Practice Problem Topic 3: Inferences for Several Population Means

Continuing on the Theme of Different Factors/Variables Affecting Blood Pressure

Doe exercise level and sodium intake level (explanatory variables) have an effect on blood pressure (response variable)?

Comparing Several Population Means Using an Independent Sample Design
A sample of 48 people was randomly selected from a population that had high blood pressure (141-170 mm Hg), got very little exercise, and were 50-60 years of age. Then, from this sample, researchers randomly assigned 8 participants to six groups based on 3 levels of exercise and 2 levels of sodium (Na) intake.

The 6 groups and their parameters are defined as follows:

μ_{NT} = No additional exercise, typical level of Na intake (i.e., control group)

μ_{MT} = Moderate exercise (30 min/day of fast walking), typical level of Na intake

μ_{VT} = Vigorous exercise (30 min/day of running), typical level of Na intake

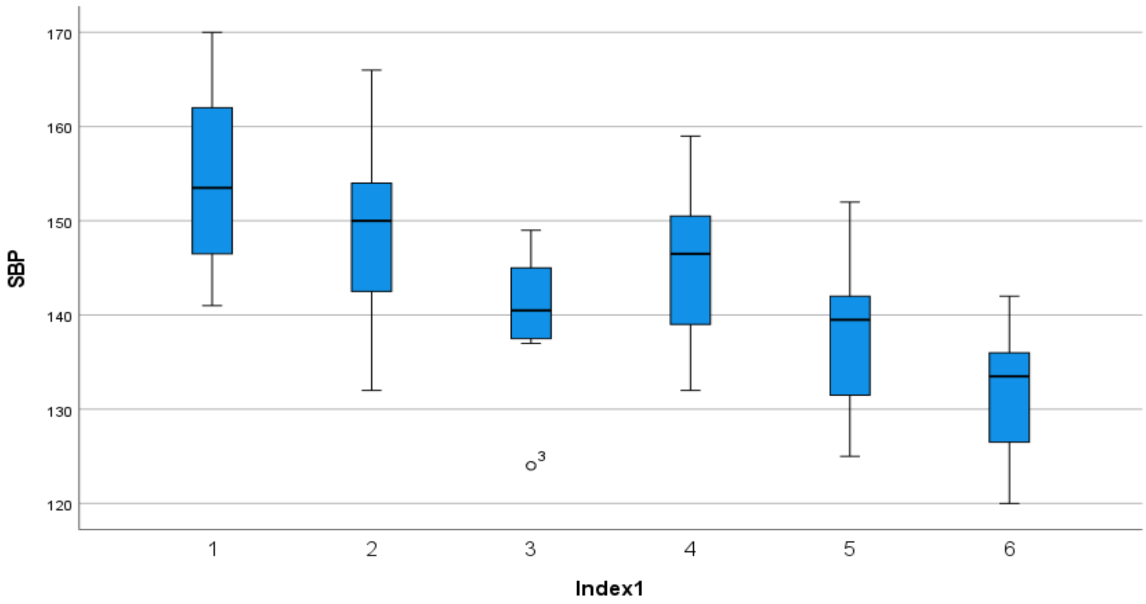
μ_{NR} = No additional exercise, reduced level of Na intake

μ_{MR} = Moderate exercise (30 min/day of fast walking), reduced level of Na intake

μ_{VR} = Vigorous exercise (30 min/day of running), reduced level of Na intake

After a period of 2 months, the systolic blood pressure (SBP) of participants in all 6 groups was recorded, obtaining data as shown below. Note: this is a post-test design. Use the SPSS output below to answer parts (a) to (g). Assume that all the assumptions are met for the required analysis.

No Ex-Typ Na	**ModEx-Typ Na**	**Vig Ex-Typ Na**	**No Ex-Red Na**	**ModEx-RedNa**	**Vig Ex-Red Na**
145	154	124	132	142	136
156	166	138	152	137	131
160	132	142	137	125	122
170	138	141	141	126	120
141	154	149	159	142	136
164	147	148	149	138	132
148	153	140	149	141	135
151	147	137	144	152	142



Tests of Normality							
		Kolmogorov-Smirnov[a]			Shapiro-Wilk		
	Index1	Statistic	df	Sig.	Statistic	df	Sig.
SBP	1	.133	8	.200*	.975	8	.937
	2	.188	8	.200*	.957	8	.782
	3	.230	8	.200*	.895	8	.261
	4	.162	8	.200*	.985	8	.985
	5	.211	8	.200*	.914	8	.385
	6	.210	8	.200*	.915	8	.393

*. This is a lower bound of the true significance.

a. Lilliefors Significance Correction

Note: If the P-value of the Shapiro-Wilk Test is **greater than 0.05**, the distribution is not significantly different from a normal distribution, so the data are considered normal. If the P-value is below 0.05, the data significantly deviate from a normal distribution.

