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CHAPTER 7

The Two-Parameter Portfolio Model

I. Introduction

In Chapter 2 the two-parameter portfolio model was introduced. We now consider the model in more detail. We proceed in three steps. This chapter discusses portfolio decisions by investors in a two-parameter world. The model is credited to Markowitz (1952; 1959), who is rightfully regarded as the founder of modern portfolio theory. The next chapter considers the implications of the portfolio model for the pricing of securities. That is, if investors make portfolio decisions according to the two-parameter model, what does this imply about the way securities are priced in the capital market? In particular, how does the capital market view risk? In the setting of prices, what is the relationship between expected return and risk? Finally, Chapter 9 presents empirical tests of the expected return-risk relationships developed in Chapter 8.

In discussing the two-parameter portfolio model in this chapter, we first give a general treatment of its major features. We then discuss different aspects of the model in more detail.

II. Normal Distributions, Risk Aversion, and the Efficient Set

A. The Framework

In the most common version of the two-parameter portfolio model, and the only version that we treat here, it is assumed that at time $t = 1$ the investor has wealth w_1 which he must allocate to current consumption c_1 and to an investment $(w_1 - c_1)$ in some portfolio of securities. The value of his portfolio at time $t = 2$ provides his consumption c_2 for time 2. Thus, the investor consumes and invests at time 1, and at time 2 he consumes the market value of the investment made at time 1. Since only one period passes between time 1 and time 2, we call this a one-period model.

The investor's wealth w_1 is the market value of all his resources at time 1. It includes the market value of securities, real estate, and any other assets purchased in previous periods, along with any income from his labor that he receives at time 1. Since we are concerned with the portfolio decision at time 1, to simplify things we assume that the investor's wealth at time 2 derives completely from the market value of the portfolio that he chooses at time 1.

It is assumed that at time 1 an investor can purchase as much or as little of any security as he sees fit, and securities are assumed to be infinitely divisible. It is also assumed that there are no transactions costs (e.g., brokerage fees) in purchasing and selling securities and that any investor can buy or sell as much as he likes of any security without affecting its price. In short, investors are atomistic competitors in frictionless markets, a statement we summarize by saying that the capital market is perfect. Throughout the rest of the book we assume that the capital market is perfect in this sense.

The problem facing the investor at time 1 is to allocate his wealth w_1 to consumption c_1 and to an investment $(w_1 - c_1)$ in some portfolio in such a way as to maximize the satisfaction or welfare that he anticipates from consumption at time 1 and time 2. Consumption at time 2 is, however, a random variable at time 1, since the return from time 1 to time 2 on a portfolio chosen at time 1 is generally unknown at time 1. In formal terms, if the investor chooses current consumption c_1 and invests $(w_1 - c_1)$ in the portfolio p, his wealth and consumption at time 2 are

$$\tilde{c}_2 = \tilde{w}_2 = (w_1 - c_1)(1 + \tilde{R}_p), \qquad (1)$$

where the random variable $\tilde{R}_p$ is the return on the portfolio p from time 1 to time 2.* Thus, the investor's problem is to choose an optimal combination of

*Since we are discussing a one-period world in this chapter, it is unnecessary to include a time subscript on the portfolio return. As usual, tildes (~) are used to denote random variables.

current consumption c_1 and a probability distribution on $\tilde{c}_2$. Equivalently, he must choose a value of c_1 and a probability distribution of portfolio return.

B. The Simplifications Obtained When Portfolio Return Distributions Are Normal

It is assumed that the investor can rank all possible combinations of c_1 and probability distributions of $\tilde{c}_2$ according to the level of welfare he perceives that they provide. This assumption is so general, however, that it yields nothing in the way of propositions about observable behavior. We would like to simplify the decision problem so that it involves only a few potentially measurable parameters and yields some simple propositions about how the typical investor behaves with respect to these parameters.

One way to accomplish this goal is to assume that the joint distribution of security returns is multivariate normal, so that probability distributions of portfolio returns are normal (see Chapter 2). The property of normal distributions that simplifies the consumption-investment decision is the fact that any normal distribution can be completely described from knowledge of its mean and standard deviation.* Thus, if the distribution of the return $\tilde{R}_p$ on any portfolio p is normal with mean $E(\tilde{R}_p)$ and standard deviation $\sigma(\tilde{R}_p)$, then from (1) and the properties of the normal distribution we know that the probability distribution of consumption at time 2 obtained with portfolio p is normal with mean and standard deviation

$$E(\tilde{c}_2) = (w_1 - c_1)[1 + E(\tilde{R}_p)] \tag{2}$$

$$\sigma(\tilde{c}_2) = (w_1 - c_1)\,\sigma(\tilde{R}_p). \tag{3}$$

In short, when all portfolio return distributions are normal, knowledge of c_1, $E(\tilde{R}_p)$, and $\sigma(\tilde{R}_p)$ is sufficient to describe completely the combination of current consumption and probability distribution on future consumption associated with any choice of c_1 and portfolio p. If all portfolio return distributions are normal, the investor can rank the different combinations of c_1 and probability distribution on $\tilde{c}_2$ available in terms of the values of c_1, $E(\tilde{R}_p)$, and $\sigma(\tilde{R}_p)$ that they imply.

C. The Simplifications Obtained When Investors Are Risk-averse

The assumption that portfolio return distributions are normal reduces the consumption-investment decision to a three-dimensional problem involving

*At this point the reader may want to review the discussion of normal distributions in Chapter 1.

the choice of c_1, $E(\tilde{R}_p)$, and $\sigma(\tilde{R}_p)$. Additional simplifications can be obtained with assumptions about the investor's tastes and, in particular, about his attitudes toward $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$. We assume that for given values of c_1 and $\sigma(\tilde{R}_p)$, the investor prefers more expected portfolio return to less. From equations (2) and (3) we can see that this is equivalent to assuming that for given values of c_1 and $\sigma(\tilde{c}_2)$, the investor prefers more expected consumption at time 2 to less. Next we assume that the investor is risk-averse in the sense that for given values of c_1 and $E(\tilde{R}_p)$, he prefers less standard deviation of portfolio return to more. From equations (2) and (3) we can again see that this is equivalent to assuming that for given values of c_1 and $E(\tilde{c}_2)$, the investor prefers less standard deviation of consumption at time 2 to more. To characterize this assumption by saying that the investor is risk-averse is to assume that the risk of a portfolio can be measured by the standard deviation of its return. This is reasonable with normal portfolio return distributions, since the dispersion of the return distribution, and thus of the distribution of consumption at time 2, can be completely described in terms of the standard deviation of the return.

The assumptions that portfolio return distributions are normal and that the investor likes expected portfolio return but is averse to standard deviation of portfolio return imply the fundamental result of the two-parameter portfolio model. The investor's optimal portfolio is an efficient portfolio, where to be classified as efficient a portfolio must have the property that no other portfolio with the same or higher expected return has lower standard deviation of return.* The argument goes as follows. With normal portfolio return distributions, the consumption-investment decision reduces to choosing c_1, $E(\tilde{R}_p)$, and $\sigma(\tilde{R}_p)$. If the investor's tastes are such that conditional on any given values of c_1 and $\sigma(\tilde{R}_p)$ more expected return is preferred to less, then among all portfolios with a given value of $\sigma(\tilde{R}_p)$ the most preferred portfolio is the one with the largest value of $E(\tilde{R}_p)$. On the other hand, if for any given values of c_1 and $E(\tilde{R}_p)$ less standard deviation of return is preferred to more, then among all portfolios with a given $E(\tilde{R}_p)$ the most preferred portfolio is the one with the smallest value of $\sigma(\tilde{R}_p)$.

Taken together, these two statements say that with normal return distributions, a risk-averse investor only considers a portfolio if it has the largest possible expected return given its standard deviation of return, and if it has the smallest possible standard deviation of return given its expected return. A portfolio that has these two properties is called efficient, and the collection of portfolios that have these two properties is called the efficient set. Alterna-

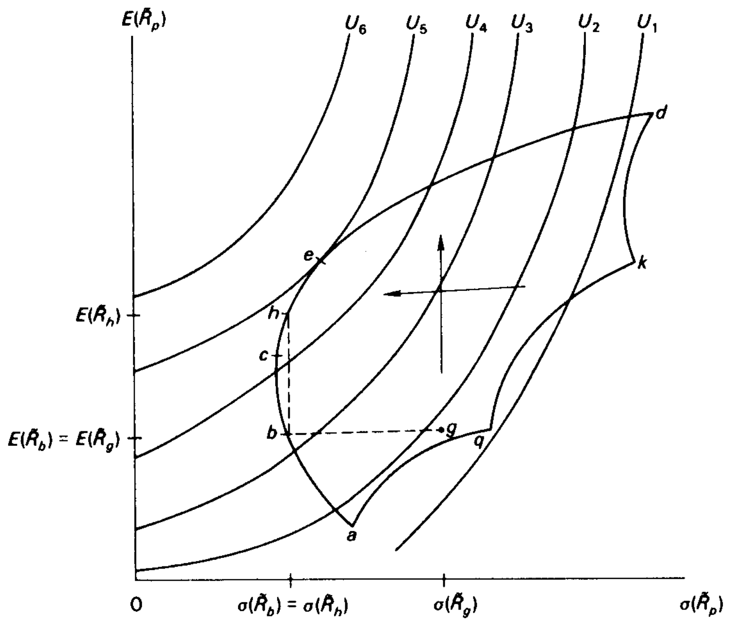
*The concept of portfolio efficiency should not be confused with the concept of market efficiency discussed in Chapters 5 and 6. The terminology, if a bit unfortunate, is standard.

tively, the two properties of an efficient portfolio can be summarized in terms of a single property which requires that for a portfolio to be efficient, there must be no other portfolio with the same or higher expected return that has lower standard deviation of return.

D. Geometric Interpretation

All of this has a convenient geometric interpretation. The curves U_1 to U_6 in Figure 7.1 give a possible representation of the investor's tastes for expected return and standard deviation of return, conditional on some value of

FIGURE 7.1
The Optimal Decision



current consumption c_1. Each of the curves represents combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ among which the investor is indifferent; no combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ along a curve is preferred to any other combination on the curve. The assumptions that the investor likes expected return but is averse to standard deviation of return imply that any such indifference curve must be positively sloped. If the investor regards two portfolios as equivalent in the sense that neither is preferred to the other, the portfolio with the higher standard deviation of return must also have higher expected return. Moreover,

with our assumptions about the investor's tastes, higher indifference curves imply more preferred combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$. Or, as suggested by the arrows in Figure 7.1, the level of the investor's satisfaction increases as we consider combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ further upward and/or further to the left in the $E(\tilde{R}_p)$, $\sigma(\tilde{R}_p)$ plane.

Having given a geometric interpretation of the investor's tastes, we now turn to the other ingredient in his decision problem, his portfolio opportunities, the feasible or attainable combinations of expected return and standard deviation of return. We spend considerable space in the rest of this chapter discussing the shapes that the opportunity set can take when plotted in the mean-standard deviation plane, but for the moment let us just assume that the possible combinations of expected return and standard deviation of return are on or within the rather irregularly shaped curve *acdkq* in Figure 7.1. All points on or within the curve are combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ attainable with some portfolio, but combinations outside the curve are not attainable.

Given our assumptions about tastes, however, only a small part of the investment opportunity set is relevant. For example, because the investor dislikes standard deviation of return, the portfolio *b* is strictly preferred to the portfolio *g*, since *g* has the same expected return as *b* but higher standard deviation of return. Indeed, the investor's aversion to standard deviation of portfolio return allows us to rule out immediately all portfolios except those along the left boundary of the opportunity set. However, many of the points along the left boundary can also be ruled out as a consequence of the fact that the investor likes expected return. Thus, the portfolio *b* is dominated by the portfolio *h*, which has the same standard deviation of return as *b* but higher expected return.

In short, the assumptions that the investor likes expected return but is averse to standard deviation of return imply that among the investment opportunities shown in Figure 7.1, the only portfolios the investor might possibly choose are those along the positively sloped segment *cd* of the left boundary of the opportunity set. Each of the portfolios along this segment has the efficiency property that no other portfolio with the same or higher expected return has lower standard deviation of return, and only portfolios along *cd* have this property. Thus the segment *cd* is the efficient set of portfolios. When we later consider the geometry of the efficient set in more detail, we find that the efficient set of portfolios is always the positively sloped segment of the upper left boundary of the opportunity set.

Given the representation of the investor's tastes and portfolio opportunities in Figure 7.1, the optimal efficient portfolio is *e*, which is the portfolio that puts the investor on the highest attainable indifference curve. Recall, however, that the indifference curves in Figure 7.1 represent the investor's tastes for $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ conditional on some assumed value of c_1. The consumption-investment decision, after all, involves choice of an optimal combination of c_1, $E(\tilde{R}_p)$, and $\sigma(\tilde{R}_p)$, and the details of the investor's tastes for $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ need not be the same for different assumed values of c_1. In terms of Figure 7.1, his indifference curves may be different depending on the value of c_1, and the portfolio that is optimal for one assumed value of c_1 is not necessarily optimal for other values of c_1. Thus, if we wish to think about the solution to the consumption-investment problem in geometric terms, we must, in the general manner of Figure 7.1, determine the optimal portfolio for each possible value of c_1 and then choose the *optimum optimorum*—that is, the value of current consumption c_1 and the associated optimal portfolio that maximizes the investor's welfare.

For our purposes, however, the important result, the one on which we build in this and the following chapters, is that in the assumed framework, the optimum portfolio for any and thus for the optimum choice of c_1 is an efficient portfolio. The next chapter considers the characteristics of market equilibrium in a world where investors make portfolio decisions according to the two-parameter model. It turns out that the measurement of risk and especially the relationships between equilibrium expected returns and risk that are relevant in a two-parameter world are direct implications of the fact that in such a world investors hold efficient portfolios. Chapter 9 then considers whether observed relationships between average returns and risk are in accordance with the predictions of the two-parameter model. These tests can be interpreted as asking whether the prices of securities are in line with the hypothesis that investors hold or attempt to hold efficient portfolios. In short, by the time discussion of the two-parameter model is finished, the dominant role played by the concept of portfolio efficiency and by the characteristics of efficient portfolios will be evident.

Finally, the preceding is an intuitive and rather general discussion of how the tastes of a risk-averse investor combine with the characteristics of his portfolio opportunities to yield the conclusion that the optimal portfolio must be efficient. The characteristics of portfolio opportunities in a world where portfolio return distributions are normal are the subject matter of the rest of this chapter. The geometry of the efficient set is developed, and then risk and the effects of diversification on portfolio risk are discussed.

