CHAPTER 9

The Two-Parameter Model: Empirical Tests

I. Introduction

In the two-parameter portfolio model of Chapter 7 the capital market is perfect, distributions of returns on all portfolios are normal, and investors are risk-averse. These assumptions imply that the optimal portfolio for any investor is efficient in the sense that no other portfolio with the same or higher expected return has lower variance of return. In Chapter 8 we studied how the presumed attempts of investors to hold efficient portfolios are reflected in the process of price formation in the capital market. In the present chapter we test these implications of the two-parameter model for the behavior of returns on securities and portfolios.

The results that we report are from Fama and MacBeth (1973). A detailed discussion of the problems that arise in tests of the two-parameter model was first given by Miller and Scholes (1972). The first empirical study to provide solutions to these problems was Black, Jensen, and Scholes (1972). The approach of Fama and MacBeth is similar to that of Black, Jensen, and Scholes.

II. Testing the Model: General Discussion

All of the testable two-parameter models of market equilibrium discussed in Chapter 8 have a common implication: Market equilibrium requires that the value-weighted market portfolio M be a minimum variance portfolio and, generally, an efficient portfolio. This means that the mathematical condition that defines a minimum variance portfolio necessarily applies to M, so that the expected return from time 1 to time 2 on any positive variance security* i is

$$E(\tilde{R}_i) = E(\tilde{R}_{0M}) + [E(\tilde{R}_M) - E(\tilde{R}_{0M})]\,\beta_{iM}, \quad i = 1, \ldots, n. \tag{1}$$

As in Chapter 8, n is the number of positive variance securities in the market; $E(\tilde{R}_i)$ and $E(\tilde{R}_M)$ are the expected returns on security i and on the market portfolio M; $E(\tilde{R}_{0M})$ is the expected return on any positive variance security or any portfolio of such securities whose return is uncorrelated with the return on M; and

$$\beta_{iM} = \frac{\operatorname{cov}(\tilde{R}_i, \tilde{R}_M)}{\sigma^2(\tilde{R}_M)} \tag{2}$$

is the risk of security i in the market portfolio M measured relative to the risk of M.

The fact that a market equilibrium requires that M be a minimum variance portfolio transforms (1) from a condition on portfolio weights that must be met if a portfolio is to minimize variance at a given level of expected return to a condition on equilibrium prices and expected returns. Since a market equilibrium requires that M be minimum variance, prices must be set in such a way that (1) holds for every positive variance security. The purpose of this chapter is to test whether actual average returns conform to the implications of this expected return-risk relationship.

A. *Hypotheses about Expected Returns*

Most of the testable implications of (1) can be noted from inspection. First, the equation says that securities are priced so that the relationship between the expected return on security i and its risk in M, β_{iM}, is linear. Second, since β_{iM} is the only measure of risk that appears in (1), it is the only measure of risk that we need in order to explain differences among the ex-

*Positive variance securities are securities whose return distributions have strictly positive variances.

pected returns on securities. Third, since the models of market equilibrium say that securities must be priced so that M is on the positively sloped segment of the boundary of minimum variance portfolios, in a market equilibrium $E(\tilde{R}_M) > E(\tilde{R}_{0M})$; that is, securities are priced so that (1) implies a positive relationship between the expected return on any security and its risk in M.

Finally, we also test the implications of the various models of market equilibrium for the expected returns on securities and portfolios that have $\beta_{iM} = 0.0$. In the Sharpe-Lintner model there is unrestricted borrowing and lending at a common risk-free rate R_F, and R_F is also the expected return $E(\tilde{R}_{0M})$ on any positive variance securities and portfolios that have $\beta_{iM} = 0.0$. In the basic Black model there are no risk-free securities, but all zero-β_{iM} securities and portfolios have the common expected return $E(\tilde{R}_{0M})$. In the variants of the Black model there are risk-free securities, but there are restrictions on the amount of borrowing at the risk-free rate. In one variant there is no borrowing at the risk-free rate, while in another borrowing is limited to a fixed fraction of the investor's portfolio funds. In either case (1) is the relevant market equilibrium condition for the expected returns on positive variance securities, so that $E(\tilde{R}_{0M})$ is the expected return on all positive variance securities and portfolios that have $\beta_{iM} = 0.0$. In these models, however, R_F is less than $E(\tilde{R}_{0M})$.

Tests of the implications of the Sharpe-Lintner model for expected returns on positive variance securities and portfolios that have $\beta_{iM} = 0.0$ against the corresponding implications of the "restricted borrowing" variants of the Black model are tests of differing implications of different models of market equilibrium. Of course, it is well to know which of the various models of market equilibrium best describes the world. However, we are more interested in the general question of whether the behavior of returns is consistent with a world where investors attempt to hold efficient portfolios. This is the fundamental empirical question of the two-parameter model. To give it testable content, some model of market equilibrium is required. We must know specifically what it means to say that the prices of securities reflect the attempts of investors to hold efficient portfolios. Finding which model of market equilibrium works best is important but secondary to the fundamental issue.

B. *Competing Hypotheses*

To test the implications drawn from (1) concerning the pricing of securities in a two-parameter world, we need an alternative hypothesis about expected returns that includes (1) as a special case but allows us to reject (1) if it is inappropriate. We propose the following:

$$E(\tilde{R}_i) = E(\tilde{R}_{0M}) + [E(\tilde{R}_M) - E(\tilde{R}_{0M})]\,\beta_{iM} + q\beta_{iM}^2 + d\sigma(\tilde{\epsilon}_i). \tag{3}$$

This representation of the expected return on security i includes two explanatory variables, β_{iM}^2 and $\sigma(\tilde{\epsilon}_i)$, that do not appear in (1). The quadratic term β_{iM}^2 is included to test the proposition of (1) that the expected return-risk relationship is linear in β_{iM}. If this proposition is true, then q in (3) is zero. Likewise, $\sigma(\tilde{\epsilon}_i)$ is meant to be some measure of the "risk" of security i which is not an exact function of β_{iM}. It is included in (3) to test the proposition of (1) that β_{iM} is the only measure of risk needed to explain the expected return on security i. If this proposition is correct, then d, the coefficient of $\sigma(\tilde{\epsilon}_i)$ in (3), is zero.

The measure of "non-β_{iM} risk," $\sigma(\tilde{\epsilon}_i)$, that we propose to use in (3) comes out of the market model of Chapters 3 and 4. The two-parameter model assumes that the joint distribution of security returns is multivariate normal. This implies that the joint distribution of the return on any security and the return on the market portfolio M is bivariate normal. Bivariate normality implies a market model relationship between $\tilde{R}_i$ and $\tilde{R}_M$ of the form

$$\tilde{R}_i = \alpha_{iM} + \beta_{iM}\tilde{R}_M + \tilde{\epsilon}_i, \quad i = 1, \ldots, n, \tag{4}$$

where

$$\alpha_{iM} = E(\tilde{R}_i) - \beta_{iM}E(\tilde{R}_M), \tag{5}$$

β_{iM} is the relative risk measure of (2), and $\tilde{\epsilon}_i$ is a random disturbance that has expected value equal to zero and is independent of $\tilde{R}_M$. Since $\tilde{\epsilon}_i$ and $\tilde{R}_M$ are independent,

$$\sigma^2(\tilde{R}_i) = \beta_{iM}^2\sigma^2(\tilde{R}_M) + \sigma^2(\tilde{\epsilon}_i). \tag{6}$$

Portfolio theory implies a world where the pricing of securities reflects the attempts of investors to hold efficient portfolios. An alternative model, completely antithetical to portfolio theory, says that the pricing of securities is dominated by investors who hold single-security portfolios. Given a market of risk-averse investors, this model says that a security's expected return is positively related to the variance of its return rather than to β_{iM}. We can see from (6) that the variance of the return on security i can be split into two components. One depends directly on β_{iM}, but the other, the disturbance variance $\sigma^2(\tilde{\epsilon}_i)$, does not. Thus, under the alternative model, $\sigma^2(\tilde{\epsilon}_i)$ or, equivalently, the standard deviation of the disturbance, $\sigma(\tilde{\epsilon}_i)$, is a measure of the non-β_{iM} risk of security i, and $\sigma(\tilde{\epsilon}_i)$ is the measure that appears in (3).

C. *The Portfolio Approach to the Tests*

To test whether the coefficients q and d in (3) are, as implied by (1), equal to zero, we need estimates of these coefficients. We obtain them by forming portfolios whose returns have expected values equal to q or d and whose expected returns reflect only the effects of β_{iM}^2 or of $\sigma(\tilde{\epsilon}_i)$ in (3). We can then compute the time series of the returns on these portfolios and test whether the mean returns are different from zero.

Specifically, multiply through (3) by the portfolio weight x_{ip} and sum over i to get

$$E(\tilde{R}_p) = \sum_{i=1}^{n} x_{ip}E(\tilde{R}_i) = E(\tilde{R}_{0M}) \sum_{i=1}^{n} x_{ip} + [E(\tilde{R}_M) - E(\tilde{R}_{0M})] \sum_{i=1}^{n} x_{ip}\beta_{iM}$$

$$+ q \sum_{i=1}^{n} x_{ip}\beta_{iM}^2 + d \sum_{i=1}^{n} x_{ip}\sigma(\tilde{\epsilon}_i). \tag{7}$$

To get a portfolio that has expected return equal to q, we choose the x_{ip} in such a way that the weighted average of β_{iM}^2 is 1.0, and all other variables that appear in (3) are "zeroed out." Thus, we choose $x_{ip}, i = 1, \ldots, n$, so that

$$\sum_{i=1}^{n} x_{ip} = 0 \tag{8a}$$

$$\sum_{i=1}^{n} x_{ip}\beta_{iM} = 0 \tag{8b}$$

$$\sum_{i=1}^{n} x_{ip}\beta_{iM}^2 = 1 \tag{8c}$$

$$\sum_{i=1}^{n} x_{ip}\sigma(\tilde{\epsilon}_i) = 0. \tag{8d}$$

To get a portfolio that has expected return equal to d, we choose another set of x_{ip} in such a way that the weighted average of the $\sigma(\tilde{\epsilon}_i)$ is 1.0 but all other variables in (3) are "zeroed out." This implies choosing the x_{ip} so that

$$\sum_{i=1}^{n} x_{ip} = 0 \tag{9a}$$

$$\sum_{i=1}^{n} x_{ip}\beta_{iM} = 0 \tag{9b}$$

$$\sum_{i=1}^{n} x_{ip}\beta_{iM}^2 = 0 \tag{9c}$$

$$\sum_{i=1}^{n} x_{ip}\sigma(\tilde{\epsilon}_i) = 1. \tag{9d}$$

From (8b) and (9b) we see that the portfolios described by equations (8) and (9) are zero-β_{pM} portfolios: that is, they are portfolios that have $\beta_{pM} = \sum_{i=1}^{n} x_{ip}\beta_{iM} = 0.0$. From (8a) and (9a) we see that they are also "zero investment" portfolios: the investor puts up no personal funds but rather obtains long positions in some securities ($x_{ip} > 0$) by taking short positions ($x_{ip} < 0$) in others. This is in contrast to a "standard portfolio," where the sum of the weights invested in individual securities is 1.

PROBLEMS II.C

1. Does condition (8d) imply a portfolio whose return is perfectly correlated with the return on M, that is, a portfolio whose market model disturbance is always zero?

2. Use equation (7) to show that equation (3) is only a legitimate representation of expected returns on securities when q and d in (3) are equal to zero.

ANSWERS

1. Equation (8d) is a condition on the weighted average of the standard deviations of market model disturbances for individual securities. It implies nothing in particular about the market model disturbance for the resulting portfolio. Thus, if we multiply through (4) by x_{ip} and sum over i, the market model disturbance for the resulting portfolio is

$$\tilde{\epsilon}_p = \sum_{i=1}^{n} x_{ip}\tilde{\epsilon}_i,$$

which has standard deviation

$$\sigma(\tilde{\epsilon}_p) = \left[\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ip}x_{jp} \operatorname{cov}(\tilde{\epsilon}_i, \tilde{\epsilon}_j)\right]^{1/2}.$$

It is then clear that the constraint on the portfolio weights described by (8d) does not imply $\sigma(\tilde{\epsilon}_p) = 0$.

In the same vein, note that equation (8c) implies nothing about the value of β_{pM}^2 that results when a portfolio is formed according to the equations of (8). Indeed, since (8b) implies $\beta_{pM} = 0.0$, it implies $\beta_{pM}^2 = 0$, while (8c) says that the x_{ip} must be chosen so that the weighted average of the β_{iM}^2 of individual securities is 1.

2. If x_{ip} in (7) is set equal to x_{iM}, we have the market portfolio M with $E(\tilde{R}_p) = E(\tilde{R}_M)$. Equation (7) will only lead us to this conclusion when q and d are both equal to zero. Thus, strictly speaking, (3) is only a legitimate representation of expected returns on securities when it reduces to (1). To make the notation in (3) rigorous, we could change $E(\tilde{R}_{0M})$ to, say, h and $[E(\tilde{R}_M) - E(\tilde{R}_{0M})]$ to, say, k. The notation in (3), however, maintains a convenient analogy with (1), and switching to a more rigorous notation would not change anything that follows.

The equations of (8) impose four constraints on the weights given to individual securities in the portfolio whose expected return concentrates on the effects of nonlinearities. Since the number of available securities n is much greater than four, there are many portfolios that satisfy (8). We are not indifferent among them. We use the average return on the chosen portfolio to test the hypothesis that the nonlinearity coefficient q in (3) is zero. To make the most reliable possible inference, we want the portfolio whose average return gives the most reliable estimate of q. This means choosing the portfolio that has the smallest variance of return among all portfolios that satisfy (8). Likewise, to make the most reliable inference about the hypothesis that the coefficient d of $\sigma(\tilde{\epsilon}_i)$ in (3) is zero, we want the portfolio that has the smallest variance of return among all portfolios where the weights applied to individual securities satisfy (9).

D. Least Squares Coefficients as Portfolio Returns

Finding the portfolios that have these properties seems a formidable task, but the approach generally taken is straightforward.

INFORMAL DISCUSSION

Let us recognize that we use data for many time periods (actually months) to test the implications of (1). Let us represent the return on security i for month t as

$$\tilde{R}_{it} = \tilde{\gamma}_{1t} + \tilde{\gamma}_{2t}\beta_{iM} + \tilde{\gamma}_{3t}\beta_{iM}^2 + \tilde{\gamma}_{4t}\sigma(\tilde{\epsilon}_i) + \tilde{\eta}_{it}, \quad i = 1, \ldots, n. \tag{10}$$

Suppose we know the values of β_{iM}, β_{iM}^2 and $\sigma(\tilde{\epsilon}_i)$ for each of the securities in the market, and suppose we take $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, $\tilde{\gamma}_{3t}$, and $\tilde{\gamma}_{4t}$ to be the least squares coefficients from a multiple regression of the n security returns for month t on the n combinations of β_{iM}, β_{iM}^2, and $\sigma(\tilde{\epsilon}_i)$. Then $\tilde{\gamma}_{3t}$ and $\tilde{\gamma}_{4t}$ are the returns for month t on portfolios that, respectively, conform to the constraints of (8) and (9). Under certain assumptions about the disturbances $\tilde{\eta}_{it}$ for different securities, the least squares values of $\tilde{\gamma}_{3t}$ and $\tilde{\gamma}_{4t}$ are also the returns for month t on the portfolios with the smallest possible return variances among all portfolios that satisfy (8) and (9).

Note that, although we usually think of "least squares" as a method of estimation, we do not use the term "estimates" when talking about the least squares values of $\tilde{\gamma}_{3t}$ and $\tilde{\gamma}_{4t}$ in (10). The appropriate view is that the least squares method just provides a convenient way to obtain portfolio returns that have the desired properties.

We now discuss more formally the properties claimed for the least squares values of $\tilde{\gamma}_{3t}$ and $\tilde{\gamma}_{4t}$. In the process, we uncover interesting properties of the least squares values of $\tilde{\gamma}_{1t}$ and $\tilde{\gamma}_{2t}$ in (10). Unfortunately, any reasonably concise exposition of the arguments requires some matrix algebra. The mathematically disinclined can skip down to the discussion that follows equation (22).

FORMAL DISCUSSION

Let

$$\underline{\tilde{R}}_t = \begin{pmatrix} \tilde{R}_{1t} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{R}_{nt} \end{pmatrix} \tag{11}$$

be the $(n \times 1)$ vector of returns on individual securities for month t. Let

$$\underline{C} = \begin{pmatrix} 1 & \beta_{1M} & \beta_{1M}^2 & \sigma(\tilde{\epsilon}_1) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \beta_{nM} & \beta_{nM}^2 & \sigma(\tilde{\epsilon}_n) \end{pmatrix} \tag{12}$$

be the $(n \times 4)$ matrix of the values of the explanatory variables in (10). Let

$$\underline{\tilde{\eta}}_t = \begin{pmatrix} \tilde{\eta}_{1t} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\eta}_{nt} \end{pmatrix} \tag{13}$$

be the $(n \times 1)$ vector of security return disturbances for month t in (10), and let

$$\underline{\tilde{\gamma}}_t = \begin{pmatrix} \tilde{\gamma}_{1t} \\ \tilde{\gamma}_{2t} \\ \tilde{\gamma}_{3t} \\ \tilde{\gamma}_{4t} \end{pmatrix} \tag{14}$$

be the (4×1) vector of the least squares values of $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, $\tilde{\gamma}_{3t}$, and $\tilde{\gamma}_{4t}$ for month t. Then the matrix representation of (10) is

$$\underline{\tilde{R}}_t = \underline{C}\underline{\tilde{\gamma}}_t + \underline{\tilde{\eta}}_t. \tag{15}$$

The least squares value of $\underline{\tilde{\gamma}}_t$ is*

$$\underline{\tilde{\gamma}}_t = (\underline{C}'\underline{C})^{-1}\underline{C}'\underline{\tilde{R}}_t. \tag{16}$$

Equivalently, if we define the $(4 \times n)$ matrix

$$\underline{X} = (\underline{C}'\underline{C})^{-1}\underline{C}', \tag{17}$$

then

$$\underline{\tilde{\gamma}}_t = \underline{X}\underline{R}_t \tag{18}$$

or

$$\tilde{\gamma}_{jt} = \sum_{i=1}^{n} x_{ij}\tilde{R}_{it}, \quad j = 1, 2, 3, 4, \tag{19}$$

where x_{ij} is the element of row j ($j = 1, 2, 3, 4$) and column i ($i = 1, \ldots, n$) of $\underline{X}$. This is the reverse of the more common notation, where the first subscript on x would refer to a row of the matrix $\underline{X}$, and the second subscript would refer to a column. The choice here reflects the fact that we shall interpret x_{ij} as the portfolio weight assigned to security i to get the least squares value of the portfolio return $\tilde{\gamma}_{jt}$. In the notation for portfolio weights it is our practice to use the first subscript to refer to the security and the second to refer to the portfolio.

Equation (19) says that the least squares value of $\tilde{\gamma}_{jt}$ is the return on a portfolio where the weights assigned to the n individual securities are the n elements of the jth row of the matrix $\underline{X}$. To determine the properties of the $\tilde{\gamma}_{jt}$, we study the properties of $\underline{X}$. Note first that

$$\underline{X}\,\underline{C} = (\underline{C}'\underline{C})^{-1}\underline{C}'\underline{C} = \underline{I}, \tag{20}$$

where $\underline{I}$ is the (4×4) identity matrix; that is, $\underline{I}$ has 1's along its diagonal and 0's elsewhere. From (12), the first column of $\underline{C}$ is an $(n \times 1)$ vector of 1's. Thus from (12) and (20), it follows that

$$\sum_{i=1}^{n} x_{ij} = \begin{cases} 1 \text{ for } j = 1 \\ 0 \text{ for } j = 2, 3, 4 \end{cases}. \tag{21}$$

In words, the least squares value of $\tilde{\gamma}_{1t}$ in (19) is the return on a standard portfolio, that is, a portfolio where the sum of the weights assigned to individual securities is 1. The least squares values of $\tilde{\gamma}_{2t}$, $\tilde{\gamma}_{3t}$, and $\tilde{\gamma}_{4t}$ given by (19) are returns on zero-investment portfolios, that is, portfolios where the sum of the weights assigned to individual securities is zero.

*See, for example, Theil (1971, chap. 3).

Next note that if c_{ki} is the element of row i and column k of $\underline{C}$, then (20) can be written as

$$\sum_{i=1}^{n} x_{ij}c_{ki} = \begin{cases} 1 \text{ for } j = k \\ 0 \text{ for } j \neq k \end{cases}. \tag{22}$$

From (19), x_{ij}, $i = 1, \ldots, n$ are the weights applied to individual security returns to get the least squares values of $\tilde{\gamma}_{jt}$, $j = 1, 2, 3, 4$. If we enumerate (22) for $j = 3$ and $k = 1, 2, 3, 4$, we find that the least squares value of $\tilde{\gamma}_{3t}$ is the return on a portfolio that satisfies the constraints of (8), and thus it has expected return equal to q, the coefficient of β_{iM}^2 in (3). Likewise, enumeration of (22) for $j = 4$ and $k = 1, 2, 3, 4$ shows that the least squares value of $\tilde{\gamma}_{4t}$ is the return on a portfolio where the proportions invested in individual securities satisfy the constraints of (9), so that the portfolio has expected return equal to d, the coefficient of $\sigma(\tilde{\epsilon}_i)$ in (3). Thus, if for each month t we calculate the cross-sectional regression of security returns on β_{iM}, β_{iM}^2, and $\sigma(\tilde{\epsilon}_i)$, the means of the times series of the least squares values of $\tilde{\gamma}_{3t}$ and $\tilde{\gamma}_{4t}$ can be used to test the propositions of (1) that the relationship between the expected returns on securities and their risks in M is linear, and that β_{iM} is the only measure of risk needed to explain differences between the expected returns on different securities.

E. *Getting the Most Powerful Tests: General Discussion*

TESTING $E(\tilde{R}_M) > E(\tilde{R}_{0M})$ AND $E(\tilde{R}_{0M}) = R_F$

If we enumerate (22) for $j = 1$ and $j = 2$, we find that the least squares values of $\tilde{\gamma}_{1t}$ and $\tilde{\gamma}_{2t}$ in (10) are also the returns on portfolios that have desirable properties. The weights x_{i1}, $i = 1, \ldots, n$, applied to individual security returns to get $\tilde{\gamma}_{1t}$, satisfy

$$\sum_{i=1}^{n} x_{i1} = 1 \tag{23a}$$

$$\sum_{i=1}^{n} x_{i1}\beta_{iM} = 0 \tag{23b}$$

$$\sum_{i=1}^{n} x_{i1}\beta_{iM}^2 = 0 \tag{23c}$$

$$\sum_{i=1}^{n} x_{i1}\sigma(\tilde{\epsilon}_i) = 0. \tag{23d}$$

In words, (23a) says that the least squares value of $\tilde{\gamma}_{1t}$ is the return on a standard portfolio; (23b) says that the β_{pM} of this portfolio is zero; and (23c) and (23d) say that the weights invested in individual securities are also chosen so as to "zero out" any effects of β_{iM}^2 and $\sigma(\tilde{\epsilon}_i)$ on expected returns. In terms of (3) or (7), the least squares value of $\tilde{\gamma}_{1t}$ is the return on a portfolio where securities are weighted so that $E(\tilde{\gamma}_{1t}) = E(\tilde{R}_{0M})$. The mean of the time series of $\tilde{\gamma}_{1t}$ can be used to test the proposition of the Sharpe-Lintner model that $E(\tilde{R}_{0M})$, the expected return on any positive variance security or portfolio whose return is uncorrelated with the return on M, is equal to the risk-free rate R_F.

