow unrestricted short-selling of positive variance securities are a step away from the real world, since they reverse the real relative degrees of restrictiveness in risk-free borrowing and short-selling of positive variance securities.

The Sharpe-Lintner model and the basic Black model are much more consistent in this respect than the variants of the Black model discussed above. In the Sharpe-Lintner model, borrowing and lending at a common risk-free rate are unrestricted, and the implication of the model for market equilibrium, the expected return-risk equation (32), does not require the assumption that positive variance securities can be sold short. Thus, although real-world borrowing is certainly not unrestricted, this model is at least consistent with the fact that real-world borrowing is less restricted than real-world short-selling of positive variance securities. In the basic Black model, risk-free securities are not assumed to exist, and short-selling of positive variance securities is unrestricted. Thus, although the assumption of unrestricted short-selling is unrealistic, this model, unlike its variants, is at least consistent in that it assumes that all available securities can be sold short.

We do not, however, judge the Sharpe-Lintner and Black models on the basis of whether the assumptions of one or the other seem more appealing. Both are to some extent unrealistic, and it is best to choose between them on the basis of which does a better job explaining real-world data on average returns and risk. Indeed, in the empirical tests of the next chapter we do not hesitate to invoke implications of variants of the Black model when these help to explain what is observed in the data. The fact that we do not find the assumptions of these models as pleasing as those of the Sharpe-Lintner and Black models does not mean that the variants of the Black model cannot offer some insight into what we observe in the data. Such insight, as always, is the basis on which we judge whether a model is useful.

VII. Market Equilibrium When There Are No Risk-free Securities and Short-Selling of Positive Variance Securities Is Prohibited

A. *Preliminary Discussion*

The implications of the Sharpe-Lintner model and the Black model and its variants for equilibrium security prices and for equilibrium relationships between the expected returns on securities and their risks derive from the fact that in these models a market equilibrium requires that M, the market portfolio of positive variance securities, be a minimum variance portfolio and usually an efficient portfolio. Thus, in equilibrium, securities must be priced so that the mathematical relationship between the expected returns on securities and their risks in a minimum variance portfolio applies to M. In the Sharpe-Lintner model the efficiency of M is a consequence of the fact that with unrestricted risk-free borrowing and lending, there is only one efficient portfolio of only positive variance securities, and this portfolio is the risky component of every efficient portfolio. If the market is to clear, this "tangency" portfolio must be M. In the Black model and its variants, the minimum variance property of the market portfolio is a consequence of the facts that (a) in a market equilibrium the portfolio of the risky components of investor portfolios must be the market portfolio; (b) the risky components of investor portfolios are minimum variance portfolios; and (c) with unrestricted short-selling of positive variance securities, any portfolio of minimum variance portfolios is a minimum variance portfolio. All models are also based on the assumption that there is complete agreement among investors with respect to the joint distribution of security prices at time 2.

In addition to the assumption of complete agreement, in the Black model and its variants the key ingredient for the result that M must be a minimum variance portfolio is the assumption that short-selling of positive variance securities is unrestricted, while the key ingredient in the Sharpe-Lintner model is the assumption that risk-free borrowing and lending is unrestricted.

We now examine what we can say about market equilibrium when there are no risk-free securities and there is no short-selling of positive variance securities. We find that in general the market portfolio is neither efficient nor minimum variance.

This result is not necessarily troublesome. In the various two-parameter models, the implications of a model for equilibrium prices of securities are obtained by showing that a market equilibrium requires that some portfolio, with proportions invested in individual securities that are known, is a minimum variance portfolio. Then we say that when applied to this portfolio, equation (20), which otherwise is just a mathematical condition that is met by securities in a minimum variance portfolio, can be interpreted as a condition on security prices that must be met in a market equilibrium. Since it contains all securities and in the proportions in which they are outstanding, the market portfolio is a convenient and appealing candidate for this market equilibrium interpretation of the minimum variance condition. However, the same interpretation of (20) would apply to any portfolio that must be a minimum variance portfolio in a market equilibrium. Unfortunately, at least as the state of the art stands now, when there are no risk-free securities and there is no short-selling of positive variance securities, we cannot identify such portfolios.

This result presents some cause for concern. The basic goal of the empirical work in the next chapter is to test whether security prices and returns behave as if investors choose portfolios in conformance with the two-parameter portfolio model. The goal is to test whether the behavior of prices and returns seems to reflect the attempts of investors to hold efficient portfolios. Such tests require some two-parameter model of market equilibrium. To date, the models of market equilibrium that produce testable propositions about security prices and expected returns require either the assumption that there is unrestricted risk-free borrowing and lending or the assumption that short-selling of positive variance securities is unrestricted. The portfolio model itself requires neither of these assumptions. Thus, in the empirical tests of the next chapter, we run the danger of rejecting the basic proposition that the behavior of returns is as if investors attempt to hold efficient portfolios, when the real problem is that we do not have a suitable two-parameter model of market equilibrium. On the other hand, if the tests turn out well, and we argue in Chapter 9 that they do, then they provide some vindication both for the models of market equilibrium that are used and for the more basic proposition that the prices of securities reflect the attempts of investors to hold efficient portfolios.