III. The Geometry of the Efficient Set

We first consider the geometric properties of portfolios that are combinations of two securities or portfolios. The geometric properties of the efficient set of portfolios follow almost directly. The reader is assumed to be familiar with the material of Chapter 2.

A. *The Geometry of Combinations of Two Securities or Portfolios*

Let q and s be two securities or portfolios. For later purposes, it is important to keep in mind that either or both q and s can be portfolios. Consider portfolios that are combinations of q and s with the proportion x of portfolio funds $w_1 - c_1$ invested in q and $(1 - x)$ invested in s. The return, expected return, and standard deviation of return for such portfolios are

$$\tilde{R}_p = x\tilde{R}_q + (1 - x)\tilde{R}_s \tag{4}$$

$$E(\tilde{R}_p) = xE(\tilde{R}_q) + (1 - x)E(\tilde{R}_s) \tag{5}$$

$$\sigma(\tilde{R}_p) = [x^2\sigma^2(\tilde{R}_q) + (1 - x)^2\sigma^2(\tilde{R}_s) + 2x(1 - x)\operatorname{cov}(\tilde{R}_q, \tilde{R}_s)]^{1/2}. \tag{6}$$

For current purposes, it is more convenient to express $\sigma(\tilde{R}_p)$ in terms of $\operatorname{corr}(\tilde{R}_q, \tilde{R}_s)$, the correlation coefficient between $\tilde{R}_q$ and $\tilde{R}_s$, than in terms of the covariance $\operatorname{cov}(\tilde{R}_q, \tilde{R}_s)$. Thus, since

$$\operatorname{corr}(\tilde{R}_q, \tilde{R}_s) = \frac{\operatorname{cov}(\tilde{R}_q, \tilde{R}_s)}{\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s)}, \tag{7}$$

equation (6) can be rewritten as

$$\sigma(\tilde{R}_p) = [x^2\sigma^2(\tilde{R}_q) + (1 - x)^2\sigma^2(\tilde{R}_s) + 2x(1 - x)\operatorname{corr}(\tilde{R}_q, \tilde{R}_s)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s)]^{1/2}. \tag{8}$$

POSITIVELY WEIGHTED COMBINATIONS ($0 \leq x \leq 1$)

Consider now the extreme and unrealistic case where the correlation coefficient between $\tilde{R}_q$ and $\tilde{R}_s$ is 1.0; that is, there is a perfect positive correlation between $\tilde{R}_q$ and $\tilde{R}_s$, so that the value of $\tilde{R}_q$ is perfectly predictable from the value of $\tilde{R}_s$ (and vice versa). Then (8) becomes

$$\sigma(\tilde{R}_p) = [x^2\sigma^2(\tilde{R}_q) + (1 - x)^2\sigma^2(\tilde{R}_s) + 2x(1 - x)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s)]^{1/2} \tag{9}$$

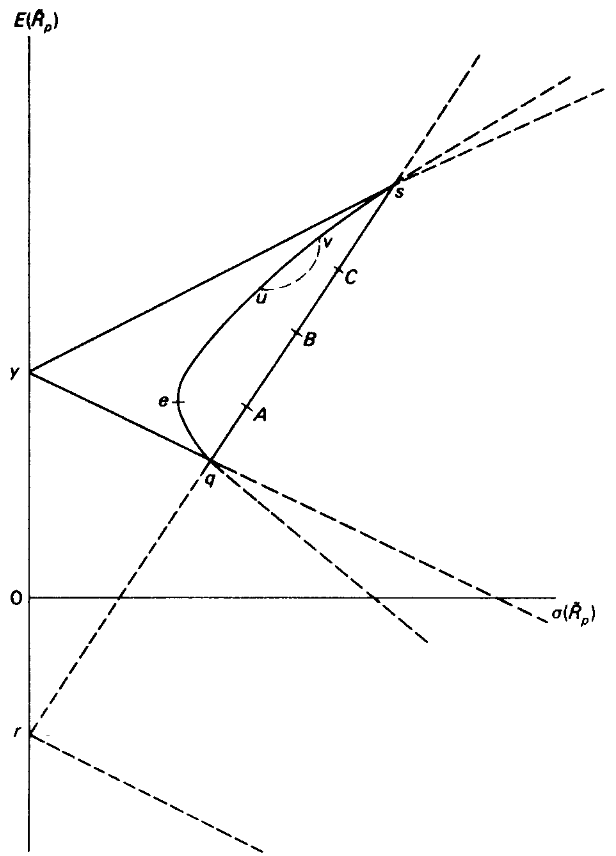
$$= |x\sigma(\tilde{R}_q) + (1 - x)\,\sigma(\tilde{R}_s)|, \quad \operatorname{corr}(\tilde{R}_q, \tilde{R}_s) = 1.0. \tag{10}$$

The absolute value sign is necessary to ensure that we take the positive root in computing the standard deviation from the variance. Standard deviations must be nonnegative. For the moment, however, we consider only nonnegative values of x and $(1 - x)$. Since $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$ are nonnegative, when x and $(1 - x)$ are nonnegative, the absolute value sign in (10) is unnecessary, and $\sigma(\tilde{R}_p)$ is just the weighted average of the component standard deviations, $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$. Since $E(\tilde{R}_p)$ is always just the weighted average of the component expected returns, $E(\tilde{R}_q)$ and $E(\tilde{R}_s)$, in the case of perfect positive correlation, both $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ are weighted averages of the expected returns and standard deviations of $\tilde{R}_q$ and $\tilde{R}_s$.

This result has a simple geometric interpretation. Suppose the expected values and standard deviations of $\tilde{R}_q$ and $\tilde{R}_s$ are as shown in Figure 7.2. In terms of equations (5) and (10), point q in Figure 7.2 corresponds to $x = 1.0$

FIGURE 7.2

Geometry of Combinations of Two Securities or Portfolios



and $(1 - x) = 0.0$, so that all portfolio funds are invested in q; point s corresponds to $x = 0.0$, and $(1 - x) = 1.0$, so that all funds are invested in s. With perfect positive correlation between $\tilde{R}_q$ and $\tilde{R}_s$, any value of x between 1.0 and 0.0 generates a combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ along the straight line between q and s. For example, point B in Figure 7.2 represents the combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained with $x = .5$ and $(1 - x) = .5$, that is, when portfolio funds are divided equally between q and s. In this case

$$E(\tilde{R}_B) = .5E(\tilde{R}_q) + .5E(\tilde{R}_s)$$

is just halfway between $E(\tilde{R}_q)$ and $E(\tilde{R}_s)$. From equation (10),

$$\sigma(\tilde{R}_B) = .5\sigma(\tilde{R}_q) + .5\sigma(\tilde{R}_s)$$

is likewise halfway between $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$. Point A in Figure 7.2 represents the combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained with $x = .75$ and $1 - x = .25$, while C is the portfolio corresponding to $x = .25$ and $1 - x = .75$. Point A is along the straight line between q and s, and it is just one-quarter of the way between q and s. Likewise, portfolio C is three-quarters of the way between q and s along the straight line between q and s.

Suppose now that the expected values and standard deviations of $\tilde{R}_q$ and $\tilde{R}_s$ are as shown in Figure 7.2 but that there is less than perfect positive correlation between the two returns. What can we say about the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained from portfolios of q and s formed in the manner of equation (4)? First of all, the correlation between $\tilde{R}_q$ and $\tilde{R}_s$ has no effect on $E(\tilde{R}_p)$. The expected return on any portfolio of q and s is always just the appropriate weighted average of $E(\tilde{R}_q)$ and $E(\tilde{R}_s)$. From (8), however, we can see that the correlation between $\tilde{R}_q$ and $\tilde{R}_s$ affects the standard deviation of the return on a portfolio of q and s. For given $x(0 < x < 1)$ and given values of $\sigma^2(\tilde{R}_q)$ and $\sigma^2(\tilde{R}_s)$, the largest possible value of $\sigma(\tilde{R}_p)$ occurs when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = 1.0$, in which case (10) holds. With less than perfect positive correlation, $\sigma(\tilde{R}_p)$ is less than the quantity given by (10),

$$\sigma(\tilde{R}_p) < x\sigma(\tilde{R}_q) + (1 - x)\sigma(\tilde{R}_s), \quad 0 < x < 1, \quad \text{corr}(\tilde{R}_q, \tilde{R}_s) < 1.0, \quad (11)$$

and for given x and given values of $\sigma^2(\tilde{R}_q)$ and $\sigma^2(\tilde{R}_s)$, $\sigma(\tilde{R}_p)$ is smaller the lower the correlation between $\tilde{R}_q$ and $\tilde{R}_s$.

These results have an important implication. Equation (10) says that when the correlation between the returns on two assets or portfolios like q and s is 1.0, diversification is ineffective in reducing dispersion: the standard deviation of the return on a portfolio which is just a weighted combination of q and s is the corresponding weighted average of $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$. On the other hand, (11) says that when $\text{corr}(\tilde{R}_q, \tilde{R}_s) < 1.0$, $\sigma(\tilde{R}_p)$ is less than the weighted average of $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$, and $\sigma(\tilde{R}_p)$ decreases as $\text{corr}(\tilde{R}_q, \tilde{R}_s)$ decreases. Thus, when there is less than perfect positive correlation between the returns on two assets or portfolios q and s, diversification is an effective way to reduce dispersion, and it is more effective the further corr $(\tilde{R}_q, \tilde{R}_s)$ is from 1.0.

The opposite extreme from perfect positive correlation is perfect negative correlation, corr $(\tilde{R}_q, \tilde{R}_s) = -1.0$; for given $x(0 < x < 1)$ and given values of $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$, perfect negative correlation gives the smallest possible value of $\sigma(\tilde{R}_p)$. With perfect negative correlation, equation (8) becomes

$$\sigma(\tilde{R}_p) = [x^2 \sigma^2(\tilde{R}_q) + (1-x)^2 \sigma^2(\tilde{R}_s) - 2x(1-x)\sigma(\tilde{R}_q)\sigma(\tilde{R}_s)]^{1/2}$$
$$= |x\sigma(\tilde{R}_q) - (1-x)\sigma(\tilde{R}_s)|, \quad \text{corr}\,(\tilde{R}_q, \tilde{R}_s) = -1.0. \tag{12}$$

One consequence of perfect negative correlation is that there is a portfolio of q and s, with both x and $(1-x)$ strictly positive, that has zero standard deviation of return. From (12) we determine that $\sigma(\tilde{R}_p) = 0.0$ when

$$x = \frac{\sigma(\tilde{R}_s)}{\sigma(\tilde{R}_q) + \sigma(\tilde{R}_s)}. \tag{13}$$

Moreover, (12) also implies that when corr $(\tilde{R}_q, \tilde{R}_s) = -1.0$,

$$\sigma(\tilde{R}_p) = x\sigma(\tilde{R}_q) - (1-x)\sigma(\tilde{R}_s), \quad \text{when } x > \frac{\sigma(\tilde{R}_s)}{\sigma(\tilde{R}_q) + \sigma(\tilde{R}_s)}, \tag{14}$$

$$\sigma(\tilde{R}_p) = (1-x)\sigma(\tilde{R}_s) - x\sigma(\tilde{R}_q), \quad \text{when } x < \frac{\sigma(\tilde{R}_s)}{\sigma(\tilde{R}_q) + \sigma(\tilde{R}_s)}. \tag{15}$$

In geometric terms, when $x = 1.0$ we are at point q in Figure 7.2. For smaller values of x, the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained are at first described by (5) and (14). Since these equations are both linear in x, we move away from q along a straight line that touches the $E(\tilde{R}_p)$ axis at y when x takes the value given by (13). For still lower values of x, the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained are given by (5) and (15), so that we move along another straight line away from y in the direction of the point s in Figure 7.2, which is reached when $x = 0.0$. Thus when corr $(\tilde{R}_q, \tilde{R}_s) = -1.0$, the two line segments that meet at y on the $E(\tilde{R})$ axis in Figure 7.2 describe the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained by varying x in (4).

For given $x(0 < x < 1)$ and given values of $\sigma^2(\tilde{R}_q)$ and $\sigma^2(\tilde{R}_s)$, the maximum possible value of $\sigma(\tilde{R}_p)$ occurs when $\tilde{R}_q$ and $\tilde{R}_s$ are perfectly positively correlated, while the minimum possible value of $\sigma(\tilde{R}_p)$ occurs when $\tilde{R}_q$ and $\tilde{R}_s$ are perfectly negatively correlated. When $-1 <$ corr $(\tilde{R}_q, \tilde{R}_s) < 1$, the value of $\sigma(\tilde{R}_p)$ is greater than the value obtained when corr $(\tilde{R}_q, \tilde{R}_s) = -1.0$ and less than the value obtained when corr $(\tilde{R}_q, \tilde{R}_s) = 1.0$. Thus when $-1 <$ corr $(\tilde{R}_q, \tilde{R}_s) < 1$, the plot of the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained by varying x between 1.0 and 0.0 in (4) is a curve that starts at point q in Figure 7.2 and ends at s. The curve must be to the left of the line

between q and s, since this line describes the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained when $\tilde{R}_q$ and $\tilde{R}_s$ are perfectly positively correlated. The curve must also be to the right of the two line segments that describe the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained with perfect negative correlation between $\tilde{R}_q$ and $\tilde{R}_s$. In short, with less than perfect correlation, the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained from portfolios of q and s must lie along a curve somewhat like qes in Figure 7.2.

In fact, the curve traced by portfolios of q and s must have the specific properties exhibited by the curve qes in Figure 7.2. Any positively sloped segment of the curve (like es) must be concave, and any negatively sloped segment (like qe) must be convex.* To establish these properties, suppose first that they do not hold. Thus, suppose that the positively sloped segment of the curve in Figure 7.2 has a convex section like the dashed curve between points u and v. Since u and v are themselves portfolios of q and s, any portfolio of q and s that gives a combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ along the dashed curve between u and v can be expressed as a portfolio of u and v. But the arguments of the preceding paragraphs imply that portfolios of u and v must plot either along a straight line between u and v (if u and v are perfectly positively correlated) or to the left of the line between u and v (if u and v are less than perfectly positively correlated). Thus, the dashed curve between u and v cannot represent portfolios of u and v, so it cannot represent portfolios of q and s. Successive application of these arguments leads to the conclusion that any positively sloped segment of the curve generated by portfolios of q and s must be concave, while any negatively sloped segment must be convex.