Likewise, (22) says that the weights applied to individual security returns to get $\tilde{\gamma}_{2t}$ in (19) satisfy

$$\sum_{i=1}^{n} x_{i2} = 0 \tag{24a}$$

$$\sum_{i=1}^{n} x_{i2}\beta_{iM} = 1 \tag{24b}$$

$$\sum_{i=1}^{n} x_{i2}\beta_{iM}^2 = 0 \tag{24c}$$

$$\sum_{i=1}^{n} x_{i2}\sigma(\tilde{\epsilon}_i) = 0. \tag{24d}$$

Thus the least squares value of $\tilde{\gamma}_{2t}$ is the return on a zero-investment portfolio that has $\beta_{pM} = 1$ and that zeroes out any effects of β_{iM}^2 and $\sigma(\tilde{\epsilon}_i)$ on expected returns. If we substitute the conditions described by (24) into (7), we find that the expected value of $\tilde{\gamma}_{2t}$ is $E(\tilde{R}_M) - E(\tilde{R}_{0M})$. We can, then, use the mean value of the time series of $\tilde{\gamma}_{2t}$ to test the proposition that $E(\tilde{R}_M) > E(\tilde{R}_{0M})$.

Tests of the hypotheses that $E(\tilde{R}_{0M}) = R_F$ and $E(\tilde{R}_M) > E(\tilde{R}_{0M})$ are not of great interest, however, unless q and d in (3) are equal to zero. The hypothesis that $E(\tilde{R}_M) > E(\tilde{R}_{0M})$ is concerned with whether M is on the positively sloped segment of the minimum variance boundary. The proposition that $E(\tilde{R}_{0M}) = R_F$ is concerned with whether M is at the point on the positively sloped segment of the boundary where a straight line from R_F is tangent to the boundary. Tests of these two propositions only make sense when the data are consistent with the hypothesis that M is somewhere on the minimum variance boundary; that is, the data must be consistent with the propositions that the relationship between the expected returns on securities and their risks in M is linear ($q = 0.0$) and that β_{iM} is the only measure of risk needed to explain expected security returns ($d = 0.0$).

If the data are consistent with the hypothesis that q and d in (3) are zero, then the time series of the least squares values of $\tilde{\gamma}_{1t}$ and $\tilde{\gamma}_{2t}$ in (10) provide the basis for tests of the hypotheses that $E(\tilde{R}_{0M}) = R_F$ and $E(\tilde{R}_M) > E(\tilde{R}_{0M})$. The inferences obtained, however, are not likely to be the most reliable possible. From (23), the least squares value of $\tilde{\gamma}_{1t}$ in (10) is the return on a standard portfolio that has $\beta_{pM} = 0.0$ and that also zeroes out any effects of β_{iM}^2 and $\sigma(\tilde{\epsilon}_i)$ on expected returns; the portfolio weights satisfy (23c) and (23d) as well as (23a) and (23b). If q and d in (3) are zero, then when testing the hypothesis that $E(\tilde{R}_{0M}) = R_F$, we do not have to worry about zeroing out the effects of β_{iM}^2 and $\sigma(\tilde{\epsilon}_i)$. If the constraints of (23c) and (23d) do not need to be met, we can probably find a standard zero-β_{pM} portfolio that has a smaller variance of return than any standard zero-β_{pM} portfolio that also satisfies (23c) and (23d). Likewise, although the least squares value of $\tilde{\gamma}_{2t}$ in (10) is the return on a zero-investment portfolio that has $\beta_{pM} = 1.0$ and expected return equal to $E(\tilde{R}_M) - E(\tilde{R}_{0M})$, it is also a portfolio that satisfies the constraints of (24c) and (24d). If the tests indicate that these constraints can be ignored, we can probably find a portfolio that has a smaller variance of return and so provides more reliable tests of the hypothesis that $E(\tilde{R}_M) > E(\tilde{R}_{0M})$.

One way to obtain portfolio returns appropriate for testing the hypotheses that $E(\tilde{R}_M) > E(\tilde{R}_{0M})$ and $E(\tilde{R}_{0M}) = R_F$, while ignoring possibly redundant constraints, is from cross-sectional regressions of security returns on their β_{iM}. Consider

$$\tilde{R}_{it} = \tilde{\gamma}_{1t} + \tilde{\gamma}_{2t}\beta_{iM} + \tilde{\eta}_{it}, \quad i = 1, \ldots, n. \tag{25}$$

If $\tilde{\gamma}_{1t}$ is defined as the intercept in the least squares regression of the n values of $\tilde{R}_{it}$ for month t on the corresponding n values of β_{iM}, then $\tilde{\gamma}_{1t}$ is the return on a standard zero-β_{pM} portfolio, but one that imposes no particular constraints on the weighted averages of β_{iM}^2 and $\sigma(\tilde{\epsilon}_i)$. The weights assigned to individual security returns satisfy (23a) and (23b) but not (23c) and (23d). If q and d in (7) are equal to zero, the expected return on this portfolio is $E(\tilde{R}_{0M})$. If we calculate the time series of cross-sectional regressions described by (25), the mean of the resulting time series of $\tilde{\gamma}_{1t}$ can be used to test the Sharpe-Lintner proposition that $E(\tilde{R}_{0M}) = R_F$. Likewise, the least squares value of $\tilde{\gamma}_{2t}$ in (25) is the return on a zero-investment portfolio that has $\beta_{pM} = 1.0$ but imposes no particular constraints on the weighted averages of β_{iM}^2 and $\sigma(\tilde{\epsilon}_i)$. The weights assigned to individual security returns satisfy (24a) and (24b) but not (24c) and (24d). If q and d in (7) are zero, the expected value of $\tilde{\gamma}_{2t}$ is $E(\tilde{R}_M) - E(\tilde{R}_{0M})$, and the mean of the time series of $\tilde{\gamma}_{2t}$ can be used to test the hypothesis that $E(\tilde{R}_M) > E(\tilde{R}_{0M})$.

PROBLEM II.E

1. The expected return-risk equation (1) holds for portfolios as well as for individual securities. (See Problem II.C.*2* of Chapter 8.) The least squares value of $\tilde{\gamma}_{2t}$ in (25) is the return on a portfolio that has $\beta_{pM} = 1.0$. Why is the expected return on this portfolio $E(\tilde{R}_M) - E(\tilde{R}_{0M})$ rather than $E(\tilde{R}_M)$, as (1) would seem to imply?

ANSWER

1. Multiplying through (1) by x_{ip} and summing over i yields

$$\sum_{i=1}^{n} x_{ip} E(\tilde{R}_i) = E(\tilde{R}_{0M}) \sum_{i=1}^{n} x_{ip} + [E(\tilde{R}_M) - E(\tilde{R}_{0M})] \sum_{i=1}^{n} x_{ip} \beta_{iM}$$

$$E(\tilde{R}_p) = E(\tilde{R}_{0M}) \sum_{i=1}^{n} x_{ip} + [E(\tilde{R}_M) - E(\tilde{R}_{0M})]\, \beta_{pM}.$$

The expected return on any standard portfolio ($\Sigma x_{ip} = 1.0$) that has $\beta_{pM} = 1.0$ is $E(\tilde{R}_M)$, but the expected return on a zero-investment portfolio ($\Sigma x_{ip} = 0$) that has $\beta_{pM} = 1.0$ is $E(\tilde{R}_M) - E(\tilde{R}_{0M})$.

TESTING FOR THE EFFECTS OF NONLINEARITIES AND NON-β_{iM} RISKS

The idea that better tests can be obtained when irrelevant constraints are ignored also applies to tests of the propositions that the coefficients q and d in (3) are equal to zero. The least squares value of $\tilde{\gamma}_{3t}$ in

$$\tilde{R}_{it} = \tilde{\gamma}_{1t} + \tilde{\gamma}_{2t}\beta_{iM} + \tilde{\gamma}_{3t}\beta_{iM}^2 + \tilde{\eta}_{it}, \quad i = 1, \ldots, n, \tag{26}$$

is the return on a zero-investment portfolio that has $\beta_{pM} = 0.0$ and where the weighted average of β_{iM}^2 is 1.0. Hence the portfolio satisfies the constraints of (8a), (8b), and (8c), but the constraint of (8d) is ignored. If d in (3) is zero, then from (7) and (8a-c) we can determine that $E(\tilde{\gamma}_{3t}) = q$. Thus, the mean of the time series of the portfolio return $\tilde{\gamma}_{3t}$ can be used to test the proposition that the nonlinearity coefficient q in (3) is zero.

Finally, if q in (3) is zero, there is no need to "zero out" the effects of β_{iM}^2 when testing the proposition that β_{iM} is the only measure of risk needed to explain differences among expected security returns. To test the proposition that d in (3) is zero, we can form a portfolio that satisfies (9a), (9b), and (9d) but ignores (9c). One such portfolio is given by the least squares value of $\tilde{\gamma}_{4t}$ in

$$\tilde{R}_{it} = \tilde{\gamma}_{1t} + \tilde{\gamma}_{2t}\beta_{iM} + \tilde{\gamma}_{4t}\sigma(\tilde{\epsilon}_i) + \tilde{\eta}_{it}, \quad i = 1, \ldots, n. \tag{27}$$

When q in (3) is zero, the mean of the time series of the $\tilde{\gamma}_{4t}$ can be used to test the hypothesis that $\sigma(\tilde{\epsilon}_i)$ contributes nothing to the explanation of differences in expected security returns.

We have claimed, without proof, some properties for the least squares values of the various $\tilde{\gamma}_{jt}$ in (25) to (27). To establish these properties, we would repeat the arguments of equations (11) to (22), deleting some explanatory variables from the matrix $\underline{C}$ of (12). For example, to establish the properties claimed for $\tilde{\gamma}_{3t}$ in (26), one would delete the column of $\underline{C}$ corresponding to $\sigma(\tilde{\epsilon}_i)$. The analysis that leads to (22) would then imply that the least squares value of $\tilde{\gamma}_{3t}$ satisfies the conditions of (8a-c); but since $\sigma(\tilde{\epsilon}_i)$ is not included as an explanatory variable, the relevant version of (22) implies no constraint, like (8d), on the weighted average of the $\sigma(\tilde{\epsilon}_i)$. In short, when one deletes $\sigma(\tilde{\epsilon}_i)$ and/or β_{iM}^2 from (10), the analysis of (11) to (22) still applies, but the least squares procedure only imposes constraints on variables that are explicitly included in the return equation.

In this respect, there is nothing special about equations (10) and (25) to (27). Anytime one does a cross-sectional regression of security returns on their β_{iM} and on other variables, an analysis similar to (11) to (22) leads to the conclusion that the least squares intercept is the return on a standard portfolio that has $\beta_{pM} = 0.0$ and that zeros out the effects of other variables. The least squares value of the coefficient of any other variable is the return on a zero-investment portfolio where the weights assigned to individual securities have the effect of setting the weighted average of the values of the variable for different securities equal to 1, while zeroing out the effects of all other variables. An excellent example of this general property of cross-sectional risk-return regressions, outside the context of the concerns of this chapter, is provided by Black and Scholes (1974).

PROBLEMS II.E

2. The least squares values of $\tilde{\gamma}_{1t}$ and $\tilde{\gamma}_{2t}$ in (25) can be written as

$$\tilde{\gamma}_{2t} = \frac{\sum_{i=1}^{n} (\tilde{R}_{it} - \tilde{\bar{R}}_t)(\beta_{iM} - \bar{\beta}_M)}{\sum_{i=1}^{n} (\beta_{iM} - \bar{\beta}_M)^2} \tag{28}$$

$$\tilde{\gamma}_{1t} = \tilde{\bar{R}}_t - \tilde{\gamma}_{2t}\bar{\beta}_M, \tag{29}$$

where $\tilde{\bar{R}}_t$ is the average of the returns on the n securities for month t and $\bar{\beta}_M$ is the average of the β_{iM}. Use (28) and (29) to show that the least squares value of $\tilde{\gamma}_{1t}$ in (25) is the return on a standard portfolio that has $\beta_{pM} = 0.0$, while $\tilde{\gamma}_{2t}$ is the return on a zero-investment portfolio that has $\beta_{pM} =$

1.0. The analysis above has already discussed these properties of $\tilde{\gamma}_{1t}$ and $\tilde{\gamma}_{2t}$. The present approach, however, does not require the matrix algebra of equations (11) to (22), and it is better for the second task of this problem, which is to interpret the weights assigned to individual securities in $\tilde{\gamma}_{1t}$ and $\tilde{\gamma}_{2t}$ of (25).

3. Tests of the propositions that q and d in (3) are zero are tests of the proposition that M is on the minimum variance boundary. Do these tests make sense if $E(\tilde{R}_M) - E(\tilde{R}_{0M}) \leq 0.0$?

ANSWERS

2. First rewrite (28) as

$$\tilde{\gamma}_{2t} = \sum_{i=1}^{n}\left(\frac{\beta_{iM} - \bar{\beta}_M}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2}\right)\tilde{R}_{it} - \sum_{i=1}^{n}\left(\frac{\beta_{iM} - \bar{\beta}_M}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2}\right)\tilde{\bar{R}}_t$$

$$= \sum_{i=1}^{n}\left(\frac{\beta_{iM} - \bar{\beta}_M}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2}\right)\tilde{R}_{it}. \quad (30)$$

With (30), (29) can be developed as follows:

$$\tilde{\gamma}_{1t} = \tilde{\bar{R}}_t - \sum_{i=1}^{n}\left(\frac{(\beta_{iM} - \bar{\beta}_M)\bar{\beta}_M}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2}\right)\tilde{R}_{it}$$

$$= \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(\beta_{iM} - \bar{\beta}_M)\bar{\beta}_M}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2}\right)\tilde{R}_{it}, \quad (31)$$

and

$$\sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(\beta_{iM} - \bar{\beta}_M)\bar{\beta}_M}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2}\right) = 1 - \sum_{i=1}^{n}\frac{(\beta_{iM} - \bar{\beta}_M)\bar{\beta}_M}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2} = 1. \quad (32)$$

Equation (31) says that $\tilde{\gamma}_{1t}$ is a weighted average of the returns on individual securities for month t, while (32) says that the sum of the weights applied to individual securities is 1.0. Thus $\tilde{\gamma}_{1t}$ is the return on a standard portfolio. As for any portfolio, this portfolio has a β_{pM} which is just a weighted average of the β_{iM} for individual securities. Since the appropriate weights are given by the expression within the brackets of (31), the value of β_{pM} for $\tilde{\gamma}_{1t}$ is

$$\sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(\beta_{iM} - \bar{\beta}_M)\bar{\beta}_M}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2}\right)\beta_{iM} = \bar{\beta}_M - \frac{\sum_{i=1}^{n}\beta_{iM}^2 - n\bar{\beta}_M^2}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2}\bar{\beta}_M = 0.$$

To interpret the weights assigned to individual security returns in $\tilde{\gamma}_{1t}$, note from (31) that the first term in any weight is $1/n$, which implies equal weights for individual securities. The second term adjusts these equal weights according to the difference between β_{iM} and $\bar{\beta}_M$. The effect is to lower the investment in securities with above-average values of β_{iM} and increase the investment in securities with below-average values of β_{iM}. Thus, securities with values of β_{iM} close to $\bar{\beta}_M$ get a weight approximately equal to $1/n$. Securities with high values of β_{iM} are sold short, with proceeds from short sales used to increase the investment in securities with low values of β_{iM}.

A similar analysis shows that $\tilde{\gamma}_{2t}$, as defined by (28), is the return on a zero-investment portfolio that has $\beta_{pM} = 1.0$. We can see from (30) that the sum of the weights assigned to individual security returns in $\tilde{\gamma}_{2t}$ is zero, so that $\tilde{\gamma}_{2t}$ is the return on a zero-investment portfolio. We can then use the weights in (30) to determine that β_{pM} for $\tilde{\gamma}_{2t}$ is

$$\sum_{i=1}^{n}\left(\frac{\beta_{iM} - \bar{\beta}_M}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2}\right)\beta_{iM} = \frac{\sum_{i=1}^{n}\beta_{iM}^2 - n\bar{\beta}_M^2}{\sum_{i=1}^{n}(\beta_{iM} - \bar{\beta}_M)^2} = 1.0.$$

Since $\tilde{\gamma}_{2t}$ is the return on a zero-investment portfolio, the investor puts up no money of his own; rather, he sells some securities short and uses the proceeds to buy others. From (30) we can see that $\tilde{\gamma}_{2t}$ involves short positions in securities with below-average values of β_{iM} and long positions in securities with above-average values of β_{iM}, which reverses the general pattern of weights assigned to individual securities in $\tilde{\gamma}_{1t}$.

3. All the models of market equilibrium say that M is on the positively sloped segment of the minimum variance boundary, so that $E(\tilde{R}_M) > E(\tilde{R}_{0M})$. If this is not true, then tests of the linearity and non-β_{iM} risk propositions implied by (1) are not of great interest.

If $E(\tilde{R}_M) > E(\tilde{R}_{0M})$ in (1), then when testing the propositions that q and d in (3) are zero, it is necessary to form portfolios that zero out the effects of β_{iM} on expected returns. Including β_{iM} in the cross-sectional regressions guarantees this result. The least squares values of $\tilde{\gamma}_{3t}$ and $\tilde{\gamma}_{4t}$ in (10), (26), and (27) are the returns on portfolios that have $\beta_{pM} = 0.0$.

F. The Reliability of the Least Squares Portfolio Returns

We have shown that the least squares values of the various $\tilde{\gamma}_{jt}$ in equations (10) and (25) to (27) are the returns on portfolios that concentrate on the effects on expected returns of the different variables in (3). We have, however, provided no justification for the presumption that the least squares values of the $\tilde{\gamma}_{jt}$ are also the returns on the portfolios that have the smallest possible return variances among portfolios that concentrate on the effects of the different variables on expected returns. Providing some justification for the "smallest variance" properties of the least squares approach is important, since it will imply that the least squares portfolio returns provide more reliable inferences about the various hypotheses than other portfolios we might use. Unfortunately, the arguments require more statistical sophistication than is generally assumed in this book. The reader should nevertheless be able to grasp the sense of the discussion, especially when we eventually make comparisons of the reliability of the least squares $\tilde{\gamma}_{jt}$ of (10), (25), (26), and (27).

The "smallest variance" property of the least squares portfolio returns follows from an assumption on the properties of the disturbances $\tilde{\eta}_{it}$ in equations (10), (25), (26), and (27). For example, if the disturbance $\tilde{\eta}_{it}$ in (10) is independent from one security to another and if the distribution of $\tilde{\eta}_{it}$ is the same for all securities, then a slightly sophisticated appeal to the Gauss-Markov theorem* can be used to imply that the least squares values of $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, $\tilde{\gamma}_{3t}$, and $\tilde{\gamma}_{4t}$ are the returns on the portfolios that have the smallest possible return variances among portfolios that satisfy, respectively, the constraints on portfolio weights described by (23), (24), (8), and (9).

The Gauss-Markov theorem concerns the properties of the least squares estimators of the coefficients in a regression where the estimators are based on a sample from the process of interest. The trick in applying the theorem in the present case is to note that although the least squares $\tilde{\gamma}_{jt}$ are portfolio returns that are computed from the entire population of returns for month t, the process itself generates returns each month, so that the population of time periods from which the $\tilde{\gamma}_{jt}$ are drawn is in principle infinite. In this view, we can regard $\tilde{\gamma}_{jt}$ as an estimator of $E(\tilde{\gamma}_j)$. Given the appropriate properties of the disturbances, the Gauss-Markov theorem then says that the least squares value of $\tilde{\gamma}_{jt}$ minimizes the variance of $\tilde{\gamma}_{jt} - E(\tilde{\gamma}_j)$, the error in $\tilde{\gamma}_{jt}$ as an estimator of $E(\tilde{\gamma}_j)$. For any given $\tilde{\gamma}_{jt}$, this is precisely the variance we wish to minimize.

*See, for example, Theil (1971, p. 119) for a general discussion of the Gauss-Markov Theorem.

With these formalities behind us, we can now look at things from a more intuitive perspective. In effect, when applied to (10), the least squares procedure attempts to find portfolio returns $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, $\tilde{\gamma}_{3t}$, and $\tilde{\gamma}_{4t}$ that have the smallest variances subject to the constraints of (23), (24), (8), and (9), but where the search is carried out under the assumption that the disturbances $\tilde{\eta}_{it}$ for different securities are independent and identically distributed.

Analogous comments apply to the least squares values of $\tilde{\gamma}_{jt}$ in (25) to (27). For example, when applied to (26), the least squares approach searches for the values of $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, and $\tilde{\gamma}_{3t}$ that have the smallest possible variances subject to the constraints of (23), (24), and (8). Since $\sigma(\tilde{\epsilon}_i)$ does not appear as an explanatory variable in (26), the constraints (23d), (24d), and (8d) are ignored. Since there are fewer constraints on the portfolio weights, we expect that the least squares portfolio returns $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, and $\tilde{\gamma}_{3t}$ obtained from (26) will have smaller variances than the $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, and $\tilde{\gamma}_{3t}$ obtained from (10). Likewise, when applied to (27), the least squares approach tries to find the smallest variance $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, and $\tilde{\gamma}_{4t}$ that satisfy the constraints of (23), (24), and (9), except that the constraints (23c), (24c), and (9c) on the weighted average of the β_{iM}^2 are ignored. Again, we expect that ignoring these constraints will lead to $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, and $\tilde{\gamma}_{4t}$ which have smaller variances than those obtained from (10). Finally, when applied to (25), the least squares approach searches for $\tilde{\gamma}_{1t}$ and $\tilde{\gamma}_{2t}$ which have the smallest possible variances, subject only to the constraints of (23a) and (23b) for $\tilde{\gamma}_{1t}$ and (24a) and (24b) for $\tilde{\gamma}_{2t}$. Because fewer constraints are imposed on the portfolio weights, we expect the $\tilde{\gamma}_{1t}$ and $\tilde{\gamma}_{2t}$ obtained from (25) to have smaller variances than the $\tilde{\gamma}_{1t}$ and $\tilde{\gamma}_{2t}$ obtained from (10), (26), and (27).

In searching for the smallest variance values of the $\tilde{\gamma}_{jt}$, the least squares approach makes the same assumption about the disturbances $\tilde{\eta}_{it}$ in (25) to (27) that is made for (10). The $\tilde{\eta}_{it}$ are assumed to be independent and identically distributed across securities i. This assumption, like any assumption, is not a completely accurate description of the world. To the extent that it is inaccurate, the least squares approach can be misled in its search for smallest variance portfolio returns. Moreover, since the extent to which the assumption is valid is likely to be different for (10), (25), (26), and (27), we might find that adding and dropping constraints does not, in fact, have the predicted effects on the variances of the resulting portfolio returns. There is, however, no need to speculate. We carry out time series of cross-sectional regressions for each of the equations (10), (25), (26), and (27). We can compute the variances of the time series of given $\tilde{\gamma}_{jt}$ in different equations and use these variances to decide which equation provides the most reliable inferences about any given hypothesis.

G. Capital Market Efficiency: The Behavior of Returns Through Time

The tests discussed so far propose to use average values of the $\tilde{\gamma}_{jt}$ in equations (10) and (25) to (27) in order to test the propositions of the two-parameter model about expected returns. It is also possible to obtain tests based on the period-by-period behavior of the $\tilde{\gamma}_{jt}$. The analysis takes us back to the market efficiency concept introduced in Chapter 5.