B. *The Efficient Set Without Short-Selling or Risk-free Securities*

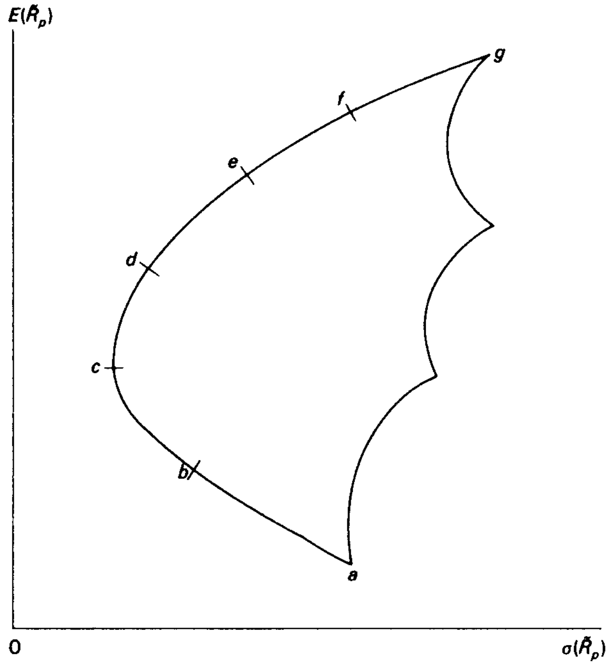
When short-selling of positive variance securities is not allowed, the set of minimum variance portfolios is generated by a sequence of "corner" portfolios. For example, in Figure 8.8 the corner portfolios are assumed to be the points a, b, c, d, e, f, g. The portfolio g is the security with the highest expected return, or, if there is more than one security with the highest expected return, g can be a portfolio of these. Moving down the boundary, the next corner portfolio, f, contains all the securities in g plus one security not contained in g. In general, adjacent corner portfolios differ by one security (that is, one of them contains a security not included in the other), but in some cases adjacent corner portfolios differ by two securities (that is, each contains a security not included in the other). Equivalently, as one moves down the minimum variance boundary, each new corner portfolio involves adding a new security to the immediately preceding corner portfolio and/or dropping a security that appears in the immediately preceding corner portfolio.

When short-selling is not allowed, portfolios on the minimum variance boundary that are between two corner portfolios are just combinations of these two adjacent corner portfolios. For example, all the points along the boundary between the corner portfolios d and e in Figure 8.8 are portfolios of d and e. In general, however, only combinations of adjacent corner portfolios are on the minimum variance boundary. For example, combinations of the nonadjacent corner portfolios d and f in Figure 8.8 are not on the minimum variance boundary. This is in contrast with the situation when short-selling is unrestricted. Then any portfolio of minimum variance portfolios is a minimum variance portfolio. In the Black model and its variants, it is this property of minimum variance portfolios that allows us to reason that in a market equilibrium the market portfolio must be a minimum variance portfolio. When short-selling is prohibited, the fact that portfolios of nonadajacent corner portfolios are not minimum variance portfolios allows us to conclude that in general the market portfolio cannot be a minimum variance portfolio.

Thus, suppose that there is complete agreement among investors with respect to the joint distribution of security prices at time 2 and that the common picture of the portfolio opportunities facing investors in a market equilibrium is Figure 8.8. Different investors choose different portfolios from along the efficient or positively sloped segment of the minimum variance boundary. Since Figure 8.8 is supposed to represent a market equilibrium, the portfolio of the portfolios chosen by investors, where the portfolio chosen by an in-

FIGURE 8.8

The Efficient Set When There Is No Short-Selling



vestor is weighted by the ratio of his portfolio funds to the portfolio funds of all investors, must be the market portfolio. Thus, the market portfolio is a portfolio of minimum variance portfolios. Unless investors all choose minimum variance portfolios between the same two adjacent corner portfolios, when short-selling is prohibited such a portfolio of minimum variance portfolios cannot itself be a minimum variance portfolio.

So much for the market portfolio! As discussed above, the cause is not lost if we can identify some portfolio or portfolios that must be of minimum variance or efficient in a market equilibrium. So far as I know, this task has met with no success. The problem is that, aside from the general comments above, we know little about the characteristics of minimum variance portfolios when short-selling is prohibited. For example, as one moves down the minimum variance boundary, minimum variance portfolios at first become more diversified; then they become less diversified as one approaches the bottom point on the boundary, which is the security with the lowest expected return or a portfolio of securities with the same lowest possible expected return. Beyond this, however, one can, at the moment, say nothing.

In short, a market equilibrium requires that aggregate investor demand for each security be equal to the outstanding supply of the security. This in turn means that securities must be priced so that each security is in some efficient portfolios that are chosen by investors. These statements fall substantially short of interesting and testable propositions about the nature of capital market equilibrium. For such propositions, we have to rely on the Sharpe-Lintner model and on the Black model and its variants.

VIII. Market Equilibrium: Mathematical Treatment

In the preceding sections, the implications of the two-parameter model for equilibrium prices of securities are derived from a combination of geometric arguments with the properties of minimum variance portfolios. This approach is best for developing an intuitive appreciation for what the portfolio model and the derived two-parameter models of capital market equilibrium are all about. In this section, we consider a more "elegant" (i.e., mathematical) approach which derives market equilibrium prices more or less directly from the solutions to the consumption-investment decision problems that investors are presumed to face. Although this approach obscures some of the more interesting aspects of the model, it is more useful for attacking some questions than the geometric approach of the preceding sections.

The nonmathematical reader who is taking a first pass at this book might skip this section, since the new insights that it provides are not critical to a good understanding of two-parameter theory. The nonmathematical reader who is interested in the new insights can, however, continue on. He can get what he wants from the verbal discussions, especially those of Section VIII.C. The mathematical reader may find the present approach more appealing than that of the preceding sections.

A. *Consumption-Investment Decisions and Equilibrium Prices*

Begin with a few definitions:

$\tilde{V}_j$ = market value of firm j at time 2. Firms are assumed to have only common stock outstanding at time 1; and $\tilde{V}_j$ is the total market value of the common stock of firm j at time 2, including any dividends paid at time 2. We assume that there is complete agreement among investors with respect to the joint distribution of $\tilde{V}_j, j = 1, \ldots, n$.