PROBLEM III.A

1. Show that any portfolio of q and s that yields a point between u and v in Figure 7.2 can be expressed as a portfolio of u and v.

ANSWER

1. Let x_u and x_v be the proportions of portfolio funds invested in q to form the portfolios u and v, so that $1 - x_u$ and $1 - x_v$ are the proportions invested in s. Consider any third portfolio k of q and s such that

$$x_u > x_k > x_v.$$

*A curve or a segment of a curve is concave if a straight line between any two points is everywhere on or below the curve. A curve or a segment of a curve is convex if a straight line between any two points is everywhere on or above the curve. With strict concavity a straight line between any two points lies below the curve, while with strict convexity a straight line between any two points is above the curve.

There is a value of y, $0 < y < 1$, such that

$$x_k = yx_u + (1 - y)x_v.$$

Thus, the return on portfolio k can be expressed as

$$\begin{aligned}\tilde{R}_k &= x_k\tilde{R}_q + (1 - x_k)\tilde{R}_s \\ &= [yx_u + (1 - y)x_v]\tilde{R}_q + \{1 - [yx_u + (1 - y)x_v]\}\tilde{R}_s \\ &= y[x_u\tilde{R}_q + (1 - x_u)\tilde{R}_s] + (1 - y)[x_v\tilde{R}_q + (1 - x_v)\tilde{R}_s] \\ &= y\tilde{R}_u + (1 - y)\tilde{R}_v.\end{aligned}$$

THE ANALYSIS OF SHORT-SELLING

In the preceding discussion, we assumed that x, the proportion of portfolio funds invested in q, was between 0.0 and 1.0, so that $(1 - x)$, the proportion invested in s, was also nonnegative. Suppose, however, that the investor can issue as well as purchase securities. For example, suppose q is the common stock of firm q. At time 1 we allow the investor to issue securities equivalent to the shares of firm q. He might do this by selling a promise to pay at time 2 whatever is the market value of a share of firm q at time 2 plus any dividends paid by the firm at time 2. If the market believes that the investor can deliver on this promise, at time 1 it will pay him the price of a share in firm q for every share that he issues. He can then use the proceeds from the securities he issues to acquire an investment in s in excess of $(w_1 - c_1)$, his own initial portfolio funds.

The mechanism whereby an investor issues a security equivalent to one already existing is a short sale. To sell short the shares of firm q, the investor borrows the shares from someone who owns them at time 1, agreeing to return the shares at time 2 along with any dividends paid at time 2. On borrowing the shares, the investor immediately sells them in the market. At this point he has issued shares in firm q, since both the lender of the shares and the person who purchases them from the short-seller receive returns at time 2 from ownership of the shares. At time 2 the investor pays his debt to the lender of the shares of firm q by repurchasing in the market the shares of q to be returned to the lender.

When he borrows the shares of firm q and sells them in the market at time 1, the investor is said to have a negative or "short" position in the shares. He "covers" his short position when he purchases shares of q at time 2 and returns them to the lender. In contrast, an investor who owns the shares of firm q has a positive or "long" position in the shares. Finally, just as an investor uses a short sale to issue a security equivalent to one that is already outstanding, we can also think of him as short-selling q or s when they are portfolios rather than individual securities. To sell short or "issue" a portfolio, the investor simply sells short each of its component securities in the appropriate proportions.

We now extend the geometric analysis of combinations of two securities or portfolios to allow for short-selling. Note first that if we always consider x to be the proportion of $w_1 - c_1$ invested in q and $(1 - x)$ to be the proportion of $w_1 - c_1$ invested in s, then (4) is always the relevant expression for the return on a portfolio of q and s, and (5) and (6) or (8) are the relevant expressions for the expected value and standard deviation of the portfolio's return.* If x and $(1 - x)$ are both between 0.0 and 1.0, positive amounts are invested in both q and s. A negative value of x implies that q is issued or sold short and the proceeds are used to get an investment in s in excess of $w_1 - c_1$. Likewise, a negative value of $(1 - x)$—and thus a value of x greater than 1.0—implies that s is issued or sold short and the proceeds used to obtain an investment in q greater than $w_1 - c_1$.

Consider now the case where the returns on q and s are perfectly negatively correlated. When corr $(\tilde{R}_q, \tilde{R}_s) = -1.0$, the expected return on a portfolio of q and s is given by (5), and equation (8) for the standard deviation of the portfolio return becomes either (14) or (15), depending on whether x, the proportion of portfolio funds invested in q, is greater or less than the strictly positive quantity given by equation (13). If x is negative—that is, if the portfolio involves issuing or short-selling of q—then (15) is relevant and in Figure 7.2 the combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained with such a portfolio is somewhere beyond the point s on the dashed extension of the line from the vertical axis through s. On the other hand, if the portfolio involves short-selling of s, so that $(1 - x) < 0$ and $x > 1$, then (14) is relevant and the portfolio plots along the dashed extension of the line from the vertical axis down through point q. Thus, when corr $(\tilde{R}_q, \tilde{R}_s) = -1.0$, all we must do in Figure 7.2 to cover the possibility of short-selling is to extend the two relevant line segments that meet at y on the $E(\tilde{R})$ axis through the points corresponding to q and s.

Consider next the case where there is perfect positive correlation between $\tilde{R}_q$ and $\tilde{R}_s$. As in the case where $0 \leq x \leq 1$, the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ given by portfolios of q and s where either q or s is sold short are described by equations (5) and (10). Moreover, if as in Figure 7.2, $E(\tilde{R}_s) > E(\tilde{R}_q)$ and $\sigma(\tilde{R}_s) > \sigma(\tilde{R}_q)$, the absolute value sign is unnecessary for $x < 0$. Thus, when q is sold short, the combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained is

*If this point is not obvious, the reader should review the development of the algebra of portfolio theory in Chapter 2. In particular, the development of expressions for the return, expected return, and standard deviation of return on a portfolio does not impose nonnegativity constraints on the proportions of portfolio funds invested in individual securities.

along the dashed extension of the straight line from q through s in Figure 7.2. On the other hand, when s is sold short, so that $x > 1$ and $(1 - x) < 0$, the absolute value sign in (10) becomes necessary when x exceeds the quantity

$$x = \frac{\sigma(\tilde{R}_s)}{\sigma(\tilde{R}_s) - \sigma(\tilde{R}_q)}, \quad \sigma(\tilde{R}_s) > \sigma(\tilde{R}_q), \tag{16}$$

and when x takes the value given in (16), $\sigma(\tilde{R}_p) = 0.0$. Thus, when s is sold short, the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained from portfolios of q and s plot at first along the dashed extension of the straight line from s through q in Figure 7.2. This line hits the $E(\tilde{R})$ axis at r when x takes the value given by (16). For values of x in excess of the quantity given by (16), the absolute value sign in (10) becomes relevant, and the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained with portfolios of q and s plot along the new dashed line that goes downward and to the right from r.

Note that for portfolios that involve short sales of either q or s, the standard deviation of the return on a portfolio of q and s is larger when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = -1.0$ than when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = 1.0$. In geometric terms, when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = -1.0$, the dashed lines in Figure 7.2 generated by portfolios that involve short sales of q or s are to the right of the corresponding dashed lines generated when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = 1.0$. This is, of course, the reverse of what is obtained when the proportions x and $(1 - x)$ invested in q and s are both positive. In that case, for given x, the largest possible value of $\sigma(\tilde{R}_p)$ occurs when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = 1.0$, while the smallest possible value occurs when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = -1.0$.

The reversal in the roles of perfect positive and negative correlation that arises with short-selling is easily explained. If two securities or portfolios are perfectly positively correlated, the return on a short position in one is perfectly negatively correlated with the return on a long position in the other. Likewise, a short sale of either q or s transforms perfect negative correlation into perfect positive correlation. In general, the correlation between the returns on a short and a long position in two securities or portfolios is the negative of the correlation between the returns obtained when both are held long.

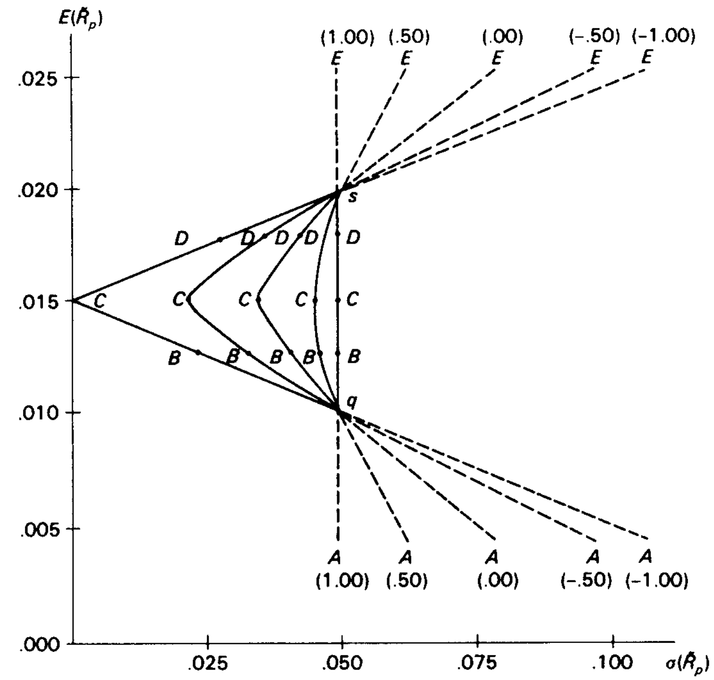
With less than perfect correlation, that is, when $-1.0 < \text{corr}(\tilde{R}_q, \tilde{R}_s) < 1.0$, then the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained when q is sold short are along the dashed extension of the solid curve through the point s in Figure 7.2, while when s is sold short, we get points along the dashed extension of the curve through q. Without going into the details, which at this point the reader should be able to provide, the dashed extensions of the curve must lie between the dashed extensions of the lines obtained with perfect positive and negative correlation. Again, any positively sloped segment of the curve must be concave, while any negatively sloped segment must be convex.

SOME NUMERICAL EXAMPLES

The preceding analysis can be made more concrete with numerical examples. We first consider portfolios of q and s when $E(\tilde{R}_q) = .01$, $E(\tilde{R}_s) = .02$, and $\sigma(\tilde{R}_q) = \sigma(\tilde{R}_s) = .05$. Plots of $E(\tilde{R}_p)$ against $\sigma(\tilde{R}_p)$ under five different assumptions about the value of $\text{corr}(\tilde{R}_q, \tilde{R}_s)$ are in Figure 7.3. The straight

FIGURE 7.3

Plots of $E(\tilde{R}_p)$ Against $\sigma(\tilde{R}_p)$ for Different Values of $\text{corr}(\tilde{R}_q, \tilde{R}_s)$ When $E(\tilde{R}_s) > E(\tilde{R}_q)$ and $\sigma(\tilde{R}_q) = \sigma(\tilde{R}_s)$



line between q and s shows the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained for values of x between 1.0 and 0.0 when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = 1.0$. The first solid curve to the left of this line shows the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained for values of x between 1.0 and 0.0 when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = .5$; the second solid curve to the left applies to the case $\text{corr}(\tilde{R}_q, R_s) = 0.0$ (that is, $\tilde{R}_q$ and $\tilde{R}_s$ are uncorrelated); the third curve applies to the case $\text{corr}(\tilde{R}_q, \tilde{R}_s) = -.5$; and the two straight lines from q and s that meet on the vertical axis show the combinations of $E(\tilde{R}_q)$ and $\sigma(\tilde{R}_p)$ obtained when $\text{corr}(\tilde{R}_q, \tilde{R}_s) = -1.0$.

Points labeled with a given uppercase letter in Figure 7.3 show the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained for fixed values of the proportions invested in q and s, but with different assumptions about the value of corr $(\tilde{R}_q, \tilde{R}_s)$. For example, in portfolio B the proportion of $w_1 - c_1$ invested in q is .75, and the proportion invested in s is .25. The expected return on the portfolio is

$$E(\tilde{R}_B) = .75E(\tilde{R}_q) + .25E(\tilde{R}_s) = .75(.01) + .25(.02) = .0125,$$

and $E(\tilde{R}_B)$ does not depend on corr $(\tilde{R}_q, \tilde{R}_s)$. However, Figure 7.3 illustrates that $\sigma(\tilde{R}_B)$ decreases as corr $(\tilde{R}_q, \tilde{R}_s)$ decreases. The same phenomenon is observed in the portfolios labeled C, where the proportions invested in q and s are always equal, and in the portfolios labeled D, where the proportions invested in q and s are always .25 and .75.

The dashed extensions of the various lines and curves in Figure 7.3 show the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained when either q is sold short (giving the dashed lines and curves above s) or s is sold short (giving the dashed lines and curves below q). Thus, the points labeled E give the combinations $E(\tilde{R}_E)$ and $\sigma(\tilde{R}_E)$, corresponding to

$$\tilde{R}_E = -.5\tilde{R}_q + 1.5\tilde{R}_s,$$

while the points labeled A give the combinations of $E(\tilde{R}_A)$ and $\sigma(\tilde{R}_A)$, corresponding to

$$\tilde{R}_A = 1.5\tilde{R}_q - .5\tilde{R}_s.$$

The values of corr $(\tilde{R}_q, \tilde{R}_s)$ are shown above the points labeled E in Figure 7.3 and below the points labeled A. We can see that when either q or s is sold short, the relationship between $\sigma(\tilde{R}_p)$ and corr$(\tilde{R}_q, \tilde{R}_s)$ is reversed; the higher the value of corr $(\tilde{R}_q, \tilde{R}_s)$, the lower the value of $\sigma(\tilde{R}_p)$.

Although a little elementary calculus is required, it is easy enough to show (and Problem III.A.3 below asks the reader to do so) that the minimum value of $\sigma(\tilde{R}_p)$ for portfolios of q and s occurs when x, the proportion of portfolio funds invested in q, is

$$x = \frac{\sigma^2(\tilde{R}_s) - \text{corr}\,(\tilde{R}_q, \tilde{R}_s)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s)}{\sigma^2(\tilde{R}_s) + \sigma^2(\tilde{R}_q) - 2\,\text{corr}\,(\tilde{R}_q, \tilde{R}_s)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s)}. \tag{17}$$

Thus, in general, the value of x that minimizes $\sigma(\tilde{R}_p)$ varies with the magnitude of $\sigma^2(\tilde{R}_s)$ relative to $\sigma^2(\tilde{R}_q)$; in general, it also depends on the value of corr $(\tilde{R}_q, \tilde{R}_s)$. But $\sigma^2(\tilde{R}_q) = \sigma^2(\tilde{R}_s)$ is the special case where for all values of corr $(\tilde{R}_q, \tilde{R}_s)$ the minimum value of $\sigma(\tilde{R}_p)$ occurs when $x = .5$, that is, when portfolio funds are split equally between q and s. Thus, in Figure 7.3 portfolio C has the minimum value of $\sigma(\tilde{R}_p)$ for all values of corr $(\tilde{R}_q, \tilde{R}_s)$, but this is a peculiar implication of the fact that $\sigma^2(\tilde{R}_q) = \sigma^2(\tilde{R}_s)$.