The models of capital market equilibrium of Chapter 8 assume a perfect capital market in which information available at any time is costlessly available to all investors. Moreover, the complete agreement assumption of these models says that at any time $t-1$, investors agree on the implications of the available information for the joint distribution of security prices at time t. From here it is a short and logical step to assume that the market is also efficient in the sense that the common assessment of the joint distribution of security prices makes full and correct use of all information available at $t-1$.*

In the notation of Chapter 5, market efficiency says that

$$\phi_{t-1}^{m} = \phi_{t-1}, \tag{32}$$

$$f_m(p_{1t}, \ldots, p_{nt} | \phi_{t-1}^{m}) = f(p_{1t}, \ldots, p_{nt} | \phi_{t-1}), \tag{33}$$

where ϕ_{t-1}^{m} is the information used by the market in setting prices at $t-1$, ϕ_{t-1} is the information available at $t-1$, $f_m(p_{1t}, \ldots, p_{nt} | \phi_{t-1}^{m})$ is the joint distribution of security prices for time t assessed by the market at $t-1$, and $f(p_{1t}, \ldots, p_{nt} | \phi_{t-1})$ is the true joint distribution implied by ϕ_{t-1}. Equations (32) and (33) say that in assessing the joint distribution of prices for t, the market correctly uses all information available at $t-1$.

If the market correctly uses all information in assessing the joint distribution of security prices for time t, then when prices are set at $t-1$,

$$E_m(\tilde{R}_{it} | \phi_{t-1}^{m}) = E(\tilde{R}_{it} | \phi_{t-1}), \quad i = 1, \ldots, n, \tag{34}$$

the market's perception of the expected return on security i from $t-1$ to t is the true expected return. If the market sets prices at $t-1$ so that the market portfolio M is on the minimum variance boundary, then

$$E_m(\tilde{R}_{it} | \phi_{t-1}^{m}) = E_m(\tilde{R}_{0Mt} | \phi_{t-1}^{m}) + [E_m(\tilde{R}_{Mt} | \phi_{t-1}^{m}) - E_m(\tilde{R}_{0Mt}^{m} | \phi_{t-1}^{m})]\, \beta_{iM}$$

*This use of the word "efficient" is not to be confused with portfolio efficiency. The terminology is unfortunately standard.

$$= E(\tilde{R}_{it} | \phi_{t-1}) = E(\tilde{R}_{0Mt} | \phi_{t-1}) + [E(\tilde{R}_{Mt} | \phi_{t-1}) - E(\tilde{R}_{0Mt} | \phi_{t-1})]\, \beta_{iM}. \tag{35}$$

In words, the market's view of the expected return-risk relationship for securities in M is the correct view.*

The general implication of (35) is that there is no way to use information available at $t-1$ to make meaningful predictions about how the returns on securities at time t will deviate from the expected return-risk relationship for securities in M that characterizes the various models of market equilibrium. Thus, if (35) is valid, then in (10) and (26), $E(\tilde{\gamma}_{3t} | \phi_{t-1}) = 0$. If the market correctly uses all the available information ϕ_{t-1} in setting prices at $t-1$, and if prices are set so that the market's expected returns conform to the two-parameter model, then there is no way to use information in ϕ_{t-1} as the basis of a correct assessment that the relationship between the expected returns on securities and their risks in M is nonlinear. Likewise, if (35) is valid, then in (10) and (27), $E(\tilde{\gamma}_{4t} | \phi_{t-1}) = 0$, and in (10) and (25) to (27), $E(\tilde{\eta}_{it} | \phi_{t-1}) = 0$; that is, there is no way to use information in ϕ_{t-1} as the basis of correct nonzero assessments of the means of the distributions of the non-β_{iM} risk coefficient $\tilde{\gamma}_{4t}$ and of the disturbances $\tilde{\eta}_{it}$.

In testing these hypotheses about $\tilde{\gamma}_{3t}$, $\tilde{\gamma}_{4t}$, and the $\tilde{\eta}_{it}$, we concentrate on one subset of ϕ_{t-1}, the time series of past values of $\tilde{\gamma}_{3t}$, $\tilde{\gamma}_{4t}$, and $\tilde{\eta}_{it}$. As summarized by (35), capital market efficiency in the two-parameter model implies that there is no way to use the period-by-period behavior of past values of $\tilde{\gamma}_{3t}$ as the basis of correct nonzero assessments of expected future values of $\tilde{\gamma}_{3t}$; the sequence of past values carries no information about expected future values. Likewise, the time series of past values of $\tilde{\gamma}_{4t}$ and $\tilde{\eta}_{it}$ have no information about the expected future values of these variables. Recall from Chapters 4 and 5 that serial correlations are a natural way to test whether the expected future values of a random variable depend on past values. If (35) is valid, the autocorrelations of $\tilde{\gamma}_{3t}$, $\tilde{\gamma}_{4t}$, and $\tilde{\eta}_{it}$ are zero for all lags. We use sample autocorrelations to test these propositions.

The hypothesis that the capital market is efficient and that the market sets prices at time $t-1$ so that the market portfolio M is on the positively sloped segment of the boundary of minimum variance portfolios has implications for the behavior of $\tilde{\gamma}_{2t}$ in (10) and (25) to (27). In each of these equations, $\tilde{\gamma}_{2t}$ is a portfolio return with expected value equal to $E(\tilde{R}_M) - E(\tilde{R}_{0M})$. If securities are priced so that the market perceives M to be on the positively sloped

*Since β_{iM} depends on the joint distribution of prices at time t, we should also note that (32) and (33) imply $\beta_{iM}(\phi_{t-1}^{m}) = \beta_{iM}(\phi_{t-1})$; that is, the market correctly uses all information in assessing the risk of any security in the market portfolio M. For the moment, however, we choose not to complicate the notation in the text in this way. A detailed discussion of problems that arise in assessing risk measures comes later.

segment of the minimum variance boundary, and if the market is efficient, then $E(\tilde{R}_{Mt} | \phi_{t-1}) - E(\tilde{R}_{0Mt} | \phi_{t-1}) = E(\tilde{\gamma}_{2t} | \phi_{t-1}) > 0.0$; there is no way to use any information available at $t - 1$ as the basis of a correct negative assessment of the expected value of $\tilde{\gamma}_{2t}$. We test only a specialized form of this hypothesis. We assume that $E(\tilde{R}_{Mt} | \phi_{t-1}) - E(\tilde{R}_{0Mt} | \phi_{t-1})$ is not only positive but constant through time. We then test this proposition with sample autocorrelations of $\tilde{\gamma}_{2t}$, all of which should be indistinguishable from zero if the proposition is valid.

Finally, in equations (10) and (25) to (27) the least squares values of $\tilde{\gamma}_{1t}$ are the returns on standard zero-β_{pM} portfolios. If the Sharpe-Lintner version of the two-parameter model is valid, $E_m(\tilde{R}_{0Mt} | \phi^m_{t-1}) = R_{Ft}$; that is the market sets prices so that it perceives the expected return on any security or portfolio whose return is uncorrelated with the return on M to be equal to R_{Ft}. If the market correctly uses available information, $E_m(\tilde{R}_{0Mt} | \phi^m_{t-1}) = E(\tilde{R}_{0Mt} | \phi_{t-1}) = R_{Ft}$; the true expected return on zero-β_{iM} securities and portfolios is R_{Ft}. One implication of this is that the time series of past values of $\tilde{\gamma}_{1t} - R_{Ft}$ cannot be used as the basis of correct nonzero assessments of expected future values of $\tilde{\gamma}_{1t} - R_{Ft}$, which in turn implies that the autocorrelations of $\tilde{\gamma}_{1t} - R_{Ft}$ are zero for all lags.

Introducing the concept of market efficiency has not produced hypotheses about expected returns that are different from those developed in the initial discussion of (1). Equation (35), after all, is just the expected return-risk equation (1) with some additional notation whose purpose is to emphasize the characteristics of the pricing process in an efficient market. Discussing market efficiency in the context of the two-parameter model has, however, made us aware of tests that were not apparent in the initial discussion of (1). The tests that came out of the discussion of (1) involve using averages of the least squares values of $\tilde{\gamma}_{1t}$, $\tilde{\gamma}_{2t}$, $\tilde{\gamma}_{3t}$, and $\tilde{\gamma}_{4t}$ in equations (10) and (25) to (27) to test the propositions of (1) about the expected values of these variables. The discussion of market efficiency leads to tests based on the period-by-period behavior of the variables.

III. Details of the Methodology

The least squares values of the $\tilde{\gamma}_{jt}$ in equations (10) and (25) to (27) give us the inputs for testing the implications of a two-parameter world for expected returns. Empirical realities, however, present us with unavoidable complications. We assume above that the values of the risk measures β_{iM} and $\sigma(\tilde{\epsilon}_i)$ of different securities are known. In fact they must be estimated from return data. We also assume that the components of the returns on the market portfolio M, the returns on all investment assets, and their corresponding value weights are available. In fact, the empirical tests deal only with common stocks on the New York Stock Exchange, and in the absence of the appropriate value weights, an equally weighted portfolio of these stocks, the portfolio we have heretofore called m, is used instead of M. We discuss first the problems that this causes and then the problems that arise from using estimates of risk measures.

A. *Application of the Approach to the Equally Weighted Market Portfolio* m

The role of the value-weighted market portfolio in the preceding analysis can be played by any portfolio that must be efficient, or at least minimum variance, when a market equilibrium is established at time $t - 1$. Although the supposition has no rigorous justification, suppose a market equilibrium requires that m, the equally weighted portfolio of NYSE stocks, be a minimum variance portfolio. Then the expected return-risk equation for a minimum variance portfolio applies to m. For any security i in m, we have

$$E(\tilde{R}_{it}) = E(\tilde{R}_{0mt}) + [E(\tilde{R}_{mt}) - E(\tilde{R}_{0mt})]\, \beta_{im}, \tag{36}$$

where

$$\beta_{im} = \frac{\mathrm{cov}(\tilde{R}_i, \tilde{R}_m)}{\sigma^2(\tilde{R}_m)} \tag{37}$$

is the risk of security i in m measured relative to the risk or variance of the portfolio's return, and $E(\tilde{R}_{0mt})$ is the expected return on any security in m, or any portfolios of the securities in m, whose return is uncorrelated with the return on m.

Like (1), equation (36) is linear in the risk measure β_{im}; β_{im} is the only measure of the risk of security i that appears in (36); and if we assume that m is along the positively sloped segment of the minimum variance boundary, then $[E(\tilde{R}_{mt}) - E(\tilde{R}_{0mt})] > 0$, so that (36) implies a positive relationship between the expected returns on securities and their risk in m. In short, the testable implications of (36) are the same as those of (1), the expected return-risk equation implied by the fact that a market equilibrium requires that the value-weighted market portfolio M be a minimum variance portfolio.

Moreover, the approach described in Section II for testing whether M is a minimum variance portfolio can be used to test whether the pricing of NYSE common stocks is consistent with the proposition that m is a minimum variance portfolio. Thus, consider

$$\tilde{R}_{it} = \tilde{\gamma}_{1t} + \tilde{\gamma}_{2t}\beta_{im} + \tilde{\gamma}_{3t}\beta_{im}^2 + \tilde{\gamma}_{4t}\sigma(\tilde{\epsilon}_i) + \tilde{\eta}_{it} \tag{38}$$

$$\tilde{R}_{it} = \tilde{\gamma}_{1t} + \tilde{\gamma}_{2t}\beta_{im} + \tilde{\gamma}_{3t}\beta_{im}^2 + \tilde{\eta}_{it} \tag{39}$$

$$\tilde{R}_{it} = \tilde{\gamma}_{1t} + \tilde{\gamma}_{2t}\beta_{im} + \tilde{\gamma}_{4t}\sigma(\tilde{\epsilon}_i) + \tilde{\eta}_{it} \tag{40}$$

$$\tilde{R}_{it} = \tilde{\gamma}_{1t} + \tilde{\gamma}_{2t}\beta_{im} + \tilde{\eta}_{it}, \tag{41}$$

where $\sigma(\tilde{\epsilon}_i)$ is now the standard deviation of the disturbance in the market model relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$,

$$\tilde{R}_{it} = \alpha_{im} + \beta_{im}\tilde{R}_{mt} + \tilde{\epsilon}_{it} \tag{42}$$

$$\alpha_{im} = E(\tilde{R}_{it}) - \beta_{i\cancel{M}m}E(\tilde{R}_{mt}), \tag{43}$$

and where the assumption that the joint distribution of security returns is multivariate normal implies all the properties of (42) discussed in Chapters 3 and 4.

Equations (38) to (41) are just (10) and (25) to (27), but with β_{im}, β_{im}^2, and the new version of $\sigma(\tilde{\epsilon}_i)$ used as explanatory variables. If the same substitution of explanatory variables is made in the matrix $\underline{C}$ of (12), then the analysis of (11) to (22) implies that the least squares values of the $\tilde{\gamma}_{jt}$ in (38) to (41), obtained from cross-sectional regressions of security returns on the relevant explanatory variables, are the returns on portfolios that have properties analogous to the least squares values of the $\tilde{\gamma}_{jt}$ in equations (10) and (25) to (27). For example, the least squares value of $\tilde{\gamma}_{1t}$ in (38) is the return on a standard portfolio that has $\beta_{pm} = 0.0$; $\tilde{\gamma}_{2t}$ is the return on a zero-investment portfolio where the weighted average value of β_{im} is 1.0; $\tilde{\gamma}_{3t}$ is the return on a zero-investment portfolio where the weighted average value of β_{im}^2 is 1.0; and $\tilde{\gamma}_{4t}$ is the return on a zero-investment portfolio where the weighted average of $\sigma(\tilde{\epsilon}_i)$ for individual securities is 1.0. As in equations (10) and (25) to (27), the least squares value of a given $\tilde{\gamma}_{jt}$ in (38) to (41) focuses on the effects of the explanatory variable of interest by choosing portfolio weights that zero out the effects of other explanatory variables in the equation. Moreover, as in equations (10) and (25) to (27), if the $\tilde{\eta}_{it}$ in (38) to (41) are independent and identically distributed for different securities i, the least squares $\tilde{\gamma}_{jt}$ are the "smallest variance" portfolios that focus on different explanatory variables in the manner described above, so they provide the most reliable tests of the proposition that securities are priced so that m is a minimum variance portfolio. The only problem with all of this is that we don't have a model of market equilibrium that tells us that m must be a minimum variance portfolio.

Given that m is to be used in the tests, there are two justifications. First, in empirical work one usually must settle for proxies for the variables called for by a theory. The equally weighted portfolio of NYSE stocks, m, might

be viewed as a proxy for M, the value-weighted portfolio of all investment assests. This tack is, however, open to valid arguments as to whether m is a reasonable proxy for M. An equally weighted portfolio is rather different from a value-weighted portfolio. More important, although investment in NYSE stocks is a large fraction of the total investment in the common stocks of publicly held companies, the NYSE does not cover investments in bonds, privately held real estate, and consumer durables, which together are a much larger fraction of invested wealth than common stocks.

The second approach to justifying the use of m in the tests is to say that since m is a diversified portfolio of many securities, perhaps it is reasonable to assume that it is "close enough" to a minimum variance portfolio to be a meaningful basis for tests of the two-parameter model. For those with tastes for rigor (and I include myself in that group), this approach is unaesthetic. Nevertheless, from the viewpoint of the empiricist (and I also include myself in that group), the approach can provide its own justification. If the testable hypotheses drawn from (36) are upheld by the data, then it seems reasonable to conclude both that the two-parameter model is a meaningful approximation of how securities are priced in the capital market and that securities are priced so that the equally weighted portfolio m of NYSE stocks is a minimum variance portfolio.

Finally, there is an important exception to the statement that (1) and (36) have the same testable implications, and the exception acquires some importance in the empirical results. The Sharpe-Lintner hypothesis that $E(\tilde{R}_{0Mt}) = R_{Ft}$ cannot be applied to the portfolio m. Even if the Sharpe-Lintner model is the relevant view of the world, it is not the case that $E(\tilde{R}_{0mt}) = R_{Ft}$. Recall that $E(\tilde{R}_{0Mt})$ is the intercept on the $E(\tilde{R}_p)$ axis of the line tangent to the boundary of minimum variance portfolios of positive variance securities at the point corresponding to the value-weighted market portfolio M. In the Sharpe-Lintner model, a market equilibrium requires that this intercept also be the risk-free rate of interest R_{Ft}. If the equally weighted portfolio m is also a minimum variance portfolio, then $E(\tilde{R}_{0mt})$ is the intercept on the $E(\tilde{R}_p)$ axis of the line tangent to the minimum variance boundary at the point corresponding to m. Since m and M are not the same portfolio, $E(\tilde{R}_{0mt}) \neq E(\tilde{R}_{0Mt}) = R_{Ft}$. We shall have more to say later about where $E(\tilde{R}_{0mt})$ is likely to be relative to R_{Ft}.

B. The Portfolio Approach to Estimating Risk Measures

To test the implications of (36), we must still face another serious problem. Equation (36) is in terms of the true values of the risk measure β_{im}, and empirical tests require that estimates b_{im} be used.

ESTIMATES OF RISK FROM THE MARKET MODEL

One approach is to build on the assumption underlying the two-parameter model that the joint distribution of security returns is multivariate normal. This means that the joint distribution of the return on any security and the return on the portfolio m is bivariate normal. If the bivariate normal joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ is also the same or stationary through time, then the market model of (42) and the methods of Chapters 3 and 4 can be used to estimate β_{im} and to assess the sampling properties of the estimates. We should note, however, that empirical realities have forced on us an assumption--that the joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ is stationary through time—which is not required by the two-parameter model. If we are to use the methods of Chapters 3 and 4 to estimate β_{im}, the stationarity assumption is required, at least for the sampling period to be used in the estimation.

It is interesting at this point to recall the discussion of the complete agreement assumption in Chapter 8. We argued there that this assumption is a sensible approximation to the world when the joint distribution of security returns is the same through time. Then history leads investors to a correct consensus about the joint distribution of future returns. We are now arguing that the stationarity assumption is a necessary ingredient for successful tests of the two-parameter model. Thus, although this assumption is not an explicit part of two-parameter theory, it makes the assumptions of the theory more palatable, and it is pretty much a precondition for tests of the theory.

THE ERRORS-IN-THE-VARIABLES PROBLEM

The most direct approach to the tests would seem to be to obtain estimates b_{im} of the β_{im} of individual securities, plug these into equations (38) to (41), and then proceed. The problem with such a brute force approach is that any estimate b_{im} differs from the true β_{im} by an estimation error. If the errors are typically large, there is a serious "errors-in-the-variables" problem.

There is a large statistical literature on the errors-in-the-variables problem that we do not need to consider in any formal way here. In intuitive terms, the problem centers on the fact that if a proxy explanatory variable is used in a least squares regression (e.g., b_{im} rather than β_{im}), the computed coefficients do not have the same properties as if the true explanatory variable were used. For example, the least squares value of $\tilde{\gamma}_{1t}$ for (41) is the return for month t on a zero-β_{pm} portfolio; the true value of β_{pm} for this portfolio is zero. Suppose, however, that we substitute estimates b_{im} for the true values of β_{im} that appear in (41) and then carry out the cross-sectional regression. It will then turn out that the least squares value of $\tilde{\gamma}_{1t}$ is the return on a standard portfolio where the weights assigned to individual securities are such that the weighted average of the b_{im} is zero; that is, $\tilde{\gamma}_{1t}$ is the return on a zero-b_{pm} portfolio. Since each of the b_{im} is just an estimate of the corresponding true β_{im}, $\tilde{\gamma}_{1t}$ is not the return on a portfolio where the true β_{pm} is zero.

The arguments are quite general. If one uses estimates instead of the true values of the explanatory variables that appear in (38) to (41), then the analysis of equations (11) to (22) will imply that the least squares values of the $\tilde{\gamma}_{jt}$ are portfolio returns that focus on the effects of given explanatory variables while zeroing out the effects of others. All the focusing and zeroing out, however, will be in terms of the estimates rather than the true values of the explanatory variables. To the extent that the estimates differ from the true values of the explanatory variables, the least squares portfolio returns are out of focus for the purpose of testing the implications of (36).

PROBLEM III.B

1. We have long known that for any portfolio p,

$$\beta_{pm} = \frac{\operatorname{cov}(\tilde{R}_p, \tilde{R}_m)}{\sigma^2(\tilde{R}_m)} = \sum_{i=1}^{n} x_{ip} \frac{\operatorname{cov}(\tilde{R}_i, \tilde{R}_m)}{\sigma^2(\tilde{R}_m)} = \sum_{i=1}^{n} x_{ip}\beta_{im}.$$

Show that the same relationship holds between the estimates b_{pm} and the b_{im} of individual securities.

ANSWER

1. For any portfolio p,

$$\bar{R}_p = \frac{\sum_{t=1}^{T} R_{pt}}{T} = \frac{\sum_{t=1}^{T}\sum_{i=1}^{n} x_{ip}R_{it}}{T} = \sum_{i=1}^{n} x_{ip}\frac{\sum_{t=1}^{T} R_{it}}{T} = \sum_{i=1}^{n} x_{ip}\bar{R}_i.$$

Thus,

$$b_{pm} = \frac{\sum_{t=1}^{T}(R_{pt} - \bar{R}_p)(R_{mt} - \bar{R}_m)}{\sum_{t=1}^{T}(R_{mt} - \bar{R}_m)^2}$$

$$= \frac{\sum_{t=1}^{T}\left(\sum_{i=1}^{n} x_{ip}R_{it} - \sum_{i=1}^{n} x_{ip}\bar{R}_i\right)(R_{mt} - \bar{R}_m)}{\sum_{t=1}^{T}(R_{mt} - R_m)^2}$$

$$= \sum_{i=1}^{n} x_{ip} \frac{\sum_{t=1}^{T} (R_{it} - \bar{R}_i)(R_{mt} - \bar{R}_m)}{\sum_{t=1}^{T} (R_{mt} - \bar{R}_m)^2}$$

$$b_{pm} = \sum_{i=1}^{n} x_{ip} b_{im}.$$

The general idea behind the solution to the errors-in-the-variables problem is direct. We try to minimize the problem by reducing the errors in the estimates of the risk measures. Since, from Chapter 3, the sampling variance of $\tilde{b}_{im}$ as an estimator of β_{im} is

$$\sigma^2(\tilde{b}_{im}) = \frac{\sigma^2(\tilde{\epsilon}_i)}{\sum_{t=1}^{T} (R_{mt} - \bar{R}_m)^2}, \quad (44)$$

it would seem that one way to accomplish this goal is to compute b_{im} from long time series of monthly returns. This ensures that the sum of squares in the denominator of (44) is large, so that $\sigma^2(\tilde{b}_{im})$ is small. We know from Chapter 4, however, that this approach leans too heavily on the assumption that the value of β_{im} is stationary through time. The values of β_{im} of individual securities do wander slightly through time, and the optimal period for estimation from monthly data is roughly 5-10 years. Recall from Chapter 4 that with 5-10 years of monthly data, the estimates b_{im} leave substantial uncertainty about the true values; that is, the errors in the b_{im} of individual securities are likely to be large relative to the true β_{im}.