Test of Homogeneity of Variance					
		Levene Statistic	df1	df2	Sig.
SBP	Based on Mean	.420	5	42	.832
	Based on Median	.404	5	42	.843
	Based on Median and with adjusted df	.404	5	39.902	.843
	Based on trimmed mean	.426	5	42	.828

Descriptives						
SBP						
	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean	
					Lower Bound	Upper Bound
No Ex-Typ Na	8	154.38	9.927	3.510	146.08	162.67
Mod Ex-Typ Na	8	148.88	10.508	3.715	140.09	157.66
Vig Ex-Typ Na	8	139.88	7.736	2.735	133.41	146.34
No Ex-Red Na	8	145.38	8.667	3.064	138.13	152.62
Mod Ex-RedNa	8	137.88	8.871	3.136	130.46	145.29
Vig Ex-Red Na	8	131.75	7.421	2.624	125.55	137.95
Total	48	143.02	11.300	1.631	139.74	146.30

ANOVA					
SBP					
	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	2657.104	5	531.421	6.675	0.000117
Within Groups	3343.875	42	79.616		
Total	6000.979	47			

(a) Although the start of the question says, "assume that all the assumptions are met for the required analysis" and SPSS ANOVA output is provided, explain in detail why single-factor ANOVA would be the correct test to use for comparing the means and verify that all the assumptions are met.

- The explanatory variable is categorical, that is, combinations of levels of exercise and Na. Although there appears to be two categorical variables, these are applied in a combined way to 6 groups. (Note: potentially, could these variables could be measured on a quantitative scale, but in this study, they are levels or categories).
- The response variable is a continuous, quantitative variable, that is, SBP.
- **Purpose** of the study: To test for a difference between several population means (mean SBP in 6 groups).

- **Assumptions:**
  1. Random selection from population and random allocation to groups
  2. Independent samples are implied in the research design.
  3. All 6 distributions are normally distributed as indicated by the Shapiro Wilk test, which results in all P-values being greater than 0.05 (the smallest one is P = 0.261), so the distributions are not significantly different from a normal distribution.
  4. Levene's test gives P = 0.832 (based on mean), which is greater than 0.05, indicating insufficient evidence of a difference in variances (or standard deviations).

(b) What is the best estimate for the common or pooled standard deviation of the 6 populations?

The best estimate of the common OR pooled standard deviation of the 6 populations is:

$$\hat{\sigma} = s_p = \sqrt{MS_{Within}} \text{ or } \sqrt{MS_{Error}} = \sqrt{\frac{SS_{Error}}{n-k}} = \sqrt{\frac{3343.875}{48-6}} = \sqrt{79.616071} = 8.922784$$

(c) At the 5% significance level, <u>carry out the most appropriate test</u> to determine if there is any significant difference in mean SBP between the 6 groups. **<u>SHOW ALL STEPS</u>**.

(Step 1 of checking the purpose and assumptions was already done in part (a).)

H_0: $\boldsymbol{\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6}$ (One-mean model)
There is no difference in mean SBP between the 6 groups.)
H_a: $\boldsymbol{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6}$ (Six-mean model)
Not all 6 means are equal. (There is a difference in the mean SBP between the 6 groups.)

k = number of populations being compare = 6
n = total sample size = 8 x 6 = 48

$$F = \frac{SS_{Treatment}/(k-1)}{SS_{Error}/(n-k)} = \frac{MS_{Treatment}}{MS_{Error}}$$

$$= \frac{2657.104/(6-1)}{3343.875/(48-6)} = \frac{531.4208}{79.616071} = 6.675$$

For df = (k – 1, n – k) = (5, 42) ≈ (5, 40)
P < 0.001 (Exact P-value = 0.000117) There is extremely strong evidence against Ho.
P-value < α (0.05), so reject Ho.

Conclusion:
At the 5% significance level, the data provide sufficient evidence to conclude that there is a difference in the mean SBP between the 6 groups (at least two means are different).