PROBLEMS III.A

2. When will the value of x in (17) be between 0.0 and 1.0? The answer to this question defines the cases where it is possible to get a portfolio of q and s, with nonnegative proportions invested in both q and s, that has a standard deviation of return less than both $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$.

3. Show that equation (17) does indeed give the value of x that minimizes $\sigma(\tilde{R}_p)$ for portfolios of q and s.

ANSWERS

2. To get

$$x = \frac{\sigma^2(\tilde{R}_s) - \text{corr}\,(\tilde{R}_q, \tilde{R}_s)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s)}{\sigma^2(\tilde{R}_s) + \sigma^2(\tilde{R}_q) - 2\,\text{corr}\,(\tilde{R}_q, \tilde{R}_s)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s)} < 1,$$

we must have

$$\sigma^2(\tilde{R}_q) - \text{corr}\,(\tilde{R}_q, \tilde{R}_s)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s) > 0.$$

Likewise, to get $x > 0$ in (17), we must have

$$\sigma^2(\tilde{R}_s) - \text{corr}\,(\tilde{R}_q, \tilde{R}_s)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s) > 0.$$

These two conditions are always satisfied when corr $(\tilde{R}_q, \tilde{R}_s) \leq 0.0$, but for given values of $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$ there will also always be some range of positive values of corr $(\tilde{R}_q, \tilde{R}_s)$ for which the conditions are satisfied. Chapter 8 will provide an application for this result.

3. The value of x that minimizes $\sigma^2(\tilde{R}_p)$ also minimizes $\sigma(\tilde{R}_p)$. From (8), we have

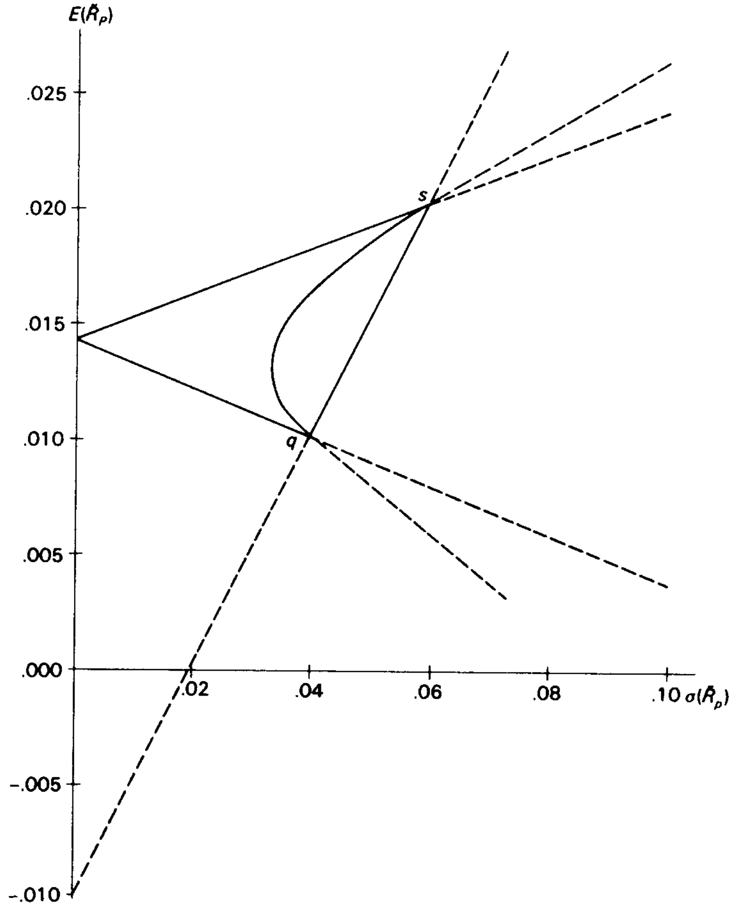
$$\frac{d\sigma^2(\tilde{R}_p)}{dx} = 2x\sigma^2(\tilde{R}_q) - 2(1-x)\,\sigma^2(\tilde{R}_s) + 2\,\text{corr}\,(\tilde{R}_q, \tilde{R}_s)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s) - 4x\,\text{corr}\,(\tilde{R}_q, \tilde{R}_s)\,\sigma(\tilde{R}_q)\,\sigma(\tilde{R}_s).$$

Setting this expression equal to 0.0 and solving for x yields (17). That (17) yields the value of x which minimizes $\sigma^2(\tilde{R}_p)$ follows from the fact (which we state without proof but which the reader can deduce from the lengthy geometric discussion above) that $\sigma^2(\tilde{R}_p)$ is a convex function of x.

The numerical example in Figure 7.3 is a special case in the sense that $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$ are assumed to be equal. Figure 7.4 shows plots of $E(\tilde{R}_p)$ against $\sigma(\tilde{R}_p)$ for three different assumed values of corr $(\tilde{R}_q, \tilde{R}_s)$ when $E(\tilde{R}_q) = .01$, $E(\tilde{R}_s) = .02$, $\sigma(\tilde{R}_q) = .04$, and $\sigma(\tilde{R}_s) = .06$. The straight line between q and s (and its dashed extensions) shows the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ ob-

FIGURE 7.4

Plots of $E(\tilde{R}_p)$ Against $\sigma(\tilde{R}_p)$ When $E(\tilde{R}_q) < E(\tilde{R}_s)$ and $\sigma(\tilde{R}_q) < \sigma(\tilde{R}_s)$

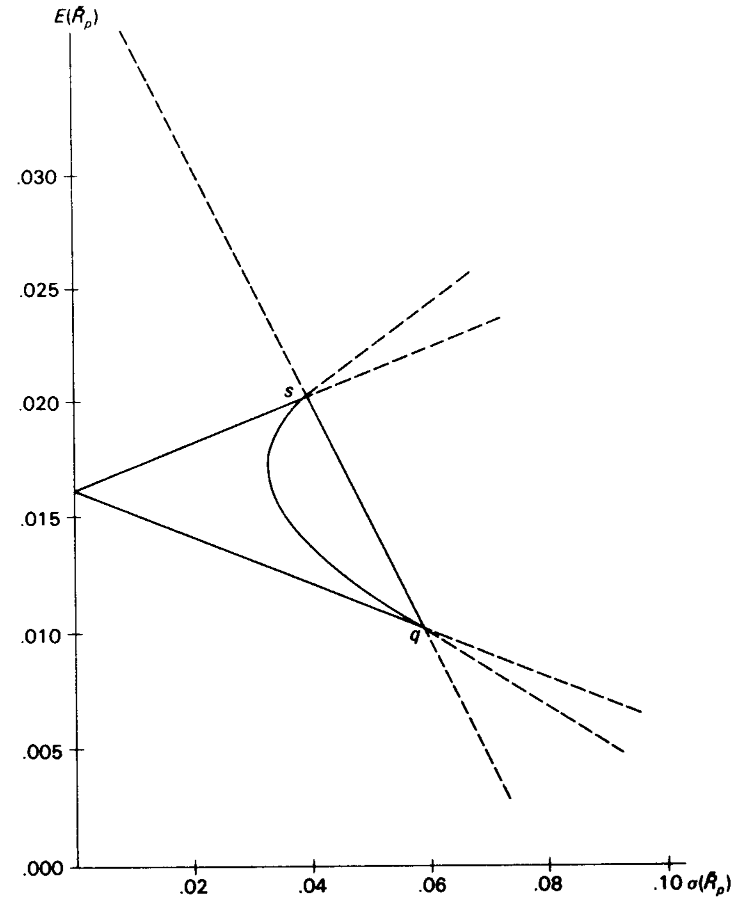


tained when there is perfect positive correlation between $\tilde{R}_q$ and $\tilde{R}_s$. The two lines from q and s that meet on the $E(\tilde{R})$ axis apply when corr $(\tilde{R}_q, \tilde{R}_s) = -1.0$, while the curve in the figure shows the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained when $\tilde{R}_q$ and $\tilde{R}_s$ are uncorrelated. Figure 7.5 shows still a different example in which, as in Figure 7.4, $E(\tilde{R}_q) = .01$ and $E(\tilde{R}_s) = .02$; but in Figure 7.5 the values of $\sigma(\tilde{R}_q)$ and $\sigma(\tilde{R}_s)$ are the reverse of those in Figure 7.4; that is, in Figure 7.5, $\sigma(\tilde{R}_q) = .06$ and $\sigma(\tilde{R}_s) = .04$. Instead of explicitly engaging in further tedium, we leave detailed scrutiny of these two examples to the reader.

Finally, in Figures 7.2-5, the plots of $E(\tilde{R}_p)$ against $\sigma(\tilde{R}_p)$ always seem

FIGURE 7.5

Plots of $E(\tilde{R}_p)$ Against $\sigma(\tilde{R}_p)$ When $E(\tilde{R}_q) < E(\tilde{R}_s)$ and $\sigma(\tilde{R}_q) > \sigma(\tilde{R}_s)$



symmetric about the portfolio that minimizes $\sigma(\tilde{R}_p)$. This is not a peculiarity of the examples. Merton (1972) shows that when $-1.0 < \text{corr}\,(\tilde{R}_q, \tilde{R}_s) < 1.0$, the plot of $E(\tilde{R}_p)$ against $\sigma(\tilde{R}_p)$ is a hyperbola. This is interesting but not important for our purposes, so we leave the reader to pursue the details in Merton's excellent treatment of the topic.

B. The Efficient Set: No Risk-free Asset

This discussion of the geometric properties of portfolios that are combinations of two securities or portfolios q and s allows us to determine easily the geometric properties of the efficient set of portfolios. Recall that to be efficient a portfolio must have the dual properties that (a) no other portfolio with the same expected return can have lower standard deviation of return, and (b) no other portfolio with the same standard deviation of return can have higher expected return. The collection of all efficient portfolios is called the efficient set. The collection of feasible portfolios—that is, portfolios that can be obtained as combinations of the n available securities—is called the feasible set or, sometimes, the investment opportunity set. Efficient portfolios are a subset of the feasible set.

Suppose now that we want to plot the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ that can be obtained with efficient portfolios. The portfolios that minimize $\sigma(\tilde{R}_p)$ at given levels of $E(\tilde{R}_p)$ are called minimum variance portfolios. In geometric terms, the set of minimum variance portfolios is the left boundary of the feasible set of portfolios; that is, minimum variance portfolios are those that are as close as possible to the $E(\tilde{R}_p)$ axis at each feasible level of $E(\tilde{R}_p)$. Since efficient portfolios minimize $\sigma(\tilde{R}_p)$ at given levels of $E(\tilde{R}_p)$, they must be along the left boundary of the feasible set; efficient portfolios are minimum variance portfolios. Since efficient portfolios also maximize $E(\tilde{R}_p)$ at given levels of $\sigma(\tilde{R}_p)$, they must plot along positively sloped segments of the left boundary of the feasible set. Thus, there are minimum variance portfolios that are not efficient. They are along negatively sloped portions of the left boundary of the feasible set.

On the basis of the earlier geometric discussion, however, we now argue that once the slope of the left boundary of the feasible set becomes positive, it remains positive at all attainable higher levels of expected return. Thus, if the left boundary of the feasible set has a positively sloped segment, there is only one such segment and it covers the range of expected returns from the point where the slope of the boundary becomes positive to the highest level of expected return on any feasible portfolio. Even more specifically, the earlier geometric discussion implies almost directly that this unique positively sloped segment of the left boundary of the feasible set, which describes the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ available from efficient portfolios, is a concave curve.

For example, the portfolio opportunities available to the investor might be as represented in Figure 7.6. The irregularly shaped solid curve is the boundary of the set of feasible portfolios. The left boundary of the feasible set, which describes the set of minimum variance portfolios, is the solid curve *abg*. The set of efficient portfolios is the positively sloped concave segment of this boundary from b to g. That the positively sloped segment of the boundary must be a concave curve follows directly from the geometric properties of combinations of two securities or portfolios. For example, the left boundary

cannot be as represented by the curves *bcd* and *deg* in Figure 7.6, since it is always possible to form portfolios of c and e, and these portfolios must plot either on or to the left of the straight line between c and e, depending on whether corr $(\tilde{R}_c, \tilde{R}_e)$ is equal to or less than 1.0. Such arguments rule out gaps in the boundary of the feasible set, and they also rule out indentations in the left boundary, such as *cde* and *hij*. Indeed, the geometry of combinations of two securities or portfolios implies that the slope of the left boundary of the feasible set of portfolios can change sign once at most. The negatively sloped segment is convex and covers lower levels of expected return, while the positively sloped segment is concave and covers higher levels of expected return.

PROBLEM III.B

1. Show that, in general, any point on the left boundary of the feasible set is unique in the sense that there is only one portfolio with the indicated combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$. What is the exception to this general rule?

ANSWER

1. Suppose there are two portfolios, call them u and v, such that $E(\tilde{R}_u) = E(\tilde{R}_v)$ and $\sigma(\tilde{R}_u) = \sigma(\tilde{R}_v)$. When corr $(\tilde{R}_u, \tilde{R}_v) < 1.0$, u and v cannot be on the left boundary of the feasible set, since positively weighted combinations of u and v have the same expected return but, from (11), lower standard deviation of return than u and v.