An alternative approach to reducing the errors in estimates of risk measures is to work on the numerator in (44), that is, to reduce $\sigma^2(\tilde{\epsilon}_i)$, the variance of the market model disturbance. The way we do this is to work with portfolios rather than individual securities. To see the basis of the approach, recall that multivariate normality of security returns implies that the joint distribution of the return on any portfolio p and the return on the portfolio m is bivariate normal, so that there are market model relationships like (42) for portfolios as well as for individual securities. For any portfolio p, we have

$$\tilde{R}_{pt} = \alpha_{pm} + \beta_{pm} \tilde{R}_{mt} + \tilde{\epsilon}_{pt}. \quad (45)$$

The portfolio return disturbance $\tilde{\epsilon}_{pt}$ in (45) is, however, just the weighted average of security return disturbances $\tilde{\epsilon}_{it}$. To the extent that the $\tilde{\epsilon}_{it}$ for different securities are less than perfectly positively correlated, there is a "diversification effect," and $\sigma^2(\tilde{\epsilon}_p)$ can be expected to be smaller than the $\sigma^2(\tilde{\epsilon}_i)$ of individual securities. The result is that $\sigma^2(\tilde{b}_{pm})$ is generally smaller for portfolios than for individual securities. This means that the errors-in-the-variables problem is likely to be less serious if tests of (36) are carried out in terms of portfolios rather than individual securities.

In brief, then, the intention is to calculate the least squares values of the coefficients $\tilde{\gamma}_{jt}$ in (38) to (41) and to use these to test the various hypotheses implied by (36) about the relationship between expected return and risk within the portfolio m. In place of the returns on individual securities that appear on the left of these equations, we substitute returns on portfolios; and the security risk measures β_{im} that appear on the right in these equations are replaced by estimates b_{pm} relevant for the portfolios that appear on the left of the equations.

CHOOSING PORTFOLIOS AND THE REGRESSION PHENOMENON

We present evidence shortly that the estimates b_{pm} for portfolios are indeed much more reliable than those for individual securities. The portfolio approach, however, also raises problems that center in large part on how the portfolios used in the analysis are chosen. When securities are combined into portfolios, some of the information in the data about the relationship between risk and expected return is lost. For example, if the allocation of securities to portfolios is random, and if the portfolios formed contain many securities, the portfolios will have b_{pm} much more closely concentrated about one than individual securities, which means that we can expect to observe only a narrow range of the expected return-risk relationship. In the extreme case where all the portfolio risk measures turn out to have about the same value, forming portfolios destroys all of the information about the expected return-risk relationship that is potentially contained in the security return data.

To reduce the loss of information caused by working with portfolios, one forms portfolios in such a way as to guarantee that a wide range of b_{pm} is obtained. This is done by allocating securities to portfolios on the basis of ranked values of b_{im}. If naïvely executed, however, such a procedure could result in what is called a "regression phenomenon." When one ranks the b_{im} of all securities, one is to some extent ranking the estimation errors in the b_{im}. A large positive estimation error is likely to result in a high b_{im}, while the reverse is true of a large negative estimation error. Forming portfolios on the basis of ranked b_{im} thus causes bunching of positive and negative sampling errors within portfolios, especially at the extremes of the b_{im} range. The result is that the larger values of b_{pm} would tend to overestimate the true β_{pm}, while the lower b_{pm} would tend to be underestimated.

The regression phenomenon can be avoided by forming portfolios from ranked b_{im} computed from data for one time period, but then using a subsequent period to obtain the b_{pm} for these portfolios that are used in the tests. The errors in the estimates from the fresh data are likely to be independent of the estimation errors for the portfolio formation period, so that in the new portfolio b_{pm} there is no regression phenomenon. We also expect that when fresh data are used, the extreme b_{pm} will be less extreme than for the period of portfolio formation. In the new data there will be some tendency for all the b_{pm} to "regress" toward 1, that is, to become less extreme. This is the basis of the term "regression phenomenon."

The errors-in-the-variables problem and the regression phenomenon that arises when the portfolio approach is used to solve it were first pointed out by Blume (1968). The portfolio approach is used by Black, Jensen, and Scholes (1972), who offer a solution to the resulting regression phenomenon which is similar to that of Fama and MacBeth (1973). The approach presented here is that of Fama and MacBeth.

DETAILS OF THE APPROACH

The specifics of the Fama-MacBeth approach are as follows. Let n be the total number of securities to be allocated to portfolios, and let int $(n/20)$ be the largest integer equal to or less than $n/20$. Using the first four years (1926–1929) of monthly return data, 20 portfolios of NYSE stocks are formed on the basis of ranked b_{im} for individual securities. The middle 18 portfolios each have int $(n/20)$ securities. If n is even, the first and last portfolios each have int $(n/20) + \frac{1}{2}\,[n - 20 \text{ int } (2/\cancel{20})]$ securities. The last (highest b_{pm}) portfolio gets an additional security if n is odd.

The following five years (1930–1934) of data are then used to recompute the b_{im}, and these are averaged across securities within portfolios to obtain 20 initial portfolio b_{pmt} for the risk-return tests. Thus, within portfolios, equal weights are applied to individual securities. The subscript t is added to b_{pmt} to indicate that for each month t of the following four years (1935–1938) these b_{pmt} are recomputed as simple averages of individual security b_{im}, thus adjusting the portfolio b_{pmt} month-by-month to allow for delisting of individual securities. The component values of b_{im} for securities are themselves updated yearly; that is, they are recomputed from monthly returns from 1930 through 1935, 1936, or 1937.

The month-by-month returns on the 20 portfolios, with equal weighting of individual securities each month, are also computed for the four-year period 1935–1938. For each month t of this period, the least squares method is used to compute γ_{jt} in*

*Since we are talking about results for a given historical period, the tildes that heretofore appeared are henceforth dropped.

$$R_{pt} = \gamma_{1t} + \gamma_{2t} b_{pm,t-1} + \gamma_{3t} b^2_{pm,t-1} + \gamma_{4t} \bar{s}_{p,t-1}(e_i) + \eta_{pt} \tag{46}$$

$$R_{pt} = \gamma_{1t} + \gamma_{2t} b_{pm,t-1} + \gamma_{3t} b^2_{pm,t-1} + \eta_{pt} \tag{47}$$

$$R_{pt} = \gamma_{1t} + \gamma_{2t} b_{pm,t-1} + \gamma_{4t} \bar{s}_{p,t-1}(e_i) + \eta_{pt} \tag{48}$$

$$R_{pt} = \gamma_{1t} + \gamma_{2t} b_{pm,t-1} + \eta_{pt}, \qquad p = 1, 2, \ldots, 20. \tag{49}$$

The explanatory variable $b_{pm,t-1}$ is the average b_{im} for securities in portfolio p discussed above; $b^2_{pm,t-1}$ is the average of the squared values of these b_{im} and is thus somewhat mislabeled; and $\bar{s}_{p,t-1}(e_i)$ is likewise the average of $s(e_i)$ for securities in portfolio p. The $s(e_i)$ are sample standard deviations of market model residuals for individual securities; that is, they are the usual estimates of $\sigma(\tilde{e}_i)$ in (42). They are computed from data for the same period as the component b_{im} of $b_{pm,t-1}$; and, like these b_{im}, they are updated annually.

Equations (46) to (49) are equations (38) to (41) averaged across the securities in a portfolio, with estimates $b_{pm,t-1}$, $b^2_{pm,t-1}$, and $\bar{s}_{p,t-1}(e_i)$ used as explanatory variables. The results from these equations, the time series of month-by-month values of γ_{1t}, γ_{2t}, γ_{3t}, and γ_{4t} for the four-year period 1935–1938 are the inputs for the tests of the implications of (36) for this period. To get results for other periods, the general steps described above are repeated. Specifically, seven years of data are used to form portfolios; the next five years are used to compute initial values of the explanatory variables; and then the least squares values of the γ_{jt} are computed month-by-month for the following four-year period.

The nine different portfolio formation periods (all except the first are seven years in length), initial five-year estimation periods, and testing periods (all except the last are four years in length) are shown in Table 9.1. Fama and MacBeth explain the choice of four-year testing periods as a balance of computation costs against the desire to reform portfolios frequently. The choice of seven-year portfolio formation periods and five- to eight-year periods for the estimates $b_{pm,t-1}$ reflects the desire to balance the statistical power obtained with a large sample from a stationary process against potential problems caused by any nonconstancy of the β_{im}. The choices here are in line with the results of Gonedes (1973). His results also led Fama and MacBeth to require that to be included in a portfolio, a security available in the first month of a testing period must also have data for all five years of the preceding estimation period and for at least four years of the portfolio formation period. The total number of securities available in the first month of each testing period and the number (n) of securities meeting the data requirement are shown in Table 9.1.

Finally, all the tests are "predictive" in the sense that the explanatory

TABLE 9.1

Portfolio Formation, Estimation, and Testing Periods

	1	2	3	4	5	6	7	8	9
Portfolio formation period	1926–29	1927–33	1931–37	1935–41	1939–45	1943–49	1947–53	1951–57	1955–61
Initial estimation period	1930–34	1934–38	1938–42	1942–46	1946–50	1950–54	1954–58	1958–62	1962–66
Testing period	1935–38	1939–42	1943–46	1947–50	1951–54	1955–58	1959–62	1963–66	1967–68
No. of stocks listed	710	779	804	908	1,011	1,053	1,065	1,162	1,261
No. of stocks (n) used	435	576	607	704	751	802	856	858	845

SOURCE: Eugene F. Fama and James D. MacBeth, "Risk, Return, and Equilibrium: Empirical Tests," *Journal of Political Economy* 71 (May–June 1973): 618–619.

variables $b_{pm,t-1}$, $b^2_{pm,t-1}$ and $\bar{s}_{p,t-1}(e_i)$ that appear in (46) to (49) are computed from a period prior to the month of the returns, the R_{pt} that appear on the left-hand side of these equations. Thus, in computing the least squares values of the γ_{jt} in (46) to (49), we are looking at the relationships between returns for month t and estimates of risk measures that were available at the beginning of the month. Having emphasized the predictive nature of the tests, we can simplify the notation by henceforth referring to the explanatory variables in (46) to (49) as b_{pm}, b^2_{pm} and $\bar{s}_p(e_i)$.

SOME EVIDENCE ON THE EFFECTIVENESS OF THE PORTFOLIO APPROACH

Table 9.2 shows the values of the 20 portfolio b_{pm} and their standard errors $s(b_{pm})$ for each of the nine five-year estimation periods of Table 9.1. Also shown are $\hat{\rho}(R_p, R_m)^2$, the coefficient of determination between R_{pt} and R_{mt}; $s(R_p)$, the standard deviation of R_{pt}; $s(e_p)$, the standard deviation of the portfolio residuals from the market model, not to be confused with $\bar{s}_p(e_i)$, the average of the residual standard deviations for individual securities in p, which is also shown. The b_{pm} and $\bar{s}_p(e_i)$ are the explanatory variables in (46) to (49) for the first month of the testing periods following the estimation periods shown.

Since the estimate of the variance of a b_{im} or a b_{pm} is

$$s^2(b) = \frac{s^2(e)}{\sum_{t=1}^{T} (R_{mt} - \bar{R}_m)^2}, \tag{50}$$

we can see that if data from a given period are used to compute b_{im} for securities and b_{pm} for portfolios, the denominator in the expression for $s^2(b)$ is the same for all of the estimates, while the numerator is just the relevant sample variance of the market model residuals for the security or portfolio. Thus, the fact that in Table 9.2 $s(e_p)$ is generally on the order of one-third to one-seventh $\bar{s}_p(e_i)$ implies that $s(b_p)$ is one-third to one-seventh $s(b_i)$. Estimates of β_{pm} for portfolios are indeed more reliable than the estimates of the β_{im} of individual securities.

Note, however, that if the market model disturbances $\tilde{e}_{it}$ were independent from security to security, the "effects of diversification" in reducing $s(e_p)$ would be about the same for all portfolios. More precisely, the ratio $s(e_p)/\bar{s}_p(e_i)$ would be about the same for all portfolios, so that the relative increase in the precision of the risk estimates obtained by using portfolios rather than individual securities would be about the same for all portfolios. We argue later, however, that the market model disturbances for securities are interdependent, and the interdependence is strongest among high-β_{im} securities and among low-β_{im} securities. For the moment we note that this shows

TABLE 9.2

Sample Statistics for Estimation Periods

STATISTIC	PORTFOLIO 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
									ESTIMATION PERIOD 1930–1934											
b_{pm}	.693	.702	.771	.840	.863	.892	.917	.935	.977	.983	1.014	1.058	1.059	1.063	1.106	1.143	1.149	1.255	1.280	1.291
$s(b_{pm})$	.031	.030	.023	.034	.031	.025	.033	.026	.024	.020	.033	.025	.037	.030	.032	.026	.040	.044	.028	.033
$\hat{\rho}(R_p, R_m)^2$	.89	.90	.95	.91	.93	.96	.93	.96	.97	.98	.94	.97	.93	.96	.95	.97	.93	.93	.97	.96
$s(R_p)$	.136	.139	.147	.163	.166	.170	.177	.178	.185	.185	.194	.200	.204	.202	.210	.216	.221	.238	.241	.245
$s(e_p)$	.045	.043	.033	.048	.044	.036	.047	.038	.034	.029	.047	.036	.053	.043	.045	.037	.057	.063	.039	.047
$\bar{s}_p(e_i)$	.173	.156	.137	.158	.143	.162	.175	.176	.148	.164	.179	.161	.203	.185	.187	.174	.203	.243	.194	.205
$s(e_p)/\bar{s}_p(e_i)$	.26	.28	.24	.30	.31	.22	.26	.21	.22	.17	.26	.22	.26	.23	.24	.21	.28	.25	.20	.22
									ESTIMATION PERIOD 1934–1938											
b_{pm}	.322	.508	.651	.674	.695	.792	.921	.942	.970	1.005	1.046	1.122	1.181	1.192	1.196	1.295	1.335	1.396	1.445	1.458
$s(b_{pm})$	.027	.027	.025	.023	.028	.026	.032	.029	.034	.027	.028	.031	.035	.028	.029	.032	.032	.053	.039	.053
$\hat{\rho}(R_p, R_m)^2$	.709	.861	.921	.936	.912	.941	.932	.946	.933	.958	.959	.956	.951	.969	.966	.966	.967	.922	.958	.927
$s(R_p)$	.040	.058	.072	.074	.077	.087	.101	.103	.106	.109	.113	.122	.128	.128	.129	.140	.144	.154	.156	.160
$s(e_p)$	.022	.022	.020	.019	.023	.021	.026	.024	.028	.022	.023	.026	.029	.023	.024	.026	.026	.043	.032	.043
$\bar{s}_p(e_i)$	.085	.075	.083	.078	.090	.095	.109	.106	.111	.097	.094	.124	.120	.122	.132	.125	.129	.158	.145	.170
$s(e_p)/\bar{s}_p(e_i)$	.259	.293	.241	.244	.256	.221	.238	.226	.252	.227	.245	.210	.242	.188	.182	.208	.202	.272	.221	.253
									ESTIMATION PERIOD 1938–1942											
b_{pm}	.335	.470	.588	.633	.768	.781	.798	.899	.981	1.057	1.084	1.129	1.206	1.241	1.255	1.257	1.261	1.331	1.560	1.627
$s(b_{pm})$	.032	.039	.038	.026	.026	.025	.034	.025	.023	.025	.015	.018	.025	.039	.038	.025	.033	.037	.050	.080
$\hat{\rho}(R_p, R_m)^2$	.65	.71	.80	.91	.94	.94	.90	.96	.97	.97	.99	.99	.97	.94	.95	.98	.96	.96	.94	.87
$s(R_p)$	.044	.059	.070	.070	.084	.085	.089	.097	.106	.114	.116	.121	.130	.135	.137	.135	.136	.144	.171	.185
$s(e_p)$	.026	.032	.031	.022	.021	.020	.028	.020	.019	.020	.012	.015	.021	.032	.031	.020	.027	.030	.041	.066
$\bar{s}_p(e_i)$	.071	.066	.083	.073	.072	.087	.083	.088	.085	.100	.098	.104	.116	.135	.123	.119	.103	.107	.148	.181
$s(e_p)/\bar{s}_p(e_i)$	.36	.48	.37	.30	.29	.22	.33	.22	.22	.20	.12	.14	.18	.23	.25	.16	.26	.28	.27	.36

STATISTIC	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
									ESTIMATION PERIOD 1942–1946											
b_{pm}	.467	.537	.593	.628	.707	.721	.770	.792	.805	.894	.949	.952	1.010	1.038	1.254	1.312	1.316	1.473	1.631	1.661
$s(b_{pm})$	.045	.041	.044	.037	.027	.032	.035	.035	.028	.040	.031	.036	.040	.030	.034	.039	.041	.084	.083	.077
$\hat{\rho}(R_p, R_m)^2$	.645	.745	.753	.829	.919	.898	.889	.898	.934	.896	.942	.923	.917	.954	.958	.951	.945	.839	.867	.887
$s(R_p)$	.035	.037	.041	.041	.044	.046	.049	.050	.050	.057	.059	.060	.063	.064	.077	.081	.081	.097	.105	.106
$s(e_p)$	.021	.019	.020	.017	.013	.015	.016	.016	.013	.018	.014	.016	.018	.014	.016	.018	.019	.039	.038	.036
$\bar{s}_p(e_i)$	.055	.055	.063	.058	.058	.063	.064	.064	.062	.069	.073	.074	.085	.077	.096	.083	.086	.134	.117	.122
$s(e_p)/\bar{s}_p(e_i)$	.382	.345	.317	.293	.224	.238	.250	.250	.210	.261	.192	.216	.212	.182	.167	.217	.221	.291	.325	.295
									ESTIMATION PERIOD 1946–1950											
b_{pm}	.538	.608	.694	.745	.810	.812	.857	.919	.937	.955	1.002	1.027	1.052	1.075	1.184	1.243	1.325	1.349	1.434	1.479
$s(b_{pm})$	.040	.041	.033	.032	.028	.029	.025	.032	.033	.028	.027	.030	.027	.037	.030	.035	.034	.049	.052	.086
$\hat{\rho}(R_p, R_m)^2$	.76	.79	.88	.90	.93	.93	.95	.93	.93	.95	.96	.95	.96	.93	.96	.96	.96	.93	.93	.83
$s(R_p)$	.032	.035	.038	.041	.043	.044	.046	.049	.050	.051	.053	.055	.056	.058	.063	.066	.070	.073	.077	.084
$s(e_p)$	.016	.016	.013	.013	.011	.012	.010	.013	.013	.011	.011	.012	.011	.015	.012	.014	.013	.020	.021	.034
$\bar{s}_p(e_i)$	.049	.052	.053	.053	.052	.059	.058	.062	.067	.064	.064	.069	.069	.069	.070	.071	.066	.082	.078	.102
$s(e_p)/\bar{s}_p(e_i)$	.32	.30	.24	.24	.21	.20	.17	.20	.19	.17	.17	.17	.15	.21	.17	.19	.19	.24	.26	.33
									ESTIMATION PERIOD 1950–1954											
b_{pm}	.418	.590	.694	.751	.777	.784	.929	.950	.996	1.014	1.117	1.123	1.131	1.134	1.186	1.235	1.295	1.324	1.478	1.527
$s(b_{pm})$	.042	.047	.045	.037	.038	.035	.050	.038	.035	.029	.039	.027	.044	.033	.037	.049	.045	.046	.058	.086
$\hat{\rho}(R_p, R_m)^2$	.629	.723	.798	.872	.878	.895	.856	.913	.933	.954	.934	.968	.919	.952	.944	.915	.933	.934	.917	.841
$s(R_p)$	.019	.025	.028	.029	.030	.030	.036	.036	.037	.038	.042	.041	.043	.042	.044	.047	.049	.050	.056	.060
$s(e_p)$	.012	.013	.013	.010	.010	.010	.014	.011	.010	.008	.011	.007	.012	.009	.010	.014	.013	.013	.016	.024
$\bar{s}_p(e_i)$	.040	.044	.046	.048	.051	.051	.052	.053	.054	.057	.066	.057	.066	.060	.064	.064	.065	.068	.076	.088
$s(e_p)/\bar{s}_p(e_i)$	.300	.295	.283	.208	.196	.196	.269	.208	.185	.140	.167	.123	.182	.150	.156	.219	.200	.192	.210	.273

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STATISTIC	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
									PORTFOLIO											
									ESTIMATION PERIOD 1954-1958											
b_{pm}	.457	.544	.614	.637	.768	.859	.924	.972	.993	1.007	1.021	1.091	1.098	1.116	1.238	1.247	1.258	1.392	1.450	1.587
$s(b_{pm})$	.042	.047	.045	.041	.034	.035	.034	.038	.037	.034	.033	.040	.040	.042	.045	.038	.042	.065	.049	.064
$\hat{\rho}(R_p, R_m)^2$	.66	.69	.76	.80	.90	.91	.92	.92	.92	.94	.94	.93	.93	.92	.93	.95	.94	.89	.94	.91
$s(R_p)$	.020	.023	.025	.025	.028	.032	.034	.036	.036	.036	.037	.040	.040	.041	.045	.045	.045	.052	.052	.058
$s(e_p)$	.011	.013	.012	.011	.009	.009	.009	.010	.010	.009	.009	.011	.011	.011	.012	.010	.011	.017	.013	.017
$\bar{s}_p(e_i)$	.045	.046	.049	.047	.053	.056	.057	.055	.065	.062	.063	.063	.067	.067	.066	.070	.064	.076	.071	.081
$s(e_p)/\bar{s}_p(e_i)$	.24	.28	.24	.23	.16	.16	.15	.18	.15	.14	.14	.17	.16	.16	.18	.14	.17	.22	.18	.20
									ESTIMATION PERIOD 1958-1962											
b_{pm}	.626	.635	.719	.801	.817	.860	.920	.950	.975	.995	1.013	1.019	1.037	1.048	1.069	1.081	1.092	1.098	1.269	1.388
$s(b_{pm})$	.043	.048	.039	.046	.047	.033	.037	.038	.032	.037	.038	.031	.036	.033	.036	.038	.045	.045	.048	.065
$\hat{\rho}(R_p, R_m)^2$	.783	.745	.851	.835	.838	.920	.913	.915	.939	.925	.922	.948	.934	.945	.936	.931	.907	.910	.922	.886
$s(R_p)$	.030	.031	.033	.037	.038	.038	.041	.042	.043	.044	.045	.045	.046	.046	.047	.048	.049	.049	.056	.063
$s(e_p)$	.014	.016	.013	.015	.015	.011	.012	.012	.011	.012	.013	.010	.012	.011	.012	.013	.015	.015	.016	.021
$\bar{s}_p(e_i)$	.049	.052	.056	.059	.064	.061	.070	.069	.068	.064	.069	.066	.067	.062	.070	.072	.076	.068	.070	.078
$s(e_p)/\bar{s}_p(e_i)$	.286	.308	.232	.254	.234	.180	.171	.174	.162	.188	.188	.152	.179	.177	.171	.180	.197	.220	.228	.269
									ESTIMATION PERIOD 1962-1966											
b_{pm}	.514	.625	.665	.697	.791	.812	.843	.888	.916	.940	.941	.943	.976	1.062	1.070	1.216	1.291	1.316	1.365	1.486
$s(b_{pm})$	.042	.046	.050	.040	.032	.032	.034	.033	.029	.034	.032	.030	.038	.036	.036	.041	.032	.046	.052	.056
$\hat{\rho}(R_p, R_m)^2$	.72	.75	.75	.84	.91	.92	.91	.92	.94	.93	.94	.94	.92	.93	.94	.94	.96	.93	.92	.92
$s(R_p)$	.025	.030	.032	.032	.035	.036	.037	.039	.039	.041	.041	.041	.043	.046	.046	.053	.055	.057	.060	.065
$s(e_p)$	.013	.015	.016	.013	.010	.010	.011	.011	.009	.011	.010	.010	.012	.012	.012	.013	.010	.015	.017	.018
$\bar{s}_p(e_i)$	.053	.046	.046	.053	.057	.057	.059	.057	.055	.060	.053	.056	.064	.062	.067	.075	.068	.071	.076	.089
$s(e_p)/\bar{s}_p(e_i)$	.24	.32	.34	.24	.17	.17	.18	.19	.16	.18	.18	.17	.18	.19	.17	.17	.14	.21	.22	.20

SOURCE: Eugene F. Fama and James D. MacBeth, "Risk, Return, and Equilibrium: Empirical Tests," *Journal of Political Economy* 71 (May-June 1973): 620-621.

up in Table 9.2 in terms of ratios $s(e_p)/\bar{s}_p(e_i)$ that are always highest at the extremes of the b_{pm} range and lowest for b_{pm} close to 1. Since these ratios are generally less than .33, however, interdependence among the market model disturbances for different securities does not destroy the value of using portfolios to reduce the dispersion of the errors in estimates of risk measures.