P_j = market value of firm j at time 1. Market equilibrium at time 1 involves determination of a set of market-clearing prices P_j for each of the $j = 1, \ldots, n$ firms in the market.

X_{ij} = fraction of P_j and thus of $\tilde{V}_j$ demanded by investor i at time 1. X_{ij} is the fraction of firm j owned by investor i after he chooses his portfolio at time 1. We assume that short-selling of all securities is unrestricted, so that X_{ij} can take positive or negative values.

c_{1i}, c_{2i} = consumption of investor i at times 1 and 2.

$E(\tilde{V}_j)$ = expected market value of firm j at time 2.

$\text{cov}(\tilde{V}_j, \tilde{V}_k)$ = covariance between the market values of firms j and k at time 2, with $\text{cov}(\tilde{V}_j, \tilde{V}_j) = \sigma^2(\tilde{V}_j)$.

The investor's consumption at time 2 is the total value of his holdings in all firms

$$\tilde{c}_{2i} = \sum_{j=1}^{n} X_{ij}\tilde{V}_j, \tag{55}$$

and $\tilde{c}_{2i}$ and $\tilde{V}_j$ are random variables at time 1. The mean and variance of the distribution of $\tilde{c}_{2i}$ are

$$E_i \equiv E(\tilde{c}_{2i}) = \sum_{j=1}^{n} X_{ij}E(\tilde{V}_j) \tag{56}$$

$$\sigma_i^2 \equiv \sigma^2(\tilde{c}_{2i}) = \sum_{j=1}^{n}\sum_{k=1}^{n} X_{ij}X_{ik}\,\text{cov}(\tilde{V}_j, \tilde{V}_k). \tag{57}$$

In the present framework, we assume that the joint distribution of firm values $\tilde{V}_j$, $j = 1, \ldots, n$, is multivariate normal. This in turn means that for any choice of X_{ij}, $j = 1, \ldots, n$, the distribution of $\tilde{c}_{2i}$ is normal. Following the arguments of Chapter 7, the investor's decision problem reduces to a choice of current consumption c_{1i}, expected future consumption E_i, and variance of future consumption σ_i^2. We assume that the tastes of investor i for c_{1i}, E_i, and σ_i^2 can be summarized by a welfare function $G_i(c_{1i}, E_i, \sigma_i^2)$ which is increasing in c_{1i} and E_i but decreasing in σ_i^2. In words, the investor likes consumption at time 1 and expected consumption at time 2, but he is averse to dispersion in the distribution of consumption at time 2. Finally, the welfare function G is assumed to be differentiable at least once in each of its arguments.

The investor's decision problem at time 1 is to choose values of c_{1i} and X_{ij}, $j = 1, \ldots, n$, that maximize his welfare subject to the constraint that the total of his current consumption and his investment in firms is equal to w_{1i}, the market value of his wealth at time 1. In formal terms, the investor must choose c_{1i} and X_{ij}, $j = 1, \ldots, n$, which

$$\text{maximize } G_i(c_{1i}, E_i, \sigma_i^2) \tag{58}$$

subject to the budget constraint

$$w_{1i} = \sum_{j=1}^{n} X_{ij}P_j + c_{1i}. \tag{59}$$

To solve the problem, we form the Lagrangian expression

$$G_i(c_{1i}, E_i, \sigma_i^2) + \lambda_i\left(w_{1i} - c_{1i} - \sum_{j=1}^{n} X_{ij}P_j\right),$$

differentiate the expression partially with respect to λ_i, c_{1i}, and X_{ij}, $j = 1, \ldots, n$, and then set these partial derivatives equal to zero. This yields (59) and the $n + 1$ equations

$$\frac{\partial G_i}{\partial c_{1i}} - \lambda_i = 0 \tag{60}$$

$$\frac{\partial G_i}{\partial E_i}E(\tilde{V}_j) + \frac{\partial G_i}{\partial \sigma_i^2}\,2\sum_{k=1}^{n} X_{ik}\,\text{cov}(\tilde{V}_j, \tilde{V}_k) - \lambda_i P_j = 0, \quad j = 1, \ldots, n, \tag{61}$$

where the partial derivatives $\partial G_i/\partial c_{1i}$, $\partial G_i/\partial E_i$ and $\partial G_i/\partial \sigma^2$ are rates of change of welfare with respect to changes in c_{1i}, E_i, and σ_i^2. Substituting (60) into (61), we get

$$\frac{\partial G_i}{\partial E_i}E(\tilde{V}_j) + \frac{\partial G_i}{\partial \sigma_i^2}\,2\sum_{k=1}^{n} X_{ik}\,\text{cov}(\tilde{V}_j, \tilde{V}_k) - \frac{\partial G_i}{\partial c_{1i}}P_j = 0, \quad j = 1, \ldots, n. \tag{62}$$

Each investor is presumed to solve a problem like that stated in (58) and (59). Thus, there is a set of equations like (59), (60), and (62) for each investor i; conditional on some set of values for firms at time 1, these equations determine the optimal values of current consumption and fractions of firms chosen by investor i at time 1. To get the implications of optimal decisions by investors for market equilibrium values of firms at time 1, we shall aggregate (62) across all investors, invoke the market-clearing constraints

$$\sum_{i=1}^{I} X_{ij} = 1, \quad j = 1, \ldots, n, \tag{63}$$

and then solve the resulting aggregate equations for P_j, $j = 1, \ldots, n$. As a first step, we divide through equation (62) by $\partial G_i/\partial \sigma_i^2$, to get

$$\frac{\partial G_i/\partial E_i}{\partial G_i/\partial \sigma_i^2} E(\tilde{V}_j) + 2 \sum_{k=1}^{n} X_{ik} \operatorname{cov}(\tilde{V}_j, \tilde{V}_k) - \frac{\partial G_i/\partial c_{1i}}{\partial G_i/\partial \sigma_i^2} P_j = 0, \quad j = 1, \ldots, n. \tag{64}$$