(d) Develop a linear combination (contrast) to test the hypothesis that there is a difference in mean SBP between the combined groups for Moderate exercise-Typical Na and Vigorous exercise-Typical Na in comparison with the combined groups for Moderate exercise-Reduced Na and Vigorous exercise-Reduced Na. Then, test this contrast at the 5% significance level. **<u>SHOW ALL STEPS</u>** of the hypothesis test.

$$\gamma = \frac{(\mu_{MT} + \mu_{VT})}{?} - \frac{(\mu_{MR} + \mu_{VR})}{?}$$

$$\gamma = \frac{(\mu_{MT} + \mu_{VT})}{2} - \frac{(\mu_{MR} + \mu_{VR})}{2}$$

Parameter: $\gamma = \frac{1}{2}\mu_{MT} + \frac{1}{2}\mu_{VT} - \frac{1}{2}\mu_{MR} - \frac{1}{2}\mu_{VR}$

$H_0 : \gamma = 0 \qquad\qquad H_a : \gamma \neq 0$

Estimate: $\hat{\gamma} = \frac{1}{2}\bar{y}_{MT} + \frac{1}{2}\bar{y}_{VT} - \frac{1}{2}\bar{y}_{MR} - \frac{1}{2}\bar{y}_{VR}$

$$= \frac{1}{2}(148.88) + \frac{1}{2}(139.88) - \frac{1}{2}(137.88) - \frac{1}{2}(131.75) = 9.565$$

$$SE(\hat{\gamma}) = s_p\sqrt{\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + ... + \frac{C_k^2}{n_k}}$$

$s_p = \sqrt{MS_{Error}} = 8.922784$ (from part (a))

$$SE(\hat{\gamma}) = 8.922784\sqrt{\frac{(1/2)^2}{8} + \frac{(1/2)^2}{8} + \frac{(-1/2)^2}{8} + \frac{(-1/2)^2}{8}}$$
$$= (8.922784)(0.353553) = 3.154681$$

The t-statistic is: $t = \frac{\hat{\gamma} - 0}{SE(\hat{\gamma})} = \frac{9.565 - 0}{3.154681} = 3.032$

$df = n - k = 48 - 6 = 42 \approx 40$

(The *df* is rounded down to the nearest one in the table to give a more conservative *P*-value.)

P-value: (0.001 < P < 0.0025) x 2 = 0.002 < P < 0.005

<u>OR</u> P-value: $2P(t_{42} > 3.032) \approx 2P(t_{40} > 3.032) \in 2(0.001, 0.0025) = (0.002, 0.005).$

There is very strong evidence against H_0. Since P < α (0.05), reject H_0.

At the 1% significance level, the data do provide sufficient evidence to conclude that there is a difference in mean SBP between the combined groups for Moderate exercise-Typical Na and Vigorous exercise-Typical Na in comparison with the combined groups for Moderate exercise-Reduced Na and Vigorous exercise-Reduced Na.

(e) Develop a linear combination (contrast) to test the hypothesis that there is a difference in mean SBP between the No exercise-Typical Na group and the Vigorous exercise-Reduced Na group. Then, test this contrast at the 5% significance level. **<u>SHOW ALL STEPS</u>** of the hypothesis test.

$$\gamma = \frac{\mu_{NT}}{?} - \frac{\mu_{VR}}{?}$$

$$\gamma = \frac{\mu_{NT}}{1} - \frac{\mu_{VR}}{1}$$

Parameter: $\gamma = \mu_{NT} - \mu_{VR}$

$H_0 : \gamma = 0 \qquad\qquad H_a : \gamma \neq 0$

Estimate: $\hat{\gamma} = \bar{y}_{NT} - \bar{y}_{VR}$
$= 154.38 - 131.75 = 22.63$

$$SE(\hat{\gamma}) = s_p \sqrt{\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \ldots + \frac{C_k^2}{n_k}}$$

$s_p = \sqrt{MS_{Error}} = 8.922784$ (from part (a))

$$SE(\hat{\gamma}) = 8.922784 \sqrt{\frac{(1)^2}{8} + \frac{(-1)^2}{8}}$$
$$= (8.922784)(0.5) = 4.461392$$

The t-statistic is: $t = \dfrac{\hat{\gamma} - 0}{SE(\hat{\gamma})} = \dfrac{22.63 - 0}{4.461392} = 5.072$

$df = n - k = 48 - 6 = 42 \approx 40$
(The *df* is rounded down to the nearest one in the table to give a more conservative *P*-value.)