PROBLEMS III.B

2. What is the formal basis for the statement that if the market model disturbances $\tilde{\epsilon}_{it}$ were independent from security to security, then the ratio $s(e_p)/\bar{s}_p(e_i)$ would be about the same for all portfolios?

3. Make some specific comments about the reliability of the b_{pm} of portfolios versus the b_{im} of individual securities. Don't be afraid to look back at the results in Chapter 4.

4. With the switch from securities to portfolios, are there any dramatic changes in the interpretation of the least squares values of $\gamma_{1t}, \gamma_{2t}, \gamma_{3t}$, and γ_{4t} in (46) to (49) as portfolio returns?

ANSWERS

2. The disturbance in the market model relationship between the return on portfolio p and the return on the portfolio m is related to the corresponding market model disturbances for the securities in p according to

$$\tilde{\epsilon}_{pt} = \sum_{i=1}^{K} x_{ip}\tilde{\epsilon}_{it},$$

where K is the number of securities in the portfolio. In the Fama-MacBeth tests, K is approximately the same for all portfolios and there is equal weighting of securities within portfolios. Thus, if the $\tilde{\epsilon}_{it}$ were independent across securities,

$$\sigma^2(\tilde{\epsilon}_p) = \sum_{i=1}^{K} x_{ip}^2\sigma^2(\tilde{\epsilon}_i) = \frac{1}{K}\,\overline{\sigma^2(\tilde{\epsilon}_i)},$$

where $\overline{\sigma^2(\tilde{\epsilon}_i)}$ is the average of the $\sigma^2(\tilde{\epsilon}_i)$ for the K securities in the portfolio p. Since this analysis would be the same for all the Fama-MacBeth portfolios if within each portfolio the $\tilde{\epsilon}_{it}$ were independent across securities, then for each portfolio $\sigma^2(\tilde{\epsilon}_p)$ would be the fraction $1/K$ of the average of the $\sigma^2(\tilde{\epsilon}_i)$ for individual securities. This statement also holds (at least approximately) for the ratio $\sigma(\tilde{\epsilon}_p)/\overline{\sigma(\tilde{\epsilon}_i)}$ if within given portfolios the values of $\sigma^2(\tilde{\epsilon}_i)$ for individual securities are not too different.

The evidence in Table 9.2 suggests, however, that the simple expression above for $\sigma^2(\tilde{\epsilon}_p)$ is not valid. Within portfolios that have b_{pm} much different from 1, there must on average be positive covariances between the $\tilde{\epsilon}_{it}$ of different securities which cause the $\sigma^2(\tilde{\epsilon}_p)$ for these portfolios to be greater than implied by the expression above.

3. To get a more direct appreciation for the effectiveness of the portfolio approach in increasing the reliability of the risk estimates, one can compare the $s(b_{pm})$ for portfolios in Table 9.2 with the $s(b_i)$ for individual securities in Tables 4.3 and 4.4. In Table 9.2, the $s(b_{pm})$ for portfolios for the five-year period 1962–1966 range from about .03, when b_{pm} is close to 1, up to .056 for the portfolio with the highest b_{pm}. Except for the portfolios with the highest b_{pm}, the values of the $s(b_{pm})$ are generally about 4 percent of the values of b_{pm}. On the other hand, in Tables 4.3 and 4.4, the $s(b_{im})$ for individual securities range from .110 to .522. In Table 4.3 $s(b_{im})$ is, on average, about 25 percent as large as b_{im}, whereas in Table 4.4, $s(b_i)$ is on average about 22 percent as large as b_{im}. In short, for individual securities the estimation error in b_{im} is likely to be a substantial fraction of the estimate, whereas for portfolios the estimation error is likely to be a small fraction of the estimate.

4. The least squares values of the γ_{jt} in (46) to (49) are the returns on portfolios, but they are portfolios of the 20 component portfolios used in the computations. Moreover, the least squares values of the γ_{jt} in (46) to (49) have the same properties as the least squares values of the γ_{jt} in (38) to (41), except that for (46) to (49), the properties of the γ_{jt} must be stated in terms of the 20 component portfolios and the b_{pm}, b^2_{pm}, and $\bar{s}_p(e_i)$ for these 20 portfolios. For example, the least squares value of γ_{1t} in (46) is the return on a standard portfolio of the 20 component portfolios, where the weights assigned to the component portfolios yield zero-weighted average values of the 20 values of b_{pm}, the 20 b^2_{pm}, and the 20 $\bar{s}_p(e_i)$. To substantiate these claims, the reader can work through the analysis of (11) to (22), substituting the estimates b_{pm}, b^2_{pm}, and $\bar{s}_p(e_i)$ for β_{iM}, β^2_{iM} and $\sigma(\tilde{\epsilon}_i)$.

IV. Results

A. *Preliminary Discussion*

THE MONTHLY RECORD

We are ready to consider the results of the tests of the two-parameter model. As a warm-up, Table 9.3 shows the month-by-month record of the least squares values of γ_{1t} and γ_{2t} in (49). The table also shows the month-by-month values of $\rho(R_{pt}, b_{pm})^2$, the coefficient of determination, adjusted

TABLE 9.3

The Month-by-Month Record of the Relationship Between Return and Risk on the NYSE, $R_{pt} = \gamma_{1t} + \gamma_{2t}b_{pm} + \eta_{pt}$

t	γ_{1t}	γ_{2t}	$\rho(R_{pt}, b_{pm})^2$	t	γ_{1t}	γ_{2t}	$\rho(R_{pt}, b_{pm})^2$
3501	0.0064	−0.0413	0.007	3903	0.0152	−0.1880	0.931
3502	0.0369	−0.0997	0.328	3904	−0.0074	0.0072	−0.033
3503	−0.0657	−0.0126	−0.045	3905	0.0683	0.0147	0.032
3504	−0.0007	0.1040	0.192	3906	0.0321	−0.1213	0.944
3505	0.1129	−0.0889	0.151	3907	0.0138	0.1202	0.834
3506	0.0277	0.0105	−0.051	3908	−0.0212	−0.0827	0.656
3507	−0.0348	0.1370	0.191	3909	−0.2040	0.6295	0.758
3508	−0.0092	0.0973	0.085	3910	0.0737	−0.0899	0.689
3509	−0.0790	0.0945	0.174	3911	0.0427	−0.1312	0.855
3510	−0.0142	0.0869	0.185	3912	0.0526	−0.0433	0.298
3511	0.1585	0.0060	−0.055	4001	0.0319	−0.0543	0.458
3512	0.0472	0.0021	−0.055	4002	0.0242	0.0069	−0.042
3601	0.0368	0.1342	0.125	4003	0.0318	−0.0076	−0.024
3602	0.0471	0.0047	−0.054	4004	0.0058	0.0038	−0.053
3603	0.0165	−0.0219	−0.039	4005	−0.1816	−0.0898	0.765
3604	0.0179	−0.1398	0.436	4006	0.0424	0.0333	0.184
3605	0.1004	−0.0433	0.024	4007	0.0346	−0.0010	−0.055
3606	0.0102	−0.0096	−0.048	4008	0.0022	0.0226	0.134
3607	0.0397	0.0292	−0.029	4009	−0.0014	0.0365	0.140
3608	0.0159	0.0098	−0.051	4010	−0.0190	0.0692	0.647
3609	0.0247	0.0113	−0.043	4011	0.0140	−0.0206	0.042
3610	0.0133	0.0458	0.042	4012	0.0232	−0.0438	0.334
3611	−0.0841	0.1763	0.341	4101	−0.0110	0.0118	−0.044
3612	−0.0626	0.0905	0.219	4102	−0.0181	−0.0010	−0.055
3701	−0.0530	0.1400	0.265	4103	0.0023	0.0165	0.045
3702	−0.0197	0.0572	0.050	4104	0.0063	−0.0609	0.594
3703	−0.0792	0.0898	0.248	4105	−0.0301	0.0432	0.504
3704	0.0170	−0.1279	0.641	4106	0.0135	0.0518	0.335
3705	0.0055	−0.0295	−0.012	4107	0.0357	0.0987	0.562
3706	−0.0599	−0.0220	−0.014	4108	0.0079	−0.0163	0.152
3707	−0.0489	0.1531	0.481	4109	0.0416	−0.0624	0.561
3708	−0.0134	−0.0449	0.129	4110	−0.0328	−0.0386	0.234
3709	−0.0738	−0.1197	0.630	4111	−0.0096	−0.0258	0.119
3710	−0.0675	−0.0333	0.061	4112	−0.0260	−0.0623	0.479
3711	−0.0380	−0.0659	0.183	4201	−0.0768	0.2445	0.426
3712	0.0110	−0.1122	0.374	4202	−0.0395	0.0240	0.165
3801	0.0305	0.0144	−0.043	4203	−0.0778	0.0266	0.019
3802	0.0200	0.0365	0.005	4204	−0.0225	−0.0226	0.066
3803	−0.1642	−0.1357	0.457	4205	0.0960	−0.0633	0.447
3804	0.1524	0.0786	−0.028	4206	0.0360	−0.0126	0.017
3805	0.0682	−0.1338	0.362	4207	0.0207	0.0274	0.273
3806	0.0405	0.2677	0.694	4208	0.0205	0.0083	−0.037
3807	0.0461	0.0746	0.091	4209	−0.0281	0.0859	0.417
3808	−0.0666	−0.0053	−0.051	4210	−0.0022	0.1036	0.635
3809	−0.0208	0.0085	−0.047	4211	0.0679	−0.0865	0.661
3810	−0.0869	0.2235	0.368	4212	0.0316	0.0001	−0.056
3811	0.0079	−0.0520	0.048	4301	−0.0064	0.1840	0.661
3812	0.0722	−0.0380	0.013	4302	−0.0967	0.2402	0.622
3901	0.0195	−0.0946	0.841	4303	0.0395	0.0746	0.352
3902	0.0406	0.0000	−0.056	4304	−0.0244	0.0631	0.667

TABLE 9.3 (CONT'D)

t	γ_{1t}	γ_{2t}	$\rho(R_{pt}, b_{pm})^2$	t	γ_{1t}	γ_{2t}	$\rho(R_{pt}, b_{pm})^2$
4305	0.0105	0.0759	0.469	4709	−0.0120	0.0188	0.265
4306	0.0441	−0.0357	0.289	4710	0.0147	0.0189	0.366
4307	0.0295	−0.0950	0.786	4711	0.0016	−0.0284	0.289
4308	0.0247	−0.0222	0.282	4712	0.0097	0.0132	0.078
4309	0.0140	0.0222	0.327	4801	−0.0435	0.0249	0.432
4310	0.0092	−0.0133	0.033	4802	−0.0109	−0.0507	0.879
4311	0.0115	−0.1024	0.847	4803	0.0307	0.0684	0.551
4312	0.0239	0.0720	0.713	4804	0.0140	0.0272	0.338
4401	−0.0025	0.0495	0.466	4805	0.0310	0.0614	0.668
4402	0.0011	0.0125	0.031	4806	−0.0226	0.0064	−0.017
4403	0.0115	0.0335	0.462	4807	−0.0272	−0.0256	0.202
4404	0.0038	−0.0348	0.694	4808	0.0095	−0.0089	0.063
4405	0.0359	0.0319	0.299	4809	−0.0106	−0.0350	0.717
4406	−0.0122	0.1177	0.779	4810	0.0306	0.0240	0.384
4407	0.0204	−0.0355	0.752	4811	−0.0182	−0.0903	0.863
4408	0.0029	0.0259	0.219	4812	0.0278	−0.0160	0.125
4409	−0.0008	−0.0019	−0.047	4901	0.0236	−0.0046	−0.032
4410	0.0229	−0.0233	0.333	4902	0.0152	−0.0582	0.736
4411	0.0103	0.0138	0.093	4903	0.0155	0.0448	0.563
4412	−0.0057	0.0778	0.679	4904	0.0265	−0.0594	0.839
4501	0.0171	0.0269	0.236	4905	0.0302	−0.0735	0.875
4502	0.0117	0.0770	0.783	4906	0.0174	−0.0230	0.430
4503	−0.0021	−0.0551	0.587	4907	0.0429	0.0190	0.201
4504	0.0332	0.0577	0.605	4908	0.0288	−0.0051	−0.029
4505	0.0137	0.0174	0.109	4909	0.0237	0.0203	0.209
4506	−0.0528	0.0877	0.688	4910	0.0088	0.0312	0.418
4507	−0.0026	−0.0290	0.570	4911	0.0208	−0.0118	0.060
4508	0.0599	0.0037	−0.051	4912	0.0237	0.0470	0.631
4509	0.0339	0.0272	0.380	5001	−0.0235	0.0603	0.799
4510	0.0622	0.0023	−0.054	5002	0.0153	−0.0009	−0.055
4511	0.0227	0.0720	0.687	5003	0.0087	−0.0105	0.072
4512	−0.0308	0.0633	0.109	5004	−0.0156	0.0646	0.587
4601	−0.0065	0.1036	0.659	5005	0.0156	0.0201	0.135
4602	−0.0169	−0.0518	0.799	5006	−0.0362	−0.0399	0.712
4603	0.0883	−0.0303	0.200	5007	−0.0933	0.1501	0.735
4604	0.0546	−0.0029	−0.052	5008	0.0487	0.0044	−0.046
4605	0.0226	0.0318	0.310	5009	0.0509	0.0037	−0.049
4606	−0.0418	−0.0070	−0.008	5010	−0.0032	−0.0001	−0.056
4607	0.0000	−0.0427	0.661	5011	−0.0068	0.0397	0.435
4608	−0.0291	−0.0477	0.695	5012	−0.0377	0.1200	0.875
4609	−0.0517	−0.0839	0.896	5101	0.0379	0.0382	0.266
4610	−0.0181	0.0104	0.036	5102	0.0331	−0.0239	0.333
4611	0.0182	−0.0190	0.192	5103	0.0190	−0.0544	0.774
4612	0.0548	−0.0116	0.004	5104	−0.0004	0.0420	0.384
4701	−0.0208	0.0543	0.653	5105	0.0162	−0.0413	0.524
4702	−0.0101	0.0026	−0.038	5106	0.0252	−0.0727	0.894
4703	0.0019	−0.0311	0.756	5107	0.0135	0.0498	0.601
4704	−0.0254	−0.0584	0.767	5108	0.0143	0.0366	0.344
4705	0.0119	−0.0494	0.858	5109	−0.0108	0.0317	0.570
4706	0.0438	0.0173	0.121	5110	−0.0105	−0.0126	0.107
4707	0.0093	0.0538	0.616	5111	0.0047	0.0018	−0.049
4708	−0.0083	−0.0082	0.028	5112	0.0347	−0.0206	0.180

TABLE 9.3 (CONT'D)

t	γ_{1t}	γ_{2t}	$\rho(R_{pt}, b_{pm})^2$	t	γ_{1t}	γ_{2t}	$\rho(R_{pt}, b_{pm})^2$
5201	0.0145	0.0013	−0.054	5605	−0.0210	−0.0207	0.398
5202	−0.0120	−0.0095	0.029	5606	0.0229	−0.0022	−0.052
5203	0.0197	0.0111	0.049	5607	0.0375	0.0019	−0.051
5204	−0.0204	−0.0267	0.377	5608	−0.0038	−0.0130	0.124
5205	0.0201	0.0036	−0.037	5609	−0.0344	0.0001	−0.056
5206	0.0030	0.0296	0.454	5610	0.0129	−0.0062	−0.028
5207	0.0199	−0.0075	−0.003	5611	0.0087	0.0042	−0.043
5208	0.0218	−0.0248	0.403	5612	0.0180	0.0002	−0.055
5209	0.0019	−0.0199	0.301	5701	−0.0035	0.0060	−0.032
5210	−0.0065	−0.0053	−0.016	5702	0.0114	−0.0346	0.500
5211	0.0338	0.0228	0.220	5703	0.0102	0.0120	0.025
5212	0.0164	0.0054	−0.033	5704	0.0254	0.0010	−0.055
5301	−0.0033	0.0291	0.274	5705	0.0131	0.0082	0.014
5302	0.0075	0.0027	−0.051	5706	0.0007	−0.0044	−0.042
5303	0.0145	−0.0261	0.180	5707	0.0184	−0.0118	0.013
5304	−0.0150	−0.0070	−0.012	5708	0.0003	−0.0506	0.740
5305	−0.0103	0.0175	0.222	5709	0.0127	−0.0646	0.747
5306	0.0039	−0.0353	0.671	5710	0.0370	−0.1023	0.824
5307	0.0289	−0.0128	0.102	5711	0.0300	−0.0119	0.058
5308	0.0452	−0.0993	0.875	5712	0.0364	−0.0876	0.833
5309	0.0259	−0.0375	0.461	5801	0.0037	0.1024	0.856
5310	0.0128	0.0261	0.368	5802	0.0400	−0.0518	0.609
5311	0.0270	−0.0067	−0.034	5803	0.0394	−0.0017	−0.053
5312	0.0462	−0.0618	0.775	5804	0.0197	0.0111	−0.019
5401	−0.0008	0.0743	0.866	5805	−0.0033	0.0404	0.410
5402	0.0127	−0.0004	−0.055	5806	0.0006	0.0317	0.497
5403	0.0282	0.0016	−0.054	5807	0.0017	0.0547	0.740
5404	0.0228	−0.0026	−0.051	5808	0.0104	0.0216	0.224
5405	0.0044	0.0388	0.431	5809	−0.0065	0.0607	0.693
5406	0.0190	−0.0065	−0.033	5810	−0.0066	0.0361	0.258
5407	0.0341	0.0359	0.280	5811	0.0384	−0.0002	−0.056
5408	−0.0070	−0.0032	−0.046	5812	0.0497	−0.0096	−0.003
5409	0.0431	0.0035	−0.048	5901	0.0157	0.0260	0.153
5410	−0.0031	−0.0026	−0.048	5902	0.0139	0.0120	0.030
5411	0.0465	0.0491	0.504	5903	0.0071	0.0038	−0.048
5412	−0.0321	0.1222	0.888	5904	0.0122	0.0125	0.043
5501	−0.0004	0.0162	0.111	5905	0.0037	0.0054	−0.030
5502	−0.0063	0.0463	0.497	5906	−0.0131	0.0177	0.119
5503	−0.0021	0.0053	−0.032	5907	0.0356	−0.0057	0.001
5504	0.0451	−0.0230	0.194	5908	0.0345	−0.0483	0.783
5505	0.0009	0.0059	−0.037	5909	−0.0247	−0.0203	0.271
5506	0.0250	0.0102	0.001	5910	0.0064	0.0145	0.128
5507	0.0107	−0.0005	−0.055	5911	0.0202	−0.0069	−0.034
5508	0.0094	−0.0057	−0.035	5912	0.0091	0.0133	0.114
5509	−0.0265	0.0187	0.113	6001	−0.0184	−0.0205	0.136
5510	0.0002	−0.0164	0.208	6002	0.0175	−0.0110	0.030
5511	0.0412	0.0139	0.040	6003	0.0270	−0.0520	0.690
5512	0.0025	0.0188	0.095	6004	0.0335	−0.0548	0.723
5601	−0.0063	−0.0152	0.113	6005	0.0216	−0.0030	−0.048
5602	0.0292	0.0056	−0.031	6006	0.0559	−0.0354	0.332
5603	0.0493	0.0039	−0.047	6007	0.0098	−0.0262	0.536
5604	−0.0035	0.0088	−0.010	6008	0.0422	−0.0059	−0.029

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TABLE 9.3 (CONT'D)

t	γ_{1t}	γ_{2t}	$\rho(R_{pt}, b_{pm})^2$	t	γ_{1t}	γ_{2t}	$\rho(R_{pt}, b_{pm})^2$
6009	−0.0146	−0.0440	0.627	6408	0.0147	−0.0236	0.193
6010	0.0156	−0.0362	0.663	6409	−0.0491	0.0914	0.748
6011	0.0435	−0.0019	−0.052	6410	0.0055	0.0122	0.009
6012	0.0477	−0.0110	0.009	6411	0.0531	−0.0543	0.414
6101	0.0269	0.0547	0.535	6412	0.0181	−0.0276	0.304
6102	0.0356	0.0243	0.222	6501	0.0344	0.0231	0.171
6103	0.0164	0.0291	0.218	6502	−0.0084	0.0373	0.266
6104	0.0158	−0.0106	−0.007	6503	−0.0006	0.0060	−0.048
6105	0.0016	0.0398	0.565	6504	−0.0198	0.0578	0.423
6106	0.0063	−0.0452	0.582	6505	−0.0019	−0.0066	−0.028
6107	0.0220	−0.0072	−0.001	6506	−0.0094	−0.0627	0.670
6108	0.0603	−0.0385	0.319	6507	−0.0445	0.0773	0.715
6109	0.0183	−0.0463	0.709	6508	−0.0201	0.0599	0.602
6110	0.0603	−0.0381	0.426	6509	−0.0161	0.0487	0.234
6111	0.0619	−0.0171	0.097	6510	−0.0496	0.0978	0.645
6112	−0.0314	0.0295	0.300	6511	−0.0429	0.0718	0.388
6201	−0.0580	0.0538	0.699	6512	−0.0530	0.0903	0.493
6202	0.0249	−0.0079	−0.016	6601	−0.0241	0.0720	0.405
6203	0.0034	−0.0087	0.002	6602	−0.0507	0.0637	0.560
6204	−0.0257	−0.0411	0.753	6603	−0.0063	−0.0139	0.068
6205	−0.0709	−0.0226	0.199	6604	−0.0602	0.0933	0.717
6206	−0.0540	−0.0277	0.373	6605	0.0257	−0.0991	0.832
6207	0.0511	0.0090	−0.009	6606	−0.0379	0.0317	0.295
6208	0.0067	0.0151	0.079	6607	0.0132	−0.0252	0.143
6209	−0.0184	−0.0392	0.595	6608	−0.0436	−0.0486	0.456
6210	−0.0207	0.0032	−0.038	6609	0.0494	−0.0638	0.679
6211	0.0305	0.1051	0.819	6610	0.0728	−0.0500	0.373
6212	0.0320	−0.0402	0.662	6611	−0.0353	0.0611	0.408
6301	0.0197	0.0615	0.461	6612	−0.0001	0.0141	0.001
6302	−0.0174	0.0061	−0.038	6701	−0.0049	0.1371	0.818
6303	−0.0053	0.0274	0.287	6702	0.0154	−0.0027	−0.050
6304	0.0225	0.0174	0.040	6703	0.0149	0.0373	0.216
6305	−0.0349	0.0695	0.610	6704	0.0292	0.0060	−0.026
6306	−0.0116	−0.0041	−0.044	6705	−0.0328	0.0154	0.024
6307	0.0142	−0.0220	0.059	6706	−0.0139	0.0640	0.621
6308	−0.0084	0.0603	0.485	6707	0.0315	0.0398	0.308
6309	0.0168	−0.0344	0.236	6708	−0.0069	0.0102	0.001
6310	−0.0011	0.0219	0.156	6709	0.0202	0.0093	−0.018
6311	−0.0273	0.0230	0.080	6710	−0.0272	−0.0125	0.014
6312	−0.0213	0.0313	0.248	6711	0.0084	−0.0032	−0.048
6401	−0.0303	0.0558	0.464	6712	0.0109	0.0445	0.368
6402	−0.0423	0.0728	0.590	6801	0.0501	−0.0561	0.414
6403	−0.0479	0.0815	0.547	6802	0.0171	−0.0577	0.549
6404	0.0405	−0.0394	0.168	6803	−0.0224	0.0146	0.101
6405	−0.0171	0.0309	0.302	6804	−0.0005	0.1127	0.838
6406	0.0095	0.0076	−0.029	6805	0.0184	0.0378	0.142
6407	0.0340	−0.0049	−0.050	6806	0.1000	−0.0809	0.730

for degrees of freedom, in the regression of the 20 portfolio returns for month t on the corresponding b_{pm}. Table 9.3 can be viewed as the monthly record of the relationship between return and risk on the New York Stock Exchange and, as such, has several points of interest.