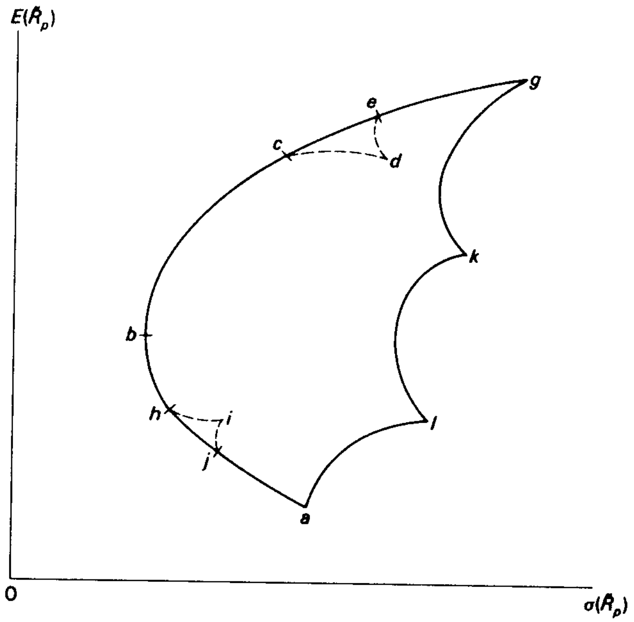
When $0 \leqslant x \leqslant 1$ and corr $(\tilde{R}_u, \tilde{R}_v) = 1.0$, from equation (10)

$$\sigma(\tilde{R}_p) = x\sigma(\tilde{R}_u) + (1 - x)\sigma(\tilde{R}_v) = \sigma(\tilde{R}_u) = \sigma(\tilde{R}_v).$$

Thus, if there are two portfolios that correspond to the same point on the left boundary of the feasible set, not only must the two portfolios have the same expected returns and standard deviations, but their returns must also be perfectly positively correlated.

There are some additional assumptions implicit in the representation of the efficient and feasible sets of portfolios as in Figure 7.6. Since there are no feasible portfolios with $\sigma(\tilde{R}_p) = 0.0$, there are no securities or portfolios whose returns are perfectly negatively correlated. Moreover, since the boundary of the feasible set has no straight line segments, there are no portfolios on the boundary whose returns are perfectly positively correlated. Finally, since the feasible set of portfolios pictured in Figure 7.6 is assumed to have a right boundary, there is an implicit assumption that there is no short-selling

FIGURE 7.6
Portfolio Opportunities When There Is No Short-Selling



or, at least, that short-selling is not unlimited.* For example, the earlier geometric analysis of portfolios formed as combinations of two securities or portfolios implies that with unlimited short-selling, combinations of portfolios b and g in which b is sold short could be used to get portfolios with arbitrarily large values of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$. From this we can conclude that if there is unlimited short-selling, the efficient set of portfolios must extend indefinitely upward and to the right in the $E(\tilde{R})$, $\sigma(\tilde{R})$ plane. Likewise, with unlimited short-selling, the left boundary of the feasible set of portfolios extends indefinitely downward and to the right in the $E(\tilde{R})$, $\sigma(\tilde{R})$ plane. One can see this in Figure 7.6 by noting that with combinations of portfolios b and a in which b is sold short, one can get portfolios with arbitrarily small values of $E(\tilde{R}_p)$ and arbitrarily large values of $\sigma(\tilde{R}_p)$.

In fact, Merton (1972) shows that if unlimited short-selling of all securities is allowed, and if it is not possible to obtain a portfolio that has $\sigma(\tilde{R}_p) = 0.0$, then the left boundary of the feasible set of portfolios, the set of minimum variance portfolios, is a hyperbola, and so has the general shape shown in

*Using the geometric properties of combinations of two securities or portfolios, the reader can determine that in the absence of short-selling, the irregular curve *alkg* is a possible shape that the right boundary of the feasible set might take.

FIGURE 7.7
Portfolio Opportunities When There Is Short-Selling

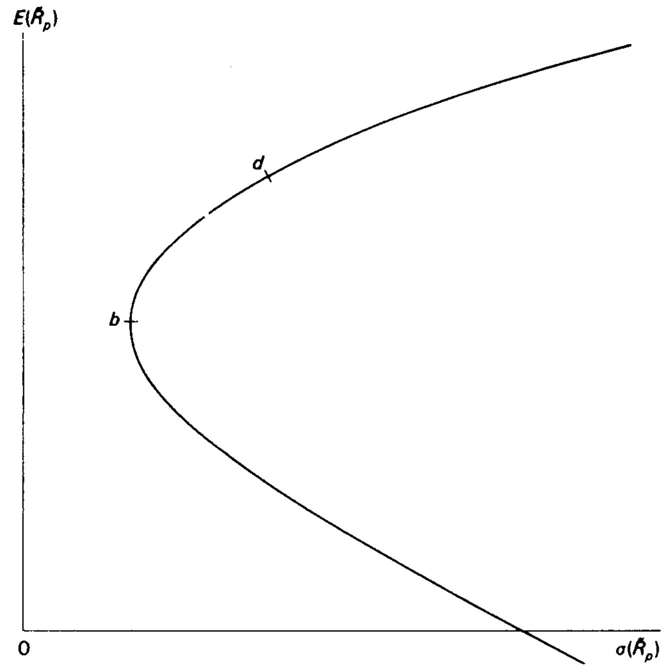


Figure 7.7. The boundary is symmetric about the point b, which is the portfolio that has the smallest possible standard deviation of return. The efficient set of portfolios covers the positively sloped segment of the boundary, the segment that starts at b and extends indefinitely upward and to the right.

From the earlier geometric discussions, the condition that it is not possible to obtain a portfolio that has $\sigma(\tilde{R}_p) = 0.0$ implies that no two securities or portfolios have perfectly negatively correlated returns. With unlimited short-selling, the condition also implies that no two securities or portfolios have perfectly positively correlated returns (see Figure 7.2). Finally, the condition implies that there is no risk-free security, that is, no security j with $\sigma(\tilde{R}_j) = 0.0$.

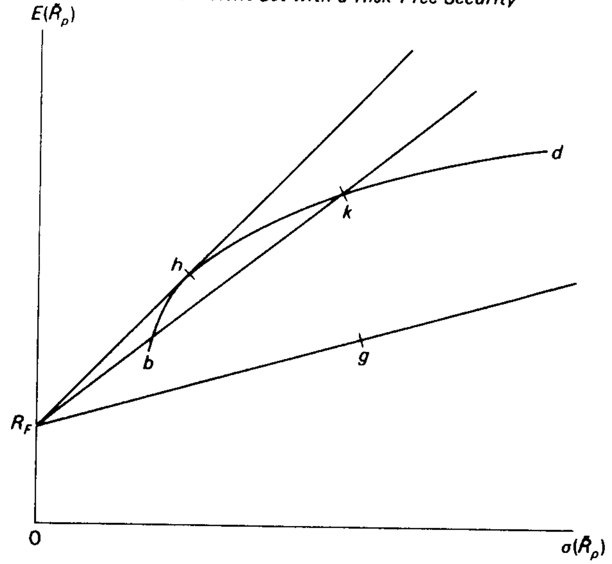
Models based on the existence of a risk-free asset play an important role in two-parameter theory and especially in the two-parameter models of capital market equilibrium presented in the next chapter. Thus, we now discuss in some detail the effect of a risk-free asset on the investment opportunity set.

C. The Efficient Set with a Risk-free Asset

Suppose the curve *bd* in Figure 7.8 represents the set of portfolios that

FIGURE 7.8

The Efficient Set with a Risk-Free Security



would be efficient in the absence of a risk-free security.* Suppose, however, that there is a risk-free security, call it F, that will pay the sure return R_F if held from time 1 to time 2. Consider portfolios of the risk-free security F and any risky security or portfolio g formed as

$$\tilde{R}_p = xR_F + (1 - x)\tilde{R}_g. \tag{18}$$

With $x = 1.0$, all portfolio funds $w_1 - c_1$ are invested in F, while with $x = 0.0$, all funds are in g. When $1.0 > x > 0.0$, positive proportions of $w_1 - c_1$ are invested in both F and g. When $x < 0.0$, the investor issues or sells F short and uses the proceeds to get an investment in g in excess of $w_1 - c_1$. Equivalently, issuing or short-selling of F can be regarded as borrowing at the rate R_F. Thus, a value of $x < 0.0$ in (18) implies that the investor borrows at the rate R_F and uses the proceeds to get an investment in g in excess of $w_1 - c_1$. In this view, $1.0 > x > 0.0$ implies that the investor lends at the rate R_F. The lending-borrowing interpretation of positions involving F is the one we use almost exclusively here.†

*Since investors only hold efficient portfolios and since we have finished our discussion of what the entire boundary of the investment opportunity set looks like when there is no risk-free asset, in the remaining geometric discussions we do not bother to show the entire boundary.

†Short-selling or issuing of a risky security can also be viewed as borrowing, but at an uncertain rate; and a long position in such a security can likewise be regarded as lending at an uncertain rate. In essence, short or long positions in one security are no different in kind from short and long positions in any other. It is, however, customary to interpret positions in bonds in terms of lending and borrowing, with the terms long and short reserved for positions in common stocks.

The expected return on any portfolio formed according to (18) is

$$E(\tilde{R}_p) = xR_F + (1 - x)E(\tilde{R}_g). \tag{19}$$

Since $\sigma(\tilde{R}_F) = 0.0$, the standard deviation of the return on any such portfolio is

$$\sigma(\tilde{R}_p) = |1 - x|\sigma(\tilde{R}_g), \tag{20}$$

and if we restrict attention to values of $x \leqslant 1.0$ (that is, if we do not consider short-selling of g), then

$$\sigma(\tilde{R}_p) = (1 - x)\sigma(\tilde{R}_g), \quad x \leqslant 1.0. \tag{21}$$

It is perhaps clear from inspection of equations (19) and (21) that the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained from portfolios of F and g plot along a straight line from R_F through g in Figure 7.8. Thus, $x = 1.0$ gives the point R_F on the $E(\tilde{R}_p)$ axis, while $x = 0.0$ is the point g. With $x = .5$,

$$E(\tilde{R}_p) = .5R_F + .5E(\tilde{R}_g)$$
$$\sigma(\tilde{R}_p) = .5\sigma(\tilde{R}_g),$$

which is the combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ halfway between R_F and g on the straight line between R_F and g. Likewise, with $x = .25$,

$$E(\tilde{R}_p) = .25R_F + .75E(\tilde{R}_g)$$
$$\sigma(\tilde{R}_p) = .75\sigma(\tilde{R}_g),$$

which is the combination of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ that is three-fourths of the way between R_F and g on the straight line between R_F and g. Finally, with $x = -.5$, that is, when the portfolio involves borrowing $.5(w_1 - c_1)$ and putting this plus $(w_1 - c_1)$ in g, we have

$$E(\tilde{R}_p) = -.5R_F + 1.5E(\tilde{R}_g)$$
$$\sigma(\tilde{R}_p) = 1.5\sigma(\tilde{R}_g),$$

which is on the extension of the line from R_F through g.

In short, the portfolios obtained from combinations of a risk~~less~~ free security F with any given risky security or portfolio g plot along a straight line from the risk~~less~~ free rate R_F through the point corresponding to g in the $E(\tilde{R}_p)$, $\sigma(\tilde{R}_p)$ plane. Lending portfolios (portfolios with $1.0 \geqslant x \geqslant 0.0$) plot on the line between R_F and g. Borrowing portfolios (portfolios with $x < 0.0$) plot along the extension of the line through g.

PROBLEMS III.C

1. Derive the exact form of the linear relationship between $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ for portfolios formed according to (18) with $x \leqslant 1.0$.

2. The linear relationships between $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ that arise from combinations of a risk-free security with a risky security or portfolio bring to mind the linear relationships obtained from combinations of two risky securities or portfolios when the returns on the two are perfectly positively correlated. Show that, in fact, with $x < 1.0$ in (18), the returns on any two portfolios of F and g are perfectly positively correlated. Interpret this result.

ANSWERS

1. First solve (19) and (21) for x to get

$$x = \frac{E(\tilde{R}_p) - E(\tilde{R}_g)}{R_F - E(\tilde{R}_g)}$$

$$x = \frac{\sigma(\tilde{R}_g) - \sigma(\tilde{R}_p)}{\sigma(\tilde{R}_g)}, \quad x \leqslant 1.0.$$

Equating these two expressions for x and solving for $E(\tilde{R}_p)$ yields

$$E(\tilde{R}_p) = R_F + \left(\frac{E(\tilde{R}_g) - R_F}{\sigma(\tilde{R}_g)}\right)\sigma(\tilde{R}_p), \quad x \leqslant 1.0. \tag{22}$$

This is indeed the equation for the line from R_F through g in Figure 7.8, since this line has intercept R_F and slope $(E(\tilde{R}_g) - R_F)/\sigma(\tilde{R}_g)$.

In intuitive terms, equation (22) describes the behavior of the expected return on portfolios formed according to (18) as the proportion x invested in the risk-free security F is decreased and the proportion $1 - x$ invested in g is increased. When one shifts from the risk-free to the risky investment, the expected portfolio return changes at the rate $E(\tilde{R}_g) - R_F$: one exchanges R_F for $E(\tilde{R}_g)$. Equation (21) tells us that such shifts toward larger holdings of the risky investment also cause $\sigma(\tilde{R}_p)$ to increase.

2. Consider two portfolios p and p' formed according to

$$\tilde{R}_p = xR_F + (1 - x)\tilde{R}_g, \quad x < 1.0$$

$$\tilde{R}_{p'} = x'R_F + (1 - x')\tilde{R}_g, \quad x' < 1.0, x \neq x'.$$

Then

$$\operatorname{corr}(\tilde{R}_p, \tilde{R}_{p'}) = \frac{\operatorname{cov}(\tilde{R}_p, \tilde{R}_{p'})}{\sigma(\tilde{R}_p)\sigma(\tilde{R}_{p'})}$$

$$= \frac{\operatorname{cov}[(1 - x)\tilde{R}_g, (1 - x')\tilde{R}_g]}{(1 - x)\sigma(\tilde{R}_g)(1 - x')\sigma(\tilde{R}_g)}$$

$$= \frac{(1 - x)(1 - x')\operatorname{cov}(\tilde{R}_g, \tilde{R}_g)}{(1 - x)(1 - x')\sigma^2(\tilde{R}_g)} = 1.0.$$

In intuitive terms, all the uncertainty in the return on a portfolio of F and g arises from g. Although the returns on two different portfolios of F and g will not be identical, if both involve positive investment in g, the returns on the portfolios are perfectly positively correlated. It is left to the reader to show that any two portfolios of F and g that both involve a short position in g (that is, $x > 1.0$) are likewise perfectly positively correlated; but if one of the portfolios involves a short position in g while the other involves a long position, their returns are perfectly negatively correlated. Completing the problem in this way completes the analogy between the linear investment opportunities obtained with a risk-free security and those obtained from two positive variance securities or portfolios whose returns are perfectly positively correlated.