P-value: (P < 0.0005) x 2 = P < 0.001
<u>OR</u> P-value: $2P(t_{42} > 5.072) \approx 2P(t_{40} > 5.072) \in 2(0.000, 0.0005) = (0.000, 0.001).$
There is extremely strong evidence against H_0. Since P < α (0.05), reject H_0.

At the 5% significance level, the data do provide sufficient evidence to conclude that there is a difference in mean SBP between the No exercise-Typical Na group and the Vigorous exercise-Reduced Na group. Then, test this contrast at the 5% significance level.

(f) Develop a linear combination (contrast) to test the hypothesis that there is a difference in mean SBP between the Vigorous exercise-Reduced Na group and the combined groups for No exercise-Typical Na, Moderate exercise-Typical Na and Vigorous exercise-Typical Na. You do <u>not</u> need to perform other steps. Just define the contrast (parameter).

$$\gamma = \frac{\mu_{VR}}{?} - \frac{(\mu_{NT} + \mu_{MT} + \mu_{VT})}{?}$$

$$\gamma = \frac{\mu_{VR}}{1} - \frac{(\mu_{NT} + \mu_{MT} + \mu_{VT})}{3}$$

$$\gamma = \mu_{VR} - \frac{1}{3}\mu_{NT} - \frac{1}{3}\mu_{MT} - \frac{1}{3}\mu_{VT}$$

(g) Examine the SPSS output below and match it up with the linear combinations obtained in parts (d)-(f).

Contrast Coefficients

Contrast	Index1: No Ex-Typ Na	Mod Ex-Typ Na	Vig Ex-Typ Na	No Ex-Red Na	Mod Ex-RedNa	Vig Ex-Red Na
1	1	0	0	0	0	-1
2	0	.5	.5	0	-.5	-.5
3	1	1	1	0	0	-3
4	3	0	0	-1	-1	-1

Contrast Tests

		Contrast	Value of Contrast	Std. Error	t	df	Sig. (2-tailed)	95% Confidence Interval Lower	Upper
SBP	Assumes equal variances	1	22.63	4.461	5.071	42	<.001	13.62	31.63
		2	9.56	3.155	3.031	42	.004	3.20	15.93
		3	47.88	10.928	4.381	42	<.001	25.82	69.93
		4	48.13	10.928	4.404	42	<.001	26.07	70.18
	Does not assume equal variances	1	22.63	4.382	5.163	12.962	<.001	13.16	32.09
		2	9.56	3.082	3.102	25.882	.005	3.23	15.90
		3	47.88	9.775	4.898	15.087	<.001	27.05	68.70
		4	48.13	11.704	4.112	10.487	.002	22.21	74.04

Part (d) is Contrast #2

Part (e) is Contrast #1

Part (f) is Contrast #3

Table showing Multiple Comparisons based on Tukey's Method

Multiple Comparisons							
Dependent Variable: SBP							
	(I) Index1	(J) Index1	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Tukey HSD	No Ex-Typ Na	Mod Ex-Typ Na	5.500	4.461	.818	-7.82	18.82
		Vig Ex-Typ Na	14.500*	4.461	.026	1.18	27.82
		No Ex-Red Na	9.000	4.461	.350	-4.32	22.32
		Mod Ex-RedNa	16.500*	4.461	.008	3.18	29.82
		Vig Ex-Red Na	22.625*	4.461	<.001	9.31	35.94
	Mod Ex-Typ Na	No Ex-Typ Na	-5.500	4.461	.818	-18.82	7.82
		Vig Ex-Typ Na	9.000	4.461	.350	-4.32	22.32
		No Ex-Red Na	3.500	4.461	.969	-9.82	16.82
		Mod Ex-RedNa	11.000	4.461	.158	-2.32	24.32
		Vig Ex-Red Na	17.125*	4.461	.005	3.81	30.44
	Vig Ex-Typ Na	No Ex-Typ Na	-14.500*	4.461	.026	-27.82	-1.18
		Mod Ex-Typ Na	-9.000	4.461	.350	-22.32	4.32
		No Ex-Red Na	-5.500	4.461	.818	-18.82	7.82
		Mod Ex-RedNa	2.000	4.461	.998	-11.32	15.32
		Vig Ex-Red Na	8.125	4.461	.464	-5.19	21.44
	No Ex-Red Na	No Ex-Typ Na	-9.000	4.461	.350	-22.32	4.32
		Mod Ex-Typ Na	-3.500	4.461	.969	-16.82	9.82
		Vig Ex-Typ Na	5.500	4.461	.818	-7.82	18.82
		Mod Ex-RedNa	7.500	4.461	.551	-5.82	20.82
		Vig Ex-Red Na	13.625*	4.461	.042	.31	26.94
	Mod Ex-RedNa	No Ex-Typ Na	-16.500*	4.461	.008	-29.82	-3.18
		Mod Ex-Typ Na	-11.000	4.461	.158	-24.32	2.32
		Vig Ex-Typ Na	-2.000	4.461	.998	-15.32	11.32
		No Ex-Red Na	-7.500	4.461	.551	-20.82	5.82
		Vig Ex-Red Na	6.125	4.461	.743	-7.19	19.44
	Vig Ex-Red Na	No Ex-Typ Na	-22.625*	4.461	<.001	-35.94	-9.31
		Mod Ex-Typ Na	-17.125*	4.461	.005	-30.44	-3.81
		Vig Ex-Typ Na	-8.125	4.461	.464	-21.44	5.19
		No Ex-Red Na	-13.625*	4.461	.042	-26.94	-.31
		Mod Ex-RedNa	-6.125	4.461	.743	-19.44	7.19