To interpret the two ratios of partial derivatives in (64), first set the differential of G_i equal to zero:

$$\frac{\partial G_i}{\partial c_{1i}} dc_{1i} + \frac{\partial G_i}{\partial E_i} dE_i + \frac{\partial G_i}{\partial \sigma_i^2} d\sigma_i^2 = 0, \tag{65}$$

where dc_{1i}, dE_i and $d\sigma_i^2$ represent "small" changes in c_{1i}, E_i, and σ_i^2. Equation (65) tells us how we can vary current consumption, expected consumption at time 2, and variance of consumption at time 2, while keeping the welfare of investor i constant. The equation allows us to determine marginal rates of substitution of one variable for another; that is, it can be used to determine how any two of the three variables can be varied while keeping the investor's welfare constant. For example, $d\sigma_i^2/dE_i$, the marginal rate of substitution of expected consumption at time 2 for variance of consumption at time 2, is obtained by setting dc_{1i} equal to zero in (65) and then solving to get

$$\frac{d\sigma_i^2}{dE_i} = -\frac{\partial G_i/\partial E_i}{\partial G_i/\partial \sigma_i^2}. \tag{66}$$

Likewise, $d\sigma_i^2/dc_{1i}$, the marginal rate of substitution of consumption at time 1 for variance of consumption at time 2, is obtained by setting dE_i equal to zero and then solving (65) to get

$$\frac{d\sigma_i^2}{dc_{1i}} = -\frac{\partial G_i/\partial c_{1i}}{\partial G_i/\partial \sigma_i^2}. \tag{67}$$

PROBLEM VIII.A

1. Give an intuitive explanation for the minus signs in (66) and (67).

ANSWER

1. The marginal rate of substitution in (66) is concerned with changes dE_i and $d\sigma_i^2$ in the mean and variance of consumption at time 2, holding consumption constant at time 1, that leave welfare unchanged. Since the investor is assumed to like expected consumption ($\partial G_i/\partial E_i > 0$) but to dislike variance of consumption ($\partial G_i/\partial \sigma_i^2 < 0$), keeping welfare constant implies that dE_i and $d\sigma_i^2$ in equation (65) must have the same sign. Since $\partial G_i/\partial E_i$ and $\partial G_i/\partial \sigma_i^2$ have opposite signs, this means that we must have a minus sign on the right of the equality in (66). Similar comments apply to (67).

We can now see that the two ratios of partial derivatives in (64) are the marginal rates of substitution of equations (66) and (67). If we interpret the n equations of (64) as the security demand equations for investor i, we can say that the fraction of the total securities of firm j demanded by investor i depends on the expected market value of the firm at time 2, the covariances of its market value at time 2 with the market values of all firms, the market value of the firm at time 1, and investor i's marginal rates of substitution between mean and variance of consumption at time 2 and between consumption at time 1 and variance of consumption at time 2.

To get an expression for the value of firm j in a market equilibrium, we aggregate the demand equations of (64) across all investors $i, i = 1, \ldots, I$, invoking the market-clearing constraints of (63). The result is

$$-\gamma E(\tilde{V}_j) + 2 \sum_{k=1}^{n} \operatorname{cov}(\tilde{V}_j, \tilde{V}_k) + \delta P_j = 0, \quad j = 1, \ldots, n, \tag{68}$$

where

$$\gamma = \sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} \quad \text{and} \quad \delta = \sum_{i=1}^{I} \frac{d\sigma_i^2}{dc_{1i}}. \tag{69}$$

Equation (68) shows the relationship between the market equilibrium value of firm j at time 1, its expected value at time 2, the covariances of its value at time 2 with the values of all firms, and the tastes of investors as summarized by γ and δ. To get an explicit equation for the market equilibrium value of firm j at time 1, we solve (68) for P_j,

$$P_j = \frac{\gamma}{\delta} E(\tilde{V}_j) - \frac{2}{\delta} \sum_{k=1}^{n} \operatorname{cov}(\tilde{V}_j, \tilde{V}_k). \tag{70}$$

A more intuitive version of this pricing equation can be obtained by developing interpretations of γ/δ and $2/\delta$. The aggregate values of all firms at time 1 and time 2 are

$$P_M = \sum_{j=1}^{n} P_j \quad \text{and} \quad \tilde{V}_M = \sum_{j=1}^{n} \tilde{V}_j. \tag{71}$$

P_M and $\tilde{V}_M$ can also be interpreted as the values of invested wealth at time 1 and time 2. The mean and variance of the distribution of $\tilde{V}_M$ are

$$E(\tilde{V}_M) = \sum_{j=1}^{N} E(\tilde{V}_j) \quad \text{and} \quad \sigma^2(\tilde{V}_M) = \sum_{j=1}^{N} \sum_{k=1}^{N} \operatorname{cov}(\tilde{V}_j, \tilde{V}_k). \tag{72}$$

Thus, if we aggregate (70) across firms, we determine that

$$P_M = \frac{\gamma}{\delta} E(\tilde{V}_M) - \frac{2}{\delta} \sigma^2(\tilde{V}_M), \tag{73}$$

$$\frac{2}{\delta} = \frac{\frac{\gamma}{\delta} E(\tilde{V}_M) - P_M}{\sigma^2(\tilde{V}_M)}. \tag{74}$$

With (74), equation (70) can be rewritten as

$$P_j = \frac{\gamma}{\delta} E(\tilde{V}_j) - \left[\frac{\frac{\gamma}{\delta} E(\tilde{V}_M) - P_M}{\sigma^2(\tilde{V}_M)}\right] \sum_{k=1}^{n} \operatorname{cov}(\tilde{V}_j, \tilde{V}_k). \tag{75}$$