How might one best take advantage of the investment opportunities provided by a risk-free security? One can get a lot better results from F than are obtained by combining it with g in Figure 7.8. For example, portfolios of F and k give combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ along the straight line from R_F through k. This line clearly dominates the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ along the line from R_F through g. At any given level of $\sigma(\tilde{R}_p)$, the portfolio of F and k has higher expected return than the portfolio of F and g. Combinations of F and k likewise do not make the best possible use of F. In the two-parameter model, since investors like maximum expected return at any given level of standard deviation of return and minimum standard deviation of return at any given level of expected return, the best way to use F is to combine it with a risky portfolio so that the resulting straight line is upward and to the left as much as possible in the $E(\tilde{R}_p)$, $\sigma(\tilde{R}_p)$ plane. In geometric terms, this involves "resting" a line on the curve bd in Figure 7.8. The point h where this line just touches bd is the portfolio of risky securities that one combines with F to get the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ along this line.

The line from R_F through h in Figure 7.8 describes the efficient set of portfolios. Except for h, portfolios along the curve bd are not efficient, since the portfolios along the line from R_F through h have higher expected returns at given levels of standard deviation of return. Thus, efficient portfolios are combinations of F and h, with returns, expected returns, and standard deviations of returns given by

$$\tilde{R}_p = xR_F + (1 - x)\tilde{R}_h \quad x \leqslant 1.0 \tag{23}$$

$$E(\tilde{R}_p) = xR_F + (1 - x)E(\tilde{R}_h) \quad x \leqslant 1.0 \tag{24}$$

$$\sigma(\tilde{R}_p) = (1 - x)\sigma(\tilde{R}_h) \quad x \leqslant 1.0. \tag{25}$$

With $x = 1.0$, all funds are invested in F and we are at the point on the $E(\tilde{R}_p)$ axis corresponding to R_F in Figure 7.8. As we consider successively lower values of x, equations (24) and (25) give combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ along the line from R_F to h, and we hit the point h when $x = 0.0$. For still lower values of x, we get combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ along the extension of the line from R_F through h.

The important point is that with risk-free borrowing and lending, all efficient portfolios are just combinations of the risk-free security F with the one portfolio of risky securities h. Some efficient portfolios, those on the line between R_F and h in Figure 7.8, are lending portfolios. Other efficient portfolios, those on the extension of the line from R_F through h, are borrowing portfolios; that is, the investor borrows at rate R_F and puts both his borrowings and his investment funds $w_1 - c_1$ into h. The only difference between one efficient portfolio and another is in how $w_1 - c_1$ is split between F and h. The portfolio h is the only efficient portfolio of only risky assets, and h is the risky component of every risky efficient portfolio.

In the next chapter, we find that these simplified characteristics of the efficient set when there is risk-free borrowing and lending are important in the analysis of market equilibrium in a world where investors make portfolio decisions in accordance with the two-parameter model.

PROBLEM III.C

3. Suppose there is risk-free lending but not borrowing; that is, suppose that in (23), x is restricted to values between 0.0 and 1.0. What then is the efficient set of portfolios in Figure 7.8?

ANSWER

3. When there is no risk-free borrowing, the portfolios on the line between R_F and h are feasible, since they are lending portfolios, but portfolios on the extension of the line through h are no longer feasible, since they are borrowing portfolios. Thus, when there is no borrowing at the risk-free rate, the efficient set includes portfolios along the line from R_F to h. Efficient portfolios with $\sigma(\tilde{R}_p) > \sigma(\tilde{R}_h)$ are those portfolios above h along the curve bd.

IV. Portfolio Risk, Security Risk, and the Effects of Diversification

Chapter 2 introduced the concepts of security risk and portfolio risk that are relevant in a two-parameter world. We now review these concepts briefly and provide empirical illustrations. The discussion should be useful background for Chapter 8, which is concerned with the relationships between expected return and risk that arise in a two-parameter world. The discussion also expands on some earlier comments concerning the effects of diversification on dispersion of returns.

A. *Portfolio Risk and Security Risk in a Two-Parameter World*

In the portfolio model, the investor is motivated by a desire to consume. He invests at time 1 in order to consume at time 2. His only concern in his portfolio decision is the probability distribution on consumption at time 2 that it implies. From equation (1) we can see that is is equivalent to say that his only concern in his portfolio decision is the probability distribution of portfolio return that it implies. The investor is only concerned with individual securities in terms of how they affect the distribution of the return on his portfolio.

In a world of normal portfolio return distributions, the distribution of the return on any portfolio can be completely characterized from knowledge of its mean and standard deviation or variance. Thus, to determine the effect of an individual security on the distribution of a portfolio's return, it is sufficient to determine the contribution of the security to the expected value and variance of the return on the portfolio. The return, expected return, and variance of return on a portfolio are

$$\tilde{R}_p = \sum_{i=1}^{n} x_{ip}\tilde{R}_i \tag{26}$$

$$E(\tilde{R}_p) = \sum_{i=1}^{n} x_{ip}E(\tilde{R}_i) \tag{27}$$

$$\sigma^2(\tilde{R}_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ip}x_{jp}\sigma_{ij}, \tag{28}$$

where $\sigma_{ij} = \text{cov}\,(\tilde{R}_i, \tilde{R}_j)$, n is the number of securities available, and one portfolio is different from another in terms of the proportions x_{ip}, $i = 1, \ldots, n$, invested in individual securities.

The contribution of security i to $E(\tilde{R}_p)$ is

$$x_{ip}E(\tilde{R}_i),$$

the expected return on the security weighted by x_{ip}, the proportion of portfolio funds $w_1 - c_1$ invested in security i to obtain portfolio p. To highlight the contribution of security i to $\sigma^2(\tilde{R}_p)$, equation (28) is rewritten as

$$\sigma^2(\tilde{R}_p) = \sum_{i=1}^{n} x_{ip}\left(\sum_{j=1}^{n} x_{jp}\sigma_{ij}\right). \tag{29}$$

Thus, the variance of the return on portfolio p is the sum of n terms,

$$x_{ip}\left(\sum_{j=1}^{n} x_{jp}\sigma_{ij}\right), \quad i = 1, \ldots, n, \tag{30}$$

each of which represents the contribution of a security to $\sigma^2(\tilde{R}_p)$. This contribution of security i to $\sigma^2(\tilde{R}_p)$ is x_{ip}, the proportion of portfolio funds invested in i, multiplied by $\left(\sum_{j=1}^{n} x_{jp}\sigma_{ij}\right)$, the weighted average of the pairwise covariances between the return on security i and the returns on each of the securities in p, where the weight applied to σ_{ij} is x_{jp}, the proportion of portfolio funds invested in security j to obtain portfolio p.

In the two-parameter model, the risk of a portfolio is measured by the variance of its return. Thus, under one interpretation, the risk of security i in portfolio p might be measured by (30), the contribution of the security to $\sigma^2(\tilde{R}_p)$. It is, however, more convenient to call the weighted average of covariances

$$\sum_{j=1}^{n} x_{jp}\sigma_{ij}, \quad i = 1, \ldots, n, \tag{31}$$

the risk of security i in portfolio p. Then, from (29), the risk of the portfolio, $\sigma^2(\tilde{R}_p)$, is just the weighted average of the risks of the individual securities.

Several comments ought to emphasize the view of security risks one gets when the object of concern to the investor is the distribution of the return on his portfolio. First, to be precise, one must always talk about the risk of security i in portfolio p, since the risk of a security is different from one portfolio to another. In formal terms, as defined by (31), the risk of security i in portfolio p depends on the return covariances σ_{ij}, $j = 1, \ldots, n$, which are parameters of the joint distribution of security returns and thus are the same from portfolio to portfolio. The risk of i in p also depends on how securities are combined to form p (that is, on the portfolio weights $x_{jp}, j = 1, \ldots, n$), and the combinations are different from one portfolio to another.

Second, an apparently "risky" security (that is, a security that has a positive variance of return) may have positive, zero, or negative risk in a portfolio. Thus, noting that one of the terms in (31) involves $\sigma_{ii} = \sigma^2(\tilde{R}_i)$, we can rewrite (31) as

$$\sum_{j=1}^{n} x_{jp}\sigma_{ij} = x_{ip}\sigma^2(\tilde{R}_i) + \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{jp}\sigma_{ij}. \tag{32}$$

Although the first term on the right of the equality is positive if x_{ip} is positive, the value of the whole expression can be positive, zero, or negative depending on the value of the weighted average of covariances in the second term on the right of the equality. In short, the risk of a security in a portfolio depends on the covariances between the return on the security and the returns on other securities in the portfolio, as well as on the variance of the return on the security. A security that apparently has high "risk" in terms of the variance of its return may have low risk when viewed as a component of a portfolio.

In fact, inspection of equation (32) suggests that in a portfolio which contains a large number of securities, and where portfolio funds are not concentrated in one or a few of the securities, the risk of a security might depend more on the covariances between the return on the security and the returns on other securities in the portfolio than on the variance of the security's return. Thus, from (32), $\sigma^2(\tilde{R}_i)$ is just one of n terms that determine the risk of security i in p, with the remaining $n - 1$ terms being the pairwise covariances. This, however, does not in itself imply that the covariances in (32) dominate the variance. For this purpose, what constitutes a "large" n depends on the magnitude of $\sigma^2(\tilde{R}_i)$ relative to the σ_{ij}, an empirical issue which the examples that follow are meant to illuminate.

PROBLEMS IV.A

1. Can a security have positive risk in a portfolio and yet make a negative contribution to the risk of the portfolio?

2. Show that the sample mean and variance of the return on a portfolio can be written in forms exactly analogous to (27) and (28). This problem is important for the numerical examples that follow.

ANSWERS

1. The contribution of a security to the variance of the return on a portfolio is given by (30). We have defined the risk of the security in the portfolio as the weighted average of covariances that appear in the parentheses in (30), that is, as (31). Thus, even if the risk of a security in the portfolio is positive, the security makes a negative contribution to the risk of the portfolio if (31) is positive and $x_{ip} < 0.0$; that is, portfolio p involves short-selling of security i.

In intuitive terms, when a security makes a negative contribution to the variance of the return on a portfolio, that security is risk-reducing as far as the portfolio is concerned. In the present example, we see that a security that would make a positive contribution to the risk of a portfolio if held long makes a negative contribution when held short. Thus, a short position in the security is risk-reducing. The sword is, however, double-edged. If the expected return on the security is positive and the security is sold short, then the security also makes a negative contribution to the expected return on the portfolio.

2. With a time series R_{pt}, $t = 1, \ldots, T$, of T observations of the return on portfolio p, the sample variance of the portfolio returns is

$$s^2(R_p) = \frac{\sum_{t=1}^{T} (R_{pt} - \bar{R}_p)^2}{T-1}, \tag{33}$$

where $\bar{R}_p$, the sample mean of the portfolio returns, is

$$\bar{R}_p = \frac{\sum_{t=1}^{T} R_{pt}}{T}. \tag{34}$$

Since

$$R_{pt} = \sum_{i=1}^{n} x_{ip} R_{it}, \tag{35}$$

$$\bar{R}_p = \frac{\sum_{t=1}^{T} \sum_{i=1}^{n} x_{ip} R_{it}}{T} = \frac{\sum_{i=1}^{n} x_{ip} \sum_{t=1}^{T} R_{it}}{T} = \sum_{i=1}^{n} x_{ip} \bar{R}_i, \tag{36}$$

where $\bar{R}_i$ is the sample mean of the returns on security i. Thus, (33) can be rewritten as

$$s^2(R_p) = \frac{\sum_{t=1}^{T} \left(\sum_{i=1}^{n} x_{ip} R_{it} - \sum_{i=1}^{n} x_{ip} \bar{R}_i \right)^2}{T-1}$$

$$= \frac{\sum_{t=1}^{T} \left(\sum_{i=1}^{n} x_{ip} [R_{it} - \bar{R}_i] \right)^2}{T-1}$$

$$= \frac{\sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ip} x_{jp} (R_{it} - \bar{R}_i)(R_{jt} - \bar{R}_j)}{T-1}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ip} x_{jp} \left(\frac{\sum_{t=1}^{T} (R_{it} - \bar{R}_i)(R_{jt} - \bar{R}_j)}{T-1} \right)$$

$$s^2(R_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ip} x_{jp} s_{ij}, \tag{37}$$

where

$$s_{ij} = \frac{\sum_{t=1}^{T} (R_{it} - \bar{R}_i)(R_{jt} - \bar{R}_j)}{T-1}$$

is the usual (see Chapter 3) sample estimate of the covariance σ_{ij}, and when $i = j$, $s_{ii} = s^2(R_i)$ is the usual sample estimate of the variance $\sigma^2(\tilde{R}_i)$.

Equation (37) says that the sample variance of the return on a portfolio is the weighted average of the sample variances and covariances of security returns, just as equation (28) says that the true variance of the return on a portfolio is the corresponding weighted average of security return variances and covariances. And (36) says that the sample mean of the return on a portfolio is the weighted average of the sample means of the security returns, just as (27) says that the expected portfolio return is the weighted average of the expected security returns.

B. Portfolio Risk and Security Risk: Empirical Examples

Using monthly data for July 1963–June 1968, Table 7.1 shows components of the average monthly return and of the variance of monthly returns on portfolios of increasing numbers of randomly selected securities. Results are presented for ten portfolios, and column (7) of the table shows that the ten portfolios include $N = 1, 2, 3, 4, 5, 6, 10, 15, 25$, and 50 securities. Columns (1) and (2) show that the number of securities in the portfolios is increased by adding securities and redistributing portfolio funds equally across securities, so that in a portfolio of N securities, each security gets the proportion $x_{ip} = 1/N$ of portfolio funds. We use upper case N now to indicate the number of securities that appear in a portfolio with nonzero weights. For a given security and portfolio, the entry in column (3) of Table 7.1 shows $x_{ip}\bar{R}_i$, which equation (36) says is the contribution of security i to the average return on the portfolio. Thus, the average return on the portfolio is the sum of the entries for individual securities in column (3), and for a given portfolio the average portfolio return is shown in column (3) in the line labeled "Totals."

For a given security and portfolio, the entries in columns (4)–(6) show components of the contribution of the security to the sample variance of the return on the portfolio. Thus in direct analogy with equations (29) to (31), equation (37) for the sample variance of the return on portfolio p can be rewritten as

$$s^2(R_p) = \sum_{i=1}^{N} x_{ip} \left(\sum_{j=1}^{N} x_{jp} s_{ij} \right), \tag{38}$$

so that the contribution of security i to the sample variance of the return on p is

$$x_{ip} \left(\sum_{j=1}^{N} x_{jp} s_{ij} \right). \tag{39}$$

If we call the weighted average of sample covariances

$$\sum_{j=1}^{N} x_{jp} s_{ij}, \tag{40}$$

the estimated risk of security i in portfolio p, then, from (38), the estimate of the risk of the portfolio, $s^2(R_p)$, is just the weighted average of the estimates of security risks.