(h) Based on the SPSS output for Tukey's multiple comparisons in the table above, performed at the 95% confidence level, firstly, construct a means comparisons diagram and, secondly, summarize the results in words.

Means Comparisons Diagram

VigEx-RedNa	ModEx-RedNa	VigEx-TypNa	NoEx-RedNa	ModEx-TypNa	NoEx-TypNa
131.75	137.88	139.88	145.38	148.88	153.38

Conclusion in words: It is estimated with 95% confidence that there is insufficient evidence of a difference in mean SBP between VigEx-RedNa, ModEx-RedNa, and VigEx-TypNa, between ModEx-RedNa, VigEx-TypNa, NoEx-RedNa, and ModEx-TypNa, and between NoEx-RedNa, ModEx-TypNa, and NoEx-TypNa. All other pairs of means can be declared different.

(i) At the 97% confidence level, perform the Bonferroni method of multiple comparisons to determine which pairs of groups have different mean SBP. **SHOW the STEPS up to and including the matrix**. Note: You only have to calculate the margin of error once since sample sizes are equal for all groups.

Number of multiple comparisons (m) is: $m = \frac{k(k-1)}{2} = \frac{6(6-1)}{2} = 15$

Individual comparison-wise error rate is: $\alpha_I = \frac{\alpha_F}{m} = \frac{0.03}{15} = 0.002$

Critical value of ***t*** at ***df = n – k = 46 – 6 = 42*** for α_I /2 = 0.001 is:

$$t_{n-k,\alpha/2} \text{ OR } t^*_{n-I,\alpha/2} = t^*_{42,0.001} \approx t_{40,0.001} = 3.307$$

Since $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = 8$ **, the margin of error for all comparisons is:**

$$ME_{ij} = t_{n-k,\alpha_I/2} \times \sqrt{MSE}\sqrt{\frac{1}{n_i} + \frac{1}{n_j}}$$

$$MSE = \frac{SS_{Error}}{n-k} = \frac{3343.875}{48-6} = 79.616071$$

$$ME = 3.307 \times \sqrt{79.616071}\sqrt{\frac{1}{8} + \frac{1}{8}} = 3.307 \times 8.922784 \times 0.5 = 14.754$$

$\mu_i - \mu_j \neq 0$ **, if** $|\bar{y}_i - \bar{y}_j| \geq ME_{ij}$ **Matrix**

	NEx-TypNa	MEx-TypNa	VEx-TypNa	NEx-RedNa	MEx-RedNa	VEx-RedNa
NEx-TypNa						
MEx-TypNa	5.50 < 14.75					
VEx-TypNa	14.50 < 14.75	9.00 < 14.75				
NEx-RedNa	9.00 < 14.75	3.50 < 14.75	5.50 < 14.75			
MEx-ReNa	16.50 > 14.75*	11.00 < 14.75	2.00 < 14.75	7.50 < 14.75		
VEx-RedNa	22.63 > 14.75*	17.13 > 14.75*	8.13 < 14.75	13.63 < 14.75	6.13 < 14.75	

* Indicates pairwise comparisons that can be declared different

(j) At the 5% significance level, perform the most appropriate test, **SHOWING ALL STEPS**, to determine whether there is a difference in mean SBP between the Typical Na groups (No Ex-Typ Na, Mod Ex-Typ Na, and Vig Ex-Typ Na) and the Reduced Na groups (No Ex-Red Na, Mod Ex-Red Na, and Vig Ex-Red Na), after accounting for the effect of exercise (No, Moderate, and vigorous). For this test, make use of the parameters defined at the start this Question. You may consider using any of the SPSS output shown in Tables 1, 2, or 3 below. [Table 1, from page 3, is repeated here below.]