First, the strength of the relationship, as measured by $\rho(R_{pt}, b_{pm})^2$, seems on average low and quite variable from month to month. There are many months when $\rho(R_{pt}, b_{pm})^2$ is negative.* Second, there are many months (149 out of 402) when γ_{1t} is negative, and, more interesting, there are many months (185 out of 402) when γ_{2t} is negative, so that for these months there is a negative relationship between return and risk. The variability of the γ_{1t} and γ_{2t} and the low $\rho(R_{pt}, b_{pm})^2$ indicate that if one were to plot risk-return lines (R_{pt} against b_{pm}), one would find that the characteristics of the lines (intercept γ_{1t} and slope γ_{2t}) change dramatically from month to month and that there is substantial dispersion of points, the 20 portfolio returns, about any given line.

None of this is particularly surprising nor, in itself, contrary to the two-parameter model. All of the hypotheses drawn from the model are statements about the relationship between expected returns and risk, and not about the relationships between return and risk. Thus, it may well be the case that risk, as measured by b_{pm}, does not account for much of the differences among the returns on the 20 portfolios for any given month. If securities are priced according to the theory, however, and if they are priced so that m is a minimum variance portfolio, then β_{pm} should be sufficient to explain differences among the expected returns on the 20 portfolios.

Moreover, it is not at all surprising that γ_{1t} and γ_{2t} are quite variable from month to month. The least squares values of γ_{1t} and γ_{2t} are the returns on portfolios of NYSE stocks. We have known since Chapter 1 that even highly diversified portfolios of NYSE stocks show substantial variability of monthly returns. Thus, we expect γ_{2t} to be quite variable through time and even negative in a large fraction of months. The hypothesis that there is a positive relationship between expected return and risk is nevertheless upheld as long as $E(\tilde{\gamma}_{2t}) > 0$, that is, as long as "on average" there is a positive relationship between return and risk.

Since the hypotheses of the model concern relationships between expected returns and risk, we have to do some manipulation of the time series of γ_{1t} and γ_{2t} in Table 9.3 to get tests of these hypotheses. In essence, average values of γ_{1t} and γ_{2t} and summary measures of the time series properties of γ_{1t} and γ_{2t} are the basis of the relevant tests. Moreover, to test all of the

*Negative values of the coefficient of determination are possible when the coefficient is adjusted for degrees of freedom.

different hypotheses, we need to do similar manipulations of the time series of the least squares values of the γ_{jt} in (46) to (48).

THE NATURE OF THE TESTS

The major tests are summarized in Table 9.4. Results are presented for ten periods: the overall period 1935–June 1968; three long subperiods, 1935–1945, 1946–1955, and 1956–June 1968; and six subperiods, which, except for the first and last, cover five years each. Results are presented for each of the equations (46) to (49). For each period and equation, the table shows: $\overline{\gamma}_j$, the average of the month-by-month least squares values of γ_{jt}; $s(\gamma_j)$, the sample standard deviation of the monthly γ_{jt}; $\overline{\rho}^2$ and $s(\rho^2)$, the mean and the standard deviation of month-by-month coefficients of determination for the regressions of the 20 portfolio returns on the relevant risk measures.

The table also shows first-order autocorrelations of the various monthly γ_{jt}, where the autocorrelations are computed either about the sample mean of γ_{jt}, in which case they are labeled $\hat{\rho}(\gamma_{jt})$, or about an assumed mean of zero, in which case they are labeled $\hat{\rho}_0(\hat{\gamma}_{jt})$. The reasons for doing this are discussed below. Finally, t-statistics for testing the hypothesis that $E(\tilde{\gamma}_{jt}) = 0$ are presented. These t-statistics are

$$t(\overline{\gamma}_j) = \frac{\overline{\gamma}_j}{s(\gamma_j)/\sqrt{T}}, \tag{51}$$

where T is the number of months in the period. If successive values of $\tilde{\gamma}_{jt}$ are independent and identically distributed normal random variables, the t-statistic of (51) is a drawing from the student distribution with $T-1$ degrees of freedom. Since the time periods in Table 9.4 are all five years or longer, $T-1$ is always greater than 59 and the student distribution is well approximated by the unit normal distribution.

B. *Tests of the Major Hypotheses*

TESTS BASED ON AVERAGE RETURNS

Consider first the proposition that if securities are priced so that m is a minimum variance portfolio, then no measure of risk, in addition to β_{im}, is needed to explain expected returns. The results in Panels C and D of Table 9.4 are consistent with this hypothesis. For both (48) and (46), the t-statistics for the mean values of γ_{4t}, the coefficient of $\bar{s}_p(e_i)$, are small, and the signs of the $t(\overline{\gamma}_4)$ for subperiods are randomly positive and negative. Thus, one cannot reject the hypothesis that in both (48) and (46), $E(\tilde{\gamma}_{4t}) = 0$.

Likewise, the results in Panels B and D of Table 9.4 do not reject the proposition that the relationship between expected return and β_{im} is linear. In

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TABLE 9.4
Tests of the Two-Parameter Model

PERIOD	$\overline{\gamma}_1$	$\overline{\gamma}_2$	$\overline{\gamma}_3$	$\overline{\gamma}_4$	$\overline{\gamma_1 - R_f}$	$s(\gamma_1)$	$s(\gamma_2)$	$s(\gamma_3)$	$s(\gamma_4)$	$\hat{\rho}_0(\gamma_1 - R_f)$	$\hat{\rho}(\gamma_2)$	$\hat{\rho}_0(\gamma_3)$	$\hat{\rho}_0(\gamma_4)$	$t(\overline{\gamma}_1)$	$t(\overline{\gamma}_2)$	$t(\overline{\gamma}_3)$	$t(\overline{\gamma}_4)$	$t(\overline{\gamma_1 - R_f})$	$\overline{\rho}^2$	$s(\rho^2)$
										STATISTIC										
										$R_{pt} = \gamma_{1t} + \gamma_{2t} b_{pm} + \eta_{pt}$										
Panel A:																				
1935–6/68	.0061	.0085			.0048	.038	.066			.15	.02			3.24	2.57			2.55	.29	.30
1935–45	.0039	.0163			.0037	.052	.098			.10	−.03			.86	1.92			.82	.29	.29
1946–55	.0087	.0027			.0078	.026	.041			.18	.07			3.71	.70			3.31	.31	.32
1956–6/68	.0060	.0062			.0034	.030	.044			.27	.15			2.45	1.73			1.39	.28	.29
1935–40	.0024	.0109			.0023	.064	.116			.07	−.09			.32	.79			.31	.23	.30
1941–45	.0056	.0229			.0054	.034	.069			.23	.15			1.27	2.55			1.22	.37	.28
1946–50	.0050	.0029			.0044	.031	.047			.20	.04			1.27	.48			1.10	.39	.33
1951–55	.0123	.0024			.0111	.019	.035			.20	.08			5.06	.53			4.56	.24	.29
1956–60	.0148	−.0059			.0128	.020	.034			.37	.18			5.68	−1.37			4.89	.22	.31
1961–6/68	.0001	.0143			−.0029	.034	.048			.22	.09			.03	2.81			−.80	.32	.27
										$R_{pt} = \gamma_{1t} + \gamma_{2t} b_{pm} + \gamma_{3t} b_{pm}^2 + \eta_{pt}$										
Panel B:																				
1935–6/68	.0049	.0105	−.0008		.0036	.052	.118	.056		.03	−.11	−.11		1.92	1.79	−.29		1.42	.32	.31
1935–45	.0074	.0079	.0040		.0073	.061	.139	.074		−.10	−.31	−.21		1.39	.65	.61		1.36	.32	.30
1946–55	−.0002	.0217	−.0087		−.0012	.036	.095	.034		.04	.00	.00		−.07	2.51	−2.83		−.38	.36	.32
1956–6/68	.0069	.0040	.0013		.0043	.054	.116	.053		.17	.07	.03		1.56	.42	.29		.97	.30	.30
1935–40	.0013	.0141	−.0017		.0012	.069	.160	.075		−.13	−.36	−.35		.16	.75	−.19		.14	.24	.30
1941–45	.0148	.0004	.0108		.0146	.050	.111	.073		−.04	−.19	−.04		2.28	.03	1.15		2.24	.39	.29
1946–50	−.0008	.0152	−.0051		−.0015	.037	.104	.032		.14	.04	.00		−.18	1.14	−1.24		−.32	.44	.32
1951–55	.0004	.0281	−.0122		−.0008	.030	.085	.035		−.17	−.14	−.01		.10	2.55	−2.72		−.20	.28	.29
1956–60	.0128	−.0015	−.0020		.0108	.030	.072	.029		.35	.11	.26		3.38	−.16	−.54		2.84	.25	.31
1961–6/68	.0029	.0077	.0034		−.0000	.066	.138	.064		.14	.06	−.01		.42	.53	.51		−.01	.34	.29

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TABLE 9.4 (CONT'D)

Tests of the Two-Parameter Model

PERIOD	STATISTIC $\bar{\gamma}_1$	$\bar{\gamma}_2$	$\bar{\gamma}_3$	$\bar{\gamma}_4$	$\overline{\gamma_1 - R_f}$	$s(\gamma_1)$	$s(\gamma_2)$	$s(\gamma_3)$	$s(\gamma_4)$	$\hat{\rho}_0(\gamma_1 - R_f)$	$\hat{\rho}_0(\gamma_2)$	$\hat{\rho}_0(\gamma_3)$	$\hat{\rho}_0(\gamma_4)$	$t(\bar{\gamma}_1)$	$t(\bar{\gamma}_2)$	$t(\bar{\gamma}_3)$	$t(\bar{\gamma}_4)$	$t(\overline{\gamma_1 - R_f})$	$\bar{\rho}^2$	$s(\rho^2)$
	$R_{pt} = \gamma_{1t} + \gamma_{2t} b_{pm} + \gamma_{4t} \bar{s}_p(e_i) + \eta_{pt}$																			
Panel C:																				
1935-6/68	.0054	.0072		.0198	.0041	.052	.065		.868	.04	-.12		-.04	2.10	2.20		.46	1.59	.32	.31
1935-45	.0017	.0104		.0841	.0015	.073	.083		.921	-.00	-.26		-.08	.26	1.41		1.05	.24	.32	.31
1946-55	.0110	.0075		-.1052	.0100	.032	.056		.609	.08	.02		-.20	3.78	1.47		-1.89	3.46	.34	.32
1956-6/68	.0042	.0041		.0633	.0016	.040	.052		.984	.12	.08		.03	1.28	.96		.79	.50	.30	.29
1935-40	.0036	.0119		-.0170	.0035	.082	.105		.744	-.03	-.26		-.18	.37	.97		-.19	.36	.25	.30
1941-45	-.0006	.0085		.2053	-.0009	.061	.052		1.091	.07	-.29		-.02	-.08	1.25		1.46	-.11	.41	.30
1946-50	.0069	.0081		-.0920	.0062	.034	.066		.504	.14	.06		-.02	1.56	.95		-1.41	1.40	.42	.33
1951-55	.0150	.0069		-.1185	.0138	.029	.043		.702	.06	-.18		-.32	4.05	1.24		-1.31	3.72	.27	.29
1956-60	.0127	-.0081		.0728	.0107	.037	.045		1.164	.15	.15		.21	2.68	-1.40		.48	2.26	.26	.30
1961-6/68	-.0014	.0122		.0570	-.0044	.042	.055		.850	.10	.00		-.19	-.32	2.12		.64	-.98	.33	.27
	$R_{pt} = \gamma_{1t} + \gamma_{2t} b_{pm} + \gamma_{3t} b_{pm}^2 + \gamma_{4t} \bar{s}_p(e_i) + \eta_{pt}$																			
Panel D:																				
1935-6/68	.0020	.0114	-.0026	.0516	.0008	.075	.123	.060	.929	-.09	-.09	-.12	-.10	.55	1.85	-.86	1.11	.20	.34	.31
1935-45	.0011	.0118	-.0009	.0817	.0010	.103	.146	.079	1.003	-.20	-.23	-.24	-.15	.13	.94	-.14	.94	.11	.34	.31
1946-55	.0017	.0209	-.0076	-.0378	.0008	.042	.096	.038	.619	-.10	-.00	-.01	-.20	.44	2.39	-2.16	-.67	.20	.36	.32
1956-6/68	.0031	.0034	-.0000	.0966	.0005	.065	.122	.055	1.061	.12	.03	.01	-.05	.59	.34	-.00	1.11	.10	.32	.29
1935-40	.0009	.0156	-.0029	.0025	.0008	.112	.171	.085	.826	-.16	-.23	-.26	-.12	.07	.78	-.29	.03	.06	.26	.30
1941-45	.0015	.0073	.0014	.1767	.0012	.092	.109	.072	1.181	-.28	-.21	-.22	-.18	.12	.52	.15	1.16	.10	.43	.31
1946-50	.0011	.0141	-.0040	-.0313	.0004	.047	.106	.042	.590	-.10	.03	-.01	-.12	.18	1.03	-.73	-.41	.07	.44	.33
1951-55	.0023	.0277	-.0112	-.0443	.0011	.037	.085	.034	.651	-.11	-.13	-.01	-.28	.48	2.53	-2.54	-.53	.23	.29	.30
1956-60	.0103	-.0047	-.0020	.0979	.0083	.049	.078	.032	1.286	-.16	.19	-.01	.02	1.63	-.47	-.49	.59	1.31	.28	.30
1961-6/68	-.0017	.0088	.0013	.0957	-.0046	.073	.144	.066	.887	.20	.00	.01	-.15	-.21	.58	.19	1.02	-.60	.35	.29

SOURCE Eugene F. Fama and James D. MacBeth, "Risk, Return, and Equilibrium: Empirical Tests," *Journal of Political Economy* 71 (May-June 1973): 622-623.

Panel B, the value of $t(\bar{\gamma}_3)$ for the overall period 1935-June 1968 is only -.29. In the subperiods, there are four positive values of $\bar{\gamma}_3$ and five negative values. In the five-year subperiods, $t(\bar{\gamma}_3)$ for 1951-1955 is approximately -2.7, but for subperiods that do not cover 1951-1955 the values of $t(\bar{\gamma}_3)$ are much closer to zero. There is likewise no systematic evidence in Panel D against the hypothesis that $E(\tilde{\gamma}_{3t}) = 0$. Thus, one cannot reject the hypothesis that b_{pm}^2 contributes nothing to the description of expected returns.

Since the evidence is consistent with (36)—that is, with the propositions that the relationship between expected return and β_{im} is linear and that β_{im} is the only measure of risk needed to explain differences among expected returns—we cannot reject the hypothesis that securities are priced so that m is a minimum variance portfolio. The data are also consistent with the hypothesis that m is on the positively sloped segment of the minimum variance boundary, so that the trade-off of expected return for risk in m is positive. For the overall period 1935-June 1968, $t(\bar{\gamma}_2)$ is large for all models. Except for 1956-1960, the values of $t(\bar{\gamma}_2)$ are also systematically positive in the subperiods, but not so systematically large.

The small values of $t(\bar{\gamma}_2)$ for subperiods reflect the substantial month-to-month variability of γ_{2t}. For example, in Panel A we find that for equation (49) during 1935-1940, $\bar{\gamma}_2 = .0109$. For this period the average incremental return per unit of b_{im} is almost 1.1 percent per month. On average, bearing risk produced substantial rewards. Nevertheless, because of the variability of γ_{2t}—in this period, $s(\gamma_2)$ is 11.6 percent per month—$t(\bar{\gamma}_2)$ is only .79. It takes the statistical power of the large sample for the overall period, that is, a large value of T in (51), before values of $\bar{\gamma}_2$ that are large in practical terms also yield large t-statistics.

At least with the sample for the overall period, $t(\bar{\gamma}_2)$ achieves values supportive of the conclusion that there is a statistically observable positive relationship between expected return and risk. This is not the case with respect to $t(\bar{\gamma}_3)$ and $t(\bar{\gamma}_4)$. Even or indeed especially for the overall period, these t-statistics are close to zero. This makes us somewhat more confident of the conclusions that the relationship between expected return and risk is linear and that β_{im} is the only measure of risk needed to explain differences among expected returns.

In short, the tests on average returns are consistent with the hypothesis that securities are priced so that the portfolio m is on the positively sloped segment of the minimum variance boundary.

TESTS BASED ON THE TIME-SERIES BEHAVIOR OF RETURNS

The autocorrelations in Table 9.4 are also consistent with what would be expected from a market where securities are priced according to (36) and where, in addition, the market is efficient. The autocorrelations of γ_{3t} and

γ_{4t}, computed about means that are assumed to be zero, test the proposition that there is no information in the time series of past values of γ_{3t} and γ_{4t} that ever warrants nonzero assessments of expected future values. Consistent with this proposition, $\hat{\rho}_0(\gamma_3)$ and $\hat{\rho}_0(\gamma_4)$ in Table 9.4 are always low in terms of explanatory power and generally low in terms of statistical significance.

Recall from Chapter 4 that the proportion of the variance of a variable explained by first-order autocorrelation is estimated by the square of the estimated first-order coefficient. In all cases, $\hat{\rho}_0(\gamma_3)^2$ and $\hat{\rho}_0(\gamma_4)^2$ are small. As for statistical significance, if the true autocorrelation is zero, the standard deviation of the sample coefficient can be approximated by $\sigma(\hat{\rho}) = 1/\sqrt{T}$. For the overall period, $\sigma(\hat{\rho})$ is approximately .05, while for the ten- and five-year subperiods, $\sigma(\hat{\rho})$ is approximately .09 and .13, respectively. Thus, the values of $\hat{\rho}_0(\gamma_{3t})$ and $\hat{\rho}_0(\gamma_{4t})$ are generally also statistically close to zero. There are exceptions to this statement, but they involve primarily periods that include the 1935–1940 subperiod, and the results for these periods are not independent. Moreover, even though the true autocorrelation may be close to zero, some autocorrelations are expected to be large on a purely chance basis when many sample autocorrelations are computed.

The proposition that securities are priced so that m is on the positively sloped segment of the minimum variance boundary only says that $E(\tilde{\gamma}_{2t}) = [E(\tilde{R}_{mt}) - E(\tilde{R}_{0mt})] > 0$; the model does not hypothesize a specific value of $E(\tilde{\gamma}_{2t})$. If we are willing to assume that the equilibrium expected value of the risk premium is constant through time, then sample autocorrelations of γ_{2t}, computed about the sample mean of γ_{2t}, test the proposition that the time series of past values of γ_{2t} never warrants an assessment of the expected future value of $\tilde{\gamma}_{2t}$, which is different from the assumed constant equilibrium expected value of γ_{2t}. Since the sample values of $\hat{\rho}(\gamma_{2t})$ in Table 9.4 are small, both statistically and in terms of explanatory power, the proposition is not rejected by the evidence.

Market efficiency in a world where securities are priced so that m is a minimum variance portfolio also implies that the disturbances $\tilde{\eta}_{pt}$ in (46) to (49) should be uncorrelated through time. If this is not the case, then the time series of past values of $\tilde{\eta}_{pt}$ can be used as the basis of correct nonzero assessments of expected future values, which means that the future expected return on the portfolio is not simply as predicted by the expected return-risk relationship (36). In this case, if the market is trying to price securities according to (36), then it is inefficient in the sense that it is ignoring information in past returns.

Fama and MacBeth do not show autocorrelations for the η_{pt}, but they report that, like the autocorrelations of the γ_{jt}, those of the η_{pt} are close to zero. They also compute higher-order autocorrelations for the γ_{jt} and η_{pt},

that is, autocorrelations for lags greater than one, and they report that these are likewise never systematically large.

EVIDENCE ON THE RELIABILITY OF THE γ_{jt} OF DIFFERENT RETURN EQUATIONS

Since the results from (46) to (49) are mutually consistent, we need not be too concerned with determining which version of a particular γ_{jt} provides the most reliable test of a specific implication of (36). Nevertheless, it is interesting to note that the results in Table 9.4 are consistent with our earlier discussion of reliability. Thus, for any given j, the least squares γ_{jt} in (46), (47), (48), or (49) is the return on a portfolio that focuses on one testable implication of (36), while "zeroing out" the effects of any other explanatory variables that appear in (46), (47), (48), or (49). The fewer the explanatory variables in an equation, the fewer "zeroing out" constraints imposed on the least squares procedure in its search for a "smallest variance" γ_{jt}. Thus, the search is likely to be more successful.

For example, in each of the equations (46) to (49), the least squares γ_{2t} is the return on a portfolio that has $b_{pm} = 1.0$ and that zeroes out the effects of any other variables that appear in the return equation. In Panel A of Table 9.4, we find indeed that the less constrained γ_{2t} of (49) has standard deviations $s(\gamma_2)$ much smaller than the $s(\gamma_2)$ from (46) or (47) in Panels B and D, and usually at least a little smaller than $s(\gamma_2)$ from (48). Similar comparisons can be carried out, by the reader, between the values of $s(\gamma_3)$ in Panels B and D of Table 9.4 and between the values of $s(\gamma_4)$ in Panels C and D.

PROBLEM IV.B

1. How can one explain the fact that $s(\gamma_2)$ increases more when one goes from Panel A to Panel B than when one goes from Panel A to Panel C?

ANSWER

1. When an additional variable is included in a return equation, its effect on $s(\gamma_2)$ depends on the strength of the relationship between the additional variable, b_{pm}^2 or $\bar{s}_p(e_i)$, and b_{pm}. If the relationship is strong, then the additional constraint that the weighted average of the 20 values of b_{pm}^2 or $\bar{s}_p(e_i)$ must be zero is a strong constraint on the way the 20 component portfolios can be combined to get the smallest variance portfolio that has $b_{pm} = 1.0$. The evidence from the $s(\gamma_2)$ in Table 9.4 is that the relationship between b_{pm} and b_{pm}^2 is (obviously) strong, but the relationship between b_{pm} and $\bar{s}_p(e_i)$ is not so strong. The statistically sophisticated reader recognizes that this is just an intuitive discussion of the statistical phenomenon called multicollinearity.

C. The Sharpe-Lintner Hypothesis

STATISTICAL CONSIDERATIONS

In the Sharpe-Lintner model of market equilibrium there is unrestricted borrowing and lending at the risk-free rate of interest R_{Ft}. The testable implication of this assumption is that the expected return on any security or portfolio whose return is uncorrelated with the return on the market portfolio M is R_{Ft}. If we are willing to assume that m, the equally weighted portfolio of NYSE stocks, is a good proxy for M, the value-weighted portfolio of all investment assets, then Table 9.4 contains tests of the Sharpe-Lintner hypothesis. Thus, the least squares value of γ_{1t} for (49) is the return on a standard portfolio where the weights assigned to the 20 component Fama-MacBeth portfolios yield a weighted average value of the 20 component b_{pm} equal to zero. The least squares values of γ_{1t} for (46), (47), and (48) are likewise the returns on zero-b_{pm} portfolios, but these portfolios are constructed under the additional constraints that the weighted average of the 20 b_{pm}^2 must be zero [(46) and (47)] and/or the weighted average of the 20 component $\bar{s}_p(e_i)$ must be zero [(46) and (48)].