Or better, using (71) we can determine that

$$\operatorname{cov}(\tilde{V}_j, \tilde{V}_M) = \operatorname{cov}\left(\tilde{V}_j, \sum_{k=1}^{n} \tilde{V}_k\right) = \sum_{k=1}^{n} \operatorname{cov}(\tilde{V}_j, \tilde{V}_k), \tag{76}$$

so that

$$P_j = \frac{\gamma}{\delta} E(\tilde{V}_j) - \left[\frac{\frac{\gamma}{\delta} E(\tilde{V}_M) - P_M}{\sigma^2(\tilde{V}_M)}\right] \operatorname{cov}(\tilde{V}_j, \tilde{V}_M). \tag{77}$$

Better yet, if we let

$$\frac{\gamma}{\delta} = \frac{1}{\theta}, \tag{78}$$

then

$$P_j = \frac{1}{\theta}\left[E(\tilde{V}_j) - \left(\frac{E(\tilde{V}_M) - \theta P_M}{\sigma^2(\tilde{V}_M)}\right) \operatorname{cov}(\tilde{V}_j, \tilde{V}_M)\right]. \tag{79}$$

It is an easy matter to develop an interpretation of θ. Suppose there is a firm, call it firm 0, whose market value at time 2 is uncorrelated with $\tilde{V}_M$, so that $\operatorname{cov}(\tilde{V}_0, \tilde{V}_M) = 0$. From (79), for this firm we have

$$P_0 = \frac{1}{\theta} E(\tilde{V}_0) \tag{80}$$

so that

$$\theta = \frac{E(\tilde{V}_0)}{P_0} = 1 + E(\tilde{R}_0) = 1 + E(\tilde{R}_{0M}). \tag{81}$$

In words, θ can be interpreted as 1 plus the expected return on the securities of a firm whose market value at time 2 is uncorrelated with the total market value of all firms at time 2. If the market value of a firm at time 2 is uncorrelated with the total market value of all firms, the return on the firm's securities is uncorrelated with the return on the market portfolio. Thus, $E(\tilde{R}_0)$ is $E(\tilde{R}_{0M})$ of preceding sections; that is, $E(\tilde{R}_0) = E(\tilde{R}_{0M})$ is the expected return on any security (or firm) whose return is uncorrelated with the return on the market portfolio M.

PROBLEM VIII.A

2. Show that $\operatorname{cov}(\tilde{V}_0, \tilde{V}_M) = 0$ implies $\operatorname{cov}(\tilde{R}_0, \tilde{R}_M) = 0$.

ANSWER

2.

$$\operatorname{cov}(\tilde{R}_j, \tilde{R}_M) = \operatorname{cov}\left(\frac{\tilde{V}_j}{P_j} - 1, \frac{\tilde{V}_M}{P_M} - 1\right) = \frac{1}{P_j P_M} \operatorname{cov}(\tilde{V}_j, \tilde{V}_M). \tag{82}$$

It follows that $\operatorname{cov}(\tilde{V}_0, \tilde{V}_M) = 0$ implies $\operatorname{cov}(\tilde{R}_0, \tilde{R}_M) = 0$.

Substituting (81) into (79), we get

$$P_j = \frac{1}{1 + E(\tilde{R}_{0M})}\left[E(\tilde{V}_j) - \left(\frac{E(\tilde{V}_M) - [1 + E(\tilde{R}_{0M})] P_M}{\sigma^2(\tilde{V}_M)}\right) \operatorname{cov}(\tilde{V}_j, \tilde{V}_M)\right]. \tag{83}$$

This equation has an interesting interpretation. From (72) and (76), we can determine that

$$\sigma^2(\tilde{V}_M) = \sum_{j=1}^{n} \sum_{k=1}^{n} \operatorname{cov}(\tilde{V}_j, \tilde{V}_k) = \sum_{j=1}^{n} \operatorname{cov}(\tilde{V}_j, \tilde{V}_M). \tag{84}$$

Since $\tilde{V}_M$ is the market value of invested wealth at time 2, $\sigma^2(\tilde{V}_M)$ can be interpreted as the risk of invested wealth. Then $\operatorname{cov}(\tilde{V}_j, \tilde{V}_M)$ can be interpreted as the risk of the market value of firm j at time 2 in the sense that it is the contribution of the securities of firm j to the risk of market wealth. Thus, (83) says that the market value of firm j at time 1 is determined by first adjusting the expected market value of the firm at time 2 for the risk of the market value of the firm at time 2, then discounting this risk-adjusted expected market value at a rate equal to the expected return on a riskless security, that is, a security which contributes nothing to the risk of market wealth at time 2.

Alternatively, (83) says that a firm sells two things in the capital market at time 1: its expected market value at time 2, $E(\tilde{V}_j)$, and the risk of its market value at time 2, $\operatorname{cov}(\tilde{V}_j, \tilde{V}_M)$. The price at time 1 of a unit of expected market value at time 2 is

$$1/[1 + E(\tilde{R}_{0M})],$$

and the price of a unit of risk is

$$-\left(\frac{E(\tilde{V}_M) - [1 + E(\tilde{R}_{0M})]P_M}{\sigma^2(\tilde{V}_M)}\right) \Big/ [1 + E(\tilde{R}_{0M})].$$

Finally, suppose there is unrestricted risk-free borrowing and lending. Then the value at time 1 of V_F dollars to be delivered for certain at time 2 is, from (83),

$$P_F = \frac{V_F}{[1 + E(\tilde{R}_{0M})]}. \tag{85}$$

Since V_F/P_F is just 1 plus the risk-free rate of interest,

$$1 + E(\tilde{R}_{0M}) = 1 + R_F. \tag{86}$$

In short, we have rederived the Sharpe-Lintner result that $E(\tilde{R}_{0M})$, the expected return on any positive variance security whose return is uncorrelated with the return on the market portfolio M, is equal to the risk-free rate of interest R_F. We simply follow a different approach to the Sharpe-Lintner model here.