Table 1: The overall ANOVA table for comparison of mean SBP between the six groups.

ANOVA					
SBP					
	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	2657.104	5	531.421	6.675	0.000117
Within Groups	3343.875	42	79.616		
Total	6000.979	47			

Table 2: ANOVA table for comparison of mean SBP between the Typical Na and Reduced Na groups (ignoring exercise level, i.e., no, moderate, and vigorous).

ANOVA					
SBP					
	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	1054.688	1	1054.688	9.80848	0.003018
Within Groups	4946.292	46	107.5281		
Total	6000.979	47			

Table 3: ANOVA table for comparison of mean SBP between different exercise groups (No, Moderate, and Vigorous) (ignoring Na level, i.e., Typical versus Reduced).

ANOVA					
SBP					
	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	1585.042	2	792.5208	8.076074	0.001007
Within Groups	4415.938	45	98.13194		
Total	6000.979	47			

Solution:
If there is no difference in mean SBP between Typical Na and Reduced Na in the No Exercise group, then: $\mu_{NT} = \mu_{NR}$
If there is no difference in mean SBP between Typical Na and Reduced Na in the Moderate Exercise group, then: $\mu_{MT} = \mu_{MR}$
If there is no difference in mean SBP between Typical Na and Reduced Na in the Vigorous Exercise group, then: $\mu_{VT} = \mu_{VR}$

$H_0 : \mu_{NT} = \mu_{NR}$ and $\mu_{MT} = \mu_{MR}$ and $\mu_{VT} = \mu_{VR}$
[Reduced model: Three-mean model]

$H_a : \mu_{NT}, \mu_{NR}, \mu_{MT}, \mu_{MR}, \mu_{VT}, \mu_{VR}$
[Full model: Six-mean model]

Using the ANOVA table for comparison of all six means (full model) (Table 1)
And the ANOVA table for comparison of exercise group (reduced model) (Table 3), we get:

$$df_E(full) = n - k = 48 - 6 = 42$$
$$df_E(reduced) = n - k = 48 - 3 = 45$$
$$F = \frac{[SS_E(reduced) - SS_E(full)]/[df_E(reduced) - df_E(full)]}{SS_E(full)/df_E(full)}$$
$$= \frac{[4415.938 - 3343.875]/[45-42]}{3343.875/42} = \frac{1072.063/3}{79.616071} = \frac{357.354333}{79.616071} = 4.488$$

F-distribution, with $df = [Extra\ df, df_E(Full)] = [df_E(reduced) - df_E(full), n-k]$
$$df = [(n-3)-(n-6), n-6] = [(48-3)-(48-6), 48-6] = [3, 42] \approx [3, 40]$$

Thus, 0.001 < P < 0.005
<u>OR</u> P-value: $P(F_{42}^{3} > 4.488) \approx P(F_{40}^{3} > 4.488) \in (0.001, 0.005).$
There is very strong evidence against the null hypothesis.
Since P < α (0.05), reject H_0.

Conclusion: At the 5% significance level, the data provide sufficient evidence to conclude that there is a difference in mean SBP between the Typical Na groups and the Reduced Na groups, <u>after accounting for</u> the effect of exercise.

(k) **(2 marks)** Suppose a pooled *t*-test was performed to test for the difference in mean SBP between the Typical Na groups (No Ex-Typ Na, Mod Ex-Typ Na, and Vig Ex-Typ Na) and the Reduced Na groups (No Ex-Red Na, Mod Ex-Red Na, and Vig Ex-Red Na). What would be the *df*, the *t*-statistic and the <u>exact</u> P-value? You may use any of the information in the previous parts of this question to determine your answer.

This *t*-test would be testing the same thing as the *F*-test shown in the SPSS output in Table 2 (in part (j)), which also tests for the difference in mean SBP between the Typical Na groups and the Reduced Na groups. Therefore, the required values can be obtained from that table. The *df* of the *t*-test would be the same as the denominator *df* of that *F*-test, that is, $df = n - k = 48 - 2 = 46$. The *t*-statistic would be $t = \sqrt{F} = \sqrt{9.80848} = 3.132$, with an exact P-value of $P = 0.003018$.