Since we could not reject the linearity and non-β risk hypotheses of the two-parameter model, we are free to choose any of the return models as the basis of the tests of the Sharpe-Lintner hypothesis. That is, in constructing a zero-b_{pm} portfolio, there is no need to "zero out" the effects of nonlinearities and non-β risk on expected returns; we can make our decision about the Sharpe-Lintner hypothesis by determining which return model seems to provide the most convincing evidence. As judged by the values of $s(\gamma_1)$, the most reliable tests are indeed from the simplest model (49), and in these the hypothesis takes a sound thumping. For the overall period, the value of $t(\overline{\gamma_1 - R_F})$ in Panel A of Table 9.4 is 2.55, which is reliably different from zero. In practical terms, the average value of $\gamma_{1t} - R_F$ for 1935-June 1968 is .0048; on average, the premium in the return on the zero-b_{pm} portfolio produced by the least squares computations is almost half of one percent per month. In the results from (49), there is only one subperiod, 1961-June 1968, when the Sharpe-Lintner hypothesis does well. In all other subperiods, and especially those covering 1951-1960, the premium of γ_{1t} over R_{Ft} is substantial.

On the basis of tests similar to those of Fama and MacBeth, Black, Jensen, and Scholes (1972) and Friend and Blume (1970) likewise come to a negative conclusion with respect to the Sharpe-Lintner hypothesis.

THEORETICAL CONSIDERATIONS

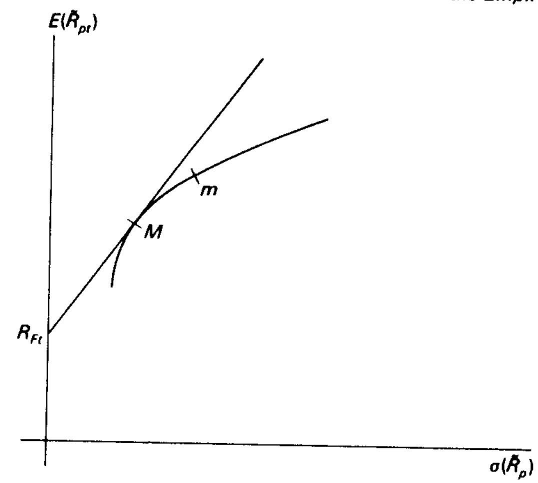
Under the assumption that m, the equally weighted portfolio of NYSE stocks, is an adequate proxy for M, the value-weighted portfolio of all investment assets, there are at least two versions of the two-parameter model consistent with the preceding evidence; that is, there are at least two models of market equilibrium in which $E(\tilde{R}_{0Mt})$, the expected return on zero-β positive variance securities and portfolios, is greater than the risk-free rate R_{Ft}.

Thus, in the restricted borrowing version of the Black model there is risk-free lending at the rate R_{Ft}, but there is no risk-free borrowing. The picture of market equilibrium is Figure 8.5. The value-weighted market portfolio M is efficient, but it is above the tangency portfolio T on the efficient boundary. As a consequence, $E(\tilde{R}_{0Mt})$, which is just the intercept on the $E(\tilde{R}_{pt})$ axis of the line tangent to the boundary at M, is greater than R_{Ft}. Similarly, in the "margin" version of the model, there is unrestricted risk-free lending, but risk-free borrowing is restricted to some fixed fraction of portfolio funds. Market equilibrium is as pictured in Figure 8.7. The market portfolio M is one of the minimum variance portfolios of positive variance securities; but since it is above the tangency portfolio T on the minimum variance boundary, we again have the condition $E(\tilde{R}_{0Mt}) > R_{Ft}$. Since there are at least two models of market equilibrium that are consistent with the evidence, rejection of the Sharpe-Lintner model is not a telling blow to two-parameter theory.

There is, however, good reason to believe that the Fama-MacBeth tests of the Sharpe-Lintner hypothesis are inappropriate, and the arguments that follow apply equally to the results of Black, Jensen, and Scholes (1972) and Friend and Blume (1970). In particular, there is good evidence that m is not

FIGURE 9.1

A View of the Sharpe-Lintner Model That Is Consistent with the Empirical Evidence



a good proxy for M. Lawrence Fisher (1966) reports that the standard deviation of the return on a value-weighted portfolio of NYSE stocks is about 80 percent as large as the standard deviation of the return on an equally weighted portfolio of NYSE stocks. NYSE stocks would be among the more risky securities in a market portfolio that included all investment assets. Thus, there is little doubt that m, the equally weighted portfolio of NYSE stocks, is substantially more risky than M, the value-weighted portfolio of all investment assets. In this light, the empirical evidence might be interpreted as consistent with the picture of market equilibrium shown in Figure 9.1. The portfolio m is along the positively sloped segment of the boundary of minimum variance portfolios of positive variance securities, but m is pictured as substantially more risky than M. The overall view of market equilibrium is that of the Sharpe-Lintner model.

In truth, all we can really say at this time is that the literature has not yet produced a meaningful test of the Sharpe-Lintner hypothesis.

V. Some Applications of the Measured Risk-Return Relationships

The general objective of this chapter is to test the implications of the portfolio model of Chapters 2 and 7 for the pricing of securities in the capital market. We want to test whether the pricing of securities reflects the attempts of investors to hold efficient portfolios. To give this hypothesis testable content, we need some specific model of market equilibrium. The testable implication of the models of market equilibrium presented in Chapter 8 is that a market equilibrium requires that the value-weighted market portfolio M be a minimum variance portfolio, and in most models M is also an efficient portfolio. Thus, securities must be priced so that the expected return-risk equation (1) of this chapter applies to securities and portfolios.

The original goal of this chapter was to test whether observed relationships between average returns and estimates of risk are consistent with (1). What we have in fact tested is whether the pricing of securities is consistent with the hypothesis that m, the equally weighted portfolio of NYSE stocks, is a minimum variance portfolio. That is, we end up testing whether (36) is an appropriate representation of the relationship between expected return and risk for NYSE stocks. The evidence seems to be consistent with this proposition, which gives us some confidence that the two-parameter model captures important aspects of the pricing of securities. On the other hand, since there is no formal model of market equilibrium that tells us that

securities are priced so that m is a minimum variance portfolio, it is not clear how much support for the two-parameter model should be imputed to the results.

Like all the empirical issues raised in this book, this is an issue that the reader can judge. My task is completed if I have presented a reasonably coherent introduction to the theory and evidence. I shall, however, now try to bias the reader's judgment with respect to the usefulness of the risk-return tests with a brief discussion of some of the applications that have been made of the results.

The applications use the month-by-month relationships between risk and return obtained from the preceding results (a) to test the efficiency of the capital market in adjusting prices to specific types of new information and (b) to measure the performance of managed portfolios. Before discussing these applications, however, we must develop a better understanding of the nature of the measured risk-return relationships that come out of the preceding results.

A. *A Two-Factor Market Model*

We found in Chapter 3 that if the joint distribution of security returns is multivariate normal, the joint distribution of the return on any security and the return on any portfolio is bivariate normal. Although we need not go into the details,* multivariate normality of security returns also implies that the joint distribution of the return on any security and the returns on any two (or three, or four or $n - 1$) portfolios is multivariate normal.

We are interested in a specific application of this result. As in Chapter 8, let Z be the minimum variance portfolio of positive variance securities whose return is uncorrelated with the return on the market portfolio M. With multivariate normality of security returns, the joint distribution of the return on any security i and the returns on Z and M is multivariate (trivariate) normal, which can be shown to imply a relationship among $\tilde{R}_{it}$, $\tilde{R}_{Zt}$, and $\tilde{R}_{Mt}$ of the form

$$\tilde{R}_{it} = \alpha_i + \beta_{iZ}\tilde{R}_{Zt} + \beta_{iM}\tilde{R}_{Mt} + \tilde{\eta}_{it}, \tag{52}$$

where

$$\alpha_i = E(\tilde{R}_{it}) - \beta_{iZ}E(\tilde{R}_{Zt}) - \beta_{iM}E(\tilde{R}_{Mt}), \tag{53}$$

$$\beta_{iZ} = \frac{\text{cov}(\tilde{R}_i, \tilde{R}_Z)}{\sigma^2(\tilde{R}_Z)}, \text{ and } \beta_{iM} = \frac{\text{cov}(\tilde{R}_i, \tilde{R}_M)}{\sigma^2(\tilde{R}_M)}, \tag{54}$$

*Those interested can consult Anderson (1958, chaps. 1-2) or Cramer (1946, chaps. 21, 24).

and where the disturbance $\tilde{\eta}_{it}$ has expected value equal to zero and is independent of $\tilde{R}_{Zt}$ and $\tilde{R}_{Mt}$.

Note that β_{iZ} is the risk of security i in the portfolio Z, measured relative to the risk or variance of the return on Z, just as β_{iM} is the risk of security i in M measured relative to the risk of M. That the coefficients in (52) correspond to these risk measures is a special consequence of the fact that $\tilde{R}_{Zt}$ and $\tilde{R}_{Mt}$ are uncorrelated. Note also that (52) can be interpreted as a "two-factor" version of the market model of (4). In (52) the return on security i is related to the returns on the portfolios Z and M, whereas in (4) only M is used. Finally, it is also important to note that if security returns have a multivariate normal distribution, then the market models of (4) and (52) are both valid representations of the return on security i.

The fact that Z is the minimum variance zero-β_{pM} portfolio will soon be shown to imply that

$$\beta_{iZ} = 1 - \beta_{iM}. \tag{55}$$

Thus, (52) can be rewritten as

$$\tilde{R}_{it} = \alpha_i + (1 - \beta_{iM})\tilde{R}_{Zt} + \beta_{iM}\tilde{R}_{Mt} + \tilde{\eta}_{it} \tag{56a}$$

$$= \alpha_i + \tilde{R}_{Zt} + \beta_{iM}(\tilde{R}_{Mt} - \tilde{R}_{Zt}) + \tilde{\eta}_{it}. \tag{56b}$$

If the world is as described by any of the models of market equilibrium of Chapter 8—that is, if the relationship between expected return and risk in M is as described by (1)—then for every security i, $\alpha_i = 0.0$ and (56a-b) become

$$\tilde{R}_{it} = (1 - \beta_{iM})\tilde{R}_{Zt} + \beta_{iM}\tilde{R}_{Mt} + \tilde{\eta}_{it} \tag{57a}$$

$$\tilde{R}_{it} = \tilde{R}_{Zt} + \beta_{iM}(\tilde{R}_{Mt} - \tilde{R}_{Zt}) + \tilde{\eta}_{it}. \tag{57b}$$

PROBLEMS V.A

1. Show that if Z is the minimum variance zero-β_{pM} portfolio, then (55) holds.

2. Show that in (56)

$$\sum_{i=1}^{n} x_{iM}\alpha_i = 0, \qquad \sum_{i=1}^{n} x_{iM}\tilde{\eta}_{it} = 0.$$

3. Interpret the constant α_i and the disturbance $\tilde{\epsilon}_{it}$ in the one-factor market model of (4) in terms of quantities from the two-factor model of (57).

4. Assume that the equally weighted portfolio m of NYSE stocks is a minimum variance portfolio, and let z be the minimum variance portfolio whose return is uncorrelated with the return on m. Show that the return on any NYSE stock can then be expressed as

$$\tilde{R}_{it} = (1 - \beta_{im})\tilde{R}_{zt} + \beta_{im}\tilde{R}_{mt} + \tilde{\eta}_{it} \tag{58a}$$

$$\tilde{R}_{it} = \tilde{R}_{zt} + \beta_{im}(\tilde{R}_{mt} - \tilde{R}_{zt}) + \tilde{\eta}_{it}, \tag{58b}$$

where $\tilde{\eta}_{it}$ has expected value equal to zero and is independent of $\tilde{R}_{zt}$ and $\tilde{R}_{mt}$.

ANSWERS

1. The minimum variance zero-β_{pM} portfolio Z is given by the set of weights x_{iZ}, $i = 1, \cdots, n$, that constitute the solution to the problem

$$\min \sigma^2(\tilde{R}_Z) = \min \sum_{i=1}^{n}\sum_{j=1}^{n} x_{iZ}x_{jZ}\sigma_{ij}$$

subject to the constraints

$$\beta_{ZM} = \sum_{i=1}^{n} x_{iZ}\beta_{iM} = 0 \text{ and } \sum_{i=1}^{n} x_{iZ} = 1.$$

The Lagrangian expression for this problem can be written as

$$\sigma^2(\tilde{R}_Z) + 2\lambda_1\left(0 - \sum_{i=1}^{n} x_{iZ}\beta_{iM}\right) + 2\lambda_2\left(1 - \sum_{i=1}^{n} x_{iZ}\right).$$

Differentiating this expression with respect to x_{iZ}, and then setting the derivative equal to zero, we get

$$\sum_{j=1}^{n} x_{jZ}\sigma_{ij} - \lambda_1\beta_{iM} - \lambda_2 = 0, \quad i = 1, \cdots, n, \tag{59}$$

or

$$\text{cov}(\tilde{R}_i, \tilde{R}_Z) = \lambda_1\beta_{iM} + \lambda_2. \tag{60}$$

If we multiply through (59) or (60) by x_{iZ} and then sum over i, we find that

$$\sigma^2(\tilde{R}_Z) = \lambda_2. \tag{61}$$

If we multiply through (60) by x_{iM} and sum over i, we find that

$$0 = \text{cov}(\tilde{R}_M, \tilde{R}_Z) = \lambda_1 + \lambda_2,$$

which, with (61), implies

$$\lambda_1 = -\lambda_2 = -\sigma^2(\tilde{R}_Z). \tag{62}$$

With (61) and (62), (60) becomes

$$\text{cov}(\tilde{R}_i, \tilde{R}_Z) = -\sigma^2(\tilde{R}_Z)\beta_{iM} + \sigma^2(\tilde{R}_Z),$$

so that

$$\beta_{iZ} = \frac{\text{cov}(\tilde{R}_i, \tilde{R}_Z)}{\sigma^2(\tilde{R}_Z)} = 1 - \beta_{iM}.$$

2. Multiply through (56) by x_{iM} and then sum over i to get

$$\tilde{R}_{Mt} = \sum_{i=1}^{n} x_{iM}\alpha_i + \tilde{R}_{Mt} + \sum_{i=1}^{n} x_{iM}\tilde{\eta}_{it}.$$

Since the α_i are constants and the $\tilde{\eta}_{it}$ are random variables, the two weighted averages must separately be equal to zero. The fact that the weighted sum of the disturbances must be zero implies that the $\tilde{\eta}_{it}$ of different securities cannot be independent. Recall the similar result that we found in Section II.C of Chapter 3 for the disturbances from the one-factor market model of (4).

3. If we take expected values in (4) and (57a), we find that the intercept α_i in the one-factor market model is

$$\alpha_i = (1 - \beta_{iM})E(\tilde{R}_Z). \tag{63}$$

If we substitute (63) into (4) and then subtract (57a) from (4), we find that the disturbance $\tilde{\epsilon}_{it}$ in the one-factor model of (4) is related to $\tilde{R}_{Zt}$ and the disturbance $\tilde{\eta}_{it}$ in the two-factor model according to

$$\tilde{\epsilon}_{it} = (1 - \beta_{iM})[\tilde{R}_{Zt} - E(\tilde{R}_{Zt})] + \tilde{\eta}_{it}. \tag{64}$$

Thus, in the one-factor market model of (4), there is an "omitted variable," $\tilde{R}_{Zt}$, which shows up in the disturbance $\tilde{\epsilon}_{it}$. Since $\tilde{R}_{Zt}$ and $\tilde{R}_{Mt}$ are uncorrelated, the omitted variable does not lead to any particular statistical problems in (4). Since $\tilde{R}_{Zt}$ and $\tilde{\eta}_{it}$ are independent, $\sigma^2(\tilde{\eta}_{it}) \leq \sigma^2(\tilde{\epsilon}_{it})$; that is, the two-factor model necessarily "explains" at least as much of $\sigma^2(\tilde{R}_{it})$ as the one-factor model. The improvement in explanatory power that one gets from including $\tilde{R}_{Zt}$ in the equation depends on the size of the variance of $\tilde{R}_{Zt}$ relative to that of $\tilde{\eta}_{it}$.

Finally, the presence of the common variable $\tilde{R}_{Zt}$ in the disturbance $\tilde{\epsilon}_{it}$ tends to produce correlation between the one-factor market model disturbances of different securities. From (64) we can see that the presence of $\tilde{R}_{Zt}$ in $\tilde{\epsilon}_{it}$ will tend to produce positive correlation between the $\tilde{\epsilon}_{it}$ of securities that have values of β_{iM} on the same side of 1.0 and negative correlation between the $\tilde{\epsilon}_{it}$ of securities that have values of β_{iM} on opposite sides of 1.0. We can also see from (64) that these effects are larger the further the values of β_{iM} are from 1. This may explain our earlier observation (Table 9.2) that when portfolios are formed from securities with very high or very low values of β, there is a smaller reduction in the variance of the one-factor market model disturbances than when portfolios are formed from securities with values of β closer to 1.

4. Multivariate normality of security returns implies that there is a relationship between $\tilde{R}_{it}$, $\tilde{R}_{zt}$, and $\tilde{R}_{mt}$ in the form of (52), with coefficients given by (53) and (54), but where one simply substitutes z for Z and m for M. The same substitutions in Problem V.A.*1* produce the conclusion that

$$\beta_{iz} = \frac{\text{cov}(\tilde{R}_i, \tilde{R}_z)}{\sigma^2(\tilde{R}_z)} = 1 - \beta_{im}, \tag{65}$$

so that the step from (52) to (56) is valid. If securities are priced so that m is a minimum variance portfolio, then (36) applies to securities in m, from which it follows that the constant α_i in the relevant version of (56) is zero for securities in m; that is, the returns on securities in m can be represented as in (58). Finally, substitution of z for Z and m for M also allows us to apply the results of Problems V.A.*2* and *3* to the two-factor market model relationship between $\tilde{R}_{it}$, $\tilde{R}_{zt}$, and $\tilde{R}_{mt}$.

We might emphasize that all of the properties ascribed to (56) are implications of multivariate normality and of the fact that Z is the minimum variance portfolio whose return is uncorrelated with the return on M. In adding the assumption that securities are priced according to some two-parameter model of market equilibrium, we get the implication that α_i is zero for all securities. In the applications, however, this additional implication is important.

B. *Market Efficiency and the Two-Factor Models*

We found in Chapter 8 that in the Black model and its variants, in a market equilibrium the returns on all minimum variance portfolios can be expressed as combinations of Z and M:

$$\tilde{R}_{et} = x_e\tilde{R}_{Zt} + (1 - x_e)\tilde{R}_{Mt}, \tag{66}$$

where x_e and $(1 - x_e)$ are the proportions invested in Z and M to get the minimum variance portfolio e. With (66) we can determine that

$$\beta_{eM} = \frac{\text{cov}(\tilde{R}_e, \tilde{R}_M)}{\sigma^2(\tilde{R}_M)}$$

$$= \frac{\text{cov}(x_e\tilde{R}_Z + (1 - x_e)\tilde{R}_M, \tilde{R}_M)}{\sigma^2(\tilde{R}_M)} = \frac{(1 - x_e)\sigma^2(\tilde{R}_M)}{\sigma^2(\tilde{R}_M)} = 1 - x_e.$$

Thus (66) can be rewritten as

$$\tilde{R}_{et} = (1 - \beta_{eM})\tilde{R}_{Zt} + \beta_{eM}\tilde{R}_{Mt} = \tilde{R}_{Zt} + \beta_{eM}(\tilde{R}_{Mt} - \tilde{R}_{Zt}). \tag{67}$$

Equation (67) says that there is an exact relationship between the return on any minimum variance portfolio e, its β_{eM}, the return on the minimum variance zero-β_{pM} portfolio Z, and the return on the market portfolio M. Equations (57) say that there is a similar risk-return relationship for any security i, except that the relationship is not exact; it is subject to a disturbance $\tilde{\eta}_{it}$. In intuitive terms, $\tilde{R}_{Zt}$ and $\tilde{R}_{Mt}$ capture the effects of marketwide factors on

$\tilde{R}_{it}$, while factors more specific to the prospects of security i show up in $\tilde{\eta}_{it}$. The applications discussed below are concerned with the efficiency of the market in adjusting security prices to information specific to the prospects of security i. Thus, they are concerned with testing the implications of market efficiency for the properties of $\tilde{\eta}_{it}$. By applying (35) to (57) we determine that if the market is efficient, and if market equilibrium is characterized by the expected return-risk equation (1), then in (57),

$$E(\tilde{\eta}_{it}|\phi_{t-1}) = 0. \tag{68}$$

In words, there is no way to use ϕ_{t-1}, the set of information available at $t-1$, or any subset of ϕ_{t-1}, as the basis of a correct nonzero assessment of the expected value of the disturbance $\tilde{\eta}_{it}$ in (57).

The applications discussed below test whether (68) is a correct description of the world. One set of tests is concerned with whether (68) holds with respect to particular items of company-specific information. For example, if a merger is announced by firm i at $t-1$, can it be used as the basis of a correct nonzero assessment of the expected value of $\tilde{\eta}_{it}$ for that firm's stock? If the market is efficient the answer to this question is, of course, no. The price of the stock of firm i will fully adjust at $t-1$ to any information in a merger, so that at time t the deviation of $\tilde{\eta}_{it}$ from zero cannot be predicted from the information available at $t-1$.

The other types of tests of market efficiency are concerned with the performance of managed portfolios. These tests ask whether portfolio managers can use any of the information available at $t-1$ to make correct nonzero assessments of the expected values of the $\tilde{\eta}_{it}$ of different securities. In practical terms, are the portfolio managers able to utilize information available at $t-1$ to choose portfolios that on average have higher returns at t than the combinations of Z and M that have the same level of risk? A positive answer to this question would imply an inefficient market: it is possible to use information available at $t-1$ to predict how the disturbance $\tilde{\eta}_{it}$ for some securities will differ from zero.

The preceding is, of course, similar in tone to most of Chapter 5. To test market efficiency, some model of market equilibrium is required. The two-parameter models of market equilibrium are more sophisticated than those used in Chapter 5, but the approach to testing market efficiency is the same.

C. *Market Efficiency and Company-Specific Information*

As usual, in carrying out the tests of market efficiency discussed above, some concessions must be made to the data. First, to date, the tests of the market's reaction to company-specific information are based on the proposition that m, the equally weighted portfolio of NYSE stocks, is a minimum variance portfolio. Thus, the expected return equation is (36), and (58) is the corresponding version of the two-factor market model. Given that m is a minimum variance portfolio, the switch from M to m is legitimate, and the approach to tests of market efficiency based on m is precisely the same as that based on M.

The second concession made to the data is in the definition of z, the minimum variance zero-β_{pm} portfolio in (58). In the tests to date, R_{zt} is taken to be γ_{1t} of (49), that is, the intercept in the least squares regression of the 20 Fama-MacBeth portfolio returns on the estimates of β_{pm} for these portfolios. Recall that the least squares γ_{1t} is the return on a portfolio that has an estimated value of β_{pm} equal to zero, but the true β_{pm} of this portfolio is not zero. Moreover, even ignoring the loss of information caused by the fact that the regressions are carried out on portfolios rather than individual securities, and even ignoring the fact that we use estimates b_{pm} rather than the true values β_{pm}, the least squares γ_{1t} is only the minimum variance zero-β_{pm} portfolio when the disturbances $\tilde{\eta}_{it}$ in (41) are independent and identically distributed across securities. In short, γ_{1t} from (49) is a proxy for R_{zt} in (58).