Substituting (86) into (83), we get

$$P_j = \frac{1}{1 + R_F}\left[E(\tilde{V}_j) - \left(\frac{E(\tilde{V}_M) - (1 + R_F)P_M}{\sigma^2(\tilde{V}_M)}\right)\operatorname{cov}(\tilde{V}_j, \tilde{V}_M)\right], \quad j = 1, \ldots, n. \tag{87}$$

In words, as in (83), the market value of firm j at time 1 is the present value of the risk-adjusted expected market value of the firm at time 2. In (87), however, the discount rate is the risk-free rate of interest R_F. Thus, in (87) we first adjust $E(\tilde{V}_j)$, the expected market value of the firm at time 2, for the risk of $\tilde{V}_j$, and then discount this risk-adjusted expected value back to time 1 at the risk-free rate.

PROBLEM VIII.A

3. Since it is based on the assumptions of complete agreement and unrestricted short-selling of all securities, (83) must be the market equilibrium pricing equation for the Black model. We earlier interpreted equation (30) in this way. Show that (30) can be obtained from (83). Likewise, show that (32), which we earlier interpreted as the market equilibrium pricing equation for the Sharpe-Lintner model, can be derived from (87).

ANSWER

3. Rearranging (83), we can determine that

$$E(\tilde{R}_j) \equiv \frac{E(\tilde{V}_j) - P_j}{P_j} = E(\tilde{R}_{0M}) + \left(\frac{E(\tilde{V}_M) - [1 + E(\tilde{R}_{0M})]P_M}{\sigma^2(\tilde{V}_M)}\right)\frac{\operatorname{cov}(\tilde{V}_j, \tilde{V}_M)}{P_j}. \tag{88}$$

Since

$$\tilde{V}_M = P_M(1 + \tilde{R}_M),$$

$$E(\tilde{V}_M) = P_M[1 + E(\tilde{R}_M)] \quad \text{and} \quad \sigma^2(\tilde{V}_M) = P_M^2\sigma^2(\tilde{R}_M).$$

Thus, equation (88) can be rewritten as

$$E(\tilde{R}_j) = E(\tilde{R}_{0M}) + \left[\frac{E(\tilde{R}_M) - E(\tilde{R}_{0M})}{\sigma^2(\tilde{R}_M)}\right]\frac{\operatorname{cov}(\tilde{V}_j, \tilde{V}_M)}{P_j P_M}.$$

Then from (82) it follows that

$$E(\tilde{R}_j) = E(\tilde{R}_{0M}) + \left[\frac{E(\tilde{R}_M) - E(\tilde{R}_{0M})}{\sigma^2(\tilde{R}_M)}\right]\operatorname{cov}(\tilde{R}_j, \tilde{R}_M),$$

which is just (30).

Thus, (83) and (30) are equivalent statements about the implications of the Black model for market equilibrium prices. Equation (83) states the equilibrium condition in terms of prices, whereas (30) states the equilibrium condition in terms of expected returns. Likewise, when the preceding analysis is applied to (87), we get (32) as the implied market equilibrium condition for expected returns in the Sharpe-Lintner model. Thus, (87) and (32) are equivalent statements of the implications of the Sharpe-Lintner model for market equilibrium prices and expected returns.

B. *Counting Equations and Unknowns*

As the first-order conditions for a solution to the maximization problem stated in (58) and (59), equations (59) to (61) determine optimal values of current consumption c_{1i} and fractions X_{ij} ($j = 1, \ldots, n$) of firms demanded by investor i at time 1, conditional on some set of market values P_j ($j = 1, \ldots, n$) of firms at time 1. Thus, equations (59) to (61) are the conditions for investor equilibrium, and there is a set of such equations for each investor. The conditions for a market equilibrium are the n market-clearing equations of (63). When these are appended to the conditions for investor equilibrium, we have a system of equations which determine the n market equilibrium values of firms at time 1, as well as the equilibrium fractions of firms demanded by individual investors and the equilibrium consumptions of individual investors at time 1.

The conditions for investor equilibrium given by (59) to (61) are $n + 2$ equations in $n + 1$ unknowns, c_{1i} and $X_{ij}(j = 1, \ldots, n)$. One of the n demand equations of (61) is redundant. Given the optimal value of c_{1i} and of $n - 1$ of the X_{ij}, the nth fraction X_{ij} can be implied from the budget constraint of (59). Thus, we can arbitrarily drop one of the n demand equations of (61), and the remaining $n - 1$ equations along with (59) and (60) determine c_{1i} and $X_{ij}(j = 1, \ldots, n)$.

There is one remaining puzzle. When the n equations of (63) are appended to the system obtained when (59) to (61) are applied to each investor, we have a sufficient number of equations to determine the market equilibrium values of each of the n firms in the market. In most models of market equilibrium, we usually can only determine relative prices. Here we seem to be able to determine all n security prices. In fact, the model does determine relative prices only. There are $n + 1$ goods in the market at time 1: the n common stocks of individual firms plus current consumption. The model determines the prices of the common stocks in terms of—that is, in units of—current consumption, so that current consumption is the numeraire.

C. *Market Equilibrium Without Complete Agreement*

The models of market equilibrium discussed so far are based on the assumption that there is complete agreement among investors with respect to the joint distribution of the market values of firms at time 2. This assumption is not a necessary ingredient for a market equilibrium. A market equilibrium simply requires a set of market-clearing values of firms at time 1. Equilibrium prices set at time 1 must cause equation (63) to be met for each security.