For a given security and portfolio, column (6) of Table 7.1 shows the quantity in expression (39) which is the contribution of the security to the sample variance of the portfolio's return; and the entry on the "Totals" line of column (6) is $s^2(R_p)$. The entries in columns (4) and (5) for a given security break the contribution of the security to $s^2(R_p)$ into a variance component and a weighted average of covariances component. That is, expression (39) is broken down as

$$x_{ip}\left(\sum_{j=1}^{N} x_{jp} s_{ij}\right) = x_{ip}^2 s^2(R_i) + x_{ip} \sum_{\substack{j=1 \\ j \neq i}}^{N} x_{jp} s_{ij}. \tag{41}$$

The entries in the "Totals" line of columns (4) and (5) thus give a breakdown of $s^2(R_p)$ into security return variances and covariances as

$$s^2(R_p) = \sum_{i=1}^{N} x_{ip}^2 s^2(R_i) + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} x_{ip} x_{jp} s_{ij}. \tag{42}$$

Several points of interest can be drawn from Table 7.1. Note that as the number of securities in the portfolios increases, the contributions of a security to average portfolio return and to sample variance of portfolio return go down. In the case of $\bar{R}_p$, since the contribution of security i to $\bar{R}_p$ is $x_{ip}\bar{R}_i$ and since we are dealing with portfolios of equally weighted securities, $x_{ip}\bar{R}_i = (1/N)\bar{R}_i$ decreases as N, the number of securities in the portfolio, is increased. Likewise, the contribution of security i to $s^2(\tilde{R}_p)$ is

$$x_{ip}\left(\sum_{j=1}^{N} x_{jp} s_{ij}\right) = \frac{1}{N}\left(\frac{\sum_{j=1}^{N} s_{ij}}{N}\right).$$

Although the estimated risk of security i in the portfolio

$$\sum_{j=1}^{N} x_{jp} s_{ij} = \frac{\sum_{j=1}^{N} s_{ij}}{N}$$

TABLE 7.1

Components of the Sample Mean and Variance of Monthly Returns on Portfolios of Increasing Numbers of Randomly Selected Securities, July 1963–June 1968

(1) SECURITIES (i)		(2) x_{ip}	(3) $x_{ip}\bar{R}_i$	(4) $x_{ip}^2 s^2(R_i)$	(5) $x_{ip}\sum_{j=1, j\neq i}^{N} x_{jp}s_{ij}$	(6) $x_{ip}\sum_{j=1}^{N} x_{jp}s_{ij}$	(7) N
IPL, Inc.		1.000	0.00701	0.012341	0.000000	0.01234	1
	Totals	1.000	0.00701	0.012341	0.000000	0.01234	
IPL, Inc.		0.500	0.00351	0.003085	0.000524	0.00361	
Lehigh Portland Cement		0.500	0.00171	0.001376	0.000524	0.00190	2
	Totals	1.000	0.00522	0.004462	0.001048	0.00551	
IPL, Inc.		0.333	0.00234	0.001371	0.000274	0.00164	
Lehigh Portland Cement		0.333	0.00114	0.000612	0.000470	0.00108	3
Hotel Corp. of America		0.333	0.01186	0.001548	0.000277	0.00183	
	Totals	1.000	0.01534	0.003531	0.001021	0.00455	
IPL, Inc.		0.250	0.00175	0.000771	0.000233	0.00100	
Lehigh Portland Cement		0.250	0.00085	0.000344	0.000432	0.00078	
Hotel Corp. of America		0.250	0.00890	0.000871	0.000370	0.00124	4
Portec, Inc.		0.250	0.00489	0.000622	0.000461	0.00108	
	Totals	1.000	0.01639	0.002608	0.001495	0.00410	
IPL, Inc.		0.200	0.00140	0.000494	0.000213	0.00071	
Lehigh Portland Cement		0.200	0.00068	0.000220	0.000296	0.00052	
Hotel Corp. of America		0.200	0.00712	0.000557	0.000280	0.00084	5
Portec, Inc.		0.200	0.00391	0.000398	0.000397	0.00080	
Bristol Myers Co.		0.200	0.00458	0.000142	0.000228	0.00037	
	Totals	1.000	0.01769	0.001811	0.001413	0.00322	
IPL, Inc.		0.167	0.00117	0.000343	0.000216	0.00056	
Lehigh Portland Cement		0.167	0.00057	0.000153	0.000265	0.00042	
Hotel Corp. of America		0.167	0.00593	0.000387	0.000234	0.00062	
Portec, Inc.		0.167	0.00326	0.000277	0.000303	0.00058	6
Bristol Myers Co.		0.167	0.00381	0.000098	0.000178	0.00028	
Van Raalte Co.		0.167	0.00347	0.000101	0.000215	0.00032	
	Totals	1.000	0.01821	0.001359	0.001411	0.00277	
IPL, Inc.		0.100	0.00070	0.000123	0.000164	0.00029	
Lehigh Portland Cement		0.100	0.00034	0.000055	0.000150	0.00021	
Hotel Corp. of America		0.100	0.00356	0.000139	0.000168	0.00031	
Portec, Inc.		0.100	0.00195	0.000100	0.000181	0.00028	
Bristol Myers Co.		0.100	0.00229	0.000035	0.000106	0.00014	10
Van Raalte Co.		0.100	0.00208	0.000036	0.000122	0.00016	
Ex-Cell-O Corp.		0.100	0.00157	0.000051	0.000164	0.00022	
Keebler Co.		0.100	0.00201	0.000056	0.000191	0.00025	
Canadian Breweries Ltd.		0.100	0.00253	0.000059	0.000018	0.00008	
Gulf, Mobile & Ohio R.R.		0.100	0.00247	0.000061	0.000187	0.00025	
	Totals	1.000	0.01949	0.000716	0.001450	0.00217	
IPL, Inc.		0.067	0.00047	0.000055	0.000115	0.00017	
Lehigh Portland Cement		0.067	0.00023	0.000024	0.000108	0.00013	
Hotel Corp. of America		0.067	0.00237	0.000062	0.000123	0.00018	

TABLE 7.1 (CONT'D)

Components of the Sample Mean and Variance of Monthly Returns on Portfolios of Increasing Numbers of Randomly Selected Securities, July 1963–June 1968

(1) SECURITIES (i)	(2) x_{ip}	(3) $x_{ip}\bar{R}_i$	(4) $x_{ip}^2 s^2(R_i)$	(5) $x_{ip}\sum_{j=1, j\neq i}^{N} x_{jp}s_{ij}$	(6) $x_{ip}\sum_{j=1}^{N} x_{jp}s_{ij}$	(7) N
Portec, Inc.	0.067	0.00130	0.000044	0.000127	0.00017	
Bristol Myers Co.	0.067	0.00153	0.000016	0.000067	0.00008	
Van Raalte Co.	0.067	0.00139	0.000016	0.000079	0.00010	
Ex-Cell-O Corp.	0.067	0.00105	0.000023	0.000109	0.00013	
Keebler Co.	0.067	0.00134	0.000025	0.000127	0.00015	
Canadian Breweries Ltd.	0.067	0.00168	0.000026	0.000031	0.00006	15
Gulf, Mobile & Ohio R.R.	0.067	0.00164	0.000027	0.000121	0.00015	
Dana Corp.	0.067	0.00044	0.000011	0.000084	0.00009	
Union Pacific R.R.	0.067	0.00057	0.000011	0.000075	0.00009	
Cyclops Corp.	0.067	0.00063	0.000017	0.000085	0.00010	
Ohio Edison Co.	0.067	0.00046	0.000007	0.000015	0.00002	
Central Foundry	0.067	0.00125	0.000101	0.000221	0.00032	
Totals	1.000	0.01635	0.000464	0.001489	0.00195	
IPL, Inc.	0.040	0.00028	0.000020	0.000059	0.00008	
Lehigh Portland Cement	0.040	0.00014	0.000009	0.000055	0.00006	
Hotel Corp. of America	0.040	0.00142	0.000022	0.000070	0.00009	
Portec, Inc.	0.040	0.00078	0.000016	0.000076	0.00009	
Bristol Myers Co.	0.040	0.00092	0.000006	0.000043	0.00005	
Van Raalte Co.	0.040	0.00083	0.000006	0.000041	0.00005	
Ex-Cell-O Corp.	0.040	0.00063	0.000008	0.000061	0.00007	
Keebler Co.	0.040	0.00080	0.000009	0.000066	0.00007	
Canadian Breweries Ltd.	0.040	0.00101	0.000009	0.000011	0.00002	
Gulf, Mobile & Ohio R.R.	0.040	0.00099	0.000010	0.000068	0.00008	
Dana Corp.	0.040	0.00026	0.000004	0.000045	0.00005	25
Union Pacific R.R.	0.040	0.00034	0.000004	0.000042	0.00005	
Cyclops Corp.	0.040	0.00038	0.000006	0.000049	0.00006	
Ohio Edison Co.	0.040	0.00027	0.000002	0.000014	0.00002	
Central Foundry	0.040	0.00075	0.000036	0.000124	0.00016	
United States Gypsum	0.040	0.00021	0.000007	0.000056	0.00006	
Eversharp Inc.	0.040	0.00013	0.000019	0.000060	0.00008	
Dayton Power & Light Co.	0.040	0.00029	0.000004	0.000034	0.00004	
Cluett, Peabody & Co.	0.040	0.00079	0.000006	0.000038	0.00004	
Washington Gas Light	0.040	−.00000	0.000001	0.000012	0.00001	
Lowenstein, M., & Sons	0.040	0.00076	0.000010	0.000067	0.00008	
International Telephone	0.040	0.00067	0.000006	0.000050	0.00006	
Carpenter Steel Co.	0.040	0.00079	0.000009	0.000045	0.00005	
Greyhound Corp.	0.040	0.00030	0.000006	0.000053	0.00006	
Allegheny Ludlum Steel	0.040	0.00057	0.000006	0.000035	0.00004	
Totals	1.000	0.01431	0.000242	0.001276	0.00152	
IPL, Inc.	0.020	0.00014	0.000005	0.000028	0.00003	
Lehigh Portland Cement	0.020	0.00007	0.000002	0.000032	0.00003	
Hotel Corp. of America	0.020	0.00071	0.000006	0.000046	0.00005	
Portec, Inc.	0.020	0.00039	0.000004	0.000042	0.00005	
Richardson Merrill Inc.	0.020	0.00026	0.000002	0.000017	0.00002	

TABLE 7.1 (CONT'D)

Components of the Sample Mean and Variance of Monthly Returns on Portfolios of Increasing Numbers of Randomly Selected Securities, July 1963–June 1968

(1) SECURITIES (i)	(2) x_{ip}	(3) $x_{ip}\bar{R}_i$	(4) $x_{ip}^2 s^2(R_i)$	(5) $x_{ip}\sum_{j=1, j\neq i}^{N} x_{jp}s_{ij}$	(6) $x_{ip}\sum_{j=1}^{N} x_{jp}s_{ij}$	(7) N
Van Raalte Co.	0.020	0.00042	0.000001	0.000021	0.00002	
Ex-Cell-O Corp.	0.020	0.00031	0.000002	0.000034	0.00004	
Keebler Co.	0.020	0.00040	0.000002	0.000031	0.00003	
Canadian Breweries Ltd.	0.020	0.00051	0.000002	0.000000	0.00000	
Gulf, Mobile & Ohio R.R.	0.020	0.00049	0.000002	0.000036	0.00004	
Dana Corp.	0.020	0.00013	0.000001	0.000025	0.00003	
Union Pacific R.R.	0.020	0.00017	0.000001	0.000020	0.00002	
Cyclops Corp.	0.020	0.00019	0.000002	0.000028	0.00003	
Ohio Edison Co.	0.020	0.00014	0.000001	0.000008	0.00001	
Central Foundry	0.020	0.00038	0.000009	0.000066	0.00007	
United States Gypsum	0.020	0.00011	0.000002	0.000030	0.00003	
Eversharp, Inc.	0.020	0.00006	0.000005	0.000033	0.00004	
Dayton Power & Light Co.	0.020	0.00014	0.000001	0.000018	0.00002	
Cluett, Peabody & Co.	0.020	0.00039	0.000002	0.000019	0.00002	
Washington Gas Light	0.020	−.00000	0.000000	0.000005	0.00001	
Lowenstein, M., & Sons	0.020	0.00038	0.000002	0.000035	0.00004	
International Telephone	0.020	0.00034	0.000001	0.000027	0.00003	50
Carpenter Steel Co.	0.020	0.00039	0.000002	0.000027	0.00003	
Greyhound Corp.	0.020	0.00015	0.000002	0.000026	0.00003	
Allegheny Ludlum Steel	0.020	0.00028	0.000001	0.000022	0.00002	
United Airlines Inc.	0.020	0.00042	0.000004	0.000037	0.00004	
Adams Express	0.020	0.00024	0.000000	0.000011	0.00001	
Ambac Industries, Inc.	0.020	0.00057	0.000005	0.000057	0.00006	
Masonite Corp.	0.020	0.00034	0.000003	0.000042	0.00005	
Lehigh Valley Industries	0.020	0.00106	0.000011	0.000040	0.00005	
American Cement Corp.	0.020	0.00031	0.000004	0.000041	0.00004	
Ebasco Industries, Inc.	0.020	0.00032	0.000001	0.000017	0.00002	
Raybestos Manhattan	0.020	0.00022	0.000001	0.000014	0.00002	
Inland Steel Co.	0.020	0.00010	0.000001	0.000028	0.00003	
Sterling Drug Inc.	0.020	0.00033	0.000002	0.000024	0.00003	
Walworth Co.	0.020	0.00040	0.000008	0.000047	0.00005	
Carborundum Co.	0.020	0.00035	0.000003	0.000036	0.00004	
Hudson Bay Mining & Smel.	0.020	0.00017	0.000001	0.000023	0.00002	
MSL Industries, Inc.	0.020	0.00022	0.000002	0.000032	0.00003	
Rohr Corp.	0.020	0.00050	0.000004	0.000027	0.00003	
Certainteed Products Corp.	0.020	0.00038	0.000005	0.000062	0.00007	
Neisner Bros.	0.020	0.00031	0.000002	0.000020	0.00002	
Rexall Drug and Chemical	0.020	0.00040	0.000003	0.000046	0.00005	
Laclede Gas Co.	0.020	0.00000	0.000000	0.000011	0.00001	
Chemetron Corp.	0.020	0.00037	0.000002	0.000038	0.00004	
Washington Gas Light	0.020	−.00000	0.000000	0.000005	0.00001	
Fiscbach and Moore Inc.	0.020	0.00038	0.000002	0.000027	0.00003	
C.I.T. Financial	0.020	0.00009	0.000001	0.000015	0.00002	
Donnelley, R. R., and Sons	0.020	0.00025	0.000001	0.000021	0.00002	
Continental Can Co. Inc.	0.020	0.00029	0.000001	0.000027	0.00003	
Totals	1.000	0.01497	0.000130	0.001426	0.00156	

need not change systematically as N is increased, the contribution of the security to $s^2(R_p)$ goes down with the weight $x_{ip} = 1/N$. In simplest terms, as portfolios become increasingly diversified, as new securities are introduced and the proportions invested in individual securities are reduced, an individual security becomes less important in terms of its effects on portfolio return distributions.