The mechanics of the approach to tests of market efficiency based on the two-factor model of (58) are in most respects identical to the mechanics of the approach based on the one-factor model, as described in Chapter 5. In brief, instead of examining the behavior of the average and cumulative average residuals from the one-factor model in months surrounding an event of interest, one examines the behavior of the average and cumulative average η_{it} from

$$R_{it} = R_{zt} + b_{im}(R_{mt} - R_{zt}) + \eta_{it}, \tag{69}$$

which is (58) but with the estimate b_{im} substituted for the true value β_{im} and with the understanding that R_{zt} is the least squares value of γ_{1t} in (49).

Thus, Ball (1972) uses average and cumulative average values of η_{it} in (69) to study the reaction of the market to changes in accounting techniques by firms. He finds that there is no unusual subsequent behavior in the returns on the shares of firms carrying out such changes, in the sense that average and cumulative average values of η_{it} do not depart much from zero in the months following the accounting change. Moreover, in cases where the accounting changes have no real effect on the net returns to the firm, there is no unusual behavior of share returns before or after the accounting change. Ball interprets this as evidence that the market reacts efficiently to any information in a change in accounting techniques, and the market is not misled when such a change has no information content.

Mandelker (1974) uses more or less the same technique to study the behavior of the returns on both acquiring and acquired firms in a merger. He finds that acquired firms experience abnormal returns when a merger is announced (cumulative average values of η_{it} that reach about .15, or 15 percent, by the time the merger takes place). The acquiring firms, however, do not on average experience abnormal returns at any time reasonably close to the release of information about the merger. Mandelker interprets this as evidence that any gains from mergers go to the acquired firms. One possible explanation is that the synergy in a merger is in many cases improved management of the acquired firm. If the acquisitions market for poorly managed firms is perfectly competitive, competition among acquiring firms will cause all the gains from the merger (removal of the poor management of the acquired firm) to be passed on to the shareholders of the acquired firm.

In another study that uses more or less the same techniques, Ibbotson (1974) finds that when a firm goes public for the first time—that is, when a firm makes its first public issue of common stock—the stock is on average underpriced by the underwriters. From the date of issue to the end of the month of issue, the average value of η_{it} in (69) for such securities is about .14. Thus, there is an "abnormal" average return of 14 percent in the month of issue. As a result, during the 1960s, the period that Ibbotson studies, new issues were typically oversubscribed and had to be rationed by underwriters to their customers. Ibbotson admits that he has no good explanation for this phenomenon. More important for the issue of capital market efficiency, he does find that once a newly public security is available in the open market (after it passes from the hands of the underwriter), the market for it seems to be efficient; that is, once they are available in the open market, the average η_{it} for these securities are not significantly different from zero.

Although it probably has no major effect on his results, there is one problem in Ibbotson's work that should be mentioned. When firms first go public, their shares are almost always traded in the over-the-counter market. Even if (58) is a valid risk-return relationship, it refers only to NYSE common stocks, and in principle it should only be the basis of tests of efficiency for NYSE stocks. For Ibbotson's data, this is probably not an important criticism. The 14 percent average initial underpricing that he observes is so large that it would probably be significant under any method of analysis. Moroever, the variability of the returns on securities newly gone public is generally so large that the method of abstracting from the effects of risk has little impact on the results.

Finally, Jaffe (1974) uses (69) to study the returns on insider trading—that is, trading by company managers and directors in the shares of their own firms. He finds that, on average, the η_{it} in (69) are negative subsequent to heavy sales by insiders and positive subsequent to heavy purchases. Strictly

speaking, this is evidence of market inefficiency: insiders typically have information that the market has not utilized in setting prices. In itself, it is not surprising that insiders can beat the market. However, Jaffe also finds that correct nonzero assessments of the expected values of future $\tilde{\eta}_{it}$ can be made on the basis of Securities and Exchange Commission publications of insider activity. This is somewhat more impressive evidence of a market inefficiency. As far as I know, Jaffe's is the only test of market efficiency that finds that the market ignores some obviously publicly available information in setting prices.

PROBLEM V.C

1. In fact, the tests of market efficiency discussed above are based on the η_{it} from

$$R_{it} = \gamma_{1t} + \gamma_{2t} b_{im} + \eta_{it}, \tag{70}$$

where γ_{1t} and γ_{2t} are the (generally Fama-MacBeth) least squares values from (49). Given that R_{zt} in (69) is taken to be γ_{1t} in (49), when will the η_{it} in (69) and (70) be identical?

ANSWER

1. Suppose the 20 Fama-MacBeth portfolios include all the stocks on the NYSE. Then, multiplying through (49) by 1/20 and summing over p leads to

$$\frac{1}{20}\sum_{p=1}^{20} R_{pt} = R_{zt} + \gamma_{2t}\frac{1}{20}\sum_{p=1}^{20} b_{pm} + \frac{1}{20}\sum_{p=1}^{20}\eta_{pt}$$

$$R_{mt} - R_{zt} = \gamma_{2t}.$$

Here we make use of the facts that (a) if the 20 portfolios include all NYSE stocks, the average of the 20 portfolio returns is R_{mt} and the average of the b_{pm} is 1; (b) the sum of the residuals in any least squares regression is zero. We find, then, that the least squares value of γ_{2t} in (49) is $R_{mt} - R_{zt}$, so that the η_{it} in (69) and (70) are identical.

In fact, because of the data requirements imposed on the securities included in the 20 portfolios, the Fama-MacBeth portfolios cover most but not all of the stocks on the NYSE for month t. Fama-MacBeth indicate, however, that the average of their 20 portfolio returns is always quite close to R_{mt}, and the average of their b_{pm} is close to 1. Thus, the relationships between (69) and (70) developed above hold to a close approximation.

D. Portfolio Selection and Performance Evaluation

GENERAL COMMENTS

In an efficient market and with securities priced according to the Black model, portfolio selection is a simple matter. All efficient portfolios are combinations of the market portfolio M and the minimum variance zero-β_{pM} portfolio Z. The investor simply chooses the combination of Z and M that has the desired combination of mean and standard deviation of return.

Some investors and portfolio managers may feel, however, that the market is inefficient. They feel that they have more information or are better able to evaluate existing information than the market. If securities are priced according to the Black model, it is, in principle, easy to test whether in fact particular investors and portfolio managers can beat the market. The returns on minimum variance portfolios are related to their β_{eM} as in (67). There is a corresponding risk-return relationship, (57), for individual securities and for nonminimum variance portfolios, which is the same as (67), except that it is subject to a disturbance $\tilde{\eta}_{it}$. If the market is efficient, (68) holds and there is no way to use information available at $t - 1$ to predict how $\tilde{\eta}_{it}$ will differ from zero. If the market is inefficient, however, and if some investors are shrewd enough to capitalize on the inefficiency, then we should find that the securities and portfolios they choose have measurably higher average returns than the combinations of Z and M that have the same level of β_{pM}. In other words, such investors should be able to use information available at $t - 1$ to assess correct nonzero expected values for the $\tilde{\eta}_{it}$ in (57).

If we take the position that m, the equally weighted portfolio of NYSE stocks, is a minimum variance portfolio, then the statements above apply, but with the usual change from Z to z and M to m, and where we talk in terms of the disturbances from (58) rather than (57). However, there is also a more substantive change. Since (58) refers to NYSE stocks, we should only use it to evaluate security and portfolio selections of NYSE stocks.

THE INVESTMENT PERFORMANCE OF MUTUAL FUNDS

The major application of these ideas, in Jensen (1968; 1969), is based on the Sharpe-Lintner model of market equilibrium. In this model, one substitutes the risk-free rate R_{Ft} for $\tilde{R}_{Zt}$ in (66) and (67), so that (67) becomes

$$\tilde{R}_{et} = (1 - \beta_{eM})R_{Ft} + \beta_{eM}\tilde{R}_{Mt} = R_{Ft} + (\tilde{R}_{Mt} - R_{Ft})\beta_{eM}. \tag{71}$$

Moreover, from Problem V.A.*3* we know that if we are in the Sharpe-Lintner world, the intercept α_i in the one-factor market model is

$$\alpha_i = R_{Ft}(1 - \beta_{iM}), \tag{72}$$

so that (4) can be rewritten as

$$\tilde{R}_{it} = R_{Ft}(1 - \beta_{iM}) + \beta_{iM}\tilde{R}_{Mt} + \tilde{\epsilon}_{it}. \tag{73}$$

If the market is efficient,

$$E(\tilde{\epsilon}_{it}|\phi_{t-1}) = 0; \tag{74}$$

that is, there is no way to use any information available at $t - 1$ as the basis of a correct nonzero assessment of the expected value of $\tilde{\epsilon}_{it}$.

Jensen tries to use the performance of mutual fund portfolios to find contradictions of (74). He tests whether mutual funds, or particular funds, seem to have access to information that allows them to earn higher average returns (positive average $\tilde{\epsilon}_{pt}$) than they would get simply by buying the combinations of F and M that have the same level of β_{pM} as their chosen portfolios. As usual, some concessions must be made to the data. He must use estimates of β_{pM} for mutual fund portfolios, and he must choose a proxy for the market portfolio M. He chooses the Standard and Poor's value-weighted index of NYSE stocks.

Jensen uses the risk-return framework described above to evaluate the performance of 115 mutual funds over the ten-year period 1955–1964. The general question to be answered is whether mutual fund managers have any special insights or information that allows them to earn average returns above the norm provided by the Sharpe-Lintner model. But Jensen attacks the question on several levels. First, can the funds generally do well enough to compensate investors for loading charges, management fees, and other costs that might be avoided by simply choosing the combination of the risk-free asset F and the market portfolio M with risk level comparable to that of the fund's actual portfolio? The answer seems to be an emphatic no. As far as net returns to investors are concerned, in 89 out of 115 cases the fund's average return for the ten-year period was below what would have been obtained from the combination of F and M with the same level of b_{pM}, and on average the investor's wealth after ten years of holding mutual funds is about 15 percent less than if he held the appropriate combinations of F and M.

The loading charge that an investor pays on buying into a fund is usually a salesman's commission that the fund itself never gets to invest. One might ask whether, if one ignores loading charges—that is, if one assumes no such charges are paid by the investor—fund managers can earn returns sufficiently above the norm to cover other expenses that are presumably more directly related to the management of the fund portfolios. Again, the answer seems to be no. When loading charges are ignored in computing returns, the average returns for 72 out of 115 funds are still below what would have been obtained from the combinations of F and M with the same level of b_{pM}, and on average the investor's wealth after ten years is about 9 percent less than if he held the appropriate combinations of F and M.

Finally, as a somewhat stronger test of market efficiency, one would like to know whether, ignoring all expenses, fund managements show any ability to pick securities that outperform the norm. Unfortunately, this question cannot be answered with precision for individual funds, since data on brokerage commissions are not published regularly. But Jensen suggests that available evidence indicates that the answer to the question is negative. Specifically, adding back all other published expenses of funds to their returns, the average returns for 58 out of 115 funds were below those for the corresponding combinations of F and M, and average wealth was about 2.5 percent less. Part of this result is due to the absence of a correction for brokerage commissions. Estimating these commissions from average portfolio turnover rates for all funds for 1953-1958 and adding them to returns for all funds just about wipes out the deficit that the funds have vis-à-vis the relevant naïve combinations of F and M, which is consistent with the proposition that the fund managers do not have access to special information.

Although mutual fund managers generally do not seem to have access to information not already fully reflected in prices, perhaps there are individual funds that consistently do better than the norm, thus providing at least some evidence against market efficiency. If there are such funds, however, they escape Jensen's search. For example, returns above the norm for individual funds in one subperiod do not seem to be associated with performance above the norm by those same funds in other subperiods.

VI. Conclusion

There is much more that could be said about how the two-parameter models of market equilibrium can and have been used to test market efficiency (cf. Ellert 1975). We could also go into much more detail on how the models can be used to evaluate the performance of managed portfolios (cf. Fama 1972). I hope, however, that the book has given the reader the base of sophistication that will enable him to continue investigation of these topics in the original literature.

References

Alexander, Sidney S. 1961. "Price Movements in Speculative Markets: Trends or Random Walks." *Industrial Management Review* 2 (May): 7-26; Reprinted in Cootner (ed.), 199-218.

——. 1964. "Price Movements in Speculative Markets: Trends or Random Walks. No. 2." In Cootner (ed.), 338-372.

Anderson, T. W. 1958. *An Introduction to Multivariate Statistical Analysis*. New York: John Wiley.

Arrow, Kenneth J. 1964. "The Role of Securities in the Optimal Allocation of Risk-Bearing." *Review of Economic Studies* 31 (April): 91-96.

Bachelier, Louis. 1900. *Théorie de la speculation*. Paris: Gauthier-Villars. Reprinted in Cootner (ed.), 17-78.

Ball, Ray. 1972. "Changes in Accounting Techniques and Stock Prices." Ph.D. dissertation, University of Chicago.

Ball, Ray, and Brown, Phillip. 1968. "An Empirical Evaluation of Accounting Income Numbers." *Journal of Accounting Research* 6 (Autumn): 159-178.

Black, Fischer. 1972. "Capital Market Equilibrium with Restricted Borrowing." *Journal of Business* 45 (July): 444-454.

Black, Fischer, and Scholes, Myron. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May-June): 637-654.

——. 1974. "The Effects of Dividend Yield and Dividend Policy on Common Stock Prices and Returns." *Journal of Financial Economics* 1 (May): 1-22.

Black, Fischer; Jensen, Michael; and Scholes, Myron. 1972. "The Capital Asset Pricing Model: Some Empirical Tests." In Jensen (ed.), 79-121.

Blattberg, Robert, and Gonedes, Nicholas. 1974. "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices." *Journal of Business* 47 (April): 244-280.

Blattberg, Robert, and Sargent, Thomas. 1971. "Regression with Non-Gaussian Disturbances: Some Sampling Results." *Econometrica* 39 (May): 501-510.

Blume, Marshall. 1968. "The Assessment of Portfolio Performance." Ph.D. dissertation, University of Chicago, Graduate School of Business.

——. 1970. "Portfolio Theory: A Step Toward Its Practical Application." *Journal of Business* 43 (April): 152-173. This is an informative summary of Blume (1968).

——. 1971. "On the Assessment of Risk." *Journal of Finance* 26 (March): 1-10.

Brennan, Michael J. 1971. "Capital Market Equilibrium with Divergent Borrowing and Lending Rates." *Journal of Financial and Quantitative Analysis* 6 (December): 1197-1205.

Cass, David, and Stiglitz, Joseph. 1970. "The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds." *Journal of Economic Theory* 2 (June): 122-160.

Cootner, Paul, ed. 1964. *The Random Character of Stock Market Prices.* Cambridge: MIT.

Cramer, Harald. 1946. *Mathematical Methods of Statistics.* Princeton: Princeton University Press.

Douglas, George W. 1969. "Risk in the Equity Markets: An Empirical Appraisal of Market Efficiency." *Yale Economic Essays* 9 (Spring): 3-45.

Ellert, James. 1975. "Anti-Trust Enforcement and Behavior of Stock Prices." Ph.D. dissertation, University of Chicago.

Fama, Eugene F. 1965. "The Behavior of Stock Market Prices." *Journal of Business* 38 (January): 34-105.

——. 1968. "Risk, Return, and Equilibrium: Some Clarifying Comments." *Journal of Finance* 23 (March): 29-40.

——. 1970*a*. "Multiperiod Consumption-Investment Decisions." *American Economic Review* 60 (March): 163-174.

——. 1970*b*. "Efficient Capital Markets: A Review of Theory and Empirical Work." *Journal of Finance* 25 (May): 383-417.

——. 1971. "Risk, Return, and Equilibrium." *Journal of Political Economy* 79 (January-February): 30-55.

——. 1972. "Components of Investment Performance." *Journal of Finance* 27 (June): 551-567.

——. 1973. "A Note on the Market Model and the Two-Parameter Model." *Journal of Finance* 28 (December): 1181-1185.

——. 1975. "Short-Term Interest Rates as Predictors of Inflation." *American Economic Review* 65 (June): 269-282.

Fama, Eugene F., and Blume, Marshall. 1966. "Filter Rules and Stock Market Trading Profits." *Journal of Business* 39 (special supp., January): 226-241.

Fama, Eugene F., and MacBeth, James D. 1973. "Risk, Return, and Equilibrium: Empirical Tests." *Journal of Political Economy* 71 (May-June): 607-636.

——. 1974*a*. "Tests of the Multiperiod Two-Parameter Model." *Journal of Financial Economics* 1 (May): 43-66.

——. 1974*b*. "Long-Term Growth in a Short-Term Market." *Journal of Finance* 29 (June): 857-885.

Fama, Eugene F., and Miller, Merton. 1972. *The Theory of Finance.* New York: Holt, Rinehart and Winston.

Fama, Eugene F., and Roll, Richard. 1968. "Some Properties of Symmetric Stable Distributions." *Journal of the American Statistical Association* 63 (September): 817-836.

——. 1971. "Parameter Estimates for Symmetric Stable Distributions." *Journal of the American Statistical Association* 66 (June): 331-338.

Fama, Eugene F.; Fisher, Lawrence; Jensen, Michael; and Roll, Richard. 1969. "The Adjustment of Stock Prices to New Information." *International Economic Review* 10 (February): 1-21.

Fisher, Irving. 1930. *The Theory of Interest.* Reprint. New York: Augustus M. Kelley, 1965.

Fisher, Lawrence. 1966. "Some New Stock Market Indices." *Journal of Business* 49 (supp., January): 191-225.

——. 1970. "The Estimation of Systematic Risk: Some New Findings." *Proceedings of the Seminar on the Analysis of Security Prices*, University of Chicago, May.

Friend, Irwin, and Blume, Marshall. 1970. "Measurement of Portfolio Performance under Uncertainty." *American Economic Review* 60 (September): 561-575.

Gonedes, Nicholas. 1973. "Evidence on the Information Content of Accounting Numbers: Accounting-Based and Market-Based Estimates of Systematic Risk." *Journal of Financial and Quantitative Analysis* 8 (June): 407-444.

Ibbotson, Roger. 1974. "Price Performance of Common Stock New Issues." Ph.D. dissertation, University of Chicago.

Jacob, Nancy. 1971. "The Measurement of Systematic Risk for Securities and Portfolios: Some Empirical Results." *Journal of Financial and Quantitative Analysis* 6 (March): 815-834.

Jaffe, Jeffrey F. 1974. "Special Information and Insider Trading." *Journal of Business* 47 (July): 410-428.

Jensen, Michael. 1968. "The Performance of Mutual Funds in the Period 1945-64." *Journal of Finance* 23 (May): 389-416.

——. 1969. "Risk, the Pricing of Capital Assets, and the Evaluation of Investment Portfolios." *Journal of Business* 42 (April): 167-247.

——. 1972. "Capital Markets: Theory and Evidence." *Bell Journal of Economics and Management Science* 3 (Autumn): 357-398.

Jensen, Michael, ed. 1972. *Studies in the Theory of Capital Markets.* New York: Praeger, 1972.

Johnston, John. 1972. *Econometric Methods.* 2nd ed. New York: McGraw-Hill.

Kendall, M. G., and Stuart, A. S. 1968. *The Advanced Theory of Statistics, vol. 3, Design & Analysis & Time Series.* 2nd ed. New York: Hafner.

King, Benjamin F. 1966. "Market and Industry Factors in Stock Price Behavior." *Journal of Business* 39 (supp., January): 139–190.

Lintner, John. 1956. "Distribution of Incomes of Corporations among Dividends, Retained Earnings, and Taxes." *American Economic Review* 46 (May): 97–113.

——. 1965*a*. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *Review of Economics and Statistics* 47 (February): 13–37.

——. 1965*b*. "Security Prices, Risk, and Maximal Gains from Diversification." Journal of Finance 20 (December): 587–616.

——. 1969. "The Aggregation of Investors' Diverse Judgments and Preferences in Purely Competitive Securities Markets." *Journal of Financial and Quantitative Analysis* 4 (December): 347–400.

Lorie, James H., and Niederhoffer, Victor. 1968. "Predictive and Statistical Properties of Insider Trading." *Journal of Law and Economics* 11 (April): 35–51.

Mandelbrot, Benoit. 1963. "The Variation of Certain Speculative Prices." *Journal of Business* 36 (October): 394–419.

——. 1966. "Forecasts of Future Prices, Unbiased Markets, and Martingale Models." *Journal of Business* 39 (special supp., January): 242–255.

Mandelker, Gershon. 1974. "Risk and Return: The Case of Merging Firms." *Journal of Financial Economics* 1 (December): 303–335.

Markowitz, Harry. 1952. "Portfolio Selection." *Journal of Finance* 7 (March): 77–91.

——. 1959. *Portfolio Selection: Efficient Diversification of Investments.* New York: John Wiley.

Mayers, David. 1972. "Nonmarketable Assets and Capital Market Equilibrium under Uncertainty." In Jensen (ed.), 223–248.

Merton, Robert C. 1972. "An Analytic Derivation of the Efficient Portfolio Frontier." *Journal of Financial and Quantitative Analysis* 7 (September): 1151–1172.

——. 1973. "An Intertemporal Capital Asset Pricing Model." *Econometrica* 41 (September): 867–887.

Miller, Merton, and Scholes, Myron. 1972. "Rates of Return in Relation to Risk: A Re-examination of Some Recent Findings." In Jensen (ed.), 47–78.

Mossin, Jan. 1966. "Equilibrium in a Capital Asset Market." *Econometrica* 34 (October): 768–783.

Niederhoffer, Victor, and Osborne, M. F. M. 1966. "Market Making and Reversal on the Stock Exchange." *Journal of the American Statistical Association* 61 (December): 897–916.

Officer, Robert R. 1971. "A Time Series Examination of the Market Factor of the New York Stock Exchange." Ph.D. dissertation, University of Chicago.

Osborne, M. F. M. 1959. "Brownian Motion in the Stock Market." *Operations Research* 7 (March–April): 145–173. Reprinted in Cootner (ed.), 100–128.

Roberts, Harry V. 1959. "Stock Market 'Patterns' and Financial Analysis:

Methodological Suggestions." *Journal of Finance* 14 (March 1959): 1–10.

Roll, Richard. 1970. *The Behavior of Interest Rates.* New York: Basic Books.

——. 1972. "Interest Rates on Monetary Assets and Commodity Price Index Changes." *Journal of Finance* 27 (May): 251–277.

Samuelson, Paul A. 1965. "Proof That Properly Anticipated Prices Fluctuate Randomly." *Industrial Management Review* 6 (Spring): 41–49.

Sargent, Thomas J. 1973. "Interest Rates and Prices in the Long Run: A Study of the Gibson Paradox." *Journal of Money, Credit, and Banking* 5 (February): 385–449.

Scholes, Myron. 1972. "The Market for Securities: Substitution versus Price Pressure and the Effects of Information on Share Prices." *Journal of Business* 45 (April): 179–211.

Sharpe, William F. 1964. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance* 19 (September). 425–442.

Theil, Henri. 1971. *Principles of Econometrics.* New York: John Wiley.

Tobin, James. 1958. "Liquidity Preference as Behavior towards Risk." *Review of Economic Studies* 25 (February): 65–85.

U. S. Bureau of Labor Statistics. 1971. *Handbook of Methods for Surveys and Studies.* BLS Bulletin no. 1458.