The method used above to derive equations (83) and (87) allows us to develop explicit pricing equations for a two-parameter world where complete agreement among investors is not assumed. Although the pricing equations are similar in form to those obtained with complete agreement, the equations obtained when complete agreement is not assumed contain parameters that cannot be estimated from market data. We conclude that meaningful empirical tests of the two-parameter model probably must be based on models of market equilibrium that assume complete agreement.

When the complete agreement assumption is dropped, the statement of the decision problem facing the investor at time 1 is to a large extent unchanged. Each investor is assumed to behave as if the joint distribution of the market values of firms at time 2 were multivariate normal, but each investor assesses

his own values for the parameters of the distribution, and the assessed values of the parameters are different from one investor to another. The investor still behaves as if he were solving the maximization problem stated in (58) and (59), but now, instead of (56) and (57), the equations for the expected value and variance of his consumption at time 2 are

$$E_i \equiv E(\tilde{c}_{2i}) = \sum_{j=1}^{n} X_{ij} E_i(\tilde{V}_j) \tag{89}$$

$$\sigma_i^2 = \sigma^2(\tilde{c}_{2i}) = \sum_{j=1}^{n} \sum_{k=1}^{n} X_{ij} X_{ik} \operatorname{cov}_i(\tilde{V}_j, \tilde{V}_k); \tag{90}$$

that is, $E_i(\tilde{V}_j)$ and $\operatorname{cov}_i(\tilde{V}_j, \tilde{V}_k)$ now contain i subscripts to indicate that they are the assessments of investor i.

The conditions for a solution to the investor's expected utility maximization problem are again (59) to (61), but in (61) we substitute $E_i(\tilde{V}_j)$ and $\operatorname{cov}_i(\tilde{V}_j, \tilde{V}_k)$ for $E(\tilde{V}_j)$ and $\operatorname{cov}(\tilde{V}_j, \tilde{V}_k)$. Then, following the steps that lead from (61) to (64) and substituting (66) and (67) into (64), in the absence of complete agreement, the security demand equations for investor i are

$$-\frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_j) + 2 \sum_{k=1}^{n} X_{ik} \operatorname{cov}_i(\tilde{V}_j, \tilde{V}_k) + \frac{d\sigma_i^2}{dc_{1i}} P_j = 0, \quad j = 1, \ldots, n. \tag{91}$$

To get an expression for the value of firm j in a market equilibrium, we aggregate the demand equations of (91) across all investors i, $i = 1, \ldots, I$, and then solve for P_j as

$$P_j = \frac{\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_j)}{\delta} - \frac{2}{\delta} \sum_{k=1}^{n} \sum_{i=1}^{I} X_{ik} \operatorname{cov}_i(\tilde{V}_j, \tilde{V}_k), \tag{92}$$

where δ is as defined in (69). Equation (92) is similar in form to the pricing equation (70) obtained when complete agreement is assumed. The common assessed expected market value $E(\tilde{V}_j)$ of firm j at time 2 that appears in (70) is replaced in (92) by a weighted average of the expected values assessed by individual investors, with $E_i(\tilde{V}_j)$ weighted by investor i's marginal rate of substitution of expected consumption at time 2 for variance of consumption at time 2. Likewise, the sum of covariances between the market value of firm j at time 2 and the market values of all firms that appears in (70) is replaced by a weighted average of investor-assessed covariances, with investor i's assessment of $\operatorname{cov}_i(\tilde{V}_j, \tilde{V}_k)$, weighted by the fraction of firm k that he holds. The similarity of (70) and (92) is apparent when one notes that (92) reduces to (70) when complete agreement is assumed; that is, (70) can be obtained from (92) by dropping the i subscripts that appear on $E_i(\tilde{V}_j)$ and $\text{cov}_i(\tilde{V}_j, \tilde{V}_k)$ in (92) and then making use of (69) and (63).

The analogy between the pricing equations obtained with and without the complete agreement assumption can be pressed even further by developing (92) into an equation similar to (79). Since

$$\text{cov}_i(\tilde{V}_j, \tilde{c}_{2i}) = \text{cov}_i\left(\tilde{V}_j, \sum_{k=1}^{n} X_{ik}\tilde{V}_k\right) = \sum_{k=1}^{n} X_{ik}\,\text{cov}_i(\tilde{V}_j, \tilde{V}_k), \tag{93}$$

we can rewrite (92) as

$$P_j = \frac{\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_j)}{\delta} - \frac{2}{\delta}\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_j, \tilde{c}_{2i}). \tag{94}$$

If we sum (94) across firms and then solve the resulting expression for $2/\delta$, we get

$$\frac{2}{\delta} = \frac{\dfrac{\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_M)}{\delta} - P_M}{\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_M, \tilde{c}_{2i})}, \tag{95}$$

where $E_i(\tilde{V}_M)$ is investor i's assessment of expected market wealth at time 2, and $\text{cov}_i(\tilde{V}_M, \tilde{c}_{2i})$ is investor i's assessment of the covariance between market wealth and his consumption at time 2. Substituting (95) into (94), we get

$$P_j = \frac{1}{\delta}\left[\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_j) - \left(\frac{\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_M) - \delta P_M}{\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_M, \tilde{c}_{2i})}\right)\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_j, \tilde{c}_{2i})\right].$$

Multiplying and dividing through this expression by γ of (69) and then making use of the definition of θ in (78), we get

$$P_j = \frac{1}{\theta}\left[\frac{\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_j)}{\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i}} - \left(\frac{\dfrac{\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_M)}{\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i}} - \theta P_M}{\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_M, \tilde{c}_{2i})}\right)\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_j, \tilde{c}_{2i})\right]. \tag{96}$$

Since market wealth at time 2 is also aggregate consumption at time 2, that is, since

$$\tilde{V}_M = \sum_{j=1}^{n} \tilde{V}_j = \sum_{i=1}^{I} \tilde{c}_{2i}, \tag{97}$$

with the assumption of complete agreement we would have

$$\sum_{i=1}^{I} \text{cov}(\tilde{V}_M, \tilde{c}_{2i}) = \text{cov}\left(\tilde{V}_M, \sum_{i=1}^{I} \tilde{c}_{2i}\right) = \sigma^2(\tilde{V}_M) \tag{98}$$

$$\sum_{i=1}^{I} \text{cov}(\tilde{V}_j, \tilde{c}_{2i}) = \text{cov}\left(\tilde{V}_j, \sum_{i=1}^{I} \tilde{c}_{2i}\right) = \text{cov}(\tilde{V}_j, \tilde{V}_M). \tag{99}$$

Thus, with complete agreement, (96) reduces to (79), and this makes it clear that the pricing equations obtained with and without complete agreement again have the same form.

In the absence of complete agreement, however, one cannot reduce the sum of covariances

$$\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_M, \tilde{c}_{2i}) \tag{100}$$

to $\sigma^2(\tilde{V}_M)$ in the manner of (98). The i subscripts that appear on the covariances in (100) prevent us from interpreting this expression as anything but the sum across investors of each investor's perceived covariance between his consumption at time 2 and aggregate consumption or market wealth at time 2. Likewise, in the absence of complete agreement, the i subscripts on the individual covariances in

$$\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_j, \tilde{c}_{2i}) \tag{101}$$

prevent us from reducing this sum to $\text{cov}(\tilde{V}_j, \tilde{V}_M)$ in the manner of (99).

Nevertheless, it is possible to develop an interpretation of (96) which is similar to the interpretation of the pricing equations obtained when complete agreement is assumed. Thus, if we interpret $\text{cov}_i(\tilde{V}_M, \tilde{c}_{2i})$ as investor i's perception of the risk of his consumption at time 2 relative to aggregate consumption, then we can interpret the sum of covariances in (100) as aggregate risk, but keeping in mind that this aggregate risk is not perceived by any particular market participant. Likewise, since

$$\sum_{j=1}^{n}\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_j, \tilde{c}_{2i}) = \sum_{i=1}^{I} \text{cov}_i\left(\sum_{j=1}^{n} \tilde{V}_j, \tilde{c}_{2i}\right) = \sum_{i=1}^{I} \text{cov}_i(\tilde{V}_M, \tilde{c}_{2i}),$$

we can interpret the sum of covariances in (101) as the risk of firm j (although again this risk is not perceived by any particular investor) since it is the contribution of the securities of firm j to what we have called aggregate risk.

With these interpretations of the two sums of covariances in (96), we can then say that this pricing equation represents the market value of firm j at time 1 as $1/\theta$ multiplied by the risk-adjusted weighted average of investor expectations of the value of the firm at time 2. Moreover, we can interpret θ as 1 plus a weighted average of investor-expected returns on a firm that is "riskless" in the sense that its risk, as measured by the sum of covariances in (101), is zero, even though none of the individual terms of (101) need be equal to zero. Specifically, with such a firm, call it firm 0, we can determine from (96) that

$$\theta = \frac{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} \frac{E_i(\tilde{V}_0)}{P_0}}{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i}} = 1 + \frac{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{R}_0)}{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i}}.$$

Better yet, if there is borrowing and lending, and all investors perceive such borrowing and lending to be risk-free, so that there is complete agreement with respect to risk-free securities, then

$$\theta = \frac{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} \frac{V_F}{P_F}}{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i}} = \frac{V_F}{P_F} = 1 + R_F.$$

Substituting $1 + R_F$ for θ in (96), we get

$$P_j = \frac{1}{1+R_F} \times \left[\frac{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_j)}{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i}} - \left(\frac{\dfrac{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i} E_i(\tilde{V}_M)}{\displaystyle\sum_{i=1}^{I} \frac{d\sigma_i^2}{dE_i}} - (1+R_F)P_M}{\displaystyle\sum_{i=1}^{I} \text{cov}_i(\tilde{V}_M, \tilde{c}_{2i})} \right) \sum_{i=1}^{I} \text{cov}_i(\tilde{V}_j, \tilde{c}_{2i}) \right]. \quad (102)$$

It is clear that (102) is an equation for the market value of firm j at time 1 which is in the same form as equation (87), the pricing equation that is obtained when there is complete agreement and risk-free borrowing and lending.

IX. Conclusions

Having pricing equations which have the same form and similar interpretations as those obtained under the assumption of complete agreement does not mean, however, that empirical tests of the implications of the two-parameter model for the process of price formation in the capital market can be based on models in which complete agreement is not assumed. Empirical tests require that the quantities that appear in pricing equations be estimable from observable market data. The expected values and covariances that appear in (96) or (102) are investor assessments of parameters that vary from one investor to another. It is not even logical to talk about estimates of these assessments obtained from market data.

In contrast, with complete agreement the assessed values of the parameters $E(\tilde{V}_j)$, $\text{cov}(\tilde{V}_j, \tilde{V}_M)$, $E(\tilde{V}_M)$, and $\sigma^2(\tilde{V}_M)$ that appear in the pricing equations (83) and (87) are common to all investors, and they are presumed to be based on correct perception of the joint distribution of the market values of firms at time 2. The fact that assessments are assumed to be common to all investors and that investor perceptions are assumed to be correct allows us to go from theory to data. This is the subject of the next and last chapter.