The decreasing importance of the role of an individual security in portfolios of increasing numbers of securities depends, however, on the presumption that smaller and smaller fractions of portfolio funds are invested in the security. If the number of securities is increased but the fraction invested in some security is not decreased, then that security will make the same contribution to average portfolio return and about the same contribution to standard deviation of portfolio return for all portfolio sizes.

One of the facts that comes out of the theoretical discussion of the risks of securities in portfolios is that in a portfolio of many securities, the variance of a security's return is only 1 of N terms that determine the contribution of the security to the variance of the return on the portfolio, with the remaining $N - 1$ terms being the pairwise covariances between the return on the security and the return on each of the other securities in the portfolio. However, we cannot conclude that for any specific "large" N the covariance terms dominate the variance term until we know something about the magnitudes of the security return variances versus the covariances. Table 7.1 provides sample information on this point.

For a given security and portfolio, columns (4) and (5) of Table 7.1 show, respectively, the contribution of the security's sample return variance to the sample variance of the portfolio's return and the contribution of its sample covariances. As one looks across portfolios of increasing numbers of securities, the numbers in both columns (4) and (5) decline, which is just a reflection of the fact that the contribution of an individual security to portfolio risk declines as the number of securities in the portfolios increases and the proportion of portfolio funds invested in an individual security decreases. As one looks across portfolios of increasing numbers of securities, however, one also observes a decline in the magnitude of the numbers in column (4) relative to those in column (5), and this is direct evidence of the declining importance of security return variances relative to covariances as the portfolios become increasingly diversified.

Perhaps the best information on this phenomenon is in the "Totals" rows in Table 7.1. For a given portfolio, the entries in columns (4) and (5) of the "Totals" row show the two components of equation (42) for $s^2(R_p)$; that is, the entry in column (4) of the "Totals" row shows the total contribution of the sample security return variances to $s^2(R_p)$, while the entry in column (5) shows the total contribution of security return covariances to $s^2(R_p)$. Since we are dealing with portfolios of equally weighted securities, equation (42) can be rewritten as

$$s^2(R_p) = \frac{1}{N}\left(\frac{\sum_{i=1}^{N} s^2(R_i)}{N}\right) + \frac{\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} s_{ij}}{N^2}$$

or

$$s^2(R_p) = \frac{1}{N}\overline{s^2(R_i)} + \frac{N-1}{N}\overline{s_{ij}}, \qquad (43)$$

where $\overline{s^2(R_i)}$ is the average of the N values of $s^2(R_i)$ in the portfolio, and

$$\overline{s_{ij}} = \frac{\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} s_{ij}}{N(N-1)} \qquad (44)$$

is the average of the $N(N-1)$ pairwise covariances s_{ij}.

When N is large, $(N-1)/N$ is close to 1, so that, from (43), the total contribution of covariances to the sample variance of the portfolio return is approximately $\overline{s_{ij}}$, and this average of pairwise covariances should not change in any particular way as a function of N. This point is well illustrated in Table 7.1 where, aside from reflecting the approach of $(N-1)/N$ to 1, the values of

$$\sum_{i=1}^{N} x_{ip} \sum_{\substack{j=1\\ j\neq i}}^{N} x_{jp}s_{ij} = \frac{N-1}{N}\overline{s_{ij}}$$

shown in column (5) of the "Totals" rows do not change with N in any particularly systematic way for portfolios of five or more securities. In short, once the portfolio becomes large, continuing to add securities has little effect on the contribution of security return covariances to the variance of the return on the portfolio.

Quite the opposite conclusion applies to the security return variances. Thus, although $\overline{s^2(R_i)}$ in (43) need not change systematically as N is increased $(1/N)\overline{s^2(R_i)}$, the total contribution of the security return variances to $s^2(R_p)$, declines inexorably as N is increased. This effect is well illustrated in Table 7.1, where the values of

$$\sum_{i=1}^{N} x_{ip}^2 s^2(R_i) = \frac{1}{N}\overline{s^2(R_i)}$$

shown in column (6) of the "Totals" rows decline more or less like $1/N$ as N is increased. For example, the values of $(1/N)\overline{s^2(R_i)}$ for the portfolios of 10, 15, 25, and 50 securities are .000716, .000464, .000242, and .000130, respectively.

The net result of the declining contribution of security return variances to $s^2(R_p)$ as N is increased, as compared to the relatively stable contribution of the security return covariances, is that $s^2(R_p)$ becomes more and more a reflection of the covariances. Thus in columns (4) and (5) of the "Totals" rows of Table 7.1, one finds that the covariances account for less than one-fifth of $s^2(R_p)$ when $N = 2$; their contribution to $s^2(R_p)$ is about equal to that of the security return variances when $N = 6$; and when $N = 50$ the security return variances account for less than one-tenth of the value of $s^2(R_p)$.

The fact that it takes a portfolio of 50 securities to get the security return variances down to accounting for less than 10 percent of the value of $s^2(R_p)$ indicates, however, that on average the pairwise covariances between security returns are small relative to security return variances. Thus, from the "Totals" row for the portfolio of 50 securities in Table 7.1 we find that

$$\frac{1}{50}\overline{s^2(R_i)} = .00013, \quad \text{so that} \quad \overline{s^2(R_i)} = .00650,$$

while

$$\frac{49}{50}\overline{s_{ij}} = .00142, \quad \text{so that} \quad \overline{s_{ij}} = .00145.$$

In short, for these 50 securities, security return variances average about 4.5 times larger than pairwise covariances between security returns. Thus, it takes a portfolio of many securities before the covariances dominate in determining the variance of the return on the portfolio. One would find similar results for other securities and portfolios.

C. *The Effects of Diversification*

The preceding analysis can be viewed from a different perspective. In particular, the analysis shows how diversification works to reduce the variance of the return on a portfolio. Thus, equation (43) and the examples in Table 7.1 tell us that, aside from reflecting the approach of $(N - 1)/N$ to 1.0, the contribution of the pairwise covariances between the returns on individual securities in a portfolio to the variance of the portfolio's return generally does not change in any systematic way as the number of securities in the portfolio is increased. Equation (43) and the examples of Table 7.1 also tell us, however, that the contribution of security return variances to the variance of the return

on the portfolio goes down as the number of securities in the portfolio is increased, and more or less in proportion to $1/N$. With the pairwise covariances making a steady contribution to the variance of the portfolio return, and with the contribution of security return variances declining as the number of securities in the portfolio is increased, the net result is a decline in the variance of the portfolio return, with the decline a direct reflection of the declining contribution of security return variances.

An illustration of the effects of diversification on the variance of portfolio returns is provided by columns (4) to (6) in the "Totals" rows of Table 7.1. For larger values of N, one observes smaller values of $s^2(R_p)$ in column (6), and it is clear from columns (4) and (5) that for $N \geq 5$ the decline in $s^2(R_p)$ as N is increased is almost entirely a reflection of the decline in the contribution of security return variances to the variance of portfolio returns.

The effects of diversification on the dispersion of distributions of portfolio returns can perhaps be better appreciated from Figure 7.9, which presents a plot of $s(R_p)$ against N for the portfolios of the random sample of stocks in Table 7.1, but for every value of N from 1 to 50. The important fact illustrated in Figure 7.9 is that most of the effects of diversification on the dispersion of the distribution of the portfolio return occur when the first few securities are added to the portfolio. Once the portfolio has 20 securities,

FIGURE 7.9

The Standard Deviation of Portfolio Return as a Function of the Number of Securities in the Portfolio: First Sample of 50 Randomly Selected Stocks

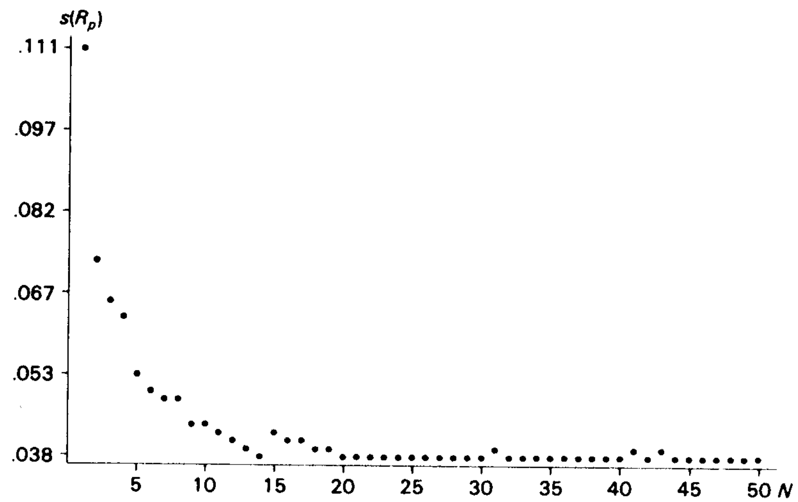
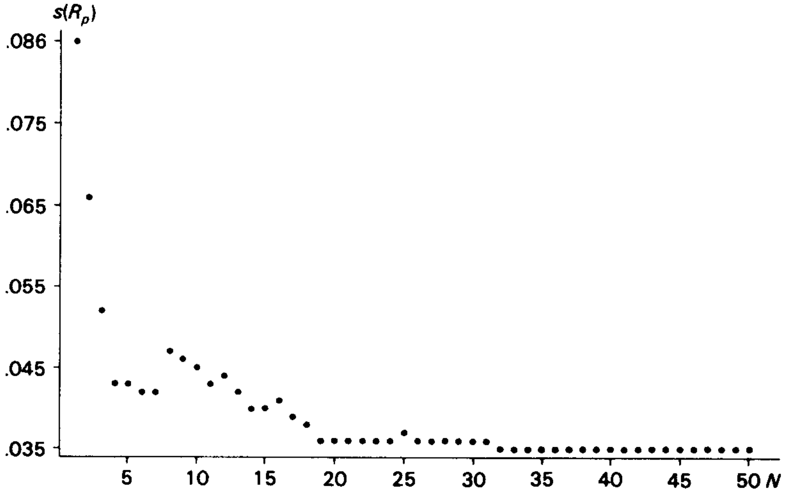


FIGURE 7.10

The Standard Deviation of Portfolio Return as a Function of the Number of Securities in the Portfolio: Second Sample of 50 Randomly Selected Stocks



further diversification has little effect. The reason for this phenomenon is evident in equation (43). With portfolios of equally weighted securities, the contribution of security return variances to the variance of portfolio return declines with N more or less in proportion to $1/N$. The function $f(N) = 1/N$ moves toward zero at an ever decreasing rate, accomplishing 95 percent of its total descent as N goes from 1 to 20.

In a sense, though, the random sample of 50 stocks in Table 7.1 might be too well behaved: that is, in Figure 7.9, $s(R_p)$ declines perhaps a little more smoothly as N is increased than we might typically expect. Although the average security return variance $\overline{s^2(R_i)}$ and the average of the security return covariances $\overline{s_{ij}}$ in (43) are not expected to change in any particular systematic way as a function of N, they will change to some extent, and this gives rise to some amount of irregularity in the behavior of $s(R_p)$ as a function of N. We can perhaps see this a little better in Figure 7.10, which is constructed in the same way as Figure 7.9 but from a different sample of 50 randomly selected stocks. In Figure 7.10 there is a noticeable jump in $s(R_p)$ when the eighth security is added to the portfolio, whereas in Figure 7.9 there is a much less noticeable jump in $s(R_p)$ when the 15th security is added.

PROBLEM IV.C

1. Compute the values of $f(N) = 1/N$ for $N = 1, 2, 3, 4, 5, 10, 25, 50, 100$, and 1,000, and make a rough sketch of the function.

ANSWER

1. Do it!

Finally, when one examines carefully columns (4) and (5) for the portfolio of 50 securities in Table 7.1, it seems that securities with large return variances generally have large average covariances with other securities. The same phenomenon would be observed for other securities and portfolios. A possible explanation is as follows.

One can argue that for measuring the association between the returns on two securities, the correlation between the returns is more relevant than the covariance, since the square of the correlation can be interpreted as the proportion of the variance of the return on either security that can be attributed to its linear association with the return on the other security. The covariance has no such direct interpretation. The correlation is, however, related to the covariance, since the former is

$$\rho(\tilde{R}_i, \tilde{R}_j) = \frac{\text{cov}(\tilde{R}_i, \tilde{R}_j)}{\sigma(\tilde{R}_i)\sigma(R_j)}. \qquad \cancel{(44)}$$

Alternatively, the covariance is

$$\text{cov}(\tilde{R}_i, \tilde{R}_j) = \rho(\tilde{R}_i, \tilde{R}_j)\sigma(\tilde{R}_i)\sigma(\tilde{R}_j). \qquad (45)$$

Consider two securities, i and k, whose returns have the same correlation with the return on security j, so that, in terms of proportion of variance explained, the returns on i and k have the same degree of association with the return on j. Nevertheless, equation (45) tells us that if $\sigma(\tilde{R}_i) > \sigma(\tilde{R}_k)$, then $\text{cov}(\tilde{R}_i, \tilde{R}_j) > \text{cov}(\tilde{R}_k, \tilde{R}_j)$. Thus, although the returns on i and k have the same correlation with the return on j, because the return on i has larger standard deviation than the return on k, it also has larger covariance with the return on j.

Suppose now that there is no systematic relationship between correlations of returns on securities and standard deviations of security returns. Then it follows from the discussion above that there will be a relationship between the standard deviations of security returns and their average covariances with the returns on other securities. As observed in Table 7.1, securities with larger standard deviations of return will tend to have larger average covariances with the returns on other